Why the naïve Derivation Recipe model cannot explain how mathematicians' proofs secure mathematical knowledge

Brendan Larvor

School of Humanities, University of Hertfordshire, Hatfield, AL10 9AB, UK

E-mail: b.p.larvor@herts.ac.uk

Abstract: The view that a mathematical proof is a sketch of or recipe for a formal derivation requires the proof to function as an argument that there is a suitable derivation. This is a mathematical conclusion, and to avoid a regress we require some other account of how the proof can establish it.

Keywords: proof, derivation

It is a common observation that the proofs that mathematicians write on blackboards and publish in journals are not like the derivations that appear as objects in proof theory. Mathematicians' proofs may make extensive use of specialist notation, but they do not employ explicitly defined logical languages; they use a mixture of algebraic notation and natural language, they leave gaps for the reader to fill, they make inferences that use implicit rules and may depend on the particular topic, and they may appeal to mathematical insight. How do such proofs satisfy mathematical standards of rigour?

There is a stock answer[1] to this, which we will call it the 'Derivation Recipe' model.[2] It depends on the idea that the derivations that appear as objects in proof theory are real proofs, because they

[1] Michael Detlefsen calls it the 'common view' (2008: 17), Jeremy Avigad calls it 'the logician's claim' (2008: 306); neither Detlefsen nor Avigad endorses it.

[2] This name owes something to Azzouni's 'Derivation-Indicator' view (2004). As he makes clear in his (2013), Azzouni does not subscribe to the Derivation Recipe model, and he has argued against it in other publications.

have no logical gaps, are expressed in a formal language, the well-formed formulae of which are recursively defined, and rely on wholly general and explicitly given rules of inference that make no reference to any particular subject matter. The Derivation Recipe model claims that mathematicians' proofs (as written on blackboards and published in journals) are rigorous because they show mathematicians how a derivation of the theorem in question might be constructed. That is to say, mathematicians' proofs are (on this model) either rough drafts of derivations, or recipes for making suitable derivations. Note (again, this is common ground) that the derivations are almost never written out. In almost all cases, the function of the mathematicians' proof (on the Derivation Recipe model) is not to provide the mathematician with a practical guide to writing out a derivation. Rather, it is to give the mathematician compelling reason for thinking that a derivation could in principle be written.

The Derivation Recipe model has prestigious backers among mathematicians. It arose in the nineteenth century, and found clear expression in the work of Moritz Pasch (see Schlimm 2010). The locus classicus for more recent discussions is probably this passage in Saunders Mac Lane's *Mathematics: Form and Function* (1986):

> A mathematical proof is rigorous when it is (or could be) written out in the first order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules… practically no one actually bothers to write out… formal proofs. In practice, a proof is a sketch, in sufficient detail to make possible a routine translation of this sketch into a formal proof. ...the test for the correctness of a proposed proof is by formal criteria and not by reference to the subject matter at issue. (Mac Lane 1986: 377-8)

This view is the target of the present note; it is the naïve Derivation Recipe model. On this view, the rigour of ordinary mathematical proofs consists entirely in their being translatable (in principle) into derivations. From the point of view of logic, there is nothing more to say. Of course, there is a great deal for psychologists and sociologists to say about how human mathematicians individually and collectively come to understand and confirm proofs, but this is not a matter for logic. One consequence of this view, as Mac Lane notes, is that there is no relation between the rigour of a proof and its subject matter, because formal logic is topic-neutral.

Some philosophers and mathematicians have argued against the Derivation Recipe model.[3] These arguments appeal to facts about mathematical practice and the finitude of mathematicians. However, the relevance of such facts to logic and epistemology is disputed within philosophy of mathematics. It would, therefore, be advantageous to offer an argument against the Derivation Recipe model that is independent of questions about human cognitive and social functioning.

The argument starts from the observation that the conclusion of a mathematician's reading of an informal proof according to the Derivation Recipe model—that a suitable derivation could be written—is itself a mathematical claim. Let *P* be a mathematician's proof for a theorem *C*. Then it follows from the Derivation Recipe model that *P* is not as it stands (before any translation) a proof of *C*, but is rather an argument to convince the reader that:

C′: there is a suitable formal system *S* such that $\vdash_S \gamma$ where γ is the formula in *S* corresponding to *C*

For the sake of clarity: this is not what proponents of the Derivation Recipe model say; rather it is what the Derivation Recipe model amounts to once we recognise that the existence of a suitable derivation is itself a mathematical claim. The Derivation Recipe model requires that *P* must be, as it stands, before any translation, a compelling, rigorous argument (epistemically equivalent to a proof) of a mathematical conclusion, namely, *C*′. Notice that *C*′ is not, as it stands, fully formalised because it includes metamathematical elements and natural language.

[3] We have already seen Avigad and Detlefsen, to whom we can add Azzouni, Goethe, Friend, Hersh, Lakatos, Nickel, Pelc, Rav, Robinson, Tanswell, Thurston, Van Bendegem and others

How can *P* work as a proof of *C′*? If it is just a recipe for a derivation, this would initiate an obvious regress. So, the Derivation Recipe picture must be that P, gappy, informal and intuitive as it may be, is an adequate proof of the mathematical claim *C′*, whereas it is not an adequate proof of the mathematical claim *C*. Compare this with the more straightforward view that *P* is as it stands, before any translation into a formal language, an adequate proof of *C*. Both of these views require *P* in its native, untranslated state to prove a mathematical result—they just differ over whether that result is *C* or *C′*. These two views are not quite equivalent, because (when formalised) *C′* is a Σ_1 sentence, regardless of the complexity of *C*. It may be possible to exploit this fact to develop a more nuanced version of the Derivation Recipe model than is currently available.[4] Perhaps defenders of the Derivation Recipe model will be satisfied to claim that a rigorous proof must be either a derivation or a proof by construction of a Σ_1 sentence. Perhaps there is a modified or hybrid version of the Derivation Recipe model that might escape the argument of this paper, which is after all directed only at the simple version that Mac Lane expressed in the quotation above.

References

Avigad, J. [2008] Computers in Mathematical Enquiry. In *The Philosophy of Mathematical Practice*, ed. P. Mancosu, 302-316. Oxford: Oxford University Press.

Azzouni, J. [2004] The Derivation-Indicator View of Mathematical Practice. *Philosophia Mathematica* (III) 12(2): 81-105.

Azzouni, J. [2009] Why do informal proofs conform to formal norms? *Foundations of Science* 14: 9-26.

[4] I am grateful to an anonymous referee for this suggestion and for some other helpful remarks. I am especially grateful because this referee has been kind enough to see this paper through several iterations. Thanks are also due to the other referee and to the Vrije Universiteit Brussel for financial support during the time when this note was written.

Azzouni, J. [2013] The Relationship of Derivations in Artificial Languages to Ordinary Rigorous Mathematical Proof. *Philosophia Mathematica* (III) 21: 247-254.

Detlefsen, M. [2008] Proof: Its Nature and Significance. In *Proof and Other Enigmas*, B. Gold and R. Simons, 3-32. Mathematical Association of America.

Goethe, N. & Friend, M. [2010] Confronting the Ideals of Proof with the Ways of Proving of the Research Mathematician. *Studia Logica* 96: 273-288.

Hersh, R. [1997) Prove—Once More and Again *Philosophia Mathematica* (III) 5(2): 153-165.

Lakatos, I. [1976] *Proofs and Refutations.* Ed. J. Worrall and E. Zahar. Cambridge: Cambridge University Press.

Mac Lane, S. [1986] *Mathematics: Form and Function.* New York: Springer.

Nickel, G. [2010] Proof: Some notes on a phenomenon between freedom and enforcement. In *PhiMSAMP Philosophy of Mathematics: Sociological Aspects and Mathematical Practices*, ed. B. Löwe & T. Müller, 281-291. London: College Publications. Texts in Philosophy Series (V.F. Hendriks, J. Symons, D. Gabbay eds).

Pelc, A. [2009] Why do we believe theorems? *Philosophia Mathematica* (III) 17(1): 84-94.

Rav, Y. [1999] Why Do We Prove Theorems? *Philosophia Mathematica* (III) 7(1): 5-41.

Robinson, J. [1997] Informal Rigor and Mathematical Understanding. In G. Gottlob, A. Leitsch & D. Mundici (eds) *Computational Logic and Proof Theory: Proceedings of the 5th annual Kurt Gödel Colloquium*. Heidelberg & New York: Springer. Series editors: G. Goos, J. Hartmanis, J. van Leeuwen.

Schlimm, D. [2010] Pasch's Philosophy of Mathematics. *The Review of Symbolic Logic*, 3: 93-118.

Tanswell, F. [2015] A Problem with the Dependence of Informal Proofs on Formal Proofs. *Philosophia Mathematica* (III) 23(3): 295-310.

Thurston, W. [1994] On Proof and Progress in Mathematics. *Bulletin of the American Mathematical Society* 30(2) 161-177.

Van Bendegem J.-P. [1988] Non-Formal Properties of Real Mathematical Proofs. *Proceedings of the Biennial Meeting of the Philosophy of Science Association*, Vol. 1: Contributed Papers (1988), pp. 249-254. University of Chicago Press.

Van Bendegem J.-P. [1990] Characteristics of real mathematical proofs. A. Diaz, J. Echeverria en A. Ibarra (eds.), *Structures in mathematical theories*, Servicio Editorial Universidad del Pais Vasco, San Sebastian, pp.333-337.