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Stationary distribution of stochastic SIRS epidemic model with standard incidence

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Abstract

We study stochastic versions of a deterministic SIRS(Susceptible, Infective, Recovered, Susceptible) epidemic model with standard incidence. We study the existence of a stationary distribution of stochastic system by the theory of integral Markov semigroup. We prove the distribution densities of the solutions can converge to an invariant density in L^1 . This shows the system is ergodic. The presented results are demonstrated by numerical simulations.

1 Introduction

Infectious diseases are the second cause of leading death worldwide, after heart disease. Therefore, it is imperative to know the dynamical behavior of such diseases and to forecast what may happen. The spread of communicable diseases is generally described mathematically by compartmental models. Most epidemiological models stimulate from the classical SIR compartmental model of Kermack and McKendrick [14] which assumes a constant population. The assumption that the population is constant or asymptotically constant is often a reasonable approximation when modeling epidemics where the disease

spreads quickly in the population and dies out within a short time (influenza, SARS, etc.) and the disease rarely causes deaths (West Nile virus in human or livestock). However, for endemic diseases in communities with changing populations (malaria), or diseases with high mortality rate (HIV/AIDS in poor countries), it is reasonable to assume varying population size and standard incidence rate in the mathematical theory of epidemiology (see, for instance, [1, 2, 3, 4, 17, 11]).

In the SIRS models, susceptible individuals may become infected by contact with infective individuals, and after some time, recovered individuals become susceptible again. Busenberg and van den Driessche [7] have proposed the continuous SIRS epidemic model in a population with varying size:

$$\begin{cases} \frac{dS_t}{dt} = bN_t - \mu S_t - \frac{\beta S_t I_t}{N_t} + \delta R_t, \\ \frac{dI_t}{dt} = \frac{\beta S_t I_t}{N_t} - (\mu + \alpha + \gamma)I_t, \\ \frac{dR_t}{dt} = \gamma I_t - (\mu + \delta)R_t. \end{cases} \quad (1)$$

Here S , I and R denote the total numbers of susceptible, infective and recovered (removed) individuals respectively, and N is the total population size,

$$dN_t/dt = (b - \mu)N_t - \alpha I_t.$$

The parameter b denotes the per capita birth rate, μ denotes the per capita disease free death rate, α denotes the disease-related per capita death rate of infected individuals. δ denotes the per capita loss of immunity rate of recovered individuals, γ denotes the per capita recovery rate of infected individuals. The force of infection is $\frac{\beta I}{N}$ with β as the effective per capita contact rate of an infective individual. All parameter values are assumed to be nonnegative. They discuss the extinction and persistence of the epidemic according to the threshold

$$R_0 = \frac{\beta}{b + \alpha + \gamma}. \quad (2)$$

As a matter of fact, population systems are often subject to environmental noise. May [21] pointed out the fact that due to environmental fluctuations, the birth rates, death rates, carrying capacity and other parameters involved with the model system exhibit random fluctuations to a greater or lesser extent. Realistic models of population dynamics must take into account both predictable and unpredictable changes in those factors. The standard technique of parameter perturbation has been used by many authors for building stochastic epidemic models [8, 9, 20, 15, 30, 28, 27].

In studying epidemic dynamical system, we are always interested in when the disease will die out or prevail. In the deterministic models, the second problem is solved by showing that the endemic equilibrium of corresponding model is a global attractor or is globally asymptotically stable. But, there may be no endemic equilibrium in stochastic systems. Therefore, it is necessary to study the existence of stationary distribution to the solution and whether the solution is ergodic. These properties can reveal the disease to persist. Mao [9] showed the existence of a unique stationary distribution of the solution of the stochastic SIS epidemic model based on the theory of Hasminskii [10]. Lahrouz and Omari [16] studied a stochastic SIRS epidemic model with general incidence rate in a population of varying size. Sufficient conditions for the extinction and the existence of

a unique stationary distribution are obtained. Lin and Jiang [18] discussed a stochastic classic SIR system with bilinear incidence. They showed sufficient conditions for the disease to extinct exponentially. In the case of persistence they proved the existence of a stationary distribution and found the support of the invariant density.

Recently, Zhao et al. [29] considered the stochastic versions of model (1) with varying population and standard incidence:

$$\begin{cases} dS_t = (bN_t - \mu S_t - \frac{\beta S_t I_t}{N_t} + \delta R_t)dt - \frac{\sigma S_t I_t}{N_t} dB_t, \\ dI_t = (\frac{\beta S_t I_t}{N_t} - (\mu + \alpha + \gamma)I_t)dt + \frac{\sigma S_t I_t}{N_t} dB_t, \\ dR_t = (\gamma I_t - (\mu + \delta)R_t)dt. \end{cases} \quad (3)$$

Here B_t is a standard Brownian motion with $B(0) = 0$, the white noise intensity is $\sigma^2 > 0$. The authors [29] obtained that, a crucial threshold

$$\tilde{R}_0 = \frac{\beta}{b + \alpha + \gamma} - \frac{\sigma^2}{2(b + \alpha + \gamma)} \quad (4)$$

is determined which indicates the differences between the deterministic and stochastic model. When the threshold is less than one or the noise intensity is large, they deduce the disease to extinct exponentially. When the threshold is more than one, sufficient conditions for persistence in the mean are established. But in the case of persistence they can not obtain the existence of stationary distribution of system (3). The aim of this paper is to fill the gap. Hence our work can be considered as the further work of Zhao et al. [29].

From system (3), the equation for the total population size is $\dot{N}(t) = (b - \mu)N_t - \alpha I_t$. Obviously, the total population is a variable. Thus it is convenient to work with proportions of susceptibles, infectives and recovered in the population (as in deterministic model [7]). Defining

$$x_t = \frac{S_t}{N_t}, \quad y_t = \frac{I_t}{N_t}, \quad \text{and} \quad z_t = \frac{R_t}{N_t}, \quad (5)$$

it gives $x_t + y_t + z_t = 1$. Then it is sufficient to study the stochastic differential equation of x_t, y_t and z_t . Using Itô's formula and the relation $z_t = 1 - x_t - y_t$, we can omit the equation of z_t and discuss the following system

$$\begin{cases} dx_t = [(b + \delta)(1 - x_t) - (\beta - \alpha)x_t y_t - \delta y_t]dt - \sigma x_t y_t dB_t, \\ dy_t = [\beta x_t y_t - (b + \alpha + \gamma)y_t + \alpha y_t^2]dt + \sigma x_t y_t dB_t, \end{cases} \quad (6)$$

with any given initial value $(x_0, y_0) \in \mathbb{R}_+^2$ and $x_0 + y_0 < 1$.

In this paper, we will study the long-time behavior of system (6). The main aim of this paper is to study the existence of a stationary distribution of system (6) and its asymptotic stability. To the best of our knowledge, there are no literatures analytically concerning the existence of a stationary distribution of epidemic models with standard incidence. This is mainly because the Fokker-Planck equation (7) corresponding to system (6) is of a degenerate type. Therefore, it is not within the scope of application of Has'minskii theorem which is used in [9, 19, 13]. The key to our analysis approach is based on the theory of integral Markov semigroups, which was used in [23, 24, 18]. The specific strategy is given section 3. In the appendix, we present some auxiliary results and the main tools concerning Markov semigroups.

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and \mathcal{F}_0 contains all P -null sets). Denote $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

2 Preliminaries

Now we introduce a Markov semigroup connected with the Fokker-Planck equation (7).

Let $X = \mathbb{R}_+^2$, Σ be the σ -algebra of Borel subsets of X , and m be the Lebesgue measure on (X, Σ) . We denote $\mathcal{P}(t, x, y, A)$ to the transition probability function for the diffusion process (x_t, y_t) of (6) with the initial condition $(x_0, y_0) = (x, y)$, i.e. $\mathcal{P}(t, x, y, A) = \text{Prob}((x_t, y_t) \in A)$. If, for $t > 0$, the distribution of (x_t, y_t) is absolutely continuous with respect to the Lebesgue measure with the density $u(t, x, y)$, then $u(t, x, y)$ satisfies the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sigma^2 \left[\frac{\partial^2(x^2 y^2 u)}{\partial x^2} - 2 \frac{\partial^2(x^2 y^2 u)}{\partial x \partial y} + \frac{\partial^2(x^2 y^2 u)}{\partial y^2} \right] - \frac{\partial(f_1(x, y)u)}{\partial x} - \frac{\partial(f_2(x, y)u)}{\partial y}, \quad (7)$$

where $f_1(x, y) = (b + \delta)(1 - x) - (\beta - \alpha)xy - \delta y$, $f_2(x, y) = \beta xy - (b + \alpha + \gamma)y + \alpha y^2$.

Let $P(t)v(x, y) = u(t, x, y)$ for any $v(x, y) \in D$ (See (33) in the Appendix). Since $P(t)$ is a contraction on D , it can be extended to a contraction on $L^1(X, \Sigma, m)$. Thus the operators $\{P(t)\}_{t \geq 0}$ form a Markov semigroup. Let \mathcal{L} be the infinitesimal generator of the semigroup $\{P(t)\}_{t \geq 0}$, i.e.

$$\mathcal{L}v = \frac{1}{2} \sigma^2 \left[\frac{\partial^2(x^2 y^2 v)}{\partial x^2} - 2 \frac{\partial^2(x^2 y^2 v)}{\partial x \partial y} + \frac{\partial^2(x^2 y^2 v)}{\partial y^2} \right] - \frac{\partial(f_1 v)}{\partial x} - \frac{\partial(f_2 v)}{\partial y}.$$

The adjoint operator of \mathcal{L} is of the form

$$\mathcal{L}^*v = \frac{1}{2} \sigma^2 x^2 y^2 \left[\frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right] + f_1 \frac{\partial v}{\partial x} + f_2 \frac{\partial v}{\partial y}. \quad (8)$$

In [29], the existence and uniqueness of positive solution of system (1.6) is given.

Theorem 2.1 *There is a unique solution $(x(t), y(t))$ of system (6) on $t \geq 0$ for any initial value $(x(0), y(0)) \in \Gamma$, and the solution will remain in Γ with probability 1, where*

$$\Gamma = \{(x, y) : x > 0, y > 0, x + y < 1\}$$

is a positively invariant set of system (6).

3 The stationary distribution of the solution

In this section, we investigate the existence for a stationary distribution of system (6).

Theorem 3.1 *Let (x_t, y_t) be a solution of system (6). Then for every $t > 0$, the distribution of (x_t, y_t) has a density $u(t, x, y)$ which satisfies Fokker-Planck equation (7). If $\tilde{R}_0 > 1$, then there exists a unique density $u_*(x, y)$ which is a stationary solution of (7) and*

$$\lim_{t \rightarrow \infty} \iint_{\Gamma} |u(t, x, y) - u_*(x, y)| dx dy = 0.$$

In addition,

$$\text{supp}u_* = \left\{ (x, y) \in \Gamma : \frac{(b + \alpha + \gamma + \delta) - \sqrt{(b + \delta - \alpha + \gamma)^2 + 4\gamma\alpha}}{2\alpha} < x + y < 1 \right\} := G. \quad (9)$$

Remark 1 By the support of a measurable function f we simply mean the set

$$\text{supp}f = \{(x, y) \in \mathbb{R}_+^2 : f(x, y) \neq 0\}.$$

This is not the customary definition of the support of a function, which is usually the closure of the set $\text{supp}f$, but this slightly unusual definition is more suitable our purposes. By Theorem 2.1, for any initial value $(x(0), y(0)) \in \Gamma$, and the solution $(x(t), y(t))$ of system (6) on $t \geq 0$ will remain in invariant set Γ with probability 1. Therefore we consider Γ is the whole space. As a result, the support of the invariant density u_* is shown in (9).

This results from the fact that the Fokker-Planck equation corresponding to system (6) is of a degenerate type. Our approach comes from Markov semigroup theory, which was used in [23, 24, 18], the specific strategy is as follows:

First, using the Hörmander condition [6] we show that the transition function of the process (x_t, y_t) is absolutely continuous (see Lemma 3.1). Then, using support theorems [26, 5, 25] we find a set G on which the density of the transition function is positive (G is given in (9)) (see Lemma 3.2 and Lemma 3.3). Next, we show that the Markov semigroup satisfies the ‘‘Foguel alternative’’ (see Appendix), i.e. it is either asymptotically stable or ‘‘sweeping’’ (see Lemma 3.3). Finally, we exclude sweeping by showing that there exists a Khasminskiĭ function (24) (see Lemma 3.5). In this way we prove the most difficult part of the paper to show asymptotic stability of system (6).

In the following, we realize this strategy by Lemma 3.2-3.6.

Lemma 3.2 *The transition probability function $\mathcal{P}(t, x_0, y_0, A)$ has a continuous density $k(t, x, y; x_0, y_0)$.*

Proof. If $a(x)$ and $b(x)$ are vector fields on \mathbb{R}^d , then the Lie bracket $[a, b]$ is a vector field given by

$$[a, b]_j(x) = \sum_{k=1}^d \left(a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x) \right), \quad j = 1, 2, \dots, d.$$

Let $a(\xi, \eta) = ((b + \delta)(1 - \xi) - (\beta - \alpha)\xi\eta - \delta\eta, \beta\xi\eta - (b + \alpha + \gamma)\eta + \alpha\eta^2)^T$ and $b(\xi, \eta) = (-\sigma\xi\eta, \sigma\xi\eta)^T$, $(\xi, \eta) \in \Gamma$. Then by direct calculating, $[a, b] = (\sigma\eta[(b + \alpha + \gamma + \delta)\xi + \delta\eta - \alpha\xi^2 - \alpha\xi\eta - (b + \delta)], \sigma\eta[(b + \delta)(1 - \xi) - \delta\eta])^T$. Consequently,

$$\begin{aligned} & \left| \begin{array}{cc} -\sigma\xi\eta & \sigma\eta[(b + \alpha + \gamma + \delta)\xi + \delta\eta - \alpha\xi^2 - \alpha\xi\eta - (b + \delta)] \\ \sigma\xi\eta & \sigma\eta[(b + \delta)(1 - \xi) - \delta\eta] \end{array} \right| \\ &= -\sigma^2\xi^2\eta^2[\gamma + \alpha(1 - \xi - \eta)] < 0 \end{aligned}$$

which means that $b, [a, b]$ are linearly independent on Γ . This implies that at any point $(\xi, \eta) \in \Gamma$, linearly independent vectors $b(\xi, \eta), [a, b](\xi, \eta)$ can span \mathbb{R}^2 . Thus the vector fields a and b satisfy the Hörmander condition [6]. Then using Hörmander Theorem we show that the transition probability function $\mathcal{P}(t, x_0, y_0, A)$ has a density $k(t, x, y; x_0, y_0)$ and $k \in C^\infty((0, \infty) \times \Gamma \times \Gamma)$. \square

Remark 2 According to Lemma 3.2, there exists a measurable function $k(t, x, y; x_0, y_0)$, such that

$$P(t)f(x, y) = \iint_{\Gamma} k(t, x, y; u, v)f(u, v)dudv$$

for every density f . By the definition (see Appendix A), the semigroup $\{P(t)\}_{t \geq 0}$ is an integral Markov semigroup.

Next, we rewrite SDE (6) of the Itô type as the SDE of the Stratonovitch type:

$$\begin{cases} dx_t = \tilde{f}_1(x_t, y_t)dt - \sigma x_t y_t \circ dB_t, \\ dy_t = \tilde{f}_2(x_t, y_t)dt + \sigma x_t y_t \circ dB_t, \end{cases}$$

where

$$\begin{aligned} \tilde{f}_1(x, y) &= (b + \delta)(1 - x) - (\beta - \alpha)xy - \delta y + \frac{1}{2}\sigma^2(x^2y - xy^2), \\ \tilde{f}_2(x, y) &= \beta xy - (b + \alpha + \gamma)y + \alpha y^2 - \frac{1}{2}\sigma^2(x^2y - xy^2). \end{aligned}$$

By the theorems [26, 5, 25], we now check where the kernel k is positive. Fixed a point $(x_0, y_0) \in \Gamma$ and a function $\psi \in L^2([0, T]; \mathbb{R})$, consider the following system of integral equations:

$$\begin{cases} x_\psi(t) = x_0 + \int_0^t [\tilde{f}_1(x_\psi(s), y_\psi(s)) - \sigma \psi x_\psi(s) y_\psi(s)] ds, \\ y_\psi(t) = y_0 + \int_0^t [\tilde{f}_2(x_\psi(s), y_\psi(s)) + \sigma \psi x_\psi(s) y_\psi(s)] ds. \end{cases} \quad (10)$$

Let $D_{x_0, y_0; \psi}$ be the Frechét derivative of the function $h \mapsto \mathbf{x}_{\psi+h}(T)$ from $L^2([0, T]; \mathbb{R})$ to \mathbb{R}^2 , where $\mathbf{x}_{\psi+h} = \begin{pmatrix} x_{\psi+h} \\ y_{\psi+h} \end{pmatrix}$. If for some $\psi \in L^2([0, T]; \mathbb{R})$ the derivative $D_{x_0, y_0; \psi}$ has rank 2, then $k(T, x, y; x_0, y_0) > 0$ for $x = x_\psi(T)$ and $y = y_\psi(T)$. The derivative $D_{x_0, y_0; \psi}$ can be found by means of the perturbation method for ODEs. Namely, let $\Gamma(t) = \mathbf{f}'(x_\psi(t), y_\psi(t)) + \mathbf{g}'(x_\psi(t), y_\psi(t))\psi$, where \mathbf{f}' and \mathbf{g}' are the Jacobians of $\mathbf{f} = \begin{pmatrix} \tilde{f}_1(x, y) \\ \tilde{f}_2(x, y) \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} -\sigma xy \\ \sigma xy \end{pmatrix}$ respectively. Let $Q(t, t_0)$, for $T \geq t \geq t_0 \geq 0$, be a matrix function such that $Q(t_0, t_0) = I$, $\partial Q(t, t_0)/\partial t = \Gamma(t)Q(t, t_0)$. Then

$$D_{x_0, y_0; \psi} h = \int_0^T Q(T, s) \mathbf{g}(s) h(s) ds.$$

Lemma 3.3 For each $(x_0, y_0) \in G$ and $(x, y) \in G$, there exists $T > 0$ such that $k(T, x, y; x_0, y_0) > 0$, where G is as in (9).

Proof. Step 1. we check that the rank of $D_{x_0, y_0; \psi}$ is 2. Let $\varepsilon \in (0, T)$ and $h(t) = \frac{\mathbf{1}_{[T-\varepsilon, T]}(t)}{x_\psi(t)y_\psi(t)}$, $t \in [0, T]$, where $\mathbf{1}_{[T-\varepsilon, T]}$ is the characteristic function of interval $[T - \varepsilon, T]$. Since $Q(T, s) = I + \Gamma(T)(T - s) + o(T - s)$, we obtain

$$D_{x_0, y_0; \psi} h = \varepsilon \mathbf{v} + \frac{1}{2} \varepsilon^2 \Gamma(T) \mathbf{v} + o(\varepsilon^2), \quad \mathbf{v} = \begin{pmatrix} -\sigma \\ \sigma \end{pmatrix},$$

$$\Gamma(T)\mathbf{v} = \begin{pmatrix} \sigma[b - (\beta - \alpha)(x - y) + \sigma(y - x)\psi + \frac{\sigma^2}{2}(x^2 + y^2 - 4xy)] \\ \sigma[\beta(x - y) - (b + \alpha + \gamma) + 2\alpha y - \sigma(y - x)\psi - \frac{\sigma^2}{2}(x^2 + y^2 - 4xy)] \end{pmatrix}.$$

Hence, \mathbf{v} and $\Gamma(T)\mathbf{v}$ are linearly independent. Thus $D_{x_0, y_0; \psi}$ has rank 2.

Step 2. we show that there exist a control function ψ and $T > 0$ such that $x_\psi(0) = x_0$, $y_\psi(0) = y_0$, $x_\psi(T) = x$, $y_\psi(T) = y$ for any two points $(x_0, y_0) \in G$ and $(x, y) \in G$.

Firstly, the system (10) can be replaced by the following system of differential equations:

$$\begin{cases} x'_\psi = \tilde{f}_1(x_\psi, y_\psi) - \sigma\psi x_\psi y_\psi, \\ y'_\psi = \tilde{f}_2(x_\psi, y_\psi) + \sigma\psi x_\psi y_\psi. \end{cases}$$

Let $u_\psi = x_\psi + y_\psi$. Then

$$\begin{cases} x'_\psi = g_1(x_\psi, u_\psi) - \sigma\psi x_\psi(u_\psi - x_\psi), \\ u'_\psi = g_2(x_\psi, u_\psi), \end{cases} \quad (11)$$

where

$$g_1(x, u) = \tilde{f}_1(x, u - x), \quad g_2(x, u) = -\gamma(u - x) + (b + \delta)(1 - u) - \alpha(u - x)(1 - u). \quad (12)$$

Then, we have

$$x_\psi = u_\psi + \frac{u'_\psi - (b + \delta)(1 - u_\psi)}{\alpha(1 - u_\psi) + \gamma}.$$

Since $0 < x_\psi < u_\psi$, we obtain

$$0 < u_\psi + \frac{u'_\psi - (b + \delta)(1 - u_\psi)}{\alpha(1 - u_\psi) + \gamma} < u_\psi,$$

then

$$\alpha u_\psi^2 - (b + \alpha + \gamma + \delta)u_\psi + (b + \delta) < u'_\psi < (b + \delta)(1 - u_\psi). \quad (13)$$

We consider the equation

$$f(x) = \alpha x^2 - (b + \alpha + \gamma + \delta)x + (b + \delta) = 0,$$

note that

$$\Delta = (b + \alpha + \gamma + \delta)^2 - 4\alpha(b + \delta) = (b + \delta - \alpha + \gamma)^2 + 4\gamma\alpha > 0,$$

so the equation has two positive roots m_1, m_2 ,

$$m_1 = \frac{(b + \alpha + \gamma + \delta) - \sqrt{(b + \delta - \alpha + \gamma)^2 + 4\gamma\alpha}}{2\alpha} < 1,$$

$$m_2 = \frac{(b + \alpha + \gamma + \delta) + \sqrt{(b + \delta - \alpha + \gamma)^2 + 4\gamma\alpha}}{2\alpha} > 1.$$

It then follows from (13) that

$$-\alpha(u_\psi(t) - m_1)(m_2 - u_\psi(t)) < u'_\psi(t) < (b + \delta)(1 - u_\psi(t)), \quad t \in [0, T]. \quad (14)$$

By the comparison principle, we can derive that $m_1 \leq \liminf_{t \rightarrow \infty} u_\psi \leq 1$. So we get

$$G_0 = \{(x, u) \in (0, 1) \times (0, 1) : 0 < x < 1, m_1 < u < 1 \text{ and } x < u\}. \quad (15)$$

Next we prove that our claim holds on G_0 . There exists a positive constant T and a differentiable function

$$u_\psi : [0, T] \rightarrow (m_1, 1),$$

such that $u_\psi(0) = u_0, u_\psi(T) = u_1, u'_\psi(0) = g_2(x_0, u_0) := u_0^d, u'_\psi(T) = g_2(x_1, u_1) := u_T^d$ and satisfied

$$-\alpha(u_\psi(t) - m_1)(m_2 - 1) < u'_\psi(t) < (b + \delta)(1 - u_\psi(t)), \quad t \in [0, T], \quad (16)$$

then u_ψ satisfies (14) for $t \in [0, T]$.

We split the construction of the function u_ψ on three intervals $[0, \tau]$, $(\tau, T - \tau)$ and $[T - \tau, T]$, where $0 < \tau < T/2$.

Let

$$\theta = \frac{1}{2} \min \{u_0 - m_1, u_1 - m_1, 1 - u_0, 1 - u_1\}.$$

When $u_\psi \in (m_1 + \theta, 1 - \theta)$, we have

$$-\alpha(u_\psi(t) - m_1)(m_2 - 1) < -\alpha\theta(m_2 - 1) < 0 \quad (17)$$

and

$$(b + \delta)(1 - u_\psi(t)) > (b + \delta)\theta > 0, \quad t \in [0, T]. \quad (18)$$

Therefore, we can construct a C^2 -function $u_\psi : [0, \tau] \rightarrow (m_1 + \theta, 1 - \theta)$ such that

$$u_\psi(0) = u_0, u'_\psi(0) = u_0^d, u'_\psi(\tau) = 0$$

and u_ψ satisfies inequality (16) for $t \in [0, \tau]$. Analogously, we construct a C^2 -function $u_\psi : [T - \tau, T] \rightarrow (m_1 + \theta, 1 - \theta)$ such that

$$u_\psi(T) = u_1, u'_\psi(T) = u_T^d, u'_\psi(T - \tau) = 0$$

and u_ψ satisfies inequality (16) for $t \in [T - \tau, T]$.

Taking T sufficiently large we can extend the function $u_\psi : [0, \tau] \cup [T - \tau, T] \rightarrow (m_1 + \theta, 1 - \theta)$ to a C^2 -function u_ψ defined on the whole interval $[0, T]$ such that

$$-\alpha\theta(m_2 - 1) \leq u'_\psi(t) \leq (b + \delta)\theta, \quad t \in [\tau, T - \tau],$$

and therefore, the function u_ψ satisfies (16) on $[0, T]$ in view of (17), (18). It follows that we can find a C^1 -function x_ψ which satisfies (11) and finally we can determine a continuous function ψ from (11).

Therefore, our claim holds. This completes our proof. \square

Lemma 3.4 *Assume $\tilde{R}_0 > 1$. For the semigroup $\{P(t)\}_{t \geq 0}$ and every density f , we have*

$$\lim_{t \rightarrow \infty} \iint_G P(t)f(x, y)dx dy = 1, \quad (19)$$

where G is given in (9).

Proof. Let $U_t = x_t + y_t$. The system (6) can be replaced by

$$\begin{aligned} dx_t &= g_1(x_t, U_t)dt - \sigma x_t(U_t - x_t)dB_t, \\ dU_t &= g_2(x_t, U_t)dt, \end{aligned}$$

where g_1 and g_2 are as in (12). Since (x_t, y_t) is a positive solution of system (6) with probability 1, from the expression of g_2 , we get

$$\alpha U_t^2 - (b + \alpha + \gamma + \delta)U_t + (b + \delta) < \frac{dU_t}{dt} < (b + \delta)(1 - U_t), \quad t \in (0, \infty), \quad a.s.$$

Similar to (14), it is rewritten as

$$\alpha(U_t - m_1)(U_t - m_2) < \frac{dU_t}{dt} < (b + \delta)(1 - U_t), \quad t \in (0, \infty), \quad a.s. \quad (20)$$

Now we claim that for almost every $\omega \in \Omega$ there exists $t_0 = t_0(\omega)$ such that

$$m_1 < U_t(\omega) < 1, \quad \text{for } t > t_0, \quad (21)$$

which completes our proof. According to the position of initial value U_0 we consider two cases:

Case 1. $U_0 \in (0, m_1]$. If our claim (21) is false, then we know that there exists $\Omega' \subset \Omega$ with $\text{Prob}(\Omega') > 0$ such that $U_t(\omega) \in (0, m_1]$, $\omega \in \Omega'$. By (20), it follows that for any $\omega \in \Omega'$, $U_t(\omega)$ is strictly increasing on $[0, +\infty)$, and therefore $\lim_{t \rightarrow \infty} U_t(\omega) = m_1$, $\omega \in \Omega'$. From the expression of g_2 , it follows that

$$\lim_{t \rightarrow \infty} x_t(\omega) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} y_t(\omega) = m_1, \quad \omega \in \Omega'. \quad (22)$$

By Itô's Formula, we get

$$d \log y_t = \left(\beta x_t - (b + \alpha + \gamma) + \alpha y_t - \frac{\sigma^2 x_t^2}{2} \right) dt + \sigma x_t dB_t,$$

which yields

$$\frac{\log y_t - \log y_0}{t} = \frac{\beta \int_0^t x_r dr}{t} - (b + \alpha + \gamma) + \frac{\alpha \int_0^t y_r dr}{t} - \frac{\sigma^2 \int_0^t x_r^2 dr}{2t} + \frac{\sigma \int_0^t x_r dB_r}{t}. \quad (23)$$

Let $M(t) = \int_0^t x_r dB_r$, by using Strong Law of Large Numbers (Lemma A.2), we obtain $\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0$ a.s. So we get

$$\begin{aligned} \lim_{t \rightarrow \infty} \left[\frac{\beta \int_0^t x_r dr}{t} - (b + \alpha + \gamma) + \frac{\alpha \int_0^t y_r dr}{t} - \frac{\sigma^2 \int_0^t x_r^2 dr}{2t} + \frac{\sigma \int_0^t x_r dB_r}{t} \right] \\ = -(b + \alpha + \gamma), \quad a.s. \quad \text{on } \Omega', \end{aligned}$$

By (22), we get

$$\lim_{t \rightarrow \infty} \frac{\log y_t - \log y_0}{t} = 0, \quad \text{on } \Omega'$$

which is a contradiction. Thus our claim holds for $U_0 \in (0, m_1]$.

Case 2. $U_0 \in (m_1, 1)$. From (20), it is obvious that our claim (21) holds and $\lim_{t \rightarrow \infty} U_t \neq m_1$, $\lim_{t \rightarrow \infty} U_t \neq 1$. By similar arguments to Case 1, we obtain $\lim_{t \rightarrow \infty} U_t \neq m_1$. If

$\lim_{t \rightarrow \infty} U_t = 1$, there exists $\Omega' \subset \Omega$ with $\text{Prob}(\Omega') > 0$ such that for any $\omega \in \Omega'$, $\lim_{t \rightarrow \infty} U_t(\omega) = 1$, $\lim_{t \rightarrow \infty} x_t(\omega) = 1$ and $\lim_{t \rightarrow \infty} y_t(\omega) = 0$. Obviously, condition $R_0 - 1 > \sigma^2/2(b + \alpha + \gamma)$ implies that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left[\frac{\beta \int_0^t x_r dr}{t} - (b + \alpha + \gamma) + \frac{\alpha \int_0^t y_r dr}{t} - \frac{\sigma^2 \int_0^t x_r^2 dr}{2t} + \frac{\sigma \int_0^t x_r dB_r}{t} \right] \\ &= \beta - \frac{1}{2} \sigma^2 - (b + \alpha + \gamma) \\ &= (b + \alpha + \gamma) \left(R_0 - 1 - \frac{\sigma^2}{2(b + \alpha + \gamma)} \right) > 0, \quad \text{a.s. on } \Omega'. \end{aligned}$$

By (23), we get $\lim_{t \rightarrow \infty} \frac{\log y_t}{t} > 0$ a.s. on Ω' , which contradicts $\lim_{t \rightarrow \infty} y_t(\omega) = 0$, $\omega \in \Omega'$. Thus our claim (21) holds for $U_0 \in (m_1, 1)$. \square

Remark 3 From lemma 3.3 and 3.4, we know that if (7) has a stationary solution u_* , then $\text{supp} u_* = G$.

Lemma 3.5 Assume $\tilde{R}_0 > 1$. The semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable or is sweeping with respect to compact sets.

Proof. Lemma 3.2 shows that $\{P(t)\}_{t \geq 0}$ is an integral Markov semigroup with a continuous kernel $k(t, x, y)$ for $t > 0$. Lemma 3.4 indicates that it is sufficient to research the restriction of the semigroup $\{P(t)\}_{t \geq 0}$ to the space $L^1(\bar{G})$, where \bar{G} denotes the closure set of G . In view of lemma 3.3 for every $f \in D$, we have

$$\int_0^\infty P(t) f dt > 0 \quad \text{a.e. on } \bar{G}.$$

So according to Lemma A.1, The semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable or is sweeping with respect to compact sets. \square

Lemma 3.6 If $\tilde{R}_0 > 1$, then the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable.

Proof. We will construct a nonnegative C^2 -function V and a closed set $U \in \Sigma$ such that

$$\sup_{x \notin U} \mathcal{L}^* V(x) < 0. \quad (24)$$

Such a function is called a Khasminskiĭ function in [22]. The existence of a Khasminskiĭ function implies that the semigroup is not sweeping from the set U .

Consider the function

$$H(x, y) = -\log x - r \left(\log y - \frac{\beta}{b + \delta} (1 - x - y) \right) - \log(1 - x - y) - \log(x + y - m_1), \quad (x, y) \in G.$$

where r is a positive constant satisfied

$$2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 - r(b + \alpha + \gamma)(\tilde{R}_0 - 1) = -2. \quad (25)$$

The function $H(x, y)$ satisfies

$$\frac{\partial H}{\partial x} = -\frac{1}{x} + \frac{1}{1-x-y} - \frac{1}{x+y-m_1} - \frac{r\beta}{b+\delta} = 0,$$

and

$$\frac{\partial H}{\partial y} = -\frac{r}{y} + \frac{1}{1-x-y} - \frac{1}{x+y-m_1} - \frac{r\beta}{b+\delta} = 0.$$

From the above equations, we can deduce $y = rx$. Then

$$f(x) = \frac{\partial H}{\partial x} = -\frac{1}{x} + \frac{1}{1-(r+1)x} - \frac{1}{(r+1)x-m_1} - \frac{r\beta}{b+\delta} = 0.$$

It is not difficult to know $f(x)$ is a monotonic increasing function and $f(\frac{m_1}{r+1}) = -\infty$, $f(\frac{1}{r+1}) = +\infty$. So $H(x, y)$ attains its minimum value at the only stable point $(\theta, r\theta)$, and

$$H(x, y) \geq H(\theta, r\theta).$$

We construct a Lyapunov function $V : G \rightarrow \bar{\mathbb{R}}_+$ by

$$\begin{aligned} V(x, y) &= H(x, y) - H(\theta, r\theta) \\ &= -\log x - r(\log y - \frac{\beta}{b+\delta}(1-x-y)) - \log(1-x-y) \\ &\quad - \log(x+y-m_1) - H(\theta, r\theta) \\ &= V_1 + rV_2 + V_3 + V_4 - H(\theta, r\theta). \end{aligned}$$

where $V_1 = -\log x$, $V_2 = -r(\log y - \frac{\beta}{b+\delta}(1-x-y))$, $V_3 = \log(1-x-y)$ and $V_4 = -\log(x+y-m_1)$. Applying the Itô's formula, we obtain

$$\begin{aligned} \mathcal{L}^*V_1 &= -\frac{b+\delta}{x_t} + (b+\delta) + (\beta-\alpha)y_t + \frac{\delta y_t}{x_t} + \frac{\sigma^2 y_t^2}{2} \\ &= -\frac{b+\delta(1-y_t)}{x_t} + (b+\delta) + (\beta-\alpha)y_t + \frac{\sigma^2 y_t^2}{2}, \end{aligned}$$

$$\begin{aligned} \mathcal{L}^*V_2 &= -r(\beta - (b+\alpha+\gamma) - \frac{\sigma^2 x_t^2}{2} + \alpha y_t - \frac{\beta(b+\alpha+\gamma+\delta)y_t}{b+\delta} + \frac{\beta\alpha y_t}{b+\delta}(x_t+y_t)) \\ &\leq -r(\beta - (b+\alpha+\gamma) - \frac{\sigma^2 x_t^2}{2} - \beta y_t(1 + \frac{\alpha+\gamma}{b+\delta})), \end{aligned}$$

$$\begin{aligned} \mathcal{L}^*V_3 &= \frac{1}{1-x_t-y_t}(-\gamma y_t + (b+\delta)(1-x_t-y_t) - \alpha y_t(1-x_t-y_t)) \\ &= -\frac{\gamma y_t}{1-x_t-y_t} + (b+\delta) - \alpha y_t, \end{aligned}$$

by (14),

$$\begin{aligned} \mathcal{L}^*V_4 &= -\frac{1}{x_t+y_t-m_1}((b+\delta)(1-x_t) - (b+\alpha+\gamma+\delta)y_t + \alpha y_t(x_t+y_t)) \\ &= -\frac{1}{x_t+y_t-m_1}(-\alpha(x_t+y_t-m_1)(m_2-x_t-y_t) + x_t(\alpha(1-x_t-y_t) + \gamma)) \\ &= \alpha(m_2-x_t-y_t) + \alpha x_t - \frac{(\alpha(1-m_1) + \gamma)x_t}{x_t+y_t-m_1} \\ &= \alpha m_2 - \alpha y_t - \frac{(\alpha(1-m_1) + \gamma)x_t}{x_t+y_t-m_1}. \end{aligned}$$

Then

$$\mathcal{L}^*V = \mathcal{L}^*V_1 + \mathcal{L}^*V_2 + \mathcal{L}^*V_3 + \mathcal{L}^*V_4.$$

Define a closed set

$$U_{\varepsilon, \kappa} = \{(x, y) \in G : \varepsilon \leq x \leq 1 - \varepsilon, \varepsilon \leq y \leq 1 - \varepsilon, m_1 + \kappa \leq x + y \leq 1 - \kappa\}, \quad (26)$$

where $\varepsilon, \kappa > 0$ are two small numbers such that

$$r\varepsilon\beta(1 + \frac{\alpha + \gamma}{b + \delta}) < -1, \quad (27)$$

$$-\frac{b}{\varepsilon} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r((b + \alpha + \gamma) + \frac{1}{2}\sigma^2\varepsilon^2 + \beta(1 + \frac{\alpha + \gamma}{b + \delta})) < -1, \quad (28)$$

$$-\frac{r}{\varepsilon} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) < -1, \quad (29)$$

and

$$-\frac{\alpha(1 - m_1) + \gamma}{\varepsilon} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) < -1. \quad (30)$$

Denote

$$D_{\varepsilon, \kappa}^1 = \{(x, y) \in G : 0 < x < \varepsilon\}, \quad D_{\varepsilon, \kappa}^2 = \{(x, y) \in G : 0 < y < \varepsilon\},$$

$$D_{\varepsilon, \kappa}^3 = \{(x, y) \in G : \varepsilon \leq x < 1, \varepsilon \leq y < 1, 1 - \kappa < x + y < 1\},$$

$$D_{\varepsilon, \kappa}^4 = \{(x, y) \in G : \varepsilon \leq x < 1, \varepsilon \leq y < 1, m_1 < x + y < m_1 + \kappa\}.$$

Then $G \setminus U_{\varepsilon, \kappa} = D_{\varepsilon, \kappa}^1 \cup D_{\varepsilon, \kappa}^2 \cup D_{\varepsilon, \kappa}^3 \cup D_{\varepsilon, \kappa}^4$. In the following, we consider four cases:

Case 1. On $D_{\varepsilon, \kappa}^1$, we have

$$\mathcal{L}^*V_1 \leq -\frac{b}{\varepsilon} + (b + \delta) + \beta + \frac{\sigma^2}{2},$$

$$\begin{aligned} \mathcal{L}^*V_2 &\leq -r(\beta - (b + \alpha + \gamma) - \frac{1}{2}\sigma^2\varepsilon^2) + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) \\ &\leq r((b + \alpha + \gamma) + \frac{1}{2}\sigma^2\varepsilon^2 + \beta(1 + \frac{\alpha + \gamma}{b + \delta})), \end{aligned} \quad (31)$$

$$\mathcal{L}^*V_3 \leq b + \delta$$

and

$$\mathcal{L}^*V_4 = \alpha m_2 - \alpha y_t - \frac{(\alpha(1 - m_1) + \gamma)x_t}{x_t + y_t - m_1} \leq \alpha m_2.$$

Then

$$\begin{aligned} \mathcal{L}^*V &= \mathcal{L}^*V_1 + \mathcal{L}^*V_2 + \mathcal{L}^*V_3 + \mathcal{L}^*V_4 \\ &\leq -\frac{b}{\varepsilon} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r((b + \alpha + \gamma) + \frac{1}{2}\sigma^2\varepsilon^2 \\ &\quad + \beta(1 + \frac{\alpha + \gamma}{b + \delta})) \end{aligned}$$

Noting that ε satisfied (28), so we can obtain

$$\mathcal{L}^*V < -1, \quad (x, y) \in D_{\varepsilon, \kappa}^1.$$

Case 2. On $D_{\varepsilon, \kappa}^2$, we get

$$\begin{aligned}\mathcal{L}^*V &\leq 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 - r(\beta - (b + \alpha + \gamma) - \frac{1}{2}\sigma^2) \\ &\quad + \frac{r\beta(b + \alpha + \gamma + \delta)\varepsilon}{b + \delta} \\ &= 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 - r(b + \alpha + \gamma)(\tilde{R}_0 - 1) + r\varepsilon\beta(1 + \frac{\alpha + \gamma}{b + \delta})\end{aligned}$$

By (25) and (27), we have

$$\mathcal{L}^*V < -1, \quad (x, y) \in D_{\varepsilon, \kappa}^2.$$

Case 3. On $D_{\varepsilon, \kappa}^3$, let $\kappa = \varepsilon^2$, from (29), then

$$\begin{aligned}\mathcal{L}^*V &\leq -\frac{r\varepsilon}{\kappa} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 - r(b + \alpha + \gamma)(\tilde{R}_0 - 1) \\ &\quad + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) \\ &\leq -\frac{r\varepsilon}{\kappa} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) \\ &\leq -\frac{r}{\varepsilon} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) \\ &< -1, \quad (x, y) \in D_{\varepsilon, \kappa}^3.\end{aligned}$$

Case 4. On $D_{\varepsilon, \kappa}^4$, in view of (30), we have

$$\begin{aligned}\mathcal{L}^*V &\leq -(\alpha(1 - m_1) + \gamma)\frac{\varepsilon}{\kappa} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) \\ &\leq -\frac{\alpha(1 - m_1) + \gamma}{\varepsilon} + 2(b + \delta) + \beta + \frac{\sigma^2}{2} + \alpha m_2 + r\beta(1 + \frac{\alpha + \gamma}{b + \delta}) \\ &< -1.\end{aligned}$$

In summary, on $G \setminus U_{\varepsilon, \kappa} = D_{\varepsilon, \kappa}^1 \cup D_{\varepsilon, \kappa}^2 \cup D_{\varepsilon, \kappa}^3 \cup D_{\varepsilon, \kappa}^4$, we get

$$\sup_{x \in G \setminus U_{\varepsilon, \kappa}} \mathcal{L}^*V(x) < 0.$$

According to Lemma A.1, the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable. In another words, the semigroup has a unique stationary solution on G . This completes the proof. \square

Remark 4 In the proof of lemma 3.6, we take $X = \Gamma$. To verify V is a Khasminskĭ function, it suffices that there exist a closed set $U \subseteq \Sigma$ (which lies entirely in Γ) such that

$$\sup_{x \notin U} \mathcal{L}^*V(x) < 0.$$

4 Simulations

Next we make numerical simulations to illustrate our results by using Milstein's Higer Order Method [12]. We assume that the unit of time is one day and the population sizes are measured in units of 1 million. The parameters in (6) are given by

$$b = 0.2, \quad \beta = 0.6, \quad \alpha = 0.1, \quad \gamma = 0.2, \quad \delta = 0.2, \quad \sigma = 0.1, \quad \Delta t = 1. \quad (32)$$

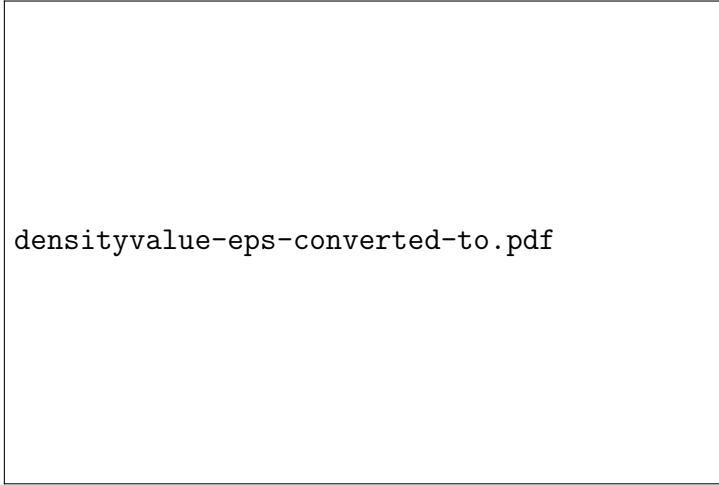


Figure 1: Based on the 10000 sample pathes, after iterating 10000 times, we get the density functions of $x(t)$ and $y(t)$ with different initial values. Here $b = 0.2$, $\beta = 0.6$, $\alpha = 0.1$, $\gamma = 0.2$, $\delta = 0.2$, $\sigma = 0.1$, $\Delta t = 1$.

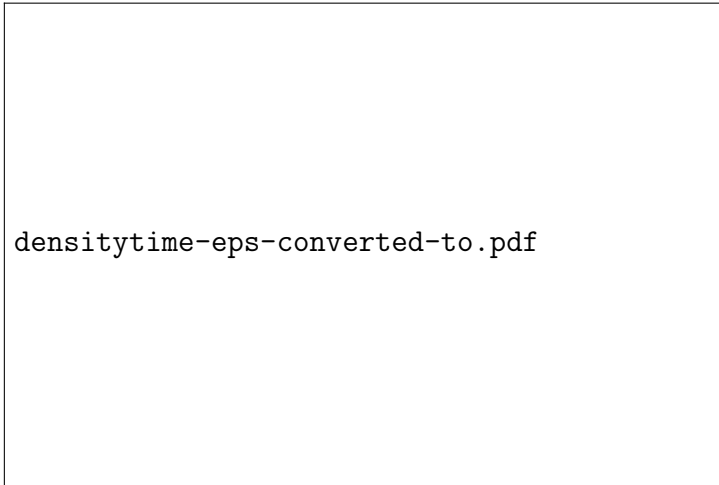


Figure 2: Based on the 10000 sample pathes, after iterating 10000 times, we get the density functions of $x(t)$ and $y(t)$ with initial value $(x(0), y(0)) = (0.4, 0.8)$. Here $b = 0.2$, $\beta = 0.6$, $\alpha = 0.1$, $\gamma = 0.2$, $\delta = 0.2$, $\sigma = 0.1$, $\Delta t = 1$.

In this case, the condition of Theorem 3.1 is satisfied. We find that these lines in Figure 1 fit very well which implies that wherever $x(t)$ and $y(t)$ start from, the density functions of $x(t)$ and $y(t)$ converge to the same functions respectively. Figure 2 indicates that there is a stationary distribution for system (6). Hence, Figure 1 and Figure 2 approve the result of theorem 3.1.

A

Since the proof of our result is based on the theory of integral Markov semigroups, we need some auxiliary definitions and results concerning Markov semigroups (see [23],[24]).

For the convenience of the reader, we present these definitions and results in the appendix. Let the triple (X, Σ, m) be a σ -finite measure space. Denote by D the subset of the space $L^1 = L^1(X, \Sigma, m)$ which contains all densities, i.e.

$$D = \{f \in L^1 : f \geq 0, \|f\| = 1\}. \quad (33)$$

A linear mapping $P : L^1 \rightarrow L^1$ is called a Markov operator if $P(D) \subset D$.

The Markov operator P is called an integral or kernel operator if there exists a measurable function $k : X \times X \rightarrow [0, \infty)$ such that

$$\int_X k(x, y)m(dx) = 1 \quad (34)$$

for all $y \in X$ and

$$Pf(x) = \int_X k(x, y)f(y)m(dy)$$

for every density f .

A family $\{P(t)\}_{t \geq 0}$ of Markov operators which satisfies conditions:

- (1) $P(0) = \text{Id}$,
- (2) $P(t+s) = P(t)P(s)$ for $s, t \geq 0$,
- (3) for each $f \in L^1$ the function $t \mapsto P(t)f$ is continuous with respect to the L^1 norm,

is called a Markov semigroup. A Markov semigroup $\{P(t)\}_{t \geq 0}$ is called integral, if for each $t > 0$, the operator $P(t)$ is an integral Markov operator.

We also need two definitions concerning the asymptotic behaviour of a Markov semigroup. A density f_* is called invariant if $P(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t \geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t \rightarrow \infty} \|P(t)f - f_*\| = 0 \quad \text{for } f \in D.$$

A Markov semigroup $\{P(t)\}_{t \geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if for every $f \in D$

$$\lim_{t \rightarrow \infty} \int_A P(t)f(x)m(dx) = 0.$$

We need some result concerning asymptotic stability and sweeping which can be found in [23] (see Corollary 1).

Lemma A.1 *Let X be a metric space and Σ be the σ -algebra of Borel sets. Let $\{P(t)\}_{t \geq 0}$ be an integral Markov semigroup with a continuous kernel $k(t, x, y)$ for $t > 0$, which satisfies (34) for all $y \in X$. We assume that for every $f \in D$ we have*

$$\int_0^\infty P(t)f dt > 0 \quad \text{a.e.}$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

The property that a Markov semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable or sweeping for a sufficiently large family of sets (e.g. for all compact sets) is called the Foguel alternative.

Lemma A.2.(Strong Law of Large Numbers) *Let $M = \{M_t\}_{t \geq 0}$ be a real-value continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty \quad a.s. \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0. \quad a.s.$$

and also

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty \quad a.s. \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0. \quad a.s.$$

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