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# Asymptotic Stability and Boundedness of Stochastic Functional Differential Equations with Markovian Switching

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## Abstract

This paper is concerned with the boundedness, exponential stability and almost sure asymptotic stability of stochastic functional differential equations (SFDEs) with Markovian switching. The key technique used is the method of multiple Lyapunov functions. We use two auxiliary functions to dominate the corresponding different Lyapunov function in different mode while the diffusion operator in different model is controlled by other multiple auxiliary functions. Our conditions on the diffusion operator are weaker than those in the related existing works.

*Keywords and phrases:* stochastic functional differential equations; Markovian switching; asymptotic stability; boundedness; generalized Itô's formula.

## 1 Introduction

Dynamic systems in many branches of science and industry do not only depend on the present states but also the past states. Stochastic functional differential equations (SFDEs), including stochastic delay differential equations (SDDEs), are often used to model these systems. The stability theory of SFDEs has received lots of attentions over past twenty years. A large number of stability results have been established (e.g. [2], [5], [6], [8], [17], [18]). In particular, many important works have been developed systematically in book [14]. On the other hand, these systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. Continuous time Markovian chains have been used to

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model these abrupt changes. Consequently, the SFDEs with Markovian switching, including SDDEs with Markovian switching, have been used to model many complicated systems.

In general, an  $n$ -dimensional SDDE with Markovian switching has the form

$$d\mathbf{x}(t) = \mathbf{f}_1(\mathbf{x}(t), \mathbf{x}(t - \tau), t, r(t))dt + \mathbf{g}_1(\mathbf{x}(t), \mathbf{x}(t - \tau), t, r(t))dB(t) \quad (1.1)$$

on  $t \geq 0$ , where  $\tau > 0$ ,  $r(t)$  is a right-continuous Markovian chain with a finite state space  $S = \{1, 2, \dots, N\}$  and  $B(t)$  is an  $m$ -dimensional Brownian motion. In the system,  $\mathbf{x}(t)$  stands for the state while  $r(t)$  for the mode. The stability problem of SDDEs with Markovian switching under different settings has been discussed, for example, in [10]-[13], [20]. Recently, Hu, Mao and Shen [3] established some new generalized theorems on the asymptotic stability and boundedness of SDDEs with Markovian switching using different types of Lyapunov functions (e.g. polynomials with different degrees) for different modes. More precisely, they assumed that there exist functions  $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ ,  $U_1, U_2 \in C(R^n \times [-\tau, \infty); R_+)$  such that

$$\begin{aligned} \liminf_{|x| \rightarrow \infty, 0 \leq t < \infty} U_1(x, t) &= \infty, \\ U_1(x, t) &\leq V(x, t, i) \leq U_2(x, t), \\ \mathcal{L}V(x, y, t, i) &\leq c_1 - c_2 U_2(x, t) + c_3 U_2(y, t - \tau) \end{aligned} \quad (1.2)$$

for all  $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$ , where  $c_j \geq 0, j = 1, 2, 3$ , with  $c_2 > c_3$ .

It is noted that the future state of system (1.1) is not only dependent on the present state  $\mathbf{x}(t)$  but also the past state  $\mathbf{x}(t - \tau)$ . However, there are many practical systems whose future state depends on the state over the whole time interval  $[t - \tau, t]$  rather than at times  $t - \tau$  and  $t$ . SFDEs with Markovian switching have therefore been developed to describe such systems. Generally speaking, an  $n$ -dimensional SFDE with Markovian switching has the form

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}_t, t, r(t))dt + \mathbf{g}(\mathbf{x}_t, t, r(t))dB(t), \quad (1.3)$$

on  $t \geq 0$ , where  $\mathbf{x}_t = \{\mathbf{x}(t + u) : u \in [-\tau, 0]\}$ . In particular, if define  $\mathbf{f}(\mathbf{x}_t, t, r(t)) = \mathbf{f}_1(\mathbf{x}(t), \mathbf{x}(t - \tau), t, r(t))$  and  $\mathbf{g}(\mathbf{x}_t, t, r(t)) = \mathbf{g}_1(\mathbf{x}(t), \mathbf{x}(t - \tau), t, r(t))$ , the SFDE (1.3) becomes the SDDE (1.1). For SFDEs with Markovian switching (1.3), Mao [9] investigated the existence, uniqueness and the Razumikhin-type theorem on exponential stability of the global solution (also see Chapter 8 in book [13]); Peng and Zhang [15] discussed some new Razumikhin-type theorems on pth moment stability; Hu and Wu [4] examined asymptotic boundedness of the global solution.

The aims of this paper are to examine the existence and uniqueness of the global solution of the SFDE (1.3) and to establish more general results on the asymptotic stability and boundedness of the solution under much weaker conditions than those imposed in the references mentioned above. To do so, we will use the idea in [3] to apply different types of Lyapunov

functions for different modes of the underlying SFDE. It is observed that the diffusion operators (1.2) in all modes, for the SDDE (1.1), are controlled by the values of  $U_2(\mathbf{x}(s), s)$  at times  $s = t - \tau$  and  $s = t$ . When dealing with the SFDE (1.3), it is natural that the diffusion operators in all modes will be influenced by the historical states on the whole interval  $[t - \tau, t]$ . In this paper, we will use multiple auxiliary functions to describe this complicated situation. More precisely speaking, the diffusion operators will be controlled by multiple auxiliary functions  $W_j$  ( $j = 1, 2, \dots, L$ ) with constant coefficients  $\alpha_j, \beta_j$  ( $j = 1, 2, \dots, L$ ) (see (3.3) in Assumptions 3.2 of Section 3) or time-varying coefficients  $b_j(t)$  ( $j = 1, 2, \dots, L$ ) (see (4.3) in Assumptions 4.1 of Section 4). As far as we know, it is first time that such general conditions are imposed to study the asymptotic boundedness and stability of SFDEs with Markovian switching.

This paper does not only present more general results on the boundedness and exponential stability of SFDEs with Markovian switching using the multiple auxiliary functions with constant coefficients but also discuss the almost sure asymptotic stability. Our results imply Theorem 3.1 in [3] and Theorem 8.4 in [13]. This paper also gives some further new results for SFDEs with Markovian switching using multiple auxiliary functions with time-varying coefficients.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations. In Section 3, we discuss the asymptotic stability and boundedness of SFDEs with Markovian switching. Some further new results for SFDEs with Markovian switching using multiple auxiliary functions with time-varying coefficients are given in Section 4. We will finally conclude our paper in Section 5.

## 2 Preliminaries

Throughout this paper, let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e., it is increasing and right continuous while  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets). If  $A$  is a matrix or vector, its transpose is denoted by  $A^T$ , and the trace norm of matrix  $A$  is denoted by  $|A| = \sqrt{\text{trace}(A^T A)}$ . Let  $B(t)$  be an  $m$ -dimensional Brownian motion defined on the probability space. Let  $\tau > 0$  and  $C([-\tau, 0]; R^n)$  denote the family of all continuous  $R^n$ -valued functions  $\varphi$  on  $[-\tau, 0]$  with the norm  $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$ . Let  $C = C_{\mathcal{F}_0}^b([-\tau, 0]; R^n)$  be the family of all bounded,  $\mathcal{F}_0$ -measurable,  $C([-\tau, 0]; R^n)$ -valued,  $\mathcal{F}_t$ -adapted stochastic processes. Let  $\eta_j$  be probability measures on  $[-\tau, 0]$ , which satisfy  $\int_{-\tau}^0 d\eta_j(\theta) = 1$  ( $j = 1, 2, \dots, L$ ). Let  $L^1(R_+; R_+)$  be the family of all Borel measurable functions  $\xi : R_+ \rightarrow R_+$  such that  $\int_0^\infty \xi(t) dt < \infty$ , and  $\Psi(R_+; R_+)$  the family of all continuous functions  $\psi : R_+ \rightarrow R_+$  such that  $\liminf_{t \rightarrow \infty} \int_t^{t+\epsilon} \psi(t) dt > 0$ , for any  $\epsilon > 0$ . A continuous function  $Q : R^n \rightarrow R_+$  is said to be positive definite if  $Q(\mathbf{x}) = 0$  iff  $\mathbf{x} = 0$ .  $\mathbf{x}(t)$  is a continuous  $R$ -valued stochastic process on  $t \in [-\tau, \infty)$ . We assume

$\mathbf{x}_t = \{\mathbf{x}(t + \theta) : -\tau \leq \theta \leq 0\}$  for all  $t \geq 0$ , which is regarded as a  $C([-\tau, 0]; R^n)$ -valued stochastic process.

Let  $r(t), t \geq 0$ , be a right-continuous Markovian chain on the probability space taking values in a finite state space  $S = \{1, 2, \dots, N\}$ , with generator  $\Gamma = (\gamma_{ij})_{N \times N}$  given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where  $\Delta > 0$ ,  $\gamma_{ij}$  is the transition rate from  $i$  to  $j$ , if  $i \neq j$ , while  $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$ .

Assume that Markovian chain  $r(t)$  is independent of Brownian motion  $B(t)$ . It is well known that almost every sample path of  $r(t)$  is a right-continuous step function.

It is useful to recall that a continuous-time Markovian chain  $r(t)$  with generator  $\Gamma = (\gamma_{ij})_{N \times N}$  can be represented as a stochastic integral with respect to a Poisson random measure

$$dr(t) = \int_R \bar{h}(r(t-), y) \nu(dt, dy), \quad t \geq 0$$

with initial value  $r(0) = i_0 \in S$ , where  $\nu(dt, dy)$  is a Poisson random measure with intensity  $dt \times m(dy)$  in which  $m$  is the Lebesgue measure on  $R$ , while the explicit definition of  $\bar{h} : S \times R \rightarrow R$  can be found in [1].

Consider an  $n$ -dimensional SFDE with Markovian switching of the form

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}_t, t, r(t))dt + \mathbf{g}(\mathbf{x}_t, t, r(t))dB(t), \quad (2.1)$$

on  $t \geq 0$  with initial data  $\{\mathbf{x}(\theta) : -\tau \leq \theta \leq 0\} = \boldsymbol{\zeta} \in C_{\mathcal{F}_0}^b([-\tau, 0]; R^n)$ ,  $i_0 \in S$ , where

$$\mathbf{f} : C([-\tau, 0]; R^n) \times R_+ \times S \rightarrow R^n, \quad \mathbf{g} : C([-\tau, 0]; R^n) \times R_+ \times S \rightarrow R^{n \times m}.$$

Let  $C(R^n \times [-\tau, \infty); R_+)$  denote the family of all continuous functions from  $R^n \times [-\tau, \infty)$  to  $R_+$ , and  $C^{2,1}(R^n \times [-\tau, \infty) \times S; R_+)$  the family of all continuous nonnegative functions  $V(x, t, i)$  on  $R^n \times [-\tau, \infty) \times S$  which are continuously twice differentiable in  $x$  and once differentiable in  $t$ . For each  $V \in C^{2,1}(R^n \times [-\tau, \infty) \times S; R_+)$ , denote an operator  $\mathcal{L}V$  from  $C([-\tau, 0]; R^n) \times R_+ \times S$  to  $R$  by

$$\begin{aligned} \mathcal{L}V(\boldsymbol{\varphi}, t, i) &= \sum_{j=1}^N \gamma_{ij} V(\boldsymbol{\varphi}(0), t, j) + V_t(\boldsymbol{\varphi}(0), t, i) + V_x(\boldsymbol{\varphi}(0), t, i) \mathbf{f}(\boldsymbol{\varphi}, t, i) \\ &\quad + \frac{1}{2} \text{trace}[\mathbf{g}^T(\boldsymbol{\varphi}, t, i) V_{xx}(\boldsymbol{\varphi}(0), t, i) \mathbf{g}(\boldsymbol{\varphi}, t, i)], \end{aligned} \quad (2.2)$$

where

$$V_t(\mathbf{x}, t, i) = \frac{\partial V(\mathbf{x}, t, i)}{\partial t}, \quad V_{xx}(\mathbf{x}, t, i) = \left( \frac{\partial^2 V(\mathbf{x}, t, i)}{\partial x_i \partial x_j} \right)_{n \times n}, \quad V_x(\mathbf{x}, t, i) = \left( \frac{\partial V(\mathbf{x}, t, i)}{\partial x_1}, \dots, \frac{\partial V(\mathbf{x}, t, i)}{\partial x_n} \right).$$

For the convenience of readers we cite the generalized Itô's formula (see [13]): if  $V \in C^{2,1}(R^n \times [-\tau, +\infty) \times S; R_+)$ , then for any  $t \geq 0$ ,

$$\begin{aligned} V(\mathbf{x}(t), t, r(t)) &= V(\mathbf{x}(0), 0, r(0)) + \int_0^t \mathcal{L}V(\mathbf{x}_s, s, r(s))ds + \int_0^t V_x(\mathbf{x}(s), s, r(s))\mathbf{g}(\mathbf{x}_s, s, r(s))dB(s) \\ &\quad + \int_0^t \int_R V(\mathbf{x}(s), s, i_0 + \bar{h}(r(s-), l)) - V(\mathbf{x}(s), s, r(s))\mu(ds, dl), \end{aligned} \quad (2.3)$$

where  $\mu(ds, dl) = \nu(ds, dl) - m(dl)$  is a martingale measure. Please note that this martingale measure is related to the Markov chain but not the Brownian motion (which forms another independent martingale in the formula above). The key benefit of this formula will allow us to apply the non-negative semi-martingale convergence theorem (see [7]) cited below as a lemma.

**Lemma 2.1.** *Let  $A_1(t)$  and  $A_2(t)$  be two continuous adapted increasing processes on  $t \geq 0$  with  $A_1(0) = A_2(0) = 0$  a.s.,  $M(t)$  a real-valued continuous local martingale with  $M(0) = 0$  a.s.,  $\zeta$  a nonnegative  $\mathcal{F}_0$ -measurable random variable such that  $E\zeta < \infty$ . Define  $X(t) = \zeta + A_1(t) - A_2(t) + M(t)$  for  $t \geq 0$ . If  $X(t)$  is nonnegative, then*

$$\{\lim_{t \rightarrow \infty} A_1(t) < \infty\} \subset \{\lim_{t \rightarrow \infty} X(t) < \infty\} \cap \{\lim_{t \rightarrow \infty} A_2(t) < \infty\} \quad a.s.,$$

where  $C \subset D$  a.s. means  $\mathbb{P}(C \cap D^c) = 0$ . In particular, if  $\lim_{t \rightarrow \infty} A_1(t) < \infty$  a.s., then, with probability one,

$$\lim_{t \rightarrow \infty} X(t) < \infty, \quad \lim_{t \rightarrow \infty} A_2(t) < \infty, \quad -\infty < \lim_{t \rightarrow \infty} M(t) < \infty.$$

For the stability purpose, we furthermore assume that  $\mathbf{f}(0, t, i) = 0$  and  $\mathbf{g}(0, t, i) = 0$  for all  $t \in R_+$ ,  $i \in S$ , so that system (2.1) admits a trivial solution  $\mathbf{x}(t) = 0$ .

### 3 General Asymptotic Stability and Boundedness of SFDEs with Markovian switching

In this section, we give the results for the existence and uniqueness, general asymptotic stability and general boundedness of the global solutions of SFDEs with Markovian switching using multiple auxiliary functions with constant coefficients.

To get our main results, we firstly put forward the following assumptions.

**Assumption 3.1.** *Both  $\mathbf{f}$  and  $\mathbf{g}$  satisfy the local Lipschitz condition. That is, for each  $k = 1, 2, \dots$ , there is a  $c_k > 0$  such that*

$$|\mathbf{f}(\varphi, t, i) - \mathbf{f}(\psi, t, i)| \vee |\mathbf{g}(\varphi, t, i) - \mathbf{g}(\psi, t, i)| \leq c_k \|\varphi - \psi\|,$$

for all  $t \in R_+$ ,  $i \in S$  and  $\varphi, \psi \in C([-\tau, 0]; R^n)$  with  $\|\varphi\| \vee \|\psi\| \leq k$ .

**Assumption 3.2.** *There exist functions  $V \in C^{2,1}(R^n \times [-\tau, \infty) \times S; R_+)$ ,  $W_0, W_j \in C(R^n \times [-\tau, \infty); R_+)$  and probability measures  $\eta_j$  on  $[-\tau, 0]$  satisfying  $\int_{-\tau}^0 d\eta_j(\theta) = 1$ , as well as nonnegative numbers  $\alpha_0, \alpha_j, \beta_j$  satisfying  $\alpha_j > \beta_j$  ( $j = 1, 2, \dots, L$ ), such that*

$$\liminf_{|x| \rightarrow \infty, 0 \leq t < \infty} W_0(x, t) = \infty, \quad (3.1)$$

$$W_0(x, t) \leq V(x, t, i) \leq W_1(x, t), \quad \forall (x, t, i) \in C([-\tau, 0]; R^n) \times R_+ \times S, \quad (3.2)$$

$$\mathcal{L}V(\varphi, t, i) \leq \alpha_0 + \sum_{j=1}^L \left[ -\alpha_j W_j(\varphi(0), t) + \beta_j \int_{-\tau}^0 W_j(\varphi(\theta), t + \theta) d\eta_j(\theta) \right], \quad (3.3)$$

for all  $i \in S$  and  $\varphi \in C([-\tau, 0]; R^n)$ ,  $t \in R_+$ .

**Remark 1.** Assumption 3.2 is a generalization of Assumption 2.2 of [3] (condition (1.2) in this paper). The generalization mainly shows in the following two aspects:

1) Assumption 3.2 introduces multiple auxiliary functions  $W_j$  ( $j = 0, 1, \dots, L$ ) instead of two functions in Assumption 2.2 of [3];

2) When the probability measure  $\eta_1(\theta)$  is a point probability measure at  $-\tau$ , Assumption 3.2 with  $\alpha_j = \beta_j = 0$  ( $j = 2, 3, \dots, L$ ) reduces to Assumption 2.2 of [3].

Since the coefficients of  $W_j$  ( $j = 1, \dots, L$ ) in (3.3) are constants, it is said to be the case of multiple auxiliary functions with constant coefficients for convenience. Now, we state our main results of SFDEs with Markovian switching in this case.

**Theorem 3.1.** *Under Assumptions 3.1 and 3.2, we have the following assertions:*

1) *For any given initial data  $\zeta \in C$  and  $i_0 \in S$ , there is a unique global solution  $\mathbf{x}(t, \zeta, i_0)$  of system (2.1) on  $t \geq -\tau$ , which satisfies the following moment properties*

$$\limsup_{t \rightarrow \infty} EW_0(\mathbf{x}(t, \zeta, i_0), t) \leq \frac{\alpha_0}{\varepsilon_0}, \quad (3.4)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t EW_j(\mathbf{x}(s, \zeta, i_0), s) ds \leq \frac{\alpha_0}{\alpha_j - \beta_j}, \quad j = 1, 2, \dots, L \quad (3.5)$$

where  $\varepsilon_0 = \sup \left\{ \varepsilon > 0 : \varepsilon + \beta_1 e^{\varepsilon \tau} - \alpha_1 \leq 0, \varepsilon \leq \frac{1}{\tau} \min_{j \geq 2} \left\{ \log \frac{\alpha_j}{\beta_j} \right\} \right\}$ .

2) *Moreover, if  $\alpha_0 = 0$ , the global solution  $\mathbf{x}(t, \zeta, i_0)$  has the following moment properties*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(EW_0(\mathbf{x}(t, \zeta, i_0), t)) \leq -\varepsilon_0, \quad (3.6)$$

$$\int_0^\infty EW_j(\mathbf{x}(s, \zeta, i_0), s) ds < \infty, \quad j = 1, 2, \dots, L \quad (3.7)$$

and the following sample properties

$$\sup_{-\tau \leq t < \infty} \mathbf{x}(t, \zeta, i_0) < \infty, \quad a.s. \quad (3.8)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(W_0(\mathbf{x}(t, \zeta, i_0), t)) \leq -\varepsilon_0, \quad a.s. \quad (3.9)$$

$$\int_0^\infty W_j(\mathbf{x}(s, \zeta, i_0), s) ds < \infty. \quad a.s. \quad j = 1, 2, \dots, L \quad (3.10)$$

3) In the case  $\alpha_0 = 0$ , and there furthermore exists a continuous positive definite function  $Q \in C(R^n; R_+)$  such that  $Q(\mathbf{x}) \leq \sum_{j=1}^L W_j(\mathbf{x}, t)$  for all  $(\mathbf{x}, t) \in R^n \times [-\tau, \infty)$ , then

$$\lim_{t \rightarrow \infty} \mathbf{x}(t, \zeta, i_0) = 0. \quad a.s. \quad (3.11)$$

*Proof.* 1) By Assumption 3.1, for any given initial data  $\zeta \in C$  and  $i_0 \in S$ , there is a unique maximal local solution  $\mathbf{x}(t, \zeta, i_0)$  on  $t \in [-\tau, \sigma_\infty)$ , where  $\sigma_\infty$  is the explosion time. Let  $k_0 > 0$  be sufficiently large satisfying  $\|\zeta\| < k_0$ . For each integer  $k \geq k_0$ , define the stopping time

$$\tau_k = \inf\{t \in [0, \sigma_\infty) : |\mathbf{x}(t)| \geq k\}.$$

Obviously,  $\tau_k$  is increasing as  $k \rightarrow \infty$ . Let  $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$ , so  $\tau_\infty \leq \sigma_\infty$  a.s.. If we can obtain that  $\tau_\infty = \infty$  a.s., then  $\sigma_\infty = \infty$  a.s.. For the sake of simplicity, write  $\mathbf{x}(t) = \mathbf{x}(t, \zeta, i_0)$ . Using the generalized Itô's formula and then taking the expectation, we have

$$\begin{aligned} EV(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k, r(t \wedge \tau_k)) &= EV(\mathbf{x}(0), 0, r(0)) + E \int_0^{t \wedge \tau_k} \mathcal{L}V(\mathbf{x}_s, s, r(s)) ds \\ &\leq EV(\mathbf{x}(0), 0, r(0)) + E \int_0^{t \wedge \tau_k} \left[ \alpha_0 + \sum_{j=1}^L [-\alpha_j W_j(\mathbf{x}(s), s) + \beta_j \int_{-\tau}^0 W_j(\mathbf{x}(s+\theta), s+\theta) d\eta_j(\theta)] \right] ds \\ &\leq EV(\mathbf{x}(0), 0, r(0)) + E \int_0^{t \wedge \tau_k} \left[ \alpha_0 + \sum_{j=1}^L [-(\alpha_j - \beta_j) W_j(\mathbf{x}(s), s) + \beta_j J_j] \right] ds, \end{aligned} \quad (3.12)$$

where  $J_j = \int_{-\tau}^0 W_j(\mathbf{x}(s+\theta), s+\theta) d\eta_j(\theta) - W_j(\mathbf{x}(s), s)$ ,  $j = 1, 2, \dots, L$ .

By virtue of the fact that

$$\begin{aligned} \int_0^{t \wedge \tau_k} J_j ds &= \int_0^{t \wedge \tau_k} \left[ \int_{-\tau}^0 W_j(\mathbf{x}(s+\theta), s+\theta) d\eta_j(\theta) - W_j(\mathbf{x}(s), s) \right] ds \\ &= \int_{-\tau}^0 \int_\theta^{(t \wedge \tau_k) + \theta} W_j(\mathbf{x}(s), s) ds d\eta_j(\theta) - \int_0^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds \\ &\leq \int_{-\tau}^0 \int_{-\tau}^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds d\eta_j(\theta) - \int_0^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds \\ &\leq \int_{-\tau}^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds - \int_0^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds \end{aligned}$$



$$\leq \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds, \quad (3.13)$$

for  $j = 1, 2, \dots, L$ , (3.12) can be rewritten as

$$\begin{aligned} EV(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k, r(t \wedge \tau_k)) &\leq a_0 E(t \wedge \tau_k) - \sum_{j=1}^L (\alpha_j - \beta_j) E \int_0^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds \\ &\quad + EV(\mathbf{x}(0), 0, r(0)) + \sum_{j=1}^L \beta_j E \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds \\ &\leq c_0 + a_0 t - \sum_{j=1}^L (\alpha_j - \beta_j) E \int_0^{t \wedge \tau_k} W_j(\mathbf{x}(s), s) ds, \end{aligned} \quad (3.14)$$

where  $c_0 = EV(\mathbf{x}(0), 0, r(0)) + \sum_{j=1}^L \beta_j E \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds$ .

From condition (3.2) and  $\alpha_j > \beta_j (j = 1, 2, \dots, L)$ , we get that

$$EW_0(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k) \leq c_0 + a_0 t. \quad (3.15)$$

Let  $\nu_k = \inf_{|x| \geq k, 0 \leq t < \infty} W_0(\mathbf{x}(t), t)$ , for  $k \geq k_0$ , then we have that

$$EW_0(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k) \geq E(W_0(\tau_k), \tau_k) I_{\{\tau_k \leq t\}} \geq \nu_k P(\tau_k \leq t).$$

Letting  $k \rightarrow \infty$  in the inequality above, we have that

$$P(\tau_\infty \leq t) = \lim_{k \rightarrow \infty} P(\tau_k \leq t) \leq \lim_{k \rightarrow \infty} \frac{EW_0(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k)}{\nu_k} = \lim_{k \rightarrow \infty} \frac{c_0 + a_0 t}{\nu_k} = 0.$$

Since  $t > 0$  is arbitrary, we must have that  $\tau_\infty = \infty$  a.s.. Hence  $\sigma_\infty = \infty$  a.s., which implies that the unique maximal local solution  $x(t)$  on  $t \in [-\tau, \sigma_\infty)$  becomes the unique global solution on  $t \in [-\tau, \infty)$ .

Using the generalized Itô's formula to  $e^{\varepsilon t} V(\mathbf{x}(t), t, r(t))$ ,  $\varepsilon \in (0, \varepsilon_0]$ , and from condition (3.2), we get that, for any  $t \geq 0$ ,

$$\begin{aligned} &e^{\varepsilon(t \wedge \tau_k)} V(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k, r(t \wedge \tau_k)) \\ &= V(\mathbf{x}(0), 0, r(0)) + \int_0^{t \wedge \tau_k} e^{\varepsilon s} (\varepsilon V(\mathbf{x}(s), s, r(s)) + \mathcal{L}V(\mathbf{x}_s, s, r(s))) ds + M(t \wedge \tau_k) \\ &\leq W_1(\mathbf{x}(0), 0) + \int_0^{t \wedge \tau_k} e^{\varepsilon s} \left[ \alpha_0 - (\alpha_1 - \varepsilon) W_1(\mathbf{x}(s), s) + \beta_1 \int_{-\tau}^0 W_1(\mathbf{x}(s + \theta), s + \theta) d\eta_1(\theta) \right. \\ &\quad \left. + \sum_{j=2}^L [-\alpha_j W_j(\mathbf{x}(s), s) + \beta_j \int_{-\tau}^0 W_j(\mathbf{x}(s + \theta), s + \theta) d\eta_j(\theta)] \right] ds + M(t \wedge \tau_k) \\ &\leq W_1(\mathbf{x}(0), 0) + \int_0^{t \wedge \tau_k} e^{\varepsilon s} \left[ \alpha_0 - (\alpha_1 - \varepsilon - \beta_1 e^{\varepsilon \tau}) W_1(\mathbf{x}(s), s) + \beta_1 J_1' \right] \end{aligned}$$

$$+ \sum_{j=2}^L [ -(\alpha_j - \beta_j e^{\varepsilon\tau}) W_j(\mathbf{x}(s), s) + \beta_j J'_j ] ds + M(t \wedge \tau_k), \quad (3.16)$$

where  $M(t \wedge \tau_k)$  is a local martingale with the initial value  $M(0) = 0$ ,  $J'_j = \int_{-\tau}^0 W_j(\mathbf{x}(s + \theta), s + \theta) d\eta_j(\theta) - e^{\varepsilon\tau} W_j(\mathbf{x}(s), s)$ ,  $j = 1, 2, \dots, L$ .

By virtue of the fact that

$$\begin{aligned} \int_0^{t \wedge \tau_k} e^{\varepsilon s} J'_j ds &= \int_0^{t \wedge \tau_k} e^{\varepsilon s} \left[ \int_{-\tau}^0 W_j(\mathbf{x}(s + \theta), s + \theta) d\eta_j(\theta) - e^{\varepsilon\tau} W_j(\mathbf{x}(s), s) \right] ds \\ &= \int_{-\tau}^0 \int_{\theta}^{(t \wedge \tau_k) + \theta} e^{\varepsilon(s - \theta)} W_j(\mathbf{x}(s), s) ds d\eta_j(\theta) - e^{\varepsilon\tau} \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_j(\mathbf{x}(s), s) ds \\ &\leq \int_{-\tau}^0 \int_{-\tau}^{t \wedge \tau_k} e^{\varepsilon(s + \tau)} W_j(\mathbf{x}(s), s) ds d\eta_j(\theta) - e^{\varepsilon\tau} \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_j(\mathbf{x}(s), s) ds \\ &\leq e^{\varepsilon\tau} \int_{-\tau}^{t \wedge \tau_k} e^{\varepsilon s} W_j(\mathbf{x}(s), s) ds - e^{\varepsilon\tau} \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_j(\mathbf{x}(s), s) ds \\ &\leq e^{\varepsilon\tau} \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds, \end{aligned} \quad (3.17)$$

for  $j = 1, 2, \dots, L$ , (3.16) can be rewritten as

$$\begin{aligned} &e^{\varepsilon(t \wedge \tau_k)} V(\mathbf{x}(t \wedge \tau_k), t \wedge \tau_k, r(t \wedge \tau_k)) \\ &\leq W_1(\mathbf{x}(0), 0) + \frac{\alpha_0}{\varepsilon} (e^{\varepsilon(t \wedge \tau_k)} - 1) - (\alpha_1 - \varepsilon - \beta_1 e^{\varepsilon\tau}) \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_1(\mathbf{x}(s), s) ds \\ &\quad - \sum_{j=2}^L (\alpha_j - \beta_j e^{\varepsilon\tau}) \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_j(\mathbf{x}(s), s) ds + \sum_{j=1}^L \beta_j e^{\varepsilon\tau} \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds + M(t \wedge \tau_k) \\ &\leq c_1 + \frac{\alpha_0}{\varepsilon} e^{\varepsilon(t \wedge \tau_k)} - (\alpha_1 - \varepsilon - \beta_1 e^{\varepsilon\tau}) \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_1(\mathbf{x}(s), s) ds \\ &\quad - \sum_{j=2}^L (\alpha_j - \beta_j e^{\varepsilon\tau}) \int_0^{t \wedge \tau_k} e^{\varepsilon s} W_j(\mathbf{x}(s), s) ds + M(t \wedge \tau_k), \end{aligned} \quad (3.18)$$

where  $c_1 = W_1(\mathbf{x}(0), 0) + \sum_{j=1}^L \beta_j e^{\varepsilon\tau} \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds$ .

Letting  $k \rightarrow \infty$  and then taking the expectation on both side of (3.18), we get

$$E(e^{\varepsilon t} W_0(\mathbf{x}(t), t)) \leq c_1 + \frac{\alpha_0}{\varepsilon} e^{\varepsilon t}, \quad (3.19)$$

from the definition of  $\varepsilon_0$  and condition (3.2). So letting  $\varepsilon \rightarrow \varepsilon_0$ , we have the assertion (3.4).

From (3.14), letting  $k \rightarrow \infty$ , we get that

$$\sum_{j=1}^L (\alpha_j - \beta_j) E \int_0^t W_j(\mathbf{x}(s), s) ds \leq c_0 + a_0 t. \quad (3.20)$$

Letting  $t \rightarrow \infty$  and using the Fubini theorem, we have the assertion (3.5).

2) Next, we prove the results in the case of  $\alpha_0 = 0$ . From (3.19) we get that

$$EW_0(\mathbf{x}(t), t) \leq c_1 e^{-\varepsilon t}.$$

So letting  $\varepsilon \rightarrow \varepsilon_0$ , we get the assertion (3.6).

Moreover, (3.20) becomes

$$\sum_{j=1}^L (\alpha_j - \beta_j) E \int_0^t W_j(\mathbf{x}(s), s) ds \leq c_0.$$

So letting  $t \rightarrow \infty$ , we get the assertion (3.7).

In the same way as (3.16), we get that

$$e^{\varepsilon t} W_0(\mathbf{x}(t), t) \leq c_1 + M(t),$$

where  $c_1$  has been defined above. By virtue of Lemma 2.1, we obtain that

$$\limsup_{t \rightarrow \infty} e^{\varepsilon t} W_0(\mathbf{x}(t), t) < \infty. \quad a.s. \quad (3.21)$$

Hence, there exists a finite positive random variable  $\zeta$  such that

$$\sup_{0 \leq t < \infty} e^{\varepsilon t} W_0(\mathbf{x}(t), t) < \zeta. \quad a.s. \quad (3.22)$$

So as  $\varepsilon \rightarrow \varepsilon_0$ , we claim the assertion (3.9).

Using the generalized Itô's formula to  $V(\mathbf{x}(t), t, r(t))$ , we obtain

$$\begin{aligned} V(\mathbf{x}(t), t, r(t)) &= V(\mathbf{x}(0), 0, r(0)) + \int_0^t \mathcal{L}V(\mathbf{x}_s, s, r(s)) ds + M'(t) \\ &\leq W_1(\mathbf{x}(0), 0) + \int_0^t \sum_{j=1}^L [-\alpha_j W_j(\mathbf{x}(s), s) + \beta_j \int_{-\tau}^0 W_j(\mathbf{x}(s+\theta), s+\theta) d\eta_j(\theta)] ds + M'(t) \\ &\leq W_1(\mathbf{x}(0), 0) + \sum_{j=1}^L \beta_j \int_{-\tau}^0 W_j(\mathbf{x}(s), s) ds - \sum_{j=1}^L (\alpha_j - \beta_j) \int_0^t W_j(\mathbf{x}(s), s) ds + M'(t) \end{aligned}$$

where  $M'(t)$  is a local martingale with the initial value  $M'(0) = 0$ . Due to Lemma 2.1, we obtain the assertions (3.10) and  $\sup_{-\tau \leq t < \infty} W_0(\mathbf{x}(t), t) < \infty, a.s.$ . From the radial unbound- edness of  $W_0(x, t)$  (see (3.1)), we directly get the assertion (3.8).

3) Finally, we prove the result (3.11). Because of the positive definiteness of  $Q(\mathbf{x})$ , the assertion (3.11) is equivalent to the following assertion

$$\lim_{t \rightarrow \infty} Q(\mathbf{x}(t)) = 0. \quad a.s. \quad (3.23)$$

Further, the assertion (3.23) is equivalent to the following assertion

$$\limsup_{t \rightarrow \infty} Q(\mathbf{x}(t)) = 0. \quad a.s. \quad (3.24)$$

Now we claim that the assertion (3.24) holds. If (3.24) is false, we must have that

$$P\left\{\limsup_{t \rightarrow \infty} Q(\mathbf{x}(t)) > 0\right\} > 0. \quad (3.25)$$

Then there exists a constant  $\delta$  such that  $P(A) > 3\delta$ , where  $A = \left\{\omega : \limsup_{t \rightarrow \infty} Q(\mathbf{x}(t)) > 2\delta\right\}$ . From the assertion (3.8), we can see that there exists a sufficiently large positive constant  $h = h(\delta)$  such that  $P(A_1) > 1 - \delta$ , where  $A_1 = \left\{\omega : \sup_{-\tau \leq t < \infty} |\mathbf{x}(t)| < h\right\}$ . So we easily obtain that  $P(A \cap A_1) > 2\delta$ .

Define a sequence of stopping times by

$$\begin{aligned} \mu_1 &= \inf\{t \geq 0 : Q(\mathbf{x}(t)) \geq 2\delta\}, \\ \mu_{2l} &= \inf\{t \geq \mu_{2l-1} : Q(\mathbf{x}(t)) \leq \delta\}, \quad l = 1, 2, \dots \\ \mu_{2l+1} &= \inf\{t \geq \mu_{2l} : Q(\mathbf{x}(t)) \geq 2\delta\}. \quad l = 1, 2, \dots \end{aligned}$$

Obviously, when  $\omega \in A \cap A_1$ , then  $\mu_l < \infty, l = 1, 2, \dots$ , i.e.

$$A \cap A_1 \subset \{\mu_l < \infty, l = 1, 2, \dots\}.$$

From Assumptions 3.1,  $\mathbf{f}(0, t, i) = 0$  and  $\mathbf{g}(0, t, i) = 0$ , we get that there exists a constant  $H_h > 0$  such that  $|\mathbf{f}(\boldsymbol{\varphi}, t, i)| \vee |\mathbf{g}(\boldsymbol{\varphi}, t, i)| \leq H_h$ , for all  $t \in R_+, i \in S$  and  $\boldsymbol{\varphi} \in C([- \tau, 0]; R^n)$  with  $\|\boldsymbol{\varphi}\| \leq h$ . Using Hölder's inequality and Burkholder-Davis-Gundy inequality, we obtain that, for any  $T_1 > 0$ ,

$$\begin{aligned} E\left(I_{A \cap A_1} \sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})|^2\right) &\leq 2E\left(I_{A \cap A_1} \sup_{0 \leq t \leq T_1} \left|\int_{\mu_{2l-1}}^{\mu_{2l-1}+t} \mathbf{f}(\mathbf{x}_s, s, r(s)) ds\right|^2\right) \\ &\quad + 2E\left(I_{A \cap A_1} \sup_{0 \leq t \leq T_1} \left|\int_{\mu_{2l-1}}^{\mu_{2l-1}+t} \mathbf{g}(\mathbf{x}_s, s, r(s)) dB(s)\right|^2\right) \\ &\leq 2T_1 H_h^2 (T_1 + 4) \end{aligned}$$

Since  $Q(\mathbf{x})$  is continuous in  $R^n$ , it is uniformly continuous in the closed ball  $\bar{B}_h = \{\mathbf{x} \in R^n : |\mathbf{x}| \leq h\}$ . Hence, we can choose a sufficiently small  $\delta' = \delta'(\delta) > 0$  such that  $|Q(\mathbf{x}) - Q(\mathbf{y})| < \delta$ , when  $|\mathbf{x}| \vee |\mathbf{y}| \leq h$  and  $|\mathbf{x} - \mathbf{y}| \leq \delta'$ .

By using Chebyshev's inequality, we get that

$$\begin{aligned} P\left(A \cap A_1 \cap \left\{\sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| < \delta'\right\}\right) \\ = P(A \cap A_1) - E\left(I_{A \cap A_1} \sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| \geq \delta'\right) \geq 2\delta - \frac{2T_1 H_h^2 (T_1 + 4)}{\delta'^2}. \end{aligned}$$

Noting that  $\left\{\sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| \leq \delta'\right\} \subset \left\{\sup_{0 \leq t \leq T_1} |Q(\mathbf{x}(\mu_{2l-1} + t)) - Q(\mathbf{x}(\mu_{2l-1}))| < \delta\right\} \subset \left\{\mu_{2l} - \mu_{2l-1} \geq T_1\right\}$ , we choose a sufficiently small  $T_1 > 0$  such that  $\frac{2T_1 H_h^2 (T_1 + 4)}{\delta'^2} < \delta$ .

So we compute that  $P\left(A \cap A_1 \cap \left\{\sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| < \delta'\right\}\right) > \delta$ .

On the other hand, from the assertion (3.7) and the relation of  $Q(\mathbf{x})$  and  $\sum_{j=1}^L W_j(\mathbf{x}, t)$ , we easily get that

$$E \int_0^\infty Q(\mathbf{x}(s))ds \leq \sum_{j=1}^L E \int_0^\infty W_j(\mathbf{x}(s, \zeta, i_0), s)ds < \infty. \quad (3.26)$$

For any  $\omega \in A \cap A_1 \cap \left\{ \sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| < \delta' \right\}$ , we have

$$\begin{aligned} E \int_0^\infty Q(\mathbf{x}(s))ds &\geq \sum_{l=1}^\infty E \left( I_{\{\mu_l < \infty\}} \int_{\mu_{2l-1}}^{\mu_{2l}} Q(\mathbf{x}(s))ds \right) \\ &\geq \delta \sum_{l=1}^\infty E \left( I_{A \cap A_1 \cap \left\{ \sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| < \delta' \right\}} (\mu_{2l} - \mu_{2l-1}) \right) \\ &\geq \delta T_1 \sum_{l=1}^\infty E \left( I_{A \cap A_1 \cap \left\{ \sup_{0 \leq t \leq T_1} |\mathbf{x}(\mu_{2l-1} + t) - \mathbf{x}(\mu_{2l-1})| < \delta' \right\}} \right) \\ &\geq \delta T_1 \sum_{l=1}^\infty \delta \\ &= \infty, \end{aligned}$$

which implies a contradiction with (3.26). So we finish the proof of (3.11).  $\square$

**Remark 2.** 1) The results obtained in Theorem 3.1 are in terms of functions  $W_j$  ( $j = 0, 1, \dots, L$ ), that are more general boundedness, general exponential stability and general  $H_\infty$  stability than usual senses. If we impose some simple conditions for these functions, we can get the usual boundedness, exponential stability and  $H_\infty$  stability of the solution. For example, if conditions

$$c|x|^p \leq W_0(x, t), \quad \bar{c}|x|^q \leq W_1(x, t), \quad (3.27)$$

hold for some positive constants  $c, p, \bar{c}, q$ , then we have

$$\limsup_{t \rightarrow \infty} E|\mathbf{x}(t, \zeta, i_0)|^p \leq \frac{\alpha_0}{c\varepsilon_0}, \quad (3.28)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t E|\mathbf{x}(s, \zeta, i_0)|^q ds \leq \frac{\alpha_0}{\bar{c}(\alpha_1 - \beta_1)}, \quad (3.29)$$

Further, let  $\alpha_0 = 0$  and (3.27) hold, we obtain

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(E|\mathbf{x}(t, \zeta, i_0)|^p) \leq -\varepsilon_0, \quad (3.30)$$

$$\int_0^\infty E|\mathbf{x}(s, \zeta, i_0)|^q ds < \infty, \quad (3.31)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(|\mathbf{x}(t, \zeta, i_0)|) \leq -\frac{\varepsilon_0}{p}, \quad a.s. \quad (3.32)$$

$$\int_0^\infty |\mathbf{x}(s, \zeta, i_0)|^q ds < \infty. \quad a.s. \quad (3.33)$$

Obviously, assertion (3.28) is the usual  $p$ th moment asymptotic boundedness of the solution (see [13], [14]), assertions (3.30) and (3.32) are the usual  $p$ th moment exponential stability and almost sure exponential stability of the solution respectively (see [13], [14]), and assertions (3.31) and (3.33) show the usual  $H_\infty$  stability of the solution (see [19]).

2) From Remark 1, the results 1) and 2) of Theorem 3.1 imply that of Theorem 3.1 in [3], and the analysis methods are similar. The main differences are that we have to deal with the terms of  $\int_{-\tau}^0 W_j(\varphi(\theta), t + \theta) d\eta_j(\theta)$  ( $j = 1, 2, \dots, L$ ) in our case. We mainly use integral inequalities (3.13) and (3.17). We also prove the almost sure asymptotic stability of the solution, which did not be studied for SDDs case in [3].

3) When  $W_0(x, t) = c|x|^p$ ,  $W_1(x, t) = \bar{c}|x|^p$ ,  $\alpha_j = \beta_j = 0$ ,  $j = 2, 3, \dots, L$ , the result 1) of Theorem 3.1 in this paper becomes that of Theorem 8.4 on asymptotic boundedness in [13].

Hu, Mao and Shen [3] introduce two auxiliary functions  $U_1, U_2$  to dominate different Lyapunov functions for different modes, and assume that the diffusion operators in all modes are controlled by function  $U_2$ . When considering different Lyapunov functions for different modes of SFDEs with Markovian switching, we may design two auxiliary functions  $W_0, W_1$  to dominate all Lyapunov functions, and we relax one function  $W_1$  to the multiple auxiliary functions  $W_j$  ( $j = 1, 2, \dots, L$ ) to dominate the diffusion operators in all modes.

Next, we give an example to illustrate our theorem.

**Example 3.1.** Let us consider a scalar SFDE with Markovian switching

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}_t, t, r(t))dt + \mathbf{g}(\mathbf{x}_t, t, r(t))dB(t), \quad (3.34)$$

where  $B(t)$  is a scalar Brownian motion,  $r(t)$  is a right-continuous Markovian chain on the state space  $S = \{1, 2\}$  with generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}, \quad (3.35)$$

and

$$\begin{aligned} \mathbf{f}(\mathbf{x}_t, t, 1) &= -4(\mathbf{x}(t) + \mathbf{x}^3(t) + \mathbf{x}^5(t)), & \mathbf{f}(\mathbf{x}_t, t, 2) &= \frac{1}{4}\mathbf{x}(t) - \mathbf{x}^3(t), \\ \mathbf{g}(\mathbf{x}_t, t, 1) &= \int_{-1}^0 |\mathbf{x}(t + \theta)|^3 d\eta(\theta) + \int_{-1}^0 |\mathbf{x}(t + \theta)|^2 d\eta(\theta), & \mathbf{g}(\mathbf{x}_t, t, 2) &= \frac{1}{2} \int_{-1}^0 |\mathbf{x}(t + \theta)| d\eta(\theta), \end{aligned}$$

where  $\eta$  is a probability measure on  $[-1, 0]$ , which satisfies  $\int_{-1}^0 d\eta(\theta) = 1$ .

As to system (3.34), take different Lyapunov functions for the two modes

$$V(x, t, i) = \begin{cases} x^2, & i = 1 \\ 2(x^2 + x^6), & i = 2 \end{cases}, \quad (3.36)$$

for all  $(x, t) \in R \times [-\tau, +\infty)$ , and we compute that

$$\begin{aligned} \mathcal{L}V(\mathbf{x}_t, t, 1) &= 2\mathbf{x}^T \mathbf{f}(\mathbf{x}_t, t, 1) + \mathbf{g}^2(\mathbf{x}_t, t, 1) + \sum_{j=1}^2 \gamma_{1j} V(\mathbf{x}(t), t, j) \\ &\leq -6(x^2 + x^6) + 2 \int_{-1}^0 (|\mathbf{x}(t+\theta)|^2 + |\mathbf{x}(t+\theta)|^6) d\eta(\theta) - 8x^4 + 2 \int_{-1}^0 |\mathbf{x}(t+\theta)|^4 d\eta(\theta), \\ \mathcal{L}V(\mathbf{x}_t, t, 2) &= 2(2x + 6x^5)^T \mathbf{f}(\mathbf{x}_t, t, 2) + \frac{1}{2} \mathbf{g}^T(\mathbf{x}_t, t, 2)(4 + 60x^4) \mathbf{g}(\mathbf{x}_t, t, 2) + \sum_{j=1}^2 \gamma_{2j} V(\mathbf{x}(t), t, j) \\ &\leq -4(x^2 + x^6) + \frac{5}{2} \int_{-1}^0 (|\mathbf{x}(t+\theta)|^2 + |\mathbf{x}(t+\theta)|^6) d\eta(\theta) - 4x^4 - 12x^8. \end{aligned}$$

That is, for  $i = 1, 2$ ,

$$\mathcal{L}V(\mathbf{x}_t, t, i) \leq -4(x^2 + x^6) + \frac{5}{2} \int_{-1}^0 (|\mathbf{x}(t+\theta)|^2 + |\mathbf{x}(t+\theta)|^6) d\eta(\theta) - 4x^4 + 2 \int_{-1}^0 |\mathbf{x}(t+\theta)|^4 d\eta(\theta).$$

Define  $W_0(x, t) = x^2$ ,  $W_1(x, t) = 2(x^2 + x^6)$  and  $W_2(x, t) = x^4$ , then

$$\begin{aligned} W_0(x, t) &\leq V(x, t, i) \leq W_1(x, t), \\ \mathcal{L}V(\mathbf{x}_t, t, i) &\leq -2W_1(x, t) + \frac{5}{4} \int_{-1}^0 W_1(\mathbf{x}(t+\theta), t+\theta) d\eta(\theta) - 4W_2(x, t) \\ &\quad + 2 \int_{-1}^0 W_2(\mathbf{x}(t+\theta), t+\theta) d\eta(\theta), \end{aligned}$$

for all  $x, x_t \in R, t \geq -1$  and  $i \in S$ . So we get the parameters in Assumption 3.2 as follows:

$$\alpha_0 = 0, \alpha_1 = 2, \beta_1 = \frac{5}{4}, \alpha_2 = 4, \beta_2 = 2.$$

And we compute that  $\varepsilon_0 = \sup\{\varepsilon > 0 : \varepsilon + \frac{5}{4}e^\varepsilon - 2 \leq 0, \varepsilon \leq \log 2\} = 0.3047$ . Hence, set positive definite function  $Q(\mathbf{x}) = \mathbf{x}^2 \leq W_1(\mathbf{x}, t) + W_2(\mathbf{x}, t)$ , by Theorem 3.1 we get that there is a unique global solution  $\mathbf{x}(t)$  of system (3.34), and that the global solution  $\mathbf{x}(t)$  has the following moment properties

$$\limsup_{t \rightarrow \infty} \frac{\log(E\mathbf{x}^2(t))}{t} \leq -0.3047, \quad (3.37)$$

$$\int_0^\infty E(\mathbf{x}^2(t)) dt < \infty, \quad \int_0^\infty E(\mathbf{x}^4(t)) dt < \infty, \quad \int_0^\infty E(\mathbf{x}^6(t)) dt < \infty, \quad (3.38)$$

and the following sample properties

$$\limsup_{t \rightarrow \infty} \frac{\log(|\mathbf{x}(t)|)}{t} \leq -0.1524, \quad a.s. \quad (3.39)$$

$$\int_0^\infty \mathbf{x}^2(t)dt < \infty, \quad \int_0^\infty \mathbf{x}^4(t)dt < \infty, \quad \int_0^\infty \mathbf{x}^6(t)dt < \infty, \quad a.s. \quad (3.40)$$

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = 0. \quad a.s. \quad (3.41)$$

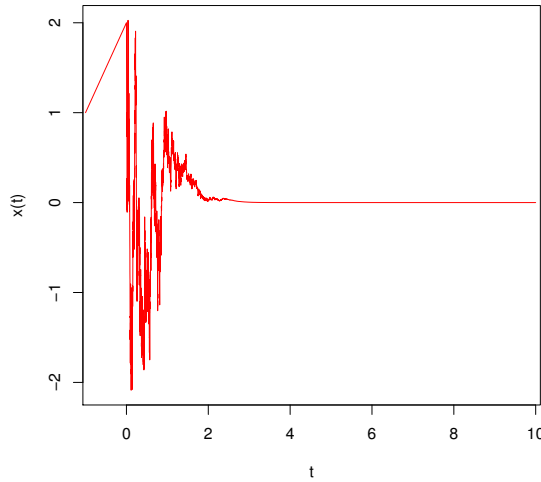


Figure 1: Simulation of the trajectory of the solution  $\mathbf{x}(t)$  of system (3.34) on  $[0,10]$ .

For system (3.34), take probability measure  $\eta(\theta) = \frac{e^\theta}{1-e^{-1}}$  and initial data  $\zeta = \{\theta+2 : -1 \leq \theta \leq 0\}$ ,  $i_0 = 1$ . Based on the Euler-Maruyama scheme with step size  $10^{-3}$ , we give a sequence of computer simulations for system (3.34) as follows. Figure 1 illustrates the assertion (3.41) by the simulation of the trajectory of the solution  $\mathbf{x}(t)$  of system (3.34). Figure 2 illustrates the assertions (3.37) and (3.38) by the simulations of  $\frac{\log(E\mathbf{x}^2(t))}{t}$ ,  $\int_0^t E(\mathbf{x}^2(t))dt$ ,  $\int_0^t E(\mathbf{x}^4(t))dt$  and  $\int_0^t E(\mathbf{x}^6(t))dt$  for the solution  $\mathbf{x}(t)$  of system (3.34). Figure 3 illustrates the assertions (3.39) and (3.40) by the simulations of the trajectories of  $\frac{\log(|\mathbf{x}(t)|)}{t}$ ,  $\int_0^t \mathbf{x}^2(t)dt$ ,  $\int_0^t \mathbf{x}^4(t)dt$  and  $\int_0^t \mathbf{x}^6(t)dt$  for the solution  $\mathbf{x}(t)$  of system (3.34).

## 4 Further Results

Assumption 3.2 in Section 3 requires that  $\alpha_0, \alpha_j, \beta_j (j = 1, 2, \dots, L)$  be constants. However, many SFDEs with Markovian switching do not satisfy these conditions. Let us see an example

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}_t, t, r(t))dt + \mathbf{g}(\mathbf{x}_t, t, r(t))dB(t), \quad (4.1)$$



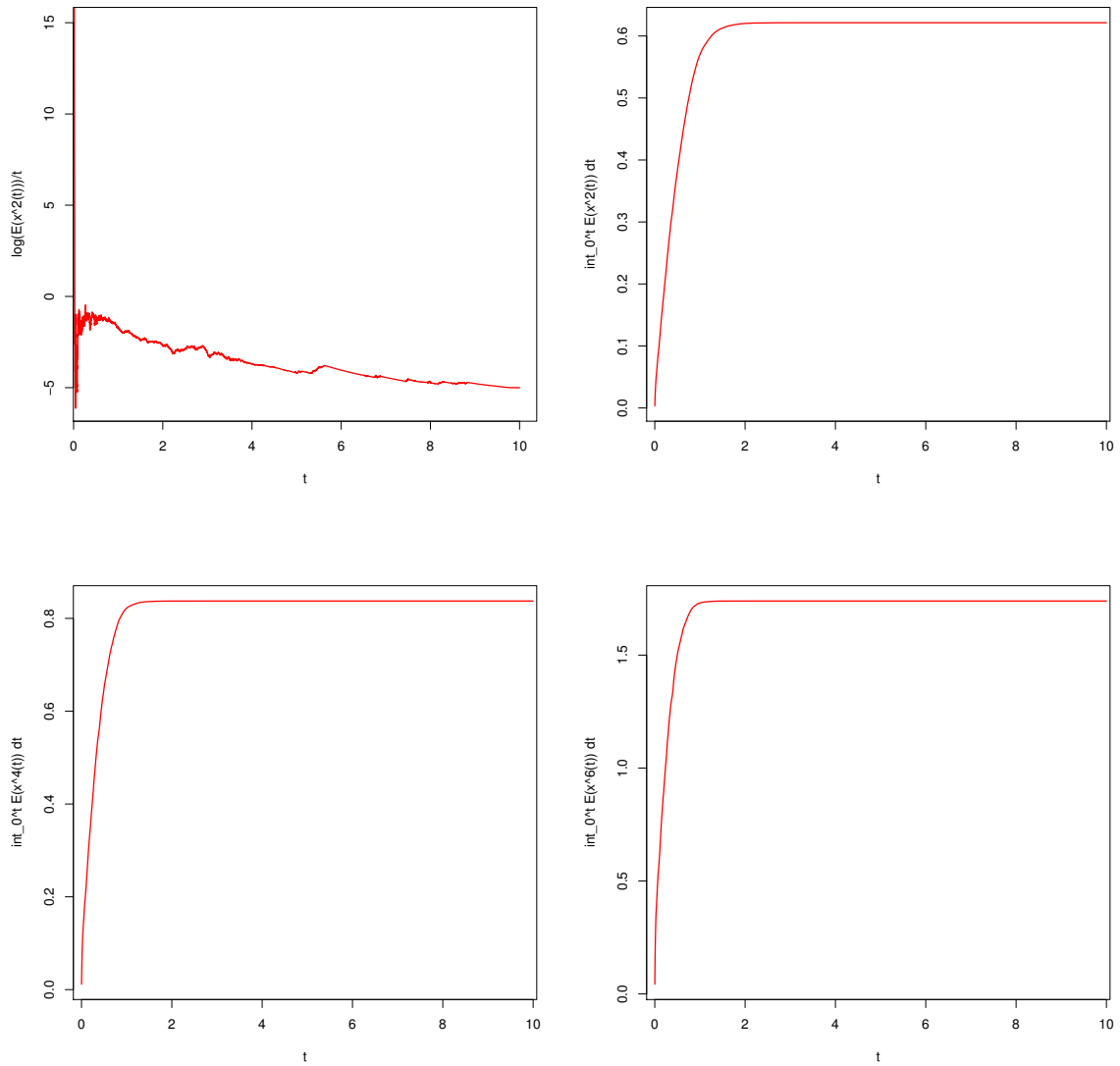


Figure 2: Simulations of  $\log(E\mathbf{x}^2(t))/t$ ,  $\int_0^t E(\mathbf{x}^2(t))dt$ ,  $\int_0^t E(\mathbf{x}^4(t))dt$  and  $\int_0^t E(\mathbf{x}^6(t))dt$  for the solution  $\mathbf{x}(t)$  of system (3.34) on  $[0, 10]$  and sample size 200.

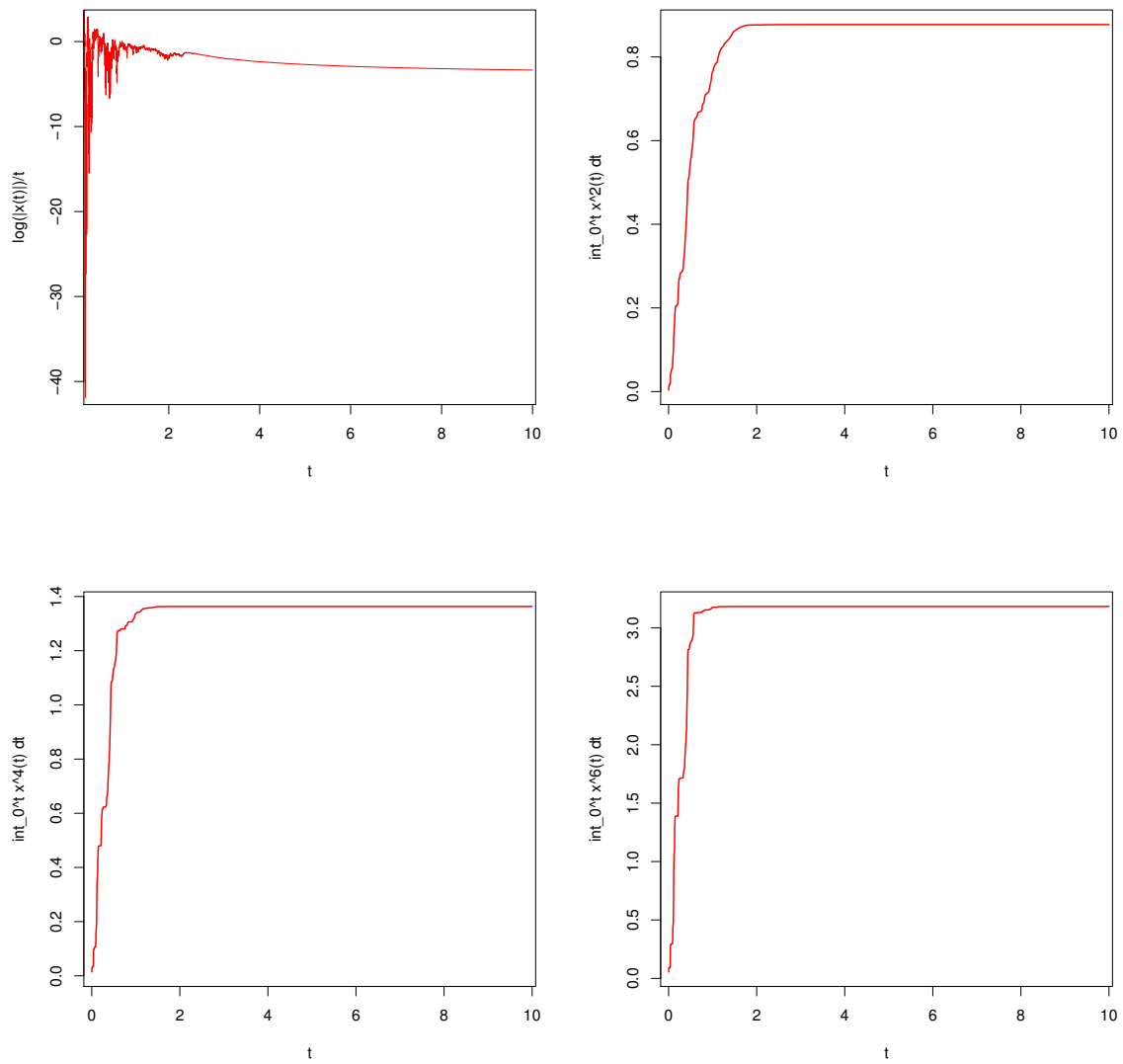


Figure 3: Simulations of the trajectories of  $\log(|\mathbf{x}(t)|)/t$ ,  $\int_0^t \mathbf{x}^2(t) dt$ ,  $\int_0^t \mathbf{x}^4(t) dt$  and  $\int_0^t \mathbf{x}^6(t) dt$  for the solution  $\mathbf{x}(t)$  of system (3.34) on  $[0, 10]$ .

where  $B(t)$ ,  $r(t)$  are the same as defined in system (3.34), and the coefficients are time-varying as follows:

$$\begin{aligned}\mathbf{f}(\mathbf{x}_t, t, 1) &= -4\left(1 + \frac{1}{1+t}\right)(\mathbf{x}(t) + \mathbf{x}^3(t) + \mathbf{x}^5(t)), & \mathbf{f}(\mathbf{x}_t, t, 2) &= \left(\frac{1}{4} - \frac{1}{1+t}\right)\mathbf{x}(t) - \frac{1}{1+t}\mathbf{x}^3(t), \\ \mathbf{g}(\mathbf{x}_t, t, 1) &= \sqrt{\frac{1}{1+t}}\left(2e^{-t} \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{\frac{1}{2}} d\eta(\theta) + \int_{-\tau}^0 |\mathbf{x}(t+\theta)|^3 d\eta(\theta)\right), \\ \mathbf{g}(\mathbf{x}_t, t, 2) &= \frac{1}{2}\sqrt{\frac{1}{1+t}} \int_{-\tau}^0 |\mathbf{x}(t+\theta)| d\eta(\theta),\end{aligned}$$

where  $\eta$  is a probability measure on  $[-\tau, 0]$ , which satisfies  $\int_{-\tau}^0 d\eta(\theta) = 1$ .

We easily see that all existing results and Theorem 3.1 in Section 3 can not be directly used to system (4.1) for the existence of time-varying coefficients.

As to system (4.1), using the same definition of  $V(x, t, i)$  as (3.36), we compute that

$$\begin{aligned}\mathcal{L}V(\mathbf{x}_t, t, 1) &= 2\mathbf{x}^T \mathbf{f}(\mathbf{x}_t, t, 1) + \mathbf{g}^2(\mathbf{x}_t, t, 1) + \sum_{j=1}^2 \gamma_{1j} V(\mathbf{x}(t), t, j) \\ &\leq -8\frac{1}{1+t}(x^2 + x^6) + 3\frac{1}{1+t} \int_{-\tau}^0 (|\mathbf{x}(t+\theta)|^2 + |\mathbf{x}(t+\theta)|^6) d\eta(\theta) - 8\frac{1}{1+t}x^4 + 3e^{-4t}, \\ \mathcal{L}V(\mathbf{x}_t, t, 2) &= 2(2x + 6x^5)^T \mathbf{f}(\mathbf{x}_t, t, 2) + \frac{1}{2}\mathbf{g}^T(\mathbf{x}_t, t, 2)(4 + 60x^4)\mathbf{g}(\mathbf{x}_t, t, 2) + \sum_{j=1}^2 \gamma_{2j} V(\mathbf{x}(t), t, j) \\ &\leq -4\frac{1}{1+t}(x^2 + x^6) + \frac{5}{2}\frac{1}{1+t} \int_{-\tau}^0 (|\mathbf{x}(t+\theta)|^2 + |\mathbf{x}(t+\theta)|^6) d\eta(\theta) - 4\frac{1}{1+t}x^4 - 12\frac{1}{1+t}x^8.\end{aligned}$$

That is, for  $i = 1, 2$ ,

$$\mathcal{L}V(\mathbf{x}_t, t, i) \leq -4\frac{1}{1+t}(x^2 + x^6) + 3\frac{1}{1+t} \int_{-\tau}^0 (|\mathbf{x}(t+\theta)|^2 + |\mathbf{x}(t+\theta)|^6) d\eta(\theta) - 4\frac{1}{1+t}x^4 + 3e^{-4t}.$$

Hence, define  $W_0(x, t) = x^2$ ,  $W_1(x, t) = 2(x^2 + x^6)$  and  $W_2(x, t) = x^4$ , then

$$\begin{aligned}W_0(x, t) &\leq V(x, t, i) \leq W_1(x, t), \\ \mathcal{L}V(\mathbf{x}_t, t, i) &\leq 3e^{-4t} - 2\frac{1}{1+t}W_1(x, t) + \frac{3}{2}\frac{1}{1+t} \int_{-\tau}^0 W_1(\mathbf{x}(t+\theta), t+\theta) d\eta(\theta) \\ &\quad - 4\frac{1}{1+t}W_2(x, t) + \frac{1}{1+t} \int_{-\tau}^0 W_2(\mathbf{x}(t+\theta), t+\theta) d\eta(\theta),\end{aligned}\tag{4.2}$$

for all  $x, x_t \in R, t \geq -\tau$  and  $i \in S$ .

Then it is natural to ask the following questions: Is there a unique global solution if a SFDE with Markovian switching obeys some conditions similar to (4.2)? If yes, whether the SFDE with Markovian switching has the properties of the asymptotic stability and boundedness? So, another aim of this paper is to answer these questions above. To do these, we extend the idea of the diffusion operators dominated by multiple auxiliary functions with constant coefficients in Theorem 3.1 to that in the case of time-varying coefficients.

Motivated by system (4.1) discussed above, we propose another new assumption.

**Assumption 4.1.** Let  $b_0(t), b_j(t) (j = 1, 2, \dots, L)$  be continuous functions from  $[-\tau, \infty)$  to  $R_+$  satisfying the following conditions:  $\int_0^\infty b_0(t)dt < \infty$ ,  $\int_0^\infty b_1(t)dt = \infty$ , functions  $b_j(t) (j = 1, 2, \dots, L)$  are monotonically non-increasing and bounded by 1. And let all conditions in Assumption 3.2 hold except (3.3) is replaced by

$$\mathcal{L}V(\varphi, t, i) \leq b_0(t) + \sum_{j=1}^L \left[ -\alpha_j b_j(t) W_j(\varphi(0), t) + \beta_j b_j(t) \int_{-\tau}^0 W_j(\varphi(\theta), t + \theta) d\eta_j(\theta) \right], \quad (4.3)$$

for all  $i \in S$  and  $\varphi \in C([-\tau, 0]; R^n)$ ,  $t \in R_+$ .

**Remark 3.** Note Assumption 4.1 is different from Assumption 3.2. The difference looks "small", but it's significant in the two following aspects:

- 1) Because of the existence of  $b_j(t) (j = 0, 1, \dots, L)$ , Assumption 4.1 is more adapt to non-autonomous systems;
- 2) The requirement of  $\alpha_0$  in Assumption 3.2 is different from that of  $b_0(t)$  in Assumption 4.1. Of course, Assumption 4.1 with  $b_0(t) = 0$  and  $b_j(t) = 1 (j = 1, 2, \dots, L)$ , reduces to the case of Assumption 3.2 with  $\alpha_0 = 0$ , namely, Assumption 3.2 with  $\alpha_0 = 0$  is only a special case of Assumption 4.1.

Since the coefficients of  $W_j (j = 1, \dots, L)$  in (4.3) are time-varying, it is said to be the case of multiple auxiliary functions with time-varying coefficients for convenience. Next, we state our main results of SFDEs with Markovian switching in this case.

**Theorem 4.1.** Under Assumptions 3.1 and 4.1, we have the following assertions:

- 1) For any given initial data  $\zeta \in C$  and  $i_0 \in S$ , there is a unique global solution  $\mathbf{x}(t, \zeta, i_0)$  of system (2.1) on  $t \geq -\tau$ , which satisfies the following moment properties

$$\sup_{-\tau \leq t < \infty} EW_0(\mathbf{x}(t, \zeta, i_0), t) < \infty, \quad (4.4)$$

$$\int_0^\infty b_j(s) EW_j(\mathbf{x}(s, \zeta, i_0), s) ds < \infty, \quad j = 1, 2, \dots, L \quad (4.5)$$

$$EW_0(\mathbf{x}(t, \zeta, i_0), t) \leq c_2 e^{-\varepsilon_0 \int_0^t b_1(h) dh} + \int_0^t e^{-\varepsilon_0 \int_s^t b_1(h) dh} b_0(s) ds, \quad (4.6)$$

$$\lim_{t \rightarrow \infty} EW_0(\mathbf{x}(t, \zeta, i_0), t) = 0. \quad (4.7)$$

and the following sample properties

$$\sup_{-\tau \leq t < \infty} \mathbf{x}(t, \zeta, i_0) < \infty, \quad a.s. \quad (4.8)$$

$$\int_0^\infty b_j(s) W_j(\mathbf{x}(s, \zeta, i_0), s) ds < \infty, \quad a.s. \quad j = 1, 2, \dots, L \quad (4.9)$$

where  $c_2$  is a finite positive constant,  $\varepsilon_0$  is the same as defined in Theorem 3.1.

2) Moreover, if  $b_0(t) = 0$ , the global solution  $\mathbf{x}(t, \boldsymbol{\zeta}, i_0)$  has the following moment property

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(EW_0(\mathbf{x}(t, \boldsymbol{\zeta}, i_0), t)) \leq -\varepsilon_0 \liminf_{t \rightarrow \infty} \frac{\int_0^t b_1(h) dh}{t}, \quad (4.10)$$

and the following sample property

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(W_0(\mathbf{x}(t, \boldsymbol{\zeta}, i_0), t)) \leq -\varepsilon_0 \liminf_{t \rightarrow \infty} \frac{\int_0^t b_1(h) dh}{t}. \quad a.s. \quad (4.11)$$

3) If furthermore there exists a continuous positive definite function  $G \in C(\mathbb{R}^n; \mathbb{R}_+)$  and a continuous function  $\psi(\cdot) \in \Psi(\mathbb{R}_+; \mathbb{R}_+)$  such that  $\psi(t)G(\mathbf{x}) \leq \sum_{j=1}^L b_j(t)W_j(\mathbf{x}, t)$  for all  $(\mathbf{x}, t) \in \mathbb{R}^n \times [-\tau, \infty)$ , then

$$\lim_{t \rightarrow \infty} \mathbf{x}(t, \boldsymbol{\zeta}, i_0) = 0. \quad a.s. \quad (4.12)$$

*Proof.* 1) The proof of the existence and uniqueness of the global solution  $\mathbf{x}(t, \boldsymbol{\zeta}, i_0)$  of system (2.1) is essentially similar to that of Theorem 3.1, so we omit it. For the sake of simplicity, write  $\mathbf{x}(t) = \mathbf{x}(t, \boldsymbol{\zeta}, i_0)$ . Using the generalized Itô's formula to  $e^{\varepsilon \int_0^t b_1(h) dh} V(\mathbf{x}(t), t, r(t))$ ,  $\varepsilon \in [0, \varepsilon_0]$ , and from condition (3.2), we get that, for any  $t \geq 0$ ,

$$\begin{aligned} & e^{\varepsilon \int_0^t b_1(h) dh} V(\mathbf{x}(t), t, r(t)) \\ &= V(\mathbf{x}(0), 0, r(0)) + \int_0^t e^{\varepsilon \int_0^s b_1(h) dh} (\varepsilon b_1(s) V(\mathbf{x}(s), s, r(s)) + \mathcal{L}V(\mathbf{x}_s, s, r(s))) ds + M_1(t) \\ &\leq W_1(\mathbf{x}(0), 0) + \int_0^t e^{\varepsilon \int_0^s b_1(h) dh} \left[ b_0(s) - (\alpha_1 - \varepsilon) b_1(s) W_1(\mathbf{x}(s), s) \right. \\ &\quad \left. + \beta_1 b_1(s) \int_{-\tau}^0 W_1(\mathbf{x}(s + \theta), s + \theta) d\eta_1(\theta) + \sum_{j=2}^L [-\alpha_j b_j(s) W_j(\mathbf{x}(s), s) \right. \\ &\quad \left. + \beta_j b_j(s) \int_{-\tau}^0 W_j(\mathbf{x}(s + \theta), s + \theta) d\eta_j(\theta)] \right] ds + M_1(t) \\ &\leq W_1(\mathbf{x}(0), 0) + \int_0^t e^{\varepsilon \int_0^s b_1(h) dh} \left[ b_0(s) - (\alpha_1 - \varepsilon - \beta_1 e^{\varepsilon \tau}) b_1(s) W_1(\mathbf{x}(s), s) + \beta_1 b_1(s) J'_1 \right. \\ &\quad \left. + \sum_{j=2}^L [-(\alpha_j - \beta_j e^{\varepsilon \tau}) b_j(s) W_j(\mathbf{x}(s), s) + \beta_j b_j(s) J'_j] \right] ds + M_1(t), \end{aligned} \quad (4.13)$$

where  $M_1(t)$  is a local martingale with the initial value  $M_1(0) = 0$ ,  $J'_j = \int_{-\tau}^0 W_j(\mathbf{x}(s + \theta), s + \theta) d\eta_j(\theta) - e^{\varepsilon \tau} W_j(\mathbf{x}(s), s)$ ,  $j = 1, 2, \dots, L$ .

By virtue of the fact that

$$\int_0^t e^{\varepsilon \int_0^s b_1(h) dh} b_j(s) J'_j ds$$

$$\begin{aligned}
&\leq \int_{-\tau}^0 \int_{-\tau}^t e^{\varepsilon \int_0^{r-\theta} b_1(h)dh} b_j(r-\theta) W_j(\mathbf{x}(r), r) dr d\eta_j(\theta) - e^{\varepsilon\tau} \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds \\
&\leq \int_{-\tau}^0 \int_{-\tau}^t e^{\varepsilon(\int_0^r b_1(h)dh + \int_r^{r-\theta} b_1(h)dh)} b_j(r) W_j(\mathbf{x}(r), r) dr d\eta_j(\theta) - e^{\varepsilon\tau} \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds \\
&\leq e^{\varepsilon\tau} \int_{-\tau}^0 e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds, \tag{4.14}
\end{aligned}$$

for  $j = 1, 2, \dots, L$ , (4.13) can be rewritten as

$$\begin{aligned}
&e^{\varepsilon \int_0^t b_1(h)dh} V(\mathbf{x}(t), t, r(t)) \\
&\leq W_1(\mathbf{x}(0), 0) + \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_0(s) ds - (\alpha_1 - \varepsilon - \beta_1 e^{\varepsilon\tau}) \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_1(s) W_1(\mathbf{x}(s), s) ds \\
&\quad - \sum_{j=2}^L (\alpha_j - \beta_j e^{\varepsilon\tau}) \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds \\
&\quad + \sum_{j=1}^L \beta_j e^{\varepsilon\tau} \int_{-\tau}^0 e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds + M_1(t) \\
&\leq c_2 + \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_0(s) ds - (\alpha_1 - \varepsilon - \beta_1 e^{\varepsilon\tau}) \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_1(s) W_1(\mathbf{x}(s), s) ds \\
&\quad - \sum_{j=2}^L (\alpha_j - \beta_j e^{\varepsilon\tau}) \int_0^t e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds + M_1(t), \tag{4.15}
\end{aligned}$$

where  $c_2 = W_1(\mathbf{x}(0), 0) + \sum_{j=1}^L \beta_j e^{\varepsilon\tau} \int_{-\tau}^0 e^{\varepsilon \int_0^s b_1(h)dh} b_j(s) W_j(\mathbf{x}(s), s) ds$ . Take the expectation on both side of (4.15), and let  $\varepsilon \rightarrow \varepsilon_0$ , we get the assertion (4.6)

$$EW_0(\mathbf{x}(t), t) \leq c_2 e^{-\varepsilon_0 \int_0^t b_1(h)dh} + \int_0^t e^{-\varepsilon_0 \int_s^t b_1(h)dh} b_0(s) ds, \tag{4.16}$$

from the definition of  $\varepsilon_0$  and condition (3.2).

From condition  $\int_0^\infty b_1(t)dt = \infty$ , it is obvious that, as  $t \rightarrow \infty$ ,

$$e^{-\varepsilon_0 \int_0^t b_1(h)dh} \rightarrow 0.$$

On the other hand, we have that

$$\begin{aligned}
\int_0^t e^{-\varepsilon_0 \int_s^t b_1(h)dh} b_0(s) ds &= \int_0^T e^{-\varepsilon_0 \int_s^t b_1(h)dh} b_0(s) ds + \int_T^t e^{-\varepsilon_0 \int_s^t b_1(h)dh} b_0(s) ds \\
&\leq e^{-\varepsilon_0 \int_T^t b_1(h)dh} \int_0^\infty b_0(s) ds + \int_T^\infty b_0(s) ds
\end{aligned}$$

For fixed  $T$ ,  $e^{-\varepsilon_0 \int_T^t b_1(h)dh} \rightarrow 0$  holds, as  $t \rightarrow \infty$ . Then letting  $T \rightarrow \infty$ , we have  $\int_T^\infty b_0(s) ds \rightarrow 0$ . So

$$\int_0^t e^{-\varepsilon_0 \int_s^t b_1(h)dh} b_0(s) ds \rightarrow 0 \tag{4.17}$$

holds, as  $t \rightarrow \infty$ . Therefore, we get the result (4.7)

$$\lim_{t \rightarrow \infty} EW_0(\mathbf{x}(t), t) = 0.$$

Letting  $\varepsilon \rightarrow 0$  in (4.15), taking the expectation and then using Fubini theorem, we have

$$EW_0(\mathbf{x}(t), t) \leq c'_2 + \int_0^t b_0(s)ds - \sum_{j=1}^L (\alpha_j - \beta_j) \int_0^t b_j(s)EW_j(\mathbf{x}(s), s)ds, \quad (4.18)$$

where  $c'_2 = EW_1(\mathbf{x}(0), 0) + \sum_{j=1}^L \beta_j E \int_{-\tau}^0 b_j(s)W_j(\mathbf{x}(s), s)ds$ . From condition  $\int_0^\infty b_0(t)dt < \infty$ , we get the assertions (4.4) and (4.5)

$$\begin{aligned} \sup_{-\tau \leq t < \infty} EW_0(\mathbf{x}(t), t) &\leq c'_2 + \int_0^\infty b_0(s)ds < \infty, \\ \int_0^\infty b_j(s)EW_j(\mathbf{x}(s), s)ds &\leq \frac{c'_2 + \int_0^\infty b_0(s)ds}{\alpha_j - \beta_j} < \infty. \quad j = 1, 2, \dots, L \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , (4.15) becomes

$$V(\mathbf{x}(t), t, r(t)) \leq c_3 + \int_0^t b_0(s)ds - \sum_{j=1}^L (\alpha_j - \beta_j) \int_0^t b_j(s)W_j(\mathbf{x}(s), s)ds + M_2(t), \quad (4.19)$$

where  $c_3 = W_1(\mathbf{x}(0), 0) + \sum_{j=1}^L \beta_j \int_{-\tau}^0 b_j(s)W_j(\mathbf{x}(s), s)ds$ ,  $M_2(t)$  is a local martingale with the initial value  $M_2(0) = 0$ . From Lemma 2.1, we obtain the assertions (4.9) and  $\sup_{-\tau \leq t < \infty} W_0(\mathbf{x}(t), t) < \infty, a.s.$ . From the radial unboundedness of  $W_0(x, t)$  (see (3.1)), we directly get the assertion (4.8).

2) Next, we prove the results in the case of  $b_0(t) = 0$ . (4.15) becomes

$$e^\varepsilon \int_0^t b_1(h)dh V(\mathbf{x}(t), t, r(t)) \leq c_2 + M_1(t), \quad (4.20)$$

where  $c_2$  has been defined above. Taking the expectation on both side of (4.20), we have

$$e^\varepsilon \int_0^t b_1(h)dh EW_0(\mathbf{x}(t), t, r(t)) \leq c_2. \quad (4.21)$$

So as  $\varepsilon \rightarrow \varepsilon_0$ , we get the result (4.10)

$$\limsup_{t \rightarrow \infty} \frac{\log(EW_0(\mathbf{x}(t), t))}{t} \leq -\varepsilon_0 \liminf_{t \rightarrow \infty} \frac{\int_0^t b_1(h)dh}{t}.$$

Applying Lemma 2.1 to (4.20), we obtain that

$$\limsup_{t \rightarrow \infty} e^\varepsilon \int_0^t b_1(h)dh W_0(\mathbf{x}(t), t) < \infty. \quad a.s. \quad (4.22)$$

Hence, there exists a finite positive random variable  $\bar{\zeta}$  such that

$$\sup_{0 \leq t < \infty} e^\varepsilon \int_0^t b_1(h)dh W_0(\mathbf{x}(t), t) < \bar{\zeta}. \quad a.s. \quad (4.23)$$

So as  $\varepsilon \rightarrow \varepsilon_0$ , we get the result (4.11)

$$\limsup_{t \rightarrow \infty} \frac{\log(W_0(\mathbf{x}(t), t))}{t} \leq -\varepsilon_0 \liminf_{t \rightarrow \infty} \frac{\int_0^t b_1(h) dh}{t}. \quad a.s.$$

3) Finally, we prove the result (4.12). Similar to the proof of (3.11), the assertion (4.12) is equivalent to the assertion

$$\limsup_{t \rightarrow \infty} G(\mathbf{x}(t)) = 0. \quad a.s.$$

If this is false, then there exists a constant  $\delta_1$  such that  $P(B) > 3\delta_1$ , where  $B = \left\{ \omega : \limsup_{t \rightarrow \infty} G(\mathbf{x}(t)) > 2\delta_1 \right\}$ . From the assertion (4.8), we can see that there exists a sufficiently large positive constant  $h = h(\delta_1)$  such that  $P(D) > 1 - \delta_1$ , where  $D = \left\{ \omega : \sup_{-\tau \leq t < \infty} |\mathbf{x}(t)| < h \right\}$ . So we easily obtain that  $P(B \cap D) > 2\delta_1$ .

For any fixed  $\bar{\theta} > 0$ , define a sequence of stopping times by

$$\begin{aligned} \sigma_1 &= \inf\{t \geq 0 : G(\mathbf{x}(t)) \geq 2\delta_1\}, \quad \sigma_2 = \inf\{t \geq \sigma_1 : G(\mathbf{x}(t)) \leq \delta_1\}, \\ \sigma_{2l+1} &= \inf\{t \geq \sigma_{2l} + \bar{\theta} : G(\mathbf{x}(t)) \geq 2\delta_1\}, \quad l = 1, 2, \dots \\ \sigma_{2l+2} &= \inf\{t \geq \sigma_{2l+1} : G(\mathbf{x}(t)) \leq \delta_1\}. \quad l = 1, 2, \dots \end{aligned}$$

Similarly, we obtain  $B \cap D \subset \{\sigma_l < \infty, l = 1, 2, \dots\}$  and that, for any  $T_2 > 0$ ,

$$E \left( I_{B \cap D} \sup_{0 \leq t \leq T_2} |\mathbf{x}(\sigma_{2l-1} + t) - \mathbf{x}(\sigma_{2l-1})|^2 \right) \leq 2T_2 H_h^2(T_2 + 4),$$

and that there exists a sufficiently small  $\delta_2 = \delta_2(\delta_1) > 0$  such that  $|G(\mathbf{x}) - G(\mathbf{y})| < \delta_1$ , when  $|\mathbf{x}| \vee |\mathbf{y}| \leq h$  and  $|\mathbf{x} - \mathbf{y}| \leq \delta_2$ .

Noting that  $\left\{ \sup_{0 \leq t \leq T_2} |\mathbf{x}(\sigma_{2l-1} + t) - \mathbf{x}(\sigma_{2l-1})| \leq \delta_2 \right\} \subset \left\{ \sup_{0 \leq t \leq T_2} |G(\mathbf{x}(\sigma_{2l-1} + t)) - G(\mathbf{x}(\sigma_{2l-1}))| < \delta_1 \right\} \subset \left\{ \sigma_{2l} - \sigma_{2l-1} \geq T_2 \right\}$ , we similarly choose a sufficiently small  $T_2 > 0$  such that  $P \left( B \cap D \cap \left\{ \sup_{0 \leq t \leq T_2} |\mathbf{x}(\sigma_{2l-1} + t) - \mathbf{x}(\sigma_{2l-1})| < \delta_2 \right\} \right) > \delta_1$ .

Since  $\psi(t) \in \Psi(R_+; R_+)$ , then for the chosen  $T_2 > 0$  there exist constants  $\rho = \rho(T_2), T_3 = T_3(\rho, T_2)$  such that  $\int_t^{t+T_2} \psi(s) ds \geq \rho$ , when  $t \geq T_3$ . From the definition of  $\sigma_l$ , there exists a positive integer  $Q_1 = [T_3/\bar{\theta}] + 2$  such that, for any  $\omega \in B \cap D$ ,  $\sigma_{2l-1} > T_3$ , when  $l \geq Q_1$ . Hence, for any  $\omega \in B \cap D$ , we have  $\int_{\sigma_{2l-1}}^{\sigma_{2l-1}+T_2} \psi(s) ds \geq \rho, l \geq Q_1$ .

On the other hand, from the assertion (4.5) and the relation of  $\psi(t)G(\mathbf{x})$  and  $\sum_{j=1}^L b_j(t)W_j(\mathbf{x}, t)$ , we easily get that

$$E \int_0^\infty \psi(s) G(\mathbf{x}(s)) ds \leq \sum_{j=1}^L E \int_0^\infty b_j(s) W_j(\mathbf{x}(s, \boldsymbol{\zeta}, i_0), s) ds < \infty. \quad (4.24)$$



For any  $\omega \in B \cap D \cap \left\{ \sup_{0 \leq t \leq T_2} |\mathbf{x}(\sigma_{2l-1} + t) - \mathbf{x}(\sigma_{2l-1})| < \delta_2 \right\}$ , we have

$$\begin{aligned}
E \int_0^\infty \psi(s)G(\mathbf{x}(s))ds &\geq E \left( \sum_{l=1}^\infty I_{\{\sigma_l < \infty\}} \int_{\sigma_{2l-1}}^{\sigma_{2l}} \psi(s)G(\mathbf{x}(s))ds \right) \\
&\geq \delta_1 E \left( \sum_{l=Q_1}^\infty I_{B \cap D \cap \left\{ \sup_{0 \leq t \leq T_2} |\mathbf{x}(\sigma_{2l-1} + t) - \mathbf{x}(\sigma_{2l-1})| < \delta_2 \right\}} \int_{\sigma_{2l-1}}^{\sigma_{2l}} \psi(s)ds \right) \\
&\geq \delta_1 E \left( \sum_{l=Q_1}^\infty I_{B \cap D \cap \left\{ \sup_{0 \leq t \leq T_2} |\mathbf{x}(\sigma_{2l-1} + t) - \mathbf{x}(\sigma_{2l-1})| < \delta_2 \right\}} \int_{\sigma_{2l-1}}^{\sigma_{2l-1} + T_2} \psi(s)ds \right) \\
&\geq \delta_1 \rho \sum_{l=Q_1}^\infty \delta_1 \\
&= \infty,
\end{aligned}$$

which implies a contradiction with (4.24). So we get the assertion (4.12).  $\square$

**Remark 4.** The results of Theorem 4.1 look similar to that of Theorem 3.1, but they are in fact significantly different. The contribution of Theorem 4.1 is significant in at least the following three aspects:

- 1) When the system is non-autonomous, it is natural to resort to Theorem 4.1 rather than Theorem 3.1;
- 2) Since  $\Psi(R_+; R_+)$  contains many functions (see [6]), the results of Theorem 4.1 can be used to much greater fields than that of Theorem 3.1 with  $\alpha_0 = 0$ ;
- 3) Because of the properties of  $\Psi(R_+; R_+)$ , the techniques used in Theorem 3.1 can not be used to Theorem 4.1 directly. So we introduce some new techniques to cope with the function in  $\Psi(R_+; R_+)$ .

**Proposition 4.1.** *Let Assumptions 3.1, 4.1 hold except (4.3) is replaced by*

$$\begin{aligned}
\mathcal{L}V(\boldsymbol{\varphi}, t, i) &\leq b_0(t) + \sum_{j=1}^L \left[ -\alpha_j b_j(t) W_j(\boldsymbol{\varphi}(0), t) + \beta_j b'_j(t) \int_{-\tau}^0 W'_j(\boldsymbol{\varphi}(\theta), t + \theta) d\eta_j(\theta) \right], \\
b_j(t) &\geq b'_j(t), \quad W_j(\boldsymbol{\varphi}, t) \geq W'_j(\boldsymbol{\varphi}, t), \quad j = 1, 2, \dots, L
\end{aligned}$$

for all  $i \in S$  and  $\boldsymbol{\varphi} \in C([-\tau, 0]; R^n)$ ,  $t \in R_+$ , where  $b'_j(t) \in C([-\tau, \infty), R_+)$ ,  $W'_j \in C(R^n \times [-\tau, \infty); R_+)$ . Then all results of Theorem 4.1 hold.

Finally, we give the following example to illustrate our theorem.

**Example 4.1.** Let us continue to use the notations of system (4.1). Further we set  $\tau = 0.2$ . By (4.2) and  $W_0(x, t) = x^2$ ,  $W_1(x, t) = 2(x^2 + x^6)$ ,  $W_2(x, t) = x^4$ , we note that  $W_0(x, t)$  satisfies (3.1), and get the parameters in Assumption 4.1 as follows:

$$b_0(t) = 3e^{-4t}, b_1(t) = b_2(t) = \frac{1}{1+t}, \alpha_1 = 2, \beta_1 = \frac{3}{2}, \alpha_2 = 4, \beta_2 = 1.$$

And we compute that  $\varepsilon_0 = \sup\{\varepsilon > 0 : \varepsilon + \frac{3}{2}e^{0.2\varepsilon} - 2 \leq 0, \varepsilon \leq 10 \log 2\} = 0.3812$ . Hence, from Theorem 4.1 there is a unique global solution  $\mathbf{x}(t)$  of system (4.1). And we get that the global solution  $\mathbf{x}(t)$  of system (4.1) has the following moment properties

$$\sup_{-0.2 \leq t < \infty} E(\mathbf{x}^2(t)) < \infty, \\ \int_0^\infty \frac{1}{1+t} E(\mathbf{x}^2(t)) dt < \infty, \int_0^\infty \frac{1}{1+t} E(\mathbf{x}^4(t)) dt < \infty, \int_0^\infty \frac{1}{1+t} E(\mathbf{x}^6(t)) dt < \infty, \quad (4.25)$$

$$E(\mathbf{x}^2(t)) \leq c_2 e^{-0.3812 \log(1+t)} + 3e^{-0.3812 \log(1+t)} \int_0^t e^{0.3812 \log(1+s) - 2s} ds,$$

$$\lim_{t \rightarrow \infty} E(\mathbf{x}^2(t)) = 0, \quad (4.26)$$

and the following sample properties

$$\sup_{-0.2 \leq t < \infty} |\mathbf{x}(t)| < \infty, \\ \int_0^\infty \frac{1}{1+t} \mathbf{x}^2(t) dt < \infty, \int_0^\infty \frac{1}{1+t} \mathbf{x}^4(t) dt < \infty, \int_0^\infty \frac{1}{1+t} \mathbf{x}^6(t) dt < \infty, \quad a.s. \quad (4.27)$$

where  $c_2$  is a finite positive constant.

Set positive definite function  $G(\mathbf{x}) = \mathbf{x}^2$  and  $\psi(t) = \frac{1}{1+t} \in \Psi(R_+; R_+)$ , then  $\psi(t)G(\mathbf{x}) \leq b_1(t)W_1(\mathbf{x}, t) + b_2(t)W_2(\mathbf{x}, t)$ , and from Theorem 4.1 we get that

$$\lim_{t \rightarrow \infty} \mathbf{x}(t, \zeta, i_0) = 0. \quad a.s. \quad (4.28)$$

For system (4.1), take probability measure  $\eta(\theta) = \frac{e^\theta - e^{-0.2}}{1 - e^{-0.2}}$  and initial data  $\zeta = \{\theta + 2 : -0.2 \leq \theta \leq 0\}$ ,  $i_0 = 1$ . Based on the Euler-Maruyama scheme with step size  $10^{-3}$ , we give a sequence of computer simulations for system (4.1) as follows. Figure 4 illustrates the assertions (4.25) and (4.26) by the simulations of  $E(\mathbf{x}^2(t))$ ,  $\int_0^t \frac{1}{1+t} E(\mathbf{x}^2(t)) dt$ ,  $\int_0^t \frac{1}{1+t} E(\mathbf{x}^4(t)) dt$  and  $\int_0^t \frac{1}{1+t} E(\mathbf{x}^6(t)) dt$  for the solution  $\mathbf{x}(t)$  of system (4.1). Figure 5 illustrates the assertions (4.27) and (4.28) by the simulations of the trajectories of the solution  $\mathbf{x}(t)$  of system (4.1),  $\int_0^t \frac{1}{1+t} \mathbf{x}^2(t) dt$ ,  $\int_0^t \frac{1}{1+t} \mathbf{x}^4(t) dt$  and  $\int_0^t \frac{1}{1+t} \mathbf{x}^6(t) dt$ .

## 5 Conclusions

Motivated by Hu, Mao and Shen [3], we have used the method of multiple Lyapunov functions to study the asymptotic boundedness and stability of the SFDE with Markovian switching. Different type of Lyapunov functions for different mode has been used to deal with the structure difference of the underlying SFDE in different mode. It is in this way that our results on the boundedness and stability are more general than the existing results in this area. Several examples and computer simulations have been used to illustrate our new results.

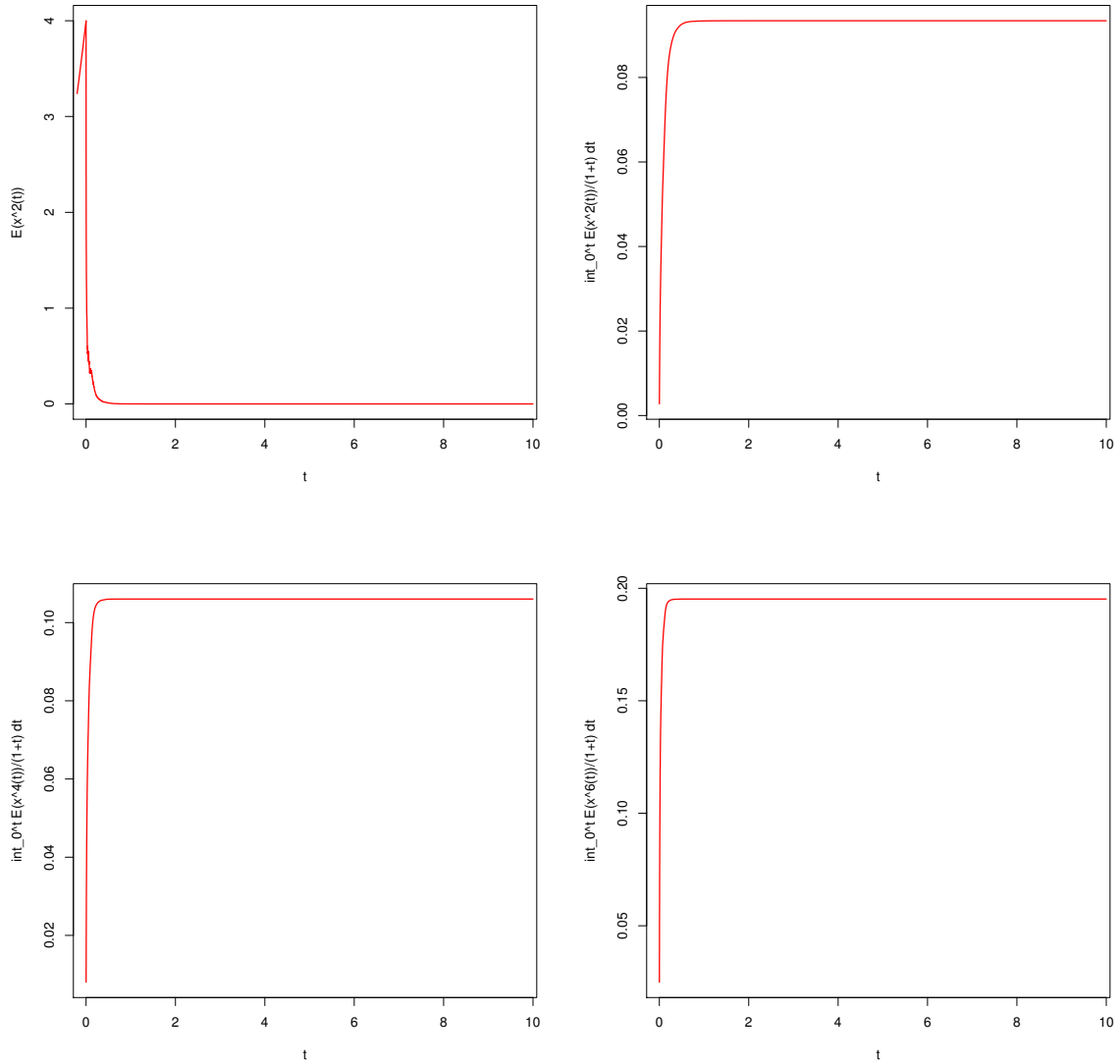


Figure 4: Simulations of  $E(\mathbf{x}^2(t))$ ,  $\int_0^t \frac{1}{1+t} E(\mathbf{x}^2(t)) dt$ ,  $\int_0^t \frac{1}{1+t} E(\mathbf{x}^4(t)) dt$ ,  $\int_0^t \frac{1}{1+t} E(\mathbf{x}^6(t)) dt$  for the solution  $\mathbf{x}(t)$  of system (4.1) on  $[0, 10]$  and sample size 200.

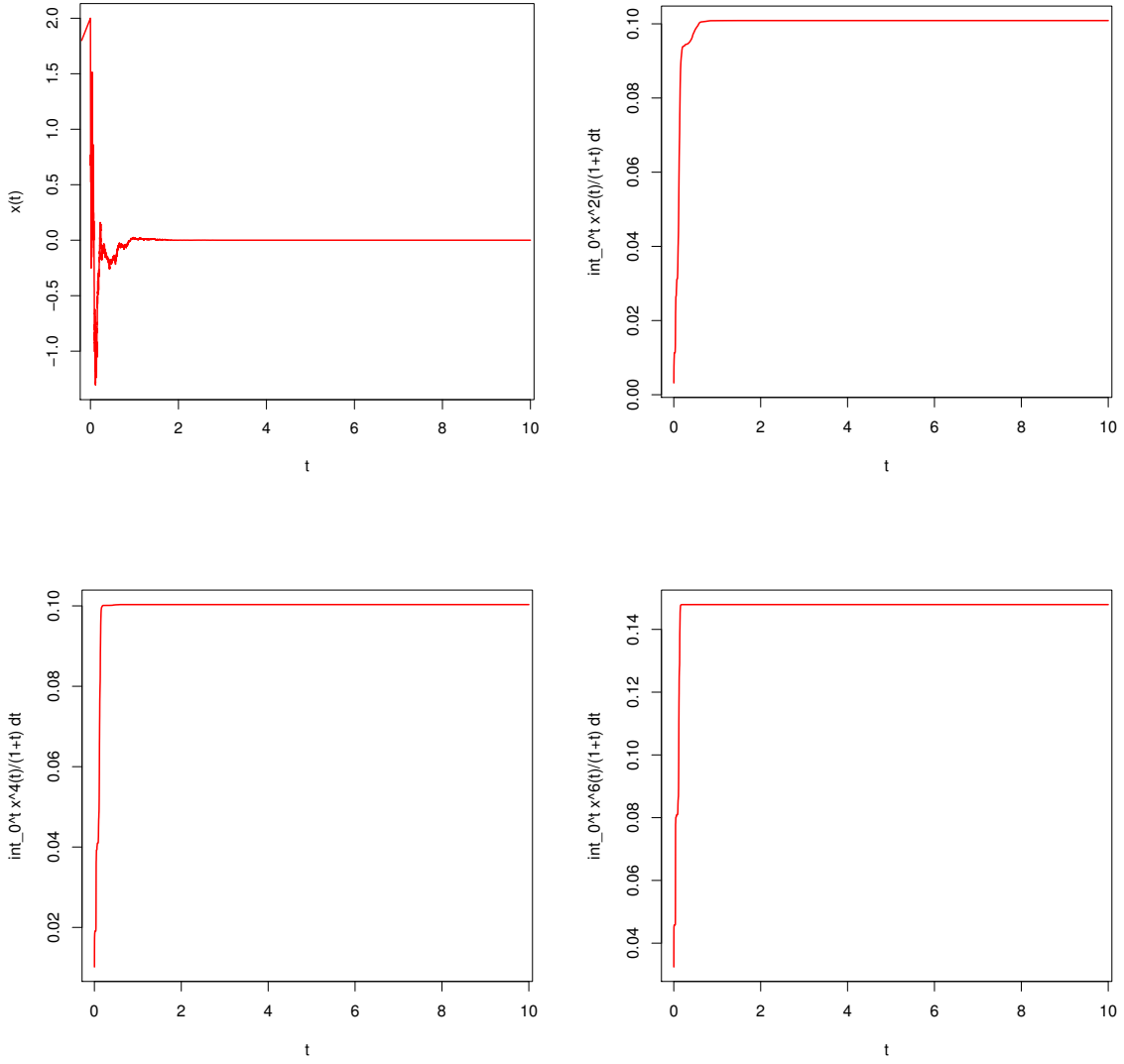


Figure 5: Simulations of the trajectories of the solution  $\mathbf{x}(t)$  of system (4.1),  $\int_0^t \frac{1}{1+t} \mathbf{x}^2(t) dt$ ,  $\int_0^t \frac{1}{1+t} \mathbf{x}^4(t) dt$  and  $\int_0^t \frac{1}{1+t} \mathbf{x}^6(t) dt$  on  $[0, 10]$ .

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