# A Theoretical Study of Two-Period Relaxations for Lot-Sizing Problems with Big-Bucket Capacities

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**Abstract**

In this paper, we study two-period subproblems proposed by [1] for lot-sizing problems with big-bucket capacities and nonzero setup times, complementing our previous work [3] investigating the special case of zero setup times. In particular, we study the polyhedral structure of the mixed integer sets related to various two-period relaxations. We derive several families of valid inequalities and investigate their facet-defining conditions. We also discuss the separation problems associated with these valid inequalities.

## 1 Introduction

In this study, we investigate multi-item production planning problems with big bucket capacities, i.e., each resource is shared by multiple items, which can be produced in a specific time period. These real-world problems are very interesting, as they remain challenging to solve to optimality and also to achieve strong bounds. The uncapacitated and single-item relaxations of the problem have been previously studied by [7]. The work of [6] introduced and studied the single-period relaxation with "preceding inventory", where a number of cover and reverse cover inequalities are defined for this relaxation. Finally, we also note the relevant study of [5], who studied a single-period relaxation and compared with other relaxations.

We present a formulation for this problem following the notation of [2]. Let $NT$, $NI$ and $NK$ indicate the number of *periods*, *items*, and *machine types*, respectively. We represent the production, setup, and inventory variables for item $i$ in period $t$ by $x_t^i$, $y_t^i$, and $s_t^i$, respectively. We note that our model can be generalized to involve multiple levels as in [1], however, we omit this for the sake of simplicity.

$$
\begin{aligned}
\min \; & \sum_{t=1}^{NT}\sum_{i=1}^{NI} f_t^i y_t^i + \sum_{t=1}^{NT}\sum_{i=1}^{NI} h_t^i s_t^i && && (1)\\
\text{s.t. } & x_t^i + s_{t-1}^i - s_t^i = d_t^i && t \in \{1,\dots,NT\}, i \in \{1,\dots,NI\} && (2)\\
& \sum_{i=1}^{NI}(a_k^i x_t^i + ST_k^i y_t^i) \le C_t^k && t \in \{1,\dots,NT\}, k \in \{1,\dots,NK\} && (3)\\
& x_t^i \le M_t^i y_t^i && t \in \{1,\dots,NT\}, i \in \{1,\dots,NI\} && (4)\\
& y \in \{0,1\}^{NT\mathrm{x}NI}; x, s \ge 0 && && (5)
\end{aligned}
$$

The objective function (1) minimizes total cost, where $f_t^i$ and $h_t^i$ indicate the setup and inventory cost coefficients, respectively. The flow balance constraints (2) ensure that the demand for each item $i$ in period $t$, denoted by $d_t^i$, is satisfied. The big bucket capacity constraints (3) ensure that the capacity $C_t^k$ of machine $k$ is not exceeded in time period $t$, where $a_k^i$ and $ST_k^i$ represent the per unit production time and setup time for item $i$, respectively. The constraints (4) guarantee that the setup variable is equal to 1 if production occurs, where $M_t^i$ represents the maximum number of item $i$ that can be produced in period $t$, based on the minimum of remaining cumulative demand and capacity available. Finally, the integrality and non-negativity constraints are given by (5).

# 2 Two-Period Relaxation

Let $I = \{1,\dots,NI\}$. We present the feasible region of a two-period, single-machine relaxation of the multi-item production planning problem, denoted by $X^{2PL}$ (see [1] for details).

$$
\begin{aligned}
& x_{t'}^i \le \widetilde{M}_{t'}^i y_{t'}^i && i \in I, t' = 1,2 && (6)\\
& x_{t'}^i \le \widetilde{d}_{t'}^i y_{t'}^i + s^i && i \in I, t' = 1,2 && (7)\\
& x_1^i + x_2^i \le \widetilde{d}_1^i y_1^i + \widetilde{d}_2^i y_2^i + s^i && i \in I && (8)\\
& x_1^i + x_2^i \le \widetilde{d}_1^i + s^i && i \in I && (9)\\
& \sum_{i\in I}(a^i x_{t'}^i + ST^i y_{t'}^i) \le \widetilde{C}_{t'} && t' = 1,2 && (10)\\
& x, s \ge 0, y \in \{0,1\}^{2\times NI} && && (11)
\end{aligned}
$$

Since we consider a single machine, we dropped the $k$ index from this formulation, however, all parameters are defined in the same lines as before. The obvious choice for the horizon would be $t+1$, in which case the definition of the parameter $\widetilde{M}^i_{t'}$ is the same as of the basic definition of $M^i_{t+t'-1}$, for all $i$ and $t' = 1, 2$. Similarly, capacity parameter $\widetilde{C}_{t'}$ is the same as $C_{t+t'-1}$, for all $t' = 1, 2$. Cumulative demand parameter $\widetilde{d}^i_{t'}$ represents simply $d^i_{t+t'-1,\, t+1}$, for all $i$ and $t' = 1, 2$, i.e., $\widetilde{d}^i_1 = d^i_{1,2}$ and $\widetilde{d}^i_2 = d^i_2$. We note the following polyhedral result for $X^{2PL}$ from [1].

**Proposition 2.1** *Assume that $\widetilde{M}^i_t > 0, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\}$ and $ST^i < \widetilde{C}_t, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\}$. Then conv($X^{2PL}$) is full-dimensional.*

For the sake of simplicity, we will drop subscript $t$ and symbol $\widetilde{}$ in the following notations. In this paper, we investigate the case of $a^i = 1, \forall i \in \{1, \ldots, NI\}$ with nonzero setups. We establish two relaxations of $X^{2PL}$ and study their polyhedral structures. For a given $t$, we define the first relaxation of $X^{2PL}$, denoted by $LR1$, as set of $(x, y) \in \mathbb{R}^{NI} \times \mathbb{Z}^{NI}$ satisfying

$$
\begin{aligned}
& x^i \leq M^i y^i, i \in I \\
& \sum_{i=1}^{NI} (x^i + ST^i y^i) \leq C \\
& x^i \geq 0, y^i \in \{0, 1\}, i \in I
\end{aligned}
$$

Next, we present a result from the literature [4] concerning this relaxation.

**Definition 2.1** *Let $S_1 \subseteq I$ and $S_2 \subseteq I$ such that $S_1 \cap S_2 = \emptyset$. We say that $(S_1, S_2)$ is a* generalized cover *of $I$ if $\sum_{i \in S_1} (M^i + ST^i) + \sum_{i \in S_2} ST^i - C = \delta > 0$.*

**Proposition 2.2** *(see [4]) Let $(S_1, S_2)$ be a generalized cover of $I$, and let $L_1 \subseteq I \backslash (S_1 \cup S_2)$ and $L_2 \subseteq I \backslash (S_1 \cup S_2)$ such that $L_1 \cap L_2 = \emptyset$. Then,*

$$
\begin{aligned}
& \sum_{i \in S_1 \cup L_1} x^i + \sum_{i \in S_1 \cup S_2 \cup L_1 \cup L_2} ST^i y^i - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ y^i - \sum_{i \in S_2} (ST^i - \delta)^+ y^i \\
& - \sum_{i \in L_1} (\overline{q}^i - \delta) y^i - \sum_{i \in L_2} (\overline{ST}^i - \delta) y^i \leq C - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ - \sum_{i \in S_2} (ST^i - \delta)^+
\end{aligned}
$$

*is valid for $LR1$, where $A \geq \max(\max_{i \in S_1}(M^i + ST^i), \max_{i \in S_2} ST^i, \delta)$, $\overline{q}^i = \max(A, M^i + ST^i)$, and $\overline{ST}^i = \max(A, ST^i)$.*

For a given $t$, second relaxation of $X^{2PL}$, denoted by $LR2$, can be defined as the set of $(x, y, s) \in \mathbb{R}^{NI} \times \mathbb{Z}^{NI} \times \mathbb{R}^{NI}$ satisfying

$$
\begin{aligned}
& x^i \leq M^i y^i, i \in I \\
& x^i \leq d^i y^i + s^i, i \in I \\
& \sum_{i=1}^{NI} (x^i + ST^i y^i) \leq C \\
& x^i \geq 0, y^i \in \{0, 1\}, s^i \geq 0, i \in I
\end{aligned}
$$

In this talk, we will present the trivial facet-defining inequalities for $LR2$, and then derive several classes of valid inequalities such as *cover* and *partition* inequalities. We will also present item- and period-extended versions of some of these families of inequalities, and we will establish facet-defining conditions for all families of inequalities. We will also discuss the separation problems associated with these valid inequalities.

## References


[1] K. Akartunalı, I. Fragkos, A.J. Miller and T. Wu, Local cuts and two-period covex hull closures for big bucket lot-sizing problems, Technical Report, Dept. of Management Science, University of Strathclyde, Available at Optimization Online, `http://www.optimization-online.org/DB_HTML/2014/07/4423.html` (2014).

[2] K. Akartunalı and A.J. Miller, A computational analysis of lower bounds for big bucket production planning problems, Computational Optimization and Applications, 53, 729-753 (2012).

[3] M. Doostmohammadi and K. Akartunalı, A Polyhedral Study of Two-Period Relaxations for Big-Bucket Lot-Sizing Problems: Zero Setup Case, Technical Report, Dept. of Management Science, University of Strathclyde, Available at Optimization Online, `http://www.optimization-online.org/DB_HTML/2015/02/4767.html` (2015).

[4] M. Goemans, Valid ineqialites and separation for mixed 0-1 constraints with variable upper bounds, Operations Research Letters, 8, 315-322 (1989).

[5] R. Jans and Z. Degraeve, Improved lower bounds for the capacitated lot sizing problem with setup times, Operations Research Letters, 32, 185-195 (2004).

[6] A.J. Miller, G.L. Nemhauser and M.W.P. Savelsbergh, Solving multi-item capacitated lot-sizing problems with setup times by branch-and-cut, Technical report CORE DP 2000/39, Université Catholique de Louvain, Louvain-la-Neuve (2000).

[7] Y. Pochet, Valid inequalities and separation for capacitated economic lot-sizing, Operations Research Letters, 7, 109-116 (1988).