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Partial Regularity of Local Minimisers in the Calculus of Variations.

Thomas James Dodd

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Declaration

I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, except where due acknowledgement is made, and not been submitted for any other degree.

Signature of Thomas James Dodd:

Signature of Supervisor:

Abstract

In this manuscript the theory of local minimisers of the general variational integral

$$\int_{\Omega} F(Du(x))\,dx$$

is discussed, where $\Omega \subset \mathbb{R}^n$ is an open bounded domain and $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$. The focus is on the partial regularity of such minimisers. Certain partial regularity results are proved for a new class of local minimisers. As background to the result a number of topics important for the result are discussed. The first of these is quasiconvexity of the integrand F, important for existence and partial regularity of minimisers of the variational integral, above. This is followed by an introduction and discussion of Morrey, Campanato and BMO spaces. Finally the regularity of $\mathbb{A}$-Harmonic functions and elliptic systems of partial differential equations with continuous coefficients is established before the results of the manuscript are presented. The results are as follows: An a priori Campanato type regularity condition is established for a class of $\mathrm{W}^1\mathrm{X}$ local minimisers $\overline{u}$ of the general variational integral above where $\Omega \subset \mathbb{R}^n$ is an open bounded domain, F is of class C^2, F is strongly quasi-convex and satisfies the growth condition

$$F(\xi) \leq c(1 + |\xi|^p)$$

for a $p > 1$ and where the corresponding Banach spaces X are the Morrey-Campanato space $\mathcal{L}^{p,\mu}(\Omega, \mathbb{R}^{N\times n})$, $\mu < n$, Campanato space $\mathcal{L}^{p,n}(\Omega, \mathbb{R}^{N\times n})$ and the space of bounded mean oscillation $\mathrm{BMO}(\Omega, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$. The admissible maps $u\colon \Omega \to \mathbb{R}^N$ are of Sobolev class $\mathrm{W}^{1,p}$, satisfying a Dirichlet boundary condition, and to help clarify the significance of the above result the sufficiency condition for $\mathrm{W}^1\mathrm{BMO}$ local minimisers is extended from Lipschitz maps to this admissible class.

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Contents

Chapter 1

Introduction.

The subject of the calculus of variations is very old. The modern name is due to Euler, who in the 18th Century gave it the present name after reading Lagrange's work [16]. The methods of the time were indirect, based on the study of the so called Euler-Lagrange equations.

In the early 19th Century interest in the Laplace and Poisson problems lead to the work of Gauss and Green on the Dirichlet integral, and to the eventual formulation of the Dirichlet principle by Riemann in his 1851 Thesis. However in the 1870's, due to a new emphasis on mathematical rigour by the likes of Weierstrass, questions arrose regarding it's validity for proving existence of harmonic functions, see BREZIS and BROWDER [11] for a full historical review.

The interest in the Dirichlet principle was re-instated by Hilbert in 1900. Pursuing a rigorous mathematical program set out by his 19th and 20th problems, Hilbert, Lebesgue, Levi, Fubini, Toneli and others [11], developed the mathematical tools necessary for the solution of the problem for the Dirchlet principle via direct methods, and in doing so setting the stage for much of modern analysis.

Of Hilbert's 23 problems the 19th and the 20th started a rigorous program for the existence and regularity of solutions to variational problems (and partial differential equations) beyond the Dirichlet principle. Hilbert's 20th problem:

> "An important problem ... is the question concerning existence of solutions of partial differential equations when the values on the boundary of the region are prescribed... Has not every regular variational problem a

solution, provided certain assumptions regarding given boundary conditions are satisfied and provided also if need be that the notion of a solution shall be suitably extended."

By "regular problems in the calculus of variations" Hilbert was referring to the minimisation of the regular variational integral

$$I[u,\Omega] = \int_\Omega F(Du)dx, \tag{1.1}$$

where

(i) $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^k for some $k \geq 2$,

(ii) $\ell|\lambda|^2 \leq F''(\xi)[\lambda,\lambda] \leq L|\lambda|^2$ for all $\xi, \lambda \in \mathbb{R}^{N\times n}$ and where $0 < \ell < L < \infty$ are constants.

The lower bound in condition (ii) follows from *strong convexity* of the integrand F and the condition (i). Convexity itself is closely related to the existence of minimisers of (1.1) in the case $N = 1$. In this case Hilbert's 20th problem has been answered in the affirmative by many authors, in particular by Hilbert, Tonelli via the direct method.

The search for suitable spaces in which to frame the problem of regularity and existence for the Dirichlet principle brought the realisation by LEVI [11] that the minimising sequence of the Dirichlet integral is a Cauchy sequence in the Dirichlet norm and thus converges in a completion space with respect to that norm. The resulting "weak" solutions belong to the space of generalised functions now known as the *Sobolev space* $W^{1,2}$. This space turns out to be the proper space in which to frame the problem for the regular integrals described above.

The program started by Hilbert has had many successes and great progress has been made. For the case $N = 1$ an important component necessary for the regularity problem was the need to show that a weak solution of a linear equation in divergence form with bounded measurable coefficients, is Hölder continuous. This proved difficult to obtain and there were many attempts to do so. However in 1957 DE GIORGI [17] and NASH [54] independently obtained the result. Later MOSER [50] came up with an entirely different proof of the same result by showing, among other things, that

the logarithm of the solution is of *bounded mean oscillation*. It is from De Giorgi and Nash's result, now known as De Giorgi-Nash-Moser theory, that LADYZHENSKAYA and URAL'TSEVA [43] finally solved Hilbert's 19th problem in the scalar case ($N = 1$) showing that solutions to regular variational problems as above, (1.1) satisfying (i) and (ii), are as regular (in the interior of Ω) as the data allow. In other words the regularity depends on order, $k \geq 2$, of continuous differentiability of F in (i).

The De Giorgi's theorem does not however transfer to the vectorial case. Previous to De Giorgi and Nash's result Morrey had proven that solutions of the regular problem are regular for the special case $n = 2$ and $N \geq 1$. However DE GIORGI's [19] 1968 counter example shows that for the case $n = N > 2$ there is no general regularity result for the critical points of the regular variational integral of the form

$$\mathcal{Q}[u, \Omega] = \int_{\Omega} F\left(x, Du(x)\right)\, dx,$$

satisfying (i) and (ii).

De Giorgi's theorem deals with the associated linear elliptic equation. Written in its weak formulation as

$$\int_{\Omega} A(x)Du(x) \cdot D\varphi(x)\, dx = 0, \text{ for all } \varphi \in C_0^1(\Omega) \tag{1.2}$$

for the homogeneous case (here $N = 1$), where $A(x)$ measurable, bounded, uniformly elliptic. The solutions of (1.2) correspond to critical points of the variational integral $\mathcal{Q}$, satisfying (ii), when $\mathcal{Q}$ is of quadratic type, i.e. when the integrand F is given by $F\left(x, Du(x)\right) = A(x)Du(x) \cdot Du(x)$, and $A(x)$ is symmetric. In particular De Giorgi's counter example relies on the construction of a functional with discontinuous x dependent coefficients independent of the gradient of the solution of (1.2). These coefficients are somehow pathologically arranged in their interaction with the gradient of the solution, causing singularities within the solution. This result leaves open the question of existence of counter examples for the regular variational problem without x, u dependent coefficients, i.e. for our original variational integral (1.1).

The first result along these lines was due to NEČAS [55]. In his example, $F : \mathbb{R}^{N \times n} \to \mathbb{R}$ is real analytic and satisfies (ii). Rather than the linear equation of (1.2), this example applies to the fully nonlinear Euler-Lagrange system of equations

derived from (1.1), satisfying (ii) (and hence the standard growth condition), which can be written in divergence form as

$$\mathrm{div} F'(Du) = 0. \tag{1.3}$$

See Chapter 2 for details on the relevance of this growth condition in the derivation of the Euler-Lagrange equation.

In Nečas' example the minimiser and the solution to (1.3) is Lipschitz continuous but not C^1 for the dimensions $n \geq 25$ and $N \geq 625$. These were later improved to $n \geq 5$ and $N \geq 25$ in [32]. The fact that in his example the minimiser $\overline{u}$ and solution to (1.3) is Lipschitz means that $\overline{u}$ is not a counter example of De Giorgi's Theorem. However it does highlight the fact that the De Giorgi-Nash-Moser approach to regularity is not an option in the vectorial case $N = n > 2$ for the regular variational problem specified in (1.1).

The fact that Nečas' example is Lipschitz opened the question of whether a non-Lipschitz solution to (1.1) satisfying (i) and (ii) exists. This was answered in the affermative by ŠVERÁK and YAN [64, 65]. They found an example of a minimiser of the regular problem with analytic F, which is not even bounded.

In their first result [64], the counterexample is non-Lipschitz and holds for dimensions $n \geq 3$ and $N \geq 5$. It is their second result [65] where they construct a counter example that is unbounded. In this case the dimensions are $n \geq 5$ and $N \geq 14$.

From Morrey's regularity result of $n = 2$ and $N \geq 1$, we see that the first result of Šverák and Yan, [64], is close to optimal (i.e. $n \geq 3$). In the case of their second result [65] we can see again that it is close to optimal (i.e. $n \geq 5$), by considering the following special cases: Due to CAMPANATO minimisers of the regular variational integral (1.1) satisfying (i) and (ii)) belong to $W^{2,2+\delta}_{\mathrm{loc}}$ for some $\delta = \delta(n, N, \frac{L}{\ell}) > 0$, thus are locally Hölder continuous for some $\alpha \in (0, 1)$ when $n \leq 4$, $N \geq 1$, see the lecture notes of KRISTENSEN [40] (for a dimension-free integrability improvement see [41]). In fact the closeness of ℓ/L to one is a factor in determining the regularity of minimisers of the regular variational integral. This is illustrated by Kristensen [40], showing we have everywhere Hölder continuity of such minimisers provided

$$n < \frac{4}{1 - \left(\frac{\ell}{L}\right)^2} \text{ and } N \geq 1.$$

The examples serve to show that for $N > 1$, quite unlike the scalar case ($n = 1$) minimisers of "regular problems" (1.1) with (i) and (ii), need not be regular everywhere. However all is not lost! Despite the above, and a considerable time after De Giorgi's and even Nečas' first counter example, EVANS' 1986 paper [22] showed the first partial regularity result for minimisers of (1.1) satisfying (i), p growth rather than the standard quadratic growth condition (that follows from (ii)), and a notion of convexity first noticed by Morrey for its relevance in the existence of minimisers but in a stronger form.

Morrey's notion of convexity, *quasiconvexity* and it's stronger form, that Evans [22] called uniformly strict quasiconvexity and we will call *strong quasiconvexity* (following [42]), proved the key for the current regularity program in the case $N > 1$. Owing in part to its close relationship with the existence of minimisers it is the natural substitute for the condition (ii) of the regular variational problem in the vectorial case $N > 1$.

Given the move away from (ii) it makes sense to consider the wider class of Sobolev spaces $W^{1,p}$, $p \geq 1$ in which to frame our minimisation problem. We consider a new set of hypotheses for the minimisation problem of the variational integral (1.1) based around quasiconvexity, where $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ for $n, N \geq 1$:

(H1) $F \in \mathrm{C}^2$;

(H2) $|F(\xi)| \leq c(1 + |\xi|^p)$ for every $\xi \in \mathbb{R}^{N\times n}$, some constant c and $p > 1$;

(H3) For some constant $\nu > 0$, every $\xi \in \mathbb{R}^{N\times n}$ and every $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{N}})$,

$$\nu \int_{\mathbb{R}^n} (|D\varphi|^2 + |D\varphi|^p) \leq \int_{\mathbb{R}^n} (F(\xi + D\varphi) - F(\xi)) \text{ when } p \geq 2 \tag{1.4}$$

$$\nu \int_{\mathbb{R}^n} (1+|\xi|^2+|D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \leq \int_{\mathbb{R}^n} (F(\xi+D\varphi)-F(\xi)) \text{ when } 1 < p < 2. \tag{1.5}$$

These three hypotheses, in one form or another, will from now on form the conditions of all minimisation problems that we discuss in this thesis. We have already mentioned (H2) in the quadratic case and point the reader in the direction of Chapter 2 for a fuller discussion. Hypothesis (H3) is the condition we will call *strong p-quasiconvexity* introduced by Evans [22] in the form of (1.4) and generalised first

by ACERBI and FUSCO [2] and later adapted to the $1 < p < 2$ case, (1.5), by CAROZZA, FUSCO and MINGIONE [14]. Often when p is clear from the context we simply speak of strong quasiconvexity. We discuss this condition in the final section of Chapter 2 and for now draw attention to the fact that as in (ii), strong quasiconvexity implies rank-one convexity and the Legendre-Hadamard condition

$$\begin{cases} F''(\xi)[\lambda,\lambda] \geq 2\nu|\lambda|^2, & p \geq 2, \\ F''(\xi)[\lambda,\lambda] \geq 2\nu(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, & 1 < p < 2, \end{cases}$$

for every $\xi \in \mathbb{R}^{N\times n}$ and all $\lambda \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\lambda) \leq 1$ (see end of Section 2.2). In the case $N = 1$ this condition is equivalent to the left-hand inequality in (ii) and associated with uniform ellipticity of (1.2) in the manner discussed immediately after our introduction of (1.2). It is a property of $\mathbb{A}$-Harmonic functions essential for ensuring their regularity (see see Chapter 3 Section 3.2, Lemma 3.4 and Theorem 3.2). In general for the existence discussion of Chapter 2 only Morrey's weaker version of quasiconvexity, $W^{1,p}$-quasiconvexity (Definition 2.2) is necessary. In Section 2.2 we discuss at some length the relationship between the notions of convexity and the implication chain

$$\text{convexity} \implies \text{polyconvexity} \implies \text{quasiconvexity} \implies \text{rank-one convexity}\,.$$

In the $N = 1$ case all these notions are equivalent. However this is not the case for $N > 1$. Showing that the reverse implications do not hold [in the case of rank-one and quasiconvexity] is not trivial matter. However ŠVERÁK provided a counter example in [63] showing that rank-one convexity does not imply quasiconvexity in the cases $n \geq 2$, $N \geq 3$.

It is important to note that for $N > 1$ strong quasiconvexity does not imply convexity of

$$u \mapsto \int_\Omega F(Du)\,dx$$

on $W^{1,p}_g := g + W^{1,p}_0$ for a given g (Dirichlet boundary condition), except in some special cases (see Proposition 2.1 of Chapter 2 and Corollary 3.2 of Chapter 3). Hence there are differences between the notions of critical points (weak solutions to the Euler-Lagrange equation) and minimisers. In fact, there are even differences between various notions of local minimisers, a point we shall be concerned with here.

First we will briefly discuss the interplay between critical points and the local minimisers of (1.1) considered in by KRISTENSEN and TAHERI in [42] and motivated by questions raised by BALL and MARSDEN [10]. This discussion will be relevant to the significance of our theorem on the positive second variation and our regularity result. In [42] it was shown that a priori Lipschitz critical points, $\overline{u}$, admitted by the Euler-Lagrange equation associated with (1.1) where I at $\overline{u}$ has strongly positive second variation, for F satisfying (H1) and (H3), are $\mathrm{W}^1\mathrm{BMO}$-local minimisers. Here, by a $\mathrm{W}^1\mathrm{BMO}$-local minimiser, we mean a minimiser $\overline{u}$ of (1.1) minimising amongst all $u \in W^{1,p}_{\overline{u}}(\Omega, \mathbb{R}^N)$ while satisfying for some $\delta > 0$ the condition

$$\|Du - D\overline{u}\|_{\mathrm{BMO}(\mathbb{R}^n, \mathbb{R}^N)} < \delta$$

where BMO denotes the space of bounded mean oscillation defined in Chapter 3, Section 3.1. Compare this with the definition for $W^{1,q}$-local minimisers, Definition 2.3 of Chapter 2, Section 2.1.1.

It had already been shown by MÜLLER and ŠVERÁK [53], that for $N, n \geq 2$ there is a very irregular Lipschitz critical point of (1.1) satisfying the hypotheses (H1)-(H3) but which is nowhere C^1. Also a recent result of SZÉKELYHIDI [59] has shown that even for polyconvex F, Lipchitz critical points of (1.1) can be similarly irregular. See Definition 2.2 and Theorem 2.6 in Chapter 2 for polyconvexity and its relation to the other notions of convexity.

Given the above Kristensen and Taheri [42] showed that Müller and Šverák's example can be used to construct an F of (1.1) still admitting a Lipschitz critical point $\overline{u}$ that is nowhere C^1 and that satisfies the same hypotheses (H1)-(H3), but with an additional condition. This extra condition is much stronger than the condition of strong positive second variation of $I[\cdot, \Omega]$ at $\overline{u}$, and so it follows by Theorem 4.2 of Chapter 4, Section 4.2, that $\overline{u}$ is actually a $\mathrm{W}^1\mathrm{BMO}$-local minimiser. Note that Theorem 4.2 is taken verbatim from Kristensen and Taheri, [42].

As a consequence the Lipschitz $\mathrm{W}^1\mathrm{BMO}$-local minimisers of Kristensen and Taheri's theorem, are not necessarily C^1 anywhere, as concluded in [42]. Further given Székelyhidi's result [59] we cannot even expect an improvement in the situation when we strengthen the notion of quasiconvexity to polyconvexity.

In tackling the regularity problem for local minimisers Kristensen and Taheri found in the same paper [42] that a partial regularity result is however possible provided a regularity condition excluding the examples of [53] and [59] is assumed a priori on the local minimiser. In the case of Lipschitz critical points that are local minimisers, this a priori condition insures that we can use comparison maps that are as irregular as the local minimiser potentially could be.

A recent result on sufficiency conditions for strong local minima with positive second variation was obtained by GRABOVSKY and MENGESHA [31], settling a conjecture of Ball [7]. Their result assumes a priori that the critical point $\overline{u} \in C^1(\Omega, \mathbb{R}^N)$, see Theorem 4.1 , Chapter 4, Section 4.1. If this is the case and the conditions of p-coercivity, (H1)-(H3) are satisfied by F and (1.1) has strong positive second variation at $\overline{u}$, then $\overline{u}$ is a strong local minimiser as defined in Definition 4.1 of the same section. An earlier result of theirs proved sufficiency for a related class of local minimisers [30]. Before these results ZHANG [66] showed that critical points of (1.1), for a certain class of F in $C^{2,\alpha}_{\text{loc}}$ satisfying (H2) and a version of (H3), strong $W^{1,p}$-quasiconvexity (compare (H3) with Definition 2.2, Chapter 2), that are C^2 on small balls with centres in Ω, are absolutely minimising on those small balls. For $\mathrm{W}^1\mathrm{BMO}$-local minimisers these results are not sufficient to show that they are strong local minimisers even when the above mentioned a priori regularity condition, allowing for partial regularity of the minimisers in the interior of Ω, is satisfied. For more on extending the Weierstrass sufficiency conditions to the vectorial case see [10,60] and the references there in. Also for further discussion on the question of existence of local minimisers see Section 2.1.1, Chapter 2 on a necessary condition for local minimisers, as well as TAHERI [62] and for a review of the problem BALL [8].

In the final two chapters of this thesis we prove our results, two theorems extending results in [42]. In the first of our two theorems, Theorem 4.3, Chapter 4, Section 4.2, we extend the result that shows Lipschitz critical points of (1.1) satisfying (H1) with strongly positive second variation are $\mathrm{W}^1\mathrm{BMO}$-local minimisers, to the non-Lipschitz case where critical points belong to $W^{1,p}(\Omega, \mathbb{R}^N)$ for $p \in [1, \infty)$. In our second theorem, Theorem 5.1, Chapter 5 on partial regularity of local minimisers, we also extend the a priori regularity condition for Lipschitz critical points from [42],

discussed above, to the non-Lipschitz case (this time $p \in (1, \infty)$) by assuming they are W^{1}BMO-local rather than $W^{1,\infty}$-local minimisers. This is appropriate for those critical points with strongly positive second variation. In actual fact we find that the partial regularity results of these local minimisers are a special case of the results for the class of $W^1\mathcal{L}^{p,\mu}$-local minimisers, where $\mathcal{L}^{p,\mu}$ denotes the *Campanato space with exponents* p *and* $\mu \geq 0$, that satisfy an a priori regularity condition which we will introduce shortly, (1.8), along with a statement of the result.

The background for our results is discussed in Chapters 2 and 3. To make this document as self contained as possible we will limit our discussion to the functional (1.1) although much of the background theory in Chapters 2 and 3 has been developed in the general case where the integrand F is also dependent on x, u and thus applies equally to that case, with suitable qualification of the conditions of F in the variables x and u. Note that the corresponding proof's are generally more technical. For a good general overview of the state of regularity theory including the x, u dependent case see [47].

In Chapter 2 we discuss conditions for existence of minimisers of (1.1) satisfying the hypotheses (H1)-(H3). In particular the importance of $W^{1,p}$-quasiconvexity for lower semicontinuity of (1.1) and the partial regularity theory of later chapters, as well as its relation to other forms of convexity.

In Chapter 3 We introduce Campanato, Morrey and BMO spaces and their relation to Hölder continuity and the regularity of the $\mathbb{A}$-Harmonic solutions to uniformly elliptic second order partial differential systems of equations with constant and continuous coefficients. These will be important in theory of the partial regularity of (1.1), see the final chapter. BMO spaces are also necessary for the result of Chapter 4.

Introducing the main results

As we have mentioned our partial regularity result is based around KRISTENSEN and TAHERI'S proof [42] of partial regularity of $W^{1,q}$-local minimisers. This was extended to the subquadratic case $1 < p < 2$ by CAROZZA and PASSARELLI DI NAPOLI [13] from CAROZZA, FUSCO and MINGIONE [14] for absolute minimisers

in the subquadratic case. We base the subquadratic part of our proof on their work. In addition to these results further strong $W^{1,\bar{q}}$-local minimiser ($1 \leq \bar{q} < \infty$) partial regularity results for (1.1) satisfying (H1) with strong $W^{1,\bar{q}}$-quasiconvex F, and for the relaxed functional strong p-quasiconvex F, but with (p,q)-growth have recently been obtain by SCHEMM and SCHMIDT [56]. Note that by the relaxed functional of (1.1) we mean the Lebesgue-Serrin extension of $I[u,\Omega]$. A further paper by SCHMIDT [57] extends the result for the relaxed functional (compare definitions 2.2 and 2.4 of Chapter 2 for the difference between strong p-quasiconvexity and strong $W^{1,\bar{q}}$-quasiconvexity). For a recent review of (p,q)-growth partial regularity results for absolute minimisers we refer the reader to [47].

The main result of this thesis is a proof of partial regularity for a special class of local minimisers $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ of the multiple integral (1.1) where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ and satisfies (H1)-(H3) for $p > 1$, see Chapter 5.

Let $(\mathrm{X}, \|\cdot\|)$ denote a normed space continuously embedded in $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$. By a $\mathrm{W}^1\mathrm{X}$-local minimiser we mean a map $\overline{u}$ for which there exists a $\delta > 0$ such that $I[\overline{u},\Omega] \leq I[u,\Omega]$ whenever

$$u \in \overline{u} + \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N) \tag{1.6}$$

and

$$\|Du - D\overline{u}\| \leq \delta. \tag{1.7}$$

In Chapters 4 and 5 we will focus on a special class of $\mathrm{W}^1\mathrm{X}$-local minimisers $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ with $\mathrm{X} = \mathcal{L}^{\mathrm{p},\mu}(\Omega,\mathbb{R}^{\mathrm{N}\times\mathrm{n}})$, the *Campanato space with exponents p and* $\mu \geq 0$, for which we prove partial regularity for $\mu \leq n$ under a δ-smallness condition of the $\mathcal{L}^{p,\mu}$-norm of $D\overline{u}$ over all open balls $B \subset \Omega$ in the limit as radius of the balls approach zero. It is important to note that the δ here is not arbitrarily small as, for example, in MOSER [51]. It is fixed by the local minimiser condition (1.7) and we impose no additional condition on its size to prove the above result.

We will show that the equivalent regularising condition for *Bounded Mean Oscillation* type local minimisers, $\mathrm{X} = \mathrm{BMO}(\Omega,\mathbb{R}^{\mathrm{N}\times\mathrm{n}})$, is (1.8) and that in the context of partial regularity such minimisers are interchangeable with $W^1\mathcal{L}^{p,n}$-local minimisers. Note that condition (1.8) was introduced in the context of partial regularity of local minimisers in a remark by Kristensen and Taheri [42]. In subsequent work

Moser [51] proved regularity of critical points, $\overline{u} \in W^{1,2}(\Omega, \mathbb{R}^N)$, of (1.1) for rank-one convex F, when the BMO-norm of the gradient $D\overline{u}$ is small (see comment of previous paragraph). We clarify the partial regularity result for the case $\mu = n$ by extending a sufficiency condition for Lipschitz critical points to be local minimisers of $\mathrm{X} = \mathrm{BMO}(\Omega, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$ type to the non-Lipschitz case, with a view to showing that there exists a local minimiser of (1.1) that is not strong in the sense of [42] and not partially regular without the regularising condition

$$\limsup_{R\to 0^+} \left(\sup_{\substack{x\in\Omega' \\ r\in(0,R)}} \fint_{\Omega(x,r)} |D\overline{u} - (D\overline{u})_{x,r}| dy \right) < \delta \tag{1.8}$$

for every open set Ω' compactly contained in Ω, and where δ corresponds to (1.7).

A regularity theorem for a new class of local minimisers.

For any normed space Y we let $\mathrm{Y}(\Omega, \mathbb{R}^N)$ denote the space of vector valued maps $u\colon \Omega \to \mathbb{R}^N$ and $\mathrm{Y}(\Omega, \mathbb{R}^{N\times n})$ the space of matrix valued maps $u\colon \Omega \to \mathbb{R}^{N\times n}$. We use $|\cdot|$ to denote the usual euclidean norms, e.g for matrices $\xi \in \mathbb{R}^{N\times n}$ we let

$$|\xi| := \sqrt{\mathrm{trace}(\xi^T\xi)}.$$

The main result of this thesis, Theorem 5.1 of Chapter 5, is a consequence of the various embeddings and isomorphisms linking Campanato, Morrey and BMO spaces on balls (see Section 3.1), Poincaré's inequality and standard compactness arguments, allowing the extension of the local minimiser version [13, 42] of the "blow up method" for quasiconvex functionals $I[\cdot, \Omega]$ [2,4,14,22], to a class of local minimisers characterised by the Morrey-Campanato metric. We state it here for the convenience of the reader:

Theorem 1.1. *Consider the functional $I[\cdot, \Omega]$ of (1.1) satisfying the hypotheses (H1)-(H3). Suppose that $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for $p \in (1, \infty)$ is a $W^1\mathcal{L}^{p,\mu}$-local minimiser of $I[\cdot, \Omega]$: There exists a $\delta > 0$ such that $I[\overline{u}, \Omega] \leq I[u, \Omega]$ whenever $u \in \overline{u} + \mathrm{W}_0^{1,p}(\Omega, \mathbb{R}^N)$ and $\|Du - D\overline{u}\|_{p,\mu;\Omega} \leq \delta$, so that $D\overline{u}$ satisfies the regularising condition*

$$\limsup_{R\to 0+} \left(\sup_{\substack{x_0\in\Omega' \\ r\in(0,R)}} \left(\frac{1}{r^\mu} \int_{\Omega(x,r)} |D\overline{u} - (D\overline{u})_{\Omega(x,r)}|^p \, dx \right)^{\frac{1}{p}} \right) < \delta \tag{1.9}$$

for every open set Ω' compactly contained in Ω. Then for $\mu \leq n$ there exists an open set $\Omega_0 \subset \Omega$ of full n-dimensional measure, such that the minimiser $\overline{u} \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, and $|\Omega \setminus \Omega_0| = 0$.

Note that condition (1.9) of the above theorem is a generalisation of (1.8), c.f. [42] Remark 4 on partial regularity for local minimisers.

Partial regularity of non-Lipschitz $\mathrm{W}^1\mathrm{BMO}$-local minimisers follows from Lemma 5.3 in the proof of the above theorem and the isomorphism $\mathcal{L}^{n,p}(B, \mathbb{R}^{N\times n}) \cong \mathrm{BMO}(\mathrm{B}, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$ on balls $B \subset \mathbb{R}^n$ (see Section 3.1.1 and Proposition 3.3 for details):

Corollary 1.1. *Consider the functional $I[\cdot, \Omega]$ of (1.1) satisfying the hypotheses (H1)-(H3). Suppose that $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for $p \in (1, \infty)$ is a $\mathrm{W}^1\mathrm{BMO}$-local minimiser of $I[\cdot]$: There exists a $\delta > 0$ such that $I[\overline{u}, \Omega] \leq I[u, \Omega]$ whenever $u \in \overline{u} + \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)$ and $\|Du - D\overline{u}\|_{*;\Omega} \leq \delta$, so that $D\overline{u}$ satisfies the regularising condition (1.8). Then there exists an open set $\Omega_0 \subset \Omega$ of full n-dimensional measure, such that the minimiser $\overline{u} \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, and $|\Omega \setminus \Omega_0| = 0$.*

As mentioned above, our proof of Theorem 1.1, will be based on the standard blow-up argument to show a decay estimate on the excess defined for every ball $B(x,r) \subset \Omega$ by

$$E(x,r) = \begin{cases} \fint_{B(x,r)} |V(D\overline{u}) - V((D\overline{u})_{x,r})|^2 & 1 < p < 2 \\ \\ \fint_{B(x,r)} \left(|D\overline{u} - (D\overline{u})_{x,r}|^2 + |D\overline{u} - (D\overline{u})_{x,r}|^p\right) & p \geq 2. \end{cases}$$

Here

$$V(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}}\xi, \qquad \xi \in \mathbb{R}^{N\times n}.$$

From this decay estimate it is well known that partial regularity follows (see Chapter 5, Section 5.3).

Significance of the regularity result

In [42] partial regularity for $\mathrm{W}^{1,q}$-local minimisers $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ $(q > p)$ was proved by assuming $D\overline{u} \in \mathrm{L}^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$. Given the Sobolev class $\mathrm{W}^{1,q}(\Omega, \mathbb{R}^N)$ for $q > p$, the inclusion $\mathrm{W}^{1,q}(\Omega, \mathbb{R}^N) \subset W^1\mathcal{L}^{p,\mu}(\Omega, \mathbb{R}^N)$ follows directly from Hölders

inequality for the exponents $\mu \leq n(1-p/q)$. Thus for each $q > p$, $W^1\mathcal{L}^{p,\mu}$ ($\mu \leq n(1-p/q)$) possess a weaker topology than $\mathrm{W}^{1,q}$ and thus in this case a $W^1\mathcal{L}^{p,\mu}$ local minimiser is a stronger notion of a local minimum than a $\mathrm{W}^{1,q}$ local minimiser. The a priori δ-smallness condition (5.1) is certainly a weaker requirement than condition $\overline{u} \in \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ when $\mu < n(1-p/q)$ as the later condition implies the arbitrary smallness condition (5.2). However it is not clear that the $\mathrm{W}^{1,q}_{\mathrm{loc}}$ condition placed on the $\mathrm{W}^{1,q}$-local minimisers of [13, 42] is necessary for partial regularity. In any case our a priori condition for the general Morrey-Campanato class of minimisers fits in neatly with previous results for weaker notions of local-minimisers, namely the results for $\mathrm{W}^1\mathrm{BMO}$, $\mathrm{W}^{1,\infty}$ local minimisers of Lipschitz class derived in [42]. In fact given the equivalence of Campanato and BMO spaces when Campanato exponent $\mu = n$ we will show that the results for $\mathrm{W}^1\mathrm{BMO}$ local minimisers follow when the minimiser $\overline{u}$ is of class $\mathrm{W}^{1,p}(\Omega)$, $1 < p < \infty$.

From previous discussion it is clear that a regularising condition like (1.8) is necessary for partial regularity for Lipschitz $\mathrm{W}^1\mathrm{BMO}$-local minimisers. The second result of the thesis justifies the regularity result for $\mathrm{W}^1\mathrm{BMO}$-local minimisers in the more general non-Lipschitz case. Following the spirit of [42] we extend the sufficiency condition for $\mathrm{W}^1\mathrm{BMO}$-local minimisers.

Positive Second Variation

It is shown in [42] that for C^2 integrands F of the functional $I[\cdot, \Omega]$ that positivity of the second variation of $I[\cdot, \Omega]$ at a given Lipschitz critical point $\overline{u}$ implies that $\overline{u}$ is not only a weak local minimiser, which is well known, but is in fact a $\mathrm{W}^1\mathrm{BMO}$ local minimiser. A similar result was also proved by FIROOZYE [26] but the proof requires stronger assumptions on the integrand F.

In the following we extend the result of [42] for critical points $\overline{u}$ of $I[\cdot, \Omega]$ that are in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for $1 \leq p < \infty$ by adding a uniform continuity condition to the second derivative of F. We assume that F'' is uniformly continuous with a modulus of continuity $\omega\colon [0, \infty) \to \mathbb{R}$, which is continuous, increasing, $\omega(0) = 0$ and

$$\sup_{t>0} \frac{\omega(2t)}{\omega(t)} < \infty. \tag{1.10}$$

This extra condition on top of the uniform continuity is not as limiting as it first appears. It excludes exponential growth of ω. However we can accommodate the subclass of piecewise polynomial growth (not necessarily increasing) that do not satisfy (1.10) but instead satisfy

$$\tilde{\omega}(t) := \sup_{s\geq 1}\left(s^{-k}\sup_{r\leq st}\omega(r)\right) < \infty,$$

(see Remark 4.2 of Chapter 4, Section 4.2 for details). The result is as follows

Theorem 1.2. *Let the integrand of (1.1), $F\colon \mathbb{R}^{N\times n}\to\mathbb{R}$ be a C^2 function, $\Omega\subset\mathbb{R}^n$ be open and bounded and $\overline{u}\in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ $1\leq p<\infty$ be a critical point of (1.1) with strong positive second variation: for some $\delta_s>0$ and all $\varphi\in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathrm{N}})\cap \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^{\mathrm{N}})$,*

$$\int_\Omega F'(D\overline{u})[D\varphi] \;=\; 0 \tag{1.11}$$

$$\int_\Omega F''(D\overline{u})[D\varphi,D\varphi] \;\geq\; \delta_s\int_\Omega |D\varphi|^2. \tag{1.12}$$

Further for $p<\infty$ assume

$$|F''(\xi)-F''(\eta)|\leq\omega(|\xi-\eta|) \tag{1.13}$$

for all $\xi,\eta\in\mathbb{R}^{N\times n}$. Then there exists a $\delta_*(n,N,c,q)>0$ such that

$$\int_\Omega F(D\overline{u}+D\varphi)\geq\int_\Omega F(D\overline{u})$$

holds for all $\varphi\in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathrm{N}})\cap \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^{\mathrm{N}})$, with $\|D\varphi\|_{\mathrm{BMO}(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathrm{N}})}\leq\delta_*$.

Finally this straight forward corollary to the above theorem gives the sufficiency conditions for non-Lipschitz critical points of $I[\cdot,\Omega]$ to be partially regular.

Corollary 1.2. *Let the integrand of $I[\cdot,\Omega]$, $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ be C^2, $\Omega\subset\mathbb{R}^n$ open and bounded Let $\overline{u}\in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, $1<p<\infty$ be a critical point of $I[\cdot,\Omega]$ with strongly positive second variation such that for some $\delta_s>0$ and all $\varphi\in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathrm{N}})\cap \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^{\mathrm{N}})$ we have (1.11) and (1.12). Suppose also that we have*

$$|F''(\xi)-F''(\eta)|\leq\omega(|\xi-\eta|) \tag{1.14}$$

such that F satisfies (H1)-(H3). Then $\overline{u}$ is partially regular in the sense of Theorem 5.1 provided $D\overline{u}$ satisfies the regularity condition (1.8) with $\delta = \delta_*$ where δ_* is given in Theorem 1.2.

We will start with the following chapter to explain the importance of quasiconvexity of F in (1.1) for the existence of minimisers in the case $N > 1$.

Chapter 2

Existence of minimisers by direct method as motivation for quasiconvexity

2.1 Quasiconvexity as necessary condition for existence of minimisers.

We consider the problem

$$\inf\left\{I[u,\Omega]=\int_\Omega F(Du(x))\,dx : u\in \overline{u}+W_0^{1,p}(\Omega,\mathbb{R}^N)\right\}. \tag{2.1}$$

As discussed in EVANS' book [23] for the case $n=N=1$, the existence of a minimum 'point' of a continuous function $F:\mathbb{R}^{N\times n}\to\mathbb{R}$ $(n,N\geq 1)$ is guaranteed by the coercivity condition

$$F(\xi)\geq \frac{1}{c}|\xi|^p - c, \qquad \text{for all } \xi\in\mathbb{R}^{N\times n},$$

and some constant $c>0$. However this does not guarantee existence of a minimum function of the functional I in the Sobolev space.

The coercivity condition gives for all $u\in W^{1,p}(\Omega,\mathbb{R}^N)$

$$\int_\Omega F(Du(x))\,dx \geq \frac{1}{c}\int_\Omega |Du(x)|^p\,dx - c|\Omega|.$$

For some fixed $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ let $W^{1,p}_g(\Omega, \mathbb{R}^N) := g + W^{1,p}_0(\Omega, \mathbb{R}^N)$. Choosing $\{u_k\} \subset W^{1,p}_g(\Omega, \mathbb{R}^N)$ as a minimising sequence:

$$I[u_k] \to \inf_{u \in W^{1,p}_g} I[u],$$

one may obtain by coercivity that such a sequence $\{u_k\}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$ and hence admits a weakly convergent subsequence (for convenience not relabelled)

$$u_k \rightharpoonup \overline{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^n).$$

We recall that when $\Omega \subset \mathbb{R}^n$ is abounded Lipschitz domain, then this amounts to that $u_k \to \overline{u}$ strongly in L^p and $Du_k \rightharpoonup \overline{u}$ weakly. It follows that $\overline{u} \in W^{1,p}_g(\Omega, \mathbb{R}^N)$ (and thus is an admissible map) by Mazur's theorem from which it follows that $W^{1,p}_0$ is weakly closed. This means that since $u_k - g \in W^{1,p}_0(\Omega, \mathbb{R}^{N\times n})$ for $\{u_k\} \subset W^{1,p}_g(\Omega, \mathbb{R}^{N\times n})$, and $u \in W^{1,p}_g(\Omega, \mathbb{R}^N)$ we have $\overline{u} - g \in W^{1,p}_0$. To guarantee existence one needs a condition to ensure that the limit of the functional

$$\liminf_{k\to\infty} I[u_k] = \inf_{u \in W^{1,p}_g} I[u]$$

is no smaller than $I[\overline{u}]$. This condition which is both necessary and sufficient for existence in this context is precisely weak sequential lower semi-continuity.

Definition 2.1 (Sequential weak lower semicontinuity)**.** *For open and bounded $\Omega \subset \mathbb{R}^N$ let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ for $p \geq 1$ and $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous. The functional $I[\cdot, \Omega]$ is said to be* sequentially weakly lower semicontinuous *in $W^{1,p}(\Omega; \mathbb{R}^N)$, $p < \infty$, if for every sequence $u_k \rightharpoonup u$ in $W^{1,p}$*

$$\liminf_{k\to\infty} I(u_k) \geq I(u).$$

If $p = \infty$ and the above inequality holds for every sequence $u_k \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}$, I is said to be sequentially weak $*$ lower semicontinuous *in $W^{1,\infty}(\Omega; \mathbb{R}^N)$.*

We next present a necessary condition for sequential weak lower semicontinuity for both the scalar ($n = 1$ or $N = 1$) and vectorial ($n, N \geq 2$) cases. First we will need the following lemma on approximation of affine functions by piecewise affine functions found in DACOROGNA'S book [15, Lemma 3.11]. By the notation $\mathrm{Aff}_{piec}(\overline{\Omega}, \mathbb{R}^N)$ we

will denote the space of piecewise affine functions over $\overline{\Omega}$ (that is, $u \in \mathrm{Aff}_{piec}(\overline{\Omega}, \mathbb{R}^N)$ if u is Lipschitz and if there exists a finite partition $\Omega = S_1 \cup \cdots \cup S_K \cup \mathcal{N}$ where S_j are open, $|\mathcal{N}| = 0$, and so u equals an affine function on each S_j).

Lemma 2.1. *For an open set $\Omega \subset \mathbb{R}^n$ with finite measure, let $\lambda \in [0,1]$ and $\alpha, \beta \in \mathbb{R}^{N\times n}$ with $\mathrm{rank}(\alpha - \beta) = 1$. Let u_ξ be such that*

$$Du_\xi(x) = \xi := \lambda\alpha + (1-\lambda)\beta, \quad \forall x \in \overline{\Omega}.$$

Then for every $\epsilon > 0$ there exist piecewise affine $u \in \mathrm{Aff}_{piec}(\overline{\Omega}, \mathbb{R}^n)$ and disjoint open sets $\Omega_\alpha, \Omega_\beta \subset \Omega$ such that

$$\begin{cases} \Big||\Omega_\alpha| - \lambda|\Omega|\Big|, \quad \Big||\Omega_\beta| - (1-\lambda)|\Omega|\Big| \leq \epsilon, \\ u \equiv u_\xi \text{ near } \partial\Omega, \quad \|u - u_\xi\|_{L^\infty} \leq \epsilon \\ Du(x) = \begin{cases} \alpha \text{ in } \Omega_\alpha \\ \beta \text{ in } \Omega_\beta, \end{cases} \\ \mathrm{dist}(\mathrm{Du(x)}, \mathrm{co}\{\alpha,\beta\}) \leq \epsilon \text{ for a.e. } \Omega \end{cases}$$

where $\mathrm{co}\{\alpha,\beta\} = [\alpha,\beta]$ is the closed segment joining α to β.

Notation. Throughout we will write $|\cdot|$ to denote either the Euclidian norm or the Lebesgue measure. The precise meaning will be clear from the context.

Thus as a consequence of the Riemann-Lebesgue Lemma one obtains the following necessary conditions for (sequential) weak lower semicontinuity of I on $W^{1,p}$, due to MORREY [48] for $p = \infty$, see also DACOROGNA [15, Theorem 3.13] and BALL & MURAT [9] for $1 \leq p < \infty$:

Theorem 2.1. *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous, bounded from below and let $\Omega \subset \mathbb{R}^n$ be open and bounded. If $I[\cdot, \Omega]$ is (sequentially) weakly lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^N)$, then:*

(i) For every bounded open set $D \in \mathbb{R}^n$ with $|\partial D| = 0$ and $\varphi \in W_0^{1,p}(D, \mathbb{R}^N)$:

$$\int_D F(\xi_0 + D\varphi(x))dx \geq |D|F(\xi_0). \tag{2.2}$$

(ii) F is rank-1 convex. Thus for $N = 1$, or $n = 1$, F is convex.

To prove the theorem we follow [15].

Proof. First we note that once (2.2) is shown to hold for a specific open bounded domain $D \subset \mathbb{R}^N$ it holds for all open and bounded domains in $\mathbb{R}^N$, (see Proposition 5.11 of [15] and Proposition 2.3 of [9] for $W^{1,p}$-Quasiconvexity).

Step 1: Let D be an open cube in Ω with faces parallel to the co-ordinate axes and let $\varphi \in W_0^{1,p}(D;\mathbb{R}^N)$. Denoting the length of the cubes edge by d we make the following periodic extension of φ to the whole of $\mathbb{R}^N$:

$$\varphi(x + dz) = \varphi(x), \quad \text{for every } x \in D,\, z \in \mathbb{Z}^n$$

and let $\varphi_k = \frac{1}{k}\varphi(kx)$. Since $\varphi = 0$ on ∂D, the extension $\varphi_k \in W^{1,p}(\Omega, \mathbb{R}^N)$, and by the Riemann-Lebesgue lemma

$$\varphi_k \rightharpoonup 0 \text{ in } W^{1,p}(\Omega, \mathbb{R}^N).$$

Now defining $\overline{u} = u_{\xi_0}$ where $u_{\xi_0} = \xi_0 x$, and letting

$$u_k := \begin{cases} u_{\xi_0}(x) & x \in \Omega \setminus D \\ u_{\xi_0}(x) + \varphi_k(x) & x \in D \end{cases}$$

we have

$$u_k \rightharpoonup u_{\xi_0} \quad \text{in } W^{1,p}(\Omega, \mathbb{R}^N). \tag{2.3}$$

We also have

$$I[u_k] = \int_\Omega F(Du_k(x))\,dx = \int_{\Omega\setminus D} F(\xi_0)\,dx + \int_D F(\xi_0 + D\varphi_k)\,dx.$$

Making the change of variables $y = kx$ we have

$$I[u_k] = |\Omega \setminus D|F(\xi_0) + \frac{1}{k^n}\int_{kD} F(\xi_0 + D\varphi(y))\,dy$$

$$= |\Omega \setminus D|F(\xi_0) + \int_D F(\xi_0 + D\varphi(y))\,dy.$$

Thus by sequentially weak lower semicontinuity

$$\liminf_{k\to\infty} I[u_k] = |\Omega \setminus D|F(\xi_0) + \int_D F(\xi_0 + D\varphi(y))\,dy$$

$$\geq |\Omega|F(\xi_0)$$

Hence we arrive at (2.2) proving part (i).

Step 2. We want to show rank-one convexity of F i.e.

$$F(\lambda\alpha + (1-\lambda)\beta) \leq \lambda F(\alpha) + (1-\lambda)F(\beta) \tag{2.4}$$

for every $\alpha, \beta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\beta - \alpha) = 1$ and $\lambda \in [0,1]$. From the affine approximation lemma, Lemma 2.1, we construct an affine piecewise function $\varphi_\epsilon \in \mathrm{Aff}_{piec}(\overline{D};\mathbb{R}^N) \subset W^{1,\infty}(D;\mathbb{R}^N)$ with disjoint open sets D_α and D_β such that

$$\begin{cases} \Big||D_\alpha| - \lambda|D|\Big|, \quad \Big||D_\beta| - (1-\lambda)|D|\Big| \leq \epsilon, \\ \varphi_\epsilon \equiv 0 \text{ near } \partial\Omega, \quad \|\varphi_\epsilon\|_{L^\infty} \leq \epsilon, \quad \|D\varphi_\epsilon\|_{L^\infty} \leq \gamma, \\ D\varphi_\epsilon(x) = \begin{cases} (1-\lambda)(\alpha-\beta) \text{ in } D_\alpha \\ -\lambda(\alpha-\beta) \text{ in } D_\beta, \end{cases} \end{cases} \tag{2.5}$$

with constant $\gamma > 0$, independent of ϵ. Put $\varphi_\epsilon(x) := u_{\mathrm{aff}}(x) - u_{\xi_0}(x)$, where $u_{\mathrm{aff}} \in \mathrm{Aff}_{piec}(\overline{D};\mathbb{R}^N)$ and u_{ξ_0} is the affine map defined as in step 1. Thus for every α and β such that $\operatorname{rank}(\alpha-\beta) = 1$, $\lambda\alpha + (1-\lambda)\beta$ corresponds to some $\xi_0 \in \mathbb{R}^{N\times n}$ for which the piecewise affine function φ_ϵ is associated. Therefore by (2.2) of (i),

$$\begin{aligned} |D|F(\lambda\alpha + (1-\lambda)\beta) &\leq \int_D F(\lambda\alpha + (1-\lambda)\beta + D\varphi_\epsilon(x))\,dx \\ &= \int_{D_\alpha} F(\alpha)\,dx + \int_{D_\beta} F(\beta)\,dx \\ &\quad + \int_{D\setminus D_\alpha\cup D_\beta} F(\lambda\alpha + (1-\lambda)\beta + D\varphi_\epsilon(x))\,dx. \end{aligned}$$

Given the continuity of F and $\|D\varphi_\epsilon\|_{L^\infty} \leq \gamma$ the right hand integrand is bounded from above for a.e. $x \in D$ and $\epsilon > 0$. Thus since $|D \setminus D_\alpha \cup D_\beta| < \epsilon$ by (2.5), taking $\epsilon \to 0$ the right hand integral converges to zero. Hence, given that in the limit $\epsilon \to 0$, $|D_\alpha| = \lambda|D|$, $|D_\beta| = (1-\lambda)|D|$, we arrive at (2.4) with $\operatorname{rank}(\alpha-\beta) = 1$. Now $\operatorname{rank}(\alpha-\beta) = 1$ for all α, β if $N = 1$ or $n = 1$ proving (ii). □

Condition (2.2) in Theorem 2.1 is known as $W^{1,p}$-*quasiconvexity*. We outline the various relevant notions of convexity including $W^{1,p}$-quasiconvexity in the following definition:

Definition 2.2 (Notions of Convexity). *(i) We say that* $F : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ *is* polyconvex *provided there exists a convex* $G : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ *such that*

$$F(\xi) = G(T(\xi))$$

where $T : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(n,N)}$ *is defined to be*

$$T(\xi) := (\xi, \operatorname{adj}_2 \xi, \dots, \operatorname{adj}_{\mathrm{n}\wedge\mathrm{N}} \xi).$$

Here $\operatorname{adj}_s \xi$ *represents the matrix of all* $s \times s$ *minors of* $\xi \in \mathbb{R}^{N\times n}$ *written as a vector in some fixed order. Accordingly* $2 \le s \le n \wedge N = \min\{n, N\}$ *and* $\tau(n, N) := \sum_{s=1}^{n\wedge N} \sigma(s)$, *where* $\sigma(s) := \frac{n!N!}{(s!)^2(n-s)!(N-s)!}$.

(ii) Let $\Omega \subset \mathbb{R}^n$ *be open and bounded with* $|\partial\Omega| = 0$, *and* $1 \le p \le \infty$. *We say that* F *is* $W^{1,p}$*-quasiconvex provided it satisfies*

$$\int_\Omega F(\xi + D\varphi(x)) - F(\xi)\, dx \ge 0, \quad \forall \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N) \text{ and } \forall \xi \in \mathbb{R}^{N\times n}. \quad (2.6)$$

When $p = \infty$ *we often merely talk about quasiconvexity.*

(iii) We say that F *is rank-one convex provided*

$$F(\lambda\alpha + (1-\lambda)\beta) \le \lambda F(\alpha) + (1-\lambda)F(\beta)$$

for $\lambda \in [0, 1]$ *and every* $\alpha, \beta \in \mathbb{R}^{N\times n}$ *such that* $\operatorname{rank}(\alpha - \beta) \le 1$.

(iv) We say a function $F\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *is separately convex if it is convex in each variable* ξ_i, *for* $i = 1, \dots, d$, *i.e.*

$$\xi_i \mapsto F(\xi_1, \xi_2 \dots \xi_i \dots \xi_{d-1}, \xi_d)$$

is convex.

(v) We say the function F *is poly-affine, quasi-affine or rank-one affine if* F *and* $-F$ *are both polyconvex, quasiconvex or rank-one convex, respectively.*

Remark 2.1. *(i)* $W^{1,p}$*-quasiconvexity is equivalent to saying each affine function, denoted* $\overline{u}_{\mathrm{Aff}}$, *minimises* $I[\cdot, E]$ *over* $W^{1,p}_{\overline{u}_{\mathrm{Aff}}}(E, \mathbb{R}^N)$ *for every open bounded* $E \subset \mathbb{R}^n$ *with* $|\partial E| = 0$. *Note that the condition of* $W^{1,p}$*-quasiconvexity becomes weaker for increasing* p *and that it really changes with* p. *Indeed on* $\mathbb{R}^{n\times n}$, $F(\xi) := |\det \xi|$ *is* $W^{1,p}$*-quasiconvex if and only if* $p \ge n$. *See [9].*

(ii) Rank-one convexity implies separate convexity, i.e. (iii) $\Longrightarrow$ *(iv). We make the obvious identification* $\mathbb{R}^{N\times n} \cong \mathbb{R}^d$ *with* $d = Nn$.

Clearly Theorem 2.1 shows that for sequentially weak lower semicontinuous I the corresponding integrand F is both quasiconvex and rank-1 convex. We will discuss the relationship between the two notions of convexity in the next section.

The above shows that $W^{1,p}$-quasiconvexity is a necessary condition for sequential weak lower semicontinuity on $W^{1,p}$. Now let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $1 \leq p \leq \infty$. In the following we will discuss a well known result of Ball and Murat [9], which supplies quasiconvexity as necessary condition for for existence of $W^{1,p}$-minimisers for the functional

$$J[u,\Omega] = \int_\Omega F(Du(x)) + \Psi(x,u(x))\,dx \tag{2.7}$$

over the set of $W^{1,p}(\Omega,\mathbb{R}^N)$ functions with affine boundary values $W^{1,p}_{\text{Aff}} = \{u : u - \xi x \in W^{1,p}_0(\Omega;\mathbb{R}^n)\}$ and arbitrary perturbation function $\Psi : \Omega \times \mathbb{R}^N \to \mathbb{R}$.

Theorem 2.2 (Ball & Murat)**.** *Suppose that* $|\partial\Omega| = 0$. *Let* $\xi \in \mathbb{R}^{N\times n}$ *and suppose that* F *is not quasiconvex at* ξ. *Let* $\Psi(x,u(x)) = \Phi(|u-\xi x|^2)$ *where* $\Phi : \mathbb{R} \to \mathbb{R}$ *is continuous and bounded such that* $\Phi(0) = 0$, *and* $\Phi(t) > 0$ *if* $t \neq 0$. *Then* $J[\cdot,\Omega]$ *does not attain a minimum on* $W^{1,p}_{\text{Aff}}$.

Proof. We follow the proof of Ball & Murat, [9]. Let $I[u,\Omega] := \int_\Omega F(Du(x))\,dx$ and $\lambda := \inf_{u\in W^{1,p}_{\text{Aff}}} I[u,\Omega]$. Since every $u \in W^{1,p}_{\text{Aff}}$ can be written as $\xi x + \varphi$ for some $\varphi \in W^{1,p}_0(\Omega,\mathbb{R}^N)$, by assumption that F is not quasiconvex at ξ we must have $\lambda < \infty$.
Step 1. We claim that

$$\inf_{u\in W^{1,p}_{\text{Aff}}} J[u,\Omega] = \lambda. \tag{2.8}$$

Let $v = \xi x + \varphi$ where $\varphi \in W^{1,p}_0(\Omega,\mathbb{R}^N)$ and for $\epsilon > 0$ let v satisfy

$$I[v,\Omega] \leq \lambda + \frac{\epsilon}{2}. \tag{2.9}$$

By Vitali, given $j \in \mathbb{N}\setminus\{0\}$ such that $0 < \epsilon_i \leq \frac{1}{j}$ there exists a finite countable disjoint sequence of closed subsets of Ω, $x_i + \epsilon_i\overline{\Omega}$, such that

$$|\Omega \setminus \cup_i x_i + \epsilon_i\overline{\Omega}| = 0.$$

Consequently as $|\partial\Omega| = 0$, we must have $\sum_i \epsilon_i^n = 1$. Now define

$$u_j(x) = \begin{cases} \xi x + \epsilon_i \varphi(\frac{x-x_i}{\epsilon_i}), & \text{if } x \in x_i + \epsilon_i \overline{\Omega} \\ \xi x, & \text{otherwise.} \end{cases}$$

Then $u_j(x) \in W^{1,p}_{\text{Aff}}$ and

$$\begin{aligned} J[u_j, \Omega] &= \sum_i \int_{x_i + \epsilon_i \overline{\Omega}} F\left(\xi + \epsilon_i D\varphi\left(\frac{x - x_i}{\epsilon_i}\right)\right) dx + \int_\Omega \Phi\left(|u_j - \xi x|^2\right) dx \\ &= \left(\sum_i \epsilon_i^n\right) \int_\Omega F\left(\xi + D\varphi\left(y\right)\right) dy + \int_\Omega \Phi\left(|u_j - \xi x|^2\right) dx \\ &= I[v, \Omega] + \int_\Omega \Phi\left(|u_j - Ax|^2\right) dx, \end{aligned}$$

where we have used $|\partial\Omega| = 0$. Now for $1 \le p < \infty$

$$\begin{aligned} \int_\Omega |u_j - \xi x|^p \, dx &= \sum_{i=1} \epsilon_i^p \int_{x_i + \epsilon \overline{\Omega}} \left|\varphi\left(\frac{x - x_i}{\epsilon_i}\right)\right|^p dx \\ &= \sum_i \epsilon_i^{n+p} \int_\Omega |\varphi(y)|^p \, dx \\ &\le j^{-p} \|\varphi\|_{L^p(\Omega, \mathbb{R}^n)}. \end{aligned}$$

Thus $\Phi(|u_j - \xi x|^2) \to 0$ in measure as $j \to \infty$ and by the dominated convergence theorem

$$\lim_{j \to \infty} \int_\Omega \Phi\left(|u_j - \xi x|^2\right) dx = 0.$$

Hence by (2.9)

$$J[u, \Omega] \le \lambda + \epsilon, \quad \forall \epsilon > 0$$

proving claim (2.8).

Step 2. We will now use the claim for a contradiction of our assumption that F is not quasiconvex at ξ. Suppose $J[\cdot, \Omega]$ attains it's minimum with some $\overline{u} \in W^{1,p}_{\text{Aff}}$, i.e. $J[\overline{u}, \Omega] = \inf_{u \in W^{1,p}_{\text{Aff}}} J[u, \Omega]$. Then by claim (2.8)

$$\begin{aligned} \lambda &= I[\overline{u}, \Omega] + \int_\Omega \Phi(|\overline{u} - \xi x|^2) \, dx \\ &\ge \lambda + \int_\Omega \Phi(|\overline{u} - \xi x|^2) \, dx. \end{aligned}$$

Therefore $\overline{u} = \xi x$ for a.e. $x \in \Omega$ and $I[\xi x] = \inf_{u \in W^{1,p}_{\mathrm{Aff}}} I[u, \Omega]$. Hence F is $W^{1,p}$-quasiconvex at ξ contradicting our assumption. □

By taking the contra positive we immediately have the following Corollary.

Corollary 2.1. *Let $|\partial\Omega| = 0$, $\xi \in \mathbb{R}^{N\times n}$. If $J[\cdot, \Omega]$ attains a minimum on $W^{1,p}_{\mathrm{Aff}}$ for all smooth nonnegative Ψ then F is $W^{1,p}$-quasiconvex.*

Of course for $\Psi \equiv 0$ the above statement is trivial if J attains an affine minimum.

The Euler-Lagrange equation and full existence for minimisers

In the following we write F', F'' for the derivatives of an integrand F. In particular, we interpret $F'(\xi)$ as an $N \times n$ matrix, and $F''(\xi)$ as a symmetric bilinear form on $\mathbb{R}^{N\times n}$.

Theorem 2.3 (Euler-Lagrange)**.** *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be C^1 and satisfy the growth condition*

$$|F'(\xi)| \leq c(1 + |\xi|^{p-1}), \quad \forall \xi \in \mathbb{R}^{N\times n}. \tag{2.10}$$

If $\overline{u} \in W^{1,p}_g(\Omega, \mathbb{R}^N)$ is a minimiser of $I[\cdot, \Omega]$, then

$$F'(D\overline{u}) \in L^{p'}(\Omega, \mathbb{R}), \quad p' = \frac{p}{1-p},$$

and

$$\int_\Omega F'(D\overline{u}(x))[D\varphi]\, dx = 0, \quad \forall \varphi \in W^{1,p}_0(\Omega, \mathbb{R}^N). \tag{2.11}$$

Proof. First part is clear from the growth condition. To prove (2.11) fix $\varphi \in W^{1,p}_0(\Omega, \mathbb{R}^N)$. Then $\overline{u} + t\varphi \in W^{1,p}_g(\Omega, \mathbb{R}^N)$ for all $t \in \mathbb{R}$. Given that $\overline{u}$ is a minimiser

$$\int_\Omega F(D\overline{u})\, dx \leq \int_\Omega F(D\overline{u} + tD\varphi)\, dx.$$

Thus by the fundamental theorem of calculus we have

$$\int_\Omega t \int_0^1 F'(D\overline{u} + stD\varphi)[D\varphi]\, ds\, dx \geq 0.$$

Since t may be either positive or negative in $\mathbb{R}$,

$$\int_\Omega \int_0^1 F'(D\overline{u} + stD\varphi)[D\varphi]\, ds\, dx = 0.$$

Finally we show that the integrand is bounded by an L^1 function. Let $|t| \leq 1$ and using the growth condition (2.10) together with $ab^{p-1} \leq a^p + b^p$

$$\begin{aligned} |F'(D\overline{u} + stD\varphi)||D\varphi| &\leq c|D\overline{u} + D\varphi|^{p-1}|D\varphi| \\ &\leq c\left(|D\varphi| + |D\overline{u}|^p + 2|D\varphi|^p\right) \\ &\leq c\left(1 + |D\overline{u}|^p + 3|D\varphi|^p\right), \end{aligned}$$

where in the final estimate we have used $|D\varphi| \leq (1 + |D\varphi|^p)$. Hence (2.11) follows by dominated convergence as $s \to 0$. □

One can show as an immediate corollary to Theorem 2.6 part (iv), stated later in Section 2.2, that

Lemma 2.2. *Let $F\colon \mathbb{R}^{nN} \to \mathbb{R}$ be separately convex and satisfy the growth condition*

$$|F(\xi)| \leq L(1 + |\xi|^p)$$

for every $\xi \in \mathbb{R}^{nN}$, $p \geq 1$ and any $L \geq 0$. Then there exists a $c \geq 0$ such that

$$|F(\xi) - F(\zeta)| \leq c(1 + |\xi|^{p-1} + |\zeta|^{p-1})|\xi - \zeta|.$$

for every $\xi, \zeta \in \mathbb{R}^{nN}$.

Thus for rank-one F, the growth condition (2.10) used in the derivation of the Euler-Lagrange system of equations (2.11) follows from the standard growth condition on F,

$$|F(\xi)| \leq L(1 + |\xi|^p) \tag{2.12}$$

for all $\xi \in \mathbb{R}^{N\times n}$ and with c in (2.10) dependent on n, N, p and L. With this the control on F we are able to state, by an amalgamation of the results of MORREY [48], MEYERS [46] and FUSCO [27], that in fact quasiconvexity is both necessary and sufficient for squential weak lower semicontinuity on $W^{1,p}_g$ for some fixed $g \in W^{1,p}(\Omega, \mathbb{R}^N)$:

Theorem 2.4 (Morrey , Meyers & Fusco)**.** *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous and*

$$|F(\xi)| \leq L(1 + |\xi|^p), \quad \forall \xi$$

where $1 \leq p < \infty$ *and* $L < \infty$. *Fix* $g \in W^{1,p}(\Omega, \mathbb{R}^N)$. *Then*

$$I[u, \Omega] = \int_\Omega F(Du)dx$$

is sequentially weakly lower semicontinuous on $W^{1,p}_g(\Omega, \mathbb{R}^n)$ *if and only if* F *is* $W^{1,p}$*-quasiconvex.*

We remark that the generalisation of this result to x, u-dependent F, $F : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ with F measurable in x and continuous in $(s, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$, was initially proved by MARCELLINI & SBORDONE [44] for the scalar case ($N = 1$), where the condition of quasiconvexity is equivalent to convexity. The result was then proved in the full generality of the vectorial case ($N > 1$) by ACERBI & FUSCO [1].

After confirming existence of global minimisers the next natural question to ask is that of uniqueness. However as it turns out global minimisers are not unique, even for problems with strongly polyconvex integrands and when the domain Ω is an open ball. Examples to this effect were obtained in [58] by modification of classical examples of non-uniqueness for minimal surfaces. The reader is referred to the aforementioned paper for the relevant details. We will now move our discussion from global minimisers to the theory of local minimisers of $I[\cdot, \Omega]$, which is the topic of this thesis.

2.1.1 A necessary condition for existence of local minimisers

It should be noted that all previously mentioned theory holds in the local case. However existence of local minimiser that are not absolute minimisers cannot be shown from the above.

Definition 2.3. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded,* $1 \leq p \leq \infty$, $1 \leq q < \infty$ *and let* $\overline{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$. *If there exists a* $\delta > 0$ *such that*

$$I[\overline{u}, \Omega] \leq I[u, \Omega]$$

whenever $u \in W^{1,p}_{\overline{u}}(\Omega, \mathbb{R}^N)$ *and*

(i) $\|u - \overline{u}\|_{W^{1,q}(\Omega,\mathbb{R}^N)} < \delta$, *then* $\overline{u}$ *is said to be a* $W^{1,q}$-local minimiser*;*

(ii) $\|u - \overline{u}\|_{W^{1,\infty}(\Omega,\mathbb{R}^N)} < \delta$, *then* $\overline{u}$ *is said to be a* $W^{1,\infty}$-local minimiser.

From [42] we have the following necessary condition for local minimisers:

Theorem 2.5 (Necessary condition for local minimisers)**.** *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be lower semicontinuous and assume that*

$$\frac{1}{c}|\xi|^p - c \leq F(\xi) \tag{2.13}$$

for all $\xi \in \mathbb{R}^{N\times n}$, where $c > 0$ is a constant and $p \in (1,\infty)$. Put

$$I[u,\Omega] := \int_\Omega F(Du)\,dx.$$

If $\overline{u} \in W^{1,p}(\Omega,\mathbb{R}^N)$ with $I[\overline{u},\Omega] < \infty$ is a $W^{1,p}$-local minimiser then there exists a $\delta > 0$ such that

$$I[\overline{u},\Omega] \leq I[u,\Omega]$$

for all $u \in W^{1,p}_{\overline{u}}(\Omega,\mathbb{R}^N)$ with $|\Omega \cap \{|Du - D\overline{u}|\} > \delta| \leq \delta$.

Proof. Suppose that the theorem is false, then there exists a sequence of $u_j \in W^{1,p}_{\overline{u}}(\Omega,\mathbb{R}^N)$ such that $Du_j \to D\overline{u}$ in measure on Ω and

$$I[u_j,\Omega] < I[\overline{u},\Omega]. \tag{2.14}$$

For a contradiction we set $f_j(x) = F(Du_j)$ and $f_\infty(x) = F(D\overline{u})$. From the coercivity assumption (2.13) it follows that $\sup_j \|Du_j\|_{L^p(\Omega,\mathbb{R}^{N\times n})} < \infty$. Thus we infer that $u_j \rightharpoonup \overline{u}$ in $W^{1,p}(\Omega,\mathbb{R}^N)$ and so $u_j \to \overline{u}$ in measure. Now given the lower-semicontinuity of F it follows that

$$\liminf_{j\to\infty} f_j(x) \geq f_\infty(x).$$

Thus by Fatou's lemma together with (2.14), we have

$$\int_\Omega f_j(x)\,dx \to \int_\Omega f_\infty(x)\,dx.$$

Thus $f_j \to f_\infty$ strongly in $L^1(\Omega)$ and once again from coercivity of F we conclude that $\{u_j\}$ is p-equiintegrable. Hence by Vitali's convergence theorem $Du_j \to D\overline{u}$ strongly in $L^p(\Omega,\mathbb{R}^{N\times n})$ and together with (2.14) this leads to a contradiction. □

Remark 2.2. *(i) This implies that all $W^{1,p}$-local minimisers, $\overline{u} \in W^{1,p}(\Omega,\mathbb{R}^N)$, of $I[\cdot,\Omega]$ with integrand F satisfying the lower semicontinuity and p-coercivity*

conditions of the theorem, are in fact global minimisers locally in Ω. The observation that C^2 critical points of I with certain strong $W^{1,p}$-quasiconvex integrands F are absolutely minimising on small balls with centres in Ω is due to ZHANG [66].

(ii) If $\overline{u} \in W^{1,q}(\Omega, \mathbb{R}^N)$ is a $W^{1,q}$-local minimiser of $I[\cdot, \Omega]$ then it is not necessarily a global minimiser locally in Ω due to the p-coercivity condition, (2.13), not guaranteeing a bound in $W^{1,q}$ for $q > p$.

Given the above, the class of local minimisers that are not subsumed in to the theory of global minimisers are those $W^{1,q}$-local minimisers, $\overline{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$, where $q > p$. Indeed by Hölder's inequality,

$$\|Du - D\overline{u}\|^q_{L^q(\Omega, \mathbb{R}^{N\times n})} \leq |\Omega|^{(1-\frac{1}{r})} \|Du - D\overline{u}\|^{qr}_{L^{qr}(\Omega, \mathbb{R}^{N\times n})}, \quad \text{for any } r \geq 1,$$

where the norm on the right hand side of the inequality is bounded by $\|Du - D\overline{u}\|^p_{L^p(\Omega, \mathbb{R}^{N\times n})}$ only if $q \leq p$. In which case there exists a $\delta > 0$ bound as dictated by the theorem. The first partial regularity results for such minimisers with the condition that the minimisers also belong to $W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)$ were also presented in Kristensen and Taheri [42] along with an example of a strong L^1- local minimiser, $\overline{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$, on an annulus, Ω, that is not also a global minimiser. In a second paper Taheri [61] proved uniqueness for stationary points with affine boundary values, and thus uniqueness of $W^{1,p}$-local minimisers, where F strongly quasiconvex and I is defined on the Dirichlet class $W^{1,p}_{u_0}(\Omega, \mathbb{R}^N)$, where Ω is a star shaped domain. In particular as a consequence of [61], a $W^{1,p}$-local minimiser on a star shaped domain with affine boundary values, u_0, is affine (and thus coincides with u_0). However for the case $p < q < \infty$, existence of $W^{1,q}$-local minimisers that are not global seems to be an open problem (see [42, pp65-66]). In the paper [20], focusing on partial regularity, we made an improvement on the additional $W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)$ condition of [42] mentioned above, and that for certain classes of minimiser showed this improvement to be necessary for their partial regularity. We will discuss this in full in Chapter 5.

2.2 Basic Properties of Quasi-Convexity.

In this section we shall discuss a collection of auxiliary background results for the various classes of convex functions encountered in the calculus of variations.

Theorem 2.6. *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous. Then*

(i) F convex $\implies$ F polyconvex $\implies$ f quasiconvex $\implies$ f rank-one convex.

(ii) If $N = 1$ or $n = 1$ all the above notions of convexity are equivalent.

(iii) If $F \in C^2(\mathbb{R}^{N\times n})$, then rank-one convexity is equivalent to the Legendre-Hadamard condition

$$\sum_{i,j,\alpha,\beta} \frac{\partial F}{\partial \xi_i^\alpha \xi_j^\beta}(D\overline{u})\eta_i\zeta^\alpha\eta_j\zeta^\beta \geq 0. \tag{2.15}$$

for every $\eta \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times n}$.

(iv) If $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is rank-one convex, F is locally Lipschitz.

Proof. We will prove (iv) here. Note that for (i) we have already shown that quasiconvexity implies rank-one in the proof of Theorem 2.1. The remaning parts are straight forward, see for example [15]. Proof of (iv): We rewrite F as a function $F\colon \mathbb{R}^d \to \mathbb{R}$ with $d := nN$. Then let F be separately convex (see Definition 2.2 (iv)). Following [15] let

$$|\xi|_\infty := \max\{\xi_i \text{ and } \alpha = 1, \ldots, nN\}$$

Step 1: We first prove that if $\xi \in \text{int}(\text{dom F})$, then F is bounded from above in a neighbourhood of ξ. Without loss of generality suppose that $\xi = 0$. Thus as $0 \in \text{int}(\text{dom F})$, there exists an $\epsilon > 0$ such that

$$\{\xi \in \mathbb{R}^{nN} : |\xi|_\infty \leq \epsilon\} \subset \text{dom F}. \tag{2.16}$$

Now setting

$$a := \max\{F(\epsilon_1, \epsilon_2, \ldots, \epsilon_{nN}) : \epsilon_i = -\epsilon, 0, \epsilon, \text{ for every } i = 1, \ldots, nN\}$$

we find from (2.16) that $a < +\infty$. We now claim that

$$|\xi|_\infty \leq \epsilon \implies F(\xi) \leq a. \tag{2.17}$$

In order to prove the claim (2.17), observe that if $0 \leq \xi_{nN} \leq \epsilon$ and $\epsilon_i = -\epsilon, 0, \epsilon$ then the separate convexity of F with respect to the last variable implies that

$$F(\epsilon_1, \epsilon_2, \ldots, \epsilon_{nN-1}, \xi_{nN}) \leq \frac{\xi_{nN}}{\epsilon} F(\epsilon_1, \ldots, \epsilon_{nN-1}, \epsilon) + (1 - \frac{\xi_{nN}}{\epsilon}) F(\epsilon_1, \ldots, \epsilon_{nN-1}, 0)$$

$$\leq \frac{\xi_{nN}}{\epsilon} a + (1 - \frac{\xi_{nN}}{\epsilon}) a = a$$

Using the above inequality and the separate convexity of F with respect to ξ_{nN-1} and letting $0 \leq \xi_{nN-1} \leq \epsilon$ we have

$$F(\epsilon_1, \epsilon_2, \ldots, \epsilon_{nN-2}, \xi_{nN-1}, \xi_{nN}) \leq \frac{\xi_{nN-1}}{\epsilon} F(\epsilon_1, \ldots, \epsilon_{N-2}, \epsilon, \xi_{nN})$$

$$+ (1 - \frac{\xi_{nN-1}}{\epsilon}) F(\epsilon_1, \ldots, \epsilon_{N-2}, 0, \xi_{nN})$$

$$\leq a.$$

Thus iterating the process with respect to all the variables we arrive at (2.17) provided $\xi_i \geq 0$. We can use a similar argument If any ξ_i are negative. Hence we have claim (2.17) implying that if $\xi \in \text{int}(\text{dom F})$ then F is bounded from above in a neighbourhood of ξ completing step 1.

Step 2: We next claim that if $\xi \in \text{int}(\text{dom F})$ then F is continuous at ξ. Once again without loss of generality assume $\xi = 0$ and $F(0) = 0$. Since F is bounded above in a neighbourhood of $\xi = 0$, there exists a $\lambda > 0$ and $a > 0$ such that

$$|\xi|_\infty \leq \lambda \implies F(\xi) \leq a. \tag{2.18}$$

Fix $\epsilon > 0$ and without loss of generality assume that $\epsilon \leq anN2^{nN}$ (otherwise choose a even larger). We now show that

$$|\xi|_\infty \leq \frac{\epsilon}{anN2^{nN}} \lambda \implies |F(\xi)| \leq \epsilon. \tag{2.19}$$

We let

$$\delta := \frac{\epsilon}{anN2^{nN}} \leq 1.$$

Using the separate convexity of F, we have

$$F(\xi) = (\xi_1, \ldots, \xi_{nN}) = F\left(\delta(\frac{\xi_1}{\delta}, \xi_2, \ldots, \xi_{nN}) + (1 - \delta)(0, \xi_2, \ldots, \xi_{nN})\right)$$

$$\leq \delta F\left(\frac{\xi_1}{\delta}, \xi_2, \ldots, \xi_{nN}\right) + (1 - \delta) F\left(0, \xi_2, \ldots, \xi_{nN}\right)$$

Repeating the process on the second term on the right with the second variable

$$F(\xi) \leq \delta F\left(\frac{\xi_1}{\delta}, \xi_2, \ldots, \xi_{nN}\right) + (1-\delta)\delta F\left(0, \frac{\xi_2}{\delta}, \ldots, \xi_{nN}\right) + (1-\delta)^2 F(0, 0, \xi_3, \ldots, \xi_{nN}).$$

Thus by iteration we obtain

$$F(\xi) \leq \delta \sum_{i=1}^{nN} (1-\delta)^{i-1} F(0, \ldots, 0, \frac{\xi_i}{\delta}, \xi_{i+1} \ldots, \xi_{nN}) + (1-\delta)^{nN} f(0, \ldots, 0).$$

If we now assume that

$$|\xi|_\infty \leq \delta\lambda = \frac{\epsilon\lambda}{anN2^{nN}} \leq \lambda, \tag{2.20}$$

we find given $F(0) = 0$ and the fact that F is bounded from above, (2.18), that

$$F(\xi) \leq \delta a \sum_{i=1}^{nN} (1-\delta)^{1-1} \leq \delta anN \leq \epsilon$$

which is the bound in the inequality bounding F from above in (2.19). To obtain (2.19) completely it remains to show that $F(\xi) \geq -\epsilon$. In a similar way to the above

$$\begin{aligned} 0 &= F(0, \ldots, 0) \\ &= F\left(\frac{1}{1+\delta}(0, \ldots, 0, \xi_{nN}) + \frac{1}{1+\delta}(0, \ldots, 0, \frac{-\xi_{nN}}{\delta}\right) \\ &\leq \frac{1}{1+\delta}\left[F(0, \ldots, 0, \xi_{nN}) + \delta F(0, \ldots, 0, \frac{-\xi_{nN}}{\delta})\right]. \end{aligned}$$

Thus proceeding with the ξ_{nN-1} variable we get

$$\begin{aligned} F(0, \ldots, 0, \xi_{nN}) &= F\left(\frac{1}{1+\delta}(0, \ldots, 0, \xi_{nN-1}, \xi_{nN}) + \frac{\delta}{1+\delta}(0, \ldots, 0, \frac{-\xi_{nN-1}}{\delta}, \xi_{nN})\right) \\ &\leq \frac{1}{1+\delta}\left[F(0, \ldots, 0, \xi_{nN-1}, \xi_{nN}) + \frac{\delta}{1+\delta}(0, \ldots, 0, \frac{-\xi_{nN-1}}{\delta}, \xi_{nN}))\right] \end{aligned}$$

and thus combining the two estimates we obtain

$$\begin{aligned} 0 \leq \frac{1}{(1+\delta)^2} F(0, \ldots, 0, \xi_{nN-1}, \xi_{nN}) + \frac{\delta}{(1+\delta)^2} F(0, \ldots, 0, -\frac{\xi_{nN-1}}{\delta}, \xi_{nN}) \\ + \frac{\delta}{1+\delta} F(0, \ldots, 0, \frac{-\xi_{nN}}{\delta}). \end{aligned}$$

Iterating we deduce that

$$0 \leq \frac{1}{(1+\delta)^N} F(\xi_1, \ldots, \xi_{nN}) + \sum_{i=1}^{nN} \frac{\delta}{(1+\delta)^{N-i+1}} F(0, \ldots, 0, \frac{-\xi_i}{\delta}, \xi_{i+1}, \ldots, \xi_{nN}).$$

So if

$$|\xi|_\infty \leq \delta\lambda = \frac{\epsilon\lambda}{anN2^{nN}} \leq \lambda,$$

by (2.18) we have

$$F(\xi_1, \ldots, \xi_{nN}) \geq -\delta \sum_{i=1}^{nN} (1+\delta)^{i-1} F(0, \ldots, 0, -\frac{\xi_i}{\delta}, \xi_{i+1}, \ldots, \xi_{nN})$$

$$\geq -\delta a \sum_{i=1}^{nN} (1+\delta)^{i-1} \geq -\delta anN2^N = -\epsilon.$$

From the above inequality we infer that

$$|\xi|_\infty \leq \frac{\epsilon}{anN2^{nN}}\lambda \implies F(\xi) \geq -\epsilon.$$

Thus (2.19) holds implying the continuity of F at $\xi = 0$.

Step 3. Finally we show that F is locally Lipschitz in the interior of the domain F. Let $\xi \in \text{int}(\text{dom F})$. From continuity of F at x, there exists an $\alpha, \beta > 0$ such that

$$|\xi - \zeta|_\infty \leq 2\beta \implies |F(\zeta)| \leq \alpha < +\infty. \tag{2.21}$$

Let z and z_1 be such that

$$|z_1 - z|_\infty, \, |z_1 - \xi|_\infty \leq \beta, \tag{2.22}$$

implying that $|z - \xi|_\infty \leq 2\beta$. Therefore (2.21) and (2.22) lead to

$$|z_1 - z|_\infty, \, |z_1 - \xi|_\infty \leq \beta \implies F(z) - F(z_1) \leq 2\alpha. \tag{2.23}$$

Let $\epsilon > 0$ be chosen later. Combining (2.23) and (2.19) of step 2 we have

$$|z_1 - z|_\infty, \, |z_1 - \xi|_\infty \leq \frac{\beta\epsilon}{2\alpha nN2^{nN}} \implies |F(z) - F(z_1)| \leq \epsilon. \tag{2.24}$$

Choosing

$$\epsilon := \frac{2\alpha nN2^{nN}}{\beta} |z_1 - z|_\infty \tag{2.25}$$

we obtain from (2.22) and (2.24) the following

$$|z_1 - z|_\infty,\, |z_1 - \xi|_\infty \le \beta \implies |F(z) - F(Z_1)| \le \frac{2\alpha n N 2^{nN}}{\beta} |z_1 - z|_\infty. \tag{2.26}$$

Now let z_2 be such that $|z_2 - \xi|_\infty \le \beta$. Let

$$u_1, u_2, \ldots, u_M \in [z_1, z_2]$$

(the segment in $\mathbb{R}^{nN}$ with endpoints z_1 and z_2) be such that

$$u_1 = z_1, u_2, \ldots, u_M = z_2 \text{ and } |u_m - u_{m+1}|_\infty \le \beta, \quad m = 1, \ldots, M-1.$$

Since $|z_1 - \xi|_\infty,\, |z_2 - \xi|_\infty \le \beta$, then

$$|u_m - x|_\infty \le \beta, \quad m = 1 \ldots M.$$

Thus using (2.26), we immediately get

$$|u_m - u_{m+1}|_\infty \le \beta \implies |F(u_m) - F(u_{m+1})| \le \frac{2\alpha n N 2^{nN}}{\beta} |u_m - u_{m+1}|_\infty.$$

Hence summing the above inequalities, we obtain

$$|z_1 - \xi|_\infty,\, |z_2 - \xi|_\infty \le \beta \implies |F(z_1) - F(z_2)| \le \frac{2\alpha n N 2^{nN}}{\beta} |z_1 - z_2|_\infty$$

proving the result. □

From Lemma 3.1 of the next chapter and its corollary we can conclude that for any $I[\cdot;\Omega]$ with quadratic rank-one convex integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$, F is $W^{1,2}$-quasiconvex and the integral functional $I[u;\Omega]$ is convex on the Dirichlet class $u \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$. However in a fundamental result by ŠVERÁK [63] it was shown that rank-one convexity does not in general imply quasiconvexity. In particular [63] provides a counter example to the hypothesis that rank-one convexity implies quasiconvexity, in the form of a quartic polynomial on $\mathbb{R}^{N\times n}$ when $N \ge 3$, $n \ge 2$ which is rank-one convex but not quasi-convex. The question of the validity of the hypothesis remains open for the case $N = 2$, $n \ge 2$.

In fact it is known that for $N \ge 3$, $n \ge 2$ there can not even be a local condition which is equivalent to quasiconvexity within the class of C^∞ functions, see [38]. The same is true for polyconvexity, see [39].

The notion of quasiconvexity therefore is a bit of a mystery. There are many functions in the literature that are known to be rank-one convex, but not polyconvex, and where the issue of their quasiconvexity is completely open and related to some deep questions in harmonic analysis and geometric function theory [6,34,35]. However, for $N = n = 2$ there are some interesting recent positive results, see [52] and [24].

Despite the underlying mystery around quasiconvexity of many rank-one functions, we are able to present a simple positive result that appears to be neglected in the literature.

Proposition 2.1. *Rank-one convexity $\implies$ quasiconvexity for rank-one polynomials $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ of degree 3.*

Proof. Let $t > 0$. Then $\xi \mapsto t^{-3}F(\pm t\xi)$ are rank-one convex. As $t \to \infty$,

$$t^{-3}F(\pm t\xi) \to F_3(\pm\xi) = \pm F_3(\xi)$$

point-wise in ξ, where

$$F(\xi) = \sum_{|\alpha|\leq 3} c_\alpha \xi^\alpha$$

and

$$F_3(\xi) = \sum_{|\alpha|=3} c_\alpha \xi^\alpha.$$

Thus both $\pm F_3$ are rank-one convex. But then F_3 is rank-one affine and hence poly-affine (see [15]). In conclusion $F(\xi) = \sum_{|\alpha|\leq 2} c_\alpha \xi^\alpha + F_3(\xi)$ is quasiconvex by Theorem 2.6, (i). □

For the purposes of regularity we need to strengthen the quasiconvexity condition. Accordingly we define:

Definition 2.4 (Strong Quasiconvexity)**.** *If F is called strongly p-quasiconvex for some constant $\nu > 0$, every $\xi \in \mathbb{R}^{N\times n}$ and every $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{N}})$,*

$$\nu \int_{\mathbb{R}^n} (|D\varphi|^2 + |D\varphi|^p)\, dx \leq \int_{\mathbb{R}^n} (F(\xi + D\varphi) - F(\xi))\, dx \text{ when } p \geq 2 \tag{2.27}$$

$$\nu \int_{\mathbb{R}^n} (1 + |\xi|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2\, dx \leq \int_{\mathbb{R}^n} (F(\xi + D\varphi) - F(\xi))\, dx \text{ when } 1 < p < 2. \tag{2.28}$$

The conditions (2.27) and (2.28) of the definition are known as strong quasiconvexity and were first introduced by EVANS in his paper on partial regularity of absolute minimisers of $I[\cdot]$ ($p \geq 2$), [22]. He called it uniform strict quasiconvexity. Note that for $p \geq 2$ (2.27) is the weaker of the two conditions. Strong quasiconvexity in the form of (2.28) was used to prove partial regularity of absolute minimisers in the subquadratic, $1 < p < 2$, case, by CAROZZA, FUSCO and MINGIONE [14] and later in [13] for local minimisers.

It follows that if F is strongly quasiconvex then it is strongly rank-one convex and the associated Legendre-Hadamard condition is:

$$\begin{cases} F''(\xi)[\lambda,\lambda] \geq \nu|\lambda|^2, & p \geq 2, \\ F''(\xi)[\lambda,\lambda] \geq \nu(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, & 1 < p < 2, \end{cases} \tag{2.29}$$

for every $\xi \in \mathbb{R}^{N\times n}$ and all $\lambda \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\lambda) \leq 1$.

Chapter 3

Hölder regularity, Morrey and Campanato spaces.

3.1 Morrey-Campanato spaces and spaces of bounded mean oscillation.

In this section we will introduce three closely related spaces and summarise how they can be used to characterise Hölder continuity within the space of Lebesgue integrable functions. A more general discussion can be found in [28]. The first of these is due to Morrey [49]:

Definition 3.1 (Morrey Space)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded define $\Omega(x_0, R) := \Omega \cap B(x_0, R)$. Then for $p > 1$ and $\mu \geq 0$ the Morrey space $L^{p,\mu}(\Omega)$ [12, 28], consists of all $f \in L^p_{\text{loc}}(\Omega)$ such that*

$$\|f\|_{p,\mu,\Omega} := \sup_{\substack{x_0 \in \Omega \\ 0<R<\text{diam}(\Omega)}} \left(\frac{1}{R^\mu} \int_{\Omega(x_0,R)} |f|^p dx \right)^{\frac{1}{p}} < \infty.$$

We say that f is locally $L^{p,\mu}$ in Ω, denoted $f \in L^{p,\mu}_{\text{loc}}(\Omega)$, if for each open Ω' compactly contained in Ω, $\|f\|_{p,\mu,\Omega'} < \infty$.

Remark 3.1. *$L^{p,n}$ is isomorphic to L^∞.*

To understand how this space relates to Hölder continuity we introduce the following space due to Campanato:

Definition 3.2 (Campanato Space)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and define $\Omega(x_0, R) := \Omega \cap B(x_0, R)$. Then for $p > 1$ and $\mu \geq 0$ the Campanato space $\mathcal{L}^{p,\mu}(\Omega)$ [12, 28], consists of all $f \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ such that*

$$[f]_{p,\mu,\Omega} := \sup_{\substack{x_0 \in \Omega \\ 0<R<\mathrm{diam}(\Omega)}} \left(\frac{1}{R^\mu} \int_{\Omega(x_0,R)} |f - f_{x_0,R}|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The $\mathcal{L}^{p,\mu}(\Omega)$-norm is given by

$$\|f\|_{p,\mu,\Omega} \equiv \|f\|_{\mathrm{L}^p} + [f]_{p,\mu,\Omega}.$$

We say that f is locally $\mathcal{L}^{p,\mu}$ in Ω, denoted $f \in \mathcal{L}^{p,\mu}_{\mathrm{loc}}(\Omega)$, if for each open Ω' compactly contained in Ω, $[f]_{p,\mu,\Omega'} < \infty$.

Using these definitions we can proceed to describe the so called Campanato characterisation of Hölder continuity which will be central to the regularity proof for linear elliptic systems, the Schauder estimates, discussed in full in the next section and a fundamental component of our regularity program for nonlinear systems (linearisation, perturbation, comparison). These spaces together with the space of functions of bounded mean oscillation defined below, which is closely related to a special case of Campanato (see Proposition 3.3 of this section), also play a central role in our main result discussed in the final chapter. It turns out that they are deeply involved in the partial regularity of local minimisers in the vectorial case. The definition of the space of functions of bounded mean oscillation on an open and bounded set $\Omega \subset \mathbb{R}^n$ or the entire space $\mathbb{R}^n$ is as follows:

Definition 3.3 (BMO Space)**.** *Let Ω be open and bounded or the entire space $\mathbb{R}^n$. Then the John-Nirenberg space* $\mathrm{BMO}(\Omega)$ *[28, 36] consists of all $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that*

$$[f]_{*,\Omega} := \sup_{B \subset \Omega} \left(\fint_B |f - f_B| \, dx \right) < \infty,$$

where the supremum is taken over all open balls contained in Ω. The $\mathrm{BMO}(\Omega)$*-norm is given by*

$$\|f\|_{*,\Omega} \equiv \|f\|_{\mathrm{L}^1(\Omega)} + [f]_{*,\Omega}.$$

We say that f is locally BMO *in Ω if for each open Ω' compactly contained in Ω, $[f]_{*,\Omega'} < \infty$.*

Notation We have used $f_{x_0,R}$ to denote the integral average of f over $\Omega(x_0, R)$

$$f_{x_0,R} = \fint_{\Omega(x_0,R)} f = \frac{1}{|\Omega(x_0, R)|} \int_{\Omega(x_0,R)} f(x)\, dx.$$

Depending on the context we may also write $f_{x,r}$ for the average over the ball $B = B(x, r)$. Alternatively we may write this as f_B and we will denote the unit ball as $B_1 = B(0, 1)$ to avoid confusion with B. We may also drop the x_0 in $\Omega(x_0, R)$ and $f_{x_0,R}$ and use the short hand Ω_R and f_R where appropriate. Note that the same definitions apply verbatim to vector valued functions with $|\cdot|$ denoting the Euclidian norm.

The relationship between Morrey and Campanato spaces can be summarised as follows. For $\mu < n$ and a sufficiently regular boundary $\partial\Omega$, the Campanato space $\mathcal{L}^{p,\mu}(\Omega)$ is equivalent to the Morrey space $\mathrm{L}^{p,\mu}(\Omega)$ In this case we refer to the space as *Morrey-Campanato space*. The inclusion $\mathrm{L}^{p,\mu}(\Omega) \hookrightarrow \mathcal{L}^{p,\mu}(\Omega)$ is a trivial result of

$$\fint_{\Omega(x_0,R)} |f - f_{x_0,R}|^p \leq 2^p \inf_{\xi\in\mathbb{R}^n} \fint_{\Omega(x_0,R)} |f - \xi|^p \tag{3.1}$$

and holds for all open Ω. Note the inequality (3.1) follows directly from the Minkowski inequality. For the opposite inclusion some work is required to derive the relevant inequality,

$$\|f\|_{p,\mu,\Omega} \leq c(n, \Omega, p, \mu) \left(\mathrm{diam}(\Omega)^{-\frac{\mu}{p}} \|f\|_{p,\Omega} + [f]_{p,\mu,\Omega}\right) \tag{3.2}$$

which only holds for exponents $0 \leq \mu < n$ and for domains without external cusps, e.g domains with Lipschitz boundary (see [28, §2.3]). To properly frame the conditions necessary for (3.2) we must define what we mean by sufficient regularity of the boundary for the open and bounded set $\Omega \subset \mathbb{R}^n$. We do this with the following measure density condition:

Definition 3.4 (No External Cusps.)**.** *We say that the set Ω has no external cusps if there exists a constant $A > 0$ such that for every $x_0 \in \overline{\Omega}$ and every $r \in (0, \mathrm{diam}(\Omega)]$ we have*

$$|\Omega(x_0, r)| \geq A|B(x_0, r)|.$$

We now can state the following proposition

Proposition 3.1. *Let $f \in L^p(\Omega, \mathbb{R}^N)$ and suppose Ω has no external cusps. Then for $0 \leq \mu < n$*

$$\|f\|_{p,\mu,\Omega} \leq c(n, A, p, \mu) \left(\operatorname{diam}(\Omega)^{-\frac{\mu}{p}} \|f\|_{p,\Omega} + [f]_{p,\mu,\Omega} \right). \tag{3.3}$$

As a direct result of (3.1) and the above proposition we have the following corollary:

Corollary 3.1. *If Ω is bounded open, has no external cusps, and if $0 \leq \mu < n$, then $L^{p,\mu}(\Omega, \mathbb{R}^N)$ is isomorphic to $\mathcal{L}^{p,\mu}(\Omega, \mathbb{R}^N)$.*

Proof of Proposition 3.1.

$$\left(\frac{1}{r^\mu} \int_{\Omega_r} |f|^p dx \right)^{\frac{1}{p}} \leq \frac{1}{r^{\frac{\mu}{p}}} \left(\left(\int_{\Omega_r} |f - f_r|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega_r} |f_r|^p dx \right)^{\frac{1}{p}} \right) \tag{3.4}$$

Estimating the second term using $|\Omega_r| \leq |B_r|$,

$$\begin{aligned} \left(\frac{1}{r^\mu} \int_{\Omega_r} |f_r|^p dx \right)^{\frac{1}{p}} &= \frac{1}{r^{\frac{\mu}{p}}} |\Omega_r|^{\frac{1}{p}} |f_r| \\ &\leq r^{\frac{n-\mu}{p}} |B_1|^{\frac{1}{p}} |f_r|. \end{aligned}$$

Let $R = \operatorname{diam}(\Omega)$. Introducing f_R into the above we have

$$\left(\frac{1}{r^\mu} \int_{\Omega_r} |f_r|^p dx \right)^{\frac{1}{p}} \leq r^{\frac{n-\mu}{p}} |B_1|^{\frac{1}{p}} \left(|f_R - f_r| + |f_R| \right). \tag{3.5}$$

To estimate $|f_R - f_r|$ we split it further

$$|f_R - f_r| \leq |f_{2^{-k}R} - f_R| + |f_r - f_{2^{-k}R}| \tag{3.6}$$

for some $k = 0, 1 \ldots$. Estimating $|f_{2^{-k}R} - f_R|$ we have

$$\begin{aligned} |f_{2^{-k}R} - f_R| &\leq \sum_{i=1}^{k} |f_{2^{-i}R} - f_{2^{-i+1}R}| \\ &\leq \sum_{i=1}^{k} \left(\fint_{\Omega_{2^{-i}R}} |f - f_{2^{-i+1}R}|^p dx \right)^{\frac{1}{p}}. \\ &\leq \sum_{i=1}^{k} \left(\frac{|\Omega_{2^{-i+1}R}|}{|\Omega_{2^{-i}R}|} \fint_{\Omega_{2^{-i+1}R}} |f - f_{2^{-i+1}R}|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Next using Definition 3.4 for sets without external cusps,

$$\begin{aligned} |f_{2^{-k}R} - f_R| &\leq \sum_{i=1}^{k} A^{-\frac{1}{p}} 2^{\frac{n}{p}} [u]_{p,\alpha p+n;\Omega} (2^{-i+1}R)^{\alpha} (A|B_1|)^{-\frac{1}{p}} \\ &= c(A,n,p,|B_1|)[u]_{p,\alpha p+n;\Omega} \sum_{i=1}^{k} (2^{-i+1})^{\alpha} R^{\alpha}. \end{aligned} \tag{3.7}$$

Referring back to the second term of (3.5)

$$|B_1|^{\frac{1}{p}} r^{\frac{n-\mu}{p}} |f_R| \leq |B_1|^{\frac{1}{p}} r^{\frac{n-\mu}{p}} \fint_{\Omega_R} |f| dx \leq |B_1|^{\frac{1}{p}} (A|B_1|)^{-\frac{1}{p}} r^{\frac{n-\mu}{p}} R^{-\frac{n}{p}} \|f\|_{p;\Omega},$$

which provided $\mu \leq n$, is bounded by

$$c(A,p,|B_1|)\mathrm{diam}(\Omega)^{-\frac{\mu}{p}} \|f\|_{p;\Omega}. \tag{3.8}$$

Now setting $\mu = \alpha p + n$, implies $\alpha = \frac{\mu - n}{p} \leq 0$. Referring back to (3.7) we see that when $\alpha = 0$, the inequality cannot provide a uniform bound for $|f_{2^{-k}R} - f_R|$ over k. Since in the following we must allow for k to be arbitrarily large we are forced to set $\alpha < 0$ translating to $\mu < n$ (indeed for $\mu = n$ the proposition is false see Remark 3.2). Now noting that $r < 2^{-k}R$ for some $k = 1, 2, \ldots$, otherwise $|f_{2^{-k}R} - f_R| = 0$ (case $k = 0$), we have

$$R^{\alpha} < 2^{k\alpha} r^{\alpha}.$$

Therefore

$$|f_{2^{-k}R} - f_R| \leq c[f]_{p,\mu;\Omega} \sum_{i=1}^{k} (2^{k-i+1})^{\alpha} r^{\alpha},$$

where the sum

$$\sum_{i=1}^{k} (2^{k-i+1})^{\alpha} = 2^{\alpha} \sum_{j=1}^{k} (2^{\alpha})^{j} = 2^{\alpha} \frac{1 - (2^{\alpha})^{k}}{1 - 2^{\alpha}} < \frac{2^{\alpha}}{1 - 2^{\alpha}}.$$

Thus

$$|f_{2^{-k}R} - f_R| \leq c(A,n,p,|B_1|,\mu)[f]_{p,\mu;\Omega} \cdot r^{\frac{\mu - n}{p}} \tag{3.9}$$

for arbitrary $k \in \mathbb{N}_0$.

Next to estimate $|f_r - f_{2^{-k}R}|$ in (3.5) we choose $k \in \mathbb{N}_0$ such that $2^{-k-1}R \leq r < 2^{-k}R$. We have

$$\begin{aligned} |f_r - f_{2^{-k}R}| &\leq \left(\fint_{\Omega_r} |f - f_{2^{-k}R}|^p \right)^{\frac{1}{p}} \\ &\leq \left(\frac{|\Omega_{2^{-k}R}|}{|\Omega_r|} \fint_{\Omega_{2^{-k}R}} |f - f_{2^{-k}R}|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Thus by our choice of k, $|\Omega_r| \geq |\Omega_{2^{-k-1}R}|$ and

$$|f_r - f_{2^{-k}R}| \leq \left(\frac{|\Omega_{2^{-k}R}|}{|\Omega_{2^{-k-1}R}|}\right)^{\frac{1}{p}} \cdot (A|B_1|)^{-\frac{1}{p}}[f]_{p,\alpha p+n;\Omega} \cdot (2^{-k}R)^{\alpha}$$

$$\leq 2^{\frac{n}{p}} A^{-\frac{1}{p}} \cdot (A|B_1|)^{-\frac{1}{p}}[f]_{p,\mu;\Omega} \cdot (2^{-k}R)^{\alpha}$$

where we have used Definition 3.4 as before. Finally given $\alpha < 0$ and our choice of k, $(2^{-k}R)^{\alpha} < r^{\alpha}$ and

$$|f_r - f_{2^{-k}R}| \leq c(A, n, p, |B_1|)[f]_{p,\mu;\Omega} \cdot r^{\frac{\mu-n}{p}} \tag{3.10}$$

for $k \in \mathbb{N}_0$ such that $2^{-k-1}R \leq r < 2^{-k}R$.

Hence setting $k \in \mathbb{N}_0$ such that $2^{-k-1}R \leq r < 2^{-k}R$, combining (3.9) and (3.10) in (3.6) with (3.5),

$$\left(\frac{1}{r^{\mu}} \int_{\Omega_r} |f_r|^p dx\right)^{\frac{1}{p}} \leq c(A, n, p, |B_1|, \mu)[f]_{p,\mu;\Omega} + r^{\frac{n-\mu}{p}} |B_1|^{\frac{1}{p}} |f_R|.$$

Thus, given the L^p-norm bound (3.8) on the second right hand term above, the result follows from (3.4) by taking the supremum over $0 < r < R := \operatorname{diam}(\Omega)$ and $x_0 \in \Omega$. □

Remark 3.2. *The isomorphism does not hold in general for the case $\mu = n$. For a counter example take $n = N = 1$. Then $log(x)$ belongs to $\mathcal{L}^{1,1}((0,1))$ but not to $L^{1,1}((0,1))$ since $L^{1,1}((0,1))$ is isomorphic to $L^{\infty}((0,1))$.*

3.1.1 Morrey, Campanato embeddings and BMO.

We now discuss some more well known relationships between Morrey, Campanato and BMO spaces important for our main result descussed in the final chapter (for further reading see [28, §2.3-2.4]). The following proposition provides the inequality between Morrey space norms (Campanato space semi-norms) of different exponents and is easily derived with Hölders inequality:

Proposition 3.2 (Morrey-Campanato embeddings)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $1 \leq p \leq q < \infty$ and $\frac{n-\mu}{p} - \frac{n-\nu}{q} \geq 0$ then $\mathrm{L}^{q,\nu}(\Omega)$ is continuously embedded in $\mathrm{L}^{p,\mu}(\Omega)$ and $\mathcal{L}^{q,\nu}(\Omega)$ is continuously embedded in $\mathcal{L}^{p,\mu}(\Omega)$ with*

$$\|f\|_{p,\mu,\Omega} \leq c \cdot \operatorname{diam}(\Omega)^{\frac{n-\mu}{p} - \frac{n-\nu}{q}} \|f\|_{q,\nu,\Omega} \quad , \quad f \in \mathrm{L}^{q,\nu}(\Omega) \tag{3.11}$$

and

$$[f]_{p,\mu,\Omega} \leq c \cdot \operatorname{diam}(\Omega)^{\frac{n-\mu}{p}-\frac{n-\nu}{q}} [f]_{q,\nu,\Omega} \quad , \quad f \in \mathcal{L}^{q,\nu}(\Omega) \tag{3.12}$$

respectively, where c is a positive constant depending only on n, μ, ν, p and q.

Proof. By Hölder with $p', q' \geq 1$

$$\frac{1}{p'} + \frac{1}{q'} = 1$$

and $p' = q/p$ $(q \geq p)$ we have

$$\begin{aligned}
\rho^{-\mu} \int_{\Omega(x_0,\rho)} |f|^p &\leq c(n,\mu) |\Omega(x_0,\rho)|^{-\frac{\mu}{n}} |\Omega(x_0,\rho)|^{\left(1-\frac{p}{q}\right)} \left(\int_{\Omega(x_0,\rho)} |f|^q \right)^{\frac{p}{q}} \\
&= c(n,\mu,\nu,p,q) |\Omega(x_0,\rho)|^{\frac{p\nu}{nq}-\frac{\mu}{n}+(1-p/q)} \left(\rho^{-\nu} \int_{\Omega(x_0,\rho)} |f|^q \right)^{\frac{p}{q}} \\
&= c \left(|\Omega(x_0,\rho)|^{\frac{1}{n}} \right)^{\frac{p\nu}{q}-\mu+n-\frac{pn}{q}} \left(\rho^{-\nu} \int_{\Omega(x_0,\rho)} |f|^q \right)^{\frac{p}{q}} \\
&= c \left(|\Omega(x_0,\rho)|^{\frac{1}{n}} \right)^{(n-\mu)-\frac{p(n-\nu)}{q}} \left(\rho^{-\nu} \int_{\Omega(x_0,\rho)} |f|^q \right)^{\frac{p}{q}}.
\end{aligned}$$

where c depends only on n, μ, ν, p and q. Given that

$$|\Omega|^{\frac{1}{n}} \geq |\Omega(x_0,\rho)|^{\frac{1}{n}}, \quad \forall \Omega(x_0,\rho)$$

we have, provided $\frac{n-\mu}{p} \geq \frac{n-\nu}{q}$, that

$$\rho^{-\mu} \int_{\Omega(x_0,\rho)} |f|^p \leq c \left(|\Omega|^{\frac{1}{n}} \right)^{(n-\mu)-\frac{p(n-\nu)}{q}} \left(\rho^{-\nu} \int_{\Omega(x_0,\rho)} |f|^q \right)^{\frac{p}{q}}.$$

Hence taking the supremum over all $\Omega(x_0,\rho)$ on both sides concludes the proof of (3.11). The Campanato space equivalent (3.12) follows in the same way. □

The next proposition summarises the relationships between Campanato and BMO spaces:

Proposition 3.3 (Campanato-BMO Isometry)**.** *Let $1 \leq p < \infty$:*

(i) For general Ω open and bounded in $\mathbb{R}^n$, $\mathcal{L}^{p,n}(\Omega)$ is continuously embedded in $\mathrm{BMO}(\Omega)$.

(ii) If $\Omega = B_0$ where B_0 is an arbitrary ball in $\mathbb{R}^n$, $\mathcal{L}^{p,n}(\Omega)$ is isomorphic to $\mathrm{BMO}(\Omega)$.

Proof. Given the open bounded set $\Omega \subset \mathbb{R}^n$, it follows from definitions 3.2 and 3.3 and Proposition 3.2 that

$$[f]_{*,\Omega} \leq \frac{1}{|B_1|}[f]_{1,n,\Omega} \leq \frac{1}{|B_1|}[f]_{p,n,\Omega}$$

for $f \in \mathcal{L}^{p,n}(\Omega)$ proving (i). Given

$$|B_1| \left(\frac{1}{2}r\right)^n \leq |B_0 \cap B|$$

for B of radius $0 < r \leq \operatorname{diam}(B_0)$, centre $x_0 \in B_0$ we may use of a result of [37] that shows $[f]^p_{*,B_0}$ is equivalent to

$$\sup_{B \in \mathbb{R}^n} \fint_{B \cap B_0} |f - f_{B \cap B_0}|\, dx.$$

Thus part (ii) follows from the inequality, bounding $L^p(B, \mathbb{R}^{N \times n})$ by $\mathrm{BMO}(\mathrm{B}_0, \mathbb{R}^{\mathrm{N} \times \mathrm{n}})$,

$$\fint_B |f - f_B|^p \leq c[f]^p_{*,B_0} \tag{3.13}$$

for all $B \subset B_0$. This inequality can be shown with a well known argument, reproduced here for the convenience of the reader, that uses the celebrated result of John and Nirenberg [36]. This result states that for every $f \in \mathrm{BMO}(\mathrm{B}_0)$ and $\sigma > 0$ there exists a positive A and α that are independent of f and σ such that

$$|\lambda_{\sigma,B}| \leq A \exp\left(-\frac{\alpha\sigma}{[f]_{*,B_0}}\right)|B|,$$

where $\lambda_{\sigma,B} := \{x \in B : |f - f_B| > \sigma\}$. Given this we have by standard formula for integrals in terms of distribution functions

$$\begin{aligned}
\int_B |f - f_B|^p &= p \int_0^\infty \sigma^{p-1} |\lambda_{\sigma,B}| d\sigma \\
&\leq pA \int_0^\infty \sigma^{p-1} \exp\left(-\frac{\alpha\sigma}{[f]_{*,B_0}}\right)|B| d\sigma \\
&= A \cdot \left(\frac{[f]_{*,B_0)}}{\alpha}\right)^p |B| \cdot p \int_0^\infty t^{p-1} e^{-t} dt \\
&\leq c_* |B| [f]_{*,B_0},
\end{aligned}$$

where the improper integral of the penultimate estimate is equal to the Gamma function of p. Thus c_* is dependent on p, α and A proving (3.13). □

3.1.2 Campanato Characterisation of Hölder Continuity.

The main interest in Morrey and Campanato spaces centres around the following Lebesgue integral characterisation of Hölder continuity, due to Campanato [12], and Meyers [45]:

Theorem 3.1. *Let Ω be a bounded open set and $f \in \mathcal{L}^{p,\alpha p+n}(\Omega,\mathbb{R}^N)$ for $1 \le p < \infty$, $0 < \alpha \le 1$, then the* precise representative *f^* belongs to $C^{0,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. Further more assuming that Ω is without external cusps $\mathcal{L}^{p,\alpha p+n}(\Omega,\mathbb{R}^N)$ is isomorphic to $C^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$.*

Proof. Step 1. We claim that if $f \in \mathcal{L}^{p,\alpha p+n}(\Omega,\mathbb{R}^N)$, then

$$|f(x_1) - f(x_2)| \le c|x_1 - x_2|^\alpha \tag{3.14}$$

for any Lebesgue points of f, $x_1, x_2 \in B(x, \frac{R}{2})$ such that $B(x,2R) \subset \Omega$.

Let $x_1, x_2 \in B(x,r) \subset \Omega$ be Lebesgue points and estimate $|f(x_1) - f(x_2)|$ by introducing the integral average f_{x_0,r_1} for a second ball $B(x_0,r_1) \subset \Omega$,

$$|f(x_1) - f(x_2)| \le |f(x_1) - f_{x_0,r_1}| + |f(x_2) - f_{x_0,r_1}|. \tag{3.15}$$

It is then a case of estimating $|f(y) - f_{x_0,r_1}|$ for the Lebesgue points $y \in B(x,r)$. Once again to estimate $|f(y) - f_{x_0,r_1}|$ we split it using the triangle inequality,

$$|f(y) - f_{x_0,r_1}| \le |f(y) - f_{y,\frac{r_1}{2}}| + |f_{y,\frac{r_1}{2}} - f_{x_0,r_1}|. \tag{3.16}$$

Starting with the second term on the right hand side we set $B(y,\frac{r_1}{2}) \subset B(x_0,r_1)$ and make the estimate

$$\begin{aligned} |f_{y,\frac{r_1}{2}} - f_{x_0,r_1}| &\le \left(\fint_{B(y,\frac{r_1}{2})} |f - f_{x_0,r_1}|^p \right)^{\frac{1}{p}} \\ &\le \left(\frac{|B(x_0,r_1)|}{|B(y,\frac{r_1}{2})|} \fint_{B(x_0,r_1)} |f - f_{x_0,r_1}|^p \right)^{\frac{1}{p}} \\ &\le 2^{\frac{n}{p}} [f]_{p,\alpha p+n;\Omega} \cdot r_1^\alpha. \end{aligned} \tag{3.17}$$

We are left with the task of estimating the first term of (3.16). To start with we estimate $|f_{y,2^{-j}\frac{r_1}{2}} - f_{y,\frac{r_1}{2}}|$ in a similar fashion to (3.10) in the proof of Proposition 3.1,

$$|f_{y,2^{-k}\frac{r_1}{2}} - f_{y,\frac{r_1}{2}}| \le 2^{\frac{n}{p}} |B_1|^{-\frac{1}{p}} [f]_{p,\alpha p+n;\Omega} \sum_{i=1}^{k} 2^{(-i+1)\alpha} \left(\frac{r_1}{2}\right)^\alpha.$$

Taking $k \to \infty$ the sum on the right hand side converges provided $\alpha > 0$. Thus for every Lebesgue point of f, $y \in B(x,r)$ we have

$$|f(y) - f_{y,\frac{r_1}{2}}| \leq c[f]_{p,\alpha p+n;\Omega} \cdot r_1^{\alpha}, \quad \alpha > 0, \tag{3.18}$$

with $c = \frac{2^{\frac{n}{p}}|B_1|^{-\frac{1}{p}}}{2^{\alpha}-1}$. Thus the result of (3.17), (3.18) and (3.16) is

$$|f(x_1) - f(x_2)| \leq c(n,p,|B_1|,\alpha)[f]_{p,\alpha p+n;\Omega} \cdot r_1^{\alpha}$$

for $\alpha > 0$ and the Lebesgue points of f, x_1 and x_2. Set $r_1 = |x_1 - x_2|$. From the above we must have $B(x_i, \frac{r_1}{2}) \subset B(x_0, r_1) \subset \Omega$ for $x_1, x_2 \in B(x,r) \subset \Omega$, $i = 1,2$. Set $x_0 = \bar{x}$ where $\bar{x} = \frac{x_1+x_2}{2}$ and fix $r = \frac{R}{2}$ then figure 3.1 clearly shows the possible case where $B(x_0, r_1) \nsubseteq B(x,R)$ but that $B(x_0,r_1) \subset B(x, \frac{3\sqrt{2}R}{4}) \subsetneq B(x,2R)$. Thus claim (3.14) follows with c dependent only on n, p, α and $[f]_{p,\alpha p+n;\Omega}$.

Step 2. We will prove all points in $B(x, \frac{R}{2})$ are Lebesgue points of f. Fix any $y \in B(x, \frac{R}{2})$ and for $B(y,r) \subset B(y,s) \subset B(x, \frac{R}{2})$ choose a Lebesgue point $\overline{x} \in B(y, \frac{R}{2})$. Then

$$\begin{aligned} |f_{y,s} - f_{y,r}| &\leq |f_{y,s} - f(\overline{x})| + |f(\overline{x}) - f_{y,r}| \\ &\leq c[f]_{p,\alpha p+n;\Omega} \cdot s^{\alpha} + c[f]_{p,\alpha p+n;\Omega} \cdot r^{\alpha} \end{aligned}$$

i.e. $(f_{y,r})_{r>0}$ is Cauchy in $\mathbb{R}^n$.

Step 3. Finally to extend the result to any pair $x, y \in \Omega' \subset\subset \Omega$ we note that for each Ω' there exists a sufficiently small $R_{\Omega'} > 0$ dependent on $\operatorname{dist}(\Omega', \partial\Omega)$ and a finite covering of balls $B(y_i, \frac{R_{\Omega'}}{4})$ with the property that

$$\overline{\Omega'} \subset \bigcup_{i=1}^{k} B(y_i, \frac{R_{\Omega'}}{4}) \subsetneq \bigcup_{i=1}^{k} B(y_i, R_{\Omega'}) \subset \Omega,$$

and $x_i, x_{i+1} \in B(y_i, \frac{R_{\Omega'}}{4})$, $i = 1 \ldots k$, with $x = x_1$ and $y = x_{k+1}$ ($r_i = |x_i - x_{i+1}| < \frac{R_{\Omega'}}{2}$).

$$|f^*(x) - f^*(y)| \leq \sum_{i=1}^{k} |f^*(x_i) - f^*(x_{i+1})|.$$

Assuming $|x-y| > \frac{R_{\Omega'}}{2}$, otherwise (3.14) trivially holds, we have

$$\sum_{i=1}^{k} |x_i - x_{i+1}|^{\alpha} \leq k|x-y|^{\alpha}.$$

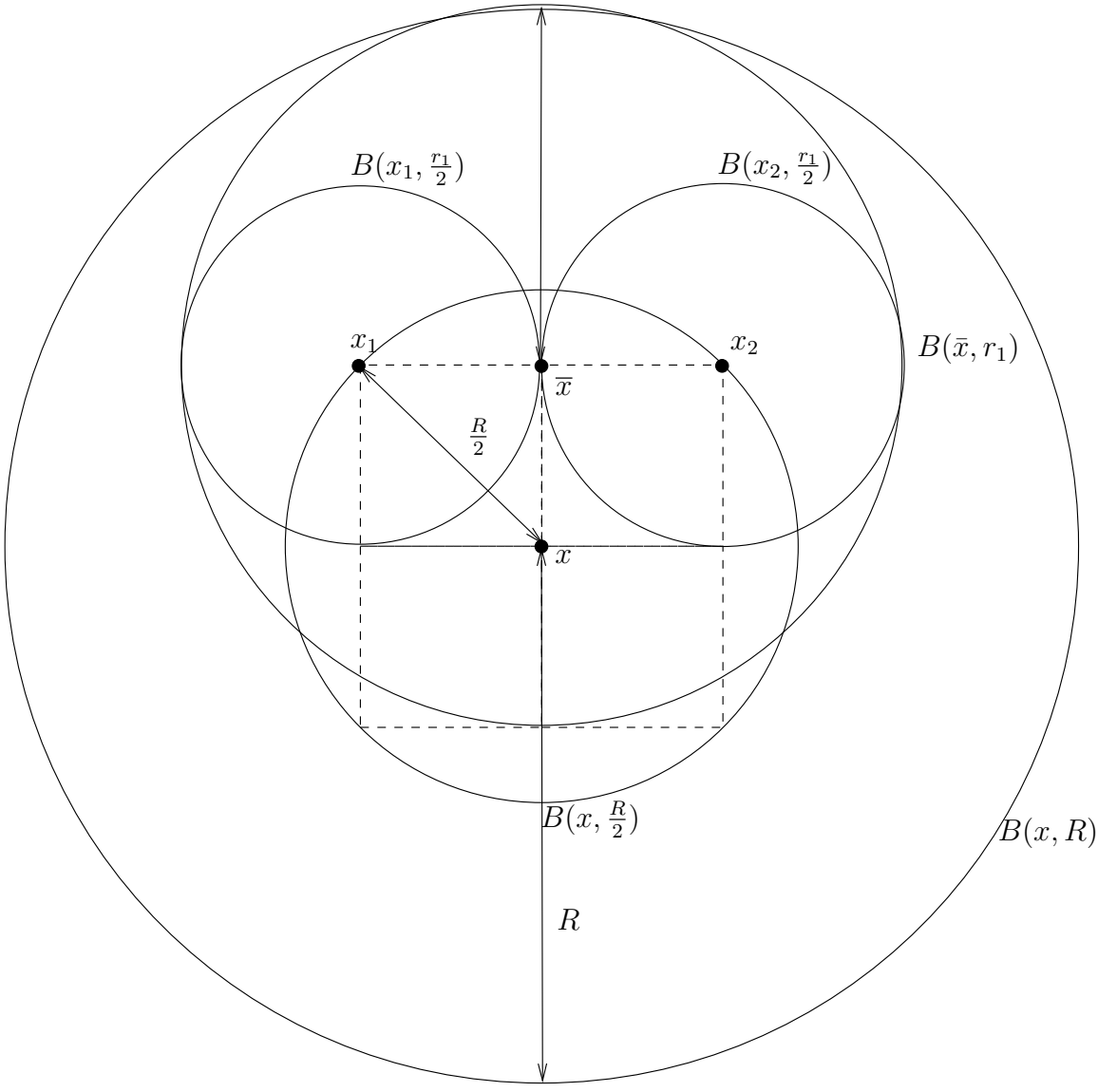

Figure 3.1: The diagram shows a case when $B(\overline{x}, r_1)$ is not quite contained in $B(x, R)$.

for some k dependent on Ω' and Ω, completing the proof for the local Hölder continuity result.

Step 4. To prove $f^* \in C^{0,\alpha}(\overline{\Omega})$ for Ω without external cusps let $x_1, x_2 \in \Omega$. As above set $\bar{x} = \frac{x_1+x_2}{2}$ and $r_1 = |x_1 - x_2|$. It follows that $x_1, x_2 \in \Omega(\bar{x}, r_1)$. In addition $\Omega(x_i, \frac{r_1}{2}) \subset \Omega(\bar{x}, r_1)$ and $\Omega(x_i, 2^{-k}\frac{r_1}{2}) \subset \Omega(x_i, \frac{r_1}{2})$ for $i = 1, 2$. Thus using $|\Omega(x,r)|^{-1} \leq A^{-1}|B(x,r)|^{-1}$ from Definition 3.4 for domains without external cusps, we can proceed by making comparable estimates to those of (3.17) and (3.18). Hence we find

$$|f^*_{x_i,\frac{r_1}{2}} - f^*_{\bar{x},r_1}| \leq 2^{\frac{n}{p}} A^{-\frac{2}{p}} |B_1|^{-\frac{1}{p}} [f]_{p,\alpha p+n;\Omega} \cdot r_1^{\alpha}, \quad i = 1, 2$$

and (3.18) with $y = x_i$, $i = 1, 2$ and $c = \frac{2^{\frac{n}{p}} A^{-\frac{2}{p}} |B_1|^{-\frac{1}{p}}}{2^{\alpha}-1}$. The conclusion then follows from (3.15) and (3.16). □

Remark 3.3. *From the Hölder characterisation of Campanato spaces it is clear that for $\mu > n+p$, the Campanato space $\mathcal{L}^{p,\mu}(\Omega)$ corresponds to the set of constant functions on Ω.*

3.2 Minimisers, Elliptic systems, Regularity of $\mathbb{A}$-Harmonic functions and the Schauder estimates.

In the regularity theory of minimisers of strongly convex or quasiconvex variational integrals a crucial step is to compare the minimiser with the solution to a linear homogeneous elliptic equation or system of equations with constant coefficients. These class of solutions to elliptic systems satisfying the Legendre-Hadamard condition (2.15) of Chapter 2.1, with F'' constant and defined to equal the tensor A, associated with symmetric bilinear form $\mathbb{A} \in \mathcal{L}_s(\mathbb{R}^{N\times n})$, are known as $\mathbb{A}$-Harmonic functions.

Definition 3.5 ($\mathbb{A}$-Harmonic functions)**.** *Let $\mathbb{A} : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a symmetric bilinear form satisfying the Legendre-Hadamard condition*

$$\mathbb{A}[\lambda, \lambda] \geq \nu|\lambda|^2, \quad \text{rank}(\lambda) \leq 1 \tag{3.19}$$

with $\nu \geq 0$. Let $\Omega \subset \mathbb{R}^n$, then $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is $\mathbb{A}$-Harmonic if and only if

$$\int_\Omega \mathbb{A}[Du, D\varphi]dx = 0, \quad \forall \varphi \in C_c^1(\Omega, \mathbb{R}^N). \tag{3.20}$$

Notation. We will write $\mathcal{L}_s^2(\mathbb{R}^{N\times n})$ to denote the space of symmetric bilinear forms on $\mathbb{R}^{N\times n}$. We may also write $\mathbb{A}[\lambda, \eta]$ as the bilinear form $\langle A\lambda, \eta\rangle$, for all $\lambda, \eta \in \mathbb{R}^{N\times n}$, where $A \in L_s^2(\mathbb{R}^{N\times n})$ and $\langle\cdot,\cdot\rangle$ denotes the inner product over $\mathbb{R}^{N\times n}$.

It is straightforward to show that for strongly convex functions the associated elliptic operator satisfies

$$\nu|\lambda|^2 \leq \langle A\lambda, \lambda\rangle \leq L|\lambda|^2, \quad \forall \lambda \in \mathbb{R}^{N\times n}. \tag{3.21}$$

See the following Proposition for details. We can use the above estimate to prove regularity of $\mathbb{A}$-Harmonic functions. However in the case that the associated F is only rank-one convex then the left hand side of (3.21) only holds for rank-one $\lambda \in \mathbb{R}^{N\times n}$, i.e. the Legendre Hadamard condition (3.19). In this case we cannot use the lefthand side of (3.21) directly and we need the following classical lemma saying that for quadratic forms, rank-one convexity implies quasiconvexity. Here we follow the presentation found in [28]:

Lemma 3.1 (The Gårding Inequality)**.** *Let $A \in L_s^2(\mathbb{R}^{N\times n})$ be a constant tensor satisfying Legendre-Hadamard condition (3.19) with $\nu \geq 0$. Then for every $\zeta \in W_0^{1,2}$*

$$\int \mathbb{A}[D\zeta, D\zeta]dx \geq \nu \int |D\zeta|^2 dx. \tag{3.22}$$

The proof uses the Fourier transform.

Proof. In (3.19) we make the extension to complex rank-one matrices of the form $\eta = A \otimes a$ where $A \in \mathbb{C}^N$ and $a \in \mathbb{R}^n$. Thus $\mathbb{A}[\eta, \overline{\eta}] \geq |\eta|^2$ where $\overline{\eta}$ denotes the complex conjugate of η with $|\eta|^2 = \langle \eta, \overline{\eta}\rangle$. We estimate the integral as follows

$$\int \mathbb{A}[D\zeta(x), D\zeta(x)]\, dx = \sum_{i,j,\alpha,\beta} A_{\alpha\beta}^{ij} \int \frac{\partial \zeta^\alpha(x)}{\partial x_i} \frac{\partial \zeta^\beta(x)}{\partial x_j}\, dx,$$

where according to the Plancherel Theorem for the Fourier transformation

$$\int \frac{\partial \zeta^\alpha(x)}{\partial x_i} \frac{\partial \zeta^\beta(x)}{\partial x_j}\, dx = \int y_i y_j [\mathcal{F}\zeta^\alpha]\overline{[\mathcal{F}\zeta^\beta]}\, dy.$$

Thus applying the Legendre-Hadamard condition (3.19) to the complex rank-one matrix $\eta = \mathcal{F}\zeta \otimes y = \{y_i \mathcal{F}\zeta^\beta\}$, we have

$$\int \mathbb{A}[D\zeta(x), D\zeta(x)]\, dx \geq \nu \int |\mathcal{F}\zeta|^2 |y|^2\, dx.$$

Hence the result follows from the Plancherel formula. □

Corollary 3.2. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a rank-one convex quadratic form, and let $g \in W^{1,2}(\Omega, \mathbb{R}^N)$. Then the functional*

$$I[u;\Omega] = \int_\Omega F(Du)\, dx$$

is convex on the Dirichlet class $W^{1,2}_g(\Omega, \mathbb{R}^N)$.

Proof. Take $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ and $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N)$. Then we have

$$\frac{d^2}{dt^2}\Big|_{t=0} I[u + t\varphi;\Omega] = 2I[\varphi;\Omega] \geq 0,$$

where the last inequality follows from Lemma 3.1. □

We require the following standard iteration result that we state as a lemma:

Lemma 3.2. *let $f : [r, R] \to [0, +\infty)$ be bounded and such that for all t and s satisfying $r \leq t < s \leq R$,*

$$f(t) \leq \vartheta f(s) + \left[\frac{A}{(s-t)^\alpha} + \frac{B}{(s-t)^\beta} + C\right] \tag{3.23}$$

with constants $A, B, C \geq 0$, $\alpha > \beta > 0$ and $0 \leq \vartheta < 1$. Then

$$f(\rho) \leq c(\alpha, \vartheta)\left[\frac{A}{(R-\rho)^\alpha} + \frac{B}{(R-\rho)^\beta} + C\right]. \tag{3.24}$$

Proof. Let $\lambda \in (0, 1)$ and choose the sequence t_i such that $t_0 = r$ and

$$t_{i+1} - t_i = (1-\lambda)\lambda^i (R - r).$$

Then by iteration of (3.23) we get

$$f(r) \leq \vartheta^k f(t_k) + \left[\frac{A}{(1-\lambda)^\alpha (R-r)^\alpha} + \frac{B}{(1-\lambda)^\beta (R-r)^\beta}\right].$$

Choosing λ such that $\lambda^{-\alpha}\vartheta < 1$, the partial sum on the right hand side converges as $k \to \infty$. Hence passing to the limit we get the conclusion with $c = \frac{1}{(1-\lambda)^\alpha (1-\vartheta\lambda^{-\alpha})}$. □

Definition 3.6 (Difference Quotients)**.** *For open $\Omega \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$ let $f \in L^p(\Omega)$, $h \in \mathbb{R}$ and x_s be the sth component of $x \in \Omega$. Then the* difference quotient *of f with respect x_s is defined to be the function*

$$\Delta_{s,h} f(x) = \frac{f(x + he_s) - f(x)}{h}$$

where e_s denotes the unit vector taking the direction of the x_s axis.

We can use difference quotients to characterise $W^{1,p}$ with the following classical result, see [28] for a proof.

Lemma 3.3 (Difference quotient characterisation of Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $\Sigma \subset\subset \Omega$ and $|h| < h_0 = \frac{1}{10\sqrt{n}} \mathrm{dist}(\Sigma, \partial\Omega)$. Then for $v \in \mathrm{W}^{1,p}(\Omega)$ there exists a positive constant $c(n)$ such that*

$$\|\Delta_{s,h} v\|_{p,\Sigma} \leq c \|D_s v\|_{p,\Omega}.$$

We will now prove that $\mathbb{A}$-Harmonic functions are smooth. Note that (3.20) is merely the weak formulation of the second order elliptic partial differential equation in divergence form with constant coefficients

$$\mathrm{div}(ADu) = 0, \quad \text{in } \Omega. \tag{3.25}$$

Here the divergence is taken row-wise.

The following lemma is a generalisation of Weyl's Lemma for $\mathbb{A}$-Harmonic functions extended to $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. The extension follows the observations made in [14].

Lemma 3.4 (Generalised Weyl's Lemma)**.** *Let $\mathbb{A}$ be a symmetric bilinear form satisfying (3.19) with $\nu > 0$ and*

$$|\mathbb{A}(\xi, \eta)| \leq L|\xi||\eta|, \quad \forall \xi, \eta \in \mathbb{R}^{N \times n}.$$

Suppose $u \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ is a solution to the variational system

$$\int_\Omega \mathbb{A}(Du, D\varphi)\, dx = 0, \quad \forall \varphi \in \mathrm{C}_\mathrm{c}^1(\Omega, \mathbb{R}^\mathrm{N}) \tag{3.26}$$

Then $u \in \mathrm{C}^\infty(\Omega, \mathbb{R}^\mathrm{N})$ and for any $B(x_0, R) \subset \Omega$ and $0 < r \leq R$ the following estimates hold

$$\sup_{B(x_0, R/2)} |Du| \leq c \fint_{B(x_0, R)} |Du|\, dx, \tag{3.27}$$

$$\int_{B(x_0,r)} |Du - (Du)_{x_0,r}|^2\,dx \le c\left(\frac{r}{R}\right)^{n+2} \int_{B(x_0,R)} |Du - (Du)_{x_0,R}|^2\,dx, \tag{3.28}$$

where c is dependent only on n, N, ν and L.

Proof. Step 1. We start by proving the first inequality for $\mathbb{A}$-Harmonic $v \in W^{1,2}(\Omega, \mathbb{R}^N)$. Let $B_R \subset \Omega$ and take a smooth cut off function ρ satisfying $1_{B_{\tau R}} \le \rho \le 1_{B_R}$ and $|D\rho| \le \frac{2}{R(1-\tau)}$. Note that as $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ the identity (3.26) holds for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$. Now choosing $\varphi = \rho^2 v$ then $\varphi \in W_0^{1,2}(B_R, \mathbb{R}^N)$, and hence

$$\begin{aligned} 0 &= \int_{B_R} \mathbb{A}(Dv, D\varphi)dx \\ &= \int_{B_R} \left\langle ADv, 2\rho v \otimes D\rho + \rho^2 Dv \right\rangle dx \\ &= \int_{B_R} \langle A\rho Dv, \rho Dv\rangle\,dx + 2\int_{B_R} \langle A\rho Dv, v \otimes D\rho\rangle\,dx. \end{aligned} \tag{3.29}$$

Re-writing the first term above,

$$\begin{aligned} \int_{B_R} \mathbb{A}[D(\rho v) - v \otimes D\rho, D(\rho v) - v \otimes D\rho]\,dx &= \int_{B_R} \mathbb{A}[D(\rho v), D(\rho v)]\,dx \\ &\quad - 2\int_{B_R} \mathbb{A}[D(\rho v), v \otimes D\rho]\,dx \\ &\quad + \int_{B_R} \mathbb{A}[v \otimes D\rho, v \otimes D\rho]\,dx. \end{aligned}$$

Re-writing the integral of the second term

$$\begin{aligned} &\int_{B_R} \mathbb{A}(\rho Dv, v \otimes D\rho)dx \\ &\quad = \int_{B_R} \mathbb{A}(D(\rho v), v \otimes D\rho)dx - \int_{B_R} \mathbb{A}(v \otimes D\rho, v \otimes D\rho)dx. \end{aligned}$$

Thus

$$\int_{B_R} \mathbb{A}(D(\rho v), D(\rho v))dx - \int_{B_R} \mathbb{A}(v \otimes D\rho, v \otimes D\rho)dx = 0.$$

By Lemma 3.1 and the definition of ρ

$$\nu \int_{B_{\tau R}} |Dv|^2 \le L \int_{B_R} |v|^2 |D\rho|^2 dx.$$

Now since $|D\rho| \le \frac{2}{(1-\tau)R}$, we have

$$\int_{B_{\tau R}} |Dv|^2 \le c(\nu, L)\frac{1}{(1-\tau)^2 R^2} \int_{B_R} |v|^2 \tag{3.30}$$

for all $B_R \subset \Omega$ with $c(\nu, L) = \frac{4L}{\nu}$. Next we use the difference-quotient method to prove

$$\int_{B_{\tau^{k-1}R}} |D^k v|^2 \leq c(\nu, L) \frac{1}{(1-\tau)^{2(k-1)} \tau^{2(k-2)} R^{2(k-1)}} \int_{B_R} |Dv|^2 dx. \tag{3.31}$$

Accordingly we set $\varphi = \Delta_{-h} \rho^2 \Delta_h v$. Thus

$$\begin{aligned} 0 &= \int_{B_R} \mathbb{A}(Dv, D\varphi) dx \\ &= \int_{B_R} \left\langle A \Delta_h Dv, 2\rho \Delta_h v \otimes D\rho + \rho^2 D \Delta_h v \right\rangle dx \\ &= \int_{B_R} \langle A \rho D \Delta_h v, \rho D \Delta_h v \rangle \, dx + 2 \int_{B_R} \langle A \rho D \Delta_h v, \Delta_h v \otimes D\rho \rangle \, dx. \end{aligned} \tag{3.32}$$

Comparing with (3.29) we see that by replacing v with $\Delta_h v$ in the derivation of (3.30) and then taking $h \to 0$ we obtain,

$$\int_{B_{\tau R}} |D D_s v|^2 \leq c(\nu, L) \frac{1}{(1-\tau)^2 R^2} \int_{B_R} |D_s v|^2.$$

Therefore by summation of $D D_s v$ from $s = 1, \ldots, n$

$$\int_{B_{\tau R}} |D^2 v|^2 \leq c(\nu, L) \frac{1}{(1-\tau)^2 R^2} \int_{B_R} |Dv|^2.$$

Thus $D_s v \in W^{1,2}_{\text{loc}}(\Omega)$ and by integration by parts $w = D_s v$ satisfies

$$\operatorname{div}(ADw) = 0, \quad \text{for a.e. } \Omega' \subset\subset \Omega \tag{3.33}$$

w is $\mathbb{A}$-Harmonic and we may preceded as above with $\varphi = \Delta_{-h} \rho^2 \Delta_h w \in W^{1,2}_0(\Omega)$. Thus by iteration with τ arbitrarily close to 1, $v \in W^{k,2}_{\text{loc}}(\Omega)$ for every $k = 1, 2, \ldots$ hence $v \in C^\infty(\Omega)$ and we have inequality (3.31) for every $k = 1, 2, \ldots$. Given that $v - \xi$, for any $\xi \in \mathbb{R}^N$, is a solution of (3.33), by combining (3.30) and (3.31),

$$\int_{B_{\tau^k R}} |D^k v|^2 \leq c(\nu, L) \frac{1}{(1-\tau)^{2k} \tau^{2(k-2)} \tau^{2(k-1)} R^{2k}} \int_{B_R} |v - (v)_{B_R}|^2. \tag{3.34}$$

When $2(k-1) > n$, i.e. $k > \frac{n}{2} + 1$ we have By the Sobolev embedding $W^{k,2}(B) \hookrightarrow W^{1,\infty}(B)$, so for $0 < r < R$,

$$\begin{aligned} \sup_{B_r} |Dv|^2 \leq & \frac{c}{r^2} \left(r^{2k-n} \int_{B_r} |D^k v|^2 \, dx \right. \\ & \left. + r^{-n} \int_{B_r} |v - (v)_{B_r}|^2 \, dx \right). \end{aligned}$$

Thus in light of (3.34) with $r \leq \tau^k R$ and $s = R$ we have

$$\sup_{B_r} |Dv|^2 \leq c(\tau^k R)^{-(n+2)} \int_{B_R} |v - (v)_{B_R}|^2 \, dx,$$

where $c = c(\tau, k, \nu, L)$. Thus by Poincaré's inequality

$$\sup_{B_r} |Dv| \leq c \left(\fint_{B_R} |Dv|^2 \, dx \right)^{\frac{1}{2}}$$

for $c = c(\tau, k, \nu, L)$. Applying Hölder's inequality followed by Young's inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ we arrive at

$$\sup_{B_r} |Dv| \leq \frac{1}{2} \sup_{B_R} |Dv| + \frac{1}{2} \frac{c}{R^n} \int_{B_R} |Dv| dx.$$

Thus by Lemma 3.2 with $\frac{R}{2} \leq r \leq \tau^k R < R$ we have

$$\sup_{B_{\frac{R}{2}}} |Dv| \leq \frac{c}{R^n} \int_{B_R} |Dv| dx \tag{3.35}$$

for fixed $\tau \in (\frac{1}{2^{\frac{1}{k}}}, 1)$ where c otherwise depends on k, ν and L. Finally by setting $k = n$ we have $c(n, \nu, L)$.

Step 2. Suppose now that $v = u * \rho_\epsilon$ for $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ satisfying (3.26) on $\Omega_\epsilon := \{x \in \Omega : \text{dist}(\text{x}, \partial\Omega) > \epsilon\}$ where ρ is a symmetric mollifier $\rho : \mathbb{R}^n \to \mathbb{R}$. Then v is smooth and $\mathbb{A}$-Harmonic and satisfies inequality (3.35). Taking $\epsilon \to 0$ in (3.35) we arrive at (3.27). Consequently $u \in C^\infty(\Omega)$.

Step 3. To prove the second inequality of the lemma, (3.28), we start by applying Poincaré's inequality:

$$\begin{aligned} \int_{B_{\tau R}} |Du - (Du)_{\tau R}|^2 dx \leq & c(\tau R)^2 \int_{B_{\tau R}} |D^2 u|^2 dx \\ & \leq c(\tau R)^2 |B_{\tau R}| \sup_{B_{\tau R}} |D^2 u|^2. \end{aligned} \tag{3.36}$$

Next given that $Du - \xi$, $\xi \in \mathbb{R}^{N \times n}$ is also a solution of (3.33), (3.31) can be written as

$$\int_{B_{\frac{R}{2}}} |D^k u|^2 \leq c(\nu, L, k) \frac{1}{R^{2(k-1)}} \int_{B_R} |Du - (Du)_{B_R}|^2 dx.$$

Thus for $k > \frac{n}{2} + 2$ by the Sobolev embedding $W^{k,2}(B) \hookrightarrow W^{2,\infty}(B)$

$$\sup_{B_{\frac{R}{2}}} |D^2 u|^2 \leq \frac{c}{R^4} \left(R^{2k-n} \int_{B_{\frac{R}{2}}} |D^k u|^2 \, dx \right.$$

$$\left. + \, R^{2-n} \int_{B_{\frac{R}{2}}} |Du - (Du)_{B_{\frac{R}{2}}}|^2 \, dx \right),$$

and it follows that

$$\sup_{B_{\frac{R}{2}}} |D^2 u|^2 \leq cR^{-(n+2)} \int_{B_R} |Du - (Du)_{B_R}|^2 \, dx.$$

Combining with (3.36) and taking $\tau = \frac{r}{R}$ we obtain the result.

□

Now from Schauder estimates and the results of the previous section, namely Campanato's characterisation of Hölder continuity, Theorem 3.1, together with Morrey's embedding we obtain the following regularity result for continuous coefficients.

Theorem 3.2 (The Schauder estimates.)**.** *Let $k \in \mathbb{N}$ and $0 < \alpha < 1$ and $\mathbb{A} \in \mathrm{C}^{\mathrm{k-1},\alpha}_{\mathrm{loc}}(\Omega, \mathcal{L}^2(\mathbb{R}^{\mathrm{N}\times\mathrm{n}}))$ with $\mathbb{A}$ satisfying the Legendre-Hadamard condition (3.19) for ever $x \in \Omega$ and $f \in \mathrm{C}^{\mathrm{k-1},\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and*

$$\int_\Omega \left(\mathbb{A}(x)[Du, D\varphi] + \langle f, D\varphi \rangle\right) dx, \quad \forall \varphi \in \mathrm{C}^1_{\mathrm{c}}(\Omega, \mathbb{R}^{\mathrm{N}}), \tag{3.37}$$

then $u \in C^{k,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

For the proof of the theorem we need the following iteration lemma

Lemma 3.5. *Let $\Phi : (0, R_0] \to [0, \infty)$ be non-decreasing and assume that for some constants $A, B, \alpha, \beta > 0$,*

$$\Phi(r) \leq A \left[\epsilon + \left(\frac{r}{R}\right)^\alpha\right] \Phi(R) + BR^\beta$$

for all $0 < r < R \leq R_0$. If $\alpha > \beta$ there exists an $\epsilon_0 = \epsilon_0(A, \alpha, \beta) > 0$ such that if $\epsilon \leq \epsilon_0$, then

$$\Phi(r) \leq c \left[\left(\frac{r}{R}\right)^\beta \Phi(R) + Br^\beta\right]$$

for all $0 < r < R \leq R_0$, where $c(\alpha, \beta, A)$ is a constant.

Proof. Let $0 < \tau < 1$ and $0 < R \leq R_0$. Then

$$\Phi(\tau R) \leq A\left(\epsilon + \tau^{\alpha}\right)\Phi(R) + BR^{\beta}$$

$$= A\tau^{\alpha}\left(\epsilon\tau^{-\alpha} + 1\right)\Phi(R) + BR^{\beta}.$$

For $\gamma \in (\beta, \alpha)$, we take τ such that

$$2A\tau^{\alpha} = \tau^{\gamma}.$$

Thus by setting $\epsilon_0 := \frac{1}{2}\tau^{\alpha} < \tau$,

$$\Phi(\tau R) \leq \tau^{\gamma}\Phi(R) + BR^{\beta}.$$

So by iteration

$$\begin{aligned}\Phi(\tau^{k+1}R) &\leq \tau^{\gamma}\Phi(\tau^{k}R) + B\tau^{k\beta}R^{\beta} \\ &\leq \tau^{(k+1)\gamma}\Phi(R) + B\tau^{k\beta}R^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)} \\ &\leq \tau^{(k+1)\gamma}\Phi(R) + B\frac{\tau^{k\beta}R^{\beta}}{1-\tau^{\gamma-\beta}}\end{aligned} \tag{3.38}$$

where $c := \frac{1}{\tau^{\beta}-\tau^{\gamma}}$. Given $r \in (0, R)$ we choose $k \in \mathbb{N}$ such that $\tau^{k+1}R < r \leq \tau^{k}R$ to arrive at the conclusion. □

Proof of Theorem 3.2. Step 1. We prove the theorem for $k = 1$ using a perturbation argument around the fixed point $x_0 \in \Omega$. For $\mathbb{A}(x_0)$ there exists a unique $\mathbb{A}(x_0)$-Harmonic function $h \in W_u^{1,2}(B_R, \mathbb{R}^N)$ which we compare with u. Let $\varphi = u - h$, then

$$\begin{aligned}\int_{B_R}\mathbb{A}[Du, D\varphi] + \langle f, D\varphi\rangle\, dx &= \int_{B_R}\mathbb{A}(x_0)[Du - Dh, Du - Dh]\, dx \\ &\quad + \int_{B_R}(\mathbb{A} - \mathbb{A}(x_0))[Du, Du - Dh]\, dx \\ &\quad + \int_{B_R}\mathbb{A}(x_0)[Dh, Du - Dh]\, dx \\ &\quad + \int_{B_R}\langle f, D\varphi\rangle\, dx.\end{aligned}$$

By the assumed properties of h the penultimate integral is clearly zero. Thus

$$\int_{B_R} \mathbb{A}(x_0)[Du - Dh, Du - Dh] = \int_{B_R} (\mathbb{A}(x_0) - \mathbb{A})[Du, Du - Dh] \\ - \int_{B_R} \langle f - \xi, D\varphi \rangle \, dx$$

for arbitrary fixed $\xi \in \mathbb{R}^{N\times n}$. Thus estimating the left hand side using Lemma 3.1 and the right hand side by the Hölder continuity of $\mathbb{A}$ we have

$$\nu \int_{B_R} |Du - Dh|^2 \, dx \leq 2^\alpha [\mathbb{A}]_{0,\alpha;B_R} R^\alpha \int_{B_R} |Du||Du - Dh| \, dx \\ + \int_{B_R} |f - \xi||Du - Dh| \, dx.$$

Next using Cauchy's inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ within the integral we have

$$\int_{B_R} |Du - Dh|^2 \, dx \leq \frac{1}{2} \left(2^{2\alpha-1} \nu^{-2} [\mathbb{A}]_{0,\alpha;B_R} R^{2\alpha} \int_{B_R} |2Du|^2 \, dx \right. \\ \left. + \int_{B_R} |Du - Dh|^2 \, dx + \frac{1}{2} \int_{B_R} |2f - 2\xi|^2 \, dx \right).$$

Since $f \in \mathcal{L}^{2,2\alpha'+n}(\Omega, \mathbb{R}^{N\times n})$ for $\alpha' \leq \alpha$ we set $\xi = f_{B_R}$. Thus

$$\int_{B_R} |Du - Dh|^2 \, dx \leq cR^{2\alpha} \int_{B_R} |Du|^2 \, dx + \frac{1}{2} [f]^2_{2,2\alpha'+n;B_R} R^{2\alpha'+n}. \tag{3.39}$$

Step 2: Starting from the above we aim to prove

$$\int_{B_r} |Du|^2 \, dx \leq c \left(R^{2\alpha} + \left(\frac{r}{R} \right)^n \right) \int_{B_R} |Du|^2 \, dx + c[f]^2_{2\alpha'+n;B_R} R^{2\alpha'+n}. \tag{3.40}$$

from which $Du \in L^{2,\lambda}_{\text{loc}}(\Omega)$ for $\lambda < n$ follows from Lemma 3.5. We start by showing that for $h \in \mathrm{W}^{1,2}_u(B_R)$, $\|Dh\|_{L^2(B_R)} \leq c\|Du\|_{L^2(B_R)}$, see Remark 3.4, then from inequality (3.27) of the generalised Weyl's Lemma, Lemma 3.4,

$$⨍_{B_r} |Dh|^2 \leq \sup_{B_{\frac{R}{2}}} |Dh|^2 \leq c⨍_{B_R} |Du|^2. \tag{3.41}$$

We can then use the triangle inequality, estimating $\|Du\|_{L^2(B_R)}$ from above by the left hand sides of (3.39) and (3.41), to deduce (3.40) as required.

Showing $\|Dh\|_{L^2(B_R)} \leq c\|Du\|_{L^2(B_R)}$: The definition of $\mathbb{A}$-Harmonic functions with $\varphi := u - h$ implies

$$\int_{B_r} \mathbb{A}[Dh, Dh] \, dx = \int_{B_R} \mathbb{A}[Dh, Du] \, dx.$$

Thus by the Gårding inequality, Lemma 3.1,

$$\begin{aligned}\nu\int_{B_R}|Du-Dh|^2 &\le \int_{B_R}\mathbb{A}[Du-Dh,Du-Dh]\,dx\\ &\le \int_{B_R}\left(\mathbb{A}[Du,Du]-2\mathbb{A}[Dh,Du]+\mathbb{A}[Dh,Dh]\right)\,dx\\ &= \int_{B_R}\left(\mathbb{A}[Du,Du]-\mathbb{A}[Dh,Du]\right)\,dx\\ &\le L\int_{B_R}|Du|^2\,dx + L\int_{B_R}|Dh||Du|\,dx.\end{aligned}$$

Thus

$$\nu\int_{B_R}\left(|Du|^2-2|\left\langle Du,Dh\right\rangle|+|Dh|^2\right)\,dx \le L\int_{B_R}\left(|Du|^2+|Dh||Du|\right)\,dx.$$

Once again by Cauchy's inequality from within the integral

$$\begin{aligned}\nu\int_{B_R}|Dh|^2 &\le (L-\nu)\int_{B_R}|Du|^2\,dx+\frac{(L+2\nu)}{l^{\frac{1}{2}}}\int_{B_R}|l^{\frac{1}{2}}Dh||Du|\,dx\\ &\le \frac{\nu}{2}\int_{B_R}|Dh|^2\,dx+\left(L-\nu+\frac{(L+2\nu)^2}{2l}\right)\int_{B_R}|Du|^2\,dx.\end{aligned}$$

Thus the inequality follows with $c=\frac{2}{\nu}\left(L-\nu+\frac{(L+2\nu)^2}{2l}\right)$.

Step 3: We will now prove

$$\begin{aligned}\int_{B_r}|Du-(Du)_{B_r}|^2\,dx \le& c\left(\frac{r}{R}\right)^{n+2}\int_{B_R}|Du-(Du)_{B_R}|^2\\ &+2^{2\alpha-1}\frac{[\mathbb{A}]_{0,\alpha;B_R}}{\nu^2}R^{2\alpha+\lambda}\|Du\|^2_{2,\lambda;B_R}+[f]_{2,2\alpha'+n}R^{2\alpha'+n}.\end{aligned}\tag{3.42}$$

From the triangle inequality and (3.39) we deduce

$$\begin{aligned}\int_{B_r}|Du-(Du)_{B_r}|^2\,dx &\le 2\int_{B_r}|Du-Dh|^2\,dx+2\int_{B_r}|Dh-(Dh)_{B_r}|^2\,dx\\ &\le 2^{2\alpha-1}\frac{[\mathbb{A}]_{0,\alpha;B_R}}{\nu^2}R^{2\alpha+\lambda}\|Du\|_{2,2\alpha+n;B_R}\\ &\quad+[f]^2_{2,2\alpha'+n;B_R}R^{2\alpha'+n}+\int_{B_r}|Dh-(Dh)_{B_r}|^2\,dx.\end{aligned}\tag{3.43}$$

In an analogous way to (3.41) we calculate,

$$\int_{B_r} |Dh - (Dh)_{B_r}|^2 \, dx \leq c \left(\frac{r}{R}\right)^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 \, dx \tag{3.44}$$

using (3.28) and $\|Dh - (Dh)_{B_R}\| \leq c\|Du - (Du)_{B_R}\|$ which, since $Dh - \xi_h$ is also $\mathbb{A}$-harmonic and $Du - \xi_u$ is a solution of (3.37) for any $\xi_h, \xi_u \in \mathbb{R}^{N\times n}$, follows from $\|Dh\|_{L^2(B_R)} \leq \|Du\|_{B_R}$of Step 2. Thus (3.42) follows from (3.43) and (3.44) as required. Finally we set $2\alpha' + n = 2\alpha + \lambda$. Thus, given $\lambda < n$ from step 2, $\alpha' < \alpha$ and once again by the iteration Lemma 3.5,

$$\int_{B_r} |Du - (Du)_{B_r}|^2 \, dx \leq c \left(\frac{r}{R}\right)^{2\alpha'+n} \int_{B_R} |Du - (Du)_{B_R}|^2 \, dx$$

$$+ \left(2^{2\alpha-1} \frac{[\mathbb{A}]_{0,\alpha;B_R}}{\nu^2} \|Du\|_{2,\lambda;B_R} + [f]^2_{2,2\alpha'+n;B_R}\right) r^{2\alpha'+n}.$$

Thus $Du \in \mathrm{C}^{0,\alpha'}_{\mathrm{loc}}(\Omega)$ follows immediately from The Campanato characterisation of Hölder continuous functions, Theorem 3.1 of the previous section. Local Hölder continuity implies that $Du \in L^\infty_{\mathrm{loc}}(\Omega)$ thus we may take $\lambda = n$ in (3.43) whilst keeping $\|Du\|_{2,\lambda}$ finite, ($L^\infty(\Omega) \equiv L^{p,n}(\Omega)$ for all $p \geq 1$). Hence it follows that we may take $\alpha = \alpha'$. As a consequence we can apply the iteration lemma for any $\alpha < 1$, applying Theorem 3.1 once more to obtain the result for $k = 1$.

Step 4. In the case $k > 1$ we take a multi-index β of length $k-1$, let $\varphi \in C^\infty_c(\Omega, \mathbb{R}^N)$ and proceed with $D^\beta\varphi \in C^\infty_c(\Omega, \mathbb{R}^N)$ in place of φ in (3.37). Thus

$$0 = \int_\Omega \left(\mathbb{A}(x)[Du, D(D^\beta\varphi)] + \left\langle f, D(D^\beta\varphi)\right\rangle\right) \, dx$$

$$= (-1)^{k-1} \int_\Omega \left(\left\langle D^\beta(A(x)Du), D\varphi\right\rangle + \left\langle D^\beta f, D\varphi\right\rangle\right) \, dx \tag{3.45}$$

$$= (-1)^{k-1} \int_\Omega \left(\mathbb{A}(x)[D(D^\beta u), D\varphi] + \left\langle D^\beta(A(x)Du + D^\beta f, D\varphi\right\rangle\right) \, dx.$$

Therefore we may set $F(x) := (-1)^{k-1} D^\beta(A(x))Du + D^\beta f$. Since we have shown Du is smooth in the previous steps, $F \in C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$. Hence substituting u for $D^\beta u$ we are back to the case $k = 1$. □

Remark 3.4. *$\|Dh\|_{L^2(B_R)} \leq c\|Du\|_{L^2(B_R)}$ implies that $\mathbb{A}$-harmonic functions are Q minimisers of the Dirichlet integral. See for example [28] for the definition of Q minimisers.*

A particular consequence of the Schauder estimates is, once the regularity of u is determined it is possible, for $u \in C^1$ say, to rewrite the Euler Lagrange system of F, provided F is regular enough, as a system of continuous coefficients to obtain further regularity of u. This is sometimes referred to as boot strapping, [23]. We can apply this to the partial regularity results of the final chapter, Chapter 5, provided we make suitable assumptions on the continuity of F.

Chapter 4

Positive second variation and an improved sufficiency result for $\mathrm{W}^1\mathrm{BMO}$-local minimisers.

4.1 Positive second variation: A sufficiency theorem for the existence strong local minimisers.

In this section we will state the recent result of GRABOVSKY and MENGESHA [31], which as we have mentioned in the introduction settles a conjecture of BALL [7] for the vectorial case $N > 1$. First we clarify the classical notion of strong and weak local minimisers with the following definition:

Definition 4.1 (Strong and weak local minimisers)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $\overline{u} \in C^1(\overline{\Omega}, \mathbb{R}^N)$. If there exists a $\delta > 0$ such that*

$$I[\overline{u}, \Omega] \leq I[u, \Omega],$$

whenever $u \in W^{1,\infty}_{\overline{u}}(\Omega, \mathbb{R}^N)$ satisfies

(i) $\|u - \overline{u}\|_{L^\infty(\Omega, \mathbb{R}^{N\times n})} < \delta$*, then $\overline{u}$ is said to be a* strong local minimiser*.*

(ii) $\|Du - D\overline{u}\|_{L^\infty(\Omega, \mathbb{R}^{N\times n})} < \delta$*, then $\overline{u}$ is said to be a* weak local minimiser*.*

Remark 4.1. *Strong local minimisers are $W^{1,q}$-local minimisers in the sense of Definition 2.3 when $q > n$. However $W^{1,q}$-local minimisers with $q > n$ are not necessarily strong local minimisers.*

Before stating the theorem we refer once again to the previous result of ZHANG [66] who showed that critical points of (1.1), for a certain class of F in $C^{2,\alpha}_{\mathrm{loc}}$ satisfying p-growth and strong $W^{1,p}$-quasiconvexity, that are C^2 on small balls with centres in Ω, are absolutely minimising on those small balls. In the following theorem of Grabovsky and Mengesha [31] the critical point $\overline{u}$ is assumed to be $C^1(\Omega, \mathbb{R}^N)$ up to the boundary of Ω. The result is for the general x, u dependent case but we state the theorem in the x, u independent case $F = F(Du)$ and make the slightly stronger assumption that F is C^2 everywhere in $\mathbb{R}^{N\times n}$. It is also assumed that F is strongly p-quasiconvex with $p = 2$, has p-coercivity and p-growth for $p \geq 2$ and has a strong positive second variation at $\overline{u}$. Note that the result shows that critical points $\overline{u}$ satisfying the above are strong local minimisers of the functional I on the whole of Ω. The precise statement of the theorem is as follows:

Theorem 4.1. *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be C^2 and satisfy for some $p \geq 2$ and $c > 0$ the growth condition*

$$|F(\xi)| \leq c(1 + |\xi|^p)$$

the coercivity condition

$$\frac{1}{c}|\xi|^p - c \leq F(\xi)$$

and strong quasiconvexity

$$\int_B F(\xi + D\varphi) - F(\xi)\, dx \geq \beta \int_B |D\varphi|^2\, dx$$

for some $\beta > 0$, all $\xi \in \mathbb{R}^{N\times n}$ and every $\varphi \in C^\infty_c(B, \mathbb{R}^n)$, where B denotes the open unit ball in $\mathbb{R}^n$. Assume $\overline{u} \in C^1(\overline{\Omega}, \mathbb{R}^N)$,

$$\int_\Omega F'(D\overline{u})[D\varphi] = 0$$

and

$$\int_\Omega F''(D\overline{u})[D\varphi, D\varphi] \geq 2\beta \int_\Omega \left(|\varphi|^2 + |D\varphi|^2\right).$$

for every $\varphi \in C^1_c(\Omega, \mathbb{R}^N)$. Then $\overline{u}$ is a strong local minimiser of $I[\cdot, \Omega]$.

In the following section we consider a sufficiency theorem for the existence of a separate class of local minimisers, the $\mathrm{W}^1\mathrm{BMO}$-local minimisers extending the Lipschitz case first presented by KRISTENSEN and TAHERI in [42], for which there is a very irregular example of MÜLLER and ŠVERÁK, [53], to the non-Lipschitz case $1 \leq p < \infty$ from the paper of DODD [20].

4.2 An improved sufficiency result for the existence of $\mathrm{W}^1\mathrm{BMO}$-local minimisers.

Our theorem, extends the result of Kristensen and Taheri [42] for the Lipschitz case to the non-Lipschitz case, $1 \leq p < \infty$. We state their theorem here for completeness:

Theorem 4.2 (Kristensen and Taheri)**.** *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 function, $\Omega \subset \mathbb{R}^n$ be open and bounded and $\overline{u} \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^N)$ be a critical point of (1.1) with strong positive second variation: for some $\delta_s > 0$ and all $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$,*

$$\int_\Omega F'(D\overline{u})[D\varphi] = 0$$

and

$$\int_\Omega F''(D\overline{u})[D\varphi, D\varphi] \geq \delta_s \int_\Omega |D\varphi|^2.$$

Then for every $M < \infty$, there exists $\delta_M > 0$ such that

$$\int_\Omega F(D\overline{u} + D\varphi) \geq \int_\Omega F(D\overline{u})$$

holds for all $\varphi \in W_0^\infty(\Omega, \mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(\Omega,\mathbb{R}^{N\times n})} \leq M$ and $\|D\varphi\|_{\mathrm{BMO}(\mathbb{R}^n,\mathbb{R}^N)} \leq \delta_M$, where $D\varphi$ is extended by 0 off Ω.

To make our extension of the theorem to the non-Lipschitz case we assume uniform continuity of F'' and that the modulus of continuity $\omega\colon [0,\infty) \to \mathbb{R}$, is continuous, increasing, $\omega(0) = 0$ and satisfies the doubling condition

$$\sup_{t>0} \frac{\omega(2t)}{\omega(t)} < \infty. \tag{4.1}$$

The theorem is as follows:

Theorem 4.3. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^2 function, $\Omega \subset \mathbb{R}^n$ be open and bounded and $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ $1 \le p < \infty$ be an critical point of (1.1) with strong positive second variation: for some $\delta_s > 0$ and all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^\mathrm{n}, \mathbb{R}^\mathrm{N}) \cap \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^\mathrm{N})$,*

$$\int_\Omega F'(D\overline{u})[D\varphi] \;=\; 0 \tag{4.2}$$

$$\int_\Omega F''(D\overline{u})[D\varphi, D\varphi] \;\ge\; \delta_s \int_\Omega |D\varphi|^2. \tag{4.3}$$

Further assume

$$|F''(\xi) - F''(\eta)| \le \omega(|\xi - \eta|) \tag{4.4}$$

for all $\xi, \eta \in \mathbb{R}^{N\times n}$. Then there exists a $\delta_*(n, N, c, q) > 0$ such that

$$\int_\Omega F(D\overline{u} + D\varphi) \ge \int_\Omega F(D\overline{u})$$

holds for all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^\mathrm{n}, \mathbb{R}^\mathrm{N}) \cap \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^\mathrm{N})$, with $\|D\varphi\|_{\mathrm{BMO}(\mathbb{R}^\mathrm{n}, \mathbb{R}^\mathrm{N})} \le \delta_*$.

Remark 4.2. *(i) The space $\mathrm{W}^1\mathrm{BMO}(\mathbb{R}^\mathrm{n}, \mathbb{R}^\mathrm{N}) \cap \mathrm{W}^{1,\mathrm{q}}_0(\Omega, \mathbb{R}^\mathrm{N})$ is exactly the space of $\mathrm{W}^1\mathrm{BMO}(\mathbb{R}^\mathrm{n}, \mathbb{R}^\mathrm{N})$ functions f, for which f and Df are extended by 0 outside of Ω.*

(ii) Beside excluding exponential growth of ω the doubling condition also excludes certain classes of piecewise polynomial growth. However we can accommodate the subclass of piecewise polynomials ω (not necessarily increasing) that do not satisfy (4.1) but instead satisfy

$$\tilde{\omega}(t) := \sup_{s\ge 1}\left(s^{-k} \sup_{r\le st} \omega(r)\right) < \infty$$

for some $k > 0$ and all $t > 0$. In this case one may easily show that $\omega(t) \le \tilde{\omega}(t)$ and $\tilde{\omega}(\alpha t) \le \alpha^k \tilde{\omega}(t)$ for $\alpha \ge 0$. Thus we can replace ω with $\tilde{\omega}$ in the proof of the theorem.

4.2.1 Proof of Theorem 4.3

Following [42], we use Taylors formula together with (1.11) to obtain

$$\int_\Omega \big(F(D\overline{u}+D\varphi)-F(D\overline{u})\big) = \int_\Omega \int_0^1 (1-t)\big(F''(D\overline{u}+tD\varphi) - F''(D\overline{u})\big)[D\varphi, D\varphi]dt \qquad (4.5)$$
$$+\frac{1}{2}\int_\Omega F''(D\overline{u})[D\varphi, D\varphi].$$

Thus by the uniform continuity condition (1.13) and positive second variation at $\overline{u}$, (1.12), we have

$$\int_\Omega \left(F(D\overline{u}+D\varphi)-F(D\overline{u})\right) \geq \frac{1}{2}\int_{\mathbb{R}^n} \left(\delta|D\varphi|^2 - \omega(|D\varphi|)|D\varphi|^2\right). \qquad (4.6)$$

Note that we have used the fact that $D\varphi = 0$ off Ω.

We next we use the Orlicz version of the inequality of FEFFERMAN and STEIN [25] derived in [42]. Noting that the derivation does not require f to be bounded or have compact support in $\mathbb{R}^n$ we reproduce the relevant lemma for the convenience of the reader, omitting those conditions that are not relevant here. First we introduce the required notation.

The Hardy-Littlewood and Fefferman-Stein maximal functions of $f\colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$ are respectively

$$f^*(x) = \sup_{\{B:x\in B\}} \fint_B |f(y)|dy$$

and

$$f^\#(x) = \sup_{\{B:x\in B\}} \fint_B |f(y)-f_B|dy$$

where we have taken suprema over all open balls $B\subset\mathbb{R}^n$ containing x.

Lemma 4.1. *Let $\Phi\colon [0,\infty)\to[0,\infty)$ be an increasing and continuous function with $\Phi(0)=0$ and consider the Borel map $f\colon \mathbb{R}^n\to\mathbb{R}^{N\times n}$ then*

$$\int_{\mathbb{R}^n}\Phi(|f^*|) \leq \frac{5^n}{\epsilon}\int_{\mathbb{R}^n}\Phi\left(\frac{|f^\#|}{\epsilon}\right) + 2\cdot 5^{3n}\epsilon\int_{\mathbb{R}^n}\Phi(5^n\cdot 2^{n+1}|f^*|). \qquad (4.7)$$

We include the proof of the lemma which can be found in [42] for the convenience of the reader.

Proof. As in [42] we let $\lambda^*(t) = \mathcal{L}^n(\{x : f^*(x) > t\})$ and $\lambda^\#(t) = \mathcal{L}^n(\{x : f^\#(x) > t\})$ for $t > 0$, then from (4.4) and (4.8) in [25] and explicit constants obtained in pp.305-309 of [33] we have

$$\lambda^*(t) \leq 5^n \lambda^\#(\epsilon t) + 2 \cdot 5^n \epsilon \lambda^*(2^{-n-1} \cdot 5^{-n} t) \tag{4.8}$$

for all $t > 0$. Now since f is in L^p so is f^* and $f^\#$ thus integrating (4.8) with respect to $d\Phi(t)$ over $[0, \infty)$ we obtain (4.7) by the usual formula for integrals in terms of distribution functions. □

Now returning to (4.6), by applying Lemma 4.1 to $\Phi(t) = \omega(t)t^2$ with sufficiently small ϵ together with condition (4.1), we have the following for some positive finite constant c_*

$$\int_\Omega (F(D\overline{u} + D\varphi) - F(D\overline{u})) \geq \frac{1}{2} \int_{\mathbb{R}^n} \left(\delta |D\varphi|^2 - c_* \omega(|D\varphi^\#|)|(D\varphi)^\#|^2 \right). \tag{4.9}$$

Now as in [42] we remark that by the Hardy Littlewood-Wiener maximal inequality there exists a constant $c_0(n, N) > 0$ such that

$$\int_{\mathbb{R}^n} |D\varphi|^2 \geq c_0 \int_{\mathbb{R}^n} |(D\varphi)^*|^2.$$

and since $(D\varphi)^\# \leq 2(D\varphi)^*$ we have

$$\int_\Omega (F(D\overline{u} + D\varphi) - F(D\overline{u})) \geq \frac{1}{2} \int_{\mathbb{R}^n} \left(\frac{\delta c_0}{4} - c_* \omega(|D\varphi^\#|) \right) |(D\varphi)^\#|^2. \tag{4.10}$$

The final integral is positive when

$$c_* \omega(|D\varphi^\#|) \leq \frac{\delta c_0}{4}. \tag{4.11}$$

It follows that integral is finite when

$$\sup_{\mathbb{R}^n} |(D\varphi)^\#| \leq \omega^{-1} \left(\frac{\delta c_0}{4 c_*} \right) := \delta_*. \tag{4.12}$$

□

The following sufficiency conditions for non-Lipschitz critical points of $I[\cdot, \Omega]$ to be partially regular are a result of combining the above theorem with Corollary 5.1 of Theorem 5.1 of the final chapter.

Corollary 4.1. *Let* $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *be* C^2, $\Omega \subset \mathbb{R}^n$ *open and bounded Let* $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$, $1 < p < \infty$ *be a critical point of* $I[\cdot]$ *with strongly positive second variation such that for some* $\delta_s > 0$ *and all* $\varphi \in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^n, \mathbb{R}^N) \cap \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^N)$ *we have (1.11) and (1.12). Suppose also that we have*

$$|F''(\xi) - F''(\eta)| \leq \omega(|\xi - \eta|) \tag{4.13}$$

such that F *satisfies (H1)-(H3). Then* $\overline{u}$ *is partially regular in the sense of Theorem 5.1 provided* $D\overline{u}$ *satisfies the regularity condition (1.8) with* $\delta = \delta_*$ *where* δ_* *is given in Theorem 1.2.*

We will prove this corollary at the end of the following and final chapter, after we have established our partial regularity result, Theorem 5.1.

Chapter 5

A priori Morrey-Campanato Type regularity condition for local minimisers.

In this chapter we prove our main result that can also be found in the paper of DODD [20]. The potential class of $W^1\mathcal{L}^{p,\mu}$-local minimisers with $\mu > 0$ are distinct from $W^{1,q}$-local minimisers and yet are not absolute minimisers even locally on Ω. Comparing these minimisers with $W^{1,q}$-local minimisers in the case $q > p$, by using Hölder and the various embeddings discussed in Section 3.1 of Chapter 3, we observe the following:

(a) For $0 < \mu \leq n\left(1 - \frac{p}{q}\right)$, $W^1\mathcal{L}^{p,\mu}$-local minimisers are a stronger notion of local minimisers than $W^{1,q}$-local minimisers (but a weaker notion than $W^{1,p}$-local minimisers).

(b) For $\mu = n$ we have locally and for domains Ω satisfying the measure density condition, Definition 3.4, Section 3.1 of Chapter 3, that $W^1\mathcal{L}^{p,\mu}$-local minimisers are equivalent to $\mathrm{W}^1\mathrm{BMO}$-local minimisers (in context of our interior regularity result they are essentially no different from $\mathrm{W}^1\mathrm{BMO}$-local minimisers). Clearly $\mathrm{W}^1\mathrm{BMO}$-local minimisers are weaker than $W^{1,q}$-local minimisers for any $q < \infty$, but stronger than $W^{1,\infty}$-local minimisers.

In the context of the partial regularity result for $W^{1,q}$-local minimisers, the real value of looking at this new class of local minimisers is related to the condition that the local minimiser is in $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ currently used to prove partial regularity for local minimisers. It is not clear that this condition is necessary for the proof of partial regularity of $W^{1,q}$-local minimisers. For our class of minimisers, in the case $\mu \leq n\left(1-\frac{p}{q}\right)$, the $W^{1,q}_{\mathrm{loc}}$ condition implies (5.2). Thus in this case our condition (5.1), below, is weaker than the assumption that the minimiser is in $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. In the special case of $\mathrm{W}^1\mathrm{BMO}$ ($\mu = n$) we have already remarked that this condition is necessary to allow for weak $\mathrm{W}^1\mathrm{BMO}$-local minimisers that are also Lipschitz continuous. In particular to exclude the irregular examples of critical points with positive second variation discussed both in the introduction and the previous chapter.

However, as we have already remarked in Chapter 2, for p-coercive F, existence of $W^{1,q}$-local minimisers ($u \in W^{1,p}$) for $p < q < \infty$ and of $W^1\mathcal{L}^{p,\mu}$-local minimisers for $\mu < n$ appears to be an open problem regardless of the regularity conditions $W^{1,q}_{\mathrm{loc}}$ and (5.1). For $\mathrm{W}^1\mathrm{BMO}$-local minimisers we have the sufficiency theorem of the previous chapter and thus the irregular example of a Lipschitz critical point that is a weak $\mathrm{W}^1\mathrm{BMO}$ local minimiser.

In the theorem we assume that F the integrand of $I[\cdot,\Omega]$ satisfies the usual hypotheses, that $F \in C^2$, has p-growth see (2.12) and is strongly p-quasiconvex (Definition 2.4, Section 2.2 of Chapter 2). The precise statement is as follows:

Theorem 5.1. *Consider the functional $I[\cdot,\Omega]$ of (1.1) with F in C^2 and satisfying*

$$|F(\xi)| \leq c\,(1+|\xi|^p)$$

and strong p-quasiconvexity (See Definition 2.4). Suppose that $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ for $p \in (1,\infty)$ is a $W^1\mathcal{L}^{p,\mu}$-local minimiser of $I[\cdot,\Omega]$: There exists a $\delta > 0$ such that $I[\overline{u},\Omega] \leq I[u,\Omega]$ whenever $u \in \overline{u} + \mathrm{W}^{1,p}_0(\Omega,\mathbb{R}^N)$ and $\|Du - D\overline{u}\|_{p,\mu;\Omega} \leq \delta$, so that $D\overline{u}$ satisfies the regularising condition

$$\limsup_{R\to 0+}\left(\sup_{\substack{x_0\in\Omega'\\ r\in(0,R)}}\left(\frac{1}{r^\mu}\int_{\Omega(x,r)}|D\overline{u}-(D\overline{u})_{\Omega(x,r)}|^p\,dx\right)^{\frac{1}{p}}\right) < \delta \tag{5.1}$$

for every open set Ω' compactly contained in Ω. Then for $\mu \leq n$ there exists an open

set $\Omega_0 \subset \Omega$ of full n-dimensional measure, such that the minimiser $\overline{u} \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, and $|\Omega \setminus \Omega_0| = 0$.

Partial regularity of non-Lipschitz $\mathrm{W}^1\mathrm{BMO}$-local minimisers follows from Lemma 5.3 in the proof of Theorem 5.1 and the isomorphism $\mathcal{L}^{n,p}(B, \mathbb{R}^{N\times n}) \cong \mathrm{BMO}(\mathrm{B}, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$ on balls $B \subset \mathbb{R}^n$ (see Section 3.1.1 and Proposition 3.3 for details):

Corollary 5.1. *Consider the functional $I[\cdot, \Omega]$ of (1.1) with F in C^2 and satisfying*

$$|F(\xi)| \leq c\,(1 + |\xi|^p)$$

and strong p-quasiconvexity (See Definition 2.4). Suppose that $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for $p \in (1, \infty)$ is a $\mathrm{W}^1\mathrm{BMO}$-local minimiser of $I[\cdot, \Omega]$: There exists a $\delta > 0$ such that $I[\overline{u}, \Omega] \leq I[u, \Omega]$ whenever $u \in \overline{u} + \mathrm{W}^{1,p}_0(\Omega, \mathbb{R}^N)$ and $\|Du - D\overline{u}\|_{*;\Omega} \leq \delta$, so that $D\overline{u}$ satisfies the regularising condition (1.8). Then there exists an open set $\Omega_0 \subset \Omega$ of full n-dimensional measure, such that the minimiser $\overline{u} \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, and $|\Omega \setminus \Omega_0| = 0$.

Remark 5.1. *By Proposition 3.2 Section 3.1.1, the embedding inequality for Morrey and Campanato spaces, condition (5.1) is satisfied if we assume $D\overline{u} \in \mathcal{L}^{p,\nu}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ for $\nu > \mu$. In this case the condition reduces to*

$$\limsup_{R\to 0+} \left(\sup_{\substack{x_0\in\Omega \\ r\in(0,R)}} \left(\frac{1}{r^\mu} \int_{\Omega(x,r)} |D\overline{u} - (D\overline{u})_{\Omega(x,r)}|^p \, dx \right)^{\frac{1}{p}} \right) = 0 \tag{5.2}$$

for every open set Ω' compactly contained in Ω.

5.1 Preliminaries

We will use the following function in the sub-quadratic case ($1 < p < 2$);

$$V(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}} \xi, \qquad \xi \in \mathbb{R}^{N\times n}. \tag{5.3}$$

As in [13] we will use the properties of V highlighted in the following lemma. The lemma is proved in [14], for $1 < p < 2$.

Lemma 5.1. *Let $1 < p < 2$ and $V\colon \mathbb{R}^{K\times k} \to \mathbb{R}^{K\times k}$ be defined by (5.3). Then, for any $\eta, \xi \in \mathbb{R}^{K\times k}$, $t > 0$:*

(i) $2^{\frac{p-2}{4}} \min\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le \min\{|\xi|, |\xi|^{\frac{p}{2}}\}$;

(ii) $|V(t\xi)| \le \max\{t, t^{\frac{p}{2}}\}|V(\xi)|$;

(iii) $|V(\xi+\eta)| \le 2^{\frac{p}{2}}\,[|V(\xi)| + |V(\eta)|]$;

(iv) $\frac{p}{2}(1+|\xi|^2+|\eta|^2)^{\frac{(p-2)}{4}}|\xi-\eta| \le |V(\xi)-V(\eta)| \le c(1+|\xi|^2+|\eta|^2)^{\frac{(p-2)}{4}}|\xi-\eta|$;

(v) $|V(\xi)-V(\eta)| \le c|V(\xi-\eta)|$;

(vi) For each $M > 0$ there exists a $c_M < \infty$ such that
$|V(\xi-\eta)| \le c_M|V(\xi)-V(\eta)|$ *if* $|\eta| \le M$.

where c depends on k and p and c_M on M and p.

Proof. Inequalities (i) and (ii) follow easily from the definition of V. To prove inequality (iii), without loss of generality let $|\eta| \le |\xi|$. If $|\xi| \le 1$ then by (i)

$$|V(\xi+\eta)| \le |\xi+\eta| \le 2|\xi| \le c(p)|V(\xi)|.$$

If $|\xi| \ge 1$ then once again by (i)

$$|V(\xi+\eta)| \le |\xi+\eta|^{\frac{p}{2}} \le c(p)|\xi|^{\frac{p}{2}} \le c(p)|V(\xi)|.$$

Inequality (iv) implies (v). Inequality (iv) is proved in Lemma 2.2 of [4].

Inequality (vi): We have

$$|\xi-\eta|^2 = |\xi|^2 + |\eta|^2 - 2\langle\xi,\eta\rangle.$$

and for $\epsilon > 0$

$$|\xi-\epsilon\eta|^2 = |\xi|^2 + |\epsilon\eta|^2 - 2\langle\xi,\epsilon\eta\rangle.$$

Thus

$$\langle\xi,\eta\rangle \le \frac{1}{2\epsilon}|\xi|^2 + \frac{\epsilon}{2}|\eta|^2.$$

Put $\epsilon = 4$, then using $|\eta| \le M$ we have

$$|\xi-\eta|^2 \ge \frac{3}{4}|\xi|^2 - 3|\eta|^2 \ge \frac{3}{4}|\xi|^2 - 3M^2.$$

Thus

$$
\begin{aligned}
(1+|\xi-\eta|^2)^{\frac{p-2}{4}}|\xi-\eta| &= \left(\frac{1+4M^2}{1+4M^2(1+|\xi-\eta|^2)}\right)^{\frac{2-p}{4}}|\xi-\eta| \\
&\le (1+4M^2)^{\frac{2-p}{4}}\left(1+4M^2+|\xi-\eta|^2\right)^{\frac{p-2}{4}}|\xi-\eta| \\
&\le (1+4M^2)^{\frac{2-p}{4}}\left(1+M^2+\frac{3}{4}|\xi|^2\right)^{\frac{p-2}{4}}|\xi-\eta| \\
&\le (1+4M^2)^{\frac{2-p}{4}}\left(1+|\eta|^2+\frac{3}{4}|\xi|^2\right)^{\frac{p-2}{4}}|\xi-\eta| \\
&\le \left(\frac{3}{4}\right)^{\frac{p-2}{4}}(1+4M^2)^{\frac{2-p}{4}}\left(1+|\eta|^2+|\xi|^2\right)^{\frac{p-2}{4}}|\xi-\eta|.
\end{aligned}
$$

Finally setting $c=\frac{2}{p}\left(\frac{3}{4}\right)^{\frac{p-2}{4}}(1+4M^2)^{\frac{2-p}{4}}$ and using the left-hand inequality of (iv) we arrive at

$$|V(\xi-\eta)|\le c(p,M)|V(\xi)-V(\eta)|.$$

□

In the sequel we will refer to the excess of $\overline{u}$ defined for every ball $B(x,r)\subset\Omega$ by

Definition 5.1 (The Excess of $\overline{u}$)**.** *Let $\Omega\subset\mathbb{R}^n$ be open and bounded. By the excess of $\overline{u}$ over the ball $B(x,r)\subset\Omega$ we mean,*

$$
E(x,r)=\begin{cases}
\fint_{B(x,r)}|V(D\overline{u})-V((D\overline{u})_{x,r})|^2 & 1<p<2 \\
\fint_{B(x,r)}\left(|D\overline{u}-(D\overline{u})_{x,r}|^2+|D\overline{u}-(D\overline{u})_{x,r}|^p\right) & p\ge 2.
\end{cases}
\tag{5.4}
$$

Here

$$V(\xi)=(1+|\xi|^2)^{\frac{p-2}{4}}\xi,\qquad \xi\in\mathbb{R}^{N\times n}.$$

and the exponent p is understood from the context.

5.2 Proof of Theorem 5.1

The proof is based on a blow-up technique originally developed by DE GIORGI [18] and ALMGREN [5] in the context of geometric measure theory, see [28, §9.6] and the

references therein, and adapted to the setting of partial regularity for elliptic systems by GIUSTI and MIRANDA [29]. Specifically once the following proposition is proved partial regularity follows as we will show in the sequel.

Proposition 5.1. *For every $L > 0$, there exists $C = C(L) > 0$ with the property that for each $\tau \in (0, \frac{1}{2})$, there exists $\epsilon = \epsilon(L, \tau) > 0$ such that for all $B(x, r) \subset \Omega$ with $|(D\overline{u})_{x,r}| \leq L$ and $E(x, r) < \epsilon$, we have*

$$E(x, \tau r) \leq C(L)\tau^2 E(x, r).$$

The proof is indirect and was originally adapted for minimisers of the quasiconvex integral $I[\cdot, \Omega]$ by EVANS [22]. The basic idea is to assume blow up of the solution for a sequence of small balls around x and study the convergence in the unit ball of the sequence of solutions for suitably re-scaled functionals so to obtain a contradiction. This argument involves 3 main steps. In step 1 we show that the limit of the blow up sequence of solutions converges weakly in $\mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ for $1 < p < 2$ and $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$ for $p \geq 2$. In step 2 show that the weak limit of these solutions satisfies a linear uniformly elliptic system with constant coefficients. Finally in step 3, show the strong convergence of the sequence of solutions to obtain the contradiction. To show this we use the standard construction of comparison maps from a suitably rescaled version of the minimiser $\overline{u} \in \mathrm{W}^{1,p}(\Omega)$, and thus must prove that these maps satisfy the Morrey-Campanato local minimiser condition

$$\|Du - D\overline{u}\|_{p,\mu;\Omega} \leq \delta$$

for all $u \in W^{1,p}_{\overline{u}}(\Omega, \mathbb{R}^N)$. It is in showing that the local minimiser condition is satisfied, Lemma 5.3, that it is necessary to introduce the condition (5.1), a generalisation of the condition for Lipschitz maps introduced in [42]. Having verified this we can proceed with the methods of [13, 42] without modification, deriving a pre-Caccioppoli inequality and using the measure theoretic argument therein to obtain our result.

Given the growth condition (2.12) and strong quasiconvexity we have shown that growth on F', (2.10), follows (Chapter 2, section 2.1). As in [14] for the $1 < p < 2$ case, a simple manipulation of (2.10) results in

$$|F'(\xi)| \leq c_0(1 + |\xi|^2)^{\frac{p-1}{2}} \tag{5.5}$$

for $p > 1$. In the sequel we will use the following Lemma, a consolidation of Lemma 3.3 [14] and Lemma 2.3 [3] for functions satisfying the above estimate. Note that Lemma 3.3 of [14] is proved in the same way as Lemma 2.3 of [3].

Lemma 5.2. *Let $F\colon \mathbb{R}^{K\times k} \to \mathbb{R}$ be a function of class C^2 with*

$$|F'(\xi)| \leq c_0(1+|\xi|^2)^{\frac{p-1}{2}}, \quad p \geq 1.$$

Then for any $\lambda > 0$ and $\xi_0 \in \mathbb{R}^{K\times k}$ with $|\xi_0| \leq L$, setting

$$F_{\xi_0,\lambda}(\xi) = \lambda^{-2}\left[F(\xi_0+\lambda\xi) - F(\xi_0) - \lambda F'(\xi_0)\xi\right] \tag{5.6}$$

there exist constants c_1 and c_2 dependent only on c_0, L, p such that for $p \geq 1$,

$$|F_{\xi_0,\lambda}(\xi)| \leq \min\left\{c_1(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\xi|^2, c_2(|\xi|^2+\lambda^{p-2}|\xi|^p)\right\}. \tag{5.7}$$

Proof. We prove the lemma in two steps. First with $|\lambda\xi| \leq 1$, then with $|\lambda\xi| > 1$. Step 1: $|\lambda\xi| \leq 1$, $p \geq 1$. Let

$$k(\xi_0) := \sup_{|\xi|\leq 1+|\xi_0|} F''(\xi),$$

then we have

$$\begin{aligned}
|F_{\xi_0,\lambda}(\xi)| &= \lambda^{-2}\int_0^1 (1-t)F''(\xi_0+t\lambda\xi)[\lambda\xi,\lambda\xi]\,dt \\
&\leq \frac{1}{2}k(\xi_0)|\xi|^2 \\
&\leq \frac{1}{2}k(\xi_0)(1+|\lambda\xi|^2)^{\frac{p-1}{2}}|\xi|^2 \\
&\leq \frac{1}{2}k(\xi_0)(1+|\lambda\xi|^2)^{\frac{p-2}{2}}(1+|\lambda\xi|^2)^{\frac{1}{2}}|\xi|^2 \\
&\leq k(\xi_0)\sqrt{2}(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\xi|^2.
\end{aligned}$$

Step 2: For $|\lambda\xi| > 1$, $p \geq 1$ we have using (5.5) that

$$\begin{aligned}
|F_{\xi_0,\lambda}| &= \lambda^{-2}\left(F(\xi_0+\lambda\xi) - F(\xi_0) - \lambda F'(\xi_0)[\xi]\right) \\
&\leq \lambda^{-2}\int_0^1 F'(\xi_0+t\lambda\xi) - F'(\xi_0)[\lambda\xi]\,dt \\
&\leq \lambda^{-2}c_0\int_0^1 \left((1+|\xi_0+t\lambda\xi|^2)^{\frac{p-1}{2}} + (1+|\xi_0|^2)^{\frac{p-1}{2}}\right)[\lambda\xi]\,dt \\
&\leq \lambda^{-1}c_0\left[\left(1+2(|\xi_0|^2+|\lambda\xi|^2)\right)^{\frac{p-1}{2}} + (1+|\xi_0|^2)^{\frac{p-1}{2}}\right]|\xi| \\
&\leq \lambda^{-1}c_0\left(2(1+2|\xi_0|^2)\right)^{\frac{p-1}{2}}(1+|\lambda\xi|^2)^{\frac{p-1}{2}}|\xi| \\
&\leq c(p,L,c_0)\lambda^{-1}(1+|\lambda\xi|^2)^{\frac{p-1}{2}-\frac{1}{2}}(1+|\lambda\xi|^2)^{\frac{1}{2}}|\xi| \\
&\leq c\lambda^{-1}\sqrt{2}(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\lambda\xi||\xi| \\
&= c_3(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\xi|^2,
\end{aligned}$$

where c_2 depends only on p, L and c_0. Now define $c_1 := \max\{\sqrt{2}L, c_3\}$. Since $|\xi_0| \leq L$ we have

$$|F_{\xi_0,\lambda}(\xi)| \leq c_1(1+|\lambda\xi|^2)^{\frac{p-2}{2}}|\xi|^2. \tag{5.8}$$

Step 3: In general for $p \geq 1$ we have

$$A_p(1+|\xi|^2)^{\frac{p-2}{2}} \leq 1+|\xi|^{p-2} \leq B_p(1+|\xi|^2)^{\frac{p-2}{2}}$$

with the constants A_p, $B_p > 0$ dependent only on p. Thus by (5.8) from the previous steps, we have

$$|F_{\xi_0,\lambda}(\xi)| \leq c_1 A_p^{-1}(1+|\lambda\xi|^{p-2})|\xi|^2.$$

Setting $c_2 := A_p^{-1}\cdot c_1$ completes the proof. □

Proof of Proposition 5.1. Suppose the proposition is false. Then there exists an $L > 0$ and a sequence of balls $\{B(x_j, r_j)\}$ with the properties that

$$|(D\overline{u})_{x_j,r_j}| \leq L \text{ for all } j,$$

and

$$E(x_j, r_j) \to 0 \text{ as } j \to \infty$$

such that for every $C > 0$ there exists a $\tau \in (0, \frac{1}{2})$ with

$$E(x_j, r_j\tau) > C\tau^2 E(x_j, r_j) \text{ for all } j. \tag{5.9}$$

We look for a C that contradicts this.

STEP 1: We suppose the sequence of balls satisfies the above with vanishing radii, $r_j \to 0$ as $j \to \infty$. We rescale the minimiser on each ball to a sequence of maps, u_j, on the unit ball in the usual way

$$u_j(y) := \frac{\overline{u}(x_j + r_j y) - \overline{u}(x_j) - \xi_j r_j y}{\lambda_j r_j}, \qquad y \in B_1$$

where the scaling is given by $\lambda_j^2 := E(x_j, r_j)$, and $\xi_j := (D\overline{u})_{x_j, r_j}$.

By assumption $|\xi_j| \leq L$, so for a subsequence (for convenience not relabelled)

$$\xi_j \to \xi_\infty \text{ as } j \to \infty.$$

From the definition of u_j, $(u_j)_{0,1} = 0$, $(Du_j)_{0,1} = 0$, so for $p \geq 2$

$$\fint_{B_1} \left(|Du_j|^2 + \lambda_j^{p-2}|Du_j|^p\right) \leq 1 \tag{5.10}$$

and for $1 < p < 2$, utilising part vi.) of Lemma 5.1,

$$\begin{aligned} \fint_{B_1} |V(Du_j)|^2 &\leq c_0(p, L)\frac{1}{\lambda_j^2}\fint_{B_j} |V(D\overline{u}) - V((D\overline{u})_{x_j, r_j})|^2 \\ &= c(p, L). \end{aligned} \tag{5.11}$$

This implies

$$\|Du_j\|_{\mathrm{L}^{s(p)}(B_1, \mathbb{R}^{N\times n})} < c_B(p, L), \quad p > 1 \tag{5.12}$$

where $s(p) := \min\{2, p\}$. Note that part i.) of Lemma 5.1 is used in the derivation for $1 < p < 2$. Thus by weak compactness (5.12) implies for a further subsequence (again not relabelled)

$$Du_j \rightharpoonup Du \text{ in } \mathrm{L}^{s(p)}(B_1, \mathbb{R}^{N\times n}). \tag{5.13}$$

Now setting $F_j := F_{\xi_j, \lambda_j}$ in (5.6) of Lemma 5.2, so that F_j satisfies the associated growth estimates, we replace the integral (1.1) with the sequence of integrals

$$I_j[u] = \int_{B_1} F_j(Du). \tag{5.14}$$

It follows using strong quasiconvexity of F that each F_j satisfies a quasi-convexity condition

$$\nu \int_{B_1} (|D\varphi|^2 + \lambda_j^{p-2}|D\varphi|^p) \leq \int_{B_1} (F_j(\xi + D\varphi) - F_j(\xi))$$

for all $\varphi \in \mathrm{W}_0^{1,p}(B_1, \mathbb{R}^N)$ when $p \geq 2$ and

$$\nu \int_{B_1} (1 + |\xi_j + \lambda_j \xi|^2 + |\lambda_j D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \leq \int_{B_1} (F_j(\xi + D\varphi) - F_j(\xi)) \tag{5.15}$$

for all $\varphi \in \mathrm{W}_0^{1,p}(B_1, \mathbb{R}^N)$ when $1 < p < 2$. Finally using the local minimality of $\overline{u}$ it follows that u_j is a $\mathrm{W}^1\mathrm{X}$-local minimiser of I_j defined at (5.14). Precisely, $I_j[u_j] \leq I_j[u]$ whenever

$$\|Du - Du_j\| \leq \delta_j := \begin{cases} \frac{\delta}{\lambda_j r_j^{\frac{n-\mu}{p}}}, & \mathrm{X} = \mathcal{L}^{\mathrm{p},\mu}(\mathrm{B}_1), \quad \mu \leq \mathrm{n} \\ \\ \frac{\delta}{\lambda_j}, & \mathrm{X} = \mathrm{BMO}(\mathrm{B}_1), \end{cases} \tag{5.16}$$

with

$$u \in u_j + \mathrm{W}_0^{1,p}(B_1, \mathbb{R}^N). \tag{5.17}$$

STEP 2 (u solves linear elliptic system) : We wish to show that the limit u satisfies

$$\int_{B_1} F''(\xi_\infty)\,[Du, D\varphi] = 0, \quad \forall \varphi \in \mathrm{C}_0^1(\Omega, \mathbb{R}^\mathrm{N}) \tag{5.18}$$

since it then follows (given $F \in C^2$ and strongly quasiconvex) that u is $\mathbb{A}$-Harmonic and by Lemma 3.4 of Section 3.2 Chapter 3 that it is C^∞ and

$$⨍_{B(0,\tau)} |Du - (Du)_{0,\tau}|^2 dy \leq C^* \tau^2 \qquad (p > 1). \tag{5.19}$$

From this we may use part (i) of Lemma 5.1 to attain

$$⨍_{B(0,\tau)} |V(Du - (Du)_{0,\tau})|^2 dy \leq C^* \tau^2 \tag{5.20}$$

for the case $1 < p < 2$. The proof of (5.18) is given in [13] for $1 < p < 2$ and [42] for $p \geq 2$ and remains unchanged in this case. It only uses the following properties: that $\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is a critical point of I ; $F \in \mathrm{C}^2$ and satisfies growth condition (5.5). As a consequence of the growth estimate (5.7) on F_j (Lemma 5.2) we are able, using

the dominated convergence theorem, to take the first variation of I_j (5.14). Writing in terms of F this results in

$$\frac{1}{\lambda_j}\int_B \left(F'(\xi_j+\lambda_j Du_j(x))-F'(\xi_j)\right)[D\varphi]=0, \tag{5.21}$$

for all $\varphi\in W_0^{1,p}(B,\mathbb{R}^N)$ satisfying $\|D\varphi\|\le\delta_j$. In the following, so that we may use Lemma 3.4, we fix $\varphi\in C_0^1(B,\mathbb{R}^N)$ and note that this implies $\varphi\in W^{1,\infty}(B,\mathbb{R}^N)$, then we aim to show that taking $j\to\infty$ results in the elliptic system (5.18). We cannot use the dominated convergence argument on (5.21). Instead, following [13] we split the domain of integration in (5.21) into the sets

$$B_j^-:=\{x\in B:\lambda_j|Du_j|\le 1\}\quad,\quad B_j^+:=\{x\in B:\lambda_j|Du_j|>1\}.$$

Our proof of (5.18) has two parts. In the first part we consider the set B_j^+. We will show that

$$\frac{1}{\lambda_j}\int_{B_j^+}\left(F'(\xi_j+\lambda_j Du_j(x))-F'(\xi_j)\right)[D\varphi]\to 0,\quad \forall\varphi\in C_0^1(B,\mathbb{R}^N). \tag{5.22}$$

that is the B_j^+ contribution of the integral has no effect in the limit. This allows one to complete the proof by showing

$$\frac{1}{\lambda_j}\int_{B_j^-}\left(F'(\xi_j+\lambda_j Du_j(x))-F'(\xi_j)\right)[D\varphi]\to\int_B F''(\xi_\infty)[Du,D\varphi] \tag{5.23}$$

for all $\varphi\in C_0^1(B,\mathbb{R}^N)$.

Part I. Showing (5.22). By inequality (5.12) of step 1,

$$|B_j^+|<\lambda_j^s c_B(p,L). \tag{5.24}$$

We now make the following estimate using (5.24), the growth estimate for F' (5.5), the elementary inequality $ab^{p-1}\le a^p+b^p$ followed by $(a+b)^p\le 2^{p-1}(a^p+b^p)$, $|\xi_j|\le L$ and once again by inequality (5.12),

$$\begin{aligned}
&\frac{1}{\lambda_j}\Big|\int_{B_j^+}\left(F'(\xi_j+\lambda_j Du_j(x))-F'(\xi_j)\right)[D\varphi]\Big|\\
&\qquad\le c\frac{1}{\lambda_j}\left(|B_j^{+1}|+\left(\int_{B_j^+}|\lambda_j Du_j|^s dx\right)\right)\|D\varphi\|_{L^\infty}\\
&\qquad\le c_B(p,L)\lambda_j^{s-1}\|D\varphi\|_{L^\infty}.
\end{aligned}$$

Noting that $s \geq 2$, this proves (5.22) completing part I.

Part II. To show (5.23) we re-write the B_j^- part of integral on the left hand side of (5.23) as follows

$$\frac{1}{\lambda_j}\int_{B_j^-}\left(F'(\xi_j+\lambda_j Du_j(x))-F'(\xi_j)\right)[D\varphi]dt$$

$$=\int_{B_j^-}\int_0^1\left(F''(\xi_j+t\lambda_j Du_j)-F''(\xi_j)\right)[Du_j,D\varphi]dt$$

$$+\int_{B_j^-}F''(\xi_j)[Du_j,D\varphi].$$

We want the first term on the right hand side to tend to zero as $j\to\infty$. Let l be such that $\frac{1}{s}+\frac{1}{l}=1$, then we have

$$\int_{B_j^-}\int_0^1\left(F''(\xi_j+t\lambda_j Du_j)-F''(\xi_j)\right)[Du_j,D\varphi]dt$$

$$\leq\left(\int_{B_j^-}\left(\int_0^1|F''(\xi_j+t\lambda_j Du_j)-F''(\xi_j)|dt|D\varphi|\right)^l\right)^{\frac{1}{l}}$$

$$\cdot\left(\int_{B_j^-}|Du_j|^s\right)^{\frac{1}{s}}.$$

Clearly, by (5.12) we can bound the Du_j term by $c_B(p,L)$. Thus we would like to show

$$1_{B_j^{-1}}\int_0^1|F''(\xi_j+t\lambda_j Du_j)-F''(\xi_j)|dt\|D\varphi\|_{L^\infty}\to 0 \tag{5.25}$$

in measure and boundedly. Indeed it clearly follows from our estimate (5.24) on B_j^+, that $1_{B_j^-}\to 1_B$ in measure and boundedly. Thus (5.23) is true provided

$$1_B\lim_{j\to\infty}\int_0^1|F''(\xi_j+t\lambda_j Du_j)-F''(\xi_j)|dt\|D\varphi\|_{L^\infty}=0 \tag{5.26}$$

for $\mathcal{L}^n$ a.e. $x\in B$. This follows from the dominated convergence theorem. Thus we have (5.26), implying (5.25) and (5.23), completing part II.

Having shown parts I and II we conclude that

$$\int_B F''(\xi_\infty)[Du, D\varphi] = 0, \quad \forall \varphi \in C_0^1(B, \mathbb{R}^{N\times n})$$

as required.

Having shown (5.18) we note as a consequence of strong p-quasiconvexity and the continuity of F that F is strongly rank-1-convex, i.e. F'' satisfies the strong Legendre-Hadamard condition, $F''(\xi_\infty)[\eta,\eta] \geq 2\nu|\eta|^2$ with $\text{rank}(\eta) \leq 1$. Further by the continuity of F'', we have $|F''(\xi_\infty)| \leq M(L)$ where $M(L) := \sup_{|\xi|\leq L} |F''(\xi)|$. Thus the coefficients of the Legendre-Hadamard condition are finite (and constant) and we may apply Lemma 3.4, Section 3.2 of Chapter 3, to the system (5.18), obtaining immediately that $u \in \mathrm{C}^\infty(\mathrm{B}_1, \mathbb{R}^\mathrm{N})$, and by (3.28) and (3.27) of the same lemma,

$$\begin{aligned}
\fint_{B(0,\tau)} |Du - (Du)_\tau|^2 dy \leq & c\tau^2 \fint_{B(0,\frac{1}{2})} |Du - (Du)_{\frac{1}{2}}|^2 \\
& \leq c\tau^2 \fint_{B(0,\frac{1}{2})} |Du|^2 \\
& \leq c\tau^2 \left(\sup_{B(0,\frac{1}{2})} |Du| \right)^2 \\
& \leq c_1\tau^2 \left(\fint_{B(0,1)} |Du|^{s(p)} \right)^{\frac{2}{s(p)}}.
\end{aligned}$$

Finally, by $\|Du_j\|_{\mathrm{L}^{s(p)}(B_1,\mathbb{R}^{N\times n})} < c_B$ for all j , inequality (5.19) follows. Hence we have the estimate (5.20) for a constant C^* that only depends on ν and L (and n, N, F'').

As we mentioned earlier we are looking for a constant C that contradicts (5.9). By part (v) of Lemma 5.1 and the definition of u_j we find,

$$\limsup_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \leq \text{RHS} \tag{5.27}$$

where

$$\text{RHS} \leq \begin{cases} \limsup_{j\to\infty} \frac{c}{\lambda_j^2} \fint_{B(0,\tau)} |V(\lambda_j(Du_j - (Du_j)_{0,\tau}))|^2 & 1 < p < 2 \\ \limsup_{j\to\infty} c \fint_{B(0,\tau)} \Big(|Du_j - (Du_j)_{0,\tau}|^2 & p \geq 2 \\ \qquad + \lambda_j^{p-2} |Du_j - (Du_j)_{0,\tau}|^p \Big). \end{cases}$$

We will show at the end of step 3, with a simple argument, that if Du_j converges strongly in $\mathrm{L}^{s(p)}(B_1, \mathbb{R}^{N\times n})$, (5.20) together with (5.27) gives the desired contradiction (recall $\lambda_j^2 := E(x_j, r_j)$). Therefore our third and final step in proving proposition 5.1 is to show suitable strong convergence of Du_j in $\mathrm{L}^{s(p)}(B_1, \mathbb{R}^{N\times n})$ as defined below.

Step 3 (Strong convergence of u_j): In this step we will show that, for every $\sigma < 1$.

$$\lim_{j\to\infty} \int_{B(0,\sigma)} \frac{1}{\lambda_j^2} |V(\lambda_j(Du_j - Du))|^2 = 0 \tag{5.28}$$

for $1 < p < 2$ and similarly

$$\lim_{j\to\infty} \int_{B(0,\sigma)} \left(|Du_j - Du|^2 + \lambda_j^{p-2} |Du_j - Du|^p \right) = 0 \tag{5.29}$$

for $p \geq 2$. The standard way to obtain (5.28)-(5.29) for global minimisers is by use of a Caccioppoli inequality. In the local minimiser case we can not use the standard method to obtain an inequality of full Caccioppoli type (see [42]). Instead we stop short of deriving the full inequality and use direct techniques introduced in [42] and modified for $1 < p < 2$ in [13] to complete our proof. This 'pre-Caccioppoli' inequality is proved as in the global minimiser case with the construction of suitable comparison maps.

Fix $\alpha \in (0,1)$, $B(x_0, r) \subset B(0,1)$ and let $a_j \colon \mathbb{R}^n \to \mathbb{R}^N$ be the affine map such that $Da_j = (Du_j)_{x_0,r}$ and $(u_j - a_j)_{x_0,r} = 0$. It follows from (5.12) that there exists a constant M such that

$$|Da_j| \leq M, \quad \text{for all } j. \tag{5.30}$$

Now let $\rho \colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz cut off function satisfying $1_{B(x_0,\alpha r)} \leq \rho \leq 1_{B(x_0,r)}$ and $|D\rho| \leq \frac{2}{(1-\alpha)r}$. The standard comparison maps φ_j and ψ_j are defined by

$$\varphi_j := \rho(u_j - a_j) \quad \text{and} \quad \psi_j := (1-\rho)(u_j - a_j).$$

We prove that $u := a_j + \psi_j$ satisfies the local minimiser condition (5.16) according to the following lemma

Lemma 5.3. *Define ψ_j as above and I_j as in (5.14). Let $B_j = B(x_j, r_j)$ and assume that*

$$\limsup_{j\to\infty} [D\overline{u}]_{p,\mu,B_j} < \delta \tag{5.31}$$

where $\delta > 0$ is given by (5.1). Then if $\mu \leq n$, $u := a_j + \psi_j$ satisfies the $W^1\mathcal{L}^{p,\mu}$-local minimiser condition i.e. condition (5.16) with $\mathrm{X} = \mathrm{W}^1\mathcal{L}^{\mathrm{p},\mu}(\mathrm{B}_1)$, so that $I_j[u_j] \leq I_j[a_j + \psi_j]$.

Corollary 5.2. *Let*

$$\limsup_{j\to\infty} [D\overline{u}]_{*,B_j} < \delta$$

Then $u := a_j + \psi_j$ satisfies the $\mathrm{W}^1\mathrm{BMO}$-local minimiser condition i.e. condition (5.16) with $\mathrm{X} = \mathrm{W}^1\mathrm{BMO}(\mathrm{B}_1)$, so that $I_j[u_j] \leq I_j[a_j + \psi_j]$.

Proof. First note $u_j - a_j = \varphi_j + \psi_j$, thus

$$\begin{aligned}[Du - Du_j]_{p,\mu,B_1} =& [D\varphi_j]_{p,\mu,B_1} \\ =& [\rho(Du_j - Da_j) + D\rho \otimes (u_j - a_j)]_{p,\mu,B_1}.\end{aligned}$$

For $\mu \leq n$,

$$[Du_j - Da_j]_{p,\mu,B_1} = \sup_{\substack{x\in B_1 \\ R\in(0,2)}} \frac{1}{\lambda_j} \left(\frac{r_j^\mu}{r_j^n R^\mu} \int_{B(x,R)} |D\overline{u} - (D\overline{u})_{B(x,R)}|^p \right)^{\frac{1}{p}}. \tag{5.32}$$

Therefore it follows that

$$[Du - Du_j]_{p,\mu,B_1} \leq \frac{1}{\lambda_j r_j^{\frac{n-\mu}{p}}} \left([D\overline{u}]_{p,\mu,B_j} + \mathcal{R}_j[\overline{u}, \alpha, r] \right), \tag{5.33}$$

where

$$\mathcal{R}_j[\overline{u}, \alpha, r] := \frac{\lambda_j r_j^{\frac{n-\mu}{p}}}{(1-\alpha)r} [1_{B(x_0,r)}(u_j - a_j)]_{p,\mu,B_1}. \tag{5.34}$$

Clearly the first term in (5.33) is bounded by $\delta/(\lambda_j r_j^{\frac{n-\mu}{p}})$ for sufficiently large $j \geq J$ as a result of (5.31). To show that u satisfies (5.16) we must show that $\mathcal{R}_j[\overline{u}, \alpha, r] \to 0$ as $j \to \infty$ for arbitrarily fixed $\alpha, r \in (0, 1)$. Although it is only necessary in the proof

of Theorem 5.1 for a subsequence of $\{\mathcal{R}_j\}$ to converge to zero, we prove that the full sequence converges to zero in the case $\mu < n$.

Case $\mu < n$: For convenience we rewrite the sequence of functionals $\mathcal{R}_j$ as the functional $\overline{\mathcal{R}}_{\alpha,r}$ of the sequence of functions f_j^r i.e. we set $\overline{\mathcal{R}}_{\alpha,r}[f_j^r] := \mathcal{R}_j[\overline{u}, \alpha, r]$ where f_j^r is given by

$$f_j^r := \lambda_j^p r_j^{n-\mu} 1_{B(x_0,r)}(u_j - a_j). \tag{5.35}$$

Our strategy is to show first that $\{f_j^r\}$ is bounded in $\mathrm{W}^{1,p}(B_1)$ as are all subsequences (it is actually uniformly bounded in r but this is not important here). Then show the full sequence $\{f_j^r\}$ converges strongly to zero in $\mathrm{L}^p(B_1)$, $1 < p < \infty$. We do this by using Rellich-Kondrackov to show that given any subsequence of $\{f_j^r\}$ a further subsequence converges strongly to zero in $\mathrm{L}^p(B_1)$, $1 < p < \infty$. Following from the boundedness of $\{f_j^r\}$ in $\mathrm{W}^{1,p}(B_1)$ we then show that $\{f_j^r\}$ is also bounded in $W^1\mathcal{L}^{p,\mu}(B_1)$. This allows the use of strong convergence to zero in $\mathrm{L}^p(B_1)$ to prove $[f_j^r]_{1,p,\mu} \to 0$ for the full sequence $\{f_j^r\}$.

In particular for the first step using $(u_j - a_j)_{B(x_0,r)} = 0$ and (5.31), it follows by Poincarés inequality on balls that $\{f_j^r\}$ and any subsequence is bounded in $\mathrm{W}^{1,p}(B_1)$ for $1 < p < \infty$. Thus for any subsequence $\{f_{j_k}^r\}$, using $\lambda_j Du_j \to 0$ $\mathcal{L}^n$ a.e. and once again $(u_j - a_j)_{B(x_0,r)} = 0$, we have by Rellich-Kondrachov

$$f_{j_k}^r \to 0 \text{ in } \mathrm{L}^p(B_1), \quad 1 < p < \infty$$

for a further (suitably relabelled) subsequence. Therefore the full sequence $\{f_j^r\}$ converges strongly to zero in $\mathrm{L}^p(B_1)$, $1 < p < \infty$. Next, given boundedness of the full sequence $\{f_j^r\}$ in $\mathrm{W}^{1,p}(B_1)$ we use the following estimate derived from Poincarés inequality and the Morrey-Campanato inclusion (3.2) applicable to bounded domains Ω without external cusps and valid for Morrey-Campanato exponent $0 < \mu < n$,

$$[f_j^r]_{p,\mu,\Omega} \leq \begin{cases} c(p,\mu,\Omega)\|Df_j^r\|_{p,\Omega}, & \mu \leq p \\ \\ c(n,p,\mu,\Omega)\left(|\Omega|^{\frac{-\mu}{np}}\|Df_j^r\|_{p,\Omega} + [Df_j^r]_{p,\mu-p,\Omega}\right), & \mu > p. \end{cases} \tag{5.36}$$

This gives us boundedness of $\{f_j^r\}$ in $W^1\mathcal{L}^{p,\mu}(B_1)$ since

$$[Df_j^r]_{p,\mu,B_1} \leq [D\overline{u}]_{p,\mu,B_j}$$

and

$$[Df_j^r]_{p,\mu-p,B_1} \leq c[Df_j^r]_{p,\mu,B_1}, \quad \mu > p$$

by the Campanato embedding (3.12). Finally to prove $[f_j^r]_{p,\mu,B_1} \to 0$ we split the family of intersections of balls with B_1 over which we take the supremum in the semi-norm $[\cdot]_{p,\mu,B_1}$ into the family of balls with radius $s \in (S, \mathrm{diam}(B_1))$ and $s \in (0, S)$. We deal with these two cases separately. In the first case $\mathrm{diam}(B_1) > s > S$, by strong convergence of $\{f_j^r\}$ to zero in $\mathrm{L}^p(B_1)$,

$$\begin{aligned} s^{-\mu} \int_{B_1(x,s)} |f_j^r - (f_j^r)_{x,r}|^p &< c(S) \int_{B_1(x,s)} |f_j^r - (f_j^r)_{x,r}|^p \\ &< 2^{p-1} c(S) \left(\int_{B_1} |f_j^r|^p + \int_{B_1} |f_j^r|^p \right) \to 0 \end{aligned}$$

as $j \to \infty$. For the second case the boundedness of $\{f_j^r\}$ in $W^1\mathcal{L}^{p,\mu}(B_1)$ allows us to write the following. Given $\epsilon > 0$, take S such that $cS < \epsilon$ where c is a constant defined according to the inequality

$$\int_{B_1(x,s)} |f_j^r - (f_j^r)_{x,s}|^p \leq c s^{p+\mu}.$$

Using Poincarés inequality for balls the above inequality follows from the Morrey-Campanato isomorphism (on balls and their intersections) and the boundedness of $\{f_j^r\}$ in $W^1\mathcal{L}^{p,\mu}(B_1)$. Hence given any $\epsilon > 0$ there exists a J such that for $j \geq J$

$$[f_j^r]_{p,\mu,B_1} < \epsilon$$

for the full sequence defined in (5.35). We remark that J is independent of r since convergence is uniform in r. However this is not the case for $\overline{\mathcal{R}}_{\alpha,r}[f_j^r]$ which converges to zero for each pair (α, r) as required, but not uniformly in either α or r.

Case $\mu = n$: By the Campanato-BMO isometry, Proposition 3.2, Chapter 3, Section 3.1.1 there exists a $c \in [|B_1|, 2^{p+n}|B_1|c_*]$ such that

$$[u_j - a_j]_{p,n,B_1} = c[u_j - a_j]_{*,B_1}. \tag{5.37}$$

We estimate the above semi-norm using the L^∞ norm,

$$[u_j - a_j]_{*,B_1} \leq \sup_{B \subset B_1} \left(\operatorname*{ess\,.sup}_{x \in B} |(u_j - a_j)(x) - (u_j - a_j)_B| \right). \tag{5.38}$$

To make sense of this estimate we use the fact that $\mathrm{W}^1\mathrm{BMO}(\Omega) \hookrightarrow \mathrm{W}^{1,\mathrm{q}}(\Omega)$ for all $1 \leq q < \infty$ and general open and bounded Ω. We set $q > n$, then make use of Morrey's inequality. Our aim is to show that the sequence

$$\left\{ \operatorname*{ess\,.sup}_{x \in B_1} |\lambda_j (u_j - a_j)(x)| \right\} \tag{5.39}$$

converges to zero as $j \to \infty$ (note that direct estimation of (5.38) results in $\sup_j [u_j - a_j]_{*,B_1} \leq \infty$, not sufficient to show $\lambda_j [u_j - a_j]_{*,B_1} \to 0$). We start by showing that the sequence is bounded. By Morrey's inequality

$$|(u_j - a_j)(x) - (u_j - a_j)(y)| \leq c R_{x,y} \Big(\fint_{B(0,R_{x,y})} |Du_j - Da_j|^q \Big)^{\frac{1}{q}} \tag{5.40}$$

for $\mathcal{L}^n$-a.e. $x, y \in B_1$ and every $R_{x,y} \geq 1$. The integral on the right may be estimated as follows

$$\Big(\fint_{B_1} |Du_j - Da_j|^q \Big)^{\frac{1}{q}} \leq |(Du_j - Da_j)_{B_1}| + \Big(\fint_{B_1} |Du_j - Da_j - (Du_j - Da_j)_{B_1}|^q \Big)^{\frac{1}{q}}.$$

By noting that $(Du_j)_{B_1} = 0$ and $|Da_j| < M$ we see immediately that the first term on the right is uniformly bounded. For the remainder we apply the equality of (5.32) for change of variables. Thus

$$\begin{aligned} \Big(\fint_{B_1} |Du_j - Da_j|^q \Big)^{\frac{1}{q}} &\leq M + [Du_j - Da_j]_{p,n,B_j} \\ &= M + \frac{1}{\lambda_j} [D\overline{u}]_{p,n,B_j}. \end{aligned} \tag{5.41}$$

Therefore, given that we can extend $Du_j - Da_j = 0$ off B_j, choosing $R_{x,y} = 2|x - y|$ (so that $B(0, R_{x,y}) \subset B(0,4)$), we find that $\lambda_j (u_j^* - a_j)$ where u_j^* denotes the precise representative of u_j, has a uniformly bounded $(1 - \frac{n}{q})$th-Hölder semi-norm over B_1. Thus by the implied continuity of u_j^* there exists for each component $(u_j - a_j)^{(k)}$, $k = 1, \ldots, N$, a point $y_k \in B_1$ such that $(u_j^* - a_j)^{(k)}(y_k) = (u_j^* - a_j)^{(k)}_{x_0,r} = (u_j - a_j)^{(k)}_{x_0,r} = 0$ and so

$$|(u_j^* - a_j)^{(k)}(x)| \leq |(u_j^* - a_j)(x) - (u_j^* - a_j)(y_k)|. \tag{5.42}$$

Therefore by taking $R_{x,y} = 1$ and substituting u_j^* for u_j in (5.40) it follows from (5.42) and (5.41) that the sequence $\{\lambda_j (u_j^* - a_j)^{(k)}\}$ is bounded uniformly on B_1 for each $k = 1, \ldots, N$. Thus the whole sequence (5.39) is bounded as required. It

now follows that $\lambda_j(u_j^* - a_j)$ has a uniformly bounded $(1 - \frac{n}{q})$th-Hölder norm over B_1 and thus the sequence $\{\lambda_j(u_j^* - a_j)\}$ is Hölder equicontinuous on B_1. Therefore $\{\lambda_j(u_j^* - a_j)\} \subset \mathrm{C}(B_1)$ and by its boundedness can uniquely be extended to $\mathrm{C}(\overline{B_1})$ as can any subsequence $\{\lambda_{j_k}(u_{j_k}^* - a_{j_k})\}$. Hence, after extracting a further subsequence if required, by Arzel-Ascoli combined with the properties $\lambda_j Du_j \to 0$ $\mathcal{L}^n$- a.e. and $(u_j - a_j)_{x_0,r} = 0$,

$$\lambda_{j_k}(u_{j_k}^* - a_{j_k}) \to 0$$

uniformly on B_1. This means, after extracting to a subsequence where necessary, that (5.39) tends to zero as required and $\mathcal{R}_{j_k}[\overline{u}, r, \alpha] \to 0$ then follows from (5.38). □

Now it is straight forward to prove the Corollary to Lemma 5.3:

Proof of Corollary 5.2. From the proof of Lemma 5.3 it is clear, as a result of equivalence of $\mathcal{L}^{p,n}$ and BMO on B_1 and in particular equivalence relation (5.37), that we may replace $[\cdot]_{p,n,B_j}$ and $[\cdot]_{p,n,B_1}$ semi-norms with $[\cdot]_{*,B_j}$ and $[\cdot]_{*,B_1}$ semi-norms in the proof of the Lemma. □

Using Lemma 5.3/Corollary 5.2 we can now follow the method of [13] and derive an inequality of pre-Caccioppoli type presented here for $1 < p < 2$:

$$\begin{aligned}
&\int_{B(x_0,\alpha r)} |V(\lambda_j(Du_j - Du))|^2 \\
&\qquad \leq \theta \int_{B(x_0,r)} |V(\lambda_j(Du_j - Du))|^2 + c \int_{B(x_0,r)} |V(\lambda_j(Du - Da_j))|^2 \qquad (5.43) \\
&\qquad + c \int_{B(x_0,r)} \frac{|V(\lambda_j(u_j - a_j))|^2}{(1-\alpha)^2 r^2} + c \int_{B(x_0,r)\setminus B(x_0,\alpha r)} |V(\lambda_j(Da_j))|^2
\end{aligned}$$

With $\theta < 1$. In the case $p \geq 2$ one simply replaces the function $V(\xi)$ with $|\xi|^2 + |\xi|^p$. We summarise the proof of (5.43) given in [13, 14]. To start we estimate

$$\frac{1}{\lambda_j^2} \int_{B(x_0,\alpha r)} |V(\lambda_j(Du_j - Da_j))|^2 = \int_{B(x_0,\alpha r)} (1 + |\lambda_j D\varphi_j|^2)^{\frac{p-2}{2}} |D\varphi_j|^2$$

in terms of F_j using quasiconvexity of F_j, (5.15). Given $|\xi_j| \leq L$ and (5.30) for all j, there exists a constant $c_J > 0$ dependent only on p, L and ν of (5.15) such that for

$j \geq J$ (J sufficiently large), $1 \leq c_J\nu(1+|\xi_j+\lambda_j Da_j|^2)^{\frac{p-2}{2}}$. Thus

$$\frac{1}{\lambda_j^2}\int_{B(x_0,\alpha r)}|V(\lambda_j(Du_j-Da_j))|^2 \leq c_J\nu\int_{B(x_0,r)}(1+|\xi_j+\lambda_j Da_j|^2+|\lambda_j D\varphi_j|^2)^{\frac{p-2}{2}}|D\varphi_j|^2$$

$$\leq c_J\int_{B(x_0,r)}(F_j(Da_j+D\varphi_j)-F_j(Da_j)). \tag{5.44}$$

To guarantee $\theta<1$ in (5.43) we estimate the right hand integral in such a way that we may remove $B(x_0,\alpha r)$ from the domain of integration $B(x_0,r)$. By construction, $Da_j+D\varphi_j=Du_j$ on $B(x_0,\alpha r)$, thus

$$\int_{B(x_0,r)}(F_j(Da_j+D\varphi_j)-F_j(Da_j))$$

$$\leq\int_{B(x_0,r)\setminus B(x_0,\alpha r)}F_j(Da_j+D\varphi_j)-F_j(Du_j))$$

$$+\int_{B(x_0,r)}(F_j(Du_j)-F_j(Da_j)).$$

Now given Lemma 5.3/Corollary 5.2 (implying that for sufficiently large j, $I_j[u_j]\leq I_j[u]$ where $u:=a_j+\psi_j$) and using $D\psi_j=0$ on $B(x_0,\alpha r)$ we obtain

$$\int_{B(x_0,r)}(F_j(Da_j+D\varphi_j)-F_j(Da_j))$$

$$\leq\int_{B(x_0,r)\setminus B(x_0,\alpha r)}F_j(Da_j+D\varphi_j)-F_j(Du_j))$$

$$+\int_{B(x_0,r)\setminus B(x_0,\alpha r)}(F_j(Da_j+D\psi_j)-F_j(Da_j)).$$

Next by (5.7) of Lemma 5.2 and properties of V, Lemma 5.1 (and $|D\rho|\leq 2/(1-\alpha)r$)

$$\int_{B(x_0,r)}(F_j(Da_j+D\varphi_j)-F_j(Da_j))$$

$$\leq\frac{c(c_1,p)}{\lambda_j^2}\int_{B(x_0,r)\setminus B(x_0,\alpha r)}\Big(|V(\lambda_j(Du_j-Da_j))|^2 \tag{5.45}$$

$$+\left|\frac{V(\lambda_j(u_j-a_j)}{(1-\alpha)r}\right|^2+|V(\lambda_j Da_j)|^2\Big).$$

Finally to obtain (5.43) with $\theta<1$ we first add and subtract Du within the first instance of V on the right hand side of (5.45). Thus using Lemma 5.1, combining the

result with (5.44) and then adding

$$\frac{1}{\lambda_j^2}\int_{B(x_0,\alpha r)}|V(\lambda_j(Du-Da_j))|^2$$

to both sides, we obtain

$$\begin{aligned}\frac{1}{\lambda_j^2}\int_{B(x_0,\alpha r)}&\Big(|V(\lambda_j(Du_j-Da_j))|^2+|V(\lambda_j(Du-Da_j))|^2\Big)\\ &\leq\frac{c}{\lambda_j^2}\int_{B(x_0,r)\setminus B(x_0,\alpha r)}\left(|V(\lambda_j(Du-Da_j))|^2+|V(\lambda_j(Du_j-Du))|^2\right) \qquad (5.46)\\ &\quad+\frac{c}{\lambda_j^2}\int_{B(x_0,r)\setminus B(x_0,\alpha r)}\left(|\frac{V(\lambda_j(u_j-a_j)}{(1-\alpha)r}|^2+|V(\lambda_j Da_j)|^2\right)\end{aligned}$$

where the constant c depends only on p, c_1 and c_J. Now using Lemma 5.1

$$\begin{aligned}\frac{1}{\lambda_j^2}\int_{B(x_0,\alpha r)}|V(\lambda_j(Du_j-Du))|^2\leq\frac{2^{p+1}}{\lambda_j^2}\int_{B(x_0,\alpha r)}&\left(|V(\lambda_j(Du_j-Da_j))|^2\right.\\ &\left.+|V(\lambda_j(Du-Da_j))|^2\right).\end{aligned}$$

Thus by multiplying (5.46) through by 2^{p+1} and combining with the above we finalise the calculation by filling the hole. I.e. by adding

$$\frac{\tilde{c}}{\lambda_j^2}\int_{B(x_0,\alpha r)}|V(\lambda_j(Du_j-Du))|^2$$

to both sides (where $\tilde{c}:=2^{p+1}\cdot c$). Hence obtaining (5.43) with $\theta=\frac{\tilde{c}}{\tilde{c}+1}$.

Weak Convergence of measures: We follow precisely the argument of [13] for $1<p<2$ and [42] for the case $p\geq 2$. Once again we reproduce it here for the convenience of the reader. In the case $1<p<2$ [13] required a Sobolev-Poincaré type inequality for the auxiliary function V as introduced in [14]. We present a refined version of this inequality proved in [21]:

Lemma 5.4. *Let $p\in(1,2)$, $B(x_0,r)\subset\mathbb{R}^n$ with $n\geq 2$ and set $p^{\#}:=\frac{2n}{n-p}$. Then*

$$\left(\fint_{B(x_0,r)}|V(\frac{u-u_{x_0,r}}{r})|^{p^{\#}}\,dx\right)^{\frac{1}{p^{\#}}}\leq c\left(\fint_{B(x_0,r)}|V(Du)|^2\,dx\right)^{\frac{1}{2}} \qquad (5.47)$$

for any $u\in \mathrm{W}^{1,p}(B(x_0,r),\mathbb{R}^N)$ and where c depends only on n, N, and p.

Unlike the inequality of [14], the radius of the ball is not increased on the right hand side but is kept the same. Note that this refinement marginally simplifies, but is not critical for, the proceeding proof.

First we claim that

$$\frac{1}{\lambda_j^2}|V(\lambda_j(Du_j - Du))|^2\mathcal{L}^n \rightharpoonup \mu \text{ in } \mathrm{C}_0(\overline{B})^* \tag{5.48}$$

for $1 < p < 2$ and

$$\left(|Du_j - Du|^2 + \lambda_j^{p-2}|Du_j - Du|^p\right)\mathcal{L}^n \rightharpoonup \mu \text{ in } \mathrm{C}_0(\overline{B})^* \tag{5.49}$$

for $p \geq 2$ where μ is a Radon measure.

As in [13], this claim follows from the bound imposed on the sequence of measures in (5.48) by

$$\int_B \frac{1}{\lambda_j^2}|V(\lambda_j(Du_j - Du))|^2 \leq 2^{p+1}c_0(p,L)|B|\fint_{B_j}\frac{1}{\lambda_j^2}|V(D\overline{u}) - V((D\overline{u})_{x_j,r_j})|^2 + 2^{p+1}\int_B |Du|^2$$

and estimate (5.11). Similarly the bound for the sequence in (5.49) follows from (5.10).

It is now straightforward to show that limit form of the pre-Caccioppoli inequality matches that of [42]. For $1 < p < 2$ using properties of V as in [13]

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{\lambda_j^2}\int_{B(x_0,r)}|V(\lambda_j(Du - Da_j))|^2 &\leq \int_{B(x_0,r)}|Du - Da|^2 \\ &= \epsilon_1(r)r^n\end{aligned}$$

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{\lambda_j^2}\int_{B(x_0,r)\setminus B(x_0,\alpha r)}|V(\lambda_j Da_j)|^2 &\leq c|Da|^2 r^n(1-\alpha)^n \\ &= \epsilon_2(r)r^n(1-\alpha)^n\end{aligned}$$

The final estimate follows from the Sobolev Poincaré inequality (5.47) of Lemma 5.4, Rellich-Kondrachov compactness theorem and Vitali's Lemma.

From Sobolev Poincaré inequality (5.47)

$$\int_{B(x_0,r)}|\frac{1}{\lambda_j}V(\lambda_j(u_j - a_j))|^{p^\#} \leq c_1$$

and since $p^{\#} > 2$,

$$\int_{B(x_0,r)} |\frac{1}{\lambda_j} V(\lambda_j(u_j - a_j))|^2 \leq c_2.$$

Thus given $\frac{2n}{n-p} > 1$, the sequence $\{v_j\}$ defined by

$$v_j(x) := \frac{1}{\lambda_j} V(\lambda_j(u_j - a_j))$$

is eqi-integrable. Now by Rellich-Kondrachov compactness theorem $u_j \to u$ in $\mathrm{L}^1(\mathrm{B}_1)$. Thus for a suitably relabelled subsequence it follows from the definition of V that

$$v_j(x) \to (u - a)(x) \text{ for } \mathcal{L}^n\text{-a.e. } x \in B_1.$$

Hence by Vitali's lemma

$$\lim_{j\to\infty} \frac{1}{\lambda_j^2} \int_{B(x_0,r)} \frac{|V(\lambda_j(u_j - a_j))|^2}{(1-\alpha)^2 r^2} = \frac{1}{(1-\alpha)^2 r^2} \int_{B(x_0,r)} |u - a|^2$$

$$= \epsilon_3(r) \frac{1}{(1-\alpha)^2} r^n$$

for a suitably relabelled subsequence, where

$$\epsilon_1 := \frac{1}{r^n} \int_{B(x_0,r)} |Du - Da|^2,$$

$$\epsilon_2 := c|Da|^2,$$

$$\epsilon_3 := \frac{1}{r^{n+2}} \int_{B(x_0,r)} |Du - Da|^2.$$

If we make the transformation $V(\xi) \mapsto |\xi|^2 + |\xi|^p$ it is easily verified that these limits hold for $p \geq 2$. Thus by the pre-Caccioppoli inequality (5.43)

$$\mu(B[x_0, \alpha r]) \leq \theta \mu(B[x_0, r]) + \left(\frac{\epsilon_3(r)}{(1-\alpha)^2} + \epsilon_2(r)(1 - \alpha^n) + \epsilon_1(r) \right) r^n$$

for $p > 1$, and following the direct methods of [42] and [13] we obtain

$$\liminf_{r\to 0^+} \frac{\mu(B[x_0, r])}{r^n} = 0.$$

Hence by Vitali's covering theorem

$$\mu(B[0, \sigma]) = 0$$

for each fixed $\sigma \in (0,1)$ implying (5.28) and (5.29), completing step 3.

We finish by recalling the estimate (5.27) from which

$$\lim_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \leq \lim_{j\to\infty} \frac{c}{\lambda_j^2} \fint_{B(0,\tau)} \Big[|V(\lambda_j(Du_j - Du))|^2 + |V(\lambda_j(Du - (Du)_{0,\tau}))|^2 + |V(\lambda_j((Du)_{0,\tau} - (Du_j)_{0,\tau}))|^2 \Big]$$

by iii.) of Lemma 5.1, and $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Thus by (5.20), (5.28) and (i) of the same lemma

$$\lim_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \leq C^*(p, L)\tau^2 + \lim_{j\to\infty} |(Du)_{0,\tau} - (Du_j)_{0,\tau}|^2. \tag{5.50}$$

Similarly we show for $p \geq 2$ that

$$\lim_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \leq C^*(p, L)\tau^2 + \lim_{j\to\infty} \big(|(Du)_{0,\tau} - (Du_j)_{0,\tau}|^2 + \lambda_j^{p-2} |(Du)_{0,\tau} - (Du_j)_{0,\tau}|^p \big). \tag{5.51}$$

Now since $Du_j \rightharpoonup Du$ weakly in $\mathrm{L}^{s(p)}(B(0,1), \mathbb{R}^{N\times n})$ ($s(p) = \min\{2, p\}$) the right hand limits in (5.50) and (5.51) are zero.

Thus

$$\lim_{j\to\infty} \frac{E(x_j, \tau r_j)}{\lambda_j^2} \leq C^*(p, L)\tau^2$$

which contradicts (5.9) with $C_L = 2C^*(p, L)$. $\square$

5.3 Proof from Blowup

Having proved the proposition we are in a position to prove Theorem 5.1, using the well established method first used in this context by EVANS [22] for the case $p \geq 2$.

Extending the exposition in [3] to the case $1 < p < 2$ we will prove the following lemma

Lemma 5.5. *Let $\overline{u}$ satisfy Proposition 5.1, $0 < \alpha < 1$ and take constant $C(L)$ of the proposition and constant $c(p, L)$ of Lemma 5.1. If for each $L > 0$ and $\tau \in (0, \frac{1}{4})$ such*

that $C(L)\tau^2 \leq \tau^{2\alpha}$*, there exists an* $\epsilon_0(L,\tau) \in (0, \min\{\tau^n(1-\tau^{2\alpha}), c(p,L)^{-2}\tau^{\frac{2n}{p}}(1-\tau^\alpha)^2\})$*, such that for* $B(x_0,R) \subset \Omega$ *with*

$$|(D\overline{u})_{x_0,R}| < L \tag{5.52}$$

and

$$E(x_0,R) < \epsilon_0(L,\tau), \tag{5.53}$$

then for every $k \in \mathbb{N}$ *we have*

$$E(x_0,\tau^k R) \leq \tau^{2k\alpha} E(x_0,R), \text{ and } |(D\overline{u})_{x_0,\tau^k R}| < L+1.$$

Proof. By iteration of the triangle inequality

$$\begin{aligned}
|(D\overline{u})_{x_0,\tau^l R}| &\leq |(D\overline{u})_{x_0,R}| + \sum_{k=1}^{l} |(D\overline{u})_{x_0,\tau^k R} - (D\overline{u})_{x,\tau^{k-1}R}| \\
&\leq L + \sum_{k=1}^{l} \left(\fint_{B(x_0,\tau^k R)} |D\overline{u} - (D\overline{u})_{x_0,\tau^{k-1}R}|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq L + \sum_{k=1}^{l} \left(\tau^{-n} \fint_{B(x_0,\tau^{k-1} R)} |D\overline{u} - (D\overline{u})_{x_0,\tau^{k-1}R}|^2 \, dx \right)^{\frac{1}{2}} \\
&= L + \tau^{-\frac{n}{2}} \sum_{k=1}^{l} \left(E(x_0,\tau^{k-1}R) \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.54}$$

Let $l=1$ and assume that

$$|(D\overline{u})_{x_0,R}| < L, \quad E(x_0,R) < \epsilon_0.$$

Then (5.55) implies

$$|(D\overline{u})_{x_0,\tau R}| \leq L + \tau^{-\frac{n}{2}} \epsilon_0^{\frac{1}{2}}.$$

Put $\epsilon_0^{\frac{1}{2}} \leq \tau^{\frac{n}{2}} \epsilon^{\frac{1}{2}}$ to get $|(D\overline{u})_{x_0,\tau R}| \leq L+1$. Let $l=2$ then (5.55) implies

$$|(D\overline{u})_{x_0,\tau^2 R}| \leq L + C(L)^{\frac{1}{2}} \tau^{-\frac{n}{2}} \epsilon_0^{\frac{1}{2}}.$$

Put $\epsilon_0^{\frac{1}{2}} \leq \tau^{\frac{n}{2}+\alpha-1} \epsilon^{\frac{1}{2}}$ to get $|(D\overline{u})_{x_0,\tau R}| \leq L+1$.

Now assume that

$$|(D\overline{u})_{x_0,R}| < L, \quad |(D\overline{u})_{x_0,\tau^k R}| < L+1.$$

for $k < l$ and

$$E(x,\tau^k R) \le (C(L)\tau^2)^k E(x,R) \le \tau^{2k\alpha} E(x,R), \quad \forall k < l.$$

Then for the lth iteration

$$|(D\overline{u})_{x,\tau^l R}| \le L + \tau^{-\frac{n}{2}} \sum_{k=1}^{l} \left(\tau^{2\alpha(k-1)} E(x,R)\right)^{\frac{1}{2}}$$

$$< L + \frac{\tau^{-\frac{n}{2}}}{(1-\tau^{2\alpha})^{\frac{1}{2}}} \epsilon_0^{\frac{1}{2}}$$

Thus $|(D\overline{u})_{x,\tau^l R}| \le L+1$ provided $\epsilon_0 < \tau^n(1-\tau^{2\alpha}) < \frac{1}{4^n}$. Thus by induction, Proposition 5.1 implies the result for $p \ge 2$.

Case $1 < p < 2$: We will need to split the domain of integration into two parts. Let $B_r^+(x_0) := B(x_0,r) \cap \{x : |D\overline{u} - (D\overline{u})_{x_0,r}| \le 1\}$ and $B_r^-(x_0) := B(x_0,r) \cap \{x : |D\overline{u} - (D\overline{u})_{x_0,r}| > 1\}$ and proceed by induction. Assume that (5.52) and (5.53) hold, then

$$\begin{aligned}
|(D\overline{u})_{x_0,\tau R}| &\le |(D\overline{u})_{x_0,R}| + |(D\overline{u})_{x_0,\tau R} - (D\overline{u})_{x_0,R}| \\
&\le L + \fint_{B^+_{\tau R}(x_0)} |D\overline{u} - (D\overline{u})_{x_0,R}|\, dx + \fint_{B^-_{\tau R}(x_0)} |D\overline{u} - (D\overline{u})_{x_0,R}|\, dx \\
&\le L + 2^{-\frac{p-2}{4}} \fint_{B^+_{\tau R}(x_0)} |V(D\overline{u} - (D\overline{u})_{x_0,R})|\, dx \\
&\quad + \left(\fint_{B^-_{\tau R}(x_0)} |D\overline{u} - (D\overline{u})_{x_0,R}|^p\, dx\right)^{\frac{1}{p}} \qquad (5.55) \\
&\le L + 2^{-\frac{p-2}{4}} \left[\left(\fint_{B(x_0,\tau R)} |V(D\overline{u} - (D\overline{u})_{x_0,R})|^2\, dx\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \left(\fint_{B(x_0,\tau R)} |V(D\overline{u} - (D\overline{u})_{x_0,R}|^2\, dx\right)^{\frac{1}{p}}\right]
\end{aligned}$$

where Lemma 5.1 part (i) has been applied in the last two inequalities. Using the bound on the integral average of the gradient (5.52) we may apply part (vi) of the

Lemma 5.1. Thus

$$|(D\overline{u})_{x_0,\tau R}| \le L + c(p,L)\left((\tau^{-n}E(x_0,R))^{\frac{1}{2}} + (\tau^{-n}E(x_0,R))^{\frac{1}{p}}\right)$$

$$\le L + c(p,L)(\tau^{-\frac{n}{2}}\epsilon_0^{\frac{1}{2}} + \tau^{-\frac{n}{p}}\epsilon_0^{\frac{1}{p}})$$

Since $\epsilon_0^{\frac{1}{p}} \le \epsilon_0^{\frac{1}{2}}$, put $\epsilon_0^{\frac{1}{2}} \le c(p,L)^{-1}\tau^{\frac{n}{2}}$ to get $|(D\overline{u})_{x_0,\tau R}| \le L+1$. Now assume

$$|(D\overline{u})_{x_0,\tau^{k-1}R}| \le L+1$$

for $k \le l$. Then in a similar way to (5.55) case $p \ge 2$

$$|(D\overline{u})_{x_0,\tau^l R}| \le |(D\overline{u})_{x_0,R}| + \sum_{k=1}^{l} |(D\overline{u})_{x_0,\tau^k R} - (D\overline{u})_{x,\tau^{k-1}R}|$$

$$\le L + \sum_{k=1}^{l} \fint_{B(x_0,\tau^k R)} |D\overline{u} - (D\overline{u})_{x_0,\tau^{k-1}R}|\,dx$$

$$\le L + c(p,L+1)\sum_{k=1}^{l}\left((\tau^{-n}E(x_0,\tau^{k-1}R))^{\frac{1}{2}} + (\tau^{-n}E(x_0,\tau^{k-1}R))^{\frac{1}{p}}\right)$$

Where we have used Lemma 5.1 as in (5.55). Assume also that

$$E(x_0,\tau^{k-1}R) \le \tau^{2\alpha(k-1)}E(x_0,R), \quad \forall k \le l.$$

Then

$$|(D\overline{u})_{x_0,\tau^l R}| \le c\left(\tau^{-\frac{n}{2}}\sum_{k=1}^{l}\tau^{\alpha(k-1)}\epsilon_0^{\frac{1}{2}} + \tau^{-\frac{n}{p}}\sum_{k=1}^{l}\tau^{\frac{2\alpha}{p}(k-1)}\epsilon_0^{\frac{1}{p}}\right)$$

$$< c\left(\frac{\tau^{-\frac{n}{2}}}{1-\tau^{\alpha}}\epsilon_0^{\frac{1}{2}} + \frac{\tau^{-\frac{n}{p}}}{1-\tau^{\frac{2\alpha}{p}}}\epsilon_0^{\frac{1}{p}}\right).$$

Thus, given

$$\epsilon_0^{\frac{1}{2}} \le c^{-1}\tau^{\frac{n}{p}}(1-\tau^{\alpha}) \le c^{-1}\tau^{\frac{n}{p}}(1-\tau^{\frac{2\alpha}{p}}) \le c^{-1}\tau^{\frac{n}{2}}(1-\tau^{\alpha}) \le c^{-1}\tau^{\frac{n}{2}}(1-\tau^{\frac{2\alpha}{p}}),$$

we have $|(D\overline{u})_{x_0,\tau^l R}| \le L+1$ provided $\epsilon_0^{\frac{1}{2}} \le c^{-1}\tau^{\frac{n}{p}}(1-\tau^{\alpha})$. Thus once again by induction, Proposition 5.1 implies the result. □

Proof of Theorem 5.1. Set

$$\Omega_0 = \Big\{x \in \Omega : \lim_{r\to 0^+}(D\overline{u})_{x,r} = D\overline{u}(x), \ \ \lim_{r\to 0^+}\fint_{B(x,r)}|D\overline{u} - (D\overline{u})_{x,r}|^p dy = 0\Big\}.$$

By the first condition within the braces we have that the precise representative of $D\overline{u}$ coincides with $D\overline{u}$ on Ω_0. Therefore given the first condition, points that satisfy the second condition are Lebesgue points and $|\Omega \setminus \Omega_0| = 0$. Note that from the definition of the Excess in the $p \geq 2$ we simply have by Jensen's inequality

$$E(x,r) \leq 2 \left(\fint_{B(x,r)} |D\overline{u} - (D\overline{u})_{x,r}|^p \, dy \right)^{\frac{1}{p}},$$

and in the case $1 < p < 2$ by Lemma 5.1,

$$E(x_0,r) \leq c \fint_{B(x_0,r)} |V(D\overline{u} - (D\overline{u})_{x_0,r})|^2 \, dx \leq c \fint_{B(x_0,r)} |D\overline{u} - (D\overline{u})_{x_0,r}|^p \, dx.$$

Let $x_0 \in \Omega_0$ then for each $L, \tau > 0$ of Lemma 5.5 we can fix a sufficiently small $R \in (0, \mathrm{dist}(\mathrm{x}_0, \Omega))$ so that (5.52) and (5.53) are satisfied with $\epsilon_0 \leq \min\{\tau^n(1 - \tau^{2\alpha}, c(p,L)^{-2}\tau^{\frac{2n}{p}}(1-\tau^\alpha)^2\}$. Thus in view of Lemma 5.5 let $x_0 \in \Omega_0$, $r \in (0, \frac{R}{4})$ and $0 < \alpha < 1$. It then follows that there exists a unique $k \in \mathbb{N}$ such that for $\tau^k R < r \leq \tau^{k-1} R$,

$$\begin{aligned} E(x_0,r) &\leq \tau^{-n} E(x, \tau^{k-1}R) \\ &\leq \tau^{-n}\tau^{2\alpha(k-1)} E(x,R) \\ &= \tau^{-n}\tau^{2\alpha(k-1)} E(x,R) \\ &< \tau^{-n} \left(\frac{r}{R}\right)^{2\alpha} E(x,R), \end{aligned} \tag{5.56}$$

where R depends on L and τ. Therefore

$$E(x_0,r) \leq c(\tau, L, n) \cdot r^{2\alpha}.$$

For $p \geq 2$, direct application of Jensen's inequality implies

$$\fint_{B(x_0,r)} |D\overline{u} - (D\overline{u})_{B(x_0,r)}|^2 \, dx \leq \left(\fint_{B(x_0,r)} |D\overline{u} - (D\overline{u})_{B(x_0,r)}|^2 \, dx \right)^{\frac{1}{2}} \leq c r^\alpha$$

with $0 < \alpha < 1$ and $r \in (0, \frac{R}{4})$ where $R \in (0, \mathrm{dist}(\mathrm{x}_0, \mathrm{R})$ is sufficiently small. Hence applying Campanato's characterisation of Hölder continuous functions, Theorem 3.1, we find the precise representative $D\overline{u} \in C^{0,\alpha}(B(x_0,R))$, for any $0 < \alpha < 1$. Left with

the case $1 < p < 2$ we see once again by Lemma 5.1

$$\begin{aligned}
\fint_{B(x_0,r)} |D\overline{u} - (D\overline{u})_{x_0,r}|\,dx &\le \fint_{B^+_{\tau R}(x_0)} |D\overline{u} - (D\overline{u})_{x_0,R}|\,dx + \fint_{B^-_{\tau R}(x_0)} |D\overline{u} - (D\overline{u})_{x_0,R}|\,dx \\
&\le 2^{-\frac{p-2}{4}} \fint_{B^+_{\tau R}(x_0)} |V(D\overline{u} - (D\overline{u})_{x_0,R})|\,dx \\
&\quad + \left(\fint_{B^-_{\tau R}(x_0)} |D\overline{u} - (D\overline{u})_{x_0,R}|^p\,dx \right)^{\frac{1}{p}} \\
&\le 2^{-\frac{p-2}{4}} \left[\left(\fint_{B(x_0,\tau R)} |V(D\overline{u} - (D\overline{u})_{x_0,R})|^2\,dx \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\fint_{B(x_0,\tau R)} |V(D\overline{u} - (D\overline{u})_{x_0,R}|^2\,dx \right)^{\frac{1}{p}} \right] \\
&\le c(p,L)[E(x_0,r)^{\frac{1}{2}} + E(x_0,r)^{\frac{1}{p}}] \\
&\le c(p,n,L,\tau)\cdot r^{\alpha}
\end{aligned}$$

where we have used (5.56) and assumed that $\epsilon_0^{\frac{1}{2}} \le 1$ and $r \le 1$ in the final inequality. Again this applies for $0 < \alpha < 1$ and $r \in (0, \frac{R}{4})$ where $R \in (0, \mathrm{dist}(\mathrm{x_0}, \mathrm{R})$ is sufficiently small, whence $D\overline{u} \in C^{0,\alpha}(B(x_0,R))$, for any $0 < \alpha < 1$. This completes the proof. □

Finally we can now prove Corollary 4.1 of Theorem 4.3 and Corollary 5.1. The proof is straight forward and requires one to take note of the distinction between $\|\cdot\|_{\mathrm{BMO}}$ and $[\cdot]_{*,\Omega}$. We restate the Corollary here for the convenience of the reader:

Corollary 5.3. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be C^2, $\Omega \subset \mathbb{R}^n$ open and bounded Let $\overline{u} \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$, $1 < p < \infty$ be a critical point of $I[\cdot]$ with strongly positive second variation such that for some $\delta_s > 0$ and all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathrm{N}}) \cap \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^{\mathrm{N}})$ we have (1.11) and (1.12). Suppose also that we have*

$$|F''(\xi) - F''(\eta)| \le \omega(|\xi - \eta|) \tag{5.57}$$

such that F satisfies (H1)-(H3). Then $\overline{u}$ is partially regular in the sense of Theorem 5.1 provided $D\overline{u}$ satisfies the regularity condition (1.8) with $\delta = \delta_*$ where δ_* is given in Theorem 1.2.

For $f \in \mathrm{BMO}(\Omega, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$ we clearly have the inequality

$$[f]_{*,\Omega} \leq \|f\|_{\mathrm{BMO}}.$$

Obtaining a reverse inequality for functions of the type $\mathrm{BMO}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{N}\times\mathrm{n}})$ restricted to zero off Ω, is not so easy and depends on the boundary of Ω. Luckily the latter inequality is not required here.

Proof. By Theorem 4.3 we have $D\overline{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ is a $\mathrm{W}^1\mathrm{BMO}$-local minimiser of $I[\cdot, \Omega]$ for all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{N}}) \cap \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^{\mathrm{N}})$ (for any $1 \leq p < \infty$) with $\|D\varphi\|_{\mathrm{BMO}} \leq \delta_*$. This implies $[D\varphi]_{*,\Omega} \leq \delta_*$ and therefore is true for all $\varphi \in \mathrm{W}^1\mathrm{BMO}(\Omega, \mathbb{R}^{\mathrm{N}}) \cap \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^{\mathrm{N}})$ with $[D\varphi]_{*,\Omega} \leq \delta_*$. Hence all conditions of Corollary 5.1 are satisfied. □

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