Model theory of finite and pseudofinite rings

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The limits of my language means the limits of my world.

L. Wittgenstein.

Abstract

The model theory of finite and pseudofinite fields as well as the model theory of finite and pseudofinite groups have been and are thoroughly studied. A close relation has been found between algebraic and model theoretic properties of pseudofinite fields and psedudofinite groups.

In this thesis we present results contributing to the beginning of the study of model theory of finite and pseudofinite rings.

In particular we classify the theory of ultraproducts of finite residue rings in the context of generalised stability theory. We give sufficient and necessary conditions for the theory of such ultraproducts to be NIP, simple, NTP_2 but not simple nor NIP, or TP_2 .

Further, we show that for any fixed positive $l \in \mathbb{N}$ the class of finite residue rings $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$ forms an l-dimensional asymptotic class. We discuss related classes of finite residue rings in the context of R-multidimensional asymptotic classes.

Finally we present a classification of simple and semisimple (in the algebraic sense) pseudofinite rings, we study NTP_2 classes of J-semisimple rings and we discuss NIP classes of finite rings and ultraproducts of these NIP classes.

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Chapter 1

Introduction

1.1 Brief overview

In this work we deal with finite and pseudofinite rings, so it is natural to start with some background comments on pseudofinite structures. We first discuss pseudofinite fields and we use this to introduce pseudofinite structures.

We say M is a **pseudofinite** field (resp. ring, group) if it is an infinite field (resp. ring, group) which satisfies every first order sentence which is true in every finite field (resp. ring, group).

Pseudofinite fields were thoroughly studied in [2] where Ax provided a classification for these fields. Namely the following was shown.

Theorem 1.1.1 (Ax, [2])**.** *A field F is pseudofinite if and only if the following holds.*

i) F is perfect. I.e. is of characteristic 0 or if it has characteristic p then every element has a p-th root.

ii) F is quasifinite. This means, within a fixed algebraic closure, for every n there is a unique field extension of degree n.

iii) F is pseudo-algebraically closed. That is, each absolutely irreducible variety V defined over F has an F-rational point.

Ax also showed in [2] that the theory of finite fields is decidable, i.e., there is an algorithm to decide whether or not a given sentence holds for all finite fields.

Examples of pseudofinite fields are mentioned in 1.3.1 below.

Pseudofinite groups form another class of pseudofinite structures that has been widely studied and for which a rich theory has been found. Notably we find the main result in [49] in which Wilson nicely relates simple pseudofinite groups and pseudofinite fields, see Theorem 1.3.4 in Subsection 1.3.2. Also, as noted in [49] it follows from a result of F. Point that Chevalley groups over a pseudofinite field are pseudofinite groups.

Another important result from Wilson is the fact that a certain important radical of a group can be nicely described. Namely in [50] we find Theorem 1.3.6, where Wilson finds that the soluble radical, see Subsection 1.3.2, can be defined uniformly in finite groups using a first order formula.

Finally, it is the aim of this work to contribute towards the study of pseudofinite rings.

1.2 Model theory

1.2.1 Preliminaries on model theory

We now present some model theoretic notions that have an important role in the present work.

Notation throughout the thesis is mostly standard.

Typically we use $\mathcal{L}$ to denote a first order language, $\mathcal{M}$ to denote an $\mathcal{L}$-structure, and T for a complete first order theory T. Sometimes we use $Th(\mathcal{M})$ to denote the complete first order theory of $\mathcal{M}$. We always work with a first order structure. Given a first order language $\mathcal{L}$, a model $\mathcal{M}$ and $A \subseteq \mathcal{M}$ we denote by $\mathcal{L}_A$ the language obtained by adding to $\mathcal{L}$ a constant c_a for each element a in A. Then, $\mathcal{M}$ is an $\mathcal{L}_A$ structure by interpreting each constant c_a by $a \in A$. We call a set $\mathbf{p}(\bar{x}) = \{\varphi_i(\bar{x})\}$ of $\mathcal{L}_A$-formulas consistent with $Th(\mathcal{M}, a)_{a \in A}$ a *partial* $|\bar{x}|$*-type over* A. If the set $\mathbf{p}(\bar{x})$ is maximal we call it a *complete type*, and we usually drop the adjective 'complete' and call $\mathbf{p}$ a *type*. Further, given an element $b \in \mathcal{M}$ we denote by $tp(b/A)$ the set of $\mathcal{L}_A$- formulas that are satisfied in $\mathcal{M}$ by the element b.

Also we write x, y, ... for variables and a, b, ... for parameters (possibly tuples). If we want to emphasise that a variable or parameter is a tuple we will write $\bar{x}$, $\bar{y}$ or $\bar{a}$, $\bar{b}$ respectively.

We use T to denote a complete first order theory and $\overline{\mathcal{M}}$ to denote a sufficiently saturated model of T.

For the sake of completeness we present here a couple of model theoretic notions. For a more in-depth, thorough treatment the reader may check [46], [33] or the classic text [22].

Definition 1.2.1. *Let I be a linear order and $\mathcal{M}$ an $\mathcal{L}$ structure. We say a sequence $(a_i : i \in I)$ of elements (possibly tuples) in $\mathcal{M}$ is* **indiscernible over** *A if and only if for all $\mathcal{L}_A$-formulas $\varphi(x_1, \dots, x_n)$ and all $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ from I we have $\mathcal{M} \models \phi(a_{i_1}, \dots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_n})$.*

Let $\{(a_{i,j} : j \in I_i) : i \in S\}$ be a family of indiscernible sequences over A. We say the sequences are **mutually indiscernible over** *A if and only if for each $k \in S$ the sequence $(a_{k,j} : j \in I_k)$ is indiscernible over $A \cup \big(\bigcup_{i \neq k} \{(a_{i,j} : j \in I_i)\}\big)$.*

Definition 1.2.2. *1. We say a formula $\varphi(x, a)$* **divides** *over a set B of parameters if there is an indiscernible sequence $(a_i : i \in \omega)$ of realisations of $tp(a/B)$ such that*

$\{\varphi(x, a_i) : i \in \omega\}$ is k-inconsistent for some $k \in \mathbb{N}$, i.e. any collection of k many formulas from $\{\varphi(x, a_i) : i \in \omega\}$ is inconsistent.

2. *We say a formula $\varphi(x, a)$* **forks** *over B if and only if it implies a disjunction $\bigvee_1^n \varphi_l(x, a_l)$ of formulas $\varphi_l(x, a_l)$ each of them dividing over B.*

Definition 1.2.3. *We will say that a first order structure $\mathcal{M}$ in the language $\mathcal{L}$ has* **quantifier elimination** *if and only if any $\mathcal{L}$-formula ϕ is equivalent in $\mathcal{M}$ to a formula ψ without quantifiers, cf. [46].*

For more on quantifier elimination see [22] or [46].

1.2.2 Ultraproducts, pseudofinite structures

As usual, given a set I of indices, a family $\{\mathcal{M}_i : i \in I\}$ of first order $\mathcal{L}$-structures and $\mathcal{U}$ an ultrafilter on I, we define an equivalence relation $\sim$ between infinite sequences in the cartesian product $\prod_{i \in I} \mathcal{M}_i$ where $(a_i) \sim (b_i)$ if and only if $\{i \in I : a_i = b_i\} \in \mathcal{U}$. Then we denote by either $\prod_{\mathcal{U}} \mathcal{M}_i$ or $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ the set of equivalence classes from the equivalence relation $\sim$. We often write $[a_i]_{\mathcal{U}}$ for an element of $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$; we may write $[a_i]$ if the ultrafilter $\mathcal{U}$ is clear from the context. Furthermore $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ is itself an $\mathcal{L}$-structure, interpreting constants $c \in \mathcal{L}$ as $[c^{\mathcal{M}_i}]_{\mathcal{U}}$, making $f([t_i]_{\mathcal{U}}) = [f(t_i)]_{\mathcal{U}}$ and defining for each relation R in $\mathcal{L}$ that $R([t_{1,i}]_{\mathcal{U}}, \ldots, [t_{n,i}]_{\mathcal{U}})$ holds in $\prod_{\mathcal{U}} \mathcal{M}_i$ if and only if

$$\{i \in I : \mathcal{M}_i \models R(t_{1,i}, \ldots, t_{n,i})\} \in \mathcal{U}.$$

We call $\prod_{\mathcal{U}} \mathcal{M}_i$ the *ultraproduct* of the structures $\mathcal{M}_i$ with respect to the ultrafilter $\mathcal{U}$. We say an ultraproduct is non-principal if the ultrafilter considered contains no finite set. We will write $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, ... for ultrafilters, and U, V, W, ... for elements of ultrafilters. Unless specified otherwise, "$\mathcal{U}$ is an ultrafilter" means $\mathcal{U}$ is a non-principal ultrafilter. For

V an element in a non-principal ultrafilter $\mathcal{U}$ we call $\{V \cap U : U \in \mathcal{U}\}$ the *induced ultrafilter on V by $\mathcal{U}$*.

The following result relates the first order behaviour of the ultraproduct to the behaviour of the factors.

Theorem 1.2.4 (Łoś's theorem)**.** *If $\{\mathcal{M}_i : i \in I\}$ is a family of $\mathcal{L}$-structures, φ is an $\mathcal{L}$-formula and $[a_i]_{\mathcal{U}} \in \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ then*

$$\prod_{i \in I} \mathcal{M}_i/\mathcal{U} \models \varphi([a_i]_{\mathcal{U}}) \text{ if and only if } \{i \in I : \mathcal{M}_i \models \varphi(a_i)\} \in \mathcal{U}.$$

Further details about ultraproducts can be found in [5].

We now connect the notion of pseudofiniteness and the construction of ultraproducts. Namely a structure $\mathcal{M}$ is pseudofinite if and only if it is elementarily equivalent to a non-principal ultraproduct of finite structures. See Theorem 1.3.2 in Subsection 1.3.1.

A more in depth survey about pseudofinite structures can be found in [29].

1.2.3 Stable theories

We now present a very brief overview of the generalised stability theory concepts we will use in the course of the thesis. For more about these topics the reader can check [46] or [45].

It can be argued that stability theory started with the work of Morley in [36] when he studied the number of possible models of a given cardinality for a given theory, proving the following.

Theorem 1.2.5 (Morley [36])**.** *Let κ be an uncountable cardinal. If a theory in a countable language $\mathcal{L}$ is κ-categorical i.e., there is only one, up to isomorphism, model of size κ then it is λ-categorical for all uncountable λ.*

Morley's result gives a strong result concerning theories with very few models, up to isomorphism, of an uncountable cardinality.

Later work of Shelah and others gave an appropriate context for distinguishing when theories have few or plenty of models of a given cardinality. It is in this context that the next definition was first considered.

The notion of stable theory was introduced in [42]. These theories provide a nice context for talking about the possible number of distinct models of a given cardinality of a theory.

Definition 1.2.6. *We say an $\mathcal{L}$-formula $\varphi(x, y)$ has the **order property** in $\mathcal{M}$ if there are sequences $(a_i : i \in \omega)$ and $(b_j : j \in \omega)$ of elements (possibly tuples) in $\mathcal{M}$ such that $\varphi(a_i, b_j)$ holds in $\mathcal{M}$ if and only if $i < j$.*

We say a theory T is **stable** *if no formula has the order property in any model of T. We say the structure $\mathcal{M}$ is stable if $Th(\mathcal{M})$ is stable.*

Theorem 1.2.7. *An unstable theory over a countable language has 2^λ models of cardinality λ for any uncountable cardinal λ.*

For example algebraically closed fields (ACF) are stable.

1.2.4 NIP theories.

We follow the notational conventions of 1.2.1.

Definition 1.2.8. *An $\mathcal{L}$-formula $\varphi(x, y)$ has the **independence property** for T if there are $(a_i : i \in \mathbb{N})$ and $(b_J : J \subseteq \mathbb{N})$ in $\overline{\mathcal{M}}$ such that $\overline{\mathcal{M}} \models \varphi(a_i, b_J)$ if and only if $i \in J$.*

*We say that a theory T is **NIP** if no formula satisfies the independence property in any model of T. Moreover we say a structure $\mathcal{M}$ is NIP if $Th(\mathcal{M})$ is NIP.*

The useful notion of **dp-rank** is defined as follows in [45].

Consider p to be a partial type, and A to be a set of parameters.

Definition 1.2.9 (Definition 4.12, [45]). *We say a (partial) type p has **dp-rank less than** α in symbols **dp-rk**$(p/A) < \alpha$, if for every family $(I_t : t < \alpha)$ of mutually indiscernible sequences over A and $b \models p$, there is $t < \alpha$ such that I_t is Ab-indiscernible.*

*We say a partial type has **dp-rank equal to** α, in symbols **dp-rk**$(p/A) = \alpha$, if **dp-rk**$(p/A) < \alpha^+$ but is not the case that **dp-rk**$(p/A) < \alpha$.*

*Given $a \in \mathcal{M}$ we write **dp-rk**(a/A) for **dp-rk**$(tp(a)/A)$.*

*When there is no ambiguity we will write **dp-rk**(p) for **dp-rk**(p/A)*

Example 1.2.10. *In [3] it is shown that for any prime p the p-adic numbers $\mathbb{Q}_p$ has NIP theory. Furthermore in [16] it is shown that it is dp-minimal, i.e. has dp-rank 1. In Section 6 of [16] the authors prove that every sufficiently saturated elementary extension of $\mathbb{Q}_p$ is dp-minimal.*

For further introduction and overview of NIP theories the reader can check [45].

1.2.5 NSOP theories

The notion of *NSOP* theory was first introduced in [43].

Definition 1.2.11. *An $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ has **the strict order property, SOP** (with respect to a complete theory T), if there are $(\bar{a}_i : i < \omega)$ in $\overline{\mathcal{M}}$ such that*

$$\overline{\mathcal{M}} \models \exists \bar{x}(\varphi(\bar{x}, \bar{a}_j) \wedge \neg\varphi(\bar{x}, \bar{a}_i))$$

*if and only if $i < j$. We say that a theory T is **NSOP** if no formula has SOP in any model of T.*

Equivalently, a theory is NSOP if for each model $\mathcal{M}$, there is no definable partial order with arbitrarily long chains.

As before we say a structure $\mathcal{M}$ is NSOP if $Th(\mathcal{M})$ is NSOP.

Example 1.2.12. *It can be found in [11] that triangle-free homogeneous universal graphs are NSOP. More generally K_n-free homogeneous universal graphs are NSOP.*

We can now mention the following well-known characterisation of stability.

Theorem 1.2.13 (Theorem 4.1 [43] or Theorem 2.67 from [45])**.** *A theory is stable if and only if is both NIP and NSOP.*

1.2.6 Simple theories

Simple theories are a generalisation of stable theories in the context of the NSOP theories.

Definition 1.2.14. *We say that an $\mathcal{L}$-formula $\psi(\bar{x}, \bar{y})$ has the **tree property** for T if there are $(\bar{a}_\eta : \eta \in \omega^{<\omega})$ in $\overline{\mathcal{M}}$ and some $k \geq 2$ such that:*

a) For every $\delta \in \omega^{<\omega}$ the set of formulas $\{\psi(\bar{x}, \bar{a}_{\delta^\frown l}) : l \in \omega\}$ is k-inconsistent;

b) If $\gamma \in \omega^\omega$ then the set $\{\psi(\bar{x}, \bar{a}_{\gamma\restriction_n}) : n \in \omega\}$ is consistent.

*A theory is said to be **simple** if no formula has the tree property.*

*Alternatively, a theory is simple if and only if for any type **p** $\in S_n(B)$ over B there is a set $A \subseteq B$ with $|A| < |T|$ such that p does not divide over A.*

*Furthermore we say a simple theory is **supersimple** if and only if for every type **p** $\in S_n(B)$ we can find a finite $A \subseteq B$ such that p does not fork (divide) over A.*

We say a structure $\mathcal{M}$ is simple (resp. supersimple) if $Th(\mathcal{M})$ is simple (resp. supersimple).

Remark 1.2.15. *It is straightforward to check that any simple theory is NSOP.*

We now present the definition of SU-rank. For more details see Chapter 13 from [7].

Definition 1.2.16 (Definition 13.5, [7]). *Let* $\mathbf{p}$ *be a complete type over* A *in a model* $\mathcal{M}$ *of a theory* T*. We define* $SU(p)$ *the* **SU-rank** *of* $\mathbf{p}$ *recursively for ordinals.*

1. $SU(p) \geq 0$.
2. $SU(p) \geq \alpha + 1$, *for an ordinal* α, *if and only if there is a forking extension* ***q*** *of* ***p*** *such that* $SU(q) \geq \alpha$.
3. $SU(p) \geq \lambda$, *for a limit ordinal* λ, *if and only if* $SU(p) \geq \beta$ *for all* $\beta < \lambda$.

We say $SU(p) = \alpha$ *if* $SU(p) \geq \alpha$ *but* $SU(p) \ngeq \alpha + 1$. *If* $SU(p) \geq \alpha$ *for every ordinal* α *we write* $SU(p) = \infty$.

An example which is quite relevant for this work is the following.

Example 1.2.17. *Pseudofinite fields are simple, in fact, supersimple of SU-rank 1, cf. [48].*

When we say that simple theories are a generalisation of stable theories we mean that the non-forking relation behaves rather nicely. The reader can refer to [25] or [48].

1.2.7 NTP_2 theories.

The notion of an NTP_2 theory was first introduced in [44] as a common generalisation of both NIP and simple theories. Further discussion and advances in the study of these theories appear in [12] and [11].

Definition 1.2.18. *We say that an* $\mathcal{L}$*-formula* $\varphi(\bar{x}, \bar{y})$ *has the* ***tree property of the second kind****, in short* TP_2 *, if there are* $\{\bar{b}_{i,j} : i, j < \omega\}$ *in* $\overline{\mathcal{M}}$ *such that*

(1) The set $\{\varphi(\bar{x}, \bar{b}_{i,j}) : j \in \omega\}$ *is* 2*-inconsistent for all* $i \in \omega$.

(2) For all $\xi \in \omega^\omega$ the set $\{\varphi(\bar{x}, \bar{b}_{i,\xi(i)}) : i \in \omega\}$ is consistent.

We say a theory T is $\mathbf{NTP_2}$*, or does not have* TP_2 *, if no formula satisfies* TP_2 *.*

We say a structure $\mathcal{M}$ is NTP_2 *if $Th(\mathcal{M})$ is* NTP_2 *.*

The definition of burden can be found in [11] Definition 2.1.

Definition 1.2.19 (Definition 2.1, [11])**.** *An **inp-pattern** in a partial type $p(\bar{x})$ of depth κ consists of an array $(\bar{a}_{\alpha,i} : \alpha < \kappa, i < \omega)$, formulas $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ and $k_\alpha < \omega$ such that:*

i) The collection $\{\varphi_\alpha(\bar{x}, \bar{a}_{\alpha,i}) : i < \omega\}$ is k_α-inconsistent, for each $\alpha < \kappa$,

ii) The set $\{\varphi_\alpha(\bar{x}, \bar{a}_{\alpha,f(\alpha)}) : \alpha < \kappa\} \cup p(\bar{x})$ is consistent for any $f \in \omega^\kappa$.

*The burden of $p(\bar{x})$, denoted by **bdn**(p), is the supremum of the depths of all inp-patterns in $p(x)$.*

Hence if a formula φ has TP_2 , then $\mathbf{bdn}(\varphi) \geq \omega$.

The next example is quite relevant for the present work.

Example 1.2.20 (Example 7.7, [11])**.** *Any (infinite) ultraproduct of fields of the form $\mathbb{Q}_p$ is an* NTP_2 *structure.*

As mentioned earlier we have the following remark.

Remark 1.2.21. NTP_2 *theories are a common generalisation of NIP and simple theories, that is, any simple and any NIP theory is* NTP_2 *.*

This follows from the fact that in a complete theory T, the existence of a formula with the tree property of the second kind implies the existence of a formula with the tree property and the existence of a formula with the independence property.

More examples of NTP_2 theories are mentioned in [11], [35] and [14].

1.2.8 Asymptotic classes

We present here the following definition from [17], which extends a definition from [30]. This will be the basis for Chapter 3 below.

Definition 1.2.22. *Let $\mathcal{L}$ be a first order language, and $N \in \omega$. We say a class $\mathcal{C}$ of finite structures is an N-***dimensional asymptotic class** *if for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where the length of $\bar{x}$ is n and the length of $\bar{y}$ is m, the following hold.*

1. *There is a finite set of pairs $D \subseteq (\{0, \ldots, Nn\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$ and a partition $\{\Phi_{\delta,\mu} : (\delta\mu) \in D\}$ of the set $\{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in \mathcal{M}^m\}$ such that for each $(\delta, \mu) \in D$*
$$\big||\varphi(\mathcal{M}^n, \bar{a})| - \mu|\mathcal{M}|^{\frac{\delta}{N}}\big| = o(|\mathcal{M}|^{\frac{\delta}{N}})$$
for all $(\mathcal{M}, \bar{a}) \in \Phi_{(\delta,\mu)}$ as $|\mathcal{M}| \to \infty$.

2. *Moreover, each element $\Phi_{(\delta,\mu)}$ of the partition is definable. This means, for each (δ, μ) there is a formula $\psi_{(\delta,\mu)}(\bar{y})$ such that*
$$\mathcal{M} \models \psi_{(\delta,\mu)}(\bar{a}) \text{ if and only if } (\mathcal{M}, \bar{a}) \in \Phi_{(\delta,\mu)}.$$

Here we consider formulas $\varphi(\bar{x}, \bar{y})$ on $|\bar{x}|$ free variables. In fact this is equivalent to the above conditions for formulas $\varphi(x, \bar{y})$ with just one free variable. Namely the following is true.

Lemma 1.2.23 ([17], Lemma 2.2)**.** *Suppose $\mathcal{C}$ is a class of finite structures which satisfies Definition 1.2.22 for formulas $\varphi(\bar{x}, \bar{y})$ where $|\bar{x}| = 1$. Then $\mathcal{C}$ is an N-dimensional asymptotic class.*

The seminal example is the main theorem from [9], where the authors proved the following.

Theorem 1.2.24 (Chatzidakis, van den Dries, Macintyre [9])**.** *Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language $\mathcal{L}_{rings}$, with $l(\bar{x}) = n$ and $l(\bar{y}) = m$. Then there is a positive constant C, a* *finite set D of pairs (δ,μ) with $\delta \in \{0,\ldots,n\}$ and μ a non-negative rational number, such that for each finite field $\mathbb{F}_q$ and $\bar{a} \in \mathbb{F}_q^m$, $\big||\varphi(\mathbb{F}_q^n,\bar{a})| - \mu q^\delta\big| \leq Cq^{\delta-\frac{1}{2}}$ for some $(\delta,\mu) \in D$.*

Furthermore, for each $(\delta,\mu) \in D$, there is a formula $\varphi_{(\delta,\mu)}(\bar{y})$ which defines in each finite field $\mathbb{F}_q$ the set of tuples $\bar{a}$ such that $\big||\varphi(\mathbb{F}_q^n,\bar{a})| - \mu q^\delta\big| \leq Cq^{\delta-\frac{1}{2}}$.

In the asymptotic classes terminology, this means that the class of finite fields is a 1-dimensional asymptotic class.

It is worth mentioning that in the original presentation of 1-dimensional asymptotic classes in [30], there is a more explicit error term much in the spirit of that one given in 1.2.24. More examples and proper development of the theory of asymptotic classes is done in [30] and [17].

Example 1.2.25.

- *Finite cyclic groups form a 1-dimensional asymptotic class. See [30].*
- *Fix a prime p, and integers m,n co-primes. Put*

$$C_{m,n,p} := \{(\mathbb{F}_{p^{kn+m}}, Frob^k) : k \in \mathbb{N}\}.$$

  Here $(\mathbb{F}_{p^{kn+m}}, Frob^k)$ is the difference field where $Frob$ denotes the Frobenius automorphism mapping $x \mapsto x^p$. Then, the class $C_{m,n,p}$ is a 1-dimensional asymptotic class. See [41] or Theorem 5.8 from [18].

One relation between N-dimensional asymptotic classes and the generalised stability notion of simplicity is given in [17], Corollary 2.6.

Proposition 1.2.26 (Corollary 2.6, [17])**.** *Any infinite ultraproduct of elements in an N-dimensional asymptotic class has supersimple SU-rank $\leq N$ theory.*

The more general notion of *multidimensional asymptotic class* is studied by S. Anscombe, D. Macpherson, C. Steinhorn and D. Wolf in [1] (in preparation). Multidimensional asymptotic classes extend the notion of N-dimensional asymptotic class, and deal with classes of finite structures where there is a strong uniformity in the asymptotic cardinalities of definable sets, in terms of the cardinalities of certain sorts or other features, which may vary freely.

We mention briefly an example, from [20], of a class that exhibits this behaviour.

For this we consider the two sorted language $\mathcal{L}'$, where we include the vector space sort $\mathcal{V}$ including a binary function symbol "$\dot{+}$", a unary function symbol "$\dot{-}$", and a constant symbol "$\dot{0}$"; the field sort $\mathcal{K}$ in $\mathcal{L}_{rings}$; and a function symbol for scalar multiplication $K \times V \to V$. Form the language $\mathcal{L}_{vs}$ by adding to $\mathcal{L}'$, for each $n > 0$, an n-ary relation symbol θ_n. In a k-vector space V interpret $\theta_n(v_1, \dots, v_n)$ as saying the vectors $v_1, \dots, v_n$ are linearly independent.

Theorem 1.2.27 (Theorem 4.1, [20])**.** *Let $\mathcal{C}$ be the class of all $\mathcal{L}_{vs}$ structures (V, F) where F is a finite field and V is a finite-dimensional F-vector space. Let $\varphi(\bar{x}, \bar{y})$ be a formula, where the length of $\bar{x}$ is r and the length of $\bar{y}$ is s, and let $\mathbf{V}, \mathbf{F}$ be indeterminates. Then there is a finite set E of polynomials $p(\mathbf{V}, \mathbf{F}) \in \mathbb{Q}[\mathbf{V}, \mathbf{F}]$, such that for every $\mathcal{M} = (V, F) \in \mathcal{C}$ and every $\bar{a} \in \mathcal{M}^s$, there is some $p(\mathbf{V}, \mathbf{F}) \in E$ such that*

$$\big||\varphi(\mathcal{M}^r, \bar{a})| - p(|V|, |F|)\big| = o(p(|V|, |F|)).$$

Furthermore, for every $p \in E$ there is a formula $\psi_p(\bar{y})$ such that if $\mathcal{M}$ is sufficiently large and $\bar{a} \in \mathcal{M}^s$ then $\mathcal{M} \models \psi_p(\bar{a})$ if and only if $\big||\varphi(\mathcal{M}^r, \bar{a})| - p(|V|, |F|)\big| = o(p(|V|, |F|))$.

1.3 Algebra

1.3.1 Fields and pseudofinite fields

We denote by $\mathbb{F}_q$ the finite field with q elements. Also we denote by $\mathcal{F}^{alg}$ the algebraic closure of a field $\mathcal{F}$.

Example 1.3.1. *Some examples of pseudofinite fields include the following.*

- *Ultraproduct of distinct finite fields, cf. [2].*
- *Infinite subfield of* $\mathbb{F}_p^{alg}$ *generated by* $\mathbb{F}_{p^2}, \mathbb{F}_{p^3}, \mathbb{F}_{p^5}, \ldots$ *for all prime powers of* p*. Cf. the remark before Theorem 1 in [8].*

We will first repeat that pseudofinite fields have supersimple theory of rank 1. This follows e.g. from the main theorem and Proposition 4.11 of [9] (or from Theorem 1.2.24 and Proposition 1.2.26 above).

We can now connect the notion of pseudofinite and infinite ultraproducts.

Theorem 1.3.2. *A field (resp. group, ring) is pseudofinite if and only if it is elementarily equivalent to an infinite ultraproduct of finite fields (resp. groups, rings).*

1.3.2 Groups and pseudofinite groups

We now briefly mention some classical group-theoretic notions.

Definition 1.3.3. *We say a group is simple if it has no proper non-trivial normal subgroups.*

Theorem 1.3.4 (Wilson [49])**.** *A pseudofinite group* G *is simple (in the group theoretic sense) if and only if* G *is elementarily equivalent to a Chevalley group over a pseudofinite field.*

Furthermore by work of Ryten [41] the following is true: a pseudofinite group G is simple if and only if it is *isomorphic* to a simple group of (possibly twisted) Lie type over a pseudofinite field.

Definition 1.3.5. *We say a group* G *is soluble if there exists* $n \in \mathbb{N}$ *such that the* n*-th derived subgroup is trivial, i.e.* $G^{(n)} = \{e_G\}$*.*

By the soluble radical of G, in symbols $R(G)$, we mean the largest soluble normal subgroup of G.

Theorem 1.3.6 (Wilson [50])**.** *The soluble radical of a group is uniformly first order definable in finite groups; that is, there is a formula $\varphi(x)$ such that for any finite group G, the soluble radical $R(G)$ equals $\{x \in G : \varphi(x) \text{ holds}\}$.*

Below we use the following terminology.

Definition 1.3.7. *We say a group G is nilpotent-by-finite (resp. soluble-by-finite) if and only if there exists a nilpotent (resp. soluble) normal subgroup N such that $|G : N|$ is finite.*

Definition 1.3.8. *We call a class $\mathcal{C}$ an NIP (resp. simple, stable, NTP_2) class of groups (rings) if all non-principal ultraproducts of elements of $\mathcal{C}$ are NIP (resp. simple, stable, NTP_2).*

Further results about stable and NIP classes of groups can be found in [31] and [32] respectively. Here we mention a couple of these results.

H. D. Macpherson and K. Tent showed in [31] that if G is a pseudofinite group with stable theory then it has a definable soluble normal subgroup of finite index.

Theorem 1.3.9 ([31])**.** *Stable pseudofinite groups are soluble-by-finite.*

Later, Macpherson and Tent proved a generalisation of the previous result in [32]. Namely they showed

Theorem 1.3.10 (Theorem 1.1 , [32])**.** *Suppose that G is such that: (1) There is a natural number $n = n(G)$ such that there do not exists $F_1, \ldots, F_{n+1} \subset G$ with $C_G(F_1) < \ldots < C_G(F_{n+1})$, and (2) G is pseudofinite with NIP theory. Then G has a soluble definable normal subgroup of finite index. Hence G is soluble-by-finite.*

In particular, (Corollary 1.4, [32]) pseudofinite groups with NIP rosy theory are soluble-by-finite.

Furthermore it is shown (in [32], Theorem 3.1) that condition (1) is necessary.

Also they show the following.

Proposition 1.3.11 (Proposition 1.2 of [32])**.** *If $\mathcal{C}$ is a NIP class of finite groups, then there is $d = d(\mathcal{C}) \in \mathbb{N}$ such that $|G : R(G)| \leq d$ for every $G \in \mathcal{C}$.*

Combining results from J.S. Wilson, E. Hrushovski and M. Ryten, pseudofinite groups which are simple in the sense of group theory have supersimple finite rank theory.

In [13] the authors investigate groups and fields with NTP_2 . In particular they prove the following lemma.

Lemma 1.3.12. *Let T be NTP_2 , G a definable group in $\mathcal{M} \models T$, and $(H_i)_{i\in\omega}$ a uniformly definable family of normal subgroups of G, with $H_i = \varphi(x, a_i)$. Let $H = \bigcap_{i\in\omega} H_i$, and $H_{\neq j} = \bigcap_{i\in\omega\setminus\{j\}} H_i$. Then there is some $i^* \in \omega$ such that, $[H_{\neq i^*}, H]$ is finite.*

Finally, we mention Remark 4.2.10 from [29]. Using Theorem 4.2.9 in [29] H. D. Macpherson shows that if G is a pseudofinite group with NTP_2 theory then G has a definable proper normal subgroup H such that the quotient G/H has a definable normal subgroup $K \leq G/H$ such that K is a direct product of finitely many definable finite or pseudofinite simple groups.

1.3.3 Simple, semisimple and definable rings

Throughout the thesis we will work in the language of rings $\mathcal{L}_{rings} := \{+, \cdot, -, 0, 1\}$. In particular unless otherwise stated all our rings have 1.

We denote by $\mathbb{P}$ the set of prime numbers. Whenever we talk about a finite residue ring we mean a ring obtained by taking a quotient of $\mathbb{Z}$ of the form $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}^{>0}$.

Given any ring R we denote by $M_n(R)$ the set of $n \times n$ matrices with entries from R. Then $M_n(R)$ is a ring with the usual matrix addition and multiplication. We denote by $GL_n(R)$ the general linear group, and if R is commutative by $SL_n(R)$ the special linear group in $M_n(R)$.

In this subsection we recall the definitions of the main concepts we will use later in Chapter 4.

Definition 1.3.13 (Jacobson radical. See Section 4 from [27])**.** *Given a ring R with 1, define $a \circ b := a + b - ab$. Then $(R, \circ)$ is a monoid with 0 as the identity element. We say $a \in R$ is **left** (resp. right) **quasi-regular** if a has a left (resp. right) inverse in the monoid $(R, \circ)$. Furthermore we say a is **quasi-regular** if it is both left and right quasi-regular. A set $I \subseteq R$ is called (left, right) quasi regular if every element is (left, right) quasi-regular. Whenever I is a left quasi-regular left ideal then it is quasi-regular.*

The **Jacobson radical** *$J(R)$ of R is defined as* $J(R) := \{a \in R : Ra \text{ is quasi-regular}\}$.

Remark 1.3.14. *We see from the definition above that the Jacobson radical is first order definable. However, the classical definition is equivalent to the one we just presented. Namely if R is a ring with 1, then the Jacobson radical is given by the intersection of the maximal (right) ideals of R. For example, see Section 4 from [27].*

We now recall the following classical ring theoretic notions (and a module theoretic consideration).

Definition 1.3.15. *1. We say a ring R is **simple** if R has no non-trivial two sided ideal.*

2. *Let S be a ring. A left S-module M is called a **semisimple** S**-module** if every S-submodule of M is an S-module direct summand of M.*

3. *We call a ring R **semisimple** if all left R-modules are semisimple R-modules.*

4. *We say a ring R is **J-semisimple** if the Jacobson radical is trivial, i.e. $J(R) = 0$.*

We present below Theorem 1.3.19, which characterises semisimple rings in terms of matrix rings over division rings. The reader could take the characterisation in 1.3.19 as the working definition for semisimple rings.

There are a lot of interesting J-semisimple rings. For example freely generated rings by a set of indeterminates $\{x_i\}$ over a division ring K are J-semisimple, see Corollary 4.16 in [27]. Any polynomial ring over a division ring K in commuting variables $\{x_i\}$ is J-semisimple, cf. 4.17 in [27].

We mention below some properties of the Jacobson radical of a ring that we will use later.

Remark 1.3.16. *i) From example (7) after Theorem 4.15 in [27], for any R ring with identity, we have $J(M_n(R)) = M_n(J(R))$.*

ii) For any direct product $\prod R_i$ of a family of rings $\{R_i\}$ we have $J(\prod R_i) = \prod J(R_i)$. In particular, the product of any family of J-semisimple rings is J-semisimple. See end of Section 4 from [27].

iii) Because $J(R) = 0$ is first order expressible, any ultraproduct of rings $\prod_{\mathcal{U}} R_i$ is J-semisimple if and only if $\mathcal{U}$-many of the rings are J-semisimple.

The next definition is standard.

Definition 1.3.17. *1. We say an element x of a ring R is **nilpotent** of nilpotent exponent n if $n \in \mathbb{N}$ is the minimum such that $x^n = 0$.*

2. *We say a (left/right/ two sided) ideal I of R is **nil** if every element in I is nilpotent.*

3. *We call a (left/right/ two sided) ideal I a **nilpotent ideal** of exponent n if $n \in \mathbb{N}$ is the minimum such that $I^n = 0$. I.e. for every n elements $a_1, \ldots, a_n$ in I we have $a_1 \cdot \ldots \cdot a_n = 0$.*

4. *We say an element x of a ring R possibly without 1 is **null** if we have $xR = Rx = \{0\}$.*

5. *We call a ring R' possibly without 1, a **null ring** if every element in R' is null. Similarly for **null ideal**.*

6. *We say a ring R is **nilpotent-by-finite** (respectively **null-by-finite**) whenever R has a two-sided nilpotent (respectively null) ideal I such that the quotient R/I is finite.*

The following is also classical.

Definition 1.3.18. *We call a ring (left/right) **Artinian** if it satisfies the descending chain condition (DCC) on (left/right respectively) ideals, i.e. for any sequence $I_1 \supseteq I_2 \supseteq \ldots$ of (left/right) ideals there is $N \in \mathbb{N}$ such that for all $j, k > N$ we have $I_j = I_k$.*

*On the other hand, we call a ring (left/right) **Noetherian** if it satisfies the ascending chain condition (ACC) on (left/right) ideals, i.e. for any sequence $I_1 \subseteq I_2 \subseteq \ldots$ of (left/right) ideals there is $N \in \mathbb{N}$ such that for all $j, k > N$ we have $I_j = I_k$.*

Recall the Wedderburn-Artin theorem.

Theorem 1.3.19 (Wedderburn-Artin Theorem, see 3.5 from [27])**.** *Let R be a semisimple ring. Then R is isomorphic to a finite direct product of rings of the form $M_n(D)$ for some division ring D and $n \in \mathbb{N}$.*

Recall also the following characterisation of simple artinian rings.

Theorem 1.3.20 (Theorem 3.10, [27])**.** *Let R be a simple ring, then the following are equivalent.*

1. *R is left artinian.*

2. *R is semisimple.*

3. *R has a minimal left ideal.*

4. *R is isomorphic to some $n \times n$ matrix ring over a division ring, i.e. $R \cong \mathbb{M}_n(D)$ for some $n \in \mathbb{N}$ and some division ring D. Furthermore n and D are uniquely determined.*

The following is folklore.

Proposition 1.3.21 (Theorem 4.14, [27])**.** *For any ring R, the following are equivalent:*

1. *R is semisimple.*

2. *R is J-semisimple and left artinian.*

3. *R is J-semisimple, and satisfies DCC on principal left ideals.*

We finish this short overview with the following result from Krupinski, [26], on the relation between algebraic properties of definable rings and algebraic properties of definable groups being defined in a given first order structure.

Theorem 1.3.22. *[Theorem 2.1, [26]] Let $\mathcal{M}$ be a first order structure.*

i) If every group definable in $\mathcal{M}$ is soluble-by-finite then every ring with identity, or of finite characteristic, definable in $\mathcal{M}$ is nilpotent-by-finite.

ii) If every nilpotent group definable in $\mathcal{M}$ is abelian-by-finite, then every ring definable in $\mathcal{M}$ is null-by-finite.

iii) If every nilpotent group definable in $\mathcal{M}$ is (finite central)-by-abelian-by-finite, then every ring definable in $\mathcal{M}$ is (finite null)-by-null-by-finite. Definitions for (finite central)-by-abelian-by-finite groups and (finite null)-by-null-by-finite rings can be found in [26].

1.3.4 Valued fields

Throughout this thesis we regularly work in the context of valued fields. Namely we heavily use definability of certain pseudofinite residue rings in well known valued fields. When handling valued fields, we will usually work in the multisorted Denef-Pas language of valued fields $\mathcal{L}_{valf}$ defined with more detail in Definition 1.3.25.

For the sake of completeness we mention briefly some notions we will use further on in the manuscript. A complete treatment and exposition can be found in the book [40] or the survey [47].

Definition 1.3.23. *Consider a field K and an ordered abelian group Γ to which we add a symbol ∞. We extend the operations and the order from Γ so ∞ is the maximum element of $\Gamma \cup \{\infty\}$. We call a function $v : K \to \Gamma \cup \{\infty\}$ a* **valuation** *on K if and only if the following hold.*

i) $v(a) = \infty$ *if and only if* $a = 0$,

ii) $v(a + b) \geq \min\{v(a), v(b)\}$,

iii) $v(a \cdot b) = v(a) + v(b)$.

We call the pair (K, v) a **valued field**. *We denote the* **valuation ring** *by $\mathcal{O} := \{x \in K : v(x) \geq 0\}$. Let $\mathfrak{M} := \{x \in K_v(x) > 0\}$ denote the unique* **maximal ideal of** $\mathcal{O}$. *The quotient $\mathrm{k} = \mathcal{O}/\mathfrak{M}$ is called the* **residue field of** (K, v). *We let res : $\mathcal{O} \to \mathrm{k}$ be the canonical projection and call it the* **residue map**.

Definition 1.3.24. *An* **angular component map**, *$\bar{ac}$, in a valued field (K, v) is a map from K to the residue field k which satisfies the following.*

- $\bar{ac}(0) = 0$;

- *The restriction of $\bar{ac}$ to K^* has image in $\mathbf{k}^*$ and it is a morphism of multiplicative groups;*

- *For any $x \in K$ of valuation 0, $\bar{ac}(x) = res(x)$.*

It is worth noting that angular component maps need not exist for arbitrary valued fields (and are not in general definable). However, (see Corollary 5.18 from [47]) any valued field has an elementary extension that has an angular component map on it.

We now define the 3-sorted Denef-Pas valued fields language $\mathcal{L}_{valf}$.

Definition 1.3.25. *The **Denef-Pas language** is a three sorted language, with a sort for the valued field in the language of rings $\mathcal{L}_{rings} = \{\cdot, +, -, 0, 1\}$, a sort for the ordered abelian group, in the language of ordered abelian groups $\mathcal{L}_{ogps} = \{+, 0, <\}$ together with an extra symbol ∞, and a sort for the residue field in the language of rings, $\mathcal{L}_{rings}$. We also include symbols for a valuation map $v : K \to \Gamma$ and an angular component map $\bar{ac} : K \to \mathbf{k}$.*

Further we use the following.

Definition 1.3.26 (Valued fields in the Denef-Pas language)**.** *A valued field in the three sorted Denef-Pas language consists of $(K, \Gamma, \mathbf{k}, v, \bar{ac})$ where K is the valued field, Γ is the value group and $\mathbf{k}$ is the residue field, v is a valuation map and $\bar{ac}$ is an angular component map as above.*

We often work in the context of p-adic numbers, which we define below.

Definition 1.3.27 (The p-adic numbers, $\mathbb{Q}_p$)**.** *For a given prime p we define the p-adic valuation in $\mathbb{Z}$, in symbols v_p. For $a \in \mathbb{Z}$ assign $v_p(a) \in \mathbb{N} \cup \infty$ as follows. $v_p(0) = \infty$ and if $a \neq 0$, then $v_p(a)$ is the natural number such that $a = p^{v_p(a)}b$ where $b \in \mathbb{Z}$ and p does not divide b. Then, for all $a, b \in \mathbb{Z}$ we get:*

1. $v_p(a+b) \geq min\{v_p(a), v_p(b)\}$,

2. $v_p(ab) = v_p(a) + v_p(b)$,

3. $v_p(1) = 0$.

Also, putting $|a|_p = p^{-v_p(a)}$ we get an absolute value on $\mathbb{Z}$. This absolute value putting $v_p(\frac{a}{b}) = v_p(a) - v_p(b)$ extends uniquely to $\mathbb{Q}$ and it is called the p-adic absolute value. The completion, as a metric space, with respect to the p-adic absolute value is denoted by $\mathbb{Q}_p$ and it is called the **field of p-adic numbers**. *By extension we denote by $||_p$ the absolute value in $\mathbb{Q}_p$. The integral closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$ and called the ring of p-adic integers.*

Remark 1.3.28. *Alternatively we can view the field of p-adic numbers as the set of formal power series $\{\sum_{i=z}^{\infty} a_i p^i : z \in \mathbb{Z}, a_i \in \mathbb{F}_q\}$. Here the addition and multiplication are done "carrying over" depending on whether the coordinatewise operation done in the usual way in $\mathbb{Z}$ gives a result greater than p.*

We use the following result for valued fields in the Denef-Pas language. See [37], or [38]. See also [47] for a thorough treatment.

Theorem 1.3.29. *Given $\widetilde{K} = (K, \Gamma, \mathbf{k}, v, \bar{ac})$ a Henselian valued field with residue field of characteristic 0 in the three sorted Denef-Pas language, then $Th(\widetilde{K})$ eliminates field quantifiers.*

A consequence of the elimination of quantifiers is the well known Ax-Kochen, Ershov theorem.

Theorem 1.3.30 (Ax-Kochen, Ershov)**.** *Two henselian valued fields of residue characteristic 0, $(K, \Gamma, \mathbf{k}, v, \bar{ac})$, and $(K', \Gamma', \mathbf{k}', v', \bar{ac}')$ are elementarily equivalent if and only if both $\Gamma \cong \Gamma'$ and $\mathbf{k} \cong \mathbf{k}'$.*

It is worth noting that similar results of quantifier elimination have been proved also for the mixed characteristic case. For example, in [28] it is shown that $\mathbb{Q}_p$ admits quantifier elimination in the one sorted language of rings together with a binary symbol "|" interpreted as $a|b \leftrightarrow v(a) \leq v(b)$ and a family of unary predicates $\{P_n : n \in \omega\}$ interpreted as the sets of n-th powers in $\mathbb{Q}_p$.

1.4 Main results

Here we present the results found in this work which contribute towards the study of pseudofinite rings and classes of finite rings.

In chapter 2 we will present the classification of pseudofinite residue rings in terms of generalised stability. Namely we present the following theorem.

Theorem 1.4.1 (See 2.1.1, 2.1.2, 2.1.3 and 2.1.4)**.** *An ultraproduct of finite residue rings of the form* $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ *has* NTP_2 *theory if and only if there exists* $d \in \mathbb{N}$ *and* $U \in \mathcal{U}$ *such that every* $n \in U$ *is a product of fewer than* d *prime powers. Moreover if it has* NTP_2 *theory then it has finite burden.*

Furthermore the following holds for NTP_2 *pseudofinite residue rings.*

i) The ultraproduct $\prod \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ *has NIP theory if and only if there exists* $V \in \mathcal{U}$ *and* $d \in \mathbb{N}$ *such that every* $n \in V$ *is a product of fewer than* d *prime powers each of the primes being less than* d*. Furthermore if the ultraproduct has NIP theory then it has finite dp-rank.*

ii) The ultraproduct $\prod \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ *has supersimple theory if and only if there exists* $V \in \mathcal{U}$ *and* $d \in \mathbb{N}$ *such that every* $n \in V$ *is a product of fewer than* d *prime powers and whenever a prime power* p^l *divides* n *we have that* $l \leq d$*. Furthermore, when the ultraproduct has supersimple theory then it has finite SU-rank.*

We will also show the following.

Proposition 1.4.2 (See 2.5.3). *If the ultraproduct* $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ *is not supersimple then its theory is SOP.*

From the theorem above we find that ultraproducts of elements of the class of finite residue rings of the form $\{\mathbb{Z}/n\mathbb{Z} : n \in U\}$ where there is a bound on the number and the exponent of prime divisors for all $n \in U$ are supersimple. This leads to ask whether the class of such rings has an asymptotic behaviour in the sense of Section 1.2.8 above. To answer this question we present the following result.

Theorem 1.4.3 (See 3.2.1). *Let* $l \in \mathbb{N}$. *Then the class of finite residue rings of the form* $\{\mathbb{Z}/p^l\mathbb{Z} : p \in \mathbb{P}\}$ *is an* l*-dimensional asymptotic class.*

Furthermore we mention the connection of this result with work of Daniel Wolf to obtain that if $d \in \mathbb{N}^{>0}$ and $l_1, \ldots, l_d \in \mathbb{N}$, then classes of the form $\{\mathbb{Z}/n\mathbb{Z} : n = p_1^{l_1} \cdot \ldots \cdot p_d^{l_d}, p_1 < \ldots < p_d \in \mathbb{P}\}$ are interpretable in a finite disjoint union of elements of N_i-dimensional asymptotic classes and hence after expanding the language by unary predicates form a multidimensional asymptotic class in the sense of Section 1.2.8 above.

Finally, in chapter 4 we will address some general observations about pseudofinite rings, ultraproducts of elements of classes of finite rings under some generalised stability properties, and some remarks about what algebraic behaviour can we obtain from the generalised stability conditions in the spirit discussed in Subsections 1.3.1 and 1.3.2.

Among other results we show the following for NIP and NTP_2 classes of finite rings.

Proposition 1.4.4 (See 4.3.12). *If* $\mathcal{C}$ *is an NIP class of finite rings then there exists* d *such that every* $R \in \mathcal{C}$ *has a nilpotent two-sided ideal of index bounded by* d.

Proposition 1.4.5 (See 4.3.7). *Let* $\mathcal{C}$ *be an* NTP_2 *class of finite* J*-semisimple rings. Then there is* $N \in \mathbf{N}$ *such that the rings in the class are of the form* $\prod_i^k M_{n_i}(\mathbb{F}_{q_i})$ *for some* k *and* $n_1, \ldots, n_k$ *in* $\mathbb{N}$ *and* $q_1, \ldots, q_k$ *primes where* $n_1, \ldots, n_k$ *and* k *are bounded by* N.

Also, we make general observations about pseudofinite rings like the following.

Proposition 1.4.6 (See 4.2.2)**.** *Pseudofinite simple rings are of the form* $M_n(\mathcal{F})$ *where* $\mathcal{F}$ *is a pseudofinite field.*

Chapter 2

Ultraproducts of finite residue rings

2.1 Introduction

As mentioned in 1.3.1 above any pseudofinite field has supersimple rank 1 theory. In this chapter we investigate generalised stability properties of arbitrary pseudofinite residue rings. More specifically, we describe non-principal ultrafilters on $\mathbb{N}$ such that the ultraproduct $\prod \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is supersimple, or NIP but non-simple, or NTP_2 but not NIP or simple, or TP_2, noting that all these possibilities occur. The information is depicted in the following diagram.

- Bounded number of prime divisors
  - Bounded exponents, (e.g. $\prod_{p\in\mathbb{P}} \mathbb{Z}/p^b\mathbb{Z}/\mathcal{U}$, for fixed b) Simple case
  - Unbounded exponents
    - Bounded set of primes, (e.g. $\prod_{n\in\mathbb{N}} \mathbb{Z}/q^n\mathbb{Z}/\mathcal{U}$, for fixed prime q) NIP case
    - Unbounded set of primes, (e.g. $\prod_{p\in\mathbb{P},n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}/\mathcal{U}$) NTP_2 case
- Unbounded number of prime divisors, (e.g. $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$) TP_2 case

We see this work as the beginning of a structure theory for pseudofinite rings with generalised stability properties. We take a moment here to mention that there is some overlap between work done in this chapter and independent work of Paola D'Aquino and Angus Macintyre. However, D'Aquino and Macintyre's point of view is different from ours.

We now state the main theorems that make up this chapter and indicate where they can be found below.

Theorem 2.1.1 (Corollary 2.2.3)**.** *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}\setminus\{0\}$ and let $R = \prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$. Then the following are equivalent.*

1. *The theory $Th(R)$ of R is NIP.*
2. *There is $U \in \mathcal{U}$ and $b \in \mathbb{N}$ such that every $n \in U$ is a product of primes each one being less than b.*

Theorem 2.1.2 (Corollary 2.2.10)**.** *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that there exists $b \in \mathbb{N}$ and $U \in \mathcal{U}$ such that for every $n \in U$ there are at most b prime divisors. Then $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is* NTP_2 .

Theorem 2.1.3 (Corollary 2.3.7)**.** *Consider a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and let $R = \prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$. Then the following are equivalent.*

1. *The theory $Th(R)$ is supersimple.*
2. *There exists $b \in \mathbb{N}$ and $U \in \mathcal{U}$ such that if $n \in U$ then n is a product of fewer than b primes and if p^l divides n then $l < b$.*

Theorem 2.1.4 (Proposition 2.4.2)**.** *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that for every $b \in \mathbb{N}$ the set*

$$\{n \in \mathbb{N} : \text{ there are at least } b \text{ distinct prime divisors of } n\}$$

is in $\mathcal{U}$. Then $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ has TP_2 *theory.*

Proposition 2.1.5 (Proposition 2.5.3)**.** *If the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is not simple, then it has SOP (the strict order property - see Definition 1.2.11).*

In the second section we present both Theorem 2.1.1 and Theorem 2.1.2, concerning the NIP and NTP_2 cases respectively.

In the third section we present Theorem 2.1.3 about the simple theory case, and also mention some results on coordinatisation, as found in [21].

In the fourth section we present Theorem 2.1.4 for the TP_2 case.

Finally in the fifth section we address questions raised by A. Chernikov and make some comments about *SOP*, how it is related to the previous cases and prove Proposition 2.1.5.

We heavily use definability of certain pseudofinite residue rings in well known valued fields. When handling valued fields, we will usually work in the multisorted Denef-Pas language of valued fields $\mathcal{L}_{valf}$ defined with more detail in Definition 1.3.25.

Below we deal with theories satisfying Definition 1.2.18, 1.2.14, 1.2.8, and 1.2.11.

As mentioned in Section 1.2.7, by Example 7.7 of [11] any ultraproduct $\prod_{p\in\mathbb{P}} \mathbb{Q}_p/\mathcal{U}$ of p-adic fields, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{P}$, has NTP_2 theory. This follows from the more general (Ax-Kochen,Ershov)-like result found in Theorem 7.6 in [11]. Paraphrased, this says the following.

Proposition 2.1.6 (Theorem 7.6, [11]). *Let $\overline{K} = (K, \Gamma, \mathbf{k}, v : K \to \Gamma, \bar{ac} : K \to \mathbf{k})$ be a Henselian valued field of characteristic $(0, 0)$ in the Denef-Pas language. Then the depth λ of an array of parameters $\{\bar{b}_{i,j} : j < \omega, i < \lambda\}$ in K which satisfies clauses (1) and (2) from Definition 1.2.18 for some formula with a single free variable is less than the depth of arrays of parameters either in $\mathbf{k}$ or in Γ which satisfy clauses (1) and (2) from Definition 1.2.18 for some formulas with a single free variable.*

In particular we have that if $\mathbf{k}$ has NTP_2 theory in $\mathcal{L}_{rings}$, then so does $\overline{K}$.

Observe that in Proposition 2.1.6, if we consider $\overline{K} := (\prod \mathbb{Q}_p/\mathcal{U}, \Gamma, \mathbf{k}, v, \bar{ac})$ then $\overline{K}$ is strictly NTP_2 , in the sense that since $\mathbf{k}$ is a pseudofinite field it has IP and so $\overline{K}$ has IP, and Γ has SOP so $\overline{K}$ has SOP.

2.2 NIP, and NTP_2 cases

First, for a fixed prime p we consider the ring $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}/\mathcal{U}$.

Proposition 2.2.1. *Fix a prime p. Ultraproducts of the form $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}/\mathcal{U}$ with $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{N}$ are interpretable in the ultrapower $\prod_{n\in\mathbb{N}} \mathbb{Q}_p/\mathcal{U}$ of $\mathbb{Q}_p$ and hence are NIP.*

Proof. We first recall that $\mathbb{Q}_p$ has NIP theory, c.f. [4], [15] or [34]. We will show that uniformly in n, $\mathbb{Z}/p^n\mathbb{Z}$ is interpretable in $\mathbb{Q}_p$, the p-adic numbers. We know that the valuation ring $\mathbb{Z}_p$ is definable inside the valued field $\mathbb{Q}_p$. Also we can use a parameter $a \in \mathbb{Q}_p$ with $v(a) = n$ to define $p^n\mathbb{Z}_p$, since $p^n\mathbb{Z}_p = \{x \in \mathbb{Z}_p : v(x) \geq v(a)\}$. Hence the structure $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$ is interpretable in $\mathbb{Q}_p$, uniformly in n (a parameter varying through the value group $\mathbb{Z}$). Furthermore the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}/\mathcal{U}$ is interpretable by the same formula in the ultrapower $\prod_{n\in\mathbb{N}} \mathbb{Q}_p/\mathcal{U}$ which is still NIP. Since being NIP is preserved under interpretability we conclude that ultraproducts of the form $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}/\mathcal{U}$ are NIP. □

We present now a lemma that will be useful further on. Here for j in an index set J, and a collection of structures $(A_j)_{j\in J}$ we denote by π_j the usual projection map from $\prod_{k\in J} A_k$ to A_j. We extend this notation to ultrafilters, i.e. if we consider $\mathcal{U}$ an ultrafilter on $\prod_{k\in J} I_k$, where each I_k is an index set, we will denote by $\pi_j(\mathcal{U})$ the ultrafilter

$$\{V \subseteq I_j : \exists U \in \mathcal{U}\big(\pi_j(U) = V\big)\}$$

induced by π_j on I_j.

Lemma 2.2.2. *Let $\{I_k : 1 \leq k \leq n\}$ be a family of index sets, for each I_k let $\{R_i^k : i \in I_k\}$ be a family of rings indexed by I_k, and $\mathcal{U}$ an ultrafilter on $\prod_{k=1}^{n} I_k$. Then*

$$\prod_{(i_1,\ldots,i_n)} (R_{i_1}^1 \times \cdots \times R_{i_n}^n)/\mathcal{U} \cong (\prod_{i_1} R_{i_1}^1/\pi_1(\mathcal{U})) \times \cdots \times (\prod_{i_n} R_{i_n}^n/\pi_n(\mathcal{U})).$$

Proof. We can show that the assignment φ given by sending $[(a)_{(k_1,\ldots,k_n)}]]_{\mathcal{U}}$ to $\big([(a_{k_1})]_{\pi_1(\mathcal{U})}, \ldots, [(a_{k_n})]_{\pi_n(\mathcal{U})}\big)$ is an isomorphism.

The assignment φ is a well defined function. We can see that if

$$[(a_{k_1}, \ldots, a_{k_n})_{(k_1,\ldots,k_n)}]_{\mathcal{U}} = [(\beta_{k_1}, \ldots, \beta_{k_n})_{(k_1,\ldots,k_n)}]_{\mathcal{U}},$$

then there exists $U \in \mathcal{U}$ such that $a_{k_i} = \beta_{k_i}$ for all n-tuples $(k_i)_{i\leq n} \in U$. Hence $a_{k_i} = \beta_{k_i}$ for all $k_i \in \pi_i(U)$ and so $\big([(a_{k_1})]_{\pi_1(\mathcal{U})}, \ldots, [(a_{k_n})]_{\pi_n(\mathcal{U})}\big) = \big([(\beta_{k_1})]_{\pi_1(\mathcal{U})}, \ldots, [(\beta_{k_n})]_{\pi_n(\mathcal{U})}\big)$. To show that φ is injective we consider $[(a_{k_1}, \ldots, a_{k_n})]_{\mathcal{U}}$ and $[(b_{k_1}, \ldots, b_{k_n})]_{\mathcal{U}}$ such that

$\varphi([(a_{k_1},\dots,a_{k_n})]) = \varphi([(b_{k_1},\dots,b_{k_n})])$. Note that since $([(a_{k_i})]_{\pi_i(\mathcal{U})}) = ([(b_{k_i})]_{\pi_i(\mathcal{U})})$ there are $U_i \in \pi_i(\mathcal{U})$ (for $i \in \{1,\dots,n\}$) that witness this equality. For each U_i put $U_i' := I_1 \times \dots \times I_{i-1} \times U_i \times I_{i+1} \times \dots \times I_n$. Taking $\bigcap_{i=1}^{n} U_i'$ we have an element in $\mathcal{U}$ that witness the equality of $[(a_{k_1},\dots,a_{k_n})]_{\mathcal{U}}$ and $[(b_{k_1},\dots,b_{k_n})]_{\mathcal{U}}$.

To show that φ is surjective note that for

$$\alpha_k := \big([(a_{k_1})]_{\pi_1(\mathcal{U})},\dots,[(a_{k_n})]_{\pi_n(\mathcal{U})}\big) \in \left(\prod_{i_1} R^1_{i_1}\big/\pi_1(\mathcal{U})\right) \times \cdots \times \left(\prod_{i_n} R^n_{i_n}\big/\pi_n(\mathcal{U})\right)$$

we have that $\varphi([(a_{k_i},\dots,a_{k_n})]) = \alpha_k$.

Furthermore φ is an homomorphism. First note that $\varphi([(1_{k_i})]_{\mathcal{U}}) = ([1_{i_k}]_{\pi_k(\mathcal{U})})_k$. Since the operations on

$$\prod_{(i_1,\dots,i_n)} (R^1_{i_1} \times \cdots \times R^n_{i_n})\big/\mathcal{U} \cong (\prod_{i_1} R^1_{i_1}\big/\pi_1(\mathcal{U})) \times \cdots \times (\prod_{i_n} R^n_{i_n}\big/\pi_n(\mathcal{U}))$$

are defined coordinatewise we have that

$$\varphi([(a_{k_i})]_{\mathcal{U}} + [(b_{k_i})]_{\mathcal{U}}) = \varphi([(a_{k_i} + b_{k_i})]_{\mathcal{U}}) = \big([a_{k_i} + b_{k_i}]_{\pi_i(\mathcal{U})}\big)_{i \le n} = \dots$$
$$\dots = \big([a_{k_i}]_{\pi_i(\mathcal{U})} + [b_{k_i}]_{\pi_i(\mathcal{U})}\big)_{i \le n} = \varphi([(a_{k_i})]_{\mathcal{U}}) + \varphi([(b_{k_i})]_{\mathcal{U}})$$

and similarly $\varphi([(a_{k_i})]_{\mathcal{U}} \cdot [(b_{k_i})]_{\mathcal{U}}) = \varphi([(a_{k_i})]_{\mathcal{U}}) \cdot \varphi([(b_{k_i})]_{\mathcal{U}})$.

□

Corollary 2.2.3. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $U \in \mathcal{U}$ and $b \in \mathbb{N}$ be such that every $n \in U$ is a product of powers of fewer than b primes each prime being less than b. Then $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}\big/\mathcal{U}$ is NIP.*

Proof. Put $R' := \prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}\big/\mathcal{U}$. Let $U \in \mathcal{U}$ be as in the hypothesis, and consider the induced ultrafilter on U by $\mathcal{U}$, namely $\mathcal{V} := \{U \cap V : V \in \mathcal{U}\}$. Put $R := \prod_{n\in U} \mathbb{Z}/n\mathbb{Z}\big/\mathcal{V}$. Then $R \cong R'$.

Furthermore we can find $V \in \mathcal{V}$ such that every $n \in V$ has the same d prime factors, $p_1, p_2, \ldots, p_d$. Considering $\mathcal{W}$ the induced ultrafilter on V by $\mathcal{V}$ and using Lemma 2.2.2 and the first paragraph using V in place of U we have that $R \cong \left(\prod_m (\mathbb{Z}/p_1^m\mathbb{Z})\diagup\mathcal{W}_1\right) \times \cdots \times \left(\prod_m (\mathbb{Z}/p_d^m\mathbb{Z})\diagup\mathcal{W}_d\right)$, for some ultrafilters $\mathcal{W}_1, \ldots, \mathcal{W}_d$ on $\mathbb{N}$. Since each of $\prod_m (\mathbb{Z}/p_k^m\mathbb{Z})\diagup\mathcal{W}_k$ is NIP by Proposition 2.2.1 we can conclude that R' is NIP. □

We take a moment here to note that we are using and will use the following result throughout the chapter.

Proposition 2.2.4. *Each of the rings $R_1, \ldots, R_n$ is NIP (respectively, simple, NTP_2) if and only if the algebraic direct product $R_1 \times \ldots \times R_n$ is NIP (respectively, simple, NTP_2).*

This is proven using the following two lemmas.

Lemma 2.2.5. *Let $\mathcal{L} := \mathcal{L}_1 \sqcup \mathcal{L}_2$ be the disjoint union of $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}$, where $\bar{x} \in \mathcal{L}_1$ and $\bar{y} \in \mathcal{L}_2$. Then $\varphi(\bar{x}, \bar{y})$ is equivalent to a finite disjunction of formulas of the form $\theta(\bar{x}) \wedge \psi(\bar{y})$, where $\theta(\bar{x}) \in \mathcal{L}_1$ and $\psi(\bar{y}) \in \mathcal{L}_2$.*

Proof. By induction on the length of the formula φ. See for example section 9.6 of [22]. □

Lemma 2.2.6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be disjoint languages. Let A be an $\mathcal{L}_1$-structure, and B be an $\mathcal{L}_2$-structure. If A, B are NIP, $simple$ or NTP_2 then so respectively is the disjoint union $A \sqcup B$ when considered in a multisorted context.*

Proof. We present here the proof for the NTP_2 case.

Note that if either A or B have TP_2 then $A \sqcup B$ the disjoint union has TP_2 witnessed by the same formula and array. Now assume that $A \sqcup B$ has TP_2 . Then TP_2 is witnessed by a formula $\varphi(x, \bar{y})$ with x a single variable and an array $(\bar{b}_{i,j})$ from $A \sqcup B$. But by Lemma 2.2.5 φ is a finite disjunction of formulas of the form $\theta_l \wedge \psi_l$. From [11] (Lemma

7.1) we know that disjunction preserves NTP_2 . Hence, for a particular index k, there is a subarray from $(\bar{b}_{i,j})$ that witnesses the tree property of the second kind for $\theta_k \wedge \psi_k$. Furthermore, since $|x| = 1$ we can find an array of parameters only in A such that $\theta_k(x, \bar{y})$ holds, or only in B such that $\psi_k(x, \bar{y})$ holds. This means that either A or B have the tree property of the second kind. □

Remark 2.2.7. *The previous Lemma 2.2.6 can be generalised for finitely many structures $A_1, \ldots, A_n$ in disjoint languages $\mathcal{L}_1, \ldots, \mathcal{L}_n$ when considered in a multisorted context.*

We now turn back to the proof of Proposition 2.2.4.

(Proof of Proposition 2.2.4). We present here the NIP case. The other cases are done similarly.

First, assume that at least one of $R_1, \ldots, R_n$ has the independence property, without loss of generality say R_1 has IP. Since R_1 is isomorphic to the $\emptyset$-definable substructure $R_1 \times \mathbf{0_{R_2}} \times \ldots \times \mathbf{0_{R_n}}$ then the direct product has also the independence property.

Finally, assume that all $R_1, \ldots, R_n$ are NIP and assume that the direct product $R_1 \times \ldots \times R_n$ has IP. Since the direct product is definable over the disjoint union $R_1 \sqcup \ldots \sqcup R_n$ we get that the disjoint union also has the independence property. However, this is impossible by Lemma 2.2.6, since each one of $R_1, \ldots, R_n$ is NIP. □

Now let both $p \in \mathbb{P}$, and $n \in \mathbb{N}$ vary in $\prod_{(p,n) \in \mathbb{P} \times \omega} \mathbb{Z}/p^n\mathbb{Z}/\mathcal{U}$ with $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{P} \times \omega$. Consider the following class of residue rings

$$\mathcal{C} := \{\mathbb{Z}/p^n\mathbb{Z} : p \in \mathbb{P}, n \in \omega\}.$$

Proposition 2.2.8. *Any ultraproduct of rings in $\mathcal{C} = \{\mathbb{Z}/p^n\mathbb{Z} : p \in \mathbb{P}, n \in \omega\}$ has NTP_2 theory.*

Proof. We first note that $\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{P}\times\omega$. For each $(p,e)\in\mathbb{P}\times\omega$ choose an element $a_{p^e}\in\mathbb{Q}_p$ such that $v(a_{p^e})=e$. This defines $p^e\mathbb{Z}_p$ in $\mathbb{Q}_p$, as the set of elements in $\mathbb{Z}_p$ with value greater or equal to the value of a_{p^e}. Hence $\mathbb{Z}_p/p^e\mathbb{Z}_p$ is interpretable in $(\mathbb{Q}_p, a_{p^e})$, and $R := \prod_{(p,e)\in\mathbb{P}\times\omega}(\mathbb{Z}_p/p^e\mathbb{Z}_p)/\mathcal{U}$ is interpretable in $\prod_{(p,e)\in\mathbb{P}\times\omega}(\mathbb{Q}_p,\overline{a_{p^e}})/\mathcal{U}$, where $\overline{a_{p^e}}$ is an element in $\prod_{(p,e)\in\mathbb{P}\times\omega}\mathbb{Q}_p$ with (p,e)-projections equal to a_{p^e}. By Proposition 2.1.6 we have that any ultraproduct $\prod_{p\in\mathbb{P}}\mathbb{Q}_p/\mathcal{W}$ is NTP_2 , so $\prod_{(p,e)\in\mathbb{P}\times\omega}(\mathbb{Q}_p, a_{p^e})/\mathcal{U}$ is also NTP_2 .

By the above observations, R is NTP_2 . So every ultraproduct in $\mathcal{C}$ has NTP_2 theory. □

Remark 2.2.9. *Furthermore, in Proposition 2.2.8 R need not be simple or NIP. If the ultrafilter concentrates on a prime p then R is NIP but not simple (see Remark 2.5.1), and if it concentrates on prime powers with exponent 1 then R is supersimple since the ultraproduct is then a pseudofinite field. Since pseudofinite fields have the independence property, R is not NIP.*

Corollary 2.2.10. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that there exist $b\in\mathbb{N}$ and $U\in\mathcal{U}$ such that every $n\in U$ has at most b prime divisors.*

i) Then $\prod_n\mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is NTP_2 .

ii) If for each e there is $U_e\in\mathcal{U}$ such that for every n in U_e some prime divides n with exponent at least e, then the theory is not NSOP, in particular not simple.

Proof. i) Let R' be such an ultraproduct and U an element of the ultrafilter as in the hypothesis. Consider $\mathcal{V} := \{V\cap U : V\in\mathcal{U}\}$. Put $R := \prod_{n\in U}\mathbb{Z}/n\mathbb{Z}/\mathcal{V}$. Furthermore there is a $V\in\mathcal{V}$ such that every $n\in V$ has exactly d prime divisors. Choose as earlier an ultrafilter $\mathcal{W}$ on V such that $R\cong\prod_{n\in V}(\mathbb{Z}/p_{n(1)}^{e_{n(1)}}\mathbb{Z}\times\cdots\times\mathbb{Z}/p_{n(d)}^{e_{n(d)}}\mathbb{Z})/\mathcal{W}$. Using Lemma 2.2.2 we have that

$$R\cong\Big(\prod_{(p_{n(1)},e_{n(1)})}\mathbb{Z}/p_{n(1)}^{e_{n(1)}}\mathbb{Z}/\mathcal{W}_1\Big)\times\cdots\times\Big(\prod_{(p_{n(d)},e_{n(d)})}\mathbb{Z}/p_{n(d)}^{e_{n(d)}}\mathbb{Z}/\mathcal{W}_d\Big)$$

for some $\mathcal{W}_i$. Hence by Proposition 2.2.8 we have that R (and therefore R') has NTP_2 theory.

ii) See Remark 2.5.1.

□

2.2.1 Dp-rank and burden

At the end of this section we turn to investigate the dp-rank and burden of the ultraproducts in the NIP and NTP_2 case.

Recall Definition 1.2.9 from Subsection 1.2.4

As we mentioned in Subsection 1.2.4, since the ultrapower $\prod_n \mathbb{Q}_p/\mathcal{U}$ is an elementary extension of the p-adics, it is dp-minimal.

The corresponding notion to dp-rank in the more general context of NTP_2 structures is that of burden presented above in Definition 1.2.19 in Subsection 1.2.7.

Furthermore, A. Chernikov and P. Simon showed in [14] that ultraproducts of p-adics (in the language $\mathcal{L}_{RV}$) are inp-minimal, i.e. have burden 1. The language $\mathcal{L}_{RV}$ is a three sorted language for valued fields viewed as $\bar{K} = (K, RV, \Gamma, \mathrm{val}_{rv})$ obtained from $\mathcal{L}_{rings}$ by adding the quotient group $K^{\times}/(1 + \mathfrak{M})$ as a sort with multiplicative group structure "$\cdot, 1$", plus a constant 0, a predicate for the residue field $\mathbf{k} \subseteq RV$ together with an addition $\widehat{+}$ on $\mathbf{k}$ and a map $\mathrm{val}_{rv} : RV \to \Gamma$ between the RV and Γ sort. In $\mathcal{L}_{RV}$ the valuation is definable, although not necessarily the angular component map. See Sections 2 and 3 from [14] for more details.

Remark 2.2.11. *Given two* NTP_2 *structures* $\mathcal{M}$, $\mathcal{N}$ *and* $a \in \mathcal{M}$, $b \in \mathcal{N}$ *with* ***bdn***$(a) = r$ *in* $\mathcal{M}$ *and* ***bdn***$(b) = s$ *in* $\mathcal{N}$*, then the burden of* (a, b) *in the disjoint union of* $\mathcal{M}$ *and* $\mathcal{N}$ *is less than or equal to* $r \cdot s$*. I.e.* ***bdn***$((a, b)) \leq r \cdot s$ *in* $\mathcal{M} \sqcup \mathcal{N}$*.*

Proof. First note that from the definition of burden we have that for two types p, q if $p \subseteq q$ then **bdn**$(p) \geq$ **bdn**(q). By theorem 2.5 in [11] we have that **bdn**$(ab) \leq rs$.

□

We now mention the following result from [11].

Proposition 2.2.12 (Corollary 2.6, [11])**.** *"Sub-multiplicativity" of burden. If* ***bdn***$(a_i) < k_i$ *for* $i < n$ *with* $k_i \in \mathbb{N}$*, then* ***bdn***$(a_0, \ldots, a_{n-1}) < \prod_{i<n} k_i$*.*

Remark 2.2.13. *In the context of the ultraproducts of finite residue rings, when considering the strict* NTP_2 *case we note that an ultraproduct of the form*

$$\prod_{(p_{n(1)}, e_{n(1)})} \mathbb{Z}/p_{n(1)}^{e(1)}\mathbb{Z}/\mathcal{W}_1 \times \ldots \times \prod_{(p_{n(d)}, e_{n(d)})} \mathbb{Z}/p_{n(d)}^{e(d)}\mathbb{Z}/\mathcal{W}_d$$

would have burden bounded by the product of the burdens of each structure, given that each structure has finite burden. From [14] we have that an ultraproduct $\prod \mathbb{Q}_p/\mathcal{U}$ *of the p-adic numbers has burden 1 (in* $\mathcal{L}_{RV}$*). Since we only use the valuation to interpret the corresponding ultraproduct of finite residue rings in the ultraproduct of p-adic numbers we have that each factor also has burden 1. This implies that the product would have burden at most* 2^d*. Furthermore it is conjectured, e.g. Conjecture 2.7 in [11], that burden is sub-additive in* NTP_2 *theories. Hence, if the conjecture holds then the burden would be bounded by* d*.*

Remark 2.2.14. *In the NIP case, it is known that* $\mathbb{Q}_p$ *is dp-minimal and dp-rank is a particular case of burden in NIP theories, and moreover it is shown in [24] that dp-rank is sub-additive. Hence we get that an ultraproduct of finite residue rings of the form*

$$\prod_m \mathbb{Z}/p_1^m\mathbb{Z}/\mathcal{W}_1 \times \ldots \times \prod_m \mathbb{Z}/p_d^m\mathbb{Z}/\mathcal{W}_d$$

has dp-rank bounded by d*.*

2.3 Simple case

Now fix $b \in \mathbb{N}$ and consider the following ultraproduct, $\prod_{p\in\mathbb{P}} \mathbb{Z}/p^b\mathbb{Z}/\mathcal{U}$.

In [10] we find the following definition.

Definition 2.3.1 (Definition 2.1.9, [10]). *Let $D \subseteq N$ be structures possibly in different languages with D definable in N, and let $a \in N^{eq}$ be a canonical parameter for D.*

1. *D is **canonically embedded** in N if the $\emptyset$-definable relations of D are the relations on D which are a-definable in the sense of N.*
2. *D is **stably embedded** in N if every N-definable relation on D is D, a-definable, uniformly, in the structure N. The uniformity can be expressed by requiring that the form of the definition over D be determined by the form of the definition over N.*
3. *D is **fully embedded** in N if it is both canonically and stably embedded in N.*

In [11] it is mentioned in the proof of Theorem 7.6 that if $\bar{K} = (K, \Gamma, \mathbf{k}, v, \bar{ac})$ is a henselian valued field of characteristic $(0, 0)$ in the three sorted Denef-Pas language then Γ and k are stably embedded with no new induced structure so are fully embedded. For completeness we include a proof. We first recall the definition of the Denef-Pas language for Henselian valued fields, see Definition 1.3.25.

Proposition 2.3.2. *Let $\bar{K} = (K, \Gamma, \mathbf{k}, v, \bar{ac})$ be a Henselian valued field of characteristic $(0, 0)$ in the Denef-Pas language. Then the value group Γ and the residue field k are fully embedded.*

Proof. In the Denef-Pas language we have elimination of field quantifiers, cf. [37], or [38].

Let us show first that Γ is stably embedded. Consider a $\bar{K}$-definable relation R on Γ, defined by $\varphi(\bar{x}, \bar{\alpha}, \bar{\beta})$. By Denef-Pas quantifier elimination we may assume φ has the form

$\bar{Q}(\bar{a},\bar{b})\psi(\bar{x},\bar{\alpha},\bar{a},\bar{\beta},\bar{b})$ where ψ is a quantifier free formula, $\bar{Q}$ is a tuple of quantifiers on the group and residue field sorts, $\bar{x}$ is a tuple of free variables of the valued field sort, $\bar{\alpha}$ and $\bar{a}$ are tuples of free variables and bound variables respectively of the ordered group sort, and $\bar{\beta}$ and $\bar{b}$ are tuples of free and bound variables respectively from the residue field sort.

Furthermore we may assume ψ is a disjunction of formulas of the form $\psi_1(\bar{x})\wedge\psi_2\big(v(t_2(\bar{x})),\bar{\alpha},\bar{a}\big)\wedge\psi_3\big(\bar{ac}(t_3(\bar{x})),\bar{\beta},\bar{b}\big)$, where $\psi_1(\bar{x})$ is a formula without quantifiers on the valued field sort, $\psi_2\big(v(t_2(\bar{x})),\bar{\alpha},\bar{a}\big)$ is a formula without quantifiers on the value group sort, and $\psi_3\big(\bar{ac}(t_3(\bar{x})),\bar{\beta},\bar{b}\big)$ is a quantifier free formula from the residue field sort, also $t_2(\bar{x})$ and $t_3(\bar{x})$ are terms obtained from the variables $\bar{x}$ via the operations from the valued field sort. We may assume the variables from $\bar{a}$ only appear in formulas like ψ_2, and the variables from $\bar{b}$ only appear in formulas like ψ_3. Hence φ is equivalent to a disjunction of formulas of the form $\psi_1(\bar{x})\wedge\varphi_2\big(v(t_2(\bar{x})),\bar{\alpha}\big)\wedge\varphi_3\big(\bar{ac}(t_3(\bar{x})),\bar{\beta}\big)$. Here ψ_1 is a quantifier free formula on the sort of valued fields, $\varphi_2\big(v(t_2(\bar{x})),\bar{\alpha}\big)$ is a (quantified) formula from the sort of ordered groups where the bound variables are among $\bar{a}$, and $\varphi_3\big(\bar{ac}(t_3(\bar{x})),\bar{\beta}\big)$ is a (quantified) formula from the residue field sort where the bound variables are from $\bar{b}$.

Since the formula φ defines a relation on Γ we end up with a formula made up with a disjunction of formulas of the form of φ_2 and the parameters involved are all from Γ, possibly of the form $v(\bar{p})$ for some $\bar{p}\in K$.

Next we show that Γ is canonically embedded. Consider now S an $\emptyset$-definable relation in Γ, defined by $\varphi(\bar{x},\bar{\alpha},\bar{\beta})$. By the above argument we end up with S being definable by a disjunction of formulas of the form $\varphi_2\big(v(t_2(\bar{x})),\bar{\alpha}\big)$ with no parameters. Hence S is $\emptyset$-definable and so Γ is canonically embedded in $\bar{K}$.

In an analogous way we have that when a formula defines a subset of k the only part of the formulas in the disjunction of formulas of the form

$$\psi_1(\bar{x}) \wedge \varphi_2\big(v(t_2(\bar{x})), \bar{\alpha}\big) \wedge \varphi_3\big(\bar{ac}(t_3(\bar{x})), \bar{\beta}\big)$$

we are interested in is that corresponding to $\varphi_3(\bar{ac}(t_3(\bar{x})), \bar{\beta})$ and every parameter used can be taken to be from $\mathbf{k}$, where some may be of the form $\bar{ac}(\bar{p})$ for $\bar{p} \in K$. Hence $\mathbf{k}$ is stably embedded. Furthermore an $\emptyset$-definable relation in $\mathbf{k}$ defined by a formula $\varphi(\bar{x}, \bar{\alpha}, \bar{\beta})$ ends up being $\emptyset$-definable by a formula only in the residue field sort. Hence $\mathbf{k}$ is canonically embedded in $\bar{K}$.

□

Definition 2.3.3 ([21], Definition 4.1)**.** *In the next definition and proposition, taken from [21], we work in a saturated model $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{eq}$ of $T = T^{eq}$. (Here $\mathcal{M}^{eq}$ stands as usual for imaginaries over $\mathcal{M}$, cf. [7] .)*

- *Suppose that $\mathcal{P}$ is a class of (partial) types (over a small subset of $\overline{\mathcal{M}}$) closed under automorphisms. We say that T is coordinatised by $\mathcal{P}$ if for every $a \in \overline{\mathcal{M}}$ there is $n \in \omega$ and a_i for $i \leq n$ such that $a_n = a$ and $\mathrm{tp}(a_i/a_{i-1}) \in \mathcal{P}$ for all $i \leq n$, with $a_{-1} = \emptyset$. The sequence $(a_i : i \leq n)$ is called a coordinatising sequence.*

- *A type q is said to be simple if for each extension $p' \in S(B)$ there is a subset A of B with $|A| \leq |T|$ such that p' does not divide over A.*

- *A type q is said to be supersimple if for each extension $p' \in S(B)$ there is a subset A of B with $|A| < \aleph_0$ such that p' does not divide over A.*

Proposition 2.3.4 ([21], Proposition 4.2)**.** *If a type p is coordinatised by simple types then p is simple. Furthermore if p is coordinatised by supersimple types then p is supersimple.*

We turn now to prove the following. We will work in $\prod\limits_{p \in \mathbb{P}} \mathbb{Q}_p / \mathcal{U}$.

Proposition 2.3.5. *Fix $b \in \mathbb{Z}^+$ and let $\mathbf{R}$ be the ultraproduct $\prod_{p\in\mathbb{P}} \mathbb{Z}/p^b\mathbb{Z}/\mathcal{U}$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{P}$. Then $\mathbf{R}$ has supersimple theory.*

Proof. Since $\prod \mathbb{Z}_p/p^b\mathbb{Z}_p/\mathcal{U} \cong \prod \mathbb{Z}/p^b\mathbb{Z}/\mathcal{U}$, we will think about $\mathbf{R}$ inside the valued field structure $\mathbf{Q} = (\prod_{p\in\mathbb{P}} \mathbb{Q}_p/\mathcal{U}, \Gamma, \prod \mathbb{F}_p/\mathcal{U}, v, \bar{ac})$, where Γ is the corresponding ultrapower of $\mathbb{Z}$. We will use below the definable function $\overline{\phi}$ on $\prod_{p\in\mathbb{P}} \mathbb{Q}_p/\mathcal{U}$ where $\overline{\phi}([x_p]_{\mathcal{U}}) = [px_p]_{\mathcal{U}}$. This is definable using only the parameter $[p]_{\mathcal{U}}$. Thus we want to show $\mathbf{R}' := \prod_{p\in\mathbb{P}} \mathbb{Z}_p/p^b\mathbb{Z}_p/\mathcal{U}$ has supersimple theory with the induced structure in which the $\emptyset$-definable relations on $\mathbf{R}'$ are all those arising from $\emptyset$-definable relations on $\mathbf{Q}$.

The key idea is to use Proposition 2.3.2 and Proposition 2.3.4, by showing that types from $\mathbf{R}'$ are coordinatised by types in the residue field, which is a fully embedded pseudofinite field and hence is supersimple. Thus, it suffices to coordinatise $\mathbf{R}'$ by types in definable bijection with those in the residue field. We will consider the class of types $\mathcal{P}$ of the form $\mathrm{tp}([p^i a_p + (p^b)]/[p^{i+1}a_p + (p^b)])$, for $[a_p + (p^b)] \in \mathbf{R}'$.

For any given $[a_p + (p^b)] \in \mathbf{R}'$ we have the following coordinatising sequence

$$[p^{b-1}a_p + (p^b)], \ldots, [p^i a_p + (p^b)], \ldots, [pa_p + (p^b)], [a_p + (p^b)].$$

Furthermore $\mathrm{tp}([p^i a_p + (p^b)]/[p^{i+1}a_p + (p^b)])$ contains the formula $[p]x = [p^{i+1}a_p + (p^b)]$. Put $S_i = \{x \in \mathbf{R}' : [p]x = [p^{i+1}a_p + (p^b)]\}$. Then

$$S_i = [p^i a_p + p^{b-1}\mathbf{R}' + (p^b)] := \{[p^i a_p + p^{b-1}x + (p^b)] : x \in \mathbf{R}'\}.$$

To show that $\mathrm{tp}([p^i a_p + (p^b)]/[p^{i+1}a_p + (p^b)])$ is simple it is enough to find a definable bijection φ_i (using just one parameter $w \in S_i$) over $[p^i a_p + (p^b)]$ between $\prod \mathbb{Z}_p/p\mathbb{Z}_p/\mathcal{U}$ and S_i. To see this, note that for $c_p \in \mathbb{Z}_p/p^b\mathbb{Z}_p$ written as

$$c_p := c_{p(0)} + pc_{p(1)} + \ldots + p^{b-1}c_{p(b-1)} + (p^b)$$

we can define, uniformly in p, a bijection $\varphi_{i,p}$ from $\mathbb{Z}_p/p\mathbb{Z}_p$ to

$$S_{i,p} := \{z \in \mathbb{Z}_p/p^b\mathbb{Z}_p : pz = p^{i+1}c_p + (p^b)\} = p^i c_p + p^{b-1}\mathbb{Z}_p/p^b\mathbb{Z}_p + (p^b)$$

as follows $\varphi_{i,p}(x) := p^{b-1}x + p^i c_p + (p^b)$. Therefore we can find the wanted definable bijection φ_i as the induced by $\varphi_{i,p}$ sending $[x_p] \in \prod \mathbb{Z}_p/p\mathbb{Z}_p/\mathcal{U}$ to $[p^i a_p + p^{b-1}x_p + (p^b)]$ (otherwise $[w + p^{b-1}x_p + (p^b)]$).

Moreover $\prod \mathbb{Z}_p/p\mathbb{Z}_p/\mathcal{U}$ is a pseudofinite field stably embedded in $\mathbf{Q}$, so all the types realised in $\prod \mathbb{Z}_p/p\mathbb{Z}_p/\mathcal{U}$ are supersimple. Hence all the types of the form

$$\mathrm{tp}([p^i a_p + (p^b)]/[p^{i+1}a_p + (p^b)])$$

are also supersimple.

Finally we note that $\mathcal{P}$ is closed under automorphisms of $\mathbf{R}'$.

Now we can apply Proposition 2.3.4 to conclude that since $\mathbf{R}'$ is coordinatised by supersimple types then it is supersimple.

□

Remark 2.3.6. *It is noted in Remark 4.3 of [21] that if* $(a_i : i \leq n)$ *is a coordinatising sequence then* $\mathrm{SU}(a_n) \leq \mathrm{SU}(a_n/a_{n-1}) \oplus \ldots \oplus \mathrm{SU}(a_0)$. *Hence the structure* $\prod_{p \in \mathbb{P}} \mathbb{Z}/p^d\mathbb{Z}/\mathcal{U}$ *has finite* SU*-rank and this rank is at most* d *since the coordinatisatising sequence used in the proof of Proposition 2.3.5 has length* d *and each of the types of the sequence has* SU*-rank* 1*, cf. [23]. For a particular* p*, the sequence of ideals*

$$p^{d-1}\mathbb{Z}_p/p^d\mathbb{Z}_p < \ldots < p\mathbb{Z}_p/p^d\mathbb{Z}_p < \mathbb{Z}_p/p^d\mathbb{Z}_p$$

has successive indices equal to p*. Hence in the ultraproduct the successive indices are infinite, and the ideals in the chain are uniformly definable in* p*. This implies that* $\mathrm{SU} \prod \mathbb{Z}_p/p^d\mathbb{Z}_p/\mathcal{U} \geq d$*.*

We can now use Proposition 2.3.5 together with Lemma 2.2.2 to cover the following more general case.

Corollary 2.3.7. *Consider an ultrafilter on $\mathbb{N}$ such that there exists $b \in \mathbb{N}$ and $U \in \mathcal{U}$ such that if $n \in U$ then n is a product of fewer than b primes and if p^l divides n then $l < b$. Then $R' := \prod \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is supersimple, of finite* SU*-rank.*

First note that for example the ultrafilter $\mathcal{U}$ will include the set

$$U = \{n \in \mathbb{N} : \text{ If } p^e \mid n \text{ then } e \leq b, \text{ and } n \text{ has fewer than } b \text{ prime divisors}\}.$$

Proof. Let $U \in \mathcal{U}$ be as in the hypothesis, and consider $\mathcal{V}$ the induced ultrafilter on U by $\mathcal{U}$. We put $R := \prod_{n \in U} \mathbb{Z}/n\mathbb{Z}/\mathcal{V}$, so $R \cong R'$. There is $V \in \mathcal{V}$ such that every $n \in V$ has the same number d of prime factors, $p_{n(1)}, p_{n(2)}, \cdots p_{n(d)}$ and every $p_{n(i)}$ has the same exponent e_i. Hence using Lemma 2.2.2 on $\mathcal{W}$ the induced ultrafilter on V by $\mathcal{V}$ we have that $R \cong (\prod_m (\mathbb{Z}/p^{e_1}_{m(1)}\mathbb{Z})/\mathcal{W}_1) \times \cdots \times (\prod_m (\mathbb{Z}/p^{e_d}_{m(d)}\mathbb{Z})/\mathcal{W}_d)$, for some $\mathcal{W}_i$. Since each one of $\prod_m (\mathbb{Z}/p^{e_i}_{m(i)}\mathbb{Z})/\mathcal{W}_i$ is supersimple by Proposition 2.3.5, R' is supersimple.

□

Remark 2.3.8. *Suppose that for each k there is $U_k \subseteq U$ such that $U_k \in \mathcal{U}$, for each $n \in U_k$ we have that n has d prime divisors, and each prime divisor is greater than or equal to k. Then, the ring*

$$\prod_m \mathbb{Z}/p^{e_1}_{m(1)}\mathbb{Z}/\mathcal{W}_1 \times \ldots \times \prod_m \mathbb{Z}/p^{e_d}_{m(d)}\mathbb{Z}/\mathcal{W}_d$$

has SU*-rank exactly $e_1 + \ldots + e_d$.*

Remark 2.3.9. *There is an alternative, maybe more direct, way of proving Proposition 2.3.5. We present here a brief sketch of the proof for $R := \prod_p \mathbb{Z}_p/p^2\mathbb{Z}_p/\mathcal{U}$, but the argument also yields Corollary 2.3.7.*

We note by the Ax-Kochen-Eršov theorem that $\prod_{p \in \mathbb{P}} \mathbb{Z}/p^2\mathbb{Z}/\mathcal{U}$ is elementarily equivalent to $R' := \prod_{p \in \mathbb{P}} \mathbb{F}_p[[t]]/(t^2)/\mathcal{U}$ in the language of rings, cf. Proposition 2.4.10 of [47].

We have that R' is interpretable in $k := \prod_p \mathbb{F}_p/\mathcal{U}$. In order to see this we only need to note that for any prime q the ring $\mathbb{F}_q[[t]]/(t^2)$ is uniformly (in q) interpretable in $\mathbb{F}_q$ since we can identify $a + bt + (t^2) \in \mathbb{F}_q[[t]]/(t^2)$ with the pair (a, b) and interpret addition, ($\oplus$), and multiplication, ($*$), from $\mathbb{F}_q[[t]]/(t^2)$ inside $\mathbb{F}_q \times \mathbb{F}_q$ in the following way.

- *For pairs $(a, b), (c, d)$ we put $(a, b) \oplus (c, d) := (a + c, b + d)$;*
- *For pairs $(a, b), (c, d)$ we put $(a, b) * (c, d) = (ac, ad + bc)$.*

Since k is supersimple of SU-rank 1, R' is supersimple of SU-rank 2 and hence R is also supersimple of SU-rank 2.

Although k is interpretable in R' the isomorphism from $(k^2, \oplus, *)$ to R' is not definable in R'. Otherwise, it would be also definable in R and thus we would get that for some $U \in \mathcal{U}$ if $p \in \mathcal{U}$ then the ring $\mathbb{Z}_p/p^2\mathbb{Z}_p$ is isomorphic to the characteristic p ring $((\mathbb{Z}/p\mathbb{Z})^2, \oplus, *)$ but this is a contradiction.

In Proposition 2.3.5 we chose to give the proof using coordinatisation through supersimple types because we believe it provides added information for the class of finite rings of the form $\mathbb{Z}/p^n\mathbb{Z}$. In particular, it motivates showing that for a fixed d the class of rings $\{\mathbb{Z}/p^d\mathbb{Z} : p \in \mathbb{P}\}$ is a d-dimensional asymptotic class in the sense of [17]; see also [18]. See Chapter 3 below.

2.4 TP_2 case

Not every ultraproduct of finite residue rings is NTP_2 . First, we note the following. The proof is routine.

Claim 2.4.1. *If $\{R_i : i \in I\}$ is a collection of commutative rings, then*

$$SL_2(\prod_{i\in I} R_i/\mathcal{U}) \cong \prod_{i\in I}(SL_2(R_i))/\mathcal{U}.$$

Proof. This follows from Łoś theorem. Alternatively, define $\beta : SL_2(\prod_{i\in I} R_i/\mathcal{U}) \to \prod_{i\in I}(SL_2(R_i))/\mathcal{U}$ as follows

$$\beta\left(\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\right)_i.$$

Here $(a)_i$, $(b)_i$, $(c)_i$ and $(d)_i$ stand for representative tuples of an equivalence class in the ultraproduct $\prod R_i/\mathcal{U}$. Also a_i refers to the i-th coordinate of $(a)_i$, similarly for b_i, c_i and d_i. Let us first consider two elements in $SL_2(\prod R_i/\mathcal{U})$ such that $\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right) = \left(\begin{smallmatrix}(a')_i & (b')_i\\ (c')_i & (d')_i\end{smallmatrix}\right)$. This means that $(a)_i = (a')_i$, $(b)_i = (b')_i$, $(c)_i = (c')_i$ and $(d)_i = (d')_i$. Hence there are U_a, U_b, U_c, U_d such that $\forall k \in U_a(a_k = a'_k)$, similarly for U_b, b, b'; U_c, c, c' and U_d, d, d'. Therefore, $\forall k \in U' = U_a \cap U_b \cap U_c \cap U_d$ we have $\left(\begin{smallmatrix}a_k & b_k\\ c_k & d_k\end{smallmatrix}\right) = \left(\begin{smallmatrix}a'_k & b'_k\\ c'_k & d'_k\end{smallmatrix}\right)$, and so $\beta\left(\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right)\right) = \beta\left(\left(\begin{smallmatrix}(a')_i & (b')_i\\ (c')_i & (d')_i\end{smallmatrix}\right)\right)$. Hence, β is a well defined function.

Also β is injective because whenever $\beta\left(\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right)\right) = \beta\left(\left(\begin{smallmatrix}(a')_i & (b')_i\\ (c')_i & (d')_i\end{smallmatrix}\right)\right)$ we have that $\left(\left(\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\right)_i = \left(\left(\begin{smallmatrix}a'_i & b'_i\\ c'_i & d'_i\end{smallmatrix}\right)\right)_i$. This means there is $U \in \mathcal{U}$ such that $\forall k \in U$ we have $\left(\begin{smallmatrix}a_k & b_k\\ c_k & d_k\end{smallmatrix}\right) = \left(\begin{smallmatrix}a'_k & b'_k\\ c'_k & d'_k\end{smallmatrix}\right)$ which in turn means $\{i \in I : (a)_i = (a')_i\} \in \mathcal{U}$, similarly for $(b), (b')$; $(c), (c')$ and $(d), (d')$. So $\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right) = \left(\begin{smallmatrix}(a')_i & (b')_i\\ (c')_i & (d')_i\end{smallmatrix}\right)$.

To see that β is surjective we note that given $\left(\left(\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\right)_i$ we consider $(a_i)_i$, $(b_i)_i$, $(c_i)_i$ and $(d_i)_i$ elements of $\prod R_i/\mathcal{U}$. Then $\beta\left(\left(\begin{smallmatrix}(a_i)_i & (b_i)_i\\ (c_i)_i & (d_i)_i\end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\right)_i$.

To check that β is a homomorphism we note that $\beta\left(\left(\begin{smallmatrix}(1)_i & (0)_i\\ (0)_i & (1)_i\end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix}1_i & 0_i\\ 0_i & 1_i\end{smallmatrix}\right)\right)_i$ and that $\beta\left(\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}(a')_i & (b')_i\\ (c')_i & (d')_i\end{smallmatrix}\right)\right) = \beta\left(\left(\begin{smallmatrix}(a)_i(a')_i+(b)_i(c')_i & (a)_i(b')_i+(b)_i(d')_i\\ (c)_i(a')_i+(d)_i(c')_i & (c)_i(b')_i+(d)_i(d')_i\end{smallmatrix}\right)\right)$ which is equal to $\left(\left(\begin{smallmatrix}a_ia'_i+b_ic'_i & a_ib'_i+b_id'_i\\ c_ia'_i+d_ic'_i & c_ib'_i+d_id'_i\end{smallmatrix}\right)\right)_i$ and this in turn is equal to

$$\left(\left(\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}a'_i & b'_i\\ c'_i & d'_i\end{smallmatrix}\right)\right)_i = \left(\left(\begin{smallmatrix}a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\right)_i \cdot \left(\left(\begin{smallmatrix}a'_i & b'_i\\ c'_i & d'_i\end{smallmatrix}\right)\right)_i = \beta\left(\left(\begin{smallmatrix}(a)_i & (b)_i\\ (c)_i & (d)_i\end{smallmatrix}\right)\right) \cdot \beta\left(\left(\begin{smallmatrix}(a')_i & (b')_i\\ (c')_i & (d')_i\end{smallmatrix}\right)\right).$$

□

For an analogous result (over fields, but with arbitrary groups of Lie type) see Proposition

1 of [39].

Recall now Lemma 1.3.12, presented in Section 1.3.2. Below we use this necessary condition for a theory to be NTP_2 from [13].

Proposition 2.4.2. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that for every $b \in \mathbb{N}$ the set*

$$\{n \in \mathbb{N} : \textit{ there are at least } b \textit{ prime divisors of } n\}$$

is in $\mathcal{U}$. Then $\prod \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ has TP_2 theory.

First we note these ultrafilters exist, since the collection

$$\big\{\{n \in \mathbb{N} : \text{ there are at least } b \text{ prime divisors of } n\} : b \in \mathbb{N}\big\}$$

has the finite intersection property.

Proof of Proposition 2.4.2. Let $\mathcal{R} := \prod(\mathbb{Z}/n\mathbb{Z})/\mathcal{U}$. We want to find a definable group G in $\mathcal{R}$ and a uniformly definable family of normal subgroups $\{H_i\}_{i<\omega}$ that contradicts the conclusion from Lemma 1.3.12, and to do this we consider for each $j < \omega$ the group

$$SL_2(\mathbb{Z}/j\mathbb{Z}) \cong G_j := F_j \times SL_2\left(\mathbb{Z}/p_{j,1}^{e_{j,1}}\mathbb{Z}\right) \times \cdots \times SL_2\left(\mathbb{Z}/p_{j,b_j}^{e_{j,b_j}}\mathbb{Z}\right)$$

such that F_j is isomorphic to $SL_2(\mathbb{Z}/n\mathbb{Z})$ where n can only have prime factors smaller than $p_{j,1}$. The condition on the ultrafilter allow us to consider the case when both b_i and $p_{i,1}$ increase as $i \to \infty$, in such a way that $p_{j,k} > p_{j',k'}$ whenever $j > j'$ or both $j = j'$ and $k > k'$.

We can now for a given j and k with $1 \leq k \leq b_j$ find non-central elements $A_{j,k} = \left(\begin{smallmatrix} a_{j,k} & 0 \\ 0 & b_{j,k} \end{smallmatrix}\right) \in SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)$ with $a_{j,k} \neq b_{j,k}$. We may assume that $a_{j,k}, b_{j,k} < p_{j,k}$. Consider $A_{j,k}$ inside G_j occurring as the $k+1$-th entry in

$$\overline{A_{j,k}} = \left(Id_{F_j}, Id_{SL_2\left(\mathbb{Z}/p_{j,1}^{e_{j,1}}\mathbb{Z}\right)}, \ldots, A_{j,k}, \ldots, Id_{SL_2\left(\mathbb{Z}/p_{j,b_j}^{e_{j,b_j}}\mathbb{Z}\right)}\right)$$

Now let $\mathcal{F}_{j,k}$ be the conjugacy class in G_j of $\overline{A_{j,k}}$. This is uniformly definable (across the class of groups G_j) using $\overline{A_{j,k}}$ as a parameter. Elements in $\mathcal{F}_{j,k}$ are of the form $(1, Id, \ldots, gA_{j,k}g^{-1}, \ldots, Id)$ where $g \in SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)$.

Consider now $N_{j,k} = C_{G_j}(\mathcal{F}_{j,k})$. Then this is a normal subgroup of G_j since for every $\alpha \in G_j$ and every $\gamma \in N_{j,k}$ we have that $\alpha\mathcal{F}_{j,k}\alpha^{-1} = \mathcal{F}_{j,k}$ so $\alpha\gamma\alpha^{-1} \in N_{j,k}$. Furthermore $N_{j,k} = C_{G_j}(\mathcal{F}_{j,k})$ is uniformly definable using $\overline{A_{j,k}}$ through the formula defining the centralizer, namely $\varphi(x, \overline{A_{j,k}}) := \forall g \in G_j(xg\overline{A_{j,k}}g^{-1}x^{-1} = g\overline{A_{j,k}}g^{-1})$. We have

$$\begin{aligned} N_{j,k} =& F_j \times SL_2\left(\mathbb{Z}/p_{j,1}^{e_{j,1}}\mathbb{Z}\right) \times \ldots \times C_{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\left(A_{j,k}^{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\right) \times \ldots \\ & \ldots \times SL_2\left(\mathbb{Z}/p_{j,b_j}^{e_{j,b_j}}\mathbb{Z}\right). \end{aligned}$$

We just note that if for a particular index i we have $b_i < k'$ then we just put $N_{i,k'} := G_i$. Define $H_k := \prod\limits_i N_{i,k}/\mathcal{U}$. Put $H := \bigcap\limits_k H_k$ and $H_{\neq j} := \bigcap\limits_{k\neq j} H_k$.

Fix $d \in \mathbb{N}$. Since b_i are increasing there is $Q = Q(d) \in \mathbb{N}$ such that whenever $j > Q$ then $b_j > d$, i.e. there are at least d many big prime factors, so $N_{j,d}$ is not the full G_j. Also, since $p_{i,1}$ are also increasing, we may choose Q so that $p_{j,1} > d$ whenever $j > Q$.

Claim 2.4.3. *Suppose that $j > Q$, and for $k \in \{1, \ldots, b_j\}$ take $N_{j,k}$ as above. Then there are at least d elements $c_{j,1}^k, \ldots, c_{j,d}^k \in G_j$, each of these with all coordinates not in $C_{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\left(A_{j,k}^{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\right)$ equal to the corresponding identity, such that $c_{j,r}^k \cdot {c_{j,s}^k}^{-1} \notin N_{j,k}$ for any distinct $r, s \in \{1, \ldots, d\}$. In particular, $|G_j : N_{j,k}| \geq d$.*

It is enough to show that if $j > Q$ we have

$$|G_j : N_{j,k}| = |SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right) : C_{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\left(A_{j,k}^{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\right)| \geq p_{j,k}.$$

To see this, let $|G_j : N_{j,k}| = \lambda$. Then for all $B \in SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)$ we have that

$$B^\lambda \in C_{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\left(A_{j,k}^{SL_2\left(\mathbb{Z}/p_{j,k}^{e_{j,k}}\mathbb{Z}\right)}\right)$$

which means that in particular $B^\lambda A_{j,k} B^{-\lambda} = A_{j,k}$. Considering the matrix $D = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$D^\lambda A_{j,k} D^{-\lambda} = \begin{pmatrix} a_{j,k} & \lambda(b_{j,k}-a_{j,k}) \\ 0 & b_{j,k} \end{pmatrix} = A_{j,k}$$

only if $\lambda(b_{j,k} - a_{j,k})$ is a multiple of $p_{j,k}^{e_{j,k}}$. In particular, if $\lambda < p_{j,k}$ then $p_{j,k}^{e_{j,k}} | (b_{j,k} - a_{j,k})$, which contradicts the choice of $a_{j,k}$ and $b_{j,k}$. This proves Claim 2.4.3.

Claim 2.4.4. *Let $d \in \mathbb{N}$. Then $|H_{\neq j} : H| \geq d$.*

Let $Q = Q(d)$ be as above. For $s \in \{1, \ldots, d\}$, let

$$\bar{c}_s := [(\mathbf{1}_{G_1}, \ldots, \mathbf{1}_{G_{Q-1}}, c^j_{Q,s}, c^j_{Q+1,s}, \ldots)]_{\mathcal{U}}.$$

We have $\bar{c}_s \in H_{\neq j}$ for $s \in \{1, \ldots, d\}$. This follows because we chose $c^j_{l,s} \in N_{l,k}$ for all $k \neq j$. Furthermore, for any distinct $r, s \in \{1, \ldots, d\}$, we have $\bar{c}_s \cdot \bar{c}_r^{-1} \notin H$. This holds because $c^j_{l,s} \cdot {c^j_{l,r}}^{-1} \notin N_{l,j}$. Hence the index $|H_{\neq j} : H|$ is at least d, which proves Claim 2.4.4.

From the above, since d was arbitrary, we get that the index $|H_{\neq j} : H|$ is infinite for every j.

Thus, by Lemma 1.3.12 applied to the family of groups $\{\prod_j N_{j,i}\}_{i<\omega}$ the ring $\prod \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ has the tree property of the second kind.

□

2.5 NSOP

We now turn to answer the questions raised by Artem Chernikov and prove Proposition 2.1.5. For this we recall the definition of NSOP theories. See Definition 1.2.11 in Subsection 1.2.5 above.

To prove Proposition 2.1.5, we must show that when $R := \prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is NIP but not simple, or it is NTP_2 but neither NIP nor simple, or it is TP_2 then R has the strict order property.

Let us focus first on the cases from Section 2.2.

Remark 2.5.1. *Notice that in the NIP and* NTP_2 *cases there is no bound on the exponents considered. In the NIP case from Proposition 2.2.1 fix a prime* q *and look at the ultraproduct* $R := \prod_{n\in\mathbb{N}} \mathbb{Z}/q^n\mathbb{Z}/\mathcal{U}$*. The relation* $a|b$*, "*a *divides* b*", is a definable partial order in the language of rings, and* $1, q, q^2, \ldots, q^{k-1}$ *forms a chain of length* k *in the finite ring* $\mathbb{Z}/q^k\mathbb{Z}$*. Hence we are able to find arbitrarily long chains inside the ultraproduct* R *precisely because there is no bound in the exponents. The same argument applies (with a non-standard prime in place of* q*) in the non-simple* NTP_2 *case considered in Proposition 2.2.8. Easily, this argument applies in the cases with boundedly many prime divisors and unbounded exponents presented in Corollary 2.2.3 and 2.2.10. In particular if* R *is non-simple* NTP_2 *then it is SOP.*

Next we deal with rings discussed in Section 2.4.

Remark 2.5.2. *Consider now ultraproducts of the form* $R := \prod_n \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ *where the ultraproduct* $\mathcal{U}$ *satisfies the conditions of Proposition 2.4.2. For every* $k \in \mathbb{N}$ *the set*

$$U_k := \{n \in \mathbb{N} : \text{ there are at least } k \text{ prime divisors of } n\}$$

is in the ultrafilter. We can write elements $n \in U_k$ *as*

$$n = p_{n(1)}^{e_{n(1)}} \cdot p_{n(2)}^{e_{n(2)}} \cdot \ldots \cdot p_{n(k)}^{e_{n(k)}} \cdot m(n)$$

where $m(n) \in \mathbb{N}$ *only has prime divisors greater than* $p_{n(k)}$*. If we consider the relation given by divisibility we have, in each such* $\mathbb{Z}/n\mathbb{Z}$*, the following chain of length* k*,*

$$p_{n(1)} | p_{n(1)} \cdot p_{n(2)} | \ldots | p_{n(1)} \cdot \ldots \cdot p_{n(k)}.$$

Hence we can define arbitrarily long chains in R and this implies that R has SOP.

Finally, we note that Proposition 2.1.5 follows from the two previous remarks.

Proposition 2.5.3. *If the ultraproduct $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}/\mathcal{U}$ is not simple, then it has SOP.*

Proof. Indeed, a non-simple ultraproduct will either have unbounded exponents involved in which case Remark 2.5.1 shows it is SOP, or it will have unbounded number of prime divisors involved so Remark 2.5.2 shows that it is SOP. □

Chapter 3

Asymptotic classes of finite residue rings

We turn now to address the asymptotic behaviour of the class of finite residue rings studied in Section 2.3; that is, the class of rings $\mathcal{R}_U := \{\mathbb{Z}/n\mathbb{Z} : n \in U\}$ where U is an element in a non-principal ultrafilter on $\mathbb{N}$ for which there is a bound b on the number and on the exponents of prime divisors of every $n \in U$.

In Proposition 2.3.7 in Section 2.3 we showed that ultraproducts of rings of any class of the form $\mathcal{R}_U$, where U is as above, have a supersimple finite rank theory.

As mentioned in Subsection 1.2.8, ultraproducts of elements of N-dimensional asymptotic classes are supersimple of rank $\leq N$. Furthermore the proof of Proposition 2.3.7 hinted towards a close relation between the ultraproducts of rings of the class $\mathcal{R}_U$ and ultraproducts of finite fields.

The aim of this chapter is to show that for any $d \in \mathbb{N}$ and $l_1, \ldots, l_d \in \mathbb{N}$ the class of rings $\{\mathbb{Z}/n\mathbb{Z} : n = p_1^{l_1} \cdot \ldots \cdot p_d^{l_d}, p_1 < \ldots < p_d \in \mathbb{P}\}$ is indeed a multidimensional asymptotic class in an expansion of $\mathcal{L}_{rings}$ by unary predicates in the sense of [1] as mentioned in 1.2.8 above. See Definition 3.3.1.

As we noted in Section 2.3, ultraproducts of rings of the class $\mathcal{R}_U$ can be viewed as finite direct sums of ultraproducts of rings of the class $\{\mathbb{Z}/p^l\mathbb{Z} : p \in \mathbb{P}\}$ for some fixed $l \in \mathbb{N}^{>0}$.

It is natural then to start by studying the latter classes. For this we present the notion of *asymptotic fragment*, the main tool of the chapter to deal with classes of rings of the form $\{\mathbb{Z}/p^l\mathbb{Z} : p \in \mathbb{P}\}$.

3.1 Asymptotic fragments

The overall strategy to deal with the class of rings $\{\mathbb{Z}/p^l\mathbb{Z} : p \in \mathbb{P}\}$ is the following. Inside each ring of the form $\mathbb{Z}_p/p^l\mathbb{Z}_p$ we find a definable subset A in correspondence with the finite field $\mathbb{Z}_p/p\mathbb{Z}_p$ and we note that we can "reflect" the asymptotic behaviour from the finite field onto our definable set A. Next we define, using elements in A as parameters, a partition of $\mathbb{Z}_p/p^l\mathbb{Z}_p \setminus A$ that has a nice asymptotic behaviour in $\mathbb{Z}_p/p^l\mathbb{Z}_p$. Then, we note that given a formula $\varphi(x, \bar{y})$ and some parameters $\bar{a} \in \mathbb{Z}_p/p^l\mathbb{Z}_p$ the solutions can be found separately in the base A and in the elements of the partition. Finally the asymptotic behaviour of both of these parts gives us the wanted asymptotic behaviour of $\mathbb{Z}_p/p^l\mathbb{Z}_p$. We present this in Theorem 3.2.1 below.

We work in the context of the valued fields $\mathbb{Q}_p$ for primes p in the three sorted Denef-Pas language, as mentioned in Definition 1.3.25 in 1.3.4.

Hence, we think about $\mathbb{Z}/p^l\mathbb{Z}$ as $\mathbb{Z}_p/p^l\mathbb{Z}_p$. Further $\mathbb{Z}_p/p^l\mathbb{Z}_p$ is interpretable in $\mathbb{Q}_p$ over a parameter p^l of value l, this is needed in the ultraproduct although not at each p. For $\mathbb{Z}_p$ is definable in $\mathbb{Q}_p$ using only the valuation and for every $k \in \mathbb{Z}^{>0}$ we have $p^k\mathbb{Z}_p = \{x \in \mathbb{Z}_p : v(x) \geq k\}$. Hence every definable set in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ is interpretable in $\mathbb{Q}_p$.

We first present the following lemma that, loosely put, allows us to reflect some of the asymptotic behaviour of the class of finite fields to the collection of uniformly definable subsets of $\mathbb{Z}_p/p^l\mathbb{Z}_p$ which are in bijective correspondence with finite fields.

Lemma 3.1.1. *Put $l \in \mathbb{N}$ and for every $p \in \mathbb{P}$ consider $\mathbb{Z}_p/p^l\mathbb{Z}_p$.*

For every $p \in \mathbb{P}$ let B_p be a set uniformly $\mathcal{L}_{rings}$-definable across the class $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$, possibly with parameters. If B_p is in uniform $\mathcal{L}_{rings}$-definable (possibly using parameters s_p from $\mathbb{Z}_p/p^l\mathbb{Z}_p$) bijective correspondence with $\mathbb{Z}_p/p\mathbb{Z}_p$, then for any $\mathcal{L}_{rings}$-formula $\varphi(x, \bar{y})$ the following hold.

i) There exists a constant C and a finite set D of pairs $(\delta, \mu) \in (\{0, 1\} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ such that for any $p \in \mathbb{P}$ and any $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$,

$$\left||\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p) \cap B_p| - \mu p^\delta\right| < Cp^{\delta-\frac{1}{2}}$$

holds for some $(\delta, \mu) \in D$ as $p \to \infty$.

ii) Furthermore, for each pair (δ, μ) in D there exists an $\mathcal{L}_{rings}$-formula $\tau_{(\delta,\mu)}(\bar{y}, \bar{z})$ such that $\tau_{(\delta,\mu)}(\bar{a}_p, s_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if

$$\left||\varphi(B_p, \bar{a}_p)| - \mu p^\delta\right| < Cp^{\delta-\frac{1}{2}}.$$

Proof. For any $p \in \mathbb{P}$ let f_{s_p} be the definable bijection (defined possibly with parameters s_p) from B_p to $\mathbb{Z}_p/p\mathbb{Z}_p$.

Put $\varphi(B_p, \bar{a}_p) := \varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p) \cap B_p$ the trace of the definable set $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)$ in B_p. The set $\varphi(B_p, \bar{a}_p)$ corresponds uniformly in p to a definable set in $\mathbb{Z}_p/p\mathbb{Z}_p$ using f_{s_p}, namely $f_{s_p}(\varphi(B_p, \bar{a}_p))$. Recall that the residue field $\mathbb{Z}_p/p\mathbb{Z}_p$, the ring $\mathbb{Z}_p/p^l\mathbb{Z}_p$ and $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)$ are interpretable in $\mathbb{Q}_p$.

Moreover, the corresponding set in $\mathbb{Z}_p/p\mathbb{Z}_p$ is definable uniformly in p with parameters $\bar{a}_p, s_p$ from $\mathbb{Z}_p/p^l\mathbb{Z}_p$. Hence we obtain a definable set in any ultraproduct $\prod_{\mathcal{U}} \mathbb{Z}_p/p\mathbb{Z}_p$ using parameters $[\bar{a}_p], [s_p]$ in $\prod_{\mathcal{U}} \mathbb{Z}_p/p^l\mathbb{Z}_p$, namely $f_{[s_p]}(\varphi(\prod_{\mathcal{U}} B_p, [\bar{a}_p]))$. We work inside the valued field $\mathbb{Q}_p$ in the three sorted Denef-Pas language setting. Then $\prod_{\mathcal{U}} \mathbb{Z}_p/p\mathbb{Z}_p$ is fully embedded in $\prod_{\mathcal{U}} \mathbb{Q}_p$, see Proposition 2.3.2. Hence, there exists an $\mathcal{L}_{rings}$-formula

$\varphi_\pi(x, \bar{w})$ and parameters $[\bar{e}_p] \in \prod_{\mathcal{U}} \mathbb{Z}_p/p\mathbb{Z}_p$ such that

$$f_{[s_p]}\left(\varphi(\prod_{\mathcal{U}} B_p, [\bar{a}_p])\right) = \varphi_\pi\left(\prod_{\mathcal{U}} \mathbb{Z}_p/p\mathbb{Z}_p, [\bar{e}_p]\right).$$

Since $\varphi_\pi(\prod_{\mathcal{U}} \mathbb{Z}_p/p\mathbb{Z}_p, [\bar{e}_p])$ is a definable set in an ultraproduct of finite fields, and the ultrafilter was arbitrary, when going back to the factors we get finitely many $\mathcal{L}_{rings}$-formulas $\psi_1, \ldots, \psi_{r_\varphi}$ such that for any particular p there is some $j \in \{1, \ldots, r_\varphi\}$ such that

$$f_{s_p}(\varphi(B_p, \bar{a}_p)) = \psi_j(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p).$$

Let $\mathcal{J}_p^*$ be the set of all $j \in \{1, \ldots, r_\varphi\}$ such that $f_{s_p}(\varphi(B_p, \bar{a}_p)) = \psi_j(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p)$. Let j_p^* be the least element in $\mathcal{J}_p^*$ and consider the $\mathcal{L}_{rings}$-formula $\psi_{j_p^*}(x, \bar{w})$

By Theorem 1.2.24 from [9], mentioned in Subsection 1.2.8 above, there exists a constant $C_{j_p^*}$ and a finite set $D_{j_p^*} \subseteq (\{0,1\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$ of pairs $(\delta_{j_p^*,k}, \mu_{j_p^*,k})$ for $k \in \{1, \ldots, t_{j_p^*}\}$ and a partition $\{\Phi_{(\delta,\mu)} : (\delta, \mu) \in D_{j_p^*}\}$ of the set $\{(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p) : p \in \mathbb{P}, \bar{e}_p \in \mathbb{Z}_p/p\mathbb{Z}_p\}$ such that for each $(\delta_{j_p^*,k}, \mu_{j_p^*,k}) \in D_{j_p^*}$

$$\left||\psi_{j_p^*}(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p)| - \mu_{j_p^*,k}|\mathbb{Z}_p/p\mathbb{Z}_p|^{\delta_{j_p^*,k}}\right| < C_{j_p^*} p^{\delta_{j_p^*,k} - \frac{1}{2}}$$

for all $(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p) \in \Phi_{(\delta_{j_p^*,k}, \mu_{j_p^*,k})}$ as $p \to \infty$. This means, after translating by the function f_{s_p} there are finitely many possible approximate sizes for $\varphi(B_p, \bar{a}_p)$ when we let p and $\bar{a}_p$ vary.

Furthermore, by Theorem 1.2.24 there are $\mathcal{L}_{rings}$-formulas $\chi_{j_p^*,k}(\bar{v})$ for $k \in \{1, \ldots, t_{j_p^*}\}$ such that $\chi_{j_p^*,k}(\bar{e}_p)$ holds in $\mathbb{Z}_p/p\mathbb{Z}_p$ if and only if the pair $(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p) \in \Phi_{(\delta_{j_p^*,k}, \mu_{j_p^*,k})}$. That is, if $\left||\psi_{j_p^*}(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{e}_p)| - \mu_{j_p^*,k}|\mathbb{Z}_p/p\mathbb{Z}_p|^{\delta_{j_p^*,k}}\right| < C_{j_p^*} p^{\delta_{j_p^*,k} - \frac{1}{2}}$.

Put the $\mathcal{L}_{rings}$-formula $\rho_{\bar{z}, j_p^*}(\bar{v}, \bar{y})$ to say $f_{\bar{z}}(\varphi(B_p, \bar{y})) = \psi_{j_p^*}(\mathbb{Z}_p/p\mathbb{Z}_p, \bar{v})$.

Also put the $\mathcal{L}_{rings}$-formula $\tau_{j_p^*,k}(\bar{y}, \bar{z})$ to say

$$\exists \bar{e} \in \mathbb{Z}_p/p\mathbb{Z}_p \exists \bar{z} \in \mathbb{Z}_p/p^l\mathbb{Z}_p(\rho_{\bar{z}, j_p^*}(\bar{e}, \bar{y}) \wedge \chi_{j_p^*,k}(\bar{e})).$$

Hence $\tau_{j_p^*,k}(\bar{a}_p, s_p)$ holds if and only if $\big||\varphi(B_p,\bar{a}_p)| - \mu_{j_p^*,k}p^{\delta_{j_p^*,k}}\big| < C_{j_p^*}p^{\delta_{j_p^*,k}-\frac{1}{2}}$ as $p \to \infty$.

Moreover, since $j \in \{1,\dots,r_\varphi\}$ and $k \in \{1,\dots,t_j\}$ we get finitely many pairs $(\delta_{j,k},\mu_{j,k})$ that represent possible approximate sizes for $\varphi(B_p,\bar{a}_p)$ (with respect to $|\mathbb{Z}_p/p\mathbb{Z}_p|$). Hence we get finitely many formulas $\tau_{(j,k)}(\bar{y},s_p)$ that pick out the parameters $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$ such that $|\varphi(B_p,\bar{a}_p)|$ is approximated by the pair $(\delta_{(j,k)},\mu_{(j,k)})$.

□

We present now the following proposition.

Proposition 3.1.2 (Asymptotic fragment)**.** *Take $s,m,l \in \mathbb{N}$ such that $m > 0$ and $s < s+m < l$. Put $\mathcal{C}_l = \{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$*

Then for any $\mathcal{L}_{rings}$-formula $\varphi(x,\bar{y})$ the following hold.

1. *There exists a constant C, a finite set D of pairs $(\delta,\mu) \in (\{0,\dots,m\}\times\mathbb{R}^{>0}) \cup \{(0,0)\}$ and a partition $\{\Phi_{(\delta,\mu)} : (\delta,\mu) \in D\}$ of the set $\{(\mathbb{Z}_p/p^l\mathbb{Z}_p,\bar{a}_p,b_p) : \mathbb{Z}_p/p^l\mathbb{Z}_p \in \mathcal{C}_l, \bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p, b_p \in p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p\}$ such that for each pair (δ,μ)*

$$\big||\varphi(F(p^m,b_p),\bar{a}_p)| - \mu p^\delta\big| < Cp^{\delta-\frac{1}{2}}$$

   for all $(\mathbb{Z}_p/p^l\mathbb{Z}_p,\bar{a}_p,b_p) \in \Phi_{(\delta,\mu)}$ as $p \to \infty$. Here $F(p^m,b_p) = \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^m x = b_p\}$.

2. *Furthermore, for each $(\delta,\mu) \in D$ from above there is an $\mathcal{L}_{rings}$-formula $\chi_{(\delta,\mu)}(\bar{y},z)$ such that $\chi_{(\delta,\mu)}(\bar{a}_p,b_p)$ holds if and only if $(\mathbb{Z}_p/p^l\mathbb{Z}_p,\bar{a}_p,b_p) \in \Phi_{(\delta,\mu)}$.*

Proof. We prove this by induction on m.

First take $m = 1$. Let $s,l \in \mathbb{N}$ be such that $s < s+1 < l$. Let $\varphi(x,\bar{y})$ be an $\mathcal{L}_{rings}$-formula.

Put $\varphi(F(p^1, b_p), \bar{a}_p) := \varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p) \cap F(p^1, b_p)$ to denote the trace of the definable set $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)$ in $F(p^1, b_p)$.

We want to find a constant C and a finite set of pairs (δ, μ) such that for any given $p \in \mathbb{P}$, $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$ and $b_p \in p^{s+1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ the size of $\varphi(F(p^1, b_p), \bar{a}_p)$ is approximated by one of these pairs.

Now, $F(p^1, b_p) = \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^1 x = b_p\}$ is in definable bijection uniformly in p with $\mathbb{Z}_p/p\mathbb{Z}_p$. Indeed, for any $p \in \mathbb{P}$ the map $\theta : \mathbb{Z}_p \to F(p^1, b_p)$ that maps z to $p^{l-1}z + p^s\beta$ induces a definable bijection $f_{\beta_p}^{-1}$ from $\mathbb{Z}_p/p\mathbb{Z}_p$ to $F(p^1, b_p)$ that sends $z + (p\mathbb{Z}_p)$ to $\theta(z)$. Furthermore, even when f_{β_p} depends on $p^s\beta$, any β' such that $p^{s+1}\beta' = b_p$ induces a bijection from $F(p^1, b_p)$ to $\mathbb{Z}_p/p\mathbb{Z}_p$ in the same way. Hence when we fix $b_p \in p^{s+1}\mathbb{Z}_p/p^l\mathbb{Z}_p$, i.e. when $\exists\beta \in \mathbb{Z}_p/p^l\mathbb{Z}_p(p^{s+1}\beta = b_p)$ holds the choice of β doesn't affect the fact that f_{β_p} is a bijection.

The set $\varphi(F(p^1, b_p), \bar{a}_p)$ corresponds uniformly in p to a definable set in $\mathbb{Z}_p/p\mathbb{Z}_p$ using f_{β_p}, namely $f_{\beta_p}\big(\varphi(F(p^1, b_p), \bar{a}_p)\big)$.

By Lemma 3.1.1, taking the parameters to be $\{\beta_p\}$, we obtain a constant C and a finite set E of pairs $(\delta, \mu) \in (\{0, 1\} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ such that for all $b_p \in p^{s+1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ and $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$

$$\big||\varphi(F(p^1, b_p), \bar{a}_p)| - \mu p^\delta\big| < Cp^{\delta - \frac{1}{2}}$$

holds for some pair (δ, μ) as $p \to \infty$.

Furthermore for each (δ, μ), after quantifying out the parameters β_p, there exists $\tau_{(\delta,\mu)}(\bar{y}, z)$ such that $\tau_{(\delta,\mu)}(\bar{a}_p, b_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if $\big||\varphi(F(p^1, b_p), \bar{a}_p)| - \mu p^\delta\big| < Cp^{\delta - \frac{1}{2}}$ as $p^l \to \infty$.

This shows that the statement in Proposition 3.1.2 holds when $m = 1$. The finite set of pairs is E and the formula $\tau_{(\delta,\mu)}$ defines each element of the partition. Note that each $\delta \in \{0, 1\}$.

Now assume that the statement in Proposition 3.1.2 holds for some m. That is, for any $s, l \in \mathbb{N}$ such that $s < s+m < l$ for any $\mathcal{L}_{rings}$-formula $\varphi(x, \bar{y})$ there exists a constant C and a finite set D of pairs $(\delta, \mu) \in (\{0, \ldots, m\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$ and a partition $\{\Phi_{(\delta,\mu)} : (\delta, \mu) \in D\}$ of the set $\{(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p, b_p) : \mathbb{Z}_p/p^l\mathbb{Z}_p \in \mathcal{C}_l, \bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p, b_p \in p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p\}$ such that for each pair (δ, μ)

$$\left| |\varphi(F(p^m, b), \bar{a}_p)| - \mu p^\delta \right| < C p^{\delta - \frac{1}{2}}$$

for all $(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p, b_p) \in \Phi_{(\delta,\mu)}$ as $p \to \infty$. Recall $F(p^m, b_p) = \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^m x = b_p\}$.

Furthermore, for each pair (δ, μ) there is an $\mathcal{L}_{rings}$-formula $\chi_{(\delta,\mu)}(\bar{y}, z)$ such that $\chi_{(\delta,\mu)}(\bar{a}_p, b_p)$ holds if and only if $(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p, b_p) \in \Phi_{(\delta,\mu)}$.

We then want to show that the statement of Proposition 3.1.2 holds for $m+1$.

For this we let s and l be such that $s < s+(m+1) < l$.

Fix a formula $\varphi(x, \bar{y})$. For each $p \in \mathbb{P}$ consider parameters $\bar{a}_p$ in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ and let b_p be an element in $p^{s+(m+1)}\mathbb{Z}_p/p^l\mathbb{Z}_p$ (where $b_p = p^{s+(m+1)}\beta$ for some $\beta \in \mathbb{Z}_p/p^l\mathbb{Z}_p$).

Consider the set $F(p^{m+1}, b_p) = \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^{m+1}x = b_p\}$. We want to approximate the size of $\varphi(F(p^{m+1}, b_p), \bar{a}_p)$. For this consider the set $\widehat{S}_{b_p} = \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^1 x = b_p\}$. Note that we have $\widehat{S}_{b_p} \subseteq p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p$ and that $\widehat{S}_{b_p} = p^m \cdot F(p^{m+1}, b_p)$.

Take $c_p \in \widehat{S}_{b_p}$. Put $F'(p^m, c_p) := \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^m x = c_p\}$. Note that the collection $\{F'(p^m, c) : c \in \widehat{S}_{b_p}\}$ partitions the set $F(p^{m+1}, b_p)$.

Also $s < s+m < l$ (for $s+m < s+m+1$). Hence by inductive hypothesis if we put $R_{l,p} = \mathbb{Z}_p/p^l\mathbb{Z}_p$ and consider the same formula $\varphi(x, \bar{y})$ we find a constant C, a finite set E of pairs $(\delta_i, \mu_i) \in (\{0, \ldots, m\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$, with $i \in \{1, \ldots, r\}$ and a partition $\{\Phi'_{(\delta_i,\mu_i)} : (\delta_i, \mu_i) \in E\}$ of the set $\{(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p, c') : p \in \mathbb{P}, \bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p, c' \in$

$p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p\}$ such that for each $(\delta_i, \mu_i) \in E$

$$\left||\varphi(F'(p^m, c), \bar{a}_p)| - \mu_i p^{\delta_i}\right| < Cp^{\delta_i - \frac{1}{2}}$$

for all $(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p, c) \in \Phi'_{(\delta_i,\mu_i)}$ as $p \to \infty$. Note that this induces a partition of the set $\{(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p, c) : p \in \mathbb{P}, \bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p, c \in \widehat{S}_{b_p}\}$ since $\widehat{S}_{b_p} \subseteq p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p$.

Furthermore since we assume Proposition 3.1.2 holds for m, there are $\mathcal{L}_{rings}$-formulas $\chi_i(w, \bar{y})$ such that for $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$ and $c \in \widehat{S}_{b_p}$ we have that $\chi_i(c, \bar{a}_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if

$$\left||\varphi(F'(p^m, c), \bar{a}_p)| - \mu_i p^{\delta_i}\right| < Cp^{\delta_i - \frac{1}{2}}.$$

Recall $\varphi(F'(p^m, c), \bar{a}_p)$ denotes $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p) \cap F'(p^m, c)$.

Note that each formula $\chi_i(x, \bar{a}_p)$ defines a set in $\widehat{S}_{b_p} = p^m F(p^{m+1}, b_p)$.

Since $\widehat{S}_{b_p} = \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : p^1 x = b_p\}$, $(s+m) < (s+m)+1 < l$ and Proposition 3.1.2 holds when $m = 1$ we can now find an approximation for $|\chi_i(\widehat{S}_{b_p}, \bar{a}_p)|$.

Since each formula $\chi_i(x, \bar{a}_p)$ defines a set in $\widehat{S}_{b_p} = F(p^1, b_p)$ then for each $i \in \{1, \ldots, r\}$ the statement of Proposition 3.1.2 gives a constant C_i, and a finite set E_i of pairs $(\varepsilon_{i,j}, \nu_{i,j}) \in (\{0,1\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$ where $j \in \{1, \ldots, t_i\}$, such that

$$\left||\chi_i(\widehat{S}_{b_p}, \bar{a}_p)| - \nu_{i,j} p^{\varepsilon_{i,j}}\right| < C_i p^{\varepsilon_{i,j} - \frac{1}{2}}.$$

Recall that $\chi_i(\widehat{S}_{b_p}, \bar{a}_p) = \chi_i(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p) \cap F(p^1, b_p)$.

Furthermore there are $\mathcal{L}_{rings}$-formulas $\eta_{i,j}(\bar{y}, x)$ for $j \in \{1, \ldots, t_i\}$ such that $\eta_{i,j}(\bar{a}_p, b_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if $\left||\chi_i(F(p^1, b_p), \bar{a}_p)| - \nu_{i,j} p^{\varepsilon_{i,j}}\right| < C_i p^{\varepsilon_{i,j} - \frac{1}{2}}$.

We follow ideas from [30] (also [17]). Note that for each $b_p, \bar{a}_p$, there is a unique function $h = h_{(\bar{a}_p, b_p)} : \{1, \ldots, r\} \to \omega$ such that for all $i \in \{1, \ldots, r\}$ we have $h(i) \in \{1, \ldots, t_i\}$ and $\eta_{i,h(i)}(\bar{a}_p, b_p)$ holds. Since $h(i) \in \{1, ..., t_i\}$ we get that the set of such possible functions is finite.

Consider $\bar{a}_p$, b_p as above and a compatible h.

Define $T_i(\bar{a}_p, b_p) := \{z \in \mathbb{Z}_p/p^l\mathbb{Z}_p : \varphi(z, \bar{a}_p) \wedge z \in F(p^{m+1}, b_p) \wedge \chi_i(p^m z, \bar{a}_p)\}$. Then, the collection $\{T_i(\bar{a}_p, b_p) : 1 \leq i \leq r\}$ forms a partition of $\varphi(F(p^{m+1}, b_p), \bar{a}_p)$, i.e. we have $\varphi(F(p^{m+1}, b_p), \bar{a}_p) = \bigsqcup_i T_i(\bar{a}_p, b)$.

Furthermore, since $\left||\chi_i(F(p^1, b_p), \bar{a}_p)| - \nu_{i,h(i)} p^{\varepsilon_{i,h(i)}}\right| < C_i p^{(\varepsilon_{i,h(i)} - \frac{1}{2})}$ if we put $\mathcal{G} := \{q \in F(p^1, b_p) : \chi_i(q, \bar{a}_p)\}$ we have that $|\mathcal{G}|$ is approximately $\nu_{i,h(i)} p^{\varepsilon_{i,h(i)}}$. Also, since $\chi_i(q, \bar{a}_p)$ holds for all $q \in \mathcal{G}$ putting $\mathcal{H} := \{z \in \mathbb{Z}_p/p^l\mathbb{Z}_p : \varphi(z, \bar{a}_p) \wedge p^m z = q\}$ we get $|\mathcal{H}|$ is approximately $\mu_i p^{\delta_i}$ since $\left||\varphi(F(p^m, q), \bar{a}_p)| - \mu_i p^{\delta_i}\right| < C p^{\delta_i - \frac{1}{2}}$. Hence the following holds for $|T_i(\bar{a}_p, b_p)|$.

First

$$|T_i(\bar{a}_p, b_p)| < \left(\mu_i p^{\delta_i} + C p^{\delta_i - \frac{1}{2}}\right) \cdot \left(\nu_{i,h(i)} p^{\varepsilon_{i,h(i)}} + C_i p^{(\varepsilon_{i,h(i)} - \frac{1}{2})}\right).$$

Second

$$\left(\mu_i p^{\delta_i} - C p^{\delta_i - \frac{1}{2}}\right) \cdot \left(\nu_{i,h(i)} p^{\varepsilon_{i,h(i)}} - C_i p^{\varepsilon_{i,h(i)} - \frac{1}{2}}\right) < |T_i(\bar{a}_p, b_p)|.$$

Hence

$$\left||T_i(\bar{a}_p, b)| - \mu_i \nu_{i,h(i)} p^{\delta_i + \varepsilon_{i,h(i)}}\right| < (C\nu_{i,h(i)} + \mu_i C_i) p^{\delta_i + \varepsilon_{i,h(i)} - \frac{1}{2}} + C C_i p^{\delta_i + \varepsilon_{i,h(i)} - 1}.$$

This means that for some large enough C'_h ,

$$\left||T_i(\bar{a}_p, b)| - \mu_i \nu_{i,h(i)} p^{\delta_i + \varepsilon_{i,h(i)}}\right| < C'_h p^{\delta_i + \varepsilon_{i,h(i)} - \frac{1}{2}}$$

as $p \to \infty$.

Put $D_h := \max\{\delta_i + \varepsilon_{i,h(i)} : i \in \{1, \ldots, r\}\}$. Note $D_h \in \{0, \ldots, m+1\}$. Consider the set $A := \{i \in \{1, \ldots, r\} : \delta_i + \varepsilon_{i,h(i)} = D_h\}$ and make $M_h := \sum_{i \in A} \mu_i \nu_{i,h(i)}$.

Then,

$$\left||\varphi(F(p^{m+1}, b_p), \bar{a}_p)| - M_h p^{D_h}\right| < (\sum_{i \in A} C'_i) p^{D_h - \frac{1}{2}} + K$$

where K accounts for the terms where p has exponent less than $D_h - \frac{1}{2}$. Hence for a big enough C''_h we have

$$\left||\varphi(F(p^{m+1}, b_p), \bar{a}_p)| - M_h p^{D_h}\right| < (C''_h) p^{D_h - \frac{1}{2}}$$

as $p \to \infty$.

Since the pair (D_h, M_h) depends only on h there is a formula $\sigma_h(\bar{y}, w)$ given by $\bigwedge_{i \in A} \eta_{i,h(i)}(\bar{y}, w)$ such that $\sigma_h(\bar{a}_p, b_p)$ holds if and only if

$$\left||\varphi(F(p^{m+1}, b_p), \bar{a}_p)| - M_h p^{D_h}\right| < (C''_h) p^{D_h - \frac{1}{2}}$$

as $p \to \infty$.

As we note before, the set of possible functions h is finite. Hence the set of pairs (D_h, M_h) is finite. Therefore we find a constant $C'' = \max_h \{C''_h\}$ and finitely many formulas $\sigma_h(\bar{y}, w)$ such that $\sigma_h(\bar{a}_p, b_p)$ holds if and only if

$$\left||\varphi(F(p^{m+1}, b_p), \bar{a}_p)| - M_h p^{D_h}\right| < (C'') p^{D_h - \frac{1}{2}}.$$

Hence, the statement from Proposition 3.1.2 holds for $m + 1$.

This completes the proof by induction. Therefore Proposition 3.1.2 holds for all $m \in \mathbb{N}^{>0}$.

□

Remark 3.1.3. *Further to Proposition 3.1.2, we call the collection* $\{F(p^m, b_p) : p \in \mathbb{P}, b_p \in p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p\}$ *an* ***asymptotic fragment of*** $\mathcal{C}_l$ ***over*** $\{p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$.

3.2 The class $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$

As we have done previously, we keep working in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ in the context of the Henselian valued fields $\mathbb{Q}_p$.

We will now use Proposition 3.1.2 to show the following result.

Theorem 3.2.1. *For any fixed* $l \in \mathbb{N}^{>0}$ *the class* $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$ *is an* l*-dimensional asymptotic class.*

Proof. We will use that for all $m \in \mathbb{N}$, and all $s, l \in \mathbb{N}$ such that $s < s+m < l$ the collection $\{F(p^m, v_p) : p \in \mathbb{P}, v_p \in p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p\}$ is an asymptotic fragment of $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$ over $\{p^{s+m}\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$.

By Lemma 1.2.23 it is enough to consider formulas $\varphi(x, \bar{y})$ in just one free variable x.

Given an $\mathcal{L}_{rings}$-formula $\varphi(x, \bar{y})$, $p \in \mathbb{P}$ and $\bar{a}_p$ some parameters in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ we want to study the asymptotic behaviour of $|\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)|$ across the class $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$.

Put $R_{l,p} = \mathbb{Z}_p/p^l\mathbb{Z}_p$, and $\mathcal{C}_l = \{R_{l,p} : p \in \mathbb{P}\}$. Note that if $s = 0$ and $m = l-1$, we get that $0 < l-1 < l$ satisfies the hypothesis of Proposition 3.1.2. Then the collection $\{F(p^{l-1}, u_p) : p \in \mathbb{P}, u_p \in p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p\}$ is an asymptotic fragment of $\mathcal{C}_l$ over $\mathcal{D}_{0,l-1} = \{p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$. Hence there exists a constant C and finitely many pairs (δ_i, μ_i), where $\delta_i \in \{0, \ldots, l-1\}$ for every $i \in \{1, \ldots, r\}$, such that for the fixed $p \in \mathbb{P}$, $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$ and any $b_p \in p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ we have that

$$\big||\varphi(F(p^{l-1}, b_p), \bar{a}_p)| - \mu_i p^{\delta_i}\big| < Cp^{\delta_i - \frac{1}{2}}$$

for some $i \in \{1, \ldots, r\}$. Furthermore, there are $\mathcal{L}_{rings}$-formulas $\zeta_i(\bar{y}, w)$ such that $\zeta_i(\bar{a}_p, b_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if

$$\big||\varphi(F(p^{l-1}, b_p), \bar{a}_p)| - \mu_i p^{\delta_i}\big| < Cp^{\delta_i - \frac{1}{2}}.$$

Now, each $\zeta_i(\bar{a}_p, w)$ defines a set in $p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p$. Note that $p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ is in bijective correspondence with $\mathbb{Z}_p/p\mathbb{Z}_p$ via the map $f^{-1} : \mathbb{Z}_p/p\mathbb{Z}_p \to p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ which maps $z + (p\mathbb{Z}_p) \mapsto p^{l-1}z + (p^l\mathbb{Z}_p)$. The map f^{-1} is induced by the map $\theta : \mathbb{Z}_p \to p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ mapping z to $p^{l-1}z + (p^l\mathbb{Z}_p)$.

Since the map f is defined uniformly on p by Lemma 3.1.1 we get C_i and finitely many pairs

$$(\varepsilon_{(i,1)}, \nu_{(i,1)}), \ldots, (\varepsilon_{(i,s_i)}, \nu_{(i,s_i)}),$$

where $\varepsilon_{(i,j)} \in \{0,1\}$ for all $j \in \{1,\ldots,s_i\}$ such that for all $p \in \mathbb{P}$ and $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$

$$\big||\zeta_i(\bar{a}_p, p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p)| - \nu_{(i,j)}p^{\varepsilon_{(i,j)}}\big| < C_i p^{\varepsilon_{(i,j)}-\frac{1}{2}}$$

for some $(\varepsilon_{(i,j)}, \nu_{(i,j)})$ as $p \to \infty$

Furthermore for each pair $(\varepsilon_{(i,j)}, \nu_{(i,j)})$ there exists a formula $\chi_{(i,j)}(\bar{y})$ such that $\chi_{(i,j)}(\bar{a}_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if

$$\big||\zeta_i(\bar{a}_p, p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p)| - \nu_{(i,j)}p^{\varepsilon_{(i,j)}}\big| < C_i p^{\varepsilon_{(i,j)}-\frac{1}{2}}.$$

Note that for each $\bar{a}_p$ there is a unique function $g : \{1,\ldots,r\} \to \omega$ such that $g(i) \in \{1,\ldots,s_i\}$ and $\chi_{(i,g(i))}(\bar{a}_p)$ holds. Since $g(i) \in \{1,\ldots,s_i\}$ the set of such functions is finite.

Fix $\bar{a}_p$ and consider a compatible g.

Define the set $T_i(\bar{a}_p) := \{x \in \mathbb{Z}_p/p^l\mathbb{Z}_p : \varphi(x,\bar{a}_p) \wedge \zeta_i(\bar{a}_p, p^{l-1}x) \wedge \chi_{(i,g(i))}(\bar{a}_p)\}$. Note that $\{T_i(\bar{a}_p) : i \in \{1,\ldots,r\}\}$ forms a partition of $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p \setminus p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)$, i.e. $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p \setminus p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p) = \bigsqcup_i T_i(\bar{a}_p)$. Note also that for each $i \in \{1,\ldots,r\}$ we have

$$\big||T_i(\bar{a}_p)| - \mu_i\nu_{(i,g(i))}p^{\delta_i+\varepsilon_{(i,g(i))}}\big| < C_i' p^{\delta_i+\varepsilon_{(i,g(i))}-\frac{1}{2}}$$

for some big enough constant C_i' as $p \to \infty$.

Put $\widehat{\delta}_g := \max\{\delta_i + \varepsilon_{(i,g(i))} : i \in \{1,\ldots,r\}\}$. Observe that $\widehat{\delta}_g \in \{0,\ldots,l\}$. Consider the set $A := \{i \in \{1,\ldots,r\} : \delta_i + \varepsilon_{(i,g(i))} = \widehat{\delta}_g\}$ and put $\widehat{\mu}_g := \sum_{i\in A}\mu_i\nu_{(i,g(i))}$. Then for some big enough constant $\widehat{C}$ we have

$$\big||\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p \setminus p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \widehat{\mu}_g p^{\widehat{\delta}_g}\big| < \widehat{C}p^{\widehat{\delta}_g-\frac{1}{2}}$$

as $p^l \to \infty$.

Furthermore there is a formula $\eta_g(\bar{y})$ given by $\big(\bigwedge_{i\in A} \chi_{(i,g(i))}(\bar{y})\big)$ such that $\eta_g(\bar{a}_p)$ holds if and only if this last inequality is satisfied.

To obtain a complete approximation of $\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)$ it is only left to approximate the size of the set $\varphi(p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)$.

Again, since $p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p$ is in bijective correspondence with $\mathbb{Z}_p/p\mathbb{Z}_p$ via f defined above and the correspondence is uniform in p by Lemma 3.1.1 we get a constant C' and a finite set E' of pairs (γ_k, λ_k) with $k \in \{1, \dots, t\}$ and $\gamma_k \in \{0, 1\}$ such that for all $p \in \mathbb{P}$ and $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$

$$\big||\varphi(p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \lambda_k p^{\gamma_k}\big| < C' p^{\gamma_k - \frac{1}{2}}$$

for some pair $(\gamma_k, \lambda_k) \in E'$.

Furthermore for each pair (γ_k, λ_k) we get an $\mathcal{L}_{rings}$-formula $\tau_k(\bar{y})$ such that $\tau_k(\bar{a}_p)$ holds in $\mathbb{Z}_p/p^l\mathbb{Z}_p$ if and only if

$$\big||\varphi(p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \lambda_k p^{\gamma_k}\big| < C' p^{\gamma_k - \frac{1}{2}}.$$

Finally, we write

$$|\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| = |\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p \setminus p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| + |\varphi(p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)|.$$

The formula $\Xi_{g,k}(\bar{a}_p)$ given by $\eta_g(\bar{a}_p) \wedge \tau_k(\bar{a}_p)$ holds if and only if both

$$\big||\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p \setminus p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \widehat{\mu}_g p^{\widehat{\delta}_g}\big| < \widehat{C} p^{\widehat{\delta}_g - \frac{1}{2}}, \text{ and}$$
$$\big||\varphi(p^{l-1}\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \lambda_k p^{\gamma_k}\big| < C' p^{\gamma_k - \frac{1}{2}}$$

hold.

There are finitely many functions g and elements $k \in \{1, \dots, t\}$. Hence there are finitely many possible pairs (g,k) and $\widehat{\mu}_g, \widehat{\delta}_g$ depends only on g. Also, there is a constant $\widehat{\mathbf{C}}$ that works for all functions g simultaneously. Then for each (g,k) put:

i) $\delta_{g,k} = \max\{\widehat{\delta}_g, \gamma_k\}$.

ii) $\mu_{g,k} = \begin{cases} \widehat{\mu}_g, & \text{if } \delta_{g,k} = \widehat{\delta}_g \\ \lambda_k, & \text{if } \delta_{g,k} = \gamma_k \\ \widehat{\mu}_g + \lambda_k, & \text{if } \widehat{\delta}_g = \gamma_k. \end{cases}$

Then for a big enough constant $\mathbf{C}$ we have that

$$\left||\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \mu_{g,k}p^{\delta_{g,k}}\right| < \mathbf{C}p^{\delta_{g,k}-\frac{1}{2}}$$

for some (g,k) as $p \to \infty$.

In conclusion we have obtained that given $\varphi(x,\bar{y})$ and $\bar{a}_p \in \mathbb{Z}_p/p^l\mathbb{Z}_p$ there exists $\mathbf{C}$ a constant and finitely many pairs $(\delta_{g,k}, \mu_{g,k})$ with $\delta_{g,k} \in \{0,\ldots,l\}$ such that

$$\left||\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \mu_{g,k}p^{\delta_{g,k}}\right| < \mathbf{C}p^{\delta_{g,k}-\frac{1}{2}}$$

as $p^l \to \infty$. Furthermore we find formulas $\Xi_{g,k}(\bar{y})$ such that $\Xi_{g,k}(\bar{a}_p)$ holds if and only if

$$\left||\varphi(\mathbb{Z}_p/p^l\mathbb{Z}_p, \bar{a}_p)| - \mu_{g,k}p^{\delta_{g,k}}\right| < \mathbf{C}p^{\delta_{g,k}-\frac{1}{2}}$$

as $p \to \infty$.

We have shown that for a fixed $l \in \mathbb{N}$ the class of rings $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$ satisfies the definition of an asymptotic class for formulas with a single variable x. By Lemma 1.2.23 we conclude that $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$ is an asymptotic class. Furthermore in the argument above we obtained pairs $(\delta_{g,k}, \mu_{g,k}) \in (\{0,\ldots,l\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$ since the size of each structure in the class $\mathcal{C}_l$ as p varies is p^l we obtain that $\{\mathbb{Z}_p/p^l\mathbb{Z}_p : p \in \mathbb{P}\}$ is an l-dimensional asymptotic class.

□

We finish this section with the following remark presenting some ideas along the lines of Remark 2.3.9.

Remark 3.2.2. *By the Ax-Kochen-Eršov theorem we know that, with l fixed, $\prod_{p\in\mathbb{P}} \mathbb{Z}/p^l\mathbb{Z}/\mathcal{U}$ is elementarily equivalent to $\prod_{p\in\mathbb{P}} \mathbb{F}_p[[t]]/(t^l)/\mathcal{U}$ in the language of rings. Also, one can argue that for any $p \in \mathbb{P}$ the ring $\mathbb{F}_p[[t]]/(t^l)$ and the finite field $\mathbb{F}_p$ are bi-interpretable, uniformly in p. This together with results on bi-interpretability between classes of finite structures, see for example Lemma 3.7 from [17], gives us that the class of rings $\mathcal{F}' := \{\mathbb{F}_p[[t]]/(t^l) : p \in \mathbb{P}\}$ is an N-dimensional asymptotic class, in fact with $N = l$. However, we take a moment to note that the proof that the class $\mathcal{C}_l = \{\mathbb{Z}/p^l\mathbb{Z}\}$ is an l-dimensional asymptotic class required quite a different approach to that of the class $\mathcal{F}'$.*

3.3 Multidimensional asymptotic classes of finite residue rings

In Section 2.3 we looked at ultraproducts $\prod_{\mathcal{U}} \mathbb{Z}/n\mathbb{Z}$ of the class of finite residue rings $\{\mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\}$ where there exists $U' \in \mathcal{U}$ such that there is a bound on the number and the exponent of prime divisors of every $n \in U'$. These ultraproducts are isomorphic to a finite direct product of ultraproducts of members of classes of the form discussed in Section 3.2 above.

We now use results from Daniel Wolf's Ph.D. thesis [51] to study the class $\mathcal{C}_{d,l_1,\ldots,l_d} := \{\mathbb{Z}/n\mathbb{Z} : n = p_1^{l_1}\cdots p_d^{l_d}, p_1 < \ldots < p_d \in \mathbb{P}\}$, where $d \in \mathbb{N}^{>0}$ and $l_1, \ldots, l_d \in \mathbb{N}$, in the context of multidimensional asymptotic classes.

For this we present the following definition from [1].

Definition 3.3.1 ([1]). *Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures and let R be a class of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$. We say $\mathcal{C}$ is an R-multidimensional asymptotic class in $\mathcal{L}$, R-*

mac for short, if for every formula $\varphi(\bar{x}, \bar{y})$, where the length of $\bar{x}$ is n and the length of $\bar{y}$ is m, the following holds.

1. *There exists a finite set F of functions $h_i \in R$ and a partition $\{\Phi_{h_i} : 1 \leq i \leq |F|\}$ of the set $\{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in \mathcal{M}\}$ such that for each $i \leq R$*

$$\big||\varphi(\mathcal{M}^n, \bar{a})| - h_i(\mathcal{M})\big| = o\left(h_i(\mathcal{M})\right)$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_{h_i}$ as $|\mathcal{M}| \to \infty$.

2. *For every $h_i \in R$ there exists $\mathcal{L}$-formulas $\chi_{h_i}(\bar{y})$ such that $\mathcal{M} \models \chi_{h_i}(\bar{a})$ if and only if $(\mathcal{M}, \bar{a}) \in \Phi_{h_i}$.*

Daniel Wolf has shown in his Ph.D. thesis, Lemma 2.3.5 [51] the following result.

Theorem 3.3.2 (Daniel Wolf, [51])**.** *Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be R_i-multidimensional asymptotic classes in disjoint languages $\mathcal{L}_1, \ldots \mathcal{L}_n$. Then the disjoint union $\mathcal{C}_1 \sqcup \ldots \sqcup \mathcal{C}_n = \{\mathcal{M}_1 \sqcup \cdots \sqcup \mathcal{M}_n : \mathcal{M}_i \in \mathcal{C}_i\}$ of the R_i-MACs itself forms an R-MAC (for an appropriate R) in the disjoint union of the languages $\mathcal{L}_1 \sqcup \ldots \sqcup \mathcal{L}_n$.*

For fixed $d \in \mathbb{N}$ and $l_1, \ldots, l_d \in \mathbb{N}$ put $\mathcal{C}_{d,l_1,\ldots,l_d} := \{\mathbb{Z}/n\mathbb{Z} : n = p_1^{l_1} \cdots p_d^{l_d}, p_1 < \ldots < p_d \in \mathbb{P}\}$. Then any ring in the class $\mathcal{C}_{d,l_1,\ldots,l_d}$ can be seen as a finite direct product of fixed length d. Moreover, the factors are taken from finitely many fixed classes $\mathcal{C}_1 = \{\mathbb{Z}/p^{l_1}\mathbb{Z} : p \in \mathbb{P}\}, \ldots, \mathcal{C}_t = \{\mathbb{Z}/p^{l_t}\mathbb{Z} : p \in \mathbb{P}\}$. By Proposition 3.2.1 above each of these $\mathcal{C}_i$ is an l_i-dimensional asymptotic class.

Note that each N-dimensional asymptotic class is an R-multidimensional asymptotic class for an appropriate R consisting of functions of the form $\mathcal{M} \mapsto \mu|\mathcal{M}|^\delta$.

By Theorem 3.3.2 we get the following.

Corollary 3.3.3. *Put $l_1, \ldots, l_d \in \mathbb{N}^{>0}$, and for each $i \in \{1, \ldots, d\}$ put $\mathcal{C}_i = \{\mathbb{Z}/p^{l_i} : p \in \mathbb{P}\}$. Then the disjoint union $\mathcal{C}_1 \sqcup \ldots \sqcup \mathcal{C}_d$ is an R-multidimensional asymptotic class in the disjoint union of d many copies of the language $\mathcal{L}_{rings}$.*

Proof. This follows from Theorem 3.2.1 and the result of Daniel Wolf in Theorem 3.3.2. □

Proposition 3.3.4. *Put $d \in \mathbb{N}^{>0}$, and $l_1, \ldots, l_d \in \mathbb{N}$. Let $\mathcal{C}_{d,l_1,\ldots,l_d} := \{\mathbb{Z}/n\mathbb{Z} : n = p_1^{l_1} \cdots p_d^{l_d}, p_1 < \ldots < p_d \in \mathbb{P}\}$. Then $\mathcal{C}_{d,l_1,\ldots,l_d}$ is an R-multidimensional asymptotic class in $\mathcal{L}_{rings} + \{P_i\}_{1 \leq i \leq b}$. Here each P_i is a unary predicate that interprets the i-th factor of each element in $\mathcal{C}_{d,l_1,\ldots,l_d}$.*

Proof. By expanding the language by the unary predicates $\{P_i\}$ the class $\mathcal{C}_{d,l_1,\ldots,l_d}$ and the disjoint union from the corollary directly above are bi-interpretable. It follows from Proposition 2.4.6 in [51] that $\mathcal{C}_{d,l_1,\ldots,l_d}$ is also an R-multidimensional asymptotic class, for some R. □

Question 3.3.5. *What can be said about the class $\{\mathbb{Z}/n\mathbb{Z} : n \in U'\}$, when there is a bound b on the number and the value of the exponents of prime divisors of elements in U'? Is this also an R-multidimensional asymptotic class? Can we get the definability clause to hold? Note that we wouldn't include the unary predicates and we would consider bounded direct sums of rings of the form $\mathbb{Z}/p^{e_i}\mathbb{Z}$ where $e_i < b$ instead of the rings discussed earlier where the length of the sums and the exponents appearing for the prime power divisors of n are fixed.*

Question 3.3.6. *How can we study more examples of definable sets with an asymptotic fragment behaviour. The naive approach presented in 3.3.7 below doesn't quite seems to work but it may be improved.*

Definition 3.3.7. *Let $\mathcal{L}$ be a first order language, $\mathcal{C}$ a class of finite $\mathcal{L}$-structures and $\mathcal{A} = \{A_{\mathcal{M}} : \mathcal{M} \in \mathcal{C}\}$ where $A_{\mathcal{M}} \subseteq \mathcal{M}$ is an ordered set of fixed size n^* for all $\mathcal{M} \in \mathcal{C}$.*

Let $\Delta(x, \bar{z}_1)$ be an $\mathcal{L}$-formula where $|\bar{z}_1| = n^*$. For each $\mathcal{M} \in \mathcal{C}$ put $D(\mathcal{M}, A_{\mathcal{M}}) := \Delta(\mathcal{M}, A_M)$. Write $\mathcal{D}_{\mathcal{A}} = \{D(\mathcal{M}, A_{\mathcal{M}}) : \mathcal{M} \in \mathcal{C}\}$.

Let $\Upsilon(x, \bar{z}_1, \bar{z}_2)$ be an $\mathcal{L}$-formula. For each $\mathcal{M} \in \mathcal{C}$ and each $\bar{c} \in D(\mathcal{M}, A_{\mathcal{M}})$ put $F(\mathcal{M}, A_{\mathcal{M}}, \bar{c}) = \Upsilon(\mathcal{M}, A_{\mathcal{M}}, \bar{c})$. Write $\mathcal{F}_{\mathcal{A}} = \{F(\mathcal{M}, A_{\mathcal{M}}, \bar{c}) : \mathcal{M} \in \mathcal{C}, \bar{c} \in D(\mathcal{M}, A_{\mathcal{M}})\}$.

*We say $\mathcal{F}_{\mathcal{A}}$ is an N***-dimensional asymptotic fragment of** $\mathcal{C}$ **over** $\mathcal{D}_{\mathcal{A}}$ *if for any formula $\varphi(x, \bar{y})$ the following hold.*

1. *There is a finite set of pairs $\mathbf{D} \subseteq (\{0, \dots, N\} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ and a partition $\{\Phi_{(\delta,\mu)} : (\delta, \mu) \in \mathbf{D}\}$ of the set $\{(\mathcal{M}, \bar{a}, \bar{c}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in \mathcal{M}^{|\bar{y}|}, \bar{c} \in (D(\mathcal{M}, A_{\mathcal{M}}))^{|\bar{z}_2|}\}$ such that for each $(\delta, \mu) \in D$*

$$\left||\varphi(\mathcal{M}, \bar{a}) \cap F(\mathcal{M}, A_{\mathcal{M}}, \bar{c})| - \mu |F(\mathcal{M}, A_{\mathcal{M}}, \bar{c})|^{\frac{\delta}{N}}\right| = o\left(|F(\mathcal{M}, A_{\mathcal{M}}, \bar{c})|^{\frac{\delta}{N}}\right)$$

   for all $(\mathcal{M}, \bar{a}, \bar{c}) \in \Phi_{(\delta,\mu)}$ as $|\mathcal{M}| \to \infty$.

2. *Furthermore, for each $(\delta, \mu) \in \mathbf{D}$ there is an $\mathcal{L}$-formula $\chi_{(\delta,\mu)}(\bar{y}, \bar{z}_2)$ such that $\chi_{(\delta,\mu)}(\bar{a}, \bar{c})$ holds in $\mathcal{M}$ if and only if $(\mathcal{M}, \bar{a}, \bar{c}) \in \Phi_{(\delta,\mu)}$, i.e. if and only if*

$$\left||\varphi(\mathcal{M}, \bar{a}) \cap F(\mathcal{M}, A_{\mathcal{M}}, \bar{c})| - \mu |F(\mathcal{M}, A_{\mathcal{M}}, \bar{c})|^{\frac{\delta}{N}}\right| = o\left(|F(\mathcal{M}, A_{\mathcal{M}}, \bar{c})|^{\frac{\delta}{N}}\right).$$

Remark 3.3.8. *Note that if $\mathcal{C}$ is an N-dimensional asymptotic class, then making $\Upsilon(x, \bar{z}_1, \bar{z}_2) := (x = x)$ we get that $\mathcal{C}$ is an asymptotic fragment of itself over any $\mathcal{D}_{\mathcal{A}}$.*

Question 3.3.9. *Could we show that if a collection of structures is uniformly coordinatised (or analysed) by an asymptotic class as in this chapter, then the class is itself an asymptotic class?*

Chapter 4

Further results on pseudofinite rings

In this short chapter we will address general results on the model theory of pseudofinite rings and some results on how stability conditions imposed on a class of finite rings can ensure some algebraic behaviour on ultraproducts of these classes.

4.1 Questions

We start by describing some of the questions we answer in this chapter.

As is done in many expositions of ring theory, we start by looking at simple and semisimple rings. It is in this train of thought that we ask the following. Can we show that, up to elementary equivalence, the following holds?

- Any simple pseudofinite ring is elementarily equivalent to one of the form $\prod M_{n_i}(\mathbb{F}_{q_i})/\mathcal{U}$. See 4.2.2 below.
- Any semisimple, perhaps plus some generalised stability condition, pseudofinite ring has the form

$$\prod_i M_{n_1}(\mathbb{F}_{q_1}) \oplus \ldots \oplus M_{n_{k_i}}(\mathbb{F}_{q_{k_i}})/\mathcal{U}$$

up to elementary equivalence. See 4.2.1 below.

Furthermore, can we show what follows?

- The theory of a ring of the form $\prod_i M_{n_1}(\mathbb{F}_{q_1}) \oplus \ldots \oplus M_{n_{k_i}}(\mathbb{F}_{q_{k_i}}) / \mathcal{U}$ is supersimple if and only if $n_1, \ldots, n_k$ and k are bounded in some $U \in \mathcal{U}$; and that if $n_1, \ldots, n_k$ or k, are unbounded, the theory has TP_2 . See 4.3.7.
- An ultraproduct of increasing powers of $M_n(q)$ is TP_2 . Is its model theory tame in any sense? See 4.3.7.

To begin we note the following remarks.

Being a division ring is first order expressible. Further, Wedderburn showed that finite division rings are fields, a result often referred to as “Wedderburn’s little Theorem”, see Theorem 13.1 from [27]. Hence we have the following.

Remark 4.1.1. *Pseudofinite division rings are pseudofinite fields.*

Further, since being an integral domain is defined by a first order sentence and since a straightforward argument shows that finite integral domains are fields, we get the next remark.

Remark 4.1.2. *Pseudofinite integral domains are pseudofinite fields.*

4.2 Simple and semisimple pseudofinite rings

We have the next remark

Remark 4.2.1. *A semisimple pseudofinite ring R is elementarily equivalent to an ultraproduct of semisimple finite rings.*

Proof. Let R be a semisimple ring such that $R \equiv \prod_i R_i/\mathcal{U}$, with the R_i all finite rings. Then by 1.3.21, R is J-semisimple and (left)-Artinian. Since R is J-semisimple then $J(R) = 0$. By Remark 1.3.14 $J(R)$ is first order definable. By Łoś's theorem and the elementary equivalence between R and $\prod_{\substack{U \\ i}} R_i$ we get $J(R) = J(\prod_{\mathcal{U}} R_i) = 0$. Therefore, $J(R_i) = 0$ for $\mathcal{U}$-many of the R_i. Since each R_i is finite it is automatically Artinian, and so semisimple since it is J-semisimple. See Proposition 1.3.21. □

Further we note the following.

Proposition 4.2.2. *If R is a simple pseudofinite ring then $R \cong M_n(\mathcal{F})$ for a pseudofinite field $\mathcal{F}$.*

Proof. Let R be a simple pseudofinite ring. For $a, b \in R$ define $a < b$ if and only if $Ra \subseteq Rb$, where Ra is the principal left ideal generated by a. Then '$<$' is a definable partial order. Furthermore by pseudofiniteness there is a minimal non-zero element a_0 (every finite partial order has a minimal element). Then Ra_0 is a minimal left ideal of R by definition of $<$. By 1.3.20 the simple ring R is isomorphic to $M_n(D)$ for some $n \in \mathbb{N}$ and some division ring D. Further D is definable in $M_n(D)$. Consider E_1 an element with entries 1 in the $(1, 1)$ position and zeros elsewhere. Then the set $M_n(D)E_1 \cap E_1 M_n(D) = DE_1$ (the D multiples of E_1) with the usual matrix addition and multiplication is isomorphic to D. Since R is pseudofinite we have that D is a pseudofinite division ring. By Remark 4.1.1 D is a pseudofinite field. □

Corollary 4.2.3. *Simple pseudofinite rings have supersimple theory.*

Proof. This follows since $M_n(\mathcal{F})$ is interpretable in $\mathcal{F}$, and pseudofinite fields have supersimple theory. □

4.3 Classes of finite rings with generalised stability properties

We will now focus our attention on the following results which link pseudofinite rings and pseudofinite groups. Recall that in [26] Theorem 2.1 Krupinski proved Theorem 1.3.22 above.

Note that Theorem 1.3.22 yields at once the following.

Remark 4.3.1. *If every definable group in any ultraproduct of a class $\mathcal{C}$ is soluble-by-finite, then every ring with 1, or of finite characteristic, definable in any ultraproduct of the class $\mathcal{C}$ is nilpotent-by-finite.*

Proof. Let R be an ultraproduct of elements in $\mathcal{C}$. Assume every definable group in R is soluble-by-finite. Hence by 1.3.22 part (i) every definable ring in R is nilpotent-by-finite. □

The same proof of 1.3.22 from [26] goes through for this case. A detailed account of part of the proof from [26] is presented in Proposition 4.3.12. In particular we have the following.

Corollary 4.3.2. *Suppose $\mathcal{C} = \{R_i : i \in I\}$ is a class of rings with 1 and the formula γ defines a group $\gamma(R_i)$. Then, if every group of the form $\prod \gamma(R_i)/\mathcal{U}$ is soluble-by-finite for every such formula γ, we conclude that every ultraproduct of the form $\prod R_i/\mathcal{W}$ is nilpotent-by-finite.*

Proof. This follows from the above Remark 4.3.1. □

Recall now what we mentioned in Proposition 1.3.11 in Subsection 1.3.2. From [32] Corollary 1.4 we have that whenever a pseudofinite group is NIP and rosy then the pseudofinite group is soluble-by-finite. This yields the following result.

Proposition 4.3.3. *Let $\mathcal{C}$ be a NIP, rosy class of finite rings. Let $R = \prod_{\mathcal{U}} R_i$ be an ultraproduct of elements in the class $\mathcal{C}$. Then R is nilpotent-by-finite.*

Proof. Let $\mathcal{C}$ be a NIP, rosy class of finite rings. Let $R = \prod_{\mathcal{U}} R_i$ be an ultraproduct of elements in the class $\mathcal{C}$. Any group definable in R is pseudofinite NIP rosy, so soluble-by-finite by 1.3.10. From 1.3.22 we have that R is nilpotent-by-finite.

□

Corollary 4.3.4. *Any NIP rosy pseudofinite ring R with 1, or of finite characteristic, is nilpotent-by-finite.*

Proof. Again, any group definable in R will be NIP rosy, so soluble-by-finite by 1.3.11. Hence R is nilpotent-by-finite by 1.3.22. □

With respect to stable pseudofinite rings we mention the following.

Remark 4.3.5. *Any ω-stable pseudofinite ring of finite Morley rank is null-by-finite.*

Proof. In [32] it is noted that Khelif showed that pseudofinite ω-stable groups of finite Morley rank are abelian-by-finite. The above remark follows from 1.3.22 part (ii). □

Moreover, from [31] we have the following, which also follows directly from Corollary 4.3.4.

Remark 4.3.6. *Any stable pseudofinite ring with 1, or of finite characteristic, is nilpotent-by-finite.*

Proof. By [31] we know that any stable pseudofinite group is soluble-by-finite. Hence by 1.3.22 part(i), we get that stable pseudofinite rings are nilpotent-by-finite.

□

As mentioned earlier, we denote by $M_n(R)$ the ring of $n \times n$-matrices with entries over a ring R, and we use $M_n(q)$ as a shorthand for $M_n(\mathbb{F}_q)$.

Recall the definable subgroup condition stated in Lemma 2.1 from [13]. In this text this is Lemma 1.3.12, in Section 1.3.2.

Proposition 4.3.7. *Let $\mathcal{C}$ be an NTP_2 class of finite J-semisimple rings. Then there is $N \in \mathbb{N}$ such that the members of $\mathcal{C}$ are of the form $M_{n_1}(q_1) \times \ldots \times M_{n_k}(q_k)$ for some k and $n_1, \ldots, n_k$ in $\mathbb{N}$ and primes $q_1, \ldots, q_k$ where $n_1, \ldots, n_k$, k are bounded by N.*

Proof. By the Artin-Wedderburn theorem, Theorem 1.3.19, and Theorem 1.3.20 we know that finite (hence Artinian) J-semisimple rings are of the form $\overset{k}{\underset{j=1}{\oplus}} M_{n_j}(q_j)$ for some $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{N}$ and $q_1, \ldots, q_k$ primes.

We deal with two cases. First assume there is no bound on k. Then for any $d \in \mathbb{N}$ there are infinitely many rings in $\mathcal{C}$ isomorphic to direct sums with more than d summands. Furthermore we have a collection of rings such that the number of summands is strictly increasing.

Consider now an ultraproduct $\mathcal{R} := \prod_i R_i/\mathcal{U} = \prod_i \left(\bigoplus_{j=1}^{k_i} M_{n_{i,j}}(q_{i,j}) \right) /\mathcal{U}$. We now proceed to find for every $N \in \mathbb{N}$ an $N \times N$ array in $\mathcal{R}$ that satisfies the conditions in the TP_2 property, contradicting the fact that $\mathcal{C}$ is an NTP_2 class.

Fix $N \in \mathbb{N}$. Since there is no bound on k there exists $U \in \mathcal{U}$ such that for any $i \in U$ the ring R_i has more that N^2 summands. I.e. we have rings of the form $R_i = \bigoplus_{j=1}^{k_i} M_{n_{i,j}}(q_{i,j})$ where $k_i > N^2$. We can then assume that each R_i has more than N^2 summands. We address two possibilities.

- First, if there is $V \in \mathcal{U}$ such that every $n_{i,j} \in V$ equals 1 then $R_i \cong \bigoplus_{j=1}^{k_i} \mathbb{F}_{q_{i,j}}$ for all $i \in V$. Define (uniformly) in each R_i the group $\mathcal{G}_i = GL_2 \left(\prod_{j=1}^{k_i} \mathbb{F}_{q_{i,j}} \right)$.

A straightforward argument shows that $GL_2\left(\prod_{j=1}^{k_i}\mathbb{F}_{q_{i,j}}\right) \cong \prod_{j=1}^{k_i} GL_2(\mathbb{F}_{q_{i,j}})$. Put $G_{i,j} := GL_2(\mathbb{F}_{q_{i,j}})$ so $\mathcal{G}_i = \prod_{j=1}^{k_i} G_{i,j}$.

- Otherwise we can assume that every $n_{i,j}$ appearing in R_i is greater than 1. Define in each R_i (uniformly, as the set of invertible elements) the group $\mathcal{G}_i := \prod_{j=1}^{k_i} GL_{n_{i,j}}(q_{i,j})$. Put $G_{i,j} := GL_{n_{i,j}}(q_{i,j})$ so $\mathcal{G}_i = \prod_{j=1}^{k_i} G_{i,j}$.

In any case, for each i take non-central elements $\{a_{i,j} : 1 \leq j \leq N^2\}$ where $a_{i,j} \in G_{i,j}$.

Put for every $l \leq N$ the sequence

$$\bar{\alpha}_{i,l} := (1, \dots, 1, a_{i,l}, 1, \dots, 1, a_{i,(N+l)}, 1, \dots, 1, a_{i,((N-1)N+l)}, 1, \dots, 1).$$

I.e. $\bar{\alpha}_{i,l} = (c_{i,j})_j$ where

$$c_{i,j} = \begin{cases} a_{i,kN+l} & \text{if } j = kN + l \text{ for some } 0 \leq k \leq (N-1), 1 \leq l \leq N\ ; \\ 1_{G_{i,j}} & \text{otherwise.} \end{cases}$$

Since $a_{i,j}$ is not central, for any $j \leq d$ there is $b_{i,j} \notin C_{G_{i,j}}(a_{i,j})$. Then we have $b_{i,j}C_{G_{i,j}}(a_{i,j}) \cap C_{G_{i,j}}(a_{i,j}) = \emptyset$.

Put now $\bar{\beta}_{i,(l,k)} := (1, \dots, 1, b_{i,l}, 1, \dots, 1, a_{i,(kN+l)}, 1, \dots, 1, b_{i,((N-1)N+l)}, 1, \dots, 1)$.

This is, $\bar{\beta}_{i,(l,k)} = (d_{i,j})_j$ where

$$d_{i,j} = \begin{cases} a_{i,kN+l} & \text{if } j = kN + l\ ; \\ b_{i,mN+l} & \text{if } j = mN + l \text{ for some } 0 \leq m \leq (N-1), m \neq k\ ; \\ 1_{G_{i,j}} & \text{otherwise.} \end{cases}$$

Note that $\bar{\beta}_{i,(l,k)}C_{G_i}(\bar{\alpha}_{i,l}) = \prod_{l=1}^{k_i} H_{i,j}$ where

$$H_{i,j} = \begin{cases} C_{G_{i,kN+l}}(a_{i,kN+l}) & \text{if } j = kN + l \text{ ;} \\ b_{i,mN+l}C_{G_{i,mN+l}}(a_{i,mN+l}) & \text{if } j = mN + l \text{ for some } 0 \leq m \leq (N-1), m \neq k \text{ ;} \\ G_{i,j} & \text{otherwise.} \end{cases}$$

Finally, for any $l \leq (N-1)$ and $k, k' \leq N$ with $k \neq k'$ we have

$$\bar{\beta}_{i,(l,k)}C_{G_i}(\bar{\alpha}_{i,l}) \cap \bar{\beta}_{i,(l,k')}C_{G_i}(\bar{\alpha}_{i,l}) = \emptyset.$$

This follows since $C_{G_{i,kN+l}}(a_{i,kN+l}) \cap b_{i,kN+l}C_{G_{i,kN+l}}(a_{i,kN+l}) = \emptyset$ (likewise $C_{G_{i,k'N+l}}(a_{i,k'N+l}) \cap b_{i,k'N+l}C_{G_{i,k'N+l}}(a_{i,k'N+l}) = \emptyset$). However, for any $f : \{0, \ldots, N-1\} \to \{1, \ldots, N\}$ we get $\bigcap_{j=1}^{N} \bar{\beta}_{i,(l,f(l))}C_{G_i}(\bar{\alpha}_{i,l}) \neq \emptyset$. This is because the tuple $(c'_{i,j})$ where

$$c'_{i,j} = \begin{cases} a_{i,f(l)N+l} & \text{if } j = f(l)N + l \text{ for some } 1 \leq l \leq N \text{ ;} \\ b_{i,mN+t} & \text{if } j \leq N^2 \text{ and } j \neq f(l)N + l \text{ for any } 1 \leq l \leq N \text{ but} \\ & j = mN + t \text{ for some } 0 \leq m \leq (N-1), 1 \leq t \leq N \text{ ;} \\ 1_{G_{i,j}} & \text{otherwise.} \end{cases}$$

is an element of $\bigcap_{j=1}^{N} \bar{\beta}_{i,(l,f(l))}C_{G_i}(\bar{\alpha}_{i,l}) \neq \emptyset$.

Then $\{\bar{\alpha}_{i,l}, \bar{\beta}_{i,(l,k)} : 0 \leq l \leq (N-1), 1 \leq k \leq N\}$ is an $N \times N$ TP_2 array for the formula $\varphi(x; y, z) := x \in zC_{G_i}(y)$.

Since we can get arbitrarily deep and wide TP_2 arrays in the ultraproduct we conclude that the ultraproduct is TP_2 .

On the other hand assume that there is a bound k on the number of summands that appear in each $R_i \in \mathcal{C}$ but there is not a bound on $n_{i,1}, \ldots, n_{i,k}$. This means that for some

$j \leq k$ the set $\{n_{i,j} : i \in I\}$ is unbounded. Then in $\prod_{i\in I} M_{n_{i,j}}(q_{i,j})/\mathcal{U}$ we can define an ultraproduct elementarily equivalent to $\prod_{n\in\mathbb{N}} GL_n(F_n)/\mathcal{W}$ for some fields F_n and some ultrafilter $\mathcal{W}$.

Fix $N \in \mathbb{N}$, then for every $n > (2N)N$ we can pick in $GL_n(F_n)$ the following subgroups. For $j \leq N^2$ consider the set of $n \times n$ matrices with entries from F_n

$$\begin{aligned} G_{n,j} = \{(t_{r,s}) :& t_{2j-1,2j-1}t_{2j,2j} - t_{2j-1,2j}t_{2j,2j-1} \neq 0; \\ & t_{r,s} = 1 \text{ if } r = s \wedge r \neq 2j-1 \wedge r \neq 2j; \\ & t_{r,s} = 0 \text{ if } r \neq s \wedge \big(r < 2j-1 \vee 2j < r \vee s < 2j-1 \vee 2j < s\big)\}. \end{aligned}$$

Then $G_{n,j}$ is the group of diagonal block matrices of the form $\begin{pmatrix} H_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & H_j & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & H_{N^2} & 0 \\ 0 & \cdots & \cdots & 0 & 0 & Id_{F_n} \end{pmatrix}$ where $H_j \in GL_2(F_n)$ and $H_i = Id_{GL_2(F_n)}$ for all $i \neq j$, and Id_{F_n} is the identity in $GL_{n-2N^2}(F_n)$

Arguing as above we find an $N \times N$ TP_2 array for arbitrary N. The TP_2 formula was quantifier free, and $GL_n(F)$ contains a direct product of N^2 copies of $GL_2(F)$. Hence the ultraproduct is TP_2 .

□

Remark 4.3.8. *Furthermore for any finite field F the theory of the class $\{GL_n(F) : n \in \mathbb{N}\}$ of all general linear groups is undecidable by [19], Appendix 1 (see also section 6.3 of [6]).*

We now turn to a natural example of a pseudofinite non-commutative ring, namely a ring of matrices over a ring.

Then 4.3.7 gives us the next result.

Corollary 4.3.9. *Let R be an ultraproduct of the form $\prod_{(q,n)} (M_2(q))^n \diagup \mathcal{U}$. Assume that for any N there is $U \in \mathcal{U}$ such that for all $(q,n) \in U$, $n > N$. Then R is* TP_2 .

Proof. From the proof of 4.3.7 we get the array that witnesses TP_2 .

□

Remark 4.3.10. *From 4.2.1 and 4.3.7 if R is an NTP_2 semisimple pseudofinite ring then R is elementarily equivalent to a ring of the form $\prod_i \bigoplus_{j=1}^{k_i} M_{n_{i,j}}(q_{i,j}) \diagup \mathcal{U}$ and there is a bound on k_i and the size of the matrices $n_{i,j}$. This means $R \equiv \prod_{(q_1,\ldots,q_k)} M_{n_1}(q_{1_j}) \oplus \ldots \oplus M_{n_k}(q_{k_j}) \diagup \mathcal{V}$*

We know NIP rosy pseudofinite rings are nilpotent-by-finite, see 4.3.4. Similarly to Theorem 3.1 from [32], NIP pseudofinite rings are not necessarily nilpotent-by-finite.

Proposition 4.3.11. *There are NIP pseudofinite rings that are not nilpotent-by-finite. Moreover, ultraproducts of NIP classes of finite rings need not be nilpotent-by-finite.*

Proof. Let p be a prime. Put $\mathcal{C}_p := \{\mathbb{Z}/p^n\mathbb{Z} : n \in \mathbb{N}\}$. By 2.2.3 in Section 2.2 any ultraproduct R of elements in $\mathcal{C}_p$ is NIP. Furthermore each ring in $\mathcal{C}_p$ has a nilpotent ideal, e.g. $p\mathbb{Z}/p^n\mathbb{Z} \subseteq \mathbb{Z}/p^n\mathbb{Z}$. However R is not nilpotent-by-finite. Otherwise assume that I is a finite index ideal. Let $(1)_n$ be the multiplicative identity in R and let $(0)_n$ be the additive identity in R. Then for $k = |R : I|$, $(1)_n + \ldots + (1)_n = (k)_n = \bar{k} \in I$. If we assume that I is nilpotent then there is m such that the product of any m elements of I equals $(0)_n$. In particular $\bar{k}^m = (0)_n$. This means that $\{n : p^n | k^m\} \in \mathcal{U}$ but there is $N \in \mathbb{N}$ such that for all $n \geq N$ we get $p^n \geq k^m$. Hence a finite index ideal cannot be nilpotent. □

We can however get some information about NIP classes of finite rings. Namely, we get the following.

Proposition 4.3.12. *Let $\mathcal{C}$ be a NIP class of finite rings with 1 or of finite characteristic. Then there exists δ such that every $R \in \mathcal{C}$ has a nilpotent ideal of index bounded by δ.*

Proof. As in Theorem 2.1 from [26] it is enough to consider the case when the rings have 1. Consider a NIP class $\mathcal{C}$ of finite rings (with 1). Write $\mathcal{C} := \{R_l : l \in \mathcal{I}\}$. We will define uniformly in each R_l the group $G_l := GL_3(R_l)$. Write $\mathcal{C}_G := \{G_l : l \in \mathcal{I}\}$.

Since each ultraproduct $\mathcal{G}$ of elements in $\mathcal{C}_G$ is uniformly definable in an ultraproduct $\mathcal{R}$ of elements R in the NIP class $\mathcal{C}$ (by using the same ultrafilter) we have that $\mathcal{G}$ is NIP. Hence $\mathcal{C}_G$ is an NIP class of finite groups.

By work of H.D. Macpherson and K. Tent from [32], which is mentioned in 1.3.11 on Subsection 1.3.2, there exists $d = d(\mathcal{C}_G)$ such that for every $G_l \in \mathcal{C}_G$ we have $|G_l : S(G_l)| \leq d$, where $S(G_l)$ is the largest soluble, normal subgroup of G_l.

It is also mentioned in Subsection 1.3.2 that from [50] the largest soluble normal subgroup of a finite group is first order definable, say by the formula ψ^*.

Hence, we have that $S(G_l)$ is a definable normal, soluble, finite (bounded across $\mathcal{C}_G$) index subgroup.

We follow the argument given by K. Krupinski in [26] to prove 1.3.22. Given distinct $i, j \in \{1, 2, 3\}$ define the matrix $t_{i,j}(\alpha)$, for $\alpha \in R_l$, as the matrix having 1's in the diagonal, α in the (i, j)-th place and 0 elsewhere. It follows that $t_{i,j}(\alpha)t_{i,j}(\beta) = t_{i,j}(\alpha+\beta)$ and $[t_{i,k}(\alpha), t_{k,j}(\beta)] = t_{i,j}(\alpha\beta)$

Put $I_l := \{\alpha \in R_l : t_{i,j}(\alpha) \in S(G_l)$ for all distinct $i, j \in \{1, 2, 3\}\}$.

We have that I_l is a definable ideal of R_l.

- $0 \in I_l$ for $t_{i,j}(0) = Id_3 \in S(G_l)$ since $S(G_l)$ is a subgroup.
- Given $\alpha, \beta \in I_l$ then $t_{i,j}(\alpha)t_{i,j}(\beta) = t_{i,j}(\alpha + \beta)$, so $\alpha + \beta \in I_l$. Also we have $t_{i,j}(-\alpha) = t_{i,j}(\alpha)^{-1}$ so $-\alpha \in I_l$.

- Let $\tau \in R_l$, and $\alpha \in I_l$ so $t_{i,j}(\alpha) \in S(G_l)$ for all distinct $i, j \in \{1, 2, 3\}$. We have $t_{i,j}(\tau\alpha) = [t_{i,k}(\tau), t_{k,j}(\alpha)] = t_{i,k}(\tau)t_{k,j}(\alpha)t_{i,k}(\tau)^{-1}t_{k,j}(\alpha)^{-1}$. Since we know $t_{i,k}(\tau)t_{k,j}(\alpha)t_{i,k}(\tau)^{-1} \in S(G_l)$ by normality of $S(G_l)$ and $t_{k,j}(\alpha)^{-1} \in S(G_l)$ because $S(G_l)$ is a subgroup we get $t_{i,j}(\tau\alpha) \in S(G_l)$. Hence $\tau\alpha \in I_l$. A similar argument shows that $\alpha\tau \in I_l$.

Furthermore I_l has bounded index in R_l across $\mathcal{C}$. Moreover it is strictly bounded by $\delta(\mathcal{C}_G) := R(d+1, d+1, \ldots, d+1)$, the Ramsey number for a colouring with as many colours as pairs of distinct $i, j \in \{1, 2, 3\}$, call them $\{c_{(i,j)}\}$ (6 possible entries).

Assume on the contrary that there are $\delta = \delta(\mathcal{C}_G)$ many distinct I_l cosets in R_l. We can now pick $\{\alpha_n : 1 \leq n \leq \delta\}$ distinct coset representatives. Then $\alpha_n - \alpha_m \notin I_l$ for any distinct n, m.

Now consider the complete graph on the set of vertices $\{\alpha_n : 1 \leq n \leq \delta(\mathcal{C}_G)\}$ and colour the edge between α_n and α_m with colour $c_{(i,j)}$ if and only if $t_{i,j}(\alpha_n)t_{i,j}(\alpha_m)^{-1} \notin S(G_l)$ and $c_{(i,j)}$ is least lexicographically. Hence by applying finite Ramsey theorem there is a colour $c_{(i,j)}$ such that there is a monochromatic complete graph with edges coloured $c_{(i,j)}$ of size $d(\mathcal{C}_G) + 1$. This means that there are $d(\mathcal{C}_G) + 1$ elements of the form $t_{i,j}(\alpha_u)$ with $1 \leq u \leq d(\mathcal{C}_G) + 1$ such that $t_{i,j}(\alpha_n)t_{i,j}(\alpha_m)^{-1} \notin S(G_l)$. This implies that $|G_l : S(G_l)| \geq d(\mathcal{C}_G) + 1$ which contradicts the fact that $S(G_l)$ has index bounded by $d(\mathcal{C}_G)$.

Hence I_l has index in R_l strictly bounded by $\delta(\mathcal{C}_G)$ across $\mathcal{C}$.

Moreover $\prod_l I_l/\mathcal{U}$ is an ideal of index less than $\delta(\mathcal{C}_G)$ in the ultraproduct $\prod_l R_l/\mathcal{U}$.

Because of the solubility of $S(G_l)$ there is some n_l such that $S(G_l)^{(n_l)} = \{e\}$. I.e. the n_l-th derived subgroup is trivial.

By induction on n it is straightforward to show that if $S(G_l)$ has derived length n then for any $\alpha_1, \ldots, \alpha_{2^n} \in I_l$ we have that $t_{i,j}(\alpha_1 \cdot \ldots \cdot \alpha_{2^n}) \in \big(S(G_l)\big)^{(n)}$, which means that, $\alpha_1 \cdot \ldots \cdot \alpha_{2^n} = 0$. Hence I_l is nilpotent of nilpotent index less than or equal to 2^{n_l}.

Therefore for any $\alpha_1, \ldots, \alpha_{2^{n_l}} \in I_l$ we get $\alpha_1 \cdot \ldots \cdot \alpha_{2^{n_l}} = 0$. Hence I_l is nilpotent ideal.

We can conclude that there exists δ such that every ring in the class $\mathcal{C} = \{R_l : l \in \mathcal{I}\}$ has a definable nilpotent ideal of index bounded by δ.

□

Question 4.3.13. *Can we remove the assumption "R has a 1 or has finite characteristic" from 4.3.4?*

Question 4.3.14. *Consider commutative nilpotent pseudofinite stable rings. Are these null-by-finite?*

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