Global Analysis of Dynamical Systems On Low-Dimensional Manifolds

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To my dearly loved parents.

Abstract

The interaction of topology and dynamics has attracted a great deal of attention from numerous mathematicians. This thesis is devoted to the study of dynamical systems on low-dimensional manifolds.

In the order of dimensions, we first look at the case of two-manifolds (surfaces) and derive explicit differential equations for dynamical systems defined on generic surfaces by applying elliptic and automorphic function theory to uniformise the surfaces in the upper half of the complex plane with the hyperbolic metric. By modifying the definition of the standard theta series, we will determine general meromorphic systems on a fundamental domain in the upper half plane, the solution trajectories of which 'roll up' onto an appropriate surface of any given genus. Meanwhile, we will show that a periodic nonlinear, time-varying dissipative system that is defined on a genus-p surface contains one or more invariant sets which act as attractors. Moreover, we shall generalize a result in [Martins, 2004] and give conditions under which these invariant sets are not homeomorphic to a circle individually, which implies the existence of chaotic behaviour. This is achieved by analyzing the topology of inversely unstable solutions contained within each invariant set.

Then the thesis concerns a study of three-dimensional systems. We give an explicit construction of dynamical systems (defined within a solid torus) containing any knot (or link) and arbitrarily knotted chaos. The first is achieved by expressing the knots in terms of braids, defining a system containing the braids and extending periodically to obtain a system naturally defined on a torus and which contains the given knotted trajectories. To get explicit differential equations for dynamical systems containing the braids, we will use a certain function to define a tubular neighbourhood of the braid. The second one, generating chaotic systems, is realized by modelling the Smale horseshoe.

Moreover, we shall consider the analytical and topological structure of systems

on 2- and 3- manifolds. By considering surgery operations, such as Dehn surgery, Heegaard splittings and connected sums, we shall show that it is possible to obtain systems with 'arbitrarily strange' behaviour, i.e., arbitrary numbers of chaotic regimes which are knotted and linked in arbitrary ways.

We will also consider diffeomorphisms which are defined on closed 3-manifolds and contain generalized Smale solenoids as the non-wandering sets. Motivated by the result in [Jiang, Ni and Wang, 2004], we will investigate the possibility of generating dynamical systems containing an arbitrary number of solenoids on any closed, orientable 3-manifold. This shall also include the study of branched coverings and Reeb foliations.

Based on the intense development from four-manifold theory recently, we shall consider four-dimensional dynamical systems at the end. However, this part of the thesis will be mainly speculative.

Publication List

Journal Publications

1. S. P. Banks and Yi Song (2006), "Elliptic and automorphic dynamical systems on surfaces", Int. J. of Bifurcation and Chaos, Vol. 16, No. 4: 911-923.

2. Yi Song, S. P. Banks and David Diaz (2007), "Dynamical systems on three manifolds–Part I: knots, links and chaos", Int. J. of Bifurcation and Chaos, Vol. 17, No. 6: 2073-2084.

3. Yi Song and S. P. Banks (2007), "Dynamical systems on three manifolds–Part II: 3-manifolds, Heegaard Splittings and three-dimensional systems", Int. J. of Bifurcation and Chaos, Vol. 17, No. 6: 2085-2095.

4. Yi Song and S. P. Banks (2008), "Inversely unstable solutions of two-dimensional systems on genus-p surfaces and the topology of knotted attractors", Int. J. of Bifurcation and Chaos, Vol. 18, No. 1: 109-119.

5. Yi Song, Xu Xu and S. P. Banks, "Automorphic forms and Reeb-like foliations on three-manifolds", J. of Mathematical Analysis and Applications, under review.

6. Xu Xu, Yi Song and S. P.Banks, "On the dynamical behaviour of cellular automata", Int. J. of Bifurcation and Chaos, under review.

Conference Publications

1. Yi Song and S. P. Banks, "Dynamical systems on three manifolds", ICCC 2006, Ostrava, Czech Republic.

2. Yi Song, Xu Xu and S. P. Banks, "Optimal tracking control of nonlinear systems on manifolds", ICCC 2007, Štrbské Pleso, Slovak Republic.

3. Xu Xu, Yi Song and S. P. Banks, "Cellular automata, symbolic dynamics and artificial life", ICCC 2007, Štrbské Pleso, Slovak Republic.

4. Yi Song, Xu Xu, "From local to global control systems", SINTES 2007, Craiova, Romania.

5. Yi Song and S. P. Banks, "Smale solenoids in $(2n+1)$-manifolds", ICCC 2008, Sinaia, Romania.

6. Yi Song and S. P. Banks, "Generalized Smale solenoids on genus-p 3-manifolds", WCNA 2008, Florida, US.

7. S. P. Banks and Yi Song, "Dynamical systems on low-dimensional manifolds", WCNA 2008, Florida, US.

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Contents

Chapter 1

Introduction

1.1 Overview

Dynamical systems, as an independent discipline, has been studied for a long time. The roots of dynamical systems theory can be traced a long way back as in the *Principia Mathematica* of Issac Newton. His attempts to understand and model the motions of the bodies in the solar system eventually led to his development of the calculus. From then on the study of dynamical problems as differential equations, which gave functional descriptions of the solutions to physical problems or of the mathematical models describing the physical systems, started and has attracted more and more attention ever since.

However, it was at the end of the nineteenth century that Henri Poincaré first established the modern, geometric theory of dynamical systems when he was studying the three body problem of celestial mechanics (see [Poincaré, 1899]). The qualitative analysis of differential equations then became the centre of attention because, determined by the global phase portrait, the qualitative behaviour provides better information than just obtaining a formula since it geometrically describes the movement of every solution for all time.

In spite of the simple forms that differential equations might take, the solutions to some specific problems have been proved tremendously difficult and engaged many mathematicians. Following Poincaré 's work, the last hundred years witnessed a remarkable explosion in this subject due to the contribution of a lot of differential topologists, including Birkoff, Pontryagin, Kolmogorov, Anosov, Arnol'd, Moser, Smale, etc. To point out, in [Birkhoff, 1912], Birkhoff outlined the conjecture of Poincaré that the 'general' motion of dynamical systems in phase space was of the so-called 'discontinuous type'. Meanwhile, he introduced the beautiful idea of 'minimal' sets of motions, also known as the 'recurrent' sets, which declared the beginning of the study of symbolic dynamics; Smale, in his classic paper [Smale, 1967], outlined a number of outstanding problems and stimulated an explosion of research in this field. However, although several potential applications has been sketched, (such as in [Takens, 1980], Takens addressed the importance of 'strange attractors' in the study of turbulence,) until the mid 1970s the modern methods of qualitative analysis were mainly in the hands of pure mathematicians,

It is during the last two decades that dynamical systems theory has found a lot of exhilarating applications in the scientific world other than mathematics. For example, in the engineering community, especially solid and fluid mechanics, there is now a widespread interest in strange attractors, chaos, and dynamical systems theory; also since a knot can be obtained as the periodic solution of some differential equation, dynamical systems theory hence provides penetrating insight into the study of knot-theory-related areas. These range from molecular biology, which involves topological structures of closed DNA strands, to physics, due to its surprising connections with quantum field theory.

Much of the theory of dynamical systems in n-dimensional spaces is studied in $\mathbb{R}^n$, i.e. a 'flat' Euclidean vector space (see, e.g. [Perko, 1991]); indeed phase-space portraits are shown in this way. However, globally speaking, a dynamical system is situated on some manifold, M, which constitutes all the states of the system and hence is also known as the state space. Defined by some first order differential equation (ODE)

$$\dot{x} = f(x), \tag{1.1}$$

where $f \in C^1(M)$, the unique solution will then generates a flow $\phi_t : M \to M,\ \ t \in \mathbb{R}$. Geometrically, these curves determine the motion of all the points in the space under this specific dynamical system. It is this flow that builds up the bridge which links dynamical systems with the corresponding topology of the base manifold. Here is an example.

Example. Consider a spherical pendulum (see fig. 1.1 for illustration), it has two degrees of freedom, and the *Lagrangian* for this system is

$$L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2(\sin\theta)^2\dot{\phi}^2\right) + mgl\cos\theta.$$

The *Euler-Langrange* equations yield

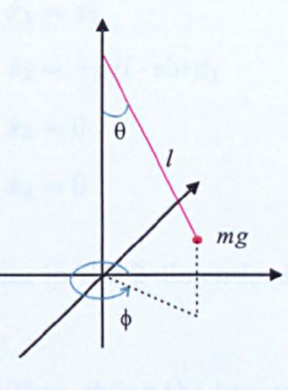


Figure 1.1: A spherical pendulum

$$\begin{cases} \frac{d}{dt}(ml^2\dot{\theta}) - ml^2 \sin\theta \cos\theta \dot{\phi}^2 + mgl \sin\theta = 0 \\ \frac{d}{dt}\left(ml^2(\sin\theta)^2\dot{\phi}\right) = 0 \end{cases},$$

so the system is given by two equations of motion, i.e.

$$\begin{cases} \ddot{\theta} = \dot{\phi}^2 \sin\theta \cos\theta - g/l \cdot \sin\theta \\ \ddot{\phi} = -(2\cos\theta \cdot \dot{\theta} \cdot \dot{\phi})/\sin\theta \end{cases}.$$

In phase-space coordinate, by setting $x_1 = \theta$, $x_2 = \dot{\theta} = \omega_\theta$, $x_3 = \phi$, $x_4 = \dot{\phi} = \omega_\phi$, we have

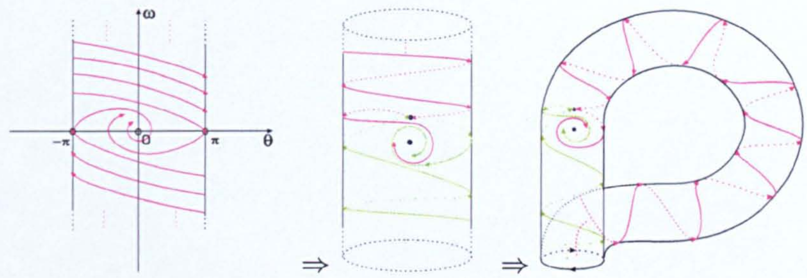
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = {x_4}^2 \sin x_1 \cos x_1 - g/l \cdot \sin x_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -(2\cos x_1 \cdot x_2 \cdot x_4)/\sin x_1 \end{cases}. \tag{1.2}$$

This is a 4-dimensional system. Now assume $x_3 = \phi = k$, where k is a constant. Consequently $\dot{x}_3 = 0$ and $\dot{x}_4 = \dot{\omega}_\phi = 0$, the system will then become

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g/l \cdot \sin x_1 \\ \dot{x}_3 = 0 \\ \dot{x}_4 = 0 \end{cases},$$

which stands for a single pendulum in the 2-dimensional case. Fig. 1.2a shows the trajectories in the phase-plane.

By identifying $-\pi$ with π and then gluing the two ends of the resulting cylinder together, we know that a single pendulum is defined on a Klein bottle (see fig. 1.2). In fact, by Liouville's theorem, any integrable Hamiltonian system has integral surfaces which are either tori or Klein bottles (see [Abraham and Marsden, 1978; Arnold, 1989]).



(a) (b) (c)

Figure 1.2: A simple pendulum is defined on a Klein bottle.

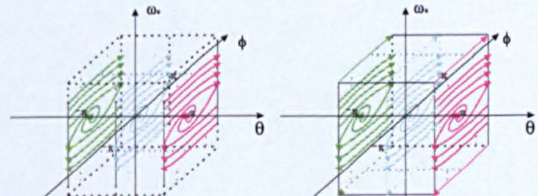
Moreover, by setting $\dot{x}_3 = \omega_\phi = k$, i.e. the system has a fixed nonzero angular velocity in ϕ, (1.2) then becomes

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_4^2 \sin x_1 \cos x_1 - g/l \cdot \sin x_1 \\ \dot{x}_3 = k \\ \dot{x}_4 = 0 \end{cases},$$

which is essentially a 3-dimensional hyperplane given by $x_4 = c$ (where c is a constant) within $\mathbb{R}^4$. Furthermore, since the vector field is periodic in both x_1- and x_3-axis with period 2π, it is naturally defined within the cube

$$C = \{(x_1, x_2, x_3) : -\pi \leq x_1 \leq \pi, -\infty < x_2 < \infty, \quad -\pi \leq x_3 \leq \pi\},$$

as shown in fig. 1.3a.



(a) (b)

Figure 1.3: Phase-plane portrait of the spherical pendulum when $\omega_\phi = k$.

Note that the phase-space portrait is a sequence of 2-dimensional single pendulums that sit on different slices defined by $\phi = k$, and since $\theta, \phi = \pi$ and $\theta, \phi = -\pi$ are physically the same, we can identify them accordingly by pairing the opposite sides.

In order to define the system on a compact manifold, we compress the infinite cube to a finite one, as shown in fig. 1.3b. Since the dynamics at the top and bottom ends are pointing the opposite directions, the appropriate identification yields a self-intersection in the 3-dimensional Euclidean space, fig. 1.4 illustrates the embedding in $\mathbb{R}^3$.

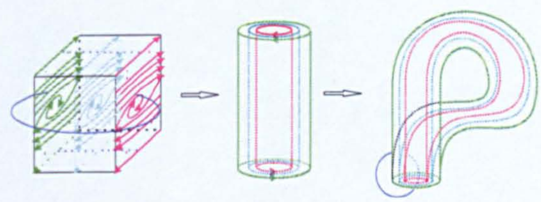


Figure 1.4: Construction of a 3-dimensional solidKlein bottle.

Thus, we obtain the 3-manifold on which this special spherical pendulum is situated. We call it 3-dimensional solid Klein bottle.

The above example suggests a potential link between dynamical systems and manifold theory. Until now, a great amount of research has been carried out to tackle the problem of how to describe the flow ϕ_t geometrically, via its action on subsets of M. This implies classification of the asymptotic behaviours of all possible solutions, by finding equilibrium points, periodic orbits, homoclinic loops, separatrix cycles and strange attractors as limit sets. In many cases, the topological changes of ϕ_t under perturbations are of the most interests. In fact, many of the exhilarating results achieved by both dynamicists and topologists have proved that this area of research is very interesting and promising, hence well worth a continued effort.

1.2 Low-Dimensional Topology

Low-dimensional topology, as an independent subject of mathematics, has been undergoing an intense development for the last century. The reason why it has received a large amount of attention is because the approach to algorithmic solutions in topology always proves to be a rich source of well-stated mathematical problems.

Manifold theory possesses a central theme of low-dimensional topology. Historically, the topology of two-manifolds is well understood with one unique invariant, the Euler characteristics, which can distinguish all different surfaces topologically. Three-manifolds theory turns out to be much more difficult since a complete set of topological invariants does not exist. However, lots of efforts from numerous topologists have contributed to a great deal of useful invariants. In [Thurston, 1979], the idea was first proposed of approaching 3-manifold topology geometrically . Then in the 1980s, low-dimensional topology encountered mathematical physics, and their exhilarating interactions then yielded the so-called quantum invariants. This wealth of invariants provides us with diverse viewpoints to study 3-manifolds from. The topology and differentiable structure of four-manifolds is probably the most difficult of any dimension. Once was almost completely ignored, 4-manifold theory has gained lots of attention for the last two decades and exhibited somewhat more strange behaviour, such as the well-known exotic $\mathbb{R}^4$. The existence of Smale, More-Smale and (pseudo) Anosov diffeomorphisms in the case of 4-manifolds then becomes more difficult because of the complex interaction of the topology and differentiable structures. In spite of all the difficulties, the differentiable structure can still be, to some extent, measured by some recently discovered invariants by Donaldson (see [Donaldson, 1983, 1990]) and simplified by Seiberg-Witten invariants (see [Moore, 1996]). With the interplays with mathematical physics, knot theory, algebraic geometry, etc., low-dimensional topology is now finding applications in diverse scientific areas.

1.2.1 Knots, Links and Surgery Operations

Knot theory has been an important subject in its own right for a long time (see [Kauffman, 1991]), and recently a great deal has been written on the connections between knot theory and dynamical systems (see, for example, [Ghrist, Holmes and Sullivan, 1997]). The key idea is this: a closed (periodic) orbit in a three-dimensional flow is an embedding of the circle, S^1, into the three-manifold that constitutes the state space of the system, hence it is a knot. Therefore, periodic solutions of dynamical systems may be knotted or linked, and in fact, a chaotic system contains any knot and link ([Birman and Williams, 1983]). A simple approach to obtaining a (non-chaotic) system which contains an arbitrary knot (even a wild knot) is given in [Banks and Diaz, 2004].

It is widely known that any knot can be expressed in terms of braids. By writing down explicit differential equations for these braids over a finite time interval and making the vector field periodic, we can glue the two ends of the phase space at successive periodic time points together (see fig. 1.5), which will yield the desired knot embedded within a solid torus. This can serve as our bridge to link knot theory and dynamical systems.

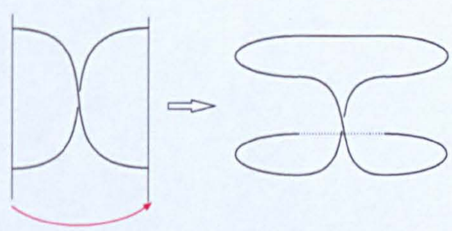


Figure 1.5: Constructing knots from braids by gluing the two ends together.

Moreover, Surgery operations play an important role in the study of 3-manifold theory. These surgeries can be regarded as decompositions of a 3-manifold in terms of simpler ones. In effect, we may construct new 3-manifolds from the existing ones. The main surgery operations include: Dehn surgery, Heegaard splittings, connected sums, branched coverings (see [Ratcliffe, 1994; Hempel, 1976; Rolfsen, 1976; Montesinos, 1974]).

1.2.2 Strange Attractors in Dissipative Systems

The general theory of dynamical systems is, of course, a subject with a long and distinguished history. In particular, the study of dynamical behaviours of non-conservative and chaotic systems, especially with the interplay of topology, has attracted a lot of attention in the past (see, for example, [Levinson, 1944; Martins, 2004; Wiggins, 1988]). Consider a system

$$\begin{cases} \dot{x} = F(x,y,t) \\ \dot{y} = G(x,y,t) \end{cases},$$

where $F(x,y,t)$ and $G(x,y,t)$ are both periodic in t. It will be called *dissipative* or *non-conservative* if there is a locally proper invariant set on the corresponding 2-manifold on which the system is defined. In fact, most real systems are of this kind. Up to present a great deal of interest has been paid to the study of the topology of this invariant set (e.g. [Levinson, 1944]).

Recently in [Martins, 2004], Martins considered the system of the form

$$\ddot{x} + h(x)\dot{x} + g(t,x) = 0,$$

where h and g are smooth functions, periodic on both x and t. It is essentially a periodic nonlinear 2-dimensional, time-varying oscillator with appropriate damping. Equivalently, we can write down the system in the following way

$$\begin{cases} \dot{y_1} = y_2 - H(y_1) \\ \dot{y_2} = -g(t,y_1) \end{cases}, \tag{1.3}$$

where $H(x) = \int_0^x h(s)ds$, h and g are smooth functions, both 1-periodic on y_1 and T-periodic in t. The Poincaré map is defined as $P(y_0) = y(T; 0, y_0)$. Since the vector field $(y_1, y_2) \to \big(y_2 - H(y_1), -g(t, y_1)\big)$ is periodic with period $R = \big(1, h(1)\big)$, the solutions y and $y + kR$ ($k \in \mathbb{Z}$) are equivalent and so the system may be defined on a cylinder, as in fig. 1.6.

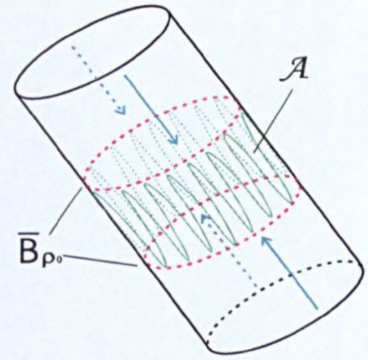


Figure 1.6: The invariant set defined on a cylinder.

Here $\mathcal{A}$ is the invariant set

$$\mathcal{A} = \bigcap_{n \in \mathbb{N}} \overline{P}^n(\overline{B}_{\rho_0}),$$

where B_{ρ_0} is some bounded set (which exists because the system is dissipative, as implied by the arrows in fig (1.6)). In [Martins, 2004], he showed that $\mathcal{A}$ is not

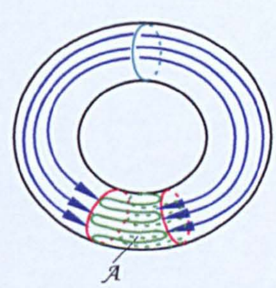


Figure 1.7: Invariant set in the torus case.

homeomorphic to a circle if there is an inversely unstable periodic orbit; we can think of the problem as sitting on a torus with one unstable cycle, as in fig (1.7). Hence the topological structures of attractors can serve as a guidance towards the understanding of geometric behaviours of differential equations.

1.3 The Contents of This Volume

This research project mainly attempts to consider dynamical systems and the related manifold theory, and we mainly concentrate on the structure of dynamical systems on manifolds of dimension 2, 3 and 4.

In Chapter 2, to follow immediately, we derive explicit differential equations for dynamical systems defined on generic surfaces by applying elliptic and automorphic function theory to the upper half plane model with hyperbolic metric. By modifying the definition of the standard theta series we will determine general analytical systems on a fundamental domain in the upper half plane where the solution trajectories 'match up' onto an appropriate surface of any given genus. The extension of this result to 3-dimensional case is concerned then and proved possible with restriction. Moreover, we will show that a periodic nonlinear, time-varying dissipative system that is defined on a genus-p surface contains one or more invariant sets which act as attractors. We shall also generalize the result in [Martins, 2004] and study the topology of these invariant sets, which might imply the existence of chaotic behaviour. This is achieved by studying the appearance of inversely unstable solutions within each invariant set.

The 3-manifold topology is much more involved than the 2-dimensional theory, largely because of the fact that there exists no complete set of topological invariants. However, we have the notions of Heegaard splittings and connected sums. Moreover, knot and link invariants turned out to be able to classify 3-manifolds to some extent (see, for example, [Rolfsen, 1976]). On the other hand, dynamicists have been seeking to use knot theory to describe periodic orbits and hence help understand the underlying ODEs better. In Chapter 3, we will give a resumé of the information mentioned above.

Chapter 4 mainly concerns the study of dynamical systems on 3-manifolds. We give an explicit construction of dynamical systems (defined within a solid torus) containing any knot (or link) and arbitrarily knotted chaos. The first is achieved by expressing the knots in terms of braids, defining a system containing the braids and extending periodically to obtain a system naturally defined on a torus which contains the given knotted trajectories. The second one, generating chaotic systems, is realized by modelling the Smale horseshoe. Moreover, from the topological point of view, we also consider analytical methods, i.e., applying surgeries such as Heegaard splittings and connected sums, to obtain dynamical systems situated on 3-manifolds. Lickorish's result (see [Lickorish, 1962]) provides us a useful tool here.

Chapter 4 is also devoted to the study of diffeomorphisms which are defined on closed 3-manifolds and contain generalized Smale solenoids as the non-wandering sets. In particular, motivated by the result in [Jiang, Ni and Wang, 2004], we will study the possibility of generating dynamical systems containing an arbitrary number of solenoids on any closed, orientable 3-manifold.

Four-manifold theory is much more involved since the topological and differentiable structure are completely different. However, we still have available generalized connected sums, 'blow-ups' and 'blow downs' which can be used to simplify the dynamics. The resulting manifold topology and differentiable structure can be measured, to some extent, by the Seiberg-Witten invariants. Chapter 5 consists of a summary of 4-manifold topology. Furthermore, we shall discuss the case of 4-dimensional dynamical systems and their defining manifolds, with an emphasis on piecing up a finite set of local dynamical systems to obtain some global result. Some open problems that pertain to 4-manifold theory will be outlined at the end.

In accordance with the understanding of this thesis, we will provide a succinct review of the principal aspects in topology and manifold theory in Appendix A. Similarly in Appendix B we recall Smale's theory of dynamical systems on manifolds and the idea of Smale solenoids, when he classified the types of basic sets (see [Smale, 1967]).

Chapter 2

Two-Dimensional Systems

The simplest non-trivial theory is in two dimensions. There exists a well-known topological invariant, genus p, which can distinguish all 2-manifolds. For an orientable one, genus p is equivalent to the number of 'holes' attached to the surface. In other words, any orientable 2-manifold is topologically a sphere with $p \geq 0$ handles attached. Any nonorientable surface can be obtained from an orientable one by cutting out a disk from the surface and sewing back in a Möbius band (which has only one edge and is topologically a circle). The Poincaré -Hopf index theorem then says that the total index I of a vector field X on a 2-manifold S must equal the *Euler characteristic*

$$I_X = \chi_S = 2(1-p). \tag{2.1}$$

Hence $\chi(S^2) = 2$, $\chi(T^2) = 0$, where T^2 denotes a 2-dimensional torus. Note that in nonorientable case, the genus $g = q/2$, where the nonorientable surface S is topologically S^2 with q holes along whose boundaries the antipodal points are identified. Thus, the characteristic of the projective plane $\chi(P) = 1$ and for Klein bottle $\chi(K) = 0$.

The relation (2.1) regulates system dynamics by topological restrictions, which can serve as a breakthrough to the study of dynamical systems on manifolds. In fact, explicit dynamical systems have been written down for the torus case. However, it turns out to be difficult to find their counterparts on higher genus surfaces due to the obstacles in finding simple explicit forms for the equations of the surfaces in three-dimensional space.

2.1 Classification of Singularities

In the case of linear (two-dimensional) systems, the standard classification by means of the eigenvalues of the system matrix recognizes three topologically distinct types - stale or unstable nodes, centres and saddle points. In the case of nonlinear systems defined by elliptic or (modified) automorphic functions, the local behaviour at a singularity is given by the local structure of a meromorphic function consisting of a pole or zero of any order, i.e., by the systems

$$\dot{z} = \frac{1}{z^k} \quad \text{or} \quad \dot{z} = z^k,$$

for any positive integer k. (We can always translate a pole or zero to the origin.) Consider first the single pole system

$$\dot{z} = z^{-k}$$

and substitute $z = re^{i\theta}$, so that

$$z^{-k} = \frac{\bar{z}^k}{r^{2k}} = \frac{e^{-ik\theta}}{r^k} = \frac{\cos k\theta - i\sin k\theta}{r^k}.$$

Since $\dot{z} = \dot{r}e^{i\theta} + rie^{i\theta}\dot{\theta}$, we obtain the radial and angular equations

$$\begin{aligned}\dot{r} &= \frac{1}{r^k}(\cos\theta\cos k\theta - \sin\theta\sin k\theta) = \frac{1}{r^k}\cos(k+1)\theta \\ \dot{\theta} &= -\frac{1}{r^{k+1}}\sin(k+1)\theta.\end{aligned}$$

Generally speaking, $\dot{\theta}$ is not identically zero, hence the solution curves given by $\dot{\theta} = 0$ divide a neighbourhood of the equilibrium point into a finite number of open regions called *sectors*. Moreover, the boundary trajectories are called *separatrices*, defined by

$$\sin(k+1)\theta = 0 \text{ or } \theta = \frac{l\pi}{k+1},\ l = 0, 1, \cdots, 2k+1. \tag{2.2}$$

Note that at these points,

$$\dot{r} = \frac{1}{r^k}\cos(l\pi), \tag{2.3}$$

which alternates in sign. In fact, there are only three types of sectors in 2-dimensional case. They are: hyperbolic sectors, elliptic sectors and parabolic sectors (see fig. 2.1).

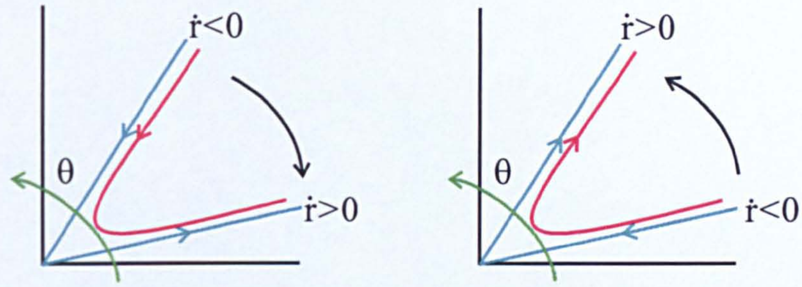
(a) Hyperbolic sector. (b) Elliptic sector. (c) Parabolic sector.

Figure 2.1: Three types of sectors in 2-dimensional systems.

For a hyperbolic sector, we have two possibilities, as shown in fig. 2.2.

From (2.2) and (2.3) it follows that for a single pole system, every sector is hyperbolic, so that a k^{th} order pole is surrounded by $2(k+1)$ hyperbolic sectors. From the index formula for a general critical point in a dynamical system (see [Perko, 1991])

$$\text{Index of a general equilibruim (critical) point } = 1 + \frac{e-h}{2},$$



(a) θ decreasing along the trajectory, i.e. $\dot{\theta} < 0$.

(b) θ increasing along the trajectory, i.e. $\dot{\theta} > 0$.

Figure 2.2: Two possibilities for hyperbolic sectors.

where e is the number of elliptic sectors, and h denotes the number of hyperbolic sectors, we see that a k^{th} order pole system has index $-k$ at the pole.

Now consider the single zero system

$$\dot{z} = z^k, \quad k > 1.$$

In this case we obtain the equations

$$\begin{aligned} \dot{r} &= r^k \cos(k-1)\theta \\ \dot{\theta} &= r^{k-1} \sin(k-1)\theta \end{aligned}$$

and so the sectors are given by

$$\sin(k-1) = 0 \text{ or } \theta = \frac{l\pi}{k-1}, \; l = 0, 1, \cdots, 2(k-1) - 1.$$

It can be seen that they are all elliptic sectors. Hence the system with a k^{th} order zero has equilibrium point surrounded by $2(k-1)$ elliptic sectors, and the index is $+k$. Therefore we have proved

Lemma 2.1.1. *Any meromorphic dynamical system on a surface which has poles at the points $P_1, \cdots, P_n$ of orders $k_1, \cdots, k_n$ and zeros at the points $Q_1, \cdots, Q_m$ of* *orders* $l_1, \cdots, l_m$ *(where all the points are distinct) has total index*

$$I = \sum_{i=1}^{m} l_i - \sum_{i=1}^{n} k_i.$$

□

Note that this index, of course, must equal the Euler characteristic of the surface.

2.2 Elliptic Functions

In this section, we shall give a brief resumé of the theory of elliptic functions which we need in the next section. All the results are well-known and can be found, for example in [Jones and Singerman, 1987].

Definition 2.2.1. *A meromorphic function* $f : \mathbb{C} \to \sum$ *($\sum$ denotes the Riemann sphere) is* **elliptic** *with respect to a lattice* $\Omega \subseteq \mathbb{C}$ *(i.e.* $\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$*,* $\omega_1, \omega_2 \in \mathbb{C}$ *where* ω_1/ω_2 *is not real) if* f *is doubly periodic with respect to* Ω*, i.e.*

$$f(z + \omega) = f(z), \ \forall \ z \in \mathbb{C}, \ \omega \in \Omega.$$

The *order* $\mathrm{ord}(f)$ of an elliptic function f is the sum of the orders of the poles of f in a fundamental parallelogram of the lattice. Note that $\mathrm{ord}(f) = 0$ if and only if f is constant, so a nonconstant elliptic function must have at least one pole. Let F be a fundamental parallelogram of the lattice of the elliptic function f.

Theorem 2.2.1. *i) The sum of the residues of* f *in* F *is zero.*

ii) If f *has order* $N > 0$*, then* f *takes every value* $c \in \sum$ *exactly* N *times.* □

Elliptic functions are uniquely determined (modulo a constant) by their zeros and poles. Given an arbitrary set of points in the fundamental parallelogram, we can find an elliptic function with poles and zeros of given orders at these points under certain conditions. Thus we have

Theorem 2.2.2. *Let $a_1, \cdots, a_r, b_1, \cdots, b_s$ be distinct points in F. Then there exists an elliptic function with zeros of orders $k_1, \cdots, k_r$ at $a_1, \cdots, a_r$, respectively, and poles of orders $l_1, \cdots, l_s$ at $b_1, \cdots, b_s$ if and only if*

i) $k_1 + \cdots + k_r = l_1 + \cdots + l_s$

ii) $(\sum k_j a_j \sim \sum l_j b_j) \bmod \Omega$. □

The most important elliptic function is the Weierstrass $\wp$ function, defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where $'$ on the summation means that we omit the origin from the lattice. Note that $\wp$ has order 2. Another important elliptic function is the Weierstrass σ function, defined by

$$\sigma(z) = z \cdot \prod_{\omega \in \Omega}{}' \left\{ \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2 \right) \right\}.$$

In order to construct an elliptic function which has poles and zeros as in theorem 2.2.2, we can use the σ-function and define

$$f(z) = \frac{\prod_{i=1}^{r} \prod_{j=1}^{k_i} \left(\sigma(z-a_i)\right)^j}{\prod_{i=1}^{s} \prod_{j=1}^{l_i} \left(\sigma(z-b_i)\right)^j}, \tag{2.4}$$

and then f will be the required elliptic function.

It is also possible to construct an elliptic function which has given principal parts at some distinct points $b_1, \cdots, b_s$ in the fundamental parallelogram F, provided they satisfy the condition of theorem 2.2.1 such that the sum of the residues is zero. Thus, let the principal part at b_j be given by

$$\sum_{k=1}^{l_i} \frac{a_{k,j}}{(z-b_j)^k},$$

and assume that

$$\sum_{j=1}^{s} a_{1,j} = 0.$$

Then we can form an elliptic function with these principal parts at $b_1, \cdots, b_s$. In order to do this, define the functions

$$\begin{aligned} F_k(z) &= \sum_{\omega} (z-\omega)^{-k}, \ k \geq 3 \\ F_2 &= \wp \end{aligned}$$

and

$$F_1(z) = \zeta(z) \triangleq \frac{1}{z} + \sum_{\omega\in\Omega}{}' \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Then we write

$$f(z) = \sum_{j=1}^{s} \sum_{k=1}^{l_j} a_{k,j} F_k(z-b_j)$$

and this is the required function. If V denotes the vector space of elliptic functions which are analytic on $F \backslash \{b_1, \cdots, b_s\}$ and are analytic or have poles of order $\leq l_j$ at b_j, then

$$\dim V = l_1 + \cdots + l_s, \tag{2.5}$$

(as a vector space over $\mathbb{C}$). (This is a simple version of the Riemann-Roch theorem.)

Note that finally the $\wp$ function generates all elliptic functions in the sense that if f is any elliptic function, then

$$f = R_1(\wp) + \wp' R_2(\wp)$$

for some rational functions R_1, R_2.

2.3 Dynamical Systems on the Torus

We now consider the dynamical systems which can be defined on a torus by the equation

$$\dot{z} = E(z)$$

where E is an elliptic function. The simplest elliptic function is, of course, a constant and the equation is $\dot{z} = c$ ($c \in \mathbb{C}$), and if $z = x + iy$,

$$\begin{cases} \dot{x} = \mathrm{Re}(c) \\ \dot{y} = \mathrm{Im}(c). \end{cases}$$

This is the well-known system which generates (p,q)-knots if $\mathrm{Re}(c)/\mathrm{Im}(c) = p/q$ (where p, q are positive integers) and has a dense orbit if $\mathrm{Re}(c)/\mathrm{Im}(c)$ is irrational.

In the following discussion, we shall choose a rectangular lattice in order to generate the elliptic functions, i.e.

$$\Omega = \{k + li : k, l \in \mathbb{Z}\}.$$

Since we are only interested in the topological structure of systems on torus, there is no loss of generality. (Of course, different lattices may generate conformally distinctly tori.)

Now consider the following question: Suppose we specify a (finite) set of distinct points $\{P_1, \cdots, P_K\}$ on a torus, which are to be the equilibria of a dynamical system and suppose that each point is surrounded by only one type of sectors (i.e. parabolic, hyperbolic or elliptic), e.g.

$$(A)\begin{cases} P_1, \cdots, P_{k_1} \text{ are each surrounded by 1 parabolic sector} \\ P_i \text{ is surrounded by } e_i \text{ elliptic sectors, for } k_1+1 \le i \le k_2 \\ P_i \text{ is surrounded by } h_i \text{ hyperbolic sectors, for } k_2+1 \le i \le K. \end{cases}$$

Then we ask - when does there exist a dynamical system on the torus with these equilibria? Recall that if any given equilibrium is surrounded by e elliptic sectors and h hyperbolic sectors, then the index of that point is given by $1 + \frac{(e-h)}{2}$. Hence we see from Poincaré's index theorem that a necessary condition for the above problem is that

$$k_1 + \frac{1}{2}\sum_{i=k_1+1}^{k_2} e_i - \frac{1}{2}\sum_{i=k_2+1}^{K} h_i = 0 \tag{2.6}$$

(since the Euler characteristic of the torus is 0).

It turns out that this condition is also sufficient, provided the points $\{P_1, \cdots, P_K\}$ can be positioned to satisfy condition (ii) of theorem 2.2.2. Thus, we have

Theorem 2.3.1. *Suppose that we choose distinct points $\{P_1, \cdots, P_K\}$ on a torus such that we may find a set of points $\{q_1, \cdots, q_K\}$ in the fundamental parallelogram in $\mathbb{C}$ for which*

$$\left(\sum_{i=1}^{k_1} q_i + \sum_{i=k_1+1}^{k_2}\left(\frac{e_i}{2}+1\right)q_i \sim \sum_{i=k_2+1}^{K}\left(\frac{h_i}{2}-1\right)q_i\right) \bmod \Omega$$

Where e_i, h_i and K satisfy (2.6). Then there exists a dynamical systems on the torus which satisfies condition (A) above. Moreover, if these conditions are satisfied, then the system can be realized by the equation

$$\dot{z} = E(z) \tag{2.7}$$

where E is an elliptic function.

Proof. The only thing left to prove is condition (i) in theorem 2.2.2. However, this is equivalent to (2.6), since by the results of section 2.1, each pole of order k is surrounded by $2(k+1)$ hyperbolic sectors and each zero of order k is surrounded by $2(k-1)$ elliptic sectors (or single parabolic sector if $k=1$). Hence condition (i) in theorem 2.2.2 becomes

$$k_1 + \sum_{i=k_1+1}^{k_2} \left(\frac{e_i}{2}+1\right) = \sum_{i=k_2+1}^{K} \left(\frac{h_i}{2}-1\right),$$

which is precisely (2.6). □

Corollary 2.3.1. *Suppose that a system on the torus (defined by an equation of the form (2.7) on a fundamental parallelogram F in $\mathbb{C}$) is analytic on $F\backslash\{p_1,\cdots,p_s\}$ and the point p_i is surrounded by $\leq h_i$ elliptic sectors (where h_i is even). Then the number of topologically independent systems on the torus of this type is given by*

$$\left(\frac{h_i}{2}-1\right).$$

Proof. We simply use the Riemann-Roch theorem in the form of (2.5). □

We can construct a system given in theorem 2.3.1 explicitly by using (2.4). Thus, given the data in condition (A), we obtain the system (assuming the conditions of theorem 2.3.1 are satisfied):

$$\dot{z} = \frac{\prod_{i=1}^{k_2}\prod_{j=1}^{e_i/2+1}\left(\sigma(z-p_i)\right)^j}{\prod_{i=k_2+1}^{K}\prod_{j=1}^{h_i/2-1}\left(\sigma(z-p_i)\right)^j}. \tag{2.8}$$

Note that the parabolic sectors are included in the elliptic sectors, in which case we take $e_i = 0$.

Example 1. Consider the system

$$\dot{z} = \wp(z),$$

i.e. where the vector field is given by the Weierstrass $\wp$ function. The dynamics are shown in fig. 2.3. Note that there is one zero and one pole in each fundamental parallelogram. The zero has two elliptic sectors (and two parabolic sectors), while the pole has four hyperbolic sectors. The total index is 0 as expected (i.e. the Euler characteristic of the torus). Note that the dynamics is periodic and so wraps onto a system on the torus. When we study systems on higher genus surfaces, it will be seen that we cannot simply use the obvious generalizations of elliptic functions (i.e. automorphic functions) to generate systems on these surfaces. This is because hyperbolic geometry is not the same as Euclidean geometry.

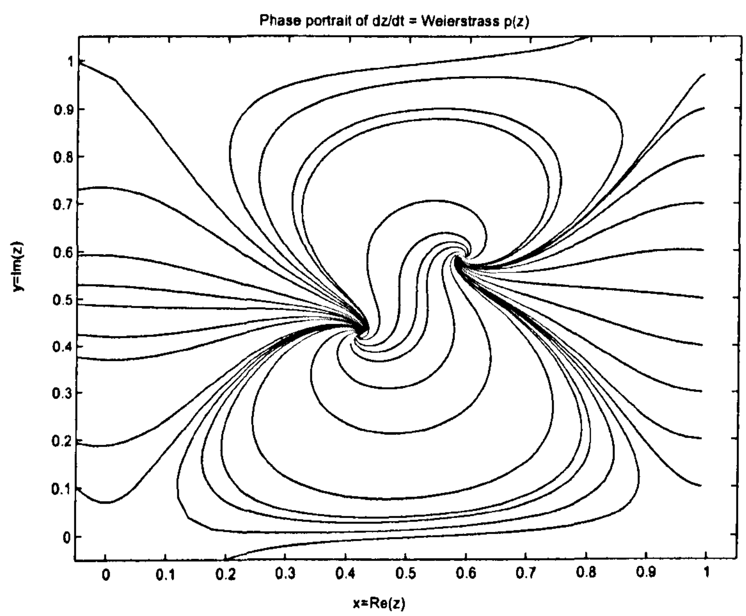


Figure 2.3: Phase portrait of the Weierstrass $\wp$ function.

Example 2. Consider now the following system of the form (2.8):

$$\dot{z} = \frac{\sigma(z - 0.25 - 0.25i)\sigma(z - 0.75 - 0.75i)}{\left(\sigma(z - 0.5 - 0.5i)\right)^2}$$

This system has two simple zeros and a pole of order 2. The phase trajectories are shown in fig. 2.4.

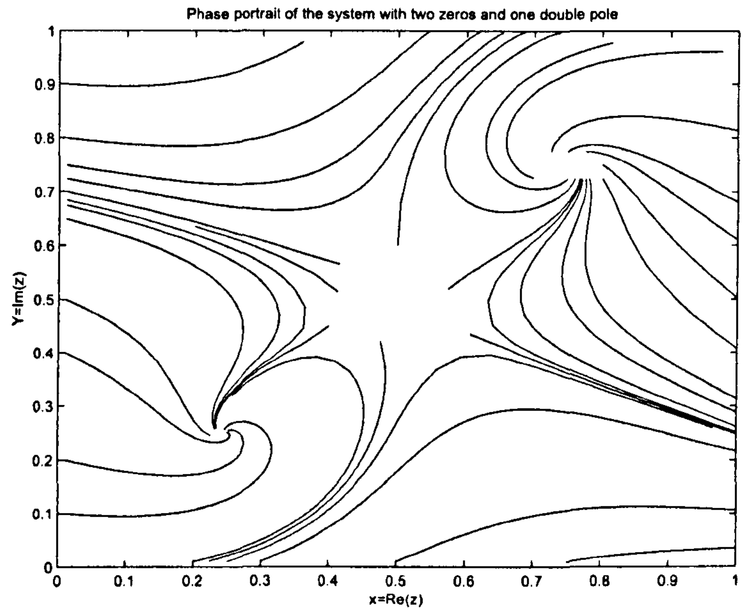


Figure 2.4: Phase portrait for a more general elliptic system.

This system has two spiral equilibria of total index 2 and an equilibrium at $0.5 + 0.5i$ surrounded by six hyperbolic sectors, which has index -2, as expected.

2.4 Automorphic Systems on Surfaces of Higher Genus

In this section we shall consider systems on surfaces of genus ≥ 2 and to do this we shall first give a brief summary of the theory of automorphic functions (essentially analytic functions, apart from isolated poles, defined on Riemann surfaces). More details can be found in [Ford, 1929; Jones and Singerman, 1987; Shimura, 1971].

We start with automorphisms of Riemann sphere Σ. These are precisely the linear fractional transformations of the form

$$T(z) = \frac{az+b}{cz+d}, \tag{2.9}$$

where $a, b, c, d \in \mathbb{C}$, $ad-bc \neq 0$. There is a group homomorphism $\mathrm{GL}(2,\mathbb{C}) \to \mathrm{Aut}(\Sigma)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to T,$$

which becomes an isomorphism on the projective special linear group $\mathrm{PSL}(2,\mathbb{C})$ (those elements of $\mathrm{GL}(2,\mathbb{C})$ of positive determinant modulo the scalar matrices).

If we restrict attention to the upper half plane, then the automorphism group is $\mathrm{PSL}(2,\mathbb{R})$ (linear fractional transformations with real coefficients). If T is a map of the form (2.9) where $a, b, c, d \in \mathbb{R}$, we classify it as follows:

i) If $|a+d| = 2$, T is **parabolic**.

ii) If $|a+d| > 2$, T is **hyperbolic**.

iii) If $|a+d| < 2$, T is **elliptic**.

We can always normalize a transformation T as follows:

$$T(z) = \frac{\frac{a}{\sqrt{\Delta}}z + \frac{b}{\sqrt{\Delta}}}{\frac{c}{\sqrt{\Delta}}z + \frac{d}{\sqrt{\Delta}}}$$

where $\Delta = ad - bc$, so that

$$\frac{a}{\sqrt{\Delta}} \cdot \frac{d}{\sqrt{\Delta}} - \frac{b}{\sqrt{\Delta}} \cdot \frac{c}{\sqrt{\Delta}} = \frac{ad-bc}{\Delta} = 1.$$

Hence without loss of generality, we can assume that

$$ad - bc = 1.$$

Note that the fixed point of T are given by

$$cz^2 + (d-a)z - b = 0,$$

if $ad - bc = 1$, then the discriminant becomes $(a+b)^2 - 4$ so that T has a single fixed point if and only if

$$(a+d)^2 = 4,$$

i.e. if and only if T is parabolic.

We shall be interested in transformations which map some region into a congruent disjoint region in hyperbolic space. We can use either of the standard models of this space (i.e. the (open) upper half plane U or the (open) unit disk D). Groups of transformations which operate like this are called *Fuchsian groups* and generalize the translation group in the case of elliptic functions. More precisely, we say that a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group. Any Fuchsian group F is properly discontinuous in the sense that if $T \in F$, $y \in U$ and V is a neighbourhood of y such that $T(V) \cap V \neq \emptyset$, then $T(y) = y$. If $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group, then the orbit $\Gamma(z)$ of any $z \in U$ is discrete. Hence any limit points must occur on the boundary of U, i.e. $\mathbb{R} \cup \{\infty\}$.

The importance of Fuchsian groups in the theory of Riemann surfaces is that they define a fundamental region of U (resp. D) which is congruent to all its transformations under the group and which, together with these congruent regions, tessellates U (resp. D). Formally, a fundamental region F in U for the group Γ is a closed set such that

i) $\bigcup_{T\in\,\Gamma} T(F) = U$

ii) $\mathring{F} \cap T(\mathring{F}) = \emptyset,\ \forall T \in \Gamma\backslash\{I\}$.

Usually we require F to be connected. The important result for us is

Theorem 2.4.1. (See [Jones and Singerman, 1987])

(a) Let Γ *be a Fuchsian group and let* $\Gamma_1 = \{T_i\}$ *be the subset of* Γ *containing maps which pair the sides of a given fundamental region of* Γ*. Then* Γ_1 *generates the group* Γ.

(b) U/Γ *is a connected Riemann surface and* $\pi : U \to U/\Gamma$ *(the canonical projection) is a holomorphic map.*

(c) If Γ_1 *and* Γ_2 *are Fuchsian groups without elliptic elements, then* U/Γ_1 *and* U/Γ_2 *are conformally equivalent if and only if there exists* $T \in \mathrm{PSL}(2, \mathbb{R})$ *such that* $T\,\Gamma_1 T^{-1} = \Gamma_2$. □

Another useful result relates to the compactness of U/Γ.

Theorem 2.4.2. (See [Jones and Singerman, 1987])

(a) U/Γ *is a compact Riemann surface if and only if there exists a compact fundamental region for* Γ.

(b) If U/Γ *is compact then* Γ *contains no parabolic elements.* □

Example. To generate the noncompact Riemann surface in fig. 2.5 (of genus 1) would require a parabolic transformation.

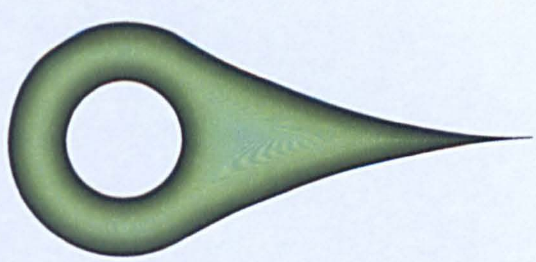


Figure 2.5: A noncompact surface of genus 1.

In order to generalize the earlier results on systems defined on tori by elliptic functions, we need their generalization to higher genus surfaces. These are functions invariant under the elements of some Fuchsian group Γ called automorphic functions. These functions 'uniformise' the Riemann surface generated by Γ, in the sense that they define meromorphic functions on U (or D) which can be used to introduce dynamical systems on the surface.

An *automorphic function* A for the Fuchsian group Γ is a meromorphic function defined on U such that

$$A(\gamma(z)) = A(z)$$

for all $\gamma \in \Gamma$ and all $z \in U$. It is tempting to generalize the elliptic (torus) case by introducing dynamical systems of the form

$$\dot{z} = A(z)$$

where A is an automorphic function. However, it can be seen that the dynamics (trajectories) generated by these systems in U are not Γ-invariant and so these systems will not 'wrap-up' into systems on higher genus surfaces. In order to generate systems $\dot{z} = f(z)$ with Γ-invariant trajectories, we require the following invariance of the vector field f.

Lemma 2.4.1. *The system*

$$\dot{z} = f(z)$$

will have Γ-invariant trajectories for any given Fuchsian group Γ, if

$$f(z) = \frac{dT^{-1}}{dz}(Tz) \cdot f(Tz), \quad \forall T \in \Gamma \tag{2.10}$$

Proof. Clearly we require the 'ends' of infinitesimal vectors in the direction of $f(z)$ to map appropriately under Γ (see fig. 2.6).

Hence we require

$$T\big(z + \varepsilon f(z)\big) = Tz + \varepsilon f(Tz)$$

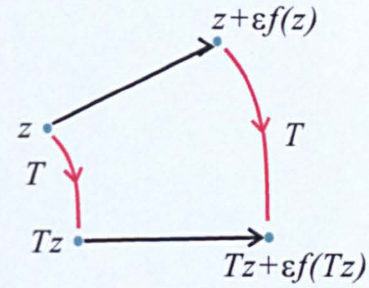


Figure 2.6: Mapping a vector field under any $T \in \Gamma$.

for sufficiently small ε. Thus,

$$f(z) = \frac{T^{-1}\big(Tz + \varepsilon f(Tz)\big) - z}{\varepsilon}$$

and the result follows from the chain rule by letting $\varepsilon \to 0$. □

If

$$T(z) = \frac{az+b}{cz+d},$$

then

$$T^{-1}(z) = \frac{dz-b}{cz-a}$$

and so

$$\frac{dT^{-1}}{dz}\big(T(z)\big) = \frac{(cz+d)^2}{ad-bc}.$$

Hence, for such a map $T \in \Gamma$, the invariance of f given by (2.10) may be written in the form

$$F(Tz) = \frac{ad-bc}{(cz+d)^2}F(z). \tag{2.11}$$

This differs from the usual invariance

$$A(Tz) = A(z), \quad T \in \Gamma$$

for any automorphic function, so we shall generate functions (vector fields) F satisfying (2.10) by modifying the usual θ-series which Poincaré used (see [Ford, 1929]) to generate automorphic functions for infinite Fuchsian groups.

Definition 2.4.1. *If H is a rational function, which has poles away from the limit points of the Fuchsian group Γ, the* ***theta series*** *(of type m) is given by*

$$\theta(z) = \sum_{i=0}^{\infty}(c_i z + d_i)^{-2m} H(z_i),$$

where $z \in U$, $m \in \mathbb{N}$, $T_0 = I, T_1, T_2, \cdots$ is an enumeration of the elements of Γ, and

$$z_i = T_i z = \frac{a_i z + b_i}{c_i z + d_i}.$$

It is easy to check that

$$\theta(z_i) = (c_i z + d_i)^{2m}\theta(z)$$

for each i, and so if we define a function

$$F(z) = \frac{\theta_1(z)}{\theta_2(z)},$$

where θ_1, θ_2 are theta series of the same type m, then

$$F(z_i) = F(z)$$

for each i, i.e. F is an automorphic function.

In the case of dynamical systems, we shall modify the definition of an automorphic function and instead use the function

$$\tilde{\theta}_1(z) = \sum \frac{(c_i z + d_i)^{2-2m}}{(a_i d_i - b_i c_i)} H_i(T_i(z))$$

and the usual theta series

$$\theta_2(z) = \sum (c_i z + d_i)^{-2m} H_2(T_i(z)).$$

Lemma 2.4.2. *The function*

$$F(z) = \frac{\tilde{\theta}_1(z)}{\theta_2(z)}$$

satisfies

$$F(T_i z) = \frac{a_i d_i - b_i c_i}{(c_i z + d_i)^2} F(z)$$

for each i and so defines a Γ invariant vector field if $m \geq 3$.

Proof. Since θ_2 is a normal theta series, we have

$$\theta_2(T_j(z)) = (c_j z + d_j)^{2m} \theta_2(z),$$

for any j. For $\tilde{\theta}_1$ we have

$$\begin{aligned}
\tilde{\theta}_1(T_j(z)) &= \sum_i \frac{1}{\left[c_i\left(\frac{a_j z + b_j}{c_j z + d_j}\right) + d_i\right]^{2m-2} (a_i d_i - b_i c_i)} H_1(T_i T_j(z)) \\
&= \sum_i \frac{(c_j z + d_j)^{2m-2}}{\left[(c_i a_j + d_i c_j) z + c_i b_j + d_i d_j\right]^{2m-2} (a_i d_i - b_i c_i)} H_1(T_i T_j(z)) \\
&= (c_j \ + d_j)^{2m-2} \\
&\quad \cdot \sum_i \frac{1}{\left[(c_i a_j + d_i c_j) z + c_i b_j + d_i d_j\right]^{2m-2} (a_i d_i - b_i c_i)} H_1(T_i T_j(z)) \\
&= (c_j \ + d_j)^{2m-2} (a_j d_j - b_j c_j) \\
&\quad \cdot \sum_i \frac{H_1(T_i T_j(z))}{\left[(c_i a_j + d_i c_j) z + c_i b_j + d_i d_j\right]^{2m-2} (a_i d_i - b_i c_i)(a_j d_j - b_j c_j)} \\
&= (c_j \ + d_j)^{2m-2} (a_j d_j - b_j c_j) \tilde{\theta}_1(z),
\end{aligned}$$

since

$$T_iT_j(z) = \frac{a_i\frac{a_j\ +b_j}{c_jz+d_j} + b_i}{c_i\frac{a_j\ +b_j}{c_jz+d_j} + d_i} = \frac{(a_ia_j + b_ic_j)z + (a_ib_j + b_id_j)}{(c_ia_j + d_ic_j)z + (c_ib_j + d_id_j)}$$

and

$$\begin{aligned}\det\big(T_iT_j(z)\big) &= (a_ia_j + b_ic_j)\cdot(c_ib_j + d_id_j) - (a_ib_j + b_id_j)\cdot(c_ia_j + d_ic_j)\\ &= (a_jd_j - b_jc_j)(a_id_i - b_ic_i).\end{aligned}$$

Hence,

$$\begin{aligned}F(z) &= \frac{\tilde{\theta}_1(z)}{\theta_2(z)} = \frac{\frac{1}{(c_jz+d_j)^{2m-2}\cdot(a_jd_j-b_jc_j)}\tilde{\theta}_1\big(T_j(z)\big)}{\frac{1}{(c_jz+d_j)^{2m}}\theta_2\big(T_j(z)\big)}\\ &= \frac{(c_jz+d_j)^2}{a_jd_j - b_jc_j}F\big(T_j(z)\big)\\ &= (T_j^{-1})'[T_j(z)]\cdot F\big(T_j(z)\big).\end{aligned}$$

The result now follows. □

Definition 2.4.2. *We shall call a meromorphic, complex-valued function F, which satisfies (2.11) for each transformation T in a given Fuchsian group Γ, an* ***automorphic vector field*** *on U.*

We have seen that such functions F give rise to dynamical systems on the Riemann surface of the Fuchsian group Γ of the form

$$\dot{z} = F(z)$$

in the sense that the trajectories of this system are Γ-invariant on any fundamental region of the group in $U \subseteq \mathbb{C}$ and so 'wrap up' into a dynamical system on the surface. We shall now give three examples to illustrate the theory developed above. Clearly, for each choice of the functions H_1, H_2 and the Fuchsian group Γ we will obtain a dynamical system, although they may not all be topologically distinct.

Example 1. We shall give an example of a system defined on the fundamental region (in U) shown in fig. 2.7, which has Fuchsian group generated by the transformations

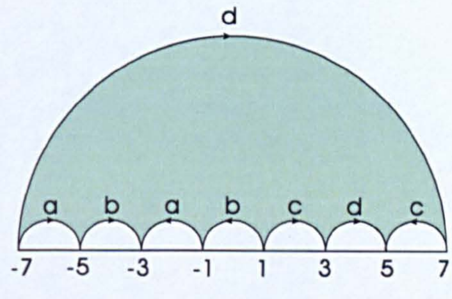


Figure 2.7: A fundamental region in U.

$$\begin{cases} T_1(z) &=& -\frac{2z+13}{z+6} \\ T_2(z) &=& -\frac{1}{z+4} \\ T_3(z) &=& \frac{6z-13}{z-2} \\ T_4(z) &=& 7z-28. \end{cases} \tag{2.12}$$

Choosing

$$H_1(z) = \frac{1}{z+2-3i},\ H_2(z) = \frac{1}{z-2-3i},$$

we obtain a system with a pole at $-2+3i$ and a zero at $2+3i$. The vector field $F(z)$ generated by these functions will have singularities at these points. However, it will also have other poles and zeros introduced by the modified theta series which, together with those at $-2+3i$ and $2+3i$, satisfy the Poincaré index theorem. The Riemann surface in this case is, of course, a two-holed torus (surface of genus 2). Fig. 2.8 shows the solution trajectories of the system (computed in MAPLE).

Example 2. (Hamiltonian Systems) It is well-known (see [Abraham and Marsden, 1978]) that a Hamiltonian system in two dimensions may be written in the form

$$\dot{z} = -2i\frac{\partial H}{\partial \bar{z}}.$$

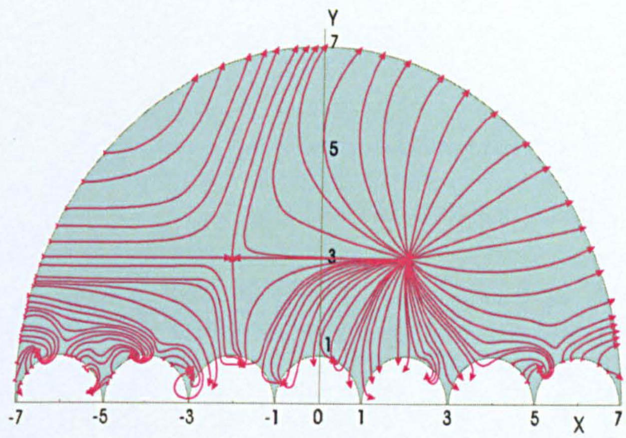


Figure 2.8: The solution trajectories for the system $\dot{z} = F(z)$, where F is generated by H_1, H_2.

This is equivalent to the system

$$\begin{cases} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q}, \end{cases}$$

if we set $z = q + ip$. Clearly, the only analytic Hamiltonian system on a surface is constant, since the system Hamiltonian is just a function of z (not $\bar{z}$). By taking the Hamiltonian

$$H(\bar{z}) = F(\bar{z})$$

where $F(z)$ is the function generated in example 1, we obtain the Hamiltonian system

$$\dot{z} = -2i\frac{\partial F(\bar{z})}{\partial \bar{z}}$$

on a surface of genus 2. Similarly, any function F generated from a Fuchsian group as above gives rise to a Hamiltonian system in this way. Note that it cannot be integrable, by Liouville's theorem.

Example 3. (Bifurcating Systems) Consider the modified fundamental region of the one in example 1, where the fundamental arcs a and b are replaced by semicircles of radius ε, as in fig. 2.9.

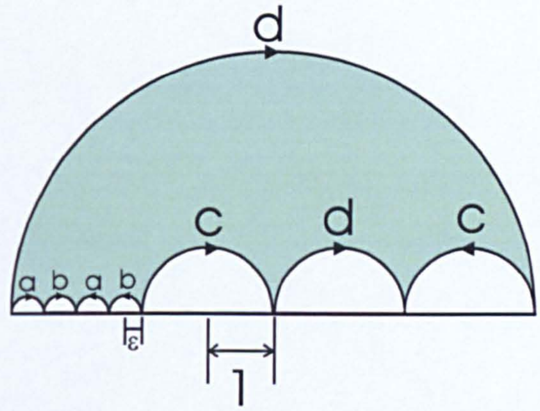


Figure 2.9: A modified fundamental region.

We can define systems on the corresponding surface as in example 1. Suppose that

$$\dot{z} = F(\varepsilon, z)$$

is such a system, depending on ε. Clearly, we will again obtain a parameterized set of systems defined on surfaces of genus 2. As $\varepsilon \to 0$ the fundamental region becomes that shown in fig. 2.10. In this case the system will bifurcate into one defined on a

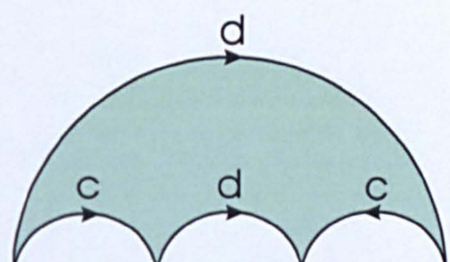


Figure 2.10: The fundamental region of fig. 2.7 as $\varepsilon \to 0$.

torus:

$$\dot{z} = F(0, z)$$

and so by the results of the last section F must be conformally equivalent to an elliptic function.

2.5 Connected Sums

Connected sums of 2- and 3-manifolds provide an efficient method of generating 'complicated' manifolds out of simpler ones. In this section we shall consider its effect to dynamical systems on 2-manifolds. Our main technical tools will be Poincaré -Hopf index theorem and the flow-box theorem. The latter may be stated as follows:

Theorem 2.5.1. *Let ϕ_t denote a dynamical system on a manifold M of dimension n. If $x \in M$ is not an equilibrium point (*i.e. $\phi_t(x) \neq x, \; \forall t \neq 0$*), then there exists a (closed) local coordinate neighbourhood U of x such that on U, ϕ_t is topologically conjugate to the dynamical system*

$$\left.\begin{aligned} \dot{x}_1 &= c \\ \dot{x}_2 &= 0 \\ &\vdots \\ \dot{x}_n &= 0 \end{aligned}\right\} \qquad x \in \{0 \leq x_i \leq 1, 1 \leq i \leq n\}$$

where c is a constant. □

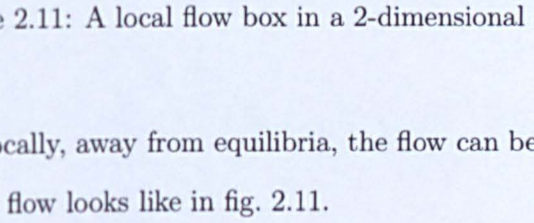
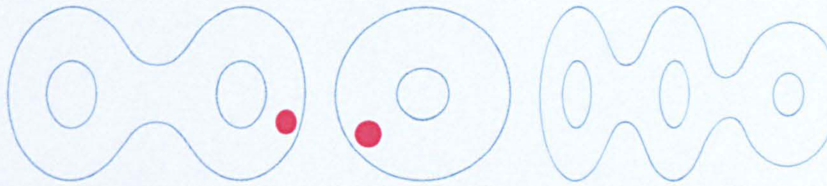


Figure 2.11: A local flow box in a 2-dimensional surface.

This says that locally, away from equilibria, the flow can be 'parallelized', e.g. in two dimensions, the flow looks like in fig. 2.11.

Given any two 2-manifolds S_1 and S_2, we define their connected sum $S_1\#S_2$ as the two-manifold obtained by cutting out disks from S_1 and S_2 and sewing together their boundaries (see fig. 2.12).



(a) S_1 (b) S_2 (c) $S_1\#S_2$

Figure 2.12: Connected sum of S_1 and S_2.

Lemma 2.5.1. *Let S_1 and S_2 be two surfaces on which dynamical systems ϕ_1 and ϕ_2 are defined. If we form the connected sum by removing discs D_1, D_2 from S_1 and S_2, where D_1 and D_2 contain no critical points, then we must introduce two hyperbolic equilibria (with index -1) on the disc boundaries.*

Proof. Since there are no equilibrium points in the discs being removed, we can find flow boxes F_1 and F_2 in S_1 and S_2, respectively, so that

$$D_i \subset F_i, \qquad i = 1, 2$$

provided D_1, D_2 are small enough. The discs can be chosen such that there are two distinct trajectories which are tangent to the discs at two points (see fig. 2.13).

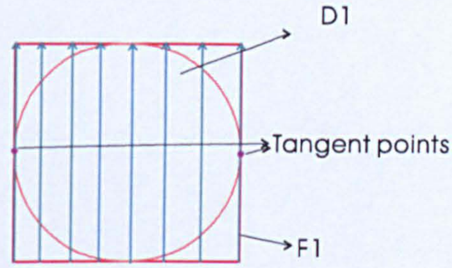


Figure 2.13: Flow box F_1 containing disc D_1.

If we now pull out tubes to form the connected sum S, these two points clearly become singular points as in fig. 2.14.

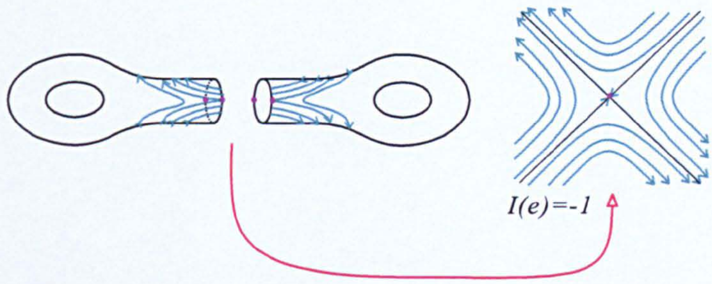


Figure 2.14: Introduction of singular points: Case I.

Suppose that one surface, say S_2, is a sphere. In this case, since $S = S_1 \# S_2 = S_1$ and $\chi(S_2) = 2$, the total index of the two singular points on the removed discs must be -2. Hence the result follows. □

Suppose next that the removed discs contain critical points. Note that we can always make D_i small enough such that it contains only one equilibrium p_i $(i = 1, 2)$. There are several situations that we need to consider.

Lemma 2.5.2. *Construct the connected sum of two surfaces S_1 and S_2, where each of them has a dynamical system situated. Suppose the removed discs D_1, D_2 contain one critical point each, p_1 and p_2. If the construction does not introduce new equilibria, then p_1 and p_2 must be 'dual' in the sense that if one equilibrium point has n_1 elliptic sectors and n_2 hyperbolic sectors, then the other must consist of precisely n_1 hyperbolic sectors and n_2 elliptic sectors.*

Proof. Suppose the genuses for S_1 and S_2 are g_1, g_2, respectively. Denote $S = (S_1 \# S_2)$, then S is a surface of genus $(g_1 + g_2)$. Hence by Poincaré index theorem, we have

$$\begin{cases} I(p_1) + I(r_{S_1}) &= 2(1-g_1) &= \chi(S_1) \\ I(p_2) + I(r_{S_2}) &= 2(1-g_2) &= \chi(S_2) \\ I(r_{S_1}) + I(r_{S_2}) &= 2(1-g_1-g_2) &= \chi(S) \end{cases}, \tag{2.13}$$

where r_{S_1} and r_{S_2} are the remaining equilibria on S_1 and S_2 respectively. Suppose

$$I(p_1) = 1 + \frac{n_1 - n_2}{2},$$

from (2.13) we have

$$I(p_2) = 2 - I(p_1) = 1 + \frac{n_2 - n_1}{2}.$$

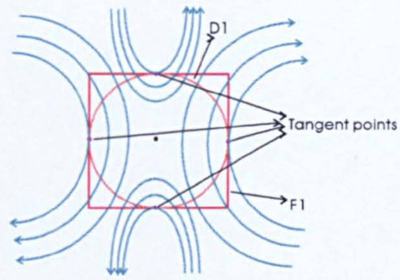
Hence the result follows. □

Clearly, after removing discs containing equilibria and gluing the rest together, we may introduce extra critical points on the disc boundaries. Stick to the notations in lemma 2.5.2, we have

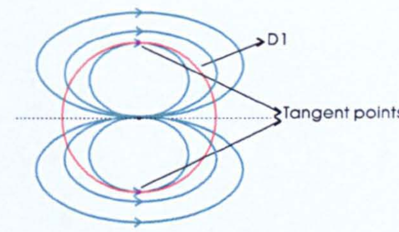
Lemma 2.5.3. *Suppose the structures of p_1 and p_2 are exactly the same (i.e., p_1 and p_2 both have n_1 elliptic sectors and n_2 hyperbolic sectors). If the construction of connected sums introduces new equilibria, then among them, there must be n_1 elliptic points (with index $+1$) and n_2 hyperbolic ones (with index -1).*

Proof. First we look at the case of hyperbolic equilibria(with index -1). It has 4 hyperbolic sectors. The removed discs D_1 and D_2 can be chosen such that there are exactly four trajectories tangent to the discs at four distinct points, as shown in fig. 2.15a. The same argument in the proof of lemma 2.5.1 applies here. Referring to fig. 2.14, one hyperbolic sector generates one hyperbolic point (with index -1) after the gluing. Since p_i has n_2 hyperbolic sectors, we end up with n_2 hyperbolic points after connected sum construction.

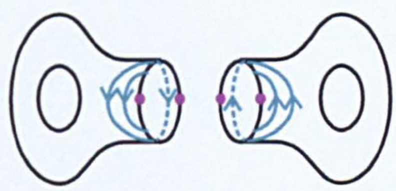
Next consider the elliptic sectors. As illustrated in fig. 2.15b, there exist two pairs of closed trajectories which are tangent to discs D_1 and D_2, respectively. By



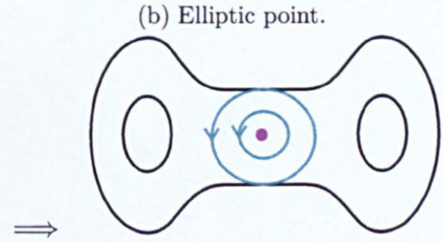
(a) Hyperbolic point.



(b) Elliptic point.



(c) Introduction of equilibria for elliptic points.



(d) Cycles around the introduced singular point.

Figure 2.15: Introduction of singular points: Case II.

pulling out the tubes to form the connected sum S, the corresponding pairs of the four tangent points are identified to yield two elliptic points which contain cycles only (see fig. 2.15c and 2.15d).

Still, we assume the genuses for S_1 and S_2 are g_1, g_2, respectively. Denote $S = (S_1 \# S_2)$, such that

$$\chi(S) = 2(1 - g_1 - g_2).$$

Also we have

$$I(p_1) = I(p_2) = 1 + \frac{n_1}{2}.$$

Suppose there are n introduced elliptic points, then

$$2(1-g_1) - (1 + \frac{n_1}{2}) + 2(1-g_2) - (1 + \frac{n_1}{2}) + n = 2(1 - g_1 - g_2),$$

which yields $n = n_1$. So there are exactly n_1 new elliptic points introduced in S. □

Certainly, the structure of p_1 can be different from that of p_2 even if there are extra equilibria introduced.

Theorem 2.5.2. *Let $D_i \subset S_i$ $(i = 1, 2)$ be two discs, $p_i \in D_i$ be two critical points, and ϕ_i be the dynamical system situated on S_i. Then it is possible to apply the connected sum operation, $S = S_1 \# S_2$, such that the dynamics given by ϕ_i match up if and only if there exists a separation to the sectors in p_1, i.e. n_{11} elliptic and n_{21} hyperbolic sectors share the same structure as those in p_2, while the rest in p_1 are 'dual' to the remaining in p_2.*

Proof. The proof follows from those of lemma 2.5.2 and 2.5.3, because these two are the only conditions that can happen to the surface dynamics when performing the connected sum, and they will match up in the following way: For p_1, split n_1 elliptic sectors to n_{11} and n_{12}, n_2 hyperbolic ones to n_{21} and n_{22}. Combine n_{11} and n_{21} sectors to the same structure on p_2, while glue n_{12} and n_{22} sectors to their 'dual', respectively.

The index also match up because in this case,

$$\begin{cases} I(p_1) &= 1 + \frac{n_{11}+n_{12}-n_{21}-n_{22}}{2} \\ I(p_2) &= 1 + \frac{n_{11}+n_{22}-n_{21}-n_{12}}{2} \end{cases}, \tag{2.14}$$

hence, we have

$$2(1-g_1) - I(p_1)) + 2(1-g_2) - I(p_2) + n_{11} - n_{21} = 2(1-g_1-g_2).$$

□

2.6 2-D Systems Containing Strange Attractors

2.6.1 General Systems on Genus-p Surfaces

In [Martins, 2004], invariant sets for 2-dimensional dynamical systems containing inversely unstable solutions were studied in the torus case. It is then natural to ask what will happen in the higher genus case?

In order to extend Matins' result, we first need to define dynamical system which contain periodic orbits. In section 2.4 we have shown how to write down analytic (or meromorphic) systems on genus-p surfaces by the use of automorphic functions. However, these systems are not general enough to include systems with knots, chaotic annuli, etc. So we must consider vector fields which are C^∞ but which are invariant under certain linear, fractional transformations. This will be the analogue of systems which are periodic in [Martins, 2004] and have inversely unstable periodic motions - the latter now becoming knots on the genus-p surface.

In order to generate the most general C^∞ systems, consider a fundamental domain F in the upper-half plane model of hyperbolic metric for the surface, then we have

Lemma 2.6.1. *There exists a map from F onto a rectangle R which is one-to-one on the interior and C^∞ apart from at the cusp points.*

Proof. We shall construct the map explicitly so that the required properties will be clear.

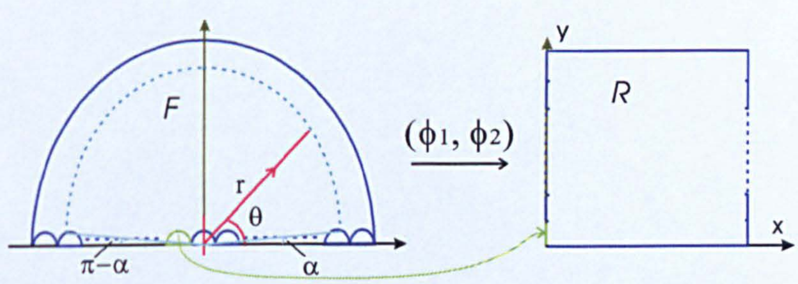


Figure 2.16: Mapping the fundamental region F onto a rectangle R

An elementary calculation shows that

$$\begin{cases} x = \phi_1(r, \theta) \\ y = \phi_2(r, \theta) \end{cases}$$

where

$$\begin{cases} \phi_1(r,\theta) &= \frac{\pi}{\pi-2\alpha}\cdot(\theta-\alpha) \\ \phi_2(r,\theta) &= r \end{cases}, \tag{2.15}$$

where α is the value of the starting angle corresponding to the curve within the fundamental region in the (r,θ)-plane (as shown in fig (2.16)). □

We shall call R the modified fundamental region, and write this map as $\phi : (r,\theta) \to (x,y)$. Let $D_i = \phi_i(C_i)$ be the edges of the curves C_i on the boundary of F. From the above remarks we see that a vector field w on R which is associated with one on the original surface must satisfy

$$w\Big(\phi(T_i(q))\Big) = \phi_*\Big((T_i)_*(w(q))\Big), \quad q \in D_{i_1} \tag{2.16}$$

where T_i pairs the segments D_{i_1} and D_{i_2}. Let $m_1, m_2, \cdots, m_{4p} \in R$ denote the points $m_i = \phi(i)$ (i.e., the ϕ-image of the cusp points on F). Then we have

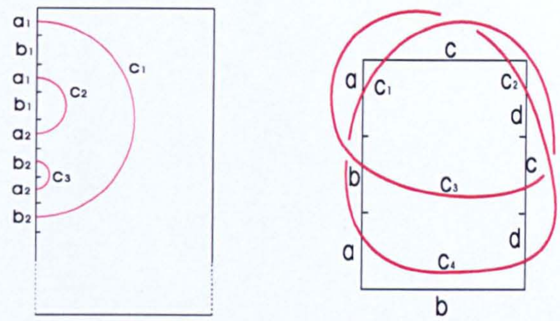
Lemma 2.6.2. *Any vector field w which is C^∞ on the interior of R and satisfies (2.16) where ϕ is given by (2.15) and such that*

$$w(m_i) = 0$$

defines a unique vector field on a genus-p $(p > 1)$ surface.

Proof. The only part left to prove is the converse. This follows from the above remarks and the Poincaré index theorem—any dynamical system on a surface of genus $p > 1$ must have at least one equilibrium point. We can choose this as the cusp points of F. □

We next consider the existence of periodic knotted trajectories on the surface. By the above results we can restrict attention to a rectangle R as shown in fig. 2.16. Any closed curve on the surface is given by a set of non-intersecting curves which 'match' in the sense of (2.16) at identified boundaries.



(a) A closed curve in R.

(b) A trefoil knot in R.

Figure 2.17: Closed curves on a surface.

For example, the set of curves shown in fig. 2.17a form a closed curve on the corresponding surface; moreover, fig. 2.17b stands for a trefoil knot on a 2-hole torus if we identify the sides in the appropriate way.

Of course, the knot type of this closed curve depends on the embedding of the surface in $\mathbb{R}^3$ (or S^3). For instance, the surface in fig. 2.17a could be embedded as in fig. 2.18, which also gives a knot diagram, therefore one can calculate a knot invariant (such as the Kauffman polynomial).

Let $\psi_i(x, y, t)$ denote the curve C_i within the modified fundamental region R, $f_i(x, y)$ and $g_i(x, y)$ be any C^∞ functions that guarantee the matching of vector fields at the identified boundaries via (2.16). Hence we have

Lemma 2.6.3. *If there are C_i $(1 \leqslant i \leqslant k)$ curves within the modified fundamental region that stand for a periodic trajectory of a dynamical system on a genus-p surface in $\mathbb{R}^3$, then this system can be defined by*

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{k} \left(\left(\frac{\partial \psi_i}{\partial y} + \psi_i f_i \right) \cdot \prod_{j \neq i} \psi_j \right) \\ \dot{y} &= \sum_{i=1}^{k} \left(\left(-\frac{\partial \psi_i}{\partial x} + \psi_i g_i \right) \cdot \prod_{j \neq i} \psi_j \right) \end{aligned} \tag{2.17}$$

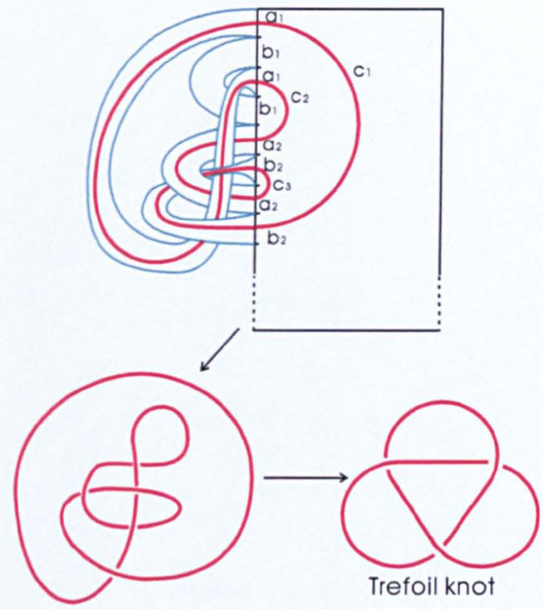


Figure 2.18: A *trefoil* knot on a surface

Proof. Since ψ_i defines the curve C_i in region R, we have

$$\psi_i(x, y, t_i) = 0 \tag{2.18}$$

$$\frac{\partial \psi_i}{\partial x} \cdot \dot{x} + \frac{\partial \psi_i}{\partial y} \cdot \dot{y} = 0. \tag{2.19}$$

For each curve C_i, ψ_i switches off all terms in (2.17) except the ith one. Substitute (2.17) into (2.19) and we have

$$\begin{aligned} \frac{\partial \psi_i}{\partial x}\Big(\frac{\partial \psi_i}{\partial y} + \psi_i f_i\Big)\prod_{j\neq i}\psi_j + \frac{\partial \psi_i}{\partial y}\Big(-\frac{\partial \psi_i}{\partial x} + \psi_i g_i\Big)\prod_{j\neq i}\psi_j \\ = \prod_{j\neq i}\psi_j\Big(\frac{\partial \psi_i}{\partial x}\psi_i f_i + \frac{\partial \psi_i}{\partial y}\psi_i g_i\Big) = 0, \end{aligned} \tag{2.20}$$

so the lemma follows. □

2.6.2 The Poincaré map and Knotted Attractors

Equation (2.17) now can be regarded as a general form of dynamical systems in the hyperbolic upper-half plane, which can be situated on a genus-p surface after identification of the corresponding sides.

Again consider the Poincaré map $P : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$P(x_0, y_0) = Y\big(T; 0, (x_0, y_0)\big),$$

where $Y(t) = Y\big(t; 0, (x_0, y_0)\big)$ is the solution of (2.17) starting from point (x_0, y_0). Because of the 'periodicity' from the automorphic form which is defined by the Fuchsian group Γ, we have

$$P\big(\Gamma_i(x_0, y_0)\big) = \Gamma_i\big(P(x_0, y_0)\big),$$

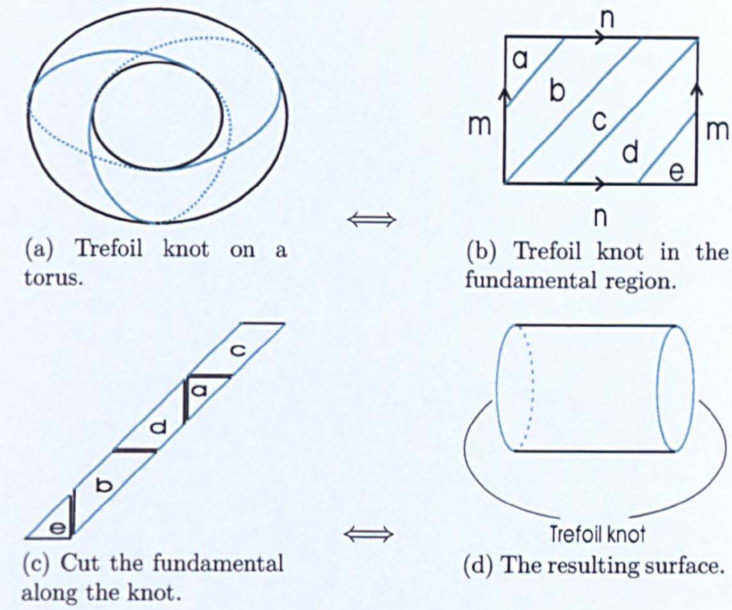
where Γ_i is the transformation from one fundamental region to another next to it. Moreover, if (x, y) is a solution of (2.17), so is $\Gamma_i^n(x, y)$ $(n \in \mathbb{Z})$. Without loss of generality, we can concentrate on one fundamental region F and consider only the dynamics within it. It is obvious that the Poincaré map is well defined on F.

If the system given by (2.17) is dissipative, then there exists an unstable periodic orbit, which means all the trajectories are pointing outward along this closed curve. Assume that it is represented by $\{\psi_i\}$ $(1 \le i \le k)$, we are mainly interested in what the dynamics will look like on the rest of the surface.

To begin with, some surgery is needed for the 2-manifold. Note that by cutting along one existing closed orbit, the genus-p surface, S, will effectively turn into another surface, S', with the genus reduced by 1 and two boundary circles being introduced.

Example. To make this statement clearer, we now look at an example of cutting along a trefoil knot which is situated on a torus.

As shown in fig. 2.19, by cutting the torus along this trefoil knot and identifying the corresponding segments on both sides, m and n, we get a cylinder (i.e., a sphere



(a) Trefoil knot on a torus.

(b) Trefoil knot in the fundamental region.

(c) Cut the fundamental along the knot.

(d) The resulting surface.

Figure 2.19: Cutting the trefoil knot on a torus.

with two holes in) with the two ends being the original trefoil knot. This surgery can always be performed on the genus-p surface such that the knot along which it is cut will generate one pair of sides in the fundamental domain. Hence this results in the constructed 2-manifold being a $(p-1)$-hole torus with two boundary cycles (as shown in fig. 2.20).

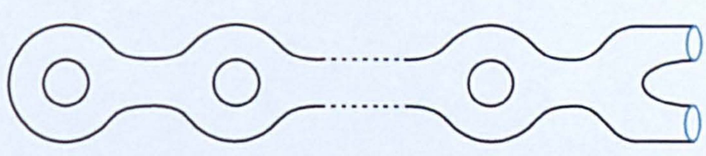


Figure 2.20: Cutting a p-hole torus open along a knot

In [Martins, 2004], Martins studied the torus case (i.e. $g = 1$), and showed that if there exists a trivial unstable periodic orbit, then an invariant set $\mathcal{A}$, i.e. a band around the tube, which may or may not be homeomorphic to a circle, must exist (see fig. 1.7 for an illustration). $\mathcal{A}$ is a compact, non-empty, connected set, and it acts as an attractor towards which all dynamics converge. It is given by the iterations of the

Poincaré map within the fundamental region to a well-defined bounded set.

In the case of genus-2 surfaces, the same argument applies for the existence of the invariant set as in [Martins, 2004], while the exact number of the invariant sets may vary. More specifically, if we cut a 2-hole torus along a knotted trajectory, topologically the surface will turn into a torus but with two boundary circles, as illustrated in fig. 2.21.

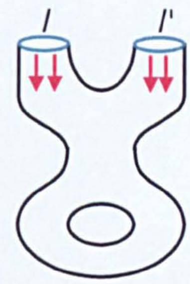


Figure 2.21: Cutting a 2-hole Torus.

Suppose that this knot is unstable. Hence after the surgery, all the dynamics are pointing outward from the two resulting boundaries. Since fig. 2.21 is essentially a cylinder with a torus attached in the middle, from [Martins, 2004], we know that there exists some invariant set $\mathcal{A}$. However, the number of invariant sets differs from that in the toroidal case. In fact, there can be three fundamental types of invariant sets, as shown in fig. 2.22. In this figure, $\mathcal{A}$ denotes the invariant set, while m and m' stand for the saddle type equilibria which have -1 index respectively. This makes sense because $I(m) + I(m') = -2$, which accounts for the correct Poincaré characteristic for a genus-2 surface.

Note that the actual invariant set may be a combination of more than on of these fundamental types, see fig. 2.23 for illustration.

As the genus of a surface increases, the number of the invariant sets may increase accordingly, but they are always based on the three fundamental types shown in fig. 2.22. Moreover, we have

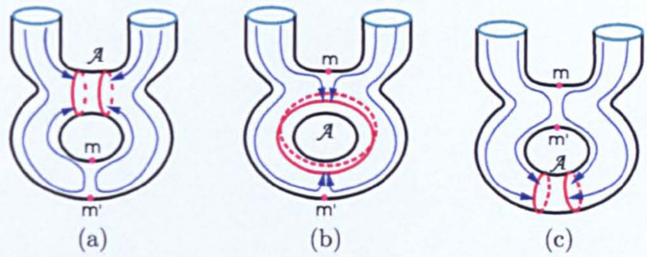


Figure 2.22: Fundamental types of invariant sets on genus-2 surfaces.

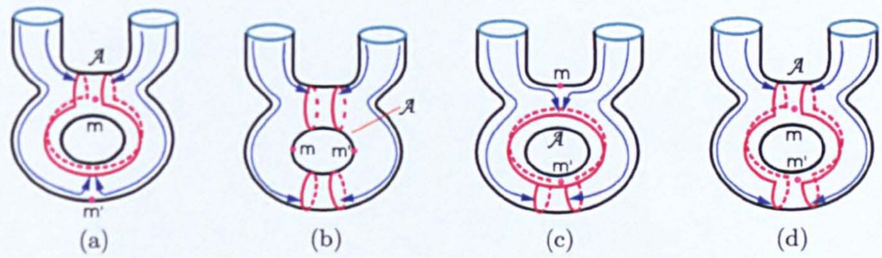


Figure 2.23: Possible combinations of invariant sets.

Lemma 2.6.4. *Given a system situated on a genus-p surface with only one unstable periodic orbit. Then there can be at most $(2p-1)$ attractors that might be knotted themselves and linked together. Moreover, a surgery can always be performed such that these attractors will be combinations of the fundamental invariant sets.*

Proof. We prove it by induction.

In the torus case, it is known that the attractor is a band as shown in fig. 1.7 ([Martins, 2004]); and on a 2-hole torus, from the discussion above, there can be at most three attractors.

Assume it is true for the genus-p surface such that it has $(2p-1)$ invariant sets at most, then by adding the genus by 1, we essentially introduce another hole to the original manifold which will yield two more attractors at most, hence proves the lemma. □

2.6.3 Inversely Unstable Solutions and Topology of Knotted Attractors

Inversely unstable solutions to a dynamical system has been studied for a long time. To be precise, we restate the main idea, which can be found in [Levinson, 1944], for example.

Denote a two-dimensional system by

$$\begin{cases} \dot{x} &= F(x,y,t) \\ \dot{y} &= G(x,y,t) \end{cases}, \tag{2.21}$$

where F and G are both T-periodic in t.

Definition 2.6.1. *Suppose $(a,b) \in \mathbb{Z} \times \mathbb{N}$, $b \geqslant 1$, a solution $z = (x,y)$ of (2.21) is **(a,b)-periodic** if and only if*

$$z(t + bT) = \Gamma_i^a\big(z(t)\big)$$

where Γ_i is the map from one fundamental region onto the other one next to it.

Note that these solutions correspond to the trajectories that 'wind around' one of the tubes in the genus-p surfaces a times within bT time interval before closing. If $\big(x(t), y(t)\big)$ is a (a,b)-periodic solution, then the initial point A, i.e. $\big(x(t_0), y(t_0)\big)$, is a fixed point of $M = P^b - \Big(\Gamma^a\big(z(0)\big) - z(0)\Big)$, where P is the Poincaré map. Assume A is an isolated fixed point, and let A_0 denote the point $\big(x(t_0) + u_0, y(t_0) + v_0\big)$ near A in the hyperbolic upper-half plane. Applying the Poincaré map once yields

$$A_1 = P(A_0)$$

where A_1 is denoted by $\big(x(t_0) + u_1, y(t_0) + v_1\big)$. By using a power series in u_0 and v_0 with coefficients function in t, we can express the solution trajectory of $\big(x(t), y(t)\big)$

starting at A_0 by

$$\begin{cases} X(t) &= x(t) + c_1(t)u_0 + c_2(t)v_0 + c_3(t)u_0^2 + c_4(t)u_0v_0 + \cdots \\ Y(t) &= y(t) + d_1(t)u_0 + d_2(t)v_0 + d_3(t)u_0^2 + d_4(t)u_0v_0 + \cdots \end{cases} \quad (2.22)$$

In particular, by setting $t = t_0 + T$, we have

$$\begin{cases} u_1 &= au_0 + bv_0 + a_1u_0^2 + b_1u_0v_0 + \cdots \\ v_1 &= cu_0 + dv_0 + c_1u_0^2 + d_1u_0v_0 + \cdots \end{cases}.$$

Denote $\big(x(t_0) + u_0, y(t_0) + v_0\big)$ and $\big(x(t_0) + u_1, y(t_0) + v_1\big)$ by (x_0, y_0) and (x_1, y_1) respectively, then

$$J(\frac{x_1, y_1}{x_0, y_0}) = J(\frac{u_1, v_1}{u_0, v_0}),$$

where J is the Jacobian of the Poincaré map of the point (x_0, y_0). For very small values u_0 and v_0, (2.22) is dominated by its linear terms. So the characteristic multiplier can be determined by

$$(a - \lambda)(d - \lambda) - bc = 0.$$

Using the notation above, we have

Definition 2.6.2. *Given an (a, b)-periodic solution $\big(x(t), y(t)\big)$ of (2.21) such that $\big(x(t_0), y(t_0)\big)$ is an isolated fixed point of M, we shall say the solution $\big(x(t), y(t)\big)$ is* ***inversely unstable*** *if and only if $\lambda_2 < -1 < \lambda_1 < 0$.*

To consider topology of the knotted attractors under the existence of inversely unstable solutions, we need

Definition 2.6.3. *A system defined on a surface S is* ***dissipative*** *relative to a knot K if there is a neighbourhood, N of K in S such that on $\partial(S/N)$, the vector field is pointing into N.*

Then we have

Theorem 2.6.1. *Given a system defined by (2.17) on a genus-p surface, which is dissipative relative to a knot K situated on this surface as well, if there exists an inversely unstable solution (x_I, y_I) within the (knotted) attractor $\mathcal{A}_I$, then $\mathcal{A}_I$ is not homeomorphic to the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.*

Proof. We shall prove this theorem in a geometrical way. Due to the dissipative nature, there exists one or more unstable periodic orbits, while each of them is equivalent to a knot on the surface that the system is defined on. By cutting the surface along one of these knots, we can reduce the surface genus by 1 with two boundary circles being introduced (see fig. 2.21 for illustration). In fact, these two circles are equivalent to the knot that was cut along. Now gluing the two circles will produce a tubular neighbourhood containing an attractor $\mathcal{A}$. Assume that there exists an inversely unstable solution in $\mathcal{A}$. Let A be a fixed point of the associated Poincaré map. It is always possible to find a neighbourhood $U \ni A$ such that A is the only fixed point in U. Suppose A_0 is a point in U close to A (see fig. 2.24 for illustration). If we apply Poincaré map once to A_0, with the dynamics being determined by the characteristic multipliers, which are $\lambda_2 < -1 < \lambda_1 < 0$, the trajectory will move from A_0 to A_1, a point lies in the other half plane with respect to y-axis and is much closer to the fixed point A. Now apply Poincaré map to A_1, note that this time the characteristic multipliers become $0 < \lambda_1^2 < 1 < \lambda_2^2$ due to the action of P^2, which gives directly unstable solution that moves the dynamics from A_1 to A_2, a point further away in the left-half plane. With iterations of the Poincaré map, the corresponding characteristic multipliers will be alternatively positive and negative. However, all neighbouring dynamics point towards the knotted attractor by dissipativity. In other words, within the invariant set near the inversely unstable solution, the dynamics tend to either get close to this periodic orbit or escape from it, while at the boundary, they are all pushed back by the external dissipative condition. This is why chaotic behaviour can happen which means that $\mathcal{A}$ is not homeomorphic to a circle. The same idea follows when there are more than one attractor containing separate inversely unstable solutions. □

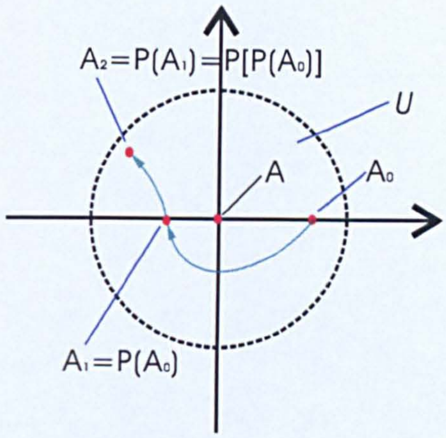


Figure 2.24: How an inversely unstable solution affects the dynamics.

So generally speaking, a dissipative system given by (2.22) situated on a genus-p surface can have at most p topologically distinct knotted attractors (see [Banks, 2002]); whether they are homeomorphic to a circle individually depends on the existence of inversely unstable solution within themselves.

It is known that any dynamical system sitting on a 2-manifold of genus p can be represented on a sphere by cutting each handle along a fundamental circuit which contains no equilibrium point and filling in the dynamics on the resulting region bounded by these curves (see [Banks, 2002]). Conversely, we can get higher genus surface systems by performing surgery on certain spherical ones. Specifically, given a spherical system, irrespective of the rest of the dynamics, provided that it contains 2 stable equilibria, E_1 and E_2, we can always choose two small neighbourhoods M_i ($i = 1, 2$) such that E_1 and E_2 are the only critical points within each region. Glue in a dissipative region with attractor $\mathcal{A}$, cut this attractor open, twist and identify the two ends together in the appropriate way, we then obtain the desired knot (as shown in fig. 2.25). If the attractor contains an inversely unstable solution, then it is not homeomorphic to a circle, which means chaotic behaviour will occur within this invariant set.

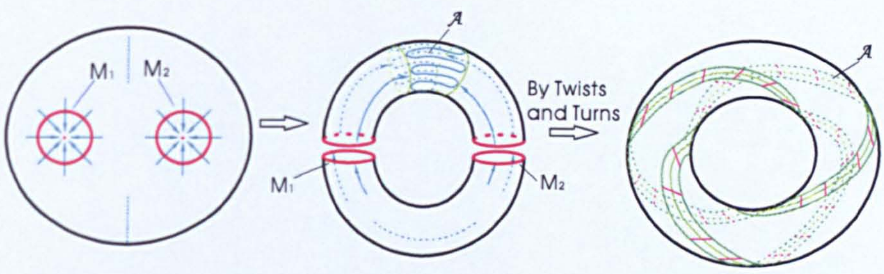


Figure 2.25: Construct a torus system from a spherical one.

Hence we have proved

Theorem 2.6.2. *Any dynamical system on a genus-p surface that contains a set of k ($k \geq 1, k \in \mathbb{N}$) (knotted) dissipative attractors, where each of them contains an inversely unstable orbit, can be represented by a system with at least $2k$ stable equilibrium points on a sphere. Conversely, starting from a spherical system that contains at least $2k$ stable equilibria, we can construct a system on a genus-p surface that contains k knotted attractors each with chaotic behaviour contained.* □

Remarks. An important consequence of this theorem is that we can determine the general structure of a system with k 'chaotic' dissipative attractors by considering systems with $2k$ stable equilibrium points on a sphere. Of course, such a system must have other singular points such that the total index adds up to 2, by the Poincaré index theorem. This implies the existence of some hyperbolic points.

2.6.4 Examples

In this section we show that we can obtain systems with dissipative chaotic behaviour by choosing stable and unstable knotted orbits, and the unstable orbit acts as the dissipative 'repeller'.

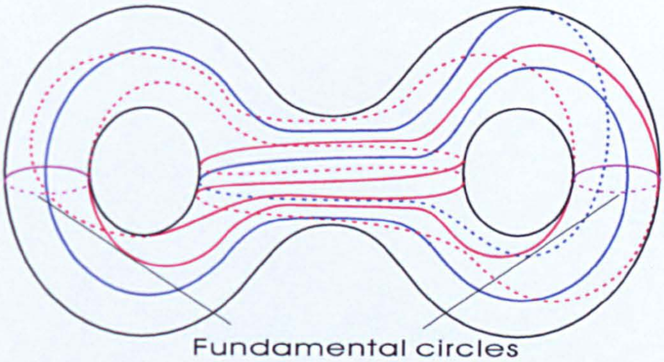


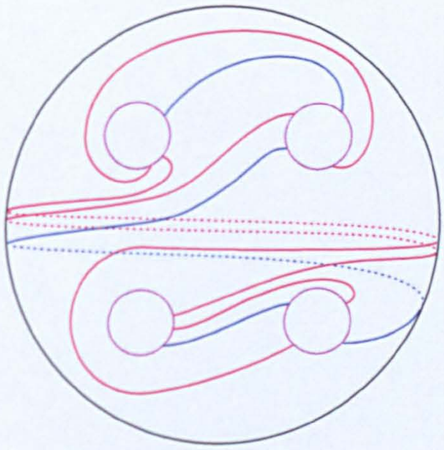
Figure 2.26: A surface of genus two carrying two distinct knot types.

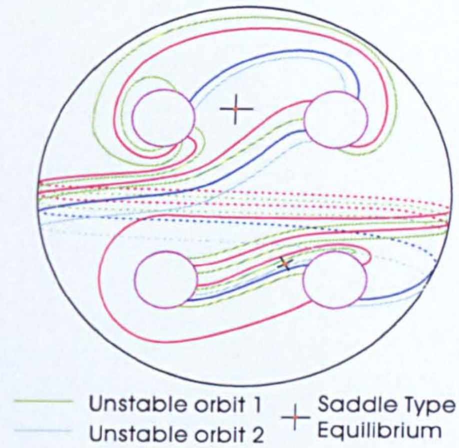
In [Banks, 2002], it is shown that for a dynamical system on a surface of genus p, it can carry at most p distinct types of (homotopically nontrivial) knots. For example, fig. 2.26 shows the two distinct knot types that a system can have on a genus-2 surface.

Suppose that these two knots act as two attractors, (the existence of chaotic behaviour will depend on whether there is an inversely unstable solution within each attractor,) then there must exist one or more unstable orbits due to which these two invariant sets are generated. To find out explicitly, we first represent the system onto a sphere with four holes, which is obtained by cutting along two fundamental circuits and opening the handles out, as shown in fig. 2.27a.

The unstable orbits therefore should bound each part of the attractors presented on the sphere such that they will push the dynamics toward the invariant sets and introduce possible chaotic behaviour. Moreover, there must exist some equilibrium points to give the correct index of a genus-2 surface, which is -2. Fig. 2.27b shows one possible scenario for the solution trajectories of two unstable orbits. Note that they are not unique.

Gluing the corresponding boundary circles, we eventually recover the original 2-manifold with a dynamical system situated. It has two unstable periodic cycles, which generate two knotted attractors with distinct types, and two saddle critical


(a) Opening out a spherical system.



(b) Possible unstable orbits.

Figure 2.27: Spherical representation for the attractors.

points which accounts for the correct index of -2 (see fig. 2.28 for an illustration).

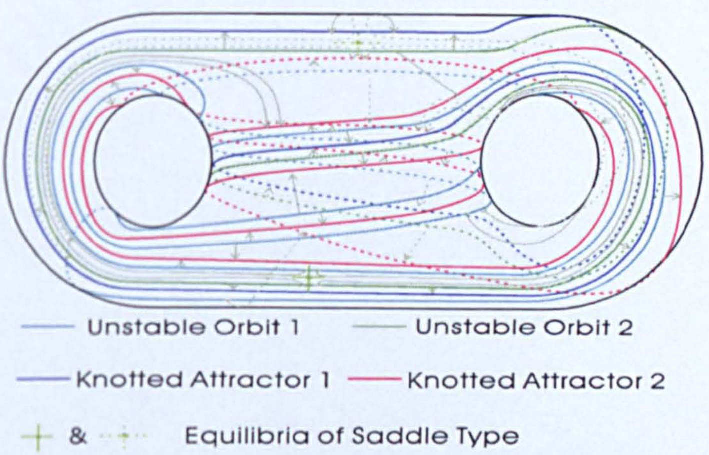


Figure 2.28: A genus-2 surface containing 2 knotted attractors.

Furthermore, as shown in fig. 2.29, if each invariant set contains an inversely unstable orbit, then around each knot there exists a band within which chaotic behaviour will occur.

Now reduce the number of invariant sets by one and assume the existence of only one unstable orbit, following the same algorithm above, we get one possible solution for the dynamics as in fig. 2.30. Note that again there are two saddle points which

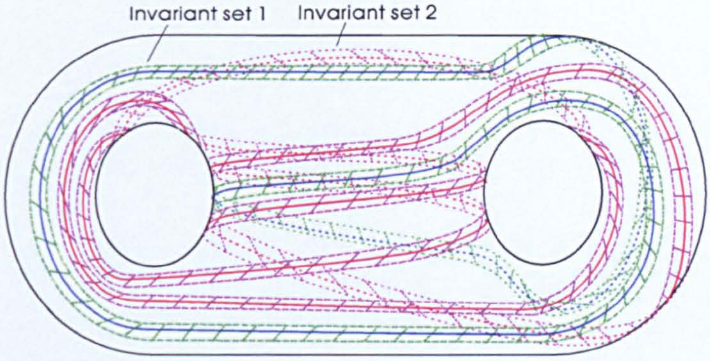


Figure 2.29: Invariant sets with inversely unstable orbits.

account for the correct index.

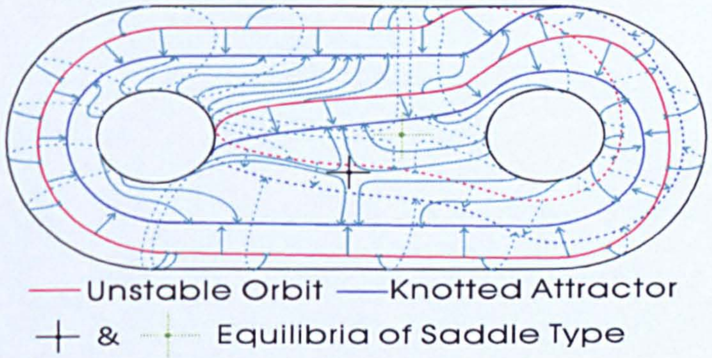


Figure 2.30: A genus-2 surface containing one Knotted Attractor.

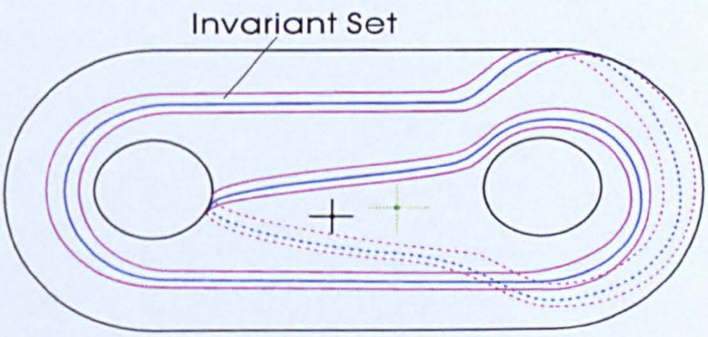


Figure 2.31: One knotted attractor with an inversely unstable solution

Under the existence of inversely unstable solution, chaotic behaviour will occur within the invariant set (see fig. 2.31).

2.7 Concluding Remarks

In this chapter, we have studied dynamical systems that are globally situated on surfaces of any genus. By using the generalized automorphic functions, we showed an algorithm to generate meromorphic systems on general surfaces. We have given some examples, including Hamiltonian systems, to illustrate such procedure. Moreover, these systems may bifurcate by shrinking meridian curves on the handles of the surfaces to zero. These global bifurcations are of a rather different type than the usual ones in that the manifold on which the dynamics are defined also changes. Of course, any local bifurcation of such type can occur on any given surface and we may generate any meromorphic system by appropriate choice of the modified theta series (based on the fundamental region) and their generating functions H_1, H_2.

On the other hand, we have looked at the topology of knotted attractors under the existence of unstable periodic orbits and proved that for a surface of genus p with only one unstable cycle, the number of invariant sets may vary while a maximum of $(2p-1)$ must not be exceeded. Furthermore, we extended the result in [Martins, 2004] and showed that for a surface of genus > 1, the individual attractor is not homeomorphic to a circle if there exists an inversely unstable solution within itself. This is purely because of the property of inversely unstable solution which can generate a local behaviour to make dynamics fight against the effect of global unstable orbit.

Chapter 3

Three-Dimensional Manifolds

A 3-manifold M is a separable metric space such that each point $x \in M$ has an open neighbourhood, which is homeomorphic to $\mathbb{R}^3$ or $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 \geq 0\}$. We can assume all the 3-manifolds we consider here are differentiable (or p.l.[1]) manifolds since any 3-manifold has a unique p.l. or differentiable structure (see [Hempel, 1976]).

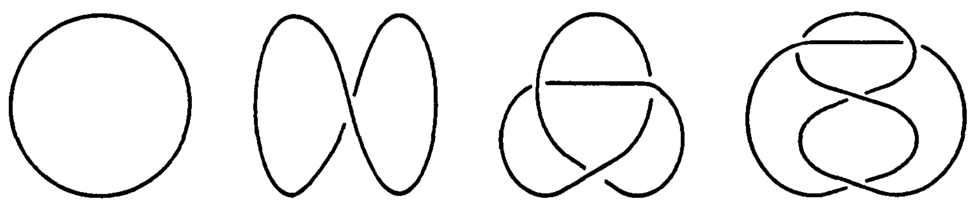
The 3-manifold topology is much more complicated than the 2-dimensional one. In fact, there is no complete set of topological invariants for 3-manifolds (see [Markov, 1958]); although a great deal of invariants have been found, with the most interesting being related to quantum groups and braided tensor categories (see [Chari and Pressley, 1994; Kassel, 1995]). Moreover, the Euler characteristic of a 3-manifold is 0 so the index theorem does not give too much information in this case. However, there are useful results in 3-manifold theory which can be used to obtain some decomposition of 3-manifolds in terms of simpler ones. These results are related to Dehn surgery, Heegaard splittings and branched covering manifolds. Also, 3-manifold theory provides a penetrating insight into the study of knot theory. We shall discuss all these topics in this section.

---
[1]piecewise linear

3.1 The Theory of Knots and Links

3.1.1 Basic Definitions

Definition 3.1.1. *A* ***knot*** *is an embedding $K : S^1 \to S^3$. More generally a* ***Link*** *is $L : \bigcup S^1 \to S^3$, i.e. a disjoint, finite collection of knots.*



(a) The unknot. (b) Trivial knot. (c) Trefoil knot. (d) Figure-8 knot.

Figure 3.1: The simplest knots.

Fig. 3.1 shows some of the simplest knots. Two knots, K and K', are equivalent if there exists a homeomorphism $h : S^3 \to S^3$ such that $h \circ K = K'$. Knots and links are usually oriented. Indeed, all spaces are endowed with orientations, all of which h is required to preserve.

Given a transverse representation of an oriented knot in the plane, each crossing point has an induced orientation, as shown in fig. 3.2.

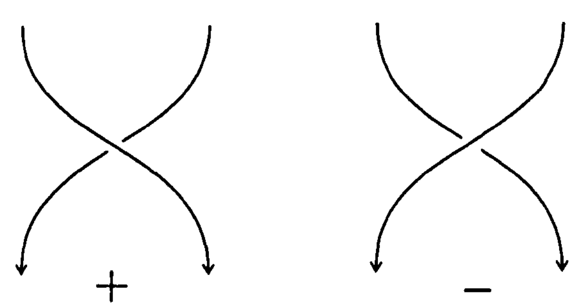
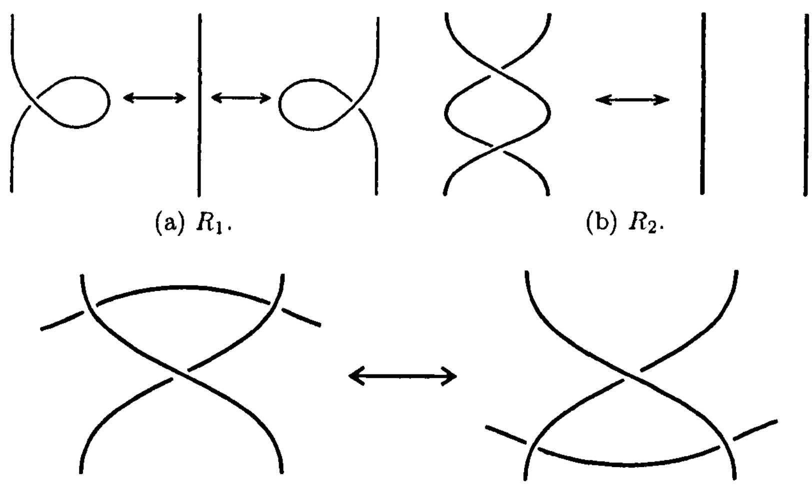


Figure 3.2: Sign convention for crossings.

Definition 3.1.2. *A homotopy $h_t : S^3 \to S^3$ $(t \in [0,1])$ is called* ***ambient isotopy*** *if h_0 is the identity and each h_t is a homeomorphism. Two knots K and K' are* ***ambient isotopic*** *if there is an h_t such that $h_1 \circ K = K'$.*

One of the fundamental problems in knot theory is to ask when two knots are isotopic. Denote knots in their corresponding planar representations, we have the following result

Theorem 3.1.1. (**Reidemeister**) *Two transverse representations denote isotopic links if and only if the diagrams are related by isotopy (fixing the crossing points) and by a finite sequence of the three Reidemeister moves, illustrated in fig. 3.3.* □



(a) R_1. (b) R_2. (c) R_3.

Figure 3.3: Three Reidemeister moves: R_1, R_2 and R_3.

3.1.2 Relation between Knots and Braids

Knot theory is studied conveniently in terms of braids, which were originally introduced in [Artin, 1925]. By definition, an *m-strand braid* is a set of m non-intersecting smooth paths connecting m points on one vertical line to m points on another vertical line (to the right of the first one) – see [Kauffman, 1991] and fig. 3.4 for illustration. By gluing the corresponding left and right hand sides of the braid together, we obtain the so-called *closure* of a braid. In fact, the closure of fig. 3.4 is actually a knot. Of course, the closure of a braid can also be a link. Usually closures of braids are taken to be oriented, all strands are oriented from left to right in this thesis.

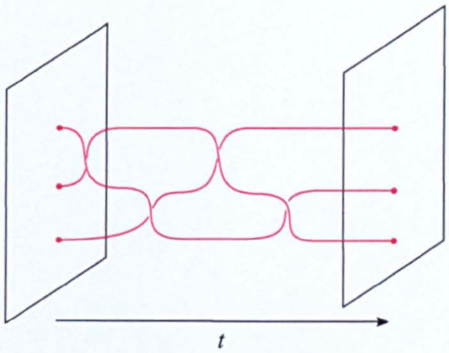


Figure 3.4: A braid in 3-space.

Theorem 3.1.2. *(***Alexander's Theorem***) Each link can be represented as the closure of a braid.* □

Since a link is composed of several knots, we have the following

Corollary 3.1.1. *Each knot can be represented as the closure of a braid.* □

Now we give an algorithm to construct braid from a given knot. We will illustrate the method by the elaboration on a figure-8 knot (see fig. 3.5) – the general case will then be clear.

Note that the braid construction has at least one strand (in the *unknot* case). The key idea will be to cut the knot in a proper way such that it will turn into several strands which yield the required braid representation.

First, we choose one part of the given knot which is away from any crossing, cut the diagram at a point from this part and straighten the two ends resulted from the cut, which gives us the first strand of the braid. Clearly it is better to cut the top or the bottom of the knot, as shown in fig. 3.5a and 3.5b.

Since the only knot with a one-string braid is the *unknot*, generally there is a need to cut the diagram several times in order to build up the braid representation. Self-crossings of any strand can be removed by Reidemeister moves. So the next step will be to choose a part of the resulting diagram which is before or after a self-crossing

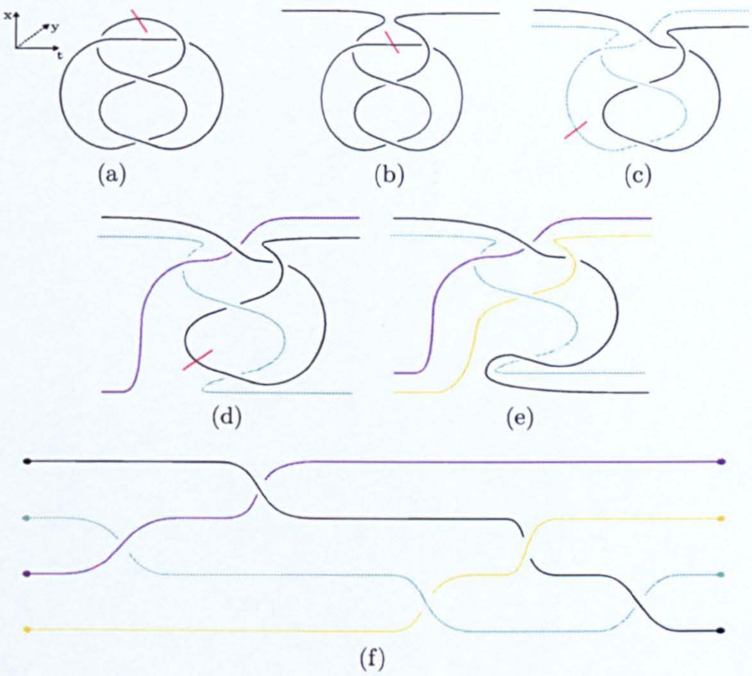


Figure 3.5: Construction of a braid from the figure-8 knot.

of the one-strand from the previous stage, cut it and straighten the diagram. Note that because a knot is homeomorphic to S^1, we must guarantee that it starts at one end and finishes at the other. The solution to this is simple: Adding another crossing if necessary to ensure that a strand coming in from one side must going out from the other (see fig. 3.5c and 3.5d for illustration). Meanwhile, for each strand, the starting and ending points cannot share the same x-value or y-value (according to the coordinate system in fig. 3.5), otherwise the braid will represent a multi-component link instead of a knot.

Reiterate the above procedures until all the strands in the braid have no self-crossings, as shown in fig. 3.5e.

Rearrange the braid so that there is at most one twist at each vertical strip (as shown in fig. 3.5f), and this will yield the braid representation eventually. Of course, there are an infinite number of braid representations of a given knot. However, we will consider the simplest one in this thesis.

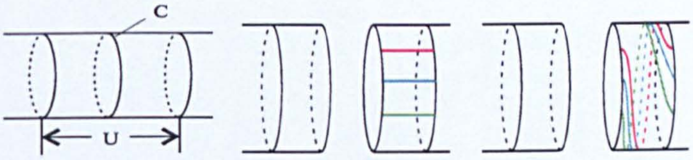
3.2 Dehn Surgery

It is known that any knot or link may be removed from its embedding space, S^3. These examples can be made compact by removing the interior of a tubular neighbourhood of the knot or link. In fact, the complement of a knot can be very enigmatic, and it turned out to be a topological invariant which can distinguish knots to a certain extent.

Moreover, it is possible to cut out a tubular neighbourhood of K in S^3 (which is topologically a solid torus) and then glue it back in by a different homeomorphism from the boundary of the excised torus to the boundary of the toroidal 'hole' in S^3. This is called *Dehn surgery.* There are many ways to do this, because the torus has many diffeomorphisms. The generator of the kernel of the inclusion map $\pi_1(T^2) \to \pi_1(\text{solid torus})$ determines the topology of the resulting 3-manifold.

It is then natural to ask what 3-manifolds can be obtained from S^3 by doing the Dehn surgery. Indeed, Lickorish proved

Theorem 3.2.1. (**[Lickorish, 1962]**) *Any closed, connected, orientable, combinatorial 3-manifold M, is piecewise linearly homeomorphic to S^3, the 3-sphere, from which have been removed a finite set of disjoint solid tori which are sewn back in a different way.* □



(a) $C \in U$. (b) Cut S along C. (c) Twist and glue.

Figure 3.6: Performing C-homeomoephism.

Moreover, Lickorish showed that such a homeomorphism can be generated (up to isotopy) by a sequence of C-homeomorphisms, which can be 'performed' in the following way on a 2-manifold S. Let C be a simple closed curve in S, $U \subset X$ be a neighbourhood of C and $U \cong S^1 \times I$, i.e. a cylinder. Cut S along C, twist one end of the cylinder through 2π, and glue them back again (see fig. 3.6 for illustration). Note that this homeomorphism leaves S fixed except in a neighbourhood of C.

As an example, fig. 3.7 shows how to obtain a trefoil knot from a trivial one by applying this surgery.

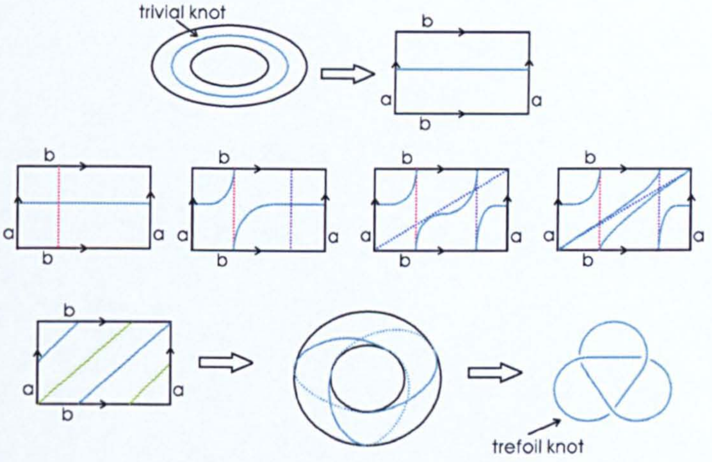


Figure 3.7: Generating a trefoil knot from a trivial one via C-homeomorphisms.

Dehn surgery and Lickorish's result then lead to the possibility of defining dynamical systems on three-manifolds: First choosing one on S^3 which has periodic solutions, and these generate some links in S^3; then we can perform Dehn surgery on the link to obtain a dynamical system on the resulting three-manifold.

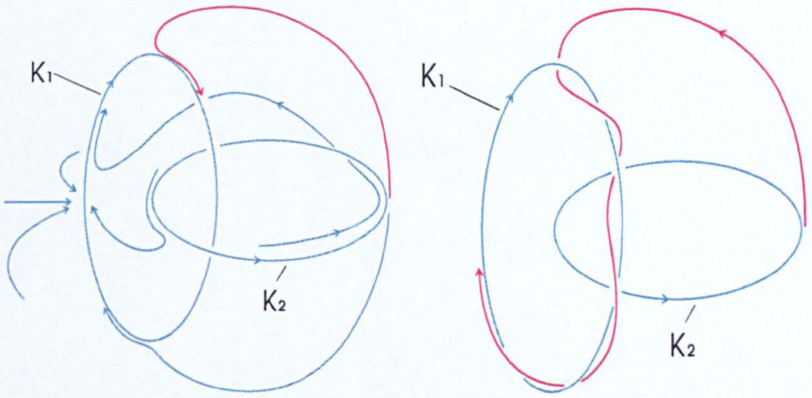
Example. It is known that by attaching two 3-balls together via any homeomorphism of their boundaries, the resulting space is S^3 (Alexander's trick). Consider the analogous construction to two solid tori, V_1 and V_2. With a homeomorphism $h : \partial V_1 \to \partial V_2$, we can form the space $M = V_1 \cup_h V_2$. Choose fixed longitude and meridian generators l_1 and m_1 for $\pi_1(\partial V_1)$, we have

$$h_*(m_1) = pl_2 + qm_2,$$

where p and q are coprime integers. Hence M is called the *Lens space* of type *(p,q)* and denoted by

$$M = L(p, q).$$

From [Rolfsen, 1976], we know that 3/4-surgery on a trivial knot in S^3 yields the lens space $L(3,4)$ ($= L(3,1)$). Consider the nonsingular Morse-Smale flow on S^3 with a Hopf link as periodic solutions given as in fig. 3.8a.



(a) A nonsingular Morse-Smale flow on S^3.

(b) A typical heteroclinic orbit in $L(3,4)$.

Figure 3.8: The 3/4-surgery on a Morse-Smale flow.

Now do 3/4-surgery on the trivial knot K_1 and we obtain a system on $L(3,4)$ with two periodic solutions such that the stable periodic solution is surrounded by stable solutions which wind around it 3 times (as illustrated in fig. 3.8b).

We have

Theorem 3.2.2. (**[Banks and Song, 2008]**) *There is a nonsingular Morse-Smale system on any lens space.* □

The next theorem is due to Papakyriakopoulos, with the importance of deriving geometric conclusion from an algebraic hypothesis.

Theorem 3.2.3. (Loop Theorem.) *Let M be a compact orientable 3-manifold and the inclusion homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$ has nontrivial kernel, then* *there exists a disk $D^2 \subset M$ such that $\partial(D^2)$, i.e. a closed curve, lying in ∂M and representing a nontrivial element of $\pi_1(\partial M)$.* □

Moreover, we lay down another fundamental result in topology for future reference.

Theorem 3.2.4. (Whitney Embedding Theorem.) *Every smooth compact manifold of dimension $n > 1$ can be embedded in $\mathbb{R}^{2n-1}$, i.e. Euclidean $(2n-1)$-space.* □

3.3 Heegaard Splittings and Connected Sums

Next we consider the concept of Heegaard splittings and connected sums (see [Hempel, 1976]).

Heegaard splittings is probably the earliest method to decompose a 3-manifold to receive attention. It has visual geometric appeal and is easy to describe.

Definition 3.3.1. *A 3-manifold, M, is called an* ***n handlebody*** *if it can be obtained from the 3-ball B^3 by attaching n distinct copies of $D^2 \times [-1, 1]$ by homeomorphisms that identify the $2n$ disks $D^2 \times \{\pm 1\}$ with a collection $\{D_1, \cdots, D_{2n}\}$ of pairwise disjoint, properly embedded 2-disks on ∂B^3. Moreover, n is called the* ***genus*** *of the handlebody M.*

$\pi_1(M)$ is a free group of rank n. Two handlebodies are homeomorphic if they have the same genus and are both orientable or both nonorientable. The boundary of a handlebody of genus n is a closed 2-manifold of genus n, which has the same orientability as the handlebody. A *Heegaard splitting* of a closed, connected 3-manifold M is a pair (H_1, H_2), where H_1 and H_2 are handlebodies of the same genus and orientation such that $M = H_1 \cup H_2$ and $H_1 \cap H_2 = \partial H_1 = \partial H_2$. The *Heegaard diagram* associated with the Heegaard splitting $M = H_1 \cup H_2$ of genus n is the identification map $h : \partial H_1 \to \partial H_2$, such that n distinct closed curves on ∂H_1 are mapped onto

n fundamental meridians of ∂H_2, respectively. For example, the trefoil knot on the torus is a Heegaard diagram for the lens space $L(3,2)$, see fig. 3.9.

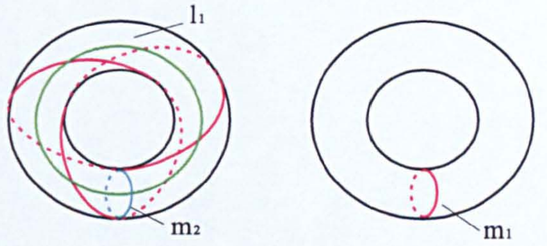


Figure 3.9: Heegaard splitting for $L(3,2)$: $h(m_1) = 3l_2 + 2m_2$.

Theorem 3.3.1. *Every closed, connected 3-manifold M has a Heegaard splitting.*

Proof. We only give an outline of the proof, for more details, refer to [Rolfsen, 1976; Hempel, 1976].

Take a triangulation K of M and let Γ_1 be the set of all 1-simplexes of K (i.e. the 1-skeleton). Let Γ_2 be the dual 1-skeleton, which is the maximal 1-subcomplex of the first derived complex K' such that K' is disjoint from Γ_1. Then if we put

$$V_i = N(\Gamma_i, K''),$$

where N is the normal neighbourhood of Γ_i with respect to K'' (the second derived of K), it can be shown that (V_1, V_2) is a Heegaard spliting of M. □

In [Lickorish, 1962], he showed that for a Heegaard diagram, the surface homeomorphism can be generated by a sequence of C-homeomorphisms as well.

The advantage of the decomposition method lies in the fact that the resulting pieces are simple. However, it is not easy to obtain useful information about the embedding of a Heegaard surface from intrinsic properties of the corresponding manifold. Hence incompressible surfaces attract a great deal of attention because they are considered to be good representatives of the properties of the corresponding manifolds. Let M be a 3-manifold and F a surface properly embedded in M. F is called *compressible* if either one of the following conditions is satisfied.

i) F bounds a 3-ball B^3.

ii) There exists a 2-disk $D \subset M$ such that $D \cap F = \partial D$, i.e. a closed curve, and ∂D is not contractible in F.

Otherwise, F is called *incompressible.* Furthermore, we have

Theorem 3.3.2. (Haken's Finiteness Theorem.) *Let M be a compact orientable 3-manifold. Then there exist a finite number of pairwise disjoint, non-parallel, closed connected incompressible surfaces in M.* □

Definition 3.3.2. *Let M, M_1 and M_2 be connected 3-manifolds. If there exist 3-balls $B_i^3 \subset \text{Int}(M_i)$ $(i = 1, 2)$ and embeddings $h_i : (M_i - B_i^3) \to M$ such that $h_1(M_1 - B_1^3) \cap h_2(M_2 - B_2^3) = h_1(\partial B_1^3) = h_2(\partial B_2^3)$ and $M = h_1(M_1 - B_1^3) \cup h_2(M_2 - B_2^3)$, then M is called a* **connected sum** *of M_1 and M_2, denoted by $M = M_1 \# M_2$.*

For orientable 3-manifolds M_1 and M_2, we require the attaching homeomorphism $h : \partial B_1^3 \to \partial B_2^3$ to be orientation reversing such that $h_i : (M_i - B_i^3) \to M$ is orientation preserving. Hence $M = M_1 \# M_2$ is an orientable 3-manifold. Under this restriction, connected sum is a well-defined associative and commutative operation in the category of oriented 3-manifolds and orientation preserving homeomorphisms.

Definition 3.3.3. *Let M, M_1 and M_2 be connected 3-manifolds. If $M = M_1 \# M_2$ implies one of M_1, M_2 is a 3-sphere, then M is called* **prime**.

Clearly, Lens spaces are prime. Large amount of interest has been focused on prime factorizations of bounded as well as closed 3-manifolds. A prime factorization $M = M_1 \# \cdots \# M_n$ is called *normal* if, when M is orientable, some of M_i $(1 \leq i \leq n)$ is $S^1 \times S^1$. In fact, any prime factorization can be replaced by a normal one. The following result is fundamental in the theory of connected sums.

Theorem 3.3.3. *Every compact 3-manifold can be expressed as a connected sum of a finite number of prime factors. Moreover, the decomposition is unique up to order and homeomorphism, i.e., let* $M = (M_1\# \cdots \# M_n) = (M'_1\# \cdots M'_{n'})$ *be two normal, prime factorizations, then* $n = n'$ *and* M_i *is homeomorphic to* M'_i *(after rearranging if necessary).* □

3.4 Branched Coverings

Definition 3.4.1. *Let* M *and* N *be two compact n-manifolds,* $A \subset M$ *and* $B \subset N$ *be two* $(n-2)$*-submanifolds that are properly embedded in* M *and* N *respectively. If there exists a continuous function* $f : M \to N$ *such that* $f(A) = B$, $f(M - A) = N - B$, *and* $(N - B)$ *is the only set which is evenly covered in* N, *i.e. for a neighbourhood* $U \subset (N - B)$, f *maps each component of* $f^{-1}(U)$ *homeomorphically onto* U, *then* f *is called a* ***branched covering*** *with branched sets* A *and* B.

The restriction $f : M - A \to N - B$ is obviously a covering space, we call it the *associated unbranched covering.* Since M is compact, it is finite-sheeted. In [Alexander, 1920], the following theorem was asserted

Theorem 3.4.1. (Alexander's Theorem.) *Every closed orientable 3-manifold is a branched covering space of* S^3 *with the branch set a link in* S^3. □

Hence branched covering is an effective method to construct 3-manifolds. Moreover, Montesinos and Hilden (independently) proved that every closed orientable 3-manifold is a 3-sheeted branched covering of S^3, branched over a knot (see [Montesinos, 1974; Hilden, 1974]). This knot is called *universal knot* and it is nontoroidal. Hence

Theorem 3.4.2. *Any three manifold is a branched covering of* S^3 *branched over a 'universal knot'.*

Proof. In fact, Hilden proved it by constructing a certain irregular 3-fold branched covering of the 3-ball, D^3, by a genus-g handlebody , called X_g, with branch set A, i.e. a set of $g+2$ proper arcs. Then for this branched covering, any homeomorphism of ∂X_g is isotopic to a homeomorphism ψ that projects to a homeomorphism of ∂D^3 (i.e. S^2). Assume X_g' is another handlebody of genus g, $h : \partial X_g \to \partial X_g'$ and $H : \partial D^3 \to \partial D^3$. (X_g, X_g', h) will represent a Heegaard diagram for a genus-g 3-manifold, M, where $M = X_g \cup_{\psi h} X_g'$ for some homeomorphism ψ of ∂X_g that projects to a homeomorphism Ψ of ∂D^3. Then we have

$$p \cup p' : X_g \cup_{\psi h} X_g' \to D^3 \cup_{\Psi H} D^3 = S^3,$$

which is a branched covering space with branch set the knot $A \cup_{\Psi H} A'$.

Meanwhile, J. Montesinos has proved the result using surgery on knots and links, which is very different from Hilden's method. □

It is worth noticing that figure-8 knot, the Borromean rings and the Whitehead link are all universal (see [Hilden, Lozano and Montesinos, 1983]).

Chapter 4

Three-Dimensional Systems

Due to the complexity of 3-manifold topology, the general theory of 3-dimensional dynamical systems is much more involved than its 2-dimensional counterpart. Among so many interesting aspects, knot theory found its close relation with the study of 3-dimensional dynamical systems, especially the periodic solutions. We shall extend our 2-manifold theory coupled with Heegaard splittings and connected sums to approach a theory of 3-dimensional dynamical systems. Moreover, this part also concerns the study of generalized Smale solenoids linked with the branched coverings of 3-manifolds.

4.1 Knots, Links and Chaos

Over the past decade, several attempts have been made to draw knot theory and dynamical systems closer together. It turned out that knot and link invariants can be used to describe periodic orbits and hence help better understand the underlying ODEs. Since any knot can be expressed in terms of braids, in this section we will consider writing down general explicit differential equations for these braids over a finite time interval, and then making the vector field periodic. In this way, we can glue the two ends in the phase space at successive periodic time points together (see fig. 4.1), which will give us the desired knot embedded within a solid torus.

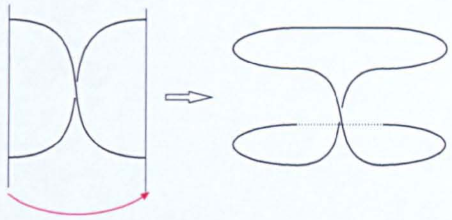


Figure 4.1: Constructing knots from braids.

4.1.1 C^∞ Functions

The twists in the braids mentioned above will be achieved by using C^∞ functions. We shall now give a brief overview of the theory of C^∞ functions which is needed later. All the results here are well-known, and can be found, for example, in [Helgason, 1978].

Let $S \subset \mathbb{R}^m$ and $S' \subset \mathbb{R}^n$ be two open subsets, and $\psi : S \to S'$ be the map between these two subsets.

Definition 4.1.1. *The mapping ψ is called **differentiable** if the coordinates $y_j(\psi(p))$ of $\psi(p)$ are differentiable functions of the coordinates $x_i(p)$ for $p \in S$.*

Definition 4.1.2. *The mapping ψ is called **analytic** if for each point $p \in S$ there exists a neighbourhood U of p and an n power series P_j $(1 \le j \le n)$ in m variables such that $y_j(\psi(q)) = P_j(x_1(q) - x_1(p), \cdots, x_m(q) - x_m(p))$ $(1 \le j \le n)$ for $q \in U$.*

Definition 4.1.3. *A differentiable mapping $\psi : O \to O'$ is called a **diffeomorphism** of O and O' if, ψ is one-to-one and onto, and the inverse mapping ψ^{-1} is differentiable.*

For an analytic function on $\mathbb{R}^m$, if it vanishes on an open set, then it is identically zero. However, for general differentiable functions and in particular C^∞ functions, the situation is completely different.

Theorem 4.1.1. *If A and B are two disjoint subsets of $\mathbb{R}^m$, then there exists an infinitely differentiable function φ which is identically 1 on A and identically 0 on B.*
□

To emphasize the dependence on A and B, we often write this as $\varphi(x; A, B)$. Obviously such a function is non-analytic, since it is identically 0 or 1 for a continuous interval; however, it is infinitely differentiable, which makes it very useful in defining twists in braids.

The standard procedure for constructing such a C^∞ function is as follows: Let $0 < a < b$ and consider the function f on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp(\frac{1}{x-b} - \frac{1}{x-a}) & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Then f is differentiable and the same holds for the function

$$F(x) = \frac{\int_x^b f(t)dt}{\int_a^b f(t)dt},$$

which has value 1 for $x \leq a$ and 0 for $x \geq b$. The C^∞ function φ defined on $\mathbb{R}^m$ is

$$\varphi(x_1, \cdots, x_m) = F(x_1^2 + \cdots + x_m^2).$$

It can be seen that φ is differentiable and has values 1 for $x_1^2 + \cdots + x_m^2 \leq a$ and 0 for $x_1^2 + \cdots + x_m^2 \geq b$, by a slight abuse of notation we shall write it as $\varphi(x; a, b)$ (see fig. 4.2). In fact we can approximate it by just using an exponential function, say $y = \exp(-x^{20})$. However, for exact matching at the boundaries we require a function which is constant on certain regions of the space.

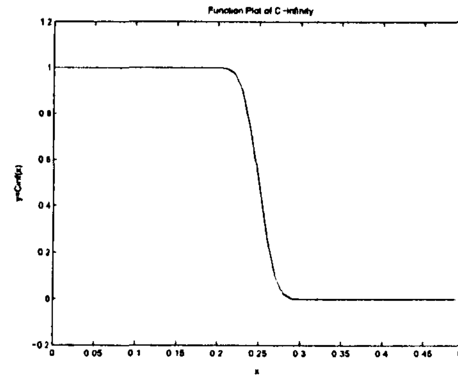


Figure 4.2: Function plot of C^∞.

4.1.2 Dynamical Systems for Braids

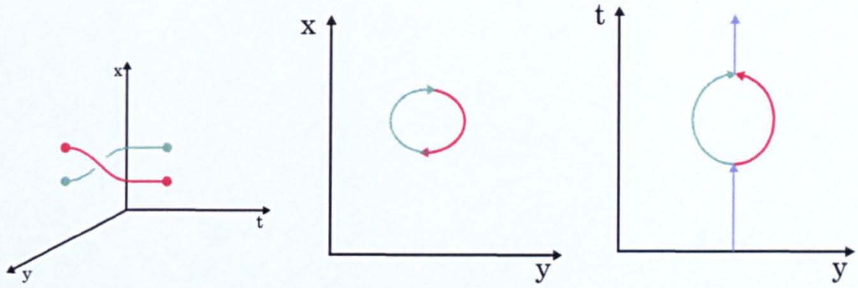
We now consider dynamical systems which contain any braids constructed from some given knots. Keep using the coordinate system shown in fig. 3.5, it is always possible to arrange the braid, which is composed of several strands, such that at an arbitrary vertical strip (an interval of the t coordinate), there is at most one twist. Hence if we can find a dynamical system which gives us the twist, then it just remains to repeat the process in order to give the appropriate number of twists necessary for the complete braid.

Each strand in the braid is given by a set of equations of the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & G_1(x,y,t) \\ \dot{y} & = & G_2(x,y,t) \\ \dot{z} & = & G_3(x,y,t) \end{array} \right. ,$$

where G_1, G_2 and G_3 are functions of x, y and t. Normally we set G_3 to be a constant such that the t-axis is effectively the time axis and periodicity of G with respect to t will then lead to a system with the required knot.

Fig. 4.3 shows a twist projected onto three different planes, namely the xt-, xy-, and ty-planes. Assume that at the two ends, all the strands are parallel to the t-axis



(a) xt-plane projection. (b) xy-plane projection. (c) ty-plane projection.

Figure 4.3: One twist of a braid.

(mathematically speaking, we need $\dot{x} = 0$ and $\dot{y} = 0$), which makes the vector fields at these connecting points (equivalently periodic points) match in order to perform the gluing. We now give explicit equations for these strings.

Note that in fig. 4.3a, the shape of the red strand is that of the C^∞ function. After studying the change of the vector field, $\dot{x}$, we have

$$\dot{x} = \varphi(t; a, b) - \varphi(t; b, c), \qquad (a < b < c). \tag{4.1}$$

In the case of an ascending strand instead of a descending one, we have

$$\dot{x} = \varphi(t; b, c) - \varphi(t; a, b), \qquad (a < b < c). \tag{4.2}$$

In the xy-plane, the transformation group brings the top strand to the bottom and the bottom one to the top without intersection (see fig. 4.3b). In the ty-plane, the trajectory for one strand is a semicircle in the middle plus two straight lines at the two ends (see fig. 4.3c). Hence a proper combination of C^∞ functions will define any desired link. Thus, for an over-crossing, such as the red strand in fig. 4.3, we have

$$\dot{y} = \varphi(t; b, c) - \varphi(t; a, b) + \varphi(t; c, d) - \varphi(t; d, e) \qquad (a < b < c < d < e),$$

while for an under-crossing, such as the blue one, it becomes

$$\dot{y} = \varphi(t;a,b) - \varphi(t;b,c) + \varphi(t;d,e) - \varphi(t;c,d) \qquad (a < b < c < d < e).$$

We assume that $\dot{t}$ = constant; then the equation for one twist in a braid is

$$\begin{aligned}
\dot{x} &= \begin{cases} \varphi(t;a,b) - \varphi(t;b,c) \\ \varphi(t;b,c) - \varphi(t;a,b) \end{cases} \qquad (a < b < c) \\
\dot{y} &= \begin{cases} \varphi(t;b,c) - \varphi(t;a,b) + \varphi(t;c,d) - \varphi(t;d,e) \\ \varphi(t;a,b) - \varphi(t;b,c) + \varphi(t;d,e) - \varphi(t;c,d) \end{cases} \\
&\qquad\qquad (a < b < c < d < e) \\
\dot{t} &= \text{constant},
\end{aligned} \tag{4.3}$$

where the choice is made depending on whether the twisted strand is ascending or descending, under-crossing or over-crossing.

Starting from (4.3), we can obtain dynamical system for just one strand, it is of the form

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{p} \pm\big(\varphi(t;a_i,b_i) - \varphi(t;b_i,c_i)\big) \ (a_i < b_i < c_i) \\
\dot{y} &= \sum_{i=1}^{p} \pm\big(\varphi(t;a_i,b_i) - \varphi(t;b_i,c_i) + \varphi(t;d_i,e_i) - \varphi(t;c_i,d_i)\big) \\
&\qquad\qquad (a_i < b_i < c_i < d_i < e_i) \\
\dot{t} &= \text{constant},
\end{aligned}$$

where p is the total number of twists in this strand, and the $\pm$ sign is taken depending on whether at the corresponding twist i, the strand ascends or descends, under-crosses or over-crosses.

The next step will be to combine all the equations for different strands together in a proper way such that they yield a system for the whole braid. By definition, an m-strand braid is a set of m non-intersecting smooth paths, so the key idea is: as long as the radius is small enough, there always exists a tubular neighbourhood, U_i, of the string, S_i, such that $U_i \cap U_j = \emptyset$ for $i \neq j$. In this way, we effectively obtain one equation for the whole braid while avoiding the intersections between different strands. Illustrated in fig. 4.4, create a tube around each string, such that within this tube, all the trajectories follow the middle strand, while outside of it, the dynamics are all zero, i.e. $\dot{x} = 0$ and $\dot{y} = 0$.

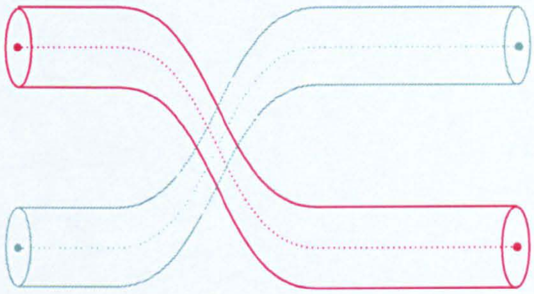


Figure 4.4: Creating a tube around each existing strand.

This is also achieved by using the C^∞ function of the form

$$\varphi = \varphi\Big(\big((x - x_1)^2 + (y - y_1)^2\big); a, b\Big), \tag{4.4}$$

where (x_1, y_1) is the coordinate of the middle strand with respect to different t value. Note that a, b need to be chosen small enough so that the tubular neighbourhood does not intersect with the others. Consequently, the dynamical system of the braid is

$$
\begin{aligned}
\dot{x} &= \sum_{j=1}^{q} \varphi_j \cdot \dot{x}_j \\
\dot{y} &= \sum_{j=1}^{q} \varphi_j \cdot \dot{y}_j \\
\dot{t} &= \text{constant},
\end{aligned}
$$

where q is the total number of strands in the braid, φ_j is the tubular function for the jth strand, and $\dot{x}_j$, $\dot{y}_j$ are the dynamics for the jth strand obtained from (4.4). As before, we set $\dot{t} = \text{constant}$.

Example. Consider the trefoil knot, we shall give two braid representations and the corresponding dynamical systems for it.

ii) As shown in fig. 4.5, the trefoil knot can be represented by a 2-strand braid. Hence the dynamics for strand 1 is of the form

$$
\begin{aligned}
\dot{x} =& \varphi(t; a, b) - \varphi(t; b, c) + \varphi(t; e, f) - \varphi(t; d, e) + \varphi(t; g, h) - \varphi(t; h, i) \\
\dot{y} =& \varphi(t; a, \frac{a+b}{2}) - \varphi(t; \frac{a+b}{2}, b) + \varphi(t; \frac{b+c}{2}, c) - \varphi(t; b, \frac{b+c}{2}) \\
&+\varphi(t; \frac{d+e}{2}, e) - \varphi(t; d, \frac{d+e}{2}) + \varphi(t; \frac{e+f}{2}, f) - \varphi(t; e, \frac{e+f}{2}) \qquad (4.5) \\
&+\varphi(t; g, \frac{g+h}{2}) - \varphi(t; \frac{g+h}{2}, h) + \varphi(t; \frac{h+i}{2}, i) - \varphi(t; h, \frac{h+i}{2}) \\
\dot{z} =& \text{constant}
\end{aligned}
$$

for some numbers $a, b, c, d, e, f, g, h, i$, while for strand 2, the equations are much the same except the change of $\pm$ sign. So let (x_1, y_1) and (x_2, y_2) stand for the x- and y-values for strands 1 and 2 respectively, we can build up a tube around each string,

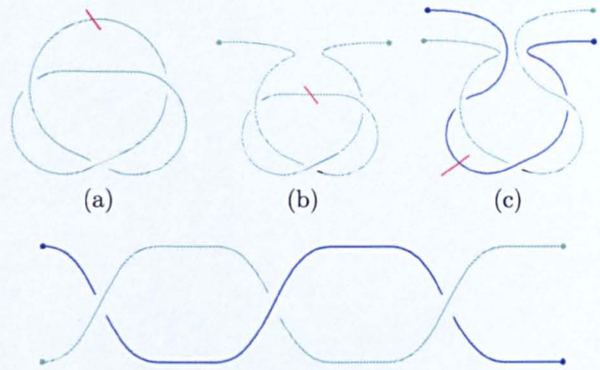


Figure 4.5: Braid construction of the trefoil knot – method 1.

and obtain the dynamical system for a braid, which is

$$
\begin{aligned}
\dot{x} &= \varphi\left((x-x_1)^2+(y-y_1)^2;\xi_1,\xi_2\right)\times \\
&\quad (\varphi(t;a,b)-\varphi(t;b,c)+\varphi(t;e,f)-\varphi(t;d,e)+\varphi(t;g,h)-\varphi(t;h,i)) \\
&\quad +\varphi\left((x-x_2)^2+(y-y_2)^2;\xi_1,\xi_2\right)\times \\
&\quad (\varphi(t;b,c)-\varphi(t;a,b)+\varphi(t;d,e)-\varphi(t;e,f)+\varphi(t;h,i)-\varphi(t;g,h)) \\
\dot{y} &= \varphi\left((x-x_1)^2+(y-y_1)^2;\xi_1,\xi_2\right)\times \\
&\quad \Big(\varphi(t;a,\frac{a+b}{2})-\varphi(t;\frac{a+b}{2},b)+\varphi(t;\frac{b+c}{2},c)-\varphi(t;b,\frac{b+c}{2}) \\
&\quad +\varphi(t;\frac{d+e}{2},e)-\varphi(t;d,\frac{d+e}{2})+\varphi(t;\frac{e+f}{2},f)-\varphi(t;e,\frac{e+f}{2}) \\
&\quad +\varphi(t;g,\frac{g+h}{2})-\varphi(t;\frac{g+h}{2},h)+\varphi(t;\frac{h+i}{2},i)-\varphi(t;h,\frac{h+i}{2})\Big) \\
&\quad +\varphi\left((x-x_2)^2+(y-y_2)^2;\xi_1,\xi_2\right)\times \\
&\quad \Big(\varphi(t;\frac{a+b}{2},b)-\varphi(t;a,\frac{a+b}{2})+\varphi(t;b,\frac{b+c}{2})-\varphi(t;\frac{b+c}{2},c) \\
&\quad +\varphi(t;d,\frac{d+e}{2})-\varphi(t;\frac{d+e}{2},e)+\varphi(t;e,\frac{e+f}{2})-\varphi(t;\frac{e+f}{2},f) \\
&\quad +\varphi(t;\frac{g+h}{2},h)-\varphi(t;g,\frac{g+h}{2})+\varphi(t;h,\frac{h+i}{2})-\varphi(t;\frac{h+i}{2},i)\Big) \\
\dot{z} &= \text{constant}
\end{aligned}
$$

where ξ_1, ξ_2 need to be chosen carefully to avoid intersections with other tubes.

Using Matlab, the phase space portrait is shown in fig. 4.6.

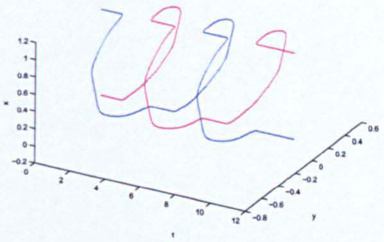
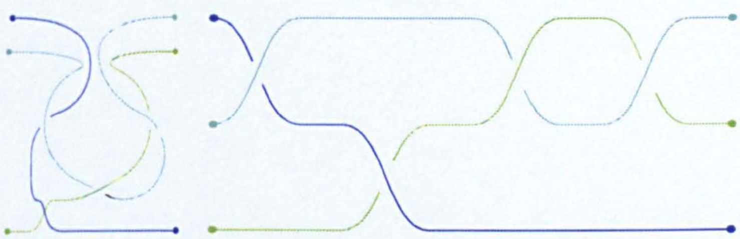


Figure 4.6: A 2-strand braid generated from Matlab.

ii) By adding another cut as shown in fig. 4.5c (the red line), a new braid representation for the same trefoil knot is introduced (see fig. 4.7).



(a) (b)

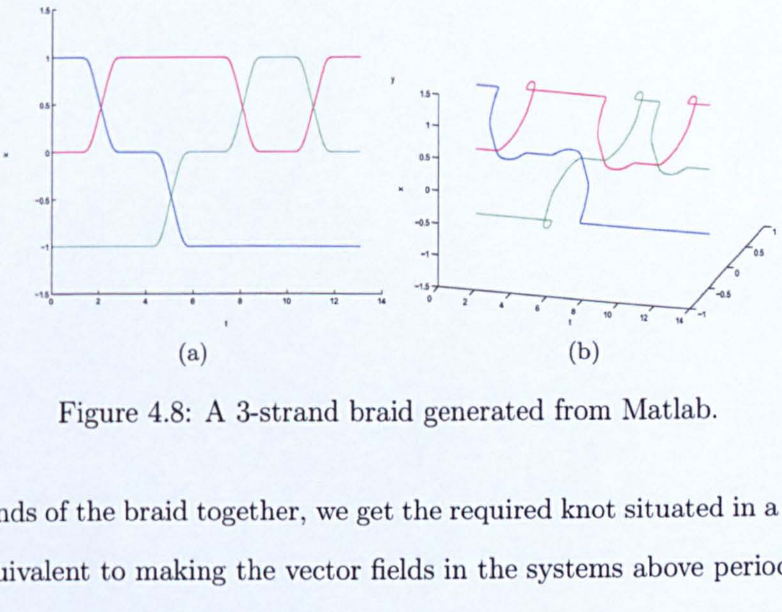
Figure 4.7: Braid construction of the trefoil knot – method 2.

In the same manner, let $(\dot{x}_1, \dot{y}_1, \dot{z}_1), (\dot{x}_2, \dot{y}_2, \dot{z}_2), (\dot{x}_3, \dot{y}_3, \dot{z}_3)$ stand for the dynamics, (x_1, y_1) ,(x_2, y_2) and (x_3, y_3) for the x- and y-value of the three strings, respectively.

Then the dynamical system for this 3-strand braid is

$$\begin{aligned}
\dot{x} &= \varphi\left((x-x_1)^2+(y-y_1)^2;\xi_1,\xi_2\right)\times \dot{x}_1 \\
&\quad +\varphi\left((x-x_2)^2+(y-y_2)^2;\xi_1,\xi_2\right)\times \dot{x}_2 \\
&\quad +\varphi\left((x-x_3)^2+(y-y_3)^2;\xi_1,\xi_2\right)\times \dot{x}_3 \\
\dot{y} &= \varphi\left((x-x_1)^2+(y-y_1)^2;\xi_1,\xi_2\right)\times \dot{y}_1 \\
&\quad +\varphi\left((x-x_2)^2+(y-y_2)^2;\xi_1,\xi_2\right)\times \dot{y}_2 \\
&\quad +\varphi\left((x-x_3)^2+(y-y_3)^2;\xi_1,\xi_2\right)\times \dot{y}_3 \\
\dot{z} &= \text{constant}
\end{aligned}$$

The phase plane portrait from Matlab is shown in fig. 4.8 After gluing the corre-



(a) (b)

Figure 4.8: A 3-strand braid generated from Matlab.

sponding ends of the braid together, we get the required knot situated in a solid torus - this is equivalent to making the vector fields in the systems above periodic.

4.1.3 Chaotic Systems

By making an extension of the methods above, we now consider constructing dynamical systems with arbitrarily knotted chaos. In particular, we shall need some elementary ideas from transformation group theory. Thus, if X is a topological space, and G is a group, we say that G is a *transformation group* on X if there is a continuous map $\varphi : G \times X \to X$ such that

i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g, h \in G$, and all $x \in X$.

ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity of G.

We usually write gx for $\varphi(g, x)$. If G is a subset of GL(n) (the general linear group), we call G a linear transformation group.

Consider now a process for modifying a given dynamical system

$$\dot{x} = f(x, t)$$

by a given function $t \to G(t)$ where $G(t)$ is an element of a (linear) transformation group for each t. We define

$$y(t) = G(t)x(t)$$

Then

$$\begin{aligned} \dot{y} &= \dot{G}x + G\dot{x} \\ &= \dot{G}G^{-1}y + Gf(x, t) \\ &= \dot{G}G^{-1}y + Gf(G^{-1}y, t) \triangleq \tilde{G}(y, t) \end{aligned}$$

Theorem 4.1.2. *Suppose that the vector field $(x, t) \to f(x, t)$ (defined on a subset $U \subseteq \mathbb{R}^n$) is periodic in t, with period π, and that the map $t \to G(t)$, where $G(t)$ belongs to some linear transformation group on U, is such that the vector field*

$$(y, t) \to \dot{G}(t)G^{-1}(t)y + G(t)f(G^{-1}(t)y, t)$$

is also periodic in t with period π, then the system

$$\begin{cases} \dot{y} &= \widetilde{G}(y,t) \\ \dot{z} &= 2\sqrt{1-z^2} \end{cases}, \qquad y \in U,\ z(0) = 0$$

is naturally defined on the torus $U \times [0,1]/\sim$, where $\sim$ is the equivalence relation such that

$$(u,t) \sim (v,t)$$

if and only if $u = v$ and $t = 0$ or π.

Proof. The result follows from the above discussion and the fact that the unique solution of the equation

$$\dot{z} = 2\sqrt{1-z^2}, \quad z(0) = 0$$

is

$$z(t) = \sin 2t$$

which is periodic with period π. □

Example. Consider the trivial system

$$\begin{cases} \dot{x}_1 &= 0 \\ \dot{x}_2 &= 0 \end{cases}, \qquad 0 \le t \le \pi,$$

which is defined on the disk $\{\|x\| < 1\}$, and let G be the orthogonal group $O(2,\mathbb{R})$. Then if

$$G(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

(i.e. counterclockwise rotation through t), we have

$$\begin{aligned}
\dot{y}(t) &= \dot{G}(t)G^{-1}(t)y(t) + G(t)\cdot\begin{pmatrix}0\\0\end{pmatrix}\\
&= \begin{pmatrix}-\sin t & \cos t\\ -\cos t & -\sin t\end{pmatrix}\cdot\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\cdot y(t)\\
&= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot y(t).
\end{aligned}$$

Hence the system

$$\begin{aligned}
\dot{y}(t) &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot y(t)\\
\dot{z} &= 2\sqrt{1-z^2}, \qquad z(0)=0
\end{aligned}$$

has trefoil knot solutions (see fig. 4.9).

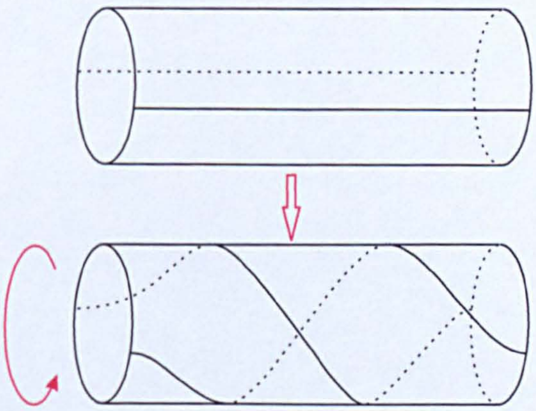


Figure 4.9: Forming a trefoil knot.

This is, of course, a trivial example and to be useful we often require to operate in different regions of the state space with different 'local' transformation groups. To do this we introduce, as in the previous sections, C^∞ functions defined on disjoint subsets of $\mathbb{R}^n$ as follows. Let U_i, $1 \le i \le K$ (for some finite K) be bounded open subsets of $\mathbb{R}^n$ such that there exist disjoint open neighbourhoods V_i of U_i for which

$$V_i \supset \overline{U_i} \text{ and } V_i \cap V_j = \emptyset, \quad \text{for } i \neq j, \quad 1 \le i, j \le K.$$

Let φ_i be a C^∞ function such that

$$\varphi_i(x) = \begin{cases} 1 & , \text{ if } x \in U_i \\ 0 & , \text{ if } x \in \mathbb{R}^n \backslash V_i. \end{cases}$$

Now let G^i, $1 \le i \le K$ be K (linear) transformation groups and let $t \to G^i(t)$ be K smooth functions with values in G^i. Then as in (4.1) we consider the system

$$\dot{x} = f(x, t)$$

and the transformed system

$$\dot{y} = \sum_{i=1}^{K} \varphi_i(y) \widetilde{G}^i(y, t),$$

where

$$\widetilde{G}^i(y, t) = \dot{G}^i(t)(G^i)^{-1}(t)y + G^i(t) f\big((G^i)^{-1}(t)y, t\big).$$

Consider the effect of G^i on V_i at $t = \pi$. Define

$$W_i = G^i(\pi) V_i$$

and let

$$X_{ij} = V_i \cap W_j, \quad 1 \le i, j \le K$$

be the K^2 intersections of the sets $\{V_i\}$ and $\{W_i\}$. We assume that the functions $t \to G^i(t)$ are chosen so that X_{ij} are mutually disjoint. Let φ_{ij} be the obvious restriction of φ_i to X_{ij} and consider the system

$$\dot{y} = \sum_{i=1}^{K} \varphi_{ij}(y)\widetilde{G}^i(y,t).$$

Theorem 4.1.3. *Using the above notation, if the function*

$$\sum_{i=1}^{K} \varphi_{ij}(y)\widetilde{G}^i(y,t)$$

is periodic with period π, then the system

$$\begin{aligned} \dot{y} &= \sum_{i=1}^{K} \varphi_{ij}(y)\widetilde{G}^i(y,t), \qquad y \in U,\ 0 \le t \le \pi \\ \dot{z} &= 2\sqrt{1-z^2}, \qquad z(0) = 0 \end{aligned}$$

(where U is a ball containing all sets V_i), is naturally defined on the torus $U \times [0,1]/\sim$, where $\sim$ is as in theorem 4.1.2. □

Example. We will use this method to generate systems with arbitrarily knotted chaos. Consider first a system with unknotted chaos. Let U_1, U_2 be the sets

$$\begin{aligned} U_1 &= \{(x_1,x_2) : 0 < x_1 < 1, 0 < x_2 < \frac{1}{3}\} \\ U_2 &= \{(x_1,x_2) : 0 < x_1 < 1, \frac{2}{3} < x_2 < 1\} \end{aligned}$$

and W_1, W_2 the sets

$$\begin{aligned} W_1 &= \{(x_1,x_2) : 0 < x_1 < \frac{1}{3}, 0 < x_2 < 1\} \\ W_2 &= \{(x_1,x_2) : \frac{2}{3} < x_1 < 1, 0 < x_2 < 1\}. \end{aligned}$$

The transformation groups G_1, G_2, G_3 correspond to:

i) the 'stretch and squeeze'

$$G_1(t) : (x_1, x_2) \rightarrow (\frac{x_1}{t}, tx_2),$$

ii) rotation

$$G_2(t) : (x_1, x_2) \rightarrow (t(x_2 - 2.5), -t(x_1 - \frac{1}{6})),$$

iii) translation

$$G_3(t) : (x_1, x_2) \rightarrow (x_1 + t, x_2 - t).$$

Finally we define

$$X_{ij} = V_i \cap W_j, \qquad 1 \leq i, j \leq 2$$

and we have the system

$$\begin{aligned} \dot{y} &= \sum_{i,j=1}^{2} \varphi_{ij}(y)\widetilde{G}^i(y,t) \\ \dot{z} &= 2\sqrt{1-z^2} \end{aligned}$$

where $\varphi_{ij}(y)$ is a C^∞ function corresponding to X_{ij},

$$G^1 = G_1, \qquad G^2 = G_3 \circ G_2 \circ G_1$$

and $\widetilde{G}^i$ is obtained from G^i as in (4.2). This system has chaotic orbits as shown in fig. 4.10. Note that if $G_i(t)$, $1 \leq i \leq 3$ are properly chosen, the system has no homoclinic orbits. (This simply implements the Smale horseshoe map.)

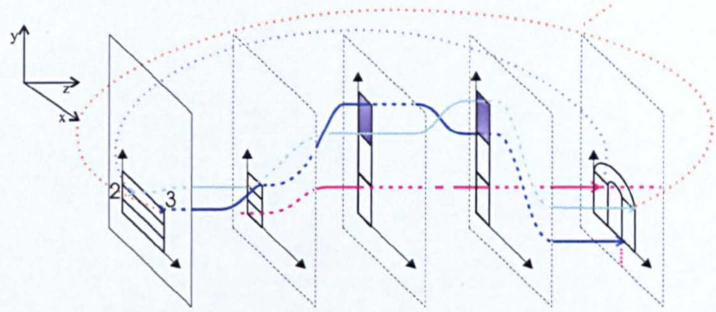


Figure 4.10: Creating a chaotic system from the Smale horseshoe.

Now consider a system of the form

$$\begin{cases} \dot{x} &= f(x,t) \\ \dot{z} &= 2\sqrt{1-z^2} \end{cases}, \tag{4.6}$$

defined for $x \in U$, where U is some bounded open set in $\mathbb{R}^n$. Let $\psi : [0,\pi] \to \mathbb{R}^n$ be any C^∞ function (which represents a strand of a braid) and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function which is 1 on U and 0 outside some neighbourhood of U. Then if we set $y = x + \psi$, the system

$$\begin{aligned} \dot{y} &= \dot{x} + \dot{\psi} = (f(x,t) + \dot{\psi})\varphi(x - \psi(t)) \\ &= \left(f(y-\psi,t) + \dot{\psi}\right)\varphi(y) \end{aligned}$$

will have trajectories like those of (4.6) in U, but 'bent' by ψ (see fig. 4.11).

More generally, if U_i $(1 \le i \le K)$ are several open (disjoint) sets in $\mathbb{R}^n$, and φ_i, ψ_i are associated functions as above, then the system

$$\begin{aligned} \dot{y} &= \sum_i \left(f(y - \psi_i(t), t) + \dot{\psi}_i(t)\right)\varphi_i(x - \psi_i(t)) \\ \dot{z} &= 2\sqrt{1-z^2} \end{aligned}$$

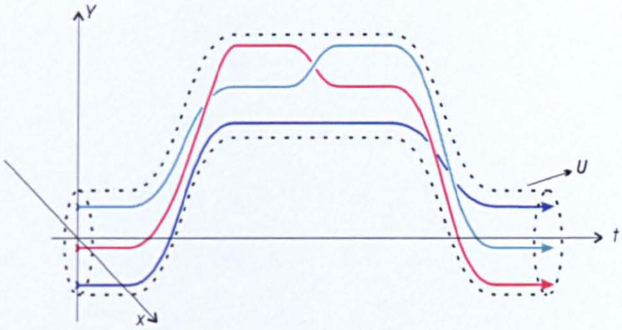


Figure 4.11: A twisted braid.

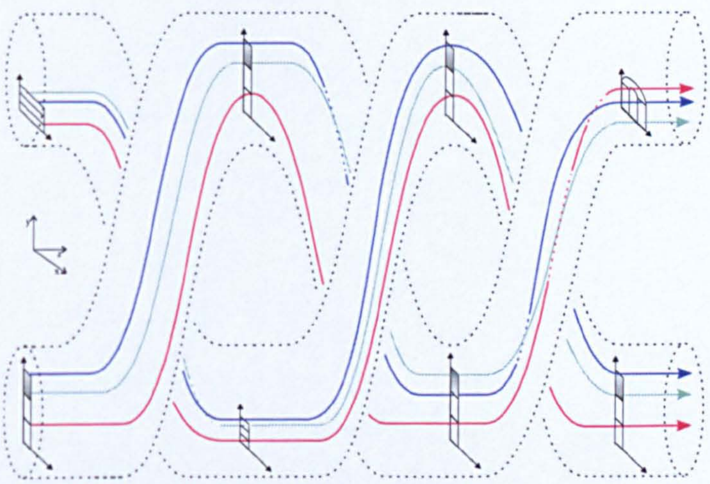


Figure 4.12: Knotted chaos.

will have trajectories similar to a given system in the regions U_i, but 'bent' by the function ψ_i. Clearly, by appropriate choice of ψ_i and φ_i we can obtain a system with arbitrarily knotted chaos, which contains no homoclinic orbits. Fig. 4.12 shows a braid representation of a trefoil knot which contains a chaotic system, Smale horseshoe, inside.

4.2 Heegaard Splittings and Connected Sums

The importance of dynamical systems theory is that, globally, they are not defined on 'flat' Euclidean spaces, but on manifolds. In fact, we have shown that the simple pendulum 'sits' naturally on a Klein bottle. Now we derive some results on the nature of three-dimensional systems and the 3-manifolds on which they 'live'.

4.2.1 Gluing Two Systems

Heegaard splittings and connected sums are two standard ways of decomposing a complicated 3-manifold as a union of 'simple' pieces. Specifically, a Heegaard diagram can be realized by a sequence of C-homeomorphisms which define a proper surface homeomorphism (see [Lickorish, 1962]). We shall use this result and consider generating a three-dimensional system by gluing together two systems defined on 'handlebodies' along specified links.

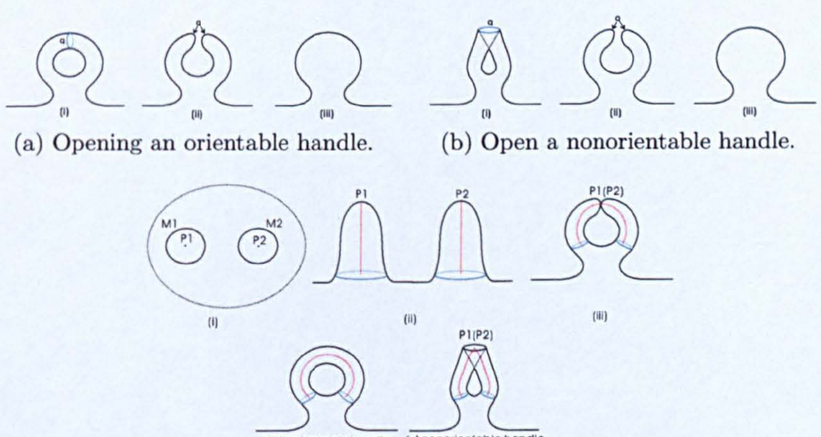
Suppose, therefore, that we wish to determine an analytic system defined on a compact 3-manifold which has an invariant surface contained in the manifold. Let M be a 3-manifold of this kind with boundary S which is a surface of genus g. From chapter 2, a dynamical system on S is given by a generalized automorphic function F, which satisfies

$$F(Tz) = \frac{ad - bc}{(cz + bd)^2} F(z), \quad T \in \Gamma, \tag{4.7}$$

where Γ is any Fuchsian group and $T \in \Gamma$ is a Möbius transformation. The following neat result shows that we can extend a meromorphic system defined on S as above to the whole of M.

Theorem 4.2.1. *Given a dynamical system on a surface S of genus g, we can extend it to a dynamical system defined throughout the solid handebody with boundary S by adding a single critical point at the 'centre' and one in the interior of each handle.*

Proof. Let $\{D_1, \cdots, D_g\}$ be a set of disjoint properly embedded 2-cells in M such that they cut M into a ball (3-cell), B_1^3 which do not contain any equilibria on S. By shrinking these 2-cells to points, we obtain a 3-cell, with $2g$ extra equilibria on the boundary. We may then regard this 3-cell as a standard ball, B_2^3, with a spherical boundary. Now extend the system defined on the surface throughout the whole of B_2^3 by simply shrinking the surface dynamics to fit on a nested set of spheres which fill out B_2^3. Thus the dynamics are foliated on a sequence of concentric spheres, and are identical on each piece. The singularity at the origin has index $2(1-g)$ by Poincaré's index theorem. To remove the equilibria inside the 3-ball apart from the one at the origin, we add a normal vector field f to the spheres such that f is zero at the origin and on the surface of B_2^3 and nonzero elsewhere. After the extension of dynamics throughout B_2^3, we can rebuild the original 3-manifold with a surface of genus-g by gluing the appropriate points of the sphere and 'blowing up' the singularities there. This can clearly be done such that each resulting handle has a single equilibrium in its interior. This process is shown in fig. 4.13. □



(a) Opening an orientable handle. (b) Open a nonorientable handle.

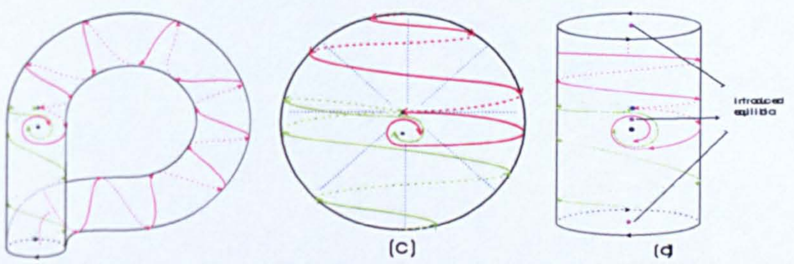
(c) Recovering the handle.

Figure 4.13: Extending the surface dynamics throughout a solid handle.

Example 1. Recall that a single pendulum is given by the following differential equations

$$\begin{cases} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{g}{l}\sin\theta \end{cases},$$

which is defined on a Klein bottle globally (see fig. 4.14a). By opening the nonorientable handle as illustrated in fig. 4.13b and shrinking the closed curve introduced by the cut, we can then extend the surface dynamics throughout the 3-ball (see fig. 4.14b). Finally after pulling and expanding the two holes as shown in fig. 4.14c, the original Klein bottle can be recovered by gluing the two ends back together properly. Note that this time the system is situated on the whole solid Klein bottle with the surface dynamics remain unchanged.



(a) Pendulum dynamics on Klein bottle. (b) Opening the handle. (c) Extending surface dynamics.

Figure 4.14: Extending the dynamics through Klein bottle.

It is known that there is exactly one nonorientable 3-manifold with a genus 1 Heegaard splitting (see [Hempel, 1976]), and since Klein bottle is such a bounding surface, the identity map will qualify to be the homeomorphism that glues the two of them together. Hence in the pendulum case, there will be exactly two such systems defined on the solid Klein bottle in the above way, and via the Heegaard diagram, these two 3-manifolds will be glued by the identity map and therefore yield a nonorientable 3-manifold.

Example 2. As shown in [Banks, 2002], a surface of genus 2 can carry at most two distinct knot types. Fig. 4.15 gives us the whole procedure of transforming a simplest knot to one type of those which can be situated on a 2-hole surface by performing

C-homeomorphisms.

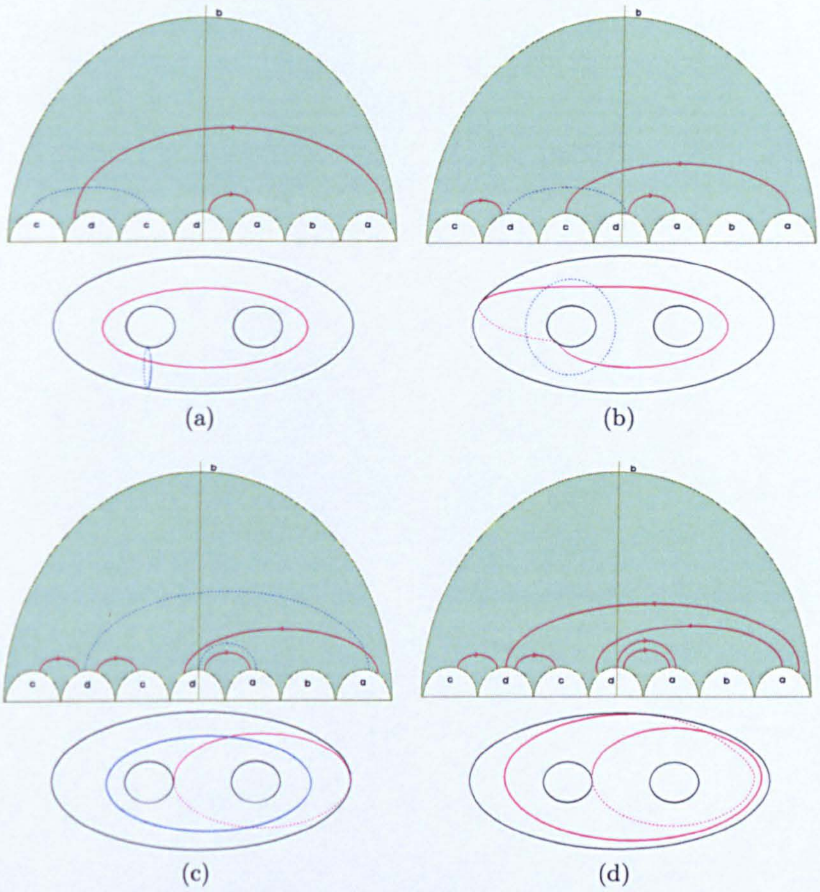
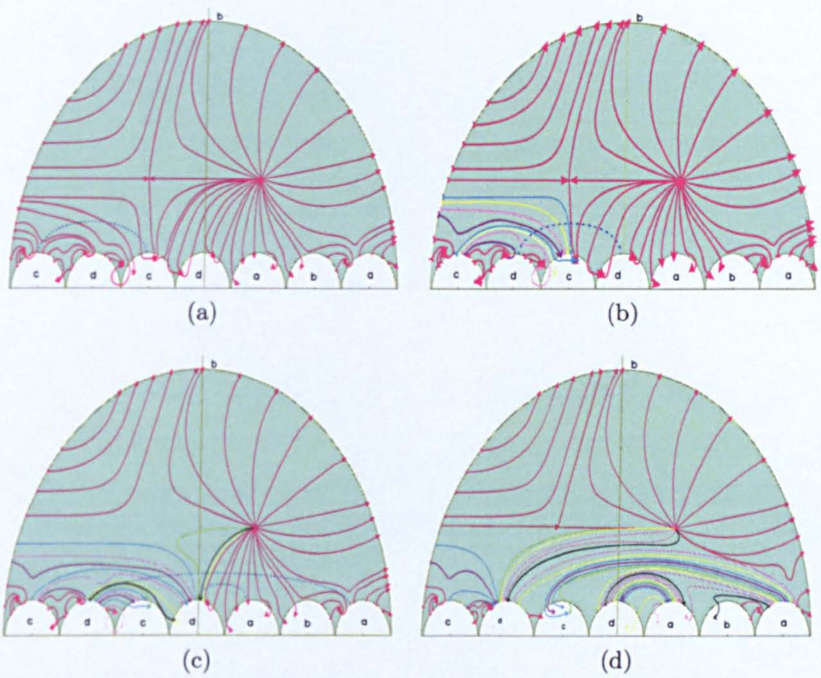


Figure 4.15: Surgery to transform one knot to another.

Also, in chapter 2, we gave explicit construction of an automorphic system generated by transformations given by (2.12). Keep using the same example, the dynamics will change according to the surgery performed on the surface. Fig. 4.16 explains this procedure.

We can then extend the system throughout the solid 2-hole torus respectively, and use C-homeomorphisms to glue the surfaces while guarantee the matching-up of the dynamics. In this way, we obtain a new system which is defined on a more complicated 3-manifold from two simpler ones.



(a) (b) (c) (d)

Figure 4.16: Surface dynamics affected by the surgery.

4.2.2 Three-dimensional systems, Heegaard Splittings and Connected Sums

In this section we consider a three-dimensional dynamical systems defined on a closed 3-manifold containing only a finite number of equilibria. We shall examine conditions under which such a system has a Heegaard splitting that respects the dynamics, i.e. contains an invariant genus-p surface, which defines a Heegaard diagram. Again the main technical tools involved will be Poincaré -Hopf index theorem and the flow-box theorem.

Since an invariant surface in M can only have those singularities of M, in order that there exists an invariant Heegaard splitting of genus $p \neq 1$, the dynamical system must have at least one equilibrium, so that systems with no equilibria can only have genus 1 Heegaard splittings, i.e. torus or Klein bottle splittings.

Theorem 4.2.2. *If a dynamical system on a closed 3-manifold M has a Heegaard* *splitting (compatible with the dynamics) of genus p, then it contains at least one equilibrium and in some subset of the equilibria, $M_1, \cdots, M_l$, there is an invariant two-dimensional local surface passing through the equilibrium with (2-dimensional) index θ_i, such that*

$$\sum_{i=1}^{l} \theta_i = 2(1-p).$$

□

Corollary 4.2.1. *A dynamical system on a compact 3-manifold which has only linearizable equilibria and a compatible Heegaard splitting of genus $p \geq 1$ must have at least $2(p-1)$ hyperbolic points.* □

The above results are only necessary, in general, to find sufficient conditions for a dynamical Heegaard splitting we first recall theorem 3.3.1, i.e. every closed, connected 3-manifold M has a Heegaard splitting. Suppose there is a splitting of a system on such a manifold M. Let K be a triangulation of M determining the splitting as in theorem 3.3.1. Hence if $V_i = N(\Gamma_i, K'')$ $(i = 1, 2)$, where Γ_1 is the set of all 1-skeletons of K, Γ_2 is the dual 1-skeleton, K'' is the second derived complex of K and N is the normal neighbourhood of Γ_i with respect to K'', then $S = (V_1 \cap V_2)$ is a surface which is invariant under the dynamics. Since M is compact, we can cover M by a finite number of open sets $\{F_1, \cdots, F_l\}$ where F_i is a flow box if it does not contain an equilibrium point of the dynamics or just a neighbourhood of such a point otherwise. Suppose that $\{p_1, \cdots, p_k\}$ are equilibrium points of the dynamics which belong to S, and $p_i \in F_i$ $(1 \leq i \leq k)$. (This can always be done by renumbering the F_i's.) Let

$$E_i^j = F_i \cap V_j, \quad 1 \leq i \leq k, \ 1 \leq j \leq 2.$$

Then we can find a refinement $\{F_1', \cdots, F_{l_1}', F_1'', \cdots, F_{l_2}''\}$ of the remaining open sets

$F_{k+1}, \cdots, F_l$ so that there exists a partition

$$\begin{aligned} \Gamma^1 &= \{E_1^1, E_2^1, \cdots, E_k^1, F_1', \cdots, F_l'\} \\ \Gamma^2 &= \{E_1^2, E_2^2, \cdots, E_k^2, F_1'', \cdots, F_l''\} \end{aligned}$$

with

$$\bigcup \Gamma^i \subseteq V_i, \quad 1 \leq i \leq 2$$

such that the sets Γ^1 and Γ^2 are invariant under the dynamics. By taking the flow boxes small enough, we can associate a triangulation of the manifold M (by taking the corners of the flow boxes away from the vertices) which is arbitrarily close to the original one. Conversely, if we can find a system of flow boxes for the dynamics on M with the above properties and the associated triangulation, then we will have a dynamical Heegaard splitting. Thus we have proved

Theorem 4.2.3. *Consider a closed 3-manifold M on which a compact dynamical system is given. Suppose there is a refinement $\Gamma^1 \cup \Gamma^2$ of a covering of M by flow boxes or neighbourhoods of equilibria, such that Γ^1 and Γ^2 are invariant under the dynamics. Let Γ^1 and Γ^2 be triangulations of $\cup\Gamma^1, \cup\Gamma^2$, respectively, such that $\Gamma^1 \cup \Gamma^2$ is a triangulation of M' (where M' is the first derived of M). Then $(\cup\Gamma^1, \cup\Gamma^2)$ is a dynamical Heegaard splitting of M if Γ^1 and Γ^2 are dual triangulations or the two-skeletons of Γ^1 and Γ^2 have equal Euler characteristics.* □

Moreover, the 'connected sum' result in chapter 2 extends to three-dimensional case. Note that the connected sum of two compact 3-manifolds M_1, M_2 is defined by removing two 3-cells from M_1 and M_2 and attaching their (spherical) boundaries together. The Euler characteristic of a compact 3-manifold is always 0, hence by Poincaré -Hopf theorem, the total index of any vector field on the manifold is zero. First we form a connected sum by removing 3-cells which contain no equilibria. This time the singular set is a (topological) circle, so we must introduce an infinite set of equilibria or a limit cycle - by twisting the cells before gluing. Note that the cycle does not change the index, as expected. If we perform the connected sum by removing cells containing equilibria without introducing new singularities, then the equilibria must be 'dual' in the sense that regions on one part which point out of the cell must be matched to those on the other part which point inwards. Clearly, the indices of such critical points in three dimensions are the inverse of each other, giving a total index change of 0, again as expected by the Poincaré -Hopf theorem. If during the procedure of removing 3-cells which contain critical points, we introduce new singularities, then from the combination of the statements above, it is known that the total change of index is still zero.

4.3 3-Dimensional Automorphic Systems

In chapter 2, we showed how to define general (analytic) systems on 2-manifolds by using the theory of automorphic functions, we shall now consider extending this result and propose to show how to generalize explicit differential equations that naturally have global behaviour on 3-manifolds. Note that commutativity is assumed in this section.

The idea follows from the 2-dimensional case, i.e. by using the generalized automorphic functions. To denote points in $\mathbb{R}^3$, we use the following coordinates:

$$\begin{aligned} \mathbb{R}^3 &= \mathbb{C} \times (-\infty, \infty) \\ &= \{(z, r) \mid z \in \mathbb{C}, r \in \mathbb{R}\} \\ &= \{(x, y, r) \mid x, y, r \in \mathbb{R}\}. \end{aligned}$$

Also, $\mathbb{R}^3$ can be regarded as a subset of Hamilton's quaternions $\mathcal{H}$, so a point p

($p \in \mathbb{R}^3$) can be expressed as a quaternion whose fourth term equals zero, i.e.

$$p = (z, r) = (x, y, r) = z + rj,$$

where $z = x + yi$ and $j = (0, 0, 1)$, then

Definition 4.3.1. *A* ***Möbius transformation*** *of* $\widehat{\mathbb{R}}^3$ *is a finite composition of reflection of* $\widehat{\mathbb{R}}^3$ *in spheres, where* $\widehat{\mathbb{R}}^3$ *is the one-point compactification of* $\mathbb{R}^3$*, i.e.*

$$\widehat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}.$$

It is exactly the same linear fractional transformation as in two dimensions of the form

$$T = \frac{ap + b}{cp + d}, \tag{4.8}$$

only where $a, b, c, d \in \mathbb{R}^3$ and $ad - bc \neq 0$.

A Möbius transformation is a conformal map of the extended 3-sphere, (i.e. Riemann 3-manifold,) denoted by $\mathbf{Aut}(\widehat{\mathbb{R}}^3)$. Moreover, (4.8) can be represented in terms of a matrix for simplicity

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In fact there exists a group homeomorphism $h : \mathrm{GL}(2, \mathbb{R}^3) \longrightarrow \mathbf{Aut}(\widehat{\mathbb{R}}^3)$ given by

$$h : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow T,$$

which becomes an isomorphism on the projective special linear group $\mathrm{PSL}(2, \mathbb{R}^3)$ (i.e. those elements of $\mathrm{GL}(2, \mathbb{R}^3)$ with positive determinant modulo the scalar matrices).

It is known that 3-dimensional hyperbolic space (or 3-dimensional hyperbolic manifold) is the unique 3-dimensional simply connected Riemannian manifold with constant sectional curvature -1 (see, e.g., [Elstrodt, 1998]). Also, since Möbius transformations are defined on Riemannian manifolds, they can then be used to generate a discrete group of discontinuous isometries, or equivalently, the transformation group Γ as in chapter 2, of the upper-half space $\mathbb{U}^3$, where

$$\begin{aligned}\mathbb{U}^3 : &= \mathbb{C} \times (0, \infty) \\ &= \{(z, r) \mid z \in \mathbb{C}, r > 0\} \\ &= \{(x, y, r) \mid x, y, r \in \mathbb{R}, r > 0\}.\end{aligned}$$

Note that like in 2-dimensional case, $\mathbb{U}^3$ is a model for hyperbolic space, so we can use Γ to tessellate $\mathbb{U}^3$ and obtain a 3-manifold, M, by Γ-side-pairing either the fundamental region or a finite collection of discrete regions which are congruent to the fundamental region. In this way, we call M is with $(\mathbb{U}^3, \Gamma)$-structure.

By restricting attention to the upper-half space, the automorphism group becomes PSL$(2, \mathbb{C})$ (linear fractional transformations with complex coefficients). If T is a map of the form (4.8), where $a, b, c, d \in \mathbb{C}$, we have

$$\begin{aligned}T(p) &= T(z + rj) \\ &= \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{\|cp + d\|^2} \\ &\quad + j\frac{r}{\|cp + d\|^2}.\end{aligned}$$

Similarly, for an element $g \in$ PSL$(2, \mathbb{C})$, it is classified as *parabolic*, *hyperbolic* or *elliptic* if $|tr(g)|$ equals, greater or less than 2, respectively, where $tr(g) \in \mathbb{R}$.

In order to obtains systems $\dot{p} = f(p)$ with Γ-invariant trajectories, we require the following invariant property of the vector field f:

Lemma 4.3.1. *The system*

$$\dot{p} = f(p)$$

will have Γ-invariant trajectories for any given discrete group Γ of isometries of hyperbolic 3-space X, if

$$f(p) = \frac{d\big(T^{-1}(T(p))\big)}{dp} \cdot f\big(T(p)\big), \qquad \forall\, T \in \Gamma. \tag{4.9}$$

Proof. This follows exactly from its 2-dimensional counterpart we stated in chapter 2.

□

Hence, for such a map $T \in \Gamma$, the invariance of the dynamical system f given by (4.9) can be written in the form

$$F\big(T(p)\big) = \frac{ad - bc}{(cp + d)^2} \cdot F(p). \tag{4.10}$$

As before, we use modified theta series to guarantee the invariant property given by (4.10). This time

$$\theta(p) = \sum_{i=0}^{\infty} (c_i p + d_i)^2 \cdot H(p_i),$$

where $p \in \mathbb{U}^3$, $I, T_1, T_2, \cdots$ are elements of Γ, and

$$p_i = T_i(p) = \frac{a_i p + b_i}{c_i p + d_i}.$$

And the modified theta series is defined as

$$\widetilde{\theta}_1(p) = \sum_{i=0}^{\infty} \frac{(c_i p + d_i)^{2-2m}}{(a_i d_i - b_i c_i)} \cdot H_1\big(T_i(p)\big).$$

Hence follow the proof in 2-dimensional case, we have

Lemma 4.3.2. *The function*

$$F(p) = \frac{\widetilde{\theta}_1(p)}{\theta_2(p)} \tag{4.11}$$

satisfies

$$F\big(T_i(p)\big) = \frac{a_i d_i - b_i c_i}{(c_i p + d_i)^2} \cdot F(p)$$

for each i and so defines a Γ-invariant dynamical system if $m \geq 2$.

Therefore, dynamical systems given by

$$\dot{p} = F(p),$$

where $F(p)$ is defined by (4.11), will be an automorphic vector field such that the trajectories are Γ-invariant on any fundamental region. We can then either 'wrap up' one of them or choose a finite number and apply the Γ-side-pairing, both of which will give rise to systems sitting on the resulting hyperbolic 3-manifold.

Example. It is known that the upper-half space $\mathbb{U}^3$ can be tessellated by hyperbolic ideal tetrahedron. Fig. 4.17 shows one particular representation.

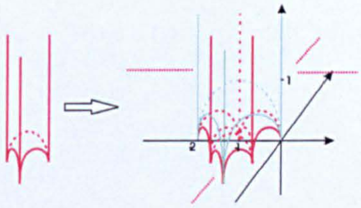


Figure 4.17: Tessellation of $\mathbb{U}^3$ by hyperbolic tetrahedra

Let the Γ-side-pairing be either translations or simple expansions and contractions. According to fig. 4.17, we have the Fuchsian group generated by the transformations

$$T_1(p) = \tfrac{p-2}{2}; \qquad T_2 = \tfrac{p}{2};$$
$$T_3(p) = \tfrac{p-1-\sqrt{3}i}{2}; \qquad T_4(p) = \tfrac{p+3+\sqrt{3}i}{2};$$
$$T_5(p) = p+2.$$

Choosing

$$H_1(p) = p + \frac{1}{2} + \frac{\sqrt{3}}{2}i + 5j, \qquad H_2(p) = 1,$$

we obtain a dynamical system by the modified automorphic functions. Note that in this example, H_1 and H_2 do not define poles within the phase-space. However, the system will have poles introduced by the modified theta series. In fact, the whole z-plane will be covered with equilibria due to the fact that it contains only cusp points. Fig. 4.18 shows one possible construction of a hyperbolic 3-manifold by translation.

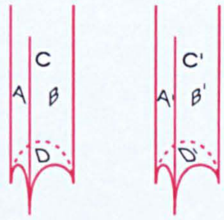


Figure 4.18: Side-pairing two tetrahedra by translation.

Moreover, fig. 4.19 illustrates the solution trajectories of the system (computed in MAPLE), and the vector fields match up perfectly at the boundaries.

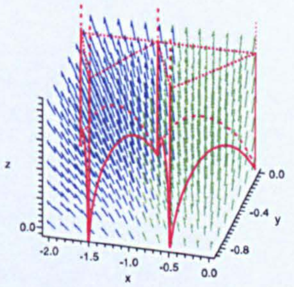


Figure 4.19: The solution trajectories for the system $\dot{p} = F(p)$.

4.4 Gluing 3-Manifolds Using the Conformal Ball Model

We now propose another way of generating dynamical systems on 3-manifolds. Instead of using the upper-half space model, we shall now investigate hyperbolic 3-manifolds under the conformal ball model. The same argument applies here, i.e.

given a group Γ of isometries of X and a proper Γ-side-pairing, we can form a 3-manifold M with an (X, Γ)-structure by gluing a finite number of disjoint convex polyhedra. Moreover, if we take into consideration of the dynamical systems naturally situated on those solid fundamental polyhedra, the Γ-side-pairing will then yield a new system defined on the resulting manifold M if and only if the trajectories match up according to the gluing pattern.

Again, as an example, we consider a regular ideal tetrahedron in $\mathbb{B}^3$, which has the shape in fig. 4.20.

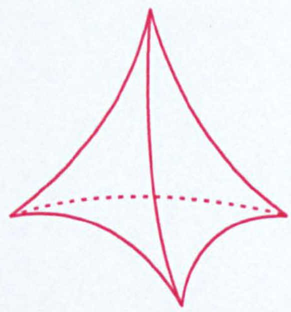


Figure 4.20: An ideal tetrahedron

Let T_1 and T_2 be two disjoint regular ideal tetrahedra in B^3, illustrated in fig. 4.21. For simplicity, we regard them as regular tetrahedra in the Euclidean space.

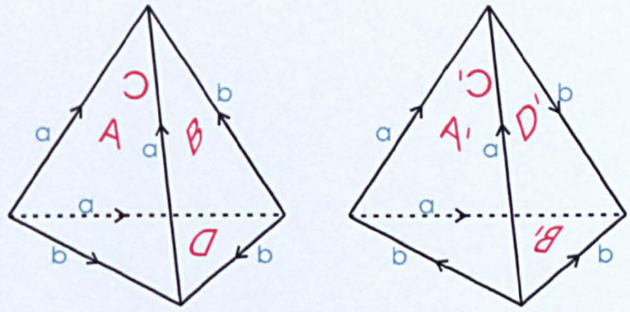


Figure 4.21: The gluing pattern of two regular ideal tetrahedra.

Because a Möbius transformation of the unit ball B^3 leaves it invariant, the permutation of the four vertices will determine the gluing pattern accordingly. Hence by labeling the sides and edges of T_1 and T_2 as in fig.4.21, there must exist an isometry of B^3 such that the sides of T_2, namely, A', B', C', D', are mapped onto those of T_1, i.e. A, B, C, D, and exactly in this order. It can be proved that this side-pairing is proper, hence implies that the resulting space will be a hyperbolic 3-manifold M. In fact, M is known as the figure-eight knot complement (see [Ratcliffe, 1994]).

Now by assuming the existence of systems on these solid tetrahedra, a new dynamical system can then be constructed on the resulting manifold via the side-pairing if and only if the trajectories match up on the corresponding boundaries of the polyhedra components. As an example, fig. 4.22 illustrates this matching up by applying the side-pairing that we mentioned above. Note that the explicit dynamics in (a) and (c) are obtained by repeating (b) ane (d) on all sides and edges of T_1 and T_2, respectively.

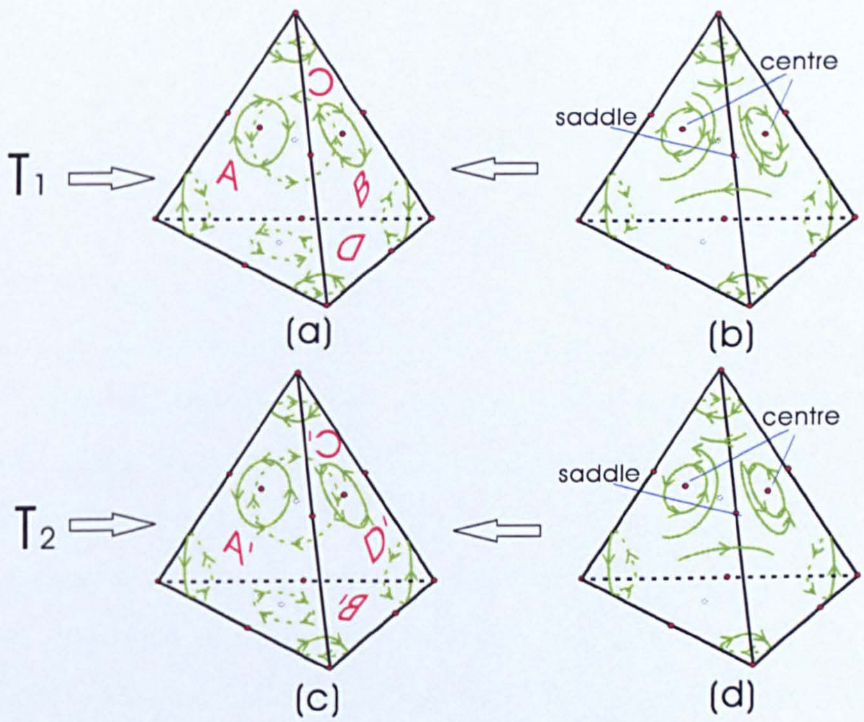


Figure 4.22: Dynamical systems on T1 and T2

4.5 Modified Reeb Foliations and Systems on 3-Manifolds

The classical Reeb foliation of the sphere and the torus are well-known (see [Candel and Conlon, 2000; Moerdijk and Mrcun, 2003]). These are obtained first from a Heegaard splitting of the sphere

$$S^3 \cong X \cup_{\partial X} X,$$

where X is a solid torus and each copy of X carries the foliation shown in fig. 4.23.

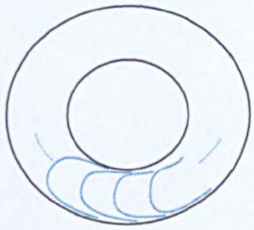


Figure 4.23: The Reeb Foliation.

Each leaf apart from the bounding torus is a plane immersed into a solid torus. We shall now show that an infinite set of dynamical systems exists on the 3-sphere which are formed by taking a genus p (for any $p \geq 1$) Heegaard splitting of S^3 and finding a generalized Reeb foliation on the solid genus p bounded 3-manifolds. Each leaf (apart from the bounding genus p surfaces and a singular leaf) will be an unbounded surface of infinite genus. Of course, it is well-known that every compact 3-manifold has a (nonsingular) foliation (see [Candel and Conlon, 2000]), essentially proved by Dehn surgery on embedded tori, each of which carries a Reeb component. However, this is an existence result and it is difficult to use to define explicit dynamical systems on 3-manifolds.

We begin by describing a simple system on the torus which can be mapped onto each leaf of the Reeb foliation to give a system on $\mathbb{R}^2$ with an infinite number of equilibria. The basic system on the torus will consist of a source, a sink and two saddles as shown in fig. 4.24. (Note that the converse of the Poincaré index theorem is not true, so it is not possible to have just a source and a saddle on the torus, although their total index would be 0.)

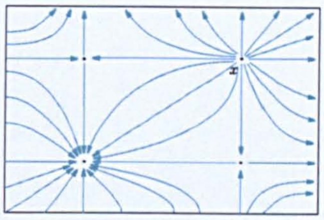


Figure 4.24: A Simple system on the torus

Consider a single noncompact leaf in the Reeb foliation consisting of a 'rolled up' plane as in fig. 4.25.

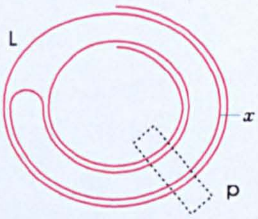


Figure 4.25: A single leaf L

The plane P cuts the leaf L into an infinite number of cylinders plus a disk. Mapping the dynamics of fig.4.24 onto each cylinder and adding a source at the origin of the disk gives the system on the plane shown in fig. 4.26.

We shall organize the dynamics on the leaf so that the sources lie 'below' the point x on the torus when the leaf is folded up. Note that the size of the shaded region in fig.4.26 depends on the leaf and shrinks to zero with origin 'below' x as in fig. 4.27.

We shall now show that there is a (singular) foliation of a 3-manifold of genus p containing a compact leaf consisting of the bounding genus p surface, an uncountable number of unbounded leaves of infinite genus and a set of one-dimensional singular leaves. Consider first the genus 2 case.

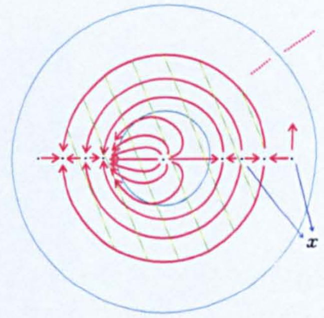


Figure 4.26: Resulting dynamics on the cylinder

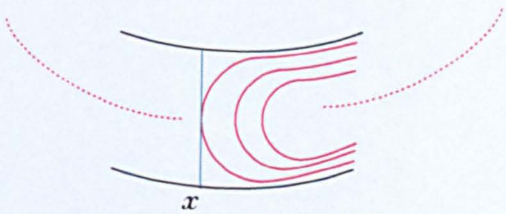


Figure 4.27: Shrinking of the leaf

Lemma 4.5.1. *Consider the orientable 3-manifold with boundary consisting of the closed surface of genus 2. There is a singular foliation of this manifold defined by a dynamical system with a singular one-dimensional invariant submanifold, an infinite number of noncompact invariant submanifolds of infinite genus and a single leaf consisting of the boundary.*

Proof. We obtain the foliation by modifying the Reeb foliation and its associated dynamical system introduced above. Hence consider two systems of the form in fig.4.26, where one has the arrows reversed (i.e. we reverse time in the corresponding dynamical system). We then form the connected sum of the bounding tori by removing a disk around the source (or link) at the point x. Then we 'plumb' each leaf in a similar way (again removing the source or sink). This will require one singular line joining the origins of the leaves which occur just 'below' x (see fig. 4.28 for illustration).

The leaves clearly have the form stated in the lemma. □

Remarks. The nonsingular leaves (apart from the genus 2 boundary surface) are

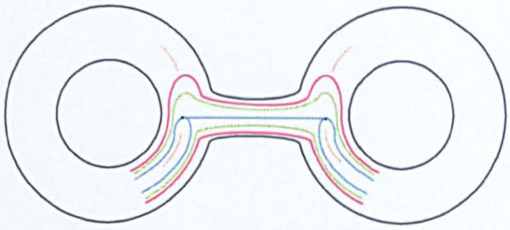


Figure 4.28: Gluing two tori via the leaves

embeddings of the surfaces shown in fig.4.29.

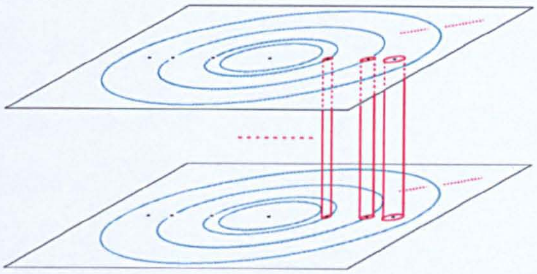


Figure 4.29: A typical leaf.

Note that we must have at least one singular fibre in order to introduce such a foliation on a higher genus surface. For we have

Theorem 4.5.1. *Any foliation of codimension 1 of a compact orientable manifold M of dimension 3 with finite fundamental groups and genus > 1, which is transversally oriented must have a singular leaf.*

Proof. By Nivikov's theorem (see [Moerdijk and Mrcun, 2003]), any codimension 1 transversely orientable foliation of M has a compact leaf and if M is orientable, this compact leaf is a torus containing a Reeb component. Thus if M contains a compact leaf of genus > 1, it is not a torus and hence there must exist a singular leaf. □

Remarks. We can find a similar singular foliation of a genus-2 3-manifold by adding a handle between the stable and unstable points on the torus in fig. 4.24. This gives a typical leaf shown in fig. 4.30, rather than the one in fig. 4.29.

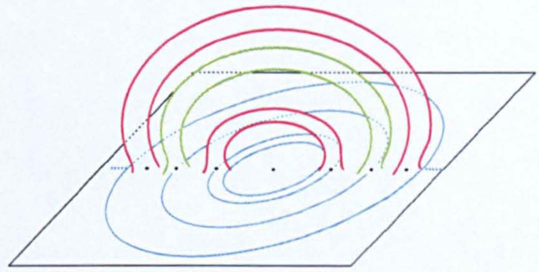


Figure 4.30: A typical leaf obtained by adding handles.

We now define systems on 3-manifolds by gluing two systems of the form above situated on solid genus-p surfaces by the use of a Heegaard diagram.

Theorem 4.5.2. *For any 3-manifold M, and any $p > 0$, there is a Reeb-like dynamical system on M given by gluing two systems of the form given in Lemma 4.5.1.*

Proof. Let (V_1, V_2) be a Heegaard splitting of M with genus p and let ϕ_1, ϕ_2 be dynamical systems defined on V_1, V_2 of the form given in Lemma 4.5.1, respectively. Let $\psi : \partial V_1 \to \partial V_2 \simeq \partial V_1$ be the homeomorphism that defines the Heegaard diagram. By using C-homeomorphisms of the type in [Lickorish, 1962], we can assume that ψ is smooth. Now let $V_2(t)$ be a solid genus-p handlebody contained within V_2 (as in fig. 4.31) so that $V_1(1) = V_2$ and $V_2(0)$ is a solid genus-p handlebody properly contained in V_2. We can extend ψ to a smooth map $\widetilde{\psi} : V_2 \to V_2$ by the homotopy

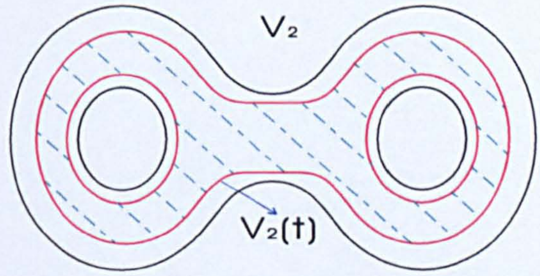


Figure 4.31: A solid genus-p handle-body contained within V_2.

$$\widetilde{\psi} = \begin{cases} (1-t)I + t\psi & \text{on } \partial V_2(t) \\ I & \text{on } V_2(0) \end{cases} . \tag{4.12}$$

Let X_2 be the vector field corresponding to ϕ_2 on ∂V_2. Then we 'twist' the dynamics on V_2 by ψ, i.e. $(\psi^{-1})_* X_2$ and extend this to V_2 in an obvious way using 4.12. Then the dynamics on ∂V_2 match those on ∂V_1 according to the Heegaard diagram and the result is proved. □

4.6 Smale Solenoids as Knotted Attractors

In [Smale, 1967], Smale classified the famous *Axiom A* systems (see Appendix B), whose non-wandering set, Ω, consists of a finite number of indecomposable subsets, i.e., the basic sets. Furthermore, he introduced the idea of solenoids into dynamical systems as hyperbolic attractors, and they have been widely studied ever since. A resumé of Smale theory can be found in Appendix B.

4.6.1 Smale Solenoids on Solid Tori

Smale solenoids in three-dimensions are defined in the following way. Let $T = S^1 \times D^2$ denote a solid torus and $e : T \to T$ be a D^2-level-preserving embedding such that the set $\bigcap_{m=0}^{\infty} \left(D^2 \cap e^m(T)\right)$ is a Cantor set.

Definition 4.6.1. *Let M be a compact, closed 3-manifold and $f : M \to M$ be a diffeomorphism. If there exists a solid torus $T \subset M$ such that $f|T$ (resp. $f^{-1}|T$) is conjugate to $e : T \to T$ above, then $S = \bigcap_{|n|>1}^{\infty} f^n(T)$ (resp. $S = \bigcap_{|n|>1}^{\infty} f^{-n}(T)$) will be called a* ***Smale solenoid****.*

Fig. 4.32 illustrates $S^1 \times D^2$ and its image under $f : S^1 \times D^2 \to S^1 \times D^2$ defined by $z \times D^2 \to z^2 \times D^2$.

In [Jiang, Ni and Wang, 2004], automorphisms containing a knotted Smale solenoid as an attractor in 3-manifolds are studied. In particular, the following theorem is proved

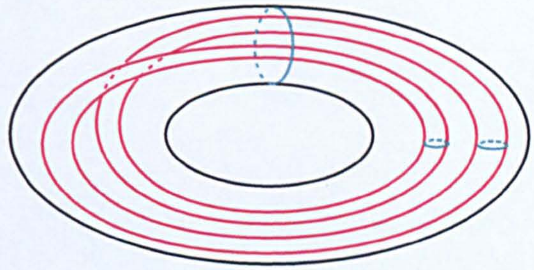


Figure 4.32: A 3-dimensional Smale solenoid.

Theorem 4.6.1. *Suppose M is a compact 3-manifold. There is a diffeomorphism $f : M \to M$ with the non-wandering set $\Omega(f)$ a union of finitely many Smale solenoids if and only if the manifold M is a lens space $L(p,q), p \neq 0$.*

It was proved by using the Loop theorem and Haken's finiteness theorem. We will now propose a simpler proof to the above result. Note that since the IF part is to find the existence result, it is only left to prove the ONLY IF part, i.e.

Lemma 4.6.1. *Suppose $f : M \to M$ is a diffeomorphism of a closed orientable 3-manifold M. If the non-wandering set $\Omega(f)$ of f contains only finitely many Smale solenoids, then it can only have two such solenoids. Moreover, one is the attractor, the other is the repeller.*

To prove the above lemma, we need the following result while keep using the notations

Lemma 4.6.2. *If the non-wandering set Ω contains only finitely many stable and unstable equilibria, then it can only have two such points. Moreover, one is stable and the other is unstable.*

Proof. Clearly, $\Omega(f)$ must contain a combination of stable and unstable nodes, due to the fact that the dynamics need to 'match up'. Furthermore, since these are the only possible structures in the basic set, we can always assume that within $M - \Omega(f)$ all the trajectories go parallel (by flow-box theorem). Suppose $\Omega(f)$ contains n_1 stable nodes and n_2 unstable nodes.

i) $n_1 = n_2 = 1$. One pair of stable and unstable points are always valid (as shown in fig. 4.33a). In fact, S^n always admits such pair of points in the corresponding dimension.

ii) $n_1, n_2 > 1$. In this case, it is necessary to have separatrices to determine individual direction of the solutions. Hence, new basic structure, such as saddle points, will be introduced, which contradicts the assumption (see fig. 4.33b-4.33e for illustration).

Hence the result follows. □

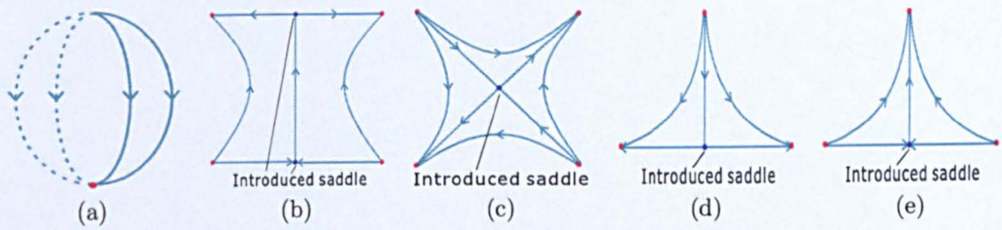


Figure 4.33: Combinations of stable and unstable nodes.

Proof to lemma 4.6.1. Suppose

$$\Omega(f) = S_1 \cup S_2 \cup \cdots \cup S_n, \quad n \geq 1$$

is the non-wandering set for f. By definition, $S_i = \bigcap_{|n|>1}^{\infty} f^n(T_i)$ is the attractor (where all dynamics enter), while $S_j = \bigcap_{|n|>1}^{\infty} f^{-n}(T_j)$ is the repeller (where all dynamics leave). Moreover, each of these solenoids is situated within the corresponding solid torus.

Without loss of generality, we can shrink S_i (which is a tubular neighbourhood of some twisted closed curve, isomorphic to S^1), to a point while leaving $T_i - S_i$ unchanged, as shown in fig. 4.34.

Note that after the shrinking, we end up with a discrete topology. Conversely, the original solenoid S_i can always be restored by 'blowing-up' p_i accordingly.

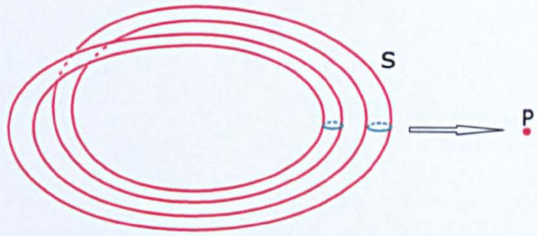


Figure 4.34: Shrinking a Smale solenoid S to a point p.

Since $\Omega(f)$ contains only solenoids, and they are either attractor or repellers, the shrinking points p_i can only be either stable or unstable critical points. Hence by lemma 4.6.2, the result follows. □

Because the proof we proposed does not involve specific 3-manifold topology, the result can then be extended to higher dimensional cases. In fact, the generalized $(2n+1)$-dimensional lens space is obtained by gluing the boundaries of two $(2n+1)$-dimensional tori, each of which has an S^n meridian and an S^n longitude.

4.6.2 Generalized Smale Solenoids On Genus-p 3-Manifolds

In this section, we are going to investigate the possibility of extending the result in [Jiang, Ni and Wang, 2004] and study generalized Smale solenoid attractors defined by automorphisms on general 3-manifolds.

Let N be a genus-p solid torus, and a_i $(1 \leq i \leq p)$ represent a sequence of trivial knots where each one of them surrounds the corresponding element in $\pi(N)$ (see fig. 4.35).

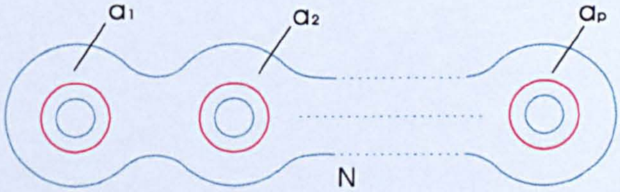
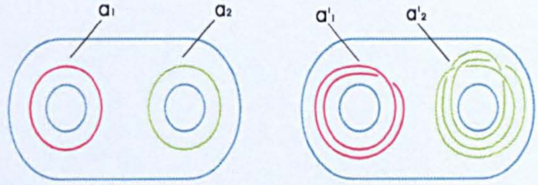


Figure 4.35: Trivial Knots $\{a_i\}$ surrounding $\pi_1(N)$.

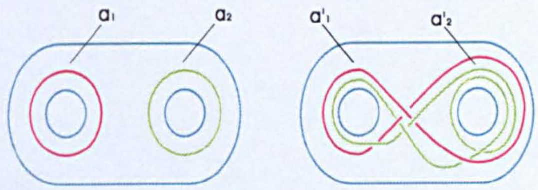
Let $e : N \to N$ be a linear embedding that maps a_i to a'_i such that

i) a'_i is a knot that involves only one generator of $\pi(N)$, and is a w-string braid with $w > 1$ (see fig. 4.36a);

ii) a'_i involves more than one generator of $\pi(N)$, in this case, a'_i can be intertwined with the image a'_j of a_j to form a link (see fig. 4.36b).



(a) a'_i only involves one generator



(b) s'_i involves more than one generator

Figure 4.36: Two possible pictures of a'_i

Hence we have

Theorem 4.6.2. *The embedding $e : N \to N$ above is a homomorphism that maps the solid genus-p torus onto a tubular neighbourhood S with the core generated by*

$$a'_1 \bigcup_{L'_{12}} a'_2 \bigcup_{L'_{23}} a'_3 \bigcup \cdots \bigcup_{L'_{(p-1)p}} a'_p,$$

where L'_{ij} is the image of the one-dimensional branch L_{ij} that joins a_i with a_j and $a \cup_L b$ denotes the disjoint union of a, b and L where the endpoints of L are identified with points of a and b (see fig. 4.37 for illustration).

Proof. We will prove the theorem for the genus-2 torus case and the general genus-p case will then follow.

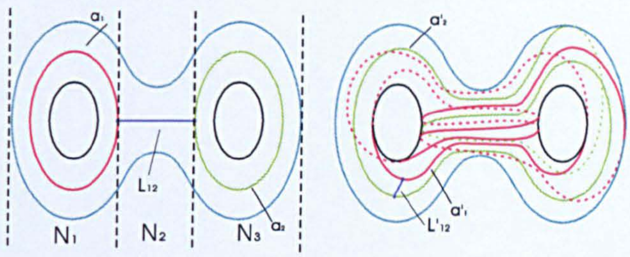


Figure 4.37: $e : a_1 \bigcup_{L_{12}} a_2 \to a'_1 \bigcup_{L'_{12}} a'_2$

Let a_1, a_2 denote the trivial knot surrounding each of the two generators of $\pi_1(N)$, hence the core of N is $a_1 \bigcup_{L_{12}} a_2$, denote it by a_c. We can decompose N into 3 parts, namely, N_1, N_2 and N_3 (as shown in fig. 4.37). With a'_1 and a'_2 being the image of a_1 and a_2 respectively, L_{12} will be mapped onto a one-dimensional curve L'_{12} that joins a'_1 with a'_2 due to the fact that e is an embedding. Denote $a'_1 \bigcup_{L'_{12}} a'_2$ by a'_c. Hence a tubular neighbourhood of a_c will be mapped onto a tubular neighbourhood of a'_c under e. And since a_c is the core of the genus-2 torus N, e actually maps N onto a tubular neighbourhood, N' of a'_c. □

Remarks. Note that by applying the embedding map e more than once, (i.e., e^2, e^3, $\cdots$,) the embedding is naturally defined within the tubular neighbourhood $N' \subset N$ of a'_c. Hence we can define the generalized Smale solenoid as follows:

Definition 4.6.2. *Let M be a compact, closed 3-manifold and $f : M \to M$ be a diffeomorphism. If there exists a solid genus-p torus $N \subset M$ such that $f|N$ (resp. $f^{-1}|N$) is conjugate to $e : N \to N$ defined above, then $S = \bigcap_{|m|>1}^{\infty} f^m(N)$ (resp. $S = \bigcap_{|m|>1}^{\infty} f^{-m}(N)$) will be called a **generalized Smale solenoid**. Moreover, if $m > 0$, S is a hyperbolic attractor of f; otherwise it is a repeller of f.*

Note that the generalized Smale solenoid carries information about the way $e(N)$ is braided in N and also how N is knotted and framed as a subset of M.

By assuming that the generalized Smale solenoid is the only type of basic set for f, we have

Theorem 4.6.3. *Suppose M is a compact, closed orientable 3-manifold, there is a diffeomorphism $f : M \to M$ such that the non-wandering set $\Omega(f)$ contains only a finite number of disjoint genus-p Smale solenoids if and only if M is a genus-p 3-manifold. Moreover, for the sufficient condition, the number of generalized Smale solenoids in $\Omega(f)$ can only be two: one is attractor, and the other is repeller.*

We will give the proof in three parts. For the necessary part,

Proof. Assume $\Omega(f)$ is a union of generalized Smale solenoids, say S_1, S_2, $\cdots$, S_n. Then by definition, it is easy to see that $\Omega(f)$ satisfies the *Axiom A* condition, i.e. $f|S_i$ is hyperbolic and the periodic points are dense in S_i.

By **spectral decomposition theorem** (see Appendix B) of diffeomorphisms, $\Omega(f)$ can be written uniquely as a finite union of disjoint, closed, invariant indecomposable subsets B_1, B_2, $\cdots$, B_k. We are now going to prove $S_i = B_l$.

Since S_i is also an f-invariant closed subset which contains a dense orbit, $S_i \subset B_l$, for $l = 1, 2, \cdots, k$. Conversely, given a point $x \in U_i$, where U_i is a closed neighbourhood of S_i, its orbit $o(x)$ is dense in B_l due to the fact that S_i is an attractor of f (or f^{-1}) and B_l contains a dense orbit. Since $x \in \Omega(f|U_i)$, and $x \in S_i$, we then have $B_l = \overline{o(x)} \in S_i$. Hence $S_i = B_l$. Therefore

$$\Omega(f) = S_1 \cup S_2 \cup \cdots \cup S_n, \quad n \geq 1$$

is a disjoint union of generalized Smale solenoids with the same genus p. Furthermore, $S_i \subset N_i$, where N_i is a solid genus-p torus.

Because of the disjoint property of S_i, it is always possible to assume $N_i \cap N_j = \emptyset$ for $i \neq j$. By rearranging the sequence $\{S_i\}$ if necessary, without loss of generality, we can assume that S_1, S_2, $\cdots$, S_k are attractors of f and S_{k+1}, $\cdots$, S_n are repellers.

Therefore,

$$f(N_i) \subset \text{Int}(N_i), \quad i = 1, \cdots, k \leq n, \tag{4.13}$$

$$\& \qquad f^{-1}(N_j) \subset \text{Int}(N_j), \quad j = k+1, \cdots, n. \tag{4.14}$$

Define $V_i = \bigcup_{m=1}^{\infty} f^{-m}(\text{Int}(N_i))$ for $i = 1, \cdots, k$. It is easy to see that V_i is open because f is a homeomorphism. Moreover, due to the unique solution of dynamical systems and the fact that generalized Smale solenoids are the only type of basic set structure, we have

$$V_i \cap V_j = \emptyset, \quad 1 \leq i < j \leq k \quad \& \quad f(V_i) = V_i \tag{4.15}$$

First let $n = 1$ and suppose $Y_1 = M - \text{Int}(N_1)$. Then Y_1 is compact, so from (4.13) and (4.14), $f^{-1}(Y_1) \subset \text{Int}(Y_1)$, which implies that there exists a repeller contained in Y_1, which contradicts the assumption that $n = 1$.

Hence $n > 1$. Denote $Y_2 = M - \bigcup_{j=k+1}^{n} S_j$ for $k \geq 1$. Since S_j are disjoint, Y_2 is connected and $V_i \subset Y_2$. If let $Y_3 = M - \left(\bigcup_{j=k+1}^{n} S_j\right) \cup \left(\bigcup_{i=1}^{k} V_i\right)$, then Y_3 is not empty. Also because $S_i \cap S_j = \emptyset$ and $V_i \cap V_j = \emptyset$, Y_3 is connected. Suppose $x \in Y_3$. Since $S_j \subset N_j$ is compact, we can choose N_j small enough such that $x \notin \bigcup_{j=k+1}^{n} \text{Int}(N_j)$. Hence by setting $Y_4 = M - \left(\left(\bigcup_{j=k+1}^{n} \text{Int}(N_j)\right) \cup \left(\bigcup_{j=1}^{k} V_j\right)\right)$, it is again compact, not empty and connected. Since N_j is a repeller for $j \geq k+1$ and $f(V_i) = V_i$, we have $f(Y_4) \subset Y_4$, which means $\Omega(f) \cap Y_4 \neq \emptyset$, which contradicts the fact the $\Omega(f)$ is a disjoint union of finite generalized Smale solenoids.

Hence we proved that f can only have one attractor, and similarly for f^{-1}, so $n = 2$ and one of them is attractor, the other is repeller of f. □

Now we are going to prove that M is a genus-p 3-manifold.

Proof. Let S_1 and S_2 both be generalized Smale solenoids. S_1 is an attractor, while S_2 is a repeller. Assume

$$S_1 = \bigcap_{m=1}^{\infty} f^m(N_1), \quad S_2 = \bigcap_{m=1}^{\infty} f^{-m}(N_2) \quad \& \quad N_1 \cap N_2 = \emptyset. \tag{4.16}$$

As S_1 and S_2 being the invariant sets, we have $\partial f^n(N_2) \subset \mathrm{Int}(N_1)$ for large enough n. Moreover, since $H_2(N, \mathbb{Z}) = 0$, $\partial f^n(N_2)$ must separate N_1 into two parts, say X_1 and X_2. Let $\partial f^n(N_2) = \partial X_1$, and ∂X_2 contains two components.

Because $\pi_1(N_1) \neq \pi_1(\partial f^n(N_2))$, it is obvious that the homomorphism induced by the immersion $i : \partial f^n(N_2) \to N_1$, which is, $i_* : \pi_1(\partial f^n(N_2)) \to \pi_1(N_1)$, is not injective. Hence by **Papakyriakopoulos' Loop Theorem**, $\partial f^n(N_2)$ is a compressible surface in N_1, i.e. there exists an embedded disc $D \subset M$ such that $D \cap \partial f^n(N_2) = \partial D$ and ∂D is an essential circle in $\partial f^n(N_2)$. Due to the irreducibility of N_1, we have $\partial f^n(N_2)$ bounds a p-hole torus N_1' in N_1, where p equals to the genus of N_1. So $N_1' = X_1$. Hence

$$\begin{aligned} M = (M - N_1) \bigcup_{\partial N_1} N_1 &= (M - N_1) \bigcup_{\partial N_1} (X_2 \bigcup_{\partial f^n(N_2)} X_1) \\ &= f^n(N_2) \bigcup_{\partial f^n(N_2)} N_1' \end{aligned} \tag{4.17}$$

Again by irreducibility (see appendix A) of N_2, we have $f^n(N_2)$ is a genus-p torus (maybe twisted in some way). Hence M is a 3-manifold which can be yielded by identifying two p-hole solid tori along their boundaries. Also because of the fact that a genus-p' splitting, where $p' < p$, cannot accommodate the genus-p generalized Smale solenoid structure on either of the components, so M is a 3-manifold of genus p. □

At the end we will prove the sufficient condition, i.e. if M is a genus-p 3-manifold, then it admits a diffeomorphism $f : M \to M$ with two generalized Smale solenoids being the only possible basic sets.

Proof. Since M admits a genus-p Heegaard splitting, we have

$$M = N_1 \cup_h N_2,$$

where N_1 and N_2 are both solid tori with the same genus p, and $h : \partial N_1 \to \partial N_2$ is the attaching homeomorphism. Hence (N_1, N_2, h) is a required genus-p Heegaard diagram for M (see fig. 4.38a and 4.38b).

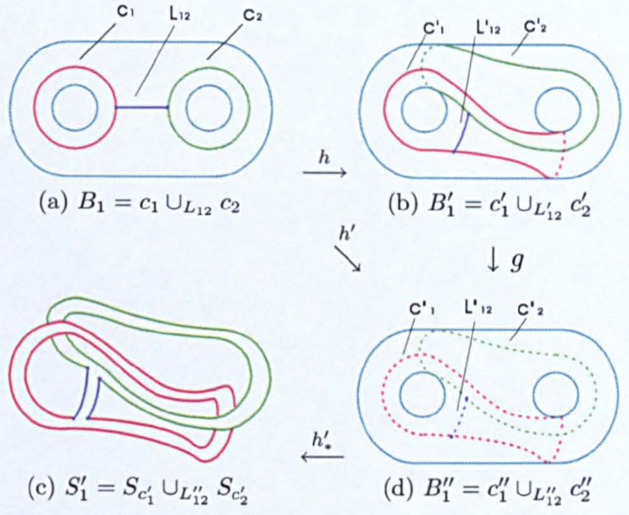


Figure 4.38: Two possible pictures of a_i'.

Without loss of generality, we can always assume that the characteristic curves on ∂N_1 are taken to be the p generators of $\pi_1(N_1)$, i.e. $\{c_1, c_2, \cdots, c_p\}$. Furthermore, under the homeomorphism h, these curves are mapped onto p distinct knots, namely, $\{c_1', c_2', \cdots, c_p'\}$. By linking c_i and c_{i+1} with a one-dimensional curve $L_{i(i+1)}$, we obtain the core of N_1, which, under h, will be mapped onto a branched 1-manifold consisting of p distinct knots c_i' $(1 \leq i \leq p)$, each linking with the neighbouring ones by $L_{i(i+1)}'$ and $L_{(i-1)i}'$. In particular, $h : L_{i(i+1)} \to L_{i(i+1)}'$ is a homeomorphism. Fig. 4.38 illustrates the operation for genus-2 case.

Now let

$$
\begin{aligned}
B_1 &= c_1 \bigcup_{L_{12}} c_2 \bigcup_{L_{23}} \cdots \bigcup_{L_{(p-1)p}} c_p \qquad \& \\
B_1' &= c_1' \bigcup_{L_{12}'} c_2' \bigcup_{L_{23}'} \cdots \bigcup_{L_{(p-1)p}'} c_p',
\end{aligned}
$$

and h' be a map induced by h but maps B_1 onto B_1'', where

$$B_1'' = c_1'' \bigcup_{L_{12}''} c_2'' \bigcup_{L_{23}''} \cdots \bigcup_{L_{(p-1)p}''} c_p''.$$

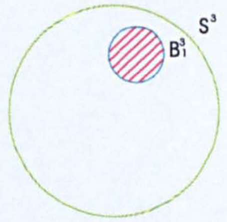
Note that B_1'' is the image of B_1' inside the solid genus-p torus N_2 (as shown in fig. 4.38b and 4.38d).

Hence h_*', induced by h', is a diffeomorphism between tubular neighbourhoods of B_1 and B_1''. Denote them by S_1 and S_1' respectively. It is always possible to take B_1'' small enough such that it is situated inside N_2, consequently, S_1' is a generalized Smale solenoid with the according genus (as in fig. 4.38c). Furthermore, since B_1 is the core of a genus-p torus, $S_1 = N_2$. Due to the property built inherently into our construction, the image of $h_*'^n$ $(n > 1)$ will always be inside S_1'. In this way, h_*' will serve as the diffeomorphism that has generalized Smale solenoid S_1' as a basic set, say the attractor. Similarly, $h_*'^{-1}$ will define the other generalized Smale solenoid S_2', which is the repeller of the system. Hence S_1' and S_2' form the only basic sets of h_*'.□

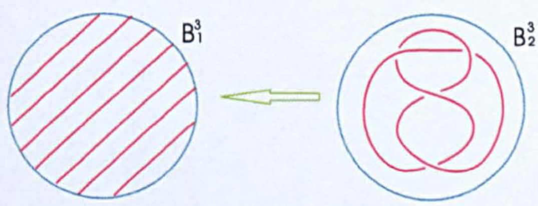
4.6.3 3-Manifolds Containing Arbitrary Generalized Smale Solenoids

So far we have considered the cases when either Smale solenoid or generalized Smale solenoid is the only type of basic set defined by $f \in \mathrm{Diff}(M)$. Clearly the genus of the solenoids needs to be commeasurable with that of the closed 3-manifold M. A natural question will be to ask whether it is possible to remove the restriction on the genus and the number of solenoids that we can put on a closed, orientable 3-manifold. The result in [Montesinos, 1974; Hilden, 1974], i.e. every closed, orientable 3-manifold is a branched covering, $P : M \to S^3$, of S^3 over a universal knot which is nontoroidal, is clearly paving the way to the solution to the above question. Note that from the perspective of dynamical systems, this universal knot, which is important for the branched covering P to be applied, must be generated by the system dynamics.

Because S^3 admits any genus Heegaard splitting, consequently we can put any pair of genus-p generalized Smale solenoids, one being the attractor while the other being the repeller, inside S^3. According to the flow-box theorem, we can cut a $B_1^3 \subset S^3$ which is far away enough from the two solenoids such that all trajectories are going parallel in B_1^3 (see fig. 4.39a). By replacing B_1^3 with another 3-ball B_2^3 which contains a nontoroidal knot, say, the figure-eight knot, we effectively change the local structure inside B_1^3 defined by f while leaving the dynamics invariant elsewhere (see fig. 4.39b for illustration). Clearly it is inevitable to introduce some new basic set structures other than solenoids. Moreover, this knot will serve as the universal knot.



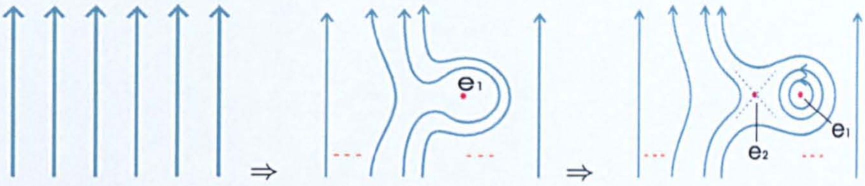
(a) S^3 containing B_1^3



(b) Replacing B_1^3 by B_2^3

Figure 4.39: Introducing the universal knot.

We now explain the surgery that changes dynamics from B_1^3 to B_2^3 in more details. One way will be to use C^∞ functions and braid representation to obtain the universal knot. Here we give a more 'natural' operation to achieve the result. By putting one nonlinear equilibrium point, e_1, the trajectories can be twisted from fig.4.40a to 4.40b. Furthermore, a periodic orbit is obtained if we add a saddle point left to e_1 as shown in fig. 4.40c.



(a) Parallel trajectories within B^3 ⇒ (b) Adding equilibrium e_1 ⇒ (c) Adding a saddle point e_2

Figure 4.40: Surgery within B^3.

Bifurcating the resulting cycle around the corresponding points, i.e. a_1 and a_2 in terms, will eventually yield the wanted figure-eight knot (see fig. 4.41 for illustration). Note that the whole operation will leave the dynamics outside B^3 unchanged.

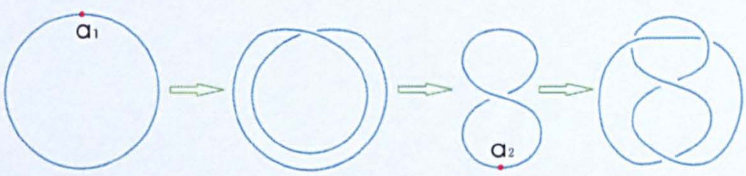


Figure 4.41: Obtaining the figure-eight knot from a periodic orbit by bifurcation.

Lifting the dynamics back to the original 3-manifold M by P^{-1} over the figure-eight knot which is constructed above, we can obtain arbitrary pair of generalized Smale solenoids on any closed, orientable 3-manifold (as shown in fig. 4.42). Hence

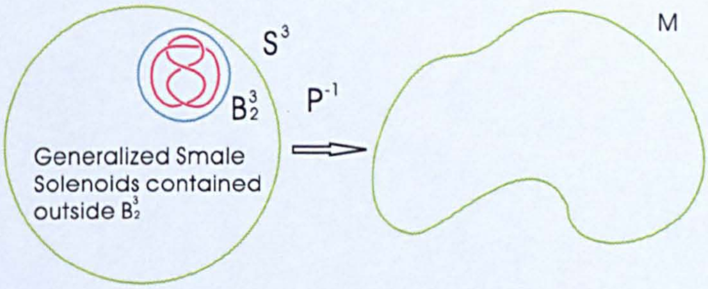


Figure 4.42: Lifting the dynamics on S^3 onto M.

we proved

Theorem 4.6.4. *By allowing basic set structure other than generalized Smale solenoid* *to exist, a closed, orientable 3-manifold M can accomodate arbitrary pair of generalized Smale solenoids with arbitrary genus p.* □

4.7 Concluding Remarks

In this chapter, we have studied knot theory from dynamical systems' point of view. Stick to the braid representations, we elaborated the procedure of generating any knot by using the theory of transformation groups and C^∞ functions. Based on these results, we then proposed an algorithm to generate 3-dimensional systems that can contain arbitrarily knotted chaos.

Inspired by the idea of generating complicated 3-manifolds from much simpler ones, (particularly Heegaard splitting and connected sums,) we considered generating new dynamical systems resulting from such surgery operations.

Moreover, we extended the result in Chapter 2, i.e. using generalized automorphic functions to yield meromorphic systems on 3-manifolds. However, because of the non-commutativity of quaternionic operations, we need to restrict the coefficients to be chosen from $\mathbb{R}$. We also studied three-dimensional systems by adopting the conformal ball model and modified Reeb foliations.

Finally, we considered automorphisms on 3-manifolds which contain generalized Smale solenoids as their basic sets and linked the dynamical behaviour thereafter defined with the topological structure of the base manifold. This can be regarded as an explicit example of how the topology interacts with the dynamics, which can be used as a kind of 'benchmarking' guiding the study of dynamical systems theory. Also, we investigated the possibility of lifting dynamics onto the original manifold from S^3 due to the fact that every closed, orientable 3-manifold is a branched covering of S^3 over some universal knot. In this way, we are able to construct systems which have arbitrary genus-p generalized Smale solenoids.

Chapter 5

Four-Dimensional Manifolds and Systems

The topology of curves and surfaces (i.e. manifolds of dimension ≤ 2) has been well understood since the nineteenth century. Three-manifold topology turns out to be much harder, however, it has been well developed and there has been steady progress in the filed during the last century. Meanwhile, the topology of high-dimensional manifolds has also been pushed forward significantly, mainly by the development of s-cobordism (see [Milnor, 1965]) and surgery theorems.

On the other hand, the topology and differentiable structure of four-manifolds is probably the most difficult of any dimension. In fact, it had remained mysterious until 1970's. However, due to the continuous efforts of numerous mathematicians, the past two decades witnessed an explosive growth in 4-manifold theory, which, surprisingly enough, has very complex interactions with various fields such as gauge theory, algebraic geometry and symplectic topology. In 1982, Donaldson introduced gauge theory, particularly the self-dual analysis of Yang-Mills equations, into the study of 4-manifold theory, and showed that smooth 4-manifolds are very much different from their high-dimensional counterparts. For example, $\mathbb{R}^4$ is the only Euclidean space that carries different differentiable structures, i.e. there are 4-manifolds homeomorphic but not differmorphic to $\mathbb{R}^4$, which are called 'exotic $\mathbb{R}^4$'. Hence the predictions about 4-manifold theory made by s-cobordism and surgery theorems have nowhere to fit in. This clearly put 4-manifold theory in a unique position, which might be regarded as a transition between low- and high-dimensional topology.

The scheme proposed by Donaldson dominated smooth 4-manifold theory for the next 12 years after its introduction. In 1994, the analysis of the Seiberg-Witten equations simplified and expanded both Donaldson's approach and results. The consequent invariants discovered by Donaldson and the Seiberg-Witten invariants thereafter can be used to provide measurement of differentiable structures to some extent.

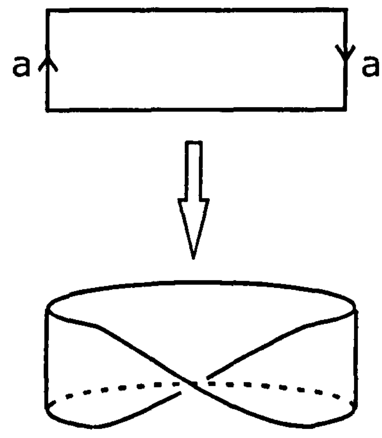
The aim of this chapter is to study the interaction between dynamical systems and four-manifold topology, and the content remains mainly speculative. However, adequate background knowledge, such as de Rham cohomology theory and differential geometry theory, is properly referred.

5.1 Vector Bundles

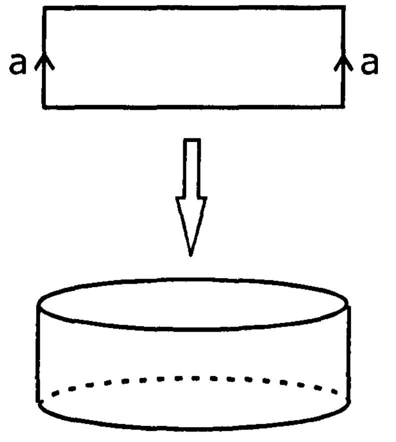
There exists an interesting connection between the vector bundles, especially tangent bundle, and the base topological spaces where they are attached. Therefore it attracted enormous attention from both topologists and geometers. Since 1930's, the so-called *characteristic classes*, i.e. methods of associating cohomology classes of base space B to the vector bundles attached, have been widely studied. The important ones are Stiefel-Whitney classes, Chern classes, Pontryagin classes and Euler classes.

We first review the basic definitions of vector bundles. The following definitions and results are well-known, and can be found, for example, [Milnor, 1965; Hatcher, 2003].

We are all familiar with the Möbius strip (see fig. 5.1a), the twisted product of a circle and a line. Moreover, an annulus is the product of a standard circle S^1 and a line (see fig. 5.1b). A natural generalization of the Möbius strip and annulus is called vector bundles.



(a) A Möbius strip.



(b) An annulus.

Figure 5.1: A Möbius strip and an annulus.

Definition 5.1.1. *An n-dimensional* ***vector bundle*** $\xi = (E, B, F, p)$ *contains the following:*

i) a topological space E which is called the ***total space****,*

ii) a topological space B which is called the ***base space****,*

iii) a continuous map $\pi : E \to B$ which is called the ***bundle projection map****,*

iv) a vector space F which is called the ***fibre****,*

v) and for each $b \in V \subset B$, there exists a homeomorphism

$$h : V \times F \to p^{-1}(V)$$

such that $\pi \circ h(b, f) = b$ for $b \in V, f \in F$.

Moreover, the homeomorphisms $\{h\}$ are taken from the general linear group of that corresponding vector space. Thus a real vector bundle has fibre $\mathbb{R}^n$ and group

$GL(n,\mathbb{R})$, a complex vector bundle has fibre $\mathbb{C}^n$ and group $GL(n,\mathbb{C})$, and a quaternionic vector bundle has fibre $\mathbb{Q}^n$ and group $GL(n,\mathbb{Q})$.

Roughly speaking, a vector bundle is a family of vector spaces, which are parameterized by a smooth base manifold B. To construct such a family, we start with an open covering of B, $\{U_\alpha : \alpha \in A\}$. Suppose the vector spaces are all isomorphic to $\mathbb{R}^m$. For every two intersecting coverings, U_α and U_β $(\alpha, \beta \in A)$, their respective vector spaces are related by

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m,\mathbb{R}),$$

where $g_{\alpha\beta}$ are called *transition functions* and satisfy the condition

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{5.1}$$

By taking a Lie group G of $GL(m,\cdot)$, we obtain a *G-vector bundle* with the corresponding transition functions taking values in G. Furthermore, if $G = O(m)$, i.e. the orthogonal group, a *fibre metric* of the bundle E is defined as the smooth function

$$<\ ,\ >_p : E_p \times E_p \to \mathbb{R} \text{ (or } \mathbb{C}),$$

where $E_p = \pi^{-1}(p)$, $p \in B$.

A *section* of a vector bundle (E, B, π) is a smooth map $\sigma : B \to E$ such that $\pi \circ \sigma =$ identity. For an open covering, $\{U_\alpha : \alpha \in A\}$, of B, we have $\sigma(p) = \{\sigma_\alpha(p)\}$, where $\sigma_\alpha : U_\alpha \to \mathbb{R}^m$ (or $\mathbb{C}^m$, $\mathbb{Q}^m$), and $p \in U_\alpha \subset B$. In fact, the smooth maps σ_α are called *local representatives* of σ and they are related by

$$\sigma_\alpha = g_{\alpha\beta}\sigma_\beta \quad \text{on } U_\alpha \cap U_\beta. \tag{5.2}$$

Suppose M is a smooth manifold, we now give two examples of vector bundles.

Example 1. The *tangent bundle* τ_M of M is a smooth vector bundle, with total space being the manifold DM, composed of all pairs (x, v) with $x \in M$ and v tangent to M at x. The projection map $\pi : DM \to M$ is given by $\pi(x, v) = x$. Moreover, the cotangent bundle T^*M is dual to the tangent bundle.

Example 2. The *normal bundle* ν of M is obtained through the tangent space DM, with the total space E being the set of all pairs (x, v) where $x \in M$ and v is orthogonal to the tangent space at x, i.e. DM_x. The projection map is defined as in the previous example.

5.2 Connections and Curvatures

One of the main geometric structures on manifolds and vector bundles is connection. In fact, the topology of 4-manifolds is strongly related to the space of connections in the tangent bundle attached.

Let $\chi(M)$ denote the set of vector fields on M, $\Gamma(E)$ be the set of smooth sections of E. By definition, a *connection* on the vector bundle is a map

$$d_A : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

which satisfies

$$d_A(f\sigma + \tau) = (df) \otimes \sigma + f d_A\sigma + d_A\tau,$$

where σ, τ are sections of E and f is a function on M (f is real-valued if E is a real vector bundle, a complex-valued if E is a complex vector bundle). As an example, any connection d_A on a trivial bundle is of the form

$$d_A\sigma = d\sigma + \omega\sigma = (d + \omega)\sigma, \tag{5.3}$$

where d is the exterior derivative and ω is a matrix of one-forms (see [Chern, 1960;

Moore, 1996]). Explicitly, it is of the form

$$d_A \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^m \end{pmatrix} = \begin{pmatrix} d\sigma^1 \\ \vdots \\ d\sigma^m \end{pmatrix} + \begin{pmatrix} \omega_1^1 & \cdots & \omega_m^1 \\ \vdots & \cdots & \vdots \\ \omega_1^m & \cdots & \omega_m^m \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^m \end{pmatrix}.$$

Because a section $\sigma \in \Gamma(E)$ consists of a collection of local representatives σ_α, $d_A\sigma$ must also be composed of $\{(d_A\sigma)_\alpha\}$, a set of local representatives on U_α. Hence in U_α, (5.3) is taking the form

$$(d_A\sigma)_\alpha = d\sigma_\alpha + \omega_\alpha\sigma_\alpha,$$

which is called the local representative of the connection d_A and ω_α is an $m \times m$ matrix of one-forms. Again, these local representatives are related by the transition functions as follows

$$d\sigma_\alpha + \omega_\alpha\sigma_\alpha = g_{\alpha\beta}(d\sigma_\beta + \omega_\beta\sigma_\beta) \quad \text{on } U_\alpha \cap U_\beta \neq \emptyset.$$

Taking into consideration of (5.2), we conclude

$$\omega_\alpha = g_{\alpha\beta} d g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1} \qquad \text{on } U_\alpha \cap U_\beta. \tag{5.4}$$

Since any vector bundle E is 'locally trivial', given a covering $\{U_\alpha : \alpha \in A\}$ of E, a connection is defined as a collection of differential operators $d + \omega_\alpha$, where ω transforms according to (5.4).

Let $\Omega^p(M) = \Gamma\big(\Lambda^p(T^*M)\big)$ be differential p-forms on M. By requiring Leibniz rule to hold and $d(dx^i) = 0$, we can obtain a cochain complex

$$\cdots \xrightarrow{d} \Omega^{p-1}(M) \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots,$$

which yields the p-th *de Rham cohomology group*

$$H^p(M;\mathbb{R}) = \frac{\text{kernel of } d : \Omega^p M \to \Omega^{p+1}(M)}{\text{image of } d : \Omega^{p-1} M \to \Omega^p(M)}.$$

Further discussion can be found in, for example, [Chern, 1960]. It is well-known that these groups are topological invariants of M, which give rise to the possibility of studying topological manifolds through the vector bundles attached to them.

By letting E be a vector field over M, a connection can be represented by

$$d_A : \Omega^0(E) \to \Omega^1(E),$$

which is a linear map from zero-forms to one-forms, and $\Omega^p(E)$ is the collection of p-forms on M with values in E. Of course, d_A can be extended according to the following

$$d_A(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge d_A \sigma, \qquad \text{for } \omega \in \Omega^p(M), \quad \sigma \in \Gamma(E).$$

Furthermore, it turns out that the space $\mathcal{S}$ of unitary connections in a given complex line bundle L over M can be classified up to isomorphism. In fact, by choosing d_{A_0}, a unitary connection on L, as a base point, any unitary connection on L can be represented as $d_A = d_{A_0} - ia$, where $a \in \Omega^1(M)$ (see [Moore, 1996]). A *gauge transformation* of L is a smooth map $g : M \to S^1$, which acts on connections by conjugation

$$(g, d_A) \to g \circ d_A \circ g^{-1} = d_A + gd(g^{-1}).$$

Denote the space of gauge transformations by $\mathcal{G}$, the space of gauge equivalence classes of connections on L is $\mathcal{E} = \mathcal{S}/\mathcal{G}$.

Moreover,

$$(d_A)^2(f\sigma + \tau) = f(d_A \circ d_A \sigma) + d_A \circ d_A \tau,$$

which shows $(d_A)^2$ is linear over functions. Hence it is a tensor field and called the *curvature* of the connection, d_A. For example, in the case of a trivial bundle, we have

$$(d_A)^2 = \Omega\sigma,$$

where Ω is a matrix of 2-forms. By differentiating (5.3), we have

$$\begin{aligned} d_A^2(\sigma) &= (d+\omega_\alpha)(d\sigma + \omega\sigma) \\ &= (d\omega + \omega\wedge\omega)\sigma. \end{aligned}$$

For each local representative σ_α of $\sigma \in \Gamma(E)$, we then have

$$\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha, \tag{5.5}$$

where Ω_α is a matrix of local 2-forms and yields the *Bianchi identity* after differentiation

$$d\Omega_\alpha = \Omega_\alpha \wedge \omega_\alpha - \omega_\alpha \wedge \Omega_\alpha = [\Omega_\alpha, \omega_\alpha].$$

The local Ω_i's transform as

$$\Omega_\alpha = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1} \quad \text{for } U_\alpha \cap U_\beta \neq \emptyset. \tag{5.6}$$

Generally speaking, the curvature Ω is only locally defined. However, it is shown Ω_α is skew-Hermitian (see [Moore, 1996]), so $\frac{i}{2\pi}\Omega_\alpha$, $\left(\frac{i}{2\pi}\Omega_\alpha\right)^k$ ($k \in \mathbb{Z}$) are Hermitian. Therefore, the trace of the differential-form matrix is invariant under similarity. We then have

$$\text{Trace}\Big[\big(\frac{i}{2\pi}\Omega_\alpha\big)^k\Big] = \text{Trace}\Big[\big(\frac{i}{2\pi}g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}\big)^k\Big] = \text{Trace}\Big[\big(\frac{i}{2\pi}\Omega_\beta\big)^k\Big] \quad \text{for } U_\alpha \cap U_\beta \neq \emptyset.$$

Hence these locally defined forms give rise to a globally defined element of $H^{2k}(M;\mathbb{R})$ (or $\mathbb{C}$, $\mathbb{Q}$) which is independent of the (unitary) connection and Hermitian metric on E. It is called the *characteristic class* of E, denoted by $\tau_k(E)$. Note that the differential form $\tau_k(E)$ is closed and $d\tau_k(E) = 0$.

The Chern classes c_1, c_2 of E are defined by

$$c_1(E) = \tau_1(E) \quad \text{and} \quad c_2(E) = \frac{1}{2}[\tau_1(E)^2 - \tau_2(E)].$$

Moreover, for a tangent bundle to a compact oriented surface M, the Gauss-Bonnet theorem gives rise to the first Chern class based on the fundamental class of M

$$< c_1(TM), [M] >= \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

The idea of finding geometric and topological invariants of a space M from the vector bundle over a collection of local coverings can be used to obtain a global structure given a finite number of local dynamical systems. In fact, we can regard the local systems as tangent bundles attached to some manifold and hence study the connections and curvatures of these bundles, which will yield the collation of each local system in order to form a global picture.

Example. We now consider dynamical systems on S^4, regarded as the one-point compactification of the quaternion numbers, i.e. $S^4 = \mathbb{Q} \cup \{\infty\}$. Given $\mathbb{Q}$ the standard quaternion coordinate q and let $U_0 = S^4 - \{\infty\}$, $U_\infty = S^4 - \{0\}$ be the two local neighbourhoods. Since the dynamical systems are given in terms of tangent bundles, the transition map is generated by

$$g_{\infty 0} = \frac{1}{q^2}, \qquad \text{on } U_0 \cap U_\infty. \tag{5.7}$$

Given two 4-dimensional local systems, a stable node and an unstable node, taking the form

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \\ \dot{z} = z \\ \dot{r} = r \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ \dot{z} = -z \\ \dot{r} = -r \end{cases}, \tag{5.8}$$

defined on U_0 and U_∞ respectively. We now consider the possibilities of how these two local systems can be situated on S^4. Using quaternion coordinate $q = x+iy+jz+kr$, the two local sections are

$$\sigma_o = \begin{pmatrix} x & -y & -z & -r \\ y & x & -r & z \\ z & r & x & -y \\ r & -z & y & x \end{pmatrix} \quad \sigma_\infty = \begin{pmatrix} -x & y & z & r \\ -y & -x & r & -z \\ -z & -r & -x & y \\ -r & z & -y & -x \end{pmatrix}. \tag{5.9}$$

Let ω_0 be a matrix of one-forms

$$\omega_0 = dq = \begin{pmatrix} dx & -dy & -dz & -dr \\ dy & dx & -dr & dz \\ dz & dr & dx & -dy \\ dr & -dz & dy & dx \end{pmatrix}. \tag{5.10}$$

By (5.3), we then have, for σ_0

$$d_{A_0} = \begin{pmatrix} dx & -dy & -dz & -dr \\ dy & dx & -dr & dz \\ dz & dr & dx & -dy \\ dr & -dz & dy & dx \end{pmatrix} + \omega_0 \cdot \begin{pmatrix} x & -y & -z & -r \\ y & x & -r & z \\ z & r & x & -y \\ r & -z & y & x \end{pmatrix}.$$

By (5.5) we have, for local curvature representative of σ_0,

$$\Omega_0 = d\begin{pmatrix} dx & -dy & -dz & -dr \\ dy & dx & -dr & dz \\ dz & dr & dx & -dy \\ dr & -dz & dy & dx \end{pmatrix} + \omega_0 \wedge \omega_0.$$

Since ω_0 is composed of one-forms, $\omega_0 \wedge \omega_0 = 0$. By definition, $d(dx) = d(dy) = d(dz) = d(dr) = 0$. Hence Ω_0 is skew-Hermitian. Moreover,

$$\text{Trace}\left[\left(\frac{i}{2\pi}\Omega_0\right)^k\right] = 0. \tag{5.11}$$

Meanwhile, for σ_∞, the 4×4 matrix ω_∞ is given by

$$\begin{aligned} \omega_\infty &= g_{\infty 0} d(g_{\infty 0}^{-1}) + g_{\infty 0}\omega_0 g_{\infty 0}^{-1} \\ &= \frac{2}{q} \cdot dq + dq, \end{aligned}$$

where

$$\frac{2}{q} = \begin{pmatrix} \frac{2x}{|q|^2} & \frac{2y}{|q|^2} & \frac{2z}{|q|^2} & \frac{2r}{|q|^2} \\ -\frac{2y}{|q|^2} & \frac{2x}{|q|^2} & \frac{2r}{|q|^2} & -\frac{2z}{|q|^2} \\ -\frac{2z}{|q|^2} & -\frac{2r}{|q|^2} & \frac{2x}{|q|^2} & \frac{2y}{|q|^2} \\ -\frac{2r}{|q|^2} & \frac{2z}{|q|^2} & -\frac{2y}{|q|^2} & \frac{2x}{|q|^2} \end{pmatrix}, \tag{5.12}$$

and $|q|^2 = x^2 + y^2 + z^2 + r^2$. The diagonal entries for ω_∞ are

$$\omega_{\infty_{ii}} = \left(\frac{2x}{|q|^2} + 1\right)dx + \frac{2y}{|q|^2}dy + \frac{2z}{|q|^2}dz + \frac{2r}{|q|^2}dr. \tag{5.13}$$

Similarly, substitute ω_∞ into (5.3) and (5.5), we then obtain the connection d_{A_∞} and curvature Ω_∞ for section σ_∞. Likewise, Ω_∞ is skew-Hermitian, and from (5.12), we obtain

$$\text{Trace}\left[\left(\frac{i}{2\pi}\Omega_\infty\right)^k\right] = 0. \tag{5.14}$$

Compare (5.11) and (5.14), we have

$$\text{Trace}\left[\left(\frac{i}{2\pi}\Omega_0\right)^k\right] = \text{Trace}\left[\left(\frac{i}{2\pi}\Omega_\infty\right)^k\right]$$

on $U_0 \cap U_\infty$. Hence these locally defined forms fit together into a globally defined $2k$-form τ_k. So the locally defined systems given by (5.8) can fit together to be situated on S^4. In fact, a pair of n-dimensional local stable and unstable nodes can always be stratified on S^n.

The example above is very simple, and in the future we will further the research of constructing a combinatorial system from a finite number of locally defined ones by using Chern classes.

5.3 Spin and Spin^c Structures

Let $\mathbb{R}^4$ be the space $\mathbb{Q}$ of quaternions, using the matrix representation with complex entries,

$$q = \begin{pmatrix} t+iz & x+iy \\ -x+iy & t-iz \end{pmatrix}.$$

The quaternion multiplication has important applications to the geometry of 4-manifolds. In fact, the spin group in four dimensions is the product of two copies of the special unitary group, i.e.

$$\text{Spin}(4) = \text{SU}_+(2) \times \text{SU}_-(2) = \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix},$$

where $SU(2) = \{q \in \mathbb{Q} : < q, q >= 1\}$ ($<\ ,\ >$ denotes the Euclidean dot product) and $W_\pm = SU_\pm(2)$.

Suppose M is an orientable manifold with an open covering $\{U_\alpha : \alpha \in A\}$. If for a collection of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(4),$$

there exists an isomorphism $\rho : \mathrm{Spin}(4) \to GL(\mathbb{Q})$ such that $\rho \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition (5.1) is satisfied, then M is called a *spin manifold* which has a *spin structure*. It is known that M admits a spin structure if and only if the second Stiefel-Whitney class $\omega_2(TM) = 0$ (see [Lawson and Michelsohn, 1989]).

Moreover, the spin^c group,

$$\mathrm{Spin}(4)^c = \left\{ \begin{pmatrix} \lambda W_+ & 0 \\ 0 & \lambda W_- \end{pmatrix} : W_+ \in SU_+(2),\ W_- \in SU_-(2),\ \lambda \in U(1) \right\},$$

in four-manifolds is very important as well. Similarly, a spin^c structure on M is defined in the same way as spin structure except that $\rho^c : \mathrm{Spin}(4)^c \to GL(\mathbb{Q})$ and $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(4)^c$. In fact, any orientable four-manifold admits a spin^c structure.

Since the cocycle condition is satisfied by the transition functions defined above, this yields a vector bundle accordingly. Furthermore, the isomorphism classes of spin^c structures on M are in a one-to-one correspondence with complex line bundles L over M. Therefore, ρ_+^c and ρ_-^c generate $W_+ \otimes L$ and $W_- \otimes L$, which are both $U(2)$-bundles. This shows that spin^c structure gives rise to the possibility of representing the complexified tangent bundle in terms of two basic vector bundles. The local representative of the connection is

$$(d_A\sigma)_\alpha = d\sigma_\alpha + \phi_\alpha\sigma_\alpha,$$

where ϕ_α is a one-form which takes values in $\mathrm{Spin}(4)^c$.

The Seiberg-Witten invariants are based on the Dirac operators $D_A : \Gamma(W \otimes L) \to \Gamma(W \otimes L)$ given by

$$D_A(\psi) = \sum_{i=1}^{\infty} e_i \cdot d_A\psi(e_i) = \sum_{i=1}^{\infty} e_i \cdot \nabla^A_{e_i}\psi,$$

where the e_i's are the standard Dirac matrices forming a basis of a Clifford algebra. In the case of four-manifolds, $1 \leq i \leq 4$.

Furthermore, the vector bundle Laplacian $\Delta^A : \Gamma(W \otimes L) \to \Gamma(W \otimes L)$ is given by

$$\Delta^A\psi = -\sum_{i=1}^{4}[\nabla^A_{e_i} \circ \nabla^A_{e_i}\psi - \nabla^A_{\nabla_{e_i}e_i}\psi],$$

where (e_1, e_2, e_3, e_4) is an orthonormal frame. Note that the introduction of $\nabla^A_{\nabla_{e_i}e_i}\psi$ is to guarantee the independence of the choice of a particular frame.

The relationship between the Dirac and Laplacian operators is given by the Weitzenböck's formula

$$D^\alpha_A(\psi) = \Delta\psi + \frac{s}{4}\psi - \sum_{i<j} F_A(e_i, e_j)(ie_i \cdot e_j \cdot \psi), \tag{5.15}$$

where s is the scalar curvature of M and F_A is the curvature of the connection of a line bundle L.

From (5.15), it is clear to see that the Dirac operator is almost the square root of the Laplacian. Another key property of the Dirac operator is that it is self-adjoint, i.e.

$$\int_M \langle D_A(\psi), \eta \rangle dV = \int_M \langle \psi, D_A(\eta) \rangle dV.$$

Of course it is known that the Laplacian is also self-adjoint.

The importance of introducing the Dirac operator lies in the fact that D_A splits into two parts,

$$D_A^{\pm} : \Gamma(W_{\pm} \otimes L) \longrightarrow \Gamma(W_{\mp} \otimes L),$$

where $W_{\pm}$ are both $U(2)$ bundles, and $D_A^{\pm}$ are adjoints of each other. The *index* of D_A^{+} is defined as

$$\text{index of } D_A^{+} = \dim\big(\text{Ker}(D_A^{+})\big) - \dim\big(\text{Ker}(D_A^{-})\big)$$

The following result is fundamental and well-known in four-manifold theory

Theorem 5.3.1. (Atiyah-Singer Index Theorem.) *Let M be a compact orientable four-manifold and L be a line bundle. If D_A is a Dirac operator with coefficients in L on M, then*

$$\text{index of } D_A^{+} = -\frac{1}{8}\tau(M) + \frac{1}{2}\int_M c_1(L)^2,$$

where $\tau(M)$ is the signature of M, $\tau(M) = b_+ - b_-$, and $c_1(L)$ is the first chern class of L. □

Several striking applications of the Atiyah-Singer Index theorem have been studied later on and yielded topological invariants which have strong connections with the geometric structure: **Rochlin's theorem** showed that the signature of a compact orientable smooth spin 4-manifold is always divisible by 8, **Lichnerowicz's theorem** related the curvature to topology of Riemannian 4-manifolds, and in the case of L

being a complex line bundle, the *Chern character* is defined as

$$\mathrm{ch}(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2!}\left(c_1(L)\right)^2 + \cdots .$$

(More details can be found in [Chern, 1960; Gompf and Stipsicz, 1999; Moore, 1996].)

5.4 Seiberg-Witten Invariants

We now consider invariants based on the theory of nonlinear partial differential equations. For a nonlinear PDE, the space M of its solutions may be defined on a compact smooth manifold. Although M will, in general, vary depending on choices such as a specific Riemannian metric, it is possible that M's cobordism class can be a topological invariant which can distinguish between different smooth structures. In fact, this is how Donaldson approached four-manifold theory originally. Specifically, he studied the moduli space of anti-self-dual connections in an $SU(2)$ bundle over a compact orientable 4-manifold, which is itself a smooth manifold (see [Donaldson, 1983]). However, the compactness of this moduli space cannot be guaranteed, hence a great amount of effort has been focused on finding suitable compactifications, among which Seiberg-Witten invariants have been proved to be a very powerful technique.

The Seiberg-Witten equations are

$$D_A^+\psi = 0, \quad F_A^+(e_i, e_j) = \sigma(\psi) + \phi = -\frac{i}{2} < \psi, e_i \cdot e_j \cdot \psi > +\phi \quad \text{for } i < j, \tag{5.16}$$

where F_A^+ is the self-dual part of F_A, i.e. the curvature of the connection d_A, $\sigma(\psi)$ is a quadratic form in ψ, and ϕ is a given self-dual 2-form. The solutions of these equations are pairs (d_A, ψ) consisting of a connection d_A on the line bundle L and a section ψ of $W_+ \otimes L$. Note that the nonlinearity encountered in the Yang-Mills equations from gauge theory is much wilder than that in the Seiberg-Witten equations.

Recall that in general, L is not always well defined. However, any orientable four-manifold possesses a spin^c structure. Hence for such a structure defined by $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(4)^c$ and a group homomorphism $\pi : \mathrm{Spin}(4)^c \to U(1)$, the transition functions

$$\pi \circ \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$$

generate a complex line bundle L^2, which is called the *anticanonical bundle.*

Clearly, L^2 is well-defined generally. In this case, the solutions to (5.16) are pairs (d_{2A}, ψ), where d_{2A} is a connection on the line bundle L^2 and $F_A^+ = \frac{1}{2} F_{2A}^+$, where F_{2A}^+ is the self-dual part of the curvature F_{2A}.

For a unitary connection d_{A_0} on L, the configuration space of the solutions to Seiberg-Witten equations is

$$\mathcal{S} = \{(d_{A_0} - ia, \psi) : a \in \Omega^1(M), \psi \in \Gamma(W_+ \otimes L)\}.$$

Taking into account the group of gauge transformations $\mathcal{G}$, we form the moduli space U_ϕ of all gauge-equivalent solutions,

$$U_\phi = \{[d_{A_0} - ia, \psi] \in (\mathcal{S}/\mathcal{G}) : (d_{A_0} - ia, \psi) \text{ satisfies } (5.16)\}.$$

Moreover, U_ϕ turns out to be a compact manifold (see, for example [Kronheimer and Mrowka, 1994]). For a complex line bundle L over M, it is clear that the moduli space U_ϕ depends on both the Riemannian metric on M and the choice of ϕ. However, by Smale's infinite-dimensional generalization of transversality (see [Smale, 1966]), different ϕ yield cobordant moduli spaces, which can be regarded as a topological invariant. Hence the Seiberg-Witten invariants of L over M are defined by

$$SW(L) = < c_1^d, U_\phi >, \qquad d = \frac{1}{2} \dim U_\phi,$$

and c_1 is the first Chern class of a line bundle over the space of all gauge-equivalent pairs (d_A, ψ).

Using these invariants, by Witten's theorem (see [Moore, 1996]), it can be shown that the compact manifold $P^2\mathbb{C}\#_q\overline{P^2\mathbb{C}}$ has infinitely many distinct smooth structures.

5.5 Four-Dimensional Dynamical Systems

As far as dynamical systems are concerned, we can start with simple systems on S^4 and use connected sums, covering manifolds or rational blowdown techniques (see [Fintushel and Stern, 1997; Gompf and Stipsicz, 1999; Orlik, 1972]) to generate more complex systems. For example, we have the following result

Theorem 5.5.1. *Given a Morse-Smale system on a smooth simply connected 4-manifold M with $b_+ > 1$, Seiberg-Witten invariant SW_M and an embedded periodic solution K with A-polynomial $P(t)$, we can obtain a dynamical system on a 4-manifold M_p with Seiberg-Witten invariant SW_{M_p} such that*

$$SW_{M_p} = SW_M \cdot P(t).$$

Proof. This follows from the results in Fintushel and Stern [Fintushel and Stern, 1998] and Etgu [Etgu and Park, 2003], by doing Dehn surgery on the knot K. □

In [Gompf, 1995], the following construction of a generalized connected sum is given:

Let M and N be two smooth, closed, orientable manifolds of dimension n and $n-2$, respectively, and $i_j : N \to M$ $(j = 1, 2)$ be disjoint embeddings with normal bundle ν_j over N and normal Euler classes $e(\nu_j) \in H^2(N; \mathbb{Z})$ which are opposite, i.e. $e(\nu_2) = -e(\nu_1)$. This implies that ν_1 and ν_2 are isomorphic by an isomorphism $\psi : \nu_1 \to \nu_2$. Note that ψ reverses the orientataion of the fibres. Moreover, for a tubular neighbourhood V_j of $i_j(N)$, there exists an orientation preserving diffeomorphism

$\phi : \big(V_1 - i_1(N)\big) \to \big(V_2 - i_2(N)\big)$. Then we denote by $\#_\phi M$ to be the manifold obtained by taking $M - \big(j_1(N) \cup j_2(N)\big)$ with the identification of $\big(V_1 - i_1(N)\big)$ with $\big(V_2 - i_2(N)\big)$ by ϕ.

If $M = M_1 \cup M_2$, then M_ϕ is the connected sum of M_1 and M_2 along $i_j(N)$ via ϕ and denoted by $M_1 \#_\phi M_2$.

Thus if $M = M_1 \cup M_2$ is a 4-manifold with M_1,M_2 both carrying dynamical systems and the dynamics on $i_1(N)$ and $i_2(N)$ 'match up' by the induced map $\phi_* : T\big(i_1(N)\big) \to T\big(i_2(N)\big)$, then there exists a well-defined system on $M_1 \#_\phi M_2$.

Another alternative is to use the theory of covering surfaces to obtain dynamical systems on any 4-manifold. The theory of branched coverings of S^4 can be found in [Montesinos, 1985]. In particular, Piergallini shows that every closed oriented PL 4-manifold is a single 4-fold covering of S^4 branched over a transversally immersed PL surface (see [Piergallini, 1995]). Hence, similar to the 3-dimensional case, we have

Theorem 5.5.2. *A four-manifold which is simple 4-fold covering of S^4 over an immersed surface S carries a dynamical system which is the lift of a system of S^4, which has S as an invariant surface - so that the immersed double and treble points of S in S^4 are invariant sets of the flow.* □

Chapter 6

Conclusions

6.1 Main Contributions

By definition, dynamical system is a mathematical formalization for a relation that describes the time dependence of a moving point's evolution in its ambient space. Generally this relation is defined in terms of differential equations. Typical examples include the mathematical models which describe the swinging of a single pendulum, the flow of the water into a tank, and the movement of a crane hanging along a fixed track. The study of dynamical systems not only have considerable influence on theoretical and practical development in both linear and nonlinear pattern formations, but can provide a useful insight into the analysis of dynamical behaviours which have been observed in a wide range of scientific disciplines.

Because a dynamical system is a smooth evolution function defined on the phase space, i.e. a manifold M, it is natural to ask a global question where topology and dynamics interact. The work in this thesis has presented extensive research on how the global topological invariants regulate dynamical structures in low-dimensional cases.

The theory of two-manifold has been well understood for a fairly long time, with one unique topological invariant, the genus, which can distinguish all different surfaces. Stick to the upper half plane model, we approach the study of dynamical systems defined globally on surfaces of any genus by using the automorphic functions.

In the genus-1 (torus) case, because of the nature of their fundamental regions, i.e. lattice Ω consisting of parallelograms, it is clear that any doubly-periodic function can serve as the generator for systems defined on a torus. By using the Weierstrass elliptic functions in particular, we formulate an algorithm to construct toroidal systems explicitly, and a necessary and sufficient condition regulated by the topological invariant, Euler characteristic (which is zero for a torus), is presented as well.

Higher genus surfaces turn out to be more difficult mainly because hyperbolic metric is understood in this case. Moreover, since general automorphic forms only provide invariant 'scalars' instead of vector fields, some form of modification is necessary. Hence, by adopting a revised version of theta series, we are able to present the algorithm of generating 'pseudo-automorphic' functions which uniformise the vector fields and yielding analytic systems on surfaces of any genus.

In dynamical systems theory, an *attractor* (resp. *repeller*) is a set which attracts (resp. repels) dynamics and contains a dense orbit. It is a generalization of stable (resp. unstable) nodes and very important in studying the asymptotic behaviour of dynamical systems. Within the attractor itself, there exist fruitful exhilarating phenomena that gather the attention of many mathematicians. A 'strange' attractor contains a transversal homoclinic orbit, with the famous example being the Duffing equation

$$\ddot{x} + \alpha\dot{x} - x + x^3 = \beta\cos(\omega t).$$

Moreover, in the case of a periodic nonlinear 2-dimensional, time-varying oscillator with appropriate damping which is situated on a torus and contains an attractor, Martins showed that if there exists an inversely unstable solution, then the attractor is not homeomorphic to a circle.

Accordingly, this thesis presents an elucidation to the generalization of the above result to higher genus surfaces. By studying the topology of the inversely unstable solutions, we are able to show that instead of just one attractor, several invariant sets can coexist and even be knotted to form a set of linked attractors. Meanwhile, the existence of an inversely unstable solution implies that the containing attractor is not homeomorphic to a circle, with the proof bearing a resemblance to the one in [Martins, 2004].

Three-dimensional dynamical systems are more interesting than their two-dim counterparts mainly because the one more dimension of freedom provides much more variety in the corresponding dynamical behaviour. In particular, Smale, in his extensive survey of differentiable dynamical systems [Smale, 1967], addressed the importance of strange attractors, especially the 'structurally stable' ones. Note that a dynamical system is structurally stable if small C^1 perturbations yield topologically equivalent systems. Historically speaking, despite the fact that they are hard to find in physical models, structurally stable hyperbolic attractors still possess an important place in dynamical systems theory, mainly because they serve as an archetype and guide investigations on the dynamical behaviour of other strange attractors.

After Smale's classification of the Axiom A systems, the topological study of hyperbolic invariant sets has grown rapidly. One of Smale's long term program is to classify a Baire set of the diffeomorphisms which define hyperbolic structures on the non-wandering set. This program evolved from the work of many topologists and dynamicists, such as [Williams, 1967, 1974; Franks, 1985; Guckenheimer and Holmes, 1983].

As one type of hyperbolic attractors, Smale solenoids were introduced to dynamical systems. They carry more information than the general solenoids which are just topological spaces. This thesis concerns the study of automorphisms on 3-manifolds which contain generalized Smale solenoids as their basic sets. Therefore this gives us a geometric picture of necessary and sufficient conditions which must be met under the existence of such hyperbolic attractors. Moreover, since every closed, orientable 3-manifold is a branched covering of S^3 over some universal knot, we studied the possibility of lifting dynamics defined on S^3 onto the original manifold in order to have arbitrary genus Smale solenoids.

On the other hand, the enormous growth in knot theory that the last two decades has witnessed also facilitates the research in dynamical systems theory, mainly because a knot is essentially a twisted version of S^1, which can be regarded as a periodic orbit generated by differential equations. From this perspective, we have shown how to generate three-dimensional systems containing arbitrarily knotted chaos by using the theory of transformation groups and C^∞ functions. By 'twisting' a simply existing dynamical system according to local transformation groups and making the resulting system periodic, virtually any dynamical behaviour can be obtained.

Meanwhile, with the aid of surgery techniques, such as Dehn surgery, Heegaard splittings, branched covering manifolds and connected sums, we obtained a kind of decomposition of 3-dimensional dynamical systems in terms of simpler ones.

Four-manifold topology turns out to be probably the most difficult one of any dimension and exhibit somewhat very strange behaviour. However, by using the characteristic classes (of vector bundles over the original 4-manifold) developed over the last 80 years by numerous mathematicians, including Chern, Stiefel, Whitney, and Pontryagin, Donaldson, Seiberg and Witten, we still managed to investigate the problem on gluing given local systems in order to obtain a global dynamical picture.

To sum up, this thesis provides a comprehensive work of dynamical systems on low-dimensional manifolds, from both topological and geometrical points of views.

6.2 Recommendations for Future Work

The idea of generating dynamical systems by using modified automorphic functions worked perfectly in two-dimensional case, largely because of the commutativity of complex numbers $\mathbb{C}$. In three-dimensions, however, the quaternion representations do not commute any more, which becomes an obstacle and presents quite a challenge. By studying modular forms on the upper-half space of quaternions more thoroughly, we are hoping to be able to find an algorithm to obtain meromorphic, or even more general systems on 3-manifolds in the near future.

The most challenging area still lies in the four-dimensions. Over the past two decades, four-manifold theory, once in the inner sanctum of topology, has been leaking out and finding its interactions with diverse fields through several successful attempts. For example, with the novel gauge-theoretic properties, topologists are now using Kirby calculus to construct new 4-manifolds, decompose them into simpler pieces, and investigate their differentiable structures.

Moreover, in [Fintushel and Stern, 1997], the *rational blowdown* is well defined as removing the interior of a 4-manifold C_p with ∂C_p being a lens space and replacing it with a rational ball B_p. Clearly this is a technique to obtain new 4-manifolds from the existing ones. Furthermore, with the aid of Seiberg-Witten invariants, Etgu and Park proved that symplectic 4-manifolds can accommodate infinite family of non-isotopic symplectic tori (see [Etgu and Park, 2003]). Hence the above results guide the thinking about the possibility of generating dynamical systems via the 'rational blowdowns'.

In [Gompf, 1995], a new surgery construction, symplectic connected sums formed along codimension-2 submanifolds, is generalized. In the setting of 4-manifolds particularly, by defining an orientation-reversing isomorphism of the normal bundles on their respective diffeomorphic submanifolds, two smooth manifolds can be summed to yield a new 4-manifold. Provided that we are given two dynamical systems situated on different smooth 4-manifolds, this construction leads to obtaining a new system on their connected sum.

To sum up, with the intense development of low-dimensional topology, dynamical systems theory on low-dimensional manifolds is becoming an area that is really worth to exploit.

Appendix A

Topology In a Nutshell

After Poincaré established the modern method of qualitative analysis of differential equations, dynamical systems theory became more and more closely related to topology, especially manifold theory, due to the reason that topology is essentially the study of topological spaces and continuous functions between them. In fact, a lot of work is focused on how the topology, especially in low-dimensional cases, interacts with the dynamics. For example, in the 1930's, Morse theory was introduced and provided a useful insight into this interaction (see [Milnor, 1963; Morse, 1934]).

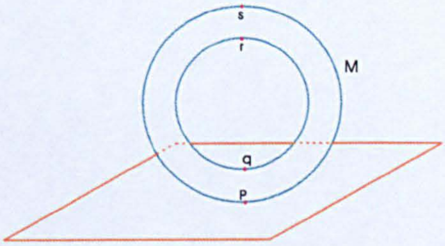


Figure A.1: Torus over a plane.

Example. Let M be a torus, tangent to a plane (as shown in fig. A.1), $f : M \to \mathbb{R}$ ($\mathbb{R}$ denotes the real numbers) be the height above a plane, and M^a be all the points $x \in M$ such that $f(x) \leq a$ ($a > 0$). Then the following statements are true:

(a) If $f(p) < a < f(q)$, then M^a is homeomorphic to a 3-cell.

(b) If $f(q) < a < f(r)$, then M^a is homeomorphic to a cylinder.

(c) If $f(r) < a < f(s)$, then M^a is homeomorphic to a compact genus-1 manifold with a circle boundary.

(d) If $f(s) < a$, then M^a is the full torus.

As illustrated in the example above, for a non-degenerate function f, the gradient vector field, grad(f), defines a dynamical system. Moreover, the global information on the singularities of f will then provide topological information on the global manifold, M.

Hence it is necessary to introduce enough topology for later requirements. Note that all the following results are well-known and can be found, for example, in [Abraham, marsden and Ratiu, 1980; Dugundji, 1966; Orlik, 1972; Spanier, 1966].

A.1 Set Theory and Group Theory

We first lay down a few definitions that are needed for future discussion.

Generally speaking, a 'set' refers to a collection of abstract objects. Given a set A, an *equivalence relation* is a relation $\sim$ between elements of A which is reflexive, symmetric and transitive. The *equivalence class* of $a \in A$ with respect to $\sim$ is the subset $\{a' \in A \mid a \sim a'\}$.

A *partial order* in A is a binary relation, denoted by '$\leq$ ' satisfying the following

i) reflexivity, i.e. $a \leq a$ for all $a \in A$,

ii) transitivity, i.e. $a \leq a'$ and $a' \leq a''$ imply $a \leq a''$ for a, a', $a'' \in A$.

A *total order* (or *simple order*) in A is a partial order such that for $a, a' \in A$ either $a \leq a'$ or $a' \leq a$ and which is antisymmetric, i.e. $a \leq a'$ and $a' \leq a$ imply $a = a'$. An *upper bound* $a \in A$ of a subset $U \subset A$ is an element such that $u \leq a$ for all $u \in U$. A *maximal element* m of an ordered set A is an element such that $a \leq m$ for all $a \in A$ and $a \neq m$. A subset $B \subset A$ is called a *chain* in A if every two elements in B are ordered. Moreover,

Definition A.1.1. *A partially ordered set A is called* **well-ordered** *(or* **ordinal***) if every nonempty subset $B \subset A$ has a first element, i.e. for each $B \neq \emptyset$, there exists a $b_0 \in B$ such that $b_0 \leq b$ for all $b \in B$.*

For example, $\emptyset$ is well-ordered; If a set contains only one element, it is then well-ordered; The nonnegative integers are well-ordered. It is evident that a well-ordered set is totally ordered, also the induced order on a subset of a well-ordered set is a well-order on that subset.

Theorem A.1.1. *Given other axioms of set theory, the following statements are equivalent*

i) **Axiom of Choice** *Given a nonempty family $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ of pairwise disjoint nonempty sets, there is a set S which is a collection of nonempty sets and consists of exactly one element from each A_α.*

ii) **Zorn's Lemma** *A partially ordered set in which every simply ordered subset has an upper bound contains maximal elements.*

iii) **Zermelo's Theorem** *Every set can be well ordered.* □

Remarks. There exists no specific construction to well order any uncountable set, even though Zermelo's theorem asserts that every set can be well ordered. Moreover, there are sets, (e.g. the set of real-valued functions of one real variable,) for which no specific construction of a total order is even known. Meanwhile, it is obvious that the well ordering guaranteed by Zermelo's theorem is not unique.

Two sets, X and Y, are *equipotent* if there exists a bijective map of X onto Y. It is an equivalence relation in the class of all sets. Hence

Definition A.1.2. *The* ***cardinal number*** *of X is the initial ordinal number of its equipotence class, denoted by $\aleph(X)$.*

There are special symbols assigned to some frequently occurred sets, e.g., $\aleph(\emptyset) = 0$, $\aleph(1, 2, \cdots, n) = n$, and $\aleph(N) = \aleph_0$. For a set X, if $\aleph(X) = n$ for some n, then it is called *finite*; otherwise it is called *infinite.* Any set X with $\aleph(X) \leq \aleph_0$ is called *countable*; otherwise it is called *uncountable.*

A group is a set G closed under a binary operation satisfying the following three axioms:

i) the operation is associative,

ii) the operation contains an identity element,

iii) every element in G has an inverse element.

Groups are fundamental in building up more elaborate algebraic structures such as rings, fields, and vector spaces.

A *homomorphism* is a map between two groups that preserves the structure imposed by the corresponding operator. A homomorphism is called a *monomorphism, epimorphism, isomorphism,* respectively, if it is injective, surjective, bijective. Given a collection of groups, $\{G_i\}_{i \in I}$, their *direct product* is defined to be the group structure on the cartesian product $\times G_i$, i.e., $(g_i)(g_i') = (g_i g_i')$.

A group G is said to be freely generated by a subgroup G_1 ($G_1 \subset G$) if, given any function $f : G_1 \rightarrow H$, where H is a group, there exists a unique homomorphism $F : G \rightarrow H$ such that F is an extension of f. Hence we have

Lemma A.1.1. *Any group is isomorphic to a quotient group of a free group.* □

A *normal* subgroup is a subgroup N of G such that it is invariant under conjugation. A *presentation* of G is composed of a set of generators, G_1, a set of relations, $B \subset F(G_1)$, where $F(G_1)$ is a group freely generated by G_1, and a function

$f : G_1 \to G$ such that the extension of f to a homomorphism $F : F(G_1) \to G$ is an epimorphism with kernel being the normal subgroup of $F(G_1)$. G is said to be *finite* if both A and B are finite sets.

A.2 Topological Spaces

Given a space X, it is geometrically evident that the points in X are arranged differently such that different subsets are 'close together'. Topology is therefore introduced to detect this inherent differences.

Definition A.2.1. *A* ***topological space*** *X is a set together with a collection $\mathscr{T}$ of subsets that are called open sets such that*

i) $\emptyset \in \mathscr{T}$ and $X \in \mathscr{T}$,

ii) if $U_1,\ U_2 \in \mathscr{T}$, then $U_1 \cap U_2 \in \mathscr{T}$,

iii) if $U_i \in \mathscr{T}$, $i \in I$, then $\cup_{i\in I} U_i \in \mathscr{T}$ for an arbitrary index set I.

Formally topological spaces are denoted by a pair $(X, \mathscr{T})$; however, we shall simply write X instead if there is no danger of confusion.

By an *open neighbourhood* of $x \in X$ is meant an element of $\mathscr{T}$ containing x. The *indiscrete topology,* also known as the *trivial topology,* on a set X is comprised of $\mathscr{T} = \{\emptyset, X\}$. The *discrete topology* on X, on the other hand, is defined as the power set $\mathscr{P}(X)$, i.e., $\mathscr{T} = \{U \mid U \subset X\}$.

Furthermore, it will simplify the specification of a topology if given just enough open sets to 'generate' all the open sets. Therefore we have

Definition A.2.2. *A* ***basis*** *for $\mathscr{T}$ is a family $\mathscr{B} \subset \mathscr{T}$ such that each element of $\mathscr{T}$ is the union of elements in $\mathscr{B}$.*

A basis for $\mathscr{T}$ completely determines $\mathscr{T}$. By specifying a basis for $\mathscr{T}$, all the open sets are generated as unions.

Theorem A.2.1. *Let $\mathscr{B} \subset \mathscr{T}$ be a basis for $\mathscr{T}$. Then A is open in $\mathscr{T}$ if and only if for each $x \in A$ there is a $U \in \mathscr{B}$ such that $x \in U \subset A$.* □

In fact, the above theorem provides a useful methods for showing that a given set A is open.

Definition A.2.3. *A set $A \subset X$ is called* ***closed*** *if its complements in X, $X \setminus A$ is open.*

Note that the concepts of 'closed' and 'open' are neither exclusive nor exhaustive: In any space X, X and $\emptyset$ are both open and closed, while $[0,1[$ is neither open nor closed. Moreover, we have the following:

(a) The intersection of any family of closed sets is a closed set.

(b) The union of finitely many closed sets is a closed set.

Definition A.2.4. *Let X be a topological space and $A \subset X$. The intersection of all closed sets containing A is called the* ***closure*** *of A, denoted by $\bar{A}$ or $\mathrm{cl}(A)$. The largest open set contained in A is called the* ***interior*** *of A, denoted by $\mathrm{Int}(A)$, i.e. $\mathrm{Int}(A) = \bigcup\{U \mid (U \text{ open}) \wedge (U \subset A)\}$. The* ***boundary*** *of A, denoted by $\mathrm{bd}(A)$ or ∂A, is defined to be $\partial A = \mathrm{cl}(A) \cap \mathrm{cl}(X \setminus A)$.*

Note that $\mathrm{cl}(A)$ is closed and $\mathrm{Int}(A)$ is open. Hence, A is closed if and only if $A = \mathrm{cl}(A)$, while A is open if and only if $A = \mathrm{Int}(A)$. Moreover, $A \subset \mathrm{cl}(A)$ for every set A. The boundary of A is a closed set, and $\mathrm{bd}(A) = \mathrm{bd}(X \setminus A)$.

For example, if $A = [0,1[$, then $\mathrm{cl}(A) = [0,1]$, $\mathrm{Int}(A) =]0,1[$, and $\mathrm{bd}(A) = \{0,1\}$.

Furthermore, if A, B, and A_i $(i \in I)$ are subsets of X, then the following statements are true

i) $A \subset B \Rightarrow \text{Int}(A) \subset \text{Int}(B)$, and $\text{cl}(A) \subset \text{cl}(B)$.

ii) $X \setminus \text{cl}(A) = \text{Int}(S \setminus A)$, and $X \setminus \text{Int}(A) = \text{cl}(S \setminus A)$.

iii) $\text{cl}(\emptyset) = \text{Int}(\emptyset) = \emptyset$.

iv) $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$, $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$,
$\text{Int}(A \cup B) \supset \text{Int}(A) \cup \text{Int}(B)$, and $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$.

v) $\text{cl}\big(\bigcup_{i\in I} A_i\big) \supset \bigcup_{i\in I} \text{cl}(A_i)$, $\text{cl}\big(\bigcap_{i\in I} A_i\big) \subset \bigcap_{i\in I} \text{cl}(A_i)$,
$\text{Int}\big(\bigcup_{i\in I} A_i\big) \supset \bigcup_{i\in I} \text{Int}(A_i)$, and $\text{Int}\big(\bigcap_{i\in I} A_i\big) \subset \bigcap_{i\in I} \text{Int}(A_i)$.

Definition A.2.5. *Let $A \subset X$. A point $x \in X$ is called a* ***cluster point*** *of A if each neighbourhood of x contains at least one point of A distinct from x. The set $A' = \{x \in X \mid \forall\, U(x) : U(x) \cap (A - x) \neq \emptyset\}$ of all cluster points of A is called the* ***derived set*** *of A.*

In particular, $\text{cl}(A) = A \cup A'$, and A is closed if and only if $A' \subset A$.

Definition A.2.6. *$D \subset X$ is called* ***dense*** *in X if $\text{cl}(D) = X$, and is called* ***nowhere dense*** *if $X \setminus \text{cl}(D)$ is dense in X.*

Clearly, X is dense in X, and in fact, X is the only closed set dense in X.

Definition A.2.7. *Let u_n be a sequence of points in a space X. If there is a point $u \in X$ such that for every neighbourhood U of u, there is an N such that $n \geq N$ implies $u_n \in U$, then this sequence is said to* ***converge****. u is called a* ***limit point*** *of $\{u_n\}$.*

For example, the sequence $\{\frac{1}{n^2}\}$ in $\mathbb{R}$ converges to 0. Clearly the limit point of sequence u_n is the cluster point of $\{u_n\}$. Moreover, in a countable space X any cluster point of a set A is a limit point of a sequence of elements in A, and a point $u \in \text{cl}(A)$ if and only if there is a sequence of points of A that converges to u.

Definition A.2.8. *A topological space X is called* **reducible** *if it can be written as a union $X = X_1 \cup X_2$ of two closed proper subsets X_1, X_2 of X. A topological space is* ***irreducible*** *if it not reducible.*

In fact, a topological space X is irreducible if and only if all open subsets of X are dense. Moreover, an irreducible component is closed because if a subset is irreducible, so is its closure.

After considering topologies on one given set, we are now going to relate different topological spaces.

Definition A.2.9. *Let X and Y be topological spaces. A map $f : X \to Y$ is called* ***continuous*** *at $u \in X$ if for each neighbourhood V of $f(u)$ there exists a neighbourhood U of u such that $f(U) \subset V$. If the inverse image of each open set V in Y, $f^{-1}(V)$, is open in X, then f is continuous.*

For example, a constant map $f : X \to Y$ is always continuous. Therefore, any map from an arbitrary topological space to the trivial topological space is continuous. Moreover, let X be any set, $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on X. The bijective map $f : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous if and only if $\mathscr{T}_2 \subset \mathscr{T}_1$.

Proposition A.2.1. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is also continuous.* □

Theorem A.2.2. *Let X, Y be two topological spaces and $f : X \to Y$. The following statements are equivalent:*

i) f is continuous;

ii) the inverse image of every closed set is closed;

iii) $f\big(\mathrm{cl}(A)\big) \subset \mathrm{cl}\big(f(A)\big)$ for every $A \subset X$;

iv) $\mathrm{cl}\big(f^{-1}(B)\big) \subset f^{-1}\big(\mathrm{cl}(B)\big)$ for every $B \subset Y$. □

Continuity is defined by putting requirement on the inverse image. It is also important to start with the image.

Definition A.2.10. *A map $f : X \to Y$ is called* **open** *(resp.* **closed***) if the image of every open (resp. closed) set in X is open (resp. closed) in Y.*

Remarks. A continuous map needs not be an open map, conversely, an open map needs not be continuous. Also in general, an open map needs not be a closed map. Hence the concepts of 'open map', 'closed map', and 'continuous map' are independent.

For example, let $A \subset X$ and $i : A \to X$ be the inclusion map. It is obvious that i is continuous. Moreover, i is open (resp. closed) if and only if A is open (resp. closed) in X. If $f : X \to Y$ is bijective, then the two situations, f is closed and f is open, are in fact equivalent.

Here are some characterizations of open maps and closed maps.

Theorem A.2.3. *The following properties of a map $f : X \to Y$ are equivalent:*

i) f is an open map.

ii) $f[\mathrm{Int}(A)] \subset \mathrm{Int}[f(A)]$ for every $A \subset X$.

iii) f sends every member of a basis for X to an open set in Y.

iv) For every $x \in U(x)$ where $U(x)$ is an open neighbourhood of x and $U(x) \subset X$, there exists an open neighbourhood $W \subset Y$ such that $f(x) \in W \subset f(U)$. □

Theorem A.2.4. *$f : X \to Y$ is a closed map if and only if $\mathrm{cl}(f(A)) \subset f(\mathrm{cl}(A))$ for every set $A \subset X$.* □

Definition A.2.11. *A continuous bijective map $f : X \to Y$ is called a* **homeomorphism** *if its inverse $f^{-1} : Y \to X$ is also continuous, denoted by $f : X \cong Y$.*

Two spaces X, Y are homeomorphic, written $X \cong Y$, if there is a homeomorphism $f : X \cong Y$.

Homeomorphism is an equivalence relation in the class of all topological spaces. In general, if a property is true for one space X, then it is also true for every space homeomorphic to X, we call it a topological invariant. For example, the Euler characteristic is a topological invariant for 2-dimensional manifolds. Hence from this perspective, homeomorphic spaces share the same topological invariants, and topology can be described, to some extent, as the study of topological invariants. However, note that it is usually very difficult to show two spaces are homeomorphic, with construction of a homeomorphism being the general method.

Here are some properties of homeomorphisms.

Theorem A.2.5. *Let $f : X \to Y$ be bijective. The following properties of f are equivalent:*

i) f is a homeomorphism.

ii) f is continuous and open.

iii) f is continuous and closed.

iv) $f\big(\mathrm{cl}(A)\big) = \mathrm{cl}\big(f(A)\big)$ for every $A \subset X$. □

To establish that a given $f : X \to Y$ is a homeomorphism, one often uses the following

Theorem A.2.6. *Let $f : X \to Y$ and $g : Y \to X$ be continuous, $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then f is a homeomorphism.* □

Moreover, the set of all homeomorphisms of a topological spaces X to itself forms a group under composition.

A.3 Subspaces, Products and Quotients

This section concerns the construction of new topological spaces from old ones. Let $Y \subset X$, we will begin with defining the relative topology on Y.

Definition A.3.1. *Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. The relative (or induced) topology is $\mathscr{T}_Y = \{Y \cap U \mid U \in \mathscr{T}\}$. $(Y, \mathscr{T}_Y)$ is called a* ***subspace*** *of $(X, \mathscr{T})$. Y is called a discrete subspace of X if $\mathscr{T}_Y$ is the discrete topology.*

For example, adjoin two points, $\{+\infty\}$ and $\{-\infty\}$, to the set of all real numbers, $\mathbb{R}$. $(\mathbb{R} \cup \{-\infty, +\infty\})$ is called the extended real line, denoted by $\widetilde{E}^1$. With the induced topology, $\mathbb{R} \subset \widetilde{E}^1$ is the Euclidean space E^1.

Let $(X, \mathscr{T})$ be a topological space and $(Y, \mathscr{T}_Y)$ is a subspace. Then:

i) If $\{U_\alpha \mid \alpha \in A\}$ is a basis for $\mathscr{T}$, $\{Y \cap U_\alpha \mid \alpha \in A\}$ is then a basis for $\mathscr{T}_Y$.

ii) Let $Y_1 \subset Y$. Then Y_1 is $\mathscr{T}_Y$-closed if and only if $Y_1 = Y \cap F$, where F is $\mathscr{T}$-closed.

If $f : X \to Y$ is continuous and $A \subset X$ is taken with the subspace topology, then the restriction $f|A : A \to Y$ is continuous. However, the converse is false. Moreover, if $f(X)$ is taken with the subspace topology, then $f : X \to f(X)$ is continuous.

Definition A.3.2. *Let $\{Y_\alpha \mid \alpha \in A\}$ be any family of topological spaces. The* ***product topology*** *on $\prod_\alpha Y_\alpha$ consists of all subsets, $(u_1, u_2, \cdots, u_i, \cdots)$, which are the union of sets which have the form $\prod_\alpha U_\alpha$, where U_α is open in Y_α and i ranges over all elements of A.*

Clearly $X \times Y$ is homeomorphic to $Y \times X$. Hence $\prod Y_\alpha$ is unrestrictedly commutative.

Theorem A.3.1. *Let $\{Y_\alpha \mid \alpha \in A\}$ be any family of topological spaces. Then for each fixed $\beta \in A$, the canonical projection $p_\beta : \prod_\alpha Y_\alpha \to Y_\beta$ is a continuous open* *surjection. Moreover, let $f : X \to \prod_\alpha Y_\alpha$ be a map, then f is continuous if and only if $p_\beta \circ f$ is continuous for every $\beta \in A$.* □

Note that p_β in general is not closed. For example, in $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$, the set $A = \{(x, y) \mid xy = 1\}$ is closed in $\mathbb{R}^2$, however, under the projection onto the first factor, $p_1(A) = \{x \mid x \neq 0\}$, which is not closed in $\mathbb{R}^1$.

One of the main methods to construct new spaces is by 'pasting' given spaces together along preassigned subsets.

Definition A.3.3. *Let X be a set. Then the set of all equivalence classes of elements in X is called the* ***quotient set*** *of X and denoted by $X/\sim$, together with the identification topology determined by the projection $\pi : X \to X/\sim$, it is called the* ***quotient space*** *of X by $\sim$. Furthermore, the collection of sets $\{U \subset X/\sim \mid \pi^{-1}(U)$ is open in $X\}$ is called the* ***quotient topology*** *of $X/\sim$.*

For example, let I be the unit interval and $\sim$ be the equivalence relation $0 \sim 1$, $x \sim x$ $(x \neq 0, 1)$, then $(I/\sim) \cong S^1$. Also, consider $\mathbb{R}^2$ and the relation $\sim$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \quad \text{if } a_1 - b_1 \in \mathbb{Z} \text{ and } a_2 - b_2 \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the integers, then $\mathbb{T}^2 = \mathbb{R}^2/\sim$ is called a torus (see fig. A.2 for illustration).

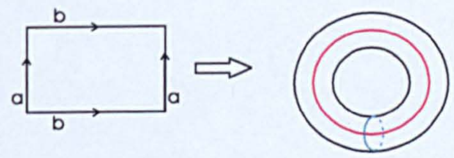


Figure A.2: Torus.

Definition A.3.4. *The set $\Gamma = \{(x, x') \mid x \sim x'\} \subset X \times X$ is called the* ***graph*** *of the equivalence relation $\sim$. The equivalence relation is called open (closed) if the canonical projection $\pi : X \to X \sim$ is open (closed).*

A map $f : X/\sim \to Y$ is continuous iff $f \circ \pi : X \to Y$ is continuous. Moreover, $\sim$ is open (closed) iff for any open (closed) subset A of X, the set $\pi^{-1}(\pi(A))$ is open (closed).

Theorem A.3.2. *Let X, Y be two spaces with equivalence relation $\sim$, and $f : X \to Y$ be a relation-preserving, continuous map. Then, the induced map to the quotient spaces, $f_* : X/\sim \to Y/\sim$ is also continuous.* □

Definition A.3.5. *For any space X, the* **cone** *CX over X is the quotient space $(X \times I)/\sim$, where $\sim$ is the equivalence relation $(x, 1) \sim (x', 1)$ for all x, x' in X.*

The elements in CX is denoted by $\langle x, t\rangle$. Intuitively, CX is obtained from $X \times I$ by pinching $X \times 1$ to a single point.

Definition A.3.6. *Let $K = [-1, 1]$. Given a space X, the* **suspension** *SX is the quotient space $(X \times K)/\sim$, where $\sim$ is the equivalence relation $(x, 1) \sim (x', 1)$, $(x, -1) \sim (x', -1)$ for all x, x' in X.*

SX is the dual to CX, and denoted by $\langle x, t\rangle$ as well. Intuitively, SX is obtained from $X \times K$ by pinching sets $X \times 1$ and $X \times (-1)$ to two points, respectively. Here are some properties:

i) $SX \cong CX/X$.

ii) CX is homeomorphic to the subspace $\{\langle x, t\rangle \in SX \mid t \geq 0\}$.

iii) If the map $f : X \to Y$ is continuous, then the induced map $Cf : CX \to CY$ ($Sf : SX \to SY$) by $\langle x, t\rangle \to \langle f(x), t\rangle$ is also continuous.

A.4 Separation Axioms and Covering Axioms

In this section, We will require a topology that 'separates' varying types of subsets and introduce several separation axioms.

Definition A.4.1. *A space Y is* ***Hausdorff*** *(or separated) if each two distinct points have disjoint neighbourhoods.*

For example, Euclidean space E^n is Hausdorff. In Hausdorff spaces, every finite set is a closed set.

Theorem A.4.1. *The following invariance properties hold for Hausdorff spaces:*

i) Hausdorff topologies are invariant under closed bijections.

ii) Each subspace of a Hausdorff space is also a Hausdorff space.

iii) The cartesian product $\prod Y_\alpha$ is Hausdorff if and only if every Y_α is Hausdorff. □

Let X be an arbitrary space, Y be Hausdorff, and $f, g : X \to Y$ be continuous. Then:

i) $\{x \mid f(x) = g(x)\}$ is closed in X.

ii) If $D \subset X$ is dense, and $f|D = g|D$, then $f = g$ on X.

iii) The graph of the continuous $f : X \to Y$ is closed in $X \times Y$.

iv) X is Hausdorff if f is injective and continuous.

Theorem A.4.2. *Let $\sim$ be an equivalence relation in X, and $\pi : X \to X/\sim$ be the projection. If the graph is closed in $X \times X$ and π is an open map, then $X/\sim$ is Hausdorff.* □

A stronger separation condition than Hausdorff yields the regular spaces.

Definition A.4.2. *A Hausdorff space Y is* **regular** *if for each $y \in Y$ and $y \notin A$, where A is a closed set and $A \subset Y$, then there exists a neighbourhood U of y and an open set $V \supset A$ such that $U \cap V = \emptyset$.*

Every regular space is a Hausdorff space, but the converse is not true. Let $\mathbb{R}$ denote the set of all real numbers, and $\mathscr{T}$ the topology having the open intervals and the set of rational numbers, $\mathbb{Q}$, as subbasis. Obviously $(\mathbb{R}, \mathscr{T})$ is a Hausdorff space. However, it is not regular. For example, $\mathbb{R}\backslash\mathbb{Q}$ is a closed set, but 1 and $\mathbb{R}\backslash\mathbb{Q}$ do not have disjoint neighbourhoods.

Properties of Hausdorff spaces also hold for regular spaces. Moreover,

Theorem A.4.3. *Let X be regular and $\pi : X \to X/\sim$ be a closed and open map, then $X/\sim$ is Hausdorff. Also if $A \subset X$ is closed, then X/A is Hausdorff.* □

Separation stronger than regularity is

Definition A.4.3. *A Hausdorff space is* ***normal*** *if every pair of disjoint closed sets have disjoint neighbourhoods.*

Most standard spaces in analysis is normal. The discrete spaces are normal, for example. It turns out that we have the following implications:

$$\text{Normal} \Longrightarrow \text{Regular} \Longrightarrow \text{Hausdorff}.$$

Theorem A.4.4. *The invariance properties of normal spaces are:*

i) Normality is invariant under continuous closed surjections.

ii) A subspace of a normal space may not be normal. However, a closed subspace is normal.

iii) The cartesian product of normal spaces may not be normal. However, if the product is normal, each factor must be normal. □

Let X, Y be normal spaces, and $A \subset X$ closed. Then X/A is normal. Furthermore, if $f : A \to Y$ is continuous, then $X \cup_f Y$ is normal.

A covering $\{U_\alpha \mid \alpha \in I\}$ of a space Y is called *point-finite* if for every $y \in Y$ there are at most finitely many indices $\alpha \in I$ such that $y \in U_\alpha$. The normal spaces are characterized by these open coverings.

Theorem A.4.5. *Let $\{U_\alpha \mid \alpha \in I\}$ be a point-finite covering of a topological space Y by open sets. Then Y is normal if and only if there exists another covering $\{V_\alpha \mid \alpha \in I\}$ of Y such that* $\mathrm{cl}(V_\alpha) \subset U_\alpha$ *for every $\alpha \in I$, and $V_\alpha \neq \emptyset$ whenever $U_\alpha \neq \emptyset$.* □

In fact, it turns out that, by relating spaces to the behaviour of their open coverings, weak separation properties can become very strong.

Definition A.4.4. *A space Y is* ***first countable*** *if for each $u \in Y$ there exists a sequence $\{U_1, U_2, \cdots\} = \{U_n\}$ of neighbourhoods of u such that for any neighbourhood U of u, there is an integer n such that $U_n \subset U$. Y is called* ***second countable*** *if it has a countable basis.*

Clearly, second countability implies first countability. $\mathbb{R}^n$ is second countable since it has the countable basis formed by rectangles. In fact, most of topological spaces of interest to us are second countable.

Theorem A.4.6. (Lindelöf) *Every open covering $\{U_\alpha\}$ in a second countable space X has a countable subcovering.* □

This is the main property of second countable spaces. It is invariant under continuous surjections.

Definition A.4.5. *A Hausdorff space is* ***separable*** *if it contains a countable dense set.*

Since the set of rational numbers $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, $\mathbb{R}$ is separable. Furthermore, if Y is a second countable space, then every subspace of Y is separable.

We now consider the countable union and intersection of sets which appear frequently in analysis.

Definition A.4.6. *A set F is called an F_σ if it is the union of at most countably many closed sets. A set G is called a G_δ if it is the intersection of at most countably many open sets.*

$\mathbb{Q}$ is an F_σ since it is countable and every point in it is a closed set. Note that the concepts of F_σ and G_δ are neither exclusive nor exhaustive. They are frequently expressed in the following manner:

i) If F is an F_σ, then there is a nondecreasing sequence $F_1 \subset F_2 \subset \cdots$ of closed sets with $F = \bigcup_{i=1}^{\infty} F_i$.

ii) If G is a G_δ, then there is a nonincreasing sequence $G_1 \supset G_2 \supset \cdots$ of open sets with $G = \bigcap_{i=1}^{\infty} G_i$.

Clearly, the complement of an F_σ is a G_δ, and vice versa. However, the properties of F_σ and G_δ are not well preserved under all countable set operations in general, hence we now define a family which is preserved under these operations.

Definition A.4.7. *A nonempty family $\Sigma \subset P(X)$, where $P(X)$ denotes the power set of X, is called as **σ-ring** if*

i) $A \in \Sigma \;\Rightarrow\; \mathrm{cl}(A) \in \Sigma$,

ii) $A_i \in \Sigma$ *for* $i = 1, 2, \cdots \;\Rightarrow\; \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

The unique smallest σ-ring containing the topology of X is called the family of *Borel sets* in X. Every F_σ and every G_δ is a Borel set. Moreover, the countable union, countable intersection and the difference of Borel sets is a Borel set.

A.5 Connectedness

Intuitively, if a space is not comprised of two separate pieces then it is connected. In topology, this simple idea has led to some very sophisticated algebraic techniques for classifying different spaces.

Definition A.5.1. *A space X is* **connected** *if it is not the union of two nonempty disjoint open sets. A subset $A \subset X$ is connected if it is connected as a subspace of Y. If A is the only nonempty connected subset of X that contains A, it is then called a component of X. Y is* **locally connected** *if the basis of Y is composed of connected open sets.*

For example, $\mathbb{R}^n$ is connected and locally connected. The trivial topology is connected and locally connected, while the discrete topology is neither. Every component of a space Y is closed in Y, and the set of all distinct components forms a partition of Y. If Y is locally connected, the components are open and closed.

Theorem A.5.1. (Intermediate Value Theorem.) *Each continuous real-valued function on a connected space X takes on all values between any two that it assumes.* □

Definition A.5.2. *A path in Y is a continuous mapping $f : I \to Y$, where $I = [0, 1]$. We say f joins $f(0)$ to $f(1)$. If each pair of points in Y can be joined by a path, the space Y is called* **path-connected.** *Y is called* **locally path-connected** *if for every $y \in Y$ and neighbourhood U of y, there exists another neighbourhood V of y such that any pair of points in V can be joined by a path in U.*

Again $\mathbb{R}^n$ is path-connected and locally path-connected. The discrete space having more than one point is never path-connected. Every path-connected space is connected ,while the converse is not necessarily true. For example, Let X be the subspace of $\mathbb{R}^2$ defined by $X = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x}$ or $x = 0, -1 \leq y \leq 1\}$, then X is connected, but not path-connected. Evidently path-connectedness is a topological invariant.

Before proceed to the notion of simple connectedness, we first give the definition of homotopy.

Definition A.5.3. *Let X, Y be two spaces, I the unit interval, and $f, g : X \to Y$ are two maps. If there is a continuous map $\Psi : X \times I \to Y$ such that $\Psi(x,0) = f(x)$ and $\Psi(x,1) = g(x)$ for every $x \in X$, then f and g are called* ***homotopic*** *(written: $f \simeq g$).*

Homotopy theory plays an important role in algebraic topology, and homotopy invariants count for most of the known algebraic invariants. Intuitively, Ψ stands for a continuous deformation of the map f to g.

Definition A.5.4. *Let X be a topological space, a closed path l is called a* ***loop*** *based at p if there exists a continuous map $l : I \to X$ such that $l(0) = l(1) = p$, where $p \in X$. The loop l is called* ***contractible*** *if it is homotopic to the constant map, i.e. $l \simeq c$, where $c : I \to p$.*

Roughly speaking, a loop is contractible if it can be shrunk continuously to the point p. The study of loops leads naturally to homotopy theory. In fact, the loops based at one point can be classified as the so called *fundamental group*, denoted by $\pi(X,p)$, which is one of the main topics in algebraic topology.

Definition A.5.5. *If a space X is connected and every loop in X is contractible, then X is* ***simply connected****.*

Clearly if X is simply connected, then $\pi_1(X,p)$ is trivial for every $p \in X$ since every loop can be shrunk to its base point.

A.6 Metric Spaces

In many cases the topology is derived from a notion of distance, for example, on $\mathbb{R}^2$ the standard distance function

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ gives rise to 2-dimensional flat disks and hence the topology.

Definition A.6.1. *A* ***metric*** *on a set M is a map $d : M \times M \to \mathbb{R}$ such that for any $x, y, z \in M$,*

i) $d(x,y) \geq 0$.

ii) $d(x,y) = 0$ if and only if $x = y$ (definiteness).

iii) $d(x,y) = d(y,x)$ (symmetry).

iv) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

With each metric d in a set M, we associate a definite topology.

Definition A.6.2. *For $\epsilon > 0$ and $a \in M$. The set $B_\epsilon(a) = \{m \in M \mid d(m,a) < \epsilon\}$ is called the open* ***ϵ-ball****. The topology with the family of all ϵ-balls in M as basis is called the* ***metric topology****, and (M,d) is called the* ***metric space****. Two metrics in a set called equivalent if they induce the same metric topology.*

The Euclidean space $\mathbb{E}^n$ is a metric space. In fact, every set M can be made into a metric space by the discrete metric defined by $d(m,n) = 1$ for $m \neq n$.

In a metric space M, the distance of a point m_0 to a nonempty set A is defined as $d(m_0, A) = \inf\{d(m_0, a) \mid a \in A\}$; the distance between two nonempty sets A and B is defined as $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$; the diameter of a nonempty set A is defined as $\delta(A) = \sup\{d(x,y) \mid x \in A, y \in A\}$. A set A is called *bounded* if $\delta(A) < \infty$.

Metrizability is a topological invariant, every subspace of a metric space is also a metric space.

Theorem A.6.1. *Every metric space is normal and first countable. Moreover, in metric spaces, the concepts of second countability, separability, and* Lindelöf *are all equivalent.* □

Definition A.6.3. *Let M be a metric space and $\{u_n\}$ a sequence in M. If for each $\varepsilon > 0$, there exists an integer N such that $n, m \geq N$ such that $d(u_n, u_m) < \varepsilon$, then $\{u_n\}$ is called a* ***Cauchy sequence****. The space M is called* ***complete*** *if every Cauchy sequence converges.*

A convergent sequence is a Cauchy sequence.

A.7 Compactness

This section concerns spaces having a strengthened version of the Lindelöf property, i.e. compactness.

Definition A.7.1. *A Hausdorff space Y is* ***compact*** *if each open covering has a finite subcovering.*

A discrete space is compact if and only if it is finite. Moreover, all finite subsets and $\emptyset$ in any space X are compact.

Theorem A.7.1. *Let Y be compact, Z be Hausdorff, and $f : Y \to Z$ continuous, then f is closed; if f is bijective, then it is a homeomorphism.* □

The continuous image of a compact set is compact; a subspace of a compact space is compact if and only if it is closed. The product space $\prod_\alpha Y_\alpha$ is compact if and only if every Y_α is compact.

Definition A.7.2. *A Hausdorff space is* ***countably compact*** *if every countable open covering has a finite subcovering.*

It is clear that every continuous real-valued function f on a countably compact space is bounded. Countable compactness is characterized by the behaviour of sequences.

Theorem A.7.2. (Bolzano-Weierstrass Theorem.) *For a compact Hausdorff space, every sequence has a convergent subsequence.* □

In metric spaces, there is no distinction between countable compactness and compactness, hence the Bolzano-Weierstrass theorem becomes very important. Moreover, the converse is also true in metric spaces. Since every compact metric space is second countable, we have

Theorem A.7.3. *A countably compact space Y is metrizable if and only if it is second countable.* □

Given a metric space Y, $A \subset Y$ compact, and $B \subset Y$, A is closed and bounded. If $A \cap B = \emptyset$, then $d(A, B) > 0$.

Theorem A.7.4. *Let Y be a compact metric space and $\{U_\alpha\}$ be an open covering of Y. Then there is a positive number $\lambda(\{U_\alpha\})$ such that every ball $B_\lambda(y)$ is contained in at least one element of $\{U_\alpha\}$, this number is called Lebesgue number of the covering.*
□

A basic application of the above theorem yields the classical theorem on the uniformity of continuity:

Theorem A.7.5. *Let Y, Z be two metric spaces, Y compact, and $f : Y \to Z$ continuous. Then for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ which depends only on ε, such that $f\big(B(y,\delta)\big) \subset B\big(f(y),\varepsilon\big)$ for every $y \in Y$, i.e. f is uniformly continuous.* □

In analysis, many of the important spaces are not compact, but have a local compactness instead. A subset A of a space is *relatively compact* if its closure is compact, and a Hausdorff space is *locally compact* if every point has a relatively compact neighbourhood. Clearly compactness implies local compactness, and it is invariant under continuous open mappings.

Moreover, some topological spaces are not compact inherently. However, there are several techniques that we can apply to compactify them. For example, $\mathbb{R}^1$ is a noncompact regular space. It can be embedded in a compact space by several methods, one of them is identifying $\mathbb{R}^1$ with S^1 by stereographic projection.

Definition A.7.3. *A* ***compactification*** *of a space X is a pair $(\widehat{X}, h)$, where $\widehat{X}$ is a compact Hausdorff space and $h : X \to \widehat{X}$ is a homeomorphism of X onto a dense subset of $\widehat{X}$.*

Since a subset of a compact Hausdorff space is necessarily completely regular, they are the only spaces that can be compactified. Among these spaces, the locally compact ones carry a property such that they are characterized by the position they must have in each compactification:

Theorem A.7.6. *X is locally compact if and only if any given compactification $(\widehat{X}, h)$, $h : X \to \widehat{X}$ is an open mapping.* □

By theorem A.7.6, a space that is not locally compact can never be compactified by adding a single point. Moreover, we have

Theorem A.7.7. *(***P. Alexanderoff***) Any locally compact space X can be embedded in a compact space by adding a single point. Given two compactifications, $\widehat{X}$ and $\widehat{Y}$, if both $\widehat{X} - X$ and $\widehat{Y} - X$ are a single point, then $\widehat{X} \cong \widehat{Y}$. In fact, there exists a homeomorphism $h : \widehat{X} \to \widehat{Y}$ such that it is the identity map on X.* □

This construction generalizes the one-point compactification of function theory, which is adding a 'point at ∞' to the plane. As a trivial example, $[0, 1]$ is the one-point compactification of $]0, 1]$. From the perspective of metrizability, we have

Theorem A.7.8. *Denote X a locally compact space. Then its one-point compactification $\widehat{X}$ is metrizable if and only if X is second countable.* □

In 1899, Baire proved that the intersection of any countable family of open dense sets in a locally compact space Y is dense. This property classified a family of spaces that has a great impact in both topology and analysis.

Definition A.7.4. *Let X be a topological space and $A \subset X$ a subset. A is called* ***nowhere dense*** *in X if the interior of its closure is empty. A is called of first category in X if $A \subset \bigcup_{n\geq 1} C_n$ where C_n is closed and* $\text{Int}(C_n) = \emptyset$*; A is called of second category if it is not of first category. X is called a* ***Baire space*** *if the intersection of each countable family of open dense sets in X is dense.*

Baire spaces are of second category, and every locally countably compact regular space is a Baire space. The significance of Baire space is that, let X be a Baire space, if $\{A_n \mid n \in \mathbb{Z}^+\}$ is any countable closed covering of X, then $\text{Int}(A_n) \neq \emptyset$ for at least one n.

Theorem A.7.9. *(***Baire Category Theorem.***)*

*i) (***BCT1***) Every complete pseudometric space is a Baire space, where pseudometric space is a generalization of metric space in which the distance between two distinct points can be zero.*

*ii) (***BCT2***) Every locally compact space is a Baire space.* □

For example, $\mathbb{R}$, the set of irrational numbers and the Cantor set are all Baire spaces. Baire condition is fundamental to the idea of 'genericity' in dynamical systems.

Appendix B

Smale's Theory

In this section, we will outline Smale's theory of dynamical systems on manifolds and the notion of basic set. All the results here are well-known, and can be found in, for example, [Smale, 1967].

B.1 Conjugacy Problems

A continuous dynamical system is defined by an $\mathbb{R}$-action, on a compact manifold M, called a *flow*, i.e. a map $\Phi : \mathbb{R} \times M \rightarrow M$ such that

(a) $\Phi(t, \cdot) : M \rightarrow M$ is a homeomorphism of M for all t,

(b) $\Phi(0, \cdot) : M \rightarrow M$ is the identity on M,

(c) $\Phi\big(t, \Phi(s, x)\big) = \Phi(t + s, x)$ for all $s, t \in \mathbb{R}$, $x \in M$.

We usually write $\phi_t(\cdot) = \Phi(t, \cdot)$.

Now let f denote such a flow, i.e. $f : M \rightarrow M$ is a diffeomorphism. The orbit of $x \in M$ relative to f, is the subset $\{f^n(x) \mid n \in Z\}$ of M. If an orbit is finite, we call it *periodic orbit.* The points on a periodic orbit are called *periodic points.* Hence $x \in M$ will be called a periodic point iff $f^n(x) = x$ and n is the period of x. Moreover, x is a *fixed point* if $n = 1$.

A subset $\Lambda \subseteq M$ is said to be *invariant* for f if $f^n(\Lambda) = \Lambda$ for all $n \in Z$. A point $x \in M$ will be called a *wandering point* if there exists a neighbourhood U of x such that $\bigcup_{|m|>0} f^n(U) \cap U = \emptyset$. A point is called *non-wandering* if it is not a wandering point. i.e. $\bigcup_{|m|>0} f^n(U) \cap U \neq \emptyset$. Clearly the set of non-wandering points form a closed invariant set, $\Omega = \Omega(f)$.

Definition B.1.1. *Suppose $f, f' \in \text{Diff}(M)$. If there exists a homeomorphism $h : \Omega(f) \to \Omega(f')$ such that $hf = f'h$, then f and f' are called* ***topologically conjugate*** *to Ω . The corresponding stability will be called* ***Ω-stability****.*

The above equivalence relation is important in Smale theory. Furthermore, if all diffeomorphisms f' in a sufficiently small neighbourhood of f in $C^1(M)$ are conjugate to f on Ω, then f will be called Ω-stable.

B.2 Stable Manifold Theorem

If a closed invariant set Λ of $f \in \text{Diff}(M)$ cannot be represented as

$$\Lambda = \Lambda_1 \cup \Lambda_2,$$

where Λ_1 and Λ_2 are nonempty disjoint invariant sets, then we say Λ is *indecomposable.*

Moreover, if the action of f on Λ can be decomposed into uniformly expanding and contracting pieces, it will, to a large extent, simplify the study of the behaviour of Λ.

Definition B.2.1. *An invariant set $\Lambda \in M$ for f is* ***hyperbolic*** *if there exists a continuous f-invariant splitting of the tangent bundle TM_Λ into stable and unstable* *bundles* $E^s_\Lambda \bigoplus E^u_\Lambda$ *with*

$$\|Df^n(v)\| \le C\lambda^{-n}\|v\| \qquad \forall\, v \in E^s_\Lambda, \forall\, n > 0$$
$$\|Df^{-n}(v)\| \le C\lambda^{-n}\|v\| \qquad \forall\, v \in E^u_\Lambda, \forall\, n > 0$$

for some fixed $C > 0,\ \lambda > 1$.

If f is hyperbolic on all of M, then f is called *Anosov.* Within this hyperbolic structure, the splitting of the tangent bundle TM_Λ into f-invariant stable and unstable bundles will hence yield the f-invariant stable and unstable manifolds in M.

Theorem B.2.1. (Stable Manifold Theorem) *Suppose* $f : M \to M$ *is a diffeomorphism having a hyperbolic structure, with the invariant set being* Λ, *for each* $x \in \Lambda$, *we then have*

$$\begin{aligned} W^s(x) &= \{y \in M : \lim_{n\to\infty} \|f^n(y) \Leftrightarrow f^n(x)\| = 0\}, \\ W^u(x) &= \{y \in M : \lim_{n\to-\infty} \|f^n(y) \Leftrightarrow f^n(x)\| = 0\}. \end{aligned}$$

Moreover, $W^s(x)$ *and* $W^u(x)$ *are smooth and injective immersions of the stable and unstable tangent bundles,* E^s_x *and* E^u_x, *respectively, and the sets* $W^s(x)$ *and* $W^u(x)$ *are the stable and unstable manifolds of* x. □

Because of its well-defined splitting structure, those $f \in \mathrm{Diff}(M)$, with M compact and satisfying the following conditions are most widely studied:

1. the non-wandering set Ω is finite,
2. the periodic points of f are hyperbolic,
3. for each $p, q \in \Omega$, $W^s(p)$ and $W^u(q)$ have transversal intersection.

The simplest examples of hyperbolic sets for diffeomorphisms are hyperbolic fixed points and hyperbolic periodic points. Clearly Anosov diffeomorphisms are defined on structures where the whole compact manifold is hyperbolic. Moreover, if the invariant set Λ is homeomorphic to a Cantor set, then it has a hyperbolic structure, in fact, this is how the Smale horseshoe is defined.

B.3 Axiom A Systems

Definition B.3.1. *Suppose $h : M \to M$ is a homeomorphism. If there exists a dense orbit defined by h, then h is said to be* ***topologically transitive.***

Clearly, the dense orbits form a Baire set of M. In [Smale, 1967], he introduced the famous Axiom A systems which satisfy the following two properties.

Definition B.3.2. (The Axiom A.) *The diffeomorphism $f : M \to M$ satisfies Axiom A if*

(a) the non-wandering set Ω is hyperbolic,

(b) the periodic points of f are dense in Ω.

Theorem B.3.1. (Spectral Decomposition Theorem.) *If $f : M \to M$ satisfies Axiom A, then there is a unique way to write $\Omega(f)$ as the finite union of disjoint, closed, invariant indecomposable subsets, on each of which f is topologically transitive:*

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k.$$

Ω_i will be called basic sets. □

In fact, the decomposition of a manifold into invariant sets of the given diffeomorphism is analogous to the decomposition of a finite dimensional vector space into eigenspaces of a linear map. Moreover, Smale introduced another main axiom

Definition B.3.3. (The Axiom B.) *Suppose that $f \in \mathrm{Diff}(M)$ satisfies Axiom A, Ω_i are the basic sets, and $W^s(\Omega_i) = \{x \in M \mid f^m(x) \to \Omega_i, m \to \infty\}$. Then there exists* *points* $p \in \Omega_i$, $q \in \Omega_j$ *such that* $W^s(p)$ *and* $W^u(q)$ *have a transversal intersection, if* $W^s(\Omega_i) \cap W^u(\Omega_j) \neq \emptyset$.

Now we state the main approximation theorem due to Pugh.

Theorem B.3.2. (Closing Lemma.) *Suppose that* $f \in \mathrm{Diff}(M)$, *and* $\phi : \mathbb{Z} \to M$ *is given by* $\phi(n) = f^n(x)$ *for* $x \in M$. *If* p *is a point such that* $\phi(m) = f^n(p)$ *is not a homeomorphism onto its image, then there exists a* C^1 *approximation* f' *of* f *where* p *is a periodic point of* f'. □

According to different dynamical behaviours with respect to various structural stabilities, we define

Definition B.3.4. *A flow* ϕ_t *is* ***Morse-Smale*** *if*

i) the basic set Ω_i *is hyperbolic,*

ii) the stable and unstable manifolds of basic sets meet transversely,

iii) each basic set is a single closed orbit or a field.

Among structurally stable flows, Morse-Smale flows have attracted special interest due to the fact that they form a dense subset among the gradient flows, regardless of the smoothness class. Moreover, Morse-Smale flows are dense in C^1 topology on compact 2-manifolds (due to the closing lemma).

Definition B.3.5. *A* ***Smale flow*** ϕ_t *on* M *is one for which*

i) the basic set Ω_i *is hyperbolic,*

ii) the basic subsets of the basic set are zero or one-dimensional,

iii) the stable manifold of any orbit in Ω *and the unstable manifold of any other orbit in* Ω *have transverse intersection.*

The importance of Smale flows on compact manifolds is that they are structurally stable under C^1 perturbations. However, they are not dense in the C^1 topology.

Moreover, given a basic set Ω_i of $f \in \text{Diff}(M)$, which has a basic set structure satisfying Axiom A and B, we say Ω_i is an *attractor* if for an open set $U \subset M$, $\bigcap_{|n|>0} f^n(U) = \Omega_i$. On the other hand, Ω_j is called a *repeller* if for an open set $U \subset M$, $\bigcap_{|n|>0} f^{-n}(U) = \Omega_j$.

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