Transformation Groups

ON BASES OF SOME SIMPLE MODULES OF SYMMETRIC GROUPS AND HECKE ALGEBRAS

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We record with deep sadness the passing of Anton Evseev on 21*st February* 2017

Abstract. We consider simple modules for a Hecke algebra with a parameter of quantum characteristic e. Equivalently, we consider simple modules D^λ, labelled by e-restricted partitions λ of n, for a cyclotomic KLR algebra $R_n^{\Lambda_0}$ over a field of characteristic $p \geqslant 0$, with mild restrictions on p. If all parts of λ are at most 2, we identify a set $\mathsf{DStd}_{e,p}(\lambda)$ of standard λ-tableaux, which is defined combinatorially and naturally labels a basis of D^λ. In particular, we prove that the q-character of D^λ can be described in terms of $\mathsf{DStd}_{e,p}(\lambda)$. We show that a certain natural approach to constructing a basis of an arbitrary D^λ does not work in general, giving a counterexample to a conjecture of Mathas.

1. Introduction

Let K be a field with a Hecke parameter $0 \neq \xi \in K$ of quantum characteristic $e \in \mathbb{Z}_{\geqslant 2}$. We consider the Iwahori–Hecke K-algebra $\mathcal{H}_n(\xi)$. An important special case occurs when $\xi = 1$ and K has characteristic e, which implies that $\mathcal{H}_n(\xi) =$

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$K\mathfrak{S}_n$ is the group algebra of a symmetric group.

The Specht $\mathcal{H}_n(\xi)$-modules $S^\lambda_{\mathcal{H}}$, parameterised by partitions λ of n, play an important role in the representation theory of $\mathcal{H}_n(\xi)$. In particular, if λ is an e-restricted partition, then $S^\lambda_{\mathcal{H}}$ has a simple head D^λ, and all simple $\mathcal{H}_n(\xi)$-modules occur in this way. The Specht module $S^\lambda_{\mathcal{H}}$ has a Murphy basis indexed by the set $\mathsf{Std}(\lambda)$ of all standard λ-tableaux. In this paper, we investigate whether there is a subset of $\mathsf{Std}(\lambda)$ that naturally labels a basis of D^λ.

This question can be made much more precise via the language of Khovanov–Lauda–Rouquier (KLR) algebras [18, 24], which is used throughout the paper. Given an arbitrary commutative ring $\mathcal{O}$, we consider the cyclotomic KLR $\mathcal{O}$-algebra $R^{\Lambda_0}_{n,\mathcal{O}}$ of type $A^{(1)}_{e-1}$, which has a natural $\mathbb{Z}$-grading, see §2.2. Brundan and Kleshchev [3] and Rouquier [24] proved that $R^{\Lambda_0}_{n,K}$ is isomorphic to $\mathcal{H}_n(\xi)$. Further, Kleshchev, Mathas and Ram [19] constructed a universal Specht $R^{\Lambda_0}_{n,\mathcal{O}}$-module $S^\lambda_{\mathcal{O}}$ by explicit generators and relations such that, in particular, S^λ_K is isomorphic to the $\mathcal{H}_n(\xi)$-module $S^\lambda_{\mathcal{H}}$. We denote by D^λ_K the (simple) head of S^λ_K if the partition λ is e-restricted and set $D^\lambda_K := 0$ otherwise.

The algebra $R^{\Lambda_0}_{n,\mathcal{O}}$ is equipped with an orthogonal family of idempotents $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^n\}$, where $I := \mathbb{Z}/e\mathbb{Z}$. The *q-character* of a finite-dimensional $R^{\Lambda_0}_{n,\mathcal{O}}$-module M is defined by

$$\operatorname{ch}_q M := \sum_{\boldsymbol{i}\in I^n} \dim_q(1_{\boldsymbol{i}} M)\cdot \boldsymbol{i} \in \langle I^n\rangle, \tag{1.1}$$

where $\langle I^n\rangle$ is the free $\mathbb{Z}[q,q^{-1}]$-module with basis I^n and $\dim_q(1_{\boldsymbol{i}}M) \in \mathbb{Z}[q,q^{-1}]$ is the graded dimension of $1_{\boldsymbol{i}}M$, see §2.1.

Let λ be a partition of n. To each standard tableau $\mathtt{t} \in \mathsf{Std}(\lambda)$ one attaches its *residue sequence* $\boldsymbol{i}^{\mathtt{t}} \in I^n$ and *degree* $\deg(\mathtt{t}) \in \mathbb{Z}$, which are both defined combinatorially, see [5] or §2.3. Then the Specht module $S^\lambda_{\mathcal{O}}$ has an $\mathcal{O}$-basis $\{v^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{Std}(\lambda)\}$ such that $1_{\boldsymbol{i}} v^{\mathtt{t}} = \delta_{\boldsymbol{i},\boldsymbol{i}^{\mathtt{t}}} v^{\mathtt{t}}$ for any $\boldsymbol{i} \in I^n$ and $v^{\mathtt{t}}$ is homogeneous of degree $\deg(\mathtt{t})$ for each $\mathtt{t}$. In particular, defining the *q-character* of any finite set $\mathcal{T}$ of standard tableaux by

$$\operatorname{ch}_q \mathcal{T} := \sum_{\mathtt{t}\in\mathcal{T}} q^{\deg(\mathtt{t})}\cdot \boldsymbol{i}^{\mathtt{t}}, \tag{1.2}$$

we have

$$\operatorname{ch}_q S^\lambda_K = \operatorname{ch}_q \mathsf{Std}(\lambda). \tag{1.3}$$

Therefore, it is reasonable to require that a desired subset of $\mathsf{Std}(\lambda)$ corresponding to a basis of D^λ_K should have q-character equal to $\operatorname{ch}_q D^\lambda_K$. Our main results give a combinatorial construction of such a subset of $\mathsf{Std}(\lambda)$ for an arbitrary field K (as above) when $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies $\lambda_1 \leqslant 2$; we refer to such partitions λ as *2-column partitions*. We refer the reader to [14, §3.3] for a further discussion of the problem in general.

In §3.3, we give a combinatorial definition of a subset $\mathsf{DStd}_e(\lambda)$ of $\mathsf{Std}(\lambda)$ for every 2-column partition λ. In order to describe $\mathsf{DStd}_e(\lambda)$, we represent standard tableaux as paths in a weight space of Dynkin type A_1 and construct a *regularisation map* reg_e on standard tableaux, which plays a key role throughout.

The following theorem shows that, when $\operatorname{char} K = 0$, the set $\mathsf{DStd}_e(\lambda)$ labels a basis of D^λ_K and, moreover, the composition series of S^λ_K can be lifted to an arbitrary commutative ring $\mathcal{O}$ in an explicit way.

Theorem 1.1. *Let λ be a 2-column partition of n.*

(i) *The $\mathcal{O}$-span $U^\lambda_{\mathcal{O}}$ of $\{v^{\mathsf{t}} \mid \mathsf{t} \in \mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)\}$ is an $R^{\Lambda_0}_{n,\mathcal{O}}$-submodule of $S^\lambda_{\mathcal{O}}$.*

(ii) *If $\operatorname{char} K = 0$ then there is an isomorphism $S^\lambda_K/U^\lambda_K \cong D^\lambda_K$ of graded $R^{\Lambda_0}_{n,K}$-modules. In particular, $\operatorname{ch}_q D^\lambda_K = \operatorname{ch}_q \mathsf{DStd}_e(\lambda)$.*

(iii) *Set $\tilde{D}^\lambda_{\mathcal{O}} := S^\lambda_{\mathcal{O}}/U^\lambda_{\mathcal{O}}$. Let $\lambda = (2^x, 1^y)$ with $y \equiv -j-1 \pmod{e}$ for some $0 \leqslant j < e$. There is an isomorphism of graded $R^{\Lambda_0}_{n,\mathcal{O}}$-modules*

$$U^\lambda_{\mathcal{O}} \cong \begin{cases} \tilde{D}^\mu_{\mathcal{O}}\langle 1\rangle & \text{if } j \neq 0 \text{ and } x \geqslant j; \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu = (2^{x-j}, 1^{y+2j})$ and $\tilde{D}^\mu_{\mathcal{O}}\langle 1\rangle$ denotes $\tilde{D}^\mu_{\mathcal{O}}$ with the grading shifted by 1.

Remarkably, the aforementioned construction of $\mathsf{DStd}_e(\lambda)$ also leads to a combinatorial description of the q-character of D^λ_K when K has positive characteristic. Indeed, given a prime p and a 2-column partition λ, define

$$\mathsf{DStd}_{e,p}(\lambda) := \bigcap_{z \in \mathbb{Z}_{\geqslant 0}} \mathsf{DStd}_{ep^z}(\lambda).$$

Theorem 1.2. *If $\operatorname{char} K = p > 0$ then $\operatorname{ch}_q D^\lambda_K = \operatorname{ch}_q \mathsf{DStd}_{e,p}(\lambda)$ for every 2-column partition λ.*

Now suppose that $\operatorname{char} K = p \geqslant 0$ and let λ be a 2-column partition. James [15], [16] and Donkin [7] determined the ungraded composition multiplicities of S^λ_K. In particular, each D^μ_K appears as a composition factor of S^λ_K with multiplicity at most 1. Moreover, in the case when $p = e$ (i.e., that of symmetric groups), Erdmann [9] has given a more direct description of the dimensions of simple modules labelled by e-restricted 2-column partitions: she proved that these dimensions are coefficients in certain explicitly determined generating functions.

Theorem 4.10 extends the results of James and Donkin to give *graded* decomposition numbers. Thus, whenever D^μ_K appears as an (ungraded) composition factor of S^λ_K, there is an explicitly described integer $r_{e,p,\lambda,\mu} \in \{0,1\}$ such that $D^\mu_K\langle r_{e,p,\lambda,\mu}\rangle$ is a graded composition factor of S^λ_K. Combining this fact with Theorems 1.1(ii) and 1.2, we obtain the character identity

$$\operatorname{ch}_q S^\lambda = \sum_\mu q^{r_{e,p,\lambda,\mu}} \operatorname{ch}_q \mathsf{DStd}_{e,p}(\mu), \tag{1.4}$$

where the sum is over 2-column partitions μ such that D^μ_K is a composition factor of S^λ_K, and where we set $\mathsf{DStd}_{e,0}(\mu) := \mathsf{DStd}_e(\mu)$. In Section 4, for any 2-column partitions λ, μ, we identify an explicit subset $\mathsf{Std}_{e,p,\mu}(\lambda)$ of $\mathsf{Std}(\lambda)$, which may be seen to correspond to the composition factors D_K^μ in S_K^λ. More precisely, $\mathsf{Std}_{e,p,\mu}(\lambda) \neq \varnothing$ if and only if D_K^μ is a composition factor of S_K^λ, and if this is the case, then

$$\mathrm{ch}_q \,\mathsf{Std}_{e,p,\mu}(\lambda) = q^{r_{e,p,\lambda,\mu}} \,\mathrm{ch}_q \,\mathsf{DStd}_{e,p}(\mu), \tag{1.5}$$

see Theorem 4.22. The identity (1.5) is proved via an explicit bijection

$$\mathrm{reg}'_{e,p,\lambda,\mu} \colon \mathsf{Std}_{e,p,\mu}(\lambda) \xrightarrow{\sim} \mathsf{DStd}_{e,p}(\mu).$$

Furthermore, there is a decomposition

$$\mathsf{Std}(\lambda) = \bigsqcup_{\mu} \mathsf{Std}_{e,p,\mu}(\lambda),$$

which may be viewed as a combinatorial lifting of the identity (1.4).

The sets $\mathsf{Std}_{e,p,\mu}(\lambda)$ are defined in terms of a map $\mathrm{reg}_{e,p}$ from the set of 2-column standard tableaux to itself, which generalises the aforementioned regularisation map reg_e. In fact, graded decomposition numbers for 2-column partitions can also be described in terms of $\mathrm{reg}_{e,p}$, see Theorem 4.2. The simple module D_K^λ is self-dual, which implies that $\mathrm{ch}_q D_K^\lambda = \mathrm{ch}_q \,\mathsf{DStd}_{e,p}(\lambda)$ is invariant under the involution given by $q \mapsto q^{-1}$. A combinatorial proof of this fact is given in Remark 4.24.

The paper is organised as follows. In Section 2, we review cyclotomic KLR algebras, their Specht modules and the connection with representations of Hecke algebras. In Section 3, we associate a path in a weight space of type A_{k-1} with every standard tableau whose shape is a k-column partition (for $k \in \mathbb{Z}_{\geqslant 2}$) and describe the degrees of standard tableaux in the language of paths. We define the aforementioned regularisation map reg_e on $\mathsf{Std}(\lambda)$ and the set $\mathsf{DStd}_e(\lambda)$ when λ has at most 2 columns.

Section 4 is combinatorial: we prove Theorem 1.2 and the results outlined after the statement of that theorem. The order in which the results are proved is different from the one above. In particular, Theorem 1.2 is obtained as a consequence of the identities (1.4) and (1.5).

In Section 5, we consider homomorphisms between 2-column Specht modules. Using a row removal result from [10], we construct a homomorphism from $S_\mathcal{O}^\mu$ to $S_\mathcal{O}^\lambda$, where λ and μ are as in Theorem 1.1(iii), and we describe explicitly the kernel and image of this homomorphism, see Theorems 5.6 and 5.14. This leads to a proof of Theorem 1.1. We also construct exact sequences of homomorphisms between 2-column Specht modules, see Corollary 5.17.

Finally, in Section 6, we remove the condition that $\lambda_1 \leqslant 2$ and consider a natural approach to extending the definition of the set $\mathsf{DStd}_e(\lambda)$ to an arbitrary partition λ of n, based on the structure of $S_\mathbb{Q}^\lambda$ and its radical $\mathrm{rad}\, S_\mathbb{Q}^\lambda$, in the spirit of [14, §3.3]. We give an example showing that in some cases the resulting set $\mathsf{DStd}_e(\lambda)$ is 'too big', which yields a counterexample to a conjecture of Mathas [23].

Throughout, given $a, b \in \mathbb{Z}$, we write $[a, b] := \{c \in \mathbb{Z} \mid a \leqslant c \leqslant b\}$. If $b \geqslant 0$, we often abbreviate $a, \dots, a$ (with b entries) as a^b. If X is a collection of elements of an $\mathcal{O}$-module, we denote the $\mathcal{O}$-span of X by $\langle X \rangle_\mathcal{O}$. The $\mathbb{Z}$-rank of a free $\mathbb{Z}$-module U of finite rank is denoted by $\dim_\mathbb{Z} U$. If $1 \leqslant r < n$ are integers, we set $s_r := (r, r+1)$ to be the corresponding elementary transposition in the symmetric group $\mathfrak{S}_n$.

2. KLR algebras and Specht modules

Fix an integer $e \geqslant 2$ throughout the paper. We set $I = \mathbb{Z}/e\mathbb{Z} = \{0, 1, \dots, e-1\}$, abbreviating $i + e\mathbb{Z}$ as i (for $0 \leqslant i < e$) when there is no possibility of confusion. For any $n \in \mathbb{Z}_{\geqslant 0}$, we write $I^n = I \times \cdots \times I$. We define $\langle I^n \rangle$ to be the free $\mathbb{Z}[q, q^{-1}]$-module with basis I^n. The symmetric group $\mathfrak{S}_n$ acts on the left on I^n by place permutations. An element of I^n denoted by $\boldsymbol{i}$ is assumed to be equal to $(i_1, \dots, i_n)$; we adopt a similar convention for other bold symbols.

2.1. Graded algebras and modules

By a graded module (over any ring) we mean a $\mathbb{Z}$-graded one. If V is a graded module and $m \in \mathbb{Z}$, we denote the m-th graded component of V by V_m. The *graded dimension* of a finite-dimensional graded vector space V is $\dim_q V := \sum_{m \in \mathbb{Z}} (\dim V_m) q^m \in \mathbb{Z}[q, q^{-1}]$.

Let A be a graded $\mathcal{O}$-algebra, where $\mathcal{O}$ is a commutative ring. If $M = \bigoplus_{m \in \mathbb{Z}} M_m$ is a graded A-module then, for any $k \in \mathbb{Z}$, we write $M\langle k \rangle$ to denote the *graded shift* of M by k, which has the same structure as M as an A-module and grading given by $M\langle k \rangle_m = M_{m-k}$ for all $m \in \mathbb{Z}$. If M and N are graded A-modules, then $\operatorname{Hom}_A(M, N)$ denotes the $\mathcal{O}$-module of A-homomorphisms from M to N as ungraded modules. Moreover, if M is finitely generated as an A-module, then $\operatorname{Hom}_A(M, N)$ is graded by the following rule: given $\varphi \in \operatorname{Hom}_A(M, N)$ and $m \in \mathbb{Z}$, $\varphi \in \operatorname{Hom}_A(M, N)_m$ if and only if $\varphi(M_k) \subseteq N_{k+m}$ for all $k \in \mathbb{Z}$. If $\mathcal{O}$ is a field, then by a *composition factor* of a finite-dimensional A-module M we mean a composition factor of M as an ungraded module, unless we explicitly specify otherwise.

For every $f = f(q) \in \mathbb{Z}[q, q^{-1}]$, we write $\bar{f}(q) := f(q^{-1})$. This yields an involution

$$\bar{\ } : \langle I^n \rangle \to \langle I^n \rangle, \quad \sum_{\boldsymbol{i} \in I^n} f_{\boldsymbol{i}} \cdot \boldsymbol{i} \mapsto \sum_{\boldsymbol{i} \in I^n} \bar{f}_{\boldsymbol{i}} \cdot \boldsymbol{i}. \tag{2.1}$$

2.2. KLR algebras

Consider the quiver Γ that has vertex set I, an arrow $i \leftarrow i+1$ for each $i \in I$ and no other arrows. We write $i \to j$ and $j \leftarrow i$ if there is an arrow from i to j but not from j to i, and we write $i \leftrightarrows j$ if there are arrows between i and j in both directions (which only happens for $e = 2$). Further, we write $i \not- j$ if $i \neq j$ and there is no arrow between i and j in either direction. The quiver Γ corresponds to the Cartan matrix $C = (\mathsf{c}_{ij})_{i,j \in I}$ of the affine type $sfA^{(1)}_{e-1}$, given by

$$\mathsf{c}_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \to j \text{ or } j \to i; \\ -2 & \text{if } i \leftrightarrows j; \\ 0 & \text{if } i \not- j. \end{cases}$$

Let $\mathcal{O}$ be a commutative ring and let $n \in \mathbb{Z}_{\geqslant 0}$. The *KLR algebra* $R_n = R_{n,\mathcal{O}}$ is the $\mathcal{O}$-algebra generated by the elements

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^n\} \cup \{\psi_r \mid 1 \leqslant r < n\} \cup \{y_r \mid 1 \leqslant r \leqslant n\}$$

subject only to the following relations:

$$1_{\boldsymbol{i}}1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}}1_{\boldsymbol{i}}, \quad \sum_{\boldsymbol{i}\in I^n} 1_{\boldsymbol{i}} = 1, \tag{2.2}$$

$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r, \tag{2.3}$$

$$\psi_r 1_{\boldsymbol{i}} = 1_{s_r\boldsymbol{i}}\psi_r, \tag{2.4}$$

$$y_r y_s = y_s y_r, \tag{2.5}$$

$$\psi_r y_{r+1} 1_{\boldsymbol{i}} = (y_r\psi_r + \delta_{i_r,i_{r+1}})1_{\boldsymbol{i}}, \tag{2.6}$$

$$y_{r+1}\psi_r 1_{\boldsymbol{i}} = (\psi_r y_r + \delta_{i_r,i_{r+1}})1_{\boldsymbol{i}}, \tag{2.7}$$

$$\psi_r y_s = y_s\psi_r \qquad \text{if } s \neq r, r+1, \tag{2.8}$$

$$\psi_r\psi_s = \psi_s\psi_r \qquad \text{if } |r-s| > 1, \tag{2.9}$$

$$\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}; \\ 1_{\boldsymbol{i}} & \text{if } i_r \not- i_{r+1}; \\ (y_{r+1}-y_r)1_{\boldsymbol{i}} & \text{if } i_r \rightarrow i_{r+1}; \\ (y_r - y_{r+1})1_{\boldsymbol{i}} & \text{if } i_r \leftarrow i_{r+1}; \\ (y_{r+1}-y_r)(y_r-y_{r+1})1_{\boldsymbol{i}} & \text{if } i_r \leftrightarrows i_{r+1}, \end{cases} \tag{2.10}$$

$$\begin{aligned}(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})1_{\boldsymbol{i}} \\ = \begin{cases} 1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \rightarrow i_{r+1}; \\ -1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}; \\ (-2y_{r+1}+y_r+y_{r+2})1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \leftrightarrows i_{r+1}; \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{2.11}$$

for all $\boldsymbol{i} \in I^n$ and all admissible r, s (see [18], [24]).

Consider a root system with Cartan matrix C, with simple coroots $\{\beta_0^\vee, \dots, \beta_{e-1}^\vee\}$, see [17]. To each fundamental dominant weight Λ of this root system, one attaches a *cyclotomic* quotient R_n^Λ of R_n. In this paper, we will only consider the *cyclotomic KLR algebra* $R_n^{\Lambda_0}$, where Λ_0 is a (level 1) weight satisfying $\langle\Lambda_0, \beta_i^\vee\rangle = \delta_{i,0}$ for all $i \in I$. The algebra $R_n^{\Lambda_0} = R_{n,\mathcal{O}}^{\Lambda_0}$ is defined as the quotient of R_n by the 2-sided ideal that is generated by the set

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^n,\ i_1 \neq 0\} \cup \{y_1\}.$$

The algebras R_n and $R_n^{\Lambda_0}$ are both $\mathbb{Z}$-graded with

$$\deg(1_{\boldsymbol{i}}) = 0, \quad \deg(y_r) = 2, \quad \deg(\psi_r 1_{\boldsymbol{i}}) = -\mathsf{c}_{i_r i_{r+1}}$$

for all $\boldsymbol{i} \in I^n$ and all admissible r.

We fix a *reduced expression* for every $w \in \mathfrak{S}_n$, i.e., a decomposition $w = s_{r_1} \dots s_{r_m}$ as a product of elementary transpositions with m as small as possible. Define

$$\psi_w := \psi_{r_1} \dots \psi_{r_m} \in R_n, \tag{2.12}$$

noting that (in general) ψ_w depends on the choice of a reduced expression for w. By definition, the *length* of w is $\ell(w) := m$.

2.3. Partitions, tableaux and Specht modules

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers. As usual, we write $|\lambda| = \sum_j \lambda_j$ and say that λ is a partition of $n := |\lambda|$. The unique partition of 0 will be denoted by $\varnothing$. We always set $\lambda_r := 0$ for all $r > l$. We say that λ is *e-restricted* if $\lambda_r - \lambda_{r+1} < e$ for all $r \in \mathbb{Z}_{>0}$. We denote the set of partitions of n by $\mathsf{Par}(n)$ and the set of e-restricted partitions of n by $\mathsf{RPar}_e(n)$. Given $k \in \mathbb{Z}_{>0}$, define

$$\mathsf{Par}_{\leqslant k}(n) := \{\lambda \in \mathsf{Par}(n) \mid \lambda_1 \leqslant k\} \quad \text{and} \quad \mathsf{RPar}_{e,\leqslant k}(n) := \mathsf{RPar}_e(n) \cap \mathsf{Par}_{\leqslant k}(n).$$

The *dominance* partial order $\trianglelefteq$ on $\mathsf{Par}(n)$ is defined as follows: for any $\lambda, \mu \in \mathsf{Par}(n)$, we set $\mu \trianglelefteq \lambda$ if $\sum_{j=1}^r \mu_j \leqslant \sum_{j=1}^r \lambda_j$ for all $r \in \mathbb{Z}_{>0}$.

The Young diagram of λ is the subset $[\![\lambda]\!] = \{(a,b) \mid 1 \leqslant a \leqslant l,\ 1 \leqslant b \leqslant \lambda_a\}$ of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. When drawing diagrams, we represent a node (a,b) as the intersection of row a and column b, with the rows numbered from the top down and the columns from left to right.

A *standard tableau* of size $n \in \mathbb{Z}_{\geqslant 0}$ is an injective map $\mathtt{t}\colon \{1,\ldots,n\} \to \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that

(i) the image of $\mathtt{t}$ is the Young diagram of some partition λ of n; and
(ii) the entries of $\mathtt{t}$ are increasing along rows and down columns, i.e., whenever $(a,b),(c,d) \in [\![\lambda]\!]$ are such that $a \leqslant c$ and $b \leqslant d$, we have $\mathtt{t}^{-1}(a,b) \leqslant \mathtt{t}^{-1}(c,d)$.

In this situation, we refer to $\mathtt{t}$ as a standard tableau of *shape* λ and write $\lambda = \mathrm{Shape}(\mathtt{t})$. If $0 \leqslant m \leqslant n$, we denote by $\mathtt{t}{\downarrow_m}$ the restriction of $\mathtt{t}$ to $\{1,\ldots,m\}$. The set of all standard tableaux of shape λ is denoted by $\mathsf{Std}(\lambda)$. For any $k \in \mathbb{Z}_{>0}$, we set $\mathsf{Std}_{\leqslant k}(n) := \bigcup_{\lambda \in \mathsf{Par}_{\leqslant k}(n)} \mathsf{Std}(\lambda)$. If $\mathtt{t}, \mathtt{s} \in \mathsf{Std}(\lambda)$, we write $\mathtt{t} \trianglerighteq \mathtt{s}$ and say that $\mathtt{t}$ dominates $\mathtt{s}$ if $\mathrm{Shape}(\mathtt{t}{\downarrow_m}) \trianglerighteq \mathrm{Shape}(\mathtt{s}{\downarrow_m})$ for all $0 \leqslant m \leqslant n$. We define $\mathtt{t}^\lambda$ as the standard λ-tableau obtained by filling each row successively, going from the top down, so that $\mathtt{t}^\lambda(\lambda_1 + \cdots + \lambda_{a-1} + b) = (a,b)$ for all $(a,b) \in [\![\lambda]\!]$. Similarly, $\mathtt{t}_\lambda \in \mathsf{Std}(\lambda)$ is obtained by successively filling each column, going from left to right, so $\mathtt{t}_\lambda(\lambda'_1 + \cdots + \lambda'_{b-1} + a) = (a,b)$, where $\lambda'_j := \#\{r \in [1,l] \mid \lambda_r \geqslant j\}$ for all $j \in \mathbb{Z}_{>0}$.

The symmetric group $\mathfrak{S}_n$ acts on the set of all bijections $\mathtt{t}\colon \{1,\ldots,n\} \to [\![\lambda]\!]$ as follows: $(g\mathtt{t})(r) = \mathtt{t}(g^{-1}r)$ for all $g \in \mathfrak{S}_n$ and $1 \leqslant r \leqslant n$. For every $\mathtt{t} \in \mathsf{Std}(\lambda)$, let $d(\mathtt{t}) \in \mathfrak{S}_n$ be the unique element such that $d(\mathtt{t})\mathtt{t}^\lambda = \mathtt{t}$.

By a *column tableau* of size n we mean an injective map $\mathtt{t}\colon \{1,\ldots,n\} \to \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that, whenever $(a,b) \in \mathtt{t}(\{1,\ldots,n\})$ and $a > 1$, we have $(a-1,b) \in \mathtt{t}(\{1,\ldots,n\})$ and $\mathtt{t}^{-1}(a-1,b) < \mathtt{t}^{-1}(a,b)$. (That is, in particular, $\mathtt{t}$ is required to increase down columns.) For any $k \in \mathbb{Z}_{>0}$, we denote by $\mathsf{CT}_{\leqslant k}(n)$ the set of column tableaux $\mathtt{t}$ of size n such that the image of $\mathtt{t}$ is contained in $\mathbb{Z}_{>0} \times \{1,\ldots,k\}$ (i.e., the entries of $\mathtt{t}$ all belong to the first k columns). Note that $\mathsf{Std}_{\leqslant k}(n) \subseteq \mathsf{CT}_{\leqslant k}(n)$.

The *residue* of a node $(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is defined as $\mathrm{res}(a,b) = b - a + e\mathbb{Z} \in I$. We refer to a node of residue i as an *i-node*. The *residue sequence* of a column tableau $\mathtt{t}$ is

$$\boldsymbol{i}^{\mathtt{t}} := \big(\mathrm{res}\big(\mathtt{t}(1)\big), \ldots, \mathrm{res}\big(\mathtt{t}(n)\big)\big) \in I^n.$$

The set of all standard λ-tableaux with a given residue sequence $\boldsymbol{i} \in I^n$ is denoted by $\mathsf{Std}(\lambda, \boldsymbol{i})$.

For each standard tableau $\mathtt{t} \in \mathsf{Std}(\lambda)$, the *degree* $\deg_e(\mathtt{t})$ of $\mathtt{t}$ is defined in [5] as follows. A node $(a,b) \in \mathbb{Z}_{>0}$ is said to be *addable* for λ if $(a,b) \notin \lambda$ and $[\![\lambda]\!] \cup \{(a,b)\}$ is the Young diagram of a partition. We say that (a,b) is a *removable* node of λ if $(a,b) \in [\![\lambda]\!]$ and $[\![\lambda]\!] \setminus \{(a,b)\}$ is the Young diagram of a partition. A node (a,b) is said to be below a node (a',b') if $a > a'$. If (a,b) is a removable i-node of λ, define

$$\begin{aligned} d_{(a,b)}(\lambda) := \#\{&\text{addable } i\text{-nodes for } \lambda \text{ below } (a,b)\} \\ &- \#\{\text{removable } i\text{-nodes of } \lambda \text{ below } (a,b)\}. \end{aligned}$$

Finally, we define the degree of the unique $\varnothing$-tableau to be 0 and define recursively

$$\deg_e(\mathtt{t}) := d_{\mathtt{t}(n)}(\lambda) + \deg_e(\mathtt{t}{\downarrow_{n-1}})$$

for $\mathtt{t} \in \mathsf{Std}(\lambda)$.

If $\mathtt{t} \in \mathsf{Std}(\lambda)$ and $1 \leqslant r \leqslant s \leqslant n$, we write $r \rightarrow_{\mathtt{t}} s$ if $\mathtt{t}(r)$ and $\mathtt{t}(s)$ are in the same row of $[\![\lambda]\!]$. We also write $\boldsymbol{i}^\lambda := \boldsymbol{i}^{\mathtt{t}^\lambda}$.

Let $\mathcal{O}$ be a commutative ring. We refer the reader to [19, Section 5] for the definition of a *Garnir node* $A \in [\![\lambda]\!]$ and the corresponding *Garnir element* $g^A \in R_n = R_{n,\mathcal{O}}$. The *universal row Specht module* $S^\lambda = S^\lambda_{\mathcal{O}}$ is defined in [19] as the left R_n-module generated by a single generator v^λ subject only to the relations

$$1_{\boldsymbol{i}} v^\lambda = \delta_{\boldsymbol{i},\boldsymbol{i}^\lambda} v^\lambda, \tag{2.13}$$

$$\psi_r v^\lambda = 0 \qquad \text{if } r \rightarrow_{\mathtt{t}^\lambda} r+1, \tag{2.14}$$

$$y_r v^\lambda = 0, \tag{2.15}$$

$$g^A v^\lambda = 0 \qquad \text{for all Garnir nodes } A \text{ of } [\![\lambda]\!], \tag{2.16}$$

for all $\boldsymbol{i} \in I^n$ and all admissible $r \in \{1, \dots, n\}$. By [19, Cor. 6.26], the action of R_n on S^λ factors through $R_n^{\Lambda_0}$, so S^λ is naturally an $R_n^{\Lambda_0}$-module. For each $\mathtt{t} \in \mathsf{Std}(\lambda)$, we set

$$v^{\mathtt{t}} := \psi_{d(\mathtt{t})} v^\lambda,$$

noting that in general $v^{\mathtt{t}}$ depends on the choice of the reduced expression for $d(\mathtt{t})$ made in (2.12). In particular, $v^{\mathtt{t}^\lambda} = v^\lambda$.

Proposition 2.1 ([19, Prop. 5.14 and Cor. 6.24])**.** *Let $\lambda \in \mathsf{Par}(n)$. The Specht module S^λ is free as an $\mathcal{O}$-module, with basis $\{v^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{Std}(\lambda)\}$. Moreover, S^λ is a graded $R_n^{\Lambda_0}$-module, with each $v^{\mathtt{t}}$ homogeneous of degree* $\deg_e(\mathtt{t})$.

Corollary 2.2. *Let $\lambda \in \mathsf{Par}(n)$. The graded $R_{n,\mathbb{Z}}^{\Lambda_0}$-module $S^\lambda_{\mathbb{Z}}$ is isomorphic to the $\mathbb{Z}$-span of $\{v^{\mathtt{t}} \otimes 1 \mid \mathtt{t} \in \mathsf{Std}(\lambda)\}$ in the $R_{n,\mathbb{Z}}^{\Lambda_0}$-module $S^\lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.*

2.4. Hecke algebras at roots of unity, a cellular basis and simple modules

Let F be a field such that—setting $p := \operatorname{char} F$—we have that $p = 0$, $p = e$ or p is coprime to e. Let the field K be an extension of F, and assume that

$\xi \in K \setminus \{0\}$ has *quantum characteristic* e, i.e., e is the smallest positive integer such that $1 + \xi + \cdots + \xi^{e-1} = 0$.

The Iwahori–Hecke algebra $\mathcal{H}_n(\xi)$ is the K-algebra generated by

$$T_1, \dots, T_{n-1}$$

subject only to the relations

$$\begin{aligned}(T_r - \xi)(T_r + 1) &= 0, \\ T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1}, \\ T_r T_s &= T_s T_r \qquad\qquad \text{if } |r-s| > 1\end{aligned}$$

for all admissible r and s. The algebra $\mathcal{H}_n(\xi)$ is cellular, with cell modules $S^\lambda_{\mathcal{H}}$ parameterised by the partitions λ of n; see [22].

By the following fundamental results, much of the modular representation theory of Iwahori–Hecke algebras at roots of unity can be phrased in terms of questions about KLR algebras and their universal row Specht modules.

Theorem 2.3 ([3, Thm. 1.1], [19, Thm. 6.23])**.** *Let K be a field of characteristic p, and suppose that $\xi \in K \setminus \{0\}$ has* quantum characteristic e, *where either $p = e$ or p is coprime to e if $p \neq 0, e$. There is an algebra isomorphism $\theta\colon \mathcal{H}_n(\xi) \xrightarrow{\sim} R^{\Lambda_0}_{n,K}$ such that if $\mathcal{H}_n(\xi)$ is identified with $R^{\Lambda_0}_{n,K}$ via θ, then the $R^{\Lambda_0}_{n,K}$-module $S^\lambda_{\mathcal{H}}$ is isomorphic to S^λ_K.*

We remark that graded modules over $R^{\Lambda_0}_{n,K}$ which can be identified with the modules $S^\lambda_{\mathcal{H}}$ were originally constructed in [5] and that the proof of the identification of Specht modules in Theorem 2.3 uses results from [5].

Simple $\mathcal{H}_n(\xi)$-modules were classified in [6], and a corresponding classification of graded simple modules $R^{\Lambda_0}_{n,K}$-modules up to isomorphism and grading shift is given by [4, Thm. 4.11]. For our purposes, it is convenient to use the description of simple $R^{\Lambda_0}_{n,F}$-modules resulting from the graded cellular basis of $R^{\Lambda_0}_{n,F}$ constructed by Hu and Mathas [13], which we now review. Let $\lambda \in \mathsf{Par}(n)$ and define $\mathscr{Y}^\lambda := \{r \in [1,n] \mid \mathtt{t}^\lambda(r) \in \mathbb{Z} \times e\mathbb{Z}\}$. Set $y^\lambda := \prod_{r \in \mathscr{Y}^\lambda} y_r \in R^{\Lambda_0}_{n,F}$. There is an anti-automorphism $*$ of $R^{\Lambda_0}_{n,F}$ defined on the standard generators by

$$1_{\boldsymbol{i}}^* = 1_{\boldsymbol{i}}, \quad \psi_r^* = \psi_r, \quad y_r^* = y_r.$$

For $\mathtt{s}, \mathtt{t} \in \mathsf{Std}(\lambda)$, set

$$\psi_{\mathtt{st}} := \psi_{d(\mathtt{s})} 1_{\boldsymbol{i}^\lambda} y^\lambda \psi^*_{d(\mathtt{t})} \in R^{\Lambda_0}_{n,F}.$$

Then $\psi^*_{\mathtt{st}} = \psi_{\mathtt{ts}}$.

Theorem 2.4 ([13, Thm. 5.8])**.** *The algebra $R^{\Lambda_0}_{n,F}$ is a graded cellular algebra with weight poset $(\mathsf{Par}(n), \trianglerighteq)$ and cellular basis $\{\psi_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathsf{Std}(\lambda),\ \lambda \in \mathsf{Par}(n)\}$.*

Let $\lambda \in \mathsf{Par}(n)$. The cell module corresponding to λ in the graded cellular structure of Theorem 2.4 is isomorphic to S^λ_F as a graded $R^{\Lambda_0}_{n,F}$-module: this follows from [13, Cor. 5.10] and the proof of [19, Thm. 6.23]. In the sequel, we identify S_F^λ with the corresponding graded cell module. The cellular structure of Theorem 2.4 yields a symmetric bilinear form $\langle \cdot\,,\cdot\rangle$ on S_F^λ. This form is determined by the equation

$$\langle v^{\mathtt{t}}, v^{\mathtt{s}}\rangle v^{\mathtt{t}^\lambda} = \psi_{\mathtt{t}^\lambda \mathtt{t}} v^{\mathtt{s}} = 1_{\boldsymbol{i}^\lambda} y^\lambda \psi^*_{d(\mathtt{t})}\psi_{d(\mathtt{s})} v^{\mathtt{t}^\lambda} \quad (\mathtt{t},\mathtt{s} \in \mathsf{Std}(\lambda)), \tag{2.17}$$

cf. [23, (3.7.2)]. The form $\langle \cdot\,,\cdot\rangle$ is homogeneous of degree 0 and satisfies $\langle xu, v\rangle = \langle u, xv\rangle$ for all $x \in R^{\Lambda_0}_{n,F}$ and $u, v \in S_F^\lambda$, see [13, §2.2].

The radical of $\langle \cdot\,,\cdot\rangle$ is the $R^{\Lambda_0}_{n,F}$-submodule

$$\operatorname{rad} S_F^\lambda := \{u \in S_F^\lambda \mid \langle u, v\rangle = 0 \text{ for all } v \in S_F^\lambda\}.$$

Define

$$D_F^\lambda := S_F^\lambda / \operatorname{rad} S_F^\lambda.$$

If $D_F^\lambda \neq 0$, then D_F^λ is an irreducible graded $R^{\Lambda_0}_{n,F}$-module. Moreover, by [6] and [13, Cor. 5.11], $D_F^\lambda \neq 0$ if and only if λ is e-restricted. Hence, by [13, Thm. 2.10], we have:

Theorem 2.5. *The family* $\{D_F^\lambda \mid \lambda \in \mathsf{RPar}_e(n)\}$ *is a complete set of graded simple* $R^{\Lambda_0}_{n,F}$*-modules up to isomorphism and grading shift.*

Let M be a finite-dimensional graded $R^{\Lambda_0}_{n,F}$-module. Given $\mu \in \mathsf{RPar}_e(n)$, the *graded composition multiplicity* of D_F^μ in M is

$$[M : D_F^\mu]_q := \sum_{k\in\mathbb{Z}} a_k q^k \in \mathbb{Z}[q, q^{-1}],$$

where a_k is the multiplicity of $D_F^\mu\langle k\rangle$ in a graded composition series of M, see [4, §2.4]. Since the algebra $R^{\Lambda_0}_{n,F}$ is cellular, it is split, so

$$[S_F^\lambda : D_F^\mu]_q = [S_K^\lambda : D_K^\mu]_q \qquad (\lambda \in \mathsf{Par}(n),\ \mu \in \mathsf{RPar}_e(n)). \tag{2.18}$$

It is well known that $[S_F^\lambda : D_F^\mu]_q = 0$ unless $\mu \trianglelefteq \lambda$, see, e.g., [22, Cor. 2.17].

Remark 2.6. Li [20] proved that $R^{\Lambda_0}_{n,\mathcal{O}}$ is cellular for any commutative ring $\mathcal{O}$, generalising Theorem 2.4. Both of these results hold with Λ_0 replaced by an arbitrary dominant integral weight Λ, as do the aforementioned results from [3–5, 19].

3. Standard tableaux and paths in the weight space

In this section we fix an integer $k \geqslant 2$. In §3.1–3.2, we attach to each standard tableau $\mathtt{t}$ with at most k columns a path $\pi_{\mathtt{t}}$ in a weight space for the Lie algebra $\mathfrak{sl}_k$. We show that the degree of $\mathtt{t}$ can be non-recursively described in terms of interactions of $\pi_{\mathtt{t}}$ with certain hyperplanes in that weight space (Lemma 3.3) and that the residue sequence of a tableau is invariant under certain reflections of the corresponding path (Lemma 3.2). In §3.3, we specialise to the case $k = 2$, which is the only one used in the rest of the paper, and define a regularisation map on paths, which plays a key role in the sequel.

3.1. The affine Weyl group

Let $A = (a_{ij})_{1\leqslant i,j\leqslant k-1}$ be the Cartan matrix of finite type A_{k-1}, so that

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } j = i \pm 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let Φ be the root system of the Lie algebra $\mathfrak{sl}_k(\mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{sl}_k(\mathbb{C})$, with $\{\alpha_1, \dots, \alpha_{k-1}\}$ being a set of simple roots. We consider the (real) weight space $V := \mathbb{R}\Phi = \mathfrak{h}^*_{\mathbb{R}}$, where $\mathfrak{h}_{\mathbb{R}}$ is the set of matrices in $\mathfrak{h}$ with real entries. For each $i = 1, \dots, k$, let $\varepsilon_i \in V$ be the weight sending a diagonal matrix $\operatorname{diag}(t_1, \dots, t_k) \in \mathfrak{h}_{\mathbb{R}}$ to t_i. Then $\varepsilon_1 + \dots + \varepsilon_k = 0$, and we may assume that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, k-1$. The set of integral weights is the $\mathbb{Z}$-span $V_{\mathbb{Z}}$ of $\varepsilon_1, \dots, \varepsilon_k$. The set of positive roots is $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant k\}$. Let $(\cdot, \cdot)$ be the symmetric bilinear form V determined by $(\alpha_i, \alpha_j) = a_{ij}$ for $i, j \in \{1, \dots, k-1\}$. Note that

$$(\varepsilon_i, \varepsilon_r - \varepsilon_t) = \begin{cases} 1 & \text{if } i = r; \\ -1 & \text{if } i = t; \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for all $1 \leqslant i \leqslant k$ and $1 \leqslant r < t \leqslant k$. The Weyl group $W \cong \mathfrak{S}_k$ of Φ is the subgroup of $\mathrm{GL}(V)$ generated by the simple reflections s_α for $\alpha \in \Phi$, where $s_\alpha v = v - (\alpha, v)\alpha$ for all $v \in V$. The $\mathbb{Z}$-submodule $\mathbb{Z}\Phi \subset V$ is W-invariant, and the corresponding *affine Weyl group* is $W_{\mathrm{aff}} := W \ltimes \mathbb{Z}\Phi$. The group W also acts on the set $\{1, \dots, k\}$ in the natural way.

Let $\rho = \sum_{i=1}^{k-1} (k-i)\varepsilon_i$, so that $(\alpha_i, \rho) = 1$ for all $i = 1, \dots, k-1$. There is a faithful action of W_{aff} on V defined by

$$w \cdot v := w(v + \rho) - \rho, \qquad \beta \cdot v := v + e\beta \qquad (w \in W,\ \beta \in \mathbb{Z}\Phi,\ v \in V).$$

If $\alpha \in \Phi$ and $m \in \mathbb{Z}$, consider the hyperplane

$$H_{\alpha,m} := \{v \in V \mid (v + \rho, \alpha) = me\} \subset V.$$

We refer to $H_{\alpha,m}$ as an *α-wall* or simply a *wall*. For each $\alpha \in \Phi$ and $m \in \mathbb{Z}$ there exists (unique) $s_{\alpha,m} \in W_{\mathrm{aff}}$ such that $s_{\alpha,m}$ acts on V by reflection with respect to $H_{\alpha,m}$, i.e.,

$$s_{\alpha,m} \cdot v = v - \big((v + \rho, \alpha) - me\big)\alpha \tag{3.2}$$

for all $v \in V$. We consider the set

$$\mathcal{C} := \{v \in V \mid (v + \rho, \alpha_i) > 0 \text{ for } 1 \leqslant i < k\},$$

which may be viewed as the dominant chamber of the Coxeter complex corresponding to Φ. Using (3.1), we see that

$$\mathcal{C} = \{c_1\varepsilon_1 + \dots + c_k\varepsilon_k \in V \mid c_1, \dots, c_k \in \mathbb{R},\ c_j > c_{j+1} - 1 \text{ for all } j = 1, \dots, k-1\}. \tag{3.3}$$

3.2. Paths

Let $r, s \in \mathbb{Z}$. Recall that $[r,s] := \{t \in \mathbb{Z} \mid r \leqslant t \leqslant s\}$. We define $\mathsf{P}_{[r,s]}$ to be the set of all maps $\pi\colon [r,s] \to V_{\mathbb{Z}}$ such that $\pi(a+1) - \pi(a) \in \{\varepsilon_1, \dots, \varepsilon_k\}$ for all $a \in [r, s-1]$. Further, set

$$\mathsf{P}^+_{[r,s]} := \{\pi \in \mathsf{P}_{[r,s]} \mid \pi(a) \in \mathcal{C} \text{ for all } a \in [r,s]\}.$$

Let $n \in \mathbb{Z}_{\geqslant 0}$. We define

$$\mathsf{P}_n := \{\pi \in \mathsf{P}_{[0,n]} \mid \pi(0) = 0\} \quad \text{and} \quad \mathsf{P}^+_n := \mathsf{P}_n \cap \mathsf{P}^+_{[0,n]},$$

so that P^+_n is the set of all maps $\pi\colon [0,n] \to V_{\mathbb{Z}} \cap \mathcal{C}$ such that $\pi(0) = 0$ and $\pi(a+1) - \pi(a) \in \{\varepsilon_1, \dots, \varepsilon_k\}$ for all $a \in [0, n-1]$.

Given $\mathtt{t} \in \mathsf{CT}_{\leqslant k}(n)$, for any $0 \leqslant a \leqslant n$ and $1 \leqslant j \leqslant k$, set

$$c_{a,j}(\mathtt{t}) := |\mathtt{t}(\{1, \dots, a\}) \cap (\mathbb{Z}_{>0} \times \{j\})|,$$

i.e., $c_{a,j}(\mathtt{t})$ is the number of elements of $\mathtt{t}(\{1, \dots, a\})$ in the jth column. Define $\pi_{\mathtt{t}} \in \mathsf{P}_n$ by

$$\pi_{\mathtt{t}}(a) := c_{a,1}(\mathtt{t})\varepsilon_1 + \cdots + c_{a,k}(\mathtt{t})\varepsilon_k \qquad (a = 0, \dots, n).$$

Note that the end-point $\pi_{\mathtt{t}}(n)$ of the path $\pi_{\mathtt{t}}$ depends only on the image of $\mathtt{t}$ (i.e., only on the shape of $\mathtt{t}$ in the case when $\mathtt{t}$ is a standard tableau).

Lemma 3.1. *The assignment $\mathtt{t} \mapsto \pi_{\mathtt{t}}$ is a bijection from $\mathsf{CT}_{\leqslant k}(n)$ onto P_n and restricts to a bijection from $\mathsf{Std}_{\leqslant k}(n)$ onto P^+_n.*

Proof. The first assertion of the lemma is clear from the definitions. For the second assertion, let $\mathtt{t} \in \mathsf{CT}_{\leqslant k}(n)$ and observe that $\mathtt{t}$ is a standard tableau if and only if $c_{a,j}(\mathtt{t}) \geqslant c_{a,j+1}(\mathtt{t})$ for all $a = 1, \dots, n$ and all $j = 1, \dots, k-1$. The lemma follows by (3.3). $\square$

Let $\pi \in \mathsf{P}_n$ and suppose that $\pi(a) \in H_{\alpha,m}$ for some $\alpha \in \Phi^+$ and $m \in \mathbb{Z}$. We define the path $s^a_{\alpha,m} \cdot \pi \in \mathsf{P}_n$ by setting

$$(s^a_{\alpha,m} \cdot \pi)(b) := \begin{cases} \pi(b) & \text{for } 0 \leqslant b \leqslant a; \\ s_{\alpha,m} \cdot \pi(b) & \text{for } a < b \leqslant n. \end{cases}$$

That is, $s^a_{\alpha,m} \cdot \pi$ is obtained by reflecting a 'tail' of π with respect to $H_{\alpha,m}$.

The residue sequence $\boldsymbol{i}^{\mathtt{t}}$ of $\mathtt{t} \in \mathsf{CT}_{\leqslant k}(n)$ may be described as follows: if we have $1 \leqslant a \leqslant n$, then

$$i^{\mathtt{t}}_a = j - c_{a,j}(\mathtt{t}) + e\mathbb{Z}, \tag{3.4}$$

where $j \in \{1, \dots, k\}$ is determined by the condition that $\mathtt{t}(a)$ is in the jth column. The following lemma shows that reflecting a tail of a path as above does not change the residue sequence of the corresponding tableau.

Lemma 3.2. *Let $\alpha \in \Phi^+$, $m \in \mathbb{Z}$ and $a \in [0,n]$. Suppose that $\mathtt{t} \in \mathsf{CT}_{\leqslant k}(n)$ and $\pi_{\mathtt{t}}(a) \in H_{\alpha,m}$. If $\mathtt{s} \in \mathsf{CT}_{\leqslant k}(n)$ is determined by $\pi_{\mathtt{s}} = s^a_{\alpha,m} \cdot \pi_{\mathtt{t}}$, then $\boldsymbol{i}^{\mathtt{s}} = \boldsymbol{i}^{\mathtt{t}}$.*

Proof. Let $1 \leqslant r < t \leqslant k$ be such that $\alpha = \varepsilon_r - \varepsilon_t$. Since $\pi_{\mathtt{t}}(a) \in H_{\alpha,m}$, we have $c_{a,r}(\mathtt{t}) - c_{a,t}(\mathtt{t}) = r - t + me$. Clearly, $i^{\mathtt{s}}_b = i^{\mathtt{t}}_b$ for $1 \leqslant b \leqslant a$. Let $b \in [a+1, n]$. For any $j \in \{1, \ldots, k\}$, we have

$$c_{b,j}(\mathtt{s}) = \begin{cases} c_{b,j}(\mathtt{t}) & \text{if } j \notin \{r,t\}; \\ c_{b,t}(\mathtt{t}) - c_{a,t}(\mathtt{t}) + c_{a,r}(\mathtt{t}) = c_{b,t}(\mathtt{t}) + r - t + me & \text{if } j = r; \\ c_{b,r}(\mathtt{t}) - c_{a,r}(\mathtt{t}) + c_{a,t}(\mathtt{t}) = c_{b,r}(\mathtt{t}) + t - r - me & \text{if } j = t. \end{cases}$$

Now, if $j \in \{1, \ldots, k\}$ is determined by the condition that $\mathtt{t}(b)$ is in the jth column, then $\mathtt{s}(b)$ is in the $(s_\alpha j)$th column, and it follows using (3.4) that $i^{\mathtt{s}}_b = i^{\mathtt{t}}_b$ in all cases. □

Let $u, v \in V_{\mathbb{Z}} \cap \mathcal{C}$ be such that $v - u = \varepsilon_i$ for some $i \in \{1, \ldots, k\}$. For every $\alpha \in \Phi^+$, set

$$\deg_{e,\alpha}(u,v) := \begin{cases} 1 & \text{if } u \in H_{\alpha,m} \text{ and } (v+\rho,\alpha) < me \text{ for some } m \in \mathbb{Z}_{>0}; \\ -1 & \text{if } v \in H_{\alpha,m} \text{ and } (u+\rho,\alpha) > me \text{ for some } m \in \mathbb{Z}_{>0}; \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Write $\deg_e(u,v) := \sum_{\alpha\in\Phi^+} \deg_{e,\alpha}(u,v)$, and for every $\pi \in \mathsf{P}^+_{[r,s]}$ define

$$\deg_e(\pi) := \sum_{a=r}^{s-1} \deg_e\big(\pi(a), \pi(a+1)\big). \tag{3.6}$$

Lemma 3.3. *For every $\mathtt{t} \in \mathsf{Std}_{\leqslant k}(n)$, we have $\deg_e(\pi_{\mathtt{t}}) = \deg_e(\mathtt{t})$.*

Proof. Let λ be the shape of $\mathtt{t}$. Arguing by induction on n, we see that it is enough to show that $d_{\mathtt{t}(n)}(\lambda) = \deg_e(\pi_{\mathtt{t}}(n-1), \pi_{\mathtt{t}}(n))$ when $n > 0$. For $j = 1, \ldots, k$, let c_j be the size of the jth column of $[\![\lambda]\!]$, i.e., $c_j = c_{n,j}(\mathtt{t})$. Let t be such that $\mathtt{t}(n) = (c_t, t)$. Then $i := t - c_t + e\mathbb{Z} \in I$ is the residue of $\mathtt{t}(n)$. It is easy to see using the definitions and (3.4) that

$$d_{\mathtt{t}(n)}(\lambda) = \#\{j \in [1, t-1] \mid j - c_j + e\mathbb{Z} = i+1\} - \#\{j \in [1,t-1] \mid j - c_j + e\mathbb{Z} = i\}.$$

Since $\pi_{\mathtt{t}}(n) = \pi_{\mathtt{t}}(n-1) + \varepsilon_t$, we have $\deg_{e,\alpha}(\pi_{\mathtt{t}}(n-1), \pi_{\mathtt{t}}(n)) = 0$ for all $\alpha \in \Phi^+$ such that $\alpha \neq \varepsilon_j - \varepsilon_t$ for any $j = 1, \ldots, t-1$. Moreover, using (3.1) we see that for each $j = 1, \ldots, t-1$,

$$\deg_{e,\,\varepsilon_j - \varepsilon_t}(\pi_{\mathtt{t}}(n-1), \pi_{\mathtt{t}}(n)) = \begin{cases} 1 & \text{if } c_j - c_t + t - j \equiv -1 \bmod e; \\ -1 & \text{if } c_j - c_t + t - j \equiv 0 \bmod e; \\ 0 & \text{otherwise.} \end{cases}$$

Since $t - c_t + e\mathbb{Z} = i$, we deduce that $d_{\mathtt{t}(n)}(\lambda) = \deg_e(\pi_{\mathtt{t}}(n-1), \pi_{\mathtt{t}}(n))$, as required. □

Remark 3.4. The correspondence between standard tableaux and paths, as above, is considered in [12, Sec. 5]. The degree function (3.6) is similar to the one defined in [2, Def. 1.4] in a somewhat different context. Also, consider a path $\pi \in P_n^+$ such that $\pi(n)$ does not belong to any wall $H_{\alpha,m}$ and $\pi(a) \notin H_{\alpha,m} \cap H_{\alpha',m'}$ for any distinct walls $H_{\alpha,m}$ and $H_{\alpha',m'}$ whenever $0 \leqslant a < n$. Then one can associate with π a *Bruhat stroll* as defined in [8, §2.4], and $\deg_e(\pi)$ is precisely the *defect* of the corresponding Bruhat stroll.

3.3. Two-column tableaux

In this subsection, we assume that $k = 2$, and we again fix $n \in \mathbb{Z}_{\geqslant 0}$. Then $\Phi^+ = \{\alpha_1\}$, and we write $\alpha = \alpha_1$. We identify V with $\mathbb{R}$ by sending ε_1 to 1 and ε_2 to -1. Then $\rho = 1$, and a wall $H_{\alpha,m}$ is the singleton set $\{me - 1\}$. Furthermore, $\mathcal{C} = \mathbb{R}_{>(-1)}$ and

$$\mathsf{P}_n^+ = \{\pi\colon [0,n] \to \mathbb{Z}_{\geqslant 0} \mid \pi(0) = 0 \text{ and } \pi(a+1) = \pi(a) \pm 1 \text{ for all } 0 \leqslant a < n\}.$$

We write s_m, s_m^a, H_m instead of $s_{\alpha,m}, s_{\alpha,m}^a, H_{\alpha,m}$ respectively.

Let
$$H = \bigcup_{m \in \mathbb{Z}_{>0}} H_m = \{me - 1 \mid m \in \mathbb{Z}_{>0}\}. \tag{3.7}$$

We define a map $\operatorname{reg}_e\colon \mathsf{P}_n^+ \to \mathsf{P}_n^+$ as follows. Given $\pi \in \mathsf{P}_n^+$, we set $\operatorname{reg}_e(\pi) = \pi$ if $\pi(a) \notin H$ for all $a \in \{0, \ldots, n\}$. Otherwise, let $a \in \{0, \ldots, n\}$ be maximal such that $\pi(a) \in H$ and, if $m \in \mathbb{Z}_{>0}$ is given by the condition that $\pi(a) \in H_m$, define

$$\operatorname{reg}_e(\pi) := \begin{cases} \pi & \text{if } \pi(n) \geqslant me - 1; \\ s_m^a \cdot \pi & \text{if } \pi(n) < me - 1. \end{cases}$$

Less formally, we consider the last point at which the path π meets a wall H_m (if such a point exists) and, if this point is greater than the endpoint of π, we get $\operatorname{reg}_e(\pi)$ by reflecting the corresponding 'tail' of π with respect to H_m. Further, we set
$$r_e(\pi) := \begin{cases} 0 & \text{if } \operatorname{reg}_e(\pi) = \pi; \\ 1 & \text{if } \operatorname{reg}_e(\pi) \neq \pi. \end{cases}$$

The following is clear from the definitions:

Lemma 3.5. *We have* $\deg_e(\operatorname{reg}_e(\pi)) = \deg_e(\pi) - r_e(\pi)$ *for all* $\pi \in \mathsf{P}_n^+$.

By Lemma 3.1, there is a well-defined map $\operatorname{reg}_e\colon \mathsf{Std}_{\leqslant 2}(n) \to \mathsf{Std}_{\leqslant 2}(n)$ determined by the condition that $\pi_{\operatorname{reg}_e(\mathtt{t})} = \operatorname{reg}_e(\pi_{\mathtt{t}})$ for all $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(n)$. We also have a map $r_e\colon \mathsf{Std}_{\leqslant 2}(n) \to \{0,1\}$ defined by $r_e(\mathtt{t}) := r_e(\pi_{\mathtt{t}})$.

For any $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$, set

$$\mathsf{DStd}_e(\lambda) := \{\mathtt{t} \in \mathsf{Std}(\lambda) \mid \operatorname{reg}_e(\mathtt{t}) = \mathtt{t}\}.$$

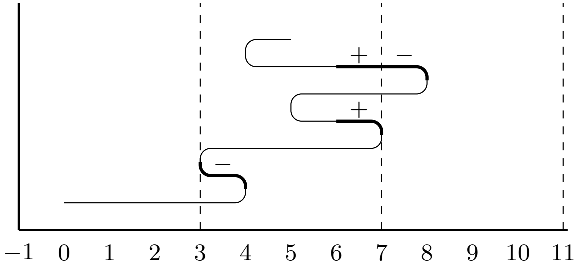
We refer to the elements of $\mathsf{DStd}_e(\lambda)$ as *e-regular* standard tableaux. The following is easily seen:

Lemma 3.6. *Let $\mu = (2^x, 1^y) \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mathtt{s} \in \mathsf{Std}(\mu)$. Write $y = me - 1 + j$, where $m \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant j < e$. If $y \geqslant e - 1$, let $a \in \{0, \dots, n\}$ be maximal such that $\pi_{\mathtt{s}}(a) = me - 1$, and let $\mathtt{t}$ be the standard tableau determined by the condition that $\pi_{\mathtt{t}} = s_m^a \cdot \pi_{\mathtt{s}}$. Then*

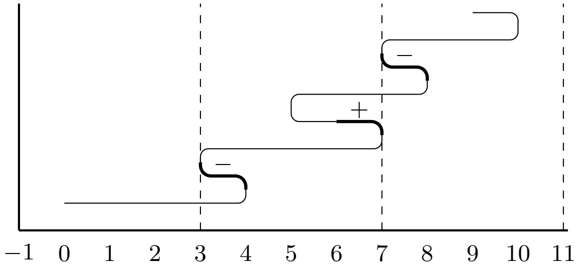
$$\mathrm{reg}_e^{-1}(\mathtt{s}) = \begin{cases} \varnothing & \text{if } \mathtt{s} \notin \mathsf{DStd}_e(\mu); \\ \{\mathtt{s}\} & \text{if } \mathtt{s} \in \mathsf{DStd}_e(\mu) \text{ and either } y < e \text{ or } j = 0; \\ \{\mathtt{s}, \mathtt{t}\} & \text{if } \mathtt{s} \in \mathsf{DStd}_e(\mu),\ y \geqslant e \text{ and } j > 0. \end{cases}$$

Moreover, if $y \geqslant e - 1$ then $\mathrm{Shape}(\mathtt{t}) = (2^{x+j}, 1^{me-1-j})$.

Example 3.7. Let $e = 4$. A path $\pi \in \mathsf{P}^+_{19}$ and the path $\mathrm{reg}_4(\pi)$ are depicted below on the left and right, respectively. The weight space $V = \mathbb{R}$ is identified with a horizontal line, but—for presentation purposes—the height at which $\pi(a)$ is drawn gradually increases as $a = 0, \dots, n$ increases. (We use this convention throughout the paper.) The vertical lines indicate the walls $H_m = \{4m - 1\}$, $m \in \mathbb{Z}_{\geqslant 0}$. In each picture, the steps from $\pi(a)$ to $\pi(a+1)$ for which $\deg_4(\pi(a), \pi(a+1))$ is 1 or -1 are marked by $+$ or $-$, respectively; these steps are highlighted for clarity. The degree of any unmarked step is 0.



$\deg_4(\pi) = 0,\ r_4(\pi) = 1$



$\deg_4(\mathrm{reg}_4(\pi)) = -1,\ r_4(\mathrm{reg}_4(\pi)) = 0$

The standard tableau $\mathtt{t}$ such that $\pi = \pi_{\mathtt{t}}$ and the tableau $\mathrm{reg}_4(\mathtt{t})$ are the transposes of

1	2	3	4	6	7	8	9	12	13	14	19
5	10	11	15	16	17	18					

and

1	2	3	4	6	7	8	9	12	13	14	16	17	18
5	10	11	15	19									

respectively.

4. Characters and graded decomposition numbers for 2-column partitions

Let F be a field of characteristic $p \geqslant 0$. We assume that $p = 0$, $p = e$ or p is coprime to e, cf. §2.4. We fix $n \in \mathbb{Z}_{\geqslant 0}$ and use the notation of §3.3 throughout the section. In particular, P_n^+ is a set of paths in a weight space of type A_1, and there is a bijection $\mathsf{Std}_{\leqslant 2}(n) \xrightarrow{\sim} \mathsf{P}_n^+$ given by $\mathtt{t} \mapsto \pi_{\mathtt{t}}$, see Lemma 3.1.

4.1. (e, p)-regularisation

We now define the (e, p)-*regularisation map* $\mathrm{reg}_{e,p} \colon \mathsf{P}_n^+ \to \mathsf{P}_n^+$, which is needed to state the main results of this section. If $p = 0$, then set $\mathrm{reg}_{e,p} := \mathrm{reg}_e$. If $p > 0$, then $\mathrm{reg}_{e,p}$ is defined recursively, as follows. For all $\pi \in \mathsf{P}_n^+$:

(1) If $\mathrm{reg}_{ep^z}(\pi) = \pi$ for all $z \in \mathbb{Z}_{\geqslant 0}$, then set $\mathrm{reg}_{e,p}(\pi) := \pi$.
(2) Otherwise, $\mathrm{reg}_{e,p}(\pi) := \mathrm{reg}_{e,p}(\mathrm{reg}_{ep^z}(\pi))$, where z is the largest non-negative integer such that $\mathrm{reg}_{ep^z}(\pi) \neq \pi$.

Note that the recursion always terminates because any map reg_m either fixes a path or increases its end-point. We also have a map

$$\mathrm{reg}_{e,p} \colon \mathsf{Std}_{\leqslant 2}(n) \to \mathsf{Std}_{\leqslant 2}(n)$$

determined by the identity $\mathrm{reg}_{e,p}(\pi_{\mathtt{t}}) = \pi_{\mathrm{reg}_{e,p}(\mathtt{t})}$ for all $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(n)$.

Given $\pi \in \mathsf{P}_n^+$, we have

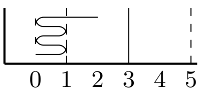
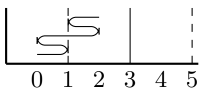
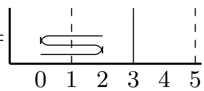
$$\mathrm{reg}_{e,p}(\pi) = \mathrm{reg}_{ep^{z_h}}(\dots \mathrm{reg}_{ep^{z_2}}(\mathrm{reg}_{ep^{z_1}}(\pi))\dots) \tag{4.1}$$

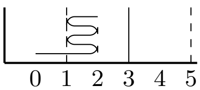
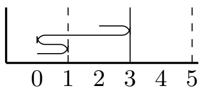
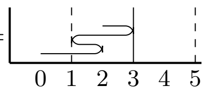
where, for each $r = 1, \dots, h$, the integer $z_r \geqslant 0$ is maximal such that the path $\mathrm{reg}_{ep^{z_{r-1}}}(\dots \mathrm{reg}_{ep^{z_1}}(\pi)\dots)$ is not an ep^{z_r}-regular path. Note that $z_1 > \cdots > z_h$. When $p = 0$, we use the convention that $ep^0 = e$; in this case, $0 \leqslant h \leqslant 1$. We refer to (4.1) as the *regularisation equation* and to $Z = \{z_1, \dots, z_h\}$ as the *regularisation set* of π. If $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(n)$, then the regularisation set of $\mathtt{t}$ is defined to be that of $\pi_{\mathtt{t}}$, and the regularisation equation of $\mathtt{t}$ is also defined to be that of $\pi_{\mathtt{t}}$, with $\pi_{\mathtt{t}}$ replaced by $\mathtt{t}$ on both sides.

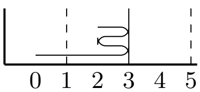
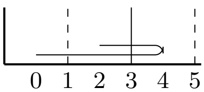
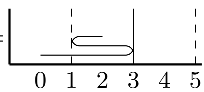
For any $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$, we set $\mathsf{DStd}_{e,0}(\lambda) := \mathsf{DStd}_e(\lambda)$ and, if $p > 0$,

$$\mathsf{DStd}_{e,p}(\lambda) := \bigcap_{z \geqslant 0} \mathsf{DStd}_{ep^z}(\lambda) = \{\mathtt{t} \in \mathsf{Std}(\lambda) \mid \mathrm{reg}_{e,p}(\mathtt{t}) = \mathtt{t}\}.$$

Example 4.1. Suppose that $e = p = 2$ and $\lambda = (2, 2, 1, 1)$. Then $\mathsf{Std}(\lambda) = \{\mathtt{t}_1, \dots, \mathtt{t}_9\}$, where

$\pi_{\mathtt{t}_1} =$  , $\pi_{\mathtt{t}_2} =$  , $\pi_{\mathtt{t}_3} =$  ,

$\pi_{\mathtt{t}_4} =$  , $\pi_{\mathtt{t}_5} =$  , $\pi_{\mathtt{t}_6} =$  ,

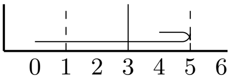
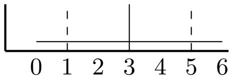
$\pi_{\mathtt{t}_7} =$  , $\pi_{\mathtt{t}_8} =$  , $\pi_{\mathtt{t}_9} =$  .

Here and in the sequel, the thicker the wall $H_m = \{em - 1\}$ is in the diagrams, the greater the p-adic valuation of m; if this valuation is 0, the wall is dashed.

We have $\mathsf{DStd}_{2,2}(\lambda) = \{\mathtt{t}_1, \mathtt{t}_2, \mathtt{t}_3, \mathtt{t}_4\}$. So Theorem 1.2 implies, in particular, that $\dim D^\lambda = 4$. The regularisation equation of $\pi_{\mathtt{t}_9}$ is

$$\mathrm{reg}_{2,2}(\pi_{\mathtt{t}_9}) = \mathrm{reg}_2(\mathrm{reg}_4(\pi_{\mathtt{t}_9})),$$

with

$\mathrm{reg}_4(\pi_{\mathtt{t}_9}) =$  and $\mathrm{reg}_{2,2}(\pi_{\mathtt{t}_9}) =$  .

Finally, the tableaux $\mathtt{t}_5, \mathtt{t}_6, \mathtt{t}_7, \mathtt{t}_8$ all have regularisation set $\{0\}$, and the images of these tableaux under $\mathrm{reg}_{2,2}$ have shape $(2, 1^4)$.

One of the main results of §4 is the following theorem, which gives a combinatorial description of the graded decomposition numbers $[S^\lambda : D^\mu]_q$ (when $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$) in terms of the map $\mathrm{reg}_{e,p}$.

Theorem 4.2. *Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$, $\mu \in \mathsf{RPar}_{\leqslant 2}(n)$, and suppose that either $p = e$ or p is coprime to e if $p \neq 0, e$. If $\mathtt{s} \in \mathsf{DStd}_{e,p}(\mu)$, then*

$$[S^\lambda_F : D^\mu_F]_q = \sum_{\substack{\mathtt{t} \in \mathsf{Std}(\lambda) \\ \mathrm{reg}_{e,p}(\mathtt{t}) = \mathtt{s}}} q^{r_e(\mathtt{t})}.$$

In particular, the right-hand side does not depend on the choice of $\mathtt{s}$.

As we will see, the sum on the right-hand side always contains at most one non-zero term.

In §4.2, we give a description of graded decomposition numbers for 2-column partitions that does not use the map $\mathrm{reg}_{e,p}$ and refines a known result on ungraded decomposition numbers, see Theorem 4.10. In §4.3, we use this description to prove Theorem 4.2.

Given $\lambda, \mu \in \mathsf{Std}_{\leqslant 2}(n)$, define the set

$$\mathsf{Std}_{e,p,\mu}(\lambda) := \{\mathtt{t} \in \mathsf{Std}(\lambda) \mid \mathrm{reg}_{e,p}(\mathtt{t}) \text{ has shape } \mu\}. \tag{4.2}$$

In §4.4 we show that the subsets $\mathsf{Std}_{e,p,\mu}(\lambda)$ satisfy the properties stated in §1 and use these properties to prove Theorem 1.2.

4.2. Decomposition numbers

Set

$$D^p(q) := ([S^\lambda_F : D^\mu_F]_q)_{\lambda \in \mathsf{Par}_{\leqslant 2}(n),\ \mu \in \mathsf{RPar}_{e,\leqslant 2}(n)},$$

so that $D^p(q)$ is the submatrix of the graded decomposition matrix of $R^{\Lambda_0}_{n,F}$ corresponding to partitions in $\mathsf{Par}_{\leqslant 2}(n)$. Let $D^p = D^p(q)|_{q=1}$ denote the corresponding *ungraded* submatrix of the decomposition matrix.

In this subsection, we prove a formula for the entries of $D^p(q)$. Recall that for $p > 0$, there is a unique square matrix A^p—known as an adjustment matrix—such that $D^p = D^0 A^p$. Similarly, there is a unique graded adjustment matrix $A^p(q)$ such that $D^p(q) = D^0(q) A^p(q)$, see [22, Thm. 6.35], [4, Thm. 5.17]. Write $A^p(q) = (a_{\lambda\mu}(q))_{\lambda,\mu \in \mathsf{RPar}_{e,\leqslant 2}(n)}$. The following is a special case of [4, Thm. 5.17]:

Theorem 4.3. *For every $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu \in \mathsf{RPar}_{e,\leqslant 2}(n)$, we have $\bar{a}_{\lambda\mu}(q) = a_{\lambda\mu}(q)$.*

Our strategy is as follows. The ungraded decomposition matrices D^p for all $p > 0$ are known by the work of James [15,16] and Donkin [7]; the graded decomposition matrices $D^0(q)$—and hence the ungraded decomposition matrices D^0—are given in [21]. As we see below, all entries in the former matrices are either 0 or 1, so it follows by Theorem 4.3 that $A^p(q) = A^p$, i.e., every entry of $A^p(q)$ is a constant (Laurent) polynomial. Hence we are able to compute the matrix $D^p(q) = D^0(q)A^p(q)$.

We begin with some notation. Given $a, b \in \mathbb{Z}_{\geqslant 0}$ and assuming that $p > 0$, we say that a *contains* b *to base* p and write $b \preceq_p a$ if, writing

$$a = a_0 + a_1 p + a_2 p^2 + \cdots \quad \text{and} \quad b = b_0 + b_1 p + b_2 p^2 + \cdots$$

(with $0 \leqslant a_i, b_i < p$ for all i), for each $i \geqslant 0$ either $b_i = 0$ or $b_i = a_i$. We write $b \preceq_0 a$ whenever $a \geqslant 0$ and $b = 0$.

For any integer $s \geqslant 0$, let s_e denote the integer part of s/e. For $p \geqslant 0$ and non-negative integers s and l, define

$$f_{e,p}(l,s) := \begin{cases} 1 & \text{if } s_e \preceq_p (l+1)_e, \text{ and either } e \mid s \text{ or } e \mid l+1-s; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

We also define a graded version of these numbers:

$$f^q_{e,p}(l,s) := \begin{cases} 1 & \text{if } s_e \preceq_p (l+1)_e \text{ and } e \mid s; \\ q & \text{if } s_e \preceq_p (l+1)_e,\ e \mid l+1-s \text{ and } e \nmid s; \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Note that $f^q_{e,p}(l,s)|_{q=1} = f_{e,p}(l,s)$ and $f^q_{e,p}(l,0) = 1$. The ungraded decomposition numbers when $p > 0$ are given by the following theorem. This theorem was proved by James in the case when $p = e$ (see [15, Thm. 24.15]) and—under certain additional conditions—when $p \neq e$ (see [16, Thm. 20.6]). The result for $p \neq e$ in full generality follows from a theorem of Donkin [7, Thm. 4.4(6)] together with (2.18). We follow the statement given by Mathas in [22, p. 127].

Theorem 4.4. *Assume that $p > 0$ and either $p = e$ or p is coprime to e. Suppose that $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$, with $u \geqslant x$. Then*

$$[S^\lambda_F : D^\mu_F] = f_{e,p}(y, u-x).$$

Our aim is to prove that $[S^\lambda_F : D^\lambda_F]_q = f^q_{e,p}(y, u-x)$ under the hypotheses of Theorem 4.4. We do so with the aid of alternative descriptions of $f^q_{e,p}(y, u-x)$, which are given by Lemmas 4.6 and 4.7 (for $p > 0$ and $p = 0$ respectively) and are used in the rest of the paper. First, we note the following elementary fact:

Lemma 4.5. *Assume that* $p > 0$. *Let* $a_0, \dots, a_s, l \in \mathbb{Z}_{\geqslant 0}$ *for some* $s \in \mathbb{Z}_{\geqslant 0}$ *satisfy* $0 \leqslant a_i < p$ *for* $i = 0, \dots, s$ *and* $0 \leqslant l < e$. *If* $\delta_0, \dots, \delta_{s+1}, \delta'_0, \dots, \delta'_{s+1} \in \{0, 1\}$ *and*

$$\Big(\sum_{i=0}^{s}(-1)^{\delta_{i+1}} a_i p^i\Big)e + (-1)^{\delta_0} l = \Big(\sum_{i=0}^{s}(-1)^{\delta'_{i+1}} a_i p^i\Big)e + (-1)^{\delta'_0} l, \tag{4.5}$$

then

(i) $\delta_{i+1} = \delta'_{i+1}$ *whenever* $0 \leqslant i \leqslant s$ *and* $a_i \neq 0$;
(ii) $\delta_0 = \delta'_0$ *if* $l \neq 0$.

Proof. Assume that (i) is false, and let i be the largest index such that $\delta_{i+1} \neq \delta'_{i+1}$ and $a_i > 0$. Without loss of generality, $\delta_{i+1} = 0$ and $\delta'_{i+1} = 1$, and hence the difference between the left-hand and the right-hand sides of (4.5) is at least

$$2a_i p^i e - \sum_{j=0}^{i-1} 2a_j p^j e - 2l \geqslant 2ep^i - 2e(p-1)\sum_{j=0}^{i-1} p^j - 2(e-1) > 0.$$

This contradiction proves (i), and (ii) follows immediately. $\square$

Definition. Let $y \in \mathbb{Z}_{\geqslant 0}$. If $p > 0$, then we define the (e, p)-*expansion* of y to be the expression

$$y = (a_s p^s + a_{s-1} p^{s-1} + \dots + a_1 p + a_0)e + l - 1,$$

where $0 \leqslant l < e$, $0 \leqslant a_i < p$ for all i, and if $y \geqslant e - 1$ then $a_s > 0$ (cf. [7, §3.4]).

Lemma 4.6. *Assume that* $p > 0$. *Let* $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$ *and* $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$. *Suppose that*

$$y = (a_s p^s + a_{s-1} p^{s-1} + \dots + a_1 p + a_0)e + l - 1$$

is the (e, p)-*expansion of* y. *Then* $f^q_{e,p}(y, u - x) \neq 0$ *if and only if* v *is of the form*

$$v = (a_s p^s \pm a_{s-1} p^{s-1} \pm \dots \pm a_1 p \pm a_0)e \pm l - 1$$

for some choice of signs. Moreover, in this case

$$f^q_{e,p}(y, u-x) = \begin{cases} 1 & \text{if } l = 0; \\ 1 & \text{if } 0 < l < e \text{ and } v = (a_s p^s \pm a_{s-1} p^{s-1} \pm \dots \pm a_0)e + l - 1; \\ q & \text{if } 0 < l < e \text{ and } v = (a_s p^s \pm a_{s-1} p^{s-1} \pm \dots \pm a_0)e - l - 1. \end{cases} \tag{4.6}$$

We note that the last two cases in (4.6) cannot occur simultaneously by Lemma 4.5.

Proof. We may assume that $u \geqslant x$. If $y < e$, then $f^q_{e,p}(y, u-x) = 0$ unless $u = x$. So we may assume that $y \geqslant e$. Writing

$$u - x = (b_s p^s + b_{s-1} p^{s-1} + \cdots + b_1 p + b_0)e + k,$$

where $0 \leqslant k < e$ and $0 \leqslant b_i < p$ for all i, we have

$$v = y - 2(u-x) = \big((a_s - 2b_s)p^s + (a_{s-1} - 2b_{s-1})p^{s-1} + \cdots + (a_0 - 2b_0)\big)e + (l - 2k) - 1.$$

Moreover,

(1) $(u-x)_e \preceq_p (y+1)_e$ if and only if $b_s = 0$ and $b_i \in \{0, a_i\}$ for all $0 \leqslant i < s$;
(2) $e \mid u - x$ if and only if $k = 0$;
(3) $e \mid y + 1 - u + x$ if and only if $k = l$.

The result now follows from (4.4). □

Lemma 4.7. *Let $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$, with $u \geqslant x$.*

(i) *We have*

$$f^q_{e,0}(y, u-x) = \begin{cases} 1 & \text{if } u - x = 0; \\ q & \text{if } 0 < u - x < e \leqslant y + 1 \text{ and } e \mid y + 1 - u + x; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *The equality $f^q_{e,0}(y, u-x) = q$ holds if and only if there exists an integer $m \geqslant 1$ such that $y = me + j - 1$ and $v = me - j - 1$ for some $1 \leqslant j < e$.*

Proof. Part (i) follows from (4.4), and (ii) follows easily from (i) because $e \mid y + 1 - u + x$ if and only if $2e \mid y + v + 2$. □

If $p = 0$, the 2-column graded decomposition numbers are given in [21, Thm. 3.1], if that result is interpreted in view of [4, Cor. 5.15]:

Theorem 4.8. *Suppose that $p = 0$. If $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$ are such that $u \geqslant x$ then*

$$[S^\lambda : D^\mu]_q = f^q_{e,0}(y, u-x).$$

The notation in [21, Thm. 3.1] is different from that of Theorem 4.8 or Lemma 4.7, but it is not difficult to see that the results are equivalent. Alternatively, a direct proof of Theorem 4.8 can be easily obtained via the fact that the decomposition numbers $[S^\lambda_F : D^\mu_F]_q$ are certain parabolic Kazhdan–Lusztig polynomials, see [26] or [12, Thm. 5.3], together with Soergel's algorithm [25] for computing those polynomials.

For $p > 0$, define a matrix $\tilde{A}^p = (\tilde{a}^p_{\lambda\mu})$ with rows and columns indexed by $\mathsf{RPar}_{e,\leqslant 2}(n)$ as follows. Let $\lambda = (2^u, 1^v), \mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$ and let

$$y = (a_s p^s + a_{s-1} p^{s-1} + \cdots + a_1 p + a_0)e + l - 1$$

be the (e,p)-expansion of y. Then we set

$$\tilde{a}^p_{\lambda\mu} := \begin{cases} 1 & \text{if } v = (a_s p^s \pm a_{s-1} p^{s-1} \pm \cdots \pm a_1 p \pm a_0)e + l - 1 \text{ for some choice of signs;} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.9. *Let $p > 0$ and $B = (b_{\lambda\mu})_{\lambda \in \mathsf{Par}_{\leqslant 2}(n),\, \mu \in \mathsf{RPar}_{e,\leqslant 2}(n)} = D^0(q)\tilde{A}^p$. If $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$, with $u \geqslant x$, then*

$$b_{\lambda\mu} = f^q_{e,p}(y, u - x).$$

Proof. In view of Lemma 4.5, the result follows from Lemmas 4.6 and 4.7 together with Theorem 4.8. □

Theorem 4.10. *Suppose that either $p = e$ or p is coprime to e if $p \neq 0, e$. If $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$ with $u \geqslant x$, then*

$$[S^\lambda_F : D^\mu_F]_q = f^q_{e,p}(y, u - x).$$

Proof. If $p = 0$, then this is Theorem 4.8, so we assume that $p > 0$. Setting $q = 1$ in Lemma 4.9, we have $D^p = D^0\tilde{A}^p$, so that $\tilde{A}^p = A^p$ is the $\mathsf{RPar}_{e,\leqslant 2}(n) \times \mathsf{RPar}_{e,\leqslant 2}(n)$-submatrix of the ungraded adjustment matrix corresponding to 2-column partitions. Since the entries of A^p are either 0 or 1, it follows from Theorem 4.3 that $\tilde{A}^p = A^p = A^p(q)$, that is, $D^p(q) = D^0(q)\tilde{A}^p$. The result then follows by another application of Lemma 4.9. □

Example 4.11. Suppose that $e = p = 2$ and $\lambda = (2, 2, 1, 1)$. We have $f^q_{2,2}(2, 0) = f^q_{2,2}(6, 2) = 1$ and $f^q_{2,2}(4, 1) = q$, so by Theorem 4.10 the composition factors of S^λ are D^λ, $D^{(2,1^4)}$, $D^{(1^6)}$ and their graded composition multiplicities are $1, q, 1$ respectively. The reader may wish to check, using Example 4.1, that Theorem 4.2 holds in this case.

4.3. Proof of Theorem 4.2

We fix a partition $\mu = (2^x, 1^y) \in \mathsf{RPar}_{e,\leqslant 2}(n)$ and prove Theorem 4.2 for this fixed μ. Recall the sets $\mathsf{Std}_{e,p,\mu}(\lambda)$ for $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$ defined by (4.2). If $p = 0$, then the statement of Theorem 4.2 is an immediate consequence of Lemma 3.6, Lemma 4.7 and Theorem 4.8. If $p > 0$, then we set

$$y = (a_s p^s + a_{s-1} p^{s-1} + \cdots + a_1 p + a_0)e + l - 1$$

to be the (e, p)-expansion of y.

Lemma 4.12. *Assume that $p > 0$. Let $\lambda = (2^u, 1^v)$ and let $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$. If*

$$\mathrm{reg}_{e,p}(\mathtt{t}) = \mathrm{reg}_{ep^{z_h}}(\ldots \mathrm{reg}_{ep^{z_2}}(\mathrm{reg}_{ep^{z_1}}(\mathtt{t}))\ldots)$$

is the regularisation equation of $\mathtt{t}$, then $a_{z_k} > 0$ for all $1 \leqslant k \leqslant h$ and

$$v = \left(a_s p^s + (-1)^{\delta_s} a_{s-1} p^{s-1} + \cdots + (-1)^{\delta_2} a_1 p + (-1)^{\delta_1} a_0\right) e + (-1)^{\delta_0} l - 1,$$

where $\delta_i = \#\{1 \leqslant k \leqslant h \mid z_k \geqslant i\}$ for all $0 \leqslant i \leqslant s$.

Proof. We use induction on h, the result being clear when $h = 0$. Assuming that $h > 0$, set $\mathtt{r} = \mathrm{reg}_{ep^{z_1}}(\mathtt{t})$ so that

$$\mathrm{reg}_{e,p}(\mathtt{t}) = \mathrm{reg}_{ep^{z_h}}(\ldots \mathrm{reg}_{ep^{z_3}}(\mathrm{reg}_{ep^{z_2}}(\mathtt{r}))\ldots)$$

is the regularisation equation of $\mathtt{r}$, and set $\nu := \mathrm{Shape}(\mathtt{r}) = (2^c, 1^d)$. By the inductive hypothesis,

$$\begin{aligned} d = \big(&a_s p^s + a_{s-1}p^{s-1} + \cdots \\ &+ a_{z_2}p^{z_2} + (-1)^{\epsilon_{z_2}} a_{z_2-1}p^{z_2-1} + \cdots + (-1)^{\epsilon_1} a_0\big)e + (-1)^{\epsilon_0} l - 1, \end{aligned}$$

where $a_{z_2}, \ldots, a_{z_h} > 0$ and $\epsilon_i = \#\{2 \leqslant k \leqslant h \mid z_k \geqslant i\}$ for all $0 \leqslant i \leqslant z_2$.

Let $m = a_s p^s + \cdots + a_{z_1}p^{z_1}$. Then $d = me + j - 1$ for some $0 \leqslant j < ep^{z_1}$. Since $\mathtt{r} = \mathrm{reg}_{ep^{z_1}}(\mathtt{t}) \neq \mathtt{t}$, we have

$$\begin{aligned} v &= 2(me-1) - d \\ &= \big(a_s p^s + a_{s-1}p^{s-1} + \cdots + a_{z_1}p^{z_1} - a_{z_1-1}p^{z_1-1} - \cdots - a_{z_2}p^{z_2} \\ &\quad - (-1)^{\epsilon_{z_2}} a_{z_2-1}p^{z_2-1} - \cdots - (-1)^{\epsilon_1} a_0\big)e - (-1)^{\epsilon_0} l - 1 \end{aligned}$$

by Lemma 3.6. Also, $a_{z_1} > 0$, for if $a_{z_1} = 0$ then either $\mathtt{t} \notin \mathsf{DStd}_{ep^{z_1+1}}(\lambda)$ or $\mathtt{t} \in \mathsf{DStd}_{ep^{z_1}}(\lambda)$, contradicting the hypothesis on the regularisation equation of $\mathtt{t}$. The result follows. □

The following lemma is an easy consequence of the definitions in §3.3.

Lemma 4.13. *Assume that $p > 0$ and let $b \in \mathbb{Z}_{\geqslant 0}$.*

(i) *Let $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(\lambda)$ and $\mathtt{s} := \mathrm{reg}_{ep^b}(\mathtt{t})$, and suppose that b is maximal such that $\mathtt{s} = \mathrm{reg}_{ep^b}(\mathtt{t})$. Let $c > b$ be an integer. Then $\mathtt{t}$ is ep^c-regular if and only if $\mathtt{s}$ is ep^c-regular.*

(ii) *Let $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(n)$. Suppose that $\mathtt{s} := \mathrm{reg}_{ep^b}(\mathtt{t}) \neq \mathtt{t}$ and $e \nmid \pi_{\mathtt{t}}(n) + 1$. Then $\mathtt{t}$ is e-regular if and only if $\mathtt{s}$ is not e-regular.*

Lemma 4.14. *Let $\lambda = (2^u, 1^v) \in \mathsf{Par}_{\leqslant 2}(n)$ be such that S^λ_F has a composition factor D^μ_F.*

(i) *If $\mathtt{s} \in \mathsf{DStd}_{e,p}(\mu)$ then there exists a unique $\mathtt{t} \in \mathsf{Std}(\lambda)$ such that $\mathrm{reg}_{e,p}(\mathtt{t}) = \mathtt{s}$.*

(ii) *All elements of $\mathsf{Std}_{e,p,\mu}(\lambda)$ have the same regularisation set.*

Proof. If $p = 0$, then the result holds by Lemma 3.6. So, assume that $p > 0$.

By Theorem 4.10 and Lemma 4.6,

$$v = \left((-1)^{\delta_{s+1}} a_s p^s + (-1)^{\delta_s} a_{s-1}p^{s-1} + \cdots + (-1)^{\delta_1} a_0\right) e + (-1)^{\delta_0} l - 1$$

for some $\delta_0, \ldots, \delta_{s+1} \in \{0, 1\}$ satisfying the following conditions:

(1) $\delta_{s+1} = 0$;
(2) $\delta_0 = \delta_1$ if $l = 0$, and $\delta_{i+1} = \delta_{i+2}$ if $a_i = 0$ and $0 \leqslant i < s$.

By Lemma 4.5, such $\delta_0, \dots, \delta_{s+1}$ are uniquely determined by v. Let

$$Z := \{i \in [0, s] \mid \delta_i \neq \delta_{i+1}\} = \{z_1 > \cdots > z_h\}.$$

By Lemma 4.12, if $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$, then the regularisation set of $\mathtt{t}$ is Z, proving (ii).

To prove (i), we argue by induction on h. If $h = 0$, then $\lambda = \mu$ and there is nothing to prove. So, assume that $h > 0$. Set

$$\begin{aligned} d = \big(&a_s p^s + a_{s-1} p^{s-1} + \cdots + a_{z_1} p^{z_1} \\ &- (-1)^{\delta_{z_1}} a_{z_1 - 1} p^{z_1 - 1} - \cdots - (-1)^{\delta_2} a_1 p - (-1)^{\delta_1} a_0\big) e - (-1)^{\delta_0} l - 1 \end{aligned}$$

and let $\nu := \big(2^c, 1^d\big) \in \mathsf{Par}_{\leqslant 2}(n)$, with $c = (n-d)/2$. By Lemma 4.6 and Theorem 4.10, D^μ_F is a composition factor of S^ν_F, so by the inductive hypothesis, there is a unique $\mathtt{r} \in \mathsf{Std}(\nu)$ such that $\mathrm{reg}_{e,p}(\mathtt{r}) = \mathtt{s}$. Moreover, by Lemmas 4.12 and 4.5,

$$\mathrm{reg}_{e,p}(\mathtt{r}) = \mathrm{reg}_{ep^{z_h}}(\dots \mathrm{reg}_{ep^{z_2}}(\mathtt{r}) \dots)$$

is the regularisation equation of $\mathtt{r}$. In particular, $\mathrm{reg}_{ep^z}(\mathtt{r}) = \mathtt{r}$ for all $z > z_2$. Let

$$m = a_s p^{s - z_1} + a_{s-1} p^{s-1-z_1} + \cdots + a_{z_1}.$$

Since $\delta_{z_1} \neq \delta_{z_1+1} = \delta_{s+1} = 0$, we have $\delta_{z_1} = 1$ and so

$$mep^{z_1} - 1 < d < (m+1)ep^{z_1} - 1.$$

By Lemma 3.6,

$$\mathrm{reg}^{-1}_{ep^{z_1}}(\mathtt{r}) = \{\mathtt{r}, \mathtt{t}\} \tag{4.7}$$

for a certain tableau $\mathtt{t} \in \mathsf{Std}(\lambda)$. By Lemma 4.13(i), $\mathrm{reg}_{ep^z}(\mathtt{t}) = \mathtt{t}$ for all $z > z_1$. Hence, $\mathrm{reg}_{e,p}(\mathtt{t}) = \mathrm{reg}_{e,p}(\mathtt{r}) = \mathtt{s}$.

To prove the uniqueness statement in (i), recall that any $\mathtt{t}' \in \mathsf{Std}(\lambda)$ such that $\mathrm{reg}_{e,p}(\mathtt{t}') = \mathtt{s}$ has regularisation set Z. By the inductive hypothesis, this implies that $\mathrm{reg}_{ep^{z_1}}(\mathtt{t}') = \mathtt{r}$, and by (4.7) we have $\mathtt{t}' = \mathtt{t}$. $\square$

Lemma 4.15. *Let $\lambda = (2^u, 1^v)$ be a partition of n such that D^μ_F is a composition factor of S^λ_F. If $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$ then $[S^\lambda_F : D^\mu_F]_q = q^{r_e(\mathtt{t})}$.*

Proof. If $p = 0$, then the result holds by Theorem 4.8 and Lemma 4.7, so we assume that $p > 0$. Consider the regularisation equation of $\mathtt{t}$:

$$\mathtt{s} := \mathrm{reg}_{e,p}(\mathtt{t}) = \mathrm{reg}_{ep^{z_h}}(\dots \mathrm{reg}_{ep^{z_2}}(\mathrm{reg}_{ep^{z_1}}(\mathtt{t})) \dots).$$

By Lemma 4.12,

$$v = \big(a_s p^s + (-1)^{\delta_s} a_{s-1} p^{s-1} + \cdots + (-1)^{\delta_2} a_1 p + (-1)^{\delta_1} a_0\big) e + (-1)^{\delta_0} l - 1,$$

where $\delta_i = \#\{1 \leqslant k \leqslant h \mid z_k \geqslant i\}$ for $0 \leqslant i \leqslant s$. Noting that $\mathtt{s} \in \mathsf{DStd}_e(\mu)$ and applying Lemma 4.13(ii) h times, we see that

$$r_e(\mathtt{t}) = \begin{cases} 0 & \text{if } l = 0; \\ 0 & \text{if } l > 0 \text{ and } h \text{ is even}; \\ 1 & \text{if } l > 0 \text{ and } h \text{ is odd}. \end{cases}$$

Observe that $\delta_0 = h$. By Theorem 4.10 and Lemma 4.6, the result follows. $\square$

Proof of Theorem 4.2. By Theorem 4.10 and Lemmas 4.6 and 4.12 (or by Lemma 3.6 and Theorem 4.8 in the case $p = 0$), if $\mathsf{Std}_{e,p,\mu}(\lambda) \neq \varnothing$ then D^μ_F is a composition factor of S^λ_F. The converse also holds, by Lemma 4.14(i). Moreover, that lemma asserts that, given $\mathtt{s} \in \mathsf{DStd}_{e,p}(\mu)$, there exists a unique $\mathtt{t} \in \mathsf{Std}(\lambda)$ such that $\mathrm{reg}_{e,p}(\mathtt{t}) = \mathtt{s}$. Further, by Lemma 4.15, we have $[S^\lambda_F : D^\mu_F]_q = q^{r_e(\mathtt{t})}$ in this case, completing the proof. □

4.4. Characters of simple modules

Recall the notation of §§3.2-3.3 applied to a weight space of type A_1. Let $u \leqslant r \leqslant s \leqslant v$ be integers and $\pi \in \mathsf{P}^+_{[u,v]}$. Denote the restriction of π to $[r,s]$ by $\pi[r,s]$, so that $\pi[r,s] \in \mathsf{P}^+_{[r,s]}$. We refer to every such restriction $\pi[r,s]$ as a *segment* of π. The following is clear:

Lemma 4.16. *If $u = c_1 \leqslant \cdots \leqslant c_l = v$ are integers, then*

$$\deg_e(\pi) = \sum_{i=1}^{l-1} \deg_e(\pi[c_i, c_{i+1}]).$$

Definition. Let $r < s$ be integers. Let $\eta \in \mathsf{P}^+_{[r,s]}$ be such that $\eta(r) = \eta(s) = me-1$ for some $m \in \mathbb{Z}_{>0}$, so that $\eta(r) \in H_m$.

- If $me-1 < \eta(c) < (m+1)e-1$ whenever $r < c < s$, then we call η a *positive arc*.
- If $(m-1)e-1 < \eta(c) < me-1$ whenever $r < c < s$, then we call η a *negative arc*.

If one of the above two statements holds, we say that η is an *arc*.

Let $m \in \mathbb{Z}_{>0}$ and $\pi \in P^+_{[u,v]}$. Recall the reflections $s_m = s_{\alpha,m}$ from (3.2). We define $s_m \cdot \pi \in \mathsf{P}_{[u,v]}$ to be the path obtained by reflecting π with respect to the wall H_m, so that

$$(s_m \cdot \pi)(a) = s_m \cdot (\pi(a)) \quad (u \leqslant a \leqslant v).$$

Lemma 4.17. *Let $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(n)$. Suppose that $0 \leqslant r < s \leqslant n$ are integers and $\pi_\mathtt{t}[r,s]$ is an arc, with $\pi_\mathtt{t}(r) = \pi_\mathtt{t}(s) = me-1$ where $m \in \mathbb{Z}_{>0}$. If $\mathtt{s} \in \mathsf{Std}_{\leqslant 2}(n)$ is defined by the conditions that*

$$\pi_\mathtt{s}[0,r] = \pi_\mathtt{t}[0,r], \quad \pi_\mathtt{s}[r,s] = s_m \cdot \pi_\mathtt{t}[r,s], \quad \pi_\mathtt{s}[s,n] = \pi_\mathtt{t}[s,n],$$

then $\boldsymbol{i}^\mathtt{s} = \boldsymbol{i}^\mathtt{t}$.

Proof. Since $\pi_\mathtt{s} = s^s_m \cdot s^r_m \cdot \pi_\mathtt{t}$, the result follows from Lemma 3.2. □

If $\pi \in P^+_{[u,v]}$, we define $\mathsf{A}^+(\pi)$ and $\mathsf{A}^-(\pi)$ to be the sets of segments of π that are positive and negative arcs, respectively. Recalling (3.7), let

$$B(\pi) = \{b_1 < \cdots < b_N\} := \{b \in [u,v] \mid \pi(b) \in H\}. \tag{4.8}$$

Then every element of $\mathsf{A}^+(\pi) \cup \mathsf{A}^-(\pi)$ is of the form $\pi[b_k, b_{k+1}]$ for some $1 \leqslant k < N$.

Lemma 4.18. *Let $\pi \in \mathsf{P}^+_{[u,v]}$ be such that $\pi(u), \pi(v) \in H$ and let $B(\pi) = \{b_1 < \cdots < b_N\}$. Then:*

(i) $\deg_e(\pi) = |\mathsf{A}^-(\pi)| - |\mathsf{A}^+(\pi)|$.

(ii) *If $m \in \mathbb{Z}_{>0}$ and $s_m \cdot \pi \in \mathsf{P}^+_{[u,v]}$, then* $\deg(s_m \cdot \pi) = -\deg(\pi)$.

Proof. It is easy to see from (3.5) and (3.6) that, whenever $1 \leqslant k < N$,

$$\deg_e(\pi[b_k, b_{k+1}]) = \begin{cases} 1 & \text{if } \pi[b_k, b_{k+1}] \text{ is a negative arc;} \\ -1 & \text{if } \pi[b_k, b_{k+1}] \text{ is a positive arc;} \\ 0 & \text{if } \pi(b_k) \neq \pi(b_{k+1}). \end{cases}$$

Part (i) follows by Lemma 4.16. Part (ii) follows from (i) because the reflection s_m transforms positive arcs into negative arcs and vice versa. □

The following result describes the degree of a path in terms of the number of its positive and negative arcs.

Corollary 4.19. *If $\pi \in \mathsf{P}^+_n$, then* $\deg_e(\pi) = |\mathsf{A}^-(\pi)| - |\mathsf{A}^+(\pi)| + r_e(\pi)$.

Proof. Let the integers $0 < b_1 < \cdots < b_N \leqslant n$ be defined as in (4.8). Then $\deg_e(\pi[0, b_1]) = 0$. Furthermore, $\deg_e(\pi[b_N, n]) = r_e(\pi)$. The result follows from Lemmas 4.16 and 4.18(i). □

Our next goal is to describe the effect of the map $\operatorname{reg}_{e,p}$ on the degree of a standard tableau. If $p > 0$, then given $\eta \in \mathsf{P}^+_n$ and a subset $Z = \{z_1 > \cdots > z_h\}$ of $\mathbb{Z}_{\geqslant 0}$, we define the tuple

$$\mathbf{w}(Z, \eta) = (w_1, \ldots, w_h) \in [0, n]^h$$

by setting w_i to be the maximal element of $[0, n]$ such that $\eta(w_i) \in \bigcup_{m \in \mathbb{Z}_{>0}} H_{mp^{z_i}}$, for each $i = 1, \ldots, h$, with $w_i = 0$ if no such element exists.

Lemma 4.20. *Assume that $p > 0$. Let $\pi \in \mathsf{P}^+_n$ and $\eta = \operatorname{reg}_{e,p}(\pi)$. Suppose that Z is the regularisation set of π and $\mathbf{w}(Z, \eta) = (w_1, \ldots, w_h)$. Let $w_0 = 0$ and $w_{h+1} \in [0, n]$ be maximal such that $\pi(w_{h+1}) \in H$, with $w_{h+1} = 0$ if no such number exists. Then*

$$\deg_e(\eta[w_i, w_{i+1}]) = (-1)^i \deg_e(\pi[w_i, w_{i+1}])$$

for all $0 \leqslant i \leqslant h$.

Proof. Let $Z = \{z_1 > \cdots > z_h\}$ and, for any $0 \leqslant i \leqslant h$, set

$$\pi_i := \operatorname{reg}_{ep^{z_i}}(\ldots(\operatorname{reg}_{ep^{z_1}}(\pi))\ldots),$$

so that $\pi_0 = \pi, \pi_1, \ldots, \pi_h = \eta$ are the 'intermediate steps' in the calculation of $\operatorname{reg}_{e,p}(\pi)$. Given $1 \leqslant i \leqslant h$, let $m_i \in p^{z_i}\mathbb{Z}_{>0}$ be defined by the condition that $\pi_{i-1}(w_i) = m_i e - 1$. Then

$$\eta[w_i, w_{i+1}] = s_{m_i} \cdot \ldots \cdot s_{m_1} \cdot \pi[w_i, w_{i+1}]$$

whenever $0 \leqslant i \leqslant h$. Using Lemma 4.18(ii), we deduce the required identity. □

Let $Z = \{z_1 > \cdots > z_h\}$ be a finite subset of $\mathbb{Z}_{>0}$. We define a map $\rho_Z \colon \mathsf{P}_n^+ \to \mathsf{P}_n^+$ as follows. If $p = 0$, we set ρ_Z to be the identity map. Assuming that $p > 0$ and given $\eta \in \mathsf{P}_n^+$, let $B(\eta) = \{b_1 < \cdots < b_N\}$ and $\mathbf{w}(Z, \eta) = (w_1, \ldots, w_h)$. Set $w_0 = 0$ and $w_{h+1} = b_N$ (with $b_N := 0$ if $B(\eta) = \varnothing$). Then we define $\rho_Z(\eta) \in P_n^+$ by the conditions that $\rho_Z(\eta)[0, b_1] = \eta[0, b_1]$, $\rho_Z(\eta)[b_N, n] = \eta[b_N, n]$ and for $1 \leqslant k < N$,

$$\rho_Z(\eta)[b_k, b_{k+1}] = \begin{cases} s_m \cdot \eta[b_k, b_{k+1}] & \text{if } \pi(b_k) = \pi(b_{k+1}) \in H_m \text{ for some } m \in \mathbb{Z}_{>0} \\ & \text{and } [b_k, b_{k+1}] \subseteq [w_i, w_{i+1}] \text{ for some odd} \\ & i \in [1, h]; \\ \eta[b_k, b_{k+1}] & \text{otherwise.} \end{cases}$$

In other words, $\rho_Z(\eta)$ is obtained from η by reflecting each arc that is a segment of $\eta[w_i, w_{i+1}]$ for *odd* i with respect to the wall where the arc begins and ends.

We consider the corresponding map

$$\rho_Z \colon \mathsf{Std}_{\leqslant 2}(n) \to \mathsf{Std}_{\leqslant 2}(n)$$

on standard tableaux, defined by the condition that $\rho_Z(\pi_{\mathtt{t}}) = \pi_{\rho_Z(\mathtt{t})}$ for all $\mathtt{t} \in \mathsf{Std}_{\leqslant 2}(n)$. Since $\rho_Z(\eta)(n) = \eta(n)$ for all $\eta \in P_n^+$, the map ρ_Z leaves $\mathsf{Std}(\mu)$ invariant for each $\mu \in \mathsf{Par}_{\leqslant 2}(n)$. Moreover, ρ_Z leaves $\mathsf{DStd}_{e,p}(\mu)$ invariant because $B(\rho_Z(\eta)) = B(\eta)$ and $\rho_Z(\eta)(b) = \eta(b)$ for all $\eta \in P_n^+$ and $b \in B(\eta)$.

Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu \in \mathsf{RPar}_{e, \leqslant 2}(n)$ be such that $\mathsf{Std}_{e,p,\mu}(\lambda) \neq \varnothing$. Define

$$r_{e,p,\lambda,\mu} := r_e(\mathtt{t}) \in \{0, 1\} \tag{4.9}$$

for any $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$. By Lemma 4.15, the right-hand side does not depend on the choice of $\mathtt{t}$ and

$$q^{r_{e,p,\lambda,\mu}} = [S_F^\lambda : D_F^\mu]_q. \tag{4.10}$$

Let Z be the regularisation set of any $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$. Note that Z does not depend on $\mathtt{t}$ by Lemma 4.14(ii). Define the map

$$\mathrm{reg}'_{e,p,\lambda,\mu} \colon \mathsf{Std}_{e,p,\mu}(\lambda) \to \mathsf{DStd}_{e,p}(\mu), \ \mathtt{t} \mapsto \rho_Z(\mathrm{reg}_{e,p}(\mathtt{t})). \tag{4.11}$$

If $p = 0$, then $\mathrm{reg}'_{e,p,\lambda,\mu}$ is simply the restriction of $\mathrm{reg}_{e,0} = \mathrm{reg}_e$ to $\mathsf{Std}_{e,p,\mu}(\lambda)$.

Lemma 4.21. *Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu \in \mathsf{RPar}_{\leqslant 2}(n)$. If $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$, then*

$$\deg_e\big(\mathrm{reg}'_{e,p,\lambda,\mu}(\mathtt{t})\big) = \deg_e(\mathtt{t}) - r_{e,p,\lambda,\mu}.$$

Proof. When $p = 0$, the result follows from Lemma 3.5, so we assume that $p > 0$. Let $\mathtt{s} = \mathrm{reg}_{e,p}(\mathtt{t})$ and Z be the regularisation set of $\mathtt{t}$, so that $\mathrm{reg}'_{e,p,\lambda,\mu}(\mathtt{t}) = \rho_Z(\mathtt{s})$. Write $\mathbf{w}(Z, \pi_{\mathtt{t}}) = (w_1, \ldots, w_h)$. Let $w_{h+1} \in [0, n]$ be maximal such that $\pi_{\mathtt{t}}(w_{h+1}) \in H$, with $w_{h+1} = 0$ if no such number exists. By Lemma 4.18(ii), for any $0 \leqslant i \leqslant h$ we have

$$\deg_e(\pi_{\rho_Z(\mathtt{s})}[w_i, w_{i+1}]) = (-1)^i \deg_e(\pi_{\mathtt{s}}[w_i, w_{i+1}]).$$

Hence,

$$\begin{aligned}\deg_e\big(\pi_{\rho_Z(\mathtt{s})}\big) &= \sum_{i=0}^{h}(-1)^i \deg_e(\pi_{\mathtt{s}}[w_i, w_{i+1}]) = \sum_{i=0}^{h} \deg_e(\pi_{\mathtt{t}}[w_i, w_{i+1}]) \\ &= \deg_e(\mathtt{t}) - r_e(\mathtt{t}) = \deg_e(\mathtt{t}) - r_{e,p,\lambda,\mu},\end{aligned}$$

where the second equality is due to Lemma 4.20 and we have used Lemma 4.16 for the first and third equalities. □

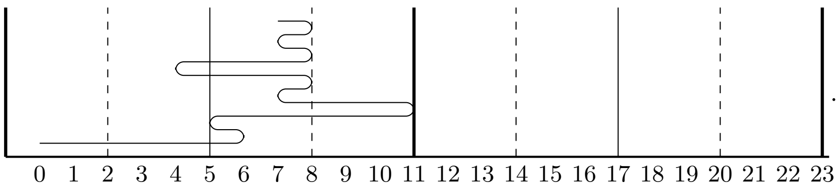
Recall the definitions (1.1) and (1.2) of q-characters.

Theorem 4.22. *Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mu \in \mathsf{RPar}_{\leqslant 2}(n)$ with $\mathsf{Std}_{e,p,\mu}(\lambda) \neq \varnothing$.*

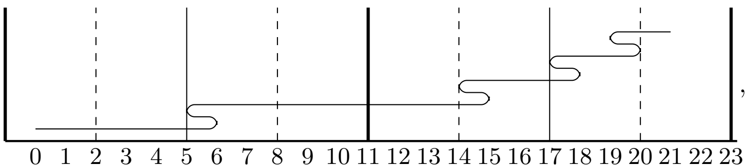
(i) *The map $\mathrm{reg}'_{e,p,\lambda,\mu} \colon \mathsf{Std}_{e,p,\mu}(\lambda) \to \mathsf{DStd}_{e,p}(\mu)$ is a bijection.*
(ii) *For all $\mathtt{t} \in \mathsf{Std}_{e,p,\mu}(\lambda)$, we have $\boldsymbol{i}^{\mathrm{reg}'_{e,p,\lambda,\mu}(\mathtt{t})} = \boldsymbol{i}^{\mathtt{t}}$.*
(iii) *We have $\mathrm{ch}_q \mathsf{Std}_{e,p,\mu}(\lambda) = q^{r_{e,p,\lambda,\mu}} \mathrm{ch}_q \mathsf{DStd}_{e,p}(\mu)$.*

Proof. By Lemma 4.14(i), $\mathrm{reg}_{e,p}$ restricts to a bijection from $\mathsf{Std}_{e,p,\mu}(\lambda)$ onto $\mathsf{DStd}_{e,p}(\mu)$. Clearly, ρ_Z restricts to an involution of $\mathsf{DStd}_{e,p}(\mu)$ for any set Z as in (4.11), and (i) follows. Part (ii) follows from Lemmas 3.2 and 4.17. Part (iii) follows immediately from (i), (ii) and Lemma 4.21. □

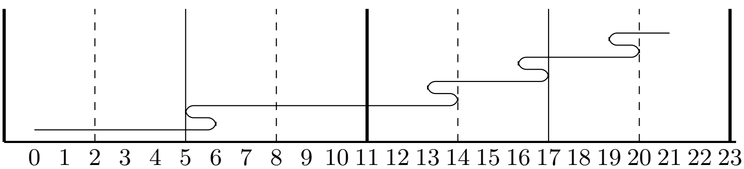
Example 4.23. Let $e = 3$, $p = 2$, $n = 29$ and $\lambda = (2^{11}, 1^7)$. Suppose that $\mathtt{t} \in \mathsf{Std}(\lambda)$ is determined from

$\pi_{\mathtt{t}} =$  .

Then the regularisation set of $\mathtt{t}$ is $Z = \{2, 1, 0\}$. Further, $\mathbf{w}(Z, \pi_{\mathtt{t}}) = (w_1, w_2, w_3)$ with $w_1 = 13$, $w_2 = 23$ and $w_3 = 28$, and $B(\pi_{\mathtt{t}}) = \{2, 5, 7, 10, 13, 16, 18, 21, 23, 26, 28\}$. Thus,

$\eta := \mathrm{reg}_{3,2}(\pi_{\mathtt{t}}) =$  ,

so that $\mathtt{t} \in \mathsf{Std}_{3,2,\mu}(\lambda)$ where $\mu = (2^4, 1^{21})$. (We use the same convention on thickness of walls as in Example 4.1.) Further,

$\pi_{\mathrm{reg}'_{3,2,\lambda,\mu}(\mathtt{t})} =$  .

Here, $\pi_{\mathrm{reg}'_{3,2,\lambda,\mu}(\mathtt{t})}$ is obtained from η by reflecting the arc $\eta[16,18]$ with respect to the wall $H_5=\{14\}$ and the arc $\eta[21,23]$ with respect to the wall $H_6=\{18\}$. We have $\deg_3(\mathtt{t})=3$, $\deg_3(\eta)=-2$ and $\deg_3\big(\mathrm{reg}'_{3,2,\lambda,\mu}(\mathtt{t})\big)=2=\deg_3(\mathtt{t})-r_{3,2,\lambda,\mu}$, cf. Lemma 4.21. Also, this example illustrates Lemma 4.20, as

$$\begin{aligned}\deg(\eta[0,w_1])&=-1=\deg(\pi_{\mathtt{t}}[0,w_1]),\\ \deg(\eta[w_1,w_2])&=-2=-\deg(\pi_t[w_1,w_2]) \quad \text{and}\\ \deg(\eta[w_2,w_3])&=1=\deg(\pi_{\mathtt{t}}[w_2,w_3]).\end{aligned}$$

Proof of Theorem 1.2. We use induction on λ with respect to the dominance order $\trianglelefteq$, recalling that $[S_F^\lambda : D_F^\mu]=0$ unless $\mu \trianglelefteq \lambda$. Observe that if $\mu \in \mathsf{Par}(n)$ and $\mu \triangleleft \lambda$, then $\mu \in \mathsf{Par}_{\leqslant 2}(n)$. Then

$$\begin{aligned}\mathrm{ch}_q\, D_F^\lambda &= \mathrm{ch}_q\, S_F^\lambda - \sum_{\mu\triangleleft\lambda}[S_F^\lambda : D_F^\mu]_q\, \mathrm{ch}_q\, D_F^\mu\\ &= \mathrm{ch}_q\, \mathsf{Std}(\lambda) - \sum_{\mu\triangleleft\lambda} q^{r_{e,p,\lambda,\mu}}\, \mathrm{ch}_q\, \mathsf{DStd}_{e,p}(\mu)\\ &= \mathrm{ch}_q\, \mathsf{Std}(\lambda) - \sum_{\mu\triangleleft\lambda} \mathrm{ch}_q\, \mathsf{DStd}_{e,p,\mu}(\lambda)\\ &= \mathrm{ch}_q\, \mathsf{DStd}_{e,p}(\lambda),\end{aligned}$$

where the first and last equalities follow from the definitions, the second equality is due to (1.3), (4.10) and the inductive hypothesis, and the third equality holds by Theorem 4.22(iii). $\square$

Remark 4.24. Recall the involution (2.1) on $\langle I^n\rangle$. Given $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$, the simple $R^{\Lambda_0}_{n,F}$-module D_F^λ is self-dual by [4, Thms. 4.11, 5.13, 5.18], which implies that $\overline{\mathrm{ch}_q\, D_F^\lambda} = \mathrm{ch}_q\, D_F^\lambda$. By Theorem 1.2 (and Theorem 1.1(ii)), it follows that

$$\overline{\mathrm{ch}_q\, \mathsf{DStd}_{e,p}(\lambda)} = \mathrm{ch}_q\, \mathsf{DStd}_{e,p}(\lambda). \tag{4.12}$$

The last identity can also be proved combinatorially via the involution $\iota\colon \mathsf{P}_n^+ \to \mathsf{P}_n^+$ defined as follows. Given $\pi \in \mathsf{P}_n^+$, let $B(\pi)=\{b_1<\cdots<b_N\}$ and define $\iota(\pi)\in\mathsf{P}_n^+$ by the conditions that

$$\begin{aligned}&\iota(\pi)[0,b_1]=\pi[0,b_1], \quad \iota(\pi)[b_N,n]=\pi[b_N,n],\\ &\iota(\pi)[b_i,b_{i+1}]=\begin{cases} s_m\cdot\pi[b_i,b_{i+1}] & \text{if } \pi[b_i,b_{i+1}] \text{ is an arc, with } \pi(b_i)=\pi(b_{i+1})\in H_m;\\ \pi[b_i,b_{i+1}] & \text{otherwise}\end{cases}\end{aligned}$$

whenever $1\leqslant i<N$. That is, $\iota(\pi)$ is obtained from π by reflecting all arcs. As usual, this yields an involution $\iota\colon \mathsf{Std}_{\leqslant 2}(n)\to\mathsf{Std}_{\leqslant 2}(n)$ determined by $\pi_{\iota(\mathtt{t})}=\iota(\pi_{\mathtt{t}})$ for all $\mathtt{t}\in\mathsf{Std}_{\leqslant 2}(n)$. Then ι restricts to an involution on $\mathsf{DStd}_{e,p}(\lambda)$. Moreover, by Lemmas 4.17 and 4.18(i), for all $\mathtt{t}\in\mathsf{DStd}_e(\lambda)$ we have

$$\boldsymbol{i}^{\iota(\mathtt{t})}=\boldsymbol{i}^{\mathtt{t}}, \quad \deg(\iota(\mathtt{t}))=-\deg(\mathtt{t}),$$

which implies (4.12). Combining (4.12) with Theorem 4.22(iii), we obtain the inequality

$$\sum_{\mathtt{t}\in\mathsf{Std}(\lambda,\boldsymbol{i})} \deg(\mathtt{t}) \geqslant 0 \tag{4.13}$$

for any fixed $\boldsymbol{i} \in I^n$. It is proved in [14, Cor. 3.16] via determinants of Gram matrices for Specht modules that this inequality holds for an *arbitrary* (multi)partition λ, but in general no combinatorial proof is known (even when λ is an e-core, in which case (4.13) becomes an equality).

5. Homomorphisms between 2-column Specht modules

Let $\mathcal{O}$ be an arbitrary (unital) commutative ring and $n \in \mathbb{Z}_{\geqslant 0}$. We recall a row removal result from [10] for homomorphisms between Specht modules, applied in the special case of the algebra $R_n^{\Lambda_0} := R_{n,\mathcal{O}}^{\Lambda_0}$.

Let $\lambda, \mu \in \mathsf{Par}(n)$. Recall from Proposition 2.1 that $S_{\mathcal{O}}^{\lambda}$ has a basis $\{v^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{Std}(\lambda)\}$. Also, recall the tableaux $\mathtt{t}^{\mu}$ and $\mathtt{t}_{\mu}$ from §2.3. A standard tableau $\mathtt{t} \in \mathsf{Std}(\lambda)$ is *μ-row-dominated* if, whenever $\mathtt{t}^{-1}(a,b) = (\mathtt{t}^{\mu})^{-1}(c,d)$ for some $(a,b) \in [\![\lambda]\!]$ and $(c,d) \in [\![\mu]\!]$, we have $a \leqslant c$. Denote the set of μ-row-dominated elements of $\mathsf{Std}(\lambda)$ by $\mathsf{Std}^{\mu}(\lambda)$. A homomorphism $\varphi \in \mathrm{Hom}_{R_n^{\Lambda_0}}(S_{\mathcal{O}}^{\mu}, S_{\mathcal{O}}^{\lambda})$ is said to be (*row-*) *dominated* if $\varphi(v^{\mu}) \in \langle v^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{Std}^{\mu}(\lambda)\rangle_{\mathcal{O}}$. We denote by $\mathrm{DHom}_{R_n^{\Lambda_0}}(S_{\mathcal{O}}^{\mu}, S_{\mathcal{O}}^{\lambda})$ the set of dominated homomorphisms lying in $\mathrm{Hom}_{R_n^{\Lambda_0}}(S_{\mathcal{O}}^{\mu}, S_{\mathcal{O}}^{\lambda})$.

If $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition and $\mathtt{t} \in \mathsf{Std}(\lambda)$, let $\bar{\lambda} := (\lambda_2, \dots, \lambda_l)$ and let $\bar{\mathtt{t}} \in \mathsf{Std}(\bar{\lambda})$ be the tableau obtained from $\mathtt{t}$ by removing the first row and decreasing all entries by λ_1.

Theorem 5.1 ([10, Thm. 4.1])**.** *Let $\lambda, \mu \in \mathsf{Par}(n)$ be such that $\lambda_1 = \mu_1$. There is an isomorphism*

$$\mathrm{DHom}_{R_n^{\Lambda_0}}\big(S_{\mathcal{O}}^{\mu}, S_{\mathcal{O}}^{\lambda}\big) \xrightarrow{\sim} \mathrm{DHom}_{R_{n-\lambda_1}^{\Lambda_0}}\big(S_{\mathcal{O}}^{\bar{\mu}}, S_{\mathcal{O}}^{\bar{\lambda}}\big), \ \varphi \mapsto \bar{\varphi}$$

of graded $\mathcal{O}$-modules. Moreover, if $\varphi \in \mathrm{Hom}_{R_n^{\Lambda_0}}(S_{\mathcal{O}}^{\mu}, S_{\mathcal{O}}^{\lambda})$ is given by

$$\varphi(v^{\mu}) = \sum_{\mathtt{t}\in\mathsf{Std}(\lambda)} a_{\mathtt{t}} v^{\mathtt{t}}$$

for some coefficients $a_{\mathtt{t}} \in \mathcal{O}$, then

$$\bar{\varphi}(v^{\bar{\mu}}) = \sum_{\mathtt{t}\in\mathsf{Std}(\lambda)} a_{\mathtt{t}} v^{\bar{\mathtt{t}}}.$$

Remark 5.2.

(i) This result is stated in [10] in terms of column Specht modules when $\mathcal{O}$ is a field. However, the proof works for an arbitrary commutative ring $\mathcal{O}$ and can be translated to the present set-up by transposing all partitions and tableaux. The second assertion of Theorem 5.1 is clear from the proof of [10, Thm. 4.1].

(ii) By [10, Thm. 3.6], whenever $e > 2$, we have

$$\mathrm{DHom}_{R_n^{\Lambda_0}}(S_{\mathcal{O}}^{\mu}, D_{\mathcal{O}}^{\lambda}) = \mathrm{Hom}_{R_n^{\Lambda_0}}(S_{\mathcal{O}}^{\mu}, S_{\mathcal{O}}^{\lambda}) \quad \text{for all } \lambda, \mu \in \mathsf{Par}(n).$$

Our next goal is to apply Theorem 5.1 to 2-column partitions.

Lemma 5.3. *Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$. Suppose that $\mathtt{t}, \mathtt{s} \in \mathsf{Std}(\lambda)$ satisfy $\mathtt{s} \trianglelefteq \mathtt{t}$, and let $w \in \mathfrak{S}_n$ be determined from $\mathtt{s} = w\mathtt{t}$. Then $\ell(d(\mathtt{s})) = \ell(d(\mathtt{t})) + \ell(w)$.*

Proof. We may assume that $\mathtt{s} \neq \mathtt{t}$. Let $r \in [1, n]$ be smallest such that $\mathtt{s}(r) \neq \mathtt{t}(r)$. Then $\mathtt{s}(r) = (a{+}1, 1)$ and $\mathtt{t}(r) = (b{+}1, 2)$ for some $a, b \in \mathbb{Z}_{\geqslant 0}$. Let $v = \mathtt{t}^{-1}(a{+}1, 1)$ and $\mathtt{u} = s_{v-1}\mathtt{t}$. Then $\ell(d(\mathtt{u})) = \ell(d(\mathtt{t})) + 1$. We claim that $\mathtt{s} \trianglelefteq \mathtt{u}$. Indeed, for $j \in [1, n] \setminus \{v - 1\}$, we have $\mathrm{Shape}(\mathtt{u}{\downarrow_j}) = \mathrm{Shape}(\mathtt{t}{\downarrow_j}) \trianglerighteq \mathrm{Shape}(\mathtt{s}{\downarrow_j})$. On the other hand, the second column of $\mathrm{Shape}(\mathtt{s}{\downarrow_{v-1}})$ has size at most $b + v - 1 - r$, which is equal to the size of the second column of $\mathrm{Shape}(\mathtt{u}{\downarrow_{v-1}})$. This proves the claim. Arguing by induction, we may assume that the lemma holds with $\mathtt{t}$ replaced by $\mathtt{u}$, whence it follows that

$$\ell(d(\mathtt{s})) = \ell(d(\mathtt{u})) + \ell(ws_{v-1}) = \ell(d(\mathtt{u})) + \ell(w) - 1 = \ell(d(\mathtt{t})) + \ell(w). \qquad \square$$

Lemma 5.4. *Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$ and $\mathtt{t} \in \mathsf{Std}(\lambda)$.*

(i) *The element $v^{\mathtt{t}} \in S^\lambda_{\mathcal{O}}$ does not depend on the choice of a reduced expression for $d(\mathtt{t}) \in \mathfrak{S}_n$.*

(ii) *Let $\mathtt{s} \in \mathsf{Std}(\lambda)$ be such that $\mathtt{s} \trianglelefteq \mathtt{t}$. If $w \in \mathfrak{S}_n$ satisfies $w\mathtt{t} = \mathtt{s}$ then $\psi_w v^{\mathtt{t}} = v^{\mathtt{s}}$.*

Proof. By (2.9), in order to prove (i) it suffices to show that $w := d(\mathtt{t})$ is fully commutative, i.e., that every reduced expression for w can be obtained from any other by using only relations of the form $s_i s_j = s_j s_i$ for $i, j \in [1, n-1]$ with $|i - j| > 1$. By [1, Thm. 2.1], this is equivalent to showing that for no triple $1 \leqslant i < j < k \leqslant n$ it is the case that $wi > wj > wk$. This last statement is clear since two of i, j, k must lie in the same column of $\mathtt{t}$. Part (ii) follows from (i) because a reduced expression for $d(\mathtt{s})$ can be obtained by concatenating reduced expressions for $d(\mathtt{t})$ and w, by Lemma 5.3. $\square$

We note that the statements of Lemmas 5.3 and 5.4 are generally false for partitions with arbitrarily many columns. An analogue of Lemma 5.4(i) for 2-row partitions is [19, Lem. 3.17].

In the rest of the section, we assume that $\mathcal{O} = \mathbb{Z}$ and write $R^{\Lambda_0}_n := R^{\Lambda_0}_{n,\mathbb{Z}}$. The results proved over $\mathbb{Z}$ below may be seen to hold over an arbitrary commutative ring by extending scalars.

Lemma 5.5. *Suppose that $n \equiv j - 1 \bmod e$ with $1 \leqslant j < e$ and $n > j - 1$. Let $\mu = (1^n)$ and $\lambda = (2^j, 1^{n-2j})$. Then there is a dominated $R^{\Lambda_0}_n$-homomorphism $\varphi \colon S^\mu_{\mathbb{Z}} \to S^\lambda_{\mathbb{Z}}$ given by $v^\mu \mapsto v^{\mathtt{t}_\lambda}$. Moreover, φ is homogeneous of degree 1.*

Proof. By Theorem 4.8 and Lemma 4.7(ii), the $R^{\Lambda_0}_{n,\mathbb{Q}}$-module $S^\lambda_{\mathbb{Q}}$ has exactly two graded composition factors, namely $D^\lambda_{\mathbb{Q}}$ and $D^\mu_{\mathbb{Q}}\langle 1\rangle$, so there is a non-zero homomorphism of degree 1 from $S^\mu_{\mathbb{Q}}$ to $S^\lambda_{\mathbb{Q}}$. It is easy to see that $\mathtt{t}_\lambda$ is the only standard λ-tableau with residue sequence $\boldsymbol{i}^\mu$, so a scalar multiple of this homomorphism sends v^μ to $v^{\mathtt{t}_\lambda}$. In view of Corollary 2.2, the lemma follows. $\square$

Consider a partition $\lambda = (2^x, 1^y) \in \mathsf{Par}_{\leqslant 2}(n)$ such that $y \equiv -j-1 \pmod e$ and $x \geqslant j$ for some $1 \leqslant j < e$. We define $\mathtt{T}^\lambda_e$ to be the tableau obtained by putting

$\mathtt{t}^{(2^{x-j})}$ on top of the tableau $\mathtt{t}_{(2^j,1^y)}$, with all entries in the latter tableau increased by $2(x-j)$. In other words, for all $(a,b) \in [\![\lambda]\!]$, we set

$$(\mathtt{T}_e^\lambda)^{-1}(a,b) := \begin{cases} 2(a-1)+b & \text{if } a \leqslant x-j; \\ x-j+a & \text{if } a > x-j \text{ and } b=1; \\ x+y+a & \text{if } a > x-j \text{ and } b=2. \end{cases}$$

Theorem 5.6. *Let $\lambda = (2^x, 1^y) \in \mathsf{Par}_{\leqslant 2}(n)$. Suppose that $y \equiv -j-1 \pmod e$ for some $1 \leqslant j < e$ and that $x \geqslant j$. Let $\mu = (2^{x-j}, 1^{y+2j})$. Then there is a dominated $R_n^{\Lambda_0}$-module homomorphism*

$$\varphi_{\lambda,\mu} \colon S_{\mathbb{Z}}^\mu \to S_{\mathbb{Z}}^\lambda,\ v^\mu \mapsto v^{\mathtt{T}_e^\lambda},$$

which is homogeneous of degree 1.

Proof. This follows from Lemma 5.5 and Theorem 5.1 applied $x-j$ times. $\square$

Remark 5.7. Up to scalar multiples, the only non-zero homomorphisms between Specht modules $S_{\mathbb{Q}}^\lambda$ and $S_{\mathbb{Q}}^\mu$ for distinct $\lambda, \mu \in \mathsf{Par}_{\leqslant 2}(n)$ are obtained from those given by Theorem 5.6 by extending scalars. This follows from a consideration of composition factors of these Specht modules given by Theorem 4.8.

The aim of the rest of this section is to describe the kernel and image of the homomorphism $\varphi_{\lambda,\mu}$ from Theorem 5.6 in terms of e-regular tableaux.

Fix $\lambda = (2^x, 1^y)$ and $j \in [1, e-1]$ satisfying the hypotheses of Theorem 5.6. Let $m \in \mathbb{Z}_{>0}$ be such that $y = me - j - 1$. Recall the set H from (3.7). Given $0 \leqslant r \leqslant x-j$, set

$$Q_r := \{\mathtt{t} \in \mathsf{Std}(\lambda) \mid \pi_{\mathtt{t}}(2r+me-1) \in H_m \text{ and } \\ \pi_{\mathtt{t}}(a) \notin H \text{ whenever } 2r+me-1 < a \leqslant n\}.$$

That is, a standard λ-tableau $\mathtt{t}$ lies in Q_r if and only if the path $\pi_{\mathtt{t}}$ hits the wall H_m at step $2r+me-1$ and does not hit any walls at later steps. Then

$$\mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda) = \bigsqcup_{r=0}^{x-j} Q_r. \tag{5.1}$$

If $e=2$, then $j=1$ and $Q_r = \varnothing$ for all $r < x-j$. If $e > 2$, then each set Q_r on the right-hand side of (5.1) is non-empty.

Lemma 5.8. *Suppose that $0 \leqslant r \leqslant x-j$ and that $\mathtt{t}, \mathtt{s} \in Q_r$ satisfy $\mathtt{t}{\downarrow_{2r+me-1}} = \mathtt{s}{\downarrow_{2r+me-1}}$. If $w \in \mathfrak{S}_n$ is such that $\mathtt{s} = w\mathtt{t}$, then $v^{\mathtt{s}} = \psi_w v^{\mathtt{t}}$.*

Proof. First, note that we can turn $\mathtt{t}$ into $\mathtt{s}$ by a series of elementary transpositions of the form s_a with $2r+me-1 < a < n$. Hence, it suffices to prove the lemma when $w = s_a = (a, a+1)$ is an elementary transposition with $a > 2r+me-1$.

If $a+1$ lies in the first column of $\mathtt{t}$ and a lies in the second, then $\mathtt{t} \rhd \mathtt{s}$, so $v^{\mathtt{s}} = \psi_a v^{\mathtt{t}}$ by Lemma 5.4(ii), as required. So we may assume that a lies in the second column of $\mathtt{s}$, whence it follows that $v^{\mathtt{t}} = \psi_a v^{\mathtt{s}}$. Let k and l be determined from $\mathtt{s}(a) = (k, 2)$ and $\mathtt{s}(a+1) = (l, 1)$, and set $\boldsymbol{i} := \boldsymbol{i}^{\mathtt{s}}$. Then

$$(m-1)e-1 < \pi_{\mathtt{s}}(a) = l-k-1 < \pi_{\mathtt{s}}(a+1) = l-k < \pi_{\mathtt{t}}(a) = l-k+1 < me-1.$$

We have $i_a = 2-k+e\mathbb{Z}$ and $i_{a+1} = 1-l+e\mathbb{Z}$, so

$$i_a - i_{a+1} = 1+l-k+e\mathbb{Z} \notin \{0, 1, -1\}.$$

Hence, $\psi_a v^{\mathtt{t}} = \psi_a^2 v^{\mathtt{s}} = v^{\mathtt{s}}$ by (2.10). $\square$

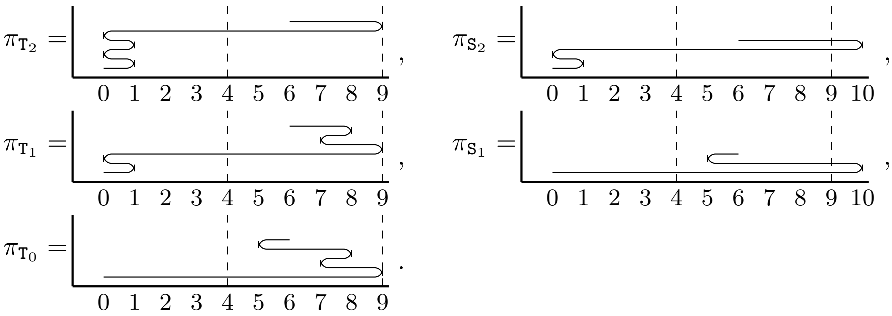
Definition. Suppose that $e > 2$. For each $r = 0, \dots, x-j$, we define a special element $\mathtt{T}_r \in Q_r$ as follows:

(a) $\mathtt{T}_{x-j} := \mathtt{T}_e^{\lambda}$.
(b) Suppose that $0 \leqslant r < x-j$. Then we set $\nu^r := (2^r, 1^{me-1}) \in \mathsf{Par}(2r+me-1)$ and define $\mathtt{T}_r$ to be an arbitrary (fixed) element of $\mathsf{Std}(\lambda)$ satisfying the following conditions:
  (i) $\mathtt{T}_r{\downarrow_{2r+me-1}} = \mathtt{t}^{\nu^r}$;
  (ii) $2r+me$ and $2r+me+1$ are in the second column of $\mathtt{T}_r$, and $2r+me+2$ is in the first column;
  (iii) whenever $2r+me+2 < a \leqslant n$, we have $(m-1)e-1 < \pi_{\mathtt{T}_r}(a) < me-1$ (or, equivalently, $\mathtt{T}_r \in Q_r$).

Further, for $0 \leqslant r < x-j$, define

$$\mathtt{S}_{r+1} := s_{2r+me} s_{2r+me+1} \mathtt{T}_r \in Q_{r+1}.$$

Example 5.9. Let $e = 5$ and $\lambda = (2^5, 1^6)$. Then $x - j = 2$, and the paths $\pi_{\mathtt{T}_r}$ and $\pi_{\mathtt{S}_r}$ (for a possible choice of $\mathtt{T}_0$ and $\mathtt{T}_1$) are as follows:



Lemma 5.10. *Assume that $e > 2$. Suppose that $0 \leqslant r \leqslant x-j$ and $\mathtt{t} \in Q_r$. If $\mathtt{t} = w\mathtt{T}_r$, where $w \in \mathfrak{S}_n$, then $v^{\mathtt{t}} = \psi_w v^{\mathtt{T}_r}$.*

Proof. Since $\mathtt{t}, \mathtt{T}_r \in Q_r$, we can write $w = w_1 w_2$, where w_1 fixes the set $[2r+me, n]$ pointwise and w_2 fixes $[1, 2r+me-1]$ pointwise. Set $\mathtt{t}_1 := w_1 \mathtt{T}_r$. By definition of $\mathtt{T}_r$, we have $\mathtt{T}_r{\downarrow_{2r+me-1}} = \mathtt{t}^{\nu^r}$, where $\nu^r = (2^r, me-1)$. Hence, $\mathtt{T}_r \trianglerighteq \mathtt{t}_1$, so by Lemma 5.4(ii) we have $\psi_{w_1} v^{\mathtt{T}_r} = v^{\mathtt{t}_1}$. Lemma 5.8 yields $\psi_{w_2} v^{\mathtt{t}_1} = v^{\mathtt{t}}$, whence the result follows. $\square$

Lemma 5.11. *If $e > 2$ and $0 \leqslant r \leqslant x-j$ then $\psi_{2r+me}v^{\mathtt{T}_r} = 0$.*

Proof. Let $\boldsymbol{i} = \boldsymbol{i}^{\mathtt{T}_r}$ and $\boldsymbol{i}' = s_{2r+me}\boldsymbol{i}$, so that $(1 - 1_{\boldsymbol{i}'})\psi_{2r+me}v^{\mathtt{T}_r} = 0$. It suffices to show that $\{\mathtt{t} \in \mathsf{Std}(\lambda) \mid \boldsymbol{i}^{\mathtt{t}} = \boldsymbol{i}'\} = \varnothing$, for then $1_{\boldsymbol{i}'}S^{\lambda}_{\mathbb{Z}} = 0$.

Suppose for contradiction that $\mathtt{t} \in \mathsf{Std}(\lambda)$ has residue sequence $\boldsymbol{i}'$. First, we claim that $\mathtt{t}(a) = \mathtt{T}_r(a)$ whenever $2r+me+2 \leqslant a \leqslant n$. Assuming the claim to be false, we choose a to be maximal such that the equality fails. By maximality of a,

$$\mathrm{Shape}(\mathtt{t}{\downarrow_a}) = \mathrm{Shape}(\mathtt{T}_r{\downarrow_a}) =: (2^c, 1^d)$$

for some $c, d \in \mathbb{Z}_{\geqslant 0}$. Since a lies in different columns in $\mathtt{t}$ and $\mathtt{T}_r$ and $i^{\mathtt{t}}_a = i^{\mathtt{T}_r}_a$, the residues of the bottom nodes of the two columns of $(2^c, 1^d)$ must be equal. However, since $(m-1)e - 1 < \pi_{\mathtt{T}_r}(a) < me - 1$, we have $d \not\equiv -1 \pmod e$, from which it follows that these two residues are not equal. This contradiction proves the claim.

Using this claim, we obtain

$$\mathrm{Shape}(\mathtt{t}{\downarrow_{2r+me+1}}) = \mathrm{Shape}(\mathtt{T}_r{\downarrow_{2r+me+1}}) = (2^{r+me-1}, 1^{r+2}) =: \gamma.$$

Hence,

$$\boldsymbol{i}'_{2r+me+1} = \boldsymbol{i}_{2r+me} = \mathrm{res}\big(\mathtt{T}_r(2r+me)\big) = \mathrm{res}(r+1, 2) = 1 - r + e\mathbb{Z}.$$

Therefore, $\mathrm{res}\,\big(\mathtt{t}(2r+me+1)\big) = 1 - r + e\mathbb{Z}$. However, $\mathtt{t}(2r+me+1)$ must be the bottom entry of either the first or the second column of γ, and these two entries have residues $2 - r + e\mathbb{Z}$ and $-r + e\mathbb{Z}$ respectively, a contradiction. $\square$

Lemma 5.12. *If $e > 2$ and $0 \leqslant r < x - j$, then $v^{\mathtt{T}_r} = -\psi_{2r+me+1}v^{\mathtt{S}_{r+1}}$.*

Proof. It is easy to see that $\mathtt{S}_{r+1} \lhd \mathtt{T}_r$. So, by Lemma 5.4(ii), we have $v^{\mathtt{S}_{r+1}} = \psi_{2r+me}\psi_{2r+me+1}v^{\mathtt{T}_r}$. Further, note that

$$\mathrm{res}\big(\mathtt{T}_r(2r+me)\big) = \mathrm{res}\big(\mathtt{T}_r(2r+me+2)\big) = \mathrm{res}\big(\mathtt{T}_r(2r+me+1)\big) + 1 = -r+1+e\mathbb{Z}.$$

Hence,

$$\begin{aligned} -\psi_{2r+me+1}v^{\mathtt{S}_{r+1}} &= -\psi_{2r+me+1}\psi_{2r+me}\psi_{2r+me+1}v^{\mathtt{T}_r} \\ &= -(\psi_{2r+me}\psi_{2r+me+1}\psi_{2r+me} - 1)v^{\mathtt{T}_r} \\ &= v^{\mathtt{T}_r}, \end{aligned}$$

where we have used (2.11) for the second equality and Lemma 5.11 for the third equality. $\square$

Proposition 5.13. *If $\mathtt{t} \in \mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)$ then $v^{\mathtt{t}}$ lies in the $R^{\Lambda_0}_n$-submodule of $S^{\lambda}_{\mathbb{Z}}$ generated by $v^{\mathtt{T}^{\lambda}_e}$.*

Proof. If $e = 2$, then $\mathtt{T}^{\lambda}_e$ dominates every element of $Q_{x-j} = \mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)$ (cf. the proof of Lemma 5.10), so the result follows from Lemma 5.4(ii).

Assume that $e > 2$. Let U be the submodule in the statement of the proposition. By Lemma 5.10, it is enough to show that $v^{\mathtt{T}_r} \in U$ for all $0 \leqslant r \leqslant x - j$. We use backward induction on r and note that $v^{\mathtt{T}_{x-j}} = v^{\mathtt{T}^{\lambda}_e} \in U$, so we may assume that $r < x - j$ and $v^{\mathtt{T}_{r+1}} \in U$. By the inductive hypothesis and Lemma 5.10 applied again, $\mathtt{S}_{r+1} \in U$. Hence, $v^{\mathtt{T}_r} \in U$ by Lemma 5.12. $\square$

The hypotheses of the following theorem are the same as in Theorem 5.6.

Theorem 5.14. *Let $\lambda = (2^x, 1^y) \in \mathsf{Par}_{\leqslant 2}(n)$. Suppose that $y \equiv -j-1 \pmod e$ for some $1 \leqslant j < e$ and that $x \geqslant j$. Let $\mu = (2^{x-j}, 1^{y+2j})$ and let $\varphi_{\lambda,\mu} \colon S^\mu_{\mathbb{Z}} \to S^\lambda_{\mathbb{Z}}$ be as in Theorem* 5.6*. Then*

$$\ker(\varphi_{\lambda,\mu}) = \langle v^{\mathsf{t}} \mid \mathsf{t} \in \mathsf{Std}(\mu) \setminus \mathsf{DStd}_e(\mu)\rangle_{\mathbb{Z}} \quad \textit{and}$$
$$\varphi_{\lambda,\mu}(S^\mu_{\mathbb{Z}}) = \langle v^{\mathsf{s}} \mid \mathsf{s} \in \mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)\rangle_{\mathbb{Z}}.$$

Proof. We use backward induction on y. Note that the kernel and image of $\varphi_{\lambda,\mu}$ are free $\mathbb{Z}$-modules by Proposition 2.1. By Proposition 5.13,

$$\varphi_{\lambda,\mu}(S^\mu_{\mathbb{Z}}) \supseteq \langle v^{\mathsf{s}} \mid \mathsf{s} \in \mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)\rangle_{\mathbb{Z}}. \tag{5.2}$$

Hence,

$$\begin{aligned} \dim_{\mathbb{Z}} \ker(\varphi_{\lambda,\mu}) &= |\mathsf{Std}(\mu)| - \dim_{\mathbb{Z}}(\varphi_{\lambda,\mu}(S^\mu_{\mathbb{Z}})) \\ &\leqslant |\mathsf{Std}(\mu)| - |\mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)| \\ &= |\mathsf{Std}(\mu)| - |\mathsf{DStd}_e(\mu)| = |\mathsf{Std}(\mu) \setminus \mathsf{DStd}_e(\mu)|, \end{aligned} \tag{5.3}$$

where the penultimate equality holds because, by Lemma 3.6, the map reg_e restricts to a bijection $\mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda) \xrightarrow{\sim} \mathsf{DStd}_e(\mu)$.

We claim that

$$\ker(\varphi_{\lambda,\mu}) \supseteq \langle v^{\mathsf{t}} \mid \mathsf{t} \in \mathsf{Std}(\mu) \setminus \mathsf{DStd}_e(\mu)\rangle_{\mathbb{Z}}. \tag{5.4}$$

Let $m \in \mathbb{Z}_{>0}$ be such that $y = me - j - 1$. If $x < e$, then for all $\mathsf{t} \in \mathsf{Std}(\mu)$ and $0 \leqslant a \leqslant n$ we have $\pi_{\mathsf{t}}(a) \notin H_{m+1}$, from which it follows that $\mathsf{t} \in \mathsf{DStd}_e(\mu)$ and (5.4) holds trivially, as its right-hand side is 0. So, suppose that $x \geqslant e$ and let $\nu := (2^{x-e}, 1^{y+2e})$, noting that $y + 2e$ is the image of $y + 2j$ under reflection with respect to H_{m+1}. By the inductive hypothesis, the right-hand side of (5.4) is exactly the image of $\varphi_{\mu,\nu}$. Now $\varphi_{\lambda,\mu}\varphi_{\mu,\nu} = 0$ because, by Theorem 4.8, $S^\lambda_{\mathbb{Q}}$ and $S^\nu_{\mathbb{Q}}$ have no composition factors in common. Thus, the claim follows.

Using (5.3) and (5.4), we obtain the first equality in the theorem. Also, (5.3) is an exact equality, so the two sides of (5.2) have the same $\mathbb{Z}$-rank. This completes the proof since the right-hand side of (5.2) is a pure $\mathbb{Z}$-submodule of $S^\lambda_{\mathbb{Z}}$. □

Remark 5.15. Let $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$. It is not difficult to show that if $\mathsf{t} \in \mathsf{DStd}_e(\lambda)$ and $\mathsf{s} \in \mathsf{Std}(\lambda) \setminus \mathsf{DStd}_e(\lambda)$ then $\boldsymbol{i}^{\mathsf{t}} \neq \boldsymbol{i}^{\mathsf{s}}$ (cf. the claim in the proof of Lemma 5.11). This leads to a more direct proof of the first equality in Theorem 5.14.

Proof of Theorem 1.1. Let $\lambda = (2^x, 1^y)$, where $y = me - 1 - j$ for some $m \in \mathbb{Z}_{>0}$ and $0 \leqslant j < e$. If $j = 0$ or $x < j$, then $\mathsf{DStd}_e(\lambda) = \mathsf{Std}(\lambda)$ and parts (i) and (iii) of the theorem hold. Also, in this case $S^\lambda_K = D^\lambda_K$ by Theorem 4.8 and Lemma 4.7, so (ii) holds as well.

On the other hand, if $j \neq 0$ and $x \geqslant j$, then (i) and (iii) follow from Theorems 5.6 and 5.14, and (ii) again follows from Theorem 4.8. □

Remark 5.16. Theorem 1.1(ii) is true with K replaced by any field of characteristic 0.

Recall the reflections s_m from §3.3: we have $s_m \cdot (me - 1 + j) = me - 1 - j$ for all $m \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}$.

Corollary 5.17. *Let* $\lambda^1 = (2^{x_1}, 1^{y_1}) \in \mathsf{Par}_{\leqslant 2}(n)$. *Suppose that* $y_1 = me - 1 + j$ *for some* $m \in \mathbb{Z}_{>0}$ *and* $1 \leqslant j < e$, *and that* $x_1 < e - j$. *For* $k = 1, \dots, m$, *let* $y_k = s_{m-k} \cdot \dots \cdot s_m \cdot y_1$ *and* $\lambda^k = (2^{x_k}, 1^{y_k}) \in \mathsf{Par}_{\leqslant 2}(n)$, *where* $x_k = (n - y_k)/2$. *Then the following is an exact sequence of* $R_n^{\Lambda_0}$*-homomorphisms:*

$$0 \to S_{\mathbb{Z}}^{\lambda^1} \xrightarrow{\varphi_{\lambda^2,\lambda^1}} S_{\mathbb{Z}}^{\lambda^2} \xrightarrow{\varphi_{\lambda^3,\lambda^2}} S_{\mathbb{Z}}^{\lambda^3} \to \cdots \to S_{\mathbb{Z}}^{\lambda^m}.$$

Proof. The hypothesis ensures that $\mathsf{DStd}_e(\lambda^1) = \mathsf{Std}(\lambda^1)$, whence the result follows from Theorems 5.6 and 5.14. □

6. Partitions with more than two columns

In this section, we outline a natural approach to extending the definition of $\mathsf{DStd}_e(\lambda)$ to the case when a partition λ has more than two columns and give an example showing that this approach does not always work. For simplicity, we consider algebras over $\mathbb{Q}$ only, though all statements below are true with $\mathbb{Q}$ replaced by any field of characteristic 0.

Fix $n \in \mathbb{Z}_{\geqslant 0}$ and $\lambda \in \mathsf{Par}(n)$. Given $\mathtt{t} \in \mathsf{Std}(\lambda)$ and $u \in S_{\mathbb{Q}}^{\lambda}$, we say that u is a $\mathtt{t}$-*element* if

$$u = v^{\mathtt{t}} + \sum_{\substack{\mathtt{s} \in \mathsf{Std}(\lambda) \\ \mathtt{s} \rhd \mathtt{t}}} a_{\mathtt{s}} v^{\mathtt{s}}$$

for some coefficients $a_{\mathtt{s}} \in \mathbb{Q}$. It is important to note that, whereas the elements $v^{\mathtt{t}}$ depend on certain choices of reduced expressions, the set of all $\mathtt{t}$-elements on $S_{\mathbb{Q}}^{\lambda}$ does not depend on such choices. This is the case because if one changes the reduced expression for $d(\mathtt{t})$, causing $v^{\mathtt{t}}$ to be replaced by $v_1^{\mathtt{t}}$, then $v_1^{\mathtt{t}} = v^{\mathtt{t}} + \sum_{\mathtt{s} \in \mathsf{Std}(\lambda),\, \mathtt{s} \rhd \mathtt{t}} a_{\mathtt{s}} v^{\mathtt{s}}$ for some coefficients $a_{\mathtt{s}} \in \mathbb{Q}$ (by [5, Prop. 4.7]).

Recall from §2.4 the bilinear form $\langle \cdot, \cdot \rangle$ on $S_{\mathbb{Q}}^{\lambda}$ and its radical $\operatorname{rad} S_{\mathbb{Q}}^{\lambda}$. Note that, by the properties of the form stated after (2.17), for any $\mathtt{t}, \mathtt{s} \in \mathsf{Std}(\lambda)$, we have $\langle v^{\mathtt{t}}, v^{\mathtt{s}} \rangle = 0$ unless $\deg(\mathtt{t}) + \deg(\mathtt{s}) = 0$ and $\boldsymbol{i}^{\mathtt{t}} = \boldsymbol{i}^{\mathtt{s}}$. These facts are used repeatedly in the sequel. Define

$$\mathsf{IStd}_e(\lambda) := \{\mathtt{t} \in \mathsf{Std}(\lambda) \mid \operatorname{rad} S_{\mathbb{Q}}^{\lambda} \text{ contains at least one } \mathtt{t}\text{-element}\}.$$

For each $\mathtt{t} \in \mathsf{IStd}_e(\lambda)$, choose a $\mathtt{t}$-element $w^{\mathtt{t}} \in \operatorname{rad} S_{\mathbb{Q}}^{\lambda}$. The set $\{w^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{IStd}_e(\lambda)\}$ is always linearly independent over $\mathbb{Q}$. In the sequel, we call the partition λ e-*agreeable* if $\{w^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{IStd}_e(\lambda)\}$ is a basis of $\operatorname{rad} S_{\mathbb{Q}}^{\lambda}$ or, equivalently, if $|\mathsf{IStd}_e(\lambda)| = \dim(\operatorname{rad} S_{\mathbb{Q}}^{\lambda})$. Whether or not $\{w^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{IStd}_e(\lambda)\}$ spans $\operatorname{rad} S_{\mathbb{Q}}^{\lambda}$ does not depend on the choice of the elements $w^{\mathtt{t}}$. Note that for each $\mathtt{t} \in \mathsf{Std}(\lambda)$ one can choose $w^{\mathtt{t}}$ in such a way that $w^{\mathtt{t}}$ is homogeneous of degree $\deg_e(\mathtt{t})$ and $1_{\boldsymbol{i}^{\mathtt{t}}} w^{\mathtt{t}} = w^{\mathtt{t}}$.

Whenever λ is e-agreeable, it is reasonable to define $\mathsf{DStd}_e(\lambda)$ as the complement $\mathsf{Std}(\lambda) \setminus \mathsf{IStd}_e(\lambda)$. Then $\operatorname{ch}_q \mathsf{DStd}_e(\lambda) = \operatorname{ch}_q D_{\mathbb{Q}}^{\lambda}$ thanks to the last observation in the previous paragraph. Theorem 1.1 shows that, if $\lambda \in \mathsf{Par}_{\leqslant 2}(n)$, then λ is e-agreeable and the definition of $\mathsf{DStd}_e(\lambda)$ just given agrees with the combinatorial one in §3.3.

Remark 6.1. If we replaced the dominance order by an arbitrary *total* order on $\mathsf{Std}(\lambda)$ in the definition of a $\mathtt{t}$-element, then $\{w^{\mathtt{t}} \mid \mathtt{t} \in \mathsf{IStd}_e(\lambda)\}$ would automatically be a basis of rad $S^\lambda_\mathbb{Q}$ by elementary linear algebra. This construction is discussed in [14, §3.3].

Hu and Mathas [14, Sec. 6] construct a distinguished homogeneous cellular basis

$$\bigsqcup_{\mu \in \mathsf{Par}(n)} \{B_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathsf{Std}(\mu)\}$$

of $R^{\Lambda_0}_{n,\mathbb{Q}}$, which does not depend on any choices of reduced expressions. This new cellular structure yields a basis $\{B_\mathtt{t} \mid \mathtt{t} \in \mathsf{Std}(\lambda)\}$ of $S^\lambda_\mathbb{Q}$. It follows from [14, Prop. 6.7] that $B_\mathtt{t}$ is a $\mathtt{t}$-element for each $\mathtt{t} \in \mathsf{Std}(\lambda)$.

A conjecture of Mathas [23, Conj. 4.4.1] implies, in particular, that for every $\lambda \in \mathsf{Par}(n)$ there is a subset $\mathscr{T}_\lambda$ of $\mathsf{Std}(\lambda)$ such that $\{B_\mathtt{t} \mid \mathtt{t} \in \mathscr{T}_\lambda\}$ is a basis of rad $S^\lambda_\mathbb{Q}$. Since each $B_\mathtt{t}$ is a $\mathtt{t}$-element, this in turn implies (via elementary linear algebra) that every partition λ is e-agreeable. However, Example 6.2 below shows that (for $e = 3$) not all partitions are e-agreeable. Hence, there is a counterexample to [23, Conj. 4.4.1].

The following example was discovered using a GAP [11] program for calculating Gram matrices of Specht modules. The program uses results from [14], especially a certain seminormal basis, and is available on the fourth author's website.[2] Below we give a self-contained and computer-independent verification.

Example 6.2. Let $e = 3$, $n = 8$ and $\lambda = (4, 3, 1)$. Consider the tuple $\boldsymbol{i} = (0, 1, 2, 2, 0, 1, 0, 1) \in I^8$. The set $\mathsf{Std}(\lambda, \boldsymbol{i})$ has exactly two elements of degree 2, namely

$$\mathtt{t}_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 7 & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \mathtt{t}_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array}\ .$$

Further, $\mathsf{Std}(\lambda, \boldsymbol{i})$ has exactly one element of degree -2, namely

$$\mathtt{s} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}\ .$$

Note that $y^\lambda = y_3 y_7$. We use the reduced expressions

$$d(\mathtt{t}_1) = s_6 s_7 s_4, \qquad d(\mathtt{t}_2) = s_6 s_5 s_4, \qquad d(\mathtt{s}) = s_6 s_7 s_5 s_3 s_4.$$

Using (2.17), we compute (cf. [23, Example 3.7.9]):

$$\begin{aligned}\langle v^{\mathtt{t}_1}, v^{\mathtt{s}} \rangle v^{\mathtt{t}^\lambda} &= 1_{\boldsymbol{i}^\lambda} y_3 y_7 \psi_4 \psi_7 \psi_6^2 \psi_7 \psi_5 \psi_3 \psi_4 v^{\mathtt{t}_\lambda} \\ &= y_3 y_7 \psi_4 \psi_7 (y_6 - y_7) \psi_7 \psi_5 \psi_3 \psi_4 v^{\mathtt{t}_\lambda}\end{aligned}$$

[2] http://www.people.virginia.edu/~ls2zz/papers.html

$$
\begin{aligned}
&= -y_3y_7\psi_4\psi_7y_7\psi_7\psi_5\psi_3\psi_4v^{\mathtt{t}^\lambda}\\
&= y_3y_7\psi_4\psi_7\psi_5\psi_3\psi_4v^{\mathtt{t}_\lambda}\\
&= -y_3\psi_4\psi_5\psi_3\psi_4v^{\mathtt{t}^\lambda}\\
&= \psi_4\psi_5\psi_4v^{\mathtt{t}^\lambda}\\
&= (\psi_5\psi_4\psi_5+1)v^{\mathtt{t}^\lambda}\\
&= v^{\mathtt{t}^\lambda},
\end{aligned}
$$

where we have repeatedly used the relations (2.13)–(2.15), and moreover we have used (2.10) for the second equality, (2.8) and (2.10) for the third equality, (2.5)–(2.8) for the fourth, fifth and sixth equalities and (2.11) for the seventh equality. Further,

$$
\begin{aligned}
\langle v^{\mathtt{t}_2}, v^{\mathtt{s}}\rangle v^{\mathtt{t}^\lambda} &= 1_{\boldsymbol{i}^\lambda}y_3y_7\psi_4\psi_5\psi_6^2\psi_7\psi_5\psi_3\psi_4v^{\mathtt{t}^\lambda}\\
&= y_3y_7\psi_4\psi_5(y_6-y_7)\psi_7\psi_5\psi_3\psi_4v^{\mathtt{t}^\lambda}\\
&= y_3y_7\psi_4\psi_5y_6\psi_7\psi_5\psi_3\psi_4v^{\mathtt{t}^\lambda}\\
&= y_3y_7\psi_4\psi_5\psi_7\psi_3\psi_4v^{\mathtt{t}^\lambda}\\
&= v^{\mathtt{t}^\lambda},
\end{aligned}
$$

where we have used (2.10) for the second equality, (2.8), (2.9) and (2.10) for the third equality, (2.5)–(2.8) for the fourth equality, and the final equality repeats the end of the previous computation.

Hence, $\langle v^{\mathtt{t}_1}, v^{\mathtt{s}}\rangle = 1 = \langle v^{\mathtt{t}_2}, v^{\mathtt{s}}\rangle$. Therefore, the degree 2 component of $1_{\boldsymbol{i}}(\operatorname{rad} S^\lambda_{\mathbb{Q}})$ is 1-dimensional and is spanned by $v^{\mathtt{t}_1} - v^{\mathtt{t}_2}$. However, since neither of $\mathtt{t}_1$ and $\mathtt{t}_2$ dominates the other, neither of these two tableaux belongs to $\mathsf{IStd}_e(\lambda)$. Hence, λ is not e-agreeable.

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