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Galilean quantum gravity in 2+1 dimensions

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Declaration

I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, except where due acknowledgement is made, and not been submitted for any other degree.

Signature of Georgios Papageorgiou:

Signature of Supervisor:

Contents

Abstract

In this thesis, we study the Galilean limit of gravity in $2+1$ dimensions and give the necessary ingredients for its quantisation. We study two groups that play fundamental role in this thesis, the two-fold central extension of the Galilei and Newton-Hooke groups in 2+1 dimensions and their corresponding Lie algebras. We construct what we call "Galilean gravity in 2+1 dimensions" as the Chern-Simons theory of the Galilei group and generalise this construction to include a cosmological constant which, in the present setting corresponds to the Chern-Simons theory of the Newton-Hooke group. Finally, we apply the combinatorial quantisation program in detail to the Galilei group: we give the irreducible, unitary representation of the relevant quantum double and fully explore Galilean quantum gravity in this setting. We highlight the associated structures for the Newton-Hooke group and provide an outline for a similar quantisation. In doing so, we provide the link between Newton-Hooke gravity, and a deformation of an extension of the Heisenberg algebra that is well-studied.

to my grandfather, Anestis Tsemperlidis

στον παππού μου, Ανέστη Τσεμπερλίδη

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Introduction

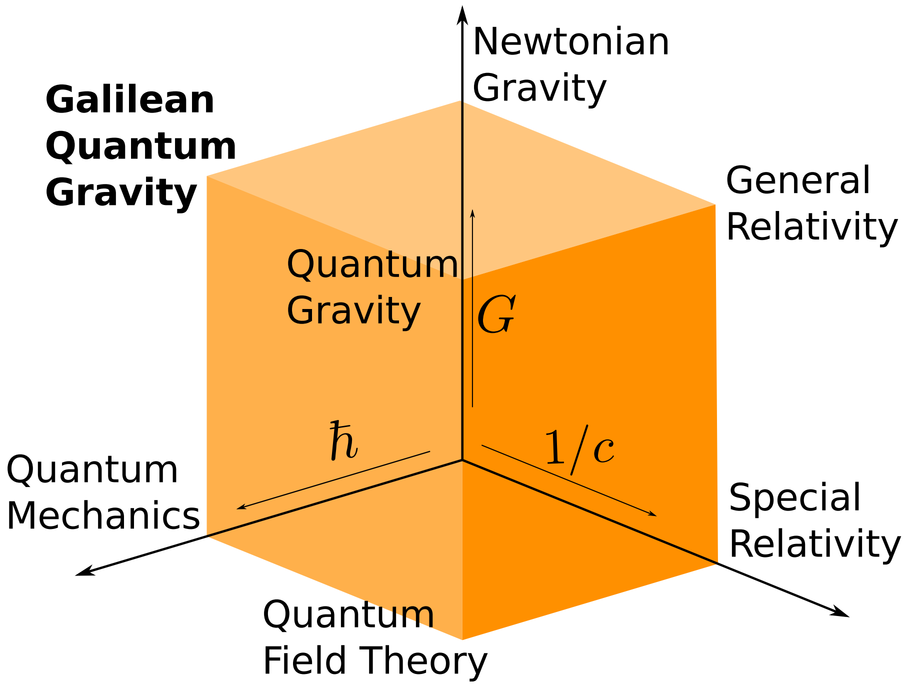

Figure 1. A cube showing the different regimes of physics, based on values of the constants c, G and $\hbar$.

The relation between the different regimes of physics and the values of the fundamental constants c, G and $\hbar$ is frequently displayed in a cube like the one shown in Figure 1. The main goal this thesis sets is to study the physical regime corresponding to the corner of this cube where G and $\hbar$ have their physical values, but $c = \infty$. As a result, one allows for quantum gravitational effects but in a framework whose relativity principle is that of Galileo rather than Einstein. One then speaks of non-relativistic or Galilean quantum gravity which explains the title of this work.

Here, we provide a detailed investigation of Galilean quantum gravity in a universe with two space and one time dimension. We construct a diffeomorphism invariant theory of Galilean gravity in this setting, and show how to quantise it.

One motivation for this work is to demonstrate that this can be done at all. A related motivation comes from a curiosity about the dependence of some of the conceptual and technical problems of quantum gravity on the speed of light. One aspect of this dependence is that Galilean gravity sits between Euclidean and Lorentzian gravity, and is therefore a good place to study the relation between the two. In the context of three-dimensional gravity, where the relative difficulty of Lorentzian to Euclidean signature is related to the non-compactness of the Lorentz group versus the compactness of the rotation group, Galilean gravity may provide a useful intermediate ground, with non-compact but commuting boost generators.

It is part of the lore of three-dimensional gravity that the Newtonian limit is trivial. Indeed, if one considers the geodesic equation for a test particle in 2+1 dimensional gravity and takes the Newtonian limit in the usual way one indeed finds that there are no Newtonian forces on such a test particle [20]. However, this does not mean that the theory becomes trivial in the Galilean limit $c \to \infty$. Gravity in 2+1 dimensions has topological degrees of freedom associated with the topology of spacetime, and topological interactions between particles. An example of the latter is a deflection of a test particle in the metric of a massive particle by an amount which is related to the mass of the latter. Such interactions should survive the Galilean limit.

The question is how to capture topological interactions as $c \to \infty$. Here we approach this problem from the point of view of the Chern-Simons formulation of 2+1 dimensional gravity. In this formulation the isometry group of a model spacetime (which depends on the signature and the value of the cosmological constant) is promoted from global to local symmetry. In the case of Lorentzian signature and vanishing cosmological constant, gravity is thus formulated as a gauge theory of the Poincaré group. One might thus attempt to take the Galilean limit by replacing the Poincaré group with the Galilei group. However, this naive strategy fails since the invariant inner product on the Poincaré Lie algebra, which is essential for writing down the Chern-Simons action, does not have a good limit as $c \to \infty$. In fact, as we shall explain in this thesis, the Lie algebra of the Galilei group in 2+1 dimensions does not possess an invariant (and non-degenerate) inner product.

In order to obtain a group of Galilean symmetries with an invariant inner product we consider central extensions of the Galilei group in two spatial dimensions. Such extensions are ubiquitous in the theoretical description of Galilei-invariant planar systems. One simple reason why central extensions are required to capture all of the physics when taking the Galilean limit is that the proportionality between rest mass and rest energy of relativistic physics is lost when $c \to \infty$. Thus, two independent generators are needed to account for mass and rest or "internal" energy in this limit. In 3+1 dimensions, the Galilei group has only one non-trivial central extension, which is related to the mass.

In 2+1 dimensions, the Lie algebra of the Galilei group has three distinct central extensions [34, 72, 86] and the Galilei group itself has two [54, 17]. In the application to non-relativistic particles, one of the central generators is related to the particle's mass and a second can be related to the particle's spin [74, 44, 45]. The interpretation in terms of spin is subtle and not unique [27, 37], but it is appropriate in the present context. In our formalism the spin parameter is naturally paired with the mass parameter, making manifest an analogy between mass and spin which was stressed in [45]. The third central extension of the Lie algebra does not exponentiate to the Galilei group and does not appear to have a clear physical interpretation [86]; it plays no role in this thesis.

Here we show that the two-fold central extensions of the Galilei group arises naturally in a framework for taking the Galilean limit which automatically preserves the invariant inner product on the Lie algebra of the Poincaré group or, more generally, on the isometry group for the relevant value of the cosmological constant. The invariant product and the associated structures do not appear to have been studied in the literature. As a by-product of working in a wider context, we obtain central extensions with invariant inner products of the so-called Newton-Hooke groups [10, 41], which are symmetries of Galilean spacetimes with a cosmological constant. Using the inner products on the Lie algebras, one can write down Chern-Simons actions for the extended Galilei and Newton-Hooke groups. Indeed we use the Chern-Simons theory for the centrally extended Galilei group with its invariant inner product as our model for classical Galilean gravity and similarly the Chern-Simons theory of the centrally extended Newton-Hooke groups as our model for "cosmological Galilean gravity".

The two-fold central extensions of the Galilei group and the Newton-Hooke groups all have a special property, which is essential for the quantisation of the associated Chern-Simons theory: equipped with their invariant inner product, the Lie algebras of these symmetry groups all have the structure of a classical double. In this thesis our strategy is to describe how the theory of Galilean/Newton-Hooke gravity can be quantised in the Chern-Simons formulation via the combinatorial quantisation method [31, 2, 3, 5] which is Hamiltonian in its approach.

For the combinatorial quantisation method, one starts by constructing the Hopf algebra or quantum group which quantises the given Poisson-Lie structure defined by a classical r-matrix. The Hilbert space of the quantum theory as well as the action of observables and of large diffeomorphisms are then constructed in terms of the representation theory of that quantum group. We carry this program out in full for Galilean gravity, while we give the necessary ingredients for this to be done in Newton-Hooke gravity.

The double structure of the two-fold central extension of the Galilei group has a particularly simple form. The group structure is that of a semi-direct product. The homogeneous part is a central extension of the homogeneous Galilei group (the group of rotations and Galilean boosts in two spatial dimensions), which has played a role in the context of lineal gravity [19] and in the string theory literature, where it is often called Nappi-Witten group [73, 30]. The inhomogeneous (normal, abelian) part is also four-dimensional, and in duality with the Lie algebra of the homogeneous group via the invariant inner product on the Lie algebra. This is precisely the Lie-algebraic data of a classical double. Moreover, classical doubles also have a Lie co-algebra structure, which can be expressed entirely in terms of a classical r-matrix. The r-matrix endows the two-fold extension of the Galilei group with the structure of a Poisson-Lie group.

The two-fold extension of the Newton-Hooke group also has the structure of a classical double. The group can be decomposed into two parts: the central extension of the homogeneous Galilei group, and a certain trivial central extension of a nilpotent group, in duality via the inner product in the Lie algebra. In fact, the classical r-matrix that induces the Newton-Hooke group with a Poisson-Lie group structure is obtained as a deformation (proportional to the cosmological constant) of the r-matrix of the Galilei group; the two differ in their antisymmetric parts. As a consequence, the coalgebra of the Newton-Hooke group is not cocommutative unlike its Galilean counterpart, which cocommutes at the classical level.

In the current context, the relevant quantum group for the quantisation of Galilean gravity is the quantum double of the centrally extended homogeneous Galilei group, which we call the Galilei double. This quantum group is a non-co-commutative deformation of the two-fold extension of the Galilei group. We provide some basic definitions on the combinatorial quantisation program inspired by the papers [78, 18, 12, 67, 68] discussing its application to three-dimensional gravity. We also study the Galilei double and its representation theory, and interpret some of its features in terms of Galilean quantum gravity. In particular we show that the non-commutative addition of momenta in Galilean quantum gravity is captured by the representation ring of the quantum double, and that distance-independent, topological interactions between massive particles can be described in terms of a braid-group representation associated to the quantum double.

Furthermore, exploiting a generic link between non-abelian momentum manifolds and non-commutative position coordinates, we comment on the non-commutative Galilean spacetime associated to the the Galilei double and define the algebra of differential operators on such a non-commutative spacetime as was done in [38].

For the quantisation of Newton-Hooke gravity, things are more complicated. The quantum group that quantises Newton-Hooke gravity, that we will call the Newton-Hooke double, is now the quantum double of a certain deformation of the extended Galilei group. The deformed extended Galilei group itself is well-studied in the literature of quantum groups(see for instance [57, 50]). It is in fact the extended Heisenberg group. In the present context, the deformation parameter has a straightforward physical interpretation in terms of the cosmological constant, much like in the relativistic case[71]. We study it here and assign it a new physical interpretation. There is little known about the quantum double of this group however and we have not dealt with its construction and representation theory.

This thesis is organised in the following way. In Chapter 1 we review the combinatorial quantisation program and give some background and definitions that we will use throughout the rest of the text. Then, in Chapter 2, we describe a unified and, to our knowledge, new framework for studying the model spacetimes of three-dimensional gravity, their isometry groups and the inner products on the corresponding Lie algebras. We use the language of Clifford algebras and show how to obtain the two-fold extension of the Galilei Lie algebra as a contraction limit of a trivial two-fold extension of the Poincaré Lie algebra. Similarly, we obtain the two-fold extensions of the Newton-Hooke Lie algebras as contractions of trivial central extensions of the de Sitter and anti-de Sitter Lie algebras. We study the corresponding spacetimes and explicitly describe them as embeddings and cosets in this language. Finally we provide a detailed account of the algebraic structures of the extended Galilei and Newton-Hooke algebras and their corresponding groups. We explain its relation to the Nappi-Witten group.

In Chapter 3 we review the conjugacy classes of the Nappi Witten group, already studied in [30]. These play an important role in many calculations in this thesis. In particular we show how to deduce coadjoint orbits of the full doubly extended Galilei group from them. We list these orbits (originally studied in [74]) in a more useful notation, and their symplectic structure. Finally, we study the coadjoint orbits of the Newton-Hooke groups and their symplectic structure. These orbits are physically interpreted as the phase spaces of free particles moving in the respective Galilei/Newton-Hooke spacetimes.

Chapter 4 contains the classical Galilean gravitational theory. We introduce the Chern-Simons theory of the doubly extended Newton-Hooke and Galilei groups as our model for classical Galilean gravity with and without a cosmological constant. We explain how to incorporate point particles by minimal coupling of the Chern-Simons actions to coadjoint orbits of the previous section. The phase space of the theory is the space of flat connections, and can be parametrised in terms of holonomies around non-contractible paths. We exhibit the classical r-matrix of the doubly extended Galilei/Newton-Hooke group, and briefly explain how this r-matrix determines the Poisson structure of the phase space in the formalism of Fock and Rosly [31].

In Chapter 5 we study applications of the quantisation of Galilean gravity. We explain the role of the Galilei double in the quantisation. We study the representation theory of the Galilei double and show how the associated braid group representation captures topological interactions between massive particles in Galilean quantum gravity. We also write down the non-commutative Galilean spacetime associated to the Galilei double and study its differential structure. Finally, we describe how the q-deformed Galilei double is the quantisation of Galilean gravity with a cosmological constant and set up the framework for the study of its representation theory, namely study its real structures.

Finally, in Chapter 6 we briefly list directions for future research and conclude.

Chapter 1

Gravity and matter in 2+1 dimensions

In this chapter we collect some important results on 2+1 gravity with a matter content, the structure of its resulting phase-space and its quantisation. We ask some questions regarding the possible limiting (in the Galilean sense) nature of the results we present in this chapter, questions that we aim to answer more concretely in the chapters that follow. In many cases we will be taking the limits $c \to \infty, \Lambda \to 0$, so we explicitly keep any dependence on fundamental constants in the formulae. The chapter is largely based on the works of Deser, Jackiw and 't Hooft [25], E. Witten [85], P. de Sousa Gerbert [23] and Meusburger and Schroers [67].

1.1 Einstein gravity in 2 + 1 dimensions

The fact that makes 3d gravity both simpler but somewhat qualitatively different in nature to its 4d version is that it is a topological theory - it has no local degrees of freedom. Consider the Einstein-Hilbert action for a universe without a cosmological constant

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}R,$$

here we denote by $\mathcal{M}$ the manifold corresponding to our 2 + 1 universe, and we have assumed on it a metric g and its associated Ricci scalar R. The topological nature of the theory can be seen by considering the Einstein equations

$$G_{\mu\nu}\,(= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\; = \frac{1}{16\pi G}T_{\mu\nu}.$$

The number of independent components of the Riemann tensor $R_{\mu\nu\kappa\lambda}$ ($3^2(3^2-1)/12 = 6$ in three spacetime dimensions) is the same as the number of independent components of the Ricci tensor $R_{\mu\nu}$ (again 6 as $R_{\mu\nu} = R_{\nu\mu}$) so the two are linearly dependent in 2 + 1 dimensions. As a consequence, the curvature tensor $R_{\mu\nu\kappa\lambda}$ and the Einstein tensor $G_{\mu\nu}$ are also linearly dependent which further means that empty space-time regions where the energy-momentum content is zero, are flat.

Hence there can be no propagating degrees of freedom; if the cosmological constant is zero, space-time can only be curved in regions where there is matter and is flat otherwise. The impact of this statement is quite strong with regards to the Galilean limit of 3d gravity. Indeed, it has been conjectured [20] that such a limit is trivial; when one tries to perform a weak-field approximation by writing the metric as "approximately flat"

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ is a perturbation to the flat space $\eta_{\mu\nu}$, then looking at the geodesic equation around the spacetime created by a uniform mass density coming from a gravitational potential, one obtains the geodesic equation independent of the gravitational potential

$$\frac{d^2x}{dt^2} = 0.$$

This means that a test mass travelling in the presence of a "weak gravitational field" experiences no gravitational force. This is expected for a topological theory though; in a topological gravitational theory there is no notion of "going far from a source". The effect of a gravitational field is the same regardless the distance (there is no inverse-square law).

The standard type of gravity-induced topological interaction that occurs in relativistic 3d gravity has no reason to vanish at $c \to \infty$, however. This type of (relativistic) interaction is described in a paper by Deser, Jackiw and 't Hooft [25] and is one of the first important results to demonstrate the topological character of 2+1 gravity, where solutions for spacetimes with localised matter content are considered. In that paper, the Einstein equations are solved for point-sources that are either static or spinning. In somewhat more detail, an ansatz for a static metric is considered[25] and the Einstein equations are solved for the classical $2 + 1$ gravity action for static, point-like sources

$$T_{00} = m_i \delta^{(2)}(\vec{r} - \vec{r}_i).$$

Solving the Einstein equations results in a spacetime that is everywhere flat but for the position of the particle

$$g_{pp} = -\mathrm{d}t^2 + \mathrm{d}r^2 + (1 - 4Gm)^2 r^2 \mathrm{d}\varphi^2.$$

This is a cone as can be demonstrated by a coordinate transformation

$$(t, r, \varphi) \longrightarrow (\bar{t}, \overline{r}, \overline{\varphi}) = (t, r, \varphi(1 - 4Gm)),$$

which transforms the metric into its Minkowskian form in polar coordinates, only now the coordinate $\overline{\varphi}$ runs from 0 to $2\pi - 8\pi Gm$. The singularity at $r = 0$ is no longer a coordinate singularity but rather a physical one. The analogous calculation for a spinning source with spin s results in a metric modified as follows:

$$g_{spp} = -(\mathrm{d}t + 4Gs\mathrm{d}\varphi)^2 + \mathrm{d}r^2 + (1 - 4Gm)^2 r^2 \mathrm{d}\varphi^2 \tag{1.1.1}$$

where now the conical structure of the spacetime can be made explicit after the coordinate transformations

$$(t, r, \varphi) \to (\hat{t}, \hat{r}, \hat{\varphi}) = (t + 4Gs\varphi, r, \varphi(1 - 4Gm))$$

so the spin results in some time-shift and the spacetime is still conical, this time with the geometry of a "helical cone". The consequence of this conical geometry, is that geodesics in the presence of a particle become straight lines on a cone i.e. they get deflected by an angle proportional to the mass of the particle. As a result, moving particles experience nontrivial gravitational scattering in the presence of each other.

This has since become one of the best known features of $2 + 1$ gravity; a point particle deforms the spacetime around it by a cone of deficit angle proportional to its mass. This is graphically depicted in Figure 1.1.

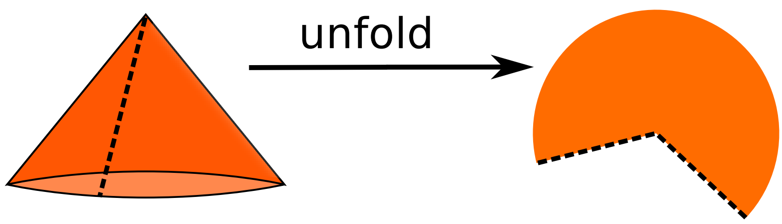

Figure 1.1. The geometry around a three-dimensional point particle; the geometry is everywhere flat but at the position of the particle (in the centre of the coordinate patch we have chosen). Unfolding the cone reveals the deficit angle proportional to the mass of the particle.

The question that arises now, is how does one incorporate such topological features as the ones we have just described, in a Galilean context since a notion of weak fields fails. To dig a little deeper into that question, at least in the approach employed in this thesis, we have to turn to another celebrated result in 3d gravity, its Chern-Simons formulation.

1.2 2+1 gravity as a Chern-Simons theory

The Chern-Simons formulation of three-dimensional gravity was discovered in [1] and elaborated in [85] for general relativity in three dimensions. The relation, at the classical level, between Chern-Simons theory and general relativity is established by writing the three dimensional Einstein-Hilbert action

$$I = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}R,$$

in the Cartan formulation. The dreibein e^a and spin connection ω^a one-forms (where we assume the indices take values on the Poincaré Lie algebra as we will soon explain) become the fundamental variables. The Einstein-Hilbert action can then be shown to be the action of a certain gauge theory [85]. To show this, one needs to write down a connection one-form

$$A = e^a P_a + \omega^a J_a, \tag{1.2.1}$$

where P_a, J_a form a basis for the Lie algebra $\mathfrak{p}_3$

$$[P_a, J_b] = \epsilon_{abc}P^c \qquad [J_a, J_b] = \epsilon_{abc}J^c \qquad [P_a, P_b] = 0, \tag{1.2.2}$$

of the Poincaré group $P_3 = ISO(2,1) = SO(2,1) \ltimes \mathbb{R}^3$, and we have defined $\epsilon_{012} = 1$.

This algebra admits a nondegenerate invariant bilinear form

$$\langle J_a\ ,\ P_b\rangle = \eta_{ab} \qquad \langle J_a\ ,\ J_b\rangle = \langle P_a\ ,\ P_b\rangle = 0, \tag{1.2.3}$$

where $\eta = \mathrm{diag}(-1/c^2, 1, 1)$. The Chern-Simons action for the gauge field A

$$\begin{aligned} I_{CS}[A] =& \frac{1}{16\pi G}\int_{\mathcal{M}} \langle A \wedge \mathrm{d}A\rangle + \frac{1}{3}\langle A\wedge[A,A]\rangle \\ =& \frac{2}{16\pi G}\int_{\mathcal{M}} e^a \wedge \big(\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\wedge\omega^c\big) \\ =& \frac{2}{16\pi G}\int_{\mathcal{M}} \omega^a \wedge \big(\mathrm{d}e_a + \frac{1}{2}\epsilon_{abc}\omega^b\wedge e^c\big), \end{aligned} \tag{1.2.4}$$

is precisely the Einstein-Hilbert action of 3d gravity (up to boundary terms).

In a more general context i.e. for the isometry group H (with corresponding Lie algebra $\mathfrak{h}$) of some three dimensional manifold $\mathcal{M}$, the Chern-Simons theory of the $\mathfrak{h}$-valued one-form A has the same action as the gravitational action on $\mathcal{M}$. Any orientation-preserving diffeomorphism of $\mathcal{M}$ leaves the Chern-Simons three-form invariant. Mathematically, A is a connection one-form of a trivial, principal $H-$bundle over $\mathcal{M}$. Depending on the isometry group considered the Chern-Simons formulation has been used in the literature to describe Euclidean, hyperbolic, spherical, deSitter, anti-deSitter and Poincaré gravity.

Note that in order to define a Chern-Simons action, one needs an invariant bilinear form $\langle\ ,\ \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ on $\mathfrak{h}$ where we have used the notation $\mathfrak{h}^*$ for the vector space dual to $\mathfrak{h}$. It should be noted however, that finding an invariant bilinear form is not always a trivial task. The Killing form of the algebra $\mathfrak{h}$ can be used when the group is semi-simple but this is degenerate for groups that are not semi-simple. This is only one of the nontrivial issues that arise in discussing the Chern-Simons formulation of 3d gravity. Let us go back to the discussion of Lorentzian $2+1$ gravity to comment further.

The equations of motion for A, in (1.2.4) read

$$F(A) = 0 \tag{1.2.5}$$

where the curvature of A is

$$F(A) = \frac{1}{2}[D, D]A = \mathrm{d}A + \frac{1}{2}[A, A], \qquad D := d + [A, -]. \tag{1.2.6}$$

We say that the equations of motion are solved for flat P_3-connections on $\mathcal{M}$. To see how this translates to Einstein gravity, we need to write the curvature of (1.2.6) as

$$F(A) = R_a(\omega)J^a + T_a(e)P^a.$$

Looking at the two different (equivalent) forms of the action (1.2.4) we can see that variation with respect to e yields the condition for zero curvature:

$$R_a = \mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c = 0 \tag{1.2.7}$$

while variation with respect to ω yields the condition for zero torsion

$$T_a = \mathrm{d}e_a + \epsilon_{abc}\omega^b \wedge e^c = 0. \tag{1.2.8}$$

Equations (1.2.7), (1.2.8) are precisely the Einstein equations in the Cartan formalism. A subtlety is that they involve the inverse of the dreibein which in turn implies that in order to employ the gauge-theoretic description of gravity, we have made the silent assumption that the dreibein is invertible.

Let us now look at the symmetries of the Chern-Simons action (1.2.4). The action is invariant under infinitesimal gauge transformations

$$\delta A = -Dw \qquad w = u^a P_a + v^a J_a, \quad D := d + [A, -] \tag{1.2.9}$$

or, explicitly

$$\begin{aligned} \delta e^a &= -\mathrm{d}u^a - \epsilon^a{}_{bc} e^b u^c - \epsilon^a{}_{bc} e^b v^c \\ \delta\omega^a &= -\mathrm{d}v^a - \epsilon^a{}_{bc}\omega^b v^c. \end{aligned} \tag{1.2.10}$$

The above can be split into two sets of gauge transformations; the local translations

$$\begin{aligned} \delta e^a &= -\mathrm{d}u^a - \epsilon^a{}_{bc} e^b u^c \\ \delta\omega^a &= 0, \end{aligned} \tag{1.2.11}$$

and the local Lorentz transformations

$$\begin{aligned} \delta e^a &= -\epsilon^a{}_{bc} e^b v^c \\ \delta\omega^a &= -\mathrm{d}v^a - \epsilon^a{}_{bc}\omega^b v^c. \end{aligned} \tag{1.2.12}$$

These coincide with the diffeomorphism invariance of Einstein gravity, when the dreibein is nondegenerate. By using Cartan's identity that relates the Lie derivative $\mathcal{L}_X$ along a vector field X with the exterior derivative, an infinitesimal diffeomorphism for the gauge field A is

$$\mathcal{L}_X A = \mathrm{d}(\iota_X A) + \iota_X \mathrm{d}A, \tag{1.2.13}$$

where we have denoted by $\iota_X B$ the contraction of the vector field X with the n-form B. Using the flatness of the curvature (equation of motion), the second term becomes $-\frac{1}{2}\iota_X[A, A] = [A, \iota_X A]$, and the diffeomorphism is an infinitesimal gauge transformation

$$\mathcal{L}_X A = \mathrm{d}(\iota_X A) + [A, \iota_X A] = D(\iota_X A), \tag{1.2.14}$$

where since we used the equations of motion, the identification holds only on-shell. One then obtains the reduced phase space of gravity as the space of physically distinct (i.e. not related by a gauge transformation) field configurations. Mathematically, this is the moduli space of flat H-connections, modulo gauge transformations.

A subtle point of much debate, first addressed in [62], arises when one considers large (not reducible to a series of infinitesimal) gauge transformations. It is argued there that there exist field configurations (multiparticle systems) where the reduced phase space does not coincide with that of Einstein gravity. This happens because for these configurations, large gauge transformations identify physically distinct system states. In particular, physical states that are separated by states where (e, ω) are non-invertible and hence not accounted for in the gauge theory. Two such states would then be related to each other via a gauge transformation and wrongly identified as describing the same physical state if one adopted the gauge-theoretic interpretation.

Having said that, it can be shown that for universes of the form $\mathbb{R} \times \Sigma$, where Σ has no punctures, the phase space one obtains from the gauge theory and Einstein gravity coincide [65]. Here, we restrict our analysis to spacetimes of topology $\mathbb{R} \times \Sigma$, where Σ is an oriented, two-dimensional manifold without boundary.

The above consideration will prove quite natural in the Galilean context as we shall see. The consequence of such an assumption is, in analogy to the ADM construction (see, for example [20]) one can write the gauge field as the sum of a one-form on $\mathbb{R}$ (here with global coordinate x^0) and a one-form on Σ, A_Σ as

$$A = A_0 \mathrm{d}x^0 + A_\Sigma,$$

where A_Σ may depend on x^0. The Chern-Simons action then becomes

$$I_{CS}[A_\Sigma, A_0] = \frac{1}{8\pi G}\int_{\mathbb{R}} \mathrm{d}x^0 \int_\Sigma \big(\langle \partial_0 A_\Sigma \wedge A_\Sigma\rangle + \langle A_0, F_\Sigma\rangle\big), \tag{1.2.15}$$

where we have denoted the curvature two-form of A_Σ as

$$F_\Sigma = \mathrm{d}_\Sigma A_\Sigma + A_\Sigma \wedge A_\Sigma.$$

The function A_0 is to be interpreted as a Lagrange multiplier, and varying (1.2.15) with respect to A_0 yields the flatness condition for F_Σ

$$F_\Sigma = 0 \tag{1.2.16}$$

while variation with respect to A_Σ yields the evolution equation

$$\partial_0 A_\Sigma = \mathrm{d}_\Sigma A^0 + [A_\Sigma, A_0]. \tag{1.2.17}$$

Together, (1.2.16), (1.2.17) imply the original equation of motion $F(A) = 0$.

1.3 Gravitating particles in the Chern-Simons formulation

As we discussed earlier, because of the topological nature of 3d gravity, the inclusion of point particles in a 3d universe (without a cosmological constant) has the effect of leaving the spacetime flat but generating a conical singularity at the position of the particle; the particle punctures the spacetime creating a cone with a deficit angle proportional to its mass. We will now view this result in a more geometric context from the point of view of Chern-Simons theory. As is usually the case in the 3d gravity literature, we will be using the terms "puncture" and "particle" interchangeably. Primarily, we will concern ourselves with interpreting the phase space of a free point particle as a coadjoint orbit in Section 1.3.2, and then couple the particle action to the Chern-Simons action by minimal coupling. The particles in this context become topological defects (punctures) of the spacelike manifold Σ, where as we said, the spacetime has the topology $\mathbb{R} \times \Sigma$.

1.3.1 Mathematical preliminary: Symplectic structures and coadjoint orbits

Adjoint and coadjoint actions

Let us give here the necessary definitions before we go on to explain Kirillov's coadjoint orbit method (see [48], [60] or for a more pedagogical discussion of the applications in 3d gravity, [23], [79]). This method relates the coadjoint orbits of a group H to the phase space where quantisation yields the irreducible unitary representations of that group. We give some formal definitions below, keeping in mind that we can apply them to the geometric and algebraic structures we will need in this thesis. However, it should not be assumed that the following analysis is complete.

For a Lie group H and its Lie algebra $\mathfrak{h}$, the group has a natural action on $\mathfrak{h}$, called the adjoint action, defined as

$$\mathrm{Ad}_g T = g\,T\,g^{-1}, \quad T \in \mathfrak{h}, g \in H.$$

The coadjoint action is defined as the dual of the map Ad

$$\langle {\mathrm{Ad}^*}_g \, \hat{S} \;,\; T \rangle = \langle \hat{S} \;,\; \mathrm{Ad}_{g^{-1}} T \rangle, \quad \hat{S} \in \mathfrak{h}^*, T \in \mathfrak{h},$$

where we have denoted the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ as $\langle\ ,\ \rangle$. Then for a given element $\hat{S} \in \mathfrak{h}^*$ the coadjoint orbits through $\hat{S}$ are obtained as the images of the map

$$g \to {\mathrm{Ad}^*}_g \, \hat{S}, \tag{1.3.1}$$

where the usual notation is

$$\mathcal{O}_{\hat{S}} = \{ {\mathrm{Ad}^*}_g \, \hat{S} | g \in H \}.$$

Consider now the derivative (push-forward) of Ad^*. This can be defined as the dual of the derivative of Ad. It is normally denoted ad and it defines a map

$$(\mathrm{Ad}_g)_* = \mathrm{ad}_S : \mathfrak{h} \to \mathfrak{h} \tag{1.3.2}$$

with $\mathrm{ad}_S T = [T, S]$, where by S we have denoted the Lie algebra element corresponding to g. Similarly, we define ${\mathrm{ad}^*}_R$ as the derivative of the map Ad^* $({\mathrm{Ad}^*}_g)_* \hat{S} :=$

${\mathrm{ad}^*}_R\,\hat{S}$. Note that

$$\langle {\mathrm{ad}^*}_R\,\hat{S}\ ,\ T\rangle = \langle \hat{S}\ ,\ -\mathrm{ad}_R T\rangle$$

$$\langle ({\mathrm{Ad}^*}_g)_*\hat{S}\ ,\ T\rangle = \langle \hat{S}\ ,\ (\mathrm{Ad}_{g^{-1}})_* T\rangle = \langle \hat{S}\ ,\ -\mathrm{ad}_R T\rangle = \langle \hat{S}\ ,\ [R,T]\rangle := \langle {\mathrm{ad}^*}_R\,\hat{S}\ ,\ T\rangle \tag{1.3.3}$$

Intrinsically, a coadjoint orbit can be characterised as the quotient

$$\mathcal{O}_{\hat{S}} = H/N_{\hat{S}}, \quad \hat{S} \in \mathfrak{h}^*, N_{\hat{S}} = \{t \in H |\, {\mathrm{Ad}^*}_t\,\hat{S} = \hat{S}\}$$

i.e. for every element $\hat{S} \in \mathfrak{h}^*$, its orbit is the quotient of the group by the centraliser group $N_{\hat{S}}$ that leaves $\hat{S}$ invariant. The fact that $\hat{S}$ is arbitrary, means that one can choose the invariants of $\hat{S}$ under the action of H to label the orbit (as we have silently done above).

The importance of coadjoint orbits of a group H is that they are symplectic manifolds and can be obtained as the reduced phase spaces of particles moving on H. Before we elaborate further, let us give some more definitions that will be of use.

Symplectic and Poisson manifolds

A Poisson manifold is a manifold $\mathcal{M}$ together with a bilinear map $\{\,,\,\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ so that $\{\,,\,\}$ satisfies the following: it is antisymmetric, it satisfies the Jacobi identity and the Leibniz rule. In other words, $C^\infty(\mathcal{M})$ forms a Lie algebra with bracket $\{\,,\,\}$, and also $\{\,,\,\}$ is a derivation in each of its factors. The map $\{\,,\,\}$ is called a Poisson bracket. Equivalently, we can think of a Poisson structure P on a manifold $\mathcal{M}$ as a bi-vector field $P : \wedge^2(T^*\mathcal{M}) \to \mathbb{R}$ so that $\{f\,,\,g\} := P(\mathrm{d}f, \mathrm{d}g)$ is a Poisson bracket on $C^\infty(\mathcal{M})$. In that sense, the data $(\mathcal{M}, P)$ define a Poisson manifold.

One could attempt to use the Poisson structure P to identify the tangent and cotangent spaces of a manifold since $P(A, -\,) \in T\mathcal{M}$ so $P : T^*\mathcal{M} \to T\mathcal{M}, A \mapsto P(A, -\,)$, for any one-form A, but unfortunately no assumption has been made on the non-degeneracy of P and it cannot always be assumed invertible.

One instance one can be guaranteed degeneracy for P is when the manifold is symplectic. Usually the definition is given in terms of a two-form ω rather than a bivector P. So we call a manifold $\mathcal{M}$ symplectic (and write $(\mathcal{M}, \omega)$) if it admits a closed, non-degenerate two form $\omega : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$. The nondegeneracy of ω now induces a truly bijective map, meaning that every vector field on $\mathcal{M}$ is mapped to a one-form on $\mathcal{M}$ through $\omega : T\mathcal{M} \to T^*\mathcal{M}$

$$X \mapsto \omega(X, -\,).$$

Of course, every symplectic manifold is also Poisson since the symplectic two-form ω defines a Poisson bracket via

$$\{f\,,\,g\} := \omega(X_f, X_g) = \mathrm{d}f(X_g) = -\mathrm{d}g(X_f).$$

It can be shown that Poisson manifolds can be partitioned into symplectic leaves - Poisson sub-manifolds which are equipped with a symplectic structure. In particular, using the Leibniz property of $\{\,,\,\}$ we can see that the map $C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), g \mapsto \{g\,,\,f\}$ is a derivation hence by picking a function h which we call a hamiltonian function, there exists a vector field H which accordingly we call a hamiltonian vector field, such that

$$H = \{h\,,\,-\}.$$

The symplectic leaves of a Poisson manifold $\mathcal{M}$ are defined via the hamiltonian vector fields since their integral curves stay on the symplectic leaves. In that sense the symplectic leaves of $\mathcal{M}$ are defined as equivalence classes of points in $\mathcal{M}$ that are joined exclusively by parts of hamiltonian integral curves. See [21], [60] for more details.

Symplectic structure of coadjoint orbits

Consider now the basis $\{T_1, \ldots, T_n\}$ of an n-dimensional Lie algebra $\mathfrak{g}$ of a Lie-group G. Its dual algebra $\mathfrak{g}^*$ spanned by the dual basis $\{T^1, \ldots, T^n\}$ is a Poisson manifold. The Poisson bracket can be identified with the Lie bracket (hence we will be using the name Poisson-Lie bracket) since we can identify the vector space $(\mathfrak{g}^*)^*$ of linear functions on $\mathfrak{g}^*$ with elements of $\mathfrak{g}$. So for any element $m = m_i T_i \in \mathfrak{g}$ we define

$f_m(\hat{S}) = \langle \hat{S} \ , \ m \rangle$ and for any two functions $f, g \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}, \hat{S} \in \mathfrak{g}^*$ the Poisson structure on $\mathfrak{g}^*$ is defined via the Lie bracket on $\mathfrak{g}$

$$\{f \ , \ g\}(\hat{S}) = \langle \hat{S} \ , \ [f, g] \rangle. \tag{1.3.4}$$

In the above basis, we therefore have

$$\{f \ , \ g\} = f^i g^j [T_i, T_j].$$

This can be extended to general functions $f_1, f_2 \in C^\infty(\mathfrak{g}^*)$ by realising that $\mathrm{d}f_1, \mathrm{d}f_2 \in \mathfrak{g}^*$. We then have for the Poisson bracket on $C^\infty(\mathfrak{g}^*)$

$$\{f_1 \ , \ f_2\}(\hat{S}) = \langle \hat{S} \ , \ [\mathrm{d}f_1, \mathrm{d}f_2] \rangle, \qquad \text{for any } \hat{S} \in \mathfrak{g}^*$$

expanding $\hat{S} = \mu_i T^i \in \mathfrak{g}^*$ we have

$$\{f_1 \ , \ f_2\} = \big[T_i \ , \ T_j\big] \frac{\partial f_1}{\partial \mu_i} \frac{\partial f_2}{\partial \mu_j}.$$

This Poisson structure is clearly antisymmetric and bilinear while it is a derivation because of the Leibniz rule for the partial derivatives. The Jacobi identity for $\{ \ , \ \}$ as defined above is slightly less trivial to see directly but can be proven by a direct calculation. The Poisson-Lie bracket above is also known as the Kirillov - Kostant bracket.

Our so far cursory description of the notions of coadjoint orbits and symplectic manifolds aims at demonstrating the following: the symplectic leaves of the dual Lie algebra $\mathfrak{g}^*$ of a group G are its coadjoint orbits. The symplectic potential $\boldsymbol{\theta}$ is induced by the Poisson-Lie structure we defined above

$$\boldsymbol{\theta} = \langle \hat{S} \ , \ g^{-1} \mathrm{d}g \rangle, \quad \hat{S} \in \mathfrak{g}^*, g \in G. \tag{1.3.5}$$

The action of the physical system in question can then be written down by considering a parameter τ along the orbit, as

$$I = \int \mathrm{d}\tau \langle \hat{S} \ , \ g^{-1} \frac{\mathrm{d}g}{\mathrm{d}\tau} \rangle$$

1.3.2 Free particle phase space as a coadjoint orbit

Let us now look at a particular application of the discussion above. Consider a particle moving in Minkowski spacetime $\mathbb{R}^{2,1}$. The Lie group of isometries of the said space is in this example the Poincaré group P_3. Note that the spacetime $\mathbb{R}^{2,1}$ can be realised as the quotient $\mathbb{R}^{2,1} \simeq P_3/SO(2,1)$. The coadjoint orbit method is used in the literature[23] to write down the action for a freely moving particle on Minkowski space. Coadjoint orbits of the Poincaré group P_3 are identified with the particle's reduced phase space i.e. they are the symplectic leaves of the Poisson manifold that is the cotangent bundle $T^*\mathbb{R}^{2,1}$. On each orbit the Casimir functions are constant.

The orbits of P_3 have been calculated and classified in various texts - in particular, using the basis for the Poincaré algebra $\mathfrak{p}_3$ introduced in (1.2.2) we can write

$$\mathcal{O}_{m,s} = \mathrm{Ad}^*{}_g\,(mP_0^* + sJ_0^*) = p^a P_a^* + j^a J_a^*, \tag{1.3.6}$$

while we write $g := (\mathbf{x}, u) \in P_3$ for the conjugating element. The different orbits can be given global coordinates defined by the two vectors $p^a (=: \mathbf{p}), j^a (=: \mathbf{j})$, If $m = 0$ and $s = 0$ they are zero-dimensional while in any other case they are four dimensional. The momenta $\mathbf{p}$ lie on two-dimensional surfaces whose geometry is determined by the values of m - it is either a future or past-directed cone or hyperboloid, or a paraboloid if $\mathbf{p}^2 < 0$. The angular momentum vector $\mathbf{j}$ has a fixed direction determined by $\hat{\mathbf{p}}$. According to the direction $\hat{\mathbf{p}}$ of the momentum, the trajectories of a free particle are represented as timelike, null or spacelike lines in configuration space. The spin acts like a proper-time translation. The phase space in each (non-degenerate) case has the geometry of a tangent bundle of a cone, hyperboloid or paraboloid respectively. For more details and a pedagogical exposition, see [79].

As we said, the Poisson-Lie brackets of the reduced phase space coordinates (p_a, j_a) are, by construction, isomorphic to the Lie algebra $\mathfrak{p}_3$ so

$$\{j_a\,,\, j_b\} = \epsilon_{abc} j^c, \quad \{p_a\,,\, p_b\} = 0, \quad \{j_a\,,\, p_b\} = \epsilon_{abc} p^c.$$

The symplectic potential $\boldsymbol{\theta}$ on the orbits is given by

$$\boldsymbol{\theta} = \langle mP_0^* + sJ_0^*\,,\; g^{-1}\mathrm{d}g\rangle \tag{1.3.7}$$

which we can use to define the action for the free point particle moving on P_3

$$I_{pp} = \int \mathrm{d}\tau \langle m P_0^* + s J_0^* \ , \ g^{-1}\frac{\mathrm{d}g}{\mathrm{d}\tau} \rangle. \tag{1.3.8}$$

Functions on the orbit play the role of observables in the quantum theory. Quantisation of the symplectic structure on the orbits will then lead to irreducible unitary representations of the group P_3.

The benefit of considering the free particle phase space as a coadjoint orbit, is that in this setting its action can easily be coupled to the Chern-Simons action. This has been done in [23] where a spinning, gravitating particle is considered in the Chern-Simons formulation. The coupling is understood as introducing a point-like topological nontriviality in the configuration space of the gauge theory - a puncture. The puncture is decorated with a coadjoint orbit labelled by the physical properties of the particle; its rest mass and spin. The setting that is most appropriate for our study of the phase space of the Chern-Simons theory is developed by Fock and Rosly. Let us now go on to discuss the phase space of Chern-Simons theory in that approach.

1.3.3 The phase space of Chern-Simons theory and the coupling to particles

The phase space of Chern-Simons theory with gauge group H on a manifold $\mathcal{M} = \mathbb{R} \times \Sigma$, is the space of physically distinguishable gauge field configurations. This is the space of flat H-connections (1.2.5) modulo gauge transformations of the form (1.2.9). Adding punctures and handles on the surface Σ, one changes the fundamental group of Σ. This moduli space has a description owing to Fock and Rosly [31] and Alekseev, Grosse and Schomerus [2], [3] that depends only on the fundamental group of the surface Σ together with an appropriate classical r-matrix defined on the quadratic tensor product $\mathfrak{h} \otimes \mathfrak{h}$ of two copies of the Lie-algebra $\mathfrak{h}$ of H. This approach, originally developed for compact gauge groups, was generalised for non-compact gauge-groups in [78] and further in [68].

In the Fock-Rosly approach therefore, the ingredient that does not depend on the topology of spacetime, and is used to determine the Poisson structure of the phase space of Chern-Simons theory, is a classical r-matrix. As we will see, the existence of a classical r-matrix endows the Chern-Simons gauge group algebra with a bialgebra structure and the corresponding group becomes a Poisson-Lie group.

Before we go on to explain further, let us give here as an aside the definition of a Poisson-Lie group. A Poisson-Lie group H is a Poisson manifold for which group multiplication is a Poisson map i.e. for $h_1, h_2 \in H$ and $f, g \in C^\infty(H)$

$$\{f\,,\,g\}(h_1 \cdot h_2) = h_1\,\cdot\,\{f\,,\,g\}(h_2) + \{f\,,\,g\}(h_1)\,\cdot\,h_2.$$

In the right hand side above, we have abbreviated the left- L_{h_1} and right- R_{h_1} multiplication by h_1, h_2 respectively and wrote $h_1 \cdot \{f\,,\,g\}(h_2)$ in place of $\{f \circ L_{h_1}\,,\,g \circ L_{h_1}\}(h_2)$ and similarly for the second term. Let us now proceed to describe in more detail the phase space of the Chern-Simons theory of the Poincaré group. For simplicity, we will assume only one puncture in the surface Σ of the universe $\mathbb{R} \times \Sigma$. The worldline of the particle stretches along the $\mathbb{R}$-direction. Let us write τ for the parameter along this worldline (proper time).

Viewed as a gauge theory, gravity can be coupled to the point-particle by minimal substitution in the particle action - i.e. the puncture on Σ is said to be "decorated with a coadjoint orbit" (see [28]). From the discussion of the previous section, in the action for a free point particle (1.3.8), we have to replace $\frac{\mathrm{d}}{\mathrm{d}\tau} \rightarrow D_\tau = \frac{\mathrm{d}}{\mathrm{d}\tau} + A_\tau$, where, A_τ stands for the component of the gauge field along the parameter τ on the worldline. Therefore, the action I describing the dynamics of a point particle interacting with the Chern-Simons field is given as a sum of two terms

$$I = I_{CS} + I'_{pp},$$

I_{CS} being the Chern-Simons action of (1.2.4) while the coupling is encoded in

$$I'_{pp} = \int \mathrm{d}\tau \langle mP_0^* + sJ_0^*\,,\,g^{-1}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau} + A_\tau\Big)g\rangle \tag{1.3.9}$$

We expect the physics of the system Chern Simons field + point particle to be the same as that of the spinning gravitating particle of Section 1.1. Let us begin to show this by considering the equations of motion for the system. Again, we write the curvature of the gauge field as

$$F(A) = R_a(\omega)J^a + T_a(e)P^a$$

where the curvature two-form R takes values in the Lorentz algebra generated by J^a, and an inhomogeneous torsion two-form T with values in the translation algebra $\mathbb{R}^3$ generated by P^a. The coupling of the particle with the gauge field now implies that m, s act as point-sources of curvature and torsion respectively. Explicitly, we can pick local coordinates $\mathbf{x}$ for Σ, in a given patch. We mark the position of the particle as $\mathbf{x}^* \in \Sigma$. The equations of motion then become

$$F(A) = -8\pi G g(mJ_0 + sP_0)g^{-1}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \delta^{(2)}(\mathbf{x} - \mathbf{x}^*). \tag{1.3.10}$$

Note that above we have identified the generators P_0^*, J_0^* with their duals $-J_0, -P_0$ via the product $\langle\ ,\ \rangle$ and we can explicitly see how the mass m that labels the coadjoint orbit, acts as a source of curvature in the coupled system, while the spin s as a source of torsion

$$R^a J_a = -\, 8\pi G\, \mathrm{Ad}_g m J_0\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \delta^{(2)}(\mathbf{x} - \mathbf{x}^*) \tag{1.3.11}$$

$$T^a P_a = -\, 8\pi G\, \mathrm{Ad}_g s P_0\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \delta^{(2)}(\mathbf{x} - \mathbf{x}^*). \tag{1.3.12}$$

The existence of a puncture at $\mathbf{x}^*$ on the surface Σ results in the introduction of a non-contractible loop on Σ. As a result, the holonomy of the gauge field surrounding the puncture becomes nontrivial. The holonomy is defined as the path ordered exponent

$$h = P\exp\big(\oint A\big),$$

and it is a group element $h \in P_3$. With respect to any basepoint $\mathbf{x}^1$ on Σ we can find a group element $g \in H$ such that the holonomy h_* with basepoint $\mathbf{x}^1$ is

$$h_* = \mathrm{Ad}_g \exp\big(-8\pi G(\ mJ_0 + sP_0)\big) \tag{1.3.13}$$

the holonomy $h_\dagger$ with basepoint $\mathbf{x}^2$ will then be related to h_* via conjugation with a group element g':

$$h^\dagger = \mathrm{Ad}_{g'} h_*$$

i.e. since $\mathcal{M}$ is connected, the holonomies will lie in fixed conjugacy classes of P_3 labelled by m, s. We have graphically depicted this in Figure 1.2.

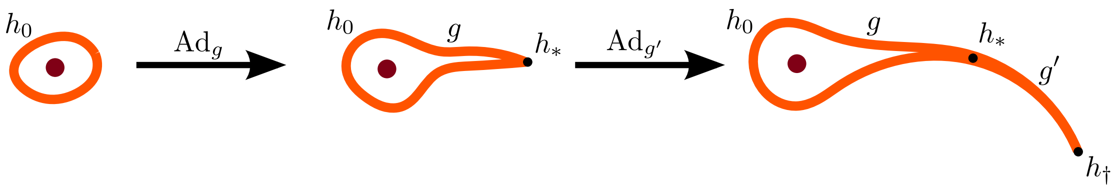

Figure 1.2. The holonomy of a puncture with respect to three different basepoints can be realised through conjugation of appropriate group elements.

As we said, Ad_g corresponds to a diffeomorphism so the infinitesimal holonomy

$$h_0 = \exp\bigl(-8\pi G(mJ_0 + sP_0)\bigr)$$

can be understood as a particle positioned at the origin while the conjugation (1.3.13) with respect to a group element g has the physical meaning of a coordinate transformation to a basis with basepoint $\mathbf{x}^*$. The phase space should be physically indistinguishable whether a particle lies at two points related to each other via a coordinate transformation (i.e. in the same conjugacy class). Therefore, the physical phase space $\mathcal{P}$ for a gravitating particle should be a holonomy conjugacy class C_* modulo conjugation [79]:

$$\mathcal{P} = C_*/\mathrm{Ad}.$$

In the Fock-Rosly formulation, one obtains the Poisson structure on the phase space by embedding a certain graph on the surface Σ. We will postpone discussing the details of this construction for later chapters, but it is worth mentioning that this Poisson structure takes a different form for handles and punctures of the surface Σ. In fact, the effects on the phase space owing to either of those topological nontrivialities, can be *decoupled* [79]. In particular, for an element of the fundamental group of Σ corresponding to a puncture, the phase space picks up a copy of the Poisson-Lie structure corresponding to the gauge group H^* *dual* to H, while for the two elements of the fundamental group corresponding to a handle, it picks up a copy of the the Poisson structure defined on the so-called Heisenberg double $D_+(H)$ of the gauge group H[68].

The Poisson-Lie group H, its dual and Heisenberg double, all take particularly simple forms if H has the group structure

$$H = F \ltimes \mathfrak{f}^*, \tag{1.3.14}$$

as is the case with the Poincaré group; it is a semidirect product of a homogeneous group (the Lorentz group $SO(2,1)$) and its dual Lie algebra (linear functions on $\mathfrak{so}(2,1)$ realised as the vector space $\mathbb{R}^3$). It is therefore instructive to study the phase space in this case. Our analysis together with more extensive applications, can be found in [68].

Note first that for any group of the form H its algebra $\mathfrak{h}$ is, as a vector space, the direct sum $\mathfrak{h} = \mathfrak{f} \oplus \mathfrak{f}^*$. As we said earlier, in the approach we take here, an additional algebraic structure is required to write down the Poisson structure on the group (1.3.14). Namely one needs an element $r \in \mathfrak{h} \otimes \mathfrak{h}$, the classical r-matrix, satisfying two constraints. On one hand, its symmetric part, should be the quadratic Casimir induced by the bilinear form $\langle\ ,\ \rangle$ where for mutually dual generators T_i, T_i^* (with respect to $\langle\ ,\ \rangle$) the Casimir invariant C is defined as $C = \sum_i T_i T_i^*$. On the other hand the r-matrix has to satisfy the so-called classical Yang-Baxter equation (we give an extensive discussion on how one picks the right r-matrix in Section 2.2). For the moment let us note that, given such a quadratic element r, the algebra $\mathfrak{h}$ acquires more structure - for instance, there is a dual operation to the commutator called the cocommutator and denoted δ, which is defined through the bilinear form $\langle\ ,\ \rangle$

$$\langle \delta(\hat{S})\ ,\ X \otimes Y \rangle := \langle \hat{S}\ ,\ [X, Y] \rangle \qquad \hat{S} \in \mathfrak{f}^*, \quad X, Y \in \mathfrak{f},$$

The classical r-matrix extends these additional structures consistently in the entire algebra $\mathfrak{h}$ - i.e. for the example above, the cocommutator is

$$\delta(T) = (\mathbf{1} \otimes \mathrm{ad}_T + \mathrm{ad}_T \otimes \mathbf{1}) r \qquad \text{for any } T \in \mathfrak{h}.$$

The algebra $\mathfrak{h}$ becomes what is known as a bialgebra. Again, we postpone the formal definitions for Section 2.2 and we turn back to the example of the Poincaré group P_3.

For P_3, the structure of its Lie algebra is $\mathfrak{p}_3 = \mathfrak{so}(2,1) \oplus \mathbb{R}^3$ with the span of $\mathfrak{so}(2,1)$ being $\{J_a\}$ while $\{P^a\}$ spans the inhomogeneous part $\mathbb{R}^3$ of $\mathfrak{p}_3$. These are in duality via the pairing (1.2.3). The r-matrix

$$r = P^a \otimes J_a \tag{1.3.15}$$

equips $\mathfrak{p}_3$ with a bialgebra structure. From the above relations, we have for the cocommutator

$$\delta(J_a) = 0 \qquad\qquad \delta(P^a) = \eta^{ab}\epsilon_{bcd}P^c \otimes P^d. \tag{1.3.16}$$

As we said, for a spacetime with one puncture and no handles, the Fock-Rosly Poisson structure for the phase space can be described in terms of the dual Poisson-Lie structure associated to the dual Lie bialgebra $\mathfrak{p}_3^*$. In this example, for the coordinate functions j^a, p^b parametrising the dual Poisson-Lie group (see [68] for details) the Poisson brackets take the particularly simple form

$$\{j_a\,,\, j_b\} = -\,\epsilon_{abc}j^c \quad \{j_a\,,\, p_b\} = -\,\epsilon_{abc}p^c \quad \{p_a\,,\, p_b\} = 0. \tag{1.3.17}$$

The general formula for the Fock-Rosly Poisson structure on the dual group depends, as we said, on the choice of the classical r-matrix. By writing X^L for the left invariant vector fields on H defined by:

$$X^L f(h) = \frac{d}{dt} f(e^{-tX}h)|_{t=0} \tag{1.3.18}$$

and similarly for the right invariant vector fields X^R:

$$X^R f(h) = \frac{d}{dt} f(he^{tX})|_{t=0}, \tag{1.3.19}$$

for any function f on an element $h \in H$ parametrising the conjugacy class that contains the nontrivial holonomy. The Fock-Rosly Poisson structure on H^* is then given in terms of the bivector

$$B_{FR} = \frac{1}{2}r^{ab}X_a^R \wedge X_b^R + \frac{1}{2}r^{ab}X_a^L \wedge X_b^L + r^{ab}X_a^R \wedge X_b^L. \tag{1.3.20}$$

It depends only on the choice of r-matrix - here (1.3.15) and it is shown in [68] that it produces (1.3.17) as a special case.

For a plethora of n gravitating particles $h_*^{(1)}, h_*^{(2)}, \ldots, h_*^{(n)}$ each lying in a conjugacy class $C_*^{(1)}, C_*^{(2)}, \ldots, C_*^{(n)}$ the phase space $\mathcal{P}_n$ is the quotient

$$\mathcal{P}_n = \{(h_*^{(1)}, h_*^{(2)}, \ldots, h_*^{(n)}) \in C_*^{(1)} \times C_*^{(2)} \times \cdots \times C_*^{(n)} | h_*^{(1)} \cdot h_*^{(2)} \cdot \ldots \cdot h_*^{(n)} = 1\}/\mathrm{Ad}.$$

Other topological "nontrivialities" like handles can be included in this scheme where, as we said, the phase space picks up a copy of the Heisenberg double of the gauge group for each handle we include. For simplicity we choose not to discuss spacetimes with handles in this thesis - it will be clear in Chapter 4 that the results of [67] can be imported directly and such a discussion will not be necessary.

1.4 Combinatorial quantisation

In the combinatorial quantisation program, one needs to construct a Hopf Algebra (quantum group) that quantises the Poisson-Lie structure defined by the classical r-matrix. It is the representations of this quantum group that one uses to define the Hilbert space of the quantum gravity theory, and the action of observables on its states.

After a short introduction on Hopf algebras, we aim to sketch the method for a group of the form $H = F \ltimes \mathfrak{f}^*$, and explain how one obtains the representations of the relevant quantum group from representations of the group itself. The combinatorial quantisation program has been carried out in detail for a group of this form in [68]. It will emerge in Chapter 5 that the results of that program are directly applicable to Galilean gravity so we will reserve technical issues for that section.

The most fundamental aspect of the combinatorial quantisation of Chern-Simons theory with gauge group $H = F \ltimes \mathfrak{f}^*$ is that the quantum group that quantises the

Poisson-Lie structure defined in the previous section, is Drinfel'd's quantum double $D(F)$.

1.4.1 Mathematical preliminary: quantum groups

First of all let us start with some fundamental definitions. The reader familiar with Hopf algebras may skip to the end of this section while more details can be found on any standard text on quantum groups like [21] or [57].

Definition 1.4.1. A *unital algebra* $\mathcal{A}$ over a field $\mathbb{F}$ is a vector space equipped with *associative multiplication* $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$:

$$m \circ (m \otimes 1) = m \circ (1 \otimes m) \tag{1.4.1}$$

and *a unit element* $\eta : \mathbb{F} \to \mathcal{A}$

$$m \circ (\eta \otimes 1) = m \circ (1 \otimes \eta) = 1. \tag{1.4.2}$$

We will use the intuitive shorthand notation

$$m(a \otimes b) = ab, \quad a, b \in \mathcal{A}.$$

Dual to the definition of an algebra is that of a coalgebra

Definition 1.4.2. A *counital colgebra* $\mathcal{C}$ over a field $\mathbb{F}$ is a vector space equipped with a *coassociative comultiplication* $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, where by coassociative it is meant that:

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta \tag{1.4.3}$$

and *a counit element* $\epsilon : \mathcal{C} \to \mathbb{F}$

$$(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta = 1. \tag{1.4.4}$$

The two definitions above can be combined into that of a bialgebra in a uniform way;

Definition 1.4.3. A *bialgebra* $\mathcal{B}$ over a field $\mathbb{F}$ is a vector space that is both an algebra and a coalgebra in a compatible way meaning that m, η are coalgebra homomorphisms, and Δ, ϵ are algebra homomorphisms:

$$\Delta \circ m = m \circ \Delta \qquad \epsilon \circ m = m \circ (\epsilon \otimes \epsilon) \qquad \Delta \circ \eta = (\eta \otimes \eta) \circ \Delta \tag{1.4.5}$$

In addition to the above, if a bialgebra admits a special linear map, it forms a Hopf algebra. So we have the following definition:

Definition 1.4.4. A *Hopf algebra* $\mathcal{H}$ over a field $\mathbb{F}$ is a bialgebra that admits an invertible linear map, *the antipode* $S : \mathcal{H} \to \mathcal{H}$, for which:

$$m \circ (S \otimes 1) \circ \Delta = m \circ (1 \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{1.4.6}$$

The above definitions can be extended to determine the dual $\mathcal{H}^*$ of a Hopf algebra $\mathcal{H}$. Let us use Greek letters to denote elements $\beta \in \mathcal{H}^*$, of the vector space $\mathcal{H}^*$, dual to $\mathcal{H}$ and let us denote the pairing between elements $\beta \in \mathcal{H}^*$ and $h \in \mathcal{H}$ as

$$\langle \beta \ , \ h \rangle := \beta(h).$$

Then we can think of $\mathcal{H}^*, \mathcal{H}$ symmetrically and thus define a Hopf algebra structure via the multiplication m, commultiplication Δ, unit η, counit ϵ and antipode S on $\mathcal{H}$ as follows:

$$\begin{aligned} \langle \Delta(\alpha) \ , \ a \otimes b \rangle =& \langle \alpha \ , \ m(a \otimes b) \rangle, \quad \langle \eta \ , \ a \rangle = \epsilon(a) \\ \langle m(\alpha \otimes \beta) \ , \ a \rangle =& \langle \alpha \otimes \beta \ , \ \Delta(a) \rangle, \qquad \epsilon(\alpha) = \langle \alpha \ , \ \eta \rangle, \end{aligned} \tag{1.4.7}$$

for $\alpha, \beta \in \mathcal{H}^*$ and $a, b \in \mathcal{H}$. Relations (1.4.7) are sufficient to determine the dual bialgebra structure of an algebra $\mathcal{H}$. If the latter is a Hopf algebra the antipode on the dual is determined by

$$\langle S\alpha \ , \ a \rangle = \langle \alpha \ , \ Sa \rangle. \tag{1.4.8}$$

The last definition we need in order to study interesting algebraic structures is that of quasitriangularity.

Definition 1.4.5. A Hopf algebra is said to be *quasitriangular* if there exists an invertible element $R \in \mathcal{H} \otimes \mathcal{H}$ called the quantum $R-$matrix, for which:

$$(\sigma \circ \Delta)(h) = R \cdot \Delta(h) \cdot R^{-1} \tag{1.4.9}$$

for any $h \in \mathcal{H}$ and

$$(\Delta \otimes 1)(R) = R_{13} \cdot R_{23} \qquad\qquad (1 \otimes \Delta)(R) = R_{13} \cdot R_{12} \tag{1.4.10}$$

where $\sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the flip map, i.e.

$$\sigma(h_{(1)} \otimes h_{(2)}) = h_{(2)} \otimes h_{(1)}, \tag{1.4.11}$$

for any $h_{(1)}, h_{(2)} \in \mathcal{H}$. Here, we have introduced the so-called Sweedler's notation where for the coproduct Δh of an element $h \in \mathcal{H}$, we do not make the choice of basis explicit in writing its expansion in $\mathcal{H} \otimes \mathcal{H}$. So in place of

$$\Delta h = \sum_i^n h^i_{(1)} \otimes h^i_{(2)}$$

we simply write

$$\Delta h = h_{(1)} \otimes h_{(2)}$$

and the summation and basis are implied. The element $R_{ij} \in \mathcal{H}^{\otimes n}, i, j \leq n$ is related to the element $R := r_{(1)} \otimes r_{(2)}$ in the following way

$$R_{ij} = \underbrace{\overbrace{\underbrace{1 \otimes 1 \otimes \cdots \otimes r_{(1)}}_{\text{i - elements}} \otimes 1 \otimes 1 \otimes \cdots \otimes r_{(2)}}^{\text{j - elements}} \otimes 1 \otimes \cdots \otimes 1}_{\text{n - elements}}\,. \tag{1.4.12}$$

An $R-$ matrix satisfying (1.4.10) automatically satisfies the quantum Yang-Baxter equation:

$$R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12}. \tag{1.4.13}$$

This is easy to show if one notes that (1.4.9) implies

$$\big((\sigma \otimes 1) \circ (\Delta \otimes 1)\big)(h) \cdot R_{12} = R_{12} \cdot (\Delta \otimes 1)(h),\ \forall h \in \mathcal{H} \tag{1.4.14}$$

so we have

$$\begin{aligned} R_{12} \cdot R_{13} \cdot R_{23} = R_{12} \cdot (\Delta \otimes 1)(R) = &\text{ from (1.4.10)} \\ = (\sigma \otimes 1) \circ (\Delta \otimes 1)(R) \cdot R_{12} = &\text{ from (1.4.14)} \\ = \big((\sigma \otimes 1)(R_{13} \cdot R_{23})\big) \cdot R_{12} = R_{23} \cdot R_{13} \cdot R_{12}. & \end{aligned} \tag{1.4.15}$$

Finally, we need to generalise the notion of unitarity by introducing a Hopf $*$-algebra before we define the quantum double. Let us first remind the reader that a $*$-algebra is an algebra A together with an involution $* : A \to A$ that is an antiautomorphism (i.e. $(hg)^* = g^* h^*$ for any $g, h \in A$). The definition of a Hopf $*$-algebra follows

Definition 1.4.6. *A Hopf $*$-algebra* is a $*$-algebra $\mathcal{H}$ such that for any $h \in \mathcal{H}$ comultiplication is a $*$-homomorphism (i.e. $\Delta h^* = (\Delta h)^{*\otimes *}$), and its counit ϵ and antipode S satisfy

$$\epsilon(h^*) = \overline{\epsilon(h)} \qquad (S \circ *)^2 = \mathrm{id}.$$

Here, conjugation $\overline{k}$ refers to an element $k \in \mathbb{F}$ in the underlying field which in all the cases we will consider will be the complex numbers $\mathbb{C}$. The duality of two Hopf $*$-algebras $\mathcal{H}_1, \mathcal{H}_2$ is established by demanding they are dual Hopf algebras and that any two elements $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$ satisfy

$$\langle h_1^* \ , \ h_2 \rangle = \overline{\langle h_1 \ , \ S h_2 \rangle^*}.$$

The quantum double of a Lie group F is a special type of Hopf algebra and it is the mathematical structure that we will be interested in. The quantum double of a group is by construction, a quasitriangular Hopf algebra whose vector space structure is that of a direct product; between functions on the group $C(F)$, and the group algebra $\mathbb{C}[F]$. We thus give the formal definition before applying it to the special case of $\mathcal{H}$ being a group.

Definition 1.4.7. Given a Hopf algebra $\mathcal{H}$ and its dual $\mathcal{H}^*_{cop}$ with flipped comultiplication $\Delta_{op} := \sigma \circ \Delta$, *the quantum double* $D(\mathcal{H})$ is the unique quasitriangular Hopf algebra with universal quantum $\mathcal{R}$-matrix $\mathcal{R} \in D(\mathcal{H}) \otimes D(\mathcal{H})$ so that:

1. As a coalgebra, $D(\mathcal{H}) = \mathcal{H}^*_{cop} \otimes \mathcal{H}$ i.e. for $\alpha \in \mathcal{H}^*_{cop}, x \in \mathcal{H}$,

$$\Delta(\alpha \otimes x) = (\alpha_{(2)} \otimes x_{(1)}) \otimes (\alpha_{(1)} \otimes x_{(2)})$$

2. As an algebra, the multiplication on $D(\mathcal{H})$ is given by

$$(\alpha \otimes x)(\beta \otimes y) = \alpha\beta_{(2)} \otimes x_{(2)} y \, \langle \beta_{(3)} \ , \ x_{(1)} \rangle \, \langle \beta_{(1)} \ , \ S^{-1} x_{(3)} \rangle$$

for $\alpha, \beta \in \mathcal{H}^*_{cop}$ and $x, y \in \mathcal{H}$

3. Its antipode is

$$S(\alpha \otimes x) = (1 \otimes S^{-1}x)(S\alpha \otimes 1)$$

4. In bases $\{e^i | i = 1, \ldots, n\}$ for $\mathcal{H}$ and $\{\phi^i | i = 1, \ldots, n\}$ for $\mathcal{H}^*_{cop}$ such that $\langle \phi^i \, , \, e^j \rangle = \delta^{ij}$ the quantum $\mathcal{R}$-matrix is

$$\mathcal{R} = \sum_i (\epsilon \otimes e^i) \otimes (\phi^i \otimes 1)$$

The definition above can be extended to include a pair of dual bialgebras $\mathcal{H}, \mathcal{H}^*$ (owing to Majid [57]). In particular it can be applied to a locally compact Lie group G and it is shown in [53] that the quantum double $D(G)$ can be realised as the set $C_0(G \times G, \mathbb{C})$ of continuous functions on $G \times G$ i.e. $D(G) = C_0(G \times G, \mathbb{C})$.

Explicitly the double structure $D(G)$ realised by its action on $G \times G$, has the following multiplication $\cdot$, unit 1, comultiplication Δ, counit ϵ, antipode and star structure $*$ (we adopt here the conventions of [68])

$$\begin{aligned}
(F_1 \cdot F_2)(g,h) &:= \int_G F_1(x,h) F_2(x^{-1}g, \mathrm{Ad}_{x^{-1}} h) \mathrm{d}x \\
1(g,h) &:= \delta_e(g) \\
(\Delta F)((g_1,h_1),(g_2,h_2)) &:= F(g_1, h_1 h_2) \delta_{g_1}(g_2) \\
\epsilon(F) &:= \int_G F(x,e) \mathrm{d}x \\
(SF)(g,h) &:= F(g^{-1}, \mathrm{Ad}_{g^{-1}} h) \\
F^*(g,h) &:= \overline{F(g^{-1}, \mathrm{Ad}_{g^{-1}} h)}
\end{aligned} \tag{1.4.16}$$

where e is the group identity and $g, h \in G$. The subtlety of including δ-functions together with the continuous functions in the definition of the double $D(G)$ is addressed in [78] and [68]. The problem is resolved simply by adjoining elements with a δ-function component to the set of continuous functions. We adopt this approach here so whenever we refer to the quantum double $D(G)$ of some group G we will mean the continuous functions $C_0(G \times G, \mathbb{C})$ together with the singular elements $(\delta_g)(f)$ whose action on $G \times G$ is given by

$$(\delta_g)(f)(g', h) = \delta_g(g')f(h), \quad g, g', h \in G. \tag{1.4.17}$$

The universal $\mathcal{R}$-matrix is

$$\mathcal{R}((g, h), (g', h')) = \delta_e(g)\delta_e(hg'^{-1}) \tag{1.4.18}$$

The quantum double $D(G)$ has quantum $\mathcal{R}$-matrix that is given as a deformation of the classical r-matrix that was used to define the Fock-Rosly dual Poisson structure.

The group $D(G)$ acts on the Hilbert space of quantum states as the relevant symmetry. The deformation parameter is proportional to $\hbar$ as expected but it deserves commenting on for different groups seeing as it is also dependent on the speed of light and the cosmological constant if present. We will therefore reserve comments on the deformation parameter for the relevant sections. The next step in our description of the combinatorial quantisation program is the setup of the Hilbert space and the study of representations of the quantum double and its action on that space.

1.4.2 Representations of semidirect products

Let us now turn to representations of quantum groups. The only kind of representation theory we will encounter in this thesis is that of (the quantum double of) a semidirect product $F \ltimes \mathfrak{f}^*$. Let us first briefly discuss the representations of the group $F \ltimes \mathfrak{f}^*$ itself.

It is known, owing to Wigner, that the fundamental properties of a particle in some spacetime $\mathcal{M}$ label the irreducible representations (irreps) of (the universal cover of) the corresponding isometry group of $\mathcal{M}$. Mackey's work [56] on induced representations, provide us with a standard way to obtain irreps of semidirect products. Irreps of groups of form $H \ltimes N$, where N is an abelian group, are labelled by H-orbits on the dual of N (the space of characters) and irreps of centraliser groups of chosen points on those orbits. In the case of interest to us, the group is $F \ltimes \mathfrak{f}^*$. The dual of the abelian group $\mathfrak{f}^*$, is the Lie algebra $\mathfrak{f}$ itself viewed as a vector space, and the irreps are labelled by adjoint orbits of F together with the associated centraliser groups of points in each orbit.

There are two equivalent ways we can describe the irreps of groups of the above form; one is in terms of functions on their homogeneous group F obeying an equivariance condition. This approach is adopted in [12], [68]. It has the benefits that it has a simpler formula for the action of $F \ltimes \mathfrak{f}^*$ elements, and easily generalises to the quantum group $D(F)$.

Alternatively, we can describe them in terms of functions directly on the orbits. In the literature for the Galilei group, that is the view point that is most commonly used and it is the one we will be adopting here as well. We will make our discussion more concrete in Chapter 5 and particularly, section 5.1. There, we will show how one easily obtains the irreps of the quantum double of our particular choice of a group of the form $F \ltimes \mathfrak{f}^*$ from irreps of the group itself.

We will not address technical issues such as defining an invariant measure on the carrier space of representations of the quantum double etc; these are both cumbersome and in detail addressed in [78] and further in [68]. It is interesting to mention, however, an effect of Poincaré quantum gravity that one expects to remain in Chern-Simons theory with any nonableian gauge group.

In Poincaré quantum gravity particles experience braid interactions. This can be easily understood even at the classical level - two particles scattering each other have initial states described by the holonomies (h_1, h_2) and final states described by (h_1', h_2') as shown in Figure 1.3. The total holonomy remains constant while the interchange of the particles is described by the action of the quantum $\mathcal{R}$-matrix. In the quantum level, this means that the wavefunctions of the particles transform in a nontrivial way upon interchanging them. Similarly, the group-like character of the holonomies means that direct product representations of two or more particles, should obey a set of fusion rules (classically, a spacelike slice with two punctures with holonomies h_1, h_2 is indistinguishable from a single puncture with holonomy their group product

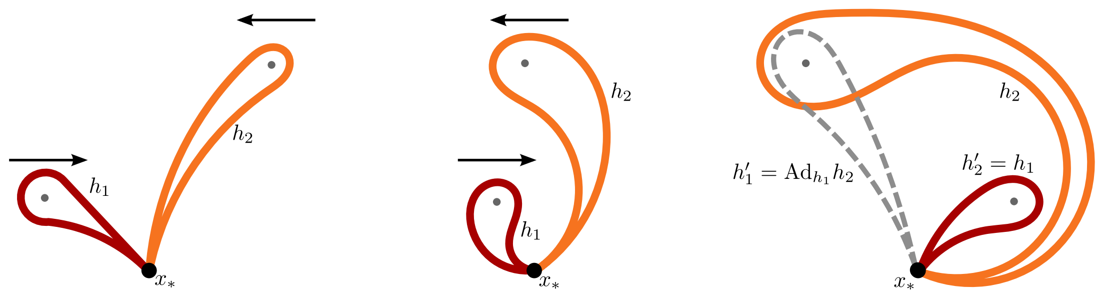

Figure 1.3. The non-contractible loops around two particles scattering. The loops correspond to group elements and the scattering is nontrivial. The particles obey braid group statistics.

$h_1 \cdot h_2$). In Poincaré gravity, this effect leads to upper bounds for the masses of the particles because of creation of Gott pairs [33].

1.5 Including a cosmological constant

Many of the results we have presented have a straightforward generalisation for spacetimes with a cosmological constant. The action for Einstein gravity with a cosmological constant again has the form of Chern-Simons theory [85]. The Einstein-Hilbert action is

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^3 x \sqrt{-g}(R - 2\Lambda). \tag{1.5.1}$$

Remarkably, it coincides with Chern-Simons theory with the same invariant product (1.2.3) with the relevant gauge group either the deSitter group $SO(3,1)$ or the anti-deSitter group $SO(2,2)$ in $2+1$ dimensions. The brackets of the (anti-)deSitter Lie algebra can be written in a uniform way

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P^c, \qquad [P_a, P_b] = -\Lambda \epsilon_{abc} J^c \tag{1.5.2}$$

where depending on the absolute value of Λ one obtains the deSitter algebra if $\Lambda < 0$ or anti-deSitter algebra when $\Lambda > 0$.

The gauge group is then given the Poisson-Lie structure of a classical double with a classical r-matrix that is a deformation of (1.3.15) (dependent on Λ). Consequently, the quantum group that quantises this Poisson-Lie structure is the quantum double of a certain q-deformation of the Lorentz group $SO(2,1)$.[1] In the general case where the gauge group in the absence of a cosmological constant is $F \ltimes \mathfrak{f}^*$, the quantum group that provides the quantisation in the presence of Λ is the quantum double $D_q(F)$ of the q-deformed homogeneous group F. This pattern holds for Euclidean as well as Lorentzian gravity with constant curvature.

A valuable computational tool, first utilised in [71], is that all the algebras of three-dimensional gravity can be realised as stemming from a single Lie algebra, their homogeneous algebra, taken not over the field of real numbers $\mathbb{R}$ but over a different commutative ring. This scheme includes the algebras of isometry groups of constant curvature spaces for both Euclidean and Lorentzian curvatures. We postpone the relevant technical discussion for Chapter 2.

1.6 Galilean gravity: a prelude

Before proceeding, let us take a minute to comment on how the Chern-Simons formulation could be of use to us in our attempt to construct a Galilean theory of 3d gravity. It should be clear from the discussion in this Chapter that the necessary ingredients for one to construct a Chern-Simons theory, are the gauge group H and an invariant bilinear form, $\langle\ ,\ \rangle$ on its Lie algebra $\mathfrak{h}$. It should also be clear that the interpretation of the Chern-Simons theory as a theory of gravity, follows from the action of the group on the spacetime $\mathcal{M}$.

Therefore a good candidate for a Galilean theory of gravity in three dimensions would be the Chern-Simons theory of the group of isometries of Galilean spacetime; the Galilei group. This can be realised as the $c \to \infty$ limit[2] of the algebra (1.2.2). The problem with naively taking the Galilean limit of Poincaré gravity and constructing a Galilean Chern-Simons theory stems from the fact that this leads to degeneracies (or infinities in the case of the inverse metric) of the metric (1.2.3) in the inner product.

[1] See also [51] and references therein for Koelink's work on $D_q(SO(2,1))$

[2] From this point on we will use the term "Galilean limit" and "$c \to \infty$ limit", interchangeably.

In fact, as we will show in Chapter 4 the Galilei algebra does not admit an invariant bilinear form at all. This difficulty can be overcome as we will see, by considering a certain central extension of the Galilei group. In fact in Section 2.1.1, we will provide the general framework by which one can uniformly describe all the algebras of the isometry groups that arise in 3d gravity and show how to construct a nondegenerate invariant bilinear form for each of these algebras. For the moment let us reassure ourselves that the degeneracy in taking the naive limit $c \to \infty$ of equation (1.2.3) does not imply that we should abandon the strategy of employing the Chern-Simons formulation to construct a theory of Galilean gravity.

Let us further comment on the physical interpretation of generators J_a of (1.2.2) that form a basis for the homogeneous algebra $\mathfrak{so}(2,1)$ of the Poincaré group P_3. Their algebraic duals P^a will form a basis for the dual algebra $\mathfrak{so}^*(2,1) \simeq \mathbb{R}^3$ of the translation (inhomogeneous) part of P_3. We point the reader's attention to the index of P^a being upstairs while the index of J_a downstairs. These are, mathematically, the true duals of each other since the bilinear form is equivalent to an inner product

$$\langle J_a \ , \ P^b \rangle = \delta_a^b.$$

One is accustomed to there always being a metric to identify the tangent and cotangent spaces of the manifold $\mathcal{M}$ and little attention is given if one defines the bilinear form of (1.2.3) as $\langle J_a \ , \ P_b \rangle = \eta_{ab}$, where $P_b = \eta_{ab} P^a$. We will see that in the Galilean theory, the absence of a metric plays a vital role in the choice of a bilinear form and assigning physical meaning to each generator - we will comment further on this point at the beginning of Chapter 4.

Chapter 2

The Galilean limit I: classical spacetimes and isometries

Obtaining the Galilean algebra through a process of reduction from the Poincaré algebra is a well-studied subject ([17, 74, 41] and references therein). It seems that the language of Clifford algebras is appropriate to describe such a setting. We further show how both the Galilei and Newton-Hooke algebras can be realised as generalised complexifications; algebras that emerge if one thinks of their homogeneous subalgebra (that plays the role of a "parent" algebra) no longer as an algebra over $\mathbb{R}$ but another field of numbers $\mathcal{R}_\lambda$. Finally we view the classical spacetimes through two different viewpoints; first through a geometrical angle we show how the Galilei spacetimes with Galilean isometry groups can be obtained as embedded hypersurfaces of a certain four-dimensional (Lorentzian) spacetime. Then we construct these spacetimes as appropriate symmetric spaces (cosets of their isometry groups) in a same way as in [32]. The results in this chapter are published in [76] and [77] (in preparation at the timing of writing this).

2.1 The Galilei and Newton-Hooke algebras

The vacuum solutions of the Einstein equations in three dimensions are locally isometric to model spacetimes which depend on the signature of spacetime and on the value of the cosmological constant.

As we shall explain below, each of the model spacetimes relevant in 3d gravity can be realised as a hypersurface in a four-dimensional embedding space with a constant metric. It turns out that these four-dimensional geometries and their Clifford algebras provide a unifying framework for discussing the isometry groups of the model spacetimes arising in 3d gravity. In particular, we focus our attention on spacetimes admitting "absolute time", i.e. spacetimes that can be given a product structure $\mathbb{T} \times \Sigma$ where the manifold $\mathbb{T}$ admits the preferred coordinate t and its one form $\mathrm{d}t$.

There are two algebras relevant to "absolute time" spacetimes. Their corresponding groups are called Galilei and oscillatory/expanding Newton-Hooke groups. As we will show, they are isometries of the classical analogues of flat space and spaces with constant curvature respectively. These have received considerable attention during the recent years mainly from a mathematical standpoint. They are algebras admitting three central extensions and that makes them more interesting to study than their four dimensional counterparts, the latter only admitting one.

Deformations of the Galilei to the Newton-Hooke algebra have been studied both from the point of view of group expansion methods (discussed in [41]) where the deformation takes place in the direction from the Galilei to the Newton-Hooke groups, and contraction (discussed in [29]) where the deformation is considered the other way around. In the same time, one can consider the Newton-Hooke algebras as appropriate contractions of their relativistic analogues; deSitter and anti-deSitter algebras in three dimensions [32].

In this section we explain how to obtain the central extensions of the Galilei and

Newton-Hooke algebras by contraction of a trivial central extension of the algebra of rotations in a 4 dimensional vector space with a special (constant) metric dependance on the cosmological constant and the speed of light. Specifically, we will use the formulation in terms of Clifford algebras in analogy to the discussion in [64], to bring out the geometric interpretation of these algebras as the isometry algebras of classical spacetimes with and without a cosmological constant.

In this approach, the possible invariant bilinear forms on the model spacetimes are naturally associated to the central elements in the even part of the Clifford algebra. For us this framework is valuable because it allows us to take the Galilean limit in such a way that we retain a non-degenerate invariant bilinear form. The construction of a Chern-Simons theory and the implementation of Fock and Rosly's approach to obtain the reduced phase space will then become straightforward and we discuss it in the following chapters.

2.1.1 Embedding spaces, Clifford algebras and symmetries

Consider a four-dimensional vector space V together with an inner product defined by a metric g

$$g_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, \frac{1}{\Lambda}), \tag{2.1.1}$$

where the parameters c, Λ are the speed of light and the cosmological constant respectively. We can define model $3D$ spacetimes as embedded hypersurfaces

$$H_{c,\Lambda} = \left\{ (t, x, y, w) \in V | - c^2 t^2 + x^2 + y^2 + \frac{1}{\Lambda} w^2 = \frac{1}{\Lambda} \right\}. \tag{2.1.2}$$

In suitable limits of the equation (2.1.2), (t, x, y, w) define the spaces of Table 2.1.

	$\Lambda < 0$	$\Lambda \to 0$	$\Lambda > 0$
$c^2 < 0$	H^3	E^3	S^3
$1/c^2 \to 0$		*Absolute time geometries*	
$c^2 > 0$	AdS^3	M^3	dS^3

Table 2.1. The Geometries of the embedding (2.1.2) for different values of Λ, c

Explicitly, the row corresponding to $c^2 < 0$ has as its entries three dimensional hyperbolic space H^3 for $\Lambda < 0$, Euclidean space E^3 for $\Lambda \to 0$ and, the three-sphere S^3 when $\Lambda > 0$. Similarly for $c^2 > 0$ one obtains the (anti-)de-Sitter spaces dS^3, AdS^3 for $\Lambda \neq 0$, while for $\Lambda = 0$ the spacetime is Minkowski space M^3. Here, note that the limit $\Lambda \to 0$, should be taken *after* multiplying the defining equation in (2.1.2) by Λ. In all cases the metric on the hypersurface is induced by the embedding and has scalar curvature equal to Λ (so that $1/\sqrt{|\Lambda|}$ is the curvature radius).

There is a large body of literature, some of it overlooked, that deals with the geometries of Table 2.1 in a unified way. One of the earliest relative references one comes across is a textbook by I. M. Yaglom [87] where the nine geometries are treated as projective geometries. Yaglom's approach makes use of generalised complex numbers and the nine geometries are referred to as Cayley-Klein geometries - we adopt our naming from more modern literature but inspired by Yaglom's approach we will see that generalised complex numbers will be of use to us in section 2.1.1 to make the semi-direct product structure of the groups we are studying more transparent. Other approaches that make use of Clifford algebras have been applied to treat these geometries uniformly (see [64] or [36] for a very pedagogical introduction).

We are interested in the middle row of Table 2.1; the limit $c \to \infty$ is somewhat pathological to obtain since the metric on the ambient space becomes not only degenerate but also infinite in that limit. Such an infinity can, however, be avoided by taking the limit of the inverse metric

$$g^{\mu\nu} = \mathrm{diag}(-\frac{1}{c^2}, 1, 1, \Lambda) \tag{2.1.3}$$

which still degenerates as $c \to \infty$ but remains well-defined as a matrix.

Let us postpone any geometric-oriented discussion associated to the embedded hypersurfaces of (2.1.2) for the end of this chapter and let us first study these hypersurfaces algebraically. The first step in looking at the Clifford algebra $Cl(V, g)$ associated to the embedding vector space (V, g) for finite c and non-zero Λ, (as we said we will work with the inverse metric since our aim is ultimately to take the limit

$c \to \infty$) is to define generators γ^a via

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}. \tag{2.1.4}$$

Lie algebra generators of the symmetry group $SO(V)$[1] of (2.1.3) can be realised as degree two elements in $Cl(V,g)$

$$M^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu], \tag{2.1.5}$$

giving six generators, $M^{12} = \frac{1}{2}\gamma^1\gamma^2$ etc. with commutation relations

$$[M^{\kappa\lambda}, M^{\mu\nu}] = g^{\kappa\mu}M^{\lambda\nu} + g^{\lambda\nu}M^{\kappa\mu} - g^{\kappa\nu}M^{\lambda\mu} - g^{\lambda\mu}M^{\kappa\nu}. \tag{2.1.6}$$

We also write 1 for the identity scalar in the Clifford algebra, and define the volume element

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.1.7}$$

Then the linear span of the even powers (with respect to Clifford multiplication) of the Clifford generators

$$\mathcal{A} = \mathrm{span}\left\{1, \gamma^5, M^{\mu\nu}\right\}_{\mu,\nu=0,\dots 3} \tag{2.1.8}$$

is an algebra with multiplication defined by the Clifford multiplication.

The elements 1 and γ^5 are central. It is easy to see using the (anti-)commutation rules of the γ's that the element γ^5 satisfies

$$(\gamma^5)^2 = -\frac{\Lambda}{c^2} = \det(g^{\mu\nu}) \tag{2.1.9}$$

and acts on $\mathcal{A}$ by Clifford multiplication. As a Lie algebra with brackets defined by commutators, $\mathcal{A}$ is a trivial two-dimensional extension of the Lie algebra $\mathfrak{so}(V)$ (spanned by $M^{\mu\nu}$).

2.1.2 The Galilean limit

For the symmetry algebras corresponding to "absolute time" spacetimes (i.e. the $c \to \infty$ limit) we switch notation to one that brings out the spacetime interpretation

[1]The notation $SO(V)$ accounts for the different signatures of the metric of V. For instance for $c, \Lambda > 0$ the degree two elements of $Cl(V,g)$ form the algebra $\mathfrak{so}(3,1)$

of (2.1.6).

We define the three-dimensional totally antisymmetric tensor with downstairs indices via $\epsilon_{012} = 1$. Note that

$$\epsilon^{abc} = -\frac{1}{c^2}\epsilon_{abc} \tag{2.1.10}$$

and that therefore

$$\epsilon_{abd}\epsilon^{aef} = -\frac{1}{c^2}(\delta^e_b\delta^f_d - \delta^e_d\delta^f_b), \qquad \epsilon_{abc}\epsilon^{dbc} = -\frac{2}{c^2}\delta^d_a. \tag{2.1.11}$$

Then define

$$J_a = \frac{1}{2}\epsilon_{abc}M^{bc}, \qquad P^a = M^{a3}, \tag{2.1.12}$$

and note that these definitions are independent of the metric $g^{\mu\nu}$. The Lie algebra brackets take the form

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P^b] = \epsilon_{a\ c}^{\ b}P^c, \quad [P^a, P^b] = -c^2\Lambda\epsilon^{abc}J_c. \tag{2.1.13}$$

Because of the identity (2.1.10), the right hand side of the last commutator is actually independent of c. As a result, there is no mathematical difficulty in taking the Galilean limit $c \to \infty$ for the Lie algebra in terms of the generators J_a (indices downstairs) and P^a (indices upstairs). However, this would not produce the Galilei Lie algebra. To see why this is so, we need to study the physical interpretation of the various generators. First we note the physical dimensions of the generators with indices upstairs and downstairs:

J_0 : dimensionless, J^0 : 1 /velocity^2, $J_i = J^i$: 1/velocity
P_0: 1/time, P^0: time/length^2, $P_i = P^i$: 1/length.

The Lie algebra with generators J_0, J_1, J_2 contracts to the algebra of rotations and Galilean boosts in the limit $c \to \infty$. However, since $\epsilon_{i\ j}^{\ 0} \to 0$ as $c \to \infty$, the brackets between P^0 and the J_i tend to zero in this limit; if P^0 were a time translation generator in the Galilean limit, then this bracket should give a spatial translation.

Thus P^0 cannot be interpreted as a time translation generator, but P_0 can (this is confirmed by dimensional analysis). Thus, in order to take the Galilean limit we need to write the Lie brackets in terms of P_a, with lowered indices. The brackets now read

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = -c^2 \Lambda \epsilon_{abc} J^c, \tag{2.1.14}$$

which is the usual way of writing down the algebra of spacetime symmetry generators in 3d gravity.

The subtleties that arise in taking the Galilean limit become manifest when one looks at the mass-energy relation in relativistic physics. In relativistic physics, mass and energy are proportional to each other, but the constant of proportionality tends to infinity in the Galilean limit.

Thus, in order to retain generators representing mass and energy we expect the need for some kind of renormalisation when taking the speed of light to infinity. To make contact with the conventions used in Galilean physics we first rename our generators, using the two-dimensional epsilon symbol with non-zero entries $\epsilon_{12} = -\epsilon_{21} = 1$:

$$\tilde{J} = -J_0, \qquad K_i = -\epsilon_{ij} J_j, \qquad \tilde{H} = -P_0. \tag{2.1.15}$$

The generator K_i, $i = 1, 2$, generate boosts in the i-th spatial direction and are widely used in place of the J_i when discussing Galilean physics. Then the algebra (2.1.13) reads

$$\begin{aligned}
&[K_i, K_j] = \epsilon_{ij} \frac{1}{c^2} \tilde{J}, \quad [K_i, \tilde{J}] = \epsilon_{ij} K_j \\
&[K_i, P_j] = \delta_{ij} \frac{1}{c^2} \tilde{H}, \quad [K_i, \tilde{H}] = P_i, \quad [P_i, \tilde{J}] = \epsilon_{ij} P_j \\
&[P_i, P_j] = -\epsilon_{ij} \Lambda \tilde{J}, \quad [\tilde{H}, P_i] = -c^2 \Lambda K_i.
\end{aligned} \tag{2.1.16}$$

Now we rescale and rename the central elements in the algebra $\mathcal{A}$

$$S = \frac{1}{2c^2}\, 1, \quad M = \frac{1}{2} \gamma^5. \tag{2.1.17}$$

As anticipated, in taking the limit $c \to \infty$ we need to renormalise the energy $\tilde{H}$ and the angular momentum $\tilde{J}$ in such a way that $P^0 = \tilde{H}/c^2$ (the rest mass) and $J^0 = \tilde{J}/c^2$ (a kind of rest spin [45]) remain finite. We do this by defining

$$H = \tilde{H} - c^2 M, \quad J = \tilde{J} - c^2 S. \tag{2.1.18}$$

With this substitution, the algebra (2.1.16) still does not have a good limit if we take $c \to \infty$ while keeping Λ constant. The problem is the appearance of c^2 in the last line of (2.1.16). If, on the other hand we let $c \to \infty$ and $\Lambda \to 0$ in such a way that $\lambda = -c^2\Lambda$ remains fixed the limit is well-defined and we recover the Lie algebra that plays a central role in this thesis:

$$\begin{aligned}
[K_i, K_j] &= \epsilon_{ij} S & [K_i, J] &= \epsilon_{ij} K_j & [M, -] &= 0 \\
[K_i, P_j] &= \delta_{ij} M, & [K_i, H] &= P_i, & [P_i, J] &= \epsilon_{ij} P_j \\
[P_i, P_j] &= \epsilon_{ij} \lambda S, & [H, P_i] &= \lambda K_i & [S, -] &= 0.
\end{aligned} \tag{2.1.19}$$

This is a centrally extended version of the so-called Newton-Hooke algebra, which describes Galilean physics with a cosmological constant λ [41].

Before we proceed it is worth recording the physical dimension of all the quantities introduced so far. For the generators we have the analogue of (2.1.2):

$$\begin{array}{lll}
J: \text{ dimensionless}, & S: 1\ /\text{velocity}^2, & K_i = K^i : 1/\text{velocity} \\
H\!: 1/\text{time}, & M\!: \text{time}/\text{length}^2, & P_i = P^i : 1/\text{length}.
\end{array}$$

while λ has the dimension $1/\text{time}^2$ and in that sense parametrises some sort of curvature along the (global) time direction of radius $1/\sqrt{\lambda}$. Note that the above formulae are symmetric with respect to $c^2 \to -c^2, \Lambda \to -\Lambda$. In that sense one can think of the Galilei algebras (and their corresponding spacetimes as we will see in a later section) as an intermediate step between Euclidean and Minkowski isometry algebras(spacetimes).

2.2 Algebraic structures on $\hat{\mathfrak{g}}, \widehat{\mathfrak{nh}}$: r-matrix and bilinear form

2.2.1 Finding a bilinear form for the Galilei algebra

Let us now go back to the definitions of section 2.1.1 and use the limiting procedure we described in section 2.1.2 to find a bilinear form for the Galilei algebra. We see that elements 1 and γ^5 are, up to scale, the unique elements of, respectively, degree 1 and 4 in $\mathcal{A}$. Having chosen them, we can define projection operators

$$\Pi^1, \Pi^5 : \mathcal{A} \to \mathbb{R}, \tag{2.2.1}$$

which project in the obvious way; i.e. for an expansion of an arbitrary element $T \in \mathcal{A}$

$$T = t_1 1 + \cdots + t_5 \gamma^5 \tag{2.2.2}$$

in any basis containing $1, \gamma^5$ and the degree two elements $M^{\mu\nu}$, Π^1, Π^5 act as follows

$$\Pi^1(T) = t_1, \qquad \Pi^5(T) = t_5. \tag{2.2.3}$$

We can therefore define bilinear forms $\langle\ ,\ \rangle$ and $(\ ,\ )$ on $\mathcal{A}$ via

$$\langle M, N \rangle = -4\Pi^5(MN), \tag{2.2.4}$$

and

$$(M, N) = -4\Pi^1(MN), \tag{2.2.5}$$

respectively. It follows immediately from the centrality of 1 and γ^5 that these bilinear forms are invariant under the conjugation action of spin groups $Sp(V)$ (which is realised as the subset of $\mathcal{A}$ that contains its invertible elements).

Note that explicitly the pairing $\langle\ ,\ \rangle$ is non-zero whenever the indices on the basis vectors are complementary:

$$\langle M^{12}, M^{03} \rangle = -1, \quad \langle 1, \gamma^5 \rangle = -4, \quad \langle M^{12}, M^{01} \rangle = 0 \quad \text{etc.}$$

By contrast, the pairing $(\ ,\ )$ of basis vectors is non-zero whenever the indices on the basis vectors match:

$$(M^{12}, M^{12}) = 1, \quad (M^{01}, M^{01}) = -\frac{1}{c^2}, \quad (M^{13}, M^{13}) = \Lambda \quad \text{etc.}$$

The bilinear form $(\ ,\ )$ is the restriction to even elements of the usual inner product on $Cl(V,g)$, induced from the inner product $g^{\mu\nu}$ of degree one elements. Not surprisingly this form degenerates in both the limits $c\to\infty$ and $\Lambda\to 0$. The inner product $\langle\ ,\ \rangle$, by contrast, remains non-degenerate even when the metric $g^{\mu\nu}$ degenerates. One can also think of it as the Grassmann integral in the Grassmann algebra generated by the γ^μ. The Grassmann algebra is isomorphic to the Clifford algebra as a vector space but it is clearly independent of the metric $g^{\mu\nu}$. The minus signs and factors in our definition of $\langle\ ,\ \rangle$ and $(\ ,\ )$ are designed to match conventions elsewhere in the literature. The physical dimensions of the pairings follow directly from the definition: the pairing $\langle\ ,\ \rangle$ has dimensions of the inverse of γ^5 i.e length^2/time while the pairing $(\ ,\ )$ is dimensionless.

We now remind ourselves of the definitions (2.1.12) to make contact with the Galilei algebra. The pairing $\langle\ ,\ \rangle$ becomes

$$\langle J_a, P_b\rangle = -\eta_{ab} \tag{2.2.6}$$

or, explicitly,

$$\langle J_0, P_0\rangle = c^2, \qquad \langle J_i, P_j\rangle = -\delta_{ij}. \tag{2.2.7}$$

Similarly $(\ ,\ )$ becomes

$$(J_a, J_b) = -\frac{1}{c^2}\eta_{ab}, \quad (P_a, P_b) = \Lambda\eta_{ab}, \tag{2.2.8}$$

or, explicitly,

$$(J_0, J_0) = 1,\ (J_i, J_j) = -\frac{1}{c^2}\delta_{ij},\ (P_0, P_0) = -c^2\Lambda,\ (P_i, P_j) = \Lambda\delta_{ij}. \tag{2.2.9}$$

It is easy to see that these pairings have potential infinities in the limit $c\to\infty$, and the pairing $(\ ,\ )$ has the additional problem of some entries becoming zero. Fortunately, if we apply the rescalings introduced in (2.1.18) the pairing $\langle\ ,\ \rangle$ remains well-defined and non-degenerate in the Galilean limit. It is clear, however, that the pairing (,) degenerates in the limit $c \to \infty$ since our re-definition (2.1.18) did not rescale the J_i. Using

$$\langle S, M\rangle = -\frac{1}{c^2}, \quad \langle \tilde{H}, \tilde{J}\rangle = c^2, \tag{2.2.10}$$

we deduce

$$\langle H, S\rangle = 1, \quad \langle M, J\rangle = 1, \quad \langle K_i, P_j\rangle = \epsilon_{ij}, \tag{2.2.11}$$

or, equivalently,

$$\langle H, S\rangle = 1, \quad \langle M, J\rangle = 1, \quad \langle J_i, P_j\rangle = -\delta_{ij}, \tag{2.2.12}$$

which is an invariant inner product on the Newton-Hooke algebra, and manifestly independent of λ. The associated quadratic Casimir is

$$C_2 = H \otimes S + S \otimes H + M \otimes J + J \otimes M - (J_i \otimes P_i + P_i \otimes J_i). \tag{2.2.13}$$

It follows that the pairing $\langle\ ,\ \rangle$ is also dimensionful with dimension length^2/time. Finally, the dimension of λ is 1/time^2. This is consistent with the discussion of Newton-Hooke spacetimes in section 2.4.1 where it is clear that λ parametrises some kind of curvature in time.

2.2.2 The Galilei and Newton-Hooke algebras as "generalised complexifications"

The smallest Lie subalgebra of (2.1.19) containing boosts and rotations is the Lie algebra with generators J, K_1, K_2, S and brackets

$$[K_1, K_2] = S \qquad [K_i, J] = \epsilon_{ij} K_j \qquad [S, -] = 0.$$

In the approach we introduce in this section and will follow throughout the thesis, it will be more natural to write the boosts in the algebras we will deal with in terms of $J_i = \epsilon_{ij} K_j$,

$$[J_1, J_2] = S \quad [J_i, J] = \epsilon_{ij} J_j. \tag{2.2.14}$$

This Lie algebra can be viewed as a central extension of the Lie algebra of the Euclidean group in two dimensions, or as the Heisenberg algebra with an outer automorphism J. In the latter interpretation, J is the number operator. In the string theory literature it is sometimes called the Nappi-Witten Lie algebra [73, 30]. In this thesis it should be interpreted as a central extension of the homogeneous part of the Galilei Lie algebra. We denote the Galilei algebra (in the absence of the central generator S) by $\mathfrak{g}_0$ while its central extension of (2.2.14) by $\hat{\mathfrak{g}}_0$. The algebra $\hat{\mathfrak{g}}_0$ also has an invariant inner product (of Lorentzian signature!) with non-zero pairings in the above basis given by

$$\langle J, S\rangle_{\hat{\mathfrak{g}}_0} = 1, \quad \langle J_i, J_j\rangle_{\hat{\mathfrak{g}}_0} = -\delta_{ij}, \tag{2.2.15}$$

and an associated quadratic Casimir

$$C_{\hat{G}_0} = J \otimes S + S \otimes J - J_i \otimes J_i. \tag{2.2.16}$$

Note that we also have $\langle K_i, K_j\rangle_{\hat{\mathfrak{g}}_0} = -\delta_{ij}$. Since this algebra is not semi-simple, its Killing form is degenerate and the invariant inner product $\langle\ ,\ \rangle_{\hat{\mathfrak{g}}_0}$ is not associated to the Killing form [73]. Its existence is a rather fortunate accident.

We will now make use of the fact that one can view the central extension of the Newton-Hooke Lie algebra (2.1.19) as a "generalised complexification" of the homogeneous algebra $\hat{\mathfrak{g}}_0$ we described above. This viewpoint has proved useful in the context of (relativistic) three-dimensional gravity, when parametrising semi-direct product groups. It is an old idea found in [87] where generalised complex numbers are employed in order to unify the geometrical properties of hyperbolic, Euclidean and elliptic spaces. In the context of 3D gravity it was introduced for vanishing cosmological constant in [61], generalised to arbitrary values of the cosmological constant in [66] and further developed in [71].

It provides both an elegant and efficient method for explicit calculations. We therefore work with the homogeneous algebra $\hat{\mathfrak{g}}_0$ over a ring of numbers $\mathcal{R}_\lambda$ of the form $a + \theta b$, with $a, b \in \mathbb{R}$ and the formal parameter θ satisfying $\theta^2 = \lambda$ where λ can be have either a negative, positive or zero value (with θ itself being nonzero in each of those cases). This amounts to taking a Lie algebra over the complex, hyperbolic or dual numbers respectively. More formally, we have the following definition [66].

Definition 2.2.1. $\mathcal{R}_\lambda = (\mathbb{R}^2, +, \cdot)$ is the commutative ring with

- multiplication $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $(a, b) \cdot (c, d) \mapsto (ac + \lambda bd, ad + bc)$

- addition $+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the usual addition on $\mathbb{R}^2$ with $(a,b)+(c,d) \mapsto (a+c, b+d)$

We can introduce the formal parameter θ and write elements of $\mathcal{R}_\lambda$ as $a+\theta b$. We can further define operators $\mathrm{Re}_\theta, \mathrm{Im}_\theta$ that act in the intuitive way, i.e. $\mathrm{Re}_\theta(a+\theta b) = a, \mathrm{Im}_\theta(a+\theta b) = b$. In this formulation from (2.2.14) by setting $H = \theta J, P_i = \theta J_i$ and $M = \theta S$, we recover the centrally extended Newton-Hooke algebra (2.1.19)

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ij} S \qquad [J_i, J] = \epsilon_{ij} J_j \\ [J_i, P_j] &= \epsilon_{ij} M, \quad [J_i, H] = \epsilon_{ij} P_j, \quad [P_i, J] = \epsilon_{ij} P_j \\ [P_i, P_j] &= \epsilon_{ij} \lambda S, \quad [P_i, H] = \epsilon_{ij} \lambda J_j \quad [J, H] = 0. \end{aligned} \tag{2.2.17}$$

We call this algebra $\hat{\mathfrak{nh}}$. In the limit $\lambda \to 0$ or if $\theta^2 = 0$, we obtain a two-fold central extension of the Galilei Lie algebra, which we denote $\hat{\mathfrak{g}}$:

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ij} S \qquad [J_i, J] = \epsilon_{ij} J_j \\ [J_i, P_j] &= \delta_{ij} M, \quad [J_i, H] = \epsilon_{ij} P_j, \quad [P_i, J] = \epsilon_{ij} P_j \\ [P_i, P_j] &= 0, \qquad [H, P_i] = 0 \qquad [J, H] = 0. \end{aligned} \tag{2.2.18}$$

In fact, the Galilei Lie algebra admits three central extensions overall. This can be demonstrated by looking at the second Lie algebra cohomology[2] group $H^2(\mathfrak{g}, \mathbb{R})$. As noted in [17], [35] $H^2(\mathfrak{g}, \mathbb{R})$ is three-dimensional which in turn means that the Galilei algebra admits three central extensions in total. Only two of those extensions however survive the exponentiation to the group. The relevant subtleties are clarified in [17] where it is pointed out that the second cohomology group is three dimensional only in the case of the universal cover of the Galilei group while for the Galilei group itself, the second cohomology group is two-dimensional.

[2]More details on the cohomology of Lie groups and algebras can be found in [22]

It is the two central generators that we have denoted S, M that survive the exponentiation. The third extension corresponds to introducing a central generator F that measures the noncommutativity between J, H i.e. $[J, H] = F$. Throughout this thesis we disregard such an extension and when we talk of the centrally extended Galilei/Newton-Hooke algebras we will always refer to the algebras $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ as defined above.

As a final note, let us point out that for both algebras $\hat{\mathfrak{nh}}, \hat{\mathfrak{g}}$, the inner product $\langle\ ,\ \rangle$ of (2.2.11) can be obtained as

$$\langle\ ,\ \rangle := \mathrm{Im}_{\theta}\langle\ ,\ \rangle_{\hat{\mathfrak{g}}_0} \tag{2.2.19}$$

i.e. the part linear in θ of the linearly extended inner product $\langle\ ,\ \rangle_{\hat{\mathfrak{g}}_0}$ of (2.2.15). The coadjoint orbits and irreducible representations (irreps) have been studied extensively for the two-fold central extension $\hat{\mathfrak{g}}$ of the inhomogeneous Galilei group associated to $\hat{\mathfrak{g}}_0$ [72, 34, 14, 74]. Similarly, for the two groups (quite different in structure depending on whether $\lambda > 0$ or $\lambda < 0$) associated to the algebra $\hat{\mathfrak{nh}}$ [7].

In Chapter 3 we need to revisit some of those results in a language that, amongst others, makes use of the formal parameter θ and of the inner product (2.2.11). Before we do so, it is worth studying the algebraic and group structures of the homogeneous algebra $\hat{\mathfrak{g}}_0$ in a little more detail seeing as it acts as a sort of "parent algebra" for $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ in the language of generalised complexifications. Another reason for our persistence in using this language is to ease the deformation of the extended Galilei group $\hat{\mathfrak{g}}$ to the Galilei quantum double in Chapter 5. It will also prove useful in the process of deforming the Newton-Hooke groups to the double of a certain deformation of the Galilei group (with deformation parameter proportional to λ) that we examine again in Chapter 5.

2.2.3 Algebraic structures for the homogeneous extended Galilei Lie algebra $\hat{\mathfrak{g}}_0$

The homogeneous Galilei Lie algebra $\hat{\mathfrak{g}}_0$ with generators J_1, J_2, J and S, and commutators (2.2.14)

$$[J_1, J_2] = S, \qquad [J_1, J] = J_2, \qquad [J_2, J] = -J_1, \tag{2.2.20}$$

can also be written in a basis $Z, \bar{Z}$ related to J_1 and J_2 via

$$Z := J_1 + \mathrm{i}J_2 \qquad \bar{Z} := J_1 - \mathrm{i}J_2.$$

The algebra $\hat{\mathfrak{g}}_0$ written in this basis will be more useful when considering the bialgebra structure of the Newton-Hooke groups. Explicitly, $\hat{\mathfrak{g}}_0$ has the following form in this basis

$$\left[Z\,,\,\bar{Z}\right] = -2\mathrm{i}S \qquad \left[Z\,,\,J\right] = -\mathrm{i}Z \qquad \left[\bar{Z}\,,\,J\right] = \mathrm{i}\bar{Z} \tag{2.2.21}$$

The non-degenerate, invariant inner product (2.2.15) on $\hat{\mathfrak{g}}_0$ can now be written

$$\langle S\,,\,J\rangle_0 = 1, \qquad \langle Z\,,\,\bar{Z}\rangle_0 = -2, \tag{2.2.22}$$

in this basis, the associated Casimir is

$$K = S \otimes J + J \otimes S - \frac{1}{2}\left(\bar{Z} \otimes Z + Z \otimes \bar{Z}\right). \tag{2.2.23}$$

The cubic Casimir

$$\Omega = \mathrm{i}\bar{Z} \wedge Z \wedge S \tag{2.2.24}$$

is related to the quadratic Casimir via the modified Yang-Baxter equation

$$[[K, K]] = \Omega, \tag{2.2.25}$$

where, for quadratic elements $A \otimes B, C \otimes D \in \hat{\mathfrak{g}}_0 \otimes \hat{\mathfrak{g}}_0$ the operation $[[-\,,\,-]]$ is defined as

$$[[A \otimes B, C \otimes D]] := [A, C] \otimes B \otimes D + A \otimes [B, C] \otimes D + A \otimes C \otimes [B, D] \in \hat{\mathfrak{g}}_0^{\otimes 3}.$$

An easy calculation shows that

$$r_A = \frac{1}{2}\bar{Z} \wedge Z \tag{2.2.26}$$

also satisfies

$$[[r_A, r_A]] = \Omega. \tag{2.2.27}$$

This means that the bialgebra $Z, \bar{Z}, J, S$ is quasitriangular (see Chari and Pressley [21]), and

$$r = K - r_A = S \otimes J + J \otimes S - \bar{Z} \otimes Z \tag{2.2.28}$$

satisfies the classical Yang-Baxter equation (CYBE)

$$[[r, r]] = 0. \tag{2.2.29}$$

In fact, the classical r-matrix in (2.2.28) is defined up to a multiplicative constant, i.e. for any $k \in \mathbb{C}$ the r-matrix $r' = kr$ still satisfies the CYBE. This in turn means that r-matrices that are linear combinations of K and r_A satisfy the modified YBE.

In either case the Lie bialgebra $\hat{\mathfrak{g}}_0$ is coboundary[57] and we can compute its co-commutators via

$$\delta(X) = (1 \otimes \mathrm{ad}_X + \mathrm{ad}_X \otimes 1)(r). \tag{2.2.30}$$

Note that it is the antisymmetric part r_A one needs to take into account in the calculations. One finds for the cocommutator:

$$\delta(Z) = \mathrm{i}S \wedge Z \quad \delta(\bar{Z}) = \mathrm{i}S \wedge \bar{Z}, \tag{2.2.31}$$

with all others being zero. As a consequence, the corresponding Lie bracket $[\ ,\ ]^*$ on the dual Lie algebra has non-vanishing commutators

$$[S^*, Z^*]^* = -\mathrm{i}Z^*. \tag{2.2.32}$$

Of course, this bracket is not unique for a coboundary Lie bialgebra since, as we argued, the r-matrix is defined up to a multiplicative constant. It is the factor multiplying the antisymmetric part of r that determines the above commutator.

2.2.4 The bialgebras $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ as classical doubles and the role of the cosmological constant

Let us now pick suitable r-matrices to describe the bialgebra structure of $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$. As we argued in Chapter 1, in order to apply the Fock-Rosly prescription, we need to pick an r-matrix whose symmetric part is the quadratic Casimir in each algebra.

Let us begin the discussion by considering the Newton-Hooke Lie algebra $\hat{\mathfrak{nh}}$ of (2.2.17). The r-matrix

$$r_{\hat{\mathfrak{nh}}} = M \otimes J + H \otimes S - P_i \otimes J_i - \sqrt{-\lambda} J_1 \wedge J_2 \tag{2.2.33}$$

satisfies the above constraint. It is easy to see that the algebra $\hat{\mathfrak{nh}}$ with the above r-matrix is coboundary; it is obtained as a linear combination of K, r_A above. Specifically,

$$r_{\hat{\mathfrak{nh}}} = (\theta \otimes 1)K - (\sqrt{-\lambda} \otimes 1) r_A. \tag{2.2.34}$$

What is more, with the r-matrix above, the Newton-Hooke Lie algebra (2.2.17) has the structure of a classical double of the Lie algebra $\hat{\mathfrak{g}}_0$. This is easier to see in the basis (2.2.21). Let us write

$$\Pi_i = P_i + \sqrt{-\lambda}\epsilon_{ij} J_j \tag{2.2.35}$$

and introduce the generators

$$\Pi = \Pi_1 + \mathrm{i}\Pi_2, \tag{2.2.36}$$

and

$$\Pi^\circ = \Pi_1 - \mathrm{i}\Pi_2. \tag{2.2.37}$$

Note that Π° may or may not be the complex conjugate of Π depending of the sign of λ. With these definitions the inner product for the span $\{H, S, M, J, Z, \bar{Z}, \Pi, \Pi^\circ\}$ of the Newton-Hooke algebra, is as before for the pairing between H, S and M, J. The additional relation is now

$$\langle Z \ , \ \Pi^\circ \rangle = \langle \bar{Z} \ , \ \Pi \rangle = -2$$

It is easy to check that the generators H, M, Π, Π° form a Lie subalgebra of the Newton-Hooke algebra. Its brackets are

$$[\Pi, \Pi^\circ] = 0, \qquad [H, \Pi] = \sqrt{-\lambda}\Pi \qquad [H, \Pi^\circ] = \sqrt{-\lambda}\Pi^\circ. \tag{2.2.38}$$

We denote this Lie algebra by $\hat{\mathfrak{g}}_0^*$. It is a trivial central extension of the Lie algebra $\mathfrak{an}$ generated by H, Π_1 and Π_2. Note that is isotropic with respect to the inner product (the inner product of any of its generators with any other of its generators is zero). It is also dual to $\hat{\mathfrak{g}}_0$ with respect to the inner product $\langle\ ,\ \rangle$.

The Lie algebra $\hat{\mathfrak{nh}}$ thus has Lie subalgebras $\hat{\mathfrak{g}}_0$ and $\hat{\mathfrak{g}}_0^*$. As a vector space $\hat{\mathfrak{nh}}$ is the direct sum $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_0^*$, it is therefore the classical double of $\hat{\mathfrak{g}}_0$. Its r-matrix is now written

$$\begin{aligned} r_{\hat{\mathfrak{nh}}} =& M \otimes J + H \otimes S - \frac{1}{2}(\Pi^\circ \otimes Z + \Pi \otimes \bar{Z}) \\ =& M \otimes J + H \otimes S - P_i \otimes J_i - \sqrt{-\lambda} J_1 \wedge J_2 \end{aligned} \tag{2.2.39}$$

and one easily checks that the two algebras $\hat{\mathfrak{g}}_0, \hat{\mathfrak{g}}_0^*$ are in duality with respect to the pairing $\langle\ ,\ \rangle$ with compatibility between commutators and cocommutators of mutually dual subalgebras of $\hat{\mathfrak{nh}}$. Compatibility here means that the commutator in $\hat{\mathfrak{g}}_0$ is dual to the cocommutator in $\hat{\mathfrak{g}}_0^*$, i.e. $\langle [A, B]\ ,\ C\rangle = \langle A \otimes B\ ,\ \delta(C)\rangle$, with $A, B \in \hat{\mathfrak{g}}_0,\ C \in \hat{\mathfrak{g}}_0^*$ and vice-versa.

Note that brackets between generators of different subalgebras are not trivial and in the present case they are given by

$$[\bar{Z}, \Pi] = 2(\sqrt{-\lambda}S + \mathrm{i}M), \qquad [Z, H] = -\mathrm{i}\Pi + \sqrt{-\lambda}Z, \qquad [\Pi, J] = -\mathrm{i}\Pi. \tag{2.2.40}$$

Let us now turn to the algebra $\hat{\mathfrak{g}}$ of (2.2.18). The r-matrix

$$r_{\hat{\mathfrak{g}}} = M \otimes J + H \otimes S - P_i \otimes J_i \tag{2.2.41}$$

is now obtained from $\hat{\mathfrak{g}}_0$ as

$$r_{\hat{\mathfrak{g}}} = (\theta \otimes 1)K. \tag{2.2.42}$$

It can also be realised as the $\lambda = 0$ limit of (2.2.33). It is clearly compatible with the inner product $\langle\ ,\ \rangle$ in the sense of Fock and Rosly: the symmetric part of r is precisely the Casimir (2.2.13). Now the algebras $\hat{\mathfrak{g}}_0 = \mathrm{span}\{S, J, J_i\}$ and $\hat{\mathfrak{g}}_0^* = \mathrm{span}\{M, H, P_i\}$ are in duality with respect to the pairing $\langle\ ,\ \rangle$ and $\hat{\mathfrak{g}}$ has the structure of a classical double $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_0^*$. Its Lie algebra structure is that of (2.2.18), while the coalgebra structure is given via

$$\delta(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)(r), \qquad X \in \hat{\mathfrak{g}} \tag{2.2.43}$$

or explicitly

$$\begin{aligned} &\delta(S) = \delta(J) = \delta(M) = 0, \quad && \delta(J_i) = 0 \\ &\delta(H) = -P_1 \wedge P_2 && \delta(P_i) = \epsilon_{ij}(M \wedge P_j). \end{aligned} \tag{2.2.44}$$

We see that in both cases, the algebras $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ are realised as classical doubles of the *same* algebra $\hat{\mathfrak{g}}_0$. They are both coboundary and the associated r-matrices $r_{\hat{\mathfrak{g}}}, r_{\hat{\mathfrak{nh}}}$ induce a different coalgebra structure in each case which is where the two (classical) doubles differ. In particular, notice how the coalgebra structure of $\hat{\mathfrak{g}}_0$ is trivial in the case of the classical double $\hat{\mathfrak{g}}$ but nontrivial in the case of $\hat{\mathfrak{nh}}$. This difference will become important when we consider the Chern-Simons theory of their associated groups in Chapter 4, and more importantly, when we consider its quantisation in Chapter 5.

2.3 The centrally extended Galilei/Newton-Hooke groups in two dimensions

Let us now discuss the group structures of the algebras we have encountered so far. In our notation, for $\mathbb{R}^2$ with its standard metric we write vectors as $\vec{a}, \vec{b}\ldots$ and ϵ for the matrix with elements ϵ_{ij} i.e.

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.3.1}$$

while we denote $R(\varphi) \in SO(2)$ the rotation matrix

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{2.3.2}$$

We use the Einstein summation convention for the trivial metric δ_{ij} and introduce the abbreviations

$$\vec{a} \times \vec{b} = a_i \epsilon_{ij} b_j, \qquad \vec{a} \cdot \vec{b} = a_i b_i \qquad \vec{a}^2 = \vec{a} \cdot \vec{a}. \tag{2.3.3}$$

2.3.1 The homogeneous Galilei group $\hat{G}_0$

The centrally extended homogeneous Galilei group $\hat{G}_0$ is a Lie group whose Lie algebra is the Lie algebra $\hat{\mathfrak{g}}_0$. As a manifold it is $S^1 \times \mathbb{R}^3$. Group elements can be parametrised in terms of tuples

$$(\varphi, \vec{w}, \zeta), \qquad \varphi \in [0, 2\pi),\ \vec{w} \in \mathbb{R}^2, \zeta \in \mathbb{R}, \tag{2.3.4}$$

which we we can identify with exponentials of the abstract generators via

$$(\varphi, \vec{w}, \zeta) \leftrightarrow \exp(\vec{w} \cdot \vec{J}) \exp(\zeta S) \exp(\varphi J). \tag{2.3.5}$$

Although this is the parametrisation we employ throughout this thesis, we list for completeness two more parametrisations found in the literature. In particular, in [38] besides the ordering (2.3.5) (there labelled "time ordering") we also find the following parametrisations:

$$(\varphi', \vec{w}', \zeta') \leftrightarrow \exp(\frac{1}{2}\varphi J) \exp(\vec{w} \cdot \vec{J}) \exp(\zeta S) \exp(\frac{1}{2}\varphi J), \tag{2.3.6}$$

labelled "symmetric time ordering", and,

$$(\varphi'', \vec{w}'', \zeta'') \leftrightarrow \exp(\vec{w} \cdot \vec{J} + \zeta S + \varphi J), \tag{2.3.7}$$

labelled "Weyl ordering". Although different parametrisations merely correspond to different group multiplication laws (always differing by a coboundary [22]), in Chapter

5 we will think of the algebra $\hat{\mathfrak{g}}_0$ as a coordinate algebra of a certain noncommutative spacetime and the derivative algebra will turn out to be relevant to each of the parametrisations we mention (in the sense described in [38]).

In addition, in the notation we have introduced for the Galilei algebra in (2.2.21), elements in the homogeneous Galilei group can be written as $(\varphi, z, \zeta) \in \hat{G}_0$ where $\varphi, \zeta \in \mathbb{R}, z \in \mathbb{C}$ (this notation -modulo a minus sign in the generator J- is used in [30]). The group law is

$$(\varphi_1, z_1, \zeta_1) \cdot (\varphi_2, z_2, \zeta_2) = (\varphi_1 + \varphi_2, z_1 + e^{\mathrm{i}\varphi_1} z_2, \zeta_1 + \zeta_2 + \frac{1}{2}\mathrm{Im}(e^{-\mathrm{i}\varphi_1} z_1 \bar{z}_2)). \tag{2.3.8}$$

And a 3×3 matrix representation for this group is the following [50] (see also [46, 47])

$$g(\varphi, z, \zeta) := \begin{pmatrix} 1 & e^{-\mathrm{i}\varphi} z & \mathrm{i}\zeta + \frac{1}{2}|z|^2 \\ 0 & e^{-\mathrm{i}\varphi} & \bar{z} \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varphi, \zeta \in \mathbb{R}, z \in \mathbb{C} \tag{2.3.9}$$

We will mainly choose conventions which allow us to make contact with the papers [72] and [34] on the Galilean group and its central extension. Using vector notation, the composition law can be written as

$$(\varphi_1, \vec{w}_1, \zeta_1)(\varphi_2, \vec{w}_2, \zeta_2) = (\varphi_1 + \varphi_2, \vec{w}_1 + R(\varphi_1)\vec{w}_2, \zeta), \tag{2.3.10}$$

where $R(\varphi)$ is an $SO(2)$ matrix given in (2.3.2) and

$$\zeta = \zeta_1 + \zeta_2 + \frac{1}{2}\vec{w}_1 \times R(\varphi_1)\vec{w}_2. \tag{2.3.11}$$

The inverse is

$$(\varphi, \vec{w}, \zeta)^{-1} = (-\varphi, -R^{-1}(\varphi)\vec{w}, -\zeta). \tag{2.3.12}$$

The formula for group conjugation in the group $\hat{G}_0$ plays a fundamental role in what follows. By explicit calculation, or by translating the result of [30] into our notation, one finds, in terms of

$$v = (\varphi, \vec{w}, \zeta), \qquad v_0 = (\varphi_0, \vec{w}_0, \zeta_0), \tag{2.3.13}$$

that

$$\begin{aligned}\mathrm{Ad}_v(v_0) = \big(&\varphi_0, (1 - R(\varphi_0))\vec{w} + R(\varphi)\vec{w}_0,\\ \zeta_0 + \frac{1}{2}(1 + R(\varphi_0))\vec{w} \times R(\varphi)\vec{w}_0 &+ \frac{1}{2}\sin\varphi_0 \vec{w}^2\big).\end{aligned} \tag{2.3.14}$$

We can obtain adjoint orbits by keeping only linear terms in $\varphi_0, \vec{w}_0, \zeta_0$, and using $R(\varphi_0) \approx 1 + \epsilon\varphi_0$ for small φ_0. Writing

$$\xi = \alpha J + \vec{a} \cdot \vec{J} + \eta S, \tag{2.3.15}$$

and with the convention

$$\vec{v} = -\epsilon\vec{w} \quad \text{(so that} \quad \vec{v} \cdot \vec{K} = \vec{w} \cdot \vec{J}) \tag{2.3.16}$$

one finds

$$\mathrm{Ad}_v(\xi) = \alpha J + \vec{a}' \cdot \vec{J} + \eta' S, \tag{2.3.17}$$

with

$$\begin{aligned}\vec{a}' &= R(\varphi)\vec{a} + \alpha\vec{v}\\ \eta' &= \eta + \vec{v} \cdot R(\varphi)\vec{a} + \frac{1}{2}\alpha\vec{v}^2.\end{aligned} \tag{2.3.18}$$

2.3.2 The centrally extended groups $\hat{G}, \hat{NH}_{\pm}$

We are now going to employ the formulation we described in section 2.2.2 and look at the groups associated to the Lie algebras $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ as the "complexifications" with respect to the formal parameter θ of the Lie algebra $\hat{\mathfrak{g}}_0$. We remind the reader that the formal parameter θ satisfies $\theta^2 = \lambda$, and use the notation (2.3.4) with entries of the form $a + \theta b$, with a and b real. We start with the pure Galilei group (i.e. $\theta^2 = 0$).

The inhomogeneous Galilei group $\hat{G}$

We can denote the two-fold central extension of the Galilei group by $\hat{G}$, and write elements in the form

$$(\varphi + \theta\alpha, \vec{w} + \theta\vec{a}, \zeta + \theta\eta), \qquad \alpha, \eta \in \mathbb{R}, \vec{a} \in \mathbb{R}^2, \tag{2.3.19}$$

with ranges for $\varphi, \vec{w}, \zeta$ as defined in (2.3.4). The element (2.3.19) can be identified with

$$\exp(\vec{w} \cdot \vec{J} + \vec{a} \cdot \vec{P}) \exp(\zeta S + \eta M) \exp(\varphi J + \alpha H). \tag{2.3.20}$$

The expansion in θ of the rotation matrix R of (2.3.2) survives only to first order, i.e.

$$R(\varphi + \theta\alpha) = R(\varphi)(1 + \epsilon\theta\alpha). \tag{2.3.21}$$

The above together with (2.3.16) can be used to compute the multiplication rule in $\hat{G}$ from the multiplication rule for $\hat{G}_0$:

$$\begin{aligned}
&(\varphi_1 + \theta\alpha_1, \vec{w}_1 + \theta\vec{a}_1, \zeta_1 + \theta\eta_1)(\varphi_2 + \theta\alpha_2, \vec{w}_2 + \theta\vec{a}_2, \zeta_2 + \theta\eta_2) \\
&= (\varphi_1 + \varphi_2 + \theta(\alpha_1 + \alpha_2), \vec{w}_1 + R(\varphi_1)\vec{w}_2 + \theta(\vec{a}_1 + R(\varphi_1)\vec{a}_2 - \alpha_1\vec{v}_2), \zeta + \theta\eta),
\end{aligned} \tag{2.3.22}$$

where ζ is given as in (2.3.11), and

$$\begin{aligned}
\eta &= \eta_1 + \eta_2 + \frac{1}{2}(\vec{a}_1 \times R(\varphi_1)\vec{w}_2 + \vec{w}_1 \times R(\varphi_1)\vec{a}_2 - \alpha_1\vec{w}_1 \cdot R(\varphi_1)\vec{w}_2) \\
&= \eta_1 + \eta_2 + \frac{1}{2}(-\vec{a}_1 \cdot R(\varphi_1)\vec{v}_2 + \vec{v}_1 \cdot R(\varphi_1)\vec{a}_2 - \alpha_1\vec{v}_1 \cdot R(\varphi_1)\vec{v}_2).
\end{aligned} \tag{2.3.23}$$

In studying the representation theory of $\hat{G}$ we will exploit the fact that $\hat{G}$ is a semi-direct product

$$\hat{G} = \hat{G}_0 \ltimes \mathbb{R}^4. \tag{2.3.24}$$

To make this manifest we write elements in factorised form

$$g = xv, \quad \text{with} \quad x = (\theta\alpha, \theta\vec{a}, \theta\tilde{\eta}), \quad v = (\varphi, \vec{w}, \zeta), \tag{2.3.25}$$

where we used the notation (2.3.19) for both elements. Note that purely inhomogeneous elements x of the group have "purely imaginary" parameters $\theta\alpha$ etc.. The factorised form can be related to the complex parametrisation (2.3.19) by multiplying out

$$(\theta\alpha, \theta\vec{a}, \theta\tilde{\eta})(\varphi, \vec{w}, \zeta) = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\eta), \tag{2.3.26}$$

where

$$\eta = \tilde{\eta} + \frac{1}{2}\vec{a} \times \vec{w} = \tilde{\eta} - \frac{1}{2}\vec{a} \cdot \vec{v}. \tag{2.3.27}$$

Using (2.3.18) one obtains the following multiplication law in the semi-direct product formulation

$$(\theta\alpha_1, \theta\vec{a}_1, \theta\tilde{\eta}_1)(\varphi_1, \vec{w}_1, \zeta_1)(\theta\alpha_2, \theta\vec{a}_2, \theta\tilde{\eta}_2)(\varphi_2, \vec{w}_2, \zeta_2)$$

$$= (\theta(\alpha_1 + \alpha_2), \theta(\vec{a}_1 + R(\varphi_1)\vec{a}_2 + \alpha_2\vec{v}_1), \theta\tilde{\eta})(\varphi_1 + \varphi_2, \vec{w}_1 + R(\varphi_1)\vec{w}_2, \zeta), \tag{2.3.28}$$

where ζ is again given by (2.3.11) and

$$\tilde{\eta} = \tilde{\eta}_1 + \tilde{\eta}_2 + \vec{v}_1 \cdot R(\varphi_1)\vec{a}_2 + \frac{1}{2}\alpha_2\vec{v}_1^2. \tag{2.3.29}$$

Neither the 'complexified' nor the semi-direct product description of $\hat{G}$ coincide with the parametrisation of the two-fold central extension of the Galilei group most commonly used in the literature. To make contact with, for example, the parametrisation in [72] we factorise $\hat{G}$ elements in the form

$$(\theta\alpha, \vec{0}, 0)(\varphi, \vec{w} + \theta\vec{a}, \zeta + \theta\eta) = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\eta). \tag{2.3.30}$$

This almost agrees with the semi-direct product parametrisation (2.3.25), except for the shift (2.3.27). The group multiplication rule in the parametrisation (2.3.30) is

$$(\theta\alpha_1, \vec{0}, 0)(\varphi_1, \vec{w}_1 + \theta\vec{a}_1, \zeta_1 + \theta\eta_1)(\theta\alpha_2, \vec{0}, 0)(\varphi_2, \vec{w}_2 + \theta\vec{a}_2, \zeta_2 + \theta\eta_2)$$

$$= \big(\theta(\alpha_1 + \alpha_2), \vec{0}, 0\big)\big((\varphi_1 + \varphi_2), \vec{w}_1 + R(\varphi_1)\vec{w}_2 + \theta(\vec{a}_1 + R(\varphi_1)\vec{a}_2 + \alpha_2\vec{v}_1), \zeta + \theta\eta\big), \tag{2.3.31}$$

where ζ is again (2.3.11) and η is as in (2.3.23). This matches the conventions used for the centrally extended Galilei group in [72] up to a sign $\eta \to -\eta$.

The Newton-Hooke groups $\hat{NH}_{\pm}$

We now turn to the centrally extended Newton-Hooke group(s). In the notation we introduced earlier, this means we no longer require the parameter θ to square to zero but rather, we have $\theta^2 = \lambda$. For calculation purposes the resulting oscillating group $\hat{NH}_+$ ($\lambda > 0$) and the expanding group $\hat{NH}_-$ ($\lambda < 0$) can be treated uniformly in the θ-formalism. When their structure is not important, we will simply denote them both as $\hat{NH}$.

Taking that into account, it should be noted that the two groups are quite different in structure. It is shown in [71] that the resulting group for $\lambda > 0$ (in the present case, $\hat{NH}_+$) is given by two copies of its homogeneous subgroup. Indeed, this is the case here; for $\lambda > 0$ and using the redefinitions below:

$$\begin{aligned} J^+ &= \frac{1}{2}(J + \frac{1}{\sqrt{\lambda}}H) & J^- &= \frac{1}{2}(J - \frac{1}{\sqrt{\lambda}}H) \\ \vec{J}^+ &= \frac{1}{2}(\vec{J} + \frac{1}{\sqrt{\lambda}}\vec{P}) & \vec{J}^- &= \frac{1}{2}(\vec{J} - \frac{1}{\sqrt{\lambda}}\vec{P}) \\ S^+ &= \frac{1}{2}(S + \frac{1}{\sqrt{\lambda}}M) & S^- &= \frac{1}{2}(S - \frac{1}{\sqrt{\lambda}}M) \end{aligned} \tag{2.3.32}$$

it is easy to see that the algebras $\mathfrak{g}_1 = \{J^+, \vec{J}^+, S^+\}$ and $\mathfrak{g}_2 = \{J^-, \vec{J}^-, S^-\}$ each satisfy the commutation relations of the extended homogeneous Galilei algebra (2.2.14). The inverse transformation is, for the case of generators in $\hat{\mathfrak{g}}_0$ by the half sum between a generator with index $+$ and $-$ i.e. $J = \frac{1}{2}(J^+ + J^-)$ and so forth. For generators $H, \vec{P}, M$ the inverse transformation is given by the half difference and a rescaling by $\sqrt{\lambda}$ so $H = \frac{\sqrt{\lambda}}{2}(J^+ - J^-)$ etc. It is straightforward to show that $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$ - the overall coefficients of each part of the algebra are chosen to be the zero divisors in the ring of dual numbers. This is pointed out in [71]; for $a_\pm := (1 \pm \frac{\theta}{\sqrt{\lambda}}) \in \mathcal{R}_\lambda$ we have $a_+ a_- = 0$. So, as an algebra, $\mathfrak{nh}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. As a group, $\hat{G}_0$ is connected so by exponentiating we get $\hat{NH}_+ \cong \hat{G}_0 \times \hat{G}_0$.

For the case of the expanding group $\hat{NH}_-$, we expect to get a complexification of the homogeneous group in the traditional sense this time; i.e. as explained in [71] we expect to get the group whose algebra is $\hat{\mathfrak{g}}_0$ over $\mathbb{C}$. Indeed by looking at the group law (2.3.10) for $\hat{G}_0$, when $\theta^2 = \lambda < 0$, we can set $\mathrm{i}\sqrt{-\lambda} = \theta$ and the transition to the inhomogeneous group amounts to taking the parameters over $\mathbb{C}$ (along with a rescaling proportional to $\sqrt{-\lambda}$). We can therefore parametrise a group element as $(q, \vec{z}, s)$ with $q, z_i, s \in \mathbb{C},\ i = 1, 2$.

The resulting group has multiplication law given by:

$$(q, \vec{z}, s) \cdot (q', \vec{z}', s') = (q + q', \vec{z} + q\vec{z}', s + s' + \vec{z} \times q\vec{z}'), \tag{2.3.33}$$

and it is clear that its group structure is $\hat{NH}_- \cong (\mathbb{C} \ltimes \mathbb{C}^2) \ltimes \mathbb{C}$ (where we view the complex numbers as a group under addition).

As explained earlier, we prefer a uniform description for the two groups $\hat{NH}_+, \hat{NH}_-$ when it comes to calculations so we will keep the dependence from the formal parameter θ explicit and follow the steps we took for the inhomogeneous Galilei group above.

The expression $R(\varphi + \theta\alpha)$ now becomes $R(\varphi + \theta\alpha) = R(\theta\alpha)R(\varphi)$ which we can write more explicitly as

$$R(\varphi + \theta\alpha) = R(\theta\alpha)R(\varphi) = (\cos\theta\alpha + \theta\epsilon\frac{\sin\theta\alpha}{\theta})R(\varphi). \tag{2.3.34}$$

The functions $\cos\theta\alpha$, $\frac{\sin\theta\alpha}{\theta}$, defined through their power series, are both real and contain non-negative, even powers of θ.

In the literature one often finds generalised trigonometric functions[39, 87, 75]

$$\mathrm{COS}_\lambda(\alpha) := \begin{cases} \cos\sqrt{\lambda}\alpha & \lambda > 0 \\ 1 & \lambda = 0 \\ \cosh\sqrt{-\lambda}\alpha & \lambda < 0 \end{cases} \quad \mathrm{SIN}_\lambda(\alpha) := \begin{cases} \sin\sqrt{\lambda}\alpha/\sqrt{\lambda} & \lambda > 0 \\ \alpha & \lambda = 0 \\ \sinh\sqrt{-\lambda}\alpha/\sqrt{-\lambda} & \lambda < 0 \end{cases} \tag{2.3.35}$$

and through them the generalised rotation matrix R_λ

$$R_\lambda(x) = \begin{pmatrix} \mathrm{COS}_\lambda(x) & \mathrm{SIN}_\lambda(x) \\ -\mathrm{SIN}_\lambda(x) & \mathrm{COS}_\lambda(x) \end{pmatrix}, \tag{2.3.36}$$

in place of $\cos\theta\alpha$, $\frac{\sin\theta\alpha}{\theta}$ and $R(\theta\alpha)$. This way of treating absolute time (also known as Cayley-Klein) groups is, to our knowledge, pioneered by I. Yaglom [87] (see also [40]).

The names for the two $\hat{NH}$ groups come, in the case of $\lambda > 0$, from the oscillatory character of the parameter α that corresponds to time, i.e. time differences $\delta\alpha$ proportional to $\frac{2\pi n}{\sqrt{\lambda}}$ are indistinguishable. If $\lambda < 0$ the trigonometric functions in (2.3.34) should be replaced with their hyperbolic counterparts $\sinh, \cosh$ (with real arguments) and the term "expanding" refers to the behaviour of those functions for large α.

For calculations, we follow the prescription of (2.3.19) and we use the homogeneous Galilei group law to obtain the Newton-Hooke group law. The exponential parametrisation is that of (2.3.20). We therefore have

$$(\varphi_1 + \theta\alpha_1, \vec{w}_1 + \theta\vec{a}_1, \zeta_1 + \theta\eta_1)(\varphi_2 + \theta\alpha_2, \vec{w}_2 + \theta\vec{a}_2, \zeta_2 + \theta\eta_2)$$

$$=((\varphi_1 + \varphi_2) + \theta(\alpha_1 + \alpha_2), \vec{w} + \theta\vec{a}, \zeta + \theta\eta), \tag{2.3.37}$$

where now

$$\vec{w} =\vec{w}_1 + \cos\theta\alpha_1 R(\varphi_1)\vec{w}_2 + \lambda\frac{\sin\theta\alpha_1}{\theta}R(\varphi_1)\epsilon\vec{a}_2 \tag{2.3.38}$$

$$\vec{a} =\vec{a}_1 - \frac{\sin\theta\alpha_1}{\theta}R(\varphi_1)\vec{v}_2 + \cos\theta\alpha_1 R(\varphi_1)\vec{a}_2 \tag{2.3.39}$$

$$\zeta =\zeta_1 + \zeta_2 + \frac{1}{2}(\vec{w}_1 \times \vec{w} + \lambda\vec{a}_1 \times \vec{a}) \tag{2.3.40}$$

$$\eta =\eta_1 + \eta_2 + \frac{1}{2}(\vec{w}_1 \times \vec{a} + \vec{a}_1 \times \vec{w}). \tag{2.3.41}$$

This expression reduces to (2.3.22) when $\lambda \to 0$ as expected.

We will work in this parametrisation for the central extension of the Newton-Hooke group throughout this thesis. A decomposition reminiscent of the semi-direct product parametrisation of the Galilei group is not unique for the Newton-Hooke group; the Newton-Hooke group, much like the AdS group, does not have the structure

Rotations $\ltimes$ Translations. To see this, let us use (2.3.25) for a group element $g \in \hat{NH}$, to write down the above result. By writing g in exponential form as:

$$\exp(\vec{a} \cdot \vec{P}) \exp(\eta M) \exp(\alpha H) \exp(\vec{w} \cdot \vec{J}) \exp(\zeta S) \exp(\phi J), \tag{2.3.42}$$

if we read the group law in an analogous manner as we did in the discussion for the Galilei group and match the group components in the new parametrisation, we find

$$(\theta\alpha, \theta\vec{\tilde{a}}, \theta\tilde{\eta})(\varphi, \vec{\tilde{w}}, \tilde{\zeta}) = (\varphi + \theta\alpha, \vec{w} + \theta\vec{a}, \zeta + \theta\eta), \tag{2.3.43}$$

where

$$\vec{w} = \cos\theta\alpha\ \vec{\tilde{w}} \tag{2.3.44}$$

$$\vec{a} = \vec{\tilde{a}} - \frac{\sin\theta\alpha_1}{\theta}\vec{\tilde{v}} \tag{2.3.45}$$

$$\eta = \tilde{\eta} - \frac{1}{2}\cos\theta\alpha\ \vec{\tilde{a}} \cdot \vec{\tilde{v}} \tag{2.3.46}$$

$$\zeta = \tilde{\zeta} + \frac{1}{2}\lambda\frac{\sin\theta\alpha_1}{\theta}\ \vec{\tilde{a}} \times \vec{\tilde{v}}. \tag{2.3.47}$$

It is clear that the transformation to the tilded coordinate functions is not everywhere invertible. The equivalent of (2.3.30) however, *is* and we can parametrise elements in $\hat{NH}$ as

$$\begin{aligned}(\theta\alpha, \vec{0}, 0)(\varphi, \vec{w} + \theta\vec{a}, \zeta + \theta\eta) = (\varphi + \theta\alpha, R(\theta\alpha)(\vec{w} + \theta\vec{a}), \zeta + \theta\eta) = \\ (\varphi + \theta\alpha, \cos\theta\alpha\vec{w} + \lambda\frac{\sin\theta\alpha}{\theta}\epsilon\vec{a} + \theta(\cos\theta\alpha\vec{a} - \frac{\sin\theta\alpha}{\theta}\vec{v}), \zeta + \theta\eta).\end{aligned} \tag{2.3.48}$$

2.4 The Galilei/NH groups as symmetries of classical spacetimes

We conclude this Chapter with a discussion of the classical spacetimes whose symmetries are the Galilei and Newton-Hooke groups that we saw earlier. We will first study them as embeddings according to (2.1.2). Then, we will employ the Cartan decomposition of the said groups and identify the spacetimes with the corresponding symmetric spaces for each case. It turns out that for such a description it is sufficient to work with the groups without the central extensions.

It should be clear by now that the spacetimes that emerge in the current study are not dependent on the relativistic cosmological constant Λ but on $\lambda = -c^2\Lambda, (c \to \infty, \Lambda \to 0)$. We will denote them, in analogy with their corresponding symmetries, as Galilean (G) and oscillatory/expanding Newton-Hooke ($\mathrm{NH}_+/\mathrm{NH}_-$) spacetime. Hence the gray-shaded row of Table 2.1 corresponding to $c \to \infty$ should have columns dependent on λ as depicted in Table 2.2.

$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
NH_-	G	NH_+

Table 2.2. The Newton-Hooke (absolute time) geometries of Table 2.1 (for $-\Lambda c^2 \to \lambda, c \to \infty, \Lambda \to 0$). For $\lambda \neq 0$ one obtains the oscillatory NH_+ and expanding NH_- Newton-Hooke spacetimes (for $\lambda > 0, \lambda < 0$ respectively) while if $\lambda = 0$ these degenerate to the Galilei spacetime G

2.4.1 Classical spacetime geometry

Let us now study in more detail the geometric structure of the spacetimes of the embedding (2.1.2) in the Galilean limit $c \to \infty$. We base our discussion on the treatment found in [6]. Such geometries are examined using various approaches in a number of textbooks [87, 59, 6]. They are always given in the form of a product $\mathbb{T} \times A^2$, where the one-dimensional space $\mathbb{T}$ is a universally defined time line whose scaling depends on the value of the cosmological constant. Looking at the embedding (2.1.2) we can speak of an "embedding time" t and an "affine time" τ.

Let us first look at the case $\Lambda \to 0$. For purely Galilean spacetime the one dimensional space $\mathbb{T}$ is affine and the spacetime is three-dimensional affine space $A^3 \simeq A \times A^2$. The affine map τ and t coincide and

$$t : A^3 \to \mathbb{R}. \tag{2.4.1}$$

After a choice of origin we can identify A^3 with a three-dimensional vector space V^3. With the standard choice of origin for $\mathbb{R}$, the map t then defines a linear map

$V^3 \to \mathbb{R}$, which we denote by t again. Coordinates in V^3 label events while the kernel of the map t is the space of all simultaneous events. It is a two-dimensional, Euclidean vector space. We will pick a basis for V^3 consisting of a vector e_0 which is normalised via $t(e_0) = 1$ and orthonormal vectors e_1, e_2 in the kernel of t. Then we can describe events in terms of coordinates t, x_1, x_2 relative to this basis, and identify V^3 with $\mathbb{R}^3$ (we have re-interpreted the map t as a coordinate here, but this should not lead to confusion). The coordinate t plays the role of absolute time for A^3. In the notation in terms of hypersurfaces within $\mathbb{R}^4$ with the metric (2.1.1), the defining equation (2.1.2) becomes, in the mentioned limits, $w = \pm 1$. The hypersurface corresponding to Galilean spacetime is then the plane $w = \pm 1$ $\mathbb{R} \times \mathbb{R}^2$, in $\mathbb{R}^4$ with global coordinates $t \in \mathbb{R}, (x, y) \in \mathbb{R}^2$.

In considering Newton-Hooke spacetimes (Galilean spacetimes with a cosmological constant λ), the time scaling is different. We assume the spacetimes (now of the form $\mathbb{T} \times \mathbb{R}^2$) have time parameters whose form is dictated by the defining equation (2.1.2). Hence we can define the so-called oscillatory and expanding Newton Hooke spacetimes (and we will write NH_+, NH_- respectively) as the hypersurfaces

$$t = \begin{cases} \pm\frac{1}{\sqrt{\lambda}}\sqrt{1 - w^2} & \lambda > 0 \\ \pm\frac{1}{\sqrt{-\lambda}}\sqrt{w^2 - 1} & \lambda < 0 \end{cases}$$

embedded in $\mathbb{R}^4$. This embedding, induces the map t

$$t_- : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R} \backslash (-\frac{1}{\sqrt{-\lambda}}, \frac{1}{\sqrt{-\lambda}}) \tag{2.4.2}$$

for the expanding Newton-Hooke spacetime, while for the oscillating Newton-Hooke spacetime

$$t_+ : \mathbb{T} \times \mathbb{R}^2 \to (-\frac{1}{\sqrt{\lambda}}, \frac{1}{\sqrt{\lambda}}). \tag{2.4.3}$$

The affine parameter along the time direction of these hypersurfaces is now τ and it is this parameter that measures time intervals for an inertial observer in these spacetimes. It is related to the maps t_+, t_- via

$$t_- = \frac{1}{\sqrt{-\lambda}} \sinh \sqrt{-\lambda}\tau \tag{2.4.4}$$

$$t_+ = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}\tau. \tag{2.4.5}$$

Note that τ can be defined up to an additive constant. It is this coordinate that we can use as a global time coordinate for these two spacetimes. The reason for the names of each of the Newton-Hooke spacetimes should be clear. We show the embedded spacetimes as hypersurfaces in $(t, \vec{x}, w) \in \mathbb{R}^4$ in Figure 2.1. For each of the spacetimes we defined in this section we depict the time-slicing t of the embedding. In the oscillating Newton-Hooke spacetime t_+ begins and ends in a finite time ($t = \pm 1/\sqrt{\lambda}$ resp.) while in the expanding Newton-Hooke spacetime their future (resp. past) worldlines begin (resp. end) at $t = 1/\sqrt{-\lambda}$ (resp. $t = -1/\sqrt{-\lambda}$). The affine time τ, not depicted here, is translational-invariant while it is clear that t is not.

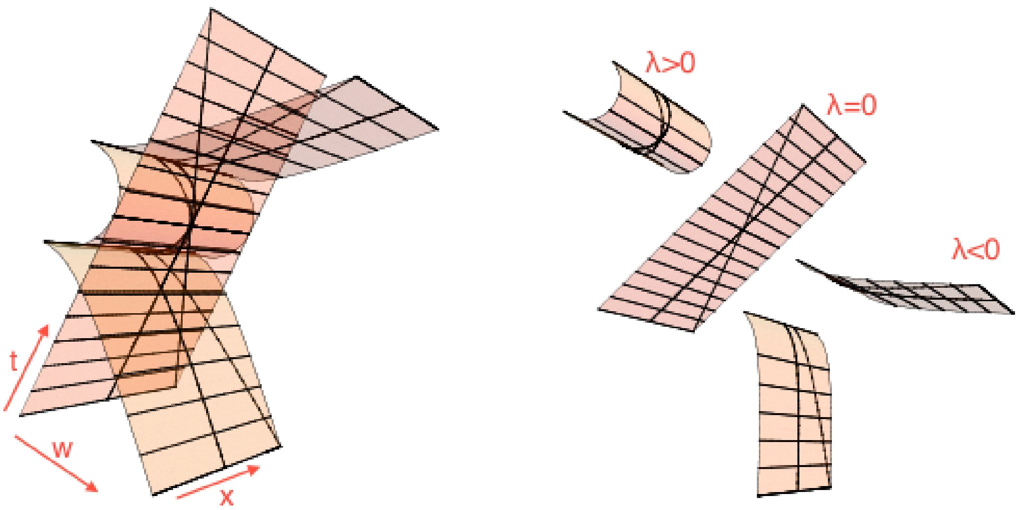

Figure 2.1. The expanding Newton-Hooke ($\lambda < 0$), Galilei ($\lambda = 0$), and oscillating Newton-Hooke spacetimes ($\lambda > 0$) as embedded hypersurfaces of (2.1.2)

2.4.2 Classical spacetimes as symmetric spaces

We can also look at the classical spacetimes not as embeddings but as symmetric spaces. This is the point we would like to elaborate on in this section. Recall that a symmetric space for a Lie group A is a homogeneous space A/H where the stabilizer H of a typical point is an open subgroup of the fixed point set of an involution (an automorphism which squares to the identity) of A. We now apply this construction to the (unextended) Newton-Hooke and Galilei groups NH, G. If we do not need to distinguish between them, we will use A to mean either of them. Using the notation

$$(\varphi, \vec{w}) \leftrightarrow \exp(\varphi J)\exp(\vec{w}\vec{J}), \qquad (\varphi, \vec{w}) \in G_0, \tag{2.4.6}$$

we write elements in A as $(\varphi + \theta\alpha, \vec{w} + \theta\vec{a})$.

We can define an involution $^{*\theta}$ that corresponds to θ-conjugation i.e. $(a + \theta b)^{*\theta} = a - \theta b$. It is straightforward to see that $^{*\theta}$ is an automorphism of A,

$$^{*\theta} : (\varphi + \theta\alpha, \vec{w} + \theta\vec{a}) \mapsto (\varphi - \theta\alpha, \vec{w} - \theta\vec{a}), \tag{2.4.7}$$

and that it squares to one. The fixed point set is the real part of A, G_0. We claim that homogeneous spaces

$$M_\lambda = NH/G_0, \quad M_0 = G/G_0, \tag{2.4.8}$$

are the Newton-Hooke/Galilei spacetimes respectively. To show this, we need a factorisation of elements $g \in A$ into elements $h \in G_0$ and (unique) coset representatives $m \in A$.

Lemma 2.4.1. Any element $g \in NH$ can be written uniquely as a product of $h \in G_0$ and an element of the form

$$m = (\theta\alpha, \theta R(\theta\alpha)\vec{a}) = (\theta\alpha, \vec{0})(0, \theta\vec{a}) \tag{2.4.9}$$

Proof: Parametrising $h \in G_0$ as

$$h = (\varphi', \vec{w}')$$

and m as

$$m = (\theta\alpha', \theta R(\theta\alpha')\vec{a}')$$

we find that

$$(\varphi + \theta\alpha, \vec{w} + \theta\vec{a}) = m \cdot h$$

iff

$$\alpha = \alpha' \qquad \vec{a} = \cos(\theta\alpha)\vec{a}' + \frac{\sin(\theta\alpha)}{\theta}\epsilon\vec{w}'$$

$$\varphi = \varphi' \qquad \vec{w} = \cos(\theta\alpha)\vec{w}' + \lambda\frac{\sin(\theta\alpha)}{\theta}\epsilon\vec{a}' \tag{2.4.10}$$

The linear relation between $\vec{w}, \vec{a}$ and $\vec{w}', \vec{a}'$ has determinant=1, for all values of λ and is therefore invertible. It follows that every element $g \in A$ can be factorised according to (2.4.9) in a unique fashion.

The factorisation allows us to define an action of the Newton-Hooke/Galilei groups on their respective spacetimes as follows. Let us introduce the simplified notation for group elements

$$(\varphi, \vec{w}; \alpha, \vec{a}) := (\theta\alpha, \theta R(\theta\alpha)\vec{a})(\varphi, \vec{w})$$

and space-time points

$$[\tau, \vec{x}] := (\theta\tau, \theta R(\theta\tau)\vec{x})$$

We then define the action ρ of G_0 elements via

$$\rho((\varphi, \vec{w}; \alpha, \vec{a})) : [\tau, \vec{x}] \mapsto [\tau', \vec{x}'], \tag{2.4.11}$$

where $[\tau', \vec{x}']$ is defined via the factorisation

$$(\theta\alpha, \theta R(\theta\alpha)\vec{a})(\varphi, \vec{w})(\theta\tau, \theta R(\theta\tau)\vec{x}) = (\theta\tau', \theta R(\theta\tau')\vec{x}')(\varphi', \vec{w}')$$

Computing the products and the factorisation one finds that

$$\rho((\varphi, \vec{w}; \alpha, \vec{a})) : [\tau, \vec{x}] \mapsto [\tau + \alpha, R(\varphi)\vec{x} + \frac{\vec{v}}{\theta}\sin\theta\tau + \vec{a}\cos\theta\tau]. \tag{2.4.12}$$

which agrees with the action for different values of λ given in [32]. For $\lambda = 0$ we recover the familiar action of the Galilei group on space-times

$$\rho((\varphi, \vec{v}; \alpha, \vec{a})) : [\tau, \vec{x}] \mapsto [\tau + \alpha, R(\varphi)\vec{x} + \vec{v}\tau + \vec{a}].$$

These actions can be realised through their matrix representation. Let us us write $(\varphi, w) \in \mathbb{R}\times\mathbb{C}$ for elements in G_0 where the components w_1, w_2 of the vector $\vec{w}$ are now realised as the real and imaginary part of a complex (in the traditional sense) number w. Elements $(\varphi, \vec{w}) \in G_0$ admit a faithful representation ρ_M in terms of complex 2×2 matrices

$$\rho_M(\varphi, \vec{w}) = \begin{pmatrix} e^{\mathrm{i}\varphi} & w \\ 0 & 1 \end{pmatrix}, \qquad \varphi \in \mathbb{R}, w = w_1 + \mathrm{i}w_2 \in \mathbb{C}. \tag{2.4.13}$$

We can use the following "complexified" matrices ρ_λ to denote a representation of $(\varphi, \vec{w}; \alpha, \vec{a}) \in G, NH$ as defined above

$$\rho_\lambda(\varphi, \vec{w}; \alpha, \vec{a}) = \begin{pmatrix} e^{\mathrm{i}\varphi} & w + \theta a \\ 0 & e^{\mathrm{i}\theta\alpha} \end{pmatrix}, \qquad \varphi, \alpha \in \mathbb{R}, w, a = a_1 + \mathrm{i}a_2 \in \mathbb{C}. \tag{2.4.14}$$

This is faithful and induces an action on classical spacetime points $[x, \tau] \in \mathbb{C}\times\mathbb{R}$. The action is realised on the "θ-imaginary" part Im_θ of column vectors $(\theta x, e^{\mathrm{i}\theta\tau}) \in G, NH$ via matrix multiplication, i.e.

$$\mathrm{Im}_\theta \begin{pmatrix} \theta x \\ e^{\mathrm{i}\theta\tau} \end{pmatrix} \to \mathrm{Im}_\theta \begin{pmatrix} e^{\mathrm{i}\varphi} & w + \theta a \\ 0 & e^{\mathrm{i}\theta\alpha} \end{pmatrix} \begin{pmatrix} \theta x \\ e^{\mathrm{i}\theta\tau} \end{pmatrix}. \tag{2.4.15}$$

One may wonder why we consumed the first half of this chapter to discuss the extended groups $\hat{G}_0, \hat{NH}$ since, as shown above, the unextended groups G, NH contain the necessary information about the flat spacetime geometries. The reason is that the corresponding Lie algebras do not admit an invariant bilinear form. Let us show this explicitly. We write $\mathfrak{g}_G, \mathfrak{g}_{NH}$ for the Lie algebras of the groups G, NH respectively. These are conventionally described in terms of a rotation generator J, boost generators K_1, K_2 and their θ counterparts: time- and space translation generators H, P_1, P_2 respectively.

As we showed, the non-vanishing bracket $[K_i, J] = \epsilon_{ij}K_j$ of the rotation algebra spanned by $\{J, K_1, K_2\}$ generates the algebra of both G, NH by "complexifying". The brackets are

$$[K_i, H] = P_i \qquad [P_i, J] = \epsilon_{ij}P_j \qquad [K_i, J] = \epsilon_{ij}K_j \qquad [P_i, H] = \lambda\epsilon_{ij}K_j \tag{2.4.16}$$

and depending on the value of λ one retrieves $\mathfrak{g}_G, \mathfrak{g}_{NH}$ accordingly. Let us collectively denote both those algebras by $\mathfrak{g}_A$ when the dependence from λ is not important.

It is easy to check that $P_i \otimes P_i$ and $\epsilon_{ij}(P_i \otimes K_j - K_i \otimes P_j)$ are both central elements in the universal enveloping algebra $U(\mathfrak{g}_A)$ and that no linear combination of them is invertible i.e. associated to an inner product on $\mathfrak{g}_A$. Since the dimension of the centre of $U(\mathfrak{g}_A)$ is bounded by the rank of $\mathfrak{g}_A$, we conclude that the Lie algebra $\mathfrak{g}_A$ cannot have an invariant, non-degenerate inner product. If it did, the associated Casimir would provide a third element of the centre of $U(\mathfrak{g}_A)$, contradicting $\mathrm{rk}(\mathfrak{g}_A) = 2$.

So, it is necessary to work with the central extension of the Galilei/Newton-Hooke algebra (2.4.16) and corresponding groups if we are to define a gauge theory for these groups. The two-fold central extensions $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ (2.2.18) we constructed via contraction earlier in this Chapter are the maximal extensions of the corresponding Lie algebras that can be exponentiated to the Galilei/Newton-Hooke groups[3]. In Chapter 4 we will see that, for both groups, the formulation of these extensions as a classical double in each case will be particularly suitable for our purposes in constructing a Galilean theory of quantum gravity.

[3]As argued this is not true for the algebra alone. The Galilei algebra $\mathfrak{g}$ for instance admits three central extensions, one for each two-cocycle in the Lie algebra cohomology [72, 34]

Chapter 3

The Galilean limit II: Galilean particles in 2+1 dimensions

In this Chapter we revisit some results on free Galilean point particles. In particular we calculate the coadjoint orbits of the relevant groups $\hat{G}, \hat{NH}$ and interpret them as phase spaces of free Galilean particles - we also provide their Poisson structure. This Chapter consists mainly of a review of the results published in [74, 7] in a notation that we will need them.

3.1 Coadjoint orbits as adjoint orbits; the phase space of Galilean particles

The coadjoint orbits of the centrally extended Galilei group $\hat{G}$ were studied in detail in [14, 74], while the equivalent study for the Newton-Hooke groups $\hat{NH}$ is made in [7] and references therein. The coadjoint orbits are identified with the phase space of free particles. Using the inner product (2.2.12) we can calculate adjoint orbits and then identify them with coadjoint orbits. In that way, we will be able to make contact with the group conjugacy classes which play an important role as the phase spaces of Galilean gravitating particles in Section 4.1.2.

As we mentioned earlier, the pairing (2.2.12) is independent of λ so we can obtain coadjoint orbits from adjoint orbits in both $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{nh}}$. Our strategy for computing these orbits is to start with adjoint orbits of the homogeneous group $\hat{G}_0$, and to obtain orbits in the full inhomogeneous groups by the complexification trick described in Section 2.3.2. Thus we start with an element

$$Q_0 = mJ - \vec{p}_0 \cdot \vec{J} + E_0 S \tag{3.1.1}$$

in the Lie algebra $\hat{\mathfrak{g}}_0$ of the homogeneous group $\hat{G}_0$. We already know the effect of conjugating with an element $v = (\varphi, \vec{w}, \zeta) \in \hat{G}_0$ from our earlier calculation (2.3.17). In the current notation the result reads

$$\mathrm{Ad}_v(Q_0) = mJ - \vec{p} \cdot \vec{J} + ES, \tag{3.1.2}$$

with

$$\begin{aligned} \vec{p} &= R(\varphi)\vec{p}_0 - m\vec{v} \\ E &= E_0 - \vec{v} \cdot R(\varphi)\vec{p}_0 + \frac{1}{2}m\vec{v}^2. \end{aligned} \tag{3.1.3}$$

Trivial orbits arise when we start with a reference element for which both mass and initial momentum vanish:

$$Q_0(E_0) = E_0 S. \tag{3.1.4}$$

The orbits consists of a point $O_{E_0} = \{E_0 S\}$ and the associated centraliser group is the entire group i.e. $N_{E_0} = \hat{G}_0$.

When the mass parameter $m \neq 0$ i.e.

$$Q_0(E_0, m) = E_0 S + mJ, \qquad m \neq 0 \tag{3.1.5}$$

the orbit is

$$O_{E_0,m} = \{mJ - \vec{p} \cdot \vec{J} + ES | E = E_0 + \frac{1}{2m}\vec{p}^2\}, \tag{3.1.6}$$

which is a paraboloid opening towards the positive or negative E-axis, depending on the sign of m. The centraliser group is

$$N_{E_0,m} = \{v = (\varphi, \vec{0}, \zeta) \in \hat{G}_0 | \varphi \in [0, 2\pi), \zeta \in \mathbb{R}\}. \tag{3.1.7}$$

Finally there are non-trivial orbits when the mass is zero, but the momentum non-zero. The reference element

$$Q_0(\vec{p}_0) = -\vec{p}_0 \cdot \vec{J} \tag{3.1.8}$$

leads to the orbit

$$O_{\vec{p}_0} = \{ES - \vec{p} \cdot \vec{J} | |\vec{p}| = |\vec{p}_0|, E \in \mathbb{R}\}, \tag{3.1.9}$$

which is a cylinder. The centraliser group is

$$N_{\vec{p}_0} = \{v = (0, \lambda\vec{p}_0, \zeta) | \lambda, \zeta \in \mathbb{R}\}. \tag{3.1.10}$$

We now turn to the inhomogeneous groups. To apply the "complexification" trick we need to replace the parameters of the homogeneous group $m, E_0 \in \mathbb{R}, \vec{p}_0 \in \mathbb{R}^2$ with parameters $m + \theta s, E_0 + \theta j_0, \in \mathcal{R}_\lambda, \vec{p}_0 + \theta \vec{j}_0 \in \mathcal{R}^2_\lambda$ (as defined in Section 2.2.2). Explicitly, adjoint orbits will then be described if we pick a reference point Q_0 in each of the relevant Lie algebras $(\hat{\mathfrak{g}}, \widehat{\mathfrak{nh}})$ through

$$Q = \mathrm{Ad}_g(Q_0). \tag{3.1.11}$$

As mentioned, we have assumed an identification of the generic element in the dual of the Lie algebra

$$Q^* = mM^* + EH^* + jJ^* + sS^* + \vec{p} \cdot \vec{P}^* + \vec{j} \cdot \vec{J}^* \tag{3.1.12}$$

with the Lie algebra element

$$Q = mJ + ES + jM + sH - \vec{p} \cdot \vec{J} - \vec{j} \cdot \vec{P} \tag{3.1.13}$$

which is valid for both $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$.

The parametrisation of the group elements is not relevant but it *does* help with the physical interpretation of the result. We therefore parametrise $g \in \hat{G}$ as is customary in the literature referring to the Galilei group (2.3.30):

$$g = (\theta\alpha, \vec{0}, 0)(\varphi, \vec{w} + \theta\vec{a}, \zeta + \theta\tau) = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\tau). \tag{3.1.14}$$

Before we carry out the computation we should clarify the physical interpretation of the various parameters we have introduced. A coadjoint orbit of the Galilei group (or its central extension) can be interpreted as the phase space of a Galilean particle. The coordinates on the phase space parametrise the states of motion of such a particle. A full justification of the interpretation requires knowledge of the Poisson structure of the phase space, which is discussed in the literature [14, 74] and which we review in Section 3.1.1. However, we give the physical picture here so that we can interpret the formulae we derive as we go along. The vectors $\vec{p}$ and $\vec{j}$ are the particle's momentum and impact vector (the quantity which is conserved because of invariance under Galilei boosts).

The parameters m and s are the mass and spin of the particle [44], E is the energy and j the angular momentum. The interpretation of the parameters in the group element (3.1.14) follows from our discussion of the Galilei group in Section 2.4.1: the vector $(\alpha, \vec{a}) \in \mathbb{R}^3$ parametrises spacetime translations, the angle φ parametrises spatial rotations, and $\vec{w}$ is related to the boost velocity via (2.3.16). The parameters τ and ζ do not have a geometrical significance. Physically, the formulae (3.1.3) give the behaviour or energy and momentum under a boost, translation and rotation of the reference frame.

We will need to know the different kinds of orbits and their associated centralisers when we study the representation theory of $\hat{G}$. Since we do not study the representation theory of $\hat{NH}$ we will not give the centralisers in $\hat{NH}$ although with proper identification of the relevant parameters, they are easy to work out from the centralisers of $\hat{G}_0$.

In the inhomogeneous groups, the fixed element of (3.1.11) is now written in the "complexified" language as:

$$Q_0 = (m + \theta s)J - \big(\vec{p}_0 + \theta\vec{j}_0\big)\vec{J} + (E_0 + \theta j_0)S. \tag{3.1.15}$$

In order to compute the effect of conjugating this element with an element in either $\hat{G}, \hat{NH}$ we simply "complexify" the result (3.1.3) i.e. we replace φ by $\varphi + \theta\alpha$, $\vec{p}_0$ by $\vec{p}_0 + \theta\vec{j}_0$ etc. Parametrising the result as

$$\mathrm{Ad}_g(Q_0) = (m' + \theta s')J - \big(\vec{p} + \theta\vec{j}\big)\vec{J} + (E + \theta j)S \tag{3.1.16}$$

we see straight away that $m = m'$ and $s = s'$ for both groups.

For the Galilei group $\hat{G}$ the formulae $\vec{p}$ and E are simply the terms of order zero in θ, and therefore given by the expressions in (3.1.3). In order to find j and $\vec{j}$ we need to compute the terms which are linear in θ. Thus j is the coefficient of θ in

$$\theta j_0 + \epsilon\big(\vec{w} + \theta(\vec{a} - \alpha\vec{v})\big)\cdot R(\varphi)(1 + \epsilon\theta\alpha)(\vec{p}_0 + \theta\vec{j}_0) + \frac{1}{2}(m + \theta s)\big(\epsilon\vec{w} + \epsilon\theta(\vec{a} - \alpha\vec{v})\big)^2. \tag{3.1.17}$$

Multiplying out and simplifying we find

$$j = j_0 - \vec{v}\cdot R(\varphi)\vec{j}_0 + \epsilon\vec{a}\cdot R(\varphi)\vec{p}_0 + \frac{1}{2}s\vec{v}^2 + m\vec{a}\times\vec{v}. \tag{3.1.18}$$

In order to compute $\vec{j}$, we need the coefficient of θ in

$$R(\varphi)(1 + \epsilon\theta\alpha)\cdot(\vec{p}_0 + \theta\vec{j}_0) + (m + \theta s)\epsilon\big(\vec{w} + \theta(\vec{a} - \alpha\vec{v})\big), \tag{3.1.19}$$

which is

$$\vec{j} = R(\varphi)\vec{j}_0 + \alpha\epsilon R(\varphi)\vec{p}_0 + m\epsilon\big(\vec{a} - \alpha\vec{v}\big) - s\vec{v}. \tag{3.1.20}$$

Having obtained explicit expressions for the adjoint orbits[1] of $\hat{\mathfrak{g}}$ it is straightforward to classify the different types of orbits which arise. This classification essentially follows from the classification of conjugacy classes in the homogeneous group $\hat{G}_0$ in [30] by linearising and using the "complexification" trick to extend to the inhomogeneous situation. These orbits are already discussed in the literature [14, 74], albeit in a different notation. We therefore simply list the orbit types in our notation in Table 3.1, giving in each case an element from which the orbit is obtained by conjugation. The non-trivial orbits are two- or four dimensional. The two-dimensional orbits are essentially the adjoint orbits of $\hat{G}_0$, which return here in the inhomogenous part of $\hat{G}$. For us, the two four-dimensional orbits are most relevant: they correspond to, respectively, massive and massless particles.

3.1.1 Poisson structure of the orbits

The canonical one-form associated to a coadjoint $\hat{G}$-orbit labelled by one of the reference elements Q_0 in Table 3.1 is of the general form [48]

$$\boldsymbol{\theta}_{Q_0} = \langle Q_0 \,,\; g^{-1}\mathrm{d}g\rangle. \tag{3.1.21}$$

To compute this, we first note the left-invariant Maurer-Cartan one-form on $\hat{G}_0$ in our parametrisation (see also [73], where this was first computed, using different coordinates). With the notation (2.3.13), it is

$$v^{-1}\mathrm{d}v = R(-\varphi)\mathrm{d}\vec{w}\cdot\vec{J} + \mathrm{d}\varphi J + (\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{w}\times\vec{w})S. \tag{3.1.22}$$

For later use we also record the expression for the right-invariant Maurer-Cartan form on $\hat{G}_0$

$$\mathrm{d}v v^{-1} = (\mathrm{d}\vec{w} - \mathrm{d}\varphi\epsilon\vec{w})\cdot\vec{J} + \mathrm{d}\varphi J + (\mathrm{d}\zeta + \frac{1}{2}\vec{w}^2\mathrm{d}\varphi - \frac{1}{2}\mathrm{d}\vec{w}\times\vec{w})S. \tag{3.1.23}$$

[1]Note that this agrees with the results in [74] but the notation is different; we use $R(\varphi)\vec{a}$ in place of $\vec{a}^{\,\varphi}$ that is used there. Also there is a relative rotation by π between the two notations so our tuple $(\vec{j},\vec{v},\vec{p},\vec{a})$is related to their $(\vec{\mathrm{k}},\vec{\mathrm{v}},\vec{\mathrm{p}},\vec{\mathrm{a}})$ in a manner similar to (2.3.16)

$$(\vec{\mathrm{k}},\vec{\mathrm{v}},\vec{\mathrm{p}},\vec{\mathrm{a}}) = (\vec{j},\epsilon\vec{v},-\epsilon\vec{p},\epsilon\vec{a}).$$

Next, we again use the "complexification" trick to obtain the left-invariant Maurer-Cartan form on the centrally extended Galilei group $\hat{G}$, which appears in (3.1.21). In terms of the parametrisation (2.3.30)

$$g = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\tau)$$

we find

$$\begin{aligned} g^{-1}\mathrm{d}g =& R(-\varphi)\mathrm{d}\vec{w}\cdot\vec{J} + \mathrm{d}\varphi J + (\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{v}\times\vec{v})S \;\; + R(-\varphi)\big(\mathrm{d}\vec{a} - \vec{v}\mathrm{d}\alpha\big)\cdot\vec{P} \\ &+ \mathrm{d}\alpha H + \big(\mathrm{d}\tau + \frac{1}{2}\vec{v}^2\mathrm{d}\alpha + \frac{1}{2}\mathrm{d}(\vec{v}\cdot\vec{a}) - \vec{v}\cdot\mathrm{d}\vec{a}\big)M. \end{aligned} \tag{3.1.24}$$

It is explained in [68], that for semi-direct product groups of the type $H \ltimes \mathfrak{h}^*$, there is an alternative formula for the symplectic structure on coadjoint orbits, which will be useful when we study Poisson-Lie group analogues of coadjoint orbits. In this alternative expression one pairs the inhomogeneous part of a generic point on the orbit with the right-invariant Maurer-Cartan form (3.1.23) on the homogeneous part

Reference element Q_0	**Generic element Q on orbit**	**Orbit geometry**
$E_0S + j_0M$	$E_0S + j_0M$	Point
$E_0S + j_0M + sH$	$E_0S + jM + sH - \vec{j}\cdot\vec{P}$ with $\vec{j}\in\mathbb{R}^2, j = j_0 + \frac{1}{2s}\vec{j}^2$	Paraboloid$\simeq\mathbb{R}^2$
$E_0S + j_0M - \vec{j}_0\cdot\vec{P}$	$E_0S + jM - \vec{j}\cdot\vec{P}$ with $j\in\mathbb{R}, \lvert\vec{j}\rvert = \lvert\vec{j}_0\rvert$	Cylinder $\simeq S^1\times\mathbb{R}$
$E_0S + mJ + j_0M + sH$	$ES + mJ - \vec{p}\cdot\vec{J} + jM - \vec{j}\cdot\vec{P} + sH$ with $\vec{p},\vec{j}\in\mathbb{R}^2$, $E = E_0 + \frac{1}{2m}\vec{p}^2, j = j_0 + \frac{1}{m}\vec{p}\cdot\vec{j} - \frac{s}{2m^2}\vec{p}^2$	Tangent bundle of paraboloid$\simeq\mathbb{R}^4$
$E_0S - \vec{p}_0\cdot\vec{J} + j_0M - \vec{j}_0\cdot\vec{P} + sH$	$ES - \vec{p}\cdot\vec{J} + jM - \vec{j}\cdot\vec{P} + sH$ with $E, j\in\mathbb{R}, \lvert\vec{p}\rvert = \lvert\vec{p}_0\rvert$, $\vec{j}\cdot\vec{p} - Es = \vec{j}_0\cdot\vec{p}_0 - E_0s_0$	Tangent bundle of cylinder $\simeq S^1\times\mathbb{R}^3$

Table 3.1. Adjoint orbits of the centrally extended Galilei group $\hat{G}$

of the group. In our case, this gives

$$\tilde{\boldsymbol{\theta}}_{Q_0} = \langle sH - \vec{j} \cdot \vec{P} + jM, \mathrm{d}vv^{-1} \rangle, \tag{3.1.25}$$

where we assume the expressions for $\vec{j}$ and j given in Table 3.1 for the generic orbit element Q on the orbit with reference element Q_0.

We briefly illustrate the formula for the symplectic potential by evaluating it for the two four-dimensional orbits in Table 3.1. Inserting the element Q_0 representing a massive particle with spin we obtain the symplectic potential for the corresponding orbit:

$$\begin{aligned} \boldsymbol{\theta}_{E_0,j_0,m,s} &= \langle E_0 S + mJ + j_0 M + sH \ , \ g^{-1}\mathrm{d}g \rangle = \\ &= j_0 \mathrm{d}\varphi + s(\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{v} \times \vec{v}) + E_0 \mathrm{d}\alpha + m\big(\mathrm{d}\tau + \frac{1}{2}\vec{v}^2 \mathrm{d}\alpha + \frac{1}{2}\mathrm{d}(\vec{v} \cdot \vec{a}) - \vec{v} \cdot \mathrm{d}\vec{a}\big), \end{aligned} \tag{3.1.26}$$

which simplifies to

$$\boldsymbol{\theta}_{E_0,j_0,m,s} = \frac{s}{2}\mathrm{d}\vec{v} \times \vec{v} + m\vec{v} \cdot \mathrm{d}(\alpha\vec{v} - \vec{a}) \quad + \text{exact term}. \tag{3.1.27}$$

The associated symplectic form only depends on m and s, and is given by

$$\omega_{ms} = \mathrm{d}\boldsymbol{\theta}_{E_0,j_0,m,s} = \frac{s}{2}\mathrm{d}(\epsilon\vec{v}) \wedge \mathrm{d}\vec{v} + m\vec{v} \cdot \mathrm{d}\vec{v} \wedge \mathrm{d}\alpha - m\mathrm{d}\vec{v} \wedge \mathrm{d}\vec{a}, \tag{3.1.28}$$

where we use the shorthand $d\vec{a} \wedge \mathrm{d}\vec{b} = da_i \wedge \mathrm{d}b_i$. In terms of the momentum vector $\vec{p} = -m\vec{v}$ we recover the formula given in [44]:

$$\omega_{m,s} = \frac{s}{2m^2}\mathrm{d}(\epsilon\vec{p}) \wedge \mathrm{d}\vec{p} - \frac{1}{m}\vec{p} \cdot \mathrm{d}\vec{p} \wedge \mathrm{d}\alpha + \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{a}. \tag{3.1.29}$$

Computing the symplectic potential via the alternative expression (3.1.25) we find

$$\tilde{\boldsymbol{\theta}}_{E_0,j_0,m,s} = j_0 \mathrm{d}\varphi + s(\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{v} \times \vec{v}) + m\big(\frac{1}{2}\vec{v}^2 \mathrm{d}\alpha + \vec{a} \cdot \mathrm{d}\vec{v}\big), \tag{3.1.30}$$

which clearly only differs from $\boldsymbol{\theta}_{E_0,j_0,m,s}$ by exact terms, and leads to the same symplectic form.

Proceeding similarly for the four-dimensional orbit corresponding to massless particles, we compute

$$\boldsymbol{\theta}_{E_0,j_0,\vec{p}_0,\vec{j}_0,s} = \langle E_0 S - \vec{p}_0 \cdot \vec{J} - \vec{j}_0 \cdot \vec{P} + j_0 M + sH \ , \ g^{-1}\mathrm{d}g \rangle \tag{3.1.31}$$

and find that the symplectic potential is

$$\boldsymbol{\theta}_{E_0,j_0,\vec{p}_0,\vec{j}_0,s} = (\vec{j} + \frac{s}{2}\vec{v}) \cdot \mathrm{d}(\epsilon\vec{v}) + \mathrm{d}\vec{p} \cdot (\alpha\vec{v} - \vec{a}) \quad + \text{exact term}, \tag{3.1.32}$$

where it is understood that $\vec{p}$ and $\vec{j}$ are given by the expressions (3.1.3) and (3.1.20). Thus we obtain the symplectic form

$$\omega_{\vec{p}_0,\vec{j}_0,s} = \mathrm{d}\vec{j} \wedge \mathrm{d}(\epsilon\vec{v}) + \frac{s}{2}\mathrm{d}\vec{v} \wedge \mathrm{d}(\epsilon\vec{v}) - \alpha \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{v} - \vec{v} \cdot \mathrm{d}\vec{p} \wedge \mathrm{d}\alpha + \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{a}. \tag{3.1.33}$$

3.2 Coadjoint orbits of the Newton-Hooke groups

For the Newton - Hooke group we proceed in a similar fashion. We again calculate $Q = \mathrm{Ad}_g(Q_0)$ where for $g \in \hat{NH}$ in the "complexified" parametrisation

$$g = (\varphi + \theta\alpha, \vec{w}' + \theta\vec{a}', \zeta + \theta\tau). \tag{3.2.1}$$

We therefore recover the element

$$Q = mJ + ES + jM + sH - \vec{p} \cdot \vec{J} - \vec{j} \cdot \vec{P}$$

This is quite straightforward to calculate. The parameters m, s corresponding to the central generators remain fixed, while

$$\vec{p} = \cos\theta\alpha R(\varphi)\vec{p}_0 + \lambda\frac{\sin\theta\alpha}{\theta}R(\varphi)\epsilon\vec{j}_0 - m\vec{v}' + \lambda s\epsilon\vec{a}' \tag{3.2.2}$$

$$\vec{j} = \frac{\sin\theta\alpha}{\theta}R(\varphi)\epsilon\vec{p}_0 + \cos\theta\alpha R(\varphi)\vec{j}_0 - s\vec{v}' + m\epsilon\vec{a}' \tag{3.2.3}$$

$$E = E_0 - \vec{v}' \cdot \vec{p} + (\lambda\vec{j} + \lambda s\vec{v}') \times \vec{a}' - \frac{1}{2}m\vec{v}'^2 - \frac{1}{2}\lambda m\vec{a}'^2 \tag{3.2.4}$$

$$j = j_0 - \vec{v}' \cdot \vec{j} + (\vec{p} + m\vec{v}') \times \vec{a}' - \frac{1}{2}s\vec{v}'^2 - \frac{1}{2}\lambda s\vec{a}'^2. \tag{3.2.5}$$

If one is to interpret the above expressions physically, or compare them to the ones for $\lambda = 0$ in (3.1.18), (3.1.20) one needs to look at the parametrisation (2.3.48). In the expression above, the components $\vec{w}', \vec{a}'$ of the conjugating elements need to be replaced by the corresponding elements

$$
\begin{aligned}
\vec{w}' &= \cos\theta\alpha\vec{w} + \lambda\frac{\sin\theta\alpha}{\theta}\epsilon\vec{a} \\
\vec{a}' &= \cos\theta\alpha\vec{a} - \frac{\sin\theta\alpha}{\theta}\vec{v}
\end{aligned}
\tag{3.2.6}
$$

one obtains from (2.3.48). One then checks that momentum, for instance, transforms as follows under conjugation:

$$
\vec{p} = \cos\theta\alpha(R(\varphi)\vec{p}_0 - m\vec{v} + \lambda s\epsilon\vec{a}) + \lambda\frac{\sin\theta\alpha}{\theta}(R(\varphi)\epsilon\vec{j}_0 - m\vec{a} - s\vec{w}).
$$

In the formula above, note how a purely spatial/time translation can boost a particle's momentum. This effect is also present in relativistic physics with a cosmological constant.

Let us now study the orbits and their geometry. We will distinguish between the oscillating and expanding groups starting by the oscillating Newton-Hooke group $\hat{NH}_+$. One can observe that by using the decomposition described in (2.3.32) it is straightforward to obtain the adjoint orbits in $\hat{NH}_+$ as orbits in each of the two copies of $\hat{G}_0$ that the group $\hat{NH}_+$ can be split into i.e.

$$
\hat{NH}_+ \simeq \hat{G}_0^{(+)} \times \hat{G}_0^{(-)}
$$

where we use a plus or minus index in accordance to (2.3.32) to distinguish between the two copies of $\hat{G}_0$ not to denote the sign of the cosmological constant which, in the present case, is positive. The orbits in each copy of $\hat{G}_0$ leave invariant the quadratic Casimirs. For the two copies of $\hat{G}_0$ these are now

$$
J^\pm, \quad C^\pm = J^\pm \otimes S^\pm + S^\pm \otimes J^\pm - \vec{J}^\pm \otimes \vec{J}^\pm, \tag{3.2.7}
$$

where note that the inverse basis transformation to (2.3.32) is

$$
J = \frac{1}{2}(J^+ + J^-), \qquad H = \frac{\sqrt{\lambda}}{2}(J^+ - J^-) \tag{3.2.8}
$$

$$
\vec{J} = \frac{1}{2}(\vec{J}^+ + \vec{J}^-), \qquad \vec{P} = \frac{\sqrt{\lambda}}{2}(\vec{J}^+ - \vec{J}^-) \tag{3.2.9}
$$

$$
S = \frac{1}{2}(S^+ + S^-) \qquad M = \frac{\sqrt{\lambda}}{2}(S^+ - S^-). \tag{3.2.10}
$$

From this we can extract the inner product between elements of each copy of $\hat{G}_0$. This is as follows

$$\langle J^{\pm}\ ,\ S^{\pm}\rangle_{\pm} = \pm\,\frac{2}{\sqrt{\lambda}}$$
$$\langle J_i^{\pm}\ ,\ J_j^{\pm}\rangle_{\pm} = \mp\,\frac{2\delta_{ij}}{\sqrt{\lambda}} \tag{3.2.11}$$

For each copy of $\hat{G}_0$ the orbits combine to produce the orbits in $\hat{NH}_+$ that are now product manifolds with coordinates $m^{\pm}, \vec{p_0}^{\pm}$. In particular, looking back to the discussion of Section 3.1, there are three classes of orbits in $\hat{G}_0^{\pm}$

$m^{\pm} = 0, \vec{p_0}^{\pm} = 0$	0D (points)
$m^{\pm} \neq 0, \vec{p_0}^{\pm} = 0$	2D (paraboloids)
$m^{\pm} = 0, \vec{p_0}^{\pm} \neq 0$	2D (cylinders)

where we have assumed the orbits in $\hat{G}_0^{\pm}$ are obtained through conjugation of two reference elements Q_0^+, Q_0^-

$$Q_0^{\pm} = m^{\pm}J^{\pm} + \vec{p_0}^{\pm}\cdot\vec{J}^{\pm} + E_0^{\pm}S^{\pm}$$

each lying in a copy of $\hat{G}_0$.

It is a trivial exercise in combinatorics to count the different orbits that arise in the product; one gets four four-dimensional orbits, four two-dimensional ones, and a point. Note however, that the physically relevant interpretation is made after one makes the following identifications

$$m^+ = \frac{1}{2}(m + \sqrt{\lambda}s) \qquad m^- = \frac{1}{2}(m - \sqrt{\lambda}s),\ \text{etc} \tag{3.2.12}$$

of the orbit coordinates with the physical parameters. We reserve this interpretation for when we discuss the Poisson structure on the orbits in the following section.[2] The different orbits are classified in [7] - we list them in our notation along with their geometry in Table 3.2.

[2]It should be pointed out that for orbits corresponding to $m^{\pm} \neq 0$, the values for which $m^+ = \pm m^-$ are special in that they result in particles with zero mass and spin

Reference element $Q_0 = m^+ J^+ + \vec{p}_0^{\,+} \cdot \vec{J}^+ + E_0^+ S^+ + m^- J^- + \vec{p}_0^{\,-} \cdot \vec{J}^- + E_0^- S^-$		
Parameters in Q_0	**Generic element Q on orbit**	**Orbit geometry**
$m^\pm = 0, \quad \vec{p}_0^{\,\pm} = \vec{0}$	$E_0^+ S^+ + E_0^- S^-$	Point
$m^+ = 0, \quad \vec{p}_0^{\,+} = \vec{0}$ $m^- = 0, \quad \vec{p}_0^{\,-} \neq \vec{0}$ and $+ \leftrightarrow -$	$E_0^+ S^+ + \vec{p}^{\,-} \cdot \vec{J}^- + E^- S^-$ with $E^- \in \mathbb{R}, \lvert\vec{p}^{\,-}\rvert = \lvert\vec{p}_0^{\,-}\rvert$ and $+ \leftrightarrow -$	Cylinder & Point $\simeq S^1 \times \mathbb{R} \cup \{E_0^+\}$
$m^+ = 0, \quad \vec{p}_0^{\,+} = \vec{0}$ $m^- \neq 0, \quad \vec{p}_0^{\,-} = \vec{0}$ and $+ \leftrightarrow -$	$E_0^+ S^+ + m^- J^- + \vec{p}^{\,-} \cdot \vec{J}^- + E^- S^-$ with $\vec{p}^{\,-} \in \mathbb{R}^2, E^- = E_0^- + \frac{1}{2m^-}(\vec{p}^{\,-})^2$ and $+ \leftrightarrow -$	Paraboloid & Point $\simeq \mathbb{R}^2 \cup \{E_0^+\}$
$m^\pm = 0, \quad \vec{p}_0^{\,\pm} \neq \vec{0}$	$\vec{p}^{\,+} \cdot \vec{J}^+ + E^+ S^+ + \vec{p}^{\,-} \cdot \vec{J}^- + E^- S^-$ with $E^\pm \in \mathbb{R}, \lvert\vec{p}^{\,\pm}\rvert = \lvert\vec{p}_0^{\,\pm}\rvert$	Cylinder & Cylinder $\simeq (S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$
$m^\pm \neq 0, \quad \vec{p}_0^{\,\pm} = \vec{0}$	$m^+ J^+ + \vec{p}^{\,+} \cdot \vec{J}^+ + E^+ S^+ + m^- J^- +$ $\vec{p}^{\,-} \cdot \vec{J}^- + E^- S^-$ with $\vec{p}^{\,\pm} \in \mathbb{R}^2, E^\pm = E_0^\pm + \frac{1}{2m^\pm}(\vec{p}^{\,\pm})^2$	Paraboloid & Paraboloid $\simeq \mathbb{R}^2 \times \mathbb{R}^2$
$m^+ \neq 0, \quad \vec{p}_0^{\,+} = \vec{0}$ $m^- = 0, \quad \vec{p}_0^{\,-} \neq \vec{0}$ and $+ \leftrightarrow -$	$m^+ J^+ + \vec{p}^{\,+} \cdot \vec{J}^+ + E^+ S^+ + \vec{p}^{\,-} \cdot$ $\vec{J}^- + E^- S^-$ with $\vec{p}^{\,+} \in \mathbb{R}^2$, $E^+ = E_0^+ + \frac{1}{2m^+}(\vec{p}^{\,+})^2$ and $+ \leftrightarrow -$	Paraboloid & Cylinder $\simeq \mathbb{R}^2 \times (S^1 \times \mathbb{R})$

Table 3.2. Adjoint orbits of the centrally extended oscillatory Newton-Hooke group $\hat{NH}_+$

The expanding group $\hat{NH}_-$ one gets for $\lambda < 0$ similarly has orbits in the complexification of the homogeneous Galilei group $(\hat{G}_0)_\mathbb{C}$. We can calculate the orbits by letting the parameters in $\hat{G}_0$ run in $\mathbb{C}$ after some rescaling. Specifically, we define

$$m_\mathbb{C} := m + \mathrm{i}\sqrt{-\lambda}\ s \tag{3.2.13}$$

$$\vec{p}_\mathbb{C} := \vec{p} + \mathrm{i}\sqrt{-\lambda}\ \vec{j} \tag{3.2.14}$$

$$E_\mathbb{C} := E + \mathrm{i}\sqrt{-\lambda}\ j \tag{3.2.15}$$

and calculate the orbits in the $\hat{G}_0$ group with the above parameters. The result only differs from the situation in the real case in that there is no longer an orbit with the geometry of a cylinder. We summarise the orbits and their geometry in Table 3.3.

Reference element $Q_0 = m_{\mathbb{C}} J + \vec{p}_{\mathbb{C}(0)} \cdot \vec{J} + E_{\mathbb{C}(0)} S$		
Parameters in Q_0	**Generic element** Q **on orbit**	**Orbit geometry**
$m_{\mathbb{C}} = 0, \quad \vec{p}_{\mathbb{C}(0)} = \vec{0}$	$E_{\mathbb{C}(0)} S$	Point
$m_{\mathbb{C}} = 0, \quad \vec{p}_{\mathbb{C}(0)} \neq \vec{0}$	$\vec{p}_{\mathbb{C}} \cdot \vec{J} + E_{\mathbb{C}} S$ with $E_{\mathbb{C}} \in \mathbb{C}, \vec{p}^{\,2}_{\mathbb{C}(0)} = \vec{p}^{\,2}_{\mathbb{C}}$	Complex two-plane $\simeq \mathbb{C}^2$
$m_{\mathbb{C}} \neq 0, \quad \vec{p}_{\mathbb{C}(0)} = \vec{0}$	$m_{\mathbb{C}} J + \vec{p}_{\mathbb{C}} \cdot \vec{J} + E_{\mathbb{C}} S$ with $\vec{p}_{\mathbb{C}} \in \mathbb{C}^2, E_{\mathbb{C}} = E_{\mathbb{C}(0)} + \frac{1}{2m_{\mathbb{C}}} (\vec{p}_{\mathbb{C}})^2$	Complex Paraboloid $\simeq \mathbb{C}^2$

Table 3.3. Adjoint orbits of the centrally extended oscillatory Newton-Hooke group $\hat{NH}_-$

Again, we stress that one should be careful in interpreting these results physically. For instance, the orbit corresponding to $m_{\mathbb{C}} \neq 0$ refers to massless particles if $m_{\mathbb{C}} \in \mathrm{i}\mathbb{R}$ so we reserve such interpretations for *after* we calculate the associated Poisson structures in this notation.

3.2.1 Poisson structure of the orbits

Our strategy for computing the Poisson structure on the $\hat{NH}$-orbits is somewhat different to the one we employed in calculating the orbits of $\hat{G}_0$.

Let us start again by calculating the Poisson structure on the orbits of the oscillatory Newton-Hooke group $\hat{NH}_+$. For calculation purposes, it is easier to continue making use of the fact that $\hat{NH}_+$ is a product manifold. We therefore compute the canonical one-forms associated to the coadjoint $\hat{G}$-orbits of each of its copies labelled by one of the reference elements Q_0 in Table 3.2. The symplectic potential is

$$\boldsymbol{\theta}_{Q_0} = \langle Q_0 \ , \ g^{-1} \mathrm{d}g \rangle. \tag{3.2.16}$$

So long as the group elements are parametrised accordingly, i.e. $v^{\pm} \in \hat{G}_0^{\pm}$, we can write for the symplectic potential on the product orbits

$$\boldsymbol{\theta}_{Q_0} = \boldsymbol{\theta}^{+}_{Q_0} + \boldsymbol{\theta}^{-}_{Q_0} = \langle Q_0\ ,\ (v^{+})^{-1}\mathrm{d}v^{+}\rangle + \langle Q_0\ ,\ (v^{-})^{-1}\mathrm{d}v^{-}\rangle. \tag{3.2.17}$$

We can very easily write elements in $\hat{NH}$ written in the "complexified" parametrisation (2.3.19), as products:

$$g = v^{+}\cdot v^{-} = v^{-}\cdot v^{+}, \qquad g \in \hat{NH}, v^{\pm} \in \hat{G}_0^{\pm}.$$

Explicitly we have

$$(\varphi + \theta\alpha, \vec{w}' + \theta\vec{a}', \zeta + \theta\tau) = (\varphi^{+}, \vec{w}^{+}, \zeta^{+})\cdot(\varphi^{-}, \vec{w}^{-}, \zeta^{-})$$

where

$$\varphi^{\pm} = \frac{1}{2}(\varphi \pm \sqrt{\lambda}\alpha) \qquad \vec{w}^{\pm} = \frac{1}{2}(\vec{w}' \pm \sqrt{\lambda}\vec{a}') \qquad \zeta^{\pm} = \frac{1}{2}(\zeta \pm \sqrt{\lambda}\tau) \tag{3.2.18}$$

It follows that the left-invariant Maurer-Cartan one-form on each $\hat{G}_0$ branch is

$$(v^{\pm})^{-1}\mathrm{d}v^{\pm} = R(-\varphi^{\pm})\mathrm{d}\vec{w}^{\pm}\cdot\vec{J}^{\pm} + \mathrm{d}\varphi^{\pm}J^{\pm} + (\mathrm{d}\zeta^{\pm} + \frac{1}{2}\mathrm{d}\vec{w}^{\pm}\times\vec{w}^{\pm})S^{\pm} \tag{3.2.19}$$

in our parametrisation. It is now straightforward to compute the Poisson structure for any orbit in $\hat{NH}_{+}$ making sure we account for the right factors from (3.2.11). Let us perform the calculation for the four dimensional orbits with geometry of a Paraboloid times a Paraboloid. The element Q_0 is now

$$Q_0 = m^{+}J^{+} + E_0^{+}S^{+} + m^{-}J^{-} + E_0^{-}S^{-} \tag{3.2.20}$$

while the symplectic potential is

$$\begin{aligned}\boldsymbol{\theta}_{m^{\pm},E_0^{\pm}} =& \frac{2}{\sqrt{\lambda}}m^{+}(\mathrm{d}\zeta^{+} + \frac{1}{2}\mathrm{d}\vec{w}^{+}\times\vec{w}^{+}) + \frac{2}{\sqrt{\lambda}}E_0^{+}\mathrm{d}\varphi^{+} - \ (+\leftrightarrow -)\\ =& \frac{1}{\sqrt{\lambda}}m^{+}(\mathrm{d}\vec{w}^{+}\times\vec{w}^{+}) - \ (+\leftrightarrow -)\ + \text{exact terms}\end{aligned} \tag{3.2.21}$$

Replacing the above from (3.2.18) we get for the symplectic potential

$$\boldsymbol{\theta}_{m^{\pm},E_0^{\pm}} = \frac{1}{2}s\mathrm{d}\vec{w}'\times\vec{w}' + \frac{\lambda}{2}s\mathrm{d}\vec{a}'\times\vec{a}' + \frac{1}{2}m\mathrm{d}\vec{w}'\times\vec{a}' + \frac{1}{2}m\mathrm{d}\vec{a}'\times\vec{w}' + \text{exact terms}. \tag{3.2.22}$$

which can be written more compactly as follows

$$\begin{aligned}\boldsymbol{\theta}_{m^\pm,E_0^\pm} =&\frac{1}{2}s\mathrm{d}\vec{w}' \times \vec{w}' - (m\vec{w}' + \frac{\lambda}{2}s\vec{a}') \times \mathrm{d}\vec{a}' + \text{exact terms} \\ =&\frac{1}{2}s\mathrm{d}\vec{v}' \times \vec{v}' - \vec{p}\cdot \mathrm{d}\vec{a}' + \text{exact terms.}\end{aligned} \tag{3.2.23}$$

The corresponding symplectic form is

$$\omega_{m^\pm,E_0^\pm} = \mathrm{d}\boldsymbol{\theta}_{m^\pm,E_0^\pm} = \frac{s}{2}\mathrm{d}(\epsilon\vec{v}') \wedge \mathrm{d}\vec{v}' - \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{a}' \tag{3.2.24}$$

Calculation of the symplectic form on the complexified $\hat{G}_0$ follows along the same lines. We have not been able to generate illuminating formulae from such calculation so we refer the reader to [7] for a more detailed analysis.

Chapter 4

The Galilean limit III: Galilean gravity in 2+1 dimensions

Our approach to formulating classical and, ultimately, quantum theory of Galilean gravity is based on the Chern-Simons formulation of three-dimensional gravity discovered in [1] and elaborated in [85] for general relativity in three dimensions. The prescription is, very simply, to pick a differentiable three-manifold $\mathcal{M}$ describing the universe and to write down the Chern-Simons action for the isometry group of the model spacetime for a given signature and value of the cosmological constant, using an appropriate inner product on the Lie algebra. Such an action is, by construction, invariant under orientation-preserving diffeomorphisms of $\mathcal{M}$. The prescription makes sense for any differentiable three-manifold $\mathcal{M}$, but for the Hamiltonian quantisation approach we need to assume $\mathcal{M}$ to be of topology $\mathbb{R}\times\Sigma$, where Σ is a two-dimensional manifold, possibly with punctures and a boundary. The fact that classical spacetimes are always of the form $\mathbb{T}\times\Sigma$ means this last assumption is identically satisfied for the spacetimes we examine. The results of this chapter have been published in [76, 77]

4.1 Galiliean gravity in three dimensions as gauge theory of the centrally extended Galilei group

4.1.1 The Chern-Simons formulation of Galilean gravity

For our model of Galilean gravity we can use again the "complexified" description of the relevant algebras to depict the general situation. We remind the reader that both $\hat{\mathfrak{g}}, \hat{\mathfrak{nh}}$ have the same invariant bilinear form (2.2.11).

The gauge field of the Chern-Simons action is locally a one-form on spacetime $\mathcal{M}$ with values in the Lie algebra of either the centrally extended Galilei group or the Newton-Hooke group.

$$A = \omega J + \omega_i J_i + \eta S + eH + e_i P_i + fM, \tag{4.1.1}$$

where $\omega, \omega_1, \omega_2, \eta, e, e_1, e_2$ and f are ordinary one-forms on $\mathcal{M}$. By the prescription that by now must seem exhausted (still useful however!) this can be written down by defining complexified combinations of one-forms more compactly as a one-form with values in the homogeneous part $\hat{\mathfrak{g}}_0$ of the above algebras:

$$A = \Omega J + \Omega_i J_i + \Psi S, \tag{4.1.2}$$

where we have defined the one forms

$$\Omega := \omega + \theta e \qquad \Omega_i := \omega_i + \theta e_i \qquad \Psi := \eta + \theta f. \tag{4.1.3}$$

The gauge field can then be written in the form $A = a + \theta b$ consisting of a homogeneous part

$$a = \omega J + \omega_i J_i + \eta S. \tag{4.1.4}$$

and an inhomogeneous part

$$\theta b = eH + e_i P_i + fM. \tag{4.1.5}$$

Then one checks straightforwardly that

$$\begin{aligned}\frac{1}{2}[A,A] =&\Omega_i\wedge\Omega\epsilon_{ij}J_j+\Omega_1\wedge\Omega_2 S\\ =&(\omega_i\wedge\omega+\lambda e_i\wedge e)\epsilon_{ij}J_j+(\omega_1\wedge\omega_2+\lambda e_1\wedge e_2)S\\ &+(e_i\wedge\omega+\omega_i\wedge e)\epsilon_{ij}P_j+(\omega_i\wedge e_j\epsilon_{ij})M.\end{aligned}\tag{4.1.6}$$

The curvature

$$F=\mathrm{d}A+\frac{1}{2}[A,A]=R+C+\theta T\tag{4.1.7}$$

is a sum of a homogeneous term R and the cosmological term C, taking values in $\hat{\mathfrak{g}}_0$,

$$\begin{aligned}R=&\mathrm{d}\omega\ J+(\mathrm{d}\eta+\omega_1\wedge\omega_2)S+(\mathrm{d}\omega_j+\omega_i\wedge\omega\epsilon_{ij})J_j\\ C=&\lambda e_i\wedge e\epsilon_{ij}J_j+\lambda e_1\wedge e_2 S\end{aligned}\tag{4.1.8}$$

while the torsion is

$$T=\mathrm{d}e\ H+(\mathrm{d}f+\omega_i\wedge e_j\epsilon_{ij})M+(\mathrm{d}e_j+\omega_i\wedge e\epsilon_{ij}+e_i\wedge\omega\epsilon_{ij})P_j.\tag{4.1.9}$$

Thus we can write the Chern-Simons action for Galilean gravity in 2+1 dimensions in the form

$$\begin{aligned}I_{CS}[A]=&\frac{1}{16\pi G}\int_{\mathcal{M}}\langle A\wedge\mathrm{d}A\rangle+\frac{1}{3}\langle A\wedge[A,A]\rangle\\ =&\frac{1}{16\pi G}\int_{\mathcal{M}}\mathrm{Im}_\theta(\Omega\wedge\mathrm{d}\Psi+\Psi\wedge\mathrm{d}\Omega-\Omega_i\wedge\mathrm{d}\Omega_i+2\Omega\wedge\Omega_1\wedge\Omega_2)\\ =&\frac{1}{16\pi G}\Bigg(\int_{\mathcal{M}}\big(\omega\wedge\mathrm{d}f+f\wedge\mathrm{d}\omega+e\wedge\mathrm{d}\eta+\eta\wedge\mathrm{d}e-\omega_i\wedge\mathrm{d}e_i-e_i\wedge\mathrm{d}\omega_i\big)\\ &+2\int_{\mathcal{M}}\big(\epsilon_{ij}\omega\wedge\omega_i\wedge e_j\epsilon_{ij}+e\wedge\omega_1\wedge\omega_2+\lambda\epsilon_{ij}e\wedge e_1\wedge\epsilon_2\big)\Bigg),\end{aligned}\tag{4.1.10}$$

where G is Newton's constant in 2+1 dimensions, with physical dimension of inverse mass. With both the connection A and the curvature being dimensionless, the physical dimension length^2/time of the pairing $\langle\ ,\ \rangle$ combines with the dimension mass of $1/G$ to produce the required dimension of an action. We note here a notational difference between the action as written above and in [76] where it was first published, namely we have defined $A = \dots + e_i J_i + \dots$ and not $A = \dots + e'_i K_i + \dots$ as is the case in that paper. The identification is by replacing $e_i \to -\epsilon_{ij} e'_j$.

In the absence of a boundary and punctures on Σ the stationary points of the Chern-Simons action are flat connections i.e. connections for which

$$F = 0 \Leftrightarrow R + C = 0 = T. \tag{4.1.11}$$

These are the vacuum field equations of classical Galilean gravity. The interpretation of solutions, i.e. flat $\hat{G}$-connections, in terms of Galilean spacetimes proceeds along the same lines as the interpretation of flat Poincaré group connections in terms of Lorentzian spacetimes, see e.g. [25, 23, 69]. The flatness condition means that the gauge field can be written in local patches as $A = g^{-1}\mathrm{d}g$ for a $\hat{G}$-valued function g in a given patch. Parametrising g in the form $g = (\theta T, \vec{0}, 0)(\phi, \epsilon\vec{V} + \theta\vec{X}, \zeta' + \theta\eta')$ the $\mathbb{R}^3$-valued function $(T, \vec{X})$ provides an embedding of the patch into our model Galilean spaces; G (which we can identify with $\mathbb{R}^3$ after a choice of origin and frame) for the Galilei and $NH_\pm$ for the Newton-Hooke group. In this way, a flat $\hat{G}$ or $\hat{NH}$ connection translates into an identification of open sets in $\mathcal{M}$ with open sets in each of the respective spacetimes - overlapping open sets are connected by Galilean transformation.

Note that coordinates η' and ζ' referring to the central generators play no role in the geometrical interpretation. This is a point we argued in discussing the spacetimes in Chapter 2. At this level, there is remarkably little difference between relativistic and Galilean gravity. More specifically, even though the model spaces have an absolute time (up to a global shift), solutions of the Galilean vacuum field equations (4.1.11) need not have an absolute time. In particular there can be time shifts when following the identifications along non-contractible paths.

A further important consequence of the field equations (4.1.11) is that a given infinitesimal diffeomorphism can be written as infinitesimal gauge transformation, as in relativistic 3d gravity [85]. As we discussed in Chapter 1, this follows from Cartan's identity applied to the $\hat{G}$-connection A:

$$\mathcal{L}_\xi A = \mathrm{d}\iota_\xi A + \iota_\xi \mathrm{d}A = D_A(\iota_\xi A), \tag{4.1.12}$$

where we used the flatness condition $\mathrm{d}A = -A \wedge A$ to write the infinitesimal diffeomorphism $\mathcal{L}_\xi$ as an infinitesimal gauge transformation with generator $\iota_\xi(A)$. Conversely we find, as in [85], that an infinitesimal local translation can be written as local diffeomorphisms provided the frame field $eH + e_i P_i$ is invertible. If that condition is met, local translations do not lead to (undesirable) gauge invariance in addition to local diffeomorphisms and homogeneous Galilei transformations. Questions regarding the role of non-invertible frames and the relation between global diffeomorphisms and large gauge transformations are difficult, as they are in the relativistic case, see [62] for a discussion and further references. However, unlike in the relativistic case, the issue of understanding the difference between the Chern-Simons and metric formulation does not arise since we do not have a metric theory of Galilean gravity in three dimensions.

For the remainder of this section let us focus on the case where $\lambda = 0$, and reserve the study of the action above *with* the cosmological term for later. The gauge group then becomes $\hat{G}$ and, as expected for gauge groups of the semi-direct product form with a pairing between the homogenous and the normal part, its action is equivalent to an action of the BF form,

$$\begin{aligned} I_{BF}[A] =& \frac{1}{8\pi G}\int_{\mathcal{M}} \langle R \wedge b \rangle \\ =& \frac{1}{8\pi G}\int_{\mathcal{M}} \big(\mathrm{d}\omega \wedge f + \mathrm{d}\eta \wedge e \\ & - \mathrm{d}\omega_i \wedge e_i + e \wedge \omega_1 \wedge \omega_2 + \epsilon_{ij}\omega \wedge e_i \wedge \omega_j \big), \end{aligned} \tag{4.1.13}$$

with I_{CS} and I_{BF} differing by an integral over an exact form i.e. a boundary term. This BF-action is closely related to the action of lineal gravity [19]: the two actions are based on the same curvature R and the same inner product (2.2.15). The difference is that the one-form b is a Lagrange multiplier *function* η in the gauge formulation of lineal gravity.

In this thesis we will not consider boundary components of Σ, although the boundary treatment in [70] includes the current situation as a special case. We will, however, couple particles to the gravitational field. This is done by marking points on the surface Σ and decorating them with coadjoint orbits of the extended Galilei group, equipped with the symplectic structures we studied in Sect. 3.1.1. In order to couple the bulk action to particles we need to switch notation which makes the splitting $\mathcal{M} = \mathbb{R} \times \Sigma$ explicit. Referring the reader to [69, 70] for details, we introduce a coordinate x^0 along $\mathbb{R}$ and coordinates x^1, x^2 on Σ, and split the gauge field into

$$A = A_0 \mathrm{d}x^0 + A_\Sigma, \tag{4.1.14}$$

where A_Σ is a one form on Σ (which may depend on x^0). For simplicity we consider only one particle, which is modelled by a single puncture of Σ at $\vec{x}_*$, decorated by a coadjoint, or equivalently adjoint, orbit. In the parametrisation of adjoint orbits of the centrally extended Galilei group in (3.1.14), the group element $g \in \hat{G}$ now becomes a function of x^0. The combined field and particle action, with Q and Q_0 defined as in (3.1.11), takes the following form:

$$\begin{aligned} I_\tau[A_\Sigma, A_0, g] =& \frac{1}{8\pi G} \int_{\mathbb{R}} \mathrm{d}x^0 \int_\Sigma \langle \partial_0 A_\Sigma \wedge A_\Sigma \rangle - \int_{\mathbb{R}} \mathrm{d}x^0 \langle Q_0 \,,\, g^{-1} \partial_0 g \rangle \\ &+ \int_{\mathbb{R}} \mathrm{d}x^0 \int_\Sigma \langle A_0 \,,\, \frac{1}{8\pi G} F_\Sigma - Q \delta^{(2)}(\vec{x} - \vec{x}_*) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \rangle. \end{aligned} \tag{4.1.15}$$

Varying with respect to the Lagrange multiplier A_0 we obtain the constraint

$$F_\Sigma(x) = 8\pi G Q \delta^{(2)}(\vec{x} - \vec{x}_*) \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{4.1.16}$$

Variation with respect to A_Σ gives the evolution equation

$$\partial_0 A_\Sigma = \mathrm{d}_\Sigma A_0 + [A_\Sigma, A_0]. \tag{4.1.17}$$

The equation obtained by varying g does not concern us here, but can be found e.g. in [70].

Together with (4.1.16) this means that the curvature F vanishes except at the "worldline" of the puncture, where it is given by the constraint (4.1.16). In order to interpret the constraint geometrically and physically we use the decomposition (4.1.7) of the curvature and parametrisation (3.1.13):

$$R^a_\Sigma = 8\pi G(mJ + ES - \vec{p}\cdot\vec{J})\delta^{(2)}(\vec{x}-\vec{x}_*)\mathrm{d}x^1\wedge\mathrm{d}x^2$$
$$T^a_\Sigma = 8\pi G(sM + jH - \vec{j}\cdot\vec{P})\delta^{(2)}(\vec{x}-\vec{x}_*)\mathrm{d}x^1\wedge\mathrm{d}x^2. \qquad (4.1.18)$$

The constraints mean that the holonomy around the puncture is related to the Lie algebra element Q via

$$h = \exp(8\pi GQ), \qquad (4.1.19)$$

where we have implicitly chosen a reference point where the holonomy begins and ends. We will return to the role of this reference point when we discuss the phase space of the theory further below. Since the element $Q = gQ_0g^{-1}$ lies in a fixed adjoint orbit, the holonomy has to lie in a fixed conjugacy class of the element $\exp(Q_0)$. As we saw in Sect. 3.1, the fixed element Q_0 combines the physical attributes of a point particle, like mass and spin. The holonomy h thus lies in fixed conjugacy class labelled by those physical attributes. As a preparation for discussing the phase space of the Chern-Simons theory we therefore study the geometry of conjugacy classes in $\hat{G}$ and their parametrisation in terms of physically meaningful parameters.

4.1.2 Conjugacy classes of the centrally extended Galilei group

We begin with a brief review of the conjugacy classes in $\hat{G}_0$ in our notation. The classification of conjugacy classes of $\hat{G}_0$ is given in [30], and follows from general formula (2.3.14). We would like to interpret this classification in terms of Galilean gravity. The key formula here is (4.1.19). Denoting the $\hat{G}_0$-valued part of the holonomy h by u we obtain

$$u = \exp(8\pi G(mJ + ES - \vec{p}\cdot\vec{J})). \qquad (4.1.20)$$

In order to understand the relation between the adjoint orbits discussed after (3.1.2) and conjugacy classes in $\hat{G}_0$, we rescale the physical parameters mass, energy and momentum which label the adjoint orbits and define

$$\mu = 8\pi Gm, \quad \vec{\pi}_0 = 8\pi G\vec{p}_0, \quad \varepsilon_0 = 8\pi GE_0. \qquad (4.1.21)$$

Then we define group elements corresponding to the special Lie algebra elements (3.1.4), (3.1.5) and (3.1.8) via

$$u_0(\varepsilon_0) = (0, \vec{0}, \varepsilon_0), \qquad u_0(\mu, \varepsilon_0) = (\mu, \vec{0}, \varepsilon_0), \qquad u_0(\vec{\pi}_0) = (0, -\vec{\pi}_0, 0), \tag{4.1.22}$$

where we assume $\mu \neq 0, 2\pi, \vec{\pi}_0 \neq \vec{0}$. When parametrising the $\hat{G}_0$-conjugacy classes obtained by conjugating these elements we could either use parameters $E, \vec{p}, m$ in the Lie algebra or a group parametrisation of the form

$$u = (\mu, -\vec{\pi}, \varepsilon). \tag{4.1.23}$$

The two are related via

$$(\mu, -\vec{\pi}, \varepsilon) = \exp(8\pi G(mJ - \vec{p} \cdot \vec{J} + ES)). \tag{4.1.24}$$

We exhibit the relationship between the two parametrisations for each type of conjugacy class separately. As we shall see, we can think of ε and $\vec{\pi}$ as energy and momentum parameters on the group, and μ as an angular mass coordinate, with range $[0, 2\pi]$. The compactification of the range of the mass from an infinite range to a finite interval is a familiar feature of three-dimensional gravity, in both the Lorentzian and Euclidean regime [84, 63].

For the trivial conjugacy classes consisting of $u_0(\varepsilon_0)$ we have simply $\varepsilon_0 = 8\pi GE$. The conjugacy class consists of a point $C_{\varepsilon_0} = \{(0, \vec{0}, \varepsilon_0)\}$; the associated centraliser group is the entire group i.e. $N_{\varepsilon_0} = \hat{G}_0$. The conjugacy class containing $u_0(\mu, \varepsilon_0)$ can be parametrised as

$$C_{\varepsilon_0,\mu} = \{(\mu, -\vec{\pi}, \varepsilon) | \varepsilon = \varepsilon_0 + \frac{1}{4} \cot\left(\frac{\mu}{2}\right) \vec{\pi}^2\}. \tag{4.1.25}$$

Provided $\mu \neq \pi$ this a paraboloid opening towards the positive or negative ε-direction, depending on the sign of μ (non-zero by assumptions). When $\mu = \pi$ we obtain a plane in the three-dimensional $(\varepsilon, \vec{\pi})$ space. Pictures of all these conjugacy classes can be found in [30].

The planar conjugacy class is the only one which looks qualitatively different from any of the adjoint orbits obtained in Sect. 3.1. The planarity means that the energy parameter ε does not depend on the momentum $\vec{\pi}$, as confirmed by the dispersion relation in (4.1.25). However, this phenomenon is not as strange as it may at first appear. A similar effect occurs when one parametrises conjugacy classes in the group $SL(2,\mathbb{R})$ (which is the relativistic analogue of $\hat{G}_0$) in the form $\exp(\varepsilon J_0)\exp(-\vec{\pi}\cdot\vec{J})$, where J_0, J_1, J_2 are the usual Lorentz generators of $SL(2,\mathbb{R})$. When plotting the conjugacy classes containing the element $\exp(\mu J_0)$ in the $(\varepsilon,\vec{\pi})$ space, they curve up or down, or are horizontal depending on the value of μ, just like for $\hat{G}_0$.

The centraliser group associated to the conjugacy classes (4.1.25) is the group $N_{E_0,m} \simeq U(1)\times\mathbb{R}$ already encountered in the discussion of adjoint orbits (3.1.7). The relation (4.1.24) to the Lie algebra parameters is

$$\vec{\pi} = \frac{\sin\frac{\mu}{2}}{\frac{m}{2}}\vec{p}, \qquad (\varepsilon-\varepsilon_0) = \frac{\sin\mu}{m}(E-E_0). \tag{4.1.26}$$

Finally, the conjugacy class containing $u_0(\vec{\pi}_0)$ is

$$C_{\vec{\pi}_0} = \{(0,\vec{\pi},\varepsilon)||\vec{\pi}| = |\vec{\pi}_0|, \varepsilon\in\mathbb{R}\}, \tag{4.1.27}$$

which is a cylinder (see again [30] for a picture). The centraliser group is the group $N_{\vec{p}_0} \simeq \mathbb{R}^2$ of (3.1.10). The relation (4.1.24) to the Lie algebra parameters is

$$\vec{\pi} = 8\pi G\vec{p}, \qquad (\varepsilon-\varepsilon_0) = 8\pi G(E-E_0). \tag{4.1.28}$$

Turning now to the conjugacy classes in the inhomogeneous group $\hat{G}$, we would like to parametrise the holonomies (4.1.19) in the form

$$h = u\cdot r, \tag{4.1.29}$$

where $u\in\hat{G}_0$ and $r\in\hat{G}$ is entirely inside the normal abelian subgroup, i.e. can be written

$$r = (\theta\sigma, -\theta\vec{\iota}, \theta\iota), \quad \sigma,\iota\in\mathbb{R}, \vec{\iota}\in\mathbb{R}^2. \tag{4.1.30}$$

The reason for this parametrisation is that the Poisson structure, to be studied further below, takes a particularly simple form in these coordinates. Note that the factorisation is similar to the semi-direct product factorisation (2.3.25), but of the opposite order.

Writing u again as in (4.1.23) we multiply out to find

$$h = (\mu + \theta\sigma, -(\vec{\pi} + \theta R(\mu)\vec{\iota}), \varepsilon + \theta(\iota + \frac{1}{2}\vec{\pi} \times R(\mu)\vec{\iota}). \tag{4.1.31}$$

In order find how the coordinates $\iota, \vec{\iota}$ transform under conjugation (σ is invariant), we use the following conjugation formula, with the conjugating element $g \in \hat{G}$ written in the semi-direct product parametrisation (2.3.25) i.e. $g = xv$:

$$\begin{aligned} u'r' &= (xv)(ur)(xv)^{-1} \\ &= x(vuv^{-1}\; vrv^{-1})x^{-1} \\ &= (vuv^{-1})\;(vu^{-1}v^{-1}\; x\; vuv^{-1}\; vrv^{-1}\; x^{-1}) \end{aligned} \tag{4.1.32}$$

or, in a more compact form

$$u' = vuv^{-1}, \qquad r' = (u')^{-1}xu'\; x^{-1}\; vrv^{-1}, \tag{4.1.33}$$

where we have used the commutativity of the inhomogeneous part of $\hat{G}$. Note that the multiplication of purely inhomogeneous elements, with all parameters proportional to θ, amounts to the *addition* of the corresponding parameters.

Using the general formula (4.1.33) we classify the conjugacy classes of $\hat{G}$. The results are summarised in Table 4.1, which should be seen as the group analogue of the adjoint orbits in Table 3.1. The two-dimensional classes are, in fact, the same as in the Lie algebra case, except that they now live in the inhomogeneous, abelian part of the group $\hat{G}$. The four-dimensional conjugacy classes are obtained by exponentiating the four-dimensional adjoint orbits into the group. Thus their topology is exactly the same as that of the corresponding adjoint orbits, but their embedding in the group, and hence their parametrisation in terms of coordinates (4.1.23) and (4.1.30) is new. In particular the "dispersion relations" between the group parameters defining each classes take an unfamiliar form. We list them in each case. We also give reference group elements from which the conjugacy classes can be obtained by conjugation. The parameters appearing in them are related to those in the reference elements of

Table 3.1 via (4.1.22) and

$$\sigma = 8\pi G s, \quad \vec{\iota}_0 = 8\pi G \vec{j}, \quad \iota_0 = 8\pi G j_0. \tag{4.1.34}$$

Reference element	**Generic element in conjugacy class**	**Geometry of conjugacy class**
$(0,\vec{0},\varepsilon_0)(0,\vec{0},\theta\iota_0)$	$(0,\vec{0},\varepsilon_0)(0,\vec{0},\theta\iota_0)$	Point
$(0,\vec{0},\varepsilon_0)(\theta\sigma,\vec{0},\theta\iota_0)$	$(0,\vec{0},\varepsilon_0)(\theta\sigma,-\theta\vec{\iota},\theta\iota)$ with $\vec{\iota}\in\mathbb{R}^2$, $\iota=\iota_0+\frac{1}{2\sigma}\vec{\iota}^2$	Paraboloid $\simeq \mathbb{R}^2$
$(0,\vec{0},\varepsilon_0)(0,\theta\vec{\iota}_0,\theta\iota_0)$	$(0,\vec{0},\varepsilon_0)(0,-\theta\vec{\iota},\theta\iota)$ with $\iota\in\mathbb{R}$, $\lvert\vec{\iota}\rvert=\lvert\vec{\iota}_0\rvert$	Cylinder $\simeq S^1\times\mathbb{R}$
$(\mu,\vec{0},\varepsilon_0)(\theta\sigma,\vec{0},\theta\iota_0)$	$(\mu,-\vec{\pi},\varepsilon)(\theta\sigma,-\theta\vec{\iota},\theta\iota)$ with $\varepsilon=\varepsilon_0+\dfrac{1}{4}\cot\frac{\mu}{2}\vec{\pi}^2$, $\iota=\iota_0+\frac{\sigma\vec{\pi}^2}{8\sin^2\frac{\mu}{2}}+\frac{1}{2}(\epsilon+\cot\frac{\mu}{2})\vec{\pi}\cdot R(\mu)\vec{\iota}$	Tangent bundle of paraboloid or plane $\simeq\mathbb{R}^4$
$(0,-\vec{\pi}_0,\varepsilon_0)(\theta\sigma,-\theta\vec{\iota}_0,\theta\iota_0)$	$(0,-\vec{\pi},\varepsilon)(\theta\sigma,-\theta\vec{\iota},\theta\iota)$ with $\lvert\vec{\pi}\rvert=\lvert\vec{\pi}_0\rvert$, $\vec{\pi}\cdot\vec{\iota}-\varepsilon\sigma=\vec{\pi}_0\cdot\vec{\iota}_0-\varepsilon_0\sigma$	Tangent bundle of cylinder $\simeq S^1\times\mathbb{R}^3$

Table 4.1. Conjugacy classes of the centrally extended Galilei group $\hat{G}$

4.1.3 The phase space of Galilean 3d gravity and the Poisson-Lie structure of $\hat{G}$

The phase space of Chern-Simons theory with gauge group H on a three-manifold of the form $\mathbb{R}\times\Sigma$ is the space of flat H connections on Σ, equipped with Atiyah-Bott symplectic structure [9, 8]. As explained in Chapter 1, this space can be parametrised in terms of holonomies around the non-contractible loops in Σ, starting and ending at an arbitrarily chosen reference point, modulo conjugation (the residual gauge freedom at the reference point). The symplectic structure on the phase space can be described in a number of ways. The one which is ideally suited to the combinatorial quantisation approach used here is a description due to Fock and Rosly [31].

As already explained, the idea is to work on the extended phase space of all holonomies around non-contractible loops on Σ (without division by conjugation), and to define a Poisson bracket on the extended phase space in terms of a classical r-matrix which solves the classical Yang-Baxter equation and which is compatible with the inner product used in the Chern-Simons action. The compatibility conditions is simply that the symmetric part of the r-matrix equals the Casimir element associated with the inner product used in defining the Chern-Simons action. As an aside, note that in the original description of the phase space due to Fock and Rosly [31], the embedded graph on Σ is a triangulation with both a cilium and an r-matrix defined on each vertex that determine the Poisson structure on the extended phase. It has been shown in [2, 3], that the construction holds when the embedded graph is the fundamental group.

The classical r-matrix endows the gauge group with a Poisson structure which is compatible with its Lie group structure, i.e. with a Poisson-Lie structure. It turns out that the Poisson structure on the extended phase space considered by Fock and Rosly can be described in terms of two standard Poisson structures associated to the Poisson-Lie group PL: one is called the Heisenberg double (a non-linear version of the cotangent bundle of PL) and the other is the dual Poisson-Lie structure (a non-linear version of the dual of the Lie algebra of PL) [4]. The extended phase space for Chern-Simons theory on $\mathbb{R} \times \Sigma_{gn}$, where Σ_{gn} is a surface of genus g with n punctures, can then be shown [4] to be isomorphic, as a Poisson manifold, to a cartesian product of g copies of the Heisenberg double of PL and a symplectic leaf of the dual Poisson Lie group for every puncture. Since the group $\hat{G}$ is of the form $PL = H \ltimes \mathfrak{h}^*$, all of the results of Chapter 1 based on [68] can be directly imported. In particular, it is shown in [68] that the Heisenberg double for groups of the form $H \ltimes \mathfrak{h}^*$ is simply the cotangent bundle of $H \times H$. The dual Poisson structure is more interesting. We now work it out in detail for the group $\hat{G}$ so we assume a universe with punctures and no handles.

As we argued in Chapter 2, the Lie algebra $\hat{\mathfrak{g}}$ has the structure of a classical double: it is the direct sum of the Lie algebra $\hat{\mathfrak{g}}_0$ (spanned by J, J_1, J_2, S) and the abelian Lie algebra spanned by H, P_1, P_2, M. These two Lie algebras are in duality via the pairing $\langle\ ,\ \rangle$, so we can identify the abelian Lie algebra with the dual vector space $\hat{\mathfrak{g}}_0^*$. It follows that $\hat{\mathfrak{g}}$ has, in fact, the structure of a Lie bi-algebra, which means that it has both commutators and co-commutators, with a compatibility between the two. The r-matrix that induces this structure is $r_{\hat{\mathfrak{g}}}$ of equation (2.2.41)

$$r_{\hat{\mathfrak{g}}} = M \otimes J + H \otimes S - P_i \otimes J_i$$

As mentioned above, the r-matrix also gives rise to a Poisson structure on the group $\hat{G}$, and to associated dual and Heisenberg double Poisson structures. For explicit formulae in a more general context we refer the reader to [68]. Here we note that the symplectic leaves of the dual Poisson structure (which are associated to the punctures in the Fock-Rosly construction) can be mapped to conjugacy classes in the original group $\hat{G}$. The symplectic potentials for the symplectic form on these leaves can be expressed very conveniently in a manner which is entirely analogous to the expression (3.1.25) for the symplectic potentials on (co)adjoint orbits. The only difference is that the inhomogeneous Lie algebra element in (3.1.25) is replaced by its group analogue in the factorisation (4.1.30), leading to the symplectic potential

$$\tilde{\boldsymbol{\theta}}_{\hat{G}^*} = \langle(\sigma H - \vec{\iota}\cdot\vec{P} + \iota M), \mathrm{d}vv^{-1}\rangle, \tag{4.1.35}$$

where $r = (\theta\sigma, -\theta\vec{\iota}, \theta\iota)$ is the inhomogeneous part of a group element in the given conjugacy class. By inserting the parameters $\sigma, \iota, \vec{\iota}$ for the various conjugacy classes given in Table 4.1 one obtains the symplectic potentials on each of the conjugacy classes listed there. One can attempt to simplify the resulting expressions using (4.1.33), but we have not been able to generate illuminating formulae in this way.

The upshot of the Fock-Rosly construction for Chern-Simons theory with gauge group $\hat{G}$ is thus as follows. For a universe of topology $\mathbb{R} \times \Sigma_{gn}$, and assuming for definiteness that the punctures are decorated with coadjoint orbits O_{m_i,s_i}, $i = 1 \ldots n$, representing particles with masses m_i and spins s_i, the extended phase space is

$$\tilde{\mathcal{P}} = \hat{G}^{2g} \times C_{\mu_n\sigma_n} \times \ldots C_{\mu_1\sigma_1}, \tag{4.1.36}$$

where we have written $C_{\mu_i\sigma_i}$ for the conjugacy classes obtained by exponentiating the orbits $O_{m_is_i}$, and assumed that none of the μ_i equal an integer multiple of 2π. We have also suppressed the labels E_0, j_0 and their group analogues ε_0, ι_0 since they do not affect the orbit geometry and symplectic structure.

The physical phase space is the quotient

$$\begin{aligned}\mathcal{P} = \{(A_g, B_g, \ldots, A_1, B_1, M_n, \ldots M_1) \in \tilde{\mathcal{P}}| \\ [A_g, B_g^{-1}] \ldots [A_1, B_1^{-1}] M_n \ldots M_1 = 1\}/\text{conjugation}.\end{aligned} \tag{4.1.37}$$

For a universe with no handles that we consider here, the non-trivial part of the phase space, apart from the quotient, are the conjugacy classes in $\hat{G}$. As explained above, these are precisely the symplectic leaves of the dual Poisson-Lie group $\hat{G}^*$, with symplectic potential (4.1.35).

4.2 Galilean gravity with a cosmological constant as the Chern-Simons theory of the Newton-Hooke group

Let us now return back to the action of Section 4.1.1:

$$\begin{aligned}I_{CS}[A] =& \frac{1}{16\pi G} \int_{\mathcal{M}} \langle A \wedge \mathrm{d}A \rangle + \frac{1}{3} \langle A \wedge [A, A] \rangle \\ =& \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{Im}_\theta(\Omega \wedge \mathrm{d}\Psi + \Psi \wedge \mathrm{d}\Omega - \Omega_i \wedge \mathrm{d}\Omega_i + 2\Omega \wedge \Omega_1 \wedge \Omega_2) \\ =& \frac{1}{16\pi G} \bigg(\int_{\mathcal{M}} \big(\omega \wedge \mathrm{d}f + f \wedge \mathrm{d}\omega + e \wedge \mathrm{d}\eta + \eta \wedge \mathrm{d}e - \omega_i \wedge \mathrm{d}e_i - e_i \wedge \mathrm{d}\omega_i\big) \\ &+ 2 \int_{\mathcal{M}} \big(\epsilon_{ij} \omega \wedge \omega_i \wedge e_j \epsilon_{ij} + e \wedge \omega_1 \wedge \omega_2 + \lambda \epsilon_{ij} e \wedge e_1 \wedge \epsilon_2\big) \bigg),\end{aligned} \tag{4.2.1}$$

The equations of motion $F = 0$ now leave the torsion part unchanged. As shown earlier in the Chapter, the curvature acquires a cosmological term proportional to λ, so (4.1.11) implies

$$R' = 0 \qquad T = 0, \tag{4.2.2}$$

where we have denoted the curvature R' to point out that it is different to R one obtains in the flat case. As a consequence we get the constraint (4.1.16)

$$F_\Sigma(\vec{x}) = 8\pi G Q \delta^{(2)}(\vec{x} - \vec{x}_*)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{4.2.3}$$

In the decomposition (4.1.7) of the curvature and parametrisation (3.1.13):

$$\begin{aligned} R^{a'}_\Sigma &= 8\pi G(mJ + ES - \vec{p}\cdot\vec{J})\delta^{(2)}(\vec{x} - \vec{x}_*)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ T^a_\Sigma &= 8\pi G(sM + jH - \vec{j}\cdot\vec{P})\delta^{(2)}(\vec{x} - \vec{x}_*)\mathrm{d}x^1 \wedge \mathrm{d}x^2. \end{aligned} \tag{4.2.4}$$

The constraints now mean that the holonomy around the puncture is related to the Lie algebra element Q via

$$h = \exp(8\pi G Q), \tag{4.2.5}$$

where now $Q \in \hat{NH}$.

Again, as in the previous section we have implicitly chosen a reference point where the holonomy begins and ends. The holonomy has to lie in a fixed conjugacy class in $\hat{NH}$ of the element $\exp(Q_0)$. The effect of gravitation is that the parameters corresponding to physical attributes are now group-like.

4.2.1 Conjugacy classes of the Newton-Hooke group

To briefly demonstrate the effect of gravitation, we give here the conjugacy classes of the Newton-Hooke groups. We use notation along the lines of the discussion of the conjugacy classes of the extended homogeneous Galilei group in Section 4.1.2. So we use Greek letters to denote the group-valued parameters we used in our discussion of coadjoint orbits.

As before, we rescale the parameters labelling the adjoint orbits. For the oscillating group $\hat{NH}_+$, we write

$$\mu^\pm = 8\pi G m^\pm, \quad \vec{\pi}_0^\pm = 8\pi G \vec{p}_0^\pm, \quad \varepsilon_0^\pm = 8\pi G E_0^\pm, \tag{4.2.6}$$

where we remind the reader that, for instance, $m^{\pm} = m \pm \sqrt{\lambda}s$, so we will denote accordingly, $\mu^{\pm} = \mu \pm \sqrt{\lambda}\sigma$ and so on. For the expanding group $\hat{NH}_-$

$$\mu_{\mathbb{C}} = 8\pi G m_{\mathbb{C}}, \quad \vec{\pi}_{\mathbb{C}(0)} = 8\pi G \vec{p}_{\mathbb{C}(0)} \quad \varepsilon_{\mathbb{C}(0)} = 8\pi G E_{\mathbb{C}(0)}, \tag{4.2.7}$$

where now $\mu_{\mathbb{C}} = \mu + \mathrm{i}\sqrt{-\lambda}\sigma$ and so on. We then write for the element labelling each conjugacy class u_0, i.e.

$$u_0 = \exp(8\pi G Q_0),$$

where Q_0 can be any of the elements labelling an adjoint orbit in $\hat{NH}_{\pm}$. We parametrise the $\hat{NH}_{\pm}$-conjugacy classes obtained by conjugating these elements using a group parametrisation of the form

$$u = (\mu, -\vec{\pi}, \varepsilon). \tag{4.2.8}$$

Here $u = \exp(8\pi G \mathrm{Ad}_g Q_0)$ and g is an element of either one of two copies $\hat{G}_0^{\pm}$ that $\hat{NH}_+$ can be decomposed to, or, in the case of $\hat{NH}_-$, g lies in the complexification of $\hat{G}_0$. We list the conjugacy classes in Tables 4.2, 4.3.

Let us discuss, the conjugacy class with the product geometry of two paraboloids again. The condition $\mu^{\pm} \neq 0$ is a constraint on the physically meaningful parameters μ, σ

$$\mu \neq \pm\sqrt{\lambda}\sigma.$$

Now, apart from μ, the group-valued spin σ also has finite range where $\sigma \in [0, \frac{2\pi}{\sqrt{\lambda}}]$

Let us note that

$$\cot\left(\frac{\mu^{\pm}}{2}\right) = -\frac{\sin(\frac{\mu}{2})\sin(\frac{\sqrt{\lambda}\sigma}{2}) \mp \cos(\frac{\mu}{2})\cos(\frac{\sqrt{\lambda}\sigma}{2})}{\cos(\frac{\sqrt{\lambda}\sigma}{2})\sin(\frac{\mu}{2}) \pm \cos(\frac{\mu}{2})\sin(\frac{\sqrt{\lambda}\sigma}{2})}.$$

It can be seen that the value $\mu^{\pm} = \pi$ is, again, special. So for the pairs of values $\mu = \pi, \sigma = 0$ and $\mu = 0, \sigma = \frac{\pi}{\sqrt{\lambda}}$, the dispersion relation for the conjugacy class becomes

$$\varepsilon = \varepsilon_0, \qquad \sigma = \sigma_0$$

regardless what $\vec{\pi}, \vec{\iota}$ is. For these values *both* energy and total spin ε, σ are left independent of the value of momentum and angular momentum $\vec{\pi}, \vec{\iota}$.

Reference element $u_0 = (\mu^+, \vec{\pi}_0^+, \varepsilon_0^+) \cdot (\mu^-, \vec{\pi}_0^-, \varepsilon_0)$		
Parameters in u_0	**Generic element** u **on conjugacy class**	**Geometry of conjugacy class**
$\mu^\pm = 0, \quad \vec{\pi}_0^\pm = \vec{0}$	$(0, \vec{0}, \varepsilon_0^+) \cdot (0, \vec{0}, \varepsilon_0^-)$	Point
$\mu^+ = 0, \quad \vec{\pi}_0^+ = \vec{0}$ $\mu^- = 0, \quad \vec{\pi}_0^- \neq \vec{0}$ and $+ \leftrightarrow -$	$(0, \vec{0}, \varepsilon_0^+) \cdot (0, \vec{\pi}^-, \varepsilon^-)$ with $\varepsilon^- \in \mathbb{R}, \lvert\vec{\pi}^-\rvert = \lvert\vec{\pi}_0^-\rvert$ and $+ \leftrightarrow -$	Cylinder & Point $\simeq S^1 \times \mathbb{R} \cup \{\varepsilon_0^+\}$ and $+ \leftrightarrow -$
$\mu^+ = 0, \quad \vec{\pi}_0^+ = \vec{0}$ $\mu^- \neq 0, \quad \vec{\pi}_0^- = \vec{0}$ and $+ \leftrightarrow -$	$(0, \vec{0}, \varepsilon_0^+) \cdot (\mu^-, \vec{\pi}^-, \varepsilon^-)$ with $\vec{\pi}^- \in \mathbb{R}^2, \varepsilon^- =$ $\varepsilon_0^- + \frac{1}{4}\cot\left(\frac{\mu^-}{2}\right)(\vec{\pi}^-)^2$ and $+ \leftrightarrow -$	Paraboloid & Point $\simeq \mathbb{R}^2 \cup \{\varepsilon_0^+\}$ and $+ \leftrightarrow -$
$\mu^\pm = 0, \quad \vec{\pi}_0^\pm \neq \vec{0}$	$(0, \vec{\pi}^+, \varepsilon^+) \cdot (0, \vec{\pi}^-, \varepsilon^-)$ with $\varepsilon^\pm \in \mathbb{R}$, $\lvert\vec{\pi}^\pm\rvert = \lvert\vec{\pi}_0^\pm\rvert$	Cylinder & Cylinder $\simeq (S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$
$\mu^\pm \neq 0, \quad \vec{\pi}_0^\pm = \vec{0}$	$(\mu^+, \vec{\pi}^+, \varepsilon^+) \cdot (\mu^-, \vec{\pi}^-, \varepsilon^-)$ with $\vec{\pi}^\pm \in \mathbb{R}^2, \varepsilon^\pm =$ $\varepsilon_0^\pm + \frac{1}{4}\cot\left(\frac{\mu^\pm}{2}\right)(\vec{\pi}^\pm)^2$	Paraboloid & Paraboloid $\simeq \mathbb{R}^2 \times \mathbb{R}^2$
$\mu^+ \neq 0, \quad \vec{\pi}_0^+ = \vec{0}$ $\mu^- = 0, \quad \vec{\pi}_0^- \neq \vec{0}$ and $+ \leftrightarrow -$	$(\mu^+, \vec{\pi}^+, \varepsilon^+) \cdot (0, \vec{\pi}^-, \varepsilon^-)$ with $\vec{\pi}^+ \in \mathbb{R}^2$, $\varepsilon^+ = \varepsilon_0^+ + \frac{1}{4}\cot\left(\frac{\mu^+}{2}\right)(\vec{\pi}^+)^2$ and $+ \leftrightarrow -$	Paraboloid & Cylinder $\simeq \mathbb{R}^2 \times (S^1 \times \mathbb{R})$

Table 4.2. Conjugacy classes of the centrally extended oscillatory Newton-Hooke group $\hat{NH}_+$

Similarly, we can look at the conjugacy class with $\mu_{\mathbb{C}} \neq 0$ in $\hat{NH}_-$. Now the cotangent is

$$\cot\left(\frac{\mu_{\mathbb{C}}}{2}\right) = -\frac{\sin(\frac{\mu}{2})\sinh(\frac{\sqrt{-\lambda}\sigma}{2}) + \mathrm{i}\cos(\frac{\mu}{2})\cosh(\frac{\sqrt{-\lambda}\sigma}{2})}{\cos(\frac{\mu}{2})\sinh(\frac{\sqrt{-\lambda}\sigma}{2}) - \mathrm{i}\cosh(\frac{\sqrt{-\lambda}\sigma}{2})\sin(\frac{\mu}{2})},$$

and it can only be zero (within the range of μ, σ) when $\mu = \pi, \sigma = 0$. As above, for these values μ, σ neither the energy nor the total spin depend on the momentum and angular momentum $\vec{\pi}, \vec{\iota}$.

Reference element $u_0 = (\mu_{\mathbb{C}}, \vec{\pi}_{\mathbb{C}(0)}, \varepsilon_{\mathbb{C}(0)})$		
Parameters in u_0	**Generic element u on conjugacy class**	**Geometry of conjugacy class**
$\mu_{\mathbb{C}} = 0, \quad \vec{\pi}_{\mathbb{C}(0)} = \vec{0}$	$(0, \vec{0}, \varepsilon_{\mathbb{C}(0)})$	Point
$\mu_{\mathbb{C}} = 0, \quad \vec{\pi}_{\mathbb{C}(0)} \neq \vec{0}$	$(0, \vec{\pi}_{\mathbb{C}}, \varepsilon_{\mathbb{C}})$ with $\varepsilon_{\mathbb{C}} \in \mathbb{C}, \vec{\pi}^2_{\mathbb{C}(0)} = \vec{\pi}^2_{\mathbb{C}}$	Complex two-plane $\simeq \mathbb{C}^2$
$\mu_{\mathbb{C}} \neq 0, \quad \vec{\pi}_{\mathbb{C}(0)} = \vec{0}$	$(\mu_{\mathbb{C}}, \vec{\pi}_{\mathbb{C}}, \varepsilon_{\mathbb{C}})$ with $\vec{\pi}_{\mathbb{C}} \in \mathbb{C}^2, \varepsilon_{\mathbb{C}} = \varepsilon_{\mathbb{C}(0)} + \frac{1}{4} \cot\left(\frac{\mu_{\mathbb{C}}}{2}\right) \vec{\pi}^2_{\mathbb{C}}$	Complex Paraboloid $\simeq \mathbb{C}^2$

Table 4.3. Conjugacy classes in the centrally extended oscillatory Newton-Hooke group $\hat{NH}_-$

We could go on to provide the symplectic potentials for the conjugacy classes of the Newton-Hooke groups but seeing as the groups are no longer of semidirect product form, such formulae are cumbersome and not illuminating so we will finish our discussion here.

Again, from the results in Chapter 2, it is clear that the Lie algebra $\hat{\mathfrak{nh}}$ also has the structure of a classical double. It is a bialgebra given as a direct sum of the Lie algebras $\hat{\mathfrak{g}}_0$ (spanned by J, J_1, J_2, S) and the abelian Lie algebra spanned by H, Π_1, Π_2, M dual to each other via the pairing $\langle \; , \; \rangle$. So we can identify the abelian Lie algebra with the dual vector space $\hat{\mathfrak{g}}_0^*$. The r-matrix that induces the bialgebra structure is now $r_{\hat{\mathfrak{nh}}}$ of equation (2.2.39)

$$r_{\hat{\mathfrak{nh}}} = M \otimes J + H \otimes S - P_i \otimes J_i - \sqrt{-\lambda} J_1 \wedge J_2.$$

Chapter 5

Galilean quantum gravity in 2+1 dimensions

Our description of classical Galilean gravity in terms of a Chern-Simons theory and a compatible r-matrix is tailor-made for the application of the combinatorial quantisation programme reviewed in Chapter 1. As explained there, all aspects of the quantisation - the construction of the Hilbert space, the representation of observables and the implementation of symmetries on the Hilbert space - are determined by a quantum group associated to the gauge group used in the Chern-Simons action and the compatible r-matrix. The centrally extended Galilei group with the classical r-matrix (2.2.41) is an example of a classical double of the form $H \ltimes \mathfrak{h}^*$, and it follows from the general results of [68] that the relevant quantum group for Galilean quantum gravity is the quantum double $D(\hat{G}_0)$ of the homogeneous part of the centrally extended Galilei group or, in short, the Galilei double. Here we work out its representation theory, discuss quantum gravity interactions and the emergence of a non-commutative spacetime if one interchanges the roles of the position/momentum algebras.

The Newton-Hooke groups, follow the pattern for quantisation of deSitter and anti-deSitter gravity. Their Poisson-Lie structure is therefore quantised by the q-deformed quantum double $G_{(q)0}$ of $\hat{G}_0$. We finish this chapter by giving a detailed description of this Hopf algebra and its real structures.

5.1 The quantum double of the Galilei group and representations

We will here define and work out the representation theory of $D(\hat{G}_0)$. As we argued in Chapter 1, this is closely related to the representation theory of the centrally extended Galilei group $\hat{G}$ itself. We therefore begin with a quick review of the latter.

5.1.1 Representation theory of the centrally extended Galilei group

The particular version of the centrally extended Galilei group studied in this thesis is one of the central extensions of the Galilei group whose representation theory is studied in [34]. We briefly review the representation theory in our notation. As explained in Chapter 1, we give particular emphasis on the fact that, for our version of the extended Galilei group, the representation theory easily follows from the general representation theory of semidirect products as developed in [56] and explained, for example, in [15], [81].

In the case at hand where the semidirect product is of the particular form $\hat{G}_0 \ltimes \hat{\mathfrak{g}}_0^*$, the irreps are labelled by $\hat{G}_0$-orbits on the dual of the abelian group $\hat{\mathfrak{g}}_0^*$, as explained in section 1.4.2. The latter is the Lie algebra $\hat{\mathfrak{g}}_0$ itself (viewed as a vector space). The relevant orbits in this case are therefore the adjoint orbits of $\hat{G}_0$ which we listed, together with associated centraliser groups, in Sect. 3.1. Since the representation theory is well-documented in the literature [34, 72], we only sketch the method and then illustrate it with one case in order to establish a clear dictionary between our notation and that used, for example, in [34].

For definiteness, let us consider the case of a massive particle with spin. The relevant adjoint orbit of $\hat{G}_0$ is the orbit $O_{E_0,m}$ (3.1.6), which we can parametrise entirely in terms of the unconstrained momentum vector $\vec{p}$, with the energy given by

$$E = E_0 + \frac{1}{2m}\vec{p}^2. \tag{5.1.1}$$

Homogeneous elements $v \in \hat{G}_0$ act on the orbit via adjoint action Ad_v (3.1.2) and, by a slight abuse of notation, we write this action as $\mathrm{Ad}_v(\vec{p})$. In other words, it is understood that E changes under Galilei boosts according to (3.1.2), which ensure that the energy-momentum relation (5.1.1) is maintained. The carrier space of the irreducible representation is then simply the space $L^2(\mathbb{R}^2)$ of square-integrable functions on the space of momenta.

In order to write down the action of a general element $g \in \hat{G}$ on states $\psi \in L^2(\mathbb{R}^2)$ we need the reference point $Q_0(E_0, m)$ (3.1.5) on the orbit, and we need to pick a map

$$S : O_{E_0,m} \to \hat{G}_0, \tag{5.1.2}$$

which associates to a given point $Q = mJ - \vec{p}\cdot\vec{J} + ES \in O_{E_0,m}$ a $\hat{G}_0$-element $S(\vec{p}) = v$ satisfying

$$\mathrm{Ad}_v(Q_0) = Q, \tag{5.1.3}$$

i.e. v is an element of the homogeneous Galilei group which rotates/boosts the reference point Q_0 to the given point Q on the adjoint orbit.

In the case at hand, we can, for example, pick

$$S(\vec{p}) = (0, -\epsilon\frac{\vec{p}}{m}, 0), \tag{5.1.4}$$

where we have again used $\vec{p}$ as a coordinate on $O_{E_0,m}$. Then we define the so-called multiplier function [15]

$$n : \hat{G}_0 \times O_{E_0,m} \to N_{E_0,m}, \qquad n(v, \vec{p}) = (S(\vec{p}))^{-1} v\, S(\mathrm{Ad}_{v^{-1}}(\vec{p})). \tag{5.1.5}$$

The key properties of this function (both easy to check) are that it takes values in the centraliser group $N_{E_0,m}$ (3.1.7) associated to the orbit $O_{E_0,m}$, and that it satisfies the cocycle condition

$$n(v_1, \vec{p})n(v_2, \mathrm{Ad}_{v_1^{-1}}\vec{p}) = n(v_1 v_2, \vec{p}).$$

Computing explicitly, we find, for $v = (\varphi, \vec{w}, \zeta)$, again with the convention $\vec{v} = -\epsilon\vec{w}$, that

$$n(v, \vec{p}) = (\varphi, \vec{0}, \zeta + \frac{1}{2m}\vec{p} \times \vec{v}). \tag{5.1.6}$$

Then, picking an irrep $\pi_{j,s}$ of the centraliser group $N_{E_0,m} \simeq U(1) \times \mathbb{R}$, labelled by an integer j and a real number s, the action of an element $g = (\theta\alpha, \theta\vec{a}, \theta\tau)(\varphi, \vec{w}, \zeta) \in \hat{G}$ parametrised in the semi-direct product form (2.3.25) on a state $\psi \in L^2(\mathbb{R}^2)$ is

$$\begin{aligned}(\Pi_{E_0,m,s,j}(g)\ \psi)\,(\vec{p}) &= \pi_{j,s}(n(v,\vec{p}))e^{i(m\tau+\vec{a}\cdot\vec{p}+E\alpha)}\ \psi(R(-\varphi)(\vec{p}-m\vec{v})) \\ &= e^{i(j\varphi+s(\zeta+\frac{1}{2m}\vec{p}\times\vec{v})+m\tau+\vec{a}\cdot\vec{p}+\alpha(E_0+\frac{1}{2m}\vec{p}^2)}\ \psi(R(-\varphi)(\vec{p}-m\vec{v})).\end{aligned} \tag{5.1.7}$$

In comparing this expression with representations in the literature readers should be aware of the different parametrisation (2.3.30) frequently used for the Galilei group, and the relation to our semi-direct parametrisation explained after (2.3.30).

It is not difficult to adapt this example to the representations corresponding to the cylindrical orbits (3.1.9). The centraliser group is $N_{\vec{p}_0} \simeq \mathbb{R} \times \mathbb{R}$, so irreps are labelled by the orbit label $\vec{p}_0$ and two real numbers s, k characterising an irrep $\mathbb{R} \times \mathbb{R}$. Finally, note that irreps of $\hat{G}$ corresponding to the vacuum orbit (3.1.4) are irreps of the group $\hat{G}_0$, which were studied in [49]

5.1.2 The Galilei double and its irreducible representations

The transition from the inhomogeneous group $\hat{G}$ to the quantum double $D(\hat{G}_0)$ can be most easily understood at the level of representations [78]. Looking at (5.1.7) we see that the translation part $(\theta\alpha, \theta\vec{a}, \theta\tau)$ acts on states by multiplication with the function $\exp(i(\alpha E + \vec{a} \cdot \vec{p} + m\tau))$ on momentum space $\hat{\mathfrak{g}}_0$. In the quantum double, these "plane waves" are replaced by general functions on the *group* $\hat{G}_0$. At the purely algebraic level, this makes no difference, but the non-abelian nature of $\hat{G}_0$ makes a crucial difference for the co-product structure. We will call the quantum double $D(\hat{G}_0)$, the Galilei double in this thesis. This is a non-co-commutative Hopf algebra, with a non-trivial $\mathcal{R}$-matrix and ribbon element.

The mathematical preliminaries on Hopf algebras have been provided in Section 1.4.1 together with the Hopf $*$-algebra structure of the double of a Lie group H. In this section we adopt a naive approach and we look at those elements of $D(H)$ that can be described as a tensor product of a group element $v \in H$ and a function f on H. In terms of functions on $H \times H$ such elements correspond to singular elements in $D(H)$ (i.e. they are given as the tensor product of a delta function with a $C_0(H, \mathbb{C})$ function) but they are convenient for summarising the algebra and co-algebra structure. The Hopf $*$-algebra structure of the double in terms of those elements can be read off from relations (1.4.16) substituting $\delta_v \otimes f$ for F. Explicitly

$$
\begin{aligned}
&(v_1 \otimes f_1) \cdot (v_2 \otimes f_2) = v_1 v_2 \otimes (f_1 f_2 \circ \mathrm{Ad}_{v_1^{-1}}) \\
&\Delta(v \otimes f)(u_1, u_2) = v \otimes v\; f(u_1 u_2) \\
&\epsilon(v \otimes f) = v \otimes f(e) \\
&S(v \otimes f) = v^{-1} \otimes f \circ \mathrm{Ad}_{v_1^{-1}} \circ (\,)^{-1} \\
&(v \otimes f)^* = v^{-1} \otimes (\overline{f} \circ \mathrm{Ad}_{v^{-1}})
\end{aligned}
\tag{5.1.8}
$$

The universal $\mathcal{R}$-matrix of $D(H)$ is an element of $D(H) \otimes D(H)$; in our singular notation it is

$$
\mathcal{R} = \int_H dv (e \otimes \delta_v) \otimes (v \otimes 1), \tag{5.1.9}
$$

where e stands for the identity element of H and 1 for the function on H which is 1 everywhere.

The irreps of the quantum double of a continuous Lie group H, classified in [53], label the H-conjugacy classes and irreps of associated centraliser groups. For the case of the extended homogeneous Galilei group $\hat{G}_0$, this is precisely the data we collected in equations (4.1.25) to (4.1.27). The study of the representation theory in [53] is in terms of equivariant functions on the group, but it is not difficult to express the irreps in terms of functions on the conjugacy classes and multipliers like (5.1.6). In fact, this is the language employed for irreps of the double of a finite group in [13].

Consider for definiteness the irreps corresponding to the conjugacy classes $C_{\varepsilon_0,\mu}$ (4.1.25), with associated centraliser group $N_{E_0,m} \simeq U(1) \times \mathbb{R}$ (3.1.7). Analogously to the adjoint orbits $O_{E_0,m}$ the conjugacy classes can be parametrised by the rescaled momentum vectors $\vec{\pi}$, with the rescaled energy determined by

$$\varepsilon = \varepsilon_0 + \frac{1}{4} \cot\left(\frac{\mu}{2}\right) \vec{\pi}^2. \tag{5.1.10}$$

The carrier space of the irrep is the space of square-integrable functions on the class $C_{\varepsilon_0,\mu}$. Using the parametrisation of that orbit in terms of the momentum $\vec{\pi}$ we could again identify the carrier space with $L^2(\mathbb{R}^2)$, but this is less convenient in this case. Instead we use the parametrisation (4.1.23) for elements $u \in C_{\varepsilon_0,\mu}$, where the relation (5.1.10) is understood. With $u_0(\mu, \varepsilon_0)$ as in (4.1.22) we have, from (2.3.14),

$$\mathrm{Ad}_v(u_0(\varepsilon_0, \mu)) = (\mu, 1 - R(\mu)\vec{w}, \varepsilon_0 + \frac{1}{2} \sin^2 \mu \vec{w}^2). \tag{5.1.11}$$

The analogue of the map (5.1.2) is now a map

$$S : C_{\varepsilon_0,\mu} \to \hat{G}_0, \tag{5.1.12}$$

which associates to $u \in C_{\varepsilon_0,\mu}$ an element $v = S(u)$ so that

$$\mathrm{Ad}_v(u_0(\varepsilon_0, \mu)) = u. \tag{5.1.13}$$

Using (5.1.11) we can pick, for example,

$$S(u) = (0, (R(\mu) - 1)^{-1}\vec{\pi}, 0), \tag{5.1.14}$$

so that the analogue of multiplier function (5.1.6) is now

$$n(v, u) = (\varphi, \vec{0}, \zeta + \frac{1}{2}(R(\mu) - 1)^{-1}\vec{\pi} \cdot \vec{v}). \tag{5.1.15}$$

Finally we again pick an irrep π_{js} of the centraliser group $N_{E_0,m} \simeq U(1) \times \mathbb{R}$, labelled by an integer j and a real number s, and denote the carrier space $L^2(C_{\varepsilon_0,\mu})$ of the associated irrep of $D(\hat{G}_0)$ by $V_{\varepsilon_0,\mu,s,j}$. Then singular elements $v \otimes f \in D(\hat{G}_0)$ act on $\psi \in V_{\varepsilon_0,\mu,s,j}$ via

$$\begin{aligned}(\Pi_{\varepsilon_0,\mu,s,j}(v \otimes f)\ \psi)\,(u) &= \pi_{js}(n(v, u)) f(u)\ \psi(\mathrm{Ad}_{v^{-1}}(u)) \\ &= e^{i(j\varphi + s(\zeta + \frac{1}{2}(R(\mu)-1)^{-1}\vec{\pi}\cdot\vec{v})} f(u)\ \psi(\mathrm{Ad}_{v^{-1}}(u)).\end{aligned} \tag{5.1.16}$$

Analogous formulae can be derived for the other conjugacy classes. For the trivial conjugacy classes C_{ε_0} , the centraliser group is the entire group $\hat{G}_0$. As in the case of the centrally extended Galilei group, the irreps of $\hat{G}_0$ again appear as special irreps of $D(\hat{G}_0)$. The irreps of $D(\hat{G}_0)$ labelled by conjugacy classes of the form $C_{\vec{\pi}_0}$ correspond to massless particles, and closely resemble the corresponding irreps of $\hat{G}$.

5.2 Aspects of Galilean quantum gravity in 2+1 dimensions

As reviewed at the beginning of this chapter, the combinatorial quantisation programme expresses all aspects of the quantisation of the phase space (4.1.37) in terms of the representation theory of $D(\hat{G}_0)$. Having studied this representation theory, it is useful to highlight some of the features of the quantum theory which can be derived without much further calculation. According to [68], the Hilbert space obtained by quantising the phase space (4.1.37) for n massive and spinning particles on a surface Σ_{gn} of genus g is

$$\mathcal{H} = \mathrm{Inv}\left(L^2(\hat{G}_0^{2g}) \otimes \bigotimes_{i=1}^{n} V_{\varepsilon_{0,i},\mu_i,\sigma_i,j_i} \right), \tag{5.2.1}$$

where Inv means the invariant part of the tensor product under the adjoint action of $D(\hat{G}_0)$. The general structure of this formula is not difficult to understand. Each Heisenberg double (the Poisson manifold associated to the handles, which is the cotangent bundle of $\hat{G}_0 \times \hat{G}_0$ in the present case) quantises to $L^2(\hat{G}_0 \times \hat{G}_0)$, and each conjugacy class (associated to punctures) quantises to an irrep of $D(\hat{G}_0)$. The division by conjugation that takes one from the extended phase space (4.1.36) to the physical phase space (4.1.37) is mirrored in the quantum theory by the restriction to the invariant part of the tensor product in (5.2.1). We refer the reader to [68] for further details, and focus on qualitative aspects here.

5.2.1 Fusion rules

In order to pick out the invariant part of the tensor product in (5.2.1) one needs to be able to decompose tensor products of representations of $D(\hat{G}_0)$ into irreps. A general method for doing this in the case of quantum doubles of compact Lie groups is explained in [52], where it is worked out in detail for $D(SU(2))$. Extending this to include the case of $D(\hat{G}_0)$ is a technical challenge, but certain aspects of the tensor product decomposition can be read off without much work.

The tensor product decomposition of products like

$$V_{\varepsilon_{0,1},\mu_1,\sigma_1,j_1} \otimes V_{\varepsilon_{0,2},\mu_2,\sigma_2,j_2}, \tag{5.2.2}$$

which occur in (5.2.1), corresponds physically to a set of fusion rules for particles in Galilean gravity. They tell us how to "add" kinetic attributes like energy and momentum of two particles. In the representation theory of the Galilei double $D(\hat{G}_0)$, the conjugacy classes labelling irreps are combined by multiplying out the group elements contained in them, and sorting the products into conjugacy classes again. The underlying rule for combining energy and momentum is thus simply to multiply out the corresponding $\hat{G}_0$ elements[1]. Thus, with

$$u_1 = (\mu_1, -\vec{\pi}_1, \varepsilon_1), \quad u_2 = (\mu_2, -\vec{\pi}_2, \varepsilon_2) \tag{5.2.3}$$

we have

$$u_1 u_2 = (\mu_1 + \mu_2, -(\vec{\pi}_1 + R(\mu_1)\vec{\pi}_2), \varepsilon_1 + \varepsilon_2 + \frac{1}{2}\vec{\pi}_1 \times R(\mu_1)\vec{\pi}_2). \tag{5.2.4}$$

This simple rule has an interesting consequence. Even if u_1 and u_2 both belong to conjugacy classes of the type $C_{E_0,\mu}$ (4.1.25) describing massive particles, it is possible for the product $u_1 u_2$ to belong to a conjugacy class of the type $C_{\vec{\pi}_0}$ (4.1.27) describing massless particles. This happens generically when the sum of the particle's (rescaled) masses is 2π and the resulting conjugacy classes have zero measure. This essentially means that, in physical units, the total mass equals the Planck mass. Such particles are the non-relativistic analogue of the so-called Gott pairs in relativistic 3d gravity [33]. However, unlike in the relativistic case there is no condition on the relative speed of the two particles. As long as the sum of the (rescaled) masses is 2π, the total energy and momentum is either of the vacuum type or, generically, of the massless type.

[1]This fact can be derived more generally for Chern-Simons descriptions of particles in 2+1 dimensions from the fact that holonomies get multiplied when the underlying loops are concatenated, see e.g. [11]

5.2.2 Braid interactions

If several punctures are present, the braid group on n strands of Σ_g acts on the Hilbert space. The action of the generators of the braid group on the Hilbert space can be expressed in terms of the $\mathcal{R}$-matrix (5.1.9) [58], and have a simple action on the $\hat{G}_0$-parts of the holonomies around any two punctures, which can again be derived from more elementary considerations, as explained in [11]. Assuming that the two punctures correspond to massive particles we denote their $\hat{G}_0$-valued energy-momenta again by u_1 and u_2 as in (5.2.3). In the quantum theory we can describe states of definite energy and momentum in terms of delta-functions δ_{u_1} and δ_{u_2} on the conjugacy classes corresponding to the particles' masses. Either by acting with the $\mathcal{R}$-matrix (5.1.9) on the the tensor product $\delta_{u_1} \otimes \delta_{u_2}$ or via the elementary considerations in [11] one finds that the action of the braid group generator is

$$(u_1, u_2) \mapsto (u_1 u_2 u_1^{-1}, u_1), \tag{5.2.5}$$

which we evaluate, using (5.2.3):

$$u_1 u_2 u_1^{-1} = (\mu_2, (1 - R(\mu_2)\vec{\pi}_1 + R(\mu_1)\vec{\pi}_2, \varepsilon_2 + \frac{1}{2}((1 + R(\mu_2))\vec{\pi}_1 \times R(\mu_1)\vec{\pi}_2 + \frac{1}{2}\sin\mu_2 \vec{\pi}_1^2). \tag{5.2.6}$$

The braid group action can be related to scattering of two massive particles [12], which was one of first aspects of 3d quantum gravity to be studied in detail in the limit where one particle is heavy, and the other can be treated as a test particle moving on the background spacetime created by the heavy particle [83, 24]. In usual 3d gravity, the scattering can be understood in terms of the test-particle's motion on the conical spacetime due to the heavy particle. The deficit angle of the conical spacetime is proportional to the mass of the heavy particle in Planck units, and this deficit angle controls the scattering cross section [84, 24]. We can recover aspects of this scattering from the formula (5.2.6). Taking the first particle to be the heavy particle, and setting its momentum to zero (so that we are in the heavy particle's rest frame), the conjugation action on the second (test) particle's momentum is simply

$$\vec{\pi}_2 \mapsto R(\mu_1)\vec{\pi}_2. \tag{5.2.7}$$

This is precisely the distance-independent deflection of the momentum direction by an angle proportional to the heavy particle's mass that is characteristic of scattering on a cone. It would be interesting to use the scattering formalism of [12], using the $\mathcal{R}$-matrix (5.1.9), to investigate the non-relativistic scattering in more detail, and to compare with [84, 24].

5.3 Noncommutative geometry, deformed derivatives and the Heisenberg double

The appearance of a non-commutative Lie group as a momentum space can be related to non-commutativity of spacetime coordinates using the representation theory of the quantum double [16]. The basic idea is to apply a nonabelian Fourier transform to switch from the momentum-space wave functions ψ in (5.1.7) to position-space representation, with a wavefunction necessarily living on a non-commutative spacetime and obeying a linear wave equation, see also [58]. Applying the reasoning of [16] and [58] to the Galilei double, one expects the relevant non-commutative space algebra in this case to be the Lie algebra $\hat{\mathfrak{g}}_0$, but with generators now interpreted as position coordinates X_1, X_2, a time coordinate T and an additional generator R:

$$[X_i, R] = 8\pi G\hbar\, \epsilon_{ij} X_j, \qquad [X_1, X_2] = 8\pi G\hbar\, T. \tag{5.3.1}$$

Note that the combination $G\hbar$ has the dimension length^2/time, so cannot simply be interpreted as a Planck length or Planck time. This dimension is consistent with the interpretation of X_i as positions and of T as time.

The algebra (5.3.1) has a number of interesting features. The operator representing Galilean time is a central element, and in that sense Galilean time remains absolute in the non-commutative regime. It is also worth noting that the algebra of spacetime coordinates closes, and is simply the universal enveloping algebra of the Heisenberg algebra. For fixed time, the non-commutativity of the space coordinates is of Moyal-type, with the deformation matrix $\Theta_{ij} = \epsilon_{ij} 8\pi G\hbar T$ proportional to time. Algebraically, this is precisely the non-commutative position algebra which was studied in the context of planar Galilean physics in [43, 26, 55]. However, the status here is different. Our non-commutativity is a quantum gravity effect (proportional to both G and $\hbar$) and thus universal. The non-commutativity of position coordinates of a particle in [55] is related to the properties of a particle moving in a classical and commutative plane. It is reasonable to expect the two effects would have to be somehow combined if such a particle was moving in the non - commutative spacetime with coordinates obeying (5.3.1).

In an attempt to study such a coupling we would need to describe calculus on the said spacetime. A set of derivative operators for (5.3.1) has been constructed in [38]. Let us begin the description of this non-commutative calculus by first implementing the change (2.2.21) in notation and write the algebra $\hat{\mathfrak{g}}_0$ in the basis $\{Z, \bar{Z}, R, T\}$, where

$$Z := X_1 + \mathrm{i}X_2 \qquad\qquad \bar{Z} := X_1 - \mathrm{i}X_2. \tag{5.3.2}$$

Furthermore, we define $\rho := 8\pi G\hbar$ so the spacetime algebra we will work on can be written as

$$\left[Z\,,\,\bar{Z}\right] = -2\mathrm{i}\rho T \qquad \left[Z\,,\,R\right] = -\mathrm{i}\rho Z \qquad \left[\bar{Z}\,,\,R\right] = \mathrm{i}\rho\bar{Z}. \tag{5.3.3}$$

The role of the generator R (the only compact generator in $\hat{\mathfrak{g}}_0$) is not clear to us. In the quantum mechanics interpretation of the algebra (5.3.3) as the extended Heisenberg algebra, it corresponds to the number operator but in the present context its meaning has yet to be clarified.

As we said, the algebra of functions on the spacetime whose coordinate functions obey (5.3.3) can be identified with the universal enveloping algebra $U(\hat{\mathfrak{g}}_0)$ of $\hat{\mathfrak{g}}_0$. The process of defining derivatives on this space in the spirit of Halliday and Szabo [38] is initiated by demanding the derivatives $\partial : U(\hat{\mathfrak{g}}_0) \to U(\hat{\mathfrak{g}}_0)$ obey three conditions:

(a) They commute amongst themselves

(b) They are consistent with the algebra relations of $\hat{\mathfrak{g}}_0$

(c) They are deformations of ordinary derivatives i.e.

$$\partial_\mu x^\nu = \delta_\mu^\nu + \mathcal{O}(\rho).$$

The technique of obtaining derivative operators obeying these conditions is explained in [38] (also [82]). After applying it we obtain the following commutation relation for the derivatives and coordinate functions

$$\begin{aligned}
\left[\partial_R\,,\,R\right] &=1 & \left[\partial_R\,,\,T\right] &=0 & \left[\partial_R\,,\,Z\right] &=0 & \left[\partial_R\,,\,\bar{Z}\right] &=0\\
\left[\partial_T\,,\,R\right] &=0 & \left[\partial_T\,,\,T\right] &=1 & \left[\partial_T\,,\,Z\right] &=\mathrm{i}\rho\partial_{\bar{Z}} & \left[\partial_T\,,\,\bar{Z}\right] &=-\,\mathrm{i}\rho\partial_Z\\
\left[\partial_Z\,,\,R\right] &=\mathrm{i}\rho\partial_Z & \left[\partial_Z\,,\,T\right] &=0 & \left[\partial_Z\,,\,Z\right] &=1 & \left[\partial_Z\,,\,\bar{Z}\right] &=0\\
\left[\partial_{\bar{Z}}\,,\,R\right] &=-\,\mathrm{i}\rho\partial_{\bar{Z}} & \left[\partial_{\bar{Z}}\,,\,T\right] &=0 & \left[\partial_{\bar{Z}}\,,\,Z\right] &=0 & \left[\partial_{\bar{Z}}\,,\,\bar{Z}\right] &=1.
\end{aligned} \tag{5.3.4}$$

Note that these relations are not unique but other solutions for the derivatives can be obtained from non-linear redefinitions of the derivatives above.

Using (5.3.4) and a simple induction argument [38], one can show that the derivatives obey a generalised Leibniz rule which is a ρ-deformation of the usual Leibniz rule. Specifically, for any $f, g \in U(\hat{\mathfrak{g}}_0)$ we have

$$\begin{aligned}
\partial_R(fg) &=(\partial_R f)g + f(\partial_R g)\\
\partial_T(fg) &=(\partial_T f)g + f(\partial_T g) + \rho(e^{-\mathrm{i}\rho\partial_R}\partial_{\bar{Z}} f)(\partial_Z g) - \rho(e^{\mathrm{i}\rho\partial_R}\partial_Z f)(\partial_{\bar{Z}} g)\\
\partial_Z(fg) &=(\partial_Z f)g + (e^{-\mathrm{i}\rho\partial_R} f)\partial_Z g\\
\partial_{\bar{Z}}(fg) &=(\partial_{\bar{Z}} f)g + (e^{\mathrm{i}\rho\partial_R} f)\partial_{\bar{Z}} g.
\end{aligned} \tag{5.3.5}$$

The (trivial) algebra and the coalgebra structure of the derivatives (as realised by its action on $U(\hat{\mathfrak{g}}_0)$) indicate that the derivatives on (5.3.3) form a bialgebra. In fact, as we will show, they form a Hopf algebra. In the basis $\{\partial_R, \partial_T, \partial_Z, \partial_{\overline{Z}}\}$ its coproduct, is given via (5.3.5)

$$
\begin{aligned}
\Delta(\partial_R) =&\partial_R \otimes 1 + 1 \otimes \partial_R \\
\Delta(\partial_T) =&\partial_T \otimes 1 + 1 \otimes \partial_T + \rho e^{-\mathrm{i}\rho\partial_R}\partial_{\bar{Z}} \otimes \partial_Z - \rho e^{\mathrm{i}\rho\partial_R}\partial_Z \otimes \partial_{\bar{Z}} \\
\Delta(\partial_Z) =&\partial_Z \otimes 1 + e^{-\mathrm{i}\rho\partial_R} \otimes \partial_Z \\
\Delta(\partial_{\bar{Z}}) =&\partial_{\bar{Z}} \otimes 1 + e^{\mathrm{i}\rho\partial_R} \otimes \partial_{\bar{Z}}.
\end{aligned}
\tag{5.3.6}
$$

with 1 being the identity on $U(\hat{\mathfrak{g}}_0)$. We denote its unit again by 1 with a slight abuse of notation, and the counit ϵ and antipode S are respectively

$$
\begin{array}{llll}
\epsilon(\partial_R) =0 & \epsilon(\partial_T) =0 & \epsilon(\partial_Z) =\epsilon(\partial_{\bar{Z}}) = 0 & \epsilon(1) =1 \\
S(\partial_R) =-(\partial_R) & S(\partial_T) =-(\partial_T) & S(\partial_Z) =-e^{\mathrm{i}\rho\partial_R}\partial_Z & S(\partial_{\bar{Z}}) =-e^{-\mathrm{i}\rho\partial_R}\partial_{\bar{Z}}.
\end{array}
\tag{5.3.7}
$$

The structure of this Hopf algebra is exactly that of the bialgebra of momenta in the momentum representation of the Galilei algebra. This can most easily be seen by looking at the coproducts (5.3.6) and taking the real and imaginary parts respectively. As an example let us look at the coproduct of P_1 or the derivative ∂_{X_1}. The cocommutator is

$$\delta(X) := \Delta(X) - \Delta^{op}(X).$$

Here we remind the reader that $\Delta^{op}(X)$ is defined as the "flipped" coproduct $\Delta^{op} := \sigma \circ \Delta$ and so if $\Delta(X) = X_{(1)} \otimes X_{(2)}$ then $\Delta^{op}(X) = X_{(2)} \otimes X_{(1)}$. So, explicitly, to first order in ρ the coproduct of ∂_{X_1} is obtained as the real part of

$$\delta(\partial_Z) = \delta(\partial_{X_1}) + \mathrm{i}\delta(\partial_{X_2}) = -\mathrm{i}\rho\partial_R \otimes \partial_{X_1} + \partial_R \otimes \partial_{X_2} + \mathrm{i}\rho\partial_{X_1} \otimes \partial_R - \partial_{X_2} \otimes \partial_R.$$

This clearly corresponds to the coproduct of the momentum generator P_1 in the momentum space representation (2.2.44). For highest orders in ρ one can see that the derivatives under comultiplication behave exactly as the group parameters in (2.3.10).

The Hopf algebra of the deformed derivatives lives on the vector space that can be thought of as the tangent space of the non-commutative manifold $U(\hat{\mathfrak{g}}_0)$. Since it can be identified with the Hopf algebra of the momenta in the momentum representation, it is dual to $U(\hat{\mathfrak{g}}_0)$ via the pairing $\langle\ ,\ \rangle$. It can also be naturally identified with the cotangent space $C^\infty(G)$. This is the viewpoint we adopt - we think of the deformed derivatives as functions $C^\infty(G)$ on the group G. Indeed the Hopf algebra of derivatives $C^\infty(G)$ acts as the symmetry of the noncommutative spacetime $U(\hat{\mathfrak{g}}_0)$. In the Hopf-algebaic sense, $C^\infty(G)$ is a left $U(\hat{\mathfrak{g}}_0)$-module algebra i.e. a left action $\triangleright : C^\infty(G) \otimes U(\hat{\mathfrak{g}}_0) \to U(\hat{\mathfrak{g}}_0)$ is defined that satisfies

$$\partial \triangleright (f \cdot g) = (\partial_{(1)} f) \cdot (\partial_{(2)} g) \quad \Delta\partial = \partial_{(1)} \otimes \partial_{(2)}, \tag{5.3.8}$$

where $\partial \in C^\infty(G)$ and $f, g \in U(\hat{\mathfrak{g}}_0)$. The data $\{C^\infty(G), U(\hat{\mathfrak{g}}_0), \langle\ ,\ \rangle\}$ are precisely those of a Heisenberg double. Formally, the Heisenberg double $D_+(\mathcal{H})$ of a Hopf algebra $\mathcal{H}$ is defined[80] as the smash product $D_+(\mathcal{H}) := \mathcal{H} \sharp \mathcal{H}^*$ with the Hopf-algebras $\mathcal{H}, \mathcal{H}^*$ being in duality via some pairing $\langle\ ,\ \rangle$. Its underlying vector space, as in the case of a quantum double, is the tensor product $\mathcal{H} \otimes \mathcal{H}^*$ and $\mathcal{H}$ is a left $\mathcal{H}^*$-module algebra with left action defined by:

$$f \triangleright h := \langle f\ ,\ h_{(2)} \rangle h_{(1)}$$

for any $f \in \mathcal{H}^*$ and $h \in \mathcal{H}$.

From the geometric viewpoint we highlighted earlier, $D_+(U(\hat{\mathfrak{g}}_0))$ corresponds to the cotangent bundle of G. The correspondence between the Heisenberg double and the deformed derivatives approach has been formalised in detail in [80].

5.4 The Newton-Hooke groups and quantisation

Let us turn now to the issue of quantising the Poisson-Lie structure of the Newton-Hooke groups. As we hinted in Chapter 1, the quantisation of the Newton-Hooke group follows the general pattern: the group is formulated as a classical double and its quantisation is given by the quantum double. From the discussion in section 2.2.4, it is clear that in order to complete combinatorial quantisation in detail we would have to study the quantum double of the deformed homogeneous group $\hat{G}_0$ and its representations. This is a challenging task for many reasons the main being that the representation theory of this Hopf algebra is not studied in the literature. One could proceed to study the said representations along the lines of [18] where the combinatorial quantisation program has been carried out in detail for spacetimes with positive cosmological constant.

The subject of this present section is much more modest, however. We will conclude our discussion of Galilean quantum gravity by studying the Hopf algebra that is a quantisation of $\hat{\mathfrak{g}}_0$ - not the *double* of that Hopf algebra.

5.4.1 The extended q-Heisenberg algebra as a quantisation of the Lie bi-algebra $\hat{\mathfrak{g}}_0$

Let us therefore proceed to introduce a quantisation for the algebra $\hat{\mathfrak{g}}_0$ (2.2.21)

$$\left[Z\,,\,\bar{Z}\right] = -\,2\mathrm{i}S \qquad \left[Z\,,\,J\right] = -\,\mathrm{i}Z \qquad \left[\bar{Z}\,,\,J\right] = \mathrm{i}\bar{Z}.$$

By "quantisation" here we mean that we seek a quantum $\mathcal{R}$- matrix that satisfies the Quantum Yang Baxter equation and is a deformation of the classical r-matrix of (2.2.28)

$$r = S \otimes J + J \otimes S - \bar{Z} \otimes Z,$$

in the sense that

$$\mathcal{R} \approx 1 + \mu r,$$

with respect to some deformation parameter μ. Fortunately, the algebra (2.2.21) is well-studied - it is the extended Heisenberg algebra. We adopt the approach found in S. Majid's textbook[57] where for the extended Heisenberg algebra defined as

$$[N, a] = -a, \qquad [N, a^\dagger] = a^\dagger, \qquad [N, H] = 0 \tag{5.4.1}$$

the following example is proved ([57], Example 3.1.2, p. 73):

Example 5.4.1. *Let q be a nonzero parameter. The q-Heisenberg algebra is defined with generators $a, a^\dagger, q^{\frac{H}{2}}, q^{-\frac{H}{2}}$ and 1 with the relations $q^{\pm\frac{H}{2}}q^{\mp\frac{H}{2}} = 1$ (...) and*

$$[q^{\frac{H}{2}}, a] = 0, \qquad [q^{\frac{H}{2}}, a^\dagger] = 0, \qquad [a, a^\dagger] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

This forms a Hopf algebra with

$$\Delta a = a \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes a, \qquad \Delta a^\dagger = a^\dagger \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes a^\dagger$$

$$\Delta q^{\pm\frac{H}{2}} = q^{\pm\frac{H}{2}} \otimes q^{\pm\frac{H}{2}}, \qquad \epsilon q^{\pm\frac{H}{2}} = 1, \qquad \epsilon a = 0 = \epsilon a^\dagger$$

$$Sa = -a, \quad Sa^\dagger = -a^\dagger, \quad Sq^{\pm\frac{H}{2}} = q^{\mp\frac{H}{2}}.$$

The extended q-Heisenberg algebra is defined with the additional mutually inverse generators q^N, q^{-N} *and relations*

$$q^N a q^{-N} = q^{-1}a, \qquad q^N a^\dagger q^{-N} = qa^\dagger, \qquad [q^N, q^{\frac{H}{2}}] = 0.$$

It forms a Hopf algebra with the additional structure

$$\Delta q^N = q^N \otimes q^N, \qquad \epsilon q^N = 1, \qquad Sq^{\pm N} = q^{\mp N}.$$

If $q = e^{\frac{t}{2}}$ *and we work over* $\mathbb{C}[[t]]$ *rather than* $\mathbb{C}$ *then we can regard* $a^\pm, H, N$ *and* 1 *as the generators. In this case, the extended q-Heisenberg Hopf algebra is quasitriangular with*

$$\mathcal{R} = q^{-(N\otimes H + H\otimes N)} e^{(q-q^{-1})q^{\frac{H}{2}} a \otimes q^{-\frac{H}{2}} a^\dagger}$$

Looking at the algebra relations (5.4.1), we see that the matching between our notation and the one we quoted is provided by identifying the pair (H, N), with our $(-2\mathrm{i}S, \mathrm{i}J)$.

Let us write $e^{\mu/2}$ for the deformation parameter, and introduce the μ-deformed Heisenberg algebra $\hat{G}_0^{(\mu)}$. In the physical interpretation we are adopting we will call it the μ-Galilei algebra. In the defining relations above, the generators (H, N) are primitive and there is no need to deform them while in the algebra (2.2.21), we need to replace the right hand side of $[Z, \bar{Z}]$ with the quantity $\frac{e^{-\mathrm{i}\mu S} - e^{\mathrm{i}\mu S}}{\mu}$. We then obtain the defining relation for the μ-deformed extended Galilei Hopf algebra as follows:

$$\big[Z\,,\,\bar{Z}\big] = \frac{e^{-\mathrm{i}\mu S} - e^{\mathrm{i}\mu S}}{\mu} \qquad \big[Z\,,\,J\big] = -\,\mathrm{i}Z \qquad \big[\bar{Z}\,,\,J\big] = \mathrm{i}\bar{Z}. \tag{5.4.2}$$

Note that $Z, \bar{Z}$, have the dimensions of inverse velocity and so S has the dimension of inverse velocity squared: $[S] = [L]^{-2}[T]^2$. This means that the combination

μS needs to be dimensionless for the exponential in (5.4.2) to make sense. It has been shown in the context of Lorentzian gravity that the dimensionless parameter that deforms the Lorentz algebra to correspond to quantisation of spaces with a cosmological constant is $\mu_{Lor} = \hbar G\sqrt{-\lambda}$.[58] Indeed if one sets $c = 1$ then μ_{Lor} is dimensionless. In Galilean gravity however one cannot set $c = 1$; not only would the limit $c \to \infty$ make no sense but also setting $c = 1$ would have to artificially introduce a preferred velocity which funnily enough is unphysical in the Galilean context. Remarkably, the combination

$$\mu = \hbar G\sqrt{-\lambda}$$

has the right dimensions to be the deformation parameter in the Galilean setting just as well. What is more, μ is a (desirably) small parameter. We will not concern ourselves with what deeper physical truth (if any) yields the particular combination $\hbar G\sqrt{-\lambda}$ as the meaningful deformation parameter when one tries to quantise spaces with a cosmological constant. We believe however, that the argument that the same combination of fundamental constants is used in the Lorentzian setting is a strong one to lead us to use μ in the present setting.

Now, let us exhibit the Hopf algebra structure of the q-Heisenberg algebra in our notation. Commultiplication Δ in $\hat{G}_0^{(\mu)}$ is defined as [57]:

$$\begin{aligned} &\Delta(Z) = Z \otimes e^{-\mathrm{i}\frac{\mu}{2}S} + e^{\mathrm{i}\frac{\mu}{2}S} \otimes Z \quad &\Delta(\bar{Z}) = \bar{Z} \otimes e^{-\mathrm{i}\frac{\mu}{2}S} + e^{\mathrm{i}\frac{\mu}{2}S} \otimes \bar{Z} \\ &\Delta(e^{\mp\mathrm{i}\frac{\mu}{2}S}) = e^{\mp\mathrm{i}\frac{\mu}{2}S} \otimes e^{\mp\mathrm{i}\frac{\mu}{2}S} \quad &\Delta(J) = 1 \otimes J + J \otimes 1 \end{aligned} \tag{5.4.3}$$

while the antipode S and counit ϵ are

$$\begin{aligned} S(Z) &= -Z \qquad & S(\bar{Z}) &= -\bar{Z} \\ S(e^{\mp\mathrm{i}\mu S}) &= e^{\pm\mathrm{i}\mu S} \qquad & S(J) &= -J \end{aligned} \tag{5.4.4}$$

$$\epsilon(Z) = \epsilon(\bar{Z}) = 0 \qquad \epsilon(e^{\mp\mathrm{i}\mu S}) = \epsilon(e^{\mp\mathrm{i}\frac{\mu}{2}J}) = 1.$$

It is straightforward to check that this Hopf algebra is a quantisation of the algebra $\hat{G}_0$ with the quantum $\mathcal{R}$-matrix being

$$\mathcal{R} = \exp\bigl(-\mu(J \otimes S + S \otimes J)\bigr)\exp\bigl(\mu e^{-\mathrm{i}\frac{\mu}{2}S} Z \otimes e^{\mathrm{i}\frac{\mu}{2}S}\bar{Z}\bigr). \tag{5.4.5}$$

5.4.2 Real structures

A $*$- structure for the Heisenberg algebra in this formulation is complex conjugation (see example 3.1.6 in [57]). We can distinguish between the following two cases:

1. $\lambda < 0$ i.e. μ is real. This makes the Hopf $*$-algebra $\hat{G}_0^{(\mu)}$, real and quasitriangular. Specifically, the $*$-operation is

$$Z^* = \bar{Z} \quad \bar{Z}^* = Z, \quad J^* = -J, \quad S^* = -S, \quad (e^{\mathrm{i}\mu S})^* = e^{\mathrm{i}\mu S}. \tag{5.4.6}$$

The $*$-operation can be easily checked to leave (5.4.2), (5.4.3), (5.4.4) invariant. Similarly, it can be shown that the r-matrix is real, i.e.

$$\begin{aligned} \mathcal{R}^{*\otimes *} =& \exp\big((e^{\frac{\mu}{2}} - e^{-\frac{\mu}{2}})^*(e^{-\mathrm{i}\mu S} Z \otimes e^{\mathrm{i}\mu S}\bar{Z})^{*\otimes *}\big) \exp\big(\mu(J \otimes S + S \otimes J)\big) \\ =& \exp\big((e^{\frac{\mu}{2}} - e^{-\frac{\mu}{2}})\bar{Z} e^{-\mathrm{i}\mu S} \otimes Z e^{\mathrm{i}\mu S}\big) \exp\big(\mu(J \otimes S + S \otimes J)\big) \\ =& \sigma(r) \end{aligned}$$

where σ is the flip operator.

2. $\lambda > 0$ i.e. $\mu \in \mathrm{i}\mathbb{R}$. The algebra $\hat{G}_0^{(\mu)}$ is now antireal, quasitriangular with the $*$-structure being

$$Z^* = Z \quad \bar{Z}^* = \bar{Z}, \quad J^* = -J, \quad S^* = -S, \quad (e^{\mathrm{i}\mu S})^* = e^{\mathrm{i}\mu S}. \tag{5.4.7}$$

Chapter 6

Conclusions-Outlook

To conclude, let us revisit the main results we came across in this thesis. We first described the groups relevant to the classical limit of gravity in a new way in Chapter 2, using Clifford algebra language. We then went on to describe the Galilean spacetimes in terms of cosets of these isometry groups. Turning to the central extension of these groups, we found an invariant bilinear form and in Chapter 4 defined Galilean gravity as the Chern-Simons theory of the centrally extended Galilei and Newton-Hooke groups. We showed that the phase space of Chern-Simons theory has the Poisson-Lie structure of a classical double in both those cases. We went on to quantise this double in full in Chapter 5 making use of the results of [68], where we provided the Hopf algebra structure and the representation theory of what we called the Galilei double - the quantum double of the Galilei group. We also set up the ground for the quantisation of Newton-Hooke gravity by describing the quantisation of the group $\hat{G}_0$.

We have therefore described the methodology and the details of Galilean quantum gravity in 2+1 dimensions and we have justified its right to be called that way. While we hope this analysis is as complete as a research program of three years permits, it would be unfair to label it fully conclusive.There are many questions that the author feels are left unanswered and deserve further researching. In particular, we would like to highlight two of them.

A "deformed Schrödinger" equation

As we said earlier, in the noncommutative spacetime (5.3.3) the derivative operators together with the coordinates form a Heisenberg double and have the bialgebra structure of momenta. This means that the momentum-space dispersion relation of (4.1.25)

$$\epsilon = \epsilon_0 + \frac{1}{4}\cot\frac{\mu}{2}\vec{\pi}^2,$$

has a position space interpretation where the group parameters are interpreted as derivatives. With the correct identifications for the derivatives, the equation above can be thought of as a linear differential equation in $\mathbb{R}^4 \simeq \hat{\mathfrak{g}}_0^*$:

$$\left(\partial_t + \frac{1}{4}\cot\frac{\partial_r}{2}\partial_z\partial_{\bar{z}}\right)\Psi = 0 \tag{6.0.1}$$

and it can be regarded as a modified Schrödinger equation. If this reasoning is correct, this equation describes the dynamics in the region around a gravitating particle. Note, however that the above equation can be linearised by a redefinition of the derivatives and is therefore dependent on the ordering of the noncommutative coordinates [38]

Non-commutative structures in Newton-Hooke gravity

Furthermore, as in Section 5.3 it would be interesting to investigate the noncommutative coordinate space corresponding to the q-deformed Heisenberg double. The commutation relations are the same as before in Section 5.3:

$$[Z, \bar{Z}] = -2\mathrm{i}\rho T \qquad\qquad [Z, R] = -\mathrm{i}\rho Z. \tag{6.0.2}$$

and in the Heisenberg double formed by the coordinate/deformed derivative algebra, one would have to relax the requirement for commutativity of the derivatives - their algebra is determined by the coalgebra structure of $\hat{\mathfrak{g}}_0^*$ which is now nontrivial. A set of derivative operators for this case would be interesting to find especially since the connection between the Heisenberg double and the deformed derivatives has been established rigorously [80].

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