STUDENTS' INFORMAL INFERENCE ABOUT THE BINOMIAL DISTRIBUTION OF "BUNNY HOPS": A DIALOGIC PERSPECTIVE

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ABSTRACT

The study explores the development of 11-year-old students' informal inference about random bunny hops through student talk and use of computer simulation tools. Our aim in this paper is to draw on dialogic theory to explain how students make shifts in perspective, from intuition-based reasoning to more powerful, formal ways of using probabilistic ideas. Findings from the study suggest that dialogic talk facilitated students' reasoning as it was supported by the use of simulation tools available in the software. It appears that the interaction of using simulation tools, talk between students, and teacher prompts helps students develop their understanding of probabilistic ideas in the context of making inferences about the distribution of random bunny hops.

Keywords: *Statistics education research; Probability distribution; Informal statistical inference; Dialogic theory; Technology*

1. INTRODUCTION

Probability and statistics have become integral parts of the school mathematics curricula starting from the primary grades for more than two decades (e.g. Australian Education Council, 1991; Department for Education and Employment, 1999; National Council of Teachers of Mathematics [NCTM], 2000). Yet, there has been a tendency that these topics are traditionally treated independently from each other, while they are in fact conceptually related. To bring probability and statistics topics together, Moore (1990) suggested that making informal conclusions based on data should be promoted at early stages. Another recommended approach is to encourage students to begin with data and then engage in probability questions in the context of analysing data (Shaughnessy, 2003). More recent research (Konold & Kazak, 2008) shows that modelling offers a way of introducing chance and data as intertwined topics with stronger connections.

The increased focus on developing young students' informal statistical inference (Paparistodemou & Meletiou-Mavrotheris, 2008; Pratt, Johnston-Wilder, Ainley, &

Mason, 2008; Watson, 2008), in combination with advances in technology tools, has drawn attention to the conceptual link between probability and statistics. Zieffler, Garfield, delMas and Reading (2008) view informal statistical inference as "the way in which students use their informal statistical knowledge to make arguments to support inferences about unknown populations based on observed samples" (p. 44). According to Makar and Rubin (2009), the three essential principles in making informal statistical inference (ISI) are (i) making generalization beyond data, (ii) using data as evidence, and (iii) using probabilistic language in describing the generalization, i.e., articulation of uncertainty and strength of evidence (Ben-Zvi, Aridor, Makar, & Bakker, 2012). Hence, probability becomes an important tool in making such connections between observed data and a population (or a process) during informal statistical inference.

Using data to make informal conclusions about particular features of an infinite population or process also involves drawing inferences about a probability distribution, such as that associated with the outcomes from rolling a die (Pratt et al., 2008). Although probability distribution is an advanced notion in statistical inference, previous research (Kazak, 2006; Lee & Lee, 2009) suggests that even young students (ages 8 to 9) begin to form intuitions about probability distributions while making sense of empirical outcomes of random events in simulation environments. Engaging students in probabilistic reasoning regarding distributions of random outcomes also helps them make the connection between data and chance. This paper focuses on a task designed to explore the development of young students' informal inference about a population model (a binomial probability distribution) based on "random bunny hops" data, through the use of technology and dialogic talk.

2. RESEARCH BACKGROUND

2.1. PROBABILISTIC REASONING

It has been well documented in the literature that students tend to use certain intuitive conceptions and strategies when making decisions or judgments about the likelihood of uncertain events. Although some of these might provide plausible reasoning in some situations, they are often in conflict with probability theory. The representativeness heuristic is one of them. When applying this heuristic, individuals often evaluate the probability of an event based on the degree to which the event represents certain essential characteristics of its parent population or the randomness of the process that generates it (Kahneman & Tversky, 1972). For example, in an experiment of flipping a coin six times, all possible sequences of tails (T) and heads (H) are equally likely according to the probability theory. However, students applying the representativeness heuristics tend to consider the sequence TTTHHH (T=tails, H=heads) more likely to happen than TTTTTT because they perceive the sequence TTTTTT less representative of the expected proportions of heads and tails (50-50 distribution) in the population model. They may also consider the sequence TTTHHH significantly less likely than THHTHT by judging the sequence TTTHHH as less random in terms of irregularity in the appearance of heads and tails.

Konold, Pollatsek, Well, Lohmeier, and Lipson (1993) observed some inconsistencies in students' responses about a coin flipping experiment. First students were asked to determine the "most likely" outcome among some possible sequences from flipping a coin five times, specifically "(a) HHHTT (b) THHTH (c) THTTT (d) HTHTH (e) All four sequences are equally likely." In contrast to the findings in Kahneman and Tversky (1972), a small percentage of students appeared to apply the representativeness heuristic and the

majority correctly answered that all given sequences were equally likely to occur. When these students were asked to choose the "least likely" result in a follow-up question, only 38% of them again responded that all four sequences were equiprobable. Konold et al. (1993) explained this inconsistency in students' responses by a switch in perspective, from an outcome approach to the representativeness heuristic. More specifically, the outcome approach (Konold, 1991) led students to interpret the first question ("most likely" outcome) as to predict what would happen. Therefore, they decided all the sequences were equally likely because "anything could happen". In the next question ("least likely" outcome), however, students using the representativeness heuristic based their response on how well a sample represented the randomness of the process from which it was generated. This inconsistency might not be problematic to the students because they made their decisions on the basis of different frameworks.

Another intuitive approach used for judging the likelihood of an event in random situations is the equiprobability bias (Lecoutre, 1992). Lecoutre described this bias as viewing all possible outcomes of random events as equally likely. For example, in an experiment with rolling a pair of dice, a majority of students tended to respond that the probability of getting a five and a six was equal to that of getting two sixes as they believed that "random events should be equiprobable by nature" or "it is a matter of chance" (Lecoutre, 1992, p. 561).

These intuitive conceptions and strategies can also pose a challenge for students in understanding probability distributions involving combined events. For example, in reasoning about the distribution of outcomes from repeated trials of a binomial experiment, the most common student error within different contexts found in several studies (e.g., Abrahamson, 2009; Kazak, 2006; Konold & Kazak, 2008) is analogous to the belief that getting THHH is more likely than getting HHHH when flipping a coin four times (Abrahamson, 2009). However, students tend to use this erroneous belief to compare the likelihood of combined events, such as Four Heads (4H) and One Tail Three Heads (1T3H), and their judgment based on this intuition tends to be plausible. What students seem to miss in their reasoning here is that the combination of 1T3H is more likely to occur than 4H because it is comprised of four times more outcomes, namely THHH, HTHH, HHTH, HHHT, than 4H (i.e. HHHH only). To promote mathematically correct reasoning about binomial probability distributions, Abrahamson (2009) proposed a sequence of tasks in which an 11-year-old student was presented with a box containing large number of green and blue marbles in equal numbers and a scooper designed to randomly draw four marbles out of this box. After making a prediction with an explanation about what would happen when one scoops from the box, the student was asked to generate all different possible outcomes that one might get at each scoop by colouring given cards using crayons. Once he completed his construction of the sample space with the cards, he was engaged in a combinatorial analysis by building a combinations tower that displayed combined outcomes— zero green, one green, two green, three green and four green— using the cards. This process supported the shift from erroneous intuition to mathematical notions in the binomial distribution.

2.2. SUPPORTING STUDENTS' PROBABILISTIC REASONING

From an educational perspective, students' intuitive strategies, mentioned above, need to be challenged in order to promote normative probabilistic reasoning. According to Fischbein (1975), intuitions that are global and immediate in nature play an important role in children's development of the concept of probability. Fischbein points out that these intuitions can be derived from individual experiences and from formal education at school.

Furthermore, he claims that the development of probabilistic conceptions can be mediated through instructional intervention and social interactions in classrooms. More recently, in the context of teaching probability concepts, Ruthven and Hofmann (2013) suggest a pedagogical model that creates a space for students to learn together, share their intuitions and engage in making sense of different viewpoints in a collaborative setting. One of the key aspects in Ruthven and Hofmann's model is adapting a dialogic approach "that takes different points of view seriously (Scott, Mortimer, & Aguiar, 2006), encouraging students to talk in an exploratory way that supports development of understanding (Mercer & Sams, 2006)" (Ruthven & Hofmann, p. 411). This approach is shown to be effective in developing children's reasoning and problem solving in mathematics (Mercer & Sams, 2006; Monaghan, 2005).

The emerging role of dialogic processes for conceptual development has been emphasized in learning mathematics (e.g., Radford, 2003). Our aim in this paper is to apply dialogic theory to the learning of probability and statistics. Dialogic theory, inspired by the work of Bakhtin (1981, 1986), argues that seeing as if from a different point of view leads to change in students' conceptions and they gain this ability to switch perspective while engaging in dialogue (Wegerif, 2011; Kazak, Wegerif, & Fujita, 2015). In real classroom dialogues students are provided with opportunities to not only experience a point of view of the specific other to whom they are talking, but also understand their own point of view as if from the outside position.

Several other studies (e.g., Ben-Zvi, 2006; Paparistodemou & Meletiou-Mavrotheris, 2008; Lee, Angotti, & Tarr, 2010) have also shown that the use of appropriate visualization tools can help students working together in small groups to develop powerful statistical and probabilistic ideas at fairly young ages. For example, Ben-Zvi (2006) found that 11-year-old students' development of informal ideas of inference was facilitated by their use of *TinkerPlots* software as a tool not only for visualization but also for explaining and arguing about data-based conclusions. Paparistodemou and Meletiou-Mavrotheris (2008) also reported similar results in their study with even younger students. The *TinkerPlots* software enabled 8-year-olds to express and develop their intuitive ideas when making informal inferential reasoning about the data they collected in their school project on nutritional, health, and safety habits. A study by Lee et al. (2010) described how 11 to 12-year-old students used a computer simulation of a dice-rolling experiment to generate data and make inferences about a probability distribution. Lee et al. argued that "well-connected and tight links that students develop between data and a model of a distribution" (p. 89), along with the recognition of the role of sample size and variability, were key to the process of informal statistical inference in the given chance context.

Most previous studies investigating probabilistic reasoning do not appear to provide insights into the combined use of technology tools and effective talk among peers. However, findings from Ben-Zvi (2006) suggest the importance of the mediated role of a combination of argumentative skills and technology use in developing students' informal statistical inferential reasoning. In the current study, we aimed to create opportunities for students to both engage in dialogue and use technology tools to support their communication and probabilistic reasoning. Moreover, incorporating an investigation of a probability distribution at early grade levels can lay the foundation for more advanced topics in statistics. For instance, students are expected to develop a probability distribution for a random variable and use probability in making decisions as part of mainstream school curriculum in the US (see High School Statistics and Probability domain in the Common Core State Standards for Mathematics, http://www.corestandards.org/Math/). Hence, we are interested in the following research question: In what ways does the combination of using *TinkerPlots* software and peer-to-peer dialogic interactions support young students'

formulation of the binomial distribution of random bunny hops on the basis of simulated data?

3. METHOD

3.1. PARTICIPANTS AND RESEARCH CONTEXT

The study took place as part of a two-year design-based research (Cobb, Confrey, diSessa, Lehrer, & Shauble, 2003) that aimed to investigate the development of young students' conceptual understanding of key ideas in statistics and probability through the mediating roles of technological tools and students' talk. This design-based research involved three teaching experiments with different groups of 10–12 year-old students. The data reported here come from the second teaching experiment with a total of six 11-year-old students, two boys and four girls (pseudonyms: Ozzy, Jake, Keyna, Flora, Gabby, and Blair), from a local primary school in Exeter, UK. The participants were recruited through their classroom teacher for the mathematics enrichment class. By following the previous National Curriculum (Department for Education and Employment, 1999), they had already experienced representing and interpreting discrete and continuous data using graphs and diagrams, as well as discussing simple probabilistic events using "equally likely," "fair," "unfair," and "certain." (Note, the new National Curriculum in England was implemented in 2014, but at that time the participants studied mathematics by following the curriculum from 1999.) In order to advance the concepts which had already been taught, the study included three sessions, each about three hours long, and focused on investigating a variety of probabilistic situations in small groups using *TinkerPlots* software.

In the study we chose to use *TinkerPlots 2.0* (Konold & Miller, 2011) software because it is developed for young students and provides both data visualization and analysis tools and simulation capabilities to explore chance events. The software allows students to construct their own graphical representations as they organize their data by ordering, stacking, and separating. With the Sampler tool, students can also build their own chance models using a range of devices that can be filled with different elements to sample from, such as mixer (see Figure 1). It also enables students to generate a large number of outcomes quickly (see Figure 2).

In order to support students' probabilistic reasoning about various random events, a dialogic way of talking in group work, based on the Thinking Together approach (Dawes, Mercer, & Wegerif, 2000), was introduced as part of the study. More specifically, a set of expectations for effective collaboration in joint activities was discussed and negotiated in class: 1) making sure that each person had an opportunity to contribute ideas; 2) asking each other "why?", listening to the explanation, and trying to understand; 3) asking others what they thought; 4) considering alternative ideas or methods; and 5) trying to reach agreement before they did anything on the computer.

3.2. TASK DESCRIPTION AND PROCEDURE

In this paper, we describe how students' reasoning about the distribution of random binomial bunny hops has developed during their small-group work. The random bunny hops task, adapted from Wilensky (1997), was previously used in a study that investigated younger students' reasoning about the way various distributions were shaped in different chance situations through concrete experiments and computer simulations in the *NetLogo* (Wilensky, 1999) environment (see Kazak, 2006). For the purpose of this study, the task

was revised to explore the combined role of the talk between paired students and the use of *TinkerPlots* modelling and simulation features.

In the teaching experiment, students engaged in a sequence of tasks involving random probabilistic events, which can be considered as extensions and enrichment from what they have already studied before, i.e. discussing simple probabilistic events. The first two sessions focused on (1) data modelling in the context of building "data factories" using the Sampler tool in *TinkerPlots* and (2) modelling of a coin flipping and some chance games. The use of probability language (e.g. likely, unlikely, equally likely, most likely, no chance, even chance, certain, uncertain, fair, and unfair) and what randomness means in chance events were also part of these early sessions. The third session began with the following problem: "Suppose there are a number of bunnies on land and each bunny can choose randomly to hop only right or left. For each hop, bunnies are just as likely to hop right as left. We want to know where a bunny is likely to be after 5 hops." First there was a class discussion about how to decide which way, right or left, the bunnies might hop. Then, students were asked to make their initial predictions about where the ten bunnies would be likely to end up after 5 random hops according to the following instruction: "Imagine that a bunny is standing on a number line at 0. You spin or flip a coin to decide which way the bunny hops. If the coin lands heads up, it hops one step right (that is, one step along the positive direction). If the coin lands tails up, it hops one step left (that is, one step along the negative direction)." They showed their predictions by marking the final locations with X for each bunny on the given number line. Moreover, in order to promote the third component of ISI (probabilistic language), students were asked to express the level of confidence or uncertainty in their prediction at each stage of the task on a 0%–100% (from "not at all confident" to "totally confident") scale. Students also watched a demonstration of five random hops in *TinkerPlots*. As seen in Figure 1, a single mixer device (on the left) in the Sampler tool is set to model equally likely random hops to right (R) or left (L). The table next to the Sampler records and shows the results of each repeat in the Outcome column. The Position attribute (created with the formula option) in the table displays the location of the bunny after each hop on the number line. For example, the first hop is to the right and the bunny ends up on 1; then the second hop is to the left and the bunny now is back at the 0 position; and so on. The graph on the right-hand side also shows each individual hop by the trial number visually. In this example, a bunny hops "right-left-right-left-left" (RLRLL) and ends up at -1 on the number line.
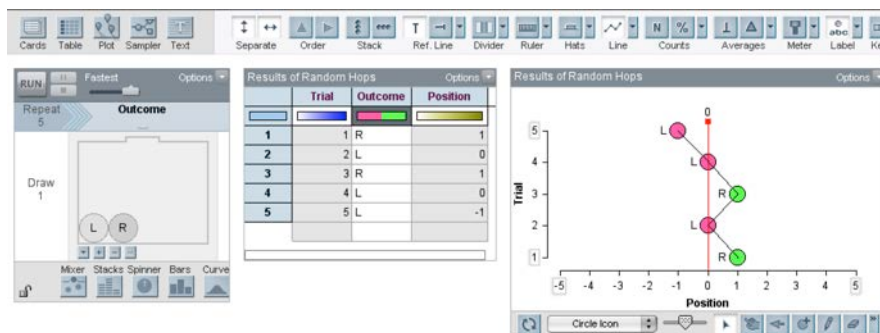

Figure 1. A model of five random hops in TinkerPlots

Next, students in pairs simulated five random hops of 10 bunnies by flipping a coin to see where they would end up. They recorded both the path each bunny took and where it ended up on the number line (e.g., the path as "R,L,L,L,L" and the final location as "-3")

in the given table on their worksheet. They were also asked to make a graph of these final locations by marking X for each bunny on the number line (see Figure 3). After collecting some data through the coin simulation, students were asked again to discuss together and decide where they think a rabbit is most likely to be after 5 hops with an explanation on the worksheet.

In the next stage, the pairs used the given *TinkerPlots* model to simulate random bunny hops to explore where the bunnies would be most likely to be after five hops. In Figure 2, the Sampler is currently set to generate 100 bunnies. Next to it the table and the plot show the results from one trial. Students used this Sampler to conduct a total of four trials and recorded the percentage of bunnies on each final position in a table on the worksheet. After these simulations, the pairs were expected to discuss where the most of the bunnies would likely to end up after five hops and why.
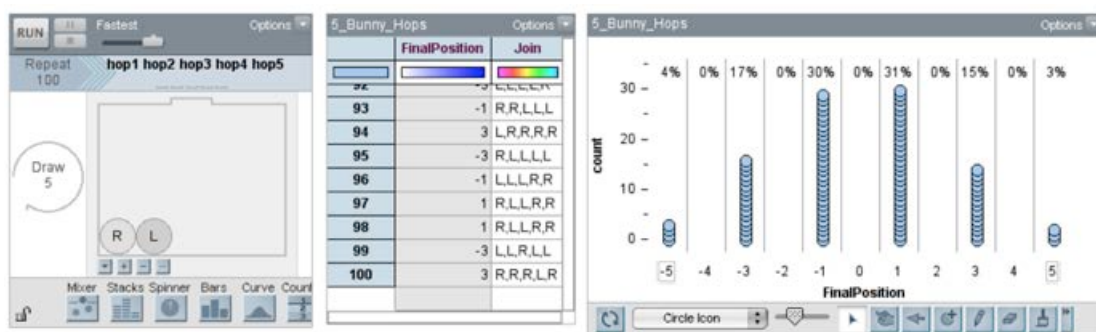

Figure 2. The Sampler students used to simulate 100 bunnies hopping randomly to right or left five times and the plot of results from a trial

3.3. DATA COLLECTION AND ANALYSIS

During the study pairs of students working around a computer were video-recorded to capture the peer-to-peer interactions. These video recordings and the written artefacts for each pair were used for subsequent analysis. In the current study we broaden our analysis of a single pair in Kazak et al. (2014) to present a trajectory of how young students' reasoning about the binomial distribution of random bunny hops develop through the use of simulated data and student talk. Drawing upon Mercer (2004), we utilized a sociocultural and dialogic discourse analysis to analyse the talk of students during their joint activity in pairs and in whole-class discussions. Additionally we examined students' written artefacts to support those analyses. The detailed analysis focused on identifying key moments of any conceptual shift in students' reasoning supported by the dialogue between participants and the use of computer tools. Then selected transcribed excerpts were analysed to show in what ways dialogic talk and the use of *TinkerPlots* enabled a switch in perspective in students' reasoning, and were evaluated based on our interpretation of existing literature.

4. RESULTS

In this section, we present episodes that illustrate how 11-year-old students developed an understanding of the binomial distribution of random bunny hops on the basis of simulated data, from a naïve perspective to a more sophisticated one. In doing so, we focus on key moments of students' reasoning during the task described above.

4.1. STUDENTS' INITIAL PREDICTIONS

Each pair's predicted results for 10 bunnies after five random hops are shown in Figure 3. One of the pairs (Keyna and Flora) expected equal number of bunnies on each side of the starting point (0) after five hops with some symmetry with respect to the final locations on either side. However, Keyna and Flora believed that 10 bunnies "can be anywhere on the number line as long as it's between 5 and -5." While they noted how far a bunny could go after five hops, they could not reason about the distribution of outcomes based on a sound probabilistic argument. Their explanation appeared to be consistent with the outcome approach, i.e. "anything can happen" (Konold, 1991).
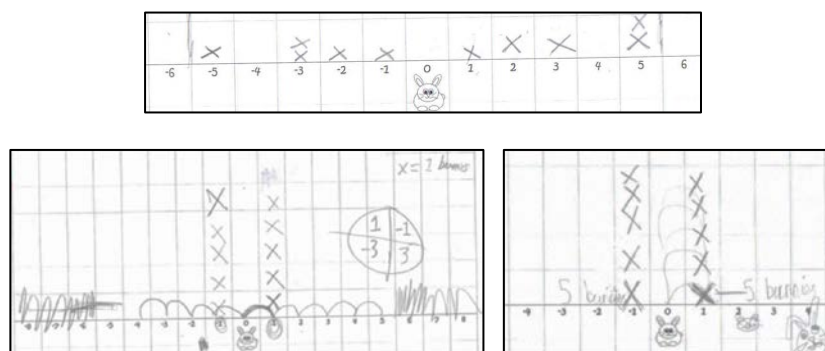

Figure 3. Each pair's initial predictions: Keyna-Flora (top), Ozzy-Jake (lower left) and Gabby-Blair (lower right)

As seen in Figure 3, the other two pairs (Ozzy and Jake; Gabby and Blair) expected five bunnies on 1 and five bunnies on -1 on the number line after five hops. Their predictions were based on symmetry around 0 and the data were only on 1 and -1. When Gabby and Blair were asked to explain their prediction, they mentioned that it would be possible to get bunnies from -5 to 5. However, their interpretation of the 50-50 chance of hopping either right or left led them to expect the outcomes to be mostly like "left-right-left-right-left" (LRLRL). Similar to the findings in previous research (Kazak, 2006), this intuitive reasoning seems to be aligned with the representativeness heuristic (Kahneman & Tversky, 1972) suggesting that students are inclined to think "LRLRL" is more representative of the expected 50-50 distribution of coin flipping results. The other group also had similar reasoning. During the whole-class discussion of each pair's predictions, Blair and Jake had different ideas about the distribution of possible outcomes if there were more bunnies. For instance, Jake thought that landing on 1 and -1 after five hops would be more likely but there could be other possibilities between -5 and 5. However, Blair expected that there would be equal number of bunnies on each possible outcome. Her expectations seemed to be based on the equiprobability bias (Lecoutre, 1992). Jake was also the first to realize that bunnies could end up on only odd numbers in the number line because of the odd number of hops when he drew paths of right or left in five hops. This discussion then helped others to note all of the possible outcomes.

4.2. STUDENTS' REASONING BASED ON EMPIRICAL DATA

After each pair completed their coin simulations for 10 bunnies and made a graph of their data, students examined how the results from each simulation were distributed during a whole-class discussion. As seen in Figure 4, simulation results overall turned out to be

mostly on 1 and -1 as expected. While two of the graphs had a mound-shaped distribution, the graph in the middle (Figure 4) showed an even distribution on every possible outcome, except -5. This "unusual" flat distribution of outcomes in Gabby and Blair's coin simulation led to a discussion about "most likely" and "less likely" outcomes in the class. While Ozzy made a claim that they "cheated" to get these results, Jake tried to explain that it was more likely to be on one of -1 and 1 than to be on 5 based on his intuitive reasoning.
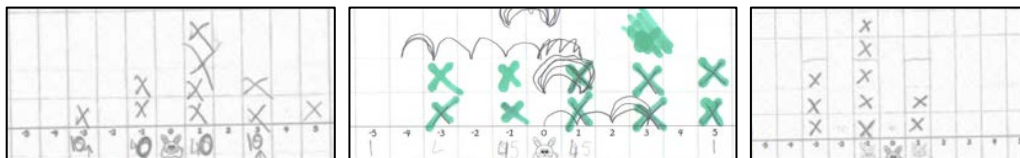
Figure 4. Each pair's coin simulation results: Ozzy-Jake (left), Gabby-Blair (middle), and Keyna-Flora (right)

Students also recorded their data in a given table (see Figure 5) as they collected them. None of the pairs made use of the different paths recorded in the table for the same outcome in their reasoning. For example, in Figure 5 Ozzy and Jake had four different paths, LRRLR, LRLRR, RRLRL and RRLLR, leading to final location 1. Even though they were quite sure that a bunny would most likely be on 1 or -1, they did not provide a solid explanation for it. Still their intuitive reasoning led students to make their next predictions about 100 bunnies. For instance, based on their reasoning about the more likely and less likely outcomes and symmetry around 0, Ozzy and Jake predicted 10 bunnies on -3, 40 on -1, 40 on 1 and 10 on 3 while Gabby and Blair expected results like 3 for -5, 10 for -3, 37 for -1, 37 for 1, 10 for 3, 3 for 5 on the number line. However, Keyna and Flora expected "10 times the results from the coin simulation", that is 30 on -3, 50 on -1 and 20 on 1.
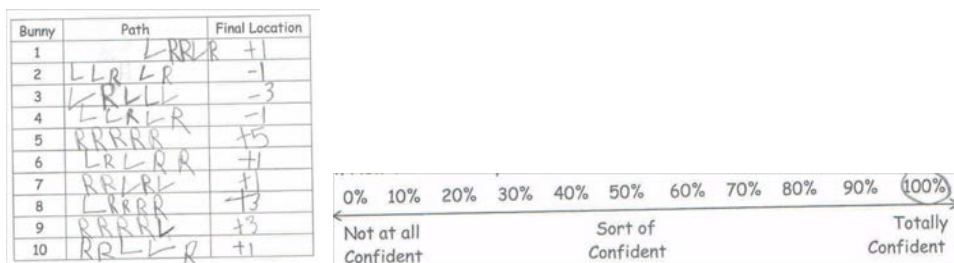

Figure 5. Table of results recorded by Ozzy and Jake during their coin simulation (on the left) and their level of confidence about where a rabbit is most likely to be after five hops after the simulation (on the right)

In terms of making a generalisation beyond the collected data, Keyna and Flora seemed to generalise from their data about 10 bunnies to 100 bunnies using the proportionality in frequencies. In contrast, the other pairs made their inferences on the basis of what they believed to be the most/least likely outcomes of five random hops, even though Gabby and Blair's simulation results (Figure 4) did not actually support that. When asked why they thought the bunnies would mostly end up on 1 and -1, Gabby expressed her strong belief that because of the 50% chance of hopping right or left the bunnies would be more likely to hop like "LRLRL" or "RLRLR". Based on this belief the two pairs appeared to have higher confidence or certainty, such as 90% and 100% (see Figure 5) in their predictions

after the data collection compared to Keyna and Flora who expressed relatively less confidence (85% and 50% respectively as they could not reach an agreement) about their predictions based on their collected data.

4.3. MAKING SENSE OF THE BINOMIAL DISTRIBUTION BASED ON COMPUTER SIMULATION RESULTS

In the later stage of the task, pairs ran their *TinkerPlots* simulations four times and recorded the results in a table (see Figure 6). Making predictions for 100 bunnies prior to these simulations seemed to help students to interpret the resulting distribution. While Gabby and Blair were conducting their trials in *TinkerPlots*, they began to compare the results with their predictions and judge the fit between the two. For example, Gabby said "it is quite close" when they got the Run4 results in their table in Figure 6. In order to test the closeness of their prediction, Gabby went further and increased the number of repeats in *TinkerPlots* to 1000 and then 10000. They had discussed this idea of a law of large numbers during the previous session where they explored the fairness of some chance games. This test helped both students strengthen their expectation of the most likely outcomes, i.e., 1 and -1, because the proportional frequencies of all possible outcomes began settling down. Accordingly, increasing the number of repeats in the simulation enabled them to express total confidence (100%) in predicting where most of the bunnies would end up after five hops.

Final Position	Run1 (%)	Run2 (%)	Run3 (%)	Run4 (%)
5	2%	3%	0%	1%
4	0%	0%	0%	0%
3	15%	14%	19%	14%
2	0%	0%	0%	0%
1	29%	33%	33%	35%
0	0%	0%	0%	0%
-1	34%	32%	29%	30%
-2	0%	0%	0%	0%
-3	19%	15%	16%	17%
-4	0%	0%	0%	0%
-5	1%	3%	3%	3%

Figure 6. The results from four different trials in TinkerPlots simulations recorded by Gabby and Blair.

After the teacher-researcher's prompt to explain why they got more on -1 and 1 than the others and less on -5 and 5, Blair started to look at a path of five hops of an individual bunny on the plot using the Sampler shown in Figure 1. Blair and Gabby then watched different paths by running the Sampler several times to get, what they called the "perfect" and the "worst" examples for the most likely and the least likely outcomes (Figure 7). Their interpretation of different paths appears to be consistent with the representativeness heuristic again.

After seeing students' focus on particular paths, the teacher-researcher posed the following question: "Did you see any other example of landing on 1 [*referring to the path they got in the plot*]?" This prompted Blair and Gabby to generate as many different paths of five random hops as they could. Then the *TinkerPlots* simulation tool became a shared space for this pair to investigate their intuitive idea of "more chance of getting bunnies on -1 and 1 than on -3 and 3" by counting each different path leading to those positions in the

plot. For example, in order to find all the possible ways to get to -1, students ran the Sampler until they thought they had found all different paths. However, by this trial and error method they were able to list only 6 different paths, as seen in Figure 8. Gabby even drew the "perfect" path that appeared in the *TinkerPlots* display.
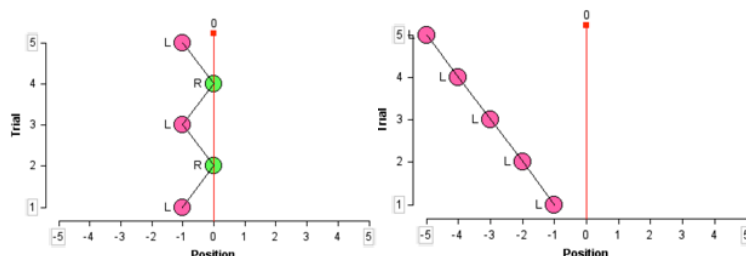

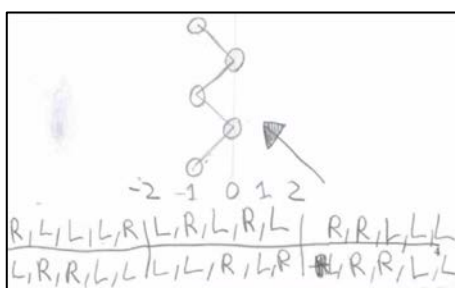*Figure 7. "Perfect" and "worst" examples according to Gabby and Blair*

Figure 8. The list of 6 different ways to get to -1

When trying to find different paths for landing on 3 in *TinkerPlots*, they started to develop a systematic way of counting all possible ways of five hops. For instance, using the path on the left in Figure 9 Blair showed how they could move the single "left" hop (the pink circle labelled L on the plot) to each of five different places on the display to get different ones. There would be five possible paths "because there are five dots [*showing on the plot*]" she said. This insight was a key moment for Blair and Gabby's formulation of all possible ways to get each outcome in the bunny hop problem. The following brief exchange around that plot on the screen illustrates how Gabby began to see the problem from Blair's perspective and became convinced that there were five different ways to land on 3.

|		
1	Blair:	If you have, okay, you have got it [*pointing to the pink circle for L*] up
2		there, you should have it there, there [*moving her finger on the dots*],
3		you can have it in five different places for one left dot.
4	Gabby:	So ten
5	Blair:	So there is
6	Gabby:	No, there is five that way
7	Blair:	Yeah because you have
8	Gabby:	Wait [*pointing to the plot*] yeah I get it, I get it [*leaning back in her
9		chair and smiling*].
10	Blair:	There is five different ways because there is five different places. Weee!
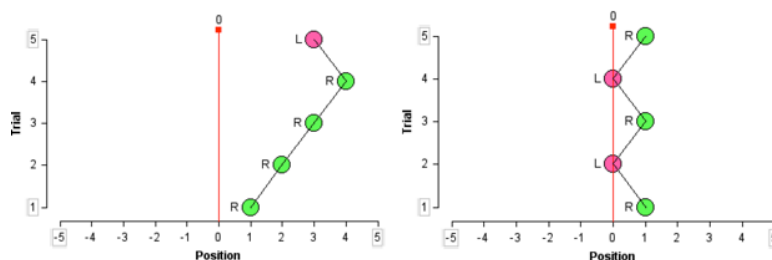
Figure 9. Representation used to count all possible ways to get to 3 (on the left) and 1 (on the right)

This strategy of counting the dots became a tool for Blair and Gabby to figure out all 10 different possible ways to get to 1. First, Gabby ran the Sampler until she found the path shown in the plot to the right in Figure 9. Then they were able to count all possible combinations of 2 Ls and 3 Rs on the display. When the teacher-researcher asked them why they got this particular distribution shape in the sample of 10000 on the graph, Blair quickly made the connection with their findings about the number of possible paths saying that "because there is only one way to get each five and then there is five ways to get three and then there is ten ways to get one."

The other two pairs tried to generate a list of different possible paths for each outcome on paper without using any particular strategy, except switching the L and R for opposite symmetric outcomes (i.e., LRLRL for -1 and RLRLR for 1). Perhaps because of that, Keyna and Flora had only a partial list of all the possibilities. Jake and Ozzy, on the other hand, listed all 32 different ways that a bunny can hop in five hops (Figure 10). After a prompt by the teacher-researcher, they even computed the theoretical probability for each possible outcome based on the number of ways each outcome could occur. This attempt was very close to generating a probability distribution of the binomial bunny hops described in the task.
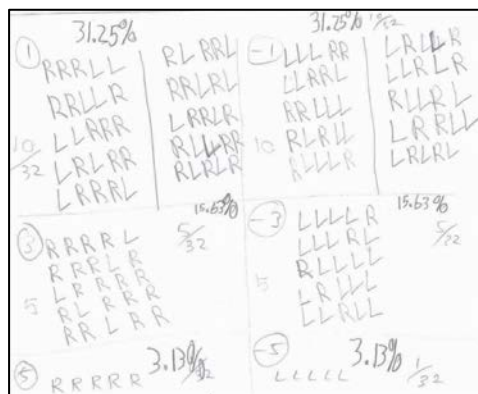

Figure 10. Ozzy and Jake's list of all different ways for each outcome in five hops

5. DISCUSSION

In response to the need for building the conceptual links between data and probability topics starting from the early grades, the current study focused on the development of 11-year-old students' informal inference about a binomial probability distribution based on

random bunny hops data. In order to support this conceptual development, the study sought to investigate the following research question: In what ways does the combination of using *TinkerPlots* software and peer-to-peer dialogic interactions support young students' formulation of the binomial distribution of random bunny hops on the basis of simulated data?

The findings in the study support Fischbein's (1975) view on children's intuitions about probabilistic ideas. Students' initial predictions about the distribution of 10 bunnies after five hops tended to be based on their intuitive ideas about where bunnies would end up by hopping only left or right with equal chance. These intuitions were mainly aligned with well-documented heuristics, beliefs or biases in dealing with chance situations, such as the representativeness heuristic (Kahneman & Tversky, 1972), outcome approach (Konold, 1991) and equiprobability bias (Lecoutre, 1992). Determining the possible outcomes of the given random event was one of the initial ideas developed by the students. While all pairs could see that no bunny could end up beyond -5 and 5, only one student initially noted that all the possible outcomes had to be an odd number because of the number of hops. After students collected some data by flipping a coin to simulate random bunny hops and graphed these results, those intuitions, particularly the representativeness heuristic, seemed plausible to them in explaining the most likely outcomes, such as -1 and 1. This result was consistent with previous research conducted with younger students on the same task (Kazak, 2006). For instance, one of the students argued (Kazak, 2006, p.193)

> [S]ee because heads and tails are both 50%. There is a 50% of a chance that either of them is going to get flipped or whatever. So, it's not very likely that you are going to get something like HHHHT because there is so many heads in it, you know compared to the tails. Now, HHHTT is pretty close to even.

In this example, the student did not focus on the number of ways to get 3 heads and 2 tails.

In the current study, collecting more data (n=100) by running the *TinkerPlots* model several times helped students to be more certain about the expected distribution, which is similar to a mound shape. However, in order to explain why it was shaped this way, students needed to focus on the individual paths comprising each possible outcome, specifically, -5, -3, -1, 1, 3 and 5. The specific example of the interaction between Gabby and Blair illustrates how students' intuitive reasoning about the distribution of random bunny hops developed into their reasoning about the number of different paths to explain the binomial distribution. This conceptual shift was facilitated primarily by the affordances and tools provided by the software and the dialogic talk between students in a shared environment. It was also supported by the teacher-researcher's questioning for getting their attention to the different examples of paths in five random hops.

In Gabby and Blair's case, the students' use of terms like the "best" (or "perfect") and "worst" examples of five hops can be interpreted as an emotional engagement that led them to consider other examples of possible outcomes in the random event. Then, after this initial encounter, the students noticed a general pattern in the stability of the distribution shape in a long run. This was facilitated by the visualization of the different paths of five random hops on the *TinkerPlots* screen. The *TinkerPlots* representation became a shared space in students' dialogue when they could point to and discuss alternative paths and figure out different ways to end up on 1 and 3. Furthermore, the extract of talk given above illustrated how Gabby moved from external dialogue (i.e., learning from the perspective of the other, Blair), to the personal appropriation in internal dialogue of "I get it" (line 8). This shift in her conceptualization of the problem was mediated by the gaze of the other.

The study shows a perspective shift between different frameworks, as seen in the findings of Konold et al. (1993). The challenge is to understand how students switch perspective in order to support it. Dialogic theory suggests that we treat this as the shift in

perspective occurring in dialogues. Based on this study, one aspect could be the way in which the representations in *TinkerPlots* and on paper enter into the dialogue like voices. They become a shared focus but are also brought on in response to questions in a dialogue between the students. The teacher-researcher here acts like the gaze of the witness or superaddressee, as Bakhtin (1986) refers to, in challenging them to think, instead of giving them an answer. Thus, we argue that it is essential to the learning process to provide an environment where the use of computer tools is combined with peer-to-peer talk and teacher scaffolding in a carefully designed sequence of activities to support the shift from intuitions to formal probabilistic ideas.

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