STOCHASTIC INVESTMENT MODELS FOR ACTUARIAL USE IN THE UK

By

SULE SAHIN

Submitted for the degree of
Doctor of Philosophy
on completion of research in the
Department of Actuarial Mathematics and Statistics,
School of Mathematical and Computer Sciences,
Heriot-Watt University

October 2010

I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, Edinburgh, except where due acknowledgement is made, and has not been submitted for any other degree.

Sule Sahin (Candidate)

Professor Andrew J.G. Cairns (Supervisor)

Doctor Torsten Kleinow (Supervisor)

Professor A. David Wilkie (Supervisor)

Date

"The truth is rarely pure and never simple"

Oscar Wilde

Abstract

The objective of this thesis is to construct a stochastic term structure model for actuarial use in the UK.

The starting point of this study is the Wilkie investment model (1995). We review the Wilkie model by updating the data and re-estimating the parameters. Then, we focus on the interest rate part of the model and construct a model for the entire term structure.

We model the UK nominal spot rates, real spot rates and implied inflation spot rates considering the linkage between their term structures and some macroeconomic variables, in particular, realised inflation and output gap.

We fit a descriptive yield curve model proposed by Cairns (1998) to fill the missing values in the yield curve data provided by the Bank of England by changing the fixed parameters (exponential rates) in the model to find the best set of parameters for each data set. Once the Cairns model is fitted to the UK yield curves we apply principal component analysis (PCA) to the fitted values to decrease the dimension of the data by extracting uncorrelated variables.

Applying PCA to the fitted values we find three principal components which correspond roughly with 'level', 'slope' and 'curvature' for each yield curve. We explore the bi-directional relations between these principal components and the macroeconomic variables to construct 'yield-only' and 'yield-macro' models. We also compare the 'yield-macro' model with the Wilkie model.

To A. D. Wilkie...

Acknowledgements

First of all, I wish to express my deepest appreciation and gratitude to my supervisors Professor Andrew Cairns, Doctor Torsten Kleinow and Professor David Wilkie for their guidance, encouragement and patience throughout this study. I have benefited greatly from their wealth of experience, wisdom and insights in the course of bringing my research to fruition. They have my utmost admiration and I feel very honoured to be their student.

I also wish to thank my examiners Professor Howard Waters and Professor Rob Thomson for being very kind to me during my viva and for their valuable comments on my work and Doctor George Streftaris for his support during these four years.

I am grateful to Ali Haydar Elveren for encouraging me to do a PhD at Heriot-Watt University, Adem Yavuz Elveren for his support and kindness since we have met and Hacettepe University for being a sponsor for my PhD.

I am immensely grateful for the support of my friends Tugba Demirezen and Işil Birengel via the emails, gtalk and skype. I am also grateful to Nneoma Ogbonna, Chenming Bao and Billy (Fei) Yu for their friendship and help in my study.

I wish to thank to Erengul Ozkok, Ugur Karabey, Georgios Papageorgiou, Dave Jones, my office mates Kokouvi Gamado, Georgios Vasilopoulos, Rebecca Noonan Heale, Zoe O'Conner and all PhD students and staff in the School of Mathematical and Computer Sciences.

Thanks also deserves to go to Demet Erbaş, Erinç Engin and Steve Tanghe for making my life much easier, being so supportive and helpful and taking care of me during my PhD. They showed me that one can meet good-hearted people at any time and any place.

I also wish to thank *Superhands*, my touch rugby team. It has been a privilege for me to be a part of the team and I did enjoy every second of our games, trainings and social events. They even made me think of staying in Edinburgh after my PhD. I feel very lucky to meet all those wonderful people and play touch rugby with them. *Superhands* is the best thing ever happened to me in Edinburgh.

Scotch malt whisky distilleries namely Ardbeg, Lagavulin, Laphroaig and Coal Ila deserve my appreciation for producing the best honey coloured spirit in the world and making my life much more tasteful both during and after my PhD.

I express my utmost appreciation to my mum Sevinç Şahin, my sister Jale Şahin and my brother Tolgahan Şahin for their unconditional love and support all this time. Their unwavering encouragement gives me the strength to achieve my dreams and for that I am indebted to them.

I would like to thank to my dad Mehmet Şahin who has the greatest influence whatever I do something useful.

And finally, I am grateful to Professor Ömer Esensoy for his valuable support since I started my academic career at Hacettepe University. I wish he could see me as a doctor...

Contents

List of Figures

List of Tables

Introduction

Stochastic investment models are important components in a variety of actuarial work. They are used for risk assessment and management, valuation of liabilities, determining mismatching reserves and setting premiums in insurance contracts. Cairns (2004a) defines the stochastic investment model as a model that incorporates some or all of the following features:

- *a model for total returns; or*
- *a model for related series, which allows one to infer total returns on the asset class (for example, dividends and the dividend yield)*
- *a model for other economic variables such as interest rates, price inflation, and wage inflation in a way that includes correlation with the assets*
- *the possibility to include more than one country or economic zone and the associated exchange rate process with correlation between different countries.*

One of the earliest models which satisfies all these features is the Wilkie model (1986). The Wilkie stochastic investment model was first introduced in 1986, and it was updated and extended in 1995. Especially in the following ten years after its publication many other stochastic investment models were developed in a variety of ways including different countries (Thomson (1994), Ranne (1998), Yakoubov, Teeger and Duval (1999), Whitten and Thomas (1999), Chan (2002)).

The main purpose of this thesis is to develop a stochastic investment model considering the term structures of interest rates and implied inflation for actuarial use in the UK. This work differs from the previous ones due to modelling the three term structures, namely nominal spot rates, implied inflation spot rates and real spot rates simultaneously with the additional macroeconomic variables such as realised inflation and output gap. As far as we know, this is the first study which incorporates a time series model for the entire market-implied term structure of the implied inflation data in the literature. Since we propose a model for actuarial use we also compare our model with the Wilkie model. It should be emphisised that being arbitrage-free is not a requirement for the models developed in this thesis.

In Chapter 1 we review the Wilkie investment model using UK data, including the Retail Prices Index, both without and with an ARCH model, the wages index, share dividend yields, share dividends and share prices, long term bond yields, short term bond yields and index-linked bond yields, in each case by updating the data to June 2009. We also estimate the values of the parameters and their confidence intervals over various sub-periods to study their stability. Furthermore, we disscuss the Wilkie model from a statistical and an economical perspective. We conclude the chapter by discussing a small number of other Wilkie-type stochastic models.

In Chapter 2 we introduce the yield curve terminology by giving some basic definitions, the data and the methodology used by the Bank of England to construct the UK yield curves. Then we discuss the Cairns model as a descriptive parametric model to fit the daily spot rates of the three term structures published on the Bank of England's web page by changing the fixed parameters (exponential rates in the model) to find the best set of parameters for each data set. We try three fixed parameter sets which have been suggested by Cairns (1998) and Cairns and Pritchard (2001) and we also find one set of optimal parameters for each yield curve data. We compare how well each parameter set fits some specific dates by examining the mean squared errors. The overall aim of fitting the Cairns model is to fill in the gaps in the yield curve data.

In Chapter 3 we describe principal component analysis (PCA) and apply PCA to the fitted values obtained from the Cairns model. Thus we reduce the dimension of the yield curves by obtaining uncorrelated variables from highly correlated data. We also examine the robustness of the principal component method to the choice of exponential parameter sets for the nominal, implied inflation and real spot rates. The first three principal components which we call ‘level’, ‘slope’ and ‘curvature’ explain more than 99% of the variability in each yield curve. We use these principal components to construct the yield curve models.

In Chapter 4 we present a brief literature review on the term structure modelling and the data we use in this study. The yield-curve models developed by macroeconomists and financial economists are quite different because of different demands and motives. While macroeconomists focus on the role of expectations of inflation and future real economic activity in the determination of yields, financial economists avoid any explicit role for such determinants. These different attitudes cause a gap between the yield curve models developed. As well as various recent papers we aim to bridge this gap by developing a yield curve model considering the bi-directional relations between these yield curves and some macroeconomic variables. We use monthly, quarterly and yearly spot rates and realised inflation and ouput gap data to construct a stochastic investment model.

In Chapter 5 we introduce the ‘yield-only’ model which is based on monthly yield curve data for the period January 1985 to December 2009. We call this model a ‘yield-only’ model because an autoregressive model of order one process fits each principal component of the yield curves quite well and we do not include any macroeconomic variables into these models. Once we estimate the parameters of the models we examine the distribution of the residuals, derive the term structures using the principal components and analyse one-month ahead forecasts by constructing 95% confidence intervals for the means. Furthermore, we check whether our one-month ahead forecasts satisfy the Fisher relation and whether we can forecast one of the yield curves using the other two.

In Chapter 6 we present two ‘yield-macro’ models using both quarterly and yearly data. When we use quarterly data we find that the output gap is significant as an explanatory variable in some of the yield-curve models. Due to the process of revision the latest output gap data available is that for the end of 2007 in OECD Economic Outlook Publications. Thus we use the data for the period 1995-2007 to construct the ‘yield-macro model-I’ based on quarterly data. According to our analysis the output gap and realised inflation affect the slope factor of the nominal interest rates while realised inflation also affects the curvature factor of the nominal interest rates. Furthermore, these two yield curve factors have been found significantly important to explain the realised inflation and output gap as well. Therefore we conclude that there is a bi-directional relation between the yield curve factors and the macroeconomic variables. Secondly, we use yearly data including the realised inflation and output gap for each month starting from January 1985 and ending with December 2009 for the 'yield-macro model-II'. We model only the 'level' factors of the yield curves on a yearly frequency and note that the realised inflation has been significant in the level factors of the three yield curves. Since we construct a model for each month we develop twelve different models at a yearly frequency. We also try to explain the economic rationale behind the correlations between the variables, examine the fitted vector autoregressive models and their residuals, and compare the models with the random walk and AR(1) process in terms of explained variability in the data as well as one-period ahead forecasts and the Fisher relation check.

In Chapter 7 we compare the quarterly yield-macro model with the Wilkie model in both philosophical and empirical ways. First, we discuss the structural similarities and differences between the models. Then we compare the models by analysing the simulated economic series, nominal and real returns based on different asset classes and the asset values and the annuity payoffs considering a hypothetical pension scheme.

Finally, In Chapter 8 we present our conclusions and ideas for further research.

Chapter 1

Revisiting the Wilkie Model

1.1 Introduction

The Wilkie stochastic investment model, developed by A. D. Wilkie, is described fully in two papers: the original version is described in 'A Stochastic Investment Model For Actuarial Use' (Wilkie, 1986) and the model is reviewed, updated and extended in 'More On A Stochastic Asset Model For Actuarial Use' (Wilkie, 1995).

The original Wilkie model (1986) was developed from U.K. data over the period 1919-1982, and was made up of four interconnected models for price inflation, share dividend yields, share dividends and long-term interest rates. Wilkie (1995) updated the original model and extended it to include an alternative autoregressive conditional heteroscedastic (ARCH) model for price inflation, and models for wage inflation, short-term interest rates, property yields and income and index-linked yields. Furthermore, these models were fitted to data from numerous developed countries and an exchange rate model was proposed.

Hardy (2003) describes the Wilkie model as a multivariate model, meaning that several related economic series are projected together. This is very useful for applications that require consistent projections of, for example, stock prices and inflation rates or fixed interest yields. It is designed for long-term actuarial applications such as simulating assets of financial institutions over many years in the future to study the risk of insolvency. Since the model is designed to be applied to annual data it is not suitable in that form for assessing short-term hedging strategies.

There is a large number of papers such as Kitts (1990), Clarkson (1991), Geohegan et al. (1992), Ludvic (1993), Harris (1995), Huber (1997), Rambaruth (2003), Hardy (2004), Nam (2004), Lee and Wilkie (2000) and books such as Daykin *et al.* (1994), Booth *et al.* (1999), Hardy (2003) which describe, compare or criticise the Wilkie Model. Furthermore, the discussions attached to Wilkie's 1986 and 1995 papers might be considered as important references for comments on the Wilkie model. Especially in the 'Abstract of Discussion' part of the 1995 paper there are various comments and criticisms about the model from twenty academics and practitioners who examined and applied the model or developed new models which followed in the footsteps of Wilkie (1986, 1995).

In this chapter, we review the Wilkie investment model only for UK data, including the Retail Prices Index, both without and with an ARCH model, the wages index, share dividend yields, share dividends and share prices, long term bond yields, short term bond yields and index-linked bond yields, in each case by updating the parameters to June 2009 in Section 1.3 to 1.10. We also estimate the values of the parameters and their confidence intervals over various sub-periods to study their stability. This chapter is based on mainly two joint papers: one is a conference paper (Sahin *et al.*, 2008), 'Revisiting the Wilkie Investment Model', which was presented in the 18th International AFIR Colloquium in Rome, September 30th - October 3rd 2008, and the other (Wilkie *et al.*, 2010), 'Yet More on a Stochastic Economic Model: Part 1: Updating and Refitting, 1995 to 2009', which has been submitted to Annals of Actuarial Science in February 2010. Additionally, we discuss the Wilkie model from a statistical and an economical perspective in Section 1.11 while omitting the forecasting performance of the models which has been discussed in the later paper. Section 1.12 introduces a number of Wilkie-type stochastic models briefly. Finally, Section 1.13 concludes the chapter.

1.2 Structure and Methodology of the Model

The Wilkie investment model is based on Box-Jenkins (1976) time series models. The parameters are estimated by using least square estimates or maximum likelihood method (which gives the same results under the 'normally distributed residuals' assumption) calculated by a non-linear optimization method, the Nelder-Mead simplex method. Almost all models are stationary or integrated of autoregressive order one, AR(1) or ARIMA(1,1,0). Some of the series are treated as if co-integrated. For example, the difference between the logarithm of the share dividends and share prices gives the logarithm of the share dividend yields, i.e. these two series are co-integrated.

The series in the Wilkie model are correlated and could be modelled simultaneously by multivariate analysis, using vector autoregressive models (VAR). The model in fact started as a straightforward VAR model but after crossing out a great many non-significant values, it was simplified to a cascade model. Figure 1.1 illustrates the cascade structure of the model where the arrows indicate the direction of influence. One can see from the figure that the complete model is wholly self-contained. The only inputs are the separate white noise series, and no exogenous variables are included.

1.3 Retail Prices

The most recent series used for the Retail Prices Index is the one called RPI, and not any of the other alternative series produced for the UK in recent years. The model for the U.K. Retail Prices Index (RPI) where $Q(t)$ is the value of a retail price index at time t, is:

$$Q(t) = Q(t-1).\exp\left(I(t)\right) \tag{1.1}$$

so that $I(t) = \ln Q(t) - \ln Q(t-1)$ is the force of inflation over the year $(t-1, t)$.

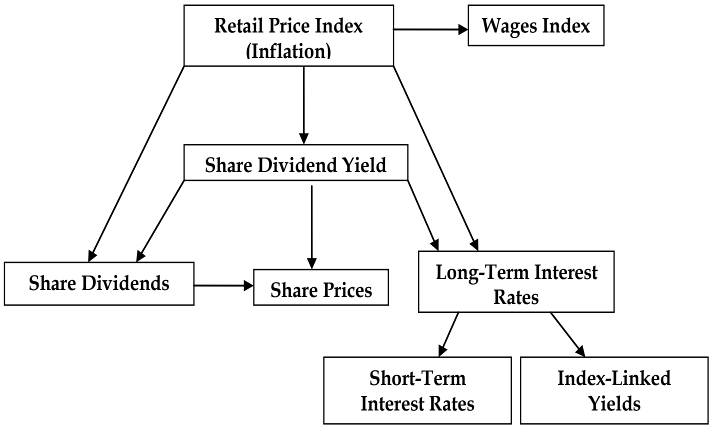

Figure 1.1: Structure of the Wilkie model

The force of inflation $I(t)$, which is defined as the difference in the logarithms of the RPI each year, is modelled as a first order autoregressive series. An AR(1) model is a statistically stationary series for suitable parameters, which means that in the long run the mean and variance are constant.

$$
\begin{aligned}
I(t) &= QMU + QA.(I(t-1) - QMU) + QE(t) \qquad (1.2)\\
QE(t) &= QSD.QZ(t)\\
QZ(t) &\sim (iid)N(0,1)
\end{aligned}
$$

that is *QZ(t)* is a series of independent, identically distributed unit normal variates.

The model states that each year the force of inflation is equal to its mean rate, QMU, plus some proportion, QA, of last year's deviation from the mean, plus a random innovation which has zero mean and a constant standard deviation, QSD.

The force of inflation, $I(t)$ from 1923 to 2009 is displayed in Figure 1.2. One can observe from the figure that there was a fall in prices after the First World War, and big rises during the Second World War and the late 1970s and early 1980s. The inflation has been positive since the 1960s and especially in the last 15 years it seems to have been low and stable. However, for the year ending June 2009 the value of I(t) was

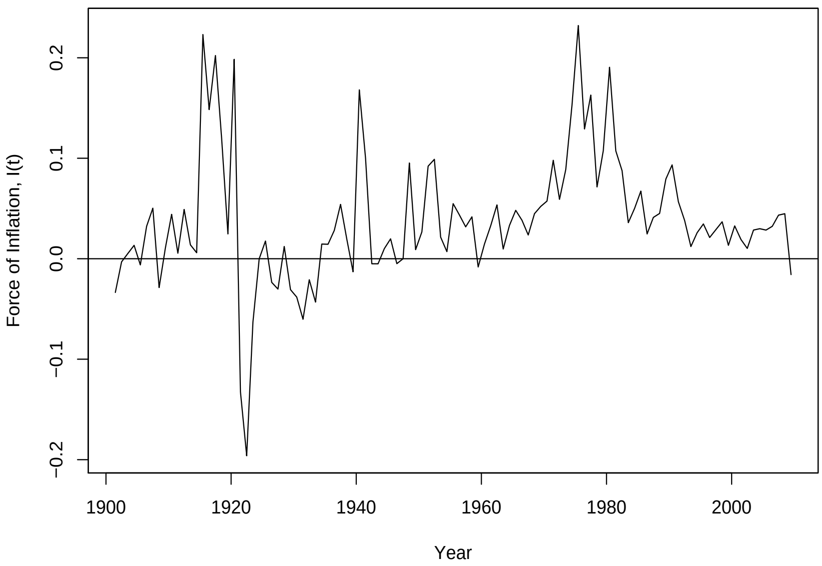

Figure 1.2: Annual force of inflation, $I(t)$, 1900-2009

negative, for the first time since 1959, and by a larger amount negative than in any year since 1933.

1.3.1 Updating and Rebasing to 1923-2009

We updated the data and re-estimated the parameters of the price inflation model for the whole period, 1923-2009. In Table 1.1 we compare these with those that were estimated in 1995. We also show some statistics from both periods: first, the first autocorrelation coefficient of the residuals, the values of $QZ(t)$, denoted $r_{(QZ)_1}$; then the first autocorrelation coefficient of the squares of the residuals, the values of $QZ(t)^2$, denoted $r_{(QZ^2)_1}$; next the skewness and kurtosis coefficients of the residuals, denoted $\sqrt{\beta_1}$ and β_2; finally the Jarque-Bera χ^2 statistic, equal to the sum of the squares of the skewness and kurtosis coefficients, in each case divided by the squares of their standard errors, together with the probability of such a large value of χ^2_2 being observed.

Table 1.1: Estimates of parameters and standard errors of AR(1) model for inflation over 1923-1994 and 1923-2009

I(t)	1923-1994	1923-2009
QMU	0.0473 (0.0119)	0.0429 (0.0101)
QA	0.5773 (0.0799)	0.5779 (0.0744)
QSD	0.0427 (0.0036)	0.0397 (0.0030)
$r_{(QZ)_1}$	-0.0057	-0.0060
$r_{(QZ^2)_1}$	0.0421	0.0691
skewness $\sqrt{\beta_1}$	1.1298	1.2521
kurtosis β_2	5.1126	5.9672
Jarque-Bera χ^2	33.09	54.65
$p(\chi^2)$	0.0000	0.0000

Possible rounded values for practical use, based on the past experience, might be:

$$QMU = 0.043; \; QA = 0.58; \; QSD = 0.04$$

.

However, the recent experience suggests that a lower mean value, such as $QMU = 0.025$, might be more appropriate for the future (Wilkie, *et.al.*, 2010). Since the path of inflation may be very uncertain in the long run, we would not recommend reducing the standard deviation except perhaps in the short term.

Table 1.1 shows that the estimated parameters over the two periods have not changed significantly. QMU and QSD have slightly decreased and QA has slightly increased. Standard errors (in brackets) show that all the parameters are significantly different from zero. When we compare these two periods by examining the diagnostic tests, it can be concluded that there is no significant improvement on the model based on the updated data. The residuals, the observed values of QE, are calculated for both periods. The autocorrelation coefficients of the residuals and squared residuals show nothing unusual, i.e. residuals can be considered to be independent and there is no simple ARCH effect. However, the skewness and kurtosis coefficients, based on the third and forth moments of the residuals, are rather large: $\sqrt{\beta_1} = 1.1298$ and 1.2521, demonstrating substantial positive skewness; and $\beta_2 = 5.1126$ and 5.9672, implying quite heavy tails in the distribution.

The Jarque-Bera test also shows significant non-normality. The test statistics are 33.09 and 54.65 for the two periods, which should be compared with a χ^2 variate with two degrees of freedom. The p-values are zero and therefore, the probability that such a result would occur at random is negligible.

1.3.2 Parameter Stability

Huber (1997) suggested that, when parameters are estimated over different periods, very different values may be obtained, indicating that the values of the parameters are not stable. We investigate the parameter constancy of the models by recursively estimating the parameters on incrementally larger data sets. Figure 1.3, 1.4 and 1.5 present these recursive estimates and 95% confidence intervals of QMU, QA and QSD, respectively, for earlier sub-periods (data sets starting in 1923) and later sub-periods (data sets ending in 2009). In the figures, solid lines show the parameter estimates and the dotted lines show the 95% confidence intervals. These are based on an assumption that the parameter value is distributed normally, and are calculated as the estimated value plus or minus 1.96 times the calculated standard error. Sub-periods with fewer than 10 observations are omitted in this case.

We explain the graphs by using Figure 1.3, for QMU, as an example. The middle bold solid line shows the estimated values of QMU for periods starting in 1923 and ending in the given year. It begins with the period ending in 1932, for which there are 10 years of data from which to estimate the parameters. Over this period we can see that the estimated value of QMU is negative for the first 10 to 17 years (1923-1939) which reflect the negative inflation of that inter-war period. We can observe that QMU tends to increase over most of the period, including two jumps in the early 1940s and the mid 1970s due to the effects of the Second World War and the oil crisis. After 1980, it drifts slightly down, ending in 2009 at 0.0429, as shown in Table 1.1. The bold dotted lines on either side of the bold solid line show approximate 95% confidence intervals for the corresponding value.

The middle thinner solid line in Figure 1.3 shows the estimated values of QMU for periods ending in 2009. This line commences in 1923 at the value 0.0429, being the value for the whole period 1923-2009. The line rises gently as the earlier years of negative or low inflation are omitted, and reaches a peak at 0.0597 in 1968. For the most recent years it declines quite sharply, ending at 0.0261 for 2000, the last year for which we have 10 years date ending in 2009. The thinner dotted lines on either side of the thinner continuous line show approximate 95% confidence intervals for the corresponding value. As the periods shorten one would expect the confidence intervals to widen, being based on fewer observations. In fact they do the opposite. At the right hand end the confidence limits are quite close to each other. This may suggest that during a period of low inflation the uncertainty of prices is lower and inflation is therefore stable. This is most obviously seen from the very low values for QSD in the recent periods seen in Figure 1.5.

Since we use the same data periods for the right hand end of the bold solid line and the left hand end of the thinner solid line (i.e. data over the period 1923 to 2009), we have exactly the same parameter values and confidence limits at these points.

It can be seen that the estimated values of QMU are fairly far apart in the earlier years and cross over in 1977. However, the confidence intervals overlap for all years from 1940 onwards. Further a value of 0.030 lies within both confidence intervals for QMU from 1949 onwards.

Figure 1.4 shows the same features for the autoregressive parameter, QA. Coming forward from 1923, we see rather low values, starting at around 0.2. The parameter value jumps in the mid-1970s, from a value of 0.4 to a value of 0.6. Before and after this period it seems stable. Then reducing the periods, but keeping the end point at 2009, the 0.6 value is apparent for a long period, but in the most recent years the value has dropped, to well below zero. The confidence intervals overlap for almost all the periods shown, but are comparatively wide, especially for the most recent years.

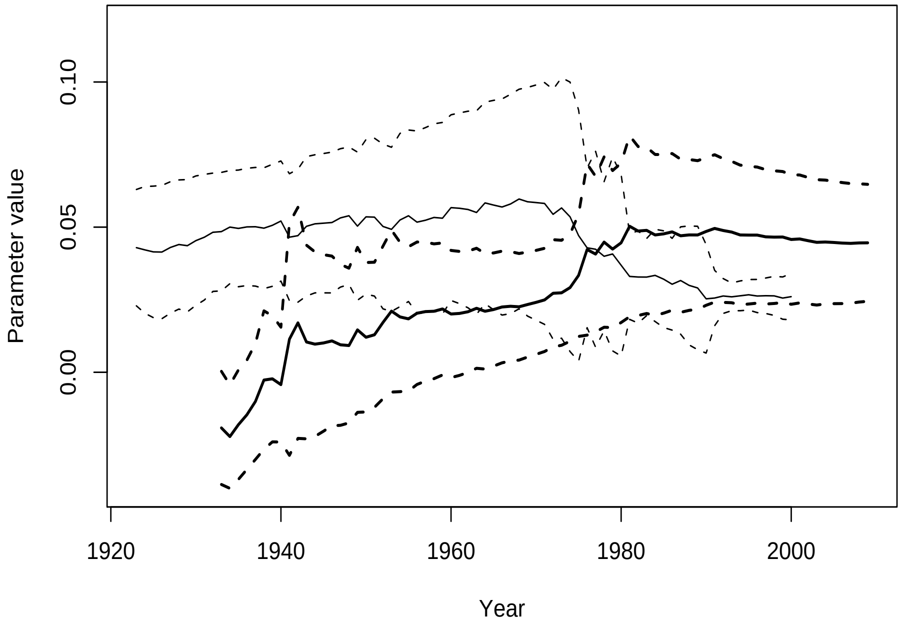

Figure 1.3: Estimates for parameter QMU for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

When inflation is very stable, and has a low variability, any autoregressive tendency that might be observed when rates are much higher cannot be identified.

Figure 1.5 shows the recursive estimates of the standard deviation, QSD, of the inflation model. The most obvious feature is how much lower the values have been in recent years, and how narrow the confidence interval has also been. This suggests that a 'regime switching' model might reflect the facts rather better than a model with fixed parameters. The parameter values estimated for the period starting in 1923 indicates that there are two jumps: one is in the early 1940s and the other is in the mid 1970s. The steadily increasing structure of the parameter values and the two jumps due to the Second World War and oil crises cause bigger jumps in the volatility of the inflation. The confidence intervals are wide around these jumps and they get smaller as we increase the data period. When we look at the right hand end, in which we used the latest years' values to estimate the parameter, we see that the standard deviation is very small (about 0.01) and the confidence limits are very close to each other. This

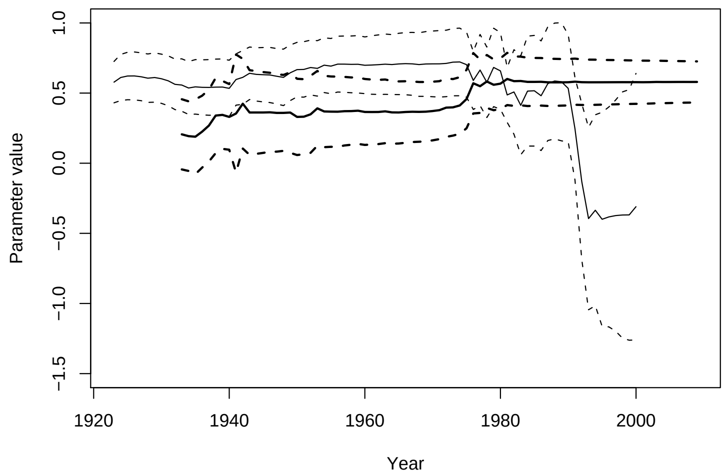

Figure 1.4: Estimates for parameter QA for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

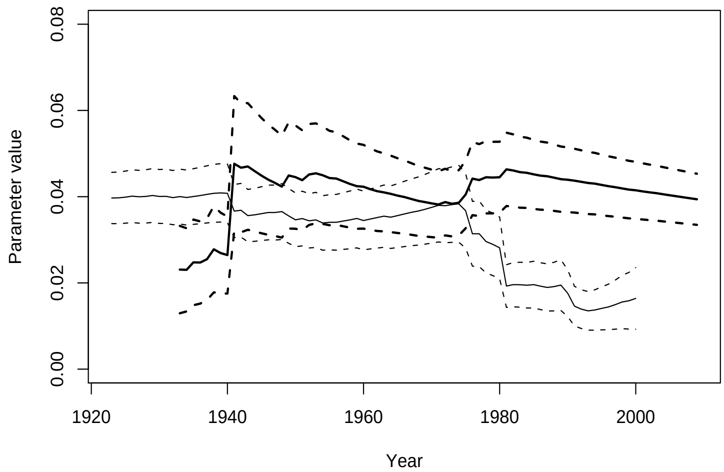

Figure 1.5: Estimates for parameter QSD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

result shows that during a low inflation period it is easier to predict the rate and there is decreased uncertainty.

1.4 An ARCH Model for Inflation

Although Wilkie (1986) initially assumed that the residuals of the inflation model were normally distributed, he observed in 1995 that they are much fatter tailed than normal distribution. One of the ways to model these fat tailed distributions is using an Autoregressive Conditional Heteroscedastic (ARCH) model (Engle, 1982). Wilkie (1995) proposed an ARCH model for the standard deviation of the inflation model. In this ARCH model the varying value of the standard deviation, $QSD(t)$, is made to depend on the previously observed value of the principal variable, $I(t-1)$, which itself is modelled by an autoregressive series. The suggested model (with a slight alteration in the notation) was:

$$\begin{aligned} I(t) &= QMU + QA.(I(t-1) - QMU) + QE(t) \qquad (1.3) \\ QE(t) &= QSD(t).QZ(t) \\ QSD(t)^2 &= QSA^2 + QSB.(I(t-1) - QSC)^2 \\ QZ(t) &\sim (iid)N(0,1) \end{aligned}$$

Thus the variance depends on how far away last year's rate of inflation, $I(t-1)$, was from some middle level, QSC (similar to the mean, QMU), but with the deviation squared, so that extreme values of inflation in either direction would increase the variance.

1.4.1 Updating and Rebasing to 1923-2009

Estimates of values of the parameters for the period from 1923 to 2009 are shown in Table 1.2, along with estimates of the values already found for the basic inflation model, in which $QSB = 0$ and QSD is a constant equalling QSA. We show two ARCH models, one with QSC free, the other with $QSC = QMU$. Since the log likelihoods and the parameter estimates for these two models are very close, we prefer the $QSC = QMU$ which has one less parameter to estimate. Although the log likelihood for the ARCH model is distinctly better than for the basic inflation model, the skewness and kurtosis are little changed. This shows that even with an ARCH model, the residuals for inflation are considerably fatter-tailed than normal.

As opposed to re-estimating the parameters, changing the structure of the Wilkie model is not an objective of this chapter (apart from the real yield, R). Therefore, we do not consider, for example, a regime switching model for inflation or an AR(1) model with fat-tailed noise, despite the evidence from the statistical tests in Table 1.1 and Table 1.2 that other models might have their merits.

Table 1.2: Estimates of parameters and standard errors of model for inflation, using an ARCH model, and relevant statistics, over different periods

$I(t)$	1923-1994 $QSC = QMU$	1923-2009 Basic $QSB = 0$	1923-1994 QSB and QSC free	1923-2009 $QSC = QMU$
QMU	0.0404	0.0429	0.0369	0.0352
	(0.0108)	(0.0101)	(0.0082)	(0.0080)
QA	0.6179	0.5779	0.5938	0.5930
	(0.1292)	(0.0744)	(0.1306)	(0.1291)
QSA $(= QSD)$	0.0256	0.0397	0.0227	0.0227
	(0.0150)	(0.0030)	(0.0032)	(0.0032)
QSB	0.5224		0.6345	0.6336
	(0.2147)		(0.2217)	(0.2149)
QSC	0.0404		0.0345	0.0352
			(0.0054)	
$r_{(QZ)_1}$		-0.0060	-0.0229	-0.0221
$r_{(QZ^2)_1}$		0.0691	0.0680	0.0674
skewness $\sqrt{\beta_1}$		1.2521	1.2303	1.2314
kurtosis β_2		5.9672	5.9294	5.9312
Jarque-Bera χ^2	5.76	54.65	53.06	53.13
$p(\chi^2)$	0.056	0.0000	0.0000	0.0000
Log likelihood		237.22	246.25	246.22

Possible rounded values for practical use, based on the past experience, might be:

$$QMU = 0.035;\ QA = 0.59;\ QSA = 0.227,\ QSB = 0.63,\ QSC = QMU = 0.035$$

.

However, a value of $QMU = 0.025$ might be preferred, as we have suggested in the previous section.

1.4.2 Parameter Stability

The parameter constancy of the models can be examined by recursively estimating the parameters on incrementally larger data sets as we have done in Section 1.3.

When we try to estimate the ARCH model of shorter subperiods, we often find that the estimated value of QSB is very small but negative. This is inconsistent because it would produce cases in simulations where the variance, QSD^2, was negative, as could happen also if QSA were negative. If the estimate of QSB is negative we can set it to zero, and revert to the non-ARCH model for inflation, with $QSD = QSA$.

In the graphs for subperiods, shown in Figures 1.6, 1.7, 1.8 and 1.9 for QMU, QA, QSA and QSB, we show the values of the non-ARCH model for the first three parameters, and omit the value of QSB if it has been set to zero. One can see that this happens for all subperiods starting in 1923 and ending before 1975, and also for the subperiod starting in 1981 and ending in 2009. However, for every subperiod starting after 1985 and ending in 2009 the estimated value of QSB is greater than 1, so the value of $QSD(t)^2$ would, in the long run, tend to infinity, and the model is unstable.

Figures 1.6 and 1.7 show that the mean and the autoregressive parameters of the ARCH model are similar to the corresponding parameters of the $AR(1)$ inflation model. When we look at Figures 1.8 and 1.9, we see two parameters which make the difference between the $AR(1)$ and ARCH inflation models. Therefore, it is useful to interpret these two parameters together. When QSB is set to 0, these two models become identical. The graph of QSA is similar to its equivalent in the $AR(1)$ model, QSD, until the 1970s. There are several jumps and two of them are significant: one is in the late 1930s and the other is in the early 1970s. On the other hand, QSB estimates for the earlier sub-periods are almost zero until the early 1970s which indicates that the $AR(1)$ model is enough to model the rate of inflation until this year. There is a sharp decrease in the QSA estimates for the earlier sub-periods after 1970s while there is significant increase in QSB estimates in early 1970s. This might indicate that after the first oil crises the $AR(1)$ model is not sufficient and through the QSB parameter the ARCH effect comes into the model. Taking this into account decreases QSA and stabilises it for the rest of the sub-periods. Besides, QSA estimates for the later sub-periods are quite stable except for two specific jumps and QSB estimates for the later sub-periods are informative just after the sub-periods including 20 or more years.

To conclude, it is only in the periods that include the 1960s and 1970s that the ARCH model is a useful description.

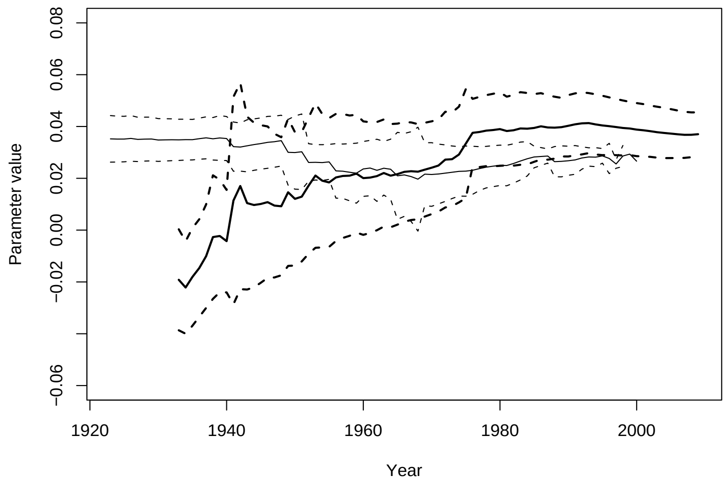

Figure 1.6: Estimates for parameter QMU for ARCH model for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

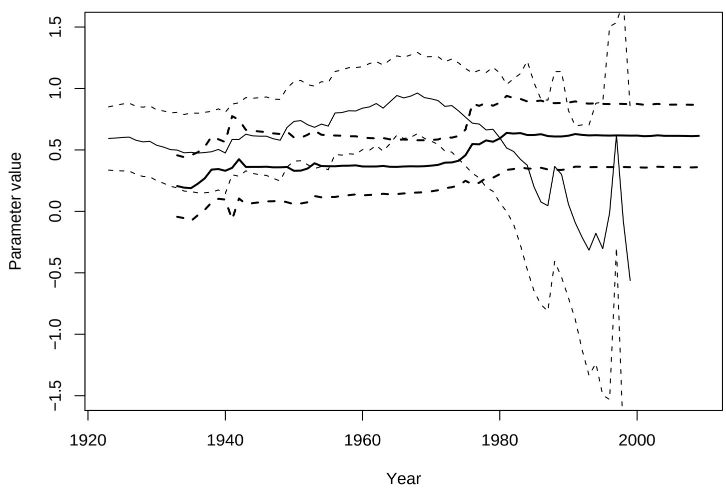

Figure 1.7: Estimates for parameter QA for ARCH model for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

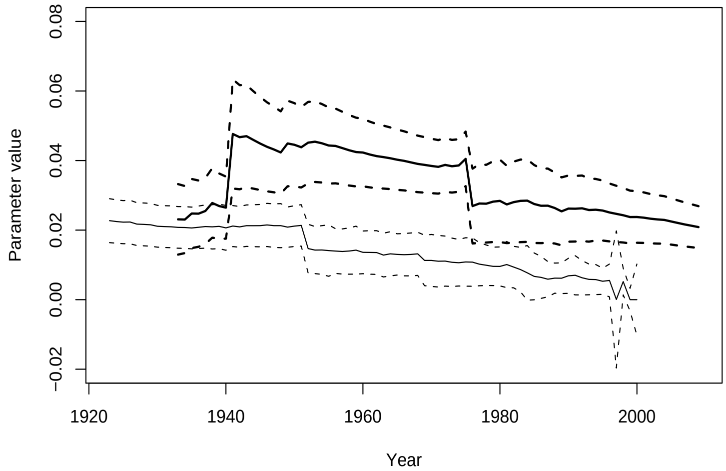

Figure 1.8: Estimates for parameter QSA for ARCH model for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

Figure 1.9: Estimates for parameter QSB for ARCH model for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

1.5 Wages Index

A series of indices, ending with the index for Monthly Earnings, All Employees, not seasonally adjusted have been used for the wages model. We use the same notation as Wilkie (1995), denoting the wages index at time t as $W(t)$, and the force of wage inflation over the year $t-1$ to t as $J(t)$, calculated as

$$J(t) = lnW(t) - lnW(t-1) \tag{1.4}$$

so that $W(t) = W(t-1). \exp J(t)$.

Figure 1.10 shows the values of both the price, $I(t)$, and the wage, $J(t)$, inflations. These two have been quite similar over the period, especially since 1923. Since 1994, like price inflation, the wage inflation has been at a much lower, and more stable, level than in previous years.

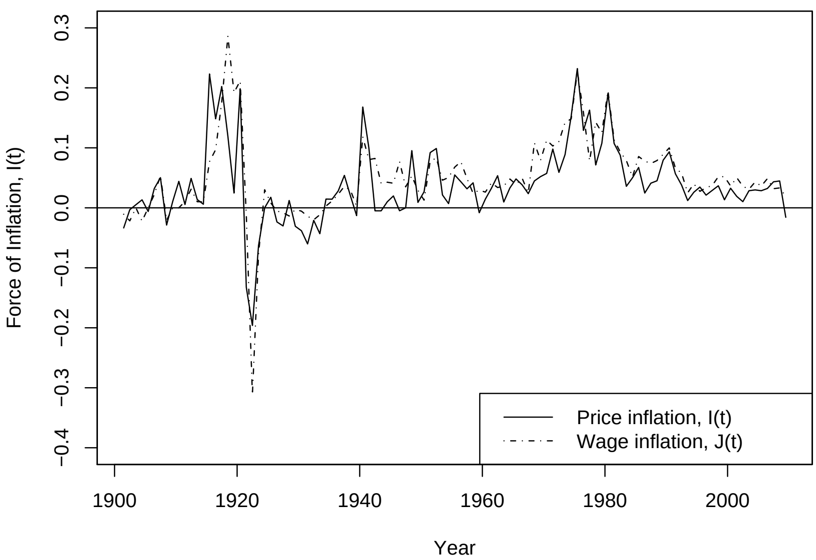

Figure 1.10: Price inflation, $I(t)$, and Wage inflation, $J(t)$, 1900-2009

1.5.1 Updating and Rebasing to 1923-2009

Wilkie (1995) proposed several models including AR(1), transfer function models and vector autoregressive models for wages. By examining all these models, he chose the transfer function model as the most suitable and we re-estimated the parameters for this model on updated data.

The model for $J(t)$ suggested in 1995 can be written as:

$$\begin{aligned} J(t) &= WW1.I(t) + WW2.I(t-1) + WMU + WN(t) \quad (1.5) \\ WN(t) &= WA.WN(t-1) + WE(t) \\ WE(t) &= WSD.WZ(t) \\ WZ(t) &\sim (iid)N(0,1) \end{aligned}$$

Two sets of values of the parameters were suggested in Wilkie (1995), based on the experience from 1923 to 1994. In both the value of WA was taken as zero. In one (Model W1) the values of the other parameters were: $WW1 = 0.60$; $WW2 = 0.27$; $WMU = 0.021$; $WSD = 0.0233$. In the other (Model W2): $WW1 = 0.69$; $WW2 = 1 - WW1 = 031$; $WMU = 0.016$; $WSD = 0.0244$. Setting $WW2 = 1 - WW1$ enables us to get 'unit gain' from prices to wages, i.e. an unexpected change in prices produces a corresponding change in wages in the long run, so that real wages are not significantly influenced by the level of inflation.

As we have done for the inflation models, we re-estimate the parameters for the whole period, 1923-2009 for wages too. We do this for four different models, with WA free or set to zero, and with $WW2$ free or set to $1 - WW1$. In Tables 1.3 and 1.4 we compare these with those that were estimated in 1995.

We can observe that the addition of the WA term improves the log likelihood by very little, and in one of the cases it increases the Jarque-Bera statistics, and further that the value of WA is not significantly different from zero. So the WA term can be omitted. We also see that there is not a very big difference between the model with

Table 1.3: Estimates of parameters and standard errors of two models for wages, with $WA = 0$, and relevant statistics, over different periods

Model W1	$WW2$ $free$		$WW2 = 1 - WW1$	
	1923-1994	1923-2009	1923-1994	1923-2009
$WW1$	0.6021	0.6020	0.6878	0.6843
	(0.0645)	(0.0592)	(0.0572)	(0.0509)
$WW2$	0.2671	0.2693	0.3122	0.3157
	(0.0577)	(0.0535)		
WMU	0.0214	0.0200	0.0159	0.0150
	(0.0035)	(0.0030)	(0.0029)	(0.0032)
WSD	0.0233	0.0219	0.0244	0.0228
	(0.0020)	(0.0017)	(0.0020)	(0.0019)
$r_{(WZ)_1}$	0.1860	0.1950	0.1780	0.1833
$r_{(WZ^2)_1}$	-0.0068	0.0407	-0.0094	0.0203
skewness $\sqrt{\beta_1}$	0.0147	0.1034	-0.3887	-0.3081
kurtosis β_2	3.6555	3.9692	4.6695	5.0074
Jarque-Bera χ^2	1.29	3.56	10.17	15.98
$p(\chi^2)$	0.52	0.17	0.0062	0.0003
Log likelihood	234.54	288.88	231.48	285.43

$WW2$ free and the one with $WW2 = 1 - WW1$ as in Wilkie (1995). However, the fit on both occasions is not so good. Therefore, there is good reason to prefer the model with $WW2$ free, even though this does not give a 'unit gain' from inflation to wages.

Possible rounded values for practical use, based on the past experience, might be:

$$WW1 = 0.60;\ WW2 = 0.27;\ WMU = 0.020;\ WSD = 0.0219$$

.

or alternatively

$$WW1 = 0.68;\ WW2 = 0.32;\ WMU = 0.015;\ WSD = 0.0228$$

.

In both suggested models, we omit the WA term. Furthermore, these are almost the same as those suggested in Wilkie (1995). In the first model, when $WW2$ is free, the kurtosis coefficient is not exceptionally large, and the Jarque-Bera statistic

Table 1.4: Estimates of parameters and standard errors of two models for wages, with WA $free$, and relevant statistics, over different periods

Model W2	$WW2$ $free$		$WW2 = 1 - WW1$	
	1923-1994	1923-2009	1923-1994	1923-2009
$WW1$	0.5824	0.5806	0.6871	0.6828
	(0.0643)	(0.0592)	(0.0554)	(0.0547)
$WW2$	0.2467	0.2495	0.3129	0.3172
	(0.0587)	(0.0543)		
WMU	0.0235	0.0220	0.0161	0.0151
	(0.0043)	(0.0037)	(0.0032)	(0.0035)
WA	0.1489	0.01525	0.0908	0.0948
	(0.0944)	(0.0873)	(0.0946)	(0.0870)
WSD	0.0229	0.0215	0.0242	0.0226
	(0.0019)	(0.0016)	(0.0020)	(0.0019)
$r_{(WZ)_1}$	0.0546	0.0603	0.0989	0.1001
$r_{(WZ^2)_1}$	0.0627	0.0991	0.0335	0.0633
skewness $\sqrt{\beta_1}$	0.1186	0.2385	-0.3447	-0.2660
kurtosis β_2	3.7418	4.0799	4.6329	4.9732
Jarque-Bera χ^2	1.82	5.05	9.42	15.14
$p(\chi^2)$	0.40	0.0800	0.0090	0.0005
Log likelihood	235.77	290.39	231.94	286.02

is acceptable. In the other model, the high value of the kurtosis coefficient indicates that the residuals are not close to being normally distributed, though they are less far away than the inflation residuals, partly because the values of inflation are already included in the formula, and the wages residuals represent variation over and above the variation due to inflation.

1.5.2 Parameter Stability

The graphs of the estimated values of the parameters over various sub-periods, those starting in 1923 and those ending in 2009, for the parameters, $WW1$, $WW2$, WMU and WSD, are displayed in Figures 1.11, 1.12, 1.13 and 1.14 respectively. We do this only for our preferred model, with $WW2$ free and $WA = 0$.

In Figure 1.11 we can see that the estimates for $WW1$ for periods starting in 1923, the bold continuous line, are reasonably constant, except for the jump at the early

1940s, whereas those for periods ending in 2009, the thinner continuous line, drop quite sharply in the most recent years. The same is true for the estimates for $WW2$, which is even more stable in the earlier years. The considerable reduction in these two factors in recent years is consistent with rather stable increases in both prices and wages, which gives the impression that the two series have little connection, even though the connection is very strong when inflation is high.

The charts for WMU are reasonably stable too, except that the estimated value has risen in the most recent years. This compensates for the reduction in $WW1$ and $WW2$; if wage increases are not dependent on inflation, from which they would obtain roughly the mean increase in prices, they must have their own, larger, mean increase.

The charts for WSD, however, are much less stable, and show much reduced values for the shorter recent periods which is consistent with the much more stable pattern of wages increases in recent years.

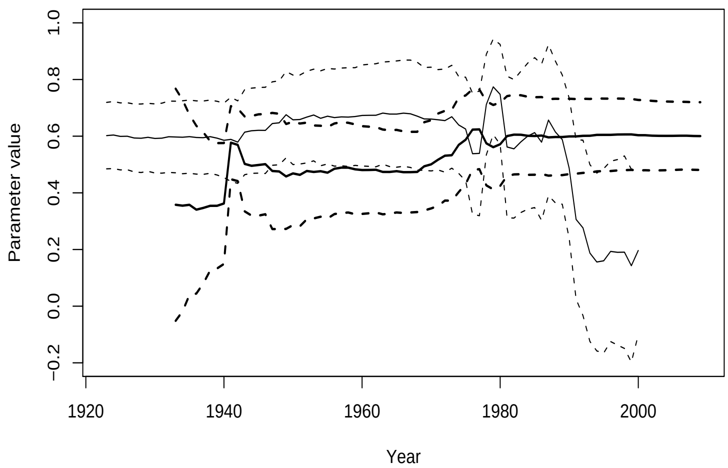

Figure 1.11: Estimates for parameter $WW1$ for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

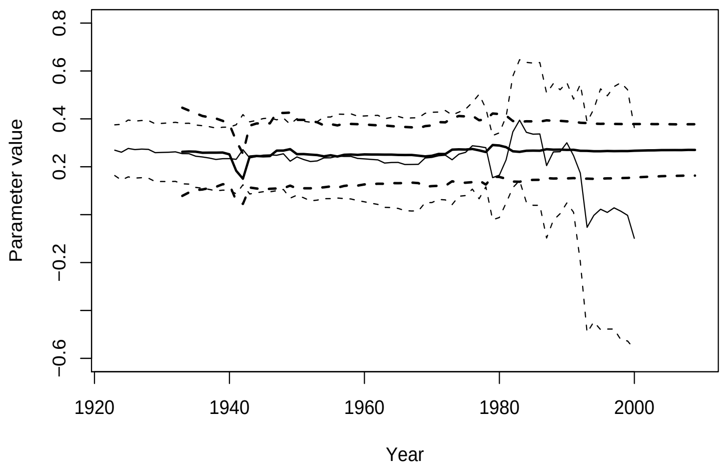

Figure 1.12: Estimates for parameter $WW2$ for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

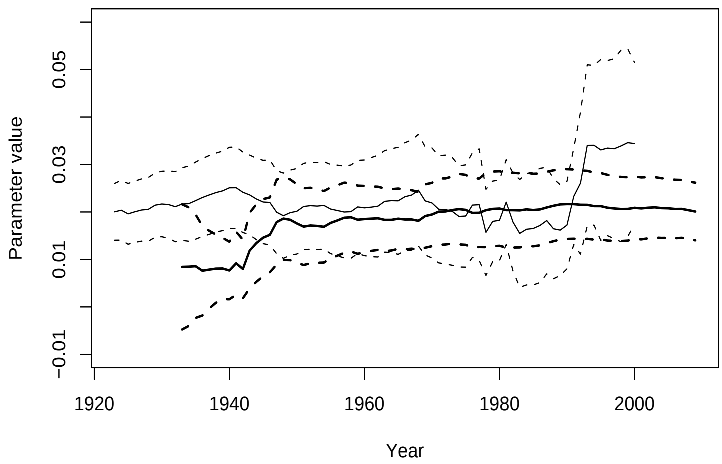

Figure 1.13: Estimates for parameter WMU for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

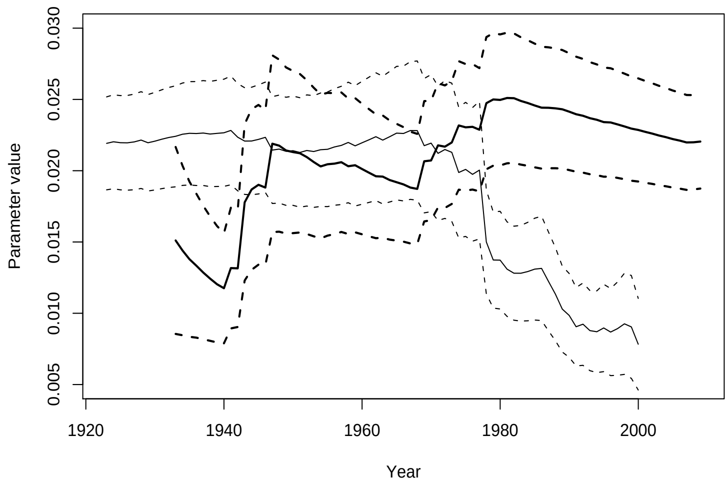

Figure 1.14: Estimates for parameter WSD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

1.6 Share Dividend Yields

The share dividend yield is based on number of indices, since 1962 on the FTSE-Actuaries All-Share Index. The yield for most of the period has been based on the gross dividend index, i.e. gross of income tax, which non-tax paying investors, such as U.K. pension funds, could reclaim (see Wilkie *et al.*, 2010 for details).

The dividend yield, $Y(t)$, is shown in Figure 1.15, at annual intervals, from 1919 to 2009. One can see that it reached very low levels during the late 1990s, but has risen recently and is now above its long run mid-point of around 4%.

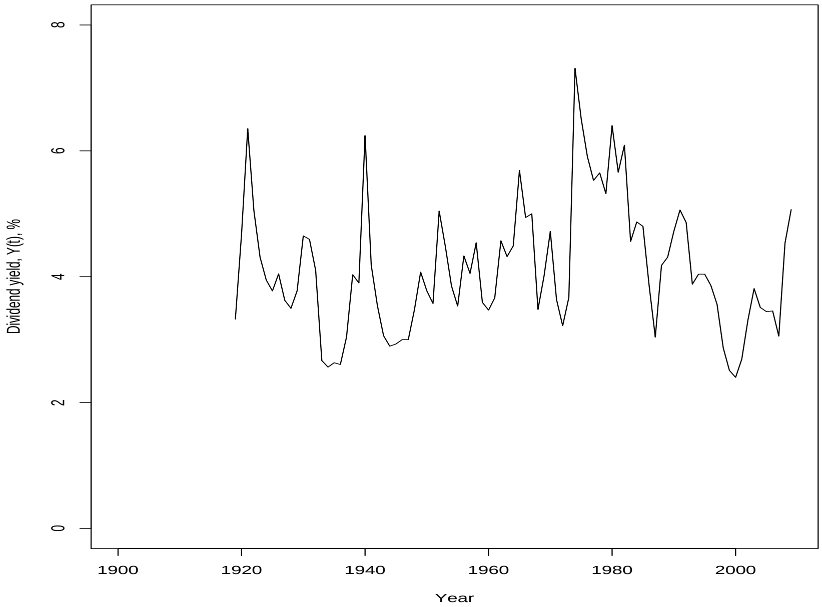

Figure 1.15: Share dividend yield, Y(t), %, 1919-2009

The original model for $Y(t)$ was:

$$\begin{aligned} \ln Y(t) &= YW.I(t) + YMU + YN(t) \qquad (1.6) \\ YN(t) &= YA.YN(t-1) + YE(t) \\ YE(t) &= YSD.YZ(t) \\ YZ(t) &\sim (iid)N(0,1) \end{aligned}$$

1.6.1 Updating and Rebasing to 1923-2007

In Table 1.5 we compare the parameters estimated for the whole period, 1923-2009, with those that were estimated in 1995, along with the usual statistics.

Table 1.5: Estimates of parameters and standard errors of models for dividend yield, and relevant statistics, over different periods

	1923-1994	1923-2009
YW	1.7940	1.5466
	(0.5862)	(0.4590)
YMU	3.77%	3.72%
	(0.18%)	(0.18%)
YA	0.5492	0.6297
	(0.1013)	(0.0854)
YSD	0.1552	0.1570
	(0.0129)	(0.0119)
$r_{(YZ)_1}$	0.0778	0.1055
$r_{(YZ^2)_1}$	0.0421	-0.0618
skewness $\sqrt{\beta_1}$	-0.1024	0.3798
kurtosis β_2	3.0944	3.3381
Jarque-Bera χ^2	0.63	2.51
$p(\chi^2)$	0.73	0.29

It can be observed that the values of the YMU and YSD are almost unchanged, while YW is reduced and YA is increased. However, all the new parameter estimates are within, or not much above, one standard deviation away from the original estimates (based on 1923-1994), so there is no strong evidence of a change in the parameters of the model on the updated data.

Diagnostic tests for both models show that the residuals appear to be independent; the autocorrelation function has no high values. The residuals appear to be normally distributed, too. The skewness and kurtosis coefficients are increased a bit but are still not far from their expected values which are zero and three respectively. The Jarque-Bera statistic increased to 2.51, giving $p(\chi^2) = 0.29$. The model is still satisfactory.

Possible rounded values for practical use, based on the past experience, might be[1]:

[1] However, because of the change in the way in which dividends are now taxed, as described in Wilkie *et al.* (2010), it might be appropriate for the future to use the 'actual yield' basis, in which case the value of YMU should be reduced by 10% to give a value of 3.375%.

$$YW = 1.55;\ YMU = 0.0375;\ YA = 0.63;\ YSD = 0.155$$

.

1.6.2 Parameter Stability

We examine the stability of the parameters by calculating the recursive estimates on incrementally larger data sets as we did in the previous sections. Figures 1.16, 1.17, 1.18, 1.19 present the recursive estimates and 95% confidence intervals of the inflation effect, YW, the mean yield, YMU, the autoregressive parameter of the yield, YA and the standard deviation, YSD, respectively, for the earlier sub-periods (data sets starting in 1923) and the later sub-periods (data sets ending in 2009). The construction of the graphs is similar to the ones in the previous sections.

In Figure 1.16, YW is the parameter which reflects the effect of inflation on dividend yields. The graph for the earlier sub-periods shows that there are two jumps in the years 1940 and 1974. These are the years in which the greatest increases in prices and in yield occurred. The graph suggests that when inflation is high, its effect on yield is also high. However, over the early and later shorter periods, the influence has been small or negative, and the confidence intervals are very wide.

When we look at the YMU graph in Figure 1.17 we can see that, as we extend the period, the confidence intervals become smaller. YMU estimates for the earlier sub-periods have a similar path to QMU estimates for the earlier sub-periods which justifies the proposition that high inflation, when it occurs, leads to a fall in share prices and hence to high dividend yields. In Figure 1.17 we see that the estimates of YMU are very stable, though the confidence intervals widen when there are fewer observations.

Figure 1.18 shows the autoregressive parameter, YA. This parameter is quite stationary except for the very short sub-periods. We should note that when inflation is high (1940 and 1974), YA decreases which means that during these years the increasing inflation effect on yields (YW increases in these years) explains most of the variability in yield and yield does not depend so much on its previous value.

The YSD graph in Figure 1.19 shows that during low, stable inflation the standard deviation of the yields is small. The confidence intervals shrink as the sub-periods extend.

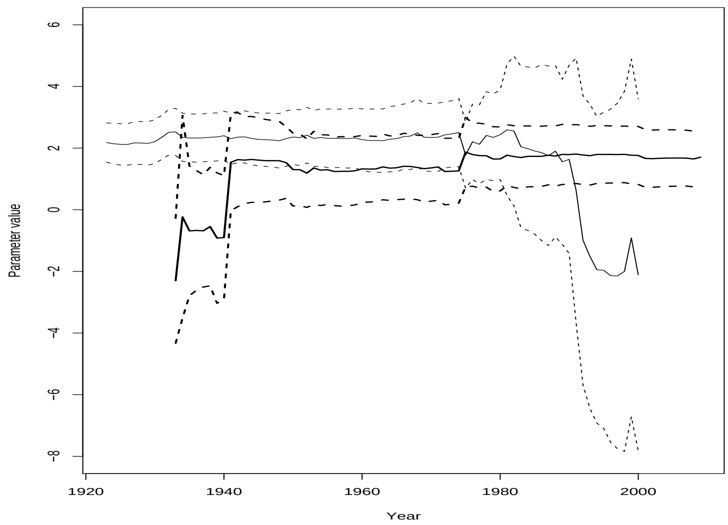

Figure 1.16: Estimates for parameter YW for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

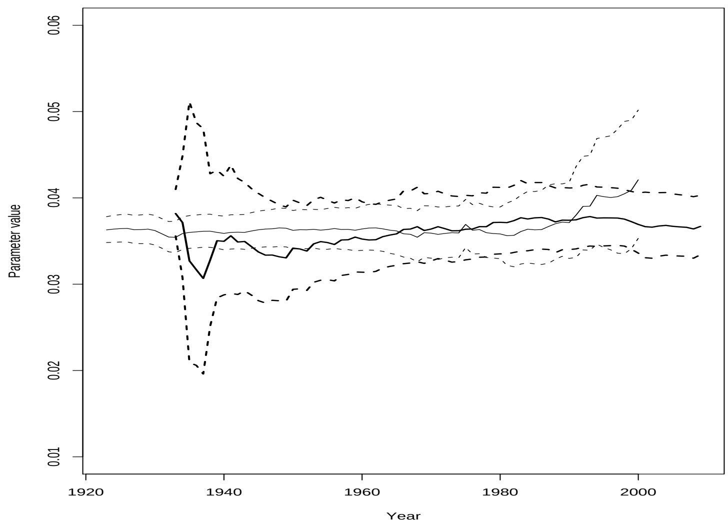

Figure 1.17: Estimates for parameter YMU for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

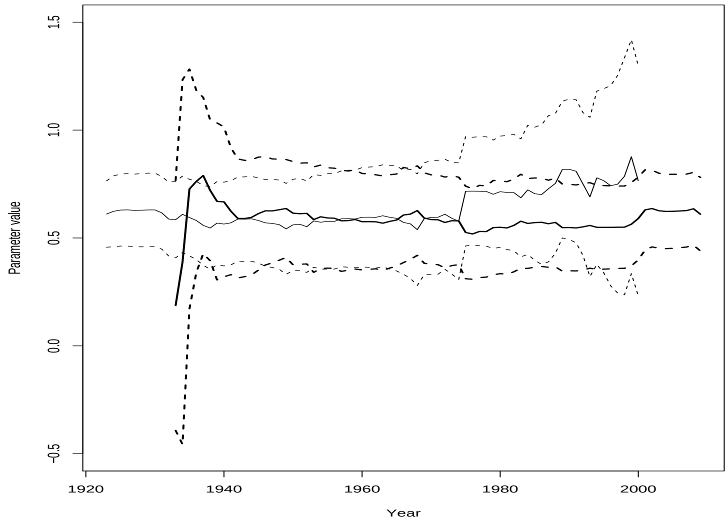

Figure 1.18: Estimates for parameter YA for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

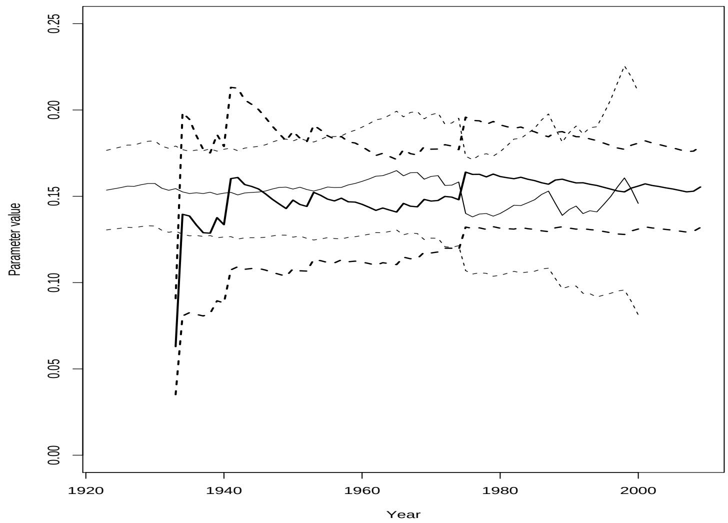

Figure 1.19: Estimates for parameter YSD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

1.7 Share Dividends

The indices for share dividends and share prices come from the same source as the share dividend yields. We calculate the 'force' of increment in the dividend index $t-1$ to t, denoted $K(t)$, as

$$K(t) = \ln D(t) - \ln D(t-1) \tag{1.7}$$

so that $D(t) = D(t-1).\exp K(t)$.

In Figure 1.20 we show the values of $K(t)$ from 1920 to 2009, along with the rate of inflation, $I(t)$.

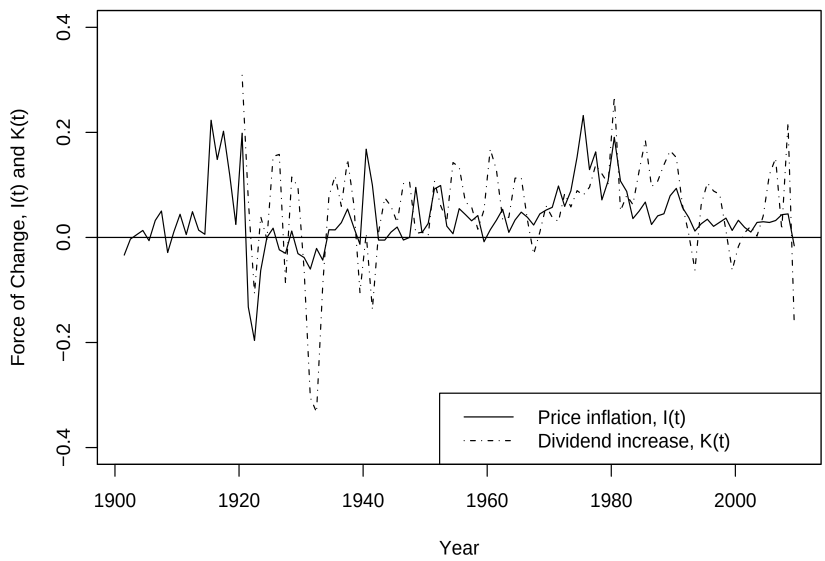

Figure 1.20: Increase in share dividends, $K(t)$, 1920-2009 and Inflation, $I(t)$, 1900-2009

The original model for share dividends, where $D(t)$ is the value of a dividend index on ordinary shares at time t and $K(t)$ is the annual change in the logarithm, is:

$$
\begin{aligned}
DM(t) &= DD.I(t) + (1 - DD).DM(t-1) \qquad (1.8)\\
DI(t) &= DW.DM(t) + DX.I(t)\\
K(t) &= DI(t) + DMU + DY.YE(t-1) + DB.DE(t-1) + DE(t)\\
DE(t) &= DSD.DZ(t)\\
DZ(t) &\sim (iid)N(0,1)
\end{aligned}
$$

In Equation 1.8 the function $DM(t)$ is an exponentially weighted moving average of inflation up to time t. $DI(t)$ takes a proportion of this and a proportion of the latest rate of inflation. DX is constrained to equal $1 - DW$, so that there is 'unit gain' from inflation to dividends. $K(t)$ is also influenced by the residuals from the previous year of dividend yields and dividends itself.

Hence, a model for $P(t)$, the value of a price index of ordinary shares at time t can be obtained as:

$$
\begin{aligned}
P(t) &= D(t)/Y(t)\\
\ln P(t) &= \ln D(t) - \ln Y(t)
\end{aligned}
$$

1.7.1 Updating and Rebasing to 1923-2009

Table 1.6 shows the estimated parameters and their standard errors. Wilkie (1995) investigated what happens if he omits the influence of inflation by setting both DW and DD to zero. Since the log likelihood is worsened substantially, and, in addition, the crosscorrelation between the residuals of dividends, DE, and the residuals from inflation, QE, is large he decided to keep these parameters. Moreover, he found it economically necessary taking into account the direct transfer from retail prices to dividends.

On the other hand, the estimated value of DMU, the mean rate of growth of real dividends, is not much more than one standard error away from zero for both of the periods. It can, therefore, be set to zero as in the model in Wilkie (1986). However, since the real rate of growth of dividends is an important element in the total return on shares Wilkie (1995) preferred keeping this parameter.

Diagnostic tests of the residuals for the model show no remaining autocorrelation. The Jarque-Bera statistics are 8.16 and 6.27, with $p(\chi^2) = 0.017$ and 0.043, so there is some evidence of fat-tailedness.

Table 1.6: Estimates of parameters and standard errors of models for dividends, and relevant statistics, over different periods

	1923-1994	1923-2009
DW	0.5793	0.4279
	(0.2157)	(0.2398)
DD	0.1344	0.1551
	(0.0800)	(0.1006)
DMU	0.0157	0.0111
	(0.0124)	(0.0110)
DY	-0.1761	-0.2142
	(0.0439)	(0.0451)
DB	0.5733	0.4477
	(0.1295)	(0.1041)
DSD	0.0671	0.0708
	(0.0056)	(0.0054)
$r_{(DZ)_1}$	-0.0338	0.0074
$r_{(DZ^2)_1}$	0.2260	0.3371
skewness $\sqrt{\beta_1}$	-0.5980	-0.5548
kurtosis β_2	4.0344	3.7066
Jarque-Bera χ^2	8.16	6.27
$p(\chi^2)$	0.017	0.043

Possible rounded values for practical use, based on the past experience, might be:

$$DW = 0.43;\ DD = 0.16;\ DX = 1 - DW = 0.57;\ DMU = 0.011;\ DY = -0.22;$$

$$DB = 0.43;\ DSD = 0.07$$

.

1.7.2 Parameter Stability

Figures 1.21, 1.22, 1.23, 1.24, 1.25, 1.26 present the recursive estimates and 95% confidence intervals of the model parameters DW, DD, DMU, DY, DB and DSD for earlier sub-periods (data sets starting in 1923) and later sub-periods (data sets ending in 2009). Since we have six parameters for this model, we have omitted the sub-periods with less than 20 years in order to use enough data to get reasonable estimates. Then for many periods, including most periods starting in or after 1971, the maximum likelihood estimate of the value of DW is negative, and sometimes also the estimated value of DD is greater than 1, which would imply that the further back we look at inflation, the greater the effect on dividend increases. This makes no sense, so we omit the values of DW and DD for these periods. The values of the other parameters, however, seem quite sensible, and we leave them in. Sometime DW is greater than 1, which implies that past inflation has a positive effect, but current inflation a negative one; this is not entirely implausible.

Where we show it, the value of DD is stable, as is the value of DMU, which is generally greater than zero, but not by much. The values of DY seem to have been increasing, and those of DB and DSD decreasing.

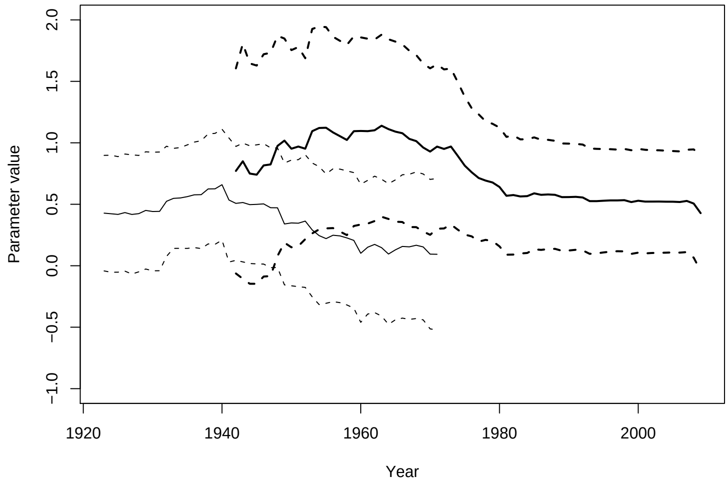

Figure 1.21: Estimates for parameter DW for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

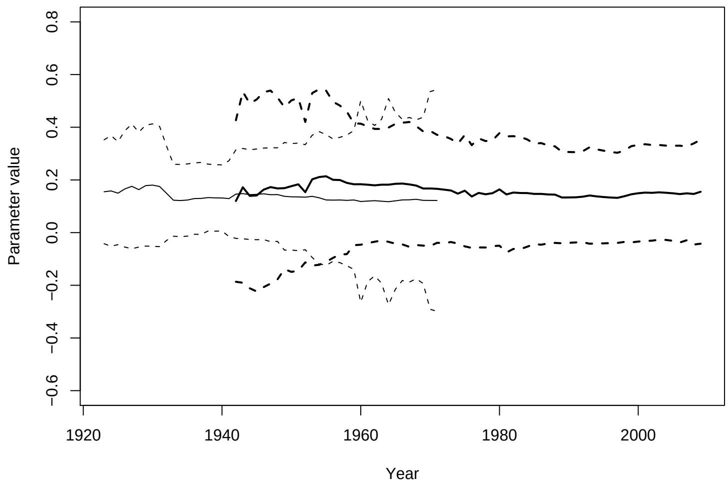

Figure 1.22: Estimates for parameter DD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

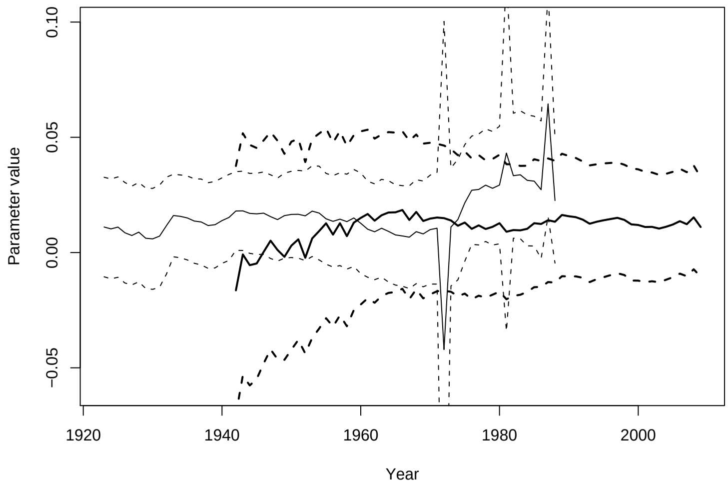

Figure 1.23: Estimates for parameter DMU for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

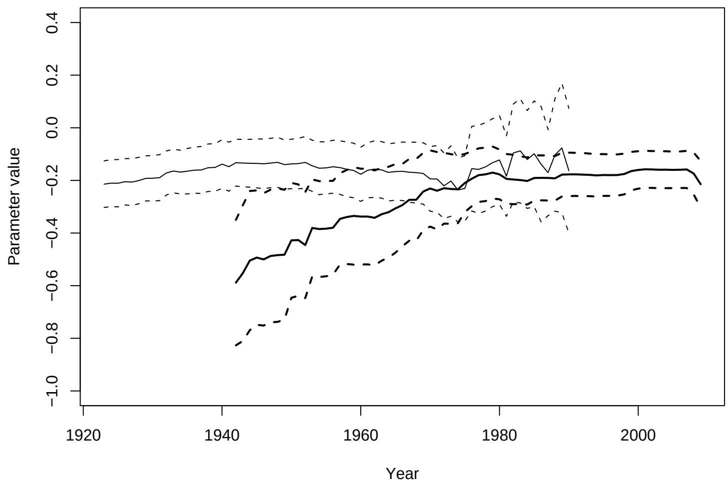

Figure 1.24: Estimates for parameter DY for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

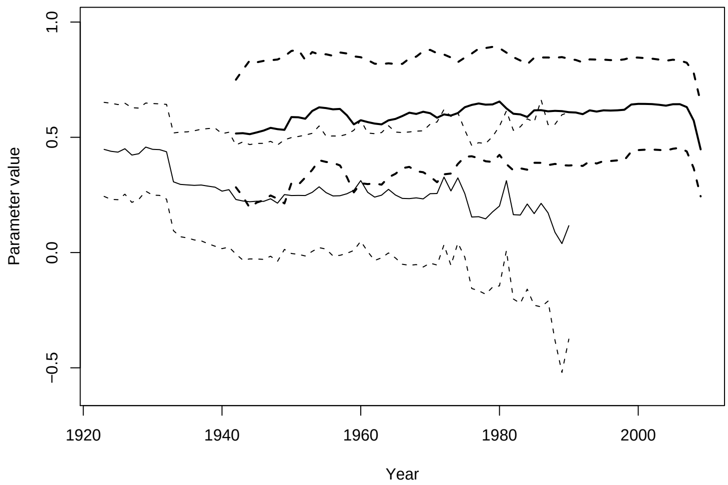

Figure 1.25: Estimates for parameter DB for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

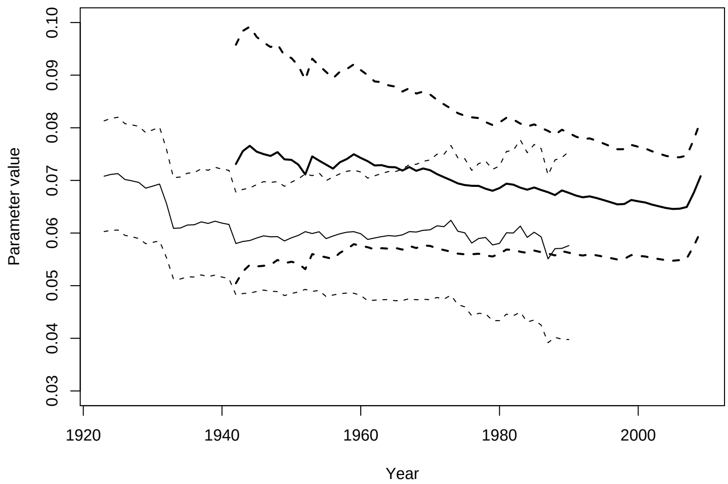

Figure 1.26: Estimates for parameter DSD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

1.8 Long-Term Interest Rates

For the long-term bond yields Wilkie (1986, 1995) used 'consols' (originally an abbreviation for Consolidated Stock) which is a form of British government bond, dating from 1756. Consols are one of the rare examples of a perpetuity, although they may be redeemed by the issuer. For long term bond yields, $C(t)$, the earlier values are the yield on $2\frac{1}{2}\%$ Consols, and the later are the yield on the FTSE-Actuaries BGS Indices irredeemables index, which is now purely the yield on $3\frac{1}{2}\%$ War Stock (War Loan).

For short-term bond yields, $B(t)$, discussed further in Section 1.9, bank rate or bank base rate has been used. This is not suitable for measuring short-term movements of yields, because it changes only occasionally, so is a step function. But this is not a problem when it is sampled at annual intervals, and it too has a very long past history, back at least to 1797.

For index-linked yields, $R(t)$, discussed further in Section 1.10, the yield from the FTSE-Actuaries BGS indices on index-linked stocks, over 5 years, is used with an assumption of 5% future inflation. This assumption is perhaps too low for the earlier period and too high for the more recent; the market presumably assumes a varying forecast future rate.

Figure 1.27 shows the long-term yield, $C(t)$, and the short-term yield, $B(t)$, from 1900 to 2009, and the index-linked yield, $R(t)$, from 1981 to 2009. One can see how the two nominal yields were low in the first part of the century, rose substantially in the 1980s, and have reduced a lot in recent years. The index-linked yield has always been lower than the nominal yields, but has fallen roughly in line with them. It can be seen that the index-linked yields, for their first few years, were not very different from the nominal yields at the beginning of the century, though they have now dropped to much lower levels.

The model for $C(t)$ proposed in Wilkie (1986) included a third order autoregressive part, but in Wilkie (1995) it was simplified to a first order one. The model became:

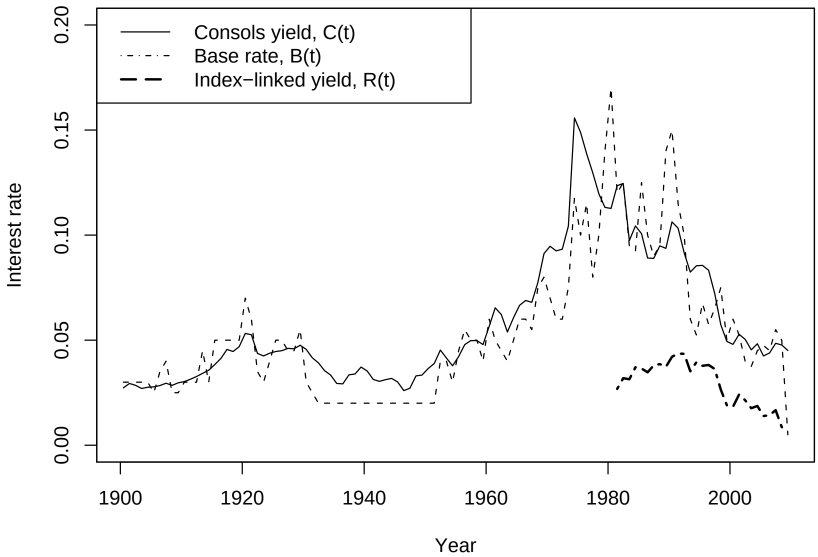

Figure 1.27: Consols yield, $C(t)$, Base rate, $B(t)$, 1900-2009 also Index-linked yield, $R(t)$, 1981-2009

$$
\begin{aligned}
C(t) &= CW.CM(t) + CMU.exp(CN(t)) \qquad (1.9)\\
CM(t) &= CD.I(t) + (1 - CD).CM(t-1)\\
CR(t) &= C(t) - CW.CM(t)\\
\ln CR(t) &= \ln CMU + CN(t)\\
CN(t) &= CA.CN(t-1) + CY.YE(t) + CE(t)\\
CE(t) &= CSD.CZ(t)\\
CZ(t) &\sim (iid)N(0,1)
\end{aligned}
$$

The model is composed of two parts: an expected future inflation, $CM(t)$, and a real yield, $CR(t)$. The inflation part of the model is a weighted moving average model. The real part is essentially an autoregressive model of order one with a contribution from the dividend yield. This model, with $CW = 1$, fully takes into account the 'Fisher effect' (Fisher, 1907, 1930), in which the nominal yield on bonds reflects both expected inflation over the life of the bond and a real rate of interest. It is assumed that there is no inflation risk premium.

1.8.1 Updating and Rebasing to 1923-2009

Wilkie (1995) fixed $CW = 1$ and $CD = 0.045$ as parameters in the consols yield model. Fixing the value of CD ensured that the values of $CR(t)$ in the period considered were never negative. However, when we updated the data using the fixed values of CW and CD, we obtained negative real interest rates, which is not allowed by the structure of the model. The reason is that the inflation has reduced since 1995, but interest rates have reduced much faster than the values of $CM(t)$, and in some years $CR(t)$ would have been negative if we had not adjusted the formula. Therefore, we modified the model by introducing a *minimum* value which is called $CMIN$ and we redefined $CM(t)$ as:

$$CM^{'}(t) = Min(CD.I(t) + (1 - CD).CM(t-1), C(t) - CMIN)$$

with still

$$CR(t) = C(t) - CM(t)$$

where $CMIN = 0.5\%$, an assumed minimum real rate of interest. If the first condition inside the Min(,) function applies, then $CM(t)$ and $CR(t)$ are calculated as before, but if the second applies, then $CR(t) = CMIN$ and the value of $CM^{'}(t)$ is reduced below what it would otherwise have been and this reduced value is carried forward to the next year. This happened in each year from 1998 to 2000 and again in 2005.

It must be noted that this adjustment does not affect the CM term before 1998 and hence does not affect the parameters previously obtained for 1923-1995.

By introducing the $CMIN$ term, we avoid negative real interest rates. However, as we will discuss below, the model standard deviation increased a lot and the residuals do not satisfy the normality assumption any more; this is an unfortunate feature.

In Table 1.7 we compare the parameters estimated for the whole period, 1923-2009, with those that were estimated in Wilkie (1995), along with the diagnostic tests. Except for CD and CW, whose values are fixed arbitrarily, the values are all somewhat different from before. The mean level of consols decreased from 3.05% to 2.23% and the dependence on the residuals of the current year's dividend yields slightly increased. Although CA remained almost the same, indicating strong autocorrelation, the standard deviation of the model increased significantly. Standard errors show that all the parameters are significantly different from zero. For the first period, the autocorrelation coefficients of the standardized residuals indicate that they are uncorrelated and we fail to reject the normality assumption for a 0.05 significance level. On the other hand, when we fit the model to updated data, though the residuals seem uncorrelated, the Jarque-Bera statistic indicates strong non-normality.

Table 1.7: Estimates of parameters and standard errors of model for 'consols', and relevant statistics, over different periods

	1923-1994	1923-2009
CW	1	1
CD	0.045	0.045
$CMU\%$	3.05%	2.23%
	(0.65%)	(0.70%)
CA	0.8974	0.9117
	(0.0442)	(0.0420)
CY	0.3371	0.3729
	(0.1436)	(0.1810)
CSD	0.1853	0.2571
	(0.0154)	(0.0195)
$r_{(CZ)_1}$	0.1313	0.0529
$r_{(CZ^2)_1}$	-0.0393	0.0724
skewness $\sqrt{\beta_1}$	-0.6662	-1.1039
kurtosis β_2	4.5425	6.3959
Jarque-Bera χ^2	4.88	59.47
$p(\chi^2)$	0.087	0.0000

Possible rounded values for practical use, based on the past experience, might be:

$$CD = 0.045;\ CW = 1;\ CMU = 2.23\%;\ CA = 0.91;\ CY = 0.37;\ CSD = 0.257$$

.

1.8.2 Parameter Stability

We examine the stability of the parameters of the modified consols yield model by calculating the recursive estimates on incrementally larger data sets as we did in the previous sections. Figures 1.28, 1.29, 1.30 and 1.31 present the recursive estimates and 95% confidence intervals of the mean level of consols yield CMU, the autoregressive parameter of the consols model, CA, the dependence on the previous year's dividend yield innovation, CY, and the standard deviation, CSD, respectively, for the earlier (data sets starting in 1923) and later sub-periods (data sets ending in 2007).

Estimates for this model are rather unstable, and we have omitted periods of less than 15 years at the beginning and end. In some cases the maximum likelihood estimate of CA is greater than one, which would give a non-stationary and unstable model for $C(t)$. Further, if $CA = 1$ the value of CMU is indeterminate, and if CA is very close to 1, the value of CMU is quite uncertain, and the standard errors cannot all be calculated because the information matrix is singular or nearly so. We have therefore omitted those few periods where this occurs, but there are still some periods where the standard errors are very high. The vertical scale has been truncated, so that not all the confidence intervals are shown.

With these caveats, the values of most of the parameters are reasonably stable, except for CMU, which jumps around a lot, and CSD, which has been increasing.

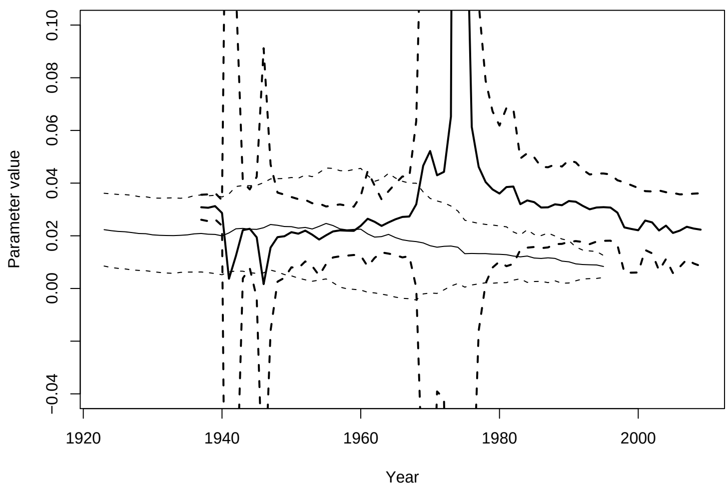

Figure 1.28: Estimates for parameter CMU for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

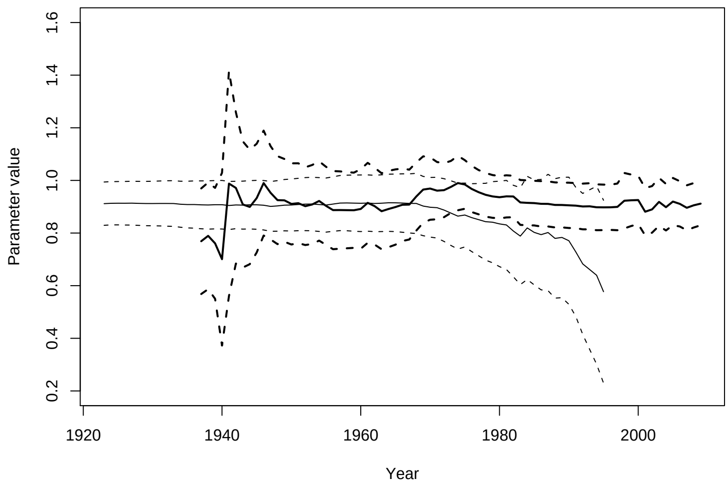

Figure 1.29: Estimates for parameter CA for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

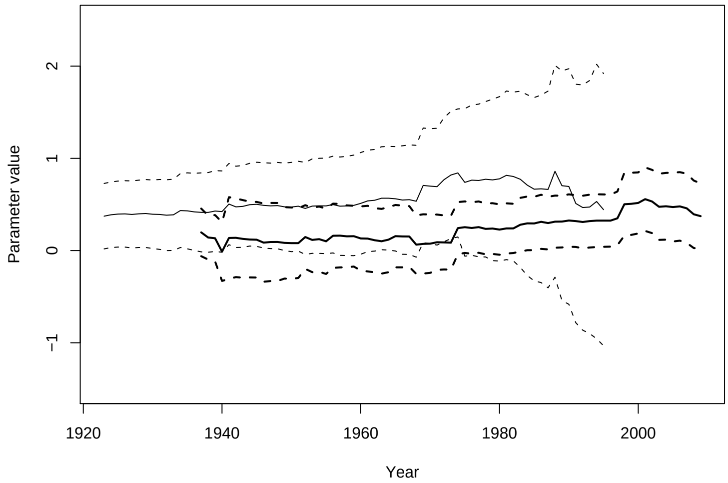

Figure 1.30: Estimates for parameter CY for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

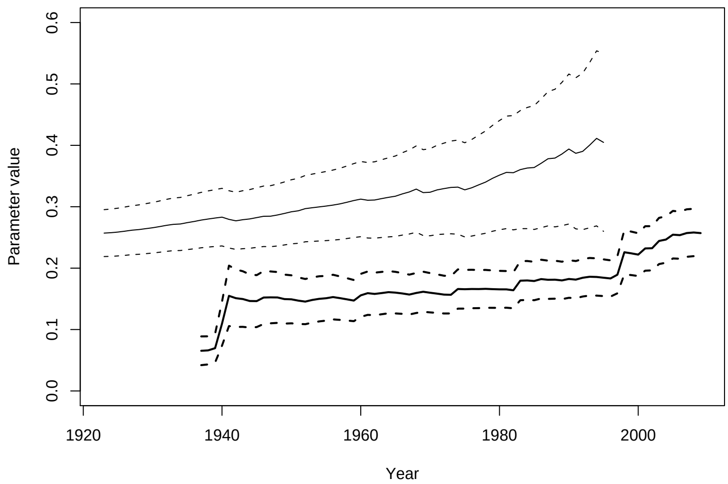

Figure 1.31: Estimates for parameter CSD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

1.9 Short-Term Interest Rates

The values for short-term interest rates from 1900 to 2009 have been displayed in Figure 1.27 along with the long-term interest rates and index-linked yields.

Short-term interest rates are clearly connected with long-term ones, as shown in Figure 1.27. Wilkie's (1995) approach was to model the difference between the logarithms of these series where $BD(t)$ is the 'log spread':

$$BD(t) = \ln C(t) - \ln B(t) \tag{1.10}$$

Values of the negative of this function from 1900 to 2009 are shown in Figure 1.32. Note that $B(t)$ is less than $C(t)$ more often than not, though sometimes it is higher, and the function has wandered around a middle level a bit below zero, like a typical first order autoregressive series, until this last year, when $B(t)$ has been reduced to an unprecedented 0.5%, without there being a corresponding fall in long-term interest rates.

The stochastic model for $BD(t)$ proposed in 1995 was:

$$\begin{aligned} BD(t) &= BMU + BA.(BD(t-1) - BMU) + BE(t) \qquad (1.11)\\ BE(t) &= BSD.BZ(t) \\ BZ(t) &\sim (iid)N(0,1) \end{aligned}$$

so that:

$$B(t) = C(t).exp(-BD(t))$$

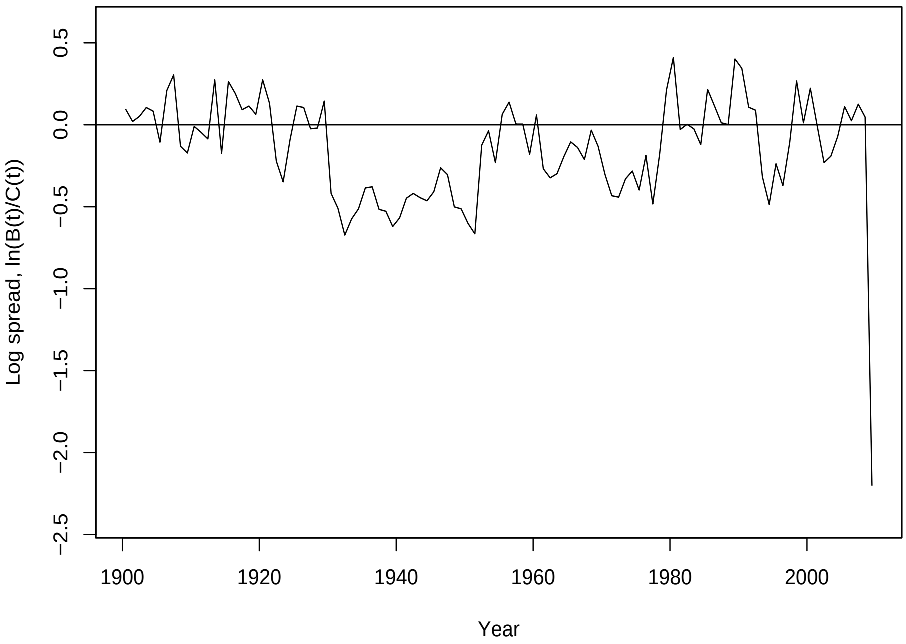

Figure 1.32: Log spread, $-BD(t) = \ln(B(t)/C(t))$, 1900-2009

1.9.1 Updating and Rebasing to 1923-2009

When we re-estimate the parameters for the whole period, 1923-2009, as shown in Table 1.8, we find that the extreme value in 2009 gives extremely high skewness and kurtosis coefficients. It is reasonable to suspect that the extreme value also distorts the estimation of the parameters. So we modify the model, introducing an 'intervention variable', $BInt(t)$, which has the value 1 in 2009 and 0 otherwise. We then modify the formula to give:

$$BD(t) = BMU + BA.(BD(t-1) - BMU) + BI.BInt(t) + BE(t)$$

and fit the parameters. The resulting value of BI is such that the residual $BE(t)$ in 2009 is zero. We show the parameter estimates also in Table 1.8. We can see that the estimated values of BA and BSD are not very different from those estimated over the period 1923 to 1994, though the value of BMU is rather different. We also see that the skewness and kurtosis are very satisfactory. The parameter values are almost the same as those we obtain when fitting 1923 to 2008, omitting the final year, but the method we have used would be more satisfactory if the outlier were an intermediate year.

We then recalculate the residuals for the period 1923 to 2009, using the values for BMU and BA that we estimated using the intervention variable, but otherwise omitting the intervention variable; we calculate the standard deviation of the residuals, thus including the extreme value; and we calculate the relevant statistics. These are shown in the final column of Table 1.8. The standard deviation is now a very little higher than it was when we did not use the intervention variable, and the statistics are similar. Estimating a higher standard deviation in this way gives some compensation for the extreme value, if we choose to simulate using normally distributed residuals. It would be better to use a different and fatter-tailed distribution.

Table 1.8: Estimates of parameters and standard errors of model for short-term interest rates, and relevant statistics, over different periods

	1923-1994	1923-2009 Without BI	1923-2009 With BI	1923-2009 Omitting BI
BMU	0.2273	0.2434	0.1699	0.1699
	(0.0797)	(0.0918)	(0.0718)	
BA	0.7420	0.6474	0.7308	0.7308
	(0.0823)	(0.1204)	(0.0738)	
BI			-2.1881	
			(0.1808)	
BSD	0.1808	0.2932	0.1790	0.2951
	(0.0151)	(0.0222)	(0.0136)	
$r_{(BZ)_1}$	0.0503	0.0346	0.0211	0.0079
$r_{(BZ^2)_1}$	0.0808	-0.0062	0.0611	-0.0066
skewness $\sqrt{\beta_1}$	0.3562	-4.3178	0.3089	-4.4117
kurtosis β_2	3.2950	33.3303	3.0506	34.1338
Jarque-Bera χ^2	1.57	3605.07	1.39	3795.98
$p(\chi^2)$	0.45	0.0000	0.50	0.0000
Log likelihood		63.24	106.18	62.69

Possible rounded values for practical use, based on the past experience, might be:

$$BMU = 0.17; BA = 0.73; BSD = 0.3$$

.

The residuals for this model are very far from being normally distributed, although the statistics are quite acceptable when the extreme value in 2009 is allowed for separately. The economic and financial circumstances in 2009 are quite exceptional, and it is most uncertain whether short-term interest rates will stay at their exceptionally low level for a long time, or whether they will revert reasonably soon to a more normal level in relation to long-term rates. But this reversion might involve long-term rates falling to very low levels too. The uncertainty is large, so a high standard deviation seems appropriate.

1.9.2 Parameter Stability

The stability of the parameters is examined using the same method as in previous sections. The values of BMU, BA and BSD over various subperiods are shown in Figures 1.33, 1.34 and 1.35. We have included the intervention variable for 2009 in every case where it is relevant, so the values of BSD are at their lower level, not the higher one when the extreme in 2009 is included. We can see that the values of all three parameters have been reducing a bit in the most recent periods, and that none shows any exceptional values.

Figure 1.33 shows that the mean rate parameter is stable over the whole period. When we look at the BA graphs in Figure 1.34, we can say that the parameter estimates for the earlier sub-periods are quite stable and the confidence interval is shrinking as larger data is considered. The right hand end of the estimates for the later sub-periods indicate a lower dependence on the previous year's ratio (i.e. $-\ln(B(t)/C(t))$).

In Figure 1.35, after a sharp decrease until the late 1940s, the estimates for the earlier sub-periods have had two jumps but still seems stable for the rest of the period and the estimates for the later sub-periods are relatively constant over the whole period.

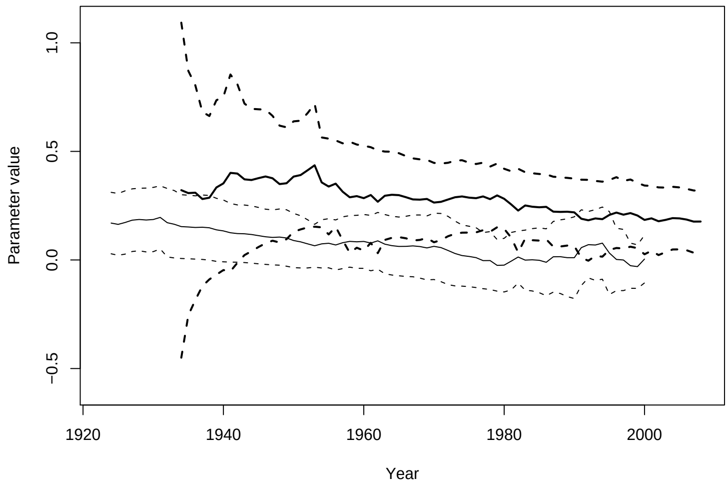

Figure 1.33: Estimates for parameter BMU for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

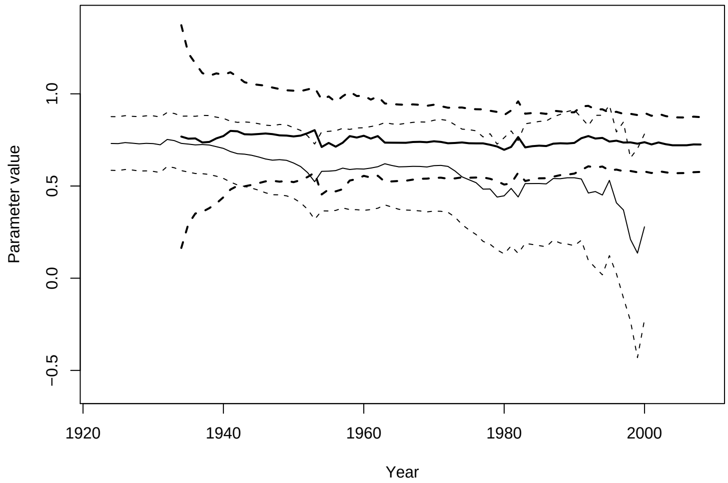

Figure 1.34: Estimates for parameter BA for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

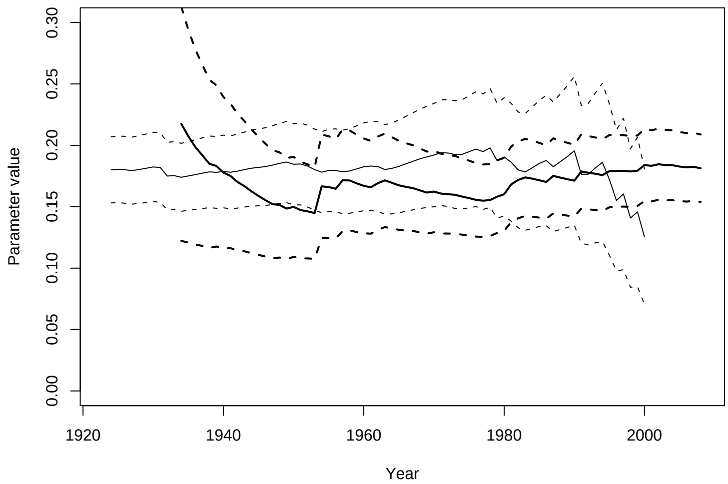

Figure 1.35: Estimates for parameter BSD for periods starting in 1923 and periods ending in 2009, with 95% confidence intervals

1.10 Index-Linked Bond Yields

As mentioned in Section 1.8 we have used the yield from the FTSE-Actuaries BGS indices on index-linked stocks, over 5 years, with an assumption of 5% future inflation, to represent the yield on index-linked stocks. We denote this as $R(t)$. It is available only since 1981. A graph is shown in Figure 1.27.

The model for $R(t)$ suggested in Wilkie (1995) was:

$$
\begin{aligned}
\ln R(t) &= \ln RMU + RA.(\ln R(t-1) - \ln RMU) + RBC.CE(t) + RE(t) \qquad (1.12)\\
RE(t) &= RSD.RZ(t)\\
RZ(t) &\sim (iid)N(0,1)
\end{aligned}
$$

The term with $CE(t)$ represents simultaneous correlation with the residuals of the consols yield model. We include also a parameter $R0 = R(1980)$, the unknown value for the year prior to 1981. Estimating this is equivalent to setting the residual, RE, for 1981 to zero.

We can observe that the UK index-linked market has perhaps been distorted in recent years. The UK government is the only issuer of such bonds, and restricts its issue to a limited proportion of all government borrowing, so the supply of these bonds is limited, in spite of their low yield and correspondingly high price. Corporations in the UK do not find it at all tax-efficient to issue such bonds. However, actuaries in the UK have been pointing out to pension fund trustees that index-linked bonds are a very satisfactory hedge against pensions wholly or partially linked to the RPI, so there has been high demand for these bonds, even at low yields, from pension funds and insurance companies that write such business. It is difficult to say whether these conditions will continue, or whether the UK government will issue many more such bonds, or whether the requirements of pension funds will be satisfied at some point (Wilkie, *et.al.*, 2010).

We can estimate the parameters for the index-linked model only over the period 1981 to 2009, which is a much shorter period than for the other series. We see from the graph in Figure 1.27 that the index-linked yield rose reasonably steadily from 1981 to 1991, and since then has fallen reasonably steadily.

1.10.1 Updating and Rebasing to 1981-2009

When we estimate parameters over the whole period for the model suggested in 1994, which are shown in Table 1.9 we find that the estimated value of RA is 1.0853, which produces an unstable model for $\ln R(t)$, in which the value of $\ln R(t)$ is certain to move in the long run towards either '+' infinity or '-' infinity. A value of '-' infinity means a long-run value $R(t)$ of zero. Wilkie (1986, 1995) originally took logarithms to avoid negative values. However, it is not impossible for the yields in index-linked to be negative.

We could think of two ways of avoiding this instability. First, we set the value of RA arbitrarily to 0.95. This is a little outside twice the estimated standard error away from 1.0853. We then estimate the other parameters. The values are shown in Table 1.9. The log likelihood is worsened by 2.21. However, the skewness and kurtosis coefficients, which were very large in our first model, are slightly higher in this. This results substantially from the fall in yields from 1.67 in 2007 to 0.87% in 2008, almost halving. Another solution we try is therefore to use the unlogged values of $R(t)$ in the formulae, instead of their logarithms. In our first trial the estimated value of RA is still greater than 1, at 1.0385, so again we fix the value of RA at 0.95 and estimate the other parameters. On this occasion the log likelihood is worsened by only 1.35, quite a small amount. However, for both the unlogged models the skewness and kurtosis coefficients are reasonably small and the Jarque-Bera probability is satisfactory.

Our preference for future use is therefore to model $R(t)$ rather than $\ln R(t)$, using the formula:

$$R(t) = RMU + RA.(R(t-1) - RMU) + RBC.CE(t) + RE(t)$$

Table 1.9: Estimates of parameters and standard errors of different models for index-linked interest rates, and relevant statistics, over different periods

	1981-1994	1981-2009 fitting $\ln R$ RA free	1981-2009 $RA = 0.95$	1981-2009 fitting R RA free	1981-2009 $RA = 0.95$
$RMU\%$	4.03 (0.17)	3.16 (1.03)	2.06 (1.16)	2.87 (1.41)	2.96 (1.14)
RA	0.5686	1.0853	0.95	1.0385	0.95
	(0.1076)	(0.0618)		(0.0526)	
RBC	0.2234	0.3527	0.3285	0.0083	0.0079
	(0.0598)	(0.0698)	(00744)	(0.0014)	(0.0015)
$R(0)\%$		2.54 (0.32)	2.53 (0.40)	2.52 (0.28)	2.49 (0.32)
RSD	0.0518	0.1348	0.1456	0.0028	0.0029
	(0.0102)	(0.0177)	(0.0191)	(0.0004)	(0.0004)
$r_{(RZ)_1}$	-0.1419	-0.1949	0.1189	-0.0486	0.1286
$r_{(RZ^2)_1}$	0.5321	-0.0087	-0.0932	-0.0920	-0.1124
skewness $\sqrt{\beta_1}$	-0.0569	-2.0519	-2.3096	-0.8737	-0.7270
kurtosis β_2	3.6306	8.2723	9.7123	3.8073	3.3183
Jarque-Bera χ^2	0.28	53.94	80.22	4.48	2.68
$p(\chi^2)$	0.86	0.0000	0.0000	0.1066	0.26
Log likelihood		43.60	41.39	155.96	154.61

with possible parameters, rounded:

$$RMU = 3\%, \; RA = 0.95, \; RBC = 0.008, \; RSD = 0.3\%$$

The period for which values of $R(t)$ are available is so short that it is not worth showing the results for shorter periods.

1.11 Comments on the Wilkie Model

The Wilkie model is a combination of statistics and economics. Hence, it has been criticised from both statistical viewpoint and an economic viewpoint. In this section we summarise the comments of various authors on the Wilkie model in these two main perspectives.

1.11.1 Statistical Review

We will summarise the statistical reviews of the Wilkie model in five subsections which consider the methodology, model and parameter uncertainty and non-stationarity, non-normality of the residuals, heteroscedasticity, and non-linearity as in Rambaruth (2003).

Methodology

Huber (1997) reviews the Wilkie Model in both empirical and theoretical sense. He expresses his reservations about the methodology proposed in specifying the Wilkie model in the discussion of 1995 paper by raising the 'data-mining' issue. He criticises Wilkie's approach in which he recommended that asset models should be developed by establishing a linear relationship based on economic theory (or 'common sense'), fitting it to the data and then testing whether this relationship satisfies various goodness-of-fit tests. If the tests are not satisfied, then parameters should be added until the tests are satisfied or the results should be ignored on theoretical grounds. This methodology ignores the problems associated with multiple hypothesis testing (which can lead to data-mining). According to Huber (1997), it basically restricts the model to the Auto-regressive Integrated Moving Average (ARIMA) class and it does not allow 'common sense' to be influenced by the data which would allow us to improve our understanding of the economy.

Hardy (2003) points out the problem of 'data-mining' by which Huber means that a statistical time-series approach, which finds a model to match the available data, cannot then use the same data to test the model. Thus, with only one data series available, all non-theory-based time-series modelling is rejected. One way around the problem is to use part of the available data fit the model, and the rest to test the fit. She emphasises that the problem for a complex model with many parameters is that data are already scarce.

Model and Parameter Uncertainty and Non-stationarity

Kitts (1990) was the first to point out that the parameters of the Wilkie inflation model (Wilkie, 1986) may not be constant over time. If the mean rate of inflation is likely to change in the future, i.e. when the current stationary sub-period ends, then the model is inadequate as it does not necessarily describe the way in which appropriate investment variables will move over the future long-term.

Huber (1997) examined the parameter constancy of the original price inflation model by recursively estimating its parameters on incrementally larger data sets. He drew the graphs of QMU and QA with 95% confidence intervals and concluded that these parameters may not be constant. However, Huber had some reservations about interpreting these results because they might simply be due to the non-normality of the residuals or they could be due to the change in the calculation of the official UK price index.

For the dividend yield model, he emphasized the sensitivity of the YW parameter to the years 1940 and 1974 as Figure 1.16 illustrates. He states that if they are excluded from the regression, then YW becomes insignificantly different from zero. The problem with including YW is that it results in a general tendency for changes in yields to be correlated with changes in inflation, but this correlation only seems to be appropriate for large increases in yields and inflation.

For the consols yield model, Wilkie (1995) noted that CY becomes insignificantly different from zero when an intervention variable for 1974 was included. Huber argues that CY appears to have a similar problem to YW because the parameter CY seems to describe mainly the event that the largest increase in interest rates coincided with the largest residual from the share dividend yield model. However, if CY is set to zero, then the model implies that there is no relationship between equity returns and real interest rates. He concludes that as this does not appear to be a reasonable assumption, it may explain why Wilkie (1995) included CY in the model.

Cairns in the discussion of Wilkie (1995) drew attention to the standard errors of parameters estimates which he found extremely important because not only is a model an approximation to reality, but it is not known what the ‘true’ set of parameters should be for this model. It is, therefore, essential as part of any simulation exercise, to repeat the exercise many times using a range of parameter values which is consistent with the past data and with the standard errors of the parameters and their correlations.

Non-normality and Non-independence of Residuals

In 1989, the Financial Management Group of the Institute and Faculty of Actuaries was assigned the task of criticising the model from a statistical viewpoint (Geohegan *et al.*, 1992). This group performed some tests of simulations and examined the standard deviations of returns and correlations between different asset classes at different time horizons on the Wilkie (1986) model. In this review, Geohegan *et al.* identified three areas of concern regarding the suitability of the model.

- The existence of burst of inflation, indicating that once an upward trend in inflation is established, there is a tendency for it to continue.
- The existence of large, irregular shocks, such as those in the mid-1970s.
- The possible skewness of residuals.

The only substantive criticism was of the inflation model. The $AR(1)$ model appeared too thin tailed, and did not reflect prolonged periods of high inflation.

In an early review of the model, Kitts (1990) reported that there is some evidence that the residuals are not independent, so that the model does not capture the frequency of the occurrence of sustained periods of extreme inflation and deflation. Moreover, the distribution of the residuals are not normal due to non-constant variance.

Finklestein, in the discussion of Wilkie (1995), expresses his concerns about the skewness of the data and the assumption of normality. He believes that the underlying probability distributions are stable non-Gaussian which are suggested for further research in Wilkie (1995).

The motivation behind introducing an ARCH model for the price inflation in Wilkie (1995) was mainly these criticisms.

Heteroscedasticity

Since it is the inflation process which drives the Wilkie model, it is crucial that this model has a good representation.

Geoghegan et al. (1992) reported the existence of bursts of inflation, indicating that once an upward trend in inflation was established, there is a tendency for it to continue as mentioned before. This leads to what was described in Engle (1982) as auto-regressive conditional heteroscedastic (ARCH) model (see also Mills, 1990). In Appendix B of Geohegan *et al.* (1992) Wilkie demonstrates how his model could be adopted to incorporate ARCH effects. In 1995 paper, he suggests using an ARCH model for inflation that would model the heavy tail. Although it has been seen that the ARCH model provided a better fit to the data for the period 1923-1995, updating the data to 2009 showed that it is not as good as it was. The recursive estimates of the ARCH model parameter, QSB also indicates that since the value is very close to zero up to early 1970s, the $AR(1)$ process is enough to model the price inflation. As mentioned in Section 1.4, the ARCH model is a useful description for the periods that include 1960s and 1970s.

A further problem that is fundamental to all ARCH models is their complex structure. Also, with small data sets the parameters are unstable which we show in Figures 1.8 and 1.9.

Non-linearity

In the discussion of Wilkie (1995), Tong comments on the several aspects of linear models which limits one's horizon and the need to use non-linear models. First of all, he criticises the linear models as not respecting the current position while making a forecast and giving exactly the same prediction interval regardless of the current position. Second, since the current position is not always known precisely, because of information delay, there is always some relevance in looking at the sensitivity of the model to the initial value (current position) which might be a trivial exercise for a linear model. As a final aspect, introducing the model with some exogenous variables, then some non-linearity may be required. In the written contribution, Wilkie (1995) refers to Tong's argument and besides stating that he found non-linearity was very much worthy of further investigation, he pointed out that the many of the data series were rather too short to allow clear phases of different types to be distinguished.

Whitten and Thomas (1999) suggest that the economy behaves differently in times of hyperinflation, than it does in times of 'normal' inflation levels. By definition, this belief cannot be incorporated into linear models. Wilkie's linear model is widely used and for the most part a good representation of its economic variables. Following the footsteps of Wilkie (1986, 1995), they thought it best to adapt his model to incorporate this non-linearity, rather than fundamentally change its formulation. Thus, they proposed a threshold non-linear model which is discussed briefly in Section 1.12.

1.11.2 Economic Review

Huber (1997) examines the Wilkie model not only in a statistical viewpoint but also in an economic (theoretical) viewpoint. Although he accepts that economic theory was considered in the development of Wilkie's model, he thinks it is inconsistent with certain orthodox financial economics theories. In this part, we summarise his comments in three subsections: Fisher relation, rational expectations hypothesis and efficient market hypothesis.

Fisher Relation

The Fisher relation (Fisher, 1907, 1930) states that expected inflation is fully reflected in nominal interest rates. As a result, this relation assumes that investors' expectations of average future inflation can be approximately determined by subtracting the average future real return required by investors from nominal interest rates.

The Fisher relation was explicitly included in the long-term interest rate model. Wilkie model assumes that the average future real return required by investors is given by $CR(t)$ and that investors' expectation of average future inflation is given by $CM(t)$. Huber (1997) shows the values of these two components, over the interval 1923-1994, calculated using Wilkie's (1995) long-term interest rate model. We present the same graph with updated data in Figure 1.36. Huber's criticism is about the required average

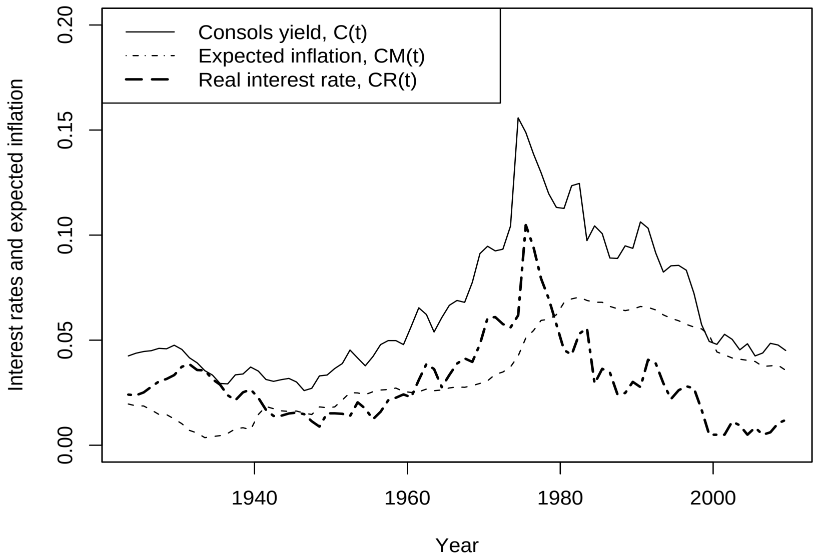

Figure 1.36: Expected price inflation and real returns, 1923-2010

future real returns by the investors in 1974 which is 10% and the returns over 5% during most of the interval 1969-1982 implied by Wilkie model. According to Huber (1997) these returns appear to be high by historical standards. Figure 1.36 shows that with the new parameters and the adjustment in consols yield model proposed by the papers Sahin *et al.* (2008) and Wilkie *et al.* (2010) the average real return implied by the Wilkie model is 2.8% for the whole period, 1923-2009 which is quite reasonable under the current economic conditions. On the other hand, as Huber emphised, even with the adjusted model, the average expected real returns during most of the interval 1969-1983 is high and the overall average real return for this period is 6.3%.

Rational Expectations Hypothesis

Another point on which the Wilkie model has been criticised is that the model is not consistent with the rational expectations hypothesis. The concept of rational expectations asserts that outcomes do not differ systematically (i.e., regularly or predictably) from what people expected them to be. It does not deny that people often make forecasting errors, but it does suggest that errors will not persistently occur on one side or the other. To assume rational expectations is to assume that agents' expectations are correct on average. In other words, although the future is not fully predictable, agents' expectations are assumed not to be systematically biased.

In economics, adaptive expectations means that people form their expectations about what will happen in the future based on what has happened in the past. Adaptive expectations principle holds that the future values of economic variables, such as future interest rates or inflation, can be predicted on the basis of previous values and their margin of error. Adaptive expectations principle is critised being underestimates or overestimates constantly changing variables, and focuses merely on past performance.

Regarding these terms, one can see that the Wilkie model has been constructed according to adaptive expectations. In the discussion of Wilkie (1995), Booth emphasises the ongoing debates about the lack of rational expectations hypothesis in the Wilkie model and suggests that it would be an interesting topic for the later studies.

Another criticism about the rational expectation hypothesis came from Huber (1997) by comparing the smoothed expected inflation, $CM(t)$, with the optimal estimate of average future inflation which is equal to QMU. Figure 1.37 illustrates the method that Huber suggested in order to compare these values. According to Huber (1997), investors consistently underestimated average future inflation over the interval 1923-1975 and overestimated average future inflation since 1975. Based on Huber's forecasts (1997), if Wilkie's model is true, then investors will continue to overestimate average future inflation by at least 0.5% until 2012. This contradicts the rational expectations hypothesis, which states that investors do not knowingly make systematic *ex ante* forecasting errors.

Efficient Market Hypothesis

In finance, the efficient market hypothesis asserts that financial markets are 'informationally efficient', or that prices on traded assets, e.g., stocks, bonds, or property, already reflect all known information and therefore are unbiased in the sense that they reflect the collective beliefs of all investors about future prospects.

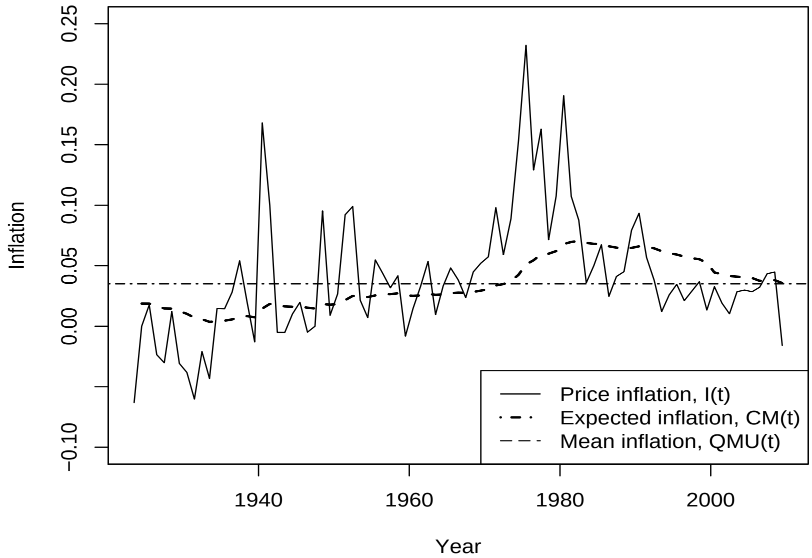

Figure 1.37: Expected price inflation, 1923-2010

Huber (1997) notes the inconsistency of the Wilkie model with the efficient market hypothesis. Hardy (2003, 2004) discusses Huber's criticisms and notes that the Wilkie model is very close to a random walk model over short terms, and the random walk model is consistent with the efficient market hypothesis. Likewise, Wilkie (1995), based on his monthly analysis, emphasises that since, in the short run, the dividend on a share index changes only very little, most of the change in share prices comes from the change in the yield, which means that this analysis of the yield transfers almost directly to the price index, and many investigators have concluded that share prices are close to a pure random walk, without relating them to dividends. Since for the monthly observations, the first autocorrelation coefficient, assuming a corresponding AR(1) model is very close to unity which is also the case for daily observations, Wilkie's annual model is quite consistent with an apparent random walk for short-term share price movements.

On the other hand, Huber (1997) mentions another implication of efficient market hypothesis which is that prices respond to information about events when this information becomes known rather than when the events occur. As a result, equity price changes are likely to anticipate future changes in equity dividends because information affecting equity dividends is often available before the dividends are declared. Hence the term $DY.YE(t-1)$ in Wilkie's dividend model which assumes that equity prices anticipate future changes in equity dividend growth rates.

1.12 Wilkie-Type Stochastic Investment Models

The Wilkie model has been a pioneer of the stochastic investment modelling. Especially after its publication (1986) many similar models have been developed for different countries. In this section, we will briefly discuss five of these models.

The Wilkie-type investment models that we will introduce are: a South African stochastic investment model (Thomson, 1996), a Finnish stochastic investment model (Ranne, 1998), an outlier adjusted multiple time-series model (Chan, 2002), TY model (Yakoubov, Teeger and Duval, 1999) and Whitten and Thomas model (Whitten and Thomas, 2000).

Thomson (1996) introduced a stochastic investment model using South African data. The series modelled by Thomson are price inflation, short-term and long-term interest rates, dividend growth rates, dividend yields, rental growth rates and rental yields. No exogenous variables are included just as in the Wilkie model, and the model was intended to be used in asset-liability modelling of South African defined benefit pension funds. Unlike Wilkie's model, Thomson's model is designed for projections of not more than ten years due to having much shorter years of data available for South Africa (for the period 1960-1993). Due to stationarity condition to apply Box & Jenkins methodology, Thomson used 'prewhitened' [2] variables for his modelling work. Although it has a cascade structure, the order of the influence is different from the one in Wilkie model. Thomson (1996) expresses his reservations about the validity of the model due to paucity of the data and he emphasises that it would be necessary to

[2] *Prewhitening* is an identification method of transfer function models proposed by Box and Jenkins (1976). If an input series is autocorrelated, the direct cross-correlation function between the input and response series gives a misleading indication of the relation between the two series. Prewhitenning is one solution of this problem. Accordingly, first an ARIMA model is fitted to the input series to reduce the residuals to white noise. Then, the response series is filtered with the same model and cross-correlate the filtered response series with the filtered input series.

modify the model as time passes.

Ranne (1998) proposed a stochastic investment model based primarily on Finnish data. The model is of the same general type as the Wilkie model but the dependencies between the variables and the equations have been selected differently. The model structure was determined on the basis of financial time series from twelve industrial countries. The variables include inflation, wage index, long-term interest rates, other interest rates, share prices, dividend yield, property prices and rental yield. They also insert a variable representing the economic cycles which are generated by interaction between the cycle variable, inflation and interest rates. This cycle variable derived from the real growth rate of the gross national product. The Finnish model, too, has several features in common with the Wilkie model; the model is discrete, inflation is the driving force and the dependencies between the variables go in one direction. The model's variables and equations are, however, generally constructed in a different way. The stochastic variation in the inflation model is divided into inflation shocks and normal variation. The shocks have been represented by the oil price inflation as two major inflation shocks in the years 1974 and 1980 (the oil crises).

Chan (2002) adopts the multiple time-series modelling approach to construct a stochastic investment model for price inflation, share dividends, share dividend yields and long-term interest rates in the UK. He considers a general VARMA (vector auto-regressive moving average) model for UK investment data by using outlier adjusted data. He proposed a VARMA (1,1) model and recommended the model for actuarial applications not involving extreme stochastic fluctuations.

Yakoubov *et al.* (1999) describes a stochastic investment model which is the first fully published model to use earnings rather than dividends to generate price returns. Another feature of the model is that the equity return is divided into three components - dividend yield, earnings growth and change in market rating. They emphasise that by modelling these components separately the model is able to capture one of the key features of the equity market, namely the high short term volatility which arises from economic fluctuations. They model price inflation, wage inflation, long and short-term interest rates, index-linked government bonds, UK equities (in three separate elements:

force of dividend yield, force of earnings growth and force of change in earnings yield) and overseas equities. They chose to adopt a cascade structure for the model by selecting the price inflation as the main driver but a different dependence structure - there is an explicit link from the 'change in gilt yield' into the 'change in equity yield'. They also use earnings growth rather than dividend growth, with a link from wage inflation rather than price inflation.

Tong (1990) in his book *Non-linear Time Series: A Dynamical System Approach* has described a class of non-linear time series models based on what he calls the 'threshold principle' which suggests that many series previously represented linearly can be modelled better by non-linear methods. Using this approach, Whitten and Thomas (1999) suggested a non-linear model. They stated two main purposes to introduce such a model. First, they introduce threshold modelling to the actuarial profession, and illustrate how this can complement or replace methods based on autoregressive conditional heteroscedasticity as suggested by Engle (1982). Second, they aim to encourage discussion and experimentation amongst actuaries on the use of non-linear models. The model considers the series for price inflation, wage inflation, share dividends, share yields, consols yield and base rates. The structure of the model is exactly the same as the Wilkie model (1995). They choose to model the investment series as a threshold autoregressive system. There are two regimes proposed for each variable, conditional on whether inflation is 'normal' or 'high' at time t. The processes in each regime (especially the 'normal' regime) is similar to those defined in Wilkie model. They suggest a threshold of 10% to partition ($I(t-1) > 10\%$) the data. They define the main disadvantage of the system as that it is more complicated than the Wilkie model, with an increased number of parameters. Furthermore, the upper regime only holds eight observation which is too few to perform any proper statistical tests.

Beside those various stochastic investment modelling works, there are several researches on comparison of these types of models such as Harris (1995), Huber and Verrall (1999), Lee and Wilkie (2000), Rambaruth (2003) and Nam (2004). All these papers follow different methods to compare the models including re-estimating the parameters on the same interval, applying some model validation tests (independence, normality, likelihood), stability of the model parameters, calculating contingency reserves for specific contracts and forecasting by simulation.

1.13 Interim Conclusion: The Wilkie Model

In this chapter we have discussed an early part of our PhD study which was presented in the 18th International AFIR Colloquium in Rome (2008) and submitted to the Annals of Actuarial Science (February 2010) as a joint paper. We have revisited the Wilkie investment model by re-estimating the parameters on updated data to 2009. We have considered models for retail prices, including an ARCH model, wage inflation, share dividend yields, share dividends and prices, long-term interest rates, short-term interest rates and index-linked bond yields. We have also recursively estimated the parameters on incrementally larger data sets and displayed those recursive estimates using graphical representation in order to analyse their stability.

The updated parameters of the retail prices model have not changed significantly. Because of low and stable inflation during last 15 years, the mean level of inflation QMU and the standard deviation QSD have decreased slightly. The model still does not satisfy the normality assumption and especially the two parameters QMU and QSD are not stable over time.

Although the ARCH model satisfies the normality assumption for the 1923-1994 data, its performance gets worse on the updated data and the residuals are not normally distributed any more. The parameters have not changed significantly. It has been seen that the suggested ARCH model is a useful description for the periods that include the 1960s and 1970s.

The parameters in the wages models have not changed significantly for the updated data. The parameter estimates over different sub-periods are quite stable except WSD.

The share dividend yield model is still satisfactory and the parameters are relatively stable over time.

The performance of the share dividend model is almost the same but its parameters are not constant over time. The DW, DD and DB parameters and their confidence intervals are highly unstable and change greatly as the sub-periods change.

We modified the long-term interest rate model to apply it to the updated data by introducing a fixed parameter called $CMIN$ which is equal to 0.5%. In order to avoid negative real interest rates, we used the modified model for 1923-2009 to estimate the parameters. The value of CMU decreased, and CY and CSD increased significantly in the model with updated data. The residuals of the modified model are not normally distributed according to the Jarque-Bera test statistic, and except for CY, the parameters are not stable either.

The short-term interest rates model is the best model among them all. It satisfies all the diagnostic tests and fits the data better over the interval 1923-2009. Moreover, its parameters are quite stable.

We have also re-estimated the parameters for the index-linked bond yields for the period 1923-2009 but could not study the stability of the parameters due to lack of data.

Finally, we have presented the comments on the Wilkie model and discussed some Wilkie type stochastic investment models briefly.

Chapter 2

A Descriptive Yield Curve Model for the UK Term Structures: The Cairns Model

2.1 Introduction

Descriptive model can be defined as a model which takes a snapshot of the bond market as it is today. The aim is to get a good description of todays prices: that is, of the rates of interest which are implicit in todays prices (Cairns, 2004b).

A descriptive model, on its own, gives no indication of how the term structure might change in the future. It is known that there is randomness in the future but this sort of model does not describe this feature.

Cairns (2004b) summarizes the number of uses descriptive models have as below:

- *They can be used to assess which bonds are over- or under-priced (so called cheap/dear analysis)*
- *They give a broad picture of market rates of interest which are implied by market prices.*
- *They can be used to price forward bond contracts.*

- *They can assist in the analysis of monetary policy.*

- *They can be used in the construction of yield indices.*

- *Finally descriptive models provide sufficient information to get a precise market value of a non-profit insurance portfolio or to price, for example, annuity contracts.*

Van Wijck (2006) discusses the methodology and the applications of the descriptive yield curve models in details.

In this chapter, we discuss the Cairns model as a descriptive parametric model to fit the daily nominal spot rates (January 1979-December 2009), implied inflation spot rates (January 1985-December 2009) and real spot rates (January 1985-December 2009) published on the Bank of England's web page by changing the fixed parameters (exponential rates in the model) to find the best set of parameters for each data set. We try three fixed parameter sets which have been suggested by Cairns (1998) and Cairns and Pritchard (2001) and then we use the least squares method with a penalty function to find the optimized set of parameters for each set of yield curve data. We compare the root mean squared errors obtained by using the four parameter sets for each yield curve to decide which set of parameters fit each yield curve data best. Once we decide these exponential rate parameters (C parameter sets), we analytically solve the equations in Cairns model as described in the following sections and fit these four different models to the data. We estimate the remaining time dependent parameters (b parameters) and find the fitted values for each day. We compare these models by examining the root mean squared errors, fitted values for some specific dates and fitted values for short, medium and long term maturities for each yield curve to choose the best set of C parameters. The overall aim of this chapter is to fill in the gaps in the nominal, implied inflation and real yield curve data provided by the Bank of England by fitting the Cairns model.

Section 2.2 introduces the yield curve terminology by giving some basic definitions and the data and the methodology used by the Bank of England to construct UK yield curves. Section 2.3 presents the Cairns model and the least squares method used, including the penalty function, to estimate the optimised C parameter set. In Section 2.4, we explore the data by looking at some descriptive statistics and estimate the time dependent b parameters for each C parameter set. We also discuss the nature of the b parameters by considering the simultaneous correlations between them. We plot and interpret the standard errors (root mean squared errors) of the residuals and also the ratios of these errors to decide the best fit in terms of the least squares method in Section 2.5. We compare how well each model fits some specific dates in Section 2.6. Similarly, we examine the short term, medium term and long term fit of the models by considering particular maturities in Section 2.7. Finally, Section 2.8 summarizes this chapter.

2.2 The Term Structure of Interest Rates and Implied Inflation

This section presents the yield curve terminology by giving some basic definitions (see Anderson *et al.*, 1996) and introduces how the Bank of England constructs UK yield curves.

2.2.1 Bond Prices and Interest Rates

A fixed-income bond is the obligation on the bond's issuer to provide one or more future cashflows on pre-specified dates. The majority of the bonds have fixed nominal interest payments and a fixed redemption or maturity date on which the issuer undertakes to repay the principal originally invested. Although the frequency at which interest payments are made varies from market to market they are mostly made either annually or semi-annually. The interest payment on a bond is referred to as a *coupon* payment.

The Bond Price Equation

The present value (PV) at rate z of an amount X due in m years time is:

$$PV(X) = \frac{X}{(1+z)^m} \tag{2.1}$$

where z is the interest rate over the period.

The interest rate z is usually referred to as the *spot interest rate* for maturity m years, because it is the interest rate that is applicable today ('spot') on an m-year loan. A bond is simply a stream of future cashflows - a series of coupon payments of size C payable at times 1,2,...,m and a redemption payment R payable on the maturity date in m years time. Supposing that the spot interest rates, z_i, for every future period, i, are known, then the present value of an m-period bond is:

$$PV(m - period\ bond) = \frac{C}{(1+z_1)} + \frac{C}{(1+z_2)^2} + ... + \frac{C+R}{(1+z_m)^m} \tag{2.2}$$

This equation is often referred to as the *bond price equation*, formalizing the relationship between spot interest rates and bond prices (Anderson *et al.*, 1996).

Discount Factors and the Discount Function

Consider an individual payment of size X due at time t. Its present value is simply:

$$PV(X) = \left[\frac{1}{(1+z(t))^t}\right] X \tag{2.3}$$

The factor by which X is multiplied to obtain its present value is called the *discount factor*. It is simply a transformation of the appropriate spot rate *z(t)*. Since time is continuous, a continuous discount function denoted $\delta(.)$ can be defined that maps time t to a discount factor. Given such a function the present value of any future cashflow can be computed by multiplying the cashflow by the appropriate point on the discount function:

$$PV(X) = \delta(t).X \tag{2.4}$$

The discount function describes the present value of one unit (e.g. 1, £1, etc.)

payable at any time in the future and so, if an instrument exists that provides a single, unit cashflow t years into the future, its price should correspond to the value of the discount function at that point, $\delta(t)$. Such an instrument would be a *zero-coupon bond*, a bond that pays no coupon payments and a unit redemption payment on the maturity date. For this reason, a discount factor is sometimes referred to as a *zero-coupon bond price* - the two are exactly equivalent (Anderson *et al.*, 1996).

Continuous Compounding

It is possible to approximate the bond price equation by assuming that coupon payments are made continuously rather than at discrete points in time, so that interest does not accrue. Under this assumption of *continuous compounding* the following equation can be written:

$$p = C \int_0^m \delta(\mu) d\mu + R\delta(m) \tag{2.5}$$

where p is the *clean* price, the price excluding any interest that has accrued since the issue or the most recent coupon payment of the bond.

Measuring the Return on a Bond

Observing the price of a bond in the market, it is straightforward to measure the *ex ante* return associated with that price. Two measures are commonly used: the *flat yield* and the *redemption yield*. The flat yield is analogous to the 'dividend yield' on a share, and is defined as:

$$Flat\ yield = \frac{Coupon}{Clean\ Price} \tag{2.6}$$

The flat yield is normally used to represent the return from holding a bond for a short period - and is often thought of as the *income* generated by the bond. The *redemption yield* (or *yield to maturity*) is the bond's internal rate of return. It is the single interest rate at which the *dirty* price (the price of a bond including the accrued interest) of a bond is equal to the present value of the stream of the cashflows discounted at that rate.

$$p + ai = \frac{C/v}{(1 + y/v)^{vt_1}} + \frac{C/v}{(1 + y/v)^{vt_2}} + ... + \frac{C/v + R}{(1 + y/v)^{vt_m}} \tag{2.7}$$

and

$$ai = t_0 C \tag{2.8}$$

where:

y = the (gross) redemption yield

p = *clean* price of bond

ai = accrued interest

t_j = maturity of bond (in years, using the appropriate day count convention)

t_0 = the proportion of a period passed since the last coupon payment was made

C = annual coupon payment

R = redemption payment

v = frequency of coupon payments (e.g. $v = 2$ for semi annual coupons)

2.2.2 The Term Structure of Interest Rates

The *term structure of interest rates*, also known as the *yield curve*, refers to the relationship between bonds of different terms and it is a very common bond valuation method. When interest rates of bonds are plotted against their terms, this is called the *yield curve*. Constructed by graphing the yield to maturities and the respective maturity dates of benchmark fixed-income securities, the yield curve is a measure of the market's expectations of future interest rates given the current market conditions.

The present value of any future cashflow can be computed by simply multiplying its nominal value by the appropriate point on the discount function. Although useful computationally, the discount function does not immediately provide a measure of the return associated with purchasing future cashflows at their present value. For this reason, the discount function is often transformed to be presented as a *spot interest rate* curve[1], a *par yield* curve[2] or an *implied forward rate* curve[3], all of which describe in different ways the return from purchasing a stream of future cashflows. Moreover, the transformation between any two of these curves is unique - given any one, the other three can be obtained (Anderson *et al.*, 1996).

2.2.3 The Bank of England UK Yield Curves

The Bank of England (2002) estimates yield curves for the United Kingdom on a daily basis. They are of two kinds. One set is based on yields on UK government bonds and on yields in the general collateral repo market. It includes nominal and real yield curves and the implied inflation term structure for the UK. The other set is based on sterling interbank rates (LIBOR) and on instruments related to LIBOR (short sterling futures contracts, forward rate aggrements and LIBOR-related interest rate swaps). These commercial bank liability curves are nominal only. The methodology used to construct the yield curves is described in the Bank of England *Quarterly Bulletin* article by Anderson and Sleath (1999) and a detailed technical description can be found in Anderson and Sleath (2001).

Anderson and Sleath (2001) presents some new estimates of the UK real and nominal yield curves for the purpose of assessing monetary conditions. These estimates differ from those presented in previous studies in a number of ways. First, the yield curves are estimated using a method put forward by Waggoner (1997) for the United States, adapted for the UK government bond market. Second, data from the generalised collateral (GC) repo market are used to provide improved estimates of the nominal yield curve at shorter maturities. Third, estimates of the real yield curve are extracted from the prices of index-linked gilts within a modified version of the framework suggested by Evans (1998).

The most basic type of information the Bank is interested in estimating is the

[1]The spot interest rate curve is the curve of gross redemption yields on zero-coupon bonds.

[2]The par yield curve specifies the interest rates at which new gilts should be priced if they are to be issued at par.

[3]The implied forward-rate curve is the curve of implied short-term interest rates in the future. It can be used to price (in a riskless way) forward bond contracts.

implied forward rates of interest at various horizons. These are important since they reflect the market's expectations about the future path of interest rates. They also provide the building-blocks for calculating other term structure variables, such as zero-coupon yields.

The government liability nominal yield curves are derived from UK gilt prices and General Collateral (GC) repo rates. The real yield curves are derived from UK index-linked bond prices. Using the Fisher relationship, the implied inflation term structure is calculated as the difference of instantaneous nominal forward rates and instantaneous real forward rates.

The spreadsheets on the Bank's website (Bank of England, 2010) provide spot rates and instantaneous forward rates for each type of curve. They also show available points on each curve out to a horizon of 25 years at half-yearly intervals. For horizons out to five years points on the curves are also available at monthly intervals.

Types of Instruments

Gilt-edged securities (gilts)

A conventional gilt is a guarantee by the Government to pay the holder of the gilt a fixed cash payment (coupon) normally every six months until the maturity date, at which point the holder receives the final coupon payment and the principal. An index-linked gilt is designed to protect of the value of the investment from erosion by inflation. This is done by adjusting coupon and principal payments to take account of accrued inflation since the gilt's issue (Bank of England, 2002).

General collateral sale and repurchase agreements (GC repo)

Gilt sale and repurchase ('gilt repo') transactions involve the temporary exchange of cash and gilts between two parties: they are a means of short-term borrowing using gilts as collateral. The lender of funds holds gilts as collateral, so is protected in the event of default by the borrower. General collateral (GC) repo rates refer to the rates for repurchase agreements in which any gilt may be used as collateral. Hence, GC repo rates should in principle be close to true risk-free rates. Repo contracts are actively traded for maturities out to one year; the rates prevailing on these contracts are very similar to the yields on comparable maturity conventional gilts.

2.2.4 Types of Yield Curve Provided

Nominal zero-coupon yields (spot interest rates)

For the data presented on the Bank's website, the nominal government spot interest rate for n years refers to the interest rate applicable today ('spot') on an n year risk-free nominal loan. It is the rate at which an individual nominal cash flow on some future date is discounted to determine its present value.

Let y_n=the n-year spot-rate of interest, and P_n be the price now of an n-year zero-coupon bond, then, for $n > 0$:

$$P_n = 1 \times (1 + y_n)^{-n} y_n = P_n^{-\frac{1}{n}} - 1 \tag{2.9}$$

By definition, it would be the yield to maturity of a nominal zero-coupon bond and can be considered as an average of single period to that maturity. Conventional dated stocks with significant amounts in issue and having more than three months to maturity, and GC repo rates (at the short end) are used to estimate these yields.

Nominal forward rates

Forward rates are the interest rates for future periods that are implicitly incorporated within today's spot interest rates for loans of different maturities. Equation 2.10 describes the relationship between the spot rate, y_t and the forward rate which is a future rate agreed now to apply from year t to $t + r$, $f_{t,t+r}$.

$$(1 + y_t)^t = (1 + f_{0,1})(1 + f_{1,1})(1 + f_{2,1})...(1 + f_{t-1,1}) \tag{2.10}$$

We can consider forward rates that rule for different periods, for example, 2-week, 3-month, 6-month or 1-year forward rates. In the limit, as the period of the loan considered tends to zero, we arrive at the *instantaneous* forward rate. Instantaneous forward rates are a stylised concept that corresponds to the notion of continuous compounding, and are commonly used measures in financial markets. Instantaneous forward rates are the building blocks of the Bank's estimated yield curves, from which other representations can be uniquely derived.

Real spot and forward rates

The return on a nominal bond can be decomposed into two components: a real rate of return and a compensation for the erosion of purchasing power arising from inflation. For conventional government nominal zero-coupon bonds, the nominal return is certain (provided that it is held to maturity) but the real return is not (because inflation is uncertain). An index-linked zero-coupon bond would have its value linked to movements in a suitable price index to prevent inflation eroding its purchasing power (so its 'real value' is protected). For such a zero-coupon bond the real return would be certain if the bond were held to maturity. A real debt market provides information on the *ex ante* real interest rates faced by borrowers and lenders who want to avoid the effects of inflation. In practice, there are factors that mean index-linked gilts do not offer exact inflation protection, and the UK index-linked gilt market is not as liquid as that for conventional UK gilts. Nevertheless, this market allows us to calculate real spot and forward rates analogous to the nominal spot and forward rates described earlier.

Implied inflation rates

As described above, the index-linked gilt market allows us to obtain real interest rates and the conventional gilt market allows us to obtain nominal interest rates. These nominal rates embody the real interest rate plus a compensation for the erosion of the purchasing power of this investment by inflation. The Bank uses this decomposition (commonly known as the Fisher relationship) and the real and the nominal yield curves to calculate the implied inflation rate factored in to nominal interest rates. This is often interpreted as a measure of inflation expectations. As with nominal and real interest rates, the 'spot' implied inflation rates are considered as the average rate of inflation expected to rule over a given period.

Similarly forward implied inflation rates can be interpreted as the rate of inflation expected to rule over a given period which begins at some future date. In the limit, instantaneous forward implied inflation rates can be calculated just as with real and nominal rates.

2.2.5 Data Coverage

The Bank of England (2010) publishes the nominal government yield curves which are available on a daily basis from 2 January 1979, and the real yield curves and implied inflation term structure are available from 2 January 1985 on their web page. The absence of data for a given day at a given maturity is due to one of the following reasons:

- There are no yield curve data for non-trading days, such as weekends and UK Bank Holidays.
- There are no data for maturities outside the range of covered by existing gilts. For example, for dates in the past where there was no bond longer than 20 years, a 20-year spot or forward rate are not provided.
- In addition, the Bank of England only provides data at maturities where they think the curve can be fitted so that it is stable and meaningful. Instability arises when small movements in bond prices lead to unrealistically large moves in the estimated yield curves, essentially because there is not enough information from observed prices at a given maturity to allow to give a robust fit in that segment of the curve. This is usually a problem at short maturities where more information is required because it is expected that the short end of the yield curve exhibits the greatest amount of structure. This is because expectations about the future path of interest rates are likely to be better informed at shorter maturities, and more likely to respond to short term news.
- In March 1997 the Bank started conducting daily money market operations in gilt

repo. Since this date the Bank has used GC repo data to estimate the short end of the nominal yield curve, and so the short end of the nominal curve is provided down to very short maturities after this date. No corresponding instrument is available to help model the short end of the real yield curve. Since implied inflation rates are calculated as the difference of the nominal and real curves, an absence of either real or nominal interest rate data at a given maturity implies an absence of corresponding implied inflation rate data at that maturity.

2.3 A Descriptive Yield Curve Model: the Cairns Model

The forward-rate curve model proposed by Cairns (1998) is designed to give an indication of what interest rates are currently implied by the market. Thus, it does not provide an arbitrage-free framework within which derivatives can be priced on their own. The curve introduced below is designed to model fixed-interest bond prices. Cairns (1998) defines $f(t, t+s)$ to be the instantaneous forward-rate curve observed at time t for payments to be made at time $t+s$.

$$f(t, t+s) = b_0(t) + b_1(t)e^{-c_1 s} + b_2(t)e^{-c_2 s} + b_3(t)e^{-c_3 s} + b_4(t)e^{-c_4 s} \tag{2.11}$$

The curve is a flexible model with four exponential terms and nine parameters in total. However, four of these parameters (the exponential rates) are fixed, which reduces the risk of multiple solutions. If the value of c_i where $i = 1, 2, 3, 4$ is small then the relevant value of b_i affects all durations whereas if c_i is large then the relevant value of b_i primarily affects the shortest durations. Considering several choices for the vector $c = (c_1, c_2, c_3, c_4)$, Cairns (1998) suggested using $c = (0.2, 0.4, 0.8, 1.6)$, values which he found to give good results over the period investigated.

Since we fit the curve on spot rates, $R(t, t+s)$ rather than forward rates, we use the representation below of the model which is specified by Cairns (1998).

$$
\begin{aligned}
R(t,t+s) &= \frac{1}{s}\int_0^s f(t,t+u)du \qquad (2.12)\\
&= b_0(t) + b_1(t)\frac{1-e^{-c_1 s}}{c_1 s} + b_2(t)\frac{1-e^{-c_2 s}}{c_2 s} + b_3(t)\frac{1-e^{-c_3 s}}{c_3 s} + b_4(t)\frac{1-e^{-c_4 s}}{c_4 s}
\end{aligned}
$$

We fit the Cairns Model on to the three daily yield curves, nominal, implied inflation and real spot rates which are published on the Bank of England's (2010) web page. We have 7838 (6320) observations for the nominal (implied inflation and real) spot rates to fit the curve and estimate the parameters. The daily nominal spot rates are based on half year maturities starting with 6 months and ending with 25 years, i.e. 50 different maturities, and the daily implied inflation and real spot rates are based on half year maturities starting with 2.5 years and ending with 25 years, i.e. 46 different maturities.

Let R_{kT} represent the daily nominal spot rates on different maturities on a single day, $k = 1, 2, ..., 7838$ (January 1979-December 2009) and T is the maturity in years, $T = 0.5, 1, ..., 25$. On some trading days, yields are not available for all maturities because the start and end points of the estimated curves depend on the shortest and longest market instruments for which reliable prices are available. Therefore, the range of maturities for which yields are available may vary according to the instruments available.

We can rewrite the model for each day as:

$$
\begin{aligned}
\widehat{R}_{kT} &= b_0(k) + b_1(k)\frac{1-e^{-c_1 T}}{c_1 T} + b_2(k)\frac{1-e^{-c_2 T}}{c_2 T} + b_3(k)\frac{1-e^{-c_3 T}}{c_3 T} \qquad (2.13)\\
&+ b_4(k)\frac{1-e^{-c_4 T}}{c_4 T}
\end{aligned}
$$

We derive the analytical solution and estimate the b parameters as below.

$$
\begin{aligned}
\widehat{R}_{kT}(b_0,b_1,b_2,b_3,b_4) &= b_0 d_0^{(T)} + b_1 d_1^{(T)} + b_2 d_2^{(T)} + b_3 d_3^{(T)} + b_4 d_4^{(T)} \\
\widehat{R}_{kT} &= \underline{d}^{(T)'}\underline{b}. \\
Let\ S(b) &= \sum_{T=1}^{50} (R_{kT} - \widehat{R}_{kT})^2 \\
&= \sum_{T=1}^{50} (R_{kT} - \underline{d}^{(T)'}\underline{b})^2 \\
&= \sum_{T=1}^{50} (R_{kT}^2 - 2R_{kT}\underline{d}^{(T)'}\underline{b} + \underline{b}^{'} D^{(T)}\underline{b}). \\
S(b)\ is\ minimised\ when\ \frac{\partial S}{\partial \underline{b}} &= \sum_{T=1}^{50} (-2R_{kT}\underline{d}^{(T)'} + 2\underline{b}^{'} D^{(T)}) = 0 \\
2\sum_{T=1}^{50} \underline{b}^{'} D^{(T)} &= 2\sum_{T=1}^{50} R_{kT}\underline{d}^{(T)'} \\
\underline{b}^{'} D &= R_{kT}\underline{d}^{'} \\
\underline{b}^{'} &= R_{kT}\underline{d}^{'} D^{-1}
\end{aligned}
$$

where:

$$
\underline{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

$$
R_{kT} = \begin{bmatrix} R_{k1} & R_{k2} & . & . & . & R_{k50} \end{bmatrix}
$$

$$
\underline{d} = \begin{bmatrix} 1 & 1 & 1 & . & . & . & . & 1 \\ \frac{1-e^{-c_1 0.5}}{c_1 0.5} & \frac{1-e^{-c_1 1}}{c_1 1} & . & . & . & . & . & \frac{1-e^{-c_1 25}}{c_1 25} \\ \frac{1-e^{-c_2 10.5}}{c_2 10.5} & \frac{1-e^{-c_2 1}}{c_2 1} & . & . & . & . & . & \frac{1-e^{-c_2 25}}{c_2 25} \\ \frac{1-e^{-c_3 0.5}}{c_3 0.5} & \frac{1-e^{-c_3 1}}{c_3 1} & . & . & . & . & . & \frac{1-e^{-c_3 25}}{c_3 25} \\ \frac{1-e^{-c_4 0.5}}{c_4 0.5} & \frac{1-e^{-c_4 1}}{c_4 1} & . & . & . & . & . & \frac{1-e^{-c_4 25}}{c_4 25} \end{bmatrix}
$$

$$
D^{(T)} = d^{(T)} d^{(T)\prime}.
$$

As we mentioned before, we will present four different C parameter sets we used to estimate the b parameters for each yield curve data. Three of these sets are proposed in Cairns (1998) and Cairns and Pritchard (2001) and the last one is obtained using the least squares method including a penalty function, $P(\underline{c})$ which is given below:

$$
Residuals = \sum (Observed - Fitted)^2 + (-\log(P(\underline{c})) \times 0.0001) \tag{2.14}
$$

where

$$
\begin{aligned} P(\underline{c}) \quad = \quad & c_1^2 \times \exp(-\beta \times c_1) \times (\frac{c_2}{c_1} - 1)^2 \times \exp(-\beta \times \frac{c_2}{c_1}) \times (\frac{c_3}{c_2} - 1)^2 \\ & \times \exp(-\beta \times \frac{c_3}{c_2}) \times (\frac{c_4}{c_3} - 1)^2 \times \exp(-\beta \times \frac{c_4}{c_3}) \times I(0 < c_1 < c_2 < c_3 < c_4) \end{aligned} \tag{2.15}
$$

with $\beta = 1$.

When we try the numerical optimization without the penalty function, we see that c values can become negative or equal to each other and the algorithm does not converge. In order to avoid these problems we add the penalty function which is designed to keep the c values positive and apart from each other. After trying different multiplication factors (0.0001, 0.00001, 0.000001 and 0.000001) to decrease the effect of the penalty function to see how much it dominates the original least square equation, we decided to use 0.0001 since decreasing the number makes the c values closer. Besides, the multiplication factor we used gives the smallest root mean square error which indicates a better fit. Furthermore, we increased the β value in $P(\underline{c})$ to see its effect on the c values when we decreased the effect of the penalty function by decreasing the multiplication function. Increasing the β value does not make a significant change in the c values.

An alternative would have been to fit the Cairns curve each day separately and estimate all nine parameters simultaneously. However, Cairns (1998) shows that there might be multiple minima on specific days and the minimisation algorithm may start at the previous minimum and stay near that minimum. On other days the chosen minimum might be only a local minimum and not the global minimum. On other days, the algorithm may jump to what an alternative local minimum. This type of discontinuity between different days can be referred to as a '*catastrophic*' jump. At the time of the catastrophic jump there might be an identifiable shift in the shape of the fitted yield curve. Fitting one set of values of the c parameters at least means that the same values are used on all days.

We have tried six different C parameter sets for the yield curve data ($C1$, $C2$, $C3$, $COpt(Nom)$, $COpt(Imp)$ and $COpt(Real)$) and estimated the b parameters for every observation using each set. By changing the C parameter sets we obtained different loadings for b parameters. The loading on b_0 is 1, for each model, a constant that does not decay to zero in the limit; hence it may be viewed as a long-term factor or overall level of the spot rate curve. Furthermore, b_1 has more influence over the long-term, while b_4 has more influence over the short term (Cairns, 1998; Diebold and Li, 2006). We plot the loadings on b parameter sets for different C sets in Figure 2.1. Beside particular influences of the b_1 and b_4 parameters because of their loadings, an overall increase in C values improves the fit for short maturities while an overall decrease improves the long maturity fit. Figure 2.1 shows the factor loadings (C parameter sets) for each model. When we look at Figure 2.1, we expect that $C1$, having the highest factor loadings, captures the short-term movement better than the others since the loadings on b parameters decay to zero faster than the other C sets. In the same way, $COpt(Imp)$ should fit the long-term maturities better due to the lower values.

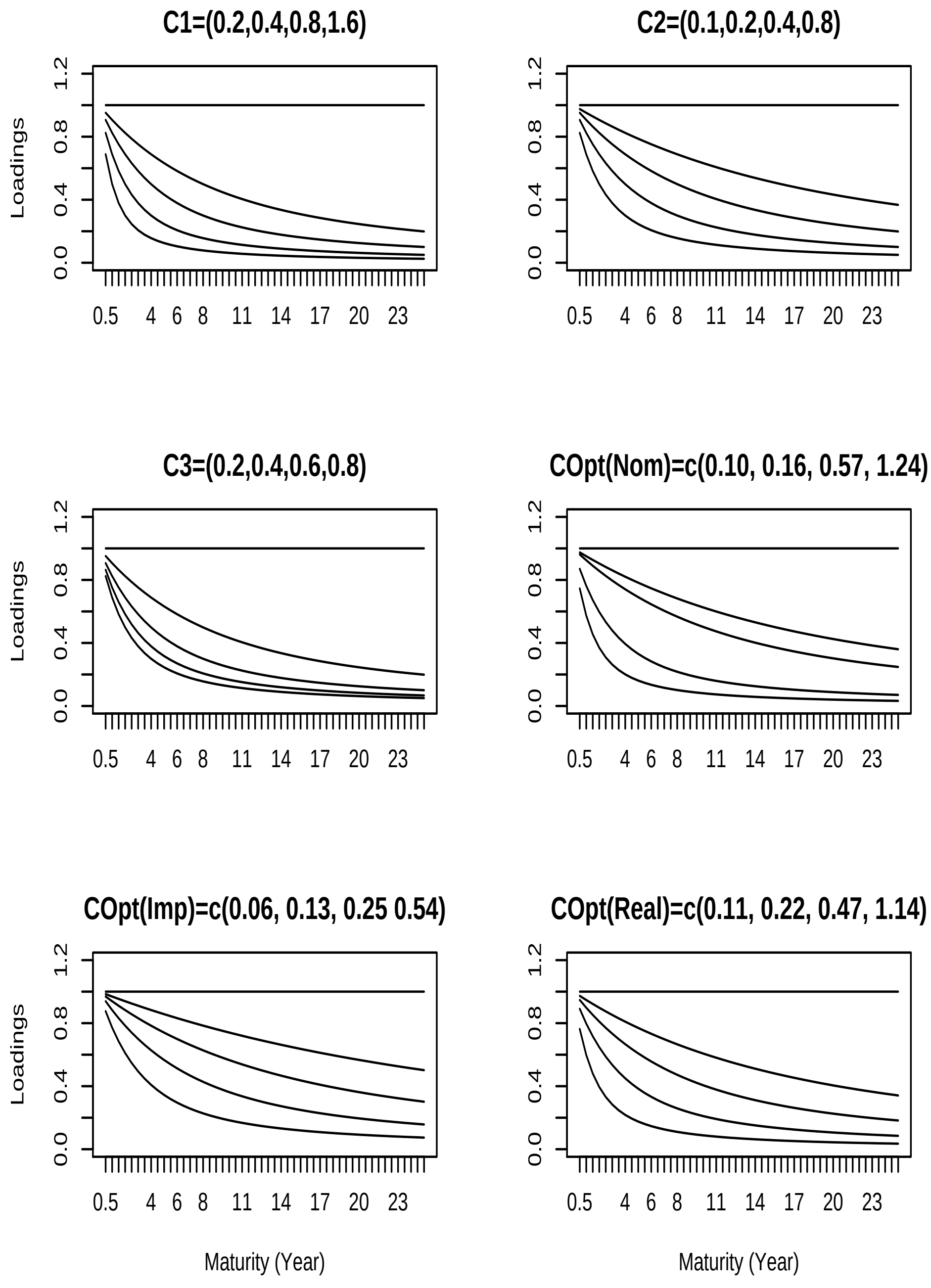

Figure 2.1: Factor Loadings

2.4 Parameter Estimates of the Yield Curves

As mentioned in Section 2.2, we use the daily spot rates, for the longest available periods published on the Bank of England's web page, to fit the Cairns Model. Therefore, we have 7838 daily observations for the nominal spot rates and 6320 daily observations for the implied inflation and real spot rates based on half year maturities. Before fitting the models with different C parameter sets, we explore the yield curves, considering some descriptive statistics. Table 2.1 shows these statistics for each yield curve. It is seen that in a typical yield curve, long rates are less volatile and more persistent than short rates. The yield curves are not upward sloping.

We display the daily yield curves for specific maturities in Figure 2.2, Figure 2.3 and Figure 2.4. Figure 2.2 shows the daily nominal spot rates for short-term, medium-term and long-term maturities. The discontinuity in the black solid lines indicates that the spot rates for those specific dates are missing. The graphs show that the nominal rates has been decreasing since 1979 independent from the maturity. There are many missing values in the data particularly in the long-term (20 and 25-year) maturities which can be explained by the lack of instruments to obtain those spot rates as we discussed in Section 2.3. Furthermore, the spot rates are quite stable for the medium and long-term (10-year to 25-year) maturities since 1998 which coincides with the inflation targeting policy of the Bank of England.

Table 2.1: Descriptive Statistics for the Daily Yield Curves

Nominal Spot Rates (%)							
Maturity	Mean	Standard deviation	Median	Minimum	Maximum	Skewness	Excess kurtosis
0.5	7.1921	3.4366	6.0192	0.3375	15.9315	0.4208	-0.5951
2.5	7.7984	3.2216	6.9589	1.3364	15.6795	0.2062	-0.9535
5	7.9544	3.1481	7.5146	2.2082	15.9370	0.2614	-1.0122
10	8.0488	3.1195	8.1250	3.0576	15.5571	0.2898	-1.0938
15	7.8749	2.9660	8.2566	3.6515	15.0450	0.3329	-0.9727
20	6.6201	2.1746	6.1600	3.7016	13.5690	0.3893	-1.0571
25	4.6714	0.9024	4.4713	3.5915	8.6602	3.3440	11.0137
Implied Inflation Spot Rates (%)							
Maturity	Mean	Standard deviation	Median	Minimum	Maximum	Skewness	Excess Kurtosis
2.5	4.1343	2.2100	3.2868	-3.0390	9.7851	0.3909	-0.6031
5	4.1446	1.8168	3.5638	-0.9900	9.1030	0.5476	-0.7532
10	4.1277	1.5485	3.5790	1.1549	8.1779	0.5798	-0.8878
15	3.9859	1.2894	3.7118	1.9805	7.4018	0.4740	-0.9583
20	3.4838	0.9207	3.2171	1.9231	6.0000	0.5565	-0.8160
25	2.9202	0.4637	2.8472	1.7835	4.0591	0.6006	-0.4605
Real Spot Rates (%)							
Maturity	Mean	Standard deviation	Median	Minimum	Maximum	Skewness	Excess Kurtosis
2.5	2.7499	1.0013	2.9619	-0.6634	5.7400	-0.5305	0.2550
5	2.7985	0.8903	2.8721	0.1205	5.1222	-0.3562	-0.5489
10	2.8697	0.9997	3.0558	0.5530	5.0887	-0.1942	-1.2295
15	2.8775	1.0907	3.1543	0.6821	4.9308	-0.1879	-1.3526
20	2.5353	1.1018	2.2311	0.5609	4.8077	0.1166	-1.3584
25	1.5806	0.6001	1.6612	0.4128	3.1241	0.1476	-0.8344

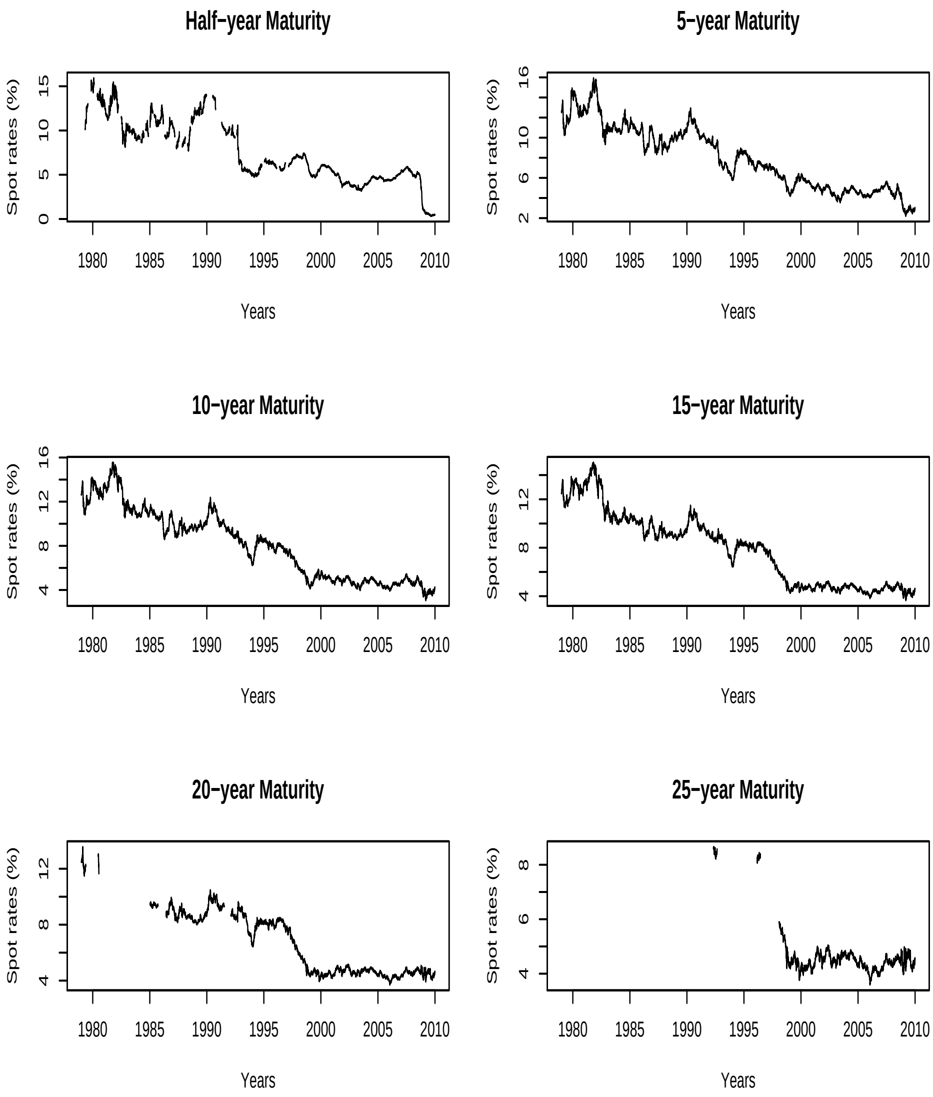

Figure 2.2: Daily Nominal Spot Rates Data for Different Maturities (1979-2009)

Figure 2.3 presents the daily implied inflation spot rates for the various maturities starting from 2.5-year and ending with 25-year maturity for the period 1985-2009. The short-term implied inflation rates have decreased (even below zero) sharply since the second half of 2008 due to financial crises experienced by most of the industrial countries. The effect of the crises is much less on the medium-term and long-term implied inflation spot rates. There are many missing values especially in the long-term implied inflation data.

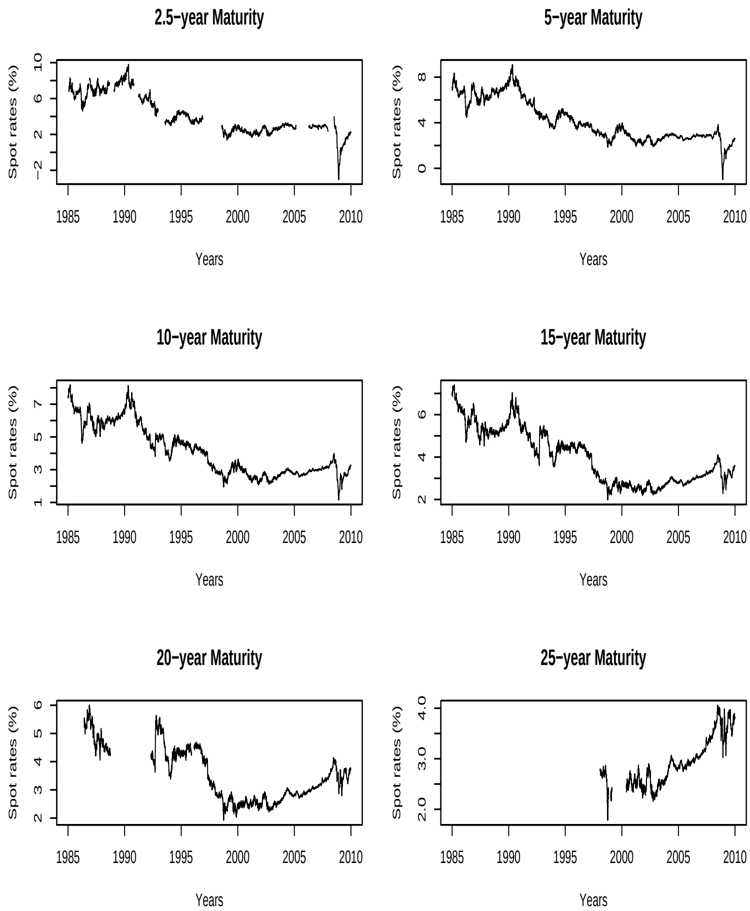

Figure 2.3: Daily Implied Inflation Spot Rates Data for Different Maturities (1985-2009)

As we see in Figure 2.4, the real spot rates (1985-2009) are much more stable except for the 2008 financial crises period than the nominal and implied inflation spot rates. The graphs show that the crises mostly affected the short-term real rates. Similar to nominal rates, there is a continuous decrease in the real spot rates for the medium and long-term maturities.

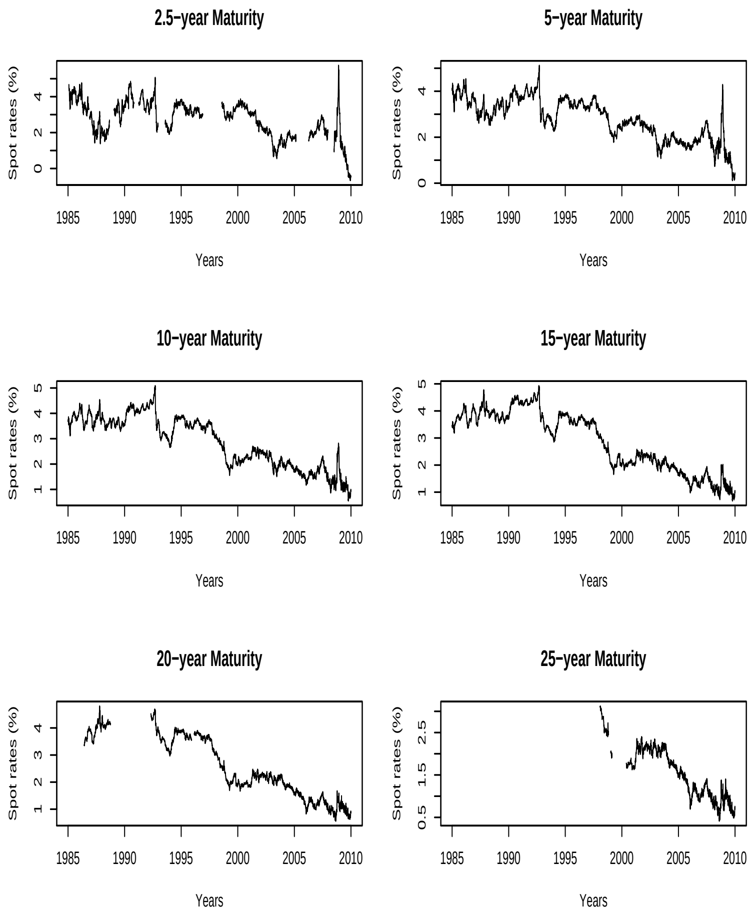

Figure 2.4: Daily Real Spot Rates Data for Different Maturities (1985-2009)

2.4.1 Model 1 with $C1 = (0.2, 0.4, 0.8, 1.6)$

The first model we fit has the same C values as Cairns (1998). Table 2.2 shows the product-movement correlation coefficients between estimated b parameters. It is clearly seen that while b_0 is not linearly related with the other parameters for the nominal and implied inflation spot rates, there are high positive or negative correlations between b_1, b_2, b_3 and b_4. However, when we look at the correlations between the b parameters for the real spot rates we see that all parameters are significantly correlated. Although we do not display them here, the autocorrelation coefficients for all the parameters decay exponentially which indicates an autoregressive effect. The partial autocorrelation plot also supports this conclusion since the first lags are significant.

As seen in Table 2.2 there are negative correlations between the lagged values of b_1 and b_2, b_1 and b_4 and positive correlations between the lagged values of b_1 and b_3, b_2 and b_4.

Table 2.2: Correlation Matrices for the b parameters for Model 1

Nominal Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	0.0223	-0.0378	0.0096	0.0197
b_1	0.0223	1.0000	-0.9763	0.9279	-0.8581
b_2	-0.0378	-0.9763	1.0000	-0.9807	0.9249
b_3	0.0096	0.9279	-0.9807	1.0000	-0.9715
b_4	0.0197	-0.8581	0.9249	-0.9715	1.0000
Implied Inflation Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.3608	0.1431	0.0114	-0.0874
b_1	-0.3608	1.0000	-0.9304	0.7749	-0.6308
b_2	0.1431	-0.9304	1.0000	-0.9418	0.8297
b_3	0.0114	0.7749	-0.9418	1.0000	-0.9625
b_4	-0.0874	-0.6308	0.8297	-0.9625	1.0000
Real Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.6557	0.4910	-0.5237	0.5644
b_1	-0.6557	1.0000	-0.9122	0.8206	-0.8112
b_2	0.4910	-0.9122	1.0000	-0.9627	0.9290
b_3	-0.5237	0.8206	-0.9627	1.0000	-0.9790
b_4	0.5644	-0.8112	0.9290	-0.9790	1.0000

2.4.2 Model 2 with $C2 = (0.1, 0.2, 0.4, 0.8)$

For the second model, we use another C parameter set published in Cairns (1998). This set has smaller values which is appropriate for fitting to the long-term rates compared with *C1*. Table 2.3 shows the correlation coefficients between b parameters. All of the parameters are significantly positively or negatively correlated with each other. b_0 has high negative correlations with b_1 and b_3, while it has high positive correlation with b_2 for the nominal yield curve. Furthermore, b_1 and b_2, b_2 and b_3, b_3 and b_4 are highly negatively correlated. We see similar high correlations between the parameters for implied and real rates as well.

Table 2.3: Correlation Matrices for the b parameters for Model 2

Nominal Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.8030	0.7890	-0.7319	0.4931
b_1	-0.8030	1.0000	-0.9804	0.8784	-0.5574
b_2	0.7890	-0.9804	1.0000	-0.9492	0.6687
b_3	-0.7319	0.8784	-0.9492	1.0000	-0.8443
b_4	0.4931	-0.5574	0.6687	-0.8443	1.0000
Implied Inflation Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.8543	0.6452	-0.3651	-0.0084
b_1	-0.8543	1.0000	-0.9191	0.6875	-0.2463
b_2	0.6452	-0.9191	1.0000	-0.9048	0.5179
b_3	-0.3651	0.6875	-0.9048	1.0000	-0.8043
b_4	-0.0084	-0.2463	0.5179	-0.8043	1.0000
Real Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.6644	0.5346	-0.6021	0.4012
b_1	-0.6644	1.0000	-0.9498	0.8316	-0.4013
b_2	0.5346	-0.9498	1.0000	-0.9186	0.4155
b_3	-0.6021	0.8316	-0.9186	1.0000	-0.6328
b_4	0.4012	-0.4013	0.4155	-0.6328	1.0000

2.4.3 Model 3 with $C3 = (0.2, 0.4, 0.6, 0.8)$

Model 3 includes the *C3* parameter set (Cairns and Pritchard, 2001) whose values are between *C1* and *C2* which means we expect it to fit the short term yield better than *C2* does and to fit the long term yield better than *C1* does. Table 2.4 displays the correlation coefficients between b parameters. Again, while b_0 is uncorrelated with the other parameters as in Model 1, b_1, b_2, b_3 and b_4 have high negative or positive correlations with each other for the nominal and real spot rates.

Table 2.4: Correlation Matrices for the b parameters for Model 3

Nominal Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.1150	0.1214	-0.1405	0.1538
b_1	-0.1150	1.0000	-0.9785	0.9464	-0.9078
b_2	0.1214	-0.9785	1.0000	-0.9908	0.9673
b_3	-0.1405	0.9464	-0.9908	1.0000	-0.9919
b_4	0.1538	-0.9078	0.9673	-0.9919	1.0000
Implied Inflation Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.4094	0.1818	-0.0703	0.0000
b_1	-0.4094	1.0000	-0.9250	0.8347	-0.7562
b_2	0.1818	-0.9250	1.0000	-0.9787	0.9372
b_3	-0.0703	0.8347	-0.9787	1.0000	-0.9878
b_4	0.0000	-0.7562	0.9372	-0.9878	1.0000
Real Spot Rates					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.6517	0.5404	-0.5645	0.5862
b_1	-0.6517	1.0000	-0.9407	0.9060	-0.8908
b_2	0.5404	-0.9407	1.0000	-0.9898	0.9698
b_3	-0.5645	0.9060	-0.9898	1.0000	-0.9915
b_4	0.5862	-0.8908	0.9698	-0.9915	1.0000

2.4.4 Model 4 with $COpt(Nom) = (0.10, 0.16, 0.57, 1.24)$, $COpt(Imp) = (0.06, 0.13, 0.25, 0.54)$ or $COpt(Real) = (0.11, 0.22, 0.47, 1.14)$

Model 4 includes the optimised parameters using the Nelder-Mead numerical optimization method adding the penalty function (see Section 2.3) which prevents the C parameters taking negative values and keep away from each other, satisfying the condition $0 < c_1 < c_2 < c_3 < c_4$. The optimization results show that the first three C values, which are small, fit the long term rates and the last one is relatively larger and fits the short term rates.

Table 2.5 displays the correlations between b parameters. As in the previous models, b_0, b_1, b_2, b_3 and b_4 have high negative or positive correlations with each other.

Table 2.5: Correlation Matrices for the b parameters for Model 4

Nominal Spot Rates $COpt(Nom)$					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.7892	0.7801	-0.7165	0.5192
b_1	-0.7892	1.0000	-0.9943	0.8642	-0.5829
b_2	0.7801	-0.9943	1.0000	-0.9008	0.6270
b_3	-0.7165	0.8642	-0.9008	1.0000	-0.8317
b_4	0.5192	-0.5829	0.6270	-0.8317	1.0000
Implied Inflation Spot Rates $COpt(Imp)$					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.9477	0.8202	-0.5907	0.2704
b_1	-0.9477	1.0000	-0.9513	0.7722	-0.4447
b_2	0.8202	-0.9513	1.0000	-0.9208	0.6458
b_3	-0.5907	0.7722	-0.9208	1.0000	-0.8778
b_4	0.2704	-0.4447	0.6458	-0.8778	1.0000
Real Spot Rates $COpt(Real)$					
	b_0	b_1	b_2	b_3	b_4
b_0	1.0000	-0.6282	0.4618	-0.5445	0.4933
b_1	-0.6282	1.0000	-0.9361	0.8205	-0.5142
b_2	0.4618	-0.9361	1.0000	-0.9250	0.5076
b_3	-0.5445	0.8205	-0.9250	1.0000	-0.6329
b_4	0.4933	-0.5142	0.5076	-0.6329	1.0000

2.5 Standard Errors Analysis

We compare the performance of these four different models in different ways. One way is to analyse the standard errors (root mean squared errors (RMSE)) of the residuals by fitting these models. We draw the graphs of RMSEs in basis points (bps) for whole maturities, then for short term, medium term and long term maturities separately and the graphs of the ratios of these standard errors by taking the best set of C set for each yield curve based on the mean RMSEs as a reference. The RMSEs are calculated using the formula below.

The mean squared error for date k is:

$$\begin{aligned} Mean\ Square\ Error\ (MSE_k) &= \frac{\sum_{t=1}^{T}(R_{kt} - \widehat{R}_{kt})^2}{T} && (2.16) \\ Root\ Mean\ Square\ Error\ (RMSE_k) &= \sqrt{MSE_k} && (2.17) \end{aligned}$$

where $\widehat{R}_{kt}$ is the fitted spot rate, R_{kt} is the observed spot rate, $T = 1, ..., 50$ ($T = 1, ..., 46$ for the implied inflation and real spot rates) is the associated maturity of the observed day $k = 1, ..., 7838$ ($k = 1, ..., 6320$ for the implied inflation and real spot rates).

To begin with, we draw the graphs of standard errors for whole maturities (6 months to 25 years or 2.5 years to 25 years) for the whole period (1979 to 2009 for nominal spot rates and 1985 to 2009 for implied inflation and real spot rates). Figure 2.5, 2.13 and 2.21 show these graphs for nominal, implied inflation and real spot rates respectively. In order to compare the RMSEs for each C parameter set, we present the graphs in the same scale. Figure 2.5 indicates that $C2$ and $COpt(Nom)$ have lower RMSEs compared to $C1$ and $C3$. Since the RMSE values seem very close to each other, we can compare the fit for these different C sets by examining Table 2.6 which gives the mean RMSEs for different maturities including overall, short-term, medium-term and long-term for nominal spot rates. The mean RMSEs are calculated as below:

$$Mean\ RMSE = \frac{\sum_{k=1}^{7838(6320)} RMSE_k}{7838(6320)} \tag{2.18}$$

Table 2.6: Mean RMSE (bps) for Different C Parameter Sets for Nominal Spot Rates

	Overall	Short-term	Medium-term	Long-term
Model 1	2.9114	3.1585	2.5175	3.1680
Model 2	1.8320	2.9340	**1.2608**	**1.2361**
Model 3	2.1852	2.8310	1.6747	2.1439
Model 4 (Opt)	**1.7297**	**2.5027**	1.3287	1.3731

Table 2.6 justifies our comment on Figure 2.5 that Model 1 has the highest mean RMSE (2.91 bps) while Model 4 has the lowest (1.73 bps) considering all maturities. Furthermore, we display the ratios of the standard errors for different models on the logarithmic scale indicating the equality line for the ratios. The reference C parameter set has been chosen as the one which produces the smallest mean squared errors for each yield curve. Figures 2.6, 2.14 and 2.22 show these ratios for the nominal, implied inflation and real spot rates for all available maturities. Figure 2.6 shows that especially Model 1 and Model 3 produce relatively higher RMSEs compared to Model 4 in which we use $COpt(Nom)$ as the C parameter set due to having more values above the equality line which is displaced with red colour. On the other hand, Model 2 performs slightly worse than Model 4 since the ratios of the RMSEs are quite close to 1 for all period. Figures 2.7 to 2.12 show the performance of these different C parameter sets for different maturities. We have decreased the C values in order to have a better fit for the long term maturities and increased to have a better short term fit. Figure 2.7 shows the standard errors of the models for short term maturities (i.e. from 6 months to 5 years). By looking at Figure 2.7, we see that Model 4 has the best fit due to its smaller RMSE. Table 2.6 indicates that Model 4 has the best fit for the short-term maturities while all of the models have relatively similar mean RMSEs.

Figure 2.8 shows the ratios of the standard errors for the short term maturities between the three C parameter sets and $COpt(Nom)$ set. These graphs also show that all models have similar RMSEs.

Figure 2.9 and Figure 2.10 show the standard errors and ratios of the standard errors of the models for medium term maturities (i.e. from 5 years to 15 years). Model 2 and Model 4 have better fits with the mean RMSE values 1.26 bps and 1.33 bps respectively than other two models.

Finally, Figure 2.11 and Figure 2.12 display the standard errors and the ratios of the standard errors of the models for long term maturities (i.e. from 15 years to 25 years). As we decrease the values of the C parameter set we get a better fit for the long term. Therefore, Model 2 and Model 4 perform very well due to producing low values. Figure 2.12 also supports our comment showing that the standard errors of these two models are quite close to each other and less than Model 1 and Model 3. Table 2.6 also shows that Model 2 has the smallest mean RMSE for the long-term maturities with 1.24 bps.

Considering all these graphs and the mean RMSEs displayed in Table 2.6, we conclude that Model 4 with $COpt(Nom)$ performs better than the other models for the overall and the short-term maturities while Model 2 with $C2$ parameter set has the smallest RMSEs for the medium and long-term maturities for the nominal spot rates. We chose $COpt(Nom)$ parameter set to fit the Cairns model on to the nominal spot rates for further work.

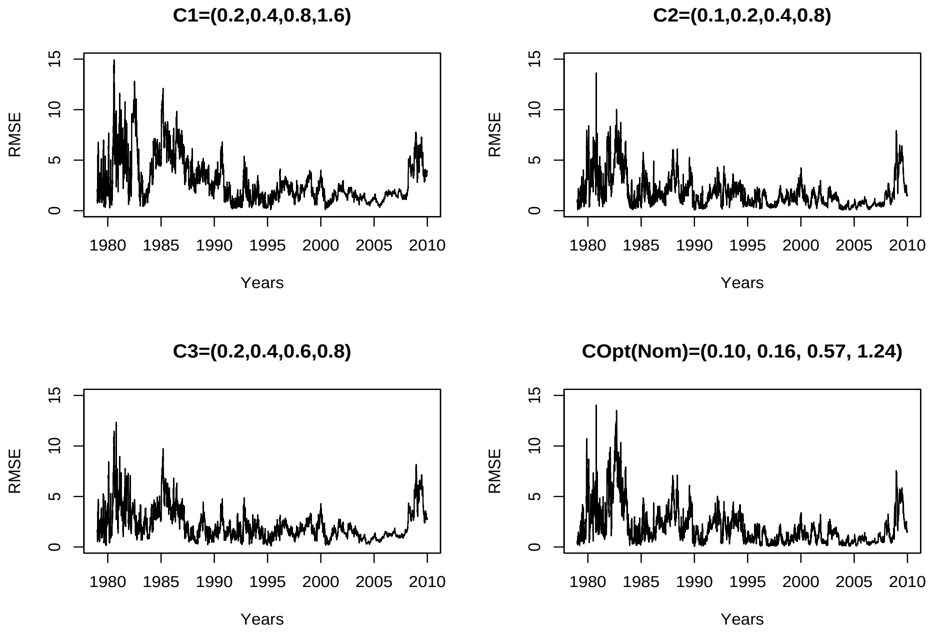

Figure 2.5: Root Mean Squared Errors for Nominal Spot Rates (in basis points)

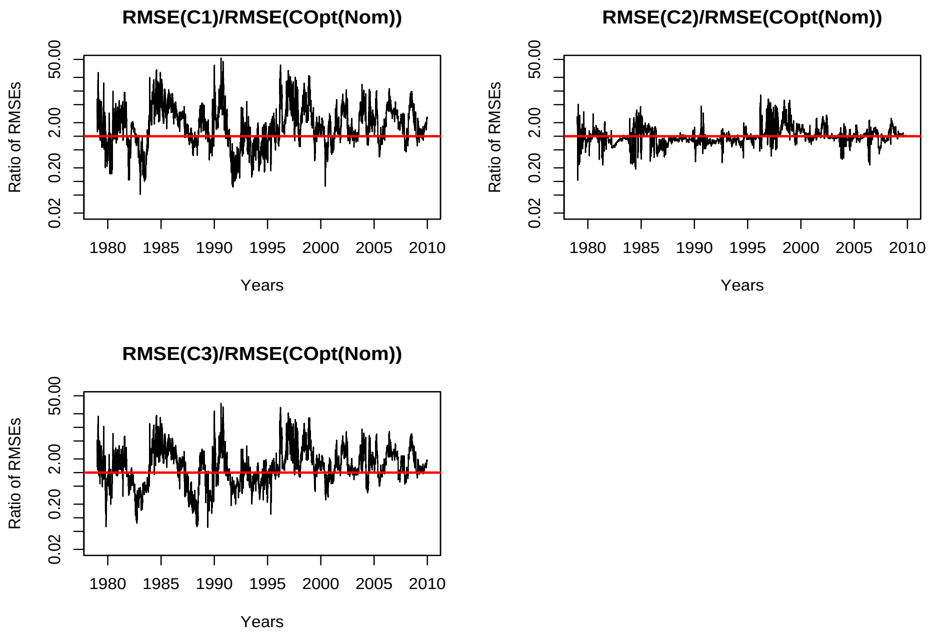

Figure 2.6: Ratios of Standard Errors for Different C Parameter Sets for Nominal Spot Rates

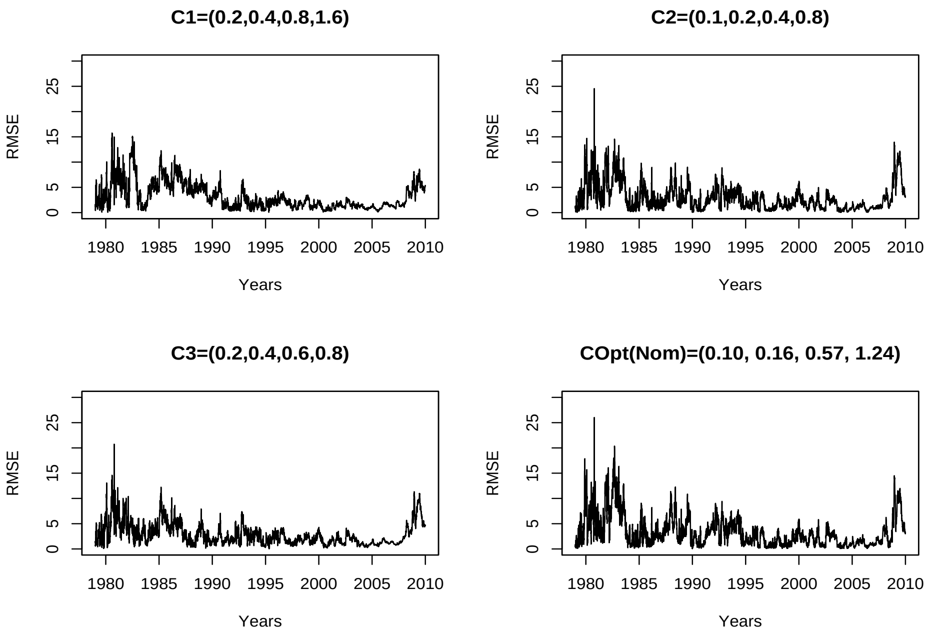

Figure 2.7: Standard Errors for Different C Parameter Sets for Short Term Nominal Spot Rates

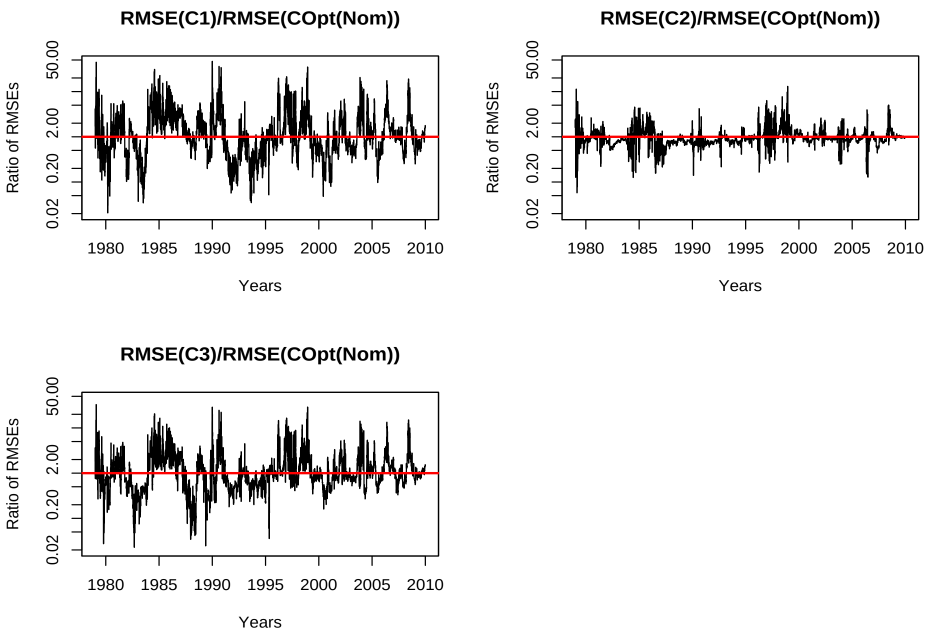

Figure 2.8: Ratios of Standard Errors for Different C Parameter Sets for Short Term Nominal Spot Rates

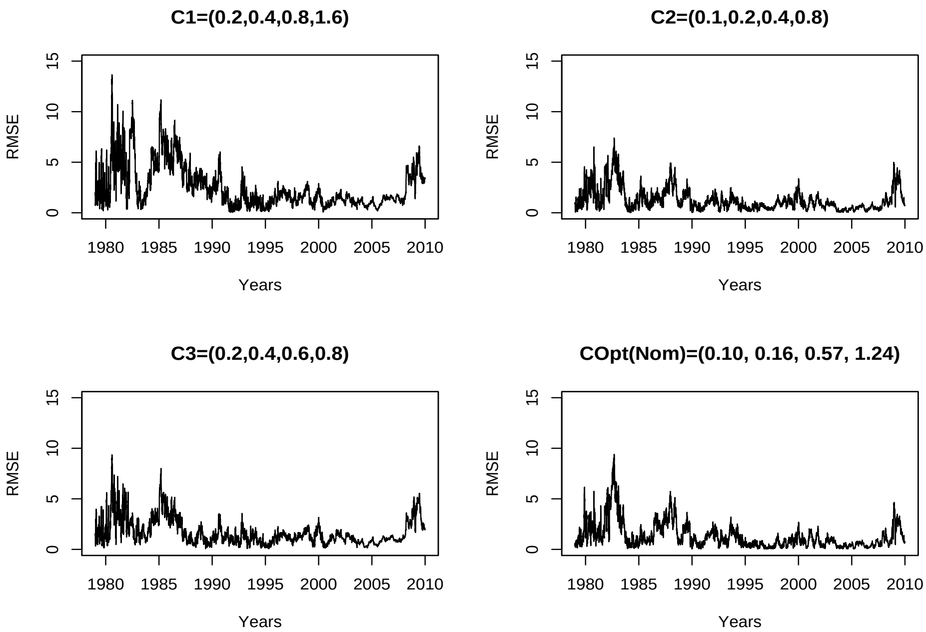

Figure 2.9: Standard Errors for Different C Parameter Sets for Medium Term Nominal Spot Rates

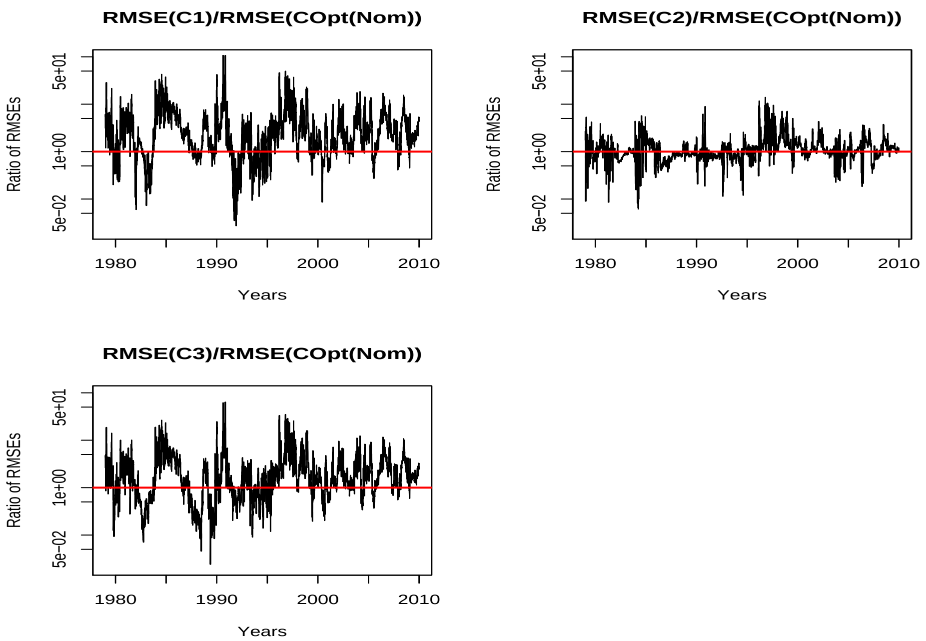

Figure 2.10: Ratios of Standard Errors for Different C Parameter Sets for Medium Term Nominal Spot Rates

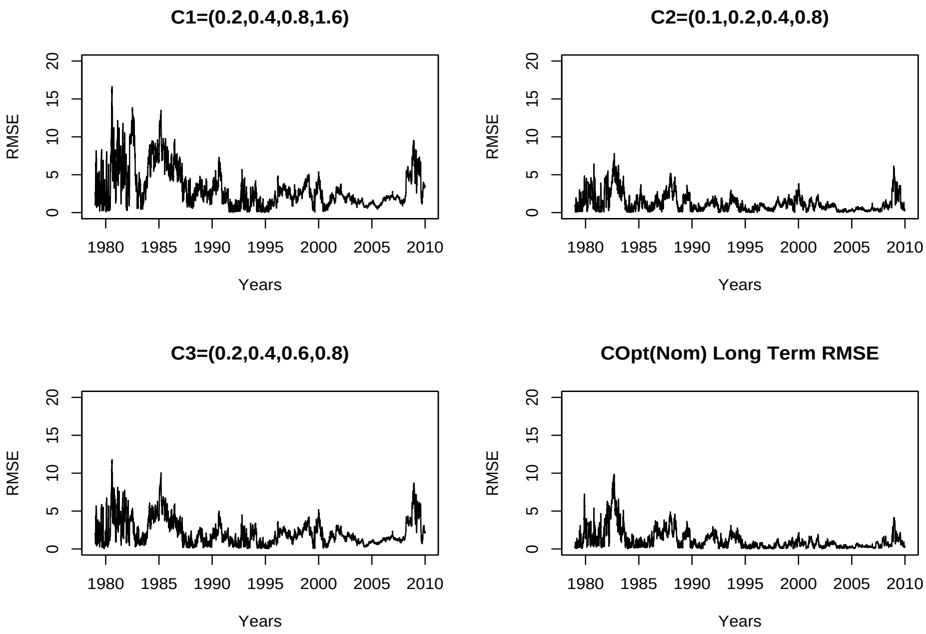

Figure 2.11: Standard Errors for Different C Parameter Sets for Long Term Nominal Spot Rates

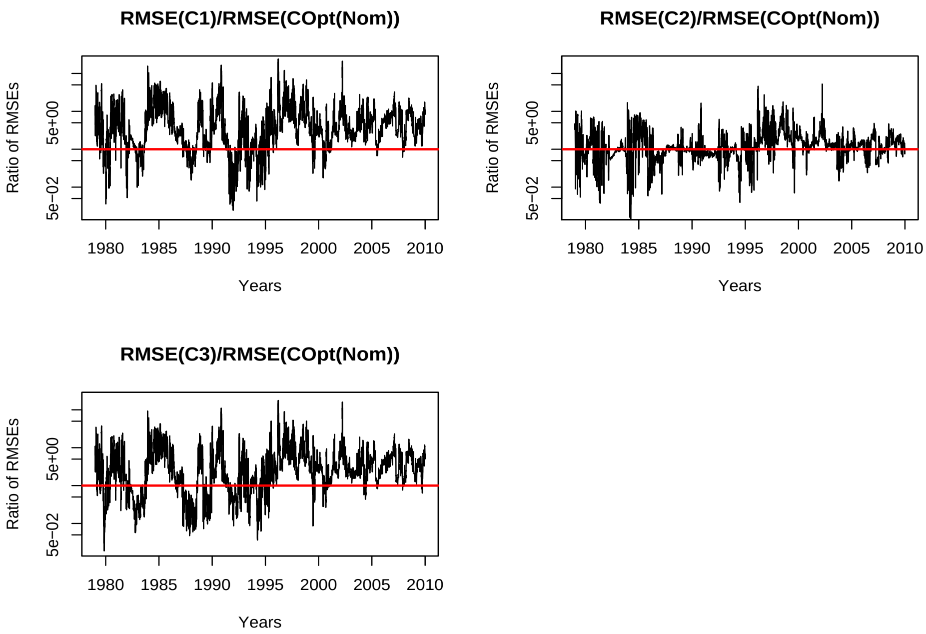

Figure 2.12: Ratios of Standard Errors for Different C Parameter Sets for Long Term Nominal Spot Rates

As for the implied inflation spot rates, Figure 2.13 and Figure 2.14 imply that $COpt(Imp)$ produces the lowest RMSEs for the whole period. Table 2.7 supports this conclusion and presents that not only for the overall maturities, for the short, medium and long-term spot rates, Model 4 produces the smallest mean RMSEs. Figures 2.15 to 2.20 show that Model 4 with the $COpt(Imp)$ parameter set is the best among the others.

Table 2.7: Mean RMSE (bps) for Different C Parameter Sets for Implied Inflation Spot Rates

	Overall	Short-term	Medium-term	Long-term
Model 1	1.0332	1.0812	0.8719	1.1839
Model 2	0.4476	0.5702	0.4266	0.3991
Model 3	0.7339	0.7444	0.6334	0.8372
Model 4 (Opt)	**0.3268**	**0.4924**	**0.3084**	**0.2355**

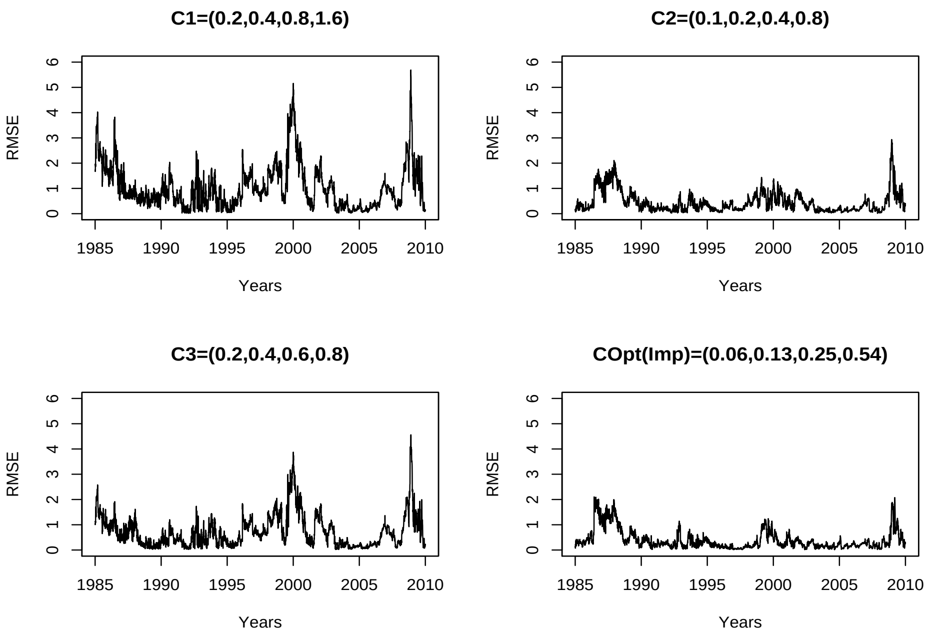

Figure 2.13: Root Mean Squared Errors for Implied Inflation Spot Rates (in basis points)

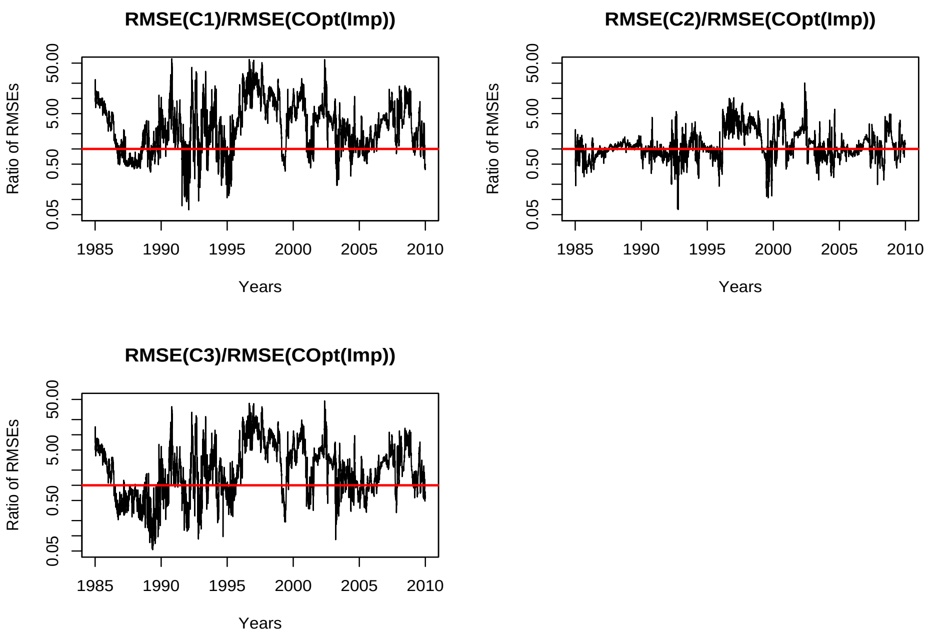

Figure 2.14: Ratios of Standard Errors for Different C Parameter Sets for Implied Inflation Spot Rates

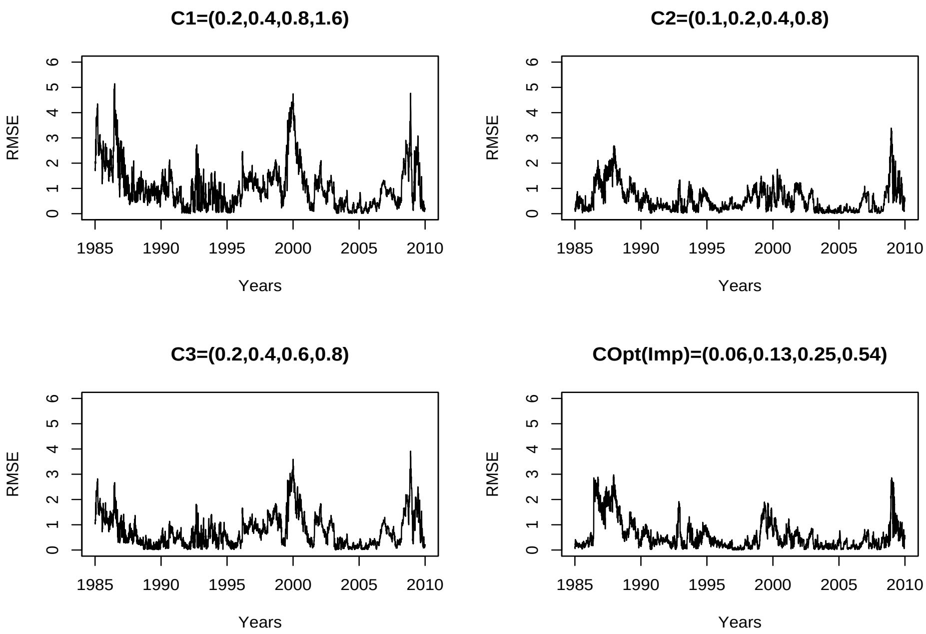

Figure 2.15: Standard Errors for Different C Parameter Sets for Short Term Implied Inflation Spot Rates

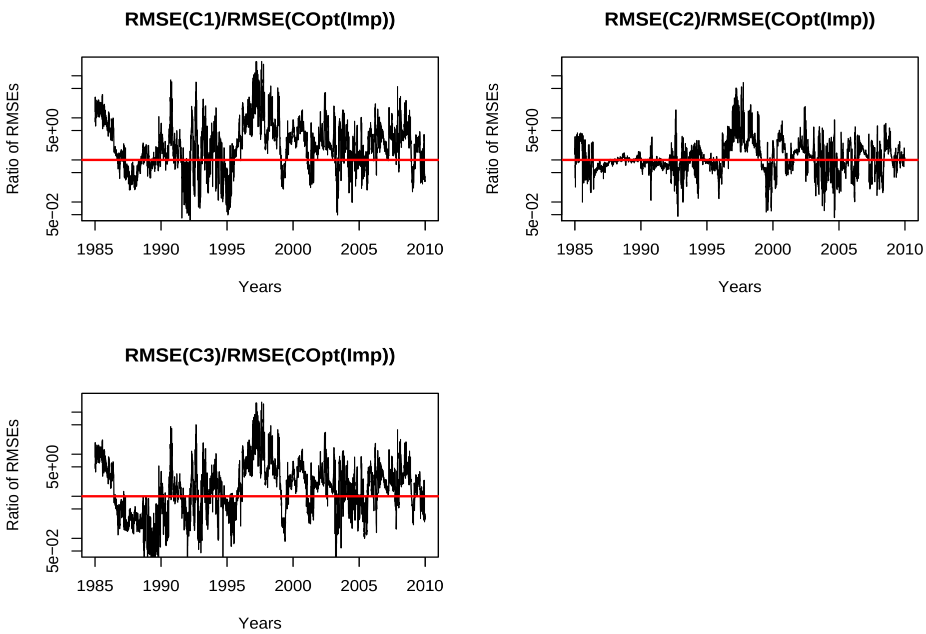

Figure 2.16: Ratios of Standard Errors for Different C Parameter Sets for Short Term Implied Inflation Spot Rates

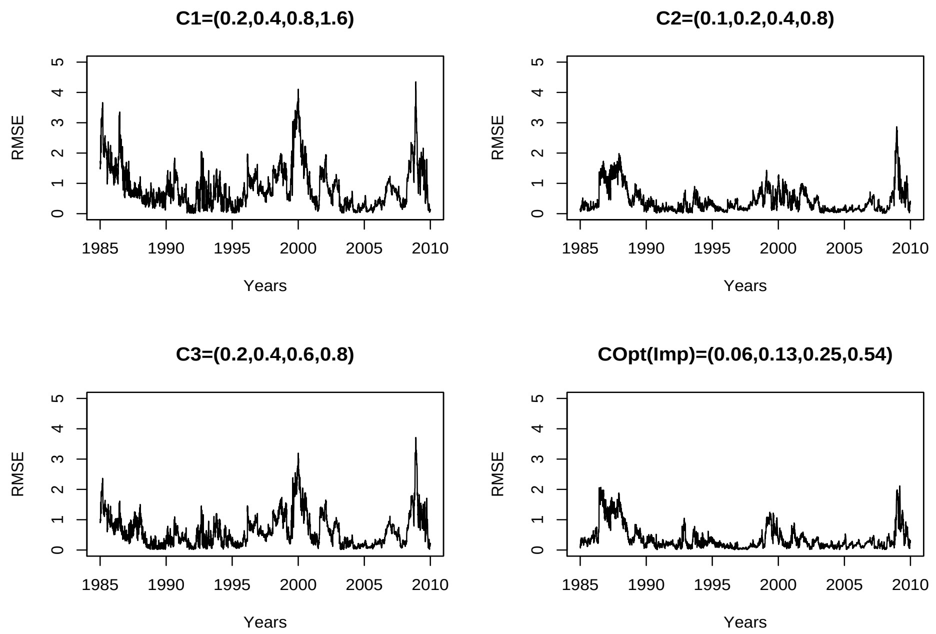

Figure 2.17: Standard Errors for Different C Parameter Sets for Medium Term Implied Inflation Spot Rates

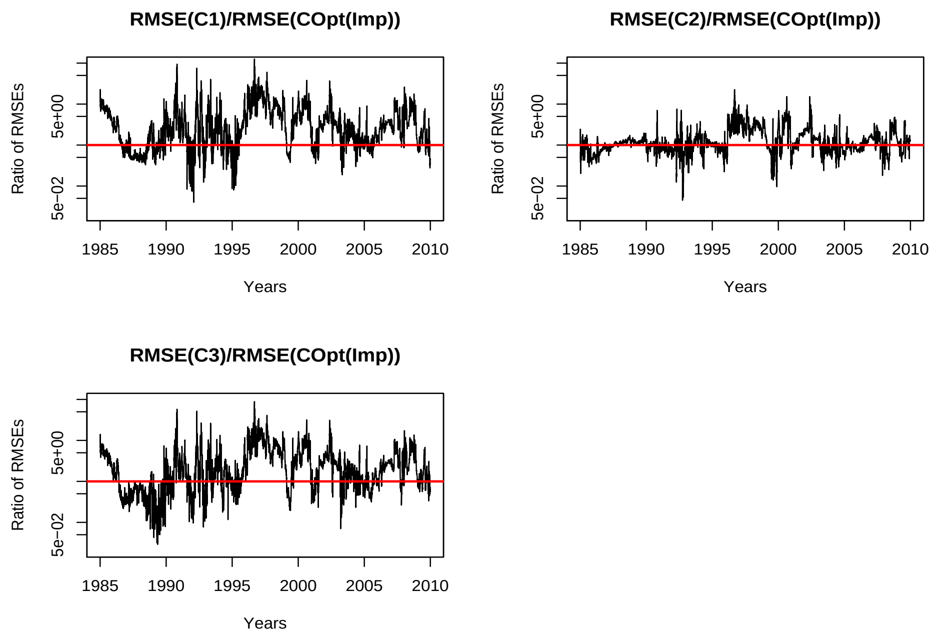

Figure 2.18: Ratios of Standard Errors for Different C Parameter Sets for Medium Term Implied Inflation Spot Rates

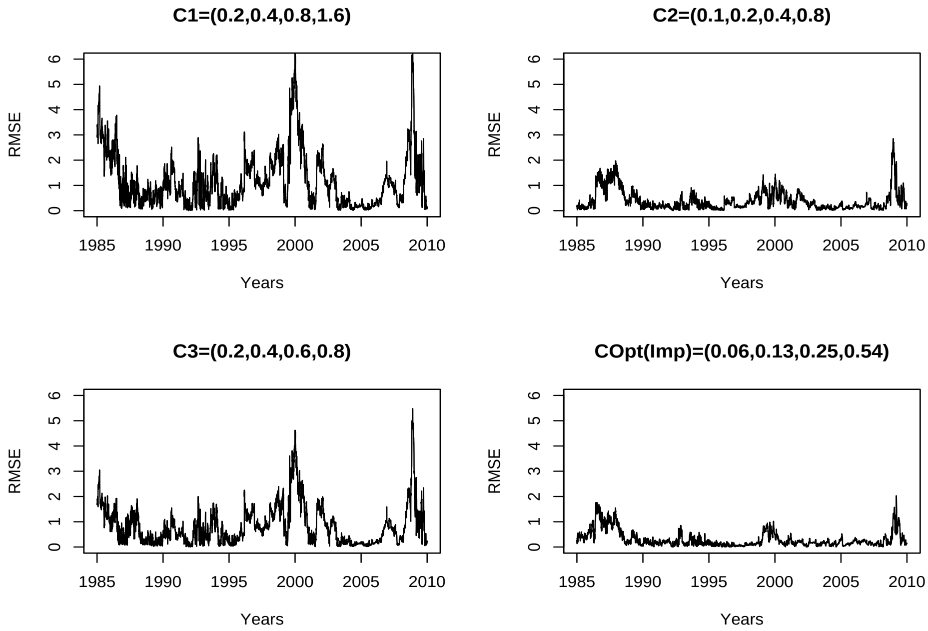

Figure 2.19: Standard Errors for Different C Parameter Sets for Long Term Implied Inflation Spot Rates

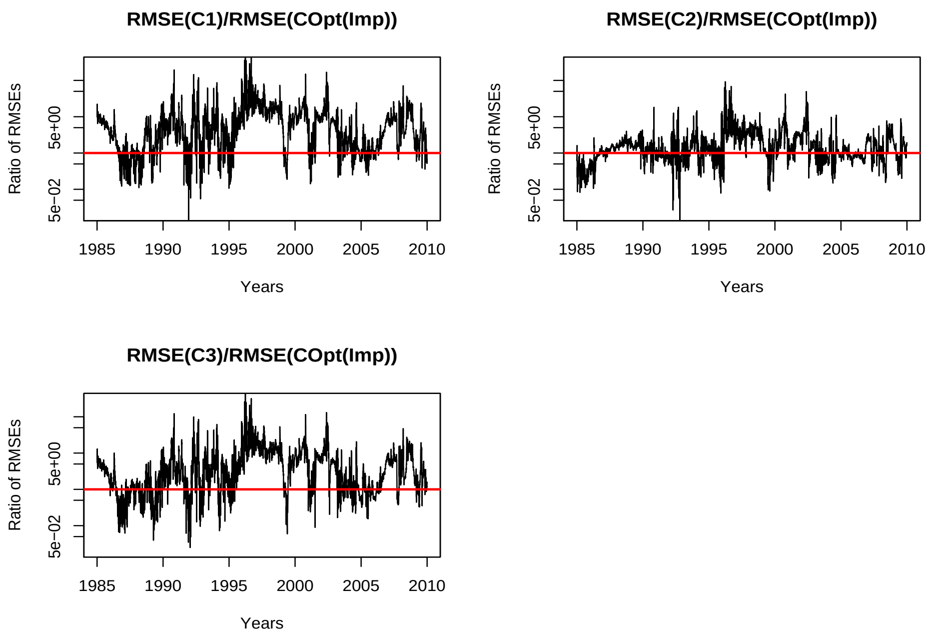

Figure 2.20: Ratios of Standard Errors for Different C Parameter Sets for Long Term Implied Inflation Spot Rates

Different from the nominal and implied inflation spot rates, we choose the $C2$ parameter set as the best to fit the Cairns model on to the real spot rates. Table 2.8 shows that Model 2 performs better than the other models for the overall, medium term and long term maturities by producing the smallest mean RMSEs while Model 4 with $COpt(Real)$ is the best for the short term maturities. Figures 2.21 to 2.28 can be interpreted in the same way as the ones for nominal and implied inflation spot rates and support our conclusion. One might think that it is contradictory if the optimised set of parameters do not produce the smallest RMSEs. Altough we expect that the $COpt$ parameter set fits the spot rates best we should consider that it is not a sole optimisation but we included a penalty function. This penalty function affects the optimisation process and it could lead to a set of parameter which is not the unique optimised one.

A general comment on the ratios of the RMSE graph is that when the RMSEs of two C parameter sets are close to each other the volatility is small. Otherwise, it is high.

Table 2.8: Mean RMSE (bps) for Different C Parameter Sets for Real Spot Rates

	Overall	Short-term	Medium-term	Long-term
Model 1	0.5071	0.5502	0.4252	0.5770
Model 2	**0.1154**	0.1707	**0.0985**	**0.1008**
Model 3	0.3436	0.3698	0.2914	0.3872
Model 4 (Opt)	0.1264	**0.1613**	0.1106	0.1249

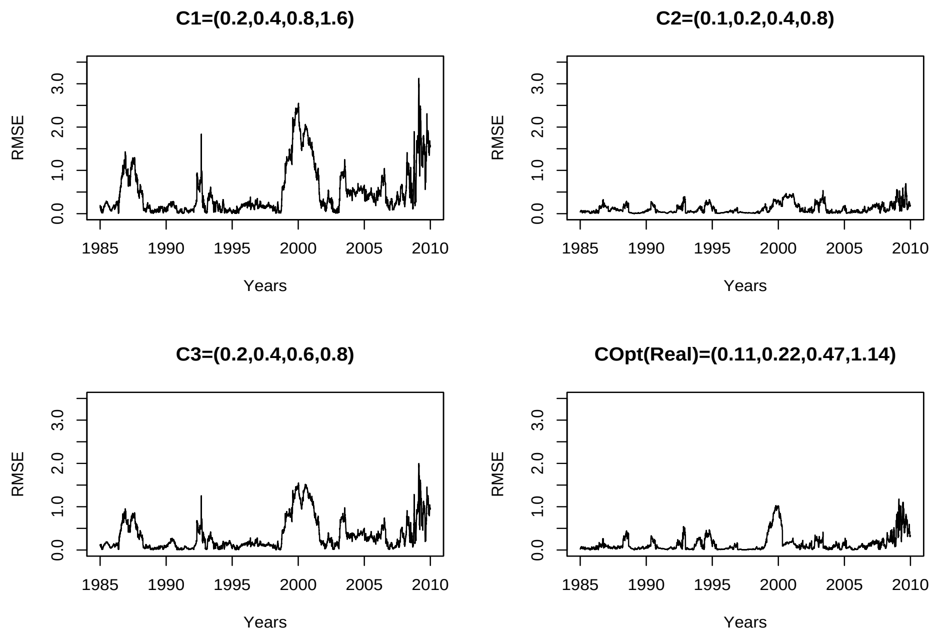

Figure 2.21: Root Mean Squared Errors for Real Spot Rates (in basis points)

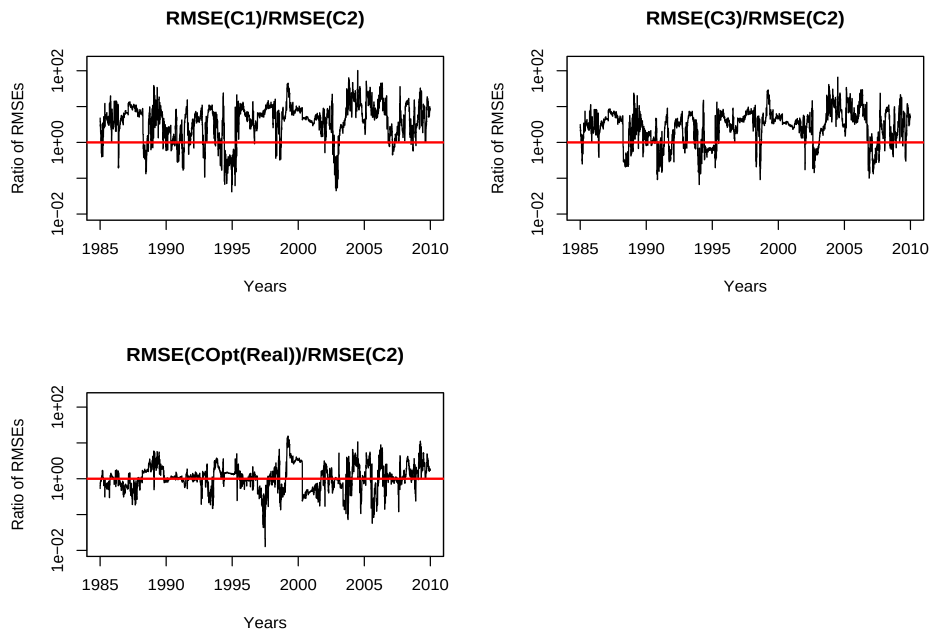

Figure 2.22: Ratios of Standard Errors for Different C Parameter Sets for Real Spot Rates

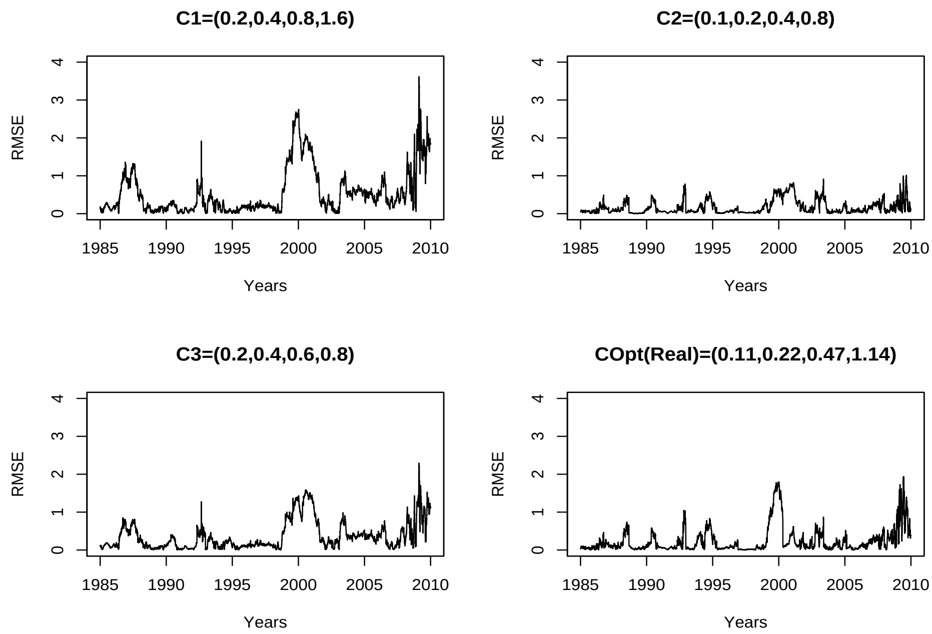

Figure 2.23: Standard Errors for Different C Parameter Sets for Short Term Real Spot Rates

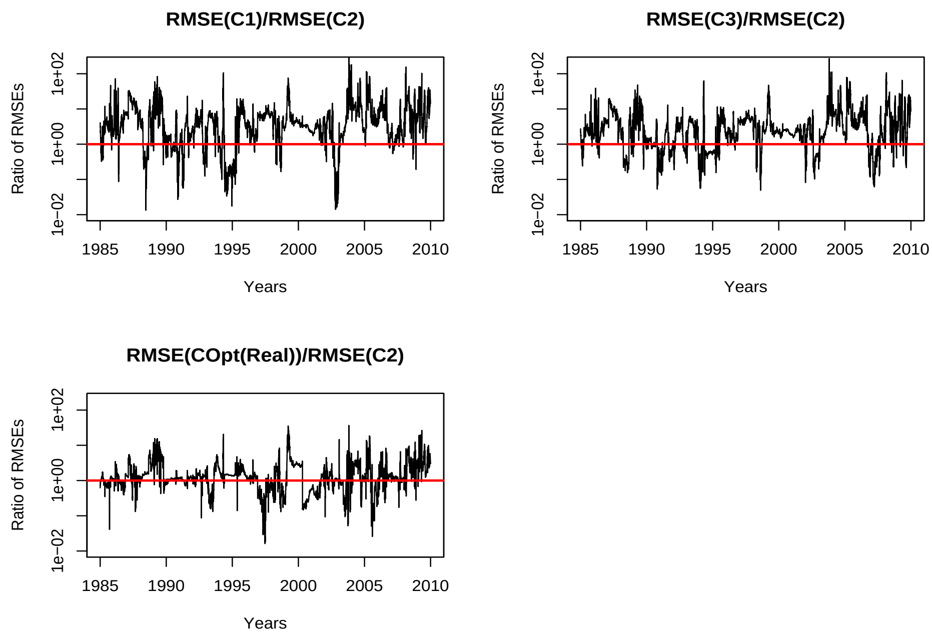

Figure 2.24: Ratios of Standard Errors for Different C Parameter Sets for Short Term Real Spot Rates

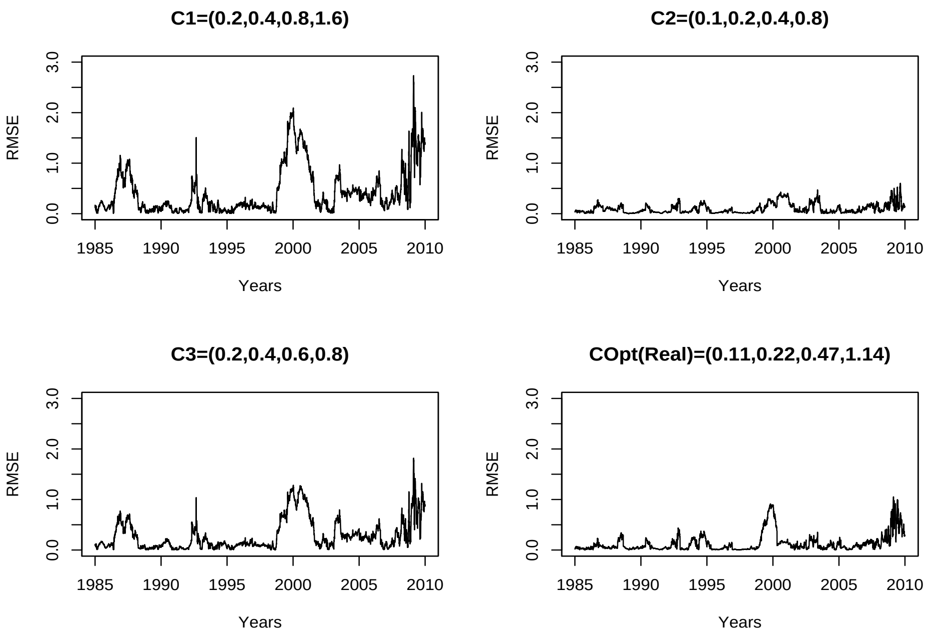

Figure 2.25: Standard Errors for Different C Parameter Sets for Medium Term Real Spot Rates

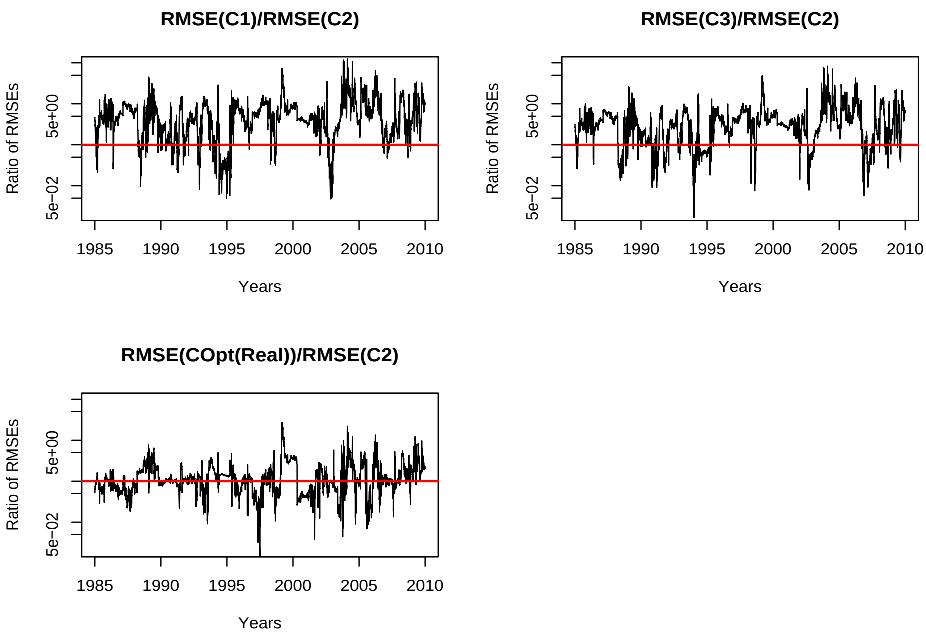

Figure 2.26: Ratios of Standard Errors for Different C Parameter Sets for Medium Term Real Spot Rates

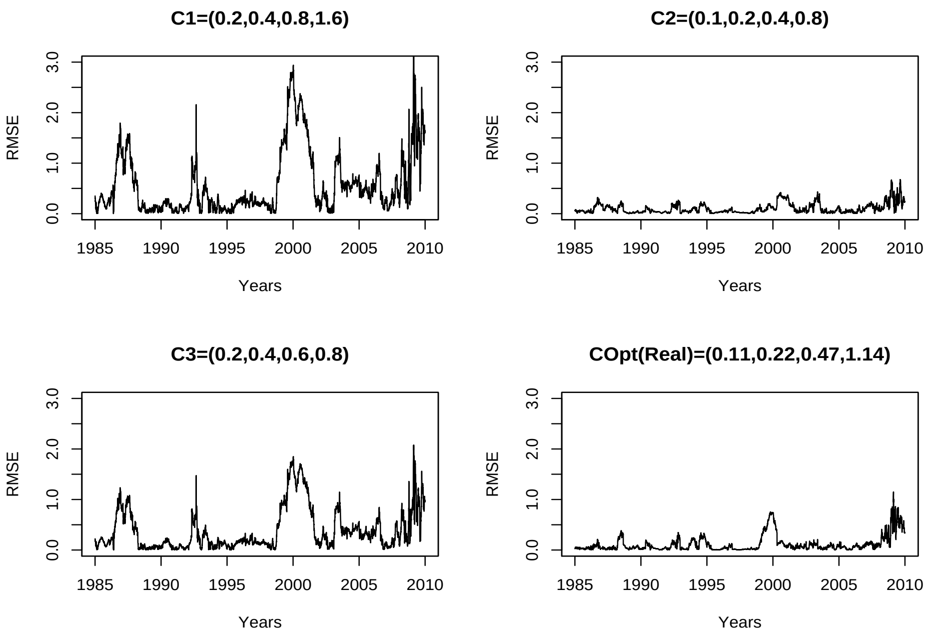

Figure 2.27: Standard Errors for Different C Parameter Sets for Long Term Real Spot Rates

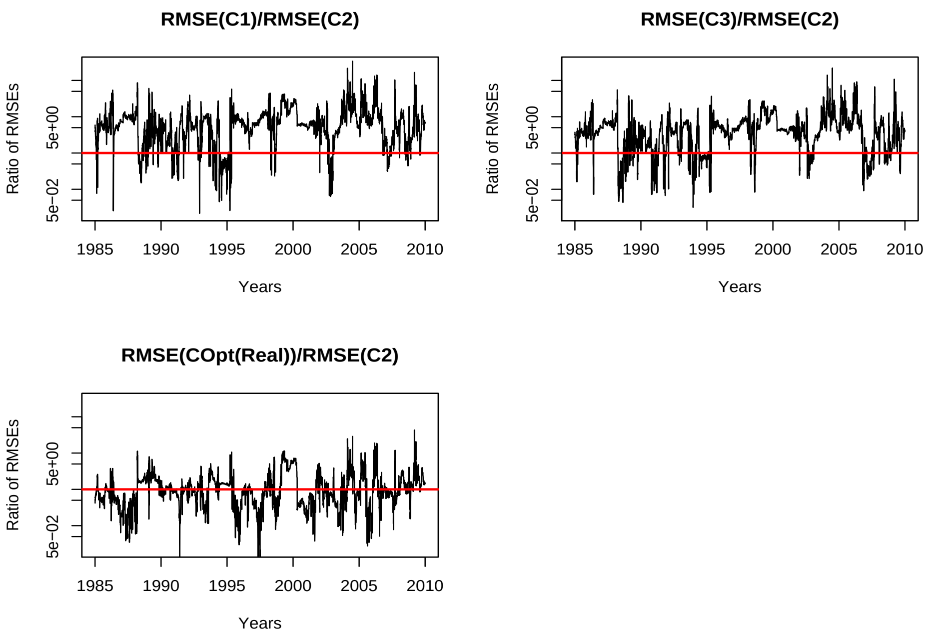

Figure 2.28: Ratios of Standard Errors for Different C Parameter Sets for Long Term Real Spot Rates

2.6 Observed and Fitted Values for Specific Days

After examining the RMSEs of each model for different maturities and yield curves we select some days randomly and draw the observed and fitted yield curves for these days using four models for each term structure. These yield curves show the observed and fitted values on that specific day and enable us to see how well the models fit the observed data for those specific dates.

We examine six different days to compare the performance of the models for the nominal yield curve. Figure 2.29 shows the yield curves for '1979-01-02' which represents a very early date in our data. Although all the models fit quite well for this specific date, it can be seen that Model 2 and Model 4 fit the long end of the curve slightly better due to the lower values of C parameter sets. On the other hand, Figure 2.30 shows that Model 1 and Model 3 fit the yield curve on '1982-12-13' since they capture the short term movements better due to the higher values of C parameter sets. Figures 2.31, 2.32, 2.33 and 2.34 display different shapes of the nominal yield curves for different dates, '1986-11-26', '2002-09-25', '2006-09-08' and '2009-11-05' respectively of which Model 4 with the optimisied C parameter set fits the observed yield curves best.

Similarly we examine four random days to see how well the fitted implied inflation and real spot rates fit the observed spot rates as we have done for the nominal spot rates. Figure 2.35, 2.36, 2.37 and 2.38 show the observed and fitted values for the implied inflation spot rates for '1985-01-02', '1988-12-12', '1992-11-25', '2008-09-25' dates. Although all the models fit the observed yield curves quite well, Model 2 and Model 4 perform slightly better.

Finally, Figure 2.39, 2.40, 2.41 and 2.42 display the observed and fitted yield curves for the real spot rates for the same dates as the implied inflation spot rates mentioned above. The graphs show that regardless of the choice of C parameter sets, the fitted values fit the observed yield curves very well for these specific dates.

To conclude, the figures displaying the observed and fitted values for different C parameter sets and different yield curves show that the Cairns model fits the different shapes of yield curves such as upward sloping, downward sloping or humped quite well independent from the choice of the exponential parameter sets.

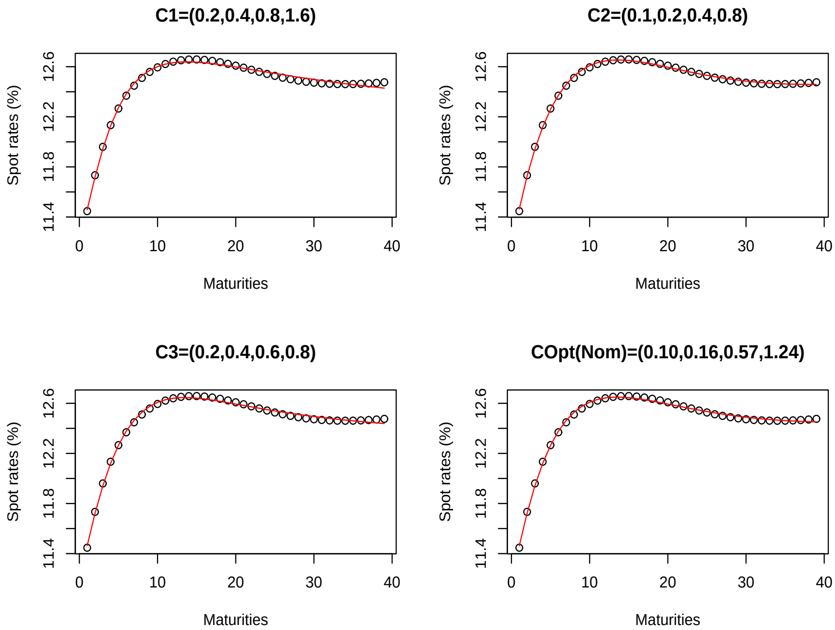

Figure 2.29: Observed and Fitted Nominal Spot Rates for '1979-01-02'

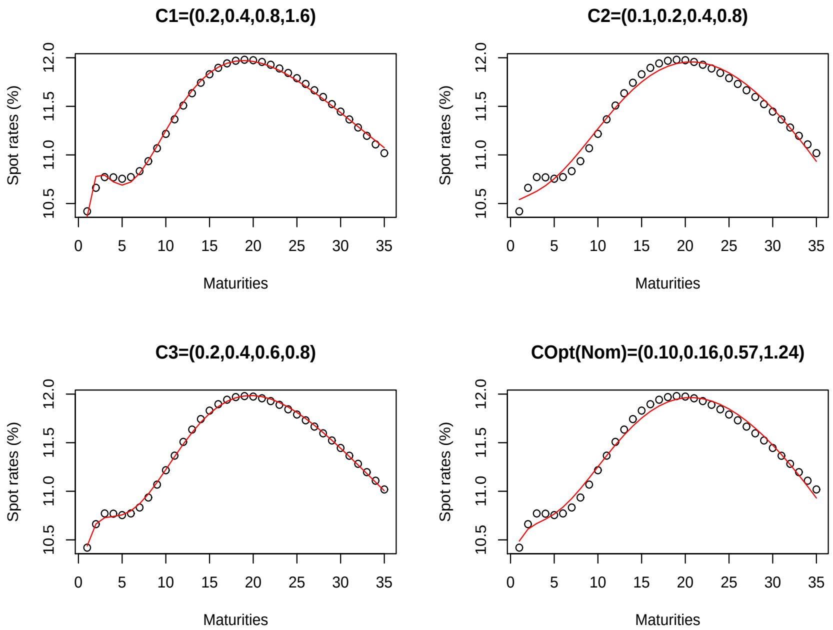

Figure 2.30: Observed and Fitted Nominal Spot Rates for '1982-12-13'

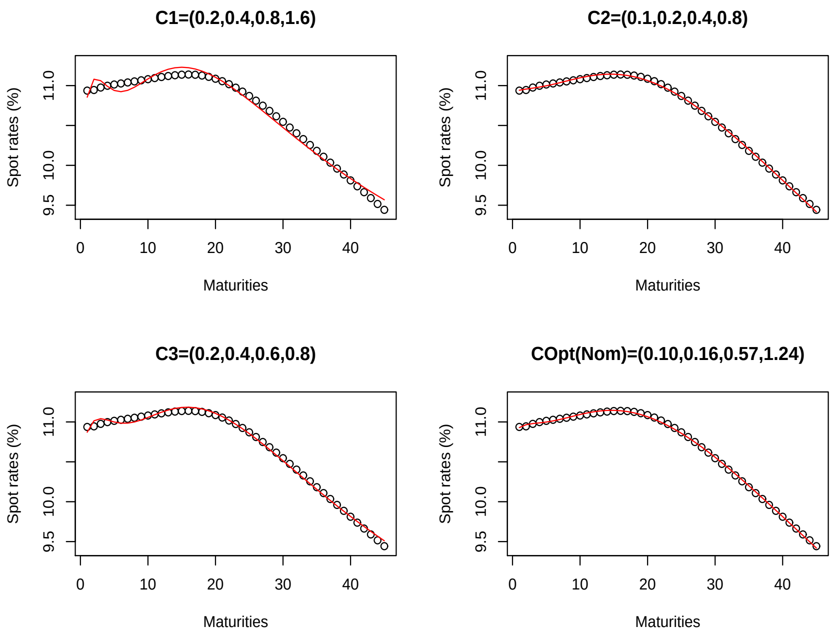

Figure 2.31: Observed and Fitted Nominal Spot Rates for '1986-11-26'

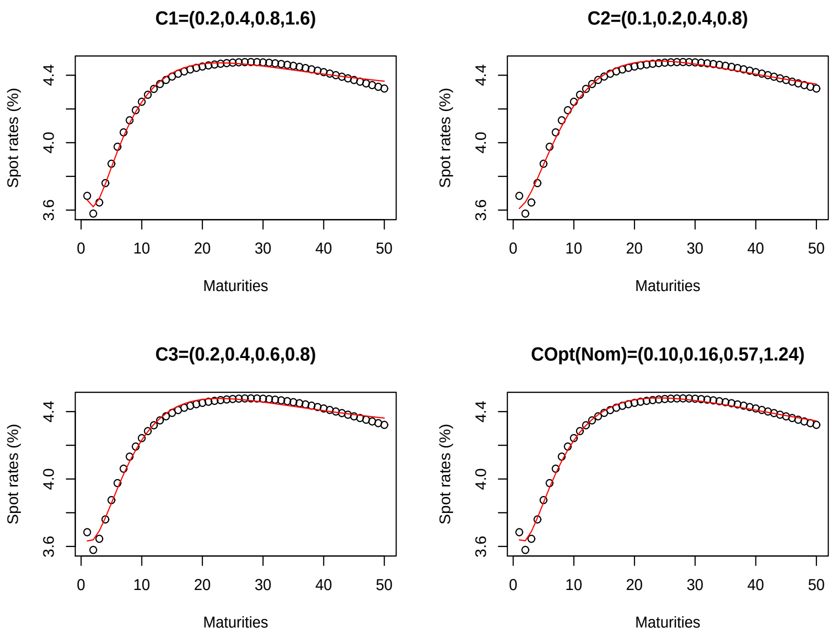

Figure 2.32: Observed and Fitted Nominal Spot Rates for '2002-09-25'

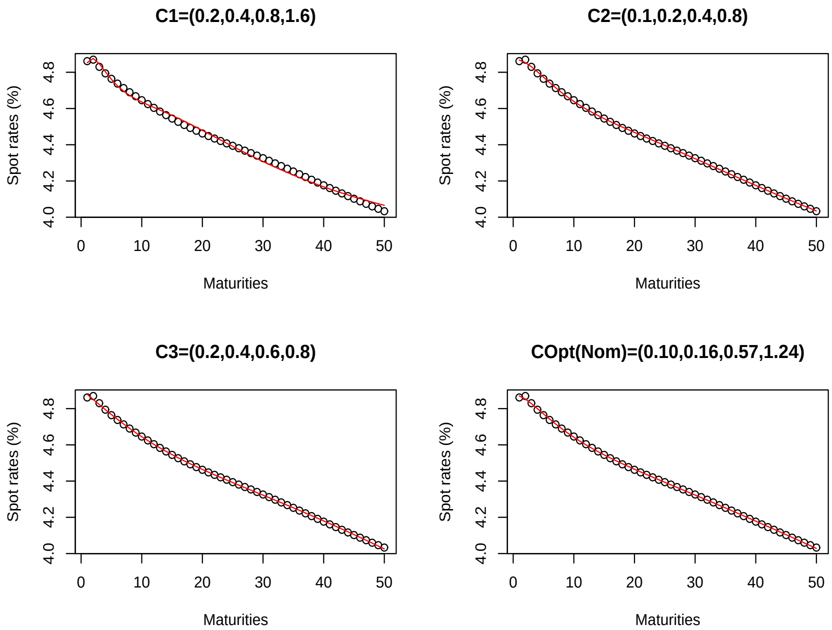

Figure 2.33: Observed and Fitted Nominal Spot Rates for '2006-09-08'

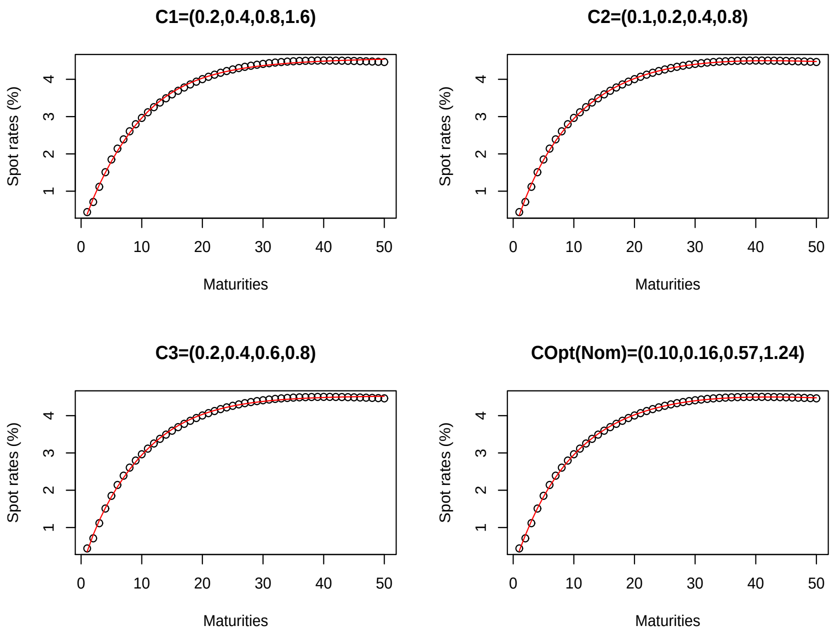

Figure 2.34: Observed and Fitted Nominal Spot Rates for '2009-11-05'

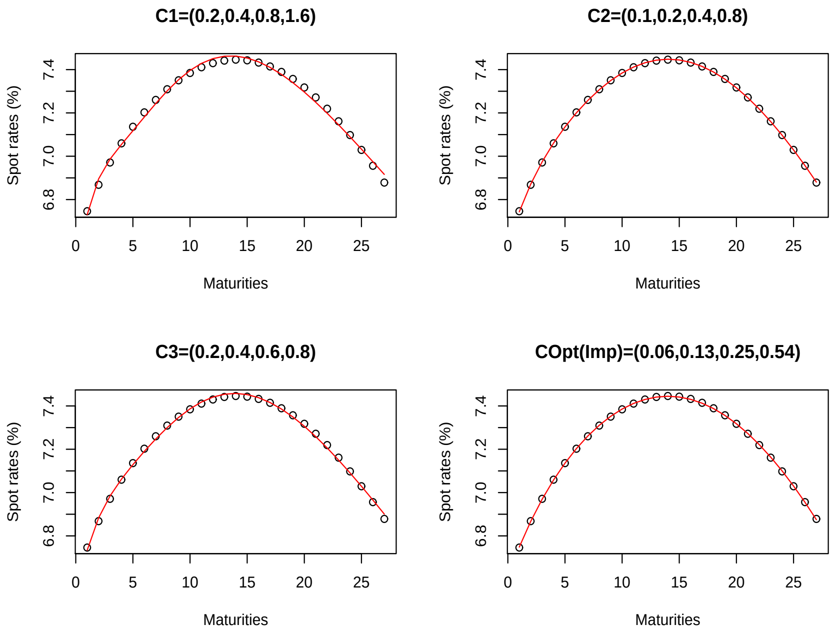

Figure 2.35: Observed and Fitted Implied Inflation Spot Rates for '1985-01-02'

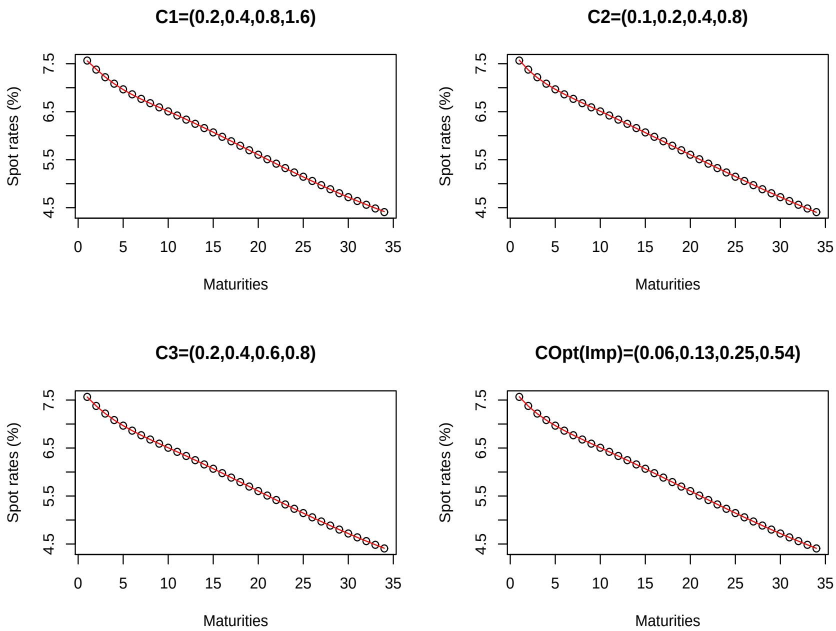

Figure 2.36: Observed and Fitted Implied Inflation Spot Rates for '1988-12-12'

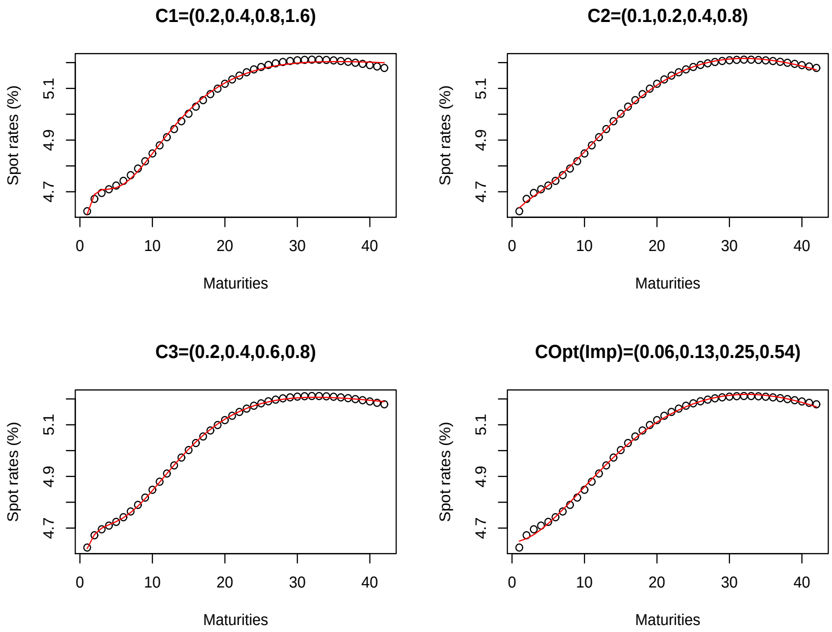

Figure 2.37: Observed and Fitted Implied Inflation Spot Rates for '1992-11-25'

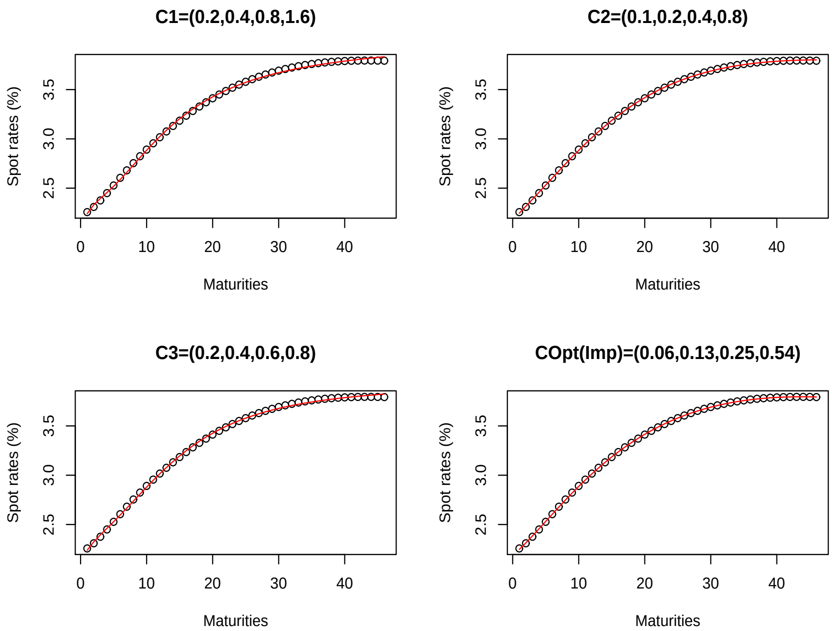

Figure 2.38: Observed and Fitted Implied Inflation Spot Rates for '2008-09-25'

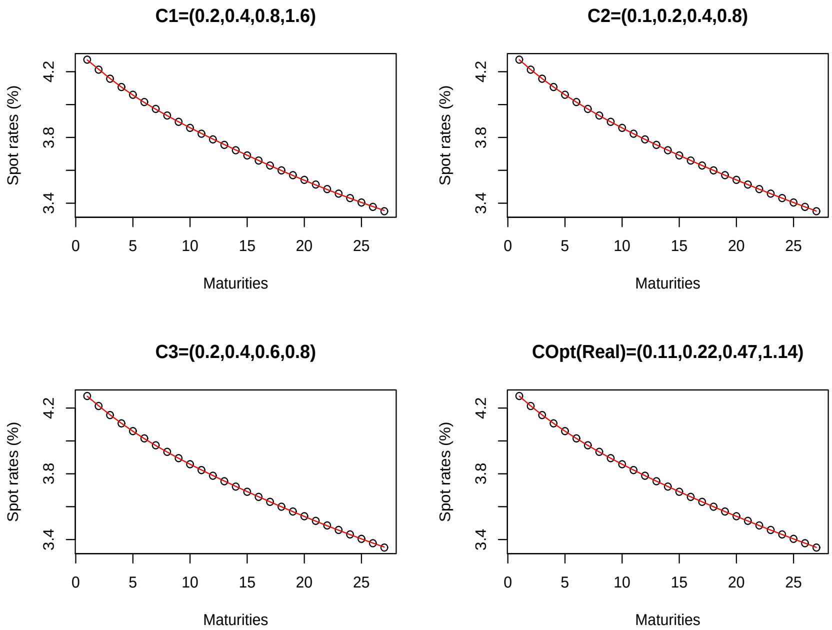

Figure 2.39: Observed and Fitted Real Spot Rates for '1985-01-02'

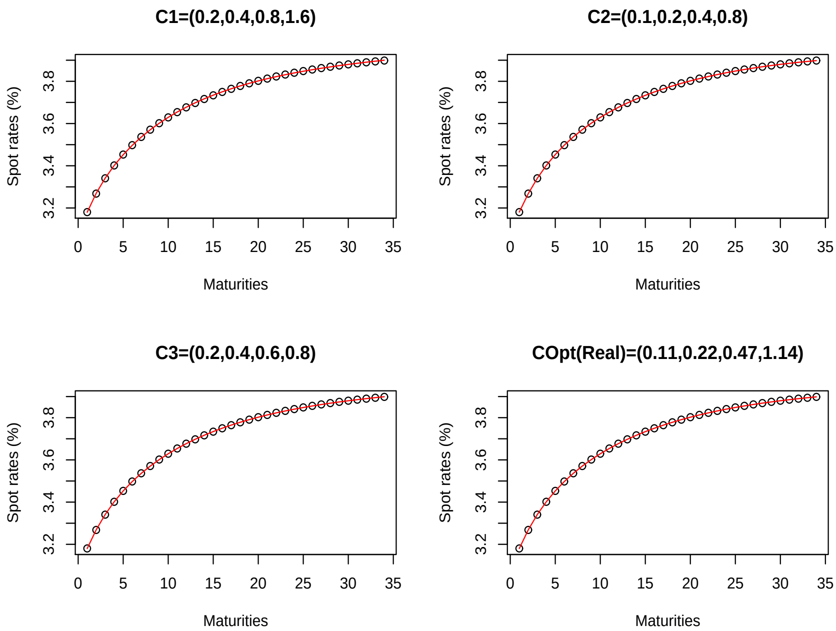

Figure 2.40: Observed and Fitted Real Spot Rates for '1988-12-12'

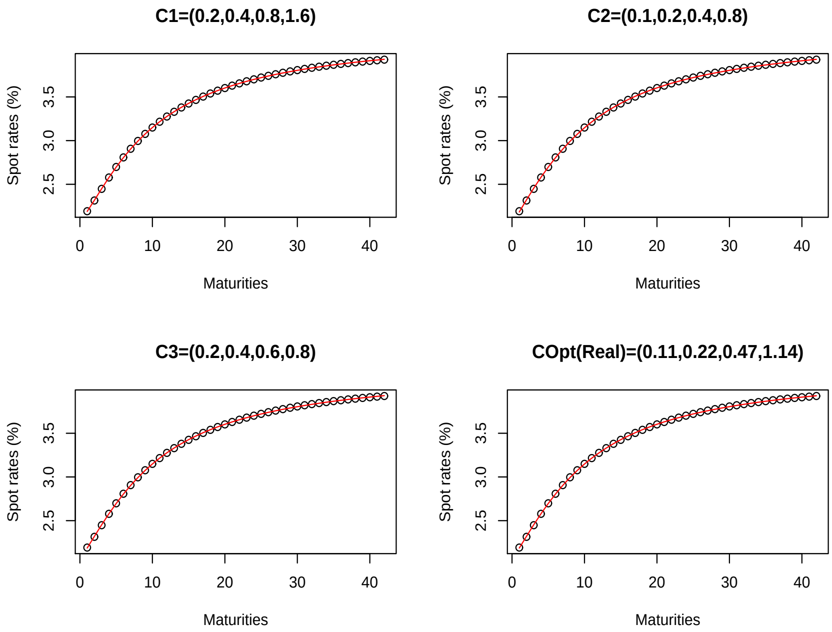

Figure 2.41: Observed and Fitted Real Spot Rates for '1992-11-25'

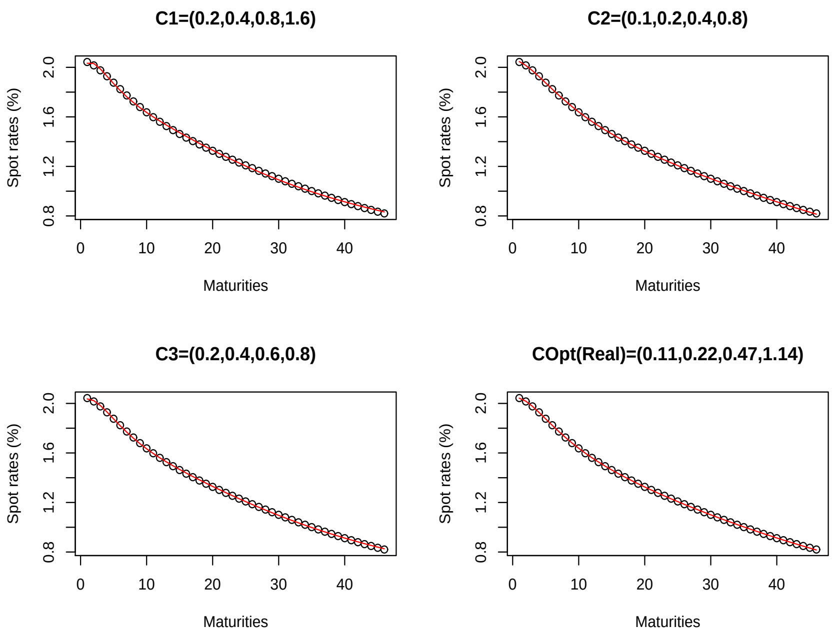

Figure 2.42: Observed and Fitted Real Spot Rates for '2008-09-25'

2.7 Fitted Values and Residuals for Specific Maturities

Another way to test which model fits best is to draw graphs of observed and fitted values for different maturities.

Figures 2.43 to 2.48 show these graphs and the residuals ($observed - fitted$) for half year (y(0.5)), 10-year (y(10)) and 25-year (y(25)) maturities for the nominal spot rates. Note that the residual graphs of the models for each maturity are drawn on the same scale to make the comparison between the difference in the observed and fitted values for each model easier. The observed spot rates are shown by black solid lines while the fitted rates are shown by red solid lines in the maturity specific yield curve graphs. Figure 2.43 indicates that all models fit well for the half-year nominal yields since the black solid line is mostly covered by the red solid line which indicates that the fitted spot rates are very close to the observed ones. However, although the differences are too small, the residual graphs in Figure 2.44 show that Model 1 fits best due to the higher values of the C parameter set whose aim is to capture the short-term volatilities in the yield curve. Model 1 and Model 3 perform better for the 10 year maturities which are shown in Figure 2.45 and Figure 2.46 while Model 2 and Model 4 fit the 25-year maturity yields much better as seen in Figure 2.47 and Figure 2.48.

Figures 2.49 to 2.54 show the graphs of the observed and fitted values and the residuals for half year, 10-year and 25-year maturities for the implied inflation spot rates. For the half year and 10-year maturities, the residuals are quite small (between -0.0006 and 0.0004) and all four models fit the yield curves equally well. On the other hand, for 25-year maturity, Model 4 with $COpt(Imp)$ parameter set fits the implied inflation spot rates best.

According to Figures 2.55 to 2.60, all four models fit the real yield curves very well while Model 2 and Model 4 produce slightly better fitted spot rates.

Note that there are missing values in the original yield curve data for some specific days and maturities due to the reasons discussed in Section 2.2.

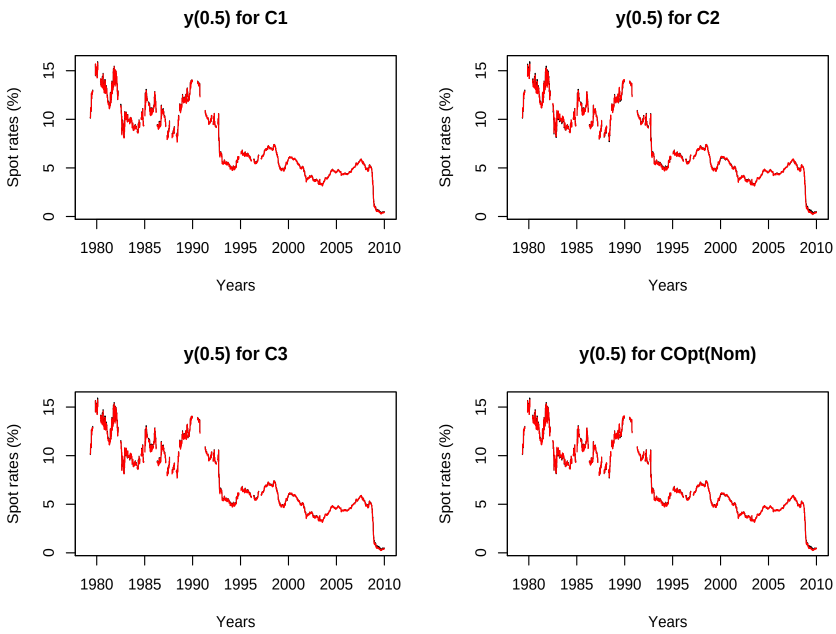

Figure 2.43: Nominal Spot Rates - 0.5-Year Maturity

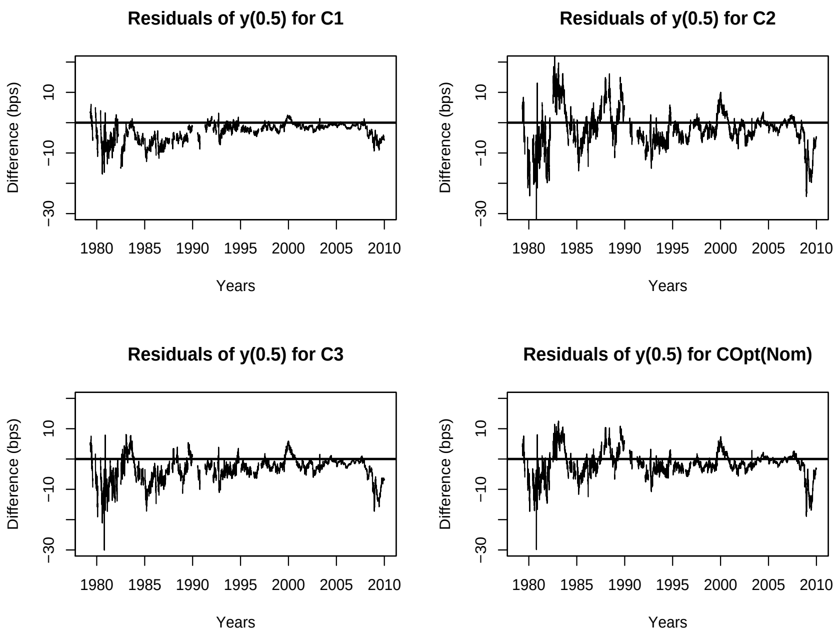

Figure 2.44: Nominal Spot Rates - Residuals for 0.5-Year Maturity

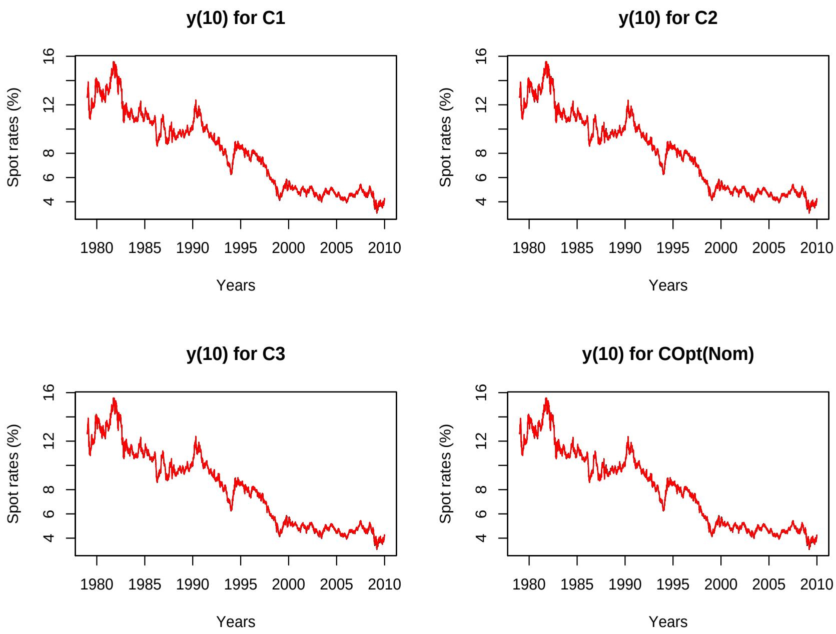

Figure 2.45: Nominal Spot Rates - 10-Year Maturity

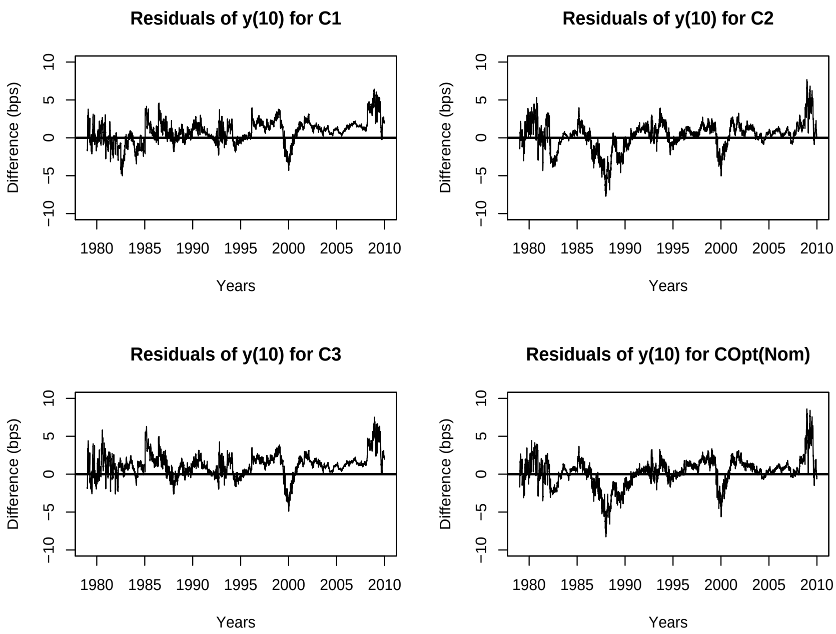

Figure 2.46: Nominal Spot Rates - Residuals for 10-Year Maturity

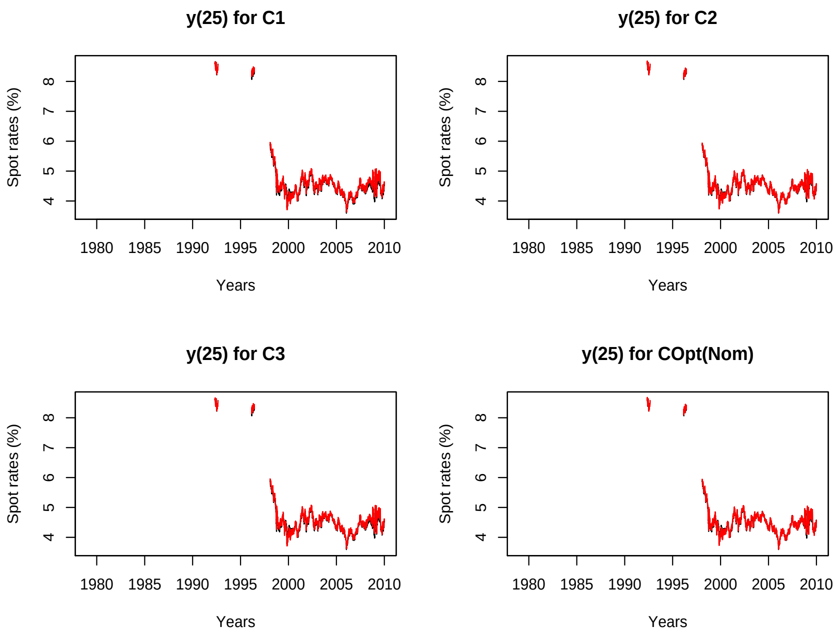

Figure 2.47: Nominal Spot Rates - 25-Year Maturity

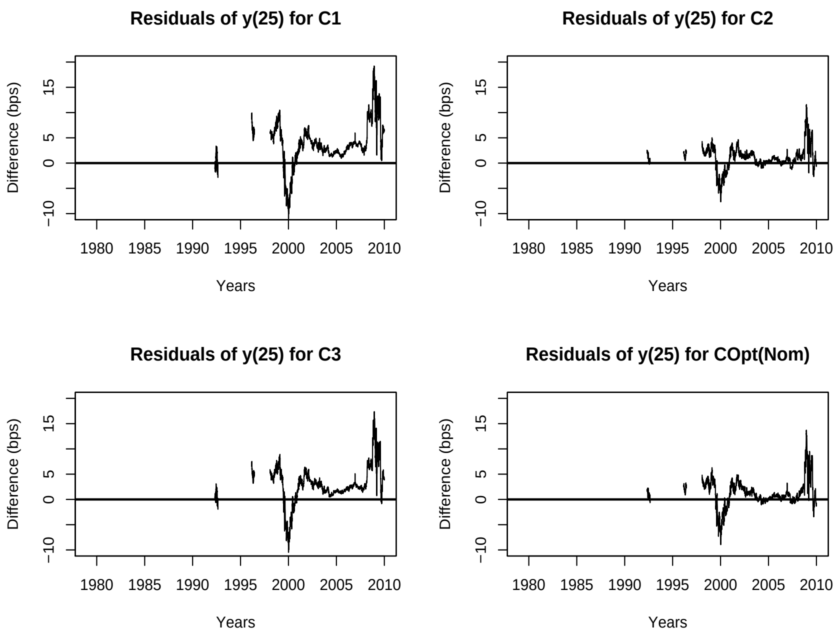

Figure 2.48: Nominal Spot Rates - Residuals for 25-Year Maturity

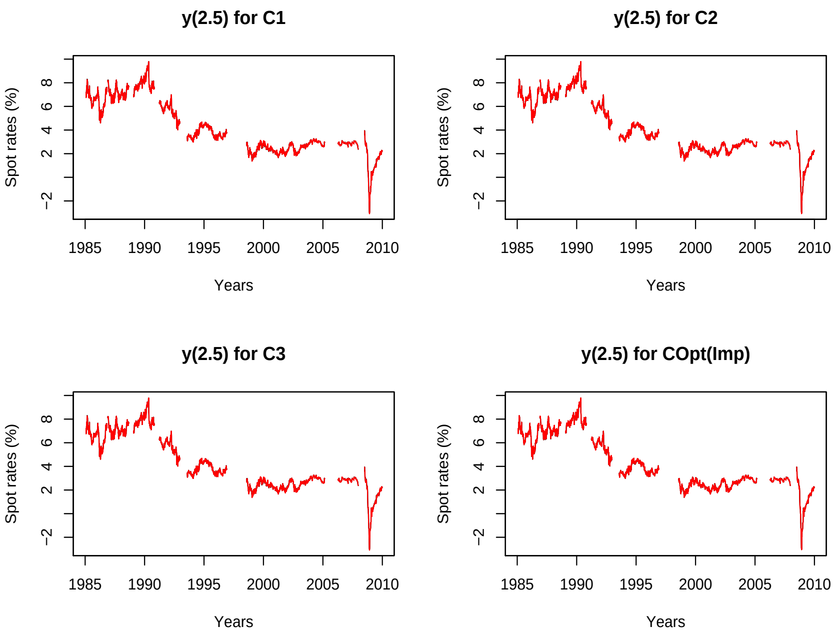

Figure 2.49: Implied Inflation Spot Rates - 0.5-Year Maturity

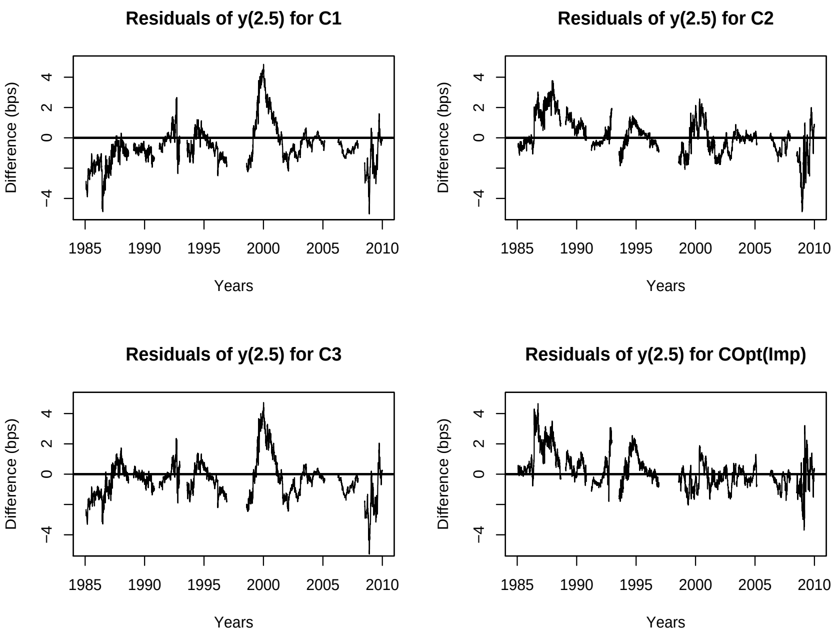

Figure 2.50: Implied Inflation Spot Rates - Residuals for 0.5-Year Maturity

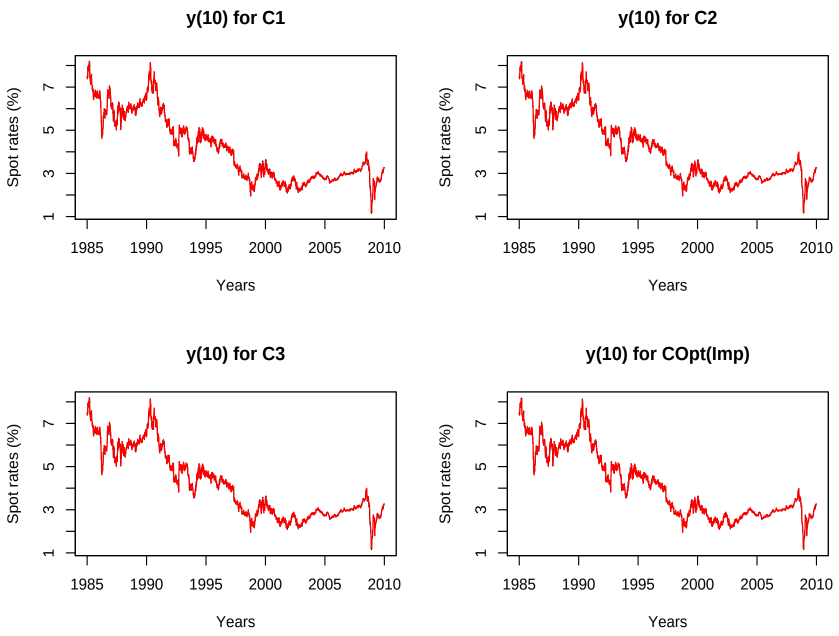

Figure 2.51: Implied Inflation Spot Rates - 10-Year Maturity

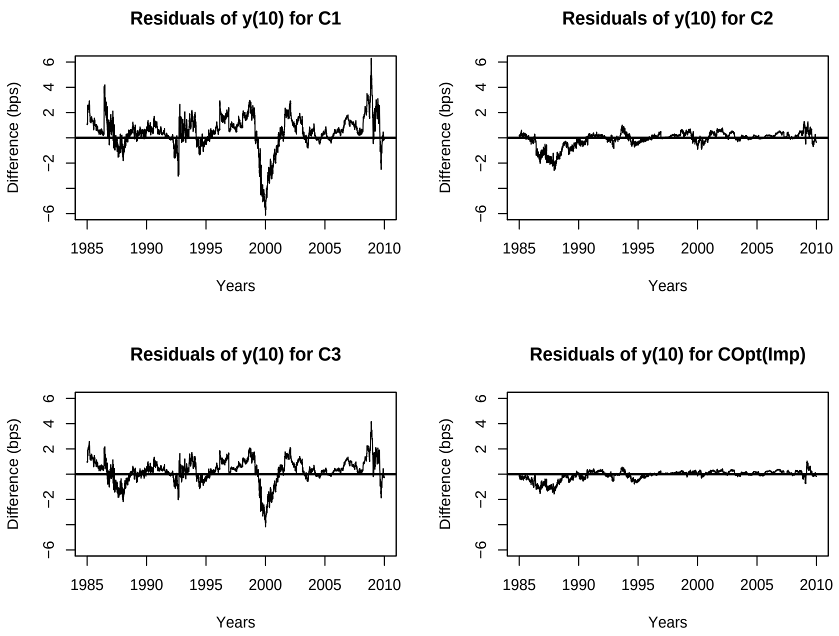

Figure 2.52: Implied Inflation Spot Rates - Residuals for 10-Year Maturity

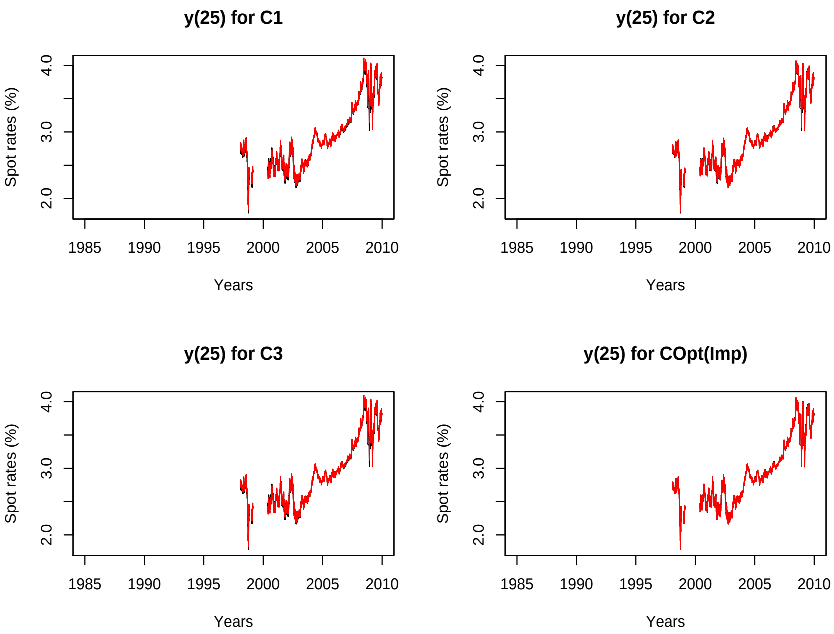

Figure 2.53: Implied Inflation Spot Rates - 25-Year Maturity

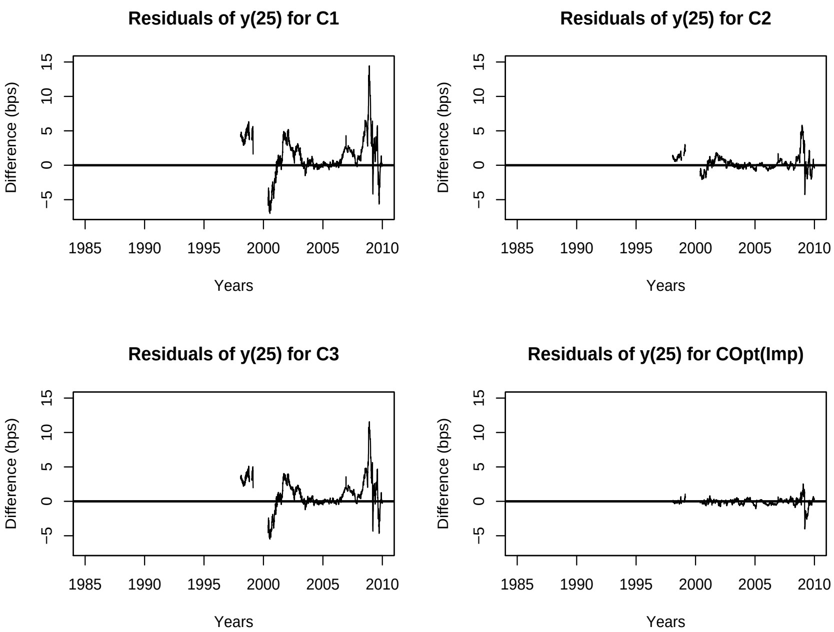

Figure 2.54: Implied Inflation Spot Rates - Residuals for 25-Year Maturity

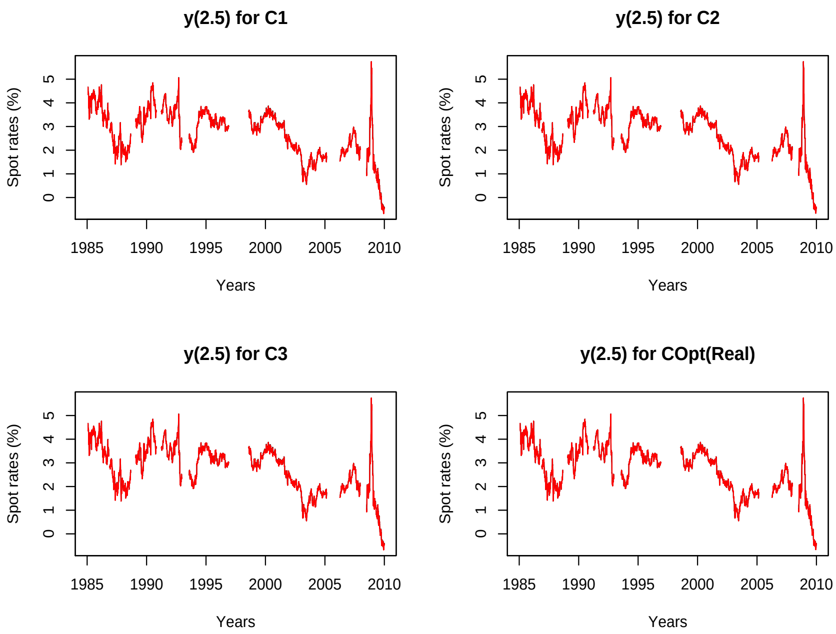

Figure 2.55: Real Spot Rates - 0.5-Year Maturity

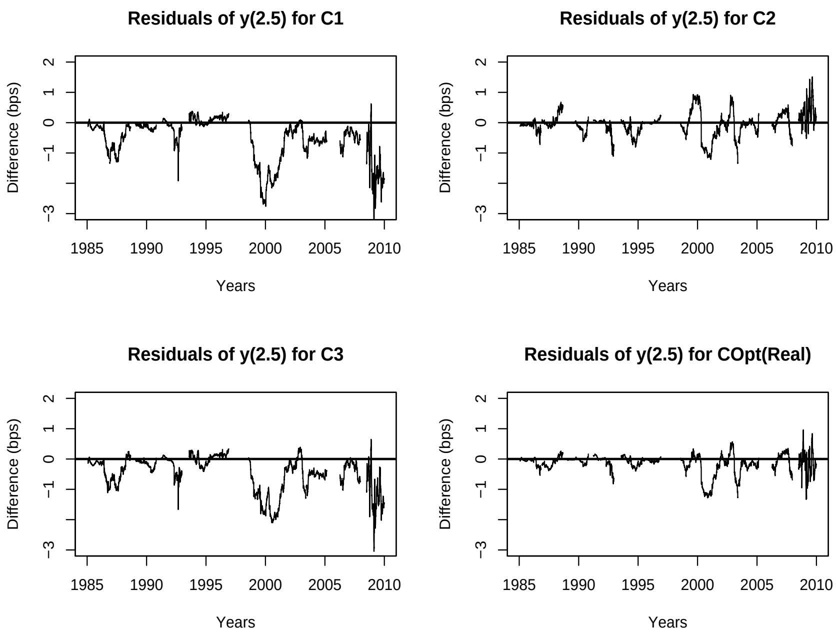

Figure 2.56: Real Spot Rates - Residuals for 0.5-Year Maturity

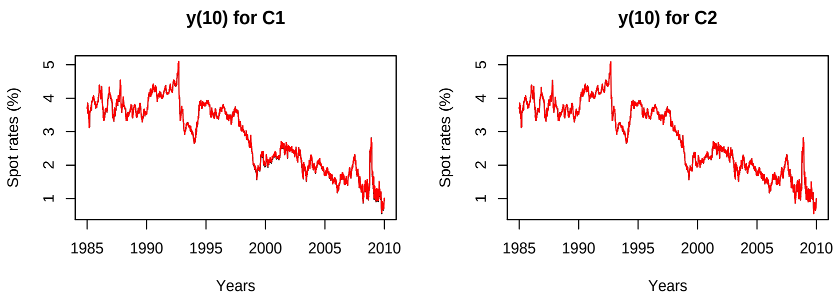

Figure 2.57: Real Spot Rates - 10-Year Maturity

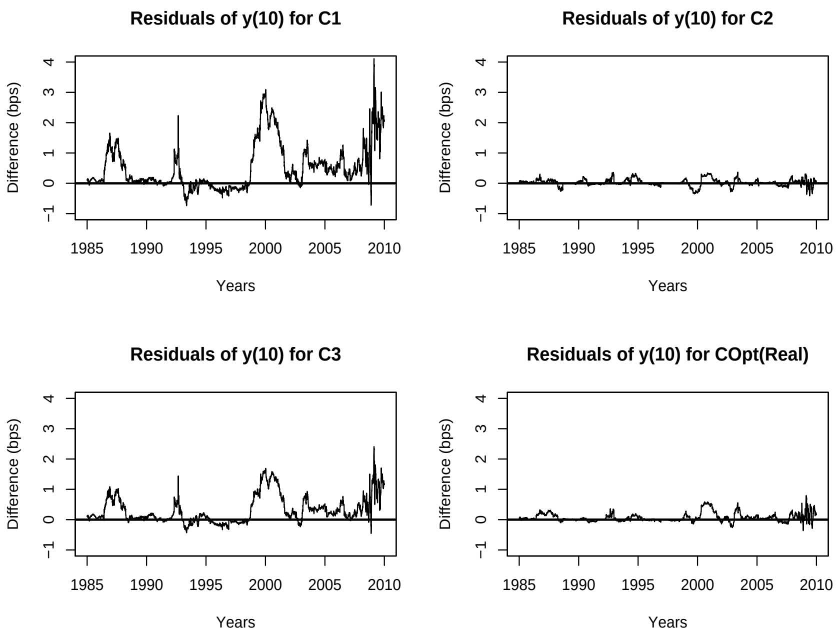

Figure 2.58: Real Spot Rates - Residuals for 10-Year Maturity

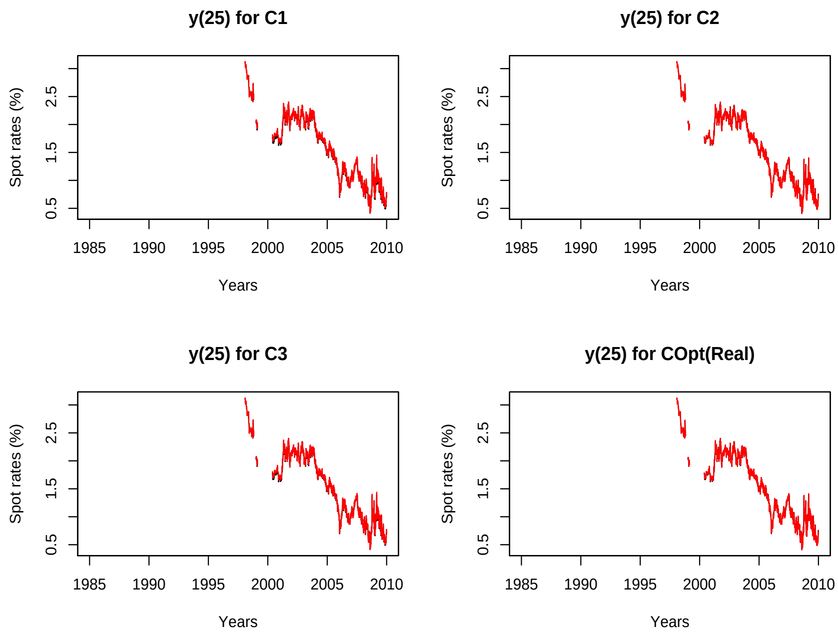

Figure 2.59: Real Spot Rates - 25-Year Maturity

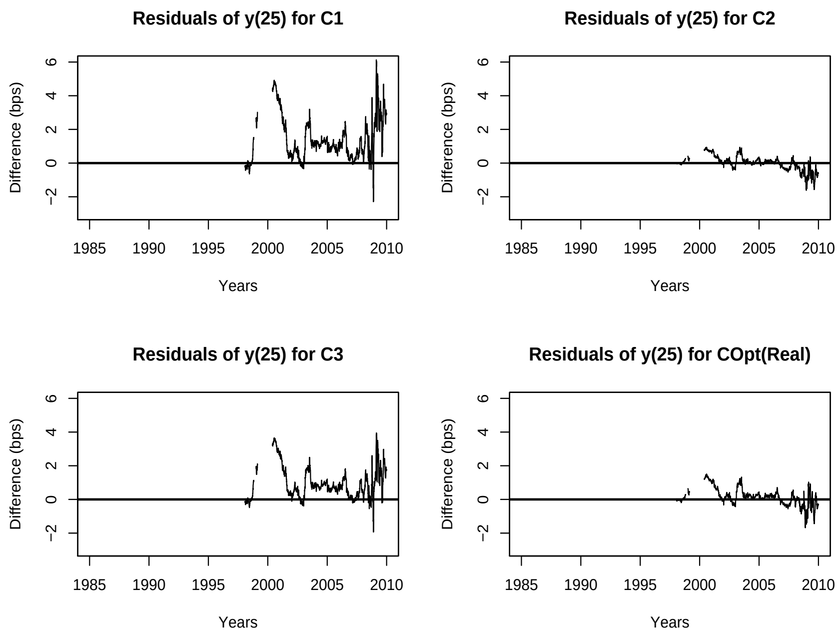

Figure 2.60: Real Spot Rates - Residuals for 25-Year Maturity

2.8 Interim Conclusion: Filling the Gaps in the UK Yield Curves

The aim of the analysis in this chapter is to fill the gaps in three UK yield curves (nominal, implied inflation and real spot rates) by fitting the Cairns model with appropriate fixed exponential parameter sets. Although the Bank of England publishes the yield curve data, there are many missing values due to the reasons discussed in Section 2.2. Since we will use all available maturities in further studies on yield curves, we need to replace these missing values by fitting a descriptive yield curve model. We have tried four different fixed parameter sets to apply the Cairns model and decide the ones which fit the yield curves best. One set of these parameters for each yield curve has been obtained by the least squares method with a penalty function. The other three parameter sets have been proposed by Cairns (1998) and Cairns and Pritchard (2001). We compared these different parameter sets by examining the root mean squared errors, how well they fit specific maturities and specific days. Based on our analysis we conclude that the parameter sets obtained from the least squares method provide the best fit for the nominal and implied inflation yield curves while one of the sets suggested by Cairns (1998) performed better on a range of criteria even than the optimised parameter set for the real spot rates.

Chapter 3

Principal Component Analysis on the Fitted UK Term Structures

3.1 Introduction

Once we fit the Cairns model to the UK yield curves we apply principal component analysis (PCA) to the fitted values to decrease the dimension of the data. The aim is to reduce the dimension of the yield curves (7838×50 for the nominal spot rates and 6320×46 for the implied inflation and the real spot rates) in order to obtain uncorrelated variables from highly correlated data to construct yield curve models which are discussed in the following chapters.

Instead of using the original Bank of England yield curve data to apply the PCA, we use fitted Cairns values in order to consider a full range of maturities in our analysis. If we used the original yield curves we would eliminate the maturities which include missing values which would lead us to continue our study without the very short end and long end of the yield curves. It is convenient to use fitted Cairns values to model the term structures as we discuss in Chapter 2 that the Cairns model fits the yield curve data quite well.

Therefore, this chapter discusses the use of the PCA. We introduce the PCA and its properties in Section 3.2. We apply the PCA on the fitted values for each model and each yield curve and present the results in Section 3.3. Then we examine the robustness of the principal component method to the choice of C parameter sets for the nominal, implied inflation and real spot rates in Section 3.4. Section 3.5 concludes the chapter.

3.2 Principal Component Analysis

The method of principal component analysis is primarily a data-analytic technique that obtains linear transformations of a group of correlated variables such that optimal conditions are achieved. The most important of these conditions is that the transformed variables are uncorrelated.

The main idea of the PCA is to reduce the dimensionality of a data set in which there are a number of interrelated variables, while retaining and explaining as much as possible of the variation present in the data set. This reduction is achieved by transforming to a new set of variables, the principal components (PCs), which are *uncorrelated*, and which are ordered so that the first *few* retain most of the variation present in all of the original variables. Computation of the principal components reduces to the solution of an eigenvalue-eigenvector problem for a positive-semidefinite symmetric matrix (Jolliffe, 1986).

The method of principal components is based on a key result from matrix algebra: a $p \times p$ symmetric matrix, such as the covariance matrix S, may be reduced to a diagonal matrix L by premultiplying and postmultiplying it by a particular orthonormal matrix U such that

$$U^{'}SU = L \tag{3.1}$$

The diagonal elements of L, $l_1, l_2, ..., l_p$ are called the *characteristic roots, latent roots* or *eigenvalues* of S. The columns of U, $u_1, u_2, ..., u_p$ are called the *characteristic vectors* or *eigenvectors* of S. The characteristic roots may be obtained from the solution of the following determinental equation, called the *characteristic equation*:

$$|S - lI| = 0 \tag{3.2}$$

where I is the identity matrix. This equation produces a pth degree polynomial in l from which the values $l_1, l_2, ..., l_p$ are obtained.

The characteristic vectors may then be obtained by the solution of the equations

$$[S - lI]\, t_i = 0 \tag{3.3}$$

and

$$u_i = \frac{t_i}{\sqrt{t_i^{'} t_i}} \tag{3.4}$$

for $i = 1, 2, ..., p$. Here, u_is are characteristic vectors which make up the matrix

$$U = [u_1 \;\; u_2 \;\; . \;\; . \;\; . \;\; u_p] \tag{3.5}$$

which is orthonormal, that is,

$$u_i^{'} u_i = 1 \qquad\qquad u_i^{'} u_j = 0$$

for $i \neq j$ (Jackson, 1991).

Geometrically, the procedure described above is nothing more than a *principal axis rotation* of the original coordinate axes about their means. The elements of the characteristic vectors are the direction cosines of the new axes related to the old.

The starting point for the PCA is the sample covariance matrix **S** (or the correlation matrix)[1]. For a p-variable problem,

$$\mathbf{S} = \begin{bmatrix} s_{11}^2 & s_{12}^2 & . & . & . & s_{1p}^2 \\ s_{12}^2 & s_{22}^2 & . & . & . & s_{2p}^2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ s_{1p}^2 & s_{2p}^2 & . & . & . & s_{pp}^2 \end{bmatrix}$$

[1]It is important to note that the PCA depends on the scale of the variables, i.e. using the covariance or the correlation matrix leads to different PCs.

where s_i^2 is the variance of the ith variable, x_i, and s_{ij} is the covariance between the ith and jth variables. If the covariances are not equal to zero, it indicates that a linear relationship exists between these two variables, the strength of that relationship being represented by the correlation coefficient.

The principal axis transformation obtained above will transform p correlated variables $x_1, x_2, ..., x_p$ into p new *uncorrelated* variables $z_1, z_2, ..., z_p$. The coordinate axes of these new variables are described by the characteristic vectors u_i which make up the matrix **U** of direction cosines used in the transformation:

$$z = U' [x - \overline{x}] \tag{3.6}$$

Here x and $\overline{x}$ are $p \times 1$ vectors of observations on the original variables and their means.

The transformed variables are called the *principal components* of x. The ith principal component is

$$z_i = u_i' [x - \overline{x}] \tag{3.7}$$

and will have mean zero and variance l_i, the ith characteristic root.

Transformations

If one wishes to transform a set of variables x by a linear transformation $z = U' [x - \overline{x}]$ whether **U** is orthonormal or not, the covariance matrix of the new variables, S_z, can be determined directly from the covariance matrix of the original observations, S by the relationship

$$S_z = U' SU \tag{3.8}$$

However, when **U** is orthonormal, this characteristic vector solution produces an S_z that is a diagonal matrix like **L** producing new variables that are uncorrelated.

Inversion of the Principal Component Model

It is possible to obtain the original data back by using all the principal components derived from that data. The equation

$$z = U' \left[x - \overline{x} \right] \tag{3.9}$$

may be inverted so that the original variables may be stated as a function of the principal components

$$x = \overline{x} + Uz \tag{3.10}$$

because U is orthonormal and hence $U^{-1} = U'$.

Residual Analysis

As described above, if one uses a full set of PCs, it is possible to invert the equation that produced the PCs from the data and, instead, determine the original data from the PCs. However, x will be determined exactly only if all the PCs are used. If $k < p$ PCs are used, only an estimate $\widehat{x}$ of x will be produced,

$$\widehat{x} = \overline{x} + Uz \tag{3.11}$$

where U is now $p \times k$ and z is $k \times 1$. The above equation can be rewritten as

$$x = \overline{x} + Uz + (x - \widehat{x}) \tag{3.12}$$

In this case, the first term on the right-hand side of the equation represents the contribution of the multivariate mean, the second term represents the contribution due to the PCs, and the final term represents the amount that is unexplained by the PC model - the residual. Wherever any PCs are deleted, some provision should be made to check the residual.

Principal Components Using a Correlation Matrix

The derivations and properties of PCs considered above have been on the eigenvectors and eigenvalues of the covariance matrix. In practice, it is more usual to define the PCs using the correlation matrix instead of the covariance matrix for the following reasons.

A major argument for using correlation matrices, rather than covariance matrices, to define PCs is that the results of analyses for different sets of random variables are more directly comparable than for analyses based on covariance matrices. A drawback of PCA based on covariance matrices is the sensitivity of the PCs to the units of measurement used for each element of $\mathbf{x}$. If there are large differences between the variances of the elements of $\mathbf{x}$, then those variables whose variances are largest will tend to dominate the first few PCs. It is unwise to use PCs on a covariance matrix when $\mathbf{x}$ consists of measurements of different types, unless there is a strong conviction that the units of measurements chosen for each element of $\mathbf{x}$ are the only ones which make sense. Even if this condition holds, using the covariance matrix will not provide very informative PCs if the variables have widely differing variances.

Another problem with the use of the covariance matrix is that it is more difficult to compare informally the results from different analyses than with correlation matrices. Sizes of variances of PCs have the same implications for different correlation matrices, but not for different covariance matrices. Also, patterns of coefficients in PCs can be readily compared for different correlation matrices which are giving similar PCs, whereas informal comparisons are often much trickier for covariance matrices (Jolliffe, 1986).

3.3 PCA on Fitted Yield Curves

As we discuss in Section 3.2, the PCA attempts to describe the behaviour of a range of correlated random variables (in this case, the various spot yields for different times to maturity) in terms of a small number of uncorrelated principal components. This type of analysis makes it possible to identify a relatively small number of factors that have affected the behaviour of the entire zero-coupon curve over the period examined.

This approach was first applied to bond yields by Litterman and Scheinkman (1991), who found three common factors that influenced the returns on all treasury bonds. They found that these three factors explained, on average, 98.4% of the observed variance in yields. The first factor, which they called *level*, represented an approximately parallel shift higher or lower in the yield curve. A shock to this factor raised or lowered all yields by roughly the same amount. Level was by far the most important factor, accounting for 89.5% of the total observed variance. The second factor was called *steepness*, since a positive shock to this factor lowered short term spot rates, while raising longer term rates. This factor was found to account for a further 8.5% of total observed variance. A positive shock to the third factor, which they called *curvature*, lowered both short and long term yields, while raising mid-term yields. This had the effect of increasing the degree of curvature in the term structure. The curvature factor accounted for 2% of the explained variance. This model has been applied to other interest rate markets with similar results, and it has become standard practice in finance to refer to shifts in yield curves as being driven by three underlying factors: *level, slope and curvature* (Johnson, 2005).

We apply the PCA to the fitted Cairns model with different C parameter sets for three yield curves: nominal, implied inflation and real spot rates. The following subsections discuss how much of the variability in the data is explained by the first five principal components for each model and each yield curve. Note that since we apply the PCA to the fitted Cairns values the first five principal components are sufficient to explain all the variability in the data. The reason is that by fitting the Cairns model on to the yield curve data we have already decreased the dimension of the nominal spot rates from 50 to 5 and the dimension of the implied inflation and the real spot rates from 46 to 5.

3.3.1 PCA on Fitted Nominal Yield Curves

Table 3.1 shows the results of the PCA of the standardized fitted nominal yield curves obtained by four different C parameter sets which have been discussed in Chapter 2 in details. The first row for each model (i.e. each C parameter sets) in Table 3.1 gives the *standard deviations* of the loadings of the principal components (i.e., the square roots of the eigenvalues of the covariance matrix). When we calculate the total of the variances of these loadings we obtain 50 for the nominal spot rates and 46 for the implied inflation and the real spot rates which are equal to the total number of PCs for the yield curves. The second row presents the *proportion of variance* which is calculated by dividing the corresponding eigenvector (variance) for each PC by the total eigenvectors (total variance) and can be interpreted as the proportion of the variance explained by that PC. The third row gives the cumulative proportion of the explained variability by the PCs.

According to Table 3.1, the first factor, *level*, accounts approximately for 97% of the explained variability for each fitted nominal yield curves. The second factor, *slope*, accounts for about 2.3% and the third factor, *curvature* accounts for 0.3% to 0.5% for the fitted yield curves. By looking at these proportions we can conclude that the choice of C parameter set does not have a significant effect on the PCs for the nominal spot rates.

Table 3.1: Importance of the PCs for the Nominal Fitted Yield Curves

Nominal Spot Rates						
		PC1	PC2	PC3	PC4	PC5
Model 1	Standard deviation	6.978	1.0559	0.38867	0.19144	0.08156
	Proportion of variance	0.974	0.0223	0.00302	0.00073	0.00013
	Cumulative proportion	0.974	0.9961	0.99913	0.99987	1.00000
Model 2	Standard deviation	6.96	1.0844	0.50555	0.2341	0.08038
	Proportion of variance	0.97	0.0235	0.00511	0.0011	0.00013
	Cumulative proportion	0.97	0.9937	0.99878	0.9999	1.00000
Model 3	Standard deviation	6.975	1.0604	0.4121	0.20809	0.07270
	Proportion of variance	0.973	0.0225	0.0034	0.00087	0.00011
	Cumulative proportion	0.973	0.9956	0.9990	0.99989	1.00000
Model 4	Standard deviation	6.97	1.0865	0.49796	0.23222	0.07728
	Proportion of variance	0.97	0.0236	0.00496	0.00108	0.00012
	Cumulative proportion	0.97	0.9938	0.99880	0.99988	1.00000

3.3.2 PCA on Fitted Implied Inflation Yield Curves

Table 3.2 shows the standard deviations, proportions and the cumulative proportions of the explained variability by the PCs for different C parameter sets for the fitted implied inflation yield curves. Since the proportions of the variability explained by the PCs are almost equal regardless of the models we can say that it does not make much difference which C parameter set we used for filling the gaps in the implied inflation spot rate data in terms of the obtained PCs.

Table 3.2: Importance of the PCs for the Implied Inflation Fitted Yield Curves

Implied Inflation Spot Rates						
		PC1	PC2	PC3	PC4	PC5
Model 1	Standard deviation	6.580	1.5970	0.34651	0.16844	0.05890
	Proportion of Variance	0.941	0.0554	0.00261	0.00062	0.00008
	Cumulative Proportion	0.941	0.9967	0.99931	0.99992	1.00000
Model 2	Standard deviation	6.564	1.6416	0.40956	0.22023	0.0490
	Proportion of Variance	0.937	0.0586	0.00365	0.00105	0.00005
	Cumulative Proportion	0.937	0.9952	0.99889	0.99995	1.00000
Model 3	Standard deviation	6.58	1.605	0.35095	0.1795	0.04068
	Proportion of Variance	0.94	0.056	0.00268	0.0007	0.00004
	Cumulative Proportion	0.94	0.997	0.99926	1.0000	1.00000
Model 4	Standard deviation	6.565	1.619	0.46583	0.2352	0.07031
	Proportion of Variance	0.937	0.057	0.00472	0.0012	0.00011
	Cumulative Proportion	0.937	0.994	0.99869	0.9999	1.00000

3.3.3 PCA on Fitted Real Yield Curves

Table 3.3 shows the standard deviations, proportions and the cumulative proportions of the explained variability by the PCs for different C parameter sets for the real spot rates. Similar to nominal and implied inflation spot rates, the proportions explained by the PCs for each model indicate that the PCA seems robust to the choice of C parameter set.

Table 3.3: Importance of the PCs for the Real Fitted Yield Curves

Real Spot Rates						
		PC1	PC2	PC3	PC4	PC5
Model 1	Standard deviation	6.596	1.5076	0.44517	0.12883	0.03918
	Proportion of Variance	0.946	0.0494	0.00431	0.00036	0.00003
	Cumulative Proportion	0.946	0.9953	0.99961	0.99997	1.00000
Model 2	Standard deviation	6.595	1.5108	0.44835	0.14407	0.03546
	Proportion of Variance	0.946	0.0496	0.00437	0.00045	0.00003
	Cumulative Proportion	0.946	0.9951	0.99952	0.99997	1.00000
Model 3	Standard deviation	6.596	1.5093	0.4445	0.13340	0.03412
	Proportion of Variance	0.946	0.0495	0.0043	0.00039	0.00003
	Cumulative Proportion	0.946	0.9953	0.9996	0.99997	1.00000
Model 4	Standard deviation	6.595	1.5114	0.44702	0.14326	0.03450
	Proportion of Variance	0.946	0.0497	0.00434	0.00045	0.00003
	Cumulative Proportion	0.946	0.9952	0.99953	0.99997	1.00000

3.4 Robustness of the Principal Components to the Choice of C=(c_1, c_2, c_3, c_4) Parameter Sets

Although the tables in Section 3.3 indicate that the PCA is robust to the choice of C parameter set we will examine it in more detail in this section.

In the previous chapter we have compared the different C parameter sets in various ways to decide the best exponential rates to be used for fitting the yield curves. The aim was to choose the one which produces the values closest to the original yield curve data. We will make a similar comparison in this section. However, the aim is to test the robustness of the PCA to the choice of C parameter sets. First, we will compare the PCs which are obtained by applying the analysis on different fitted term structures due to using different C parameter sets by deriving the Cairns fitted yield curves using these PCs. Then we calculate the residuals as the difference between the fitted and the derived yield curve data using the PCs. We have discussed how to obtain the yield curve data back by using the PCs as well as calculating the residuals in Section 3.2. Table 3.4 shows the sum of squares of the residuals for different models (different C parameter sets) for nominal, implied inflation and real spot rates. Second, we will draw the loadings of the PCs obtained from the different models for each yield curve on the same scale to see their shapes and how much they differ from each other.

Table 3.4: Residual Analysis of the PCs for Different C parameter sets for the fitted yield curves

Nominal Spot Rates				
Number of PCs	**Model 1**	**Model 2**	**Model 3**	**Model 4**
3	339.36	479.99	380.78	469.44
4	52.13	50.63	41.42	46.80
5	0.00	0.00	0.00	0.00
Implied Inflation Spot Rates				
Number of PCs	**Model 1**	**Model 2**	**Model 3**	**Model 4**
3	201.20	321.67	214.16	380.91
4	21.92	15.19	10.46	31.24
5	0.00	0.00	0.00	0.00
Real Spot Rates				
Number of PCs	**Model 1**	**Model 2**	**Model 3**	**Model 4**
3	114.57	139.11	119.81	137.21
4	9.70	7.95	7.35	7.52
5	0.00	0.00	0.00	0.00

According to the results in Table 3.4, when we use first three PCs to derive the nominal yield curve back, the PCs obtained from Model 1 gives the smallest errors while Model 2 gives the largest. Although we see a very small difference between the explained variability by the first three PCs for the different models in Table 3.1, analysing the residuals in terms of the sum of squares shows the effect of that small difference while deriving the original nominal yield curve data back. Moreover, when we use the first four PCs and calculate the sum of squares Model 3 produces the smallest values. Since the first five PCs explained all the variability in the data they enable us to obtain the original yield curves back without any errors.

For the implied inflation yield curve, the residuals obtained from Model 1 give the best result due to the highest explained variability by the first three PCs showed in Table 3.2. Although the difference is much smaller between the sum of squares obtained by using the first four PCs, Model 3 is the best among the others.

We have a similar conclusion for the real yield curve analysis as well. Despite the fact that the difference in the sum of squares is quite small, Model 1 and Model 3 perform better when we use first three and first four PCs respectively to derive the real spot rates back. These results are consistent with the proportion of the explained variability by these models in Table 3.3.

As a concluding comment, we see that Model 1 is the best for all yield curves considering the first three PCs and Model 3 is the best for all yield curves for considering the first four PCs.

As a last step of the comparison of the models we test the robustness of the principal components of the fitted yield curves to the choice of C parameter sets by displaying the loadings of the first five principal components (eigenvectors) of the four models for each yield curve.

Figure 3.1 shows the loadings of the PCs for the fitted nominal spot rates. Each graph shows the loadings for each PCs obtained from four different models. For example, the first graph displays the first PC for each model in different colours. We draw the loading graphs on the same vertical scale to see the shapes of the PCs and how they look like relative to each other. When we look at the loadings of the first PC, levels, we see that they are quite flat and overlap on this vertical scale. The second graphs represent the loadings of the second PC, slope. The lines are very close to each other for all models except for small discrepancies at the very short and very long ends. The third graphs show the loadings for the third PC which is named as curvature. Although there are some differences between the curvature component loadings we can still conclude that the choice of C parameter set does not have a significant effect on the obtained PCs. The graphs for the loadings of the forth and fifth PCs show that there are more varieties in the loadings based on different C parameter sets but since the contribution of these PCs are very small we can ignore them for our further study.

Figure 3.2 and Figure 3.3 show the loadings of the PCs for the fitted implied inflation and real spot rates respectively. The first three loadings based on the first three PCs are mostly overlapping on the displayed vertical scale both for the implied inflation and real yield curves. However the loadings for the last two PCs are not as close as the previous ones as in the nominal yield curve graphs.

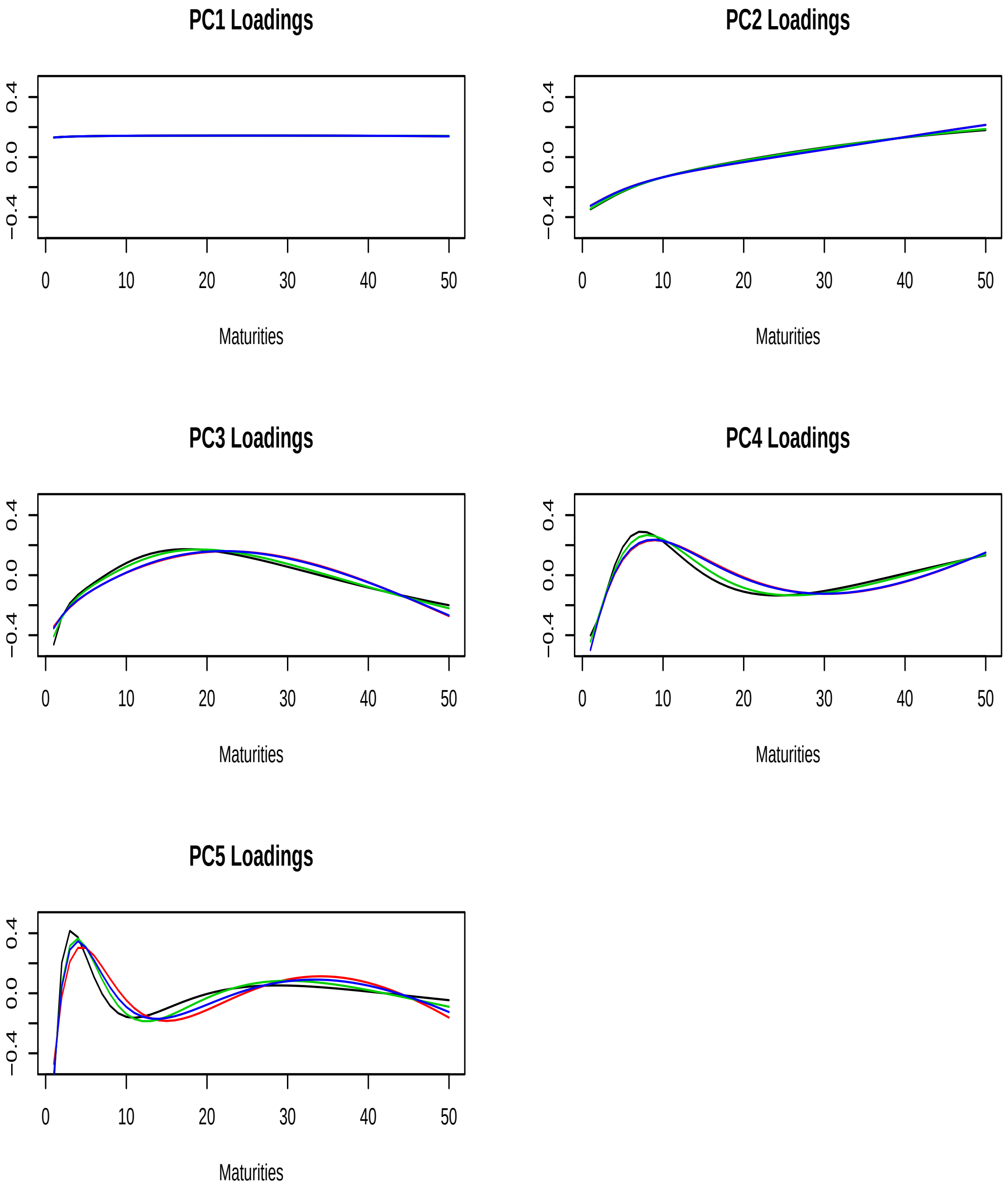

Figure 3.1: Loadings of the PCs for Nominal Spot Rates for Different C Parameter Sets

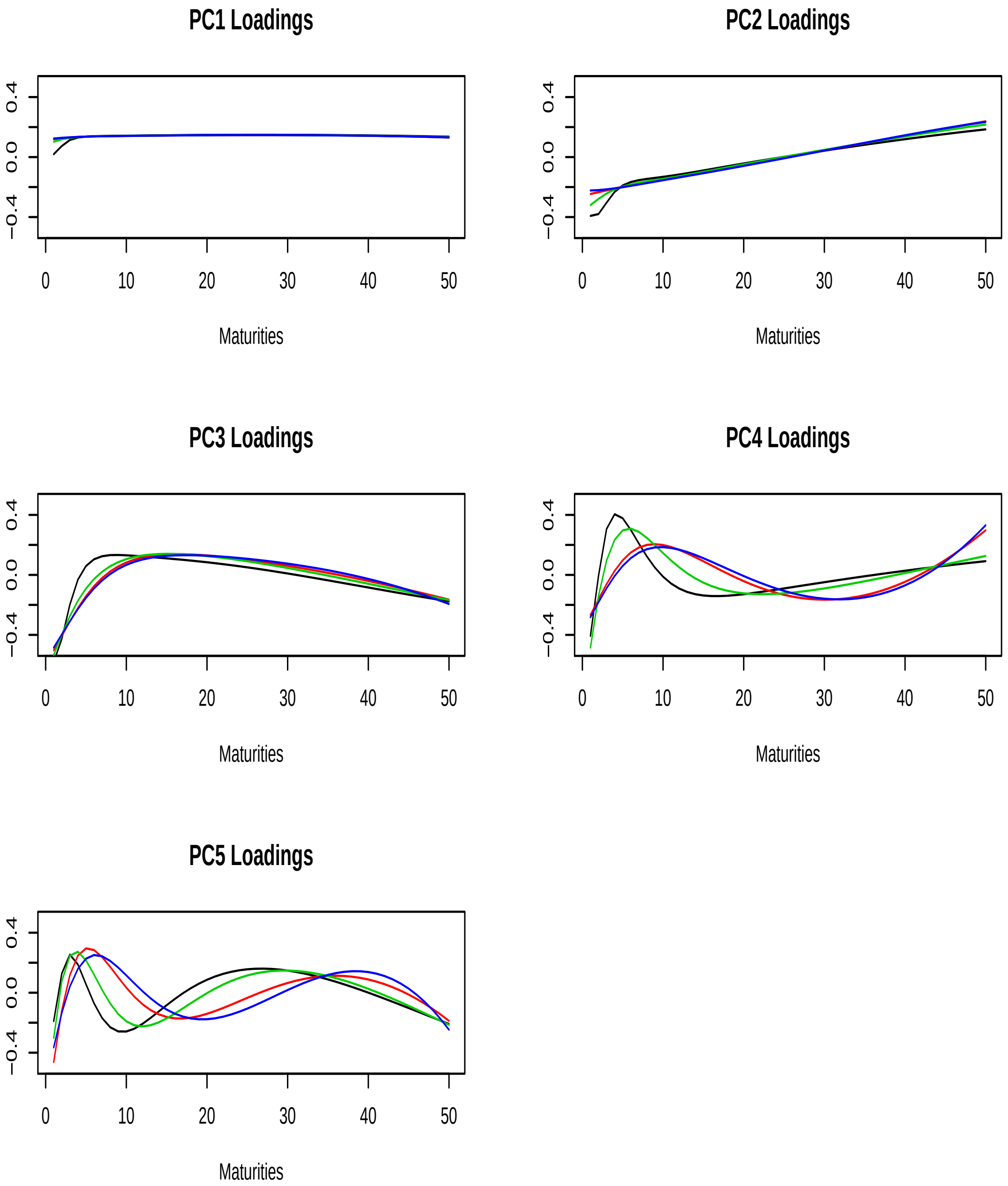

Figure 3.2: Loadings of the PCs for Implied Inflation Spot Rates for Different C Parameter Sets

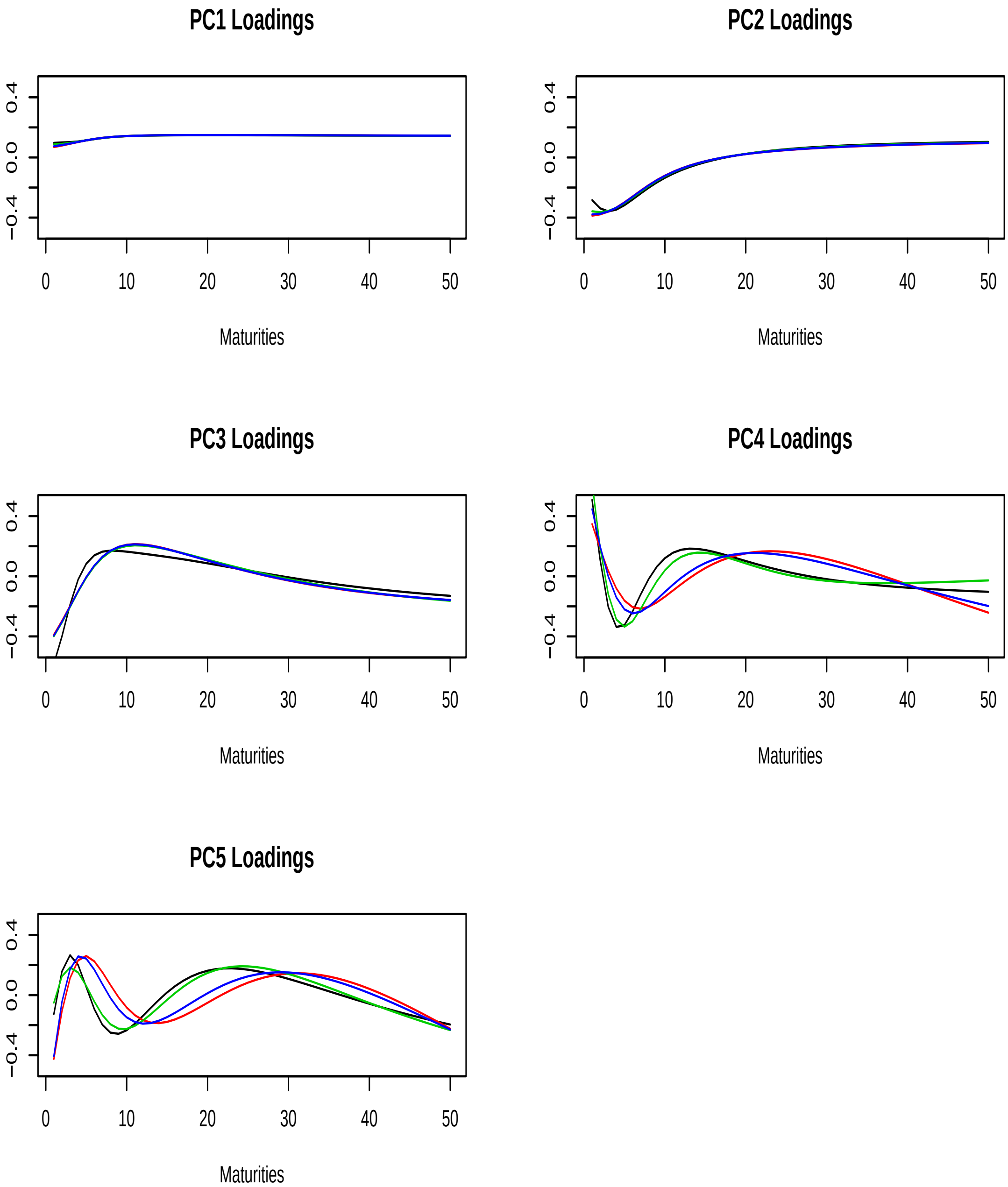

Figure 3.3: Loadings of the PCs for Real Spot Rates for Different C Parameter Sets

3.5 Interim Conclusion: Principal Component Analysis on the Fitted UK Term Structures

In this chapter we have discussed the PCA and its robustness to the choice of C parameter set. We compared the PCs obtained from the fitted nominal, implied inflation and real spot rates for the different C parameters used in Cairns parametric curve. Our analyses show that the amount of variability explained by the PCs does not change significantly for different fitted yield curves. However, even the small changes in the explained variability might affect the size of the residuals noticeably when we drive the yield curves back using those PCs. Model 1 and Model 3 perform better than the other two models in terms of producing the closest values to the fitted spot rates when we use the first three and four PCs to obtain the yield curves back. On the other hand the graphs of the loadings of the PCs show that the PCA is quite robust to the choice of C parameter set due to displaying overlapping lines for the most important components.

Chapter 4

Modelling the Term Structures

4.1 Introduction

The yield-curve models developed by macroeconomists and financial economists are very different due to particular demands and different motives. While macroeconomists focus on the role of expectations of inflation and future real economic activity in the determination of yields, financial economists avoid any explicit role for such determinants. These different attitudes cause a gap between the yield curve models developed. There are various recent papers which aim to bridge this gap by formulating and estimating a yield curve model that integrates macroeconomic and financial factors (Ang and Piazzesi (2001, 2003), Hördahl *et al.* (2006), Wu (2002), Evans and Marshall (1998, 2001), Kozicki and Tinsley (2001), Ang and Bekaert (2003), Dai and Philippon (2005), Dewachter and Lyrio (2006), Rudebusch and Wu (2004, 2008), Diebold, Piazzesi and Rudebusch (2004), Diebold, Rudebusch and Aruoba (2006), Diebold and Li (2006), Diebold, Li and Yue (2007), Lildholdt, Panigirtzoglou and Peacock (2007), Ang, Bekaert and Wei (2008), Ang, Piazzesi and Wei (2006), Kaminska (2008))

Different from the previous studies, this study aims to model the UK term structures of interest rates and the term structure of implied inflation simultaneously using the additional macroeconomic variables in a way that is consistent with macroeconomic theory. As will be introduced in Section 5.2, the related literature discusses the term structures of the interest rates but not the term structure of implied inflation. Hence, the work is important due to being the first and only study which models all three yield curves simultaneously so far.

We model the yield curve data for different frequencies. Following the previous studies on macro-finance models we start with a basic 'yield-only' model as a model of just the yield curve without macroeconomic variables. Then we model the yield curves simultaneously using the additional macroeconomic variables namely output gap and realised inflation.

This chapter aims to present a brief literature review and the data we use in the 'yield-only' and 'yield-macro' models we will discuss in Chapter 5 and Chapter 6 respectively.

4.2 Literature Review

Short-term interest rates have different meanings from a macroeconomic perspective and a finance perspective. From a macroeconomic perspective, the short-term interest rate is a policy instrument directly controlled by the central bank to achieve its economic stabilization goals. From a finance perspective, the short rate is a fundamental building block for yields of other maturities, which are just risk-adjusted averages of expected future short rates. Much recent research has pointed out that a joint macro-finance modelling strategy would provide the most comprehensive understanding of the term structure of interest rates (Diebold, Piazzesi and Rudebusch, 2004).

The previous studies on macro-finance models mostly start with a basic 'yield only' model as a model of just the yield curve without macroeconomic variables. Then they incorporate macroeconomic variables and estimate a 'yield-macro' model. The stated aim is to examine the nature of the linkage between the factors driving the yield curve and macroeconomic fundamentals.

Ang and Piazzesi (2001) is one of the earliest works which describes joint dynamics of bond yields and macroeconomic variables. They investigate how macro variables affect bond prices and the dynamics of the yield curve using a term structure model with inflation and economic growth factors, together with latent variables. They use both observed macro factors and unobserved yield variables in a Vector Autoregression with a no-arbitrage restriction.

Ang and Piazzesi (2001) use Taylor policy rules (1993) [1] to model the short term yields. Movements in the short rate r_t are traced to movements in observed macro variables f_t^o and a component which is not explained by macro variables, an orthogonal shock v_t:

$$r_t = a_0 + a_1' f_t^o + v_t \tag{4.1}$$

Taylor's original specification uses two macro variables as factors in f_t^o. The first variable is an annual inflation rate and the second variable is the output gap. Another type of policy rule that has been proposed by Clarida *et al.* (2000) is a forward-looking version of the Taylor rule. According to this rule, the central bank reacts to expected inflation and the expected output gap. This implies that any variable that forecasts inflation or output will enter the right-hand side of Equation 4.1. Thus, Ang and Piazzesi (2001) specify the short rate as affine functions of factors

$$r_t = \delta_0 + \delta_{11}' X_t^o + \delta_{12}' X_t^u \tag{4.2}$$

Their approach is to specify the latent factors X_t^u (the superscript u stands for unobserved) as orthogonal to the macro factors X_t^o (the superscript o stands for observed). In this case, the short rate dynamics of the term structure model can be interpreted as a version of the Taylor rule with the errors $v_t = \delta_{12}' X_t^u$ being unobserved factors. They use the restrictions from no-arbitrage to separately identify latent factors.

They estimate three models: The estimation based on the current values of the

[1] Taylor rule is a monetary-policy rule that stipulates how much the central bank should change the nominal interest rate in response to divergences of actual GDP from potential GDP and of actual inflation rates from a target inflation rate. Taylor (1993) showed that the behaviour of the nominal interest rate used by the Federal Reserve as its policy instrument was well described by the simple formula:

$$i_t = \pi_t + r_t^* + a_\pi(\pi_t - \pi_t^*) + a_y(y_t - \overline{y}_t)$$

In this equation, i_t is the target short-term nominal interest rate (the federal funds rate in the US), π_t is the rate of inflation, π_t^* is the desired rate of inflation, r_t^* is the assumed equilibrium real interest rate, y_t is the logarithm of real GDP, and $\overline{y}_t$ is the logarithm of potential output, as determined by a linear trend.

macro variables is called *macro model*. The version with the full lagged Taylor rule is denoted as the *macro lag model*. The estimation without any macro variables is called the *yields-only model*. They find that the forecasting performance of a VAR improves with the no-arbitrage restrictions and macro factors. Variance decompositions show that macro factors explain up to 85% of the variation in bond yields. Macro factors primarily explain movements at the short end and middle of the yield curve while unobservable factors still account for most of the movement at the long end of the yield curve.

Evans and Marshall (2001) looked at the different types of macroeconomic impulses on the nominal yield curve. They use a variety of vector autoregression approaches. They start with an atheoretical empirical exercise that simply asks whether the level, slope and curvature of the yield curve is significantly affected by the block of macroeconomic variables. The only restriction they impose is to assume (following Ang and Piazzesi (2001)) that the three yields do not feed back to the macro variables. They confirm Ang and Piazzesi's (2001) result that a substantial portion of the variability of short-and medium-term yields is driven by macroeconomic factors. Unlike those authors, they find that most of the long-run variability of long-term rates is driven by macro impulses and that the level of the yield curve responds strongly to macro factors. The strongest responses come from innovations that induce output and inflation responses in the same direction. Then they employ a structural vector autoregressive model to identify macro economic impulses.

Evans and Marshall (2001) find that macroeconomic factors have a substantial, persistent and statistically significant effect on the level of the term structure. This finding stands in contrast to Ang and Piazzesi (2001), who find that the level of the yield curve is driven only by latent variables orthogonal to their macro factors.

Ang and Bekaert (2003) develop a term structure model with regime switches, time varying prices of risk and inflation to identify the real interest rate and expected inflation components of the nominal yield curve. They find that expected inflation drives about 80% of the variation of nominal yields at both short and long maturities, but during normal times, all of the variation of nominal term spreads is due to expected inflation and inflation risk.

Rudebusch and Wu (2004) describe the economic underpinnings of the yield curve by constructing and estimating a combined macro-finance framework. They characterise the relationships between the no-arbitrage latent term structure factors and various macroeconomic variables. The level factor is given an interpretation as the perceived medium-term central bank inflation target. The slope factor is related to cyclical variation in inflation and output gaps. In particular, the slope factor varies as the central bank moves the short end of the yield curve up and down in order to achieve its macroeconomic policy goals. In their work, Rudebusch and Wu modelled macro factors as completely exogenous to the yield curve.

Dewachter and Lyrio (2006) model *consistently* long-run inflation expectations *simultaneously* with the term structure and show the importance of long-run inflation expectations in the modelling of long-term bond yields. Their paper also provides a macroeconomic interpretation for the latent factors found in standard finance models of the yield curve: the 'level' factor represents the long-run inflation expectation of agents; the 'slope' factor captures temporary business cycle conditions; and the 'curvature' factor expresses a clear independent monetary policy factor. Their method improves on the approach taken in the literature to use long-run expectations of macroeconomic variables in order to fit the yield curve. A two-step approach is used where long-run expectations are first filtered from the data using some statistical procedure, and then subsequently used to fit the term structure. A drawback of this method is that not all available information is used to filter the long-run expectations since only a subset of the data series is used.

Diebold and Li (2006) use variations on the Nelson-Siegel (1987) exponential components framework to model the entire yield curve as a three dimensional parameter evolving dynamically. They show that the three time varying parameters may be interpreted as factors corresponding to level, slope and curvature, and they may be estimated with high efficiency. They propose and estimate autoregressive models for the factors to produce term-structure forecasts at both short and long horizons.

Diebold, *et al.* (2006) estimated a model that summarises the yield curve using latent factors (level, slope and curvature) and also includes observable macroeconomic variables (real activity, inflation and the monetary policy instrument).

Referring to Diebold and Li (2006), they interpret the Nelson-Siegel (1987) curve as a latent factor model in which β_1, β_2 and β_3 are time-varying level, slope and curvature factors and the terms that multiply these factors are factor loadings. Thus, they write

$$y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{\lambda\tau}\right) \tag{4.3}$$

where L_t, S_t and C_t are the time-varying β_1, β_2 and β_3.

Starting with the 'yield-only' model, Diebold *et al.* (2006) suggest that a VAR(1) model might fit the data well by examining the autocorrelations and crosscorrelations of the three latent factors. Thus, one of the possible structures for the 'yield-only' model is as below:

If the dynamic movements of L_t (*level*), S_t (*slope*) and C_t (*curvature*) follow a vector autoregressive process of first order, then the model forms a state-space system. The transition equation, which governs the dynamics of the state vector, is

$$\begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix} \tag{4.4}$$

$t = 1, ..., T$. The measurement equation, which relates a set of N yields to the three unobservable factors, is

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ . \\ . \\ . \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\tau_1\lambda} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\tau_2\lambda} \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\tau_N\lambda} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ . \\ . \\ . \\ \epsilon_t(\tau_N) \end{pmatrix} \tag{4.5}$$

$t = 1, ..., T$.

While previous works only consider a unidirectional macro linkage, because inflation and output are assumed to be determined independently of the shape of the yield curve, but not vice versa, Diebold *et al.* (2006) are particularly interested in analyzing the potential bidirectional feedback from the yield curve to the economy and back again.

They also compare their approach with others that have been used in the literature such as an unrestricted VAR model for a set of yields (Evans and Marshall (1998, 2001)). They indicate one potential drawback of such a representation as the results may depend on the particular set of yields chosen. A factor representation, as above, can aggregate information from a large set of yields. Such an approach restricts the factors to be orthogonal to each other but does not restrict the factor loadings at all. In contrast, their model allows correlated factors but restricts the factor loadings through limitations on the set of admissible yield curves. For example, the Nelson-Siegel form guarantees positive forward rates at all horizons and a discount factor that approaches zero as maturity increases. Alternative restrictions such as no-arbitrage could also be imposed.

Given the ability of the level, slope and curvature factors to provide a good representation of the yield curve, Diebold *et al.* (2006) relate them to macroeconomic variables and construct a *yield-macro* model. They use an expanded version of the above state-space model and estimate the parameters of the new model. Their measures of the economy include three key variables: manufacturing capacity utilization, the federal fund rates and annual price inflation. These three variables represent, respectively, the level of real economic activity relative to potential, the monetary policy instrument and the inflation rate, which are widely considered to be the minimum set of fundamentals needed to capture basic macroeconomic dynamics. The measurement errors associated with the yields-macro model are essentially identical to those of the yields-only model. They find strong evidence of macroeconomic effects on the future yield curve and somewhat weaker evidence of yield curve effects on future macroeconomic developments. Hence, although bidirectional causalty is likely to be present, effects in the tradition of Ang and Piazzesi (2001) seem more important. They also relate their yield curve modelling approach to a traditional macroeconomic approach based on the expectations hypothesis. The results indicate that the expectation hypothesis [2] may hold reasonably well during certain periods, but that it does not hold across the entire sample.

Lildholdt, Panigirtzoglou and Peacock (2007) estimate yield curve models for the United Kingdom, where the underlying determinants have a macroeconomic interpretation. The first factor is an unobserved inflation target, the second factor is annual inflation and the third factor is a 'Taylor rule residual', which among other things, captures the effects of the output gap and monetary policy surprises in the Taylor rule. They find that the long end of the yield curve is primarily driven by changes in the unobserved inflation target. At shorter maturities, yield curve movements reflect short-run inflation and the Taylor rule residual including the output gap effect.

Ang, Piazzesi and Wei (2006) build a dynamic model for GDP growth and yields that completely characterizes expectations of GDP which does not permit arbitrage. Contrary to previous findings, they predict that the short rate has more predictive power than any term spread.

4.3 Data

To construct the 'yield-only' and 'yield-macro' models, we use nominal government spot interest rates extracted from the conventional gilt market, real spot interest rates and implied inflation rates extracted from the index-linked gilt market by the Bank of England (2010). We use all available maturities i.e. 50 different maturities for nominal rates (starting from 6 month and ending with 25 years) and 46 maturities for real rates and implied inflation (starting from 2.5 years and ending with 25 years). As for the macroeconomic variables we use realised inflation obtained from the Retail Price Index and output gap provided by the OECD Economic Outlook publications.

The output gap, as defined by the OECD in the Economic Outlook, is the difference between actual Gross Domestic Product (GDP) and potential GDP as a percent of potential GDP. Potential GDP has been defined as the level of output that an economy

[2]The expectations hypothesis of the term structure states that movements in long rates are due to movements in expected future short rates.

can produce at a constant inflation rate. However an economy can temporarily produce more than its potential level of output at the cost of creating inflationary pressures. Therefore, while GDP is compiled according to international guidelines and observed the same cannot be said for the potential GDP. Not only is the methodology for estimating potential GDP open to discussion with the estimate itself usually depending on the estimate of capital stock, the potential labour force (which in turn depends on the demographic factors and on the participation rates), the estimate for NAIRU (non-accelerating inflation rate of unemployment or structural rate of unemployment) and the level of labour efficiency (Tosetto, 2008).

The output gap is linked to the concepts of 'capacity' and 'demand/supply'. When actual output exceeds the economy's potential, the output gap is positive and when actual output is below potential output, the output gap is negative. A positive output gap is also referred to as excess demand, while a negative to as excess supply. Therefore in theory when spending in the economy is high in relation to capacity (positive output gap), this tends to put upward pressure on prices and, accordingly inflation will also tend to rise.

The output gap is often subject to considerable revision over time. This is due to the fact that as for any measure of the business cycle potential activity, which is, in this case potential output or potential GDP as a target variable is unobservable. So the measure of the gap between actual and potential output: is not well defined, sensitive to the choice of the estimation technique, and also sensitive to the available dataset and therefore itself often subject to considerable revision over time. However uncertainty about the size and the movements of the output gap is not the only one which policymakers have to face and it does not imply that the output gap and the potential output estimates are not useful, because they still contain information, even if measured with error (Tosetto, 2008).

As for the realised inflation, we calculate the annual inflation by taking the difference of the logged values of quarterly RPI data.

Chapter 5

Modelling the UK Term Structures: The Yield-Only Model

5.1 Introduction

We use monthly data to construct the UK 'yield-only' model. First we introduce the data by presenting some descriptive statistics in Section 5.2. Section 5.3 discuses the 'yield-only' model along with the principal component analysis applied on the data, auto- and cross-correlations among the PCs, suitable models for each variable and an analysis of the residuals respectively. Section 5.4 describes how we derive the term structures back and examine the one-month ahead forecasts by constructing 95% confidence intervals for the forecasts. Furthermore, we check whether our one-month ahead forecasts satisfy the Fisher relation and whether we can forecast one of the yield curves using the other two in Section 5.5. Finally, Section 5.6 concludes.

5.2 Data

To construct the 'yield-only' model, we use monthly UK nominal government spot interest rates extracted from the conventional gilt market, monthly real spot interest rates and monthly implied inflation rates extracted from the index-linked gilt market by the Bank of England. As we have discussed in Chapter 2, first we fit the Cairns model in order to use all available maturities, i.e. 50 different maturities for nominal rates (starting from 6 month and ending with 25 years) and 46 maturities for real rates and implied inflation (starting from 2.5 years and ending with 25 years).

In Table 5.1, we present the summary statistics for the fitted monthly nominal and real interest rates and implied inflation rates at representative maturities (in years). Although a typical yield curve is upward sloping, and the long rates are less volatile and more persistent than short rates, due to having a relatively short period of data we see that the means of the yield curves for different maturities are quite close to each other. Considering the standard deviations, although they do not change significantly, the volatilities decrease for nominal and implied inflation data as the maturities get longer. The minimum (maximum) values for the shortest maturities for all three yield curves are lower (higher) than the minimum (maximum) values for the longest maturities. The autocorrelation functions indicate significant correlations for one month, six months and twelve months (one year) lags in the yield curves. These high correlations show that the interest rates and implied inflation rates depend highly on their previous values. Although the autocorrelation functions decay very slowly for the three yield curves, which might indicate non-stationarity, we will assume that they are stationary. It is more an economic assumption rather than a statistical one. We do not have a sufficiently long period of data here to justify the stationarity of the yield curves, but observation over far longer periods shows that yields must be stationary (Homer, 1963).

5.3 The Yield-Only Model

5.3.1 PCA on the Monthly Yield Curve Data

We apply PCA on monthly values of the fitted nominal spot rates, implied inflation spot rates and real spot rates to obtain the three most important components of these yield curves.

Tables 5.2, 5.3 and 5.4 show the results of the principal component analysis based on the mean adjusted fitted yield curves. It is seen that the first five principal components explain all the variability in the data. The first factor, *level*, accounts for

Table 5.1: Descriptive Statistics for the Fitted Monthly Yield Curves

Nominal Spot Rates (%)										
Maturity	Mean	Std. Dev.	Med	Min	Max	Skewness	Kurtosis	$\rho(1)$	$\rho(6)$	$\rho(12)$
0.5	6.79	3.22	5.81	0.28	14.82	0.55	-0.25	0.98	0.86	0.73
2.5	6.81	2.68	6.29	1.55	13.49	0.32	-0.77	0.98	0.86	0.75
5	6.93	2.53	6.45	2.41	12.95	0.30	-1.13	0.98	0.88	0.79
10	6.99	2.41	6.63	3.36	12.36	0.28	-1.41	0.98	0.91	0.84
15	6.85	2.25	6.66	3.82	11.48	0.20	-1.60	0.99	0.92	0.86
20	6.64	2.08	6.62	3.86	10.43	0.09	-1.74	0.99	0.93	0.88
25	6.41	1.93	6.56	3.75	9.41	0.04	-1.78	0.99	0.94	0.88
Implied Inflation Spot Rates (%)										
Maturity	Mean	Std. Dev.	Med	Min	Max	Skewness	Kurtosis	$\rho(1)$	$\rho(6)$	$\rho(12)$
2.5	4.07	3.01	3.11	-6.37	10.72	0.45	0.06	0.97	0.85	0.77
5	4.10	2.03	3.28	-2.37	9.41	0.50	-0.52	0.98	0.87	0.80
10	4.14	1.64	3.61	0.50	8.49	0.57	-0.87	0.98	0.87	0.80
15	4.04	1.39	3.65	2.05	7.67	0.52	-0.95	0.97	0.88	0.80
20	3.86	1.14	3.74	2.16	6.64	0.32	-1.10	0.97	0.87	0.79
25	3.66	0.92	3.79	2.06	5.56	0.07	-1.26	0.97	0.85	0.77
Real Spot Rates (%)										
Maturity	Mean	Std. Dev.	Med	Min	Max	Skewness	Kurtosis	$\rho(1)$	$\rho(6)$	$\rho(12)$
2.5	2.61	1.74	2.54	-2.29	7.90	-0.04	-0.12	0.91	0.57	0.37
5	2.74	0.95	2.95	-0.23	5.26	-0.47	-0.04	0.91	0.62	0.45
10	2.85	0.97	3.02	0.55	4.94	-0.24	-1.10	0.96	0.83	0.73
15	2.88	1.07	3.13	0.72	4.90	-0.19	-1.32	0.98	0.90	0.83
20	2.86	1.14	3.16	0.63	4.74	-0.21	-1.38	0.99	0.93	0.86
25	2.82	1.21	3.21	0.49	4.72	-0.24	-1.39	0.99	0.93	0.88

96%, 95% and 95% for the nominal, implied inflation and real spot rates respectively. Slope factors account for 4%, 5% and 4% and curvatures account for less than 1% for all yield curves. Thus, the first three principal components explain more than 99% of the variability in the term structures. Although the curvature factors seem to explain very little, it is important to include this component to capture the hump shape of the yield curves for some specific dates.

Figure 5.1 shows the loadings of the first three principal components for the monthly fitted yield curves. The first factor, *level* is relatively flat and represents an approximately parallel shift in the yield curve; the second factor, *slope* takes negative values

Table 5.2: Importance of the PCs for the Fitted Nominal Spot Rates

	PC1	PC2	PC3	PC4	PC5
Standard deviation	16.249	3.3570	0.80963	0.54153	0.18836
Proportion of variance	0.956	0.0408	0.00237	0.00106	0.00013
Cumulative proportion	0.956	0.9964	0.99881	0.99987	1.00000

Table 5.3: Importance of the PCs for the Fitted Implied Inflation Spot Rates

	PC1	PC2	PC3	PC4	PC5
Standard deviation	9.340	2.0980	0.66632	0.38008	0.11611
Proportion of variance	0.946	0.0477	0.00481	0.00157	0.00015
Cumulative proportion	0.946	0.9935	0.99829	0.99985	1.00000

Table 5.4: Importance of the PCs for the Fitted Real Spot Rates

	PC1	PC2	PC3	PC4	PC5
Standard deviation	7.127	1.495	0.4556	0.16869	0.03995
Proportion of variance	0.954	0.042	0.0039	0.00053	0.00003
Cumulative proportion	0.954	0.996	0.9994	0.99997	1.00000

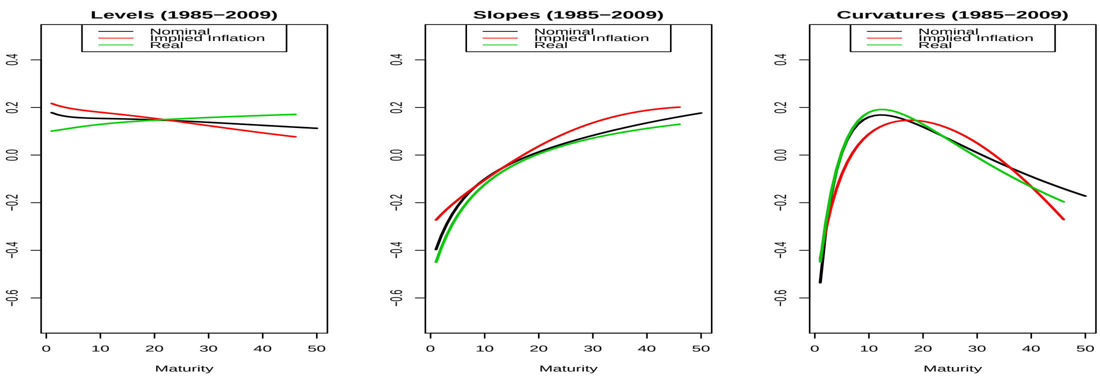

Figure 5.1: Loadings of the PCs for the Monthly Fitted Yield Curves

on the short maturities and positive values on the long maturities to capture the slope of the curve and the third factor, *curvature* takes negative values for the short and long maturities and positive values for the medium maturities to give the hump shape to the yield curve. The three components for the three yield curves have similar shapes. The slope and curvature factors of the nominal and real spot rates seem much closer than the corresponding factors of the implied inflation spot rates. The similarity between the first three principal components of the yield curves may indicate the existence of some *common principal components* which will be discussed as a further research in Chapter 8.

Figure 5.2, 5.3 and 5.4 present the time series graphs of the first three PCs of the monthly fitted yield curves for the nominal, implied inflation and real spot rates on the same scale. Drawing the time series graphs of the PCs on the same scale make it easier to see the explanatory power of these components of the variability in the data. As the percentage of the variability explained by the PC decreases, the graph becomes flatter. This explains why the graphs for the second and third PCs are much flatter than the first one for three yield curves.

The graphs of the first PCs of the nominal and real spot rates show that the levels of the interest rates are mostly decreasing since 1995 whereas the level of the implied inflation is relatively stable. This might be consistent with the "inflation targeting policy" of the bank of England after 1995. The relative stability of the implied inflation level factor after 1998 can be explained by the independence of the Bank of England to set the monetary policy in 1997[1].

[1]In 1997, as well as modifying the inflation target, the Bank of England was given independence to set interest rates by the new Government. This was a major change in the policy framework. It meant interest rates would no longer be set by politicians. The Bank would act independently of Government, though the inflation target would be set by the Chancellor. The Bank would be accountable to parliament and the wider public (Bank of England, 2010).

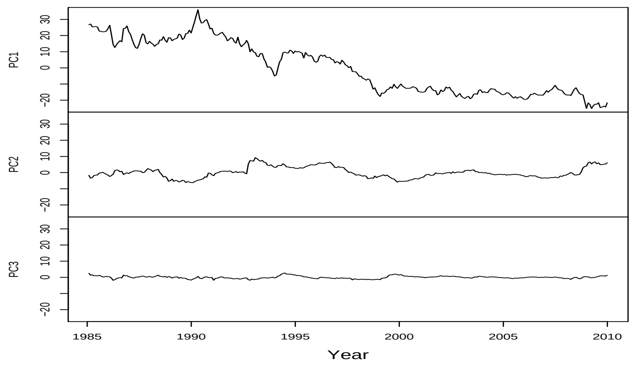

Figure 5.2: PCs of the Monthly Fitted Nominal Spot Rates

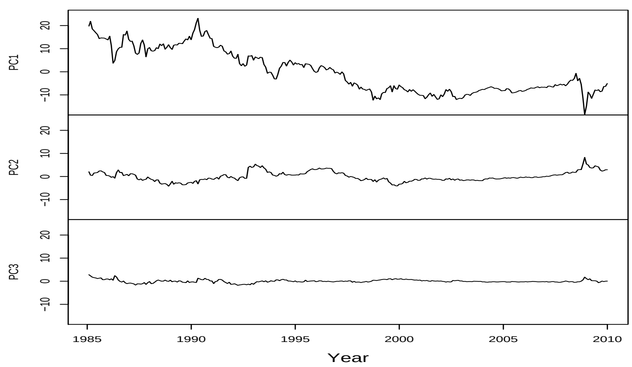

Figure 5.3: PCs of the Monthly Fitted Implied Inflation Spot Rates

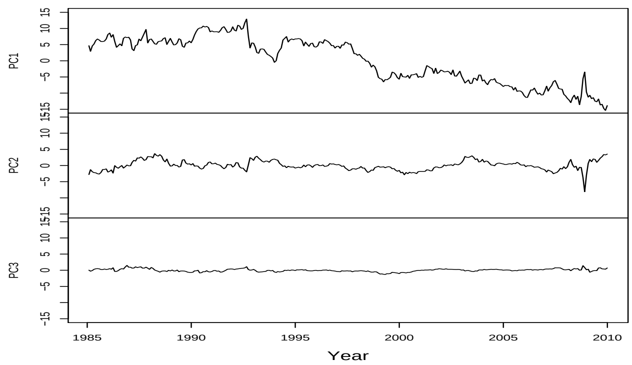

Figure 5.4: PCs of the Monthly Fitted Real Spot Rates

5.3.2 Correlations Between the Monthly Yield Factors

Table 5.5, Table 5.6 and Table 5.7 show the lagged correlations between the PCs of the three yield curves. The lag k value in the tables is the correlation between $x[t]$ and $y[t-k]$ where $x[t]$ is the variable whose autocorrelation function has been displayed by red colour and $y[t-k]$ represents all the other variables. We assume that all the variables are stationary. We use N, I and R as the abbreviations for the nominal, implied inflation and real spot rates respectively. PC represents the principal component.

Chatfield (2004) states that if a time series is completely random, and the sample size is large, the lagged-correlation coefficient is approximately normally distributed with mean 0 and variance $1/n$. Assuming normality and independence, the standard error of each autocorrelation and crosscorrelation coefficient is $1/\sqrt{n}$ where n is the number of observations in the series. Since we have 300 monthly observations, the standard error of the coefficients is equal to $1/\sqrt{300} = 0.058$. We assume that the coefficients which are greater or less than three standard errors (i.e. $3 \times 0.058 = 0.174$) are significant.

As seen from the below tables, all PCs have strong auto-correlations. The auto-correlation functions of the first PCs ($NPC1$, $IPC1$ and $RPC1$) decay very slowly and even for the *lag* 12 the auto-correlation coefficients are higher than 0.80. This might indicate non-stationarity in the data. As we have discussed previously, our analysis is based on the assumption that the spot rates are stationary. We also take the first difference of each PC and calculate the correlation coefficients. Taking the difference removes the auto-correlations and produce stationary ‘random walk’ series. Since modelling the yield curves using AR processes is economically reasonable we will continue our study by using the yield curve data themselves instead of the changes in the yield curves. Another reason to use the levels of the yield curves instead of the yield changes is that the economic theory states that the levels of the interest rates and the macroeconomic variables are connected.

The high auto-correlations in the first PCs indicate that the level of the spot rates highly depends on the level of the previous month rates. There is a significant negative simultaneous and lagged correlation between the level and slope factors of the spot rates.

The lagged cross-correlations between the first PCs of the yield curves are quite high. This is consistent with the Fisher relation which defines the nominal interest rates as the sum of the expected future inflation (implied inflation) and real interest rates. The second PCs (slope factors) and the third PCs (curvature factors) of the yield curves also have significant simultaneous and lagged cross-correlations.

Table 5.5: Lagged Correlations between the Monthly Yield Curves - I

NPC1[t] **Lag, k**	**NPC1** **[t-k]**	**NPC2** **[t-k]**	**NPC3** **[t-k]**	**IPC1** **[t-k]**	**IPC2** **[t-k]**	**IPC3** **[t-k]**	**RPC1** **[t-k]**	**RPC2** **[t-k]**	**RPC3** **[t-k]**
(0)	**1.000**	0.000	0.000	0.962	-0.039	-0.020	0.926	0.043	0.051
(1)	0.986	0.010	-0.049	0.940	-0.031	-0.049	0.924	0.044	0.053
(2)	0.970	0.027	-0.087	0.917	-0.021	-0.066	0.919	0.052	0.055
(3)	0.955	0.045	-0.120	0.898	-0.013	-0.076	0.913	0.067	0.051
(4)	0.940	0.061	-0.145	0.880	-0.004	-0.089	0.907	0.080	0.045
(5)	0.926	0.077	-0.169	0.863	0.004	-0.104	0.900	0.096	0.036
(6)	0.913	0.092	-0.191	0.848	0.008	-0.116	0.892	0.115	0.017
(7)	0.901	0.103	-0.206	0.836	0.005	-0.127	0.886	0.136	0.001
(8)	0.892	0.112	-0.216	0.825	0.003	-0.142	0.881	0.151	-0.008
(9)	0.882	0.121	-0.225	0.812	0.005	-0.156	0.876	0.158	-0.008
(10)	0.870	0.133	-0.229	0.800	0.009	-0.162	0.870	0.167	-0.008
(11)	0.858	0.143	-0.234	0.786	0.013	-0.169	0.863	0.174	-0.011
(12)	0.845	0.153	-0.237	0.771	0.017	-0.176	0.854	0.180	-0.013

NPC2[t] **Lag, k**	**NPC1** **[t-k]**	**NPC2** **[t-k]**	**NPC3** **[t-k]**	**IPC1** **[t-k]**	**IPC2** **[t-k]**	**IPC3** **[t-k]**	**RPC1** **[t-k]**	**RPC2** **[t-k]**	**RPC3** **[t-k]**
(0)	0.000	**1.000**	0.000	-0.018	0.783	-0.155	0.115	0.432	0.156
(1)	0.004	0.967	0.036	-0.014	0.733	-0.162	0.118	0.439	0.129
(2)	0.008	0.928	0.060	-0.010	0.678	-0.170	0.123	0.442	0.120
(3)	0.013	0.891	0.075	-0.006	0.629	-0.181	0.127	0.444	0.117
(4)	0.016	0.849	0.095	-0.005	0.577	-0.168	0.131	0.435	0.102
(5)	0.020	0.803	0.107	-0.006	0.526	-0.164	0.136	0.413	0.099
(6)	0.024	0.757	0.117	-0.009	0.482	-0.158	0.145	0.381	0.107
(7)	0.027	0.716	0.116	-0.013	0.448	-0.144	0.153	0.344	0.097
(8)	0.030	0.671	0.107	-0.018	0.412	-0.139	0.161	0.307	0.084
(9)	0.033	0.626	0.100	-0.019	0.370	-0.133	0.165	0.289	0.064
(10)	0.038	0.583	0.097	-0.017	0.332	-0.128	0.171	0.270	0.045
(11)	0.049	0.539	0.090	-0.011	0.295	-0.124	0.181	0.245	0.019
(12)	0.064	0.496	0.089	0.001	0.264	-0.104	0.195	0.212	-0.008

NPC3[t] **Lag, k**	**NPC1** **[t-k]**	**NPC2** **[t-k]**	**NPC3** **[t-k]**	**IPC1** **[t-k]**	**IPC2** **[t-k]**	**IPC3** **[t-k]**	**RPC1** **[t-k]**	**RPC2** **[t-k]**	**RPC3** **[t-k]**
(0)	0.000	0.000	**1.000**	0.053	-0.094	0.390	-0.046	-0.093	0.117
(1)	0.011	-0.033	0.875	0.052	-0.115	0.300	-0.027	-0.104	0.130
(2)	0.012	-0.048	0.771	0.044	-0.136	0.243	-0.018	-0.088	0.145
(3)	0.015	-0.055	0.668	0.038	-0.141	0.218	-0.008	-0.080	0.147
(4)	0.017	-0.064	0.581	0.030	-0.140	0.183	0.002	-0.090	0.166
(5)	0.021	-0.071	0.493	0.026	-0.131	0.145	0.015	-0.102	0.179
(6)	0.025	-0.067	0.412	0.027	-0.117	0.122	0.020	-0.092	0.171
(7)	0.032	-0.073	0.345	0.033	-0.124	0.105	0.026	-0.074	0.153
(8)	0.041	-0.074	0.268	0.039	-0.128	0.080	0.035	-0.052	0.149
(9)	0.051	-0.074	0.214	0.045	-0.114	0.047	0.048	-0.054	0.174
(10)	0.061	-0.065	0.172	0.052	-0.089	0.047	0.058	-0.063	0.192
(11)	0.069	-0.052	0.117	0.057	-0.054	0.024	0.068	-0.071	0.219
(12)	0.077	-0.039	0.052	0.063	-0.024	0.015	0.076	-0.071	0.250

Table 5.6: Lagged Correlations between the Monthly Yield Curves - II

IPC1[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]
(0)	0.962	-0.018	0.053	**1.000**	0.000	0.000	0.792	0.143	0.084
(1)	0.956	-0.017	0.006	0.977	0.002	-0.032	0.807	0.117	0.101
(2)	0.945	-0.007	-0.031	0.951	0.010	-0.045	0.817	0.092	0.118
(3)	0.935	0.006	-0.061	0.931	0.017	-0.045	0.820	0.089	0.119
(4)	0.925	0.020	-0.080	0.916	0.023	-0.049	0.821	0.095	0.115
(5)	0.915	0.033	-0.105	0.898	0.027	-0.064	0.822	0.099	0.107
(6)	0.904	0.047	-0.133	0.882	0.027	-0.076	0.822	0.116	0.087
(7)	0.896	0.059	-0.154	0.870	0.024	-0.088	0.823	0.134	0.070
(8)	0.890	0.069	-0.169	0.859	0.022	-0.100	0.824	0.145	0.056
(9)	0.883	0.079	-0.185	0.849	0.025	-0.122	0.824	0.151	0.056
(10)	0.875	0.094	-0.193	0.838	0.028	-0.139	0.824	0.161	0.064
(11)	0.867	0.106	-0.203	0.827	0.033	-0.155	0.822	0.169	0.068
(12)	0.857	0.120	-0.209	0.815	0.038	-0.162	0.818	0.178	0.066

IPC2[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]
(0)	-0.039	0.783	-0.094	0.000	**1.000**	0.000	-0.054	-0.011	0.264
(1)	-0.045	0.781	-0.068	-0.004	0.946	-0.016	-0.057	0.053	0.249
(2)	-0.047	0.778	-0.050	-0.008	0.895	-0.030	-0.053	0.105	0.257
(3)	-0.051	0.772	-0.038	-0.014	0.856	-0.041	-0.049	0.137	0.251
(4)	-0.054	0.761	-0.023	-0.019	0.815	-0.034	-0.047	0.160	0.231
(5)	-0.058	0.745	-0.016	-0.025	0.775	-0.045	-0.047	0.178	0.226
(6)	-0.058	0.717	-0.002	-0.033	0.726	-0.048	-0.038	0.176	0.224
(7)	-0.058	0.690	-0.001	-0.042	0.685	-0.044	-0.028	0.164	0.211
(8)	-0.058	0.665	0.004	-0.048	0.647	-0.061	-0.019	0.160	0.215
(9)	-0.057	0.639	0.028	-0.050	0.603	-0.056	-0.014	0.165	0.214
(10)	-0.053	0.612	0.051	-0.051	0.563	-0.051	-0.004	0.156	0.208
(11)	-0.044	0.584	0.075	-0.046	0.524	-0.056	0.008	0.152	0.201
(12)	-0.028	0.554	0.097	-0.033	0.489	-0.048	0.025	0.140	0.185

IPC3[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]
(0)	-0.020	-0.155	0.390	0.000	0.000	**1.000**	-0.078	-0.402	-0.473
(1)	-0.021	-0.168	0.351	-0.001	-0.005	0.826	-0.076	-0.381	-0.433
(2)	-0.018	-0.173	0.342	0.001	-0.024	0.719	-0.070	-0.358	-0.368
(3)	-0.015	-0.181	0.332	0.000	-0.043	0.653	-0.063	-0.349	-0.310
(4)	-0.010	-0.196	0.335	0.000	-0.067	0.576	-0.055	-0.352	-0.238
(5)	-0.004	-0.211	0.332	0.009	-0.087	0.493	-0.051	-0.340	-0.182
(6)	-0.004	-0.215	0.325	0.007	-0.098	0.434	-0.051	-0.333	-0.108
(7)	-0.001	-0.235	0.290	0.000	-0.111	0.356	-0.038	-0.351	-0.059
(8)	0.003	-0.241	0.244	-0.005	-0.123	0.304	-0.025	-0.356	-0.015
(9)	0.006	-0.241	0.233	-0.008	-0.134	0.260	-0.014	-0.343	0.032
(10)	0.011	-0.234	0.227	-0.008	-0.140	0.235	-0.002	-0.334	0.071
(11)	0.023	-0.235	0.190	-0.009	-0.136	0.174	0.023	-0.355	0.122
(12)	0.025	-0.222	0.146	-0.018	-0.130	0.126	0.041	-0.349	0.171

Table 5.7: Lagged Correlations between the Monthly Yield Curves - III

RPC1[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]
(0)	0.926	0.115	-0.046	0.792	-0.054	-0.078	**1.000**	0.000	0.000
(1)	0.906	0.129	-0.088	0.776	-0.045	-0.098	0.980	0.029	-0.017
(2)	0.886	0.145	-0.121	0.762	-0.041	-0.117	0.958	0.072	-0.036
(3)	0.869	0.160	-0.153	0.746	-0.037	-0.136	0.942	0.104	-0.046
(4)	0.851	0.169	-0.183	0.728	-0.030	-0.155	0.928	0.118	-0.057
(5)	0.834	0.181	-0.203	0.713	-0.023	-0.166	0.912	0.138	-0.067
(6)	0.820	0.189	-0.216	0.702	-0.017	-0.173	0.897	0.152	-0.082
(7)	0.806	0.193	-0.221	0.690	-0.020	-0.178	0.883	0.168	-0.097
(8)	0.795	0.194	-0.226	0.681	-0.024	-0.190	0.873	0.179	-0.102
(9)	0.783	0.196	-0.229	0.668	-0.026	-0.192	0.863	0.183	-0.105
(10)	0.769	0.198	-0.231	0.656	-0.025	-0.185	0.850	0.187	-0.117
(11)	0.756	0.198	-0.232	0.642	-0.024	-0.177	0.839	0.185	-0.132
(12)	0.740	0.197	-0.235	0.626	-0.023	-0.181	0.827	0.182	-0.137

RPC2[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]
(0)	0.043	0.432	-0.093	0.143	-0.011	-0.402	0.000	**1.000**	0.000
(1)	0.072	0.372	-0.032	0.152	-0.030	-0.357	0.043	0.864	0.008
(2)	0.095	0.306	0.013	0.156	-0.045	-0.312	0.078	0.720	0.006
(3)	0.117	0.250	0.059	0.171	-0.071	-0.272	0.097	0.637	0.008
(4)	0.137	0.203	0.103	0.186	-0.101	-0.225	0.116	0.578	-0.001
(5)	0.152	0.152	0.129	0.188	-0.134	-0.183	0.140	0.498	-0.012
(6)	0.162	0.119	0.139	0.189	-0.153	-0.157	0.155	0.446	-0.009
(7)	0.172	0.099	0.148	0.192	-0.151	-0.122	0.169	0.396	-0.010
(8)	0.181	0.064	0.148	0.192	-0.157	-0.075	0.182	0.326	-0.053
(9)	0.190	0.032	0.116	0.197	-0.167	-0.078	0.190	0.289	-0.092
(10)	0.199	0.002	0.088	0.203	-0.183	-0.101	0.197	0.268	-0.104
(11)	0.211	-0.029	0.059	0.211	-0.201	-0.095	0.209	0.242	-0.127
(12)	0.222	-0.054	0.042	0.221	-0.210	-0.077	0.216	0.209	-0.160

RPC3[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]
(0)	0.051	0.156	0.117	0.084	0.264	-0.473	0.000	0.000	**1.000**
(1)	0.041	0.185	0.035	0.079	0.272	-0.401	-0.011	0.061	0.857
(2)	0.028	0.221	-0.032	0.079	0.277	-0.387	-0.033	0.159	0.756
(3)	0.025	0.260	-0.088	0.081	0.296	-0.393	-0.037	0.232	0.697
(4)	0.020	0.287	-0.142	0.077	0.313	-0.378	-0.038	0.271	0.618
(5)	0.020	0.306	-0.171	0.075	0.322	-0.371	-0.033	0.298	0.580
(6)	0.027	0.325	-0.173	0.088	0.323	-0.345	-0.029	0.336	0.530
(7)	0.027	0.337	-0.152	0.101	0.315	-0.320	-0.043	0.385	0.485
(8)	0.030	0.337	-0.129	0.110	0.297	-0.308	-0.045	0.416	0.468
(9)	0.028	0.333	-0.131	0.111	0.287	-0.295	-0.050	0.428	0.446
(10)	0.019	0.329	-0.124	0.108	0.286	-0.253	-0.066	0.439	0.402
(11)	0.008	0.338	-0.116	0.111	0.292	-0.229	-0.090	0.478	0.355
(12)	0.010	0.329	-0.102	0.125	0.283	-0.207	-0.104	0.499	0.321

5.3.3 Fitting AR(1) Models to the Monthly PCs

Once we examine the correlations between the PCs of the yield curves we get an intuition for a possible vector autoregressive model for the series. Initially we start with a vector autoregressive model for each PC but after eliminating the insignificant variables we find that the AR(1) process is the most appropriate model for each PC.

Before introducing the models we describe how we obtain the PCs of the yield curves as time series in formulas.

Let X_M be the matrix of monthly yield curve data for the period 1985-2009 where:
X_{M_N}: Nominal spot rates (300×50)
X_{M_I}: Implied inflation spot rates (300×46)
X_{M_R}: Real spot rates (300×46)

As described in Chapter 3, the first three PCs can be obtained by decomposing the covariance (or correlation) matrix into the eigenvectors and eigenvalues. This decomposition can be shown for the nominal spot rates as below:

$$U_N^t C_N U_N = L_N \tag{5.1}$$

where
C_N: covariance matrix of the nominal spot rates (50×50)
U_N: matrix of eigenvector of C_N (50×3)
L_N: eigenvalues of C_N (3×3) (diagonal matrix)

The eigenvectors extracted using Equation 5.1 are called the *loadings* of the PCs. Using the first three loadings which explain more than 99% of the variability in the data and the nominal yield curve data we obtain the first three PCs for the nominal rates.

$$M_N = X_{M_N} U_N \tag{5.2}$$

where

M_N: principal components of the monthly nominal spot rates (300×3)

Let M be the matrix of the monthly PCs where:
M_{N_L}: level component of the nominal spot rates (300×1)
M_{N_S}: slope component of the nominal spot rates (300×1)
M_{N_C}: curvature component of the nominal spot rates (300×1)
M_{I_L}: level component of the implied inflation spot rates (300×1)
M_{I_S}: slope component of the implied inflation spot rates (300×1)
M_{I_C}: curvature component of the implied inflation spot rates (300×1)
M_{R_L}: level component of the real spot rates (300×1)
M_{R_S}: slope component of the real spot rates (300×1)
M_{R_C}: curvature component of the real spot rates (300×1)

The structure of the 'yield-only' model is as below:

$$M[t] - \mu_M = A \ (M[t-1] - \mu_M) + \epsilon_M[t] \tag{5.3}$$

where:

μ_M is the matrix of long run mean of the variables, A is the coefficient matrix for the first lag of the explanatory variables and $\epsilon_M[t] \sim (0, \Sigma_M)$, i.e. the residuals with zero mean and Σ_M variance-covariance matrix. The autoregressive coefficients in matrix A are very close to 1 which indicates that the models are close to RW models. However, when we examine the standard errors of the parameters presented in Appendix A we see that except for the nominal slope and real level factors, all the coefficients are significantly different from 1, i.e. they are at least two standard errors far from 1.

$$M = \begin{bmatrix} M_{N_L} \\ M_{N_S} \\ M_{N_C} \\ M_{I_L} \\ M_{I_S} \\ M_{I_C} \\ M_{R_L} \\ M_{R_S} \\ M_{R_C} \end{bmatrix} \tag{5.4}$$

$$\widehat{\mu}_M^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.5}$$

$$\widehat{A} = \begin{bmatrix} 0.992 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.88 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.98 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.83 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.993 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.88 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.86 \end{bmatrix} \tag{5.6}$$

$$\widehat{\Sigma}_M = \begin{bmatrix} 3.28 & & & & & & & & \\ -0.41 & 0.59 & & & & & & & \\ 0.38 & -0.04 & 0.14 & & & & & & \\ 1.99 & -0.07 & 0.22 & 2.49 & & & & & \\ 0.42 & -0.33 & 0.08 & 0.46 & 0.44 & & & & \\ -0.09 & -0.07 & -0.04 & -0.04 & 0.02 & 0.12 & & & \\ 1.04 & -0.22 & 0.16 & -0.43 & -0.12 & -0.04 & 1.30 & & \\ -0.09 & 0.16 & -0.03 & 0.41 & 0.24 & 0.03 & -0.44 & 0.53 & \\ 0.05 & -0.01 & 0.02 & -0.01 & 0.01 & 0.04 & 0.05 & 0.00 & 0.06 \end{bmatrix} \tag{5.7}$$

We display the correlation matrix, $\widehat{\rho}_M$, for the residuals below. As explained in the previous section, we assume that the coefficients which are greater or less than three standard errors (0.17) are significant. Therefore, we see several significant correlations between the residuals in the matrix $\widehat{\rho}_M$. These significant correlations may be caused by various reasons. One reason is that we exclude the simultaneous explanatory variables in the modelling work. As we observe in Tables 5.5, 5.6 and 5.7, there are very strong simultaneous correlations particularly between the corresponding PCs of the three yield curves. The high correlations between the residuals for the level and slope factor models may be due to these strong simultaneous correlations between the level and slope components. Another correlation that requires explanation is the one between the residuals of level and curvature models of the nominal rates. Although the PCs themselves are independent within each yield curve, there is a strong negative correlation (0.58) between the residuals. This might be some statistical artifact which does not really indicate a correlation between those two set of residuals.

$$
\widehat{\rho}_M = \begin{bmatrix}
1.00 & & & & & & & & \\
\mathbf{-0.29} & 1.00 & & & & & & & \\
\mathbf{0.58} & -0.14 & 1.00 & & & & & & \\
\mathbf{0.72} & -0.06 & 0.38 & 1.00 & & & & & \\
\mathbf{0.32} & \mathbf{-0.57} & \mathbf{0.29} & \mathbf{0.41} & 1.00 & & & & \\
-0.14 & \mathbf{-0.25} & \mathbf{-0.34} & -0.08 & 0.07 & 1.00 & & & \\
\mathbf{0.49} & \mathbf{-0.23} & \mathbf{0.36} & \mathbf{-0.24} & -0.14 & -0.09 & 1.00 & & \\
-0.08 & \mathbf{0.30} & -0.11 & \mathbf{0.41} & \mathbf{0.50} & 0.14 & \mathbf{-0.58} & 1.00 & \\
0.14 & -0.04 & \mathbf{0.19} & -0.02 & 0.04 & \mathbf{0.47} & \mathbf{0.21} & -0.03 & 1.00
\end{bmatrix} \tag{5.8}
$$

We present each AR(1) model in Appendix A with the standard errors of the parameters and the explained variabilities (R^2_{adj}).

5.3.4 Residual Analysis

Once we fit the AR(1) models we obtain the residuals using the estimated parameters and apply some statistical tests on the residuals. To begin with, we inspect whether the residuals are independent and whether there is an ARCH effect. We calculate the auto-correlation coefficients up to *lag* 36 (i.e. three years) and examine if there is any significant correlations or pattern in the auto-correlation functions. An indication of ARCH is that the residuals will be uncorrelated but the squared residuals will show auto-correlation.

Figure 5.5 shows the auto-correlation plots for the residuals of the nominal principal components. Although some of the correlation coefficients are slightly significant considering both the residuals and the squared residuals, they are not large. Therefore we can conclude that the residuals can be assumed to be independent and there is no ARCH effect in the data, noting that we use data at monthly intervals; there might be short term, e.g. daily, ARCH effect which we cannot observe.

Figure 5.6 shows the auto-correlation plots for the residuals of the implied inflation principal components. Some of the auto-correlation coefficients of the residuals are significant but not large. On the other hand, the auto-correlation coefficients of the squared residuals for the level factor display some high and significant correlations particularly for the first three lags. When we analyse the partial auto-correlation coefficients for this model, we see that for the level factor we could try to fit an ARCH model with order one. This might be a further study.

Figure 5.7 shows the auto-correlation plots for the residuals of the real principal components. The residuals seem independent although there are some significant auto-correlation coefficients as we have for the nominal and implied inflation residuals. The auto-correlation coefficients for the squared residuals of the level and slope components indicate some ARCH effects. The autocorrelation coefficient for the first lag of the slope component is quite high (0.752). The partial auto-correlation function of this component also shows two significant and high correlations. As for the other two components, the partial auto-correlation functions indicate some significant but low correlations which might be ignored.

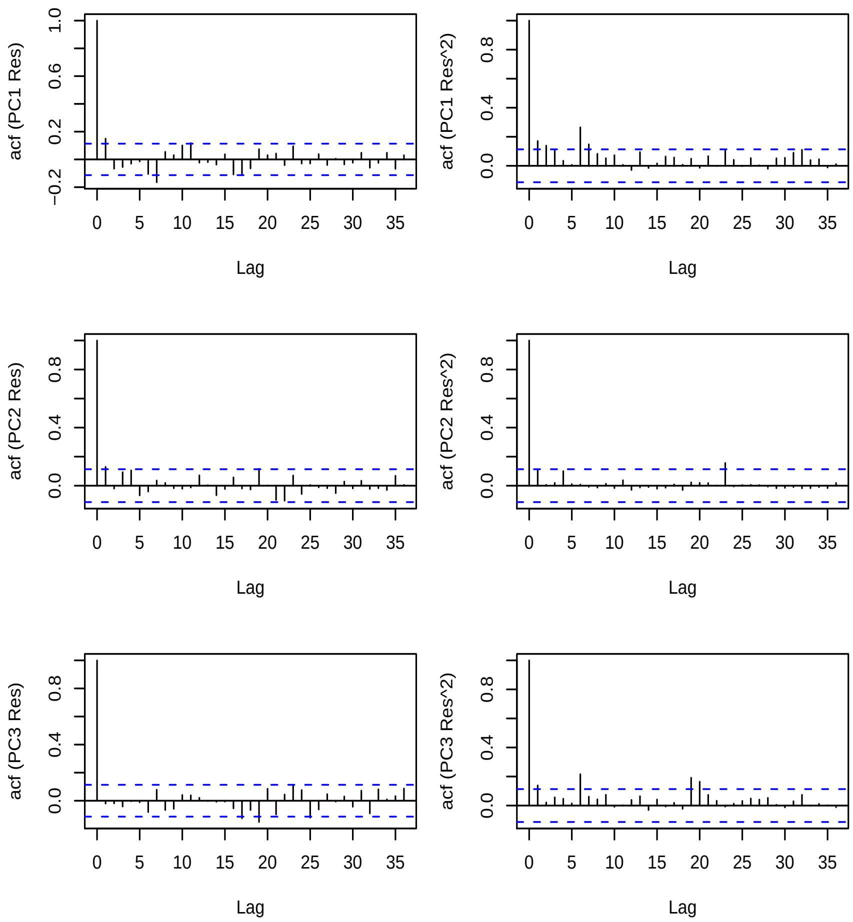

Figure 5.5: Auto-correlation Functions for the Nominal Spot Rates Residuals

Table 5.8 shows the descriptive statistics such as mean, standard deviation, skewness and excess kurtosis for each set of residuals. All the means are either zero or very close to zero while the standard deviations vary. The skewness of the slope factors residuals for the nominal and implied inflation models are relatively high. Except for the nominal level factor residuals all the kurtosis of the residuals are quite high. This might indicate a violation of the normality assumption. Since the kurtosis coefficients are high the normal distribution is not suitable to fit these residuals. The Jarque-Bera test results also show that the residuals except for the nominal level factor model are not distributed normally. According to the statistics presented in Table 5.8, we need a symmetric distribution like a normal distribution with a higher kurtosis for the residuals. We consider two distributions which might be appropriate for the

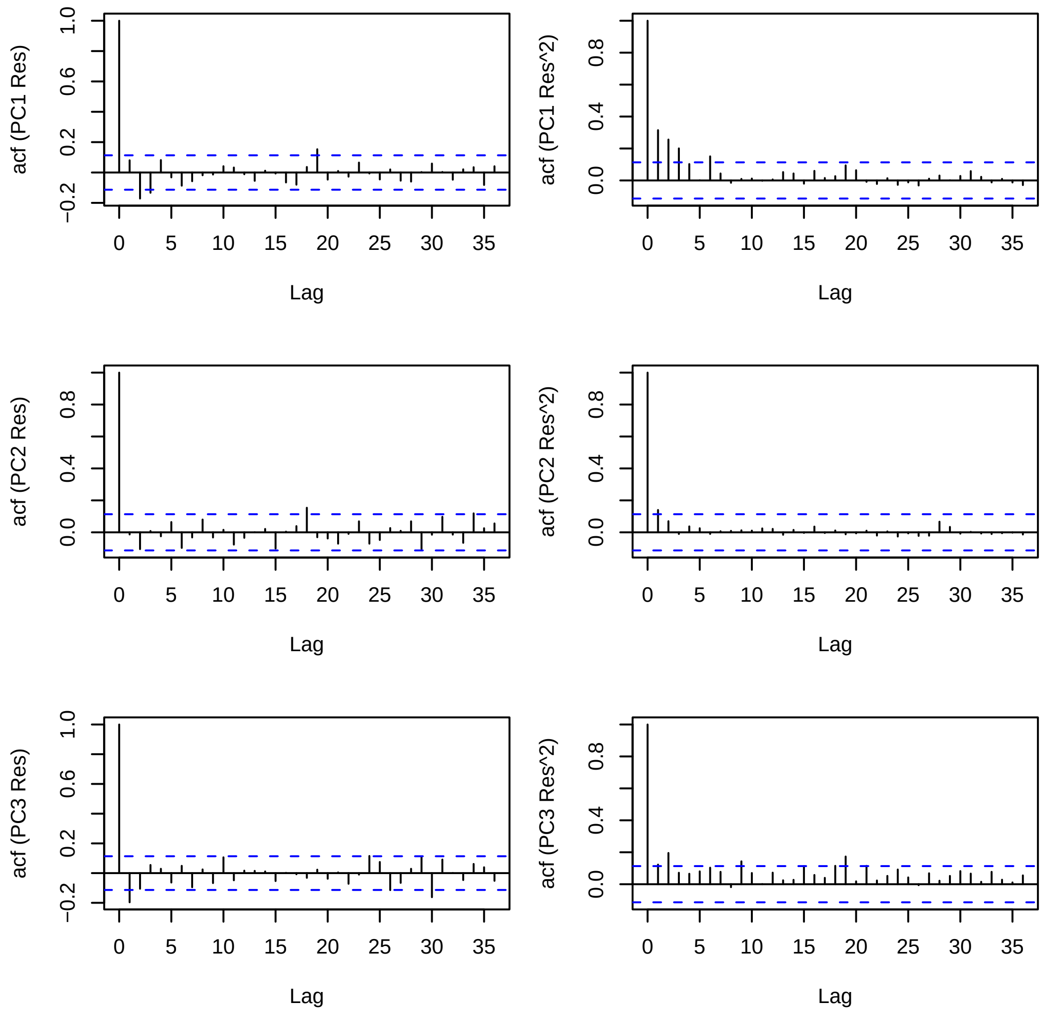

Figure 5.6: Auto-correlation Functions for the Implied Inflation Spot Rates Residuals

monthly residuals. One distribution is the *Student's t distribution* and the other is the *logistic distribution*. The Kolmogorov-Smirnov goodness of fit test[2] indicates that the *logistic distribution* fits each set of residuals with very close location (close to 0) and scale (close to 0.5) parameters at given levels in the Table.

[2]The Kolmogorov-Smirnov test is used to decide if a sample comes from a population with a specific distribution.

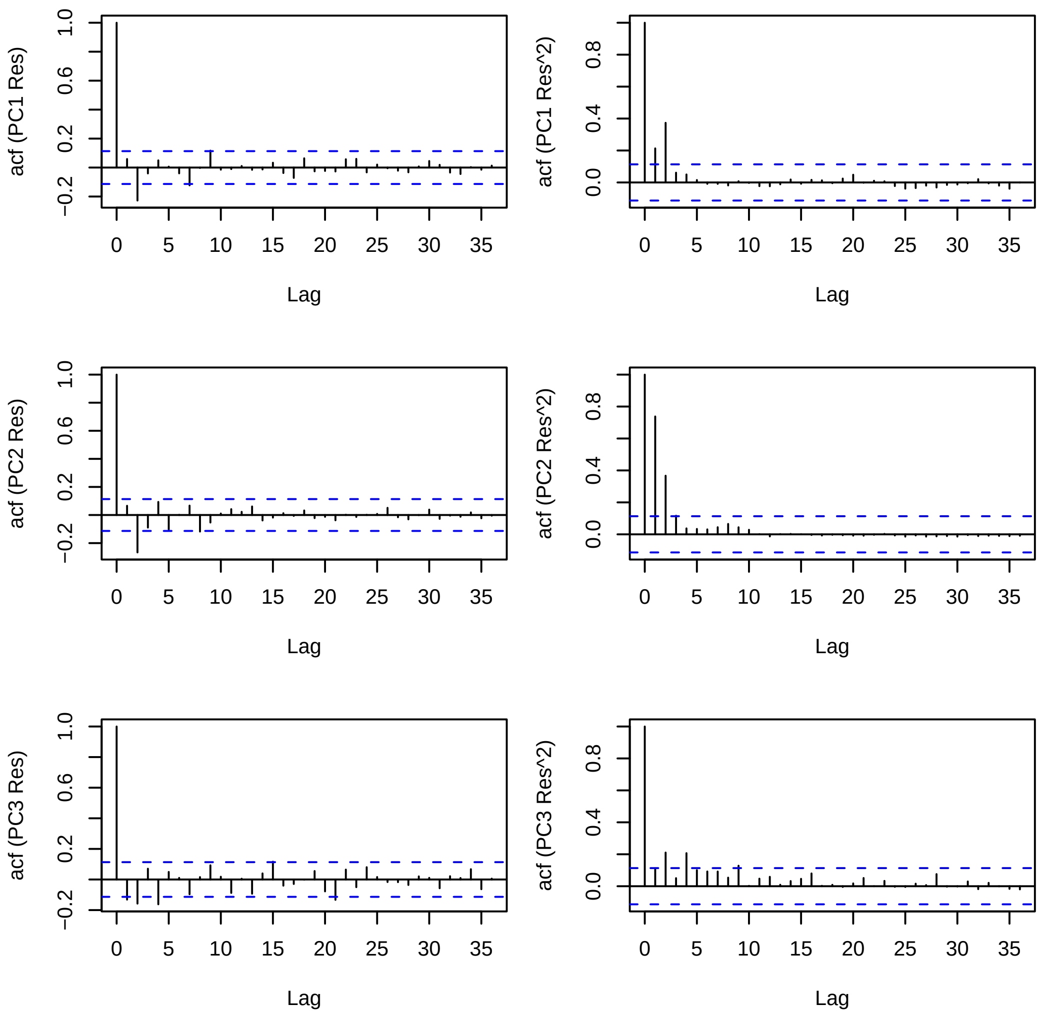

Figure 5.7: Auto-correlation Functions for the Real Spot Rates Residuals

Table 5.8: Residual Analysis of the Yield-Macro Model-I

		Residuals				**Standardised Residuals**
		Mean	Standard Deviation	Skewness	Excess Kurtosis	Logistic Distribution ($\mu = 0, \sigma = 0.5$) KS-test p-value
Nominal	Level	0.0000	1.8048	0.2032	1.7668	0.9763
	Slope	0.0167	0.7742	1.6398	11.5738	0.1663
	Curvature	-0.0039	0.3710	-0.1266	3.6505	0.5868
Implied Inflation	Level	-0.0832	1.5847	-0.6137	4.3705	0.0992
	Slope	0.0025	0.6590	1.3838	11.0622	0.0503
	Curvature	-0.0089	0.3564	0.7801	5.6449	0.2090
Real	Level	0.0000	1.1416	-0.6069	5.1429	0.8143
	Slope	0.0198	0.7325	0.7886	13.9309	**0.0396**
	Curvature	0.0021	0.2395	-0.1957	5.8755	0.0672

5.4 Forecasting

After modelling the PCs of the yield curves, we test these models by forecasting one-month ahead spot rates using the estimated parameters. In order to compare our forecasts with the fitted spot rates we have fitted the models to the data recursively; starting with the first 24 months and ending with 299 months. As we increase the data period, we apply the PCA, re-fit the model and estimate the parameters for that period. Afterwards, we use the parameters for each period to forecast the next month's level, slope and curvature factors of the spot rates. As a final step, we convert the forecasts for PCs into the spot rates, i.e. we obtain the fitted spot rates by using these three PCs. As we discuss in Chapter 3, since we use only the first three PCs to obtain the fitted spot rates there will be some error between the fitted spot rates and the converted spot rates. We obtain the fitted yield curve back as below.

$$\widehat{X}_{M_N} = M_N U_N^t \tag{5.9}$$

where
$\widehat{X}_{M_N}$: forecast for the fitted nominal spot rates $(i \times 3)$
M_N: principal components of the monthly nominal spot rates $(i \times 3)$
U_N: eigenvectors of the covariance of the nominal spot rates (50×3)
$i = 25, 26, ..., 300$

We apply the PCA on the data recursively and use only the available information up to specific time to forecast the next month's rate. It would be interesting to look at n-month ahead forecasts where $n > 1$ since the models are designed for actuarial applications. However, due to data constraints the forecasting period needs to be modest.

We also calculate the variance for forecasts for the nominal spot rates for each maturity of each observation as below:

$$
\begin{aligned}
Var(\widehat{X}_{M_N}) &= Var(M_N U_N^t) \\
&= U_N Var(M_N) U_N^t \\
&= U_N \Sigma_i U_N^t
\end{aligned} \tag{5.10}
$$

where

Σ_i: the variance-covariance matrix of the residuals for the fitted nominal spot rates (3×3)

We calculate the variance-covariance matrix of the residuals for each set of recursive estimates to construct the confidence intervals for the forecasts.

This sort of 'in-sample forecasting' enables us to compare how far our forecasts are from the fitted spot rates. Furthermore, we also calculate the 95% confidence intervals for these forecasts by assuming the residuals have a logistic distribution with the specified parameters discussed in Section 5.4 (we use ∓ 1.83 as the quantiles of the logistic distribution for the 95% confidence intervals). Figure 5.8, Figure 5.9 and Figure 5.10 show one-month ahead forecasts with 95% confidence bands for the nominal, implied inflation and the real spot rates respectively. The one-month ahead forecasts seem quite close to the fitted spot rates for all three yield curves. It is not surprising that the forecasts seem like 'random walk' forecasts since the AR(1) coefficients are very close to 1. The confidence intervals shrink as the data period extends. Due to having more information by fitting the models on to longer data sets the residuals and thus the variance of the residuals get smaller. This leads to smaller confidence interval bands. We can examine the performance of our forecasts by calculating the percentage of the fitted spot rates out of the confidence bands for each maturity and each yield curve. Since we construct the 95% confidence intervals we expect about 5% of the fitted values are out of the bands. Table 5.9 shows the number and ratio of the spot rates which are not within the upper and lower confidence bands for different maturities for the nominal, implied inflation and the real spot rates. The number of the spot rates out of the interval increase as the maturity gets longer for the nominal and real spot rates. This also increases the percentage of the observations out of the bands. The overall averages for the nominal, implied inflation and the real yield curves are 5.1%, 4.1% and 6.8% respectively. Since these percentages are not far from 5% we can conclude that our forecasts are good enough.

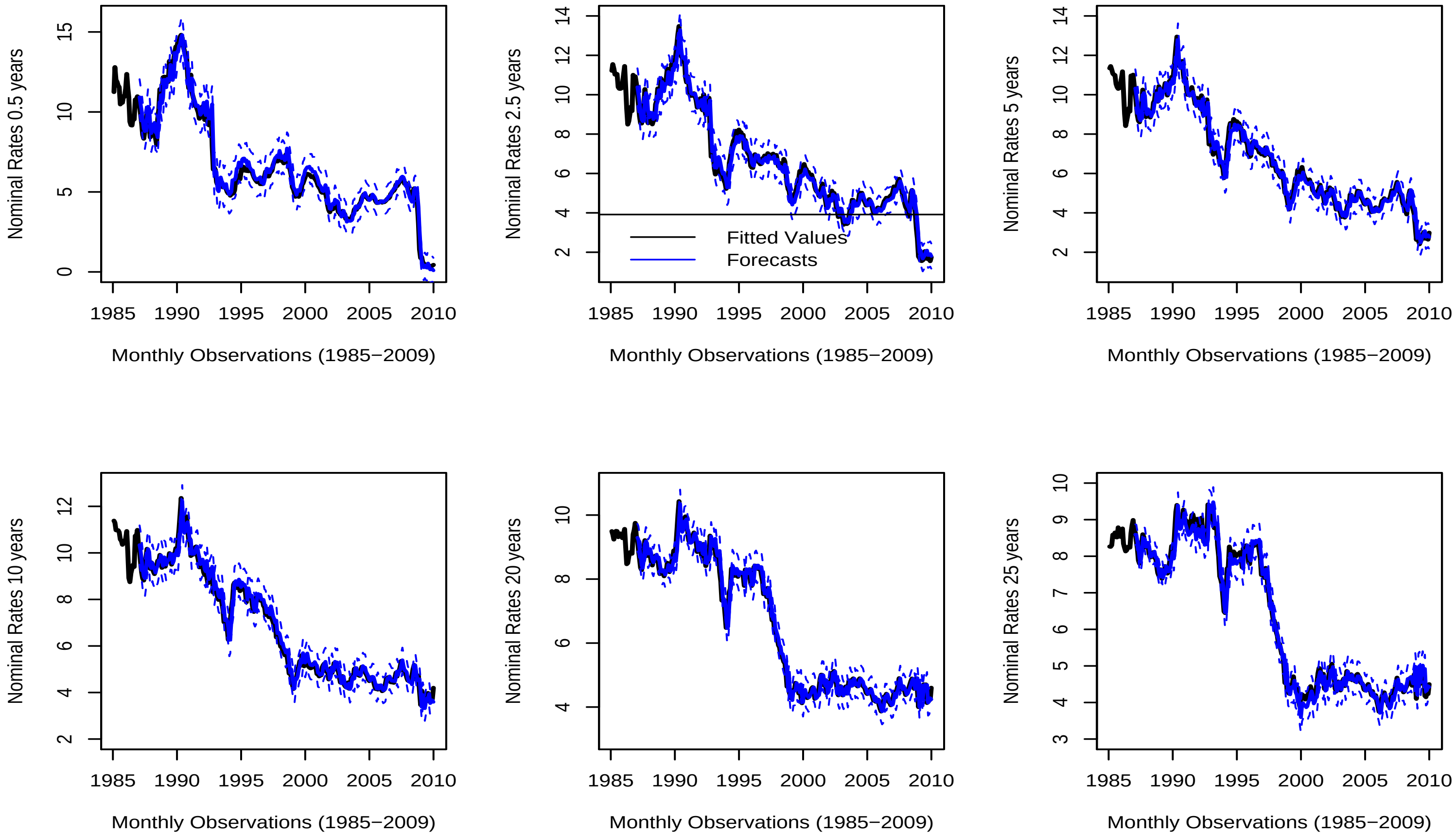

Figure 5.8: 1-Month Ahead Forecasts with Upper and Lower Confidence Limits for Nominal Spot Rates (%)

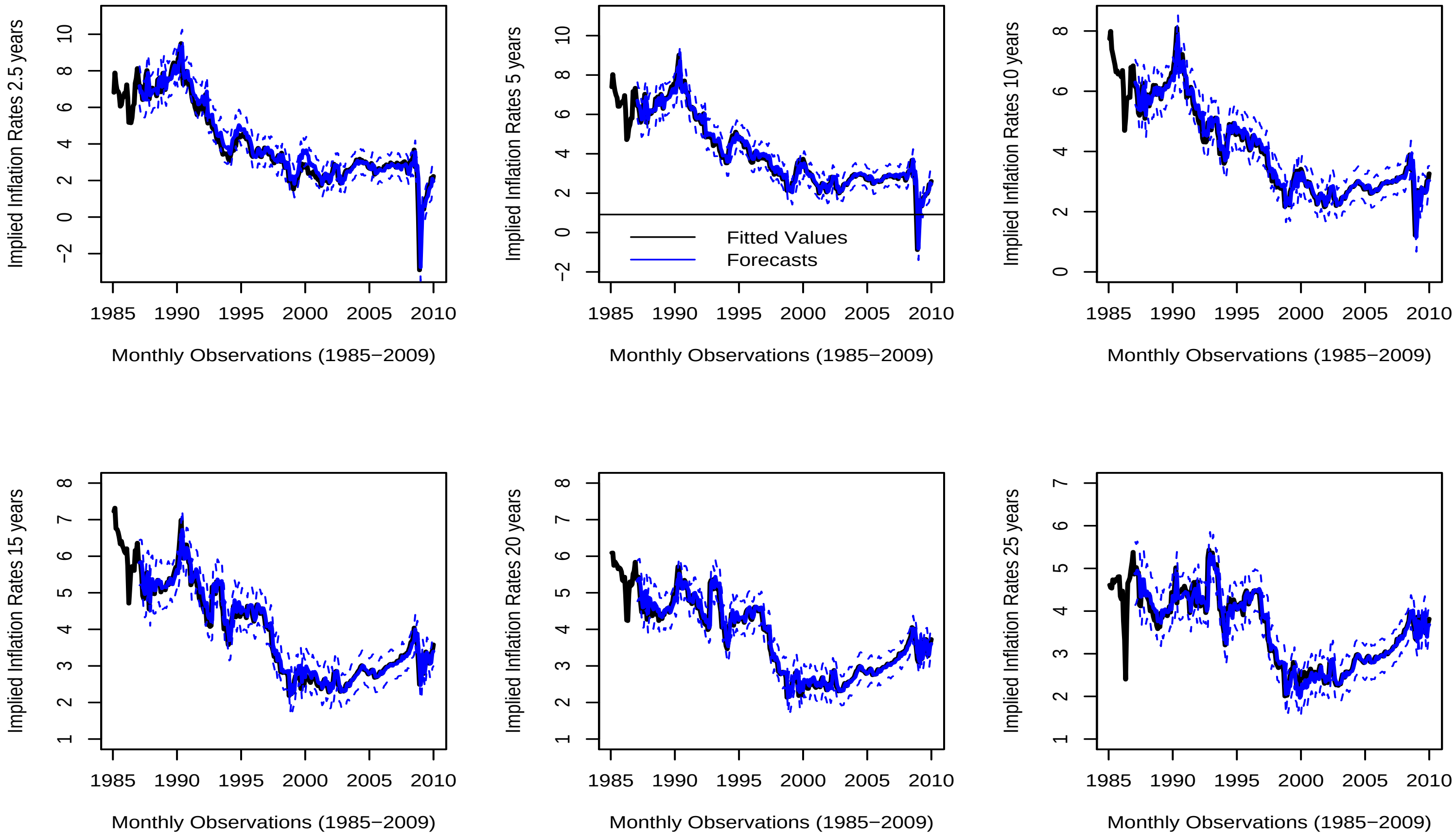

Figure 5.9: 1-Month Ahead Forecasts with Upper and Lower Confidence Limits for Implied Inflation Spot Rates (%)

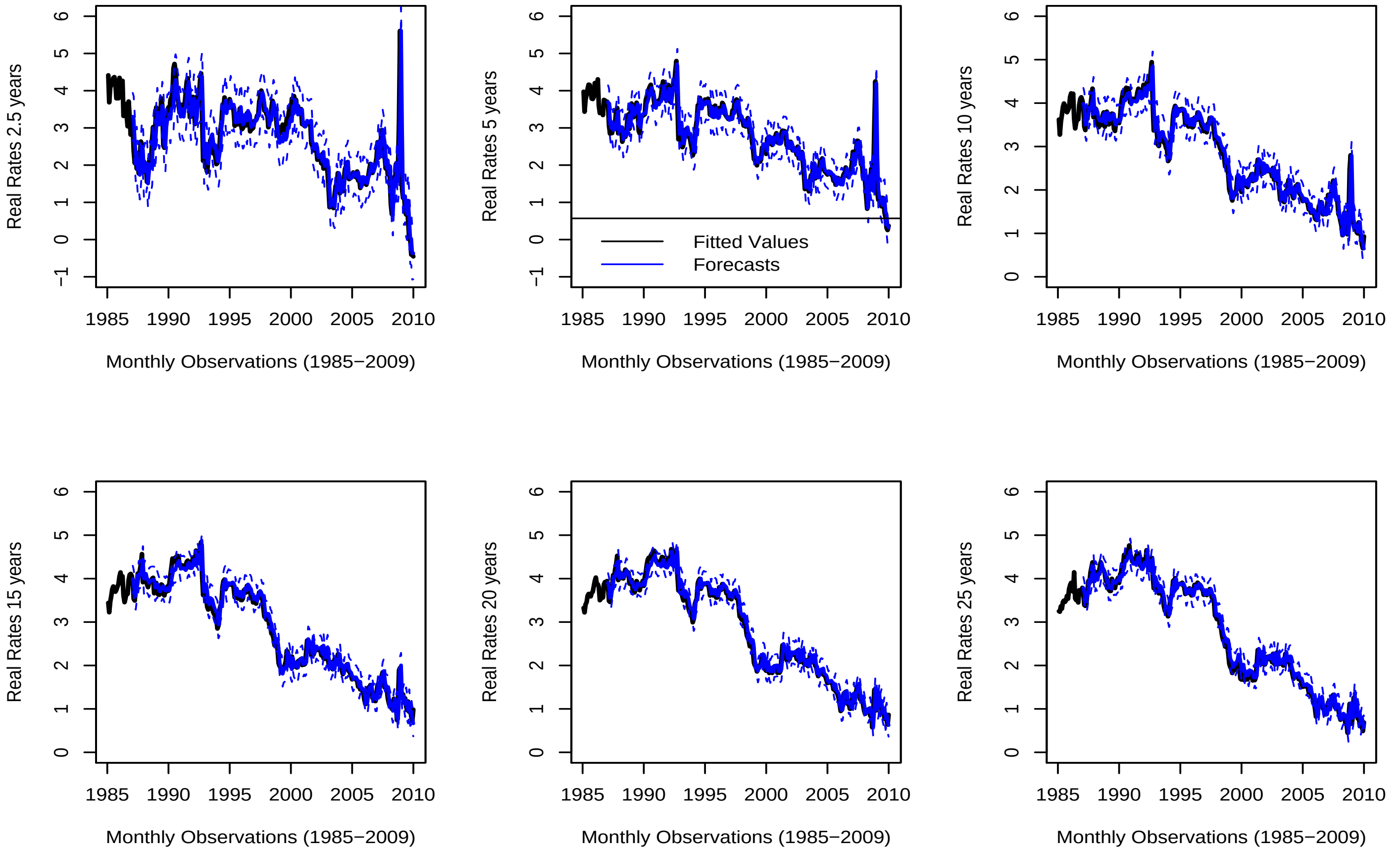

Figure 5.10: 1-Month Ahead Forecasts with Upper and Lower Confidence Limits for Real Spot Rates (%)

Table 5.9: Number and the Ratio of the Observations Outside of the 95% Confidence Bounds for the 1-Month Ahead Forecasts

Maturity	Nominal		Implied Inflation		Real	
	Number	Ratio	Number	Ratio	Number	Ratio
0.5	8	0.029				
1	10	0.036				
1.5	10	0.036				
2	9	0.033				
2.5	9	0.033	16	0.058	16	0.058
3	12	0.044	11	0.040	13	0.047
3.5	13	0.047	10	0.036	13	0.047
4	10	0.036	12	0.044	14	0.051
4.5	10	0.036	12	0.044	13	0.047
5	10	0.036	13	0.047	14	0.051
5.5	9	0.033	13	0.047	14	0.051
6	9	0.033	12	0.044	14	0.051
6.5	9	0.033	12	0.044	14	0.051
7	10	0.036	13	0.047	14	0.051
7.5	10	0.036	11	0.040	14	0.051
8	9	0.033	11	0.040	15	0.055
8.5	10	0.036	11	0.040	15	0.055
9	11	0.040	11	0.040	14	0.051
9.5	11	0.040	12	0.044	15	0.055
10	11	0.040	12	0.044	13	0.047
10.5	11	0.040	11	0.040	12	0.044
11	12	0.044	12	0.044	13	0.047
11.5	14	0.051	11	0.040	13	0.047
12	14	0.051	12	0.044	13	0.047
12.5	13	0.047	13	0.047	15	0.055
13	14	0.051	13	0.047	18	0.065
13.5	14	0.051	14	0.051	20	0.073
14	14	0.051	14	0.051	21	0.076
14.5	14	0.051	15	0.055	21	0.076
15	14	0.051	15	0.055	23	0.084
15.5	14	0.051	14	0.051	23	0.084
16	13	0.047	14	0.051	25	0.091
16.5	14	0.051	12	0.044	24	0.087
17	13	0.047	11	0.040	21	0.076
17.5	13	0.047	11	0.040	20	0.073
18	13	0.047	11	0.040	20	0.073
18.5	13	0.047	9	0.033	21	0.076
19	15	0.055	9	0.033	20	0.073
19.5	16	0.058	9	0.033	20	0.073
20	17	0.062	10	0.036	19	0.069
20.5	19	0.069	10	0.036	20	0.073
21	20	0.073	9	0.033	21	0.076
21.5	20	0.073	8	0.029	25	0.091
22	21	0.076	7	0.025	25	0.091
22.5	23	0.084	8	0.029	25	0.091
23	24	0.087	7	0.025	27	0.098
23.5	25	0.091	9	0.033	28	0.102
24	26	0.095	10	0.036	28	0.102
24.5	26	0.095	10	0.036	27	0.098
25	26	0.095	10	0.036	27	0.098
Average		**0.051**		**0.041**		**0.068**

5.5 Fisher Relation Check

As mentioned throughout the previous chapters nominal interest rates embody the real interest rates plus a compensation for the erosion of the purchasing power of this investment by inflation. The Bank of England uses this decomposition, which is also known as the Fisher relation and nominal and real yield curves to calculate the implied inflation rate factored into nominal interest rates. Since we model these three yield curves separately, we can check whether our one-month ahead forecasts satisfy the Fisher relation. This enables us to test both the consistency of the forecasts with the economic theory used in extracting the implied inflation yield curve and to eliminate one of the yield curves and derive it by only modelling the other two yield curves. To decide which one to eliminate we check for which yield curve the Fisher relation holds better. Figure 5.11, Figure 5.13 and Figure 5.15 show the fitted spot rates (black solid lines), forecasts (red solid lines) and the forecasts obtained using Fisher relation (blue solid lines) for different maturities for the nominal, implied inflation and the real yield curves separately.

We see that the fitted values and the forecasts derived by using the Fisher relation show significant differences in particular for very short and very long maturities for the three yield curves. However, the nominal yield curve forecasts seem better than the other two considering the two ends of the term structures. Since there is a significant decrease in the spot rates over the period examined (1985-2009) we have to draw the graphs on a large scale in order to display the whole period. Therefore, the overlapping solid lines in Figure 5.11, Figure 5.13 and Figure 5.15 do not tell much. Taking this drawback into account, we calculate and present the errors between the fitted yield curves and the one-month ahead forecasts and the fitted yield curves and the forecasts derived by the Fisher relation for the three term structures. Figure 5.12, Figure 5.14 and Figure 5.16 show these errors. According to Figure 5.12, the differences between the fitted nominal spot rates and forecasts (both obtained by modelling the nominal PCs and the ones derived from the Fisher relation) decrease as the maturity increases. This might be explained by the higher volatility in the short rates due to being used as a monetary policy instrument. Since the changes in the economy are reflected into the short term interest rates first the short rates are more volatile than the long rates. This feature of the short rates make it relatively difficult to obtain a good fit in terms of modelling. Regardless of maturity, the error graphs indicate that the forecasts obtained by modelling the nominal rates produce closer values than the forecasts obtained by modelling the implied inflation and real rates to derive the nominal spot rates. Figure 5.14 shows the errors for different maturities for the implied inflation spot rates. Similar to the nominal rates, the errors get smaller as the maturity increases. Different from the other maturities, the forecasts obtained from the Fisher relation (the difference between the nominal and real spot rate forecasts) are closer to the fitted implied inflation rates than the forecasts obtained from modelling the implied inflation rates themselves for the very short maturity. The model forecasts are better than the Fisher relation forecasts for the other maturities. Finally, Figure 5.16 shows that the forecasts obtained from modelling the PCs of the real rates produce a better fit than the Fisher relation even for the very short rates. Thus, we can conclude that the implied inflation model does not fit the short end very well. The Fisher relation can be useful to derive maybe not all but some part of the term structures.

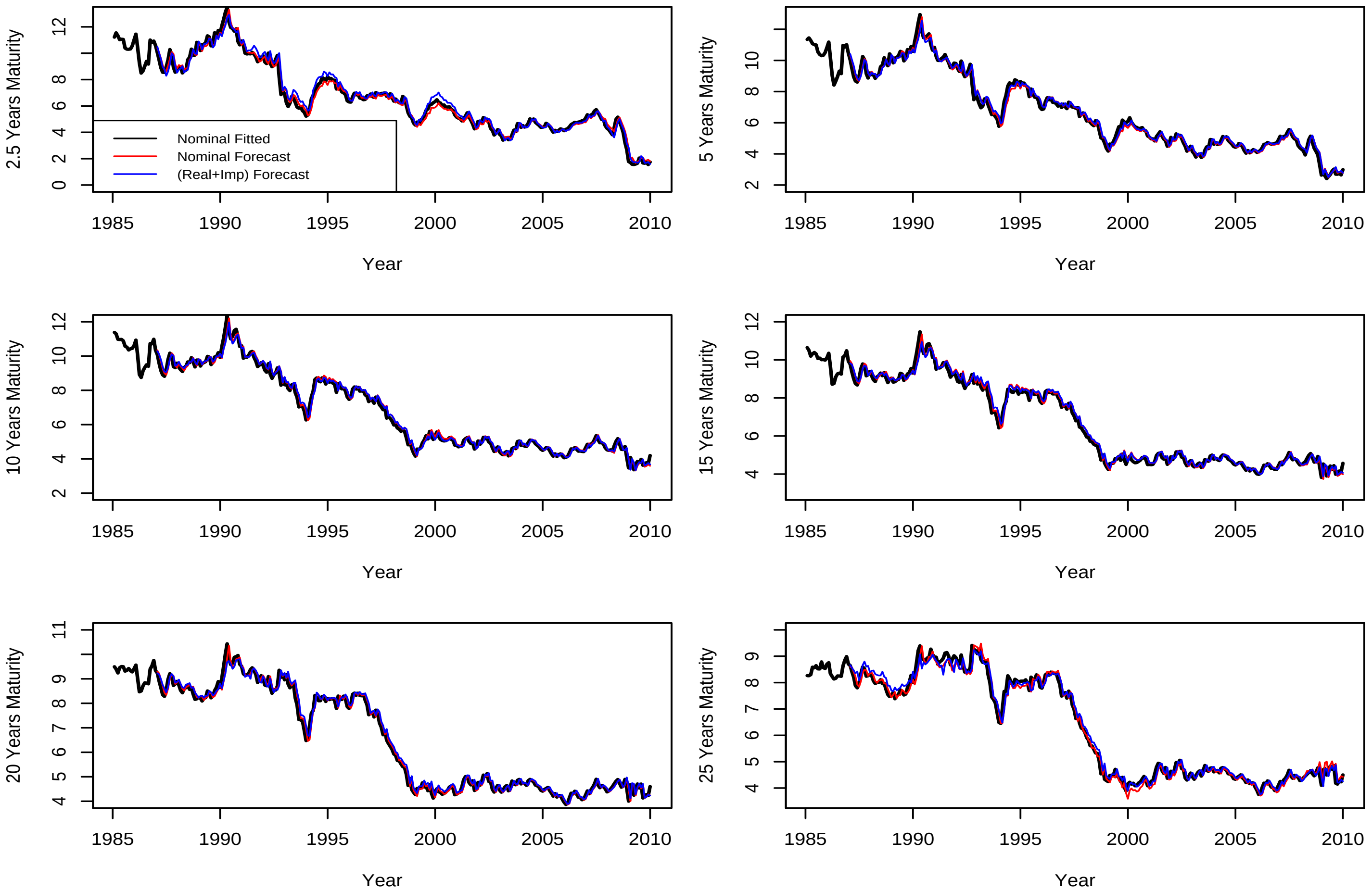

Figure 5.11: Fisher Relation Check for the 1-Month Ahead Nominal Spot Rate Forecasts (%)

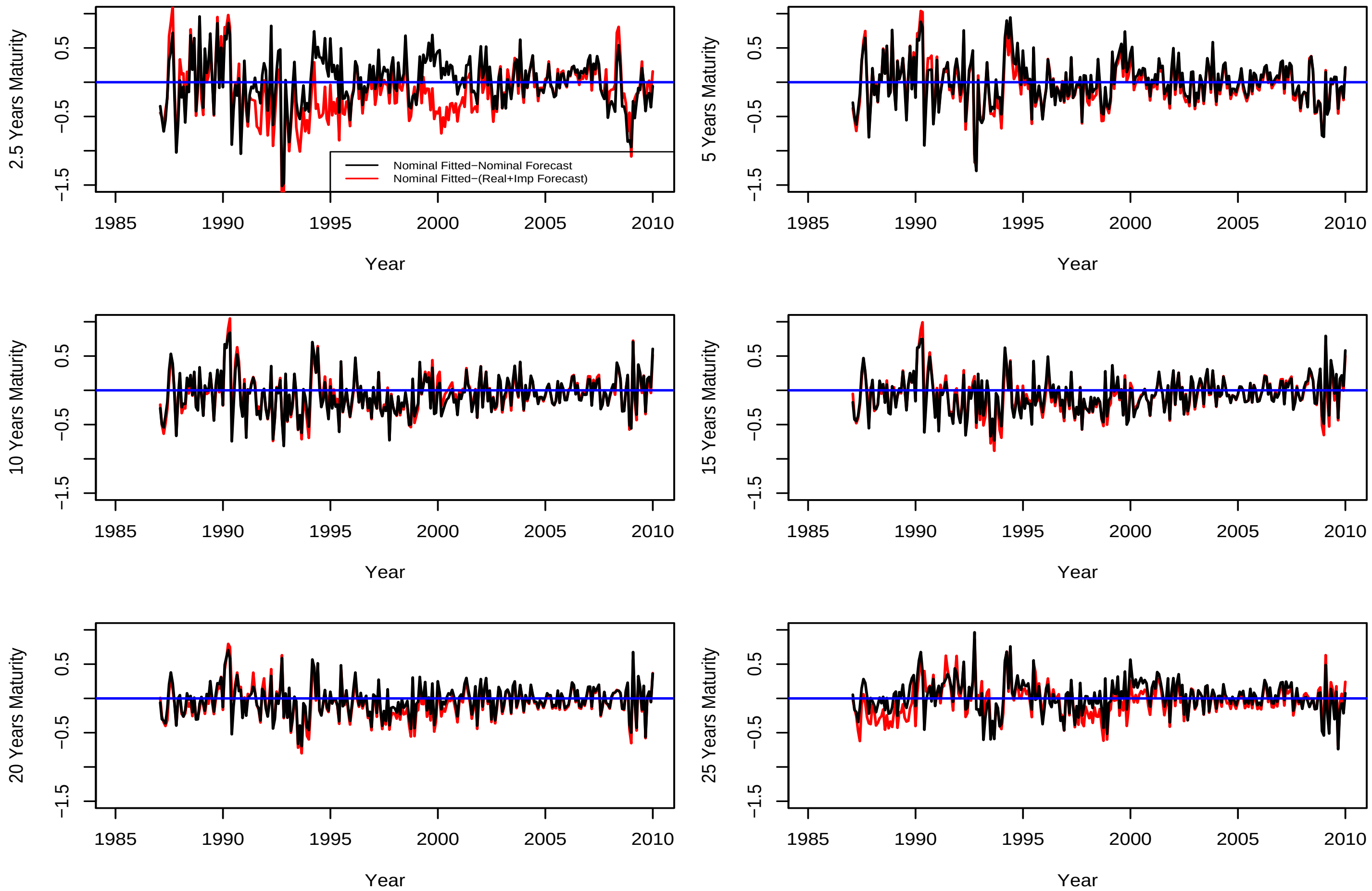

Figure 5.12: Errors for the Fisher Relation Check for the 1-Month Ahead Nominal Spot Rate Forecasts (%)

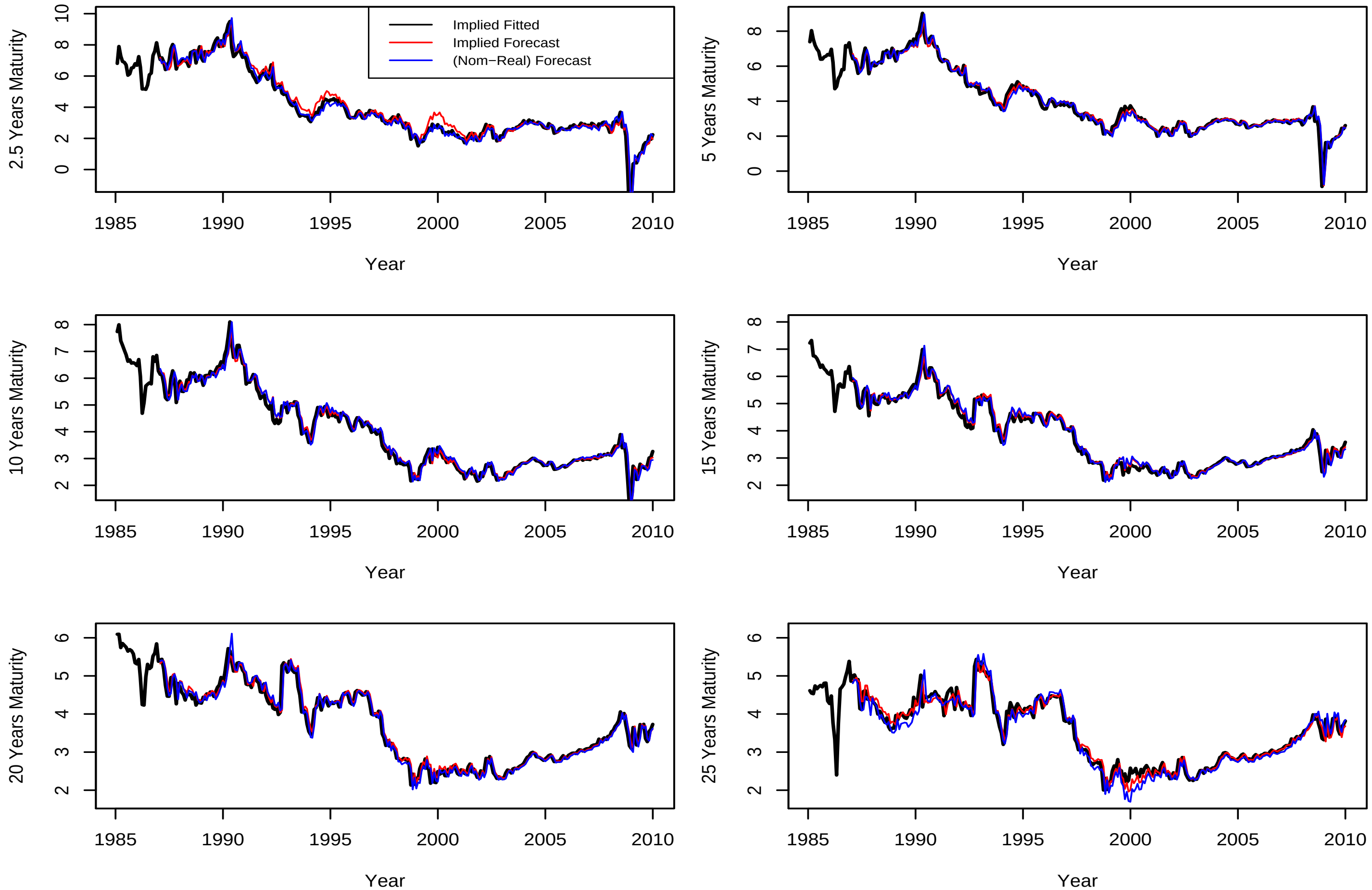

Figure 5.13: Fisher Relation Check for the 1-Month Ahead Implied Inflation Spot Rate Forecasts (%)

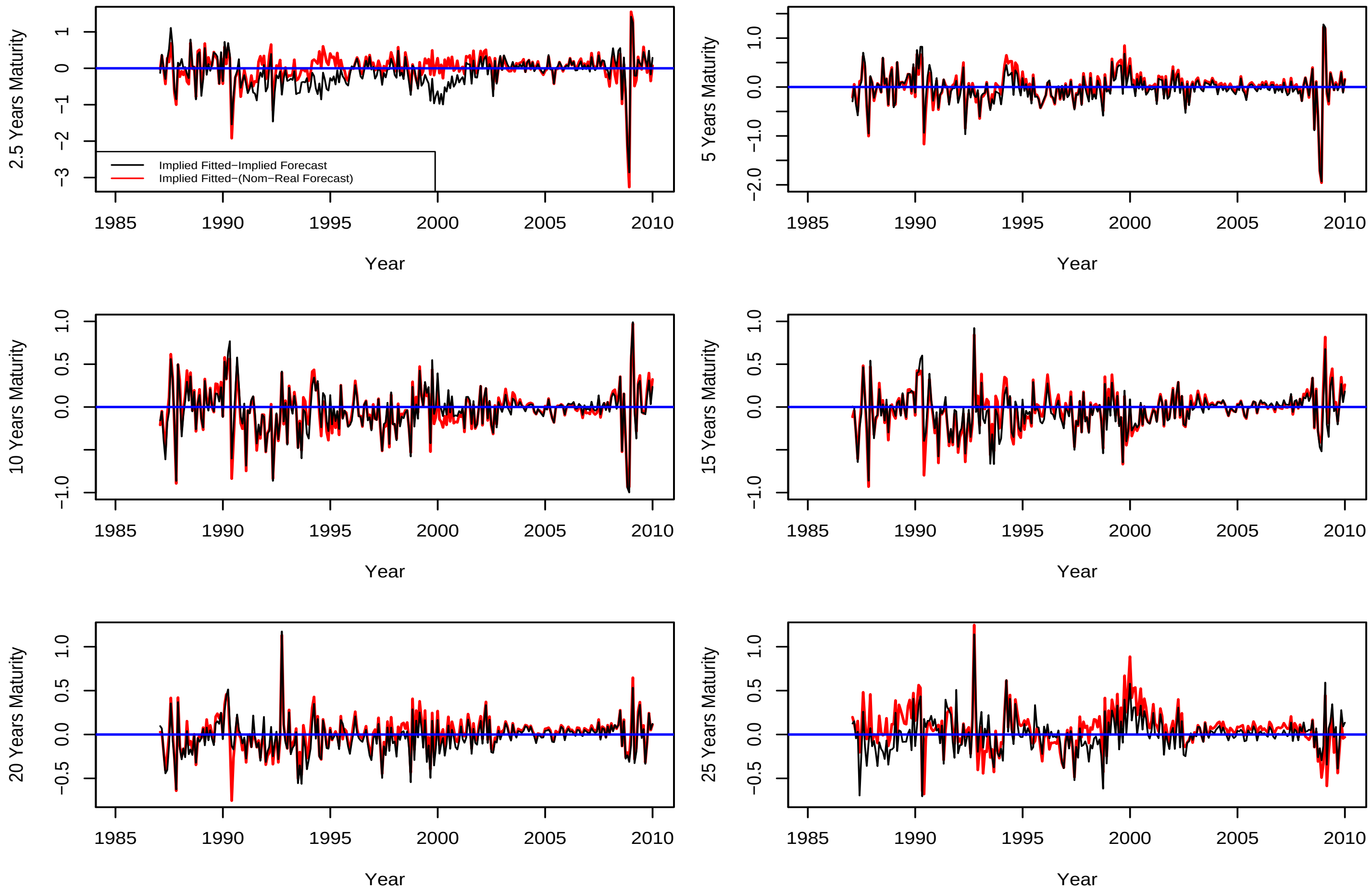

Figure 5.14: Errors for the Fisher Relation Check for the 1-Month Ahead Implied Inflation Spot Rate Forecasts (%)

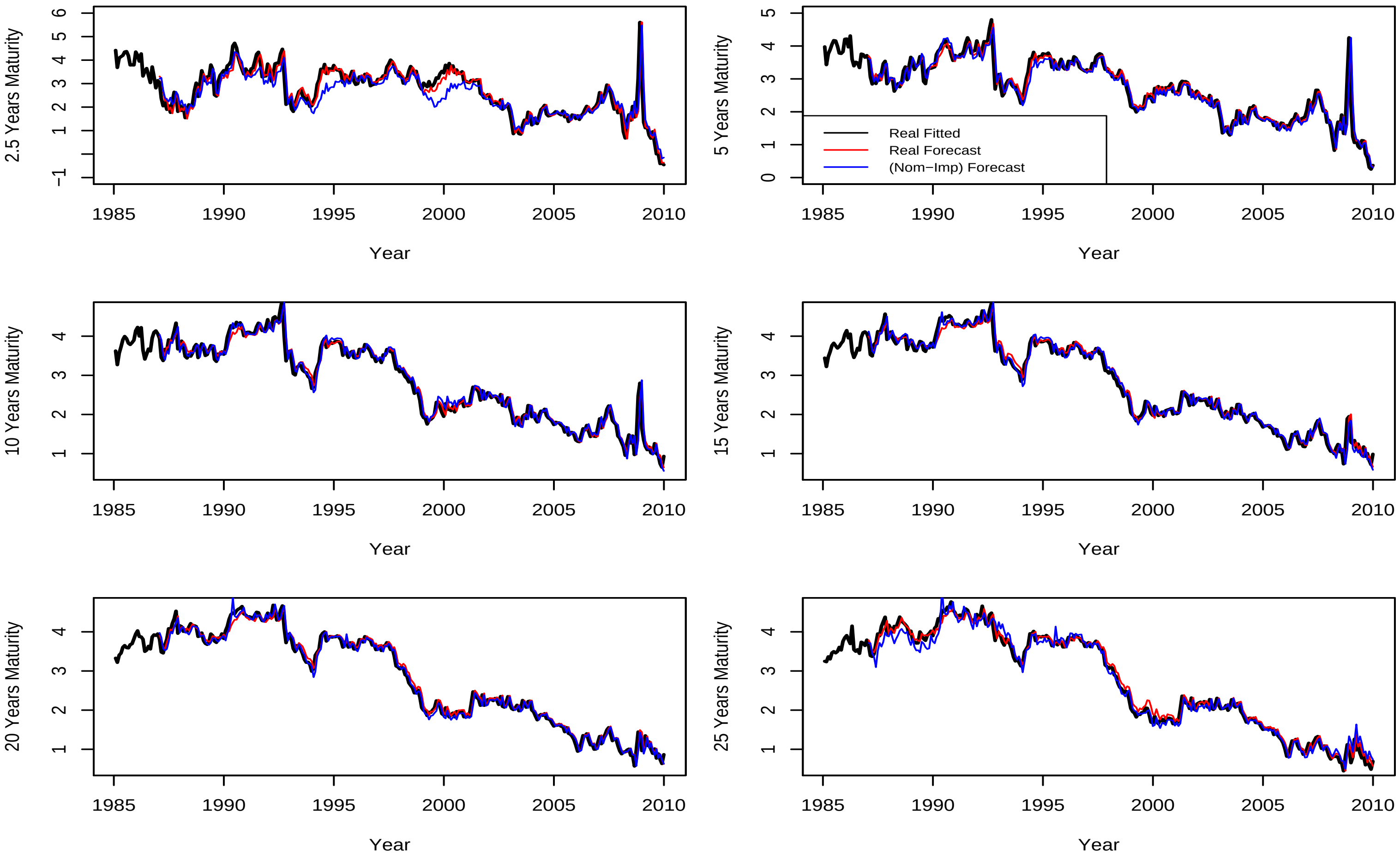

Figure 5.15: Fisher Relation Check for the 1-Month Ahead Real Spot Rate Forecasts (%)

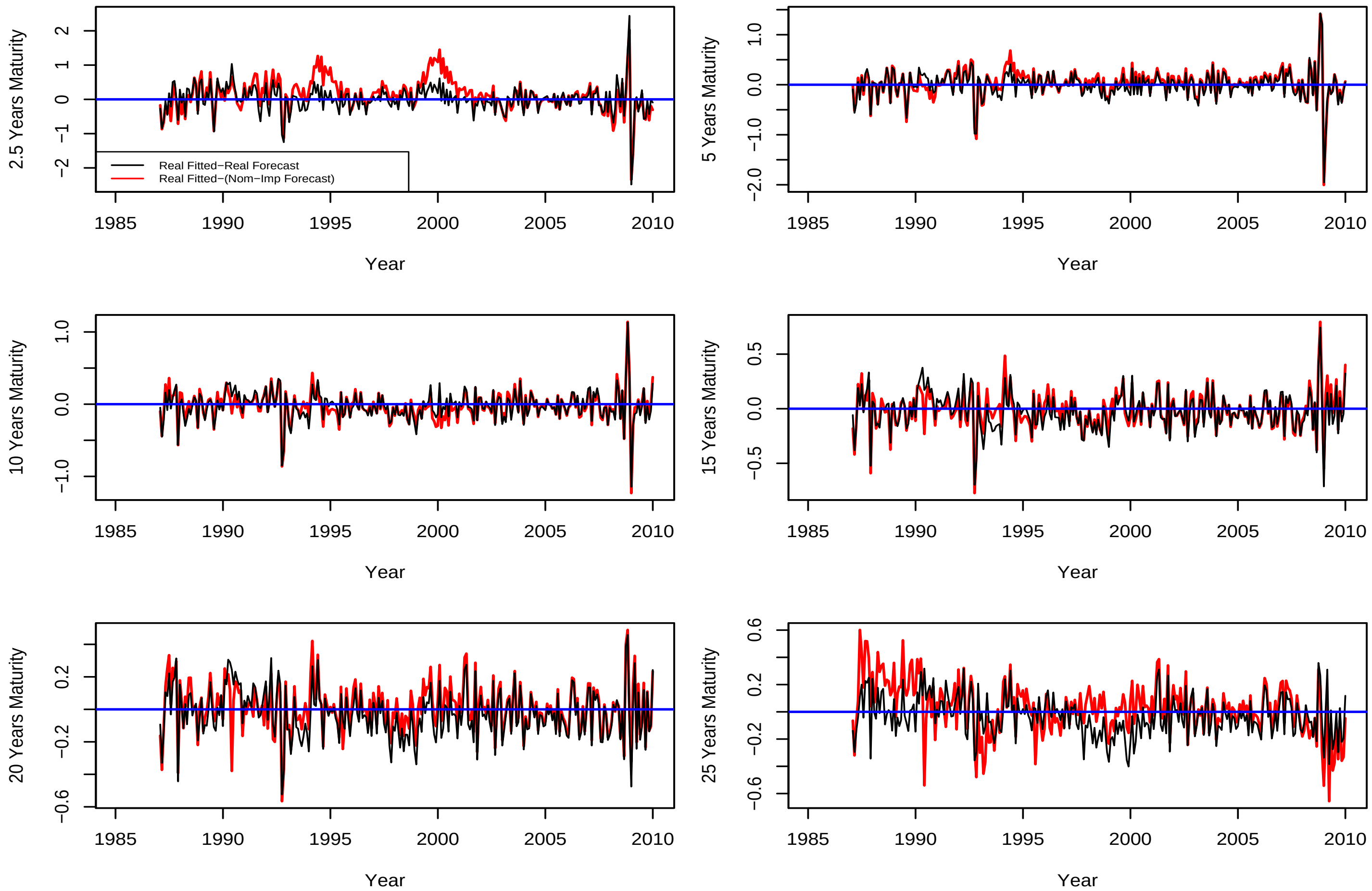

Figure 5.16: Errors for the Fisher Relation Check for the 1-Month Ahead Real Spot Rate Forecasts (%)

5.6 Interim Conclusion: The Yield-Only Model

In this chapter we have presented the ‘yield-only’ model which we construct by using the monthly UK nominal, implied inflation and real spot rates. First we apply the PCA on the three term structures and obtain the three most important components to derive the yield curves. Then we examine the relation within and between these components by analysing the auto- and cross-correlation functions. Once we try to fit vector autoregressive models to each component we see that the AR(1) model fits each variable quite well. Although the auto-correlation coefficients in the models are very high and close to 1 we find it economically reasonable to fit AR processes rather than some random walk models to the interest rates. To test our models we examine the residuals which we obtain by using the estimated parameters for each PC. The zero mean and high kurtosis of the residuals show that a distribution which is symmetric like the normal distribution but has a higher kurtosis, such as a logistic distribution, fits the residuals well. We have also found some evidence of an ARCH effect particularly in the level and slope factors of the implied inflation and the real spot rates. As a next step to test our models we have calculated the one-month ahead forecasts with the 95% confidence limits. Our analysis shows that the fitted spot rates are well within the confidence limits for all three yield curves which indicate a good forecast. As a final analysis, we check whether our forecasts satisfy the Fisher relation which might enable us to derive one of the yield curves by using the other two. We have discovered that not for all maturities but for specific ones, such as short term implied inflation, the Fisher relation can be used to forecast the spot rates.

Chapter 6

Modelling the UK Term Structures: The Yield-Macro Models

6.1 Introduction

In this chapter we present two 'yield-macro' models using the three yield curves (nominal, implied inflation and real spot rates) and two macroeconomic variables (annual realised inflation and output gap) at different frequencies. The first part of the chapter discusses the quarterly yield macro model by introducing the PCs obtained from the yield curve data. We examine the correlations between the variables and fit a VAR model. Once we estimate the parameters, we obtain the residuals to analyse their distributions. Furthermore, we compare the VAR model with the random walk and AR(1) process, calculate the one-quarter ahead forecasts and check whether the Fisher relation holds for the forecasts. Besides, we use output gap first estimate and annual GDP growth data instead of output gap latest estimate to see whether there is a significant change in the models. As for the yearly data, we can only use level factors of the yield curves and realised inflation as a macroeconomic variable due to having a very short period of data (i.e. 25 years). We examine the yearly model using the same methodology as we use for the quarterly model.

6.2 Yield-Macro Model-I

6.2.1 Data

To construct the yield-macro model-I, we use quarterly UK nominal government spot rates, real spot rates and implied inflation spot rates published on the Bank of England's web page. As for the macroeconomic variables we use annual realised inflation obtained from Retail Price Index and output gap provided by the OECD Economic Outlook publications. Due to the revision process, the latest available estimate for output gap is the end of 2007. Therefore we use the quarterly data for the period 1995-2007 for the yield-macro model-I.

In Table 6.1, we present the summary statistics for the nominal and real interest rates and implied inflation rates at representative maturities (in years). The means of the yield curves for different maturities are quite close to each other. Considering the standard deviations, although they do not change significantly, there is an increase in the volatility as the maturities get longer. One possible reason is that the instruments from which long term interest rates are obtained are not available for some periods. This causes a gap and the values before and after this gap differ significantly. This leads to an increase in the volatility. On the other hand, the autocorrelation functions indicate significant correlations for the first and fourth lags of the three yield curves.

6.2.2 PCA for the Yield-Macro Model-I

We apply PCA on quarterly values of nominal interest rates, real interest rates and implied inflation rates to obtain the three most important components of these yield curves.

Tables 6.2, 6.3 and 6.4 show the results of the principal component analysis based on the mean adjusted fitted yield curves. It is seen that the first five principal components explain all the variability in the data. The first factor, *level*, accounts for 95%, 94% and 94% of the variance for the nominal, implied inflation and real spot rates respectively. Slope factors account for 5%, and curvatures account for less than 1% for all yield curves. Thus, the first three principal components explain about 99.9% of the

Table 6.1: Descriptive Statistics for the Fitted Quarterly Yield Curves

Nominal Spot Rates (%)										
Maturity	Mean	Std. Dev.	Med	Min	Max	Skewness	Kurtosis	$\rho(1)$	$\rho(4)$	$\rho(12)$
0.5	5.20	1.05	5.21	3.31	7.34	0.11	-0.93	0.92	0.44	0.08
2.5	5.37	1.13	5.10	3.48	7.94	0.36	-0.96	0.85	0.50	0.14
5	5.44	1.19	5.05	3.83	8.38	0.78	-0.47	0.87	0.52	0.09
10	5.45	1.28	4.96	4.06	8.42	1.18	-0.04	0.90	0.56	0.01
15	5.38	1.36	4.77	4.01	8.41	1.28	0.02	0.92	0.58	-0.04
20	5.30	1.40	4.64	3.96	8.42	1.29	0.00	0.93	0.58	-0.07
25	5.22	1.43	4.55	3.92	8.36	1.28	-0.02	0.93	0.59	-0.09
Implied Inflation Spot Rates (%)										
Maturity	Mean	Std. Dev.	Med	Min	Max	Skewness	Kurtosis	$\rho(1)$	$\rho(4)$	$\rho(12)$
2.5	2.81	0.59	2.86	1.71	4.38	0.35	0.04	0.74	0.35	-0.12
5	2.98	0.62	2.87	1.99	4.60	0.82	0.23	0.78	0.33	0.03
10	3.06	0.66	2.89	2.16	4.69	1.05	0.21	0.87	0.50	0.01
15	3.07	0.68	2.81	2.18	4.68	1.19	0.24	0.90	0.61	-0.07
20	3.04	0.68	2.85	2.13	4.62	1.11	0.05	0.91	0.63	-0.11
25	3.00	0.64	2.84	2.00	4.48	1.03	0.09	0.89	0.59	-0.13
Real Spot Rates (%)										
Maturity	Mean	Std. Dev.	Med	Min	Max	Skewness	Kurtosis	$\rho(1)$	$\rho(6)$	$\rho(12)$
2.5	2.56	0.85	2.77	0.84	3.86	-0.24	-1.30	0.90	0.72	0.21
5	2.47	0.72	2.37	1.29	3.79	0.24	-1.26	0.89	0.66	0.19
10	2.38	0.74	2.19	1.35	3.86	0.62	-0.97	0.90	0.69	0.08
15	2.32	0.80	2.05	1.14	3.86	0.62	-0.86	0.91	0.72	0.07
20	2.26	0.86	2.06	0.96	3.86	0.60	-0.85	0.92	0.73	0.06
25	2.19	0.92	2.03	0.83	3.87	0.61	-0.88	0.93	0.73	0.03

variability in the term structures.

6.2.3 Loadings for the Yield-Macro Model-I

Figure 6.1 shows the loadings of the first three principal components for the quarterly yield curves. Except for the short end of the loadings of the slope factor and the long end of the curvature factor of the implied inflation, the loadings seem similar to each other. Changing the frequency of the data has not changed the structure of the

Table 6.2: Importance of the PCs for the Fitted Quarterly Nominal Spot Rates

	PC1	PC2	PC3	PC4	PC5
Standard deviation	8.976	2.0124	0.70487	0.18575	0.08495
Proportion of variance	0.946	0.0476	0.00584	0.00041	0.00008
Cumulative proportion	0.946	0.9937	0.99951	0.99992	1.00000

Table 6.3: Importance of the PCs for the Fitted Quarterly Implied Inflation Spot Rates

	PC1	PC2	PC3	PC4	PC5
Standard deviation	4.309	1.0255	0.40812	0.15057	0.05238
Proportion of variance	0.937	0.0531	0.00841	0.00114	0.00014
Cumulative proportion	0.937	0.9903	0.99872	0.99986	1.00000

Table 6.4: Importance of the PCs for the Fitted Quarterly Real Spot Rates

	PC1	PC2	PC3	PC4	PC5
Standard deviation	5.283	1.2171	0.43586	0.10470	0.02768
Proportion of variance	0.943	0.0501	0.00642	0.00037	0.00003
Cumulative proportion	0.943	0.9932	0.99960	0.99997	1.00000

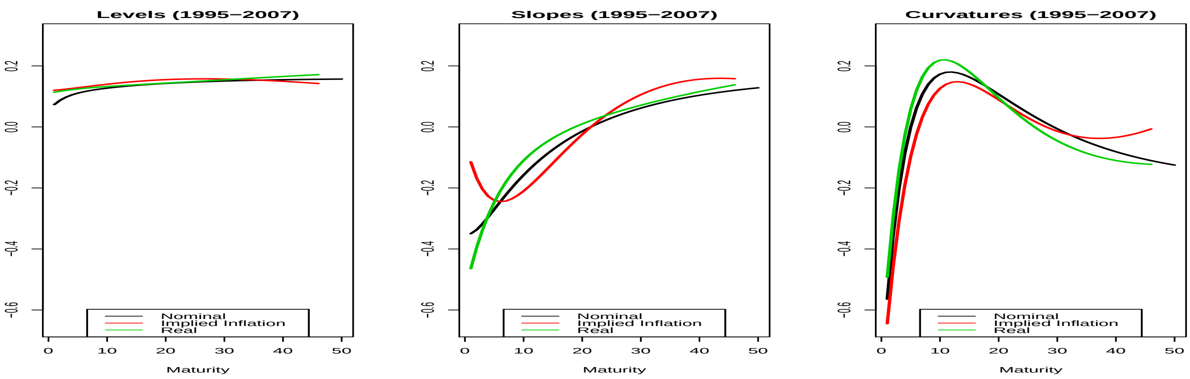

Figure 6.1: Loadings of the PCs for the Fitted Quarterly Yield Curves

loadings of the PCs significantly. Furthermore, they are still close to each other which might indicate the existence of the common PCs as we have mentioned in Chapter 5.

Figure 6.2, Figure 6.3 and Figure 6.4 present the time series graphs of the first three PCs of the quarterly yield curves for the nominal, implied inflation and real spot rates on the same scale. Similar to the monthly PCs, the graphs indicate that the levels of the nominal, implied inflation and real spot rates have decreased since 1995. However, the implied inflation level factor is relatively stable after 1998. As mentioned

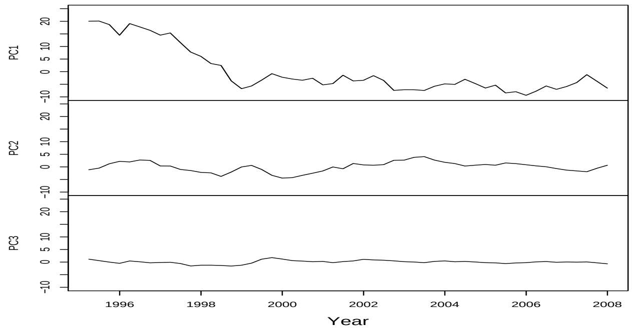

Figure 6.2: PCs of the Fitted Quarterly Nominal Spot Rates

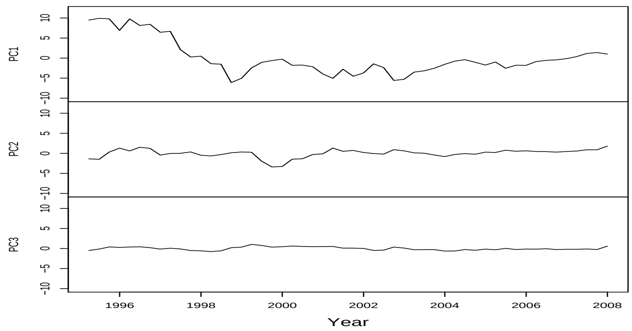

Figure 6.3: PCs of the Fitted Quarterly Implied Inflation Spot Rates

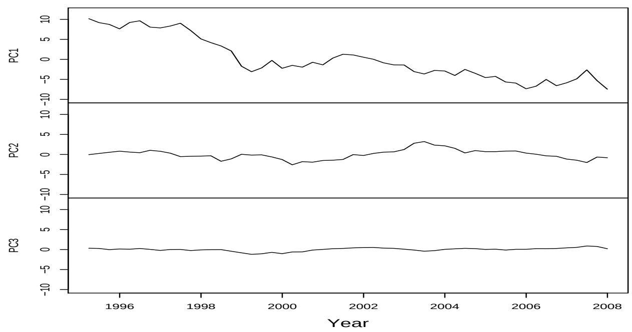

Figure 6.4: PCs of the Fitted Quarterly Real Spot Rates

in Chapter 5, the inflation targeting policy along with the independence of the Bank of England in 1997 might be the reasons for the stable implied inflation after this year.

6.2.4 Correlations Between the Quarterly Yield Factors

Table 6.5, Table 6.6 and Table 6.7 show the lagged correlations between the PCs of the three yield curves and the macroeconomic variables, annual realised inflation, output gap and annual GDP growth. The lag k value in the tables is the correlation between $x[t]$ and $y[t-k]$ where $x[t]$ is the variable whose autocorrelation function has been displayed by red colour and $y[t-k]$ represents all the other variables. We assume that all the variables are stationary. Although we try to explain what the correlations between the variables mean economically, it is important to emphasise that the short period of available data might prevent us to make some clear interpretation about the relations between the yield curves and macro variables.

Since we use quarterly data to construct the model, we have 52 observations and the standard error of the coefficients is equal to $1/\sqrt{52} = 0.14$. We assume that the coefficients which are greater or less than three standard errors (i.e. $3 \times 0.14 = 0.42$) are significant.

The level component of the nominal interest rates as a first variable in the tables shows a very high autocorrelation which decreases exponentionally. Thus, the level of the nominal interest rates highly depends on the value of the previous quarters. It has very high simultaneous and lagged correlations with the levels of the implied inflation and the real interest rates too. Since the nominal interest rates can be decomposed into two parts containing the expected future inflation (we use implied inflation as an estimate for the expected future inflation in this work) and real interest rates, the high inflation expectations or high real interest rates lead to high nominal interest rates. Although we would expect a significant lagged correlation between the levels of the nominal interest rates and the realised inflation because the level of the nominal yields is supposed to embody the inflation expectations, we could not find any correlations among these two variables. Both the frequency and the short period of data along with the relatively stable inflation rates might be the reasons for this. When we look at the correlation between the level of the nominal rates and output gap, we see negative simultaneous and lagged correlations. These correlations can be explained considering the links between the goods market and the financial markets. Equilibrium in the goods market implies that an increase in the interest rate leads to a decrease in output (IS relation)[1]. On the other hand, equilibrium in the financial markets implies that an increase in output leads to an increase in the interest rate (LM relation). Goods market determines the output and the financial markets determine the interest rates. Considering the relation between the investment, interest rate and the goods market we can explain the negative correlation between the level factors of the yield curves and the output gap. An increase in the interest rates lowers the investment and thus reduces the output. The reduction in the actual output may lead a negative output gap (output gap is defined as the difference between the actual output and the potential output divided by the potential output) and thus justifies the negative correlations.

Previous studies explained the positive correlations between the short term interest rates and output gap with the Taylor rule which says that an increase in the output gap increases the short term nominal rates. Although we discuss negative correlations between the level factors and output gap rather than the short term interest rates, looking at the correlations between different maturities (short, medium and long) for the nominal rates and the output gap, we can conclude that the spot rates themselves also have negative correlations with the output gap. Therefore, not only the PCs we use in this study but also the original data themselves have negative correlations with the output gap. The positive correlation between the short term interest rates and the output gap on which the Taylor rule is based can be explained considering the relation between the money demand, aggregate output (income) and the financial markets. Accordingly, changes in output (income), which takes place in the goods market, shift

[1]IS relation follows from the condition that the supply of goods must be equal to the demand for goods. It tells us how the interest rate affects output. The LM relation follows from the condition that the supply of money must be equal to the demand for money. It tells us how output, in turn, affects the interest rate. By putting the IS and LM relations together: at any time, the supply of goods must be equal to the demand for goods, and the supply of money must be equal to the demand for money. Both the IS and LM relations must hold. Together, they determine both output and the interest rate:

$$\begin{aligned} IS\ relation: Y &= C + I + G \\ LM\ relation: M &= \$Y\ L(i) \end{aligned}$$

In the IS relation, Y is the output, C is the consumption of the households, G is the government spending. In the LM relation, M is the money supply, $\$Y$ is the nominal income and $L(i)$ is a function which depends on the interest rate i (Blanchard, 2006).

the money demand (LM) curve and cause changes in the interest rates. Hence, when there is an increase in output (which might lead to a positive output gap), the money demand increases. Since the money supply does not change, the equilibrium can be satisfied at a higher interest rate. Since the short term interest rates are used as a monetary policy instrument, the effect of the change in the output would be observed on the short rates firstly.

The slope factor of the nominal spot rates has positive simultaneous and lagged correlations with the slope factor of the real spot rates. Although there is a simultaneous correlation between the slope factors of the nominal rates and implied inflation it is not as strong as the correlation between the nominal and real slope factors. The previous studies mostly connect the slope factor of the nominal rates with the GDP growth or output gap. The negative correlation between the slope of the nominal rates and the GDP growth indicates that the increase in the GDP growth increases the short term interest rates by much larger amounts than the long term interest rates, so that the yield curve becomes less steep and its slope decreases. This also explains the stronger correlation between the GDP growth and the slope factor than the output gap and the slope factor. Since the output gap data is the latest estimate obtained after 3 years revision since it was published, it is reasonable to see its effect on the level factors which represent the long term maturities. On the other hand, the GDP growth (or the output gap first estimate which we have examined but not displayed here) has a strong but short lived influence on the slope factor due to affecting short rates in the short run.

Furthermore, the series can be modelled by using AR processes because of the exponentially decreasing auto-correlation functions they have. Realised inflation does not have any significant simultaneous or lagged correlations with any of the variables except the nominal curvature factor. An increase in the nominal curvature factor which means that the medium term interest rates increased more than the short and long ends causes a decrease in realised inflation.

Finally, the curvature factor of the real spot rates have significant negative simultaneous and lagged correlations with the annual GDP growth.

Table 6.5: Lagged Correlations between the Quarterly Yield Factors and Macro Variables - I

NPC1[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**	**Output Gap [t-k]**	**Annual GDP Growth [t-k]**
(0)	**1.00**	0.00	0.00	0.90	-0.05	0.03	0.93	0.02	0.08	0.09	-0.66	0.20
(1)	0.91	0.02	-0.20	0.77	0.06	0.09	0.89	0.02	0.03	0.08	-0.64	0.13
(2)	0.83	0.04	-0.35	0.65	0.17	0.14	0.85	0.01	-0.05	0.08	-0.61	0.16
(3)	0.75	0.02	-0.42	0.54	0.16	0.10	0.81	0.01	-0.12	0.02	-0.63	0.19
(4)	0.69	-0.02	-0.42	0.45	0.09	0.08	0.77	0.01	-0.19	-0.04	-0.61	0.25
(5)	0.59	-0.09	-0.46	0.32	0.01	0.06	0.71	-0.01	-0.28	-0.08	-0.57	0.27
(6)	0.49	-0.19	-0.47	0.23	-0.12	0.04	0.62	-0.04	-0.40	-0.09	-0.50	0.31
(7)	0.41	-0.31	-0.41	0.14	-0.26	0.04	0.55	-0.10	-0.47	-0.09	-0.41	0.43
(8)	0.33	-0.38	-0.37	0.06	-0.31	0.09	0.48	-0.19	-0.52	-0.08	-0.32	0.51

NPC2[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**	**Output Gap [t-k]**	**Annual GDP Growth [t-k]**
(0)	0.00	**1.00**	0.00	0.08	0.57	-0.09	-0.02	0.85	0.24	-0.19	-0.25	-0.64
(1)	0.07	0.86	0.14	0.17	0.31	-0.27	0.01	0.86	0.19	-0.26	-0.33	-0.54
(2)	0.11	0.66	0.20	0.23	0.07	-0.46	0.04	0.80	0.20	-0.21	-0.32	-0.40
(3)	0.11	0.44	0.09	0.24	-0.01	-0.55	0.02	0.65	0.21	-0.03	-0.24	-0.21
(4)	0.08	0.28	-0.13	0.20	0.04	-0.57	-0.02	0.48	0.20	0.25	-0.18	-0.09
(5)	0.05	0.16	-0.31	0.18	0.09	-0.57	-0.07	0.36	0.18	0.49	-0.16	0.01
(6)	0.00	0.13	-0.46	0.12	0.18	-0.51	-0.10	0.28	0.18	0.61	-0.17	0.03
(7)	-0.04	0.08	-0.57	0.08	0.23	-0.45	-0.14	0.20	0.12	0.64	-0.20	-0.01
(8)	-0.07	0.04	-0.60	0.06	0.26	-0.34	-0.17	0.13	0.06	0.55	-0.22	-0.03

NPC3[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**	**Output Gap [t-k]**	**Annual GDP Growth [t-k]**
(0)	0.00	0.00	**1.00**	0.11	-0.50	0.18	-0.08	0.03	0.05	-0.50	0.27	-0.06
(1)	0.03	0.00	0.76	0.12	-0.33	0.12	-0.05	-0.01	0.16	-0.29	0.32	-0.11
(2)	0.03	0.06	0.44	0.10	-0.06	0.10	-0.03	-0.01	0.21	-0.03	0.34	-0.12
(3)	0.03	0.22	0.21	0.04	0.18	0.03	0.02	0.08	0.31	0.13	0.28	-0.13
(4)	0.05	0.36	0.07	0.05	0.36	-0.07	0.07	0.18	0.43	0.17	0.18	-0.21
(5)	0.05	0.47	-0.08	0.00	0.49	-0.22	0.11	0.28	0.51	0.10	0.08	-0.34
(6)	0.04	0.54	-0.19	-0.02	0.51	-0.34	0.10	0.41	0.48	-0.03	0.01	-0.44
(7)	0.04	0.56	-0.16	-0.01	0.46	-0.39	0.10	0.47	0.45	-0.15	-0.05	-0.46
(8)	0.05	0.54	-0.09	0.01	0.38	-0.44	0.10	0.48	0.42	-0.20	-0.16	-0.48

IPC1[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**	**Output Gap [t-k]**	**Annual GDP Growth [t-k]**
(0)	0.90	0.08	0.11	**1.00**	0.00	0.00	0.68	0.09	0.18	0.22	-0.66	0.08
(1)	0.85	0.06	-0.09	0.88	0.11	0.08	0.70	0.03	0.16	0.22	-0.64	0.05
(2)	0.80	0.04	-0.28	0.77	0.23	0.10	0.70	-0.03	0.11	0.25	-0.62	0.09
(3)	0.74	0.02	-0.42	0.66	0.29	0.06	0.70	-0.05	0.08	0.25	-0.64	0.12
(4)	0.69	0.00	-0.49	0.57	0.27	0.02	0.68	-0.05	0.02	0.20	-0.65	0.16
(5)	0.61	-0.08	-0.55	0.44	0.21	-0.02	0.66	-0.09	-0.04	0.15	-0.60	0.18
(6)	0.53	-0.19	-0.56	0.34	0.06	-0.08	0.60	-0.10	-0.14	0.10	-0.53	0.18
(7)	0.45	-0.29	-0.53	0.23	-0.05	-0.08	0.55	-0.17	-0.21	0.06	-0.53	0.29
(8)	0.38	-0.36	-0.51	0.14	-0.09	-0.00	0.49	-0.24	-0.31	0.03	-0.46	0.38

Table 6.6: Lagged Correlations between the Quarterly Yield Factors and Macro Variables - II

IPC2[t] Lag, k	**NPC1** [t-k]	**NPC2** [t-k]	**NPC3** [t-k]	**IPC1** [t-k]	**IPC2** [t-k]	**IPC3** [t-k]	**RPC1** [t-k]	**RPC2** [t-k]	**RPC3** [t-k]	**Realised Inflation** [t-k]	**Output Gap** [t-k]	**Annual GDP Growth** [t-k]
(0)	-0.05	0.57	-0.50	0.00	**1.00**	0.00	-0.06	0.15	0.54	0.24	-0.21	-0.57
(1)	0.00	0.56	-0.27	0.05	0.69	-0.17	-0.02	0.33	0.43	0.04	-0.37	-0.56
(2)	0.05	0.48	-0.07	0.11	0.37	-0.30	0.01	0.41	0.35	-0.12	-0.44	-0.53
(3)	0.07	0.29	0.02	0.14	0.11	-0.35	0.01	0.35	0.24	-0.15	-0.43	-0.43
(4)	0.05	0.12	-0.06	0.10	-0.01	-0.39	0.01	0.25	0.12	0.02	-0.33	-0.22
(5)	0.04	-0.01	-0.12	0.12	-0.12	-0.39	-0.03	0.19	0.02	0.22	-0.26	0.04
(6)	0.01	-0.02	-0.20	0.09	-0.06	-0.26	-0.06	0.13	-0.02	0.41	-0.19	0.20
(7)	-0.01	-0.08	-0.29	0.07	-0.05	-0.22	-0.08	0.08	-0.09	0.53	-0.14	0.27
(8)	-0.02	-0.10	-0.37	0.06	-0.05	-0.14	-0.09	0.06	-0.17	0.56	-0.06	0.29

IPC3[t] Lag, k	**NPC1** [t-k]	**NPC2** [t-k]	**NPC3** [t-k]	**IPC1** [t-k]	**IPC2** [t-k]	**IPC3** [t-k]	**RPC1** [t-k]	**RPC2** [t-k]	**RPC3** [t-k]	**Realised Inflation** [t-k]	**Output Gap** [t-k]	**Annual GDP Growth** [t-k]
(0)	0.03	-0.09	0.18	0.00	0.00	**1.00**	0.05	-0.34	-0.49	-0.28	0.08	0.13
(1)	0.11	-0.13	0.38	0.05	-0.21	0.67	0.15	-0.24	-0.41	-0.39	0.03	0.16
(2)	0.16	-0.21	0.47	0.10	-0.36	0.43	0.19	-0.23	-0.29	-0.43	0.07	0.18
(3)	0.19	-0.30	0.44	0.08	-0.36	0.29	0.24	-0.30	-0.12	-0.40	0.17	0.23
(4)	0.17	-0.34	0.28	-0.01	-0.22	0.12	0.29	-0.38	0.07	-0.24	0.27	0.22
(5)	0.14	-0.29	0.08	-0.07	-0.09	-0.04	0.28	-0.33	0.17	-0.09	0.31	0.20
(6)	0.08	-0.10	-0.07	-0.17	0.17	0.01	0.27	-0.26	0.27	-0.04	0.27	0.08
(7)	0.04	0.08	-0.14	-0.24	0.28	0.00	0.26	-0.10	0.25	-0.08	0.21	-0.09
(8)	0.01	0.22	-0.14	-0.26	0.27	-0.07	0.23	0.09	0.18	-0.17	0.12	-0.26

RPC1[t] Lag, k	**NPC1** [t-k]	**NPC2** [t-k]	**NPC3** [t-k]	**IPC1** [t-k]	**IPC2** [t-k]	**IPC3** [t-k]	**RPC1** [t-k]	**RPC2** [t-k]	**RPC3** [t-k]	**Realised Inflation** [t-k]	**Output Gap** [t-k]	**Annual GDP Growth** [t-k]
(0)	0.93	-0.02	-0.08	0.68	-0.06	0.05	**1.00**	0.00	0.00	-0.05	-0.56	0.24
(1)	0.83	0.02	-0.25	0.58	0.02	0.07	0.91	0.05	-0.08	-0.07	-0.54	0.16
(2)	0.73	0.06	-0.35	0.47	0.09	0.13	0.84	0.07	-0.18	-0.09	-0.53	0.18
(3)	0.65	0.04	-0.35	0.37	0.03	0.10	0.79	0.08	-0.25	-0.16	-0.54	0.21
(4)	0.58	-0.01	-0.31	0.29	-0.06	0.09	0.73	0.08	-0.32	-0.21	-0.49	0.27
(5)	0.47	-0.07	-0.34	0.18	-0.15	0.09	0.64	0.07	-0.42	-0.23	-0.43	0.31
(6)	0.38	-0.15	-0.34	0.11	-0.24	0.11	0.54	0.04	-0.54	-0.21	-0.34	0.37
(7)	0.31	-0.26	-0.27	0.04	-0.38	0.10	0.47	-0.02	-0.60	-0.18	-0.26	0.47
(8)	0.24	-0.34	-0.22	-0.02	-0.43	0.12	0.39	-0.10	-0.61	-0.15	-0.17	0.53

RPC2[t] Lag, k	**NPC1** [t-k]	**NPC2** [t-k]	**NPC3** [t-k]	**IPC1** [t-k]	**IPC2** [t-k]	**IPC3** [t-k]	**RPC1** [t-k]	**RPC2** [t-k]	**RPC3** [t-k]	**Realised Inflation** [t-k]	**Output Gap** [t-k]	**Annual GDP Growth** [t-k]
(0)	0.02	0.85	0.03	0.09	0.15	-0.34	0.00	**1.00**	0.00	-0.19	-0.25	-0.41
(1)	0.05	0.68	0.08	0.15	0.03	-0.36	-0.01	0.87	-0.03	-0.16	-0.25	-0.26
(2)	0.07	0.47	0.07	0.17	-0.09	-0.46	0.00	0.72	0.01	-0.05	-0.20	-0.11
(3)	0.05	0.31	-0.06	0.17	-0.06	-0.50	-0.05	0.56	0.04	0.13	-0.12	0.04
(4)	0.02	0.19	-0.21	0.17	0.00	-0.45	-0.11	0.41	0.04	0.33	-0.09	0.07
(5)	-0.01	0.11	-0.32	0.16	0.08	-0.38	-0.14	0.26	0.06	0.47	-0.10	0.04
(6)	-0.03	0.02	-0.38	0.13	0.11	-0.36	-0.16	0.15	0.07	0.51	-0.14	0.00
(7)	-0.06	-0.04	-0.46	0.10	0.17	-0.29	-0.18	0.05	0.06	0.49	-0.32	-0.06
(8)	-0.09	-0.08	-0.45	0.07	0.21	-0.18	-0.22	-0.05	0.05	0.38	-0.30	-0.06

Table 6.7: Lagged Correlations between the Quarterly Yield Factors and Macro Variables - III

RPC3[t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]	Realised Inflation [t-k]	Output Gap [t-k]	Annual GDP Growth [t-k]
(0)	0.08	0.24	0.05	0.18	0.54	-0.49	0.00	0.00	**1.00**	0.27	-0.03	-0.50
(1)	0.05	0.39	-0.17	0.15	0.66	-0.41	-0.02	0.14	0.86	0.29	-0.12	-0.63
(2)	0.04	0.53	-0.28	0.12	0.73	-0.32	-0.02	0.29	0.67	0.26	-0.27	-0.68
(3)	0.05	0.61	-0.23	0.13	0.59	-0.30	-0.00	0.46	0.48	0.16	-0.42	-0.66
(4)	0.07	0.59	-0.17	0.16	0.42	-0.33	0.01	0.54	0.32	0.10	-0.50	-0.53
(5)	0.09	0.49	-0.13	0.19	0.23	-0.35	0.01	0.56	0.17	0.07	-0.51	-0.37
(6)	0.08	0.38	-0.15	0.20	0.07	-0.35	-0.02	0.55	0.02	0.12	-0.42	-0.12
(7)	0.10	0.22	-0.16	0.24	-0.05	-0.33	-0.03	0.43	-0.08	0.24	-0.33	0.10
(8)	0.10	0.09	-0.19	0.24	-0.11	-0.28	-0.02	0.32	-0.13	0.39	-0.25	0.24

Realised Inflation [t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]	Realised Inflation [t-k]	Output Gap [t-k]	Annual GDP Growth [t-k]
(0)	0.09	-0.19	-0.50	0.22	0.24	-0.28	-0.05	-0.19	0.27	**1.00**	-0.11	0.16
(1)	0.08	-0.08	-0.55	0.19	0.35	-0.17	-0.04	-0.14	0.24	0.83	-0.15	0.07
(2)	0.10	-0.00	-0.47	0.21	0.39	-0.04	-0.02	-0.11	0.19	0.54	-0.17	0.00
(3)	0.13	0.01	-0.31	0.24	0.30	0.04	0.01	-0.11	0.12	0.21	-0.21	-0.06
(4)	0.17	-0.03	-0.10	0.27	0.14	0.13	0.05	-0.11	0.03	-0.08	-0.23	-0.06
(5)	0.20	-0.11	0.05	0.29	-0.03	0.18	0.09	-0.14	-0.07	-0.21	-0.28	-0.01
(6)	0.22	-0.20	0.13	0.29	-0.15	0.23	0.11	-0.21	-0.15	-0.17	-0.30	-0.03
(7)	0.21	-0.29	0.12	0.25	-0.22	0.21	0.14	-0.26	-0.17	-0.06	-0.32	0.06
(8)	0.19	-0.32	0.01	0.20	-0.17	0.20	0.14	-0.30	-0.17	0.11	-0.29	0.14

Output Gap [t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]	Realised Inflation [t-k]	Output Gap [t-k]	Annual GDP Growth [t-k]
(0)	-0.61	-0.56	0.02	-0.64	-0.17	0.17	-0.53	-0.60	-0.08	0.04	**1.00**	0.14
(1)	-0.58	-0.44	0.08	-0.62	-0.08	0.15	-0.49	-0.55	0.02	0.01	0.90	-0.02
(2)	-0.57	-0.30	0.13	-0.61	-0.02	0.16	-0.46	-0.44	0.07	-0.12	0.78	-0.21
(3)	-0.53	-0.15	0.23	-0.57	-0.02	0.16	-0.42	-0.29	0.08	-0.25	0.64	-0.39
(4)	-0.50	-0.00	0.31	-0.51	0.01	0.14	-0.40	-0.14	0.11	-0.38	0.49	-0.54
(5)	-0.44	0.14	0.45	-0.43	0.03	0.17	-0.37	-0.03	0.13	-0.47	0.37	-0.55
(6)	-0.38	0.24	0.55	-0.35	0.04	0.12	-0.33	0.07	0.20	-0.48	0.25	-0.54
(7)	-0.32	0.35	0.58	-0.25	0.06	0.09	-0.31	0.19	0.23	-0.43	0.16	-0.53
(8)	-0.27	0.41	0.58	-0.17	0.07	0.02	-0.28	0.29	0.26	-0.31	0.11	-0.54

Annual GDP Growth [t] Lag, k	NPC1 [t-k]	NPC2 [t-k]	NPC3 [t-k]	IPC1 [t-k]	IPC2 [t-k]	IPC3 [t-k]	RPC1 [t-k]	RPC2 [t-k]	RPC3 [t-k]	Realised Inflation [t-k]	Output Gap [t-k]	Annual GDP Growth [t-k]
(0)	0.20	-0.64	-0.06	0.08	-0.57	0.13	0.24	-0.41	-0.50	0.16	0.14	**1.00**
(1)	0.17	-0.64	-0.12	0.06	-0.45	0.22	0.21	-0.49	-0.42	0.22	0.24	0.78
(2)	0.13	-0.56	-0.15	0.00	-0.29	0.34	0.19	-0.52	-0.38	0.18	0.30	0.58
(3)	0.10	-0.47	-0.10	-0.06	-0.17	0.44	0.20	-0.52	-0.31	0.01	0.27	0.37
(4)	0.12	-0.37	0.03	-0.04	-0.11	0.52	0.21	-0.50	-0.26	-0.19	0.22	0.15
(5)	0.10	-0.27	0.15	-0.06	-0.10	0.49	0.21	-0.42	-0.21	-0.44	0.13	0.04
(6)	0.08	-0.20	0.26	-0.05	-0.13	0.46	0.18	-0.33	-0.20	-0.60	0.06	-0.02
(7)	0.08	-0.22	0.36	-0.05	-0.24	0.38	0.18	-0.29	-0.18	-0.67	0.04	-0.04
(8)	0.07	-0.17	0.37	-0.09	-0.24	0.30	0.19	-0.22	-0.14	-0.62	0.02	-0.01

6.2.5 Fitting a VAR Model to the Quarterly PCs and the Macroeconomic Variables

After examining the correlations between the yield curves and macro variables we construct a vector autoregressive model for the series. We start with including the first two lags of each variable and eliminate the insignificant ones to obtain the best model. Furthermore, we avoid including simultaneous explanatory variables in to the models because in forecasting we do not want to deal with additional uncertainty rooted by the simultaneous correlations. Appendix B introduces the models for each variable and presents the coefficients of determination.

To construct the 'yield-macro' model, we use quarterly nominal spot rates, implied inflation spot rates, real spot rates, annual realised inflation and output gap over the period 1995 to 2007 [2].

Let X_Q be the matrix of quarterly yield curve data where
X_{Q_N}: Nominal spot rates (52×50)
X_{Q_I}: Implied inflation spot rates (52×46)
X_{Q_R}: Real spot rates (52×46)

Let Q be the matrix of quarterly PCs and macroeconomic variables where:
Q_{N_L}: level component of the nominal spot rates (52×1)
Q_{N_S}: slope component of the nominal spot rates (52×1)
Q_{N_C}: curvature component of the nominal spot rates (52×1)
Q_{I_L}: level component of the implied inflation spot rates (52×1)
Q_{I_S}: slope component of the implied inflation spot rates (52×1)
Q_{I_C}: curvature component of the implied inflation spot rates (52×1)
Q_{R_L}: level component of the real spot rates (52×1)
Q_{R_S}: slope component of the real spot rates (52×1)
Q_{R_C}: curvature component of the real spot rates (52×1)

[2] Since the output gap data are subject to continuous revision which may take three years to get the latest estimate, the data period in this modelling work is restricted with 2007.

Q_{RI}: realised inflation (52×1)

Q_{OG}: output gap (52×1)

The VAR structure of the quarterly model is:

$$Q[t] - \mu_Q = B_1 \, (Q[t-1] - \mu_Q) + B_2 \, (Q[t-2] - \mu_Q) + \epsilon_Q[t]$$

where:

μ_Q is the vector of long run mean of the variables, B_1 and B_2 are the coefficient matrices for the first and second lags of the explanatory variables respectively and $\epsilon_Q[t] \sim N(0, \Sigma_Q)$, i.e. normally distributed residuals with zero mean and Σ_Q variance-covariance matrix.

$$Q = \begin{bmatrix} Q_{N_L} \\ Q_{N_S} \\ Q_{N_C} \\ Q_{I_L} \\ Q_{I_S} \\ Q_{I_C} \\ Q_{R_L} \\ Q_{R_S} \\ Q_{R_C} \\ Q_{RI} \\ Q_{OG} \end{bmatrix}$$

$$\widehat{\mu}_Q^t = \begin{bmatrix} -6.76 & 0 & 0 & -1.47 & 0 & 0 & -6.99 & 0 & 0 & 2.88 & 0 \end{bmatrix}$$

$$\widehat{B}_1 = \begin{bmatrix} 0.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.78 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.96 & 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & 0 \\ 0 & 0 & 0 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.56 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.62 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 & 0 & 0 & 0 & 0.49 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.86 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.92 & 0 \\ 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.89 \end{bmatrix}$$

$$\widehat{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.21 & 0 & -0.41 \\ 0 & 0 & -0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.32 & 0 & 0 & 0 & 1.38 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{\Sigma}_Q = \begin{bmatrix} 4.54 & & & & & & & & & & \\ -0.62 & 0.76 & & & & & & & & & \\ -0.50 & 0.07 & 0.14 & & & & & & & & \\ 2.51 & -0.22 & -0.30 & 2.28 & & & & & & & \\ -0.17 & 0.25 & 0.07 & -0.18 & 0.27 & & & & & & \\ 0.09 & -0.11 & 0.01 & 0.07 & -0.05 & 0.08 & & & & & \\ 1.88 & -0.30 & -0.19 & 0.20 & 0.05 & 0.00 & 1.61 & & & & \\ -0.53 & 0.30 & 0.05 & -0.10 & -0.02 & 0.02 & -0.40 & 0.29 & & & \\ -0.20 & 0.06 & 0.03 & -0.02 & -0.02 & -0.02 & -0.17 & 0.05 & 0.05 & & \\ 0.07 & 0.10 & -0.05 & 0.07 & 0.03 & -0.01 & 0.02 & 0.02 & -0.01 & 0.17 & \\ -0.03 & 0.03 & 0.01 & 0.04 & 0.01 & 0.01 & -0.06 & 0.02 & 0.01 & 0.02 & 0.06 \end{bmatrix}$$

The negative long run means for the level factors of the yield curves displayed in μ_Q show that these factors have been decreasing since 1995 as seen in Figure 6.2, Figure 6.3 and Figure 6.4. It should be emphasized that the series we model are not the levels of the yield curves but the factors which affect the levels of the yield curves. Thus, it is not surprising that we obtain negative values for the long run mean of these factors. On the other hand, the long run mean for the realised inflation is about 3%.

When we look at the matrix $\widehat{B}_1$, although there are some off-diagonal values, the diagonal structure of the matrix shows how strong the AR(1) effect is in the models. Similarly, few number of values in $\widehat{B}_2$ shows that the second lags are mostly insignificant.

We display the estimated correlation matrix, $\widehat{\rho}_Q$ for the residuals below. As stated previously, we assume that the coefficients which are greater or less than three standard errors (0.42) are significant. As in the 'yield-only' model, we see several significant correlations between the residuals in matrix $\widehat{\rho}_Q$. Again one reason is that we exclude the simultaneous explanatory variables in the modelling work. As we observe in Table 6.5, Table 6.6 and Table 6.7, there are very strong simultaneous correlations particularly between the corresponding PCs of the three yield curves. The high correlations between the residuals for the level and slope factor models may be due to these strong simultaneous correlations between the level and slope PCs. Although the PCs themselves are independent within each yield curve, there is a strong negative correlation between the level and the slope factors residuals of the nominal spot rates. This might be some statistical artifact which does not really indicate a correlation between those two set of residuals.

$$\widehat{\rho}_Q = \begin{bmatrix} \mathbf{1} \\ -0.34 & \mathbf{1} \\ \mathbf{-0.63} & 0.21 & \mathbf{1} \\ \mathbf{0.80} & -0.17 & \mathbf{-0.54} & \mathbf{1} \\ -0.15 & \mathbf{0.57} & 0.38 & -0.23 & \mathbf{1} \\ 0.15 & \mathbf{-0.50} & 0.03 & 0.20 & -0.41 & \mathbf{1} \\ \mathbf{0.68} & -0.28 & -0.40 & 0.10 & 0.09 & -0.06 & \mathbf{1} \\ \mathbf{-0.49} & \mathbf{0.64} & 0.24 & -0.13 & -0.07 & 0.07 & \mathbf{-0.61} & \mathbf{1} \\ -0.40 & 0.32 & 0.33 & -0.06 & -0.15 & -0.30 & \mathbf{-0.58} & \mathbf{0.48} & \mathbf{1} \\ 0.07 & 0.28 & -0.28 & 0.11 & 0.12 & -0.12 & 0.03 & 0.07 & -0.07 & \mathbf{1} \\ -0.13 & 0.14 & 0.07 & 0.10 & 0.11 & 0.07 & -0.20 & 0.12 & 0.14 & 0.16 & \mathbf{1} \end{bmatrix}$$

6.2.6 Residual Analysis

After fitting the models and estimating the parameters we obtain the residuals for each PC and the macro variables. Table 6.8 shows the descriptive statistics such as mean, standard deviation, skewness and excess kurtosis for each set of residuals. Except for the implied inflation level factor, Jarque-Bera test p-value indicates that all residuals are normally distributed with at least 13% significance level. As for the implied inflation level factor, the residuals are distributed normally with a 2% significance level. On the other hand, although not presented here, the auto-correlation functions show that the residuals are independent too.

Table 6.8: Residual Analysis of the Yield-Macro Model-I

		Residuals				
		Mean	Standard Deviation	Skewness	Excess Kurtosis	Jarque-Bera p-value
Nominal	Level	0.0000	2.1251	0.1232	0.3058	0.7607
	Slope	0.0323	0.8658	0.1079	-1.0285	0.3730
	Curvature	-0.0385	0.3736	0.4240	0.9223	0.1326
Implied Inflation	Level	0.0001	1.5058	-0.6653	1.1571	0.0210
	Slope	0.0716	0.5177	0.3523	0.3307	0.4595
	Curvature	0.0162	0.2754	0.5622	-0.0509	0.2455
Real	Level	0.0000	1.2672	-0.1597	-0.3772	0.8243
	Slope	-0.0035	0.5419	0.4712	-0.2771	0.3567
	Curvature	-0.0029	0.2220	-0.5470	-0.4608	0.2263
Realised Inflation		0.0005	0.4124	-0.1117	-0.1002	0.9459
Output Gap		0.0261	0.2375	0.3764	0.5331	0.3249

6.2.7 Model Comparisons

In order to examine the goodness of fit of our models we calculate the adjusted coefficient of determination, R^2_{adj} [3] which is given in Appendix B for each model.

Using the adjusted coefficient of determination we discuss the performance of our models with respect to random walk (RW) and autoregressive order one (AR(1)) process. Therefore, to compare our models with the RW and AR(1) we calculate the following ratios.

[3]

$$
\begin{aligned}
R^2 &= \frac{SS_{reg}}{SS_{tot}} \\
R^2_{adj} &= 1-(1-R_2)\frac{n-1}{n-p-1} \\
&= 1-\frac{SS_E}{SS_T}\frac{df_t}{df_e}
\end{aligned}
$$

In our comparisons we use adjusted coefficient of determination, R^2_{adj} rather than coefficient of determination, R^2 to take the number of explanatory variables in the models into account. It is adjusted for the number of independent variables in the regression model. Unlike the coefficient of determination, R^2_{adj} may decrease if variables are entered in the model that do not add significantly to the model fit.

$$R^2_{RW^*} = 1 - \frac{SS_{model}}{SS_{RW}} \frac{df_{RW}}{df_{model}}$$

or

$$R^2_{AR(1)^*} = 1 - \frac{SS_{model}}{SS_{AR(1)}} \frac{df_{AR(1)}}{df_{model}}$$

where

SS_{model} is the sum of squares of the residuals obtained from the yield-macro model

SS_{RW} is the sum of squares of the residuals obtained from the random walk model

$SS_{AR(1)}$ is the sum of squares of the residuals obtained from the $AR(1)$ model

As we mentioned earlier, the output gap data have been published by the OECD Economic Outlook and due to some revision process the latest available data end by the last quarter of 2007. OECD also publishes the output gap first estimate before any revision process. To see whether the output gap data provided make any difference in terms of correlations between the variables and the VAR model we examine the output gap first estimate as a macro variable too. Moreover, we also examine annual GDP growth as a replacement of output gap data.

The use of exogenous variables such as output gap and the GDP growth might be criticised in asset models. The main argument against their use is that, while they may have a significant effect on the modelled variables in the short term, in the long term they merely constitute another noise term (Thomson, 1996). However, considering the yield-macro model-I, the output gap has an autoregressive term which carries its effect on the realised inflation and the nominal slope factor many years ahead into the future.

Table 6.9 shows the increase in the explained variability in the models compared to RW and AR(1) process. Zeros in the table indicate that the fitted models are already AR(1). When we have a general look at the table we see non-negative values which indicate that our models are superior to the RW and AR(1) process. However, the improvements in the explained variability are not always significant as we see in the curvature factor of the real spot rates. Nominal spot rate models explain significant amount of variability comparing with the RW and AR(1) models. Implied inflation slope model improves the explained variability for about 51% and 43% comparing with the RW and AR(1) respectively. Real slope model shows a significant improvement too while real level and curvature do not. Realised inflation model performs better than the RW and AR(1) when it includes the nominal curvature and output gap lagged values as explanatory variables. Output gap model performs slightly better than the RW and AR(1) with the help of nominal slope factor as an explanatory variable.

When we use output gap first estimate and the annual GDP growth instead of the output gap latest estimate, we see that slope factor of nominal spot rates, realised inflation and output gap models are affected from the data change. However, these changes are mostly insignificant. When we use output gap first estimate data we see that output gap is not significant in the realised inflation model anymore. Output gap model remains the same in terms of the explanatory variables it includes. Moreover, the performance gets slightly better (explained variability with respect to RW and AR(1) models increase to 19% and 11% from 14% and 8% respectively). As for the slope factor of the nominal spot rates, output gap first estimate is not significant while output gap latest estimate was significant. The other models have not changed at all with the replacement of the latest estimate with the first estimate.

When we use annual GDP growth instead of the output gap latest estimate, we see that it is not significant in the realised inflation model anymore. Annual GDP growth includes nominal level and real curvature factors beside its lagged value as explanatory variables. There is no significant improvement in the explained variability with respect to modelling the output gap latest estimate. When we examine the slope factor of the nominal rates we see that the annual GDP growth is not a significant explanatory variable while output gap latest estimate was. On the other hand, although the slope factors of the three yield curves have significant simultaneous and lagged correlations with the annual GDP growth, we see that it is not significant anymore when we take the auto-correlations into account in the modelling work. The other models have not changed when we use annual GDP growth instead of the output gap latest estimate.

Table 6.9: Model Comparisons with the RW and AR(1) process

	Using OG latest estimate		Using OG first estimate		Using annual GDP growth	
	$R^2_{RW^*}$	$R^2_{AR(1)^*}$	$R^2_{RW^*}$	$R^2_{AR(1)^*}$	$R^2_{RW^*}$	$R^2_{AR(1)^*}$
Nominal Spot Rates						
Level	0.16	0.00	0.16	0.00	0.16	0.00
Slope	0.31	0.26	**0.28**	**0.22**	**0.27**	**0.21**
Curvature	0.30	0.20	0.30	0.20	0.30	0.20
Implied Inflation Spot Rates						
Level	0.13	0.00	0.13	0.00	0.13	0.00
Slope	0.51	0.43	0.51	0.43	0.51	0.43
Curvature	0.23	0.10	0.23	0.10	0.23	0.10
Real Spot Rates						
Level	0.10	0.00	0.10	0.00	0.10	0.00
Slope	0.26	0.21	0.26	0.21	0.26	0.21
Curvature	0.07	0.00	0.07	0.00	0.07	0.00
Realised Inflation	0.26	0.21	**0.25**	**0.19**	**0.20**	**0.14**
Output Gap/ GDP growth	0.14	0.08	**0.19**	**0.11**	**0.17**	**0.06**

6.2.8 Forecasting

After modelling the PCs along with the macroeconomic variables, we test these models by forecasting one-quarter ahead spot rates, realised inflation and the output gap using the estimated parameters. In order to compare our forecasts with the fitted spot rates and the macroeconomic variables we have fitted the models to the data recursively; starting with first 32 quarters and ending with 51 quarters. As we increase the data period, we apply the PCA, re-fit the model and estimate the parameters for that period. Afterwards, we use the parameters for each period to forecast the next quarter's level, slope and curvature factors of the spot rates. As a final step, we convert the forecasts for PCs into the spot rates, i.e. we obtain the fitted spot rates by using these three PCs. Furthermore, we calculate the variance-covariance matrix of the residuals for each set of recursive estimates to construct the 95% confidence intervals for the forecasts under the normally distributed residuals assumption.

Figure 6.5, Figure 6.6, Figure 6.7 and Figure 6.8 display the 1-quarter ahead forecasts and the 95% confidence intervals for the three yield curves in different maturities and the macroeconomic variables. Although the forecasts seem like a RW model forecasts, the models are better than RW in terms of explained variability in the data as we have examined previously. Since almost all of the observations are within the confidence bands we can conclude that the confidence intervals are too wide.

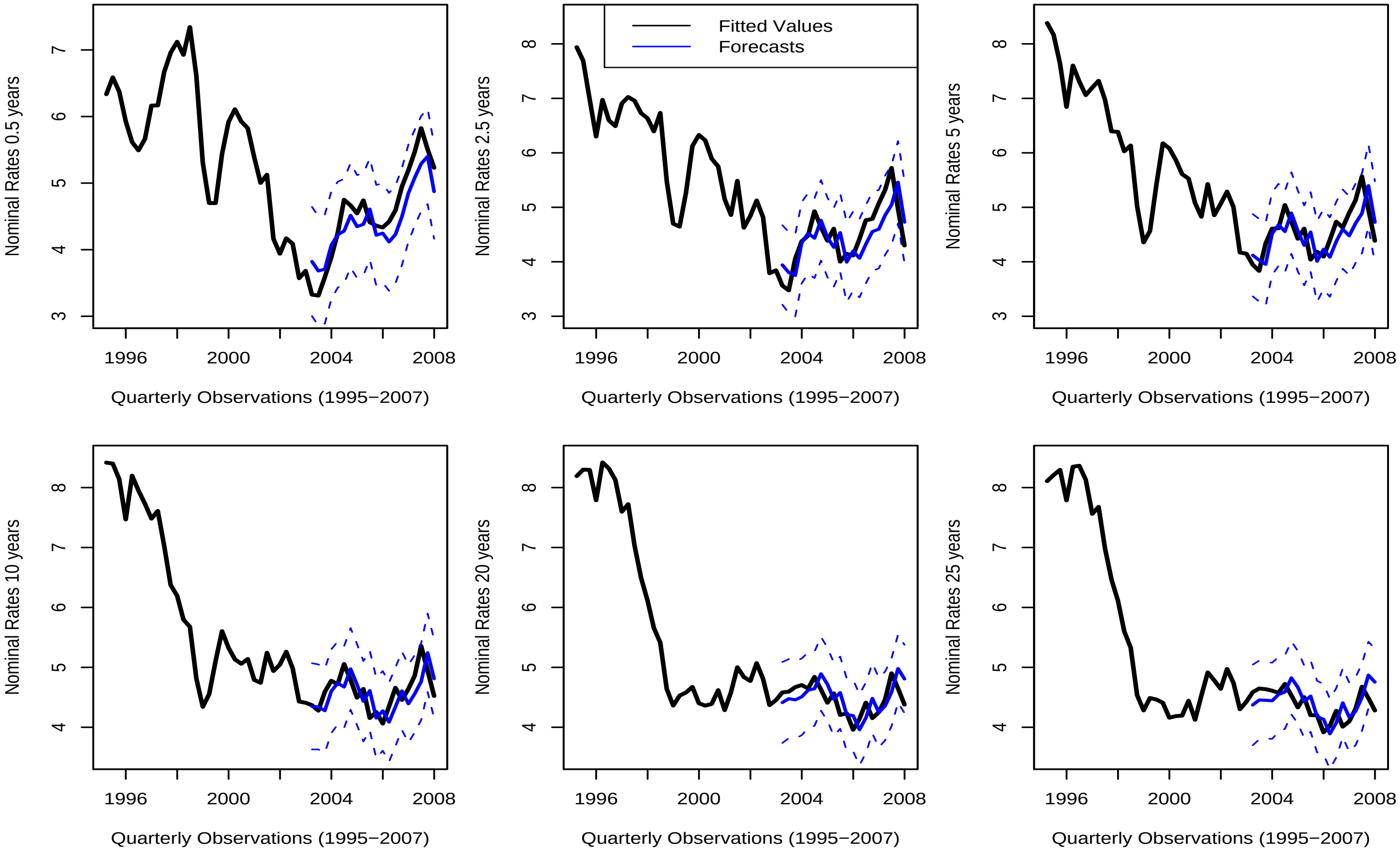

Figure 6.5: 1-Quarter Ahead Forecasts with Upper and Lower Confidence Limits for Nominal Spot Rates (%)

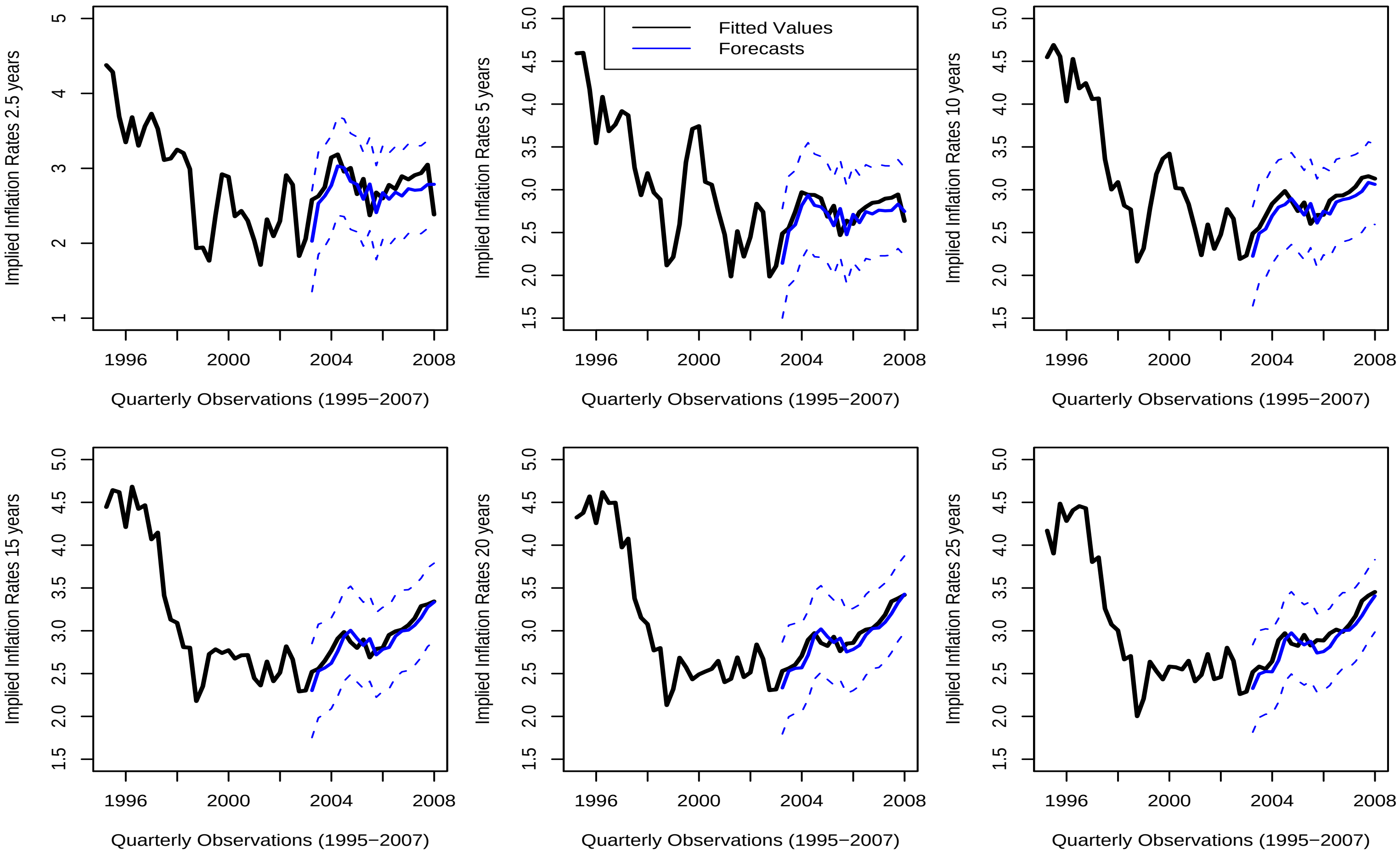

Figure 6.6: 1-Quarter Ahead Forecasts with Upper and Lower Confidence Limits for Implied Inflation Spot Rates (%)

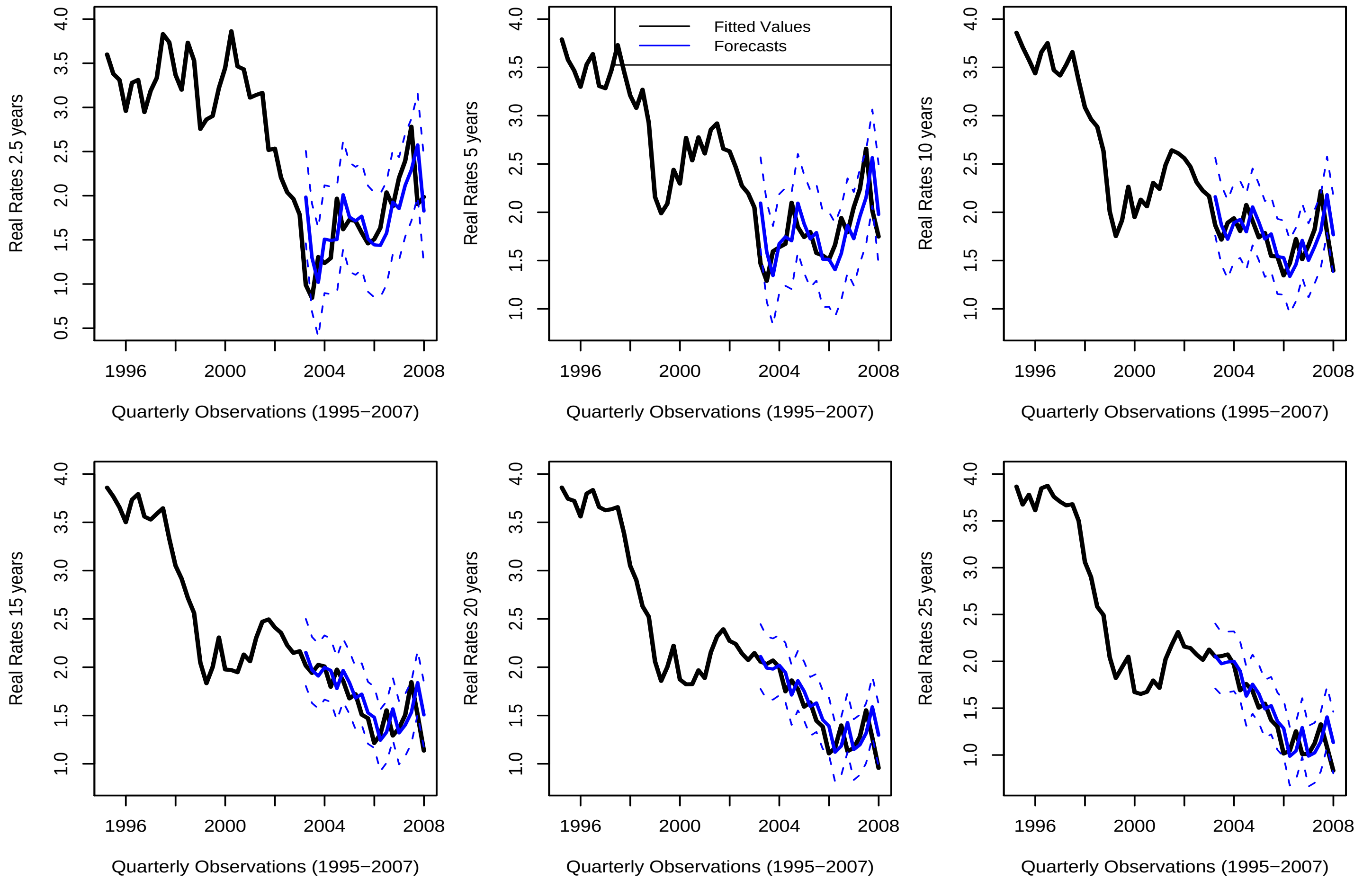

Figure 6.7: 1-Quarter Ahead Forecasts with Upper and Lower Confidence Limits for Real Spot Rates (%)

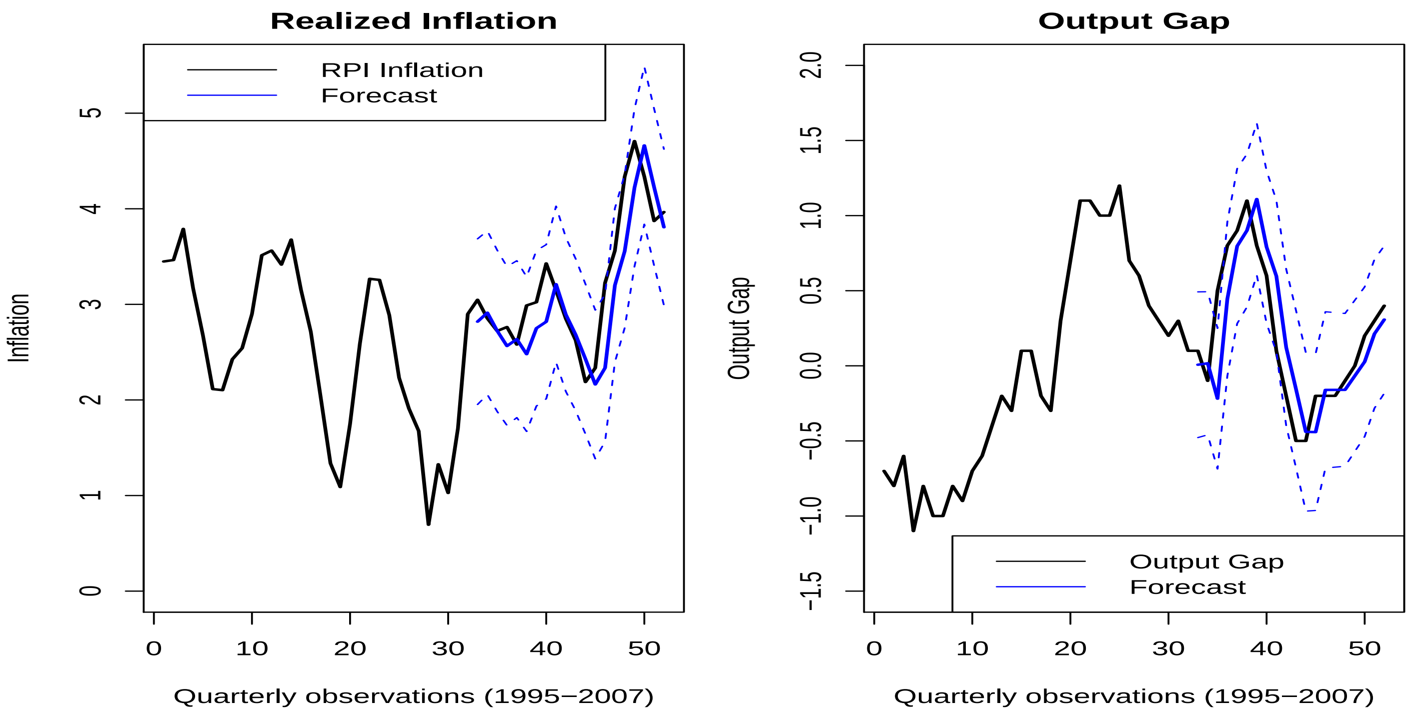

Figure 6.8: 1-Quarter Ahead Forecasts with Upper and Lower Confidence Limits for Realised Inflation (%) and Output Gap

6.2.9 Fisher Relation

As we have done for the yield-only model, we check whether the Fisher relation holds for our yield-macro model 1-quarter ahead forecasts too. Figures 6.9, 6.10, 6.11, 6.12, 6.13 and 6.14 present the graphs of the forecasts and the forecast errors for both the yield curves and the ones obtained by using the Fisher relation as explained in Chapter 5. Although the forecast graphs show that the yield curve forecast for each yield curve and the yield curves derived by the Fisher relation seem quite close, the error graphs show that they are significantly different particularly for the long ends of the yield curves.

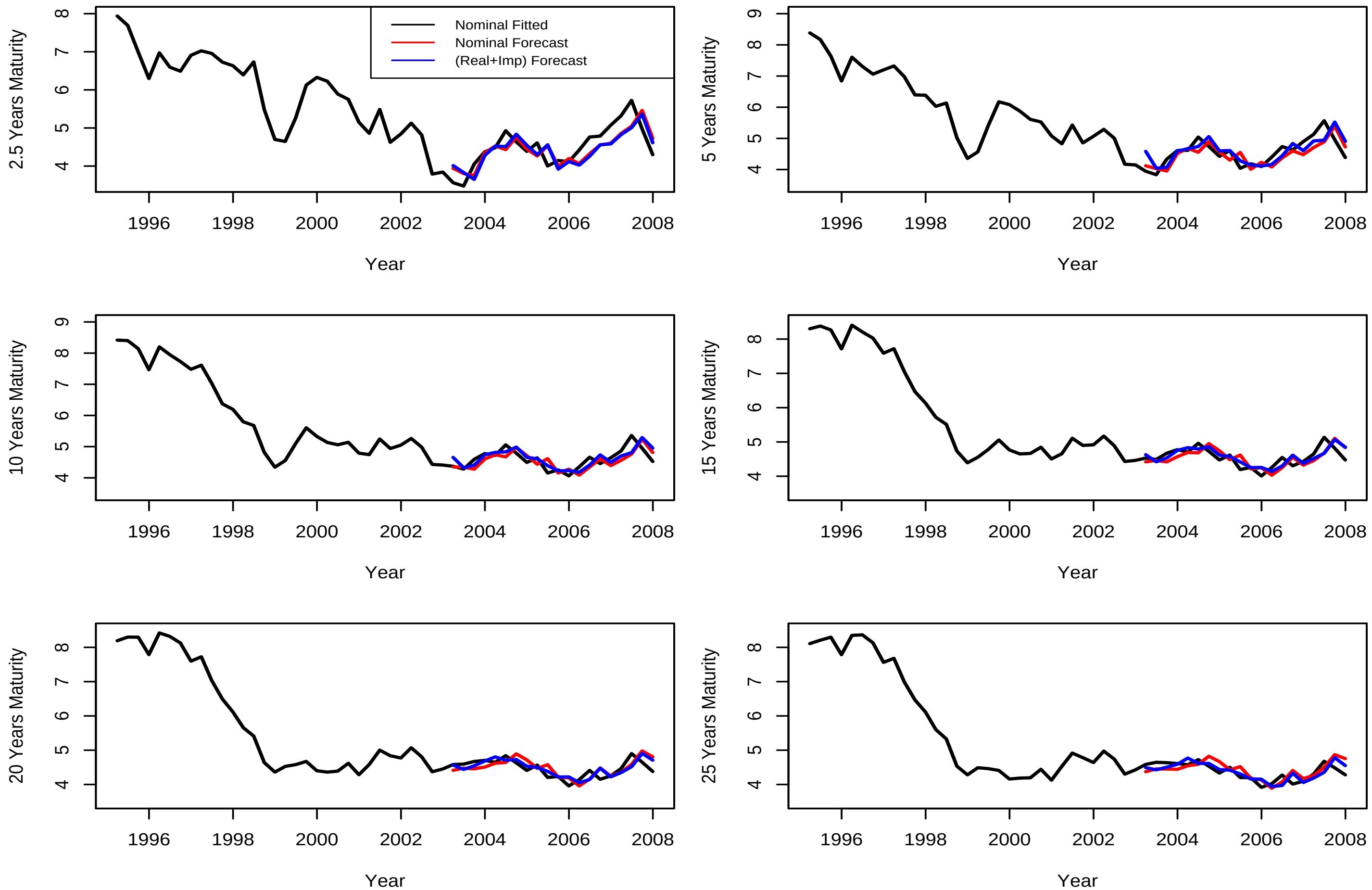

Figure 6.9: Fisher Relation Check for the 1-Quarter Ahead Nominal Spot Rate Forecasts (%)

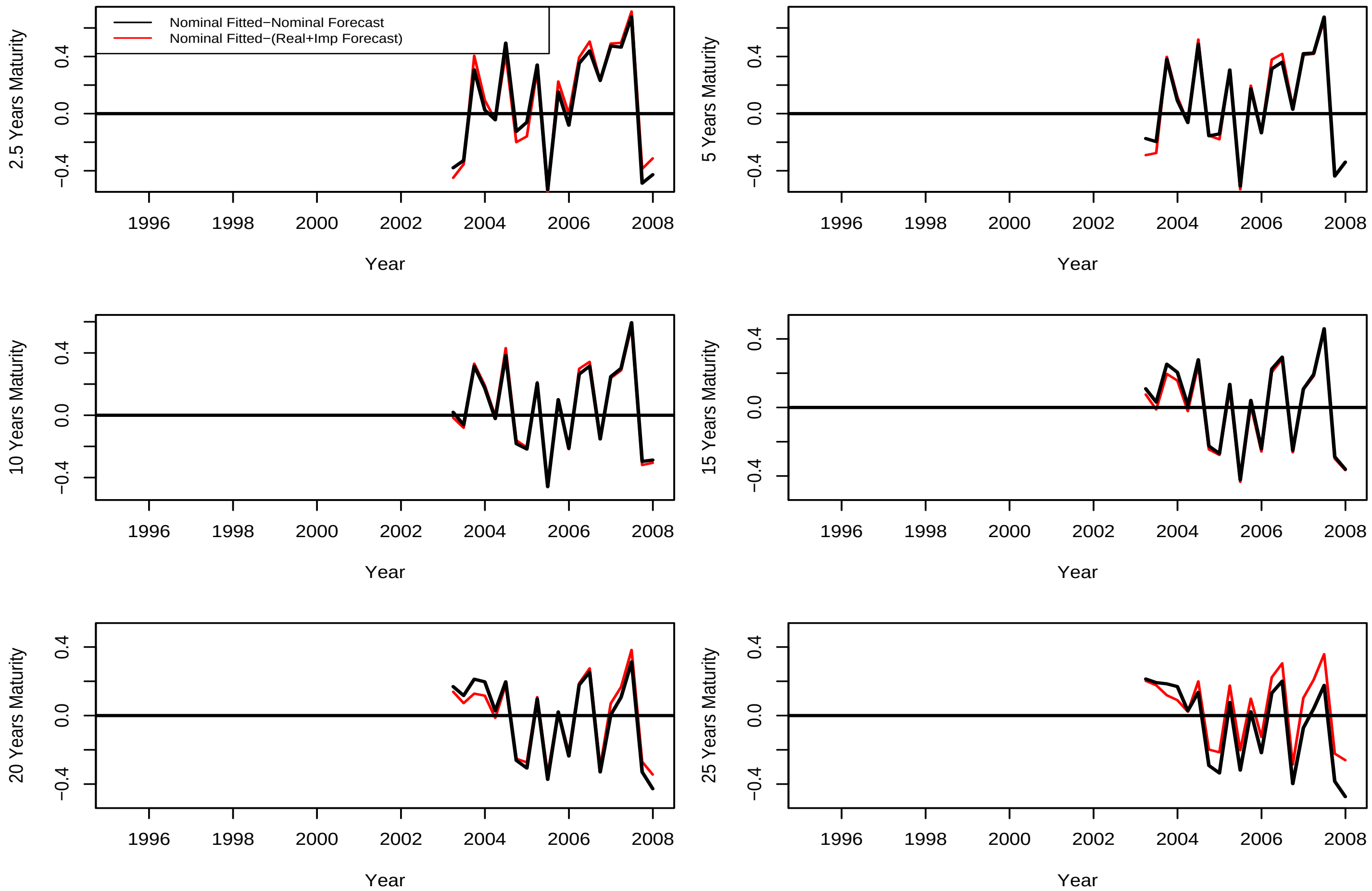

Figure 6.10: Errors for the Fisher Relation Check for the 1-Quarter Ahead Nominal Spot Rate Forecasts (%)

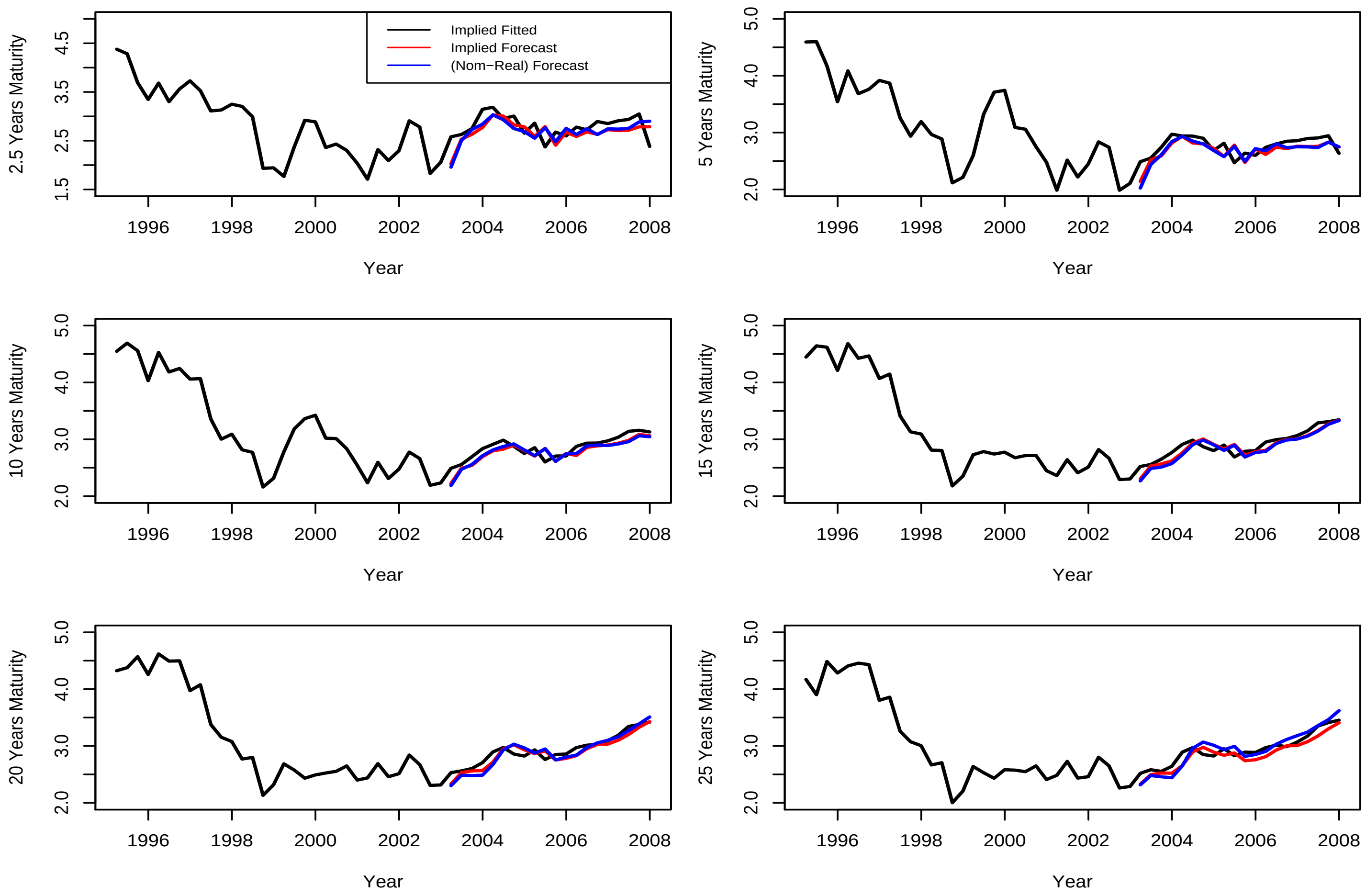

Figure 6.11: Fisher Relation Check for the 1-Quarter Ahead Implied Inflation Spot Rate Forecasts (%)

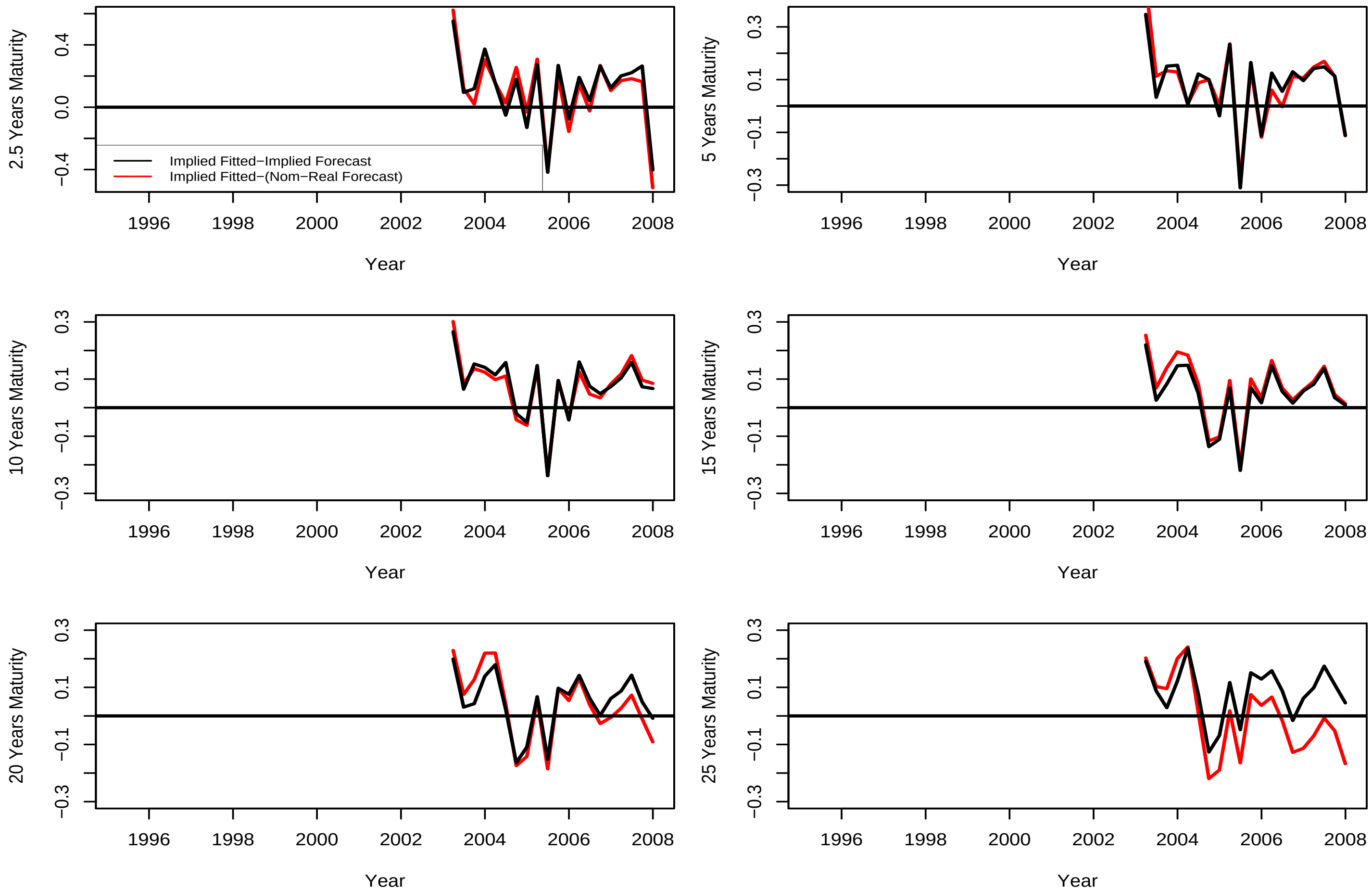

Figure 6.12: Errors for the Fisher Relation Check for the 1-Quarter Ahead Implied Inflation Spot Rate Forecasts (%)

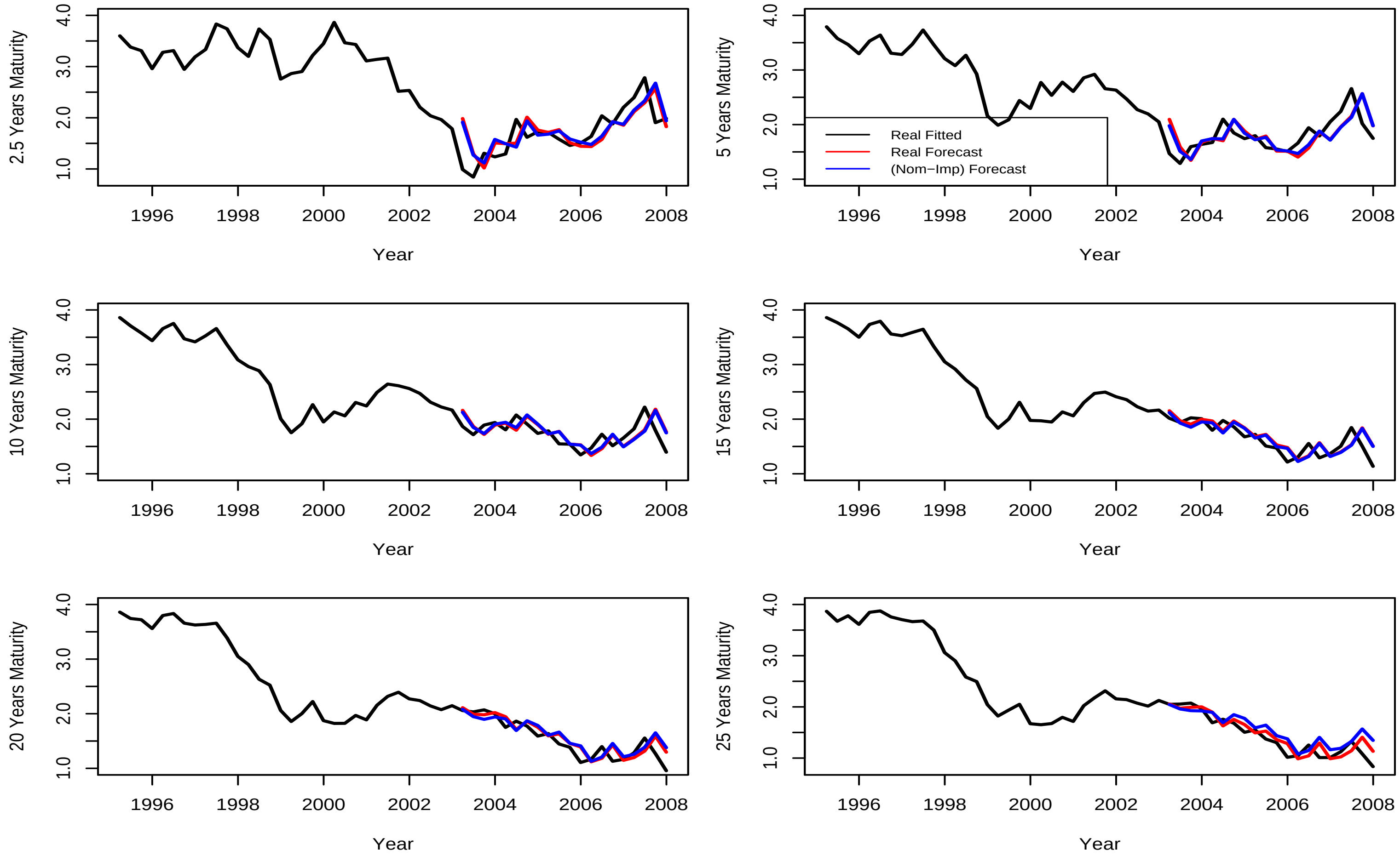

Figure 6.13: Fisher Relation Check for the 1-Quarter Ahead Real Spot Rate Forecasts (%)

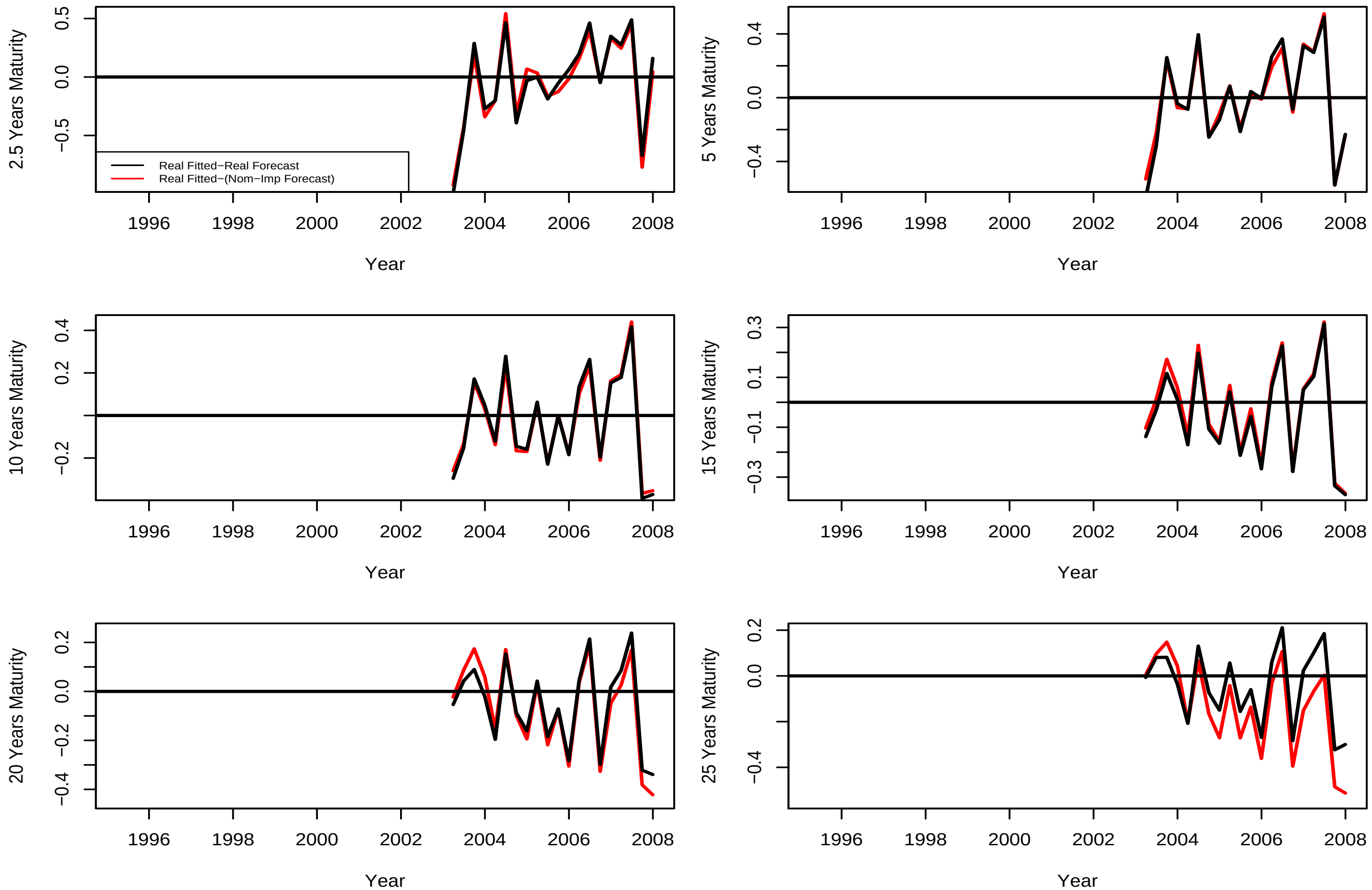

Figure 6.14: Errors for the Fisher Relation Check for the 1-Quarter Ahead Real Spot Rate Forecasts (%)

6.3 Yield-Macro Model-II

Considering previous studies on yield-macro models which we have discussed in Chapter 4, we expect that realised inflation would be involved in some of the PC models as an explanatory variable. However, our findings show that although there are some significant simultaneous and lagged correlations between the realised inflation and the yield curve factors, realised inflation has not been found significant in the models except for the curvature factor of the nominal rates. Although the macroeconomic theory suggests that the annual realised inflation should be connected with the level of the nominal spot rates, the data period and the frequency might affect this relation. Therefore we model the yield curves using yearly data to see whether we will discover such a relation between the yield curves and the annual realised inflation.

When we model yearly data we have only 25 observations for each month (from 1985 to 2009). We apply PCA on monthly data. Then we use June PCs for the yearly models and once we find the best model for each PC we apply that model to the other months and estimate the parameters. Therefore, we obtain 12 different set of parameters for the level factors at yearly frequency. Since we have very few data we only model the level factors of the yield curves. We try to fit some models to slope factors as well but the models have changed significantly when we change the month and most importantly the coefficients of determination are very low which indicate a poor fit. The curvature factors are some sort of white noise and we do not model them either. Therefore, we use only first PCs to derive the nominal, implied inflation and the real yield curves. Since we model the yearly level factors for each month, we obtain monthly yield curve data using 12 different yearly level factor models.

6.3.1 Correlations Between the Yearly Yield Factors and Realised Inflation

Table 6.10 and Table 6.11 show the simultaneous and the lagged correlations between the PCs and the annual realised inflation for the yearly data. The level factors have significant auto- and cross-correlations. They have high correlations with the realised inflation too. Since the realised inflation at time t has been defined as the difference between the logarithm of the RPI at time t and $t-1$ it is reasonable that the inflation at time t has an effect on the levels of the term structures. Accordingly, the high inflation in the previous year leads an increase in the level of the interest rates in the following year. Furthermore, there is a negative correlation between the slope factor of the implied inflation and the realised inflation. When the inflation is high the slope factor of the nominal spot rates in the following year decreases because the high inflation is followed by an increase in the short term interest rates while the long term interest rates are relatively stable. This produces a flatter slope factor which means that the slope factor decreases. Finally, none of the curvature factors have significant correlations.

Table 6.10: Lagged Correlations between the Yearly Yield Curve Factors (June Data) - I

NPC1[t] Lag, k	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	**1.000**	0.045	0.038	0.957	0.006	0.217	0.932	-0.064	-0.009	0.668
(1)	0.847	0.207	-0.136	0.776	0.085	0.036	0.850	0.071	-0.126	0.402
(2)	0.748	0.232	-0.016	0.646	-0.033	-0.067	0.817	0.145	-0.124	0.319
(3)	0.676	0.191	0.092	0.563	-0.072	0.168	0.760	0.045	-0.329	0.276

NPC2[t] Lag, k	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.045	**1.000**	0.189	0.027	0.721	-0.127	0.154	0.421	0.107	-0.573
(1)	0.151	0.477	0.163	0.045	0.183	-0.162	0.304	0.144	0.252	-0.162
(2)	0.123	0.147	-0.187	0.051	0.092	0.135	0.195	-0.051	-0.123	0.038
(3)	0.083	-0.115	0.051	0.025	-0.052	0.360	0.121	-0.290	-0.279	0.087

NPC3[t] Lag, k	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.038	0.189	**1.000**	0.100	-0.013	0.430	0.001	0.061	0.016	-0.167
(1)	0.093	0.078	-0.049	0.085	0.084	0.091	0.079	-0.044	0.151	0.056
(2)	0.108	0.317	-0.233	0.080	0.302	-0.190	0.150	0.134	0.208	-0.046
(3)	0.067	0.054	-0.132	-0.004	-0.151	-0.031	0.164	0.161	-0.069	0.008

IPC1[t] Lag, k	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.957	0.027	0.100	**1.000**	0.056	0.271	0.792	0.033	-0.010	0.683
(1)	0.858	0.198	-0.101	0.833	0.133	0.046	0.803	0.103	-0.070	0.436
(2)	0.825	0.213	-0.092	0.745	-0.005	-0.141	0.853	0.172	-0.027	0.414
(3)	0.753	0.226	0.070	0.666	-0.025	0.112	0.803	0.119	-0.298	0.328

IPC2[t] Lag, k	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.006	0.721	-0.013	0.056	**1.000**	-0.006	-0.034	-0.109	0.247	-0.391
(1)	0.049	0.556	0.169	0.010	0.482	-0.208	0.127	0.052	0.383	-0.347
(2)	0.049	0.284	-0.009	0.031	0.137	-0.060	0.089	0.172	0.078	-0.193
(3)	0.096	-0.019	0.152	0.096	-0.060	0.270	0.088	-0.017	-0.234	-0.010

Table 6.11: Lagged Correlations between the Yearly Yield Curve Factors (June Data)-II

IPC3[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.217	-0.127	0.430	0.271	-0.006	**1.000**	0.107	-0.285	-0.604	0.185
(1)	0.216	-0.223	-0.099	0.116	-0.059	0.067	0.259	-0.468	0.186	0.187
(2)	0.090	0.137	-0.349	0.011	0.001	-0.397	0.190	0.171	0.222	-0.133
(3)	-0.025	0.090	0.125	-0.034	-0.209	-0.173	0.038	0.386	-0.091	-0.183

RPC1[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.932	0.154	0.001	0.792	-0.034	0.107	**1.000**	-0.073	-0.022	0.511
(1)	0.754	0.208	-0.140	0.627	-0.008	0.018	0.839	0.047	-0.171	0.332
(2)	0.593	0.222	0.038	0.468	-0.054	0.055	0.704	0.069	-0.246	0.207
(3)	0.516	0.122	0.102	0.379	-0.108	0.243	0.636	-0.088	-0.343	0.199

RPC2[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	-0.064	0.421	0.061	0.033	-0.109	-0.285	-0.073	**1.000**	-0.113	-0.240
(1)	0.152	-0.085	0.136	0.178	-0.303	0.011	0.124	0.289	-0.045	0.294
(2)	0.273	-0.324	-0.339	0.274	-0.085	0.193	0.189	-0.261	-0.143	0.579
(3)	0.192	-0.080	-0.128	0.200	0.187	0.073	0.127	-0.294	-0.012	0.302

RPC3[t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	-0.009	0.107	0.016	-0.010	0.247	-0.604	-0.022	-0.113	**1.000**	-0.018
(1)	-0.103	0.403	-0.153	0.040	0.390	-0.145	-0.219	0.508	0.073	-0.254
(2)	-0.020	0.071	0.222	0.088	0.057	0.011	-0.125	0.233	0.064	0.025
(3)	0.179	-0.286	-0.199	0.215	-0.139	0.198	0.075	-0.118	-0.114	0.351

Realised Inflation [t] **Lag, k**	**NPC1 [t-k]**	**NPC2 [t-k]**	**NPC3 [t-k]**	**IPC1 [t-k]**	**IPC2 [t-k]**	**IPC3 [t-k]**	**RPC1 [t-k]**	**RPC2 [t-k]**	**RPC3 [t-k]**	**Realised Inflation [t-k]**
(0)	0.668	-0.573	-0.167	0.683	-0.391	0.185	0.511	-0.240	-0.018	**1.000**
(1)	0.478	-0.066	-0.243	0.513	0.151	0.128	0.358	-0.121	-0.197	0.370
(2)	0.406	0.119	-0.058	0.376	0.090	-0.274	0.407	0.038	0.098	0.164
(3)	0.383	0.299	0.065	0.355	0.139	-0.123	0.408	0.175	-0.057	0.045

6.3.2 Modelling the Yearly PCs

By using the yearly data we model 12 different sets of yield curve data from January to December. As we have mentioned earlier, once we decide the best model for each PC we apply the same model to different months and estimate the parameters. Due to the few number of observations at yearly intervals we could find significant correlations for level factors only and thus we model these factors.

Let X_Y be the matrix of yearly yield curve data for June from 1985 to 2009 where:
X_{Y_N}: Nominal spot rates (25×50)
X_{Y_I}: Implied inflation spot rates (25×46)
X_{Y_R}: Real spot rates (25×46)

Let Y be the matrix of the yearly PCs where:
Y_{N_L}: level component of the nominal spot rates (25×1)
Y_{I_L}: level component of the implied inflation spot rates (25×1)
Y_{R_L}: level component of the real spot rates (25×1)

and
Y_{RI}: annual realised inflation (25×1)

The structure of the yearly yield-macro model is:

$$Y[t] - \mu_Y = C_1 \, (Y[t-1] - \mu_Y) + C_2 \, Y_{RI}[t] + \epsilon_Y[t] \tag{6.1}$$

where:

μ_Y is the vector of long run mean of the variables, C_1 is the coefficient matrix for the first lag of the explanatory variables, C_2 is the coefficient matrix of the realised inflation and $\epsilon_Y[t] \sim N(0, \Sigma_Y)$, i.e. normally distributed residuals with zero mean and Σ_Y variance-covariance matrix. In this model, since we could not find a good model for the realised inflation using the level factors due to short period of data, we use realised inflation as an exogenous variable.

$$Y = \begin{bmatrix} Y_{N_L} \\ Y_{I_L} \\ Y_{R_L} \end{bmatrix} \tag{6.2}$$

$$\widehat{\mu}_Y^t = \begin{bmatrix} -44.18 & -17.22 & -28.05 \end{bmatrix} \tag{6.3}$$

$$\widehat{C}_1 = \begin{bmatrix} 0.82 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 0.91 \end{bmatrix} \tag{6.4}$$

$$\widehat{C}_2 = \begin{bmatrix} 1.86 & 0 & 0 \\ 0 & 1.03 & 0 \\ 0 & 0 & 0.55 \end{bmatrix} \tag{6.5}$$

$$\widehat{\Sigma}_Y = \begin{bmatrix} 12.82 & & \\ 5.38 & 6.30 & \\ 6.99 & -0.53 & 7.34 \end{bmatrix} \tag{6.6}$$

We display the correlation matrix, $\widehat{\rho}_Y$, for the residuals below. As explained in the previous section, we assume that the coefficients which are greater or less than three standard errors (0.60) are significant. Therefore, the residuals of the level factors are significantly positively correlated which can be explained by excluding the simultaneous correlations in the modelling work.

$$\widehat{\rho}_Y = \begin{bmatrix} 1.00 & & \\ \mathbf{0.60} & 1.00 & \\ \mathbf{0.71} & -0.08 & 1.00 \end{bmatrix} \tag{6.7}$$

6.3.3 Parameter Estimates for Different Months

Since we use 12 different sets of yearly data to model the level factors of the yield curves, we estimate the parameters for each model. Table 6.12 displays these parameters for each model and each month. It should be emphasised that all three parameters (long term mean, the coefficients of the first lag of the level factors and realised inflation) are significantly different from zero for each month. This is one of the reasons that we cannot model the slope components of the yield curves. We cannot find a common model for every month which fits the data well enough.

Table 6.12 shows that the autoregressive parameter of the nominal level factor changes between 0.76 and 0.84. Considering the standard error of the estimation which we present in Appendix C, the differences between the parameter value for month June with the parameter values for some other months in Table 6.12 are not high but just above three standard errors. Therefore, we might think that although the change in the parameter value is not big it might be significant. When we look at the other autoregressive parameters for the implied and the real level factors we reach a similar conclusion: the changes in the parameter values are small but significant. Therefore we need to use different parameters for different months in our further analysis.

We do not present a model for annual realised inflation in this chapter because we already have a model for the realised inflation at yearly frequency discussed in Chapter 1, i.e. the Wilkie model of price inflation.

6.3.4 Model Comparisons

We also compare our models with the RW and AR(1) processes to see how much the realised inflation contributes to the explained variability in the data. Table 6.13

Table 6.12: Parameter Estimates for the Yearly Models

Month	**Nominal Level**			**Implied Level**			**Real Level**		
	$\mu_{Y_{N_L}}$	Y_{N_L} $(t-1)$	$Y_{RI}(t)$	$\mu_{Y_{I_L}}$	Y_{I_L} $(t-1)$	$Y_{RI}(t)$	$\mu_{Y_{R_L}}$	Y_{R_L} $(t-1)$	$Y_{RI}(t)$
January	-37.96	0.78	1.85	-16.45	0.73	0.99	-28.05	0.89	0.68
February	-39.41	0.81	1.59	-15.44	0.70	1.06	-28.05	0.91	0.58
March	-39.78	0.78	2.02	-15.30	0.60	1.54	-28.05	0.89	0.66
April	-38.24	0.76	2.27	-15.53	0.61	1.57	-28.05	0.91	0.57
May	-40.18	0.80	1.85	-15.46	0.73	1.00	-28.05	0.90	0.60
June	-44.18	0.82	1.86	-17.22	0.75	1.03	-28.05	0.91	0.55
July	-45.70	0.84	1.63	-18.01	0.81	0.77	-28.05	0.90	0.60
August	-45.15	0.82	1.94	-18.83	0.75	1.12	-28.05	0.90	0.60
September	-44.18	0.84	1.54	-16.96	0.78	0.86	-28.06	0.91	0.50
October	-44.18	0.84	1.50	-13.91	0.78	0.64	-28.05	0.87	0.82
November	-44.18	0.84	1.49	-13.38	0.75	0.68	-24.50	0.84	0.91
December	-39.84	0.79	1.89	-16.85	0.67	1.32	-28.05	0.90	0.57

shows the increased percentage in the explained variability for the level factors of the yield curves. Accordingly, our models perform much better than particularly the RW models increasing the explained variability up to 50% for some months. The nominal level factor model is significantly superior to both RW and AR(1) process for every months. Implied inflation level factor model performs much better than the RW model while real level factor is still better but not by as much as the implied inflation or nominal level factor models. Therefore, as the previous studies indicate there is a significant correlation between the level factors of the yield curves and this helps to improve the yield curve modelling.

6.3.5 Residual Analysis

Table 6.14 and Table 6.15 show some descriptive statistics such as mean, standard deviation, skewness and excess kurtosis along with the Jarque-Bera test results of the residuals for each level factors for each month. All mean values are zero, while skewness and excess kurtosis values are either negative or positive for different set of residuals. Jarque-Bera test p-values are quite high indicating normally distributed residuals apart from the nominal level factor for October and December.

Table 6.13: Model Comparisons for the Yearly Level Factors

Month	**Nominal Level**		**Implied Level**		**Real Level**	
	$R^2_{RW^*}$	$R^2_{AR(1)^*}$	$R^2_{RW^*}$	$R^2_{AR(1)^*}$	$R^2_{RW^*}$	$R^2_{AR(1)^*}$
January	0.30	0.19	0.30	0.12	0.16	0.10
February	0.32	0.18	0.35	0.15	0.12	0.08
March	0.33	0.23	0.35	0.20	0.14	0.08
April	0.45	0.37	0.42	0.29	0.18	0.13
May	0.46	0.37	0.36	0.21	0.20	0.13
June	0.57	0.49	0.48	0.36	0.18	0.11
July	0.49	0.40	0.36	0.23	0.16	0.11
August	0.43	0.35	0.43	0.34	0.14	0.08
September	0.31	0.22	0.25	0.13	0.21	0.12
October	0.34	0.24	0.18	0.04	0.29	0.23
November	0.27	0.15	0.13	0.00	0.24	0.18
December	0.30	0.19	0.29	0.17	0.15	0.06

Table 6.14: Residual Analysis of the Yearly Yield Curve Models-II

January			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	5.5686	3.6069	2.9532
Skewness	-0.5702	-0.1551	-0.3223
Kurtosis	0.5906	-0.5350	0.2557
JB p-value	0.3159	0.9007	0.6807
February			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	4.8364	3.5561	2.8643
Skewness	-0.6652	-0.3947	0.0510
Kurtosis	0.4341	-0.6374	-0.8256
JB p-value	0.2784	0.6405	0.8139
March			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	5.4153	4.3605	3.1737
Skewness	0.1263	-0.1128	0.3793
Kurtosis	-0.3498	-0.3517	-1.0924
JB p-value	0.9581	0.9650	0.4710
April			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	5.1137	4.0898	2.5343
Skewness	0.2188	-0.3222	0.3447
Kurtosis	-0.2457	-0.4462	-0.8144
JB p-value	0.8969	0.7711	0.6299
May			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	4.2938	3.2743	2.6232
Skewness	0.2561	-0.2298	0.3135
Kurtosis	-0.9562	-0.4958	-0.7762
JB p-value	0.6383	0.8544	0.6766
June			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	3.5658	2.5012	2.7511
Skewness	-0.0912	-0.0199	0.4997
Kurtosis	-0.5373	-0.6600	-0.2442
JB p-value	0.9327	0.9020	0.5670

Table 6.15: Residual Analysis of the Yearly Yield Curve Models-II

July			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	3.6478	2.3898	3.0049
Skewness	-0.4022	0.7771	0.1182
Kurtosis	-0.1806	-0.2771	-0.4791
JB p-value	0.6908	0.2534	0.9377
August			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	4.7098	2.6768	3.3036
Skewness	-0.3433	0.0285	0.6269
Kurtosis	0.5626	0.3962	0.0011
JB p-value	0.5198	0.7824	0.3950
September			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	5.1341	3.5724	2.3200
Skewness	-0.4518	0.0412	-0.5572
Kurtosis	0.5391	-0.4383	0.6116
JB p-value	0.4369	0.9743	0.3197
October			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	4.5535	3.7032	2.8311
Skewness	-1.0146	0.0657	-0.1133
Kurtosis	1.5436	-1.1645	0.6731
JB p-value	0.0145	0.5997	0.5894
November			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	4.9912	4.5728	3.4243
Skewness	-0.5489	0.1121	0.1035
Kurtosis	0.8782	0.3788	1.5977
JB p-value	0.2388	0.7718	0.1304
December			
	Nom Level	Imp Level	Real Level
Mean	0.0000	0.0000	0.0000
SD	5.5863	3.8353	2.9668
Skewness	-0.9415	-0.3008	-0.7189
Kurtosis	1.9310	-0.9199	1.0099
JB p-value	0.0081	0.6214	0.1215

6.3.6 Forecasting

Similar to the monthly and quarterly yield-curve analysis we forecast one-year ahead spot rates using the level factor models for each yield-curve. We also calculate the 95% confidence intervals based on normally distributed residuals.

Figure 6.15, Figure 6.16 and Figure 6.17 show one-year ahead forecasts with 95% confidence bands for the nominal, implied inflation and the real spot rates respectively. Although the one-year ahead forecasts seem like RW forecasts we show that the models explain significantly more variability in the data with the help of realised inflation. The forecast graphs also show that the short and long end of the yield curves indicate poor forecasts relative to the medium term.

Table 6.16 shows the number and the ratio of the spot rates which are not within the upper and lower confidence bands for different maturities for the nominal, implied inflation and the real spot rates. The number of the spot rates out of the interval are high at both end of the nominal yield curve while it has been mostly decreasing for the real and implied inflation yield curves. Since we use yearly data with a few number of observations and we use only level factor to derive the yield curves back, we do not expect that the forecasts are as good as the ones we obtain using the monthly (yield-only) or quarterly (yield-macro-I) models. Furthermore, excluding two factors for each yield curve changes the variances and thus affects the width of the confidence intervals for different maturities. Therefore, the relatively poor forecasts for the nominal spot rates and implied inflation (12% and 8% of the data are out of the confidence bands for the nominal and implied inflation yield curves respectively) can be explained by these facts.

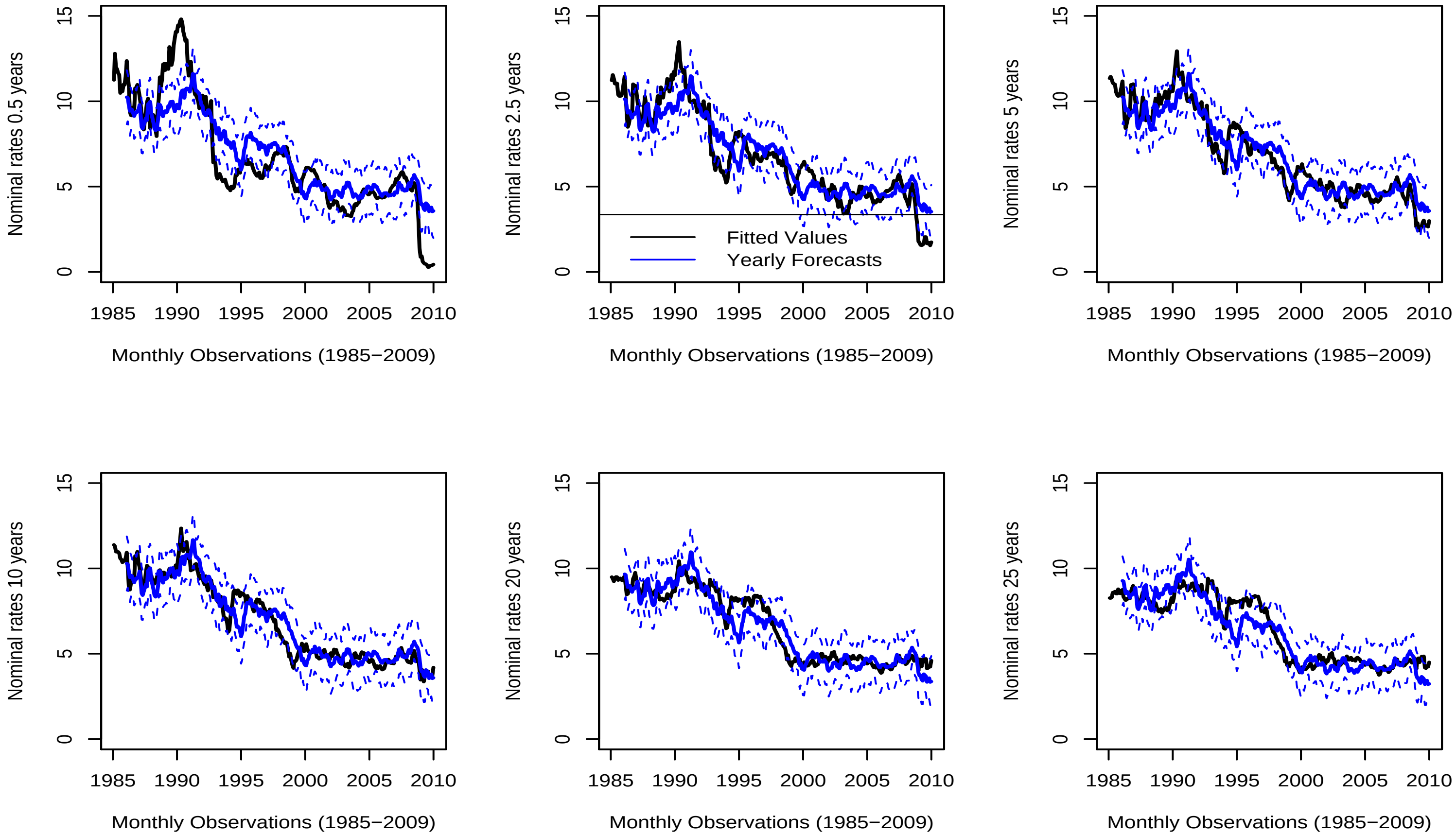

Figure 6.15: 1-Year Ahead Forecasts with Upper and Lower Confidence Limits for Nominal Spot Rates (%)

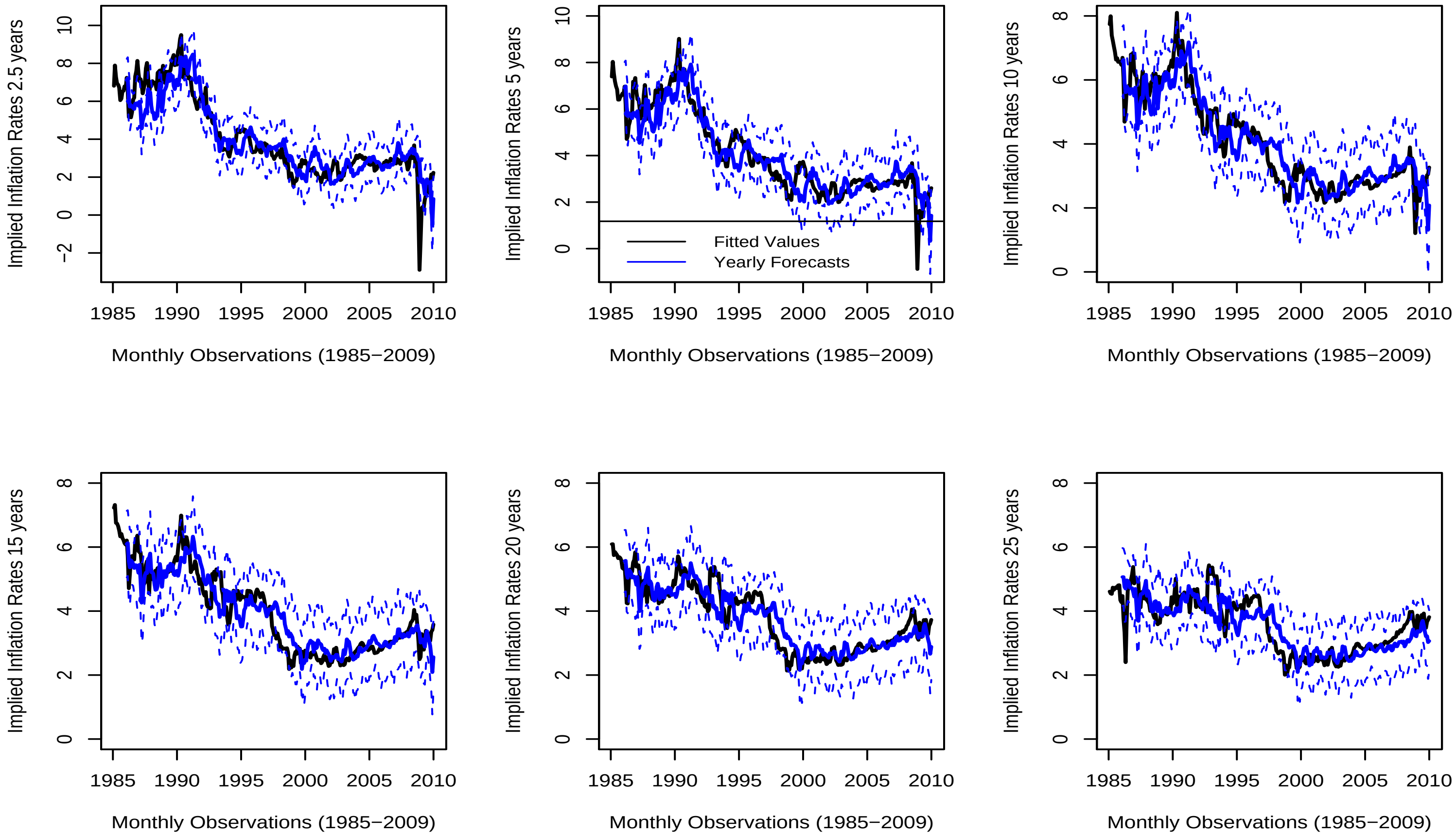

Figure 6.16: 1-Year Ahead Forecasts with Upper and Lower Confidence Limits for Implied Inflation Spot Rates (%)

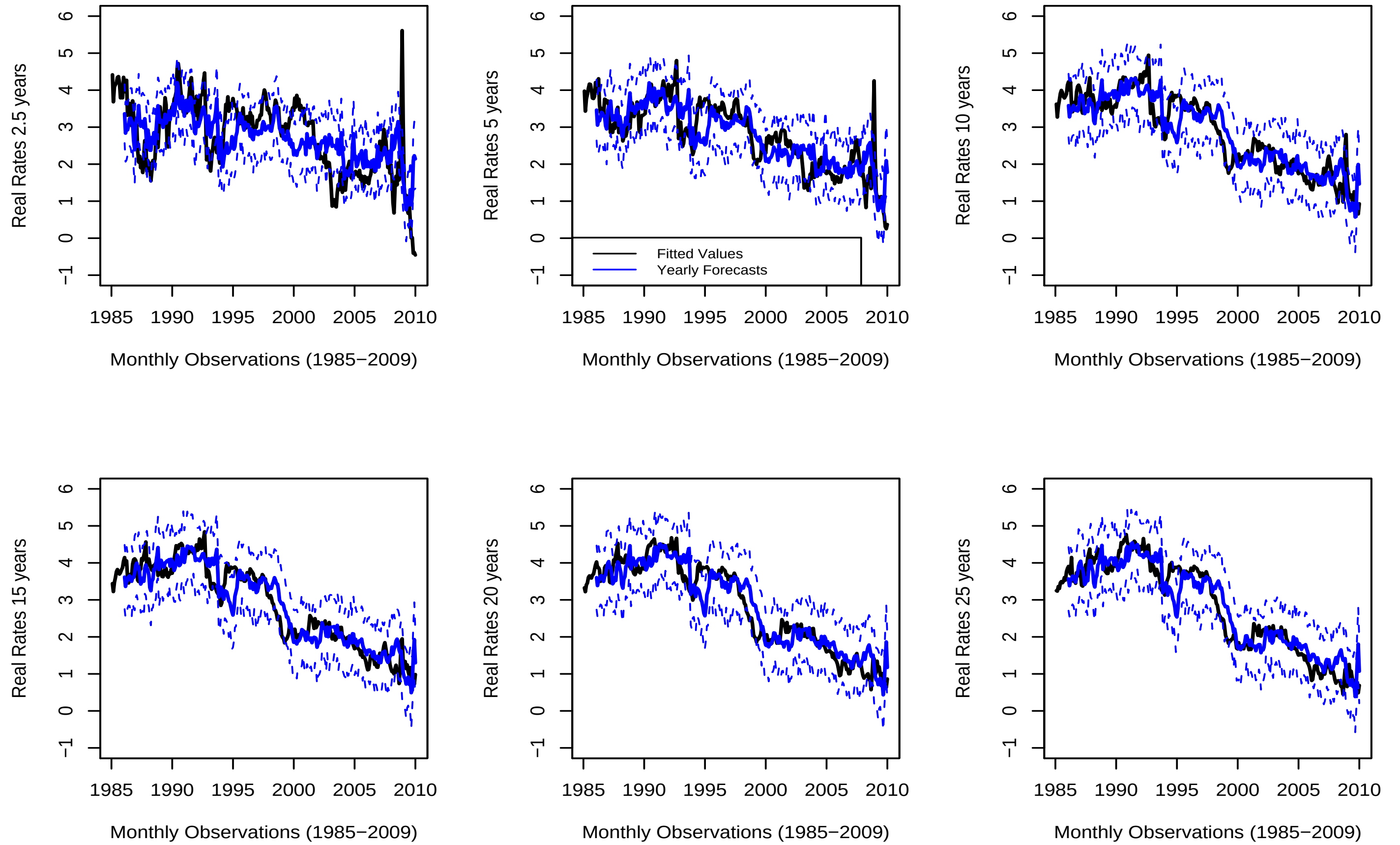

Figure 6.17: 1-Year Ahead Forecasts with Upper and Lower Confidence Limits for Real Spot Rates (%)

Table 6.16: Number and the Ratio of the Observations Outside of the 95% Confidence Bounds for the 1-Year Ahead Forecasts

Maturity	Nominal		Implied Inflation		Real	
	Number	Ratio	Number	Ratio	Number	Ratio
0.5	96	0.333				
1	87	0.302				
1.5	78	0.271				
2	71	0.247				
2.5	68	0.236	53	0.184	49	0.170
3	66	0.229	53	0.184	37	0.128
3.5	60	0.208	43	0.149	24	0.083
4	53	0.184	43	0.149	20	0.069
4.5	48	0.167	42	0.146	18	0.063
5	43	0.149	38	0.132	15	0.052
5.5	38	0.132	39	0.135	15	0.052
6	36	0.125	38	0.132	14	0.049
6.5	31	0.108	37	0.128	14	0.049
7	30	0.104	36	0.125	12	0.042
7.5	31	0.108	37	0.128	11	0.038
8	28	0.097	37	0.128	11	0.038
8.5	27	0.094	34	0.118	12	0.042
9	27	0.094	31	0.108	12	0.042
9.5	26	0.090	27	0.094	11	0.038
10	26	0.090	23	0.080	11	0.038
10.5	25	0.087	22	0.076	12	0.042
11	25	0.087	20	0.069	12	0.042
11.5	23	0.080	18	0.062	12	0.042
12	23	0.080	16	0.056	13	0.045
12.5	21	0.073	14	0.049	14	0.049
13	19	0.066	13	0.045	14	0.049
13.5	20	0.069	12	0.042	15	0.052
14	20	0.069	12	0.042	14	0.049
14.5	19	0.066	12	0.042	14	0.049
15	19	0.066	11	0.038	14	0.049
15.5	19	0.066	12	0.042	14	0.049
16	19	0.066	12	0.042	14	0.049
16.5	20	0.069	12	0.042	14	0.049
17	21	0.073	13	0.045	14	0.049
17.5	22	0.076	15	0.052	14	0.049
18	23	0.080	13	0.045	14	0.049
18.5	23	0.080	14	0.049	14	0.049
19	24	0.083	15	0.052	14	0.049
19.5	26	0.090	14	0.049	14	0.049
20	27	0.094	15	0.052	14	0.049
20.5	28	0.097	15	0.052	16	0.056
21	28	0.097	17	0.059	17	0.059
21.5	28	0.097	18	0.062	17	0.059
22	29	0.101	18	0.062	17	0.059
22.5	31	0.108	18	0.062	17	0.059
23	31	0.108	20	0.069	18	0.063
23.5	33	0.115	22	0.076	19	0.066
24	36	0.125	24	0.083	20	0.069
24.5	39	0.135	25	0.087	20	0.069
25	41	0.142	25	0.087	21	0.073
Average		**0.120**		**0.083**		**0.056**

6.3.7 Fisher Relation

Figure 6.18, Figure 6.19 and Figure 6.20 show the fitted spot rates (black solid lines), one-year ahead forecasts (red solid lines) and the forecasts obtained using Fisher relation (blue solid line) for different maturities for the nominal, implied inflation and the real yield curves separately. Different from the monthly and quarterly models, Fisher relation does not hold for the one-year ahead forecasts. Possible reasons are the poor forecasts due to few data and using only first principal component to derive the yield curves.

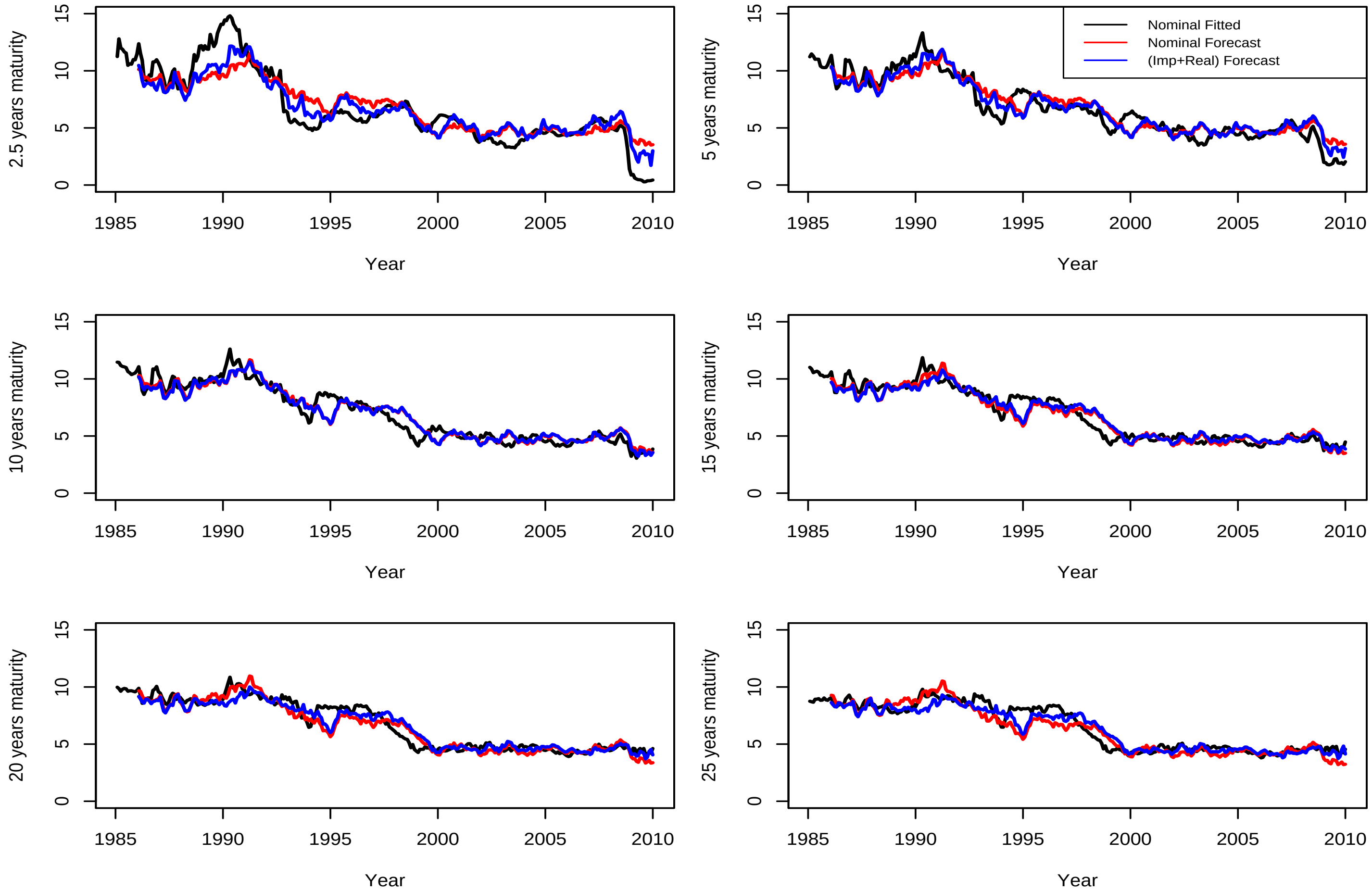

Figure 6.18: Fisher Relation Check for the 1-Year Ahead Nominal Spot Rate Forecasts (%)

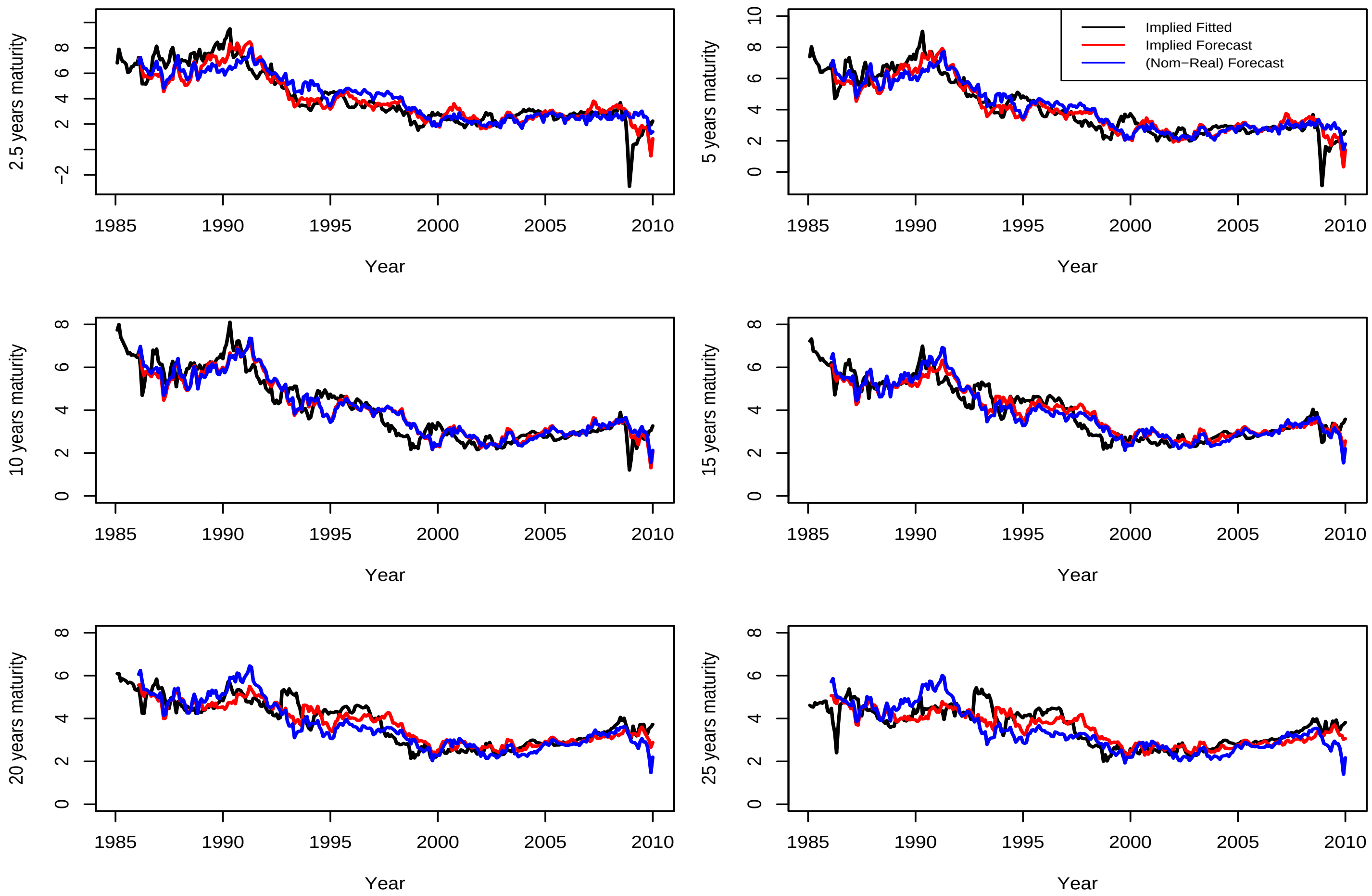

Figure 6.19: Fisher Relation Check for the 1-Year Ahead Implied Inflation Spot Rate Forecasts (%)

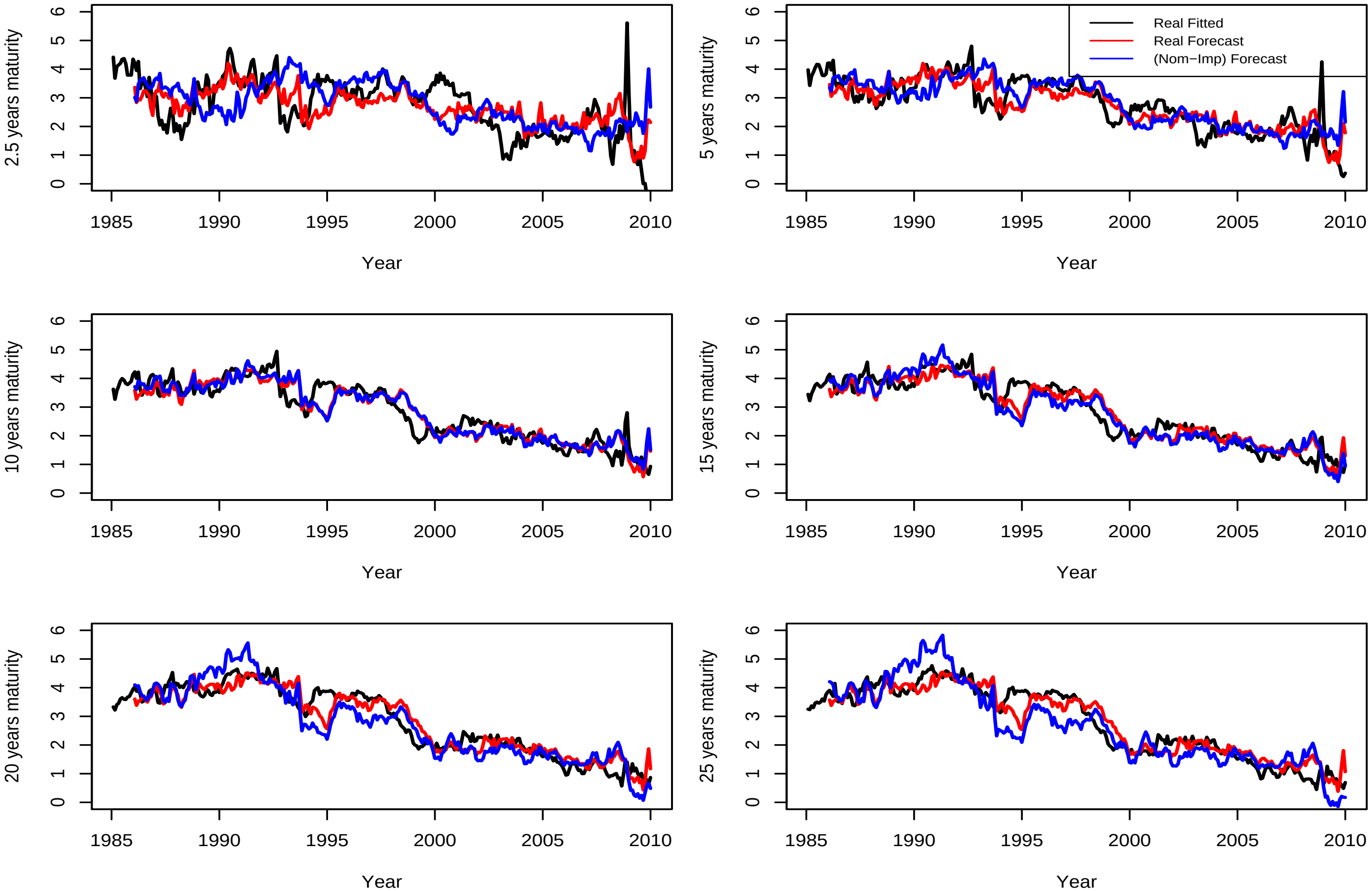

Figure 6.20: Fisher Relation Check for the 1-Year Ahead Real Spot Rate Forecasts (%)

6.4 Interim Conclusion: The Yield-Macro Models

In this chapter we have presented two 'yield-macro' models using quarterly and annual yield curve and macroeconomic variables. First we have discussed the quarterly yield macro model. According to our analysis, the macro variables and the yield curve factors are significantly correlated. Although the level factors of the yield curves are modelled as AR(1) processes, the macro variables have been significant in slope and curvature factor models. Moreover, the yield curve factors also improve the models for realised inflation and output gap. Thus, we have found a bi-directional relation between the yield curves and the macroeconomic variables. When we consider the yearly model proposed in the second half of the chapter, we see that the annual realised inflation has been found significant in the level factor models. Accordingly, the increase in the inflation leads to an increase in the level factors of the yield curves. Due to having short period of data we could not model the other factors of the yield curves at yearly intervals and this affects the forecasting performance of the models. On the other hand, although the forecasts seem close to random walk forecasts, our models perform better than the random walk and AR(1) process in terms of the explained variability in the data.

Chapter 7

Comparison of the Wilkie Model and the Yield-Macro Model

7.1 Introduction

In this chapter we compare the Wilkie model with the quarterly yield-macro model in two ways. First we compare these two models in a philosophical way. We discuss the structures of the models by considering the economic series they cover, the period examined and the nature of the relation between these economic series. Secondly, we compare the models in a variety of empirical ways. We start with the comparison of the simulated series using the models. For this analysis we only use the common economic series such as inflation, bank base rates, consols yields and nominal spot rates. We compare the total nominal and real returns obtained from 1000 simulations for each model. Finally, we consider a hypothetical pension scheme and compare the real asset values along with the annuity payoffs for different investment scenarios.

Section 7.2 discusses the structural comparison and Section 7.3 and Section 7.4 present the empirical analysis. Finally, Section 7.5 concludes the chapter.

7.2 Structural Comparison of the Models

The frequency of the data used in the models is an important feature which distinguishes the models. The Wilkie model has been constructed on yearly data while the yield-macro model is based on quarterly data. The reason for using quarterly data for our yield-macro model is that the output gap is available on a quarterly frequency. Although we develop different models based on the monthly and the yearly intervals, we would like to compare the Wilkie model with our quarterly yield-macro model because it includes all the variables we intend to model.

The historical data for the series used in the Wilkie model have been available since the 1900s while the term structures of the interest rates and implied inflation and output gap data are available since the 1980s. Using different periods of data for the two models affects the parameters estimated due to different economic conditions experienced in those periods. This also affects the simulations produced for the future years. In order to make the two models exactly comparable, we will introduce 'neutral' initial conditions and 'neutralised' parameters for the models and we will use the same initial values for our state variables to simulate the future in the next sections.

Another distinguishing feature is the output variables the models produce. Figure 7.1 and Figure 7.2 display the structures of the Wilkie model and the yield-macro model respectively. When we look at Figure 7.1 we see that the Wilkie model has a cascade structure and that price inflation is the driving force as has been discussed in Chapter 1. It includes wage inflation, share dividend yields, share dividends, share prices, long term and short term interest rates and index-linked yields. On the other hand, the yield macro model in Figure 7.2 is composed of the term structures of nominal, implied inflation and real spot rates along with the realised inflation and the output gap as macroeconomic variables. Thus, while we exclude the share dividends, dividend yields and share prices and also wage inflation we incorporate two new variables namely implied inflation and the output gap. Additionally, we model the entire term structures rather than just the two ends of the yield curves.

Incorporating new variables has also changed the structure of the model. The price inflation is not the driving force of the yield-macro model because the output gap and the nominal spot rates have influences on it. Thus, we can see that the use of different variables not only changes the structure of the models but also changes the nature of the relations between the model variables. One of the main features of the yield curve models proposed in this work is the bi-directional relations between the yield-curve factors and the macroeconomic variables.

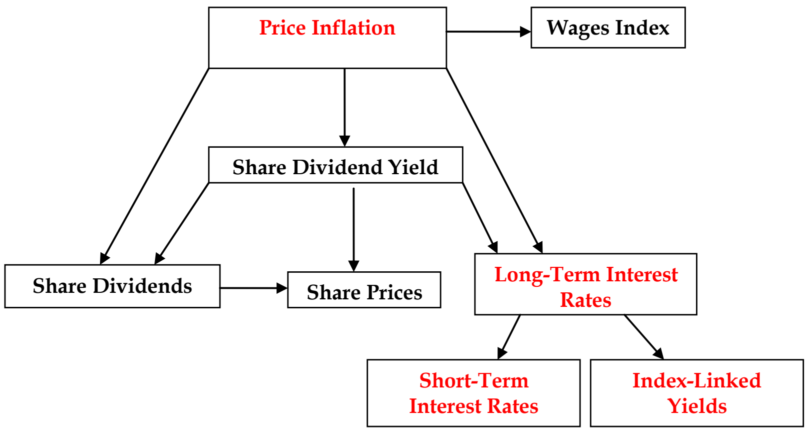

Figure 7.1: Structure of the Wilkie model

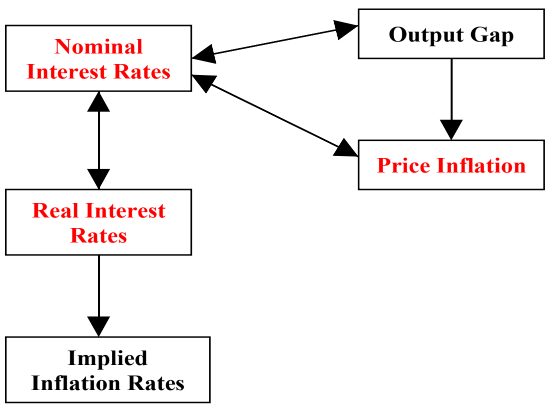

Figure 7.2: Structure of the Yield-Macro model

When we consider the similarities between these two models, besides indicating some common factors such as price inflation, nominal and index-linked yields we might go further and associate particular variables with the factors used in the yield-macro model. To begin with, both models include the nominal interest rates. The consols yield in the Wilkie model can be considered as an equivalent of the 'level' and the 'log spread', $BD(t) = \ln C(t) - \ln B(t)$, as the equivalent of the 'slope' of the nominal yield curve in the yield-macro model. However, we additionally include the 'curvature' factor of the nominal spot rates in our model.

It is possible to discuss the model formulae too. While the nominal slope factor $BD(t)$ has been modelled as an AR(1) process in the Wilkie model, the real curvature factor and output gap have been found significant in the nominal slope model as a part of the yield-macro model. Including two more explanatory variables we see that our model performs significantly better than the AR(1) model of Wilkie.

Wilkie's index-linked yield model might be compared with the 'real level factor' model of the yield-macro model. Wilkie (1995) models the index-linked yields including the residuals obtained from the consols yield model. This is consistent with the significant correlation between the residuals of the level factors of the nominal and real spot rates which has been presented in Chapter 6 and Appendix B.

7.3 Empirical Comparisons of the Models

7.3.1 Simulated Economic Series

In this section we compare the Wilkie model and the yield-macro model considering the inflation models, long-term and short-term interest rates and nominal spot rates. To begin with, we simulate the inflation index for 1000 years to study the long run auto-correlation functions of the stationary components of the models. Figure 7.3 shows the auto- and partial auto-correlation functions of the historical data and the simulated values for the two models over 1000 years in future. The auto-correlation functions decay at different speeds for each model. The auto- and partial auto-correlation functions of the historical data and the simulated values using Wilkie model look similar while the yield-macro model differs showing the first and third lags significant in the partial auto-correlation function. Besides, the auto-correlation function for the yield-macro model decays much slower than the auto-correlation functions of the other two data sets. Since the price inflation model of Wilkie is a strict AR(1) process it has a continuously decreasing auto-correlation function and only the first lag is significant in the partial auto-correlation function. On the other hand, the price inflation part of the yield-macro model incorporates some other factors namely the nominal curvature factor and the output gap as well as depending on its previous value. The nominal curvature factor is an AR(2) process including the price inflation as an explanatory variable as well. Thus relatively complex structure of the yield-macro model produces an auto-correlation function decreasing first, then increasing a little bit and then decreasing again. The significant partial auto-correlation values for the first and the third lags are caused by the structure of the model.

Although we forecast the values in Figure 7.3 by simulation, it is also possible to calculate them theoretically. The calculations are straight forward for the Wilkie model whereas many matrix multiplications are required for the yield-only model.

Since the two models are constructed based on different periods the estimated parameters are quite different from each other due to having been affected by the economic conditions of those periods. For example the long-run mean of the Wilkie price inflation model, QMU, is about 4.3% while it is equal to 2.88% for the yield-macro model. All the other means and the standard deviations are different as well. Thus, if we use the model parameters, particularly the means, as they are it is unavoidable that we would find very different economic scnearios for the two models.

All time series models need some initial conditions, that is values of the state space at time $t = 0$. Except in some special cases, the choice of initial conditions affects the short-term properties of the simulations. It is convenient therefore to start with 'unbiased' initial conditions. These unbiased initial conditions are what Wilkie (1995) and Lee and Wilkie (2000) call 'neutral' initial conditions. For a linear model, these

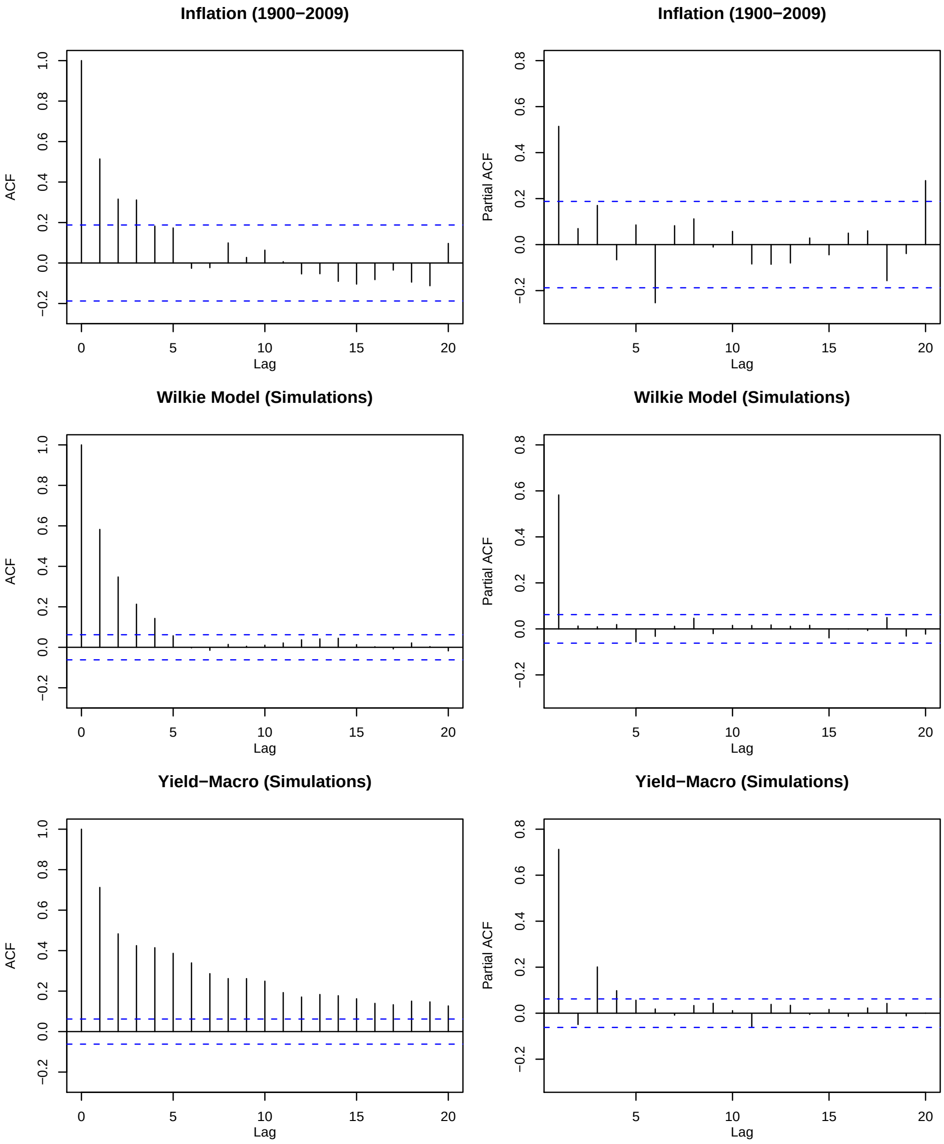

Figure 7.3: Autocorrelation functions for the historical and simulated price inflation rates

neutral conditions might be the means and for non-linear models these might be long-run expected values, or alternatively, long-run medians. It may also be interesting to see the effect of biased initial conditions, or market condition on a particular date but we do not do this here.

In order to make the two models, the Wilkie model and the yield-macro model, exactly comparable we introduce some 'neutral' initial conditions and 'neutralised' parameters (Lee and Wilkie, 2000). To begin with, we use 'neutral' initial conditions for the yield-macro model by setting the starting values at their long-run means. We obtain these long-run means by setting the standard deviations at zero. By using the neutral initial conditions for the yield-macro model we derive the zero coupon yield curves. Converting the initial zero-coupon yield curve into the par yield curve gives us the initial values for the long-term and short-term bond yields of the Wilkie model. Thus we use the same initial conditions for both models. However, while those initial conditions are neutral starting values for the yield-macro model, they are not neutral for the Wilkie model. Therefore, we adjust (or 'neutralise' (Lee and Wilkie, 2000)) the mean parameters of the Wilkie model according to the initial conditions so that those initial conditions would be neutral for the parameter-adjusted Wilkie model.

For the inflation model the initial value for the yield-macro model is the long-run mean and we use that value as the initial condition for Wilkie's inflation model. When we set the standard deviation of the Wilkie model to zero, we see that the initial condition becomes the long-run mean of the Wilkie inflation model as well. Therefore, for the inflation models, both the initial conditions and the mean parameters are the same and equal to 2.88%. We have done the same for the yield curves too. Note that we start with the same initial conditions for the two models and we adjust only the mean parameters of the Wilkie model based on these initial conditions.

After all these adjustments we can now compare these two models empirically. The economic series have been simulated for the next 35 years in this application.

Once we derive the RPI values after simulating the inflation values for both models we plot the empirical cumulative distribution functions (ECDF) for specific years to compare the distributions of the simulated values in Figure 7.4. Since the Wilkie inflation model has a higher standard deviation which has been caused by the data period including some extreme values, the distribution of the RPI values are more dispersed than the values obtained from the yield-macro model.

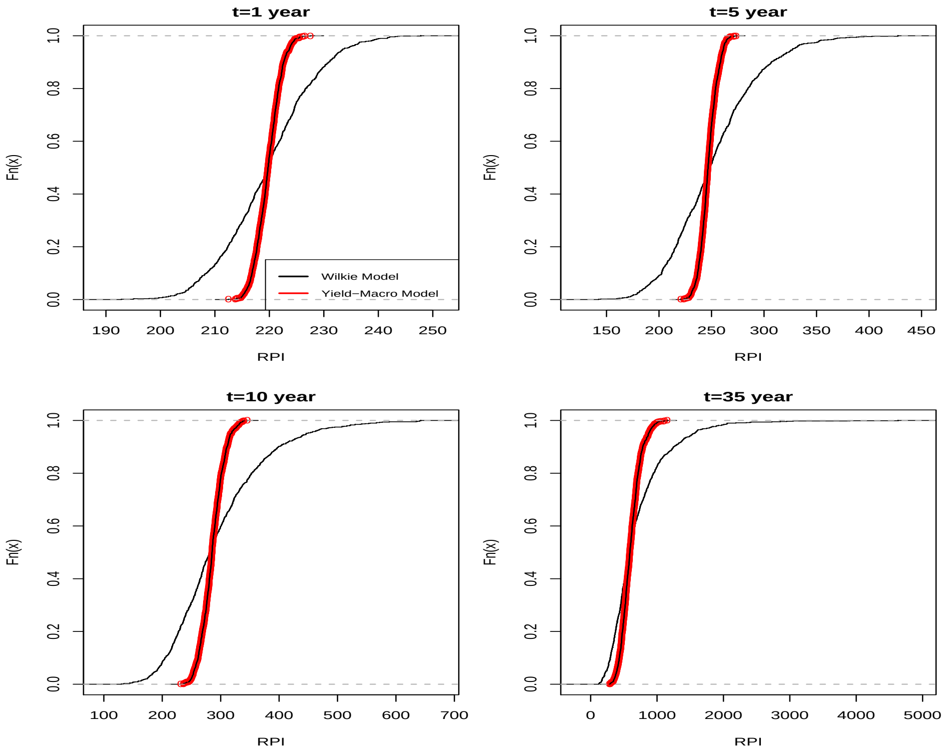

Figure 7.4: Empirical Cumulative Distribution Functions for the Simulated RPI Values over 35 Years

7.3.2 Simulated Zero-Coupon Yields

We can also compare the zero-coupon bond yields for different maturities and different forecast years obtained from the two models. In order to do such a comparison: First we simulate the short and long-term interest rates of the Wilkie model. Using these simulated values we construct the par yield curve for each year using Equation 7.1 in Lee and Wilkie (2000) and Wilkie *et al.* (2003).

$$Y(t,n) = C(t) + (B(t) - C(t)) \exp(-\beta n) \tag{7.1}$$

where $Y(t,n)$ is the par yield at time t for term n, $B(t)$ is the base rate, $C(t)$ is the consols yield from the Wilkie model and β is a constant whose value will be given later. We then derive the zero-coupon rates, at annual intervals, recursively, as follows:

Let $v(t,n)$ be the value at time t of a zero-coupon bond of term n.

Then the value of a coupon bond of term n, currently priced at par, with coupon equal to the par yield $Y(t,n)$, and redeemable at par, means that we have, for each n,

$$1 = Y(t,n)\sum_{m=1}^{n} v(t,m) + v(t,n). \tag{7.2}$$

Given the values of $Y(t,n)$, we can use Equation 7.2 to derive the $v(t,n)$ recursively. Starting with $n = 1$, we have

$$1 = Y(t,1)\sum_{m=1}^{1} v(t,m) + v(t,1)$$

whence $v(t,1) = 1/(Y(t,1)+1)$.

We continue year by year:

$$1 = Y(t,n)\sum_{m=1}^{n-1} v(t,m) + (1+Y(t,n))v(t,n)$$

whence $v(t,n) = \left(1 - Y(t,n)\sum_{m=1}^{n-1} v(t,m)\right)/(1+Y(t,n))$.

From the values of $v(t,n)$ we can derive a zero-coupon yield curve:

$$Z(t,n) = \frac{1}{v(t,n)^{1/n}} - 1 \tag{7.3}$$

Wilkie *et al.* (2003) indicate a problem about this approach which we have encountered in our calculations too. When calculating the zero-coupon discount factor $v(t,n)$, the sum of the values of the coupons from years one to $n-1$, $Y(t,n)\sum_{m=1}^{n-1} v(t,m)$, might exceed unity, so that the calculated value of the zero-coupon discount factor $v(t,n)$ is negative. This unsatisfactory condition happens when, for longer maturities, the par yield is still rising noticeably, and this happens when, with Equation 7.1, the value of β is too low for the particular values of $B(t)$ and $C(t)$. Therefore we have to choose a value of β that is large enough to prevent this anomaly from happening, at least within the first 35 years (the period for investing in zero-coupon bonds in this application). We find that a value of $\beta = 0.55$ is large enough considering the initial values and the simulations for our calculations. Indeed, Wilkie *et al.* (2003) use $\beta = 0.39$ and Yang (2001) uses a value of β of 0.5. Although we start with the value of 0.1 for β, we have had to increase it up to 0.55 to avoid negative or zero discount factors for the zero-coupon bonds. Using a high value of β produces a very flat yield curve, rather little different from using a constant interest rate of $C(t)$. However, $\beta = 0.55$ is the lowest value that does not give us inconsistencies.

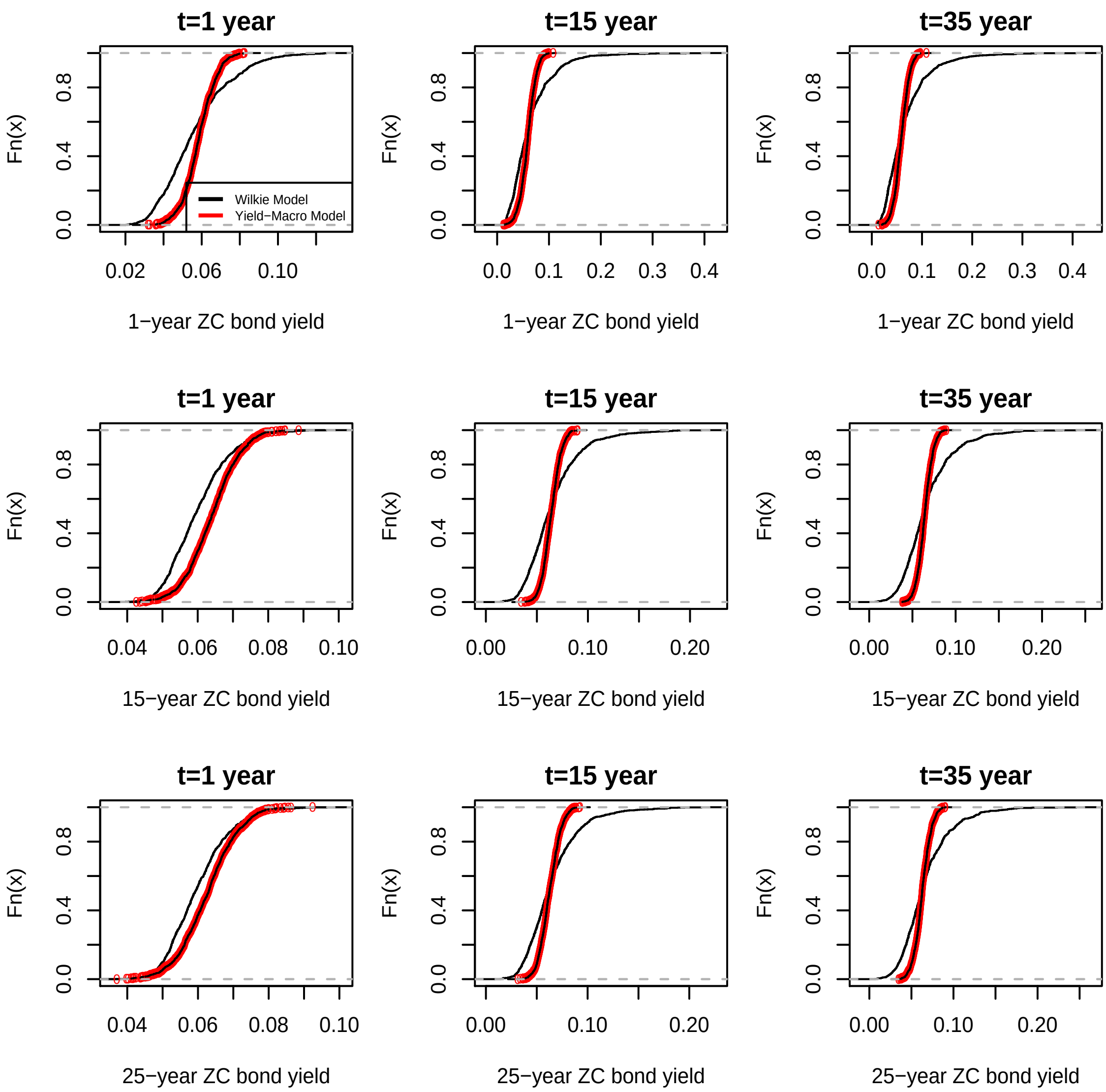

Figure 7.5: Empirical Cumulative Distribution Functions for the Simulated Zero-Coupon Bond Yields

Figure 7.5 displays the ECDFs of the zero-coupon yield curves based on 1000 simulations for different maturities and different years from the two models. The ECDFs for the zero-coupon yields for the first forecast year, $t = 1$, seem rather similar for the two models although the simulations obtained from the Wilkie model have a wider spread. At time $t = 1$, as the maturity increases the ECDFs get closer. On the other hand, as we simulate the yield curves for further years the standard deviations decrease for both models while the means remain almost the same. There are some high zero-coupon bond yields for the forecast years $t = 15$ and $t = 35$ in the simulated values using the Wilkie model. Figure 7.5 indicates that the distributions of the zero-coupon yields obtained from the two models become different as the maturity and the forecasting years increase. The calibration periods and the structures of the models might explain the differences observed in Figure 7.5. The parameters of the yield-macro model have been calculated based on a much more stable period. Therefore it is not surprising that the distributions of the ZC bond yields or any other simulated variables are less skewed or humped than the simulated Wilkie model variables. Furthermore, the structural differences between these two models also affect the simulation results. One of the main advantages of the yield-macro model over the Wilkie model is that the yield-macro model forecasts the entire yield curves. When we try to construct the ZC yield curve using the Wilkie model we see that there are some high ZC bond yields for reasons which have been discussed previously.

7.3.3 Nominal and Real Returns

After we simulate the zero-coupon yield curves for each model for the next 35 years we compare the investment returns based on these yield curves.

Tables 7.1, 7.2, 7.3 and 7.4 show numerical results from the Wilkie model and the yield-macro model on the same lines as shown in Tables 11.1 and 11.2 of Wilkie (1995) and Tables 3.1a to 3.7a of Lee and Wilkie (2000). However, we use zero-coupon bonds rather than par bonds for this application. We follow the notation of Wilkie (1995). Consider any variable $X(t)$, such as a price index or a total return index. Wilkie (1995) defines nominal returns as:

$$\begin{aligned} FX(t) &= X(t)/X(0) \\ GX(t) &= 100(FX(t)^{1/t} - 1) \end{aligned}$$

and real returns (relative to price inflation, FQ) as:

$$
\begin{aligned}
HX(t) &= FX(t)/FQ(t) \\
JX(t) &= 100(HX(t)^{1/t} - 1)
\end{aligned}
$$

Thus $FX(t)$ is the return over t years from an investment of 1 at time 0, and $GX(t)$ is the equivalent compound annual rate of return, expressed as a percentage; $HX(t)$ and $JX(t)$ are defined similarly, but based on real returns relative to the retail price index. We then denote the various series using below notation:
Q: retail price index
LR: long-term bond total return index
SR: "cash" or short-term bond total return index

Since for the two models the full yield curves are available now, we consider a rolling investment strategy and assume investment in 25-year zero-coupon bond which the following year has become a 24-year bond; it is then sold and reinvested in a new 25-year zero-coupon bond. For the short-term bond returns, we have followed the same approach, but using as a short-term rate, a 1-year zero-coupon bond compounded annually.

In Table 7.1 and Table 7.3 we show values measured in nominal terms and in Table 7.2 and Table 7.4 values measured in real terms (since the real return on price inflation is zero it is omitted). We show means, standard deviations (sd), skewnesses (skew), excess kurtosises (kurt) and correlation coefficients (cor) based on 1000 simulations.

According to Table 7.1, the mean for the inflation has not changed significantly while the standard deviation has reduced with t. The skewness and the excess kurtosis seem low and stable over the 35 years. As for the nominal returns on long-term bonds, we see almost 1% decrease in the mean over the next year but it has increased gradually since then. The standard deviation has come down significantly while the skewness and excess kurtosis vary over time displaying some high values particularly after $t = 10$.

Long-term bond returns are negatively correlated with inflation but the correlation has been decreasing slowly over time. The mean and the standard deviation of the short-term nominal returns have been increasing with t. The correlation coefficients for the inflation and the short-term nominal returns have been increasing up to 0.382 while they are negative for the first 25 years and become positive afterwards when we look at the coefficients between the long-term and short-term nominal returns.

Although the means and the standard deviations of the real returns present similar patterns to nominal returns, the correlation coefficients have changed both in terms of sign and magnitude as seen in Table 7.2.

Table 7.3 and Table 7.4 display the results for the nominal and real returns for the yield-macro model. The means and the standard deviations have similar trends as with their akins in the Wilkie model. However, they are generally lower than the values in Table 7.1 and Table 7.2. The low values of the skewness and excess kurtosis coefficients obtained from the returns for the yield-macro model are also noticeable. The nominal short-term returns are positively correlated with the inflation which has reached up to 0.913 at year 35.

We could calculate the continuously compounded rates by taking the logarithms rather than calculating the annual compounded rates of the variables in Tables 7.1, 7.2, 7.3 and 7.4. The high values of the skewness and excess kurtosis coefficients might indicate some log-normally distributed returns. When we take logarithms we expect to have approximately normally distributed returns which might produce lower values for the skewness and kurtosis coefficients.

Table 7.1: Wilkie Model: Results for Nominal Returns from 1000 Simulations

Mean rate of inflation, GQ								
Term	1	5	10	15	20	25	30	35
mean(GQ)	2.949	3.137	2.973	2.958	2.945	2.954	2.969	2.987
sd(GQ)	4.013	3.437	2.720	2.321	2.105	1.930	1.794	1.663
skew(GQ)	0.102	0.170	0.297	0.281	0.232	0.190	0.136	0.124
kurt(GQ)	-0.082	0.103	0.185	0.204	0.120	0.026	-0.014	0.126
Mean rate of growth of nominal total return on long bonds, GLR								
Term	1	5	10	15	20	25	30	35
mean(GLR)	5.917	4.952	5.142	5.463	5.714	5.786	5.969	5.903
sd(GLR)	19.544	7.778	4.613	3.353	2.455	2.309	1.881	1.904
skew(GLR)	-0.076	-0.790	-1.420	-1.351	-1.103	-1.239	-0.662	-0.746
kurt(GLR)	-0.191	0.944	5.072	3.331	2.451	3.925	2.946	3.534
cor(GQ,GLR)	-0.211	-0.384	-0.367	-0.371	-0.344	-0.222	-0.158	-0.033
Mean rate of growth of nominal total return on cash, GSR								
Term	1	5	10	15	20	25	30	35
mean(GSR)	5.855	5.986	6.174	6.265	6.303	6.341	6.370	6.397
sd(GSR)	0.000	1.832	2.163	2.227	2.239	2.201	2.151	2.087
skew(GSR)	0.000	1.091	1.084	1.299	1.299	1.328	1.181	1.048
kurt(GSR)	0.000	1.548	1.654	3.883	5.583	3.822	2.697	1.859
cor(GQ,GSR)	0.000	0.150	0.200	0.242	0.262	0.301	0.350	0.382
cor(GLR,GSR)	0.000	-0.383	-0.320	-0.287	-0.154	-0.030	0.141	0.236

Table 7.2: Wilkie Model: Results for Real Returns from 1000 Simulations

Mean rate of growth of real total return on long bonds, JLR								
Term	1	5	10	15	20	25	30	35
mean(JLR)	3.197	1.969	2.221	2.511	2.749	2.796	2.949	2.859
sd(JLR)	20.366	9.421	6.014	4.630	3.669	3.262	2.755	2.531
skew(JLR)	0.104	-0.228	-0.493	-0.469	-0.365	-0.362	-0.108	-0.059
kurt(JLR)	-0.026	0.168	1.383	0.843	0.306	0.572	0.449	1.191
cor(GQ,JLR)	-0.396	-0.667	-0.717	-0.756	-0.792	-0.741	-0.755	-0.680
Mean rate of growth of real total return on cash, JSR								
Term	1	5	10	15	20	25	30	35
mean(JSR)	2.979	2.867	3.169	3.252	3.293	3.314	3.321	3.325
sd(JSR)	4.015	3.607	3.084	2.761	2.599	2.411	2.228	2.073
skew(JSR)	0.121	0.137	0.391	0.588	0.825	0.944	0.925	0.860
kurt(JSR)	-0.068	0.166	0.750	2.258	3.119	2.993	2.671	1.950
cor(GQ,JSR)	-0.999	-0.873	-0.743	-0.650	-0.590	-0.534	-0.479	-0.431
cor(JLR,JSR)	0.399	0.455	0.407	0.382	0.431	0.417	0.485	0.472

Table 7.3: Yield-Macro Model: Results for Nominal Returns from 1000 Simulations

Mean rate of inflation, GQ								
Term	1	5	10	15	20	25	30	35
mean(GQ)	2.988	2.964	2.943	2.918	2.927	2.927	2.935	2.937
sd(GQ)	1.066	0.687	0.691	0.715	0.732	0.723	0.712	0.697
skew(GQ)	0.074	0.013	-0.005	-0.033	-0.089	-0.119	-0.098	-0.086
kurt(GQ)	-0.183	-0.087	-0.167	0.026	-0.055	-0.099	-0.148	-0.131
Mean rate of growth of nominal total return on long bonds, GLR								
Term	1	5	10	15	20	25	30	35
mean(GLR)	7.370	6.141	6.012	5.973	5.991	6.001	5.951	5.918
sd(GLR)	16.654	4.971	2.605	1.845	1.381	1.179	1.029	0.877
skew(GLR)	0.583	0.023	0.098	-0.015	0.132	-0.025	0.068	-0.029
kurt(GLR)	0.809	-0.198	-0.211	0.186	-0.073	-0.207	-0.187	-0.131
cor(GQ,GLR)	-0.182	-0.101	-0.035	-0.038	-0.194	-0.190	-0.273	-0.282
Mean rate of growth of nominal total return on cash, GSR								
Term	1	5	10	15	20	25	30	35
mean(GSR)	5.888	5.860	5.831	5.836	5.847	5.851	5.854	5.852
sd(GSR)	0.644	0.779	0.883	0.910	0.906	0.895	0.870	0.838
skew(GSR)	0.061	-0.051	-0.092	-0.075	-0.103	-0.071	-0.060	-0.080
kurt(GSR)	0.207	-0.135	-0.239	-0.263	-0.261	-0.221	-0.192	-0.134
cor(GQ,GSR)	-0.150	0.359	0.681	0.794	0.850	0.884	0.901	0.913
cor(GLR,GSR)	0.166	0.129	0.070	0.038	-0.111	-0.127	-0.207	-0.224

Table 7.4: Yield-Macro Model: Results for Real Returns from 1000 Simulations

Mean rate of growth of real total return on long bonds, JLR								
Term	1	5	10	15	20	25	30	35
mean(JLR)	4.2967	3.093	2.987	2.973	2.983	2.993	2.936	2.902
sd(JLR)	16.424	4.944	2.648	1.956	1.649	1.467	1.379	1.243
skew(JLR)	0.604	0.0151	0.140	0.004	0.086	0.0457	0.178	0.009
kurt(JLR)	0.907	-0.156	-0.150	0.107	-0.158	0.035	0.011	0.054
cor(GQ,JLR)	-0.244	-0.238	-0.294	-0.401	-0.602	-0.642	-0.716	-0.754
Mean rate of growth of real total return on cash, JSR								
Term	1	5	10	15	20	25	30	35
mean(JSR)	2.829	2.815	2.806	2.835	2.837	2.841	2.835	2.831
sd(JSR)	1.313	0.819	0.637	0.538	0.464	0.410	0.370	0.336
skew(JSR)	0.056	-0.023	-0.099	-0.041	-0.066	-0.077	-0.046	-0.012
kurt(JSR)	0.110	-0.102	-0.023	-0.110	-0.037	0.052	0.001	0.035
cor(GQ,JSR)	-0.882	-0.506	-0.166	-0.022	0.037	0.112	0.135	0.142
cor(JLR,JSR)	0.280	0.269	0.169	0.112	0.061	-0.005	-0.031	-0.051

7.4 Asset Values and Annuity Payoffs

Another way to compare the Wilkie model and the yield-macro model is to examine the asset values and the annuity payoffs under a hypothetical pension scheme. Although a more realistic application would include mortality, we ignore it during both the investment and the retirement periods for simplicity in this analysis.

We assume an employee at age 30, with an arbitrary initial salary, S. The salary increases according to the simulated RPI index for the next 35 years and the employee retires at age 65. She contributes a constant fraction of her salary, f to a pension fund which is invested into a portfolio of nominal bonds for different maturities. We ignore mortality during both the investment and the retirement period, which is taken as a fixed 25 years, and we analyse the variations in the assets and annuity payoffs.

Let $v(t,n)$ be the price of an n-year zero-coupon bond at time t.

$$v(t,n) = \frac{1}{(1+Z(t,n))^n} \tag{7.4}$$

where

$Z(t,n)$ is the n-year spot rate at time t.

Salary rises in line with $RPI(t)$ and contributions are a constant fraction, f, of salary. Thus the yearly contribution C_t is,

$$C_t = S \times f \times \frac{RPI(t)}{RPI(0)}$$

where

$S = 10000$ units

$f = 10\%$

$RPI(t)$ values are simulated using the stochastic models.

Thus, the asset value just before the contribution at time t, A_t, can be calculated as:

$$A_t = (A_{t-1} + C_{t-1}) \underbrace{\frac{v(t, n-1)}{v(t-1, n)}}_{1+R(t)} \tag{7.5}$$

where $A_0 = 0$ and $R(t)$ is the return at time t. Equation 7.5 assumes investment in a rolling n-year zero-coupon bond fund.

Once we calculate the asset values over time, we can find the annuity payoffs for the 25 years retirement period using the zero-coupon yield curves at age 65, i.e. the simulated yield curve at year 35. We assume that the annuity is paid yearly in advance.

Let ap be the annuity payoff. Then,

$$A_{35} = ap \times \ddot{a}(35, N) \tag{7.6}$$

where $\ddot{a}(35, N)$ is the annuity price for 1 unit,

$$\ddot{a}(35, N) = \sum_{m=0}^{N-1} (1 + Z(35, m))^{-m} = \sum_{m=0}^{N-1} v(35, m)$$

$N = 25$ and $Z(35, N)$ is the zero-coupon yield curve at $t = 35$.

We calculate the asset values under different investment strategies for both models. We assume rolling investments in zero-coupon bonds for specific maturities such as 5-year (F1), 10-year (F2), 15-year (F3), 20-year (F4) and 25-year (F5) ZC bonds as described in the previous section. We consider two more scenarios which we invest on decreasing maturity for some years of the investment period. First, we invest in 25-year ZC bonds for the first 10 years, then for the last 25 years instead of a rolling investment we use the zero-coupon yield curve to calculate the returns on decreasing maurities (D1). Second, we again invest in 25-year ZC bonds but for a longer period, 25 years, then for the last 10 years we invest in decreasing maturity bonds (D2). While in D1 the maturity of the assets at time $t = 35$ corresponds to the retirement date, in D2 the maturity of the assets is 15 years at the retirement date. With D2 we try to hedge the risk in the annuity price, $\ddot{a}(35, N)$. On the other hand, a more realistic strategy might be to assume deterministic mortality and an investment policy which aims to match the expected annuity payoffs more exactly by buying small fraction of bonds of different maturities.

Table 7.5 shows some descriptive statistics for the real asset values calculated using the first 'decreasing maturity' investment strategy (D1) for both models over the next 35 years. Although the mean of the real asset values obtained from the Wilkie model grows faster than the values of the yield macro model, the medians for different years are quite close to each other. The higher standard deviations, skewness and excess kurtosis coefficients indicate that Wilkie model tends to produce some extreme values relative to the yield-macro model. The minimum and maximum values displayed over the years also support this conclusion.

Table 7.5: Real Asset Values, A_t, on a Decreasing Maturity (D1) Investment

Wilkie Model Real Asset Values							
Year	Mean	Sd	Median	Min	Max	Skewness	Kurtosis
1	219.69	8.56	219.63	192.83	247.82	0.10	-0.08
5	251.82	42.45	248.45	142.52	427.77	0.59	0.64
10	295.29	81.53	281.32	130.01	642.74	1.08	1.77
20	413.52	182.87	375.90	103.06	1556.13	1.59	4.05
30	586.74	335.57	510.11	121.58	3157.36	1.95	6.78
35	699.74	444.15	593.25	120.31	4632.24	2.42	11.22
Yield-Macro Model Real Asset Values							
Year	Mean	Sd	Median	Min	Max	Skewness	Kurtosis
1	219.78	2.28	219.75	212.49	227.54	0.07	-0.18
5	247.06	8.25	246.70	220.79	273.45	0.09	-0.09
10	285.79	19.18	285.23	231.86	345.76	0.16	-0.12
20	383.65	54.43	381.28	230.74	565.17	0.30	0.06
30	518.92	107.46	508.83	269.88	961.85	0.47	0.25
35	603.76	142.94	591.03	285.74	1154.34	0.56	0.31

Figure 7.6 shows the real asset values for different investment strategies over the years. The yield-macro model produces lower mean values than the Wilkie model after the first year but while the difference is negligible for 5-year (which has not been displayed in the figure) and 10-year ZC bond investments, the difference increases as the maturity of the invested bond increases. For the investment on the 25-year ZC bond the Wilkie model produces very high values. After 15 years investment the Wilkie model asset values increase sharply which might be related with very low zero-coupon

discount factors. As we have discussed in the previous section, choosing $\beta = 0.55$ prevents negative discount factors but some of them are still very close to zero. These low values mean that the ZC bond prices are very low for some specific maturities and years and this causes extreme values in returns considering the rolling investment strategies. The last two plots in Figure 7.6 show the asset values for the decreasing maturity investments. Since we invest in 25-year ZC bonds only for 10 years, the real annuity payoffs of the models are relatively close in D1 while they are quite different in D2 as a result of much longer investment period on the 25-year ZC bonds.

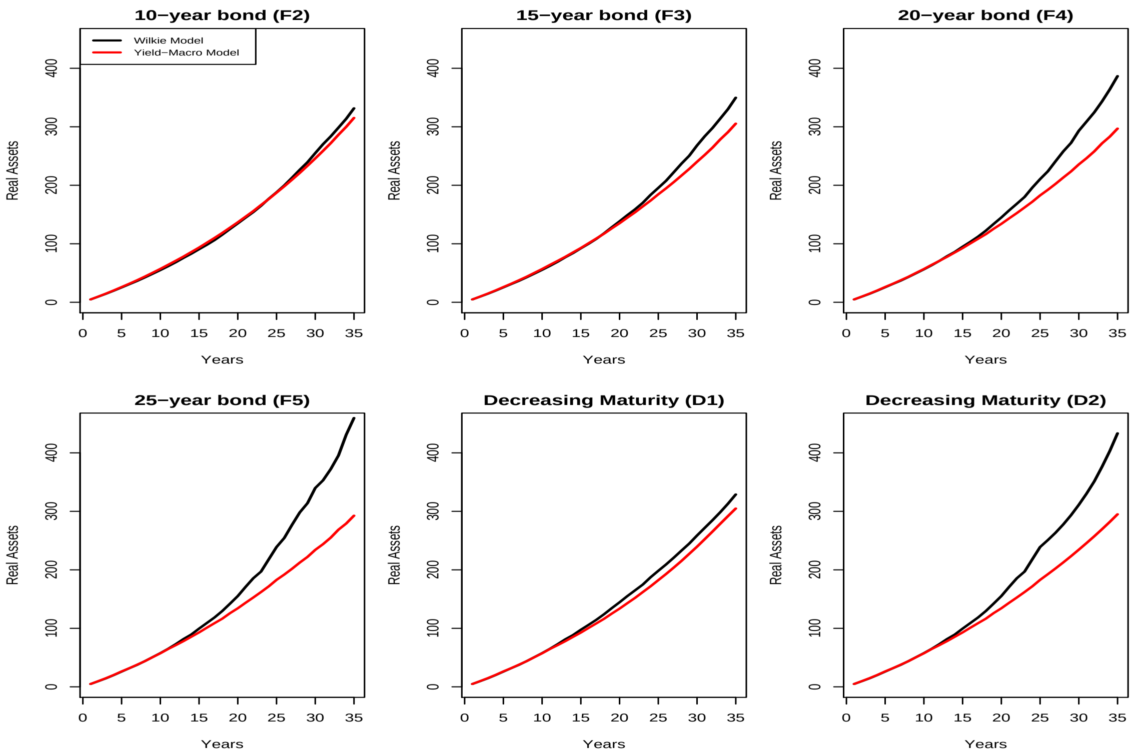

Figure 7.6: The Mean Amount of Real Assets for Different Investment Strategies

Table 7.6: Annuity Payoffs as a % of Final Salary

Wilkie Model							
	F1	F2	F3	F4	F5	D1	D2
Mean	63.17%	63.03%	66.18%	73.72%	90.02%	64.92%	88.75%
SD	51%	53%	74%	132%	300%	57%	269%
Median	50.18%	49.63%	49.91%	50.95%	52.34%	53.00	52.69%
Minimum	20.63%	17.51%	15.36%	10.80%	6.03%	20.08%	9.77%
Maximum	637.12%	886.90%	1449.79%	2971.28%	7536.14%	1005.49%	6515.18%
Skewness	5.98	7.49	10.74	15.16	19.20	9.42	17.35
Kurtosis	51.73	87.15	165.34	289.00	426.69	127.27	362.61
Yield-Macro Model							
	F1	F2	F3	F4	F5	D1	D2
Mean	55.11%	55.46%	53.71%	52.11%	51.23%	53.39%	50.61%
SD	13%	15%	16%	16%	17%	13%	12%
Median	53.43%	52.97%	51.55%	49.75%	48.88%	51.77%	48.80%
Minimum	24.94%	23.37%	23.89%	20.73%	17.90%	23.28%	25.89%
Maximum	117.54%	147.68%	155.64%	159.66%	164.92%	106.19%	100.27%
Skewness	0.87	1.14	1.21	1.24	1.29	0.87	0.80
Kurtosis	1.21	2.49	3.02	3.24	3.38	1.06	0.79

Table 7.6 presents some descriptive statistics for the nominal annuity payoffs as a percentage of final salary for both models. As for the Wilkie model, the mean and the standard deviation of the ratio have been increasing as we use a longer term bond for investment. The significant differences between the means and the medians indicate that there are some extreme values which affect the ratios. The ratios are positively skewed and the excess kurtosis coefficients are exceptionally high. On the other hand, the means and the medians for the yield-macro model are not very different from each other. The standard deviations seem stable and the ratios are slightly positively skewed. Although the excess kurtosis coefficients are much lower than the ones in the Wilkie model, they are significantly high for the ratios obtained from some of the investment strategies.

We might also compare the distributions of these ratios graphically. Figure 7.7 displays the ECDFs of the annuity payoffs as a percentage of final salary for different investment strategies for the models. Since we know that the annuity payoffs obtained from the Wilkie model have some extreme values we exclude the ratios lower than 5% and higher than 200% to draw the ECDFs. Regardless of the portfolio chosen, the payoff ratios calculated using the Wilkie model are more dispersed than the ratios obtained from the yield-macro model due to more volatile calibration period and the structure of the model.

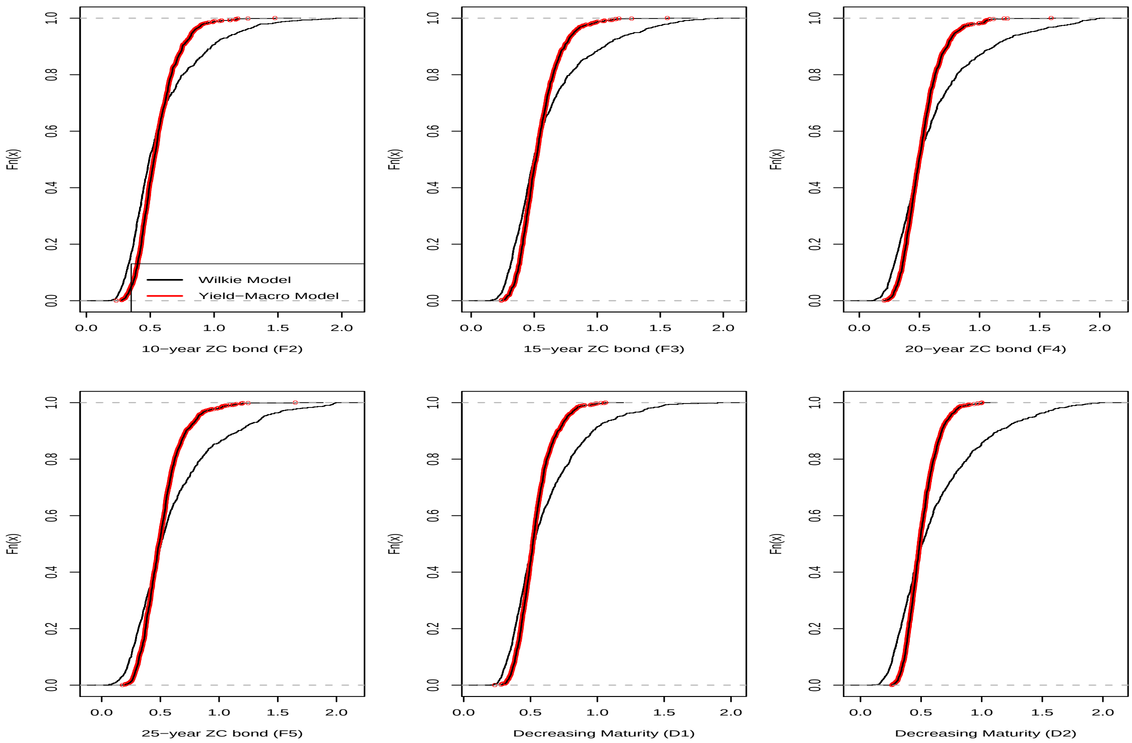

Figure 7.7: The Empirical Cumulative Distribution Functions of the Annuity Payoffs as a % of Final Salary

Figure 7.8 and Figure 7.9 show the scatter plots for the asset/salary ratios and annuity prices ($\ddot{a}(35, N)$) on a horizontal log scale for the Wilkie model and the yield-macro model respectively. We have omitted extremely high values for the Wilkie model in Figure 7.8 but there are still very high and very low values which increase the spread of the plots. As the maturity of the invested ZC bond extends the correlation between the ratios and the annuity price increases in both figures. As for the decreasing maturity investment strategies, D1 and D2, the correlations seem stronger for D2 at least for the Wilkie model. The reason is that having 15-year ZC bonds as assets at retirement hedges the risk in the annuity price, $\ddot{a}(35, N)$ better. However, the correlations are relatively weak for both D1 and D2 suggesting that this type of strategy does not work all that well, at least looking ahead from time $t = 0$.

7.5 Interim Conclusion: Comparison of the Models

In this chapter we have compared the Wilkie model and the yield-macro model in both structural and empirical ways. Due to incorporating different input variables, the models have different structures and the nature of the relations between these variables is also different. Since the two models were developed based on different periods of data we use the neutral initial conditions of the yield-macro model for the Wilkie model and we adjust the mean parameters of the inflation and interest rates models of Wilkie according to these initial conditions. Therefore, we have made the two models exactly comparable. Afterwards, we have simulated the nominal and real total returns based on a rolling investment strategy and compared the models by examining some descriptive statistics and the correlations between the outputs. Considering both the nominal and real total returns, the Wilkie model has produced higher values for the means and the standard deviations than the yield-macro model. However, the correlation coefficients between the variables vary for both models, while the yield-macro model gives higher positive correlation between the inflation and the short-term bond returns.

We have also calculated the asset values and annuity payoffs for the two models under a hypothetical pension scheme. The results show that the Wilkie model produces higher asset values (including some extreme values) for different portfolios and the volatilities have been much higher than the ones obtained from the yield-macro model. This is due to small values of the zero-coupon discount factors which have caused extremely high returns for the chosen investment strategy. The distribution of the ratios are positively skewed with very high kurtosis coefficients while the yield-macro model produce much more stable ratios. Finally, we have compared the annuity payoffs

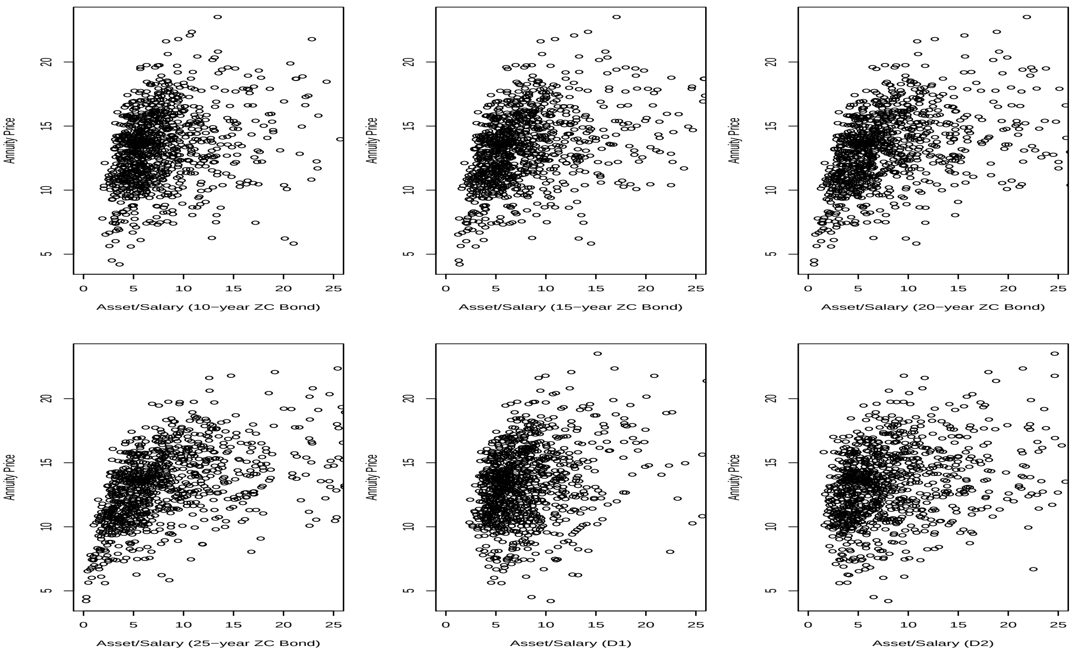

Figure 7.8: Asset/Salary vs Price (25-Year ZC Bond), Wilkie Model

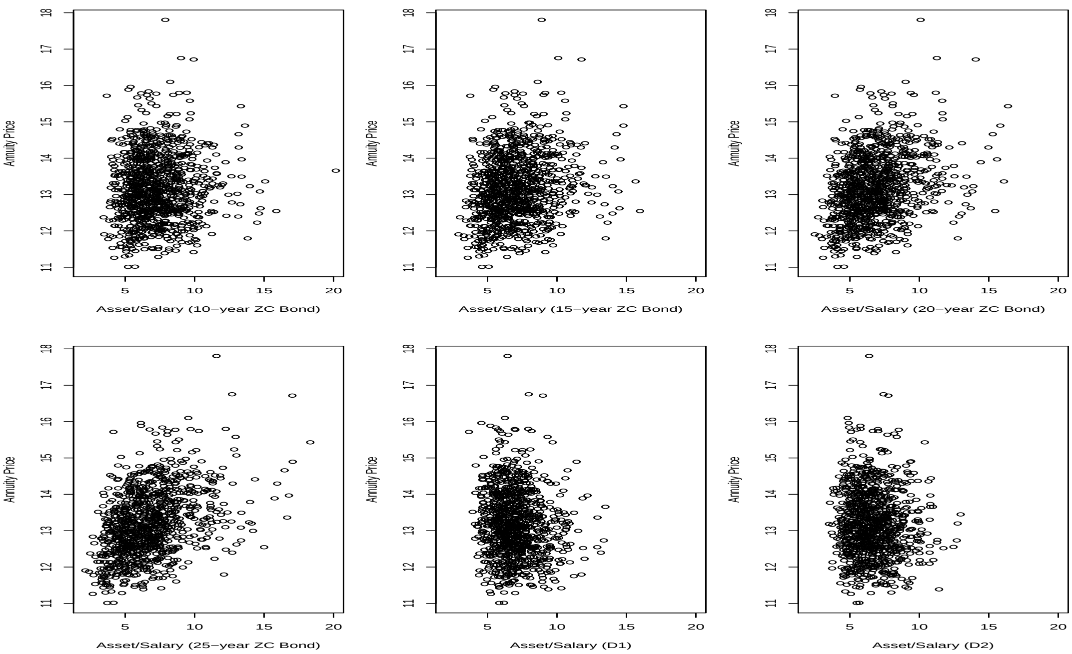

Figure 7.9: Asset/Salary vs Price (25-Year ZC Bond), Yield-Macro Model

as a percentage of final salary for each model and for each portfolio. When we omit the extreme values for the Wilkie model, the distribution of the ratios seem similar in terms of means but the standard deviations of the ratios from the Wilkie model are still higher. Furthermore the correlation between the asset/salary ratios at retirement and the annuity price increases as the maturity of the bond invested increases.

Chapter 8

Conclusions and Further Research

The purpose of this chapter is to provide an overview of the main findings of this thesis as well as some suggestions for further research.

8.1 Conclusions

The main contribution of this thesis is the construction of a stochastic investment model incorporating the term structures of the nominal, implied inflation and the real spot rates simultaneously along with the realised inflation and output gap for the UK. The work is original as it provides a model for the term structure of implied inflation for the first time. While any of the three term structures on the base data can be derived from the other two, after applying PCA the three sets of simulated values are not additive. Thus we investigate which pairs give the plausible values for the other set, checking whether the Fisher relation holds for the simulated values.

In Chapter 1 we have discussed the first comprehensive stochastic investment model, the Wilkie model, in detail. The estimated parameters based on the updated data have not changed significantly for most of the models while the recursive estimates and the confidence intervals for these estimates show that the parameters might change over time. Therefore, we have concluded that the parameters have not been stable except for the wages, dividend yields and short-term interest rates models.

Since the purpose of this study has been to propose a stochastic investment model which incorporates the term structures we have used the yield curve data provided by the Bank of England. However, the data include many missing values which prevents us from using all 50 (or 46) maturities available. In Chapter 2 we have fitted the Cairns model (Cairns, 1998) to the yield curve data in order to fill the gaps in the data. This has enabled us to make two contributions. First, instead of using some given fixed exponential rates in the descriptive parametric model of Cairns we have found a set of optimal parameters for each yield curve and two of the three sets have given better results than the other fixed parameter sets. Second, by replacing the missing values we could use the information from 50 (or 46) different maturities for our yield curve models rather than using only a small number of maturities which have been the case in other studies.

After replacing the missing values in the yield curves we have applied the PCA to the fitted values to decrease the dimension of the data by extracting some uncorrelated variables. The first three components have explained almost all the observed variability for each term structure. We have also discussed the robustness of the PCA relative to the choice of the exponential parameter sets and concluded that the analysis produces consistent results for different sets of parameters. Then, we have modelled these components in Chapter 5 on a monthly frequency. An AR(1) process has been found good enough to model all nine factors of the three yield curves. The distribution of the residuals follow the logistic distribution due to having zero mean and high kurtosis coefficients. We have also noticed that there is some evidence of ARCH effects for the implied inflation and real spot rates. One-month ahead forecasts have been satisfactory, while the Fisher relation held for some of the maturities.

Chapter 6 presents the main contributions of this work by including the vector autoregressive stochastic investment models which consist of the term structures and the macroeconomic variables. As for the quarterly yield-macro model, our analysis has shown that the yield curve factors and the macroeconomic variables are significantly correlated. The level factors of the yield curves have been modelled as AR(1) processes. The nominal slope and curvature factors are the ones which are connected with the macroeconomic variables in a bi-directional way. The yield curve factors also have been found significantly correlated for some of the models as expected.

On the other hand, the yearly yield-macro model presents a relation between the 'level' factors of the yield curves and the realised inflation. However, this is a one way relation and the realised inflation has been found to have a significant impact on the levels of the yield curves.

Furthermore, we have tried to explain the auto- and cross-correlations between the term structures and the macroeconomic variables. The nature of the correlations have been changed as we have used data on different frequencies. While we observe significant positive correlations between the level factors of the yield curves and the realised inflation on yearly data, there is no such correlation between these variables on monthly or quarterly frequencies. Besides, there is a negative correlation between the level factors of the spot rates and the output gap. The economic theory states that an increase in interest rates decreases the actual output. Since the output gap is defined as the difference between the actual output and the potential output divided by the potential output, when the actual output decreases the output gap decreases too.

We have compared our stochastic investment models with the random walk and the AR(1) process in terms of the explained variability in the data. The results have shown that including the bi-directional relation between the yield curves and the macro variables improves the performance of the models significantly. We have also concluded that the Fisher relation holds for some maturities when we examined one-period ahead forecasts for both yield-macro models.

In the final chapter we have compared our quarterly yield-macro model with the Wilkie model. The structures of the two models are quite different due to different variables included and the frequency of the data used. The distributions of the nominal and real returns produced by the Wilkie model have been found more skewed and humped relative to the yield-macro model. Besides, the real asset values and the annuity payoffs have been higher for the Wilkie model with high uncertainty. Our analyses have showed that the extreme values for the asset returns simulated from the Wilkie model have been caused by the low zero coupon bond prices. This has happened because of the neutralised parameters and the initial conditions we have chosen to make the two models comparable. The main advantage of the yield-macro model is to forecast the entire term structures rather than just the two ends of the curves as in the Wilkie model. Incorporating the three term structures provides a broader application field to the yield-macro model which consists of the interest rates forecasting. However, in Chapter 7 we have restricted ourselves with the common applications of the models due to comparison purposes.

8.2 Suggestions for Further Research

There are possible ways to carry the analyses in this thesis further.

To begin with, instead of the PCA analysis one could apply the common principal component analysis (CPCA) (Flury (1988)) to the yield curve data. The CPCA is a generalization of the PCA to several groups. The basic assumption is that the PC transformation (the *eigenvectors* or the *loadings*) is identical in all k groups considered, while the variances associated with the components (*eigenvalues*) may vary between groups. In other words, the level, slope and curvature factors are assumed to be the same for the nominal, implied inflation and real term structures. There are some studies which have investigated comovements or common features observed on several domestic bond markets by applying the CPCA (Moraux *et al.* (2002), Fengler *et al.* (2004), Perignon *et al.* (2007)). As the loading graphs in Chapter 5 and Chapter 6 indicate, there might be some common factors affecting the nominal, implied inflation and the real yield curves which is worth investigating.

According to our preliminary analysis on this method, we might encounter two problems while applying the CPCA on the three term structures. The first one is that since the successive maturities are highly correlated for the yield curves, the covariance matrices are almost positive-semi definite while the CPCA can be applicable for the positive-definite covariance matrices. Actually we have a very high dimensional data which requires a very large number of samples to avoid singular covariance matrices and zero eigenvalues. Flury's (1988) method depends on calculating a maximum likelihood value that is made up of the product of the eigenvalues. Thus, having a zero eigenvalue breaks the method. Since we do not have a large number of observations we can pick every n^{th} maturity (n=5 or higher) to eliminate the highly correlated maturities before applying the CPCA. A second and more challenging issue is that the CPCA requires independent groups. However, the term structures of the nominal, implied inflation and the real spot rates are highly correlated. It is still possible to apply the analysis on our data but the tests for the *existence of common factors* hypothesis should be adjusted for the dependent groups which might not be easy.

Another future research might be to model the time dependent b parameters discussed in Chapter 2 as an alternative to the fitted curves themselves. Diebold and Li (2006) have modelled the time varying parameters obtained by fitting a modified version of the Nelson-Siegel curve which produces uncorrelated factors. They interpret those time varying parameters as factors corresponding to level, slope and curvature. Since the b parameters are highly correlated it is possible to apply the PCA first to obtain uncorrelated variables and then to model these new variables. However, it should be noted that the PCs obtained from the b parameters cannot be named 'level', 'slope' and 'curvature'.

As we have briefly discussed in Chapter 5 there might be some ARCH effects on the factors of the term structures which is also worth investigating.

Another interesting piece of research would be to assume an inflation premium and model the term structures accordingly.

Finally, the application of the models can be extended. First, the Wilkie model and the yield-macro model could be used together in a coherent way with the inflation model and the term structures being adopted from the quarterly yield-macro model and the share dividends and dividend yields being adopted from the Wilkie model using the future inflation rates generated by the yield-macro model. Second, the yield-only model and the yield-macro models could be used for different applications. It would be interesting to focus only on the term structures and examine the annuity prices obtained from the nominal and the real yield curves, even including the implied inflation through the Fisher relation and discuss the variation in these prices.

Appendix A

Yield-Only Model

The below representation is suitable when each model has been considered separately. It should be noted that the Z noises are correlated as expressed in Equation 5.3 in Chapter 5.

Nominal Level Factor

$$M_{N_L}(t) = \underbrace{-20.68}_{13.35} + \underbrace{0.992}_{0.002}\ (M_{N_L}(t-1) + 20.68) + 1.81\ Z_{N_L}(t)$$

where $Z_{N_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.9875$

Nominal Slope Factor

$$M_{N_S}(t) = \underbrace{0.98}_{0.013}\ M_{N_S}(t-1) + 0.77\ Z_{N_S}(t)$$

where $Z_{N_S}(t) \sim N(0,1)$

$R^2_{adj} = 0.95$

Nominal Curvature Factor

$$M_{N_C}(t) = \underbrace{0.88}_{0.03}\ M_{N_C}(t-1) + 0.37\ Z_{N_C}(t)$$

where $Z_{N_C}(t) \sim N(0,1)$

$R^2_{adj} = 0.79$

Implied Inflation Level Factor

$$M_{I_L}(t) = \underbrace{0.978}_{0.009}\ M_{I_L}(t-1) + 1.58\ Z_{I_L}(t)$$

where $Z_{I_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.97$

Implied Inflation Slope Factor

$$M_{I_S}(t) = \underbrace{0.952}_{0.018}\ M_{I_S}(t-1) + 0.66\ Z_{I_S}(t)$$

where $Z_{I_S}(t) \sim N(0,1)$

$R^2_{adj} = 0.90$

Implied Inflation Curvature Factor

$$M_{I_C}(t) = \underbrace{0.826}_{0.03}\ M_{I_C}(t-1) + 0.35\ Z_{I_C}(t)$$

where $Z_{I_C}(t) \sim N(0,1)$

$R^2_{adj} = 0.71$

Real Level Factor

$$M_{R_L}(t) = \underbrace{0.993}_{0.006}\ M_{R_L}(t-1) + 1.14\ Z_{R_L}(t)$$

where $Z_{R_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.97$

Real Slope Factor

$$M_{R_S}(t) = \underbrace{0.88}_{0.027}\ M_{R_S}(t-1) + 0.73\ Z_{R_S}(t)$$

where $Z_{R_S}(t) \sim N(0,1)$

$R^2_{adj} = 0.85$

Real Curvature Factor

$$M_{R_C}(t) = \underbrace{0.864}_{0.03}\ M_{R_C}(t-1) + 0.24\ Z_{R_C}(t)$$

where $Z_{R_C}(t) \sim N(0,1)$

$R^2_{adj} = 0.80$

Appendix B

Yield-Macro Model - I

Nominal Level Factor

$$Q_{N_L}(t) = -\underbrace{6.76}_{3.93} + \underbrace{0.92}_{0.03} \left(Q_{N_L}(t-1) + 6.76\right) + 2.13\ Z_{N_L}(t)$$

where $Z_{N_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.94$

Nominal Slope Factor

$$Q_{N_S}(t) = \underbrace{0.78}_{0.06}\ Q_{N_S}(t-1) - \underbrace{1.21}_{0.29}\ Q_{R_C}(t-2) - \underbrace{0.41}_{0.19}\ Q_{OG}(t-2) + 0.87\ Z_{N_S}(t)$$

where $Z_{N_S}(t) \sim N(0,1)$

$R^2_{adj} = 0.81$

Nominal Curvature Factor

$$\begin{aligned} Q_{N_C}(t) &= \underbrace{0.96}_{0.08}\ Q_{N_C}(t-1) - \underbrace{0.34}_{0.08}\ Q_{N_C}(t-2) - \underbrace{0.15}_{0.06}\ (Q_{RI}(t-1) - 2.88) \\ &\quad +0.38\ Z_{N_C}(t) \end{aligned}$$

where $Z_{N_C}(t) \sim N(0,1)$
$R^2_{adj} = 0.69$

Implied Inflation Level Factor

$$Q_{I_L}(t) = -\underbrace{1.47}_{1.84} + \underbrace{0.89}_{0.05}\ (Q_{I_L}(t-1) + 1.47) + 1.51\ Z_{I_L}(t)$$

where $Z_{I_L}(t) \sim N(0,1)$
$R^2_{adj} = 0.87$

Implied Inflation Slope Factor

$$Q_{I_S}(t) = \underbrace{0.56}_{0.08}\ Q_{I_S}(t-1) - \underbrace{0.32}_{0.07}\ Q_{I_S}(t-2) + \underbrace{1.38}_{0.17}\ Q_{R_C}(t-2) + 0.52\ Z_{I_S}(t)$$

where $Z_{I_S}(t) \sim N(0,1)$
$R^2_{adj} = 0.66$

Implied Inflation Curvature Factor

$$Q_{I_C}(t) = \underbrace{0.62}_{0.10}\ Q_{I_C}(t-1) - \underbrace{0.09}_{0.03}\ Q_{R_S}(t-2) + 0.28\ Z_{I_C}(t)$$

where $Z_{I_C}(t) \sim N(0,1)$

$R^2_{adj} = 0.53$

Real Level Factor

$$Q_{R_L}(t) = -\underbrace{6.99}_{3.67} + \underbrace{0.95}_{0.02}\ (Q_{R_L}(t-1) + 6.99) + 1.27\ Z_{R_L}(t)$$

where $Z_{R_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.94$

Real Slope Factor

$$Q_{R_S}(t) = \underbrace{0.49}_{0.06}\ Q_{R_S}(t-1) + \underbrace{0.27}_{0.04}\ Q_{N_S}(t-1) + 0.54\ Z_{N_S}(t)$$

where $Z_{N_S}(t) \sim N(0,1)$

$R^2_{adj} = 0.80$

Real Curvature Factor

$$Q_{R_C}(t) = \underbrace{0.86}_{0.07}\ Q_{R_C}(t-1) + 0.22\ Z_{R_C}(t)$$

where $Z_{R_C}(t) \sim N(0,1)$

$R^2_{adj} = 0.74$

Realised Inflation

$$Q_{RI}(t) = \underbrace{2.88}_{0.71} + \underbrace{0.92}_{0.07} (Q_{RI}(t-1) - 2.88) + \underbrace{0.34}_{0.08} \, Q_{N_C}(t-2) - \underbrace{0.20}_{0.09} \, Q_{OG}(t-2) + 0.41 \, Z_{RI}(t)$$

where $Z_{RI}(t) \sim N(0,1)$

$R^2_{adj} = 0.77$

Output Gap

$$Q_{OG}(t) = \underbrace{0.89}_{0.053} \, Q_{OG}(t-1) - \underbrace{0.04}_{0.017} \, Q_{N_S}(t-1) + 0.24 \, Z_{OG}(t)$$

where $Z_{OG}(t) \sim N(0,1)$

$R^2_{adj} = 0.85$

Appendix C

Yield-Macro Model - II

Nominal Level Factor

$$Y_{N_L}(t) = -\underbrace{44.18}_{4.09} + \underbrace{0.82}_{0.015}\,(Y_{N_L}(t-1) + 44.18) + \underbrace{1.86}_{0.18}\; Y_{RI}(t) + 3.58\; Z_{N_L}(t)$$

where $Z_{N_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.94$

Implied Inflation Level Factor

$$Y_{I_L}(t) = -\underbrace{17.22}_{2.15} + \underbrace{0.75}_{0.03}\,(Y_{I_L}(t-1) + 17.22) + \underbrace{1.03}_{0.14}\; Y_{RI}(t) + 2.51\; Z_{I_L}(t)$$

where $Z_{I_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.90$

Real Level Factor

$$Y_{R_L}(t) = -\underbrace{28.05}_{6.29} + \underbrace{0.91}_{0.02}\,(Y_{R_L}(t-1) + 28.05) + \underbrace{0.55}_{0.15}\; Y_{RI}(t) + 2.71\; Z_{R_L}(t)$$

where $Z_{R_L}(t) \sim N(0,1)$

$R^2_{adj} = 0.86$

References

ANDERSON, N., BREEDON, F., DEACON, M., DERRY, A., MURPHY, G. (1996). *Estimating and Interpreting the Yield Curve*. John Wiley & Sons Ltd, England.

ANDERSON, N., SLEATH, J. (1999). New estimates of the UK real and nominal yield curves. *Bank of England Quarterly Bulletin*.

ANDERSON, N., SLEATH, J. (2001), New estimates of the UK real and nominal yield curves, *Bank of England Working Paper*, **126**.

ANG, A., PIAZZESI, M. (2001). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Manuscript, Columbia University*.

ANG, A., PIAZZESI, M. (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary Economics*, **50**, 745–787.

ANG, A., PIAZZESI, M., WEI, M. (2006). What does the Yield Curve Tell us about GDP Growth?. *Journal of Econometrics*, **131**, 359–403.

ANG, A., BEKAERT, G. (2003), The Term structure of Real Rates and Expected Inflation, *Columbia University and NBER Working Paper*, **12930**.

ANG, A., BEKAERT, G., WEI, M. (2008). The Term Structure of Real Rates and Expected Inflation. *Journal of Finance*, **63(2)**, 797–849.

BANK OF ENGLAND (2002). Notes on the Bank of England UK Yield Curves.

BANK OF ENGLAND (2010). http://www.bankofengland.co.uk/statistics/yieldcurve/.

BLANCHARD, O. (2006). *Macroeconomics*. Prentice Hall.

Booth, P. J., Chadburn, R., Cooper, D., Haberman, S., James, D. (1999). *Modern Actuarial Theory and Practice*. Chapman & Hall/CRC.

Box, G. E. P., Jenkins, G. M. (1976). *Time Series Analysis, Forecasting and Control*. Holdon-Day, San Francisco.

Cairns, A. J. G. (1998). Descriptive Bond Yield and Forward-Rate Models for the British Government Securities' Market. *British Actuarial Journal*, **4(2)**, 265–321 and 350–383.

Cairns, A. J. G., Pritchard, D. J. (2001). Stability of descriptive models for the term structure of interest rates with application to German market data. *British Actuarial Journal*, **7**, 467–507.

Cairns, A. J. G. (2004a). Stochastic Investment Models. *Encyclopedia of Actuarial Science*, **3**, 1602.

Cairns, A. J. G. (2004b). *Interest Rate Models: An Introduction*. Princton University Press.

Chan, W. S. (2002). Stochastic Investment Modelling: A Multiple Time Series Approach. *British Actuarial Journal*, **8(III)**, 545–591.

Chatfield, C. (2004). *The Analysis of Time Series: An Introduction*. Chapman & Hall/CRC.

Clarida, R., Gali, J., Gertler, M. (2000). Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory. *Quarterly Journal of Economics*, **65**, 147–180.

Clarkson, R. S. (1991). A Non-Linear Stochastic Model for Inflation. *Transactions of the 2nd AFIR International Colloquium, Brighton*, **3**, 233–253.

Dai, Q., Philippon, T. (2005). Government Deficits and Interest Rates: A No-Arbitrage Structural VAR Approach. *New York University Working Paper*.

Daykin, C. D., Pentikainen, T., Pesonen, M. (1994). *Practical Risk Theory for Actuaries*. Chapman & Hall.

Dewachter, H., Lyrio, M. (2006). Macro Factors and the Term Structure of Interest Rates. *Journal of Money, Credit and Banking*, **38(1)**, 119–140.

DIEBOLD, F. X., LI, C. (2006). Forecasting the Term Structure of Government Bond Yields. *Journal of Econometrics*, **130**, 337–364.

DIEBOLD, F. X., LI, C., YUE, V. Z. (2007), Global Yield Curve Dynamics and Interactions: A Dynamic Nelson-Siegel Approach, *PIER Working Paper*, **07–030**.

DIEBOLD, F. X., RUDEBUSCH, G. D., ARUOBA, S. B. (2006). The Macroeconomy and the Yield Curve: A Dynamic Latent Factor Approach. *Journal of Econometrics*, **131**, 309–338.

DIEBOLD, F. X., PIAZZESI, M., RUDEBUSCH, G. D. (2004). Modelling Bond Yields in Finance and Macroeconomics. *American Economic Review Papers and Proceedings*.

ENGLE, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, **50**, 987–1008.

EVANS, M. D. D. (1998). Real Rates, Expected Inflation and Inflation Risk Premia. *Journal of Finance*, **53**, 187–218.

EVANS, C. L. , MARSHALL, D. (1998). Monetary Policy and the Term Structure of Nominal Interest Rates: Evidence and Theory. *Carnegie-Rochester Conference Series on Public Policy*, **49**, 53–111.

EVANS, C. L. , MARSHALL, D. (2001), Economic Determinants of the Nominal Treasury Yield Curve, *FRB of Chicago Working Paper*, **16**.

FENGLER, M. R., HARDLE, W., VILLA, C. (2004). The Dynamics of Implied Volatilities: A Common Principal Component Approach. *Review of Derivatives Research*, **6(3)**, 179–202.

FISHER, I. (1907). *The Rate of Interest.* The Macmillan Company, New York.

FISHER, I. (1930). *The Rate of Interest.* The Macmillan Company, New York. Reprinted (Kelley), 1955.

FLURY, B. (1988). *Common Principal Components and Related Multivariate Models.* John Wiley & Sons, Inc., United States of America.

GEOHEGAN, T. J., CLARKSON, R. S., FELDMAN, K. S., GREEN, S. J., KITTS, A., LAVECKY, J. P., ROSS, F. J. M., SMITH, W. J., TOUTOUNCHNI, A. (1992).

Report on the Wilkie Stochastic Investment Model. *Journal of the Institute of Actuaries*, **119–II**, 173–228.

HARDY, M. (2003). *Investment Guarantees.* John Wiley & Sons, Inc.

HARDY, M. (2004). Wilkie Investment Model. *Encyclopedia of Actuarial Science*, **3**, 1750–1758.

HARRIS, G. R. (1995), A Comparison of Stochastic Asset Models for Long Term Studies, *Institute of Actuaries of Australia Quarterly Journal*, **43–75**.

HOMER, S. (1963). *A History of Interest Rates: 200 B.C. to the Present.* Rutgers University Press, New Brunswick.

HÖRDAHL, P., TRISTANI, O., VESTIN, D. (2006). A Joint Econometric Model of Macroeconomic and Term Structure Dynamics. *Journal of Econometrics*, **131 (1-2)**, 405-444.

HUBER, P. (1997). A Review of the Wilkie's Stochastic Investment Model. *British Actuarial Journal*, **3–I**, 181–210.

HUBER, P. P., VERRAL, R. J. (1999). The Need for Theory in Actuarial Economic Models. *British Actuarial Journal*, **5–II**, 377-395.

JACKSON, J. E. (1991). *A User's Guide To principal Components.* John Wiley & Sond, Inc., United states of America.

JOHNSON, G. (2005). Government of Canada Yield-Curve Dynamics, 1986-2003. *Bank of Canada Review Article.*

JOLLIFFE, I . T. (1986). *Principal Component Analysis.* Springer-Verlag New York Inc.

KAMINSKA, I. (2008), A No-Arbitrage Structural Vector Autoregressive Model of the UK Yield Curve, *Bank of England Working Paper*, **357**.

KITTS, A. (1990). Comments on A Model of Retail Price Inflation. *Journal of the Institute of Actuaries*, **117**, 407–413.

KOZICKI, S., TINSLEY, P. A. (2001). Shifting Endpoints in the Term Structure of Interest Rates. *Journal of Monetary Economics*, **47**, 613–652.

Lee, P., Wilkie, A. D. (2000), A comparison of stochastic asset models, *Proceedings of the 10th International AFIR Colloquium, Tromso, Norway*, **447–445**.

Lildholdt, P., Panigirtzoglou, N., Peacock, C. (2007), An-Affine Macro-Factor Model of the UK Yield Curve, *Bank of England Working Paper*, **322**.

Litterman, R., Scheinkman, J. (1991), Common Factors Affecting Bond Returns, *Journal of Fixed Income*, **54–61**.

Ludvic, P. M. (1993), The Wilkie Model Revisited, *Proceedings for the 3rd AFIR Colloquium*, **717**.

Mills, T. C. (1990). *Time Series Techniques for Economists.* University of Cambridge.

Moraux, F., Perignon, C., Villa, C. (2002). Simultaneous Modelling of Domestic Term Structures of Interest Rates. *To appear in Applied Microeconomics Workshop, Renne.*

Nam, N., J. (2004). Sensitivity of Results in Stochastic Asset Liability Modelling, MSc Thesis. *City University, London.*

Nelson, C. R., Siegel, A. F. (1987). Parsimonious Modelling of Yield Curves. *Journal of Business*, **60**, 473–489.

OECD Quarterly Output Revision Data Base (2010). http://www.oecd.org/.

Perignon, C., Smith, D. R., Villa, C. (2007). Why Common Factors in International Bond Returns Are Not So Common. *Journal of International Money and Finance*, **26**, 284–304.

Rambaruth, G. (2003). A Comparison of Wilkie-Type Stochastic Investment Models. *City University, London.*

Ranne, A. (1998). The Finnish Stochastic Investment Model. *Transactions of 26th International Congress of Actuaries*, **7**, 213–238.

Rudebusch, G. D., Wu, T. (2004), A Macro-Finance Model of the Term Structure, Monetary Policy and the Economy, *Federal Reserve Bank of San Francisco Working Paper Series*, **17**.

Rudebusch, G. D., Wu, T. (2008). A Macro-Finance Model of the Term Structure, Monetary Policy and the Economy. *Economic Journal*, **118**, 906–926.

Sahin, S., Cairns, A. J. G., Kleinow, T., Wilkie, A. D. (2008). Revisiting the Wilkie Investment Model. *Proceedings of the 18th AFIR Coloquium, Rome, Italy.*

Taylor, J. B. (1993). Discretion versus Policy Rules in Practice. *Carnegie-Rochester Conference Series on Public Policy*, **39**, 195–214.

Thomson, R. J. (1996). Stochastic Investment Models: The Case of South Africa. *British Actuarial Journal*, **2(3)**, 765–801.

Tong, H. (1990). *Non-Linear Time Series: A Dynamical System Approach.* Oxford University Press.

Tosetto, E. (2008), Revisions of Quarterly Output Gap Estimates for 15 OECD Member Countries, *OECDEconomicOutlookpublicationsanddatabasesfromEO*, **60–83**.

Van Wijck, T. (2006). Interest Rate Model Theory with Reference to the South African Market, Master of Commerce in Financial Risk Management. *University of Stellenbosch, Matieland.*

Waggoner, D. (1997), Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices, *Federal Reserve Bank of Atlanta, Working Paper Series*, **97–110**.

Whitten, S. P., Thomas, R. G. (1999). A Non-Linear Stochastic Model for Actuarial Use. *British Actuarial Journal*, **5(V)**, 919–953.

Wilkie, A. D. (1986). A Stochastic Investment Model for Actuarial Use. *Transactions of the Faculty of Actuaries*, **39**, 341–403.

Wilkie, A. D. (1995). More on a Stochastic Asset Model for Actuarial Use. *British Actuarial Journal*, **1**, 777–964.

Wilkie, A. D., Waters, H. R., Yang, S. (2003). Reserving, Pricing and Hedging for Policies with Guaranteed Annuity Options. *British Actuarial Journal*, **9, Part II, 41**, 263–425.

Wilkie, A. D., Sahin, S., Cairns, A.J.G., Kleinow, T. 2010. *Yet More on a Stochastic Economic Model: Part 1: Updating and Refitting, 1995 to 2009.* Annals of Actuarial Science. *Accepted.*

Wu, T. (2002), Monetary Policy and the Slope Factors in Empirical Term Structure Estimations, *Federal Reserve Bank of San Francisco Working Paper*, **2002-07**.

Yakoubov, Y. H., Teeger, M. H., Duval, D.B. (1999). A Stochastic Investment Model for Asset and Liability Management. *Proceedings of the AFIR Colloquium, Tokyo, Japan.*

Yang, S. (2001). Reserving, Pricing and Hedging for Guaranteed Annuity Options. *PhD Thesis, Heriot-Watt University, Edinburgh.*