# Essays in Environmental and Natural Resource Economics as a Contribution to Sustainable Development

by


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There are many people to whom I owe a great deal to for keeping me going throughout the last 4 years.

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# SUMMARY

This research focuses on the use of dynamic optimisation modelling techniques to describe the interactions between the economy and the environment.

The environment not only provides us with economically valuable resources but also provides us with many essential services that support human welfare. Over-exploitation of these resources and the destruction of the natural environment not only affects human welfare but may severely limit future production possibilities. For natural resources to continue to be inputs to production and to ensure equal access to environmental services by future generations, all ecological systems must remain in operation. The issue is how we treat our natural resources so that we have a sustainable economy.

In this thesis, models are formulated that combine the economic and environmental processes. Current environmental concerns are incorporated into the framework of economic optimisation problems. The issues addressed are:

1. The competition for land of preservation and development. What is the optimal balance between the two?

2. Pollution from production can have negative effects on the environment. This in turn can affect the economy through diminished resource supply. What is

the optimal use of these environmental resources so that we can sustain our productive capabilities?

3. Carbon emissions need to be controlled. A tax on emissions would encourage switching away from carbon intensive fuels. How should this tax behave over time - should it rise or fall?

4. With increasing populations, resources are being used up dramatically. Can we get to a point where the economy can be sustained while maximising human welfare?

5. What happens to a private firm's output decisions when it has to conform to environmental regulations?

The models are useful for studying sustainable development in that they provide us with the steady state relations of a sustainable economy and, in some cases, the short run dynamics.

# CHAPTER ONE

# INTRODUCTION

Before the main thesis is presented, it is important to discuss current environmental issues so that the areas the work is concerned with can be set in context. It is necessary to briefly describe the problems that we face today so as to give a background to the work and to see why it is important that environmental considerations are taken into account. It would be foolish to launch into such work without knowing what the motives are behind it and why the work is being carried out.

In Section I, there will be a discussion of the current issues that this thesis is concerned with, such as deforestation, global warming and population pressures. In Section II there will be a brief discussion about sustainable development and the importance the environment plays in supporting human life.

In Section III there will be a summary of the thesis and how this bears on the environmental and sustainable development issues that have been discussed.

## SECTION I

### Deforestation

Tropical forests cover 9% of the earth's surface or 1,260 million hectares (FAO 1988). Around 7 million hectares are lost every year and over half of that is in Latin America alone. Overall there is a deforestation rate of 0.6% and the rate of disappearance is accelerating. For example, Brazil accounts for 67% of the Amazonian tropical forest. Before 1975 there was very little loss of the forest, perhaps 0.6% of the total area, but from 1975 to 1985 the deforestation rate grew at a rate of 11%.(Barbier *et al* 1991).

The focus of attention on tropical rain forests has arisen because they have many important ecological and environmental functions to serve and there is an increasing threat to their existence. Tropical forests absorb carbon dioxide through photosynthesis. They retain and release water into the atmosphere and regulate the regional climate. The loss of forest cover will change the reflectivity of the earth's surface. If the forests are burned, then the carbon dioxide that they have absorbed is released into the atmosphere and so contributes to the greenhouse effect, (see below). Deforestation will have serious impacts on regional, global and local climates.

Tropical rain forests are crucial for the protection of biological diversity. They are home to millions of wildlife and plant species and scientists believe that only a tiny fraction of species have been identified. Deforestation will mean that millions of species will lose their homes and may become extinct. Tropical rain forests are also valued for their existence; people desire the very existence of these resources, irrespective of whether they use them or not. They are valued for educational and medicinal purposes; the forests provide timber and non-timber products such as resins, honey, nuts and the increasing loss of forest cover will destroy the essential services provided by the forest.

Tropical rain forests are relatively open access areas and so harvesting and converting them for development purposes are undertaken without any common property management of the resource. Therefore users will not take into account the full environmental costs they are imposing. The forest is not being managed in a sustainable manner.

The most important cause of forest loss is the conversion of forest land for agricultural purposes. The traditional practice of "slash and burn" agriculture has been used for thousands of years in an environmentally sound way. Most of the tropical forest soils are of low fertility and so are not suited to continuous cultivation. New forest cultivators bring with them farming practices that are not suited to the forest and pursue continuous cropping until the soil is completely degraded and unable to restore itself.

Barbier et al's paper examines deforestation in three countries; Brazil, Indonesia and Zaire. In Brazil deforestation is attributable mostly to cattle ranching, which is estimated to contribute 73 - 88% of deforestation. The establishment of settlements by farmers have also contributed to a high degree of deforestation in Brazil. Most of the deforestation is a direct consequence of deliberate Government policy. Development strategies financed by the Government through subsidies and incentives have favoured the expansion of the forest destruction. They point out that a typical subsidised cattle project in the Amazon would have a net return of 2.5 times the investors outlay. However, without subsidies the project would make a net loss of 0.9 times the outlay. There are also Government policies such as the land tax system that encourages owners of large farms to convert their forests for development purposes and in doing so they are exempt from land tax.

In Indonesia 97% of the forests are in the Outer Islands. The major source of deforestation is the migration of people from the Inner Islands. Since 1979, approximately 540,000 families have resettled in the Outer Islands. Zaire contains approximately 10% of the world's tropical forests and 180,000 hectares of forest are estimated to be cut down every year. The main cause of deforestation is the settlement of small holding farms in the forest. Demand for fuelwood is also a primary cause as this is the most common energy source. With a well managed system of forest resources, sustainable development could be achieved without the tropical forests being completely destroyed. The areas that need to be addressed are population pressure, transmigration of people and misguided government policy.

Miller et al (1993), point out many causes of deforestation. Firstly, increasing populations in developing countries places ever greater demand on their resource base, but one of the main causes they point out is misguided Government policies.

Forestlands in developing countries were owned by local tribes and local farming communities. The practices they used for hunting farming and grazing were based on traditional customs that effectively regulated the use of the forest. In the last forty years, they point out that 80% of the worlds tropical rain forests have been bought under the ownership of national governments, resulting in government having the rights of forest control. There is no longer now the incentive to conserve the forest resources. They also discuss the fact that deforestation is being triggered in many rain forest regions as title to land is granted to those who 'improve' the land by converting it to another use. Those who would exercise traditional sustainable practices would not normally be given title to the land. Also, landless people may find that their tenure is insecure and so are less likely to practice sustainable forest use than if they owned the land. They will farm intensively so as to obtain the maximum immediate financial gain in case they loose their tenure of the land.

Governments are also subsidising the exploitation of the forest through investment subsidies and tax incentives. In Brazil, corporations that invest in Amazonian development are given investment tax credits against income tax liabilities. Subsidised credit is also given to crop and livestock development of the forest, thereby diverting developments to the Amazonian region. National governments also allocate significant funds to infrastructure development, the motives for which are economic and social and also to secure national boundaries.

The third cause of deforestation is from international pressures. There is increasing pressure to finance immediate development needs by liquidating forest capital. The external debt problem exerts a powerful incentive to exploit the forests. The forests are one of the only resources that can be converted to much needed immediate revenue for debt servicing

Miller et al also point out that behind many of the flawed government policies is faulty economic analysis concerning the use of forest resources. The fourth cause they discuss of deforestation is that the true value of the forest and the true cost of its exploitation are not calculated correctly. The common value put to forests is the value of the timber or the agricultural potential of cleared land. There is a need to evaluate the unpriced services of the forests so that attention can focus on the full environmental costs of deforestation. The non-timber products sustain much of the forest population who depend on nuts berries, fish and honey. These products do not reach the market place and so the value of the products is ignored. Some products do however reach the market place for instance, resins, essential oils and medicinals and this can provide a considerable income for those who collect them. Indeed the income gained from collection of forest products can be very large. In the 1980's, Indonesia exported $125 million of non-timber products annually. These are sustainable practices that could yield sustainable incomes. A study on Peru found that the net earnings from the sale of non-timber products were 13 times greater than the net earnings from the sale of the forest timber, (Miller et al (1993)).

The costs of exploitation are not borne by those who benefit from such practices. The costs will accrue to innocent parties, usually future generations. The company that is dutting down trees in the forest will not bear the cost of the loss of watershed, local farmers and communities will be affected by flooding or increased salinisation of the waterways they depend on. The full costs and benefits need to be internalised - they need to be accounted for in the actions of those that destroy the forest. There is a need to evaluate the unpriced services of the forests and the full costs of exploitation so that attention can focus on the full environmental costs of deforestation.

However, the industrialised world has no right to criticise the developing countries for the policies that encourage deforestation. Why should the developing countries not exploit their resources, all other countries have done and still do? After all, that is how industrialised countries have developed into the economic forces they are today. Industrialised countries now call for conservation measures in the developing countries; this shows no sensitivity for their development objectives. The only way forward is for industrialised countries to encourage policies and practices that are sustainable and so sustainable economic growth can be achieved. For instance, industrialised countries should only import timber from well managed sustainable forests. Governments should be encouraged to invest in non-timber products, which have been shown capable of yielding more revenue than timber products. Government interference in the market for timber and agriculture give distorted price signals, making timber extraction and clearance more profitable. If markets were allowed to function more efficiently then it would be more profitable to exploit the forest for non-timber products.

The adverse effects that clearing land has on the economy in the relatively near future, such as reduced agricultural output due to soil degradation and thereby reduced employment need to be made clear to national governments.

Developing countries should not be criticised for investing in infrastructure, this is essential for economic growth, but should be encouraged not to clear areas and open them for up agricultural use so as to try and relieve unemployment and poverty when these areas have soil conditions that are unsuitable for agricultural practices. Other policies are needed to help reduce the external debt these countries face and to encourage overall economic growth so that subsistence agriculture is not the staple form of income.

## Global Warming

Cline (1991) gives a clear account of the greenhouse effect. Radiation is emitted from any substance in space. The wavelength of this radiation is inversely related to the temperature of that substance. The sun's radiation is therefore in the shorter wavelength band (0.2 to 0.4 micrometers) whereas the radiation emitted from the earth is in the long-wave bands (4 - 100 micrometers). The greenhouse gases, such as carbon dioxide, chlorofluorocarbons, methane,and water vapour are transparent to short-wave radiation, thus letting in the sun's radiation, but don't allow the long-wave radiation to pass through. Thus the sun heats up the earth and a portion of this heat is not able to escape. This is known as the greenhouse effect.

Most of the greenhouse effect has come from natural causes , primarily from water vapour. But since the industrial revolution anthropogenic, or man-made emissions, have increasingly added to the effect as emissions of carbon dioxide have risen by about 25%. This has been largely as a consequence of burning fossil fuels.

Cline also gives us some interesting figures. Radiation is measured in watts per square metre (W $m^{-2}$). Radiation from the sun (the incoming radiation), is 340 W $m^{-2}$. Part of the earth's surface, such as snow and clouds, reflects radiation; they reflect 100 W $m^{-2}$ back into space. There is a remaining 240 W $m^{-2}$ and it is this that warms the earth's surface from $0^{0}$C to $18^{\circ}$C. The earth emits its own radiation equal to 420 W $m^{-2}$ from the surface. The greenhouse gases deflect 180 W $m^{-2}$ back to the earth thereby warming it by another $33^{\circ}$C. The radiation that is let through (420 - 180 = 240) exactly equals the incoming radiation from the sun (340 - 100 = 240).

Cline states that the annual emissions of carbon dioxide from fossil fuel burning is 5.4 billion tonnes of carbon (GtC). Deforestation contributes approximately 1.6 GtC. Much of the emissions are absorbed by the oceans. The surface oceans hold 1,000 GtC and the deep ocean holds 38,000 GtC. The forests hold 550 GtC and the soil 1,500 GtC. The stock of carbon dioxide stored in the atmosphere amounts to 750 GtC.

The most important of the anthropogenic greenhouse gases is carbon dioxide. The burning of fossil fuels is the prime cause of emissions. In 1992 over 95% of carbon dioxide emissions in the UK came from the burning of fossil fuels. The UK contributes about 2 per cent of global emissions of $CO_2$. Emissions in the UK in 1992 were less than 1% higher than in 1982, but were 15% less than in

1970 when power stations and industry emitted more $CO_2$. Methane ($CH_4$) is the second most important gas. The global yearly emissions of $CH_4$ into the atmosphere is approximately 500 million tonnes. Out of that, man-made emissions amount to 360 million tonnes of which the UK is responsible for about 1.3%. The main sources of methane emissions are livestock (32%), landfill waste disposal sites (41%) and coal mining (14%). Emissions in 1992 were 7% less than in 1982 and also 7% lower than in 1970, (Digest of Environmental Protection and Water Statistics 1994).

The greenhouse effect is different to any other environmental problem. Where the effects of most kinds of pollution can be felt within a couple of years, or at least in one generations lifetime, the greenhouse effect may last for several centuries and so can be regarded as virtually irreversible. The Intergovernmental Panel on Climate Change (IPCC) which was set up by the World Meteorological Organisation and the United Nations Environment Programme in 1988, argue that if the atmospheric concentration of $CO_2$ were to double there would be a rise in the mean temperatures of between 1.5 and 4.5 degrees Celsius (Houghton *et al* (1990)).

They predict that the higher temperatures will cause the sea water to expand and the glaciers and ice caps to melt and so cause the sea level to rise. The IPCC report predicted that the sea level would rise by about 65 centimetres by the end of the next century. There will be other changes too: the warmer climate will result in the extinction of some species as they may well be unable to adapt to the change; rainfall may be reduced; there may be an increase in tropical storms; some regions may find they are no longer able to grow food.

The United States Environmental Protection Agency estimated the cost of protecting US cities by sea walls if there was a rise in the sea level of 1 metre, as $100 billion. They also estimated that with a 1 metre rise in sea level, Bangladesh would lose 20% of its land (Beckerman 1991). It is expected that the effects of climate change will fall unequally on the worlds regions. Places that are characteristically cold such as Russia may actually benefit from the warmer weather, but arid areas may find that marginal agricultural land become as unproductive as a desert. They will have a diminished ability to grow food.

The IPCC report only examines the effects of doubling $CO_2$ concentrations from pre-industrial levels. The known reserves of fossil fuels are substantial enough to permit burning them until well after the doubling point has been reached. Cline (1992) argues that if fossil fuel burning continued fast enough so that resources were exhausted in 300 years, then the concentration of carbon dioxide in the atmosphere would be 6 times the pre-industrial level in approximately the year 2200.

At the Rio de Janeiro "Earth Summit" in 1992 the world agreed to ward off the build up of carbon dioxide and other greenhouse gases in the atmosphere. The rich countries agreed that nations should stabilise emissions of $CO_2$ at the 1990 level by the year 2000. In 1995 a UN conference on climate change was held in Berlin to assess what progress has been made since Rio. The result was that there has been virtually no progress made to date. A group of 30 Caribbean, Pacific and Indian islands want the rich countries to reduce their $CO_2$ emission by 20% from 1990 levels by 2005 for fear of sea levels rising and flooding lowland areas. Rich countries are unlikely to even meet their Rio target. OPEC countries never agreed to anything at Rio as their economies would be badly hit by falling demand for their oil. It is likely that America, Canada and the European Union will not meet their targets as there is more concern for employment than the environment. Germany is, however, likely to meet its target and reunification has closed down antique eastern coal-burning industries (Economist,1995).

It was argued that even if these targets were met the growing emissions from poor countries would overtake the rich countries emissions by 2010. Of concern is what happens in China, with its increasing population and thirst for development. The rich countries are reluctant to make the effort to reduce emissions when poor countries have committed themselves to nothing, (Economist, 1995).

## Acid Rain

Acid rain is the term used for the deposition of acidic substances from rain and other forms of moist air. In fact these substances can be deposited by dry particles as well. Oxides of Sulphur, $SO_2$, and Nitrogen, $NO_2$, are released from both natural and man-made sources. In Europe, anthropogenic emissions of $SO_2$ account for 90% of total emissions, mainly coming from power stations. The remaining 10% come from natural sources such as sea spray, plankton and volcanoes. Half the emissions of $NO_2$ are from anthropogenic sources (Blunden and Reddish (1991)). They are dispersed into the air and then physically transported downwind where they will undergo a chemical transformation. Emissions of $SO_2$ remain in the atmosphere for very long distances and usually travel distances of hundreds of kilometres. While it is in the air it can have its proportion of oxygen increased - it can be oxidised and result in the oxide $SO_3$. If it is still a gas the reaction that takes place with the $SO_3$ is very slow, but if it is absorbed into droplets of fog or water the reaction is much faster. The $SO_3$ reacts with the water and forms sulphuric acid $H_2SO_4$. This is then deposited on the ground.

In a report funded by the US Congress called the National Acid Rain Precipitation Assessment Program (1989), it was found that 14% of 1,290 lakes in the Adirondacks in New York and 23% of 2,098 lakes in Florida were acidified. In fact the trout populations of these lakes have died due to the effects of acid rain. They also found that in many of the national parks sulphuric acid which is produced from sulphur emissions were responsible for 50% to 60% of the degradation in visibility. The deposits of acid also appeared to intensify the effects of natural stresses upon the red spruce tree in the eastern mountain-top locations.

The effects of acid rain are mainly the reduction of pH values in lakes and rivers, resulting in the loss of aquatic life in commercial fisheries, direct damage to leaf surfaces of trees and crops and the acidification of soils which can damage forest and crop cover.

## Ozone depletion

Just above the troposphere, which is the portion of the atmosphere closest to the earth, lies the stratosphere. The ozone present in this layer of the atmosphere has a crucial role to play, it absorbs the ultraviolet rays from the sun. Thus the stratospheric ozone is a shield to protect people, plants and animals from harmful radiation.

In 1985 a hole was discovered in the ozone layer over Antarctica. Since then the concentration of this protective stratospheric gas has been thinning allowing more of the sun's harmful rays to penetrate to the earth's surface. Ozone destruction is caused mainly by anthropogenic substances such as Chloroflourocarbons (CFCs), which are used as aerosol propellants, coolants, cleaners, frothers used for making plastic foam, halons used in fire extinguishers and air conditioning for cars and buildings.

The alternatives to CFCs are hydrochloroflourocarbons (HCFCs), and hydroflourocarbons (HFCs). These substitute compounds are not satisfactory substitutes for environmentalists. While HFCs do not attack the ozone, HCFCs, do albeit at a far less vicious rate than CFCs. Also all these chemicals are greenhouse gases and so trap the sun's heat and add to global warming.

Environmental groups are now pushing for the use of hydrocarbon gases such as propane and butane which had been used in refrigerators before CFCs took over. In Germany, use of these gases were introduced in refrigerators in 1992 and have since become very popular and now other companies are entering the market.

Under the Montreal Protocol, countries are phasing out production of these CFCs. This was drawn up at the United Nations Convention on the Protection of the Ozone Layer in Vienna in 1985. Under the Protocol, the use of CFC's was to be frozen at the 1986 levels by 1989, and cut to 80% of this level by 1994. The Protocol was tightened in June 1990 to achieve reductions of 50% by 1995 and 85% by 1997. Indeed the UK and United States have announced that they plan for a complete phase out by the year 2000. The Protocol gave developing countries a grace period of 10 years. Most of the industrialised countries have now signed the Protocol and these countries account for most of the global use of CFCs.

## Population

Underlying almost all environmental concerns is the relentless growth in the human population. The simple fact is that population growth puts pressure on both environmental and economic systems. The prospects for development may actually be increased by population growth. But population growth contributes to the depletion of natural resources and in so doing so will impede development and reduce environmental quality.

The population growth that the world is currently witnessing threatens the quantity and quality of natural resources and the capacity of the environment to assimilate waste from the economic process. The sheer fact that the earth has a finite amount of space indicates that a growing population must eventually lead to crowding and congestion that would one day be very unpleasant and destructive. We can already see land being increasingly eaten up for housing, development and agriculture; this can be most drastically observed in the developing countries where forests are being cut down at an alarming rate. In the rainforest areas particularly, land is increasingly being ploughed up for agricultural purposes, increasing population is placing ever greater demand on their resource base. In 1987 The World Commission on Environment and Development concluded that with the current deforestation rate and the expected growth in world population and economic activity, there would be little virgin rainforest left outside of forest preserves beyond the year 2000 (Miller et al (1993)).

The increasing demands as well as the sheer size of numbers means that population increases will limit environmental quality and our ability for economic growth. However, the absolute level of population is not the only issue, the geographic regional distribution also has important environmental implications. Any concentration of people will have important implications for the surrounding environment and the quality of life that exists there.

# SECTION II

## Sustainable Development

The environment provides us with the infrastructure that without which our economy could not survive. All economic activity is based on those resources found in nature. Current over-exploitation of the worlds natural resources and destruction of natural environments jeopardises the future possibilities of obtaining environmental services from these areas and so threatens the future world economy. For flows of these natural materials and energy to continue to be inputs in our productive processes, all ecological systems must remain in operation. Renewable resources must be able to maintain their regenerative capabilities, the environment must retain the capacity to break down waste that pollutes natural areas, and non-renewable resources must be used up at a slow enough rate so that they can be substituted for renewable resources.

We must ensure equal access to environmental services by different generations. Intergenerational equity relates to fairness and justice between different generations. It may be possible to grow more trees, set aside land to revert back to wilderness but in many cases environmental losses that occur are irreversible. This irreversibility means the removal of an option for future generations. Once a tropical rain forest has been pulled down and the land used for development, it is impossible to recreate it due to loss of species, loss of soil fertility and altering of the water table. Desertified land is very difficult to reclaim.

Apart from the productive function of the environment, which provides us with useful energy and material inputs in the economic process, the environment also provides us with important services that are essential in supporting human welfare. The natural environment provides us with recreational, cultural, educational, scientific and aesthetic services and also maintains the climate and ecological cycles and functions. The need for preserving our environment is becoming an increasingly important issue. The biosphere is home to a diverse range of species, including ourselves, and the greater it is damaged the less hospitable the environment becomes for those that live there.

What is important is to have a sustainable economy. The issue is how we should treat our natural environments in order that they can play their part in sustaining the economy as an improved source of standard of living.

A statement from the World Bank Development Report 1984 says:

> "Degradation and destruction of environmental systems and natural resources are now assuming massive proportions in some developing countries, threatening continued, sustainable development. It is now generally recognised that economic development itself can be an important contributing factor to growing environmental problems in the absence of appropriate safeguards. A greatly improved understanding of the natural resource base and environment systems that support national economies is needed if patterns of development that are sustainable can be determined and recommended to governments." (World Bank Development Report 1984).

The concept of sustainable development (SD) has become a key issue in environment-ecological science. Robert M. Solow (1991) quoted from a UNESCO document that:

> "every generation should leave water, air and soil resources as pure and unpolluted as when it came to earth."

But, as Solow argues, carrying out this obligation in unfeasible and not even desirable for this would mean that to leave the earth as we found it there could be no use of the earth's natural resources, no construction on the land and no roads could be built. This is clearly not desirable and not the basis of SD.

Solow's idea of SD is:

> "an obligation to conduct ourselves so that we leave to the future the option or the capacity to be as well off as we are." (Solow 1991)

The clearest and most widely used definition of SD is reflected in the publication " Our Common Future", (WCED 1987), or sometimes called "The Brundtland Report", by the World Commission on Environment and Development. The WCED was established by the General Assembly of the United Nations in 1983.

They define SD as:

> "Development is sustainable if it satisfies present needs without compromising the ability of future generations to meet their own needs." (WCED, 1987).

The environment and its ecosystems clearly play an important role in trying to achieve SD. The goal of SD therefore is to achieve a level of economic welfare that can be perpetuated for many generations in the future and to establish a viable path for economic development that is compatible with environmental quality.

The yield from renewable resources also needs to be maintained in such a way that they are not eliminated or degraded in some way that their usefulness for future generations is reduced. Renewable resources should be harvested so that the yield can be sustained over time rather than being extracted until they become extinct. A tropical rain forest is an example of a renewable resource that is often not harvested sustainably. Such amenities are regularly extracted to such a level that the natural regeneration potential is threatened.

The idea is that there is some minimum level at which the resource, such as a fishery, can survive. If the yield associated with this resource level is exceeded, then the resource is unsustainable. This is sometimes called the maximum sustainable yield - it is the maximum amount of the resource crop that can be safely harvested so that the resource is sustained over time. This is the principle on which international fishery agreements are based.

In the quest for SD it therefore becomes evident that a transition is required away from economic growth based on the extraction of non-renewable natural resource stocks and towards economic growth based on renewable resources. A permanently decreasing environmental stock cannot support increasing or constant levels of economic growth. SD would imply depleting the stock of a non-renewable natural resource at a slow enough rate to ensure that there is a high probability of a transition to a renewable resource producing the same economic output when the non-renewable resource becomes more costly. An example would be to extract petroleum at such a rate so that it can be substituted for a now cheaper renewable resource such as solar power. Thus we need long term planning to guide this transition from non-renewable to renewable natural resources rather than leaving it to market forces. When the price of a non-renewable resource drops, e.g. petroleum in the 80's, demand for that resource will increase, encouraging increased depletion of the resource. In such a case the market is not reflecting its future scarcity. Economic activities must be compatible with the functions of the environment. Therefore long term planning is required to achieve sustainability.

# SECTION III

## Summary of thesis

The research in this thesis is concerned with the contribution of environmental and resource economics to sustainable development. Within the work, conventional theories of natural resource scarcity are contrasted with the alternative analysis. Conventional theories treat natural resources as those resources that provide economically valuable productive services, while the alternative approach recognises the fact that there is a great deal of environmental-economic interaction. The main objective of this thesis is to demonstrate the physical dependence of economic activity on the sustainability of crucial natural resource systems.

From the previous discussions it can be seen that the environment provides us with many important functions - recreational, cultural, educational, scientific and aesthetic and also maintains the climate and ecological cycles. The focus of this thesis is to incorporate these issues into the framework of economic optimisation problems. A brief outline of the subject matter of each chapter of the thesis now follows.

***Chapter 4 - The Competition between Preservation and Development for Land Use.***

In the first chapter the issue of competition for land use is analysed. There is increasing pressure on the land to be preserved but there is also the need to develop the land for production and living. Developing the land renders our natural environment and resources more scarce and so puts our future prospects of development at risk. They are therefore in direct competition with each other for land use. In this chapter the competition between these two uses is analysed under the constraint that conversion of the natural area for development purposes represents an irreversible development. The optimal division between preservation and development is determined along with the optimal time path of investment in developing the land.

This chapter is an extension of a previous paper by Fisher et al (1972), who formulate a general model for the allocation of land between preservation and development. By applying the Maximum Principle, the investment path is chosen to maximise discounted utility subject to the constraint that investment is irreversible. In their paper, they showed that the optimal development path for a given area of land is given by a sequence of investment intervals. This chapter is an extension to their paper and shows that there is a singular arc solution to the optimal control problem and that it is optimal to reach the singular arc in the shortest amount of time possible and then freeze further investment in development. This means that all investment should be concentrated at the beginning of the plan and is in direct contrast to the conclusions reached by Fisher et al. Specific functional forms are used for the benefit functions of preservation and development so that the solution can be fully characterised. Their paper is also extended by changing the dynamic constraint on investment from one implying constant returns to increasing investment, to one of decreasing returns to investment. This illustrates that the easiest land to be developed would be developed first and then from then on the land would become more difficult to develop. Developing this marginal land would require greater investment and would imply decreasing returns to investment. The result is to change the optimal path of investment to one under which investment should be undertaken gradually over time.

***Chapter 5 - Sustainable Economic Development and Natural Resource Use in a Polluted Environment.***

We know that pollution flows from the economic process and collects into a stock in the natural environment and that this in turn lowers human welfare by affecting people's health or the aesthetic properties of the environment. Pollution can also have an effect on the regeneration of renewable resources, for instance, the effect that acid rain depositions have on tropical rain forests, or the effect of pollution in rivers and oceans on the growth rate of fish. Pollution can also negatively affect the rate at which the environment is able to clean itself up. This chapter addresses these issues in a dynamic optimisation model where the problem is to find the conditions under which the economic growth path is sustainable.

In this chapter a model is formulated in which a renewable resource is extracted from the environment and used in the production process along with capital services. Productive activity generates a flow of pollution, which in turn builds up as a stock in the environment. This stock of pollution has a negative impact on the regenerative capacity of the renewable natural resource and also affects the assimilative capacity (the natural self-purification process) of the environment. The problem is to choose a time path for harvesting the resource so as to maximise some objective functional whose arguments are the time path of consumption and the stock of pollution. Social welfare at any point in time depends on the flow of consumption and the quality of the environment. Pontryagin's Maximum Principle is used to derive the optimal solution and the steady state values of the variables of interest. The effects that changes in the parameters of the model have on the steady state solutions are also analysed. Sufficient conditions for the existence of the steady state are given and it is shown that when the discount rate is small enough all bounded solutions converge to a unique steady state.

*Chapter 6 - Population Growth and Environmental Preservation.*

This chapter concentrates on the pressure a growing population puts on the depletion of our natural resources where extraction of the resource causes irreversible damage to the environment. It deals with the scenario of an economy that possesses a single renewable resource that is extracted from a pre-existing pool. The resource is self-replenishable. This situation could apply to the cutting down of trees where the forest re-seeds itself. There are no controls on regeneration - no additional inputs, e.g. fertiliser. It is managed only by cropping.

The resource is used in the production of a single composite commodity which is either consumed or invested. Depletion of the resource, however, causes irreversible damage to the natural environment. The objective is to find the optimal extraction rate so as to maximise welfare given that the population is growing at a constant rate. The implications of making the population growth rate endogenous - a function of per capita consumption and capital per capita are examined. This is an extension of a paper by Cigno (1981) in which he does not develop this argument into a formal optimisation model or take into account environmental considerations.

It is found that an optimal and sustainable consumption and resource harvesting policy does exist and also the conditions that are necessary for the steady state to exist are given. It is shown that when the rate of discount is small enough all bounded solutions converge to a unique steady state. A comparative statics analysis shows that if there is a greater preference to deplete the resource early, i.e. there is a greater preference for current consumption, then future generations will be deprived of some output possibilities. Sustainable development requires that the options of future generations are not diminished. If we use up natural resources at too fast a rate then we are removing an option for future generations.

### *Chapter 7 - The Optimal Time Path of a Carbon Tax.*

The by product that is emitted into the atmosphere by the burning of fossil fuels is carbon dioxide, $CO_2$, and this has been discusssed in Section I. The need to control these emissions arises because of the externalities that are incurred by other members of society. These costs are not taken into consideration by individuals, therefore the objective of environmental policy should be to internalise these costs. A tax on carbon emissions would have the effect of creating the incentive to switch away from carbon intensive fuels and would encourage efficient energy use.

Much of the concern in the literature has been with the question of what level of carbon tax is required to reduce $CO_2$ emissions and it is argued by Ulph and Ulph (1994) that what matters is the time path of the tax. A falling carbon tax would encourage the delayed depletion of non-renewable resources but as the damage arising from global warming is an increasing function of the level $CO_2$ emissions, then maybe the correct policy is a rising tax rate. This chapter criticises the mathematics of Ulph and Ulph and solves their model for the optimal time path of the variables and the steady state. This will be an extension to a paper by Chappell and Dury (1994) entitled:

"On The Optimal Depletion of a Non-Renewable Natural Resource under Conditions of Increasing Marginal Extraction Costs".

The model in this chapter also extends the Ulph and Ulph analysis to incorporate increasing marginal costs of extraction and it is shown that the model can be solved in general for the time path of the carbon tax without having to make simplifying assumptions about some of the parameters as in Ulph and Ulph's analysis. The socially optimal time path for the carbon tax is found and shown to fall over time. This is in contrast to Ulph and Ulph who argue that the optimal carbon tax trajectory should be one that first rises and then falls. It is also shown that it is not optimal to completely exhaust the resource and indeed less is extracted when the environmental damage of $CO_2$ in the atmosphere is taken into account.

*Chapter 8 - An Economic Model of Open-Cast Coal Mining.*

Open-cast coal is cheaper to extract then deep mined coal and so it is generally accepted that the production of this type of coal should be maximised. However, open-cast coal mining inevitably causes adverse environmental effects in the geographical area concerned; therefore there is a need to strike a balance between the benefits of development and protecting the environment.

Any application for open-cast mining will have to conform to rigorous regulations and any proposal is subject to examination of all the possible effects that might occur, including the effects on the environment. The after-use of the land must be decided before planning permission is granted as this will affect the cause of reclamation of the land. All open-cast coal mining developments have to restore the land after the mining has taken place. The purpose of this chapter is to consider the optimal exploitation of a non-renewable resource, such as a fossil fuel, under the ownership of a monopolist who faces conditions of increasing marginal costs of extraction and regularity constraints to protect and restore the environment. This is also an extension to the paper by Chappell and Dury (1994).

A model is formulated where the mining firm is obliged to fill in the area after the mining has finished to see how the addition of this new constraint affects the optimal decision of the firm whose objective is to maximise profits. Two models are considered where different after uses of the site affect the timing of the reclamation. In the first model the land is to be used for development and so infilling is undertaken after the site has been mined. In the other, the reclaimed land is to be used for agriculture or forestry. In this case the infilling is undertaken continually to ensure that the subsoil and topsoil are replaced at the earliest opportunity to minimise deterioration of the biological value of the soil during storage.

The main conclusions from the work are that it is not optimal for the monopolist to completely exhaust the resource when there are no constraints placed on him for restoration. Also when he is faced with regularity constraints, it is optimal for him to leave more of the resource in the ground and if restoration of the site is undertaken as the mining development is going along then it is not optimal for the monopolist to shut down temporarily as restoration is taking place.

# CHAPTER TWO

# LITERATURE REVIEW

## INTRODUCTION

The aim of this chapter is to review the economic growth theory literature involving natural resources and the environment. Section I is a history of how natural resources and the environment have been viewed in economic theory and describes how this has developed over the centuries and led to our concerns of today. Section II gives a brief overview of selected articles from the conventional natural resource scarcity literature and Section III shows how this literature has taken a different route with regard to sustainability issues. It recognises that not only does the environment provide useful material and energy inputs to the economic process, but also provides us with important services that are essential in supporting the economic system and human welfare. This alternative approach to natural resource scarcity shows how concerns for the quality of the environment have developed and also shows how dependent the economic process is on the environment.

The aim of Section III is to identify the issues that seem to be of most concern in the economic analysis of sustainability and to review economic growth theory, which bears on the issue of sustainability - that which involves natural resources and the environment. This review of the literature focuses on certain issues such as environmental preservation, pollution emissions, and population growth. This section also identifies other areas that the economic literature has focused on such as global warming and computer simulation modelling of economic and environmental systems and the feedback effects that occur.

To understand the concerns of today and how economic theory has developed in analysing the interactions of the environment and the economic system, it is helpful to know how the environment has been viewed by economists over the years and how environmental issues have been incorporated into economic models.

# SECTION I

## Historical economic approaches to environmental issues

### Physiocrats

The important role that nature had to play in the economic system had already been recognised by the physiocrats. The Physiocratic school of economic thought was developed in France in the 1750's, and it had as its first principle that natural resources and fertile agricultural land were the source of material wealth. Physiocracy means literally 'rule of nature'. They held that the economic process could be understood by concentrating on a single factor: the productivity of agriculture. Adam Smith referred to the ideas of the physiocrats on the agricultural system; he wrote in *The Wealth of Nations*:

> 'That system which represents the produce of land as the sole source of the revenue and wealth of every country has, so far as I know, never been adopted by any nation, and it at present exists only in the speculations of a few men of great learning and ingenuity in France." (Smith 1776, Volume 2, Book 4, Chapter 9, pp 156 - 157).

Quesnay's Tableau Economique was at the heart of Physiocratic economics. This was the name for the visual representation of the circular flow of income and expenditure. Any policy that had the effect of increasing the circular flow was consistent with economic growth. Any policy that reduces or restricts the circular flow is inconsistent with economic growth. Quesnay then selected a key factor in the circular flow and examined the effects that various policies had on this factor and the subsequent effects on the economy as a whole. This key factor was agriculture. Manufacturing and service industries were considered *unproductive* or *sterile* - they added nothing to the *produit net* - (net product). He looked upon the net product as the only source of wealth and that it was from agriculture that additional wealth was created. He postulated that it was changes in the net product that effected the course of the economy. He held that manufacturing only changed the forms of goods, merely adding labour to the products of the soil, nothing new had emerged.

> "Agriculture is the source of all the wealth of the State and of the wealth of all the citizens." (Institut National 1958, p 102).

According to the Physiocrats, agriculture was the ultimate occupation because it alone yielded a disposable surplus over cost. The Principles of Physiocratic economics held sway for nearly two decades until Adam Smith wrote *The Wealth of Nations* in 1776.

The main criticism of the Physiocrats focuses on the argument that manufacturing was sterile, incapable of yielding a surplus over cost. They argued it was sterile in this sense only under conditions of perfect competition, thus competition would reduce the price of manufacturing goods to equal necessary costs. However they were willing to admit that a value surplus over necessary costs might result under monopoly conditions. But why does competition reduce the price of agricultural products to the level of necessary costs and wipe out rent, which was so closely associated with the Physiocratic class structure. It was the landlords and proprietors who presided over agricultural production and it was to then that the *produit net* ultimately accrued. The Physiocrats toyed with a monopoly explanation of why this surplus existed, but their main argument was that the net product was simply a gift of nature. This unsatisfactory answer does not explain why there would exist a value surplus. This can be explained by a general theory of value which the Physiocrats failed to construct. This task fell to Adam Smith.

**Malthus**

Malthus published his first *Essay on Population* in 1798. He was the first to emphasise the dependence of population growth on the food supply, he focused his attention on the limited supply of land. He made an important departure from the view shared by the Physiocrats and Smith who held that there was no limit to nature. He held that agricultural land scarcity implied strict limits on population growth and that this had powerful social implications. He argued that some part of society will live under conditions below subsistence level as a result of the finite capacity of nature to support humans and their activities. He states that the supply of food would grow at most in an arithmetic progression whereas population would grow in a geometric progression. Malthus says that even the smallest finite sum growing at the smallest compound rate must eventually overwhelm the largest finite sum growing at the highest simple rate; (consider the geometric progression, 2 + 4 + 8 + 16 + 32 +... in contrast to the arithmetic progression, 1,000 + 1,003 + 1,006 + 1,009 + ...). There would be a compounding factor on the population growth because additional people could reproduce themselves, whereas additional food can not reproduce itself.

Malthus argued that a rapid increase in food crops is out of the question since the supply of land is limited and technical improvements do not come fast enough. His predecessors made some striking statements about the explosive nature of potential population growth and the availability of land:

Quesnay assumes that only capital and entrepreneurs are needed to expand agricultural production; he states that the availability of land and labour doesn't pose a problem. Concerning land, he writes:

> 'The cultivation of corn is very expensive; we have far more land than we need for it", (Institut National d'Etudes Démographiques 1958, p473, translated by Eltis 1987).

Malthus's theory consists of three propositions; 1). man's capacity to reproduce is greater than his capacity to increase the food supply, 2). preventative or positive checks will always be in operation and 3). the ultimate check on population growth lies in the limitations of the food supply.

He states that the population will be held in check by the food supply unless other prior checks on its increase are operating. By these he classified positive and preventative checks. Positive checks such as disease would increase the death rate, preventative checks such as the foreseeable difficulties in the rearing a family would lower the birth rate.

Concerning the preventative check, Malthus states that:

> "a foresight of the difficulties attending the rearing of a family , acts as a preventative check...the labourer who earns eighteen pence per day, and lives with some degree of comfort as a single man, will hesitate a little before he divides that pittance amount between four or five...the preventative check to population in this country operates, though with varied force, through all the classes of the community." (Malthus 1798, pp 62 - 69).

Malthus confines the positive checks on population growth, such as epidemics and diseases, to the poorer members of society:

> "The positive check on population, by which I mean that represses an increase which is already begun, is confined chiefly, though not perhaps solely, to the lowest orders of society" (Malthus 1978, p 71).

Malthus states that disease is not the cause of depopulation, but it is a positive check that corrects for overpopulation. Population is determined by the limitations of the food supply and this is the ultimate check:

> "Famine seems to be the last, the most dreadful resource of nature. The power of the population is so superior to the power in the earth to

> produce subsistence for man, that premature death must in some shape or other, visit the human race....but should they fail in this war of extermination, sickly seasons, epidemics, pestilence, and plague, advance in terrific array, and sweep off their thousands and ten thousands. Should success still be incomplete; gigantic inevitable famine stalks in the rear, and with one mighty blow, levels the population with the food of the world." (Malthus 1920, pp 110 -111).

Malthus's theory focuses on limited supply of land; he states that because of the power of the population to grow in the way described previously:

> "...it is quite obvious that some limit to the production of food, or some other of the necessities of life exist." (Malthus 1920, pp 227 - 228).

He is saying that the production functions are bounded above at some point because of the physical limitations of space and the supply of land is fixed.

Malthus did not mention a tendency to diminishing returns to agriculture until the second edition of the Essay in 1803. It wasn't until he wrote *The Principles of Political Economy (1820)*, that he used his concept of limited supply of land and the consequent population theory to an examination of the long run conditions for economic growth. He assumed the land to be homogeneous in quality and as Barnett and Morse point out:

> "..the Malthusian [doctrine] rested on the assumption that the stock of agricultural land was absolutely limited; once this limit had been reached, continuing population growth would require increasing intensity of cultivation and, consequently, would bring about diminishing returns per capita." (Barnett and Morse 1963, p 51)

Under the Malthusian doctrine, it is not until the absolute limit of the available stock of natural resources (e.g. land) is reached that diminishing returns set in. Therefore if rising marginal labour-capital costs are to be associated with diminishing returns, then according to Malthus, it is only when the absolute limits to the resource stock are reached do costs start to rise. Malthusian scarcity is thus an absolute scarcity implying that the physically limited stock of resources acts to constrain the increase in output. Only when this limit is reached does the scarcity effect show up as rising prices.

So how close is Malthus to the alternative view of economic resource scarcity? A key difference is that the alternative view of natural resource scarcity assumes that the entire economic process is dependent on the limited resources of the natural environment, not just agriculture. Another difference is that the alternative view of rapid population growth is not the only condition that will generate scarcity effects. The continual depletion of resources as inputs on material and energy in the economic system may over time threaten ecological stability. The alternative view is concerned not just with resource depletion but is concerned with economic-environmental interrelationships and emphasises the important scarcity effects brought about by greater environmental disorder, ecological instability and falling environmental quality. It is not only population growth that is the source of constraint on economic growth, rather it is the physical dependency of the economic system on the environment and its resources.

**Ricardo**

David Ricardo differs to Malthus in that he viewed the scarcity of natural resources as a 'relative' scarcity phenomenon. As Barnett and Morse describe:

> "The [Ricardian] version viewed diminishing returns as a current phenomenon, reflecting decline in the quality of land as successive parcels were brought within the margin of profitable cultivation" (Barnett and Morse 1963, p 51).

In the Ricardian system then, unlike Malthus, diminishing returns to agriculture set in as soon as the land was cultivated. The better quality land would be used first and the cause of increasing resource scarcity was the declining quality of the land. In this case costs would start to rise as lower quality land was brought into cultivation, whereas for Malthus the costs do not start to rise until the absolute limit of land is reached. This increase in costs results from the rise in labour costs. As the land is declining in quality, further cultivation of land requires a more disproportionate use of labour and it is the rise in the labour cost that is the only way this scarcity effect will lead to higher prices for agricultural products. Thus, for Ricardo, the increasing scarcity of natural resources (land), will only lead to increased agricultural prices if more labour is required to work the land. Increasing natural resource scarcity alone is not enough to raise prices.

Ricardo also implies that there is not necessarily an absolute limit to the availability of resources, as postulated by Malthus. Barnett and Morse state that:

> "there is always another extensive margin, another plateau of lower quality, which will be reached before the increasing intensity of utilisation becomes intolerable" (Barnett and Morse 1963, p 63).

Ricardo also saw population expansion as the central ingredient in his system, the source of economic stagnation. The effect of increasing populations would be to push cultivation to ever poorer land and this would continue until very poor quality land was worked. This desolate soil would eventually return only the minimum necessary for the lives of those who worked on the land and this would determine their wages. Better quality land would return a surplus over cost and the system would result in differential rents being earned. Rent being defined as:

> "the portion of the produce of the earth, which is paid to the landlord for the use of the original and indestructible powers of the soil" (Ricardo 1821, p 67).

In the long run then, increased population forces the increasing use of less fertile land requiring larger quantities of labour. This results in the inevitable decline in profits as a greater proportion of output would be distributed in the form of wages. As long as profits are positive, then investment is increasing. The increased demand for labour would cause the wage rate to rise. But when wages rise above the subsistence level, then population is encouraged to increase. Eventually a minimum profit rate would be reached at which new investment and additional capital accumulation would cease. Ricardo referred to this as the stationary state.

In both Malthusian and Ricardian systems, population growth is the primary constraint on economic growth. In the alternative view it is the environmental consequences of economic exploitation of the limited supply of natural resources that is considered to be the ultimate constraint on economic growth.

**Smith**

Smith took the view of the Physiocrats that nature was in abundance and that agriculture could produce output far in excess of the inputs required. There is no suggestion by Smith that there is a finite limit to the earth's resources and so there was no threat of an absolute constraint on economic growth. Both Smith and Quesnay saw capital as the principle constraint on agricultural output. In Smith's economic system, economic growth is dependent on agricultural production but the constraint on growth did not come from the diminishing returns to agriculture arising from the absolute limits of natural resource (land) availability. Rather, that the reliance on agricultural production would eventually result in excess demand for agricultural output. This excess demand would result in a situation of profound distributional consequences.

It was these distributional consequences that would constrain growth. He did then seem to share the alternative view that there would be a constraint on growth arising from the dependency of the economic system on the environment. But to

Smith this did not result from any absolute limits of nature. He attributed this situation not only to the relative scarcity of agricultural output but to the pattern of income distribution brought about by the pattern of land ownership and tenure, the high living consumption of the landlords, the institution of rent and the distributional effects of higher agricultural prices, the consequence of excess demand. For Smith it was the combined effect of these social factors and relative scarcity of agricultural output that would lead to long run economic stagnation. Thus Smith incorporates social relations and distributional consequences into his doctrine.

**Mill**

Like Ricardo, Mill saw that one of the limiting factors of economic growth was the diminishing returns to agriculture. He also viewed the declining incentive to invest as another limit to growth. He allocated a crucial role to capital and capital accumulation for production, just as the classical economists had done. In general, Mill focused his discussion of the theory of economic development upon capital accumulation, population growth and technology. He combined this with the theory of diminishing returns to agriculture. Mill saw economic growth being a race between technical change and diminishing returns to agriculture and combined with lack of incentives to invest he argued that the economy was being driven to a stationary state. However, Mill viewed the distant prospect of this stationary state with optimism; here he differed with the other classical economists, he did not view this stationary state as being undesirable. He postulated that by the time the steady state had been reached, technical progress would have provided for much of man's needs. He viewed that once this was reached then the important social reforms could proceed, the problems of inequality of wealth and opportunity could be dealt with. He saw the steady state as a necessary precondition for social reform.

Mill criticised the idea of accumulating wealth merely for the sake of accumulation, and in this he is attacking economic growth for its own sake. He seems to be the first to take this view and has been echoed by many economists since (Galbraith (1958) and Mishan (1967)). In his doctrine, Mill is describing the problems of today and can thus be credited with great foresight.

"It is only in the backward countries of the world that increased production is still an important object: in those most advanced, what is economically needed is a better distribution, of which one indispensable means is a stricter restraint on population" (Mill 1848, p 749).

Mill saw, unlike the other classical economists, that technical progress could broaden the resource base by increasing output per unit of input. He also can be credited with providing us with quite a different view of the environment than that which had preceded him. He viewed the environment as providing society not only with the resources that are necessary for the inputs to production, but he saw the environment as a source of amenity services, surrounding us with natural beauty and providing us with a quality of life. He saw the environment as performing services to humans that are essential for human welfare. Since these services represent an alternative use of the natural resources, Mill made an important extension away from the classical economists before him who only were concerned with the allocation of land for agricultural production.

Mill saw that the increasing scarcity of natural resources brought about by the extension of economic activities, would lead to increasing scarcity of these amenity services. He goes onto say that the increasing scarcity of these essential services to mankind would have a detrimental impact on human welfare and this would occur long before diminishing returns imposes an absolute constraint. He argued that society should wish to preserve these essential amenity services and desire should lead to the steady state being viewed as a desirable outcome. Others have taken Mill's view and argued that a steady state is essential for the preservation of nature's services (Daly (1973), (1974) and (1977)).

**Jevons**

Jevons spread the concern of a constraint on economic growth to the concern over mineral deposits in Britain. In 1865 he published *The Coal Question*, in which he observed the physical limitations of the coal deposits in Britain and predicted the end of the Industrial Revolution. He saw Britain's economic growth being dependent on the reserves of coal. Given that these supplies were limited and there were no feasible substitutes and no way of effectively increasing supplies, then the inevitable increase in costs would lead to a stationary state. His theory was analogous to Malthu's theory of population where population growth outstripped food supplies.

Jevons work, like Mills, was in contrast to the classical treatment of natural resource scarcity in which economists were preoccupied with the scarcity of agricultural land. He saw the economy shifting from being agriculturally based to one that is industrially based and this meant replacing corn as the means of subsistence with coal.

> "Our subsistence no longer depends upon our produce of corn. The momentous repeal of the Corn Laws throws us from corn upon coal. It marks, at any rate, the epoch when coal was finally recognised as the staple produce of the country:...it marks the ascendancy of the manufacturing interest, which is only another name for the development of the use of coal" (Jevons 1909, p 195).

Jevons was the first to analyse the economic effects of the depletion of fossil fuel resources and he anticipated the modern concerns of the exhaustion of non renewable resources. However, Jevons did not consider the environmental effects of the economic dependency on the environment.

Jevons other contribution to the economic literature was his connection between cycles in sunspots and commercial activity, the 'sunspot theory'. The theory goes that there are rhythms of temperature caused by solar activity which would then effect crop yields and thereafter, economic activity. He put it:

> 'If the planets govern the sun, and the sun governs the vintages and harvests, and thus the prices of food and raw materials and the state of the money market, it follows that the configurations of the planets may prove to be the remote causes of the greatest commercial disasters." (Jevons 1909, p 185).

Firstly Jevons is implying that the economic system is dependent on agricultural production and secondly, he is signifying a direct link between economic activity and the state of the environment. Jevon's explanation of commercial crisis on the basis of periodic changes in sunspots was ridiculed. But in the light of concerns of global warming, the idea of a *sunspot theory* today does not seem so farfetched!

**Marshall**

The first person to approach an economic analysis of environmental problems was Alfred Marshall. His economic analysis concerns the general area of "externalities", property rights, and "market failure". He introduced the concept of external economies where by the development of certain industries had a positive effect on other firms within that industry - a positive externality. Marshall linked external economies to location of the industry. He argued that with the growth and localisation of an industry in a particular region there would become a localised market for skilled labour. As the industry expands further, then skilled labour would be encouraged to locate in that region where the demand for that service was high. The availability of specialised labour is therefore expanded. The existing firms will find that the cost of labour turnover and training would decline. Marshall talks about external economies arising from improved communication between firms. As the industry reaches a certain size then it would be feasible to publish information and this could be made cheaply available to all.

Also Marshall states that advantages would accrue to the individual firm through general industrial development such as inventions and improvements in machinery.

> 'Good work is rightly appreciated, inventions and improvements in machinery, in processes, and the general organisation of the business have their merits promptly discussed: if one man starts a new idea , it is taken up by others and combined with suggestions of their own; and thus it becomes the source of further new ideas. And presently subsidiary trades grow up in the neighborhood, supplying it with implements and materials, organising its traffic, and in many ways conducing to the economy of its material" (Marshall 1920, p 271).

Here he also mentioned that supportive industries would grow and this would create external economies for the firms in the industry. The growth of specialised industries to service the needs of the parent industry would have the effect of lowering costs for the industry as the specialised industry would gain the advantages of the division of labour. As Marshall put it:

> 'The economic use of expensive machinery can sometimes be attained in a very high degree in a district in which there is a large aggregate production of the same kind, even though no individual capital employed in the trade be very large. For subsidiary industries devoting themselves each to one small branch of the process of production, and working for a great many of their neighbors, are able to keep in constant use machinery of the most highly specialised character, and to make it pay its expenses, though its original cost may have been high, and its rate of depreciation very rapid." (Marshall 1920, p271).

Admittedly, Marshall was talking about the benefits that accrued to the firm without payment and it is from this concept that came the key to the economic analysis of environmental deterioration. Therefore, in principle the step had been taken that in the production of goods and services, it was possible for there to accrue benefits outside the market that could affect the conditions for other firms within the industry. It was from expansion of this idea that Pigou developed the idea of market failure.

**Pigou**

Pigou was Marshall's protégé. He expanded Marshall's ideas and in his publication entitled *The economics of Welfare* (1920) he distinguished between social net product and private net product. The two net products diverge because there exist costs or benefits of a transaction that are incurred or received by persons who are not involved in production and these externalities are not taken into account by the parties to the transaction. In Pigou's theory not only could the production conditions of a third party be affected but also the welfare of private individuals. An example of a negative externality would be a factory pumping out smoke or discharging effluent into a nearby stream. Fishermen downstream would be affected by lower quality, and perhaps quantity of fish, also the tourist industry could lose revenue and the general welfare of the residents would be adversely affected.

Pigou himself mentions the damage done to surrounding woods by sparks from railway engines. This damage is uncompensated as the train company would not take this cost into account as their profits are not affected by the external costs of its actions.

Pigou's remedy was to impose a tax on the offending firm that was equal to the difference between the marginal social cost and the marginal private cost. This would then internalise the externality as the producer would include the cost he was imposing on society in his private costs and so he would bear the full cost of production.

**Kapp**

In 1950 there was the first awareness of the serious adverse affects of production and consumption in the economic literature. Kapp was not given due credit for foreseeing the future debate of the far reaching consequences of environmental deterioration and natural resource depletion.

The focal point of Kapp's book *The Social Costs of Private Enterprise* (1950), is the place that social costs hold in the social system. He defines social costs as those losses either indirect or direct that are incurred by third persons in society as a consequence of uncontrolled economic activities.

Kapp gives a description of the effects of environmental deterioration and natural resource depletion. He talks about the harmful effects of air and water pollution, the health effects, effects on nature and the dangers to aquatic life.

Kapp discusses the over exploitation of renewable resources and attributes the cause of this to uncontrolled competition in the utilisation of these resources. Kapp recognised that over-hunting, over-fishing and excess timber harvesting lead to the destruction of species and the deterioration of fertile land. This irreversible destruction of the earth's renewable resources will eventually lead to an impoverished state of the world for future generations. Thus man is himself blocking his path to further economic growth. Kapp argues that the only way to prevent this from happening is to keep exploitation of these resources within close limits. He also recognised that there were severe consequences from congestion in urban environments being caused in part by the over concentration of industry.

Kapp was the first to discuss in the economic literature, the adverse effects of our economic process on the environment.

**Boulding**

In his publication *The Economics of the Coming Spaceship Earth* (1966), Boulding described the economy of yesterday and today as being an open system. Inputs to the system are taken from the earth's reservoir in the form of natural resources. This is considered unlimited and so not of economic concern. Then these are used up in the production and consumption process (the throughput sphere) and then they vanish into the reservoir as outputs. The through put is measured as gross national product. Boulding argues that economic growth and national income will have no importance in the future.

The result is that the concentrations of natural resources that are taken from the earth's supplies are used up in production and consumption and then they are diffused and scattered across the earth's surface. Boulding goes on to say that this process cannot go on forever. The earth's resources are being depleted, there is increasing environmental pollution and thus from now on, man will have to think of the economic system as closed. Here output from the production and consumption activities will be used again as an input, so that the whole economic system is a cyclical process. Boulding compares this system to that of a spaceship.

Energy that is stored mainly in fossil fuels and is not contained in the product itself, will disappear into space. This energy is lost after use and so cannot be recirculated. Therefore renewable energy that does not pollute the reservoir to such an extent that the carrying capacity of the earth's reservoir is destroyed will be crucial to the spaceship economy.

In Boulding's spaceship economy, economic growth and nation product are no longer relevant. What is important is the nature and state of the capital stock, interpreted to mean the state of society. The economy does not have to have a high national income to be successful. The concern is to minimise the gross national costs, the fewer means necessary to maintain the capital stock, the better the economy is.

Boulding was the first economist to describe the circular nature of the economic and environmental system and recognise that the one of the most important factors that will determine the possible future level of economic activity, is the input of clean energy into this cyclical system.

**Mishan**

The work of Kapp didn't receive the attention amongst economists and those outside the profession that it deserved. In 1967 Mishan wrote *The Costs of Economic Growth* and it wasn't until then that people were really interested in the economic implications of environmental deterioration. This was mainly due to the fact that at the end of the 1960s, it was becoming obvious that the environment was being over burdened and so the time was right for a publication of this subject.

According to Mishan, the cause of the adverse external effects is the uncontrolled attitudes of commercial society. He criticises the emphasis that is placed on the dissatisfaction of old products and creation of new wants thereby increasing production. Mishan does not believe that this will improve society. This is in league with other earlier economists who denounce economic growth for its own sake (Galbraith (1958) and Mill (1848)).

Mishan argues that economic policy places too high an emphasis on matters that are statistically measured, such as the price level, employment and production figures. Mishan regards the concern over current economic quantities as being greatly exaggerated. He regards the happiness of society as being far more important and this is affected by the external effects of production that are not quantitatively measured. He talks about the congestion of traffic in cities, the effect of mass tourism in historic towns and he places great emphasis on the oil and sewage pollution of beaches, air pollution and the effect of uncontrolled pesticides on wildlife and fauna.

Mishan recognises that things that were once available to the people are now becoming scarce and natural beauty is being destroyed for the current generation as well as the future. The only way to stop this happening is to direct private resources towards restoration of the environment through economic policy and to change the way people view the environment.

**Forrester and Meadows**

In the early 1970's there were some econometric studies that like Jevons, predicted an absolute scarcity constraint on economic growth. Forrester (1971) was the first to simulate the worldwide relations between a number of key variables in his work entitled *World Dynamics* (1971). The model that Meadows *et al* formulated a year later was more extensive and set up the mathematical relations between population, non-renewable resources, capital, land and pollution, (*The Limits to Growth* (1972)). Each of these is subdivided into categories, for instance land is subdivided into applications, including agriculture, development, industry; population is divided up into age groups, and so on.

There are numerous feedback loops in which variables are interlinked and have an effect on each other at some point in time and this goes on throughout the time period. These feedback loops can be negative or positive. For example, the level of pollution is influenced by the previous level and on the other assimilative capacities that the environment has of abatement, be it natural or manmade. This can affect the birth rate along with industrial production and food consumption which will cause the population to change. From this highly complicated computer model with all the changes, levels and variables linked by mathematical relations, came predictions of what was going to happen in the next 100 years or so.

Meadows *et al*, who had extended Forrester's model, shows that natural resources are being depleted at increasing rates, and that population, industrial output per capita and food per capita all increase then fall dramatically. This is attributed to the fast decline of resources and by reaching the limits of the environment. Output possibilities increase, food production falls and pollution rises and population starts to fall. Meadows and Forrester forecast a world disaster. Meadows *et al* go on to investigate the effects of new technologies such as the availability of large quantities of nuclear energy, the recycling of resources, pollution control and birth control.

They then go on to do simulations with these variations of the standard model. For instance, the effect of new technologies doubling of the resource reserves by the use of unlimited nuclear energy. They show that not a single combination of any of these new changes would avert the disaster. The conclusion is clear - growth cannot continue indefinitely in a finite world.

A more optimistic view was taken by Kahn (1976), who argued that *The Limits to Growth* was not accurate. He based his optimism on the evolution of technical progress and argued that this would push back the limits of nature.

**Daly and Cobb**

Cobb and Daly in their book *For the Common Good*, offer a critique of the standard economic doctrine on economic growth and demonstrate how economic growth can lead to environmental disaster. They disagree with the assumption of neo-classical economics that the market allocates resources in the best interests of society. Conventional economics has tended to exclude aspects of this world that are now desirable to analyse. They postulate that the increasing production of consumer goods has meant the sacrifice of the environment and supportive local communities. The cause of the destruction of the environment and the community results from the pursuit of self-interest and the unlimited production growth in a world that has a limited environment. They postulate that emphasis in economics should be moved from money to real life resource management. We should shift from individualism in favour of community commitments and there needs to be a focus on the physical realities of the environment.

The authors claim that there is an optimal size for the community; ten thousand people living in one community can be relatively self sufficient.

Daly (1977) mentions that there are ecological and environmental limits to growth and he proposes that we should strive for a steady state where economic capital and population are constant. In an earlier paper (1974) he states that:

> 'Our economy is a subsystem of the earth, and the earth is apparently a steady-state open system. The subsystem cannot grow beyond the frontiers of the total system and , if it is not to disrupt the functioning of the latter, must at some much earlier point conform to the steady state mode. The techocratic project of redesigning the world (substituting technosphere with ecosphere) so as to allow for indefinite growth is a bit of hubris that has received the insufficiently pejorative label of "growthmania"' (Daly 1974 p 17).

# SECTION II

## Literature on the optimal exploitation of natural resources

### Hotelling

Economists have been stimulated into considering the optimal extraction of resources by the necessity for natural resources as inputs to the production process. In Hotelling's classic article - *The Economics of Exhaustible Resources* (1931), the notion of "social value" of an exhaustible resource is used for judging the desirability of any extraction plan for the resource. According to Hotelling, the competitive firm's aim is to maximise present value profits. Therefore they would manage exhaustible resource stocks so as to meet this objective. He states that the gross value to society of a marginal unit of output or extraction of the resource is measured by the price society is willing to pay to bring forth that particular unit of output, and the net value to society is the gross value less the cost of extracting that unit. Competitive extraction paths would therefore be identical to that chosen by a social planner seeking to maximise intertemporal social surplus. One major conclusion of the Hotelling paper is that pure competition can yield an extraction path that matches the socially optimal one, whereas a monopolistic firm will adopt an extraction path that is more conservationist, but socially suboptimal.

Subject to a condition specifically noted by Hotelling, that social and private discount rates must be the same, the conclusion that the competitive market extraction plan and the work of a rational social planner were identical meant that the invisible hand was sufficient and that the use of policy intervention is inappropriate as the market outcome is optimal.

There was a rediscovery of the Hotelling (1931) model for the efficient depletion of a fixed homogeneous resource stock after the first oil shocks in late 1973. The now familiar 'Hotelling Rule' which governs this efficient depletion, states that net price (price less marginal cost) - should rise at the rate of interest in order for producers to be indifferent to the timing of extraction and in order for remaining reserves to be competitive asset holdings.

## Conventional view - Economic Growth and Resource use

### Dasgupta and Heal, Solow and Stiglitz

From the mid-1970s, through to the early 1980s, there has been a substantial literature on the optimal growth paths for an economy with depletable resources. The Symposium sponsored by the Review of Economic Studies in 1974 was a major contribution in this field. These papers were based on the one-sector model of growth in which a depletable resource is extracted from the environment and used along with capital and labour in the production function. Thus the resource is an essential input in the production process. Three papers by Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974a), published in the Symposium, exemplify the contribution of this literature.

The earlier literature stressed the case that in the long run the limited availability of non-renewable resources would act as a constraint on the growth potential of the economic system (Forrester (1971) and Meadows *et al* (1972)). This issue dates back to the nineteenth century to the concerns of Mill and Jevons .

Exhaustible resources would only pose a problem if they are essential to production, i.e. output is zero if none of the resource is used. The question would then be, would output fall to zero in an economy that possessed a non-renewable resource that was essential in production. The three papers by Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974a) examine the feasibility of sustained or growing per-capita consumption paths and the conditions under which such consumption paths might be achieved.

Dasgupta and Heal show that for the class of production functions with a constant elasticity of substitution between capital and resource inputs, and with a constant population, it is feasible to have a sustained consumption path across generations. This conclusion depends on the elasticity of substitution between capital services and the non-renewable resource being at least equal to one. This implies that capital services are sufficiently substitutable for the depletable resource.

Solow and Stiglitz show that if this elasticity is equal to one, as in the Cobb-Douglas technology, it must also be the case for sustained growth in per-capita consumption to be feasible, that population growth must be zero and that the elasticity of output with respect to capital exceeds the elasticity of output with respect to exhaustible resources. But there is an upper bound on the magnitude of the consumption that can be sustained in this case. Here then, it is capital accumulation that can offset the effects of declining resource inputs, so long as capital is more 'important' than the resource, i.e. the share of capital exceeds that of the natural resource.

Stiglitz shows how ever-increasing technical progress can, in theory, alleviate the resource constraints. He postulates that sustained growth in per-capita consumption, and thus utility, may be feasible and optimal with a positive rate of population growth. The necessary and sufficient condition is that the ratio of the rate of technical change to the rate of population growth must be greater than or equal to the share of natural resources.

Dasgupta and Heal, Solow, Stiglitz and Kamien and Schwartz (1978), show that it may be optimal to completely exhaust the non-renewable resource if the availability of perfect substitutes and future technology render the resource no longer essential for future production.

### Clark and Smith

The same conclusion as Dasgupta and Heal, Solow and Stiglitz came to has also been reached about renewable resources. Clark (1976) and Smith (1977) examine renewable resources and find that complete exhaustion of the resource may be optimal if the cost of harvesting the resource is low, the resource is growing at a slow rate and the value of the resource appreciates more slowly than the market rate of interest.

However, these studies do not consider another aspect of resource use and that is the environmental effects and the amenity values that the environment provides us with. The conventional way of dealing with natural resources has been to treat them as those environmental resources that provide economically valuable productive inputs. Thus natural resources were seen as having only one function, and that is as supplier of raw materials and energy used as inputs in the economic process.

But not only does our environment provide useful material and energy inputs to the economic process, but the natural environment also provides us with important services that are essential in supporting the economic system and human welfare. The following is a discussion of this alternative approach.

# SECTION III

## Alternative View - Stock effects and Preservation of 'Natural Capital'

The literature has taken a different route with regard to sustainability issues and that is that there is now concern for the preservation of natural environments. This alternative approach to natural resource scarcity recognises that the environment and its scarce resources have other valuable functions as well as providing new materials for the production of goods and services. We know that the natural environment provides us with the essential inputs for production, but it also assimilates waste that is generated by the economic process and provides us with utility yielding services. These utility yielding services range from recreational, educational, scientific, aesthetic cultural to maintenance of the ecological and climatic cycles and functions.

The economic system and human welfare are thus dependent on the environment and its scarce natural resources. Given that natural capital provides us with valuable services to society and human welfare and essential inputs in the production process, the issue of preservation is an important concern in the overall debate on sustainability. Therefore, as the environment is being increasingly exploited to provide us with the essential inputs for economic activity and as a dumping ground for waste, the quantity and quality of the environment is deteriorating. The result of this is the increasing scarcity of essential environmental and ecological services and functions, (Barbier (1986)). Hueting (1980), had previously stated that the destruction and exploitation of the natural environment is also an economic problem.

This part of the literature is concerned with the conditions under which partial preservation of natural environments is optimal.

**Amenity values and stock preservation.**

Given that the flow of amenity services that the environment provides is positively related to the stock of the preserved natural environments, a simple way of including the value of these amenity services in a growth model is to include the resource stock in the utility function. Krautkraemer (1985) includes the amenity services of the environment in this way:

> "If A, E and S denote amenity services, preserved environments, and the remaining resource stock, the U(C,A) = U(C, A(E(S)) = U(C,S)." (Krautkraemer (1985) p 169n).

Vousden (1973), was perhaps the earliest to represent the 'conservation motive' by using the resource stock in the utility function. He finds that when the conservative motive - "a tendency to value the resource for its own sake independently of its value as a source of future consumption", (p. 127) - is incorporated into the social welfare function it is no longer optimal to completely exhaust the resource.

Referring back to the conventional approach discussed earlier, technological progress and capital-resource substitution are two ways in which it is possible for an economy to maintain consumption in the presence of a non-renewable resource that is an essential input in the production process. Krautkraemer (1985), examines the effects of these factors on the permanent preservation of natural environments and examines the conditions under which it is optimal to permanently preserve natural environments that contain the natural productive resources.

Krautkraemer shows that if society has a large enough initial capital stock and that capital is sufficiently productive and substitutable for the depletable resource, some permanent preservation is optimal. Under these conditions it is possible to have sustained consumption and also permanently preserve part of the environment. However, Krautraemer also shows that it can be optimal to exhaust the resource stock even though the marginal value of consumption is falling to zero. This is because technological progress, as well as allowing growth in consumption, also increases the productivity of the resource. Consequently the marginal value of the flow of resource inputs is increasing even though the marginal value of consumption is falling. The same forces which cause consumption to grow, i.e. technical progress, also raise the marginal value product of the resource and this drives the incentive to exhaust the resource.

There is another aspect of resource exploitation which has been considered and that is the concern about the loss of amenity services associated with unspoiled environments. Krutilla (1967), was one of the first to argue that the problem of providing the amenity services that are associated with preserved natural environments has become more of a pressing issue than the problem of providing future generations with resource inputs.

While Krautraemer had broadened the conventional approach to optimal resource depletion and looked at the conditions that allow preservation of natural environments that contain productive inputs, an earlier paper by Fisher, Krutilla and Cichetti (1972), looked at the environmental costs from irreversibly transferring land for development purposes. In their paper they argue that not only is it necessary to preserve our environment but there is also the need to develop land for production and housing. Their paper emphasises the direct competition between preservation and development. As Fisher and Krutilla (1985) point out, by irreversibly converting natural areas the future possibilities of obtaining environmental services from these areas are lost. Another reason for preserving the environmental stock relates to fairness and justice to future generations; access to it by different generations must be ensured for reasons of intergenerational equity. In many cases environmental losses that occur are irreversible and this irreversibility means the removal of an option for future generations.

Fisher *et al* formulate a model for the allocation of natural environments between preservation and development. They show that the optimal development path for a given area of land is given by a sequence of investment intervals. They find that when optimal development begins to fall, implying that marginal benefits from development are less than marginal costs, there has been too much development and we need to disinvest or reverse previous development. But if development of natural environments is irreversible, they find that development should then stop short of the level that is indicated by current valuations if in the near future reduced development is desirable. Investment should cease until future values indicate that the marginal benefits of development are greater than the marginal costs. Then the optimal growth path for development would be an alternating sequence of periods of investment in development and periods of no investment. Their analysis presents a strong case for the permanent preservation of some natural environments. It may even be optimal to permanently preserve some natural environments whose current return to preservation is less than the current return to development. The optimum level of preservation, therefore, may be greater than current values would indicate.

Barrett (1992), shows that the conditions for permanently preserving natural environments are more general than Fisher et al postulated and so therefore his paper adds weight to the argument that it is optimal to permanently preserve some natural areas and protect our environment. For a more detailed discussion see Chapter 4.

The idea of including the stock of natural capital in the utility function, has also been used by Barbier and Markandya (1990). They analyse a model where substitution between capital and the natural capital is not so free. They maintain that there is a minimum positive level of natural capital that is necessary to prevent ecological catastrophe. Again Barbier and Markyanda include the stock of natural capital along with consumption in the utility function. They start with the problem of maximising utility subject to the condition that the stock of natural capital is prevented from falling below a catastrophe threshold. They conclude that where there are multiple equilibria if the initial level of natural capital is below a critical value (which is still higher than the threshold): Maximising discounted utility may cause catastrophe. A higher discount rate increases this critical value and so increases the likelihood of catastrophe. In their work, they have highlighted an important point. They show that while it might be suggested that efficiency criteria such as internalising externalities would stop natural capital falling below the critical value, if these market failures have already caused the natural capital to fall below the critical value then efficiency criteria cannot achieve sustainability. This could be important when evaluating policies that call for improved efficiency in natural resource use in developing countries (see World Bank, 1992). These actions may then not be sufficient for sustainability as these countries may have natural capital below the threshold level and will 'opt' to drive themselves to environmental catastrophe. The policy implication for this is that aid specifically for improving the natural resource base will help countries to achieve an environmentally sustainable optimal growth path.

**Pollution models**

The generation of pollution has also been included in a number of studies on optimal growth and resource depletion. (Plourde (1972), d'Arge and Kogiku (1973), Forster (1973b), Barbier (1989), Tahvonen and Kuuluvainen (1993)). For a more detailed analysis of these models see Chapter 5.

Pollution enters these models in a variety of ways, as a stock which indicates the level of environmental quality, or as a flow, to show the rate of emission. Pollution can flow from the economic system either from production or consumption. There are a diverse range of modelling possibilities and because of this the literature has produced a wide variety of results. Most of the literature argues that an economy that follows an optimal path will progress to a steady state equilibrium where marginal cost of production which includes the environmental cost of pollution, is equal to the marginal value of consumption, see for example Forster (1973a).

Forster (1973b) presented one of the first models of economic growth and pollution. In his model, the consumption of a composite commodity generates emissions and the problem for the central planner is to determine the optimal consumption plan. He shows that the equilibrium of the system is a saddle point.

In the literature some of the models include both capital accumulation and pollution stock accumulation and there can be more than one optimal steady state equilibrium. Thus models of optimal economic growth with stock pollution may lead to multiple equilibria and cyclical paths around an unstable steady state (Becker (1982), Brock (1977), Ryder and Heal (1973), Heal (1982)).

In these models, the optimal steady state depends on the initial level of resources. A capital rich economy may choose a path that will lead to a relatively clean steady state. Whereas a resource-poor economy may choose a lower level of pollution abatement activity in order to allocate capital to the production of consumption goods. This is the argument we face today in that it is the rich industrialised countries that can afford to care about the environment, they have already reached a position where industrialisation has provided for much of man's needs and so now they can direct attention to the concerns of the environment. Poor developing countries have not reached this position and they cannot afford the 'luxury' of caring about the environment and its natural resources.

Tahvonen and Kuuluvainen (1993) and Brock (1977), have shown that if the available technology allows a high level of substitution between capital and emissions, an optimal solution exists and approaches a unique steady state and the rich-poor situation described above is avoided.

Other models have analysed the case where output can be allocated to consumption, investment and abatement of the pollution stock. In a paper by

Keeler et al (1972), welfare depends on the consumption of a composite commodity and the accumulation of pollution, i.e. the pollution stock. Pollution is generated from production and builds up into a stock which decays naturally and can also be decreased by the allocation of output to abatement. The result of their analysis is that a unique steady state can exist with either positive or zero pollution abatement. However Keeler et al do not include natural resources in their model.

d'Arge and Kogiku (1973), examine a model in which a non-renewable resource is an input in the production process and pollution is a by-product of production. The authors show that in this model, it is impossible for the economy to sustain itself forever, even with recycling.

**Population models**

Another strand of the literature relevant to sustainable development has explored the consequences of a growing population in models involving extraction of non-renewable natural resources: Stiglitz (1974a), Ingham and Simmons (1975), Cigno (1981) and Stiglitz (1974b). Previously the literature has treated population as stationary, Solow (1974), Dasgupta and Heal (1974), Dasgupta and Heal (1979) and Krautkraemer (1985). Beddington, Watts and Wright (1975), look at the optimal paths of extraction of renewable resources with a stationary population.

Empirical evidence shows that the worlds population is increasing and Stiglitz (1974b), analyses the standard neo-classical growth model with constant rate of population growth and examines the implications of introducing exhaustible resources as an essential input into the production process. He shows that if a steady state exists, it is a saddle point. Hence introducing exhaustible resources into the model has the effect of making an otherwise stable system unstable. Stiglitz (1974a), shows that steadily growing per capita consumption may be feasible forever if a wasting and non-replenishable resource is an essential input into production and if the population is growing at a constant exponential rate. He shows that necessary and sufficient conditions for a steadily growing per capita consumption is that the rate of technical progress is greater than or equal to the share of the natural resources multiplied by the rate of population growth. However, in the limit, with a finite stock of the resource and a growing population, productivity would have to be infinite to mantain per-capita consumption and the only possible steady state would have zero resource and thus zero consumption.

Stiglitz assumes that there is a steady exponential improvement in the economy's technical productivity. The rate of technical change is assumed exogenous. He also assumes that the productivity of production factors are independent of the resource stock, hence the last unit of resource extracted is as easy to extract as the one previous. Susuki (1976), modifies the Stiglitz (1974a) model by making production dependent on the level of resource stock. He concludes that the necessary and sufficient condition for steady growth of per capita consumption is the same as the conclusion reached by Stiglitz. The previous model, and that of Stiglitz, assumes that the steady growing per capita consumption is not possible without sustained technical improvement and this is assumed to be costless. He goes on to modify the model further by assuming that there is no technical progress without there being a prior commitment of productive factors to research and development. He shows that if the population is growing at a constant exponential rate, then the necessary and sufficient condition for a steady growth in per capita consumption is that the share of investment in research and development is greater than the share of resource input.

Another paper by Fisher (1992), analyses economic growth in a model which exhibits over-lapping generations, i.e. where the labout force live for finitely many periods. He shows that the rate of growth of the economy depends on the marginal efficiency of investment and the share of capital devoted to investment.

Most of the past literature on economic activity and the natural environment with a growing population focuses on studies in which population of the economy is determined exogenously. Clearly, there are interactions between population, economic activity and natural resource use and this should be a central element of concerns about sustainability. Cigno (1981) presented a paper where the population growth rate was endogenously determined. He shows that an economy with non-renewable resource exploitation, and where population is a function of per-capita income and the degree of industrialisation, is capable of stable growth. However, this depends on the choice of the savings to income ratio - i.e. how much the economy saves out of income. He postulates then the economy can be put on a stable growth path if a policy maker can control the savings ratio.

**Global Warming**

Another strand of the literature has looked at the problem of global warming. There is the growing concern that global warming and other accompanying climatic changes will occur as a result of growing concentrations of greenhouse gases in the atmosphere. The main green house gas, $CO_2$, is emitted chiefly as a result of burning fossil fuels. The increasing use of fossil fuels is closely related to the growth of economic activity worldwide and since gross domestic product (GDP), a crude measure of a countries economic activity, is expected to continue to rise, the emissions of $CO_2$ and its concentration in the atmosphere will also increase.

Clearly this adds to the problem of the optimal exploitation of non-renewable resources and has a direct implication for energy use. If global warming is to be limited, then we cannot simply burn fossil fuels to our hearts content. The generation of $CO_2$ emissions and the damage caused by global warming have been incorporated into various growth models. For instance, Nordhaus (1993b) formulated a model where the optimal growth model is extended to include a climate module and a damage sector which both feed back into the economy. In the DICE (Dynamic Integrated Climate Economy) model, the objective is to maximise the discounted sum of the utilities of consumption summed over time, subject to economic, climate, emissions and damage constraints. He uses a computer programme to run several different policy scenarios; for instance, he looks at the optimal tax on carbon emissions that would be necessary to raise fossil fuel and other prices sufficiently to induce substitution between carbon-intensive goods and services for ones that are less carbon-intensive.

Peck and Tiesburg (1991) provide an assessment of what the optimal trajectory of a carbon tax might be. Both Peck and Tiesburg and Nordhaus show that the carbon tax should rise over time. For a more detailed review of these and further papers see Chapter 7. Neither of these papers include non-renewable resource constraints in their models.

A paper by Ulph and Ulph (1994) incorporates non-renewable resources as a constraint on economic growth as well as including emissions flows and a damage function. They conclude that an optimal carbon tax would rise sharply and then fall. Sinclair (1992 and 1994), formulates a model of endogenous growth, oil extraction and global warming. He show that the optimal carbon tax should fall over time.

A number of papers explore the likely impact of imposing a carbon tax at levels sufficient to reduce $CO_2$ emissions significantly and many have indicated the probability of such a tax entailing substantial economic costs. Nordhaus (1993a) has estimated the global GDP loss of US$762 billion if a US$56 per ton carbon tax was implemented to reduce $CO_2$ emissions by 20%. A number of studies have looked at the possible positive effects of using the revenue from carbon taxes to reduce distortionary taxes elsewhere in the economy; for instance Jorgenson and Wilcoxen (1993). They show that this would lower the net cost of a carbon tax by removing inefficiencies elsewhere in the economy.

## Other areas of research

There have been several studies modelling the effects of economic activity on the environment and the feedback effects that occur as a result. Many of these models are extremely complicated and require computer simulation to obtain their results. For example, Agostini et al (1992) examine the effects of introducing a carbon tax produced by the combustion processes in OECD- European countries. They formulate a model of energy consumption in different sectors and analyse the energy saving effects of introducing a carbon tax. Their simulation provides support for the role of carbon taxes to encourage energy savings and fuel substitution and thereby stabilise carbon emissions. However, there should not be a uniform tax across the OECD, but there should be a country specific tax depending on the economic situation and technological choices facing each country.

Nicoletti and Oliveira-Martins (1992) studied the potential effects of the carbon/energy tax that was proposed by the European Union on global $CO_2$ emissions and economic activity. They formulated a global dynamic applied general equilibrium model known as GREEN. Also Birkelund et al (1993) have studied the effects on energy use and $CO_2$ emissions using a multisectoral energy demand model in Western Europe.

In a paper by Walker and Birol (1992) the long run impacts on the world energy markets, of implementing a carbon tax to reduce $CO_2$ emissions is examined. To investigate this they use a long term time series CES (constant elasticity of substitution) econometric model with over 150 equations. There are five main modules that make up the model, the macroeconomic module, and energy demand module, an energy supply module, an energy pricing module and an environmental module. There are feedback mechanisms that exist between these modules. They find that by implementing a carbon tax world energy consumption by the year 2010 will be at least 20% lower than if no action was taken.

Haugland et al (1992) provide an empirical analysis of the effect on international energy markets of policy measures to curb $CO_2$ emissions. Their analysis is carried out using a global energy demand model called ECON-ENERGY. They show that if $CO_2$ emissions are to be stabalised by means of a carbon tax, then the tax level needs to be very high. A main conclusion from their work is that taxing carbon emissions alone will not stabilise $CO_2$ concentrations. Additional efforts are needed to encourage conservation of fossil fuels and this could come from supply side measures, such as incentives for renewable resource exploitation.

**Backstop Technology**

Non-renewable resources are difficult to deal with in infinite horizon problems. There has been a fair amount of work done on non-renewable resources and technological progress, but most of the literature deals with this in the sense that technological improvement will effectively increase the supply of the resource by increasing productivity of the resource. In this case the supply of the resource will never run out as progressively less of the resource will be needed to produce a given amount of output. As there is a finite limit to the amount of the non-renewable resource, it is wrong to suggest that the supply of the resource can go on forever, obviously it cannot. In the limit, with a finite stock of the resource, productivity would have to be infinite to maintain per-capita consumption and the only possible steady state would have zero resource and thus zero consumption.

Traditionally, in neo-classical economics, the effect of natural resource scarcity is reflected in rising prices. This will result in the substitution of capital for the scarce resource and thus will effectively increase its supply, and a decrease in demand for the resource since the price will have risen. The market will again be in equilibrium and thus the market for the resource will clear. Therefore in a perfectly competitive market, provided that the elasticity of substitution between capital and the resource is large enough, there will never be an actual shortage. Going back to the classical doctrine, the presence of non-renewable resources do not pose a threat to economic growth, which can, then, last forever.

Stiglitz (1974a) states, as shown previously, that the limitations imposed by natural resources can be offset by technical change, capital resource substitution and returns to scale. He analyses the conditions under which a sustainable level of per-capita consumption is feasible. He finds that sustained levels of per-capita consumption are feasible if the elasticity of substitution between capital and the resource is greater than unity.

Krautkraemer (1985) looked at the impact of technological progress and capital-resource substitution on the permanent preservation of natural environments. He found that a necessary condition for permanently preserving natural environments is that consumption must be prevented from dropping to zero as the marginal value of the extractive resource will become infinite in this case. Thus it is never optimal to leave any of the resource in the ground.

However there has been some work that doesn't treat technological improvement in this way. An optimistic view is taken of the role which technology plays in freeing us from the dependence on natural resources. This is that some almost indefinitely renewable resource will eventually take over when the exhaustible resource has run out. This in the literature is called a back-stop technology. Examples of backstop technologies are energy from converting shale oil, energy from fusion reactors or solar energy. Krautkraemer (1986) examines the optimal depletion of a non-renewable resource in the presence of a backstop technology and resource amenities. His analysis emphasises the impact the backstop technology has on the optimal preservation of natural environments.

He shows that the outlook for the permanent preservation of natural environments is more favourable if there are upper limits on the marginal productivity of the exhaustible resource and the marginal value of consumption. There will be upper limits if there is an alternative source of consumption or resource input because then it is possible to maintain the positive flow of consumption without having the marginal product of the resource increasing to infinity. Renewable resources and a backstop technology are ways in which this may occur.

Krautkraemer (1986), analyses the optimal depletion of a non-renewable resource when there are amenity values from the resource stock and there is a backstop technology. He examines the impact the backstop technology has on the the optimal preservation of natural environments. The problem is to choose the optimal extraction of the resource, use of the backstop, and optimal consumption plan which will maximise the present value of utility. Utility depends on the flow of consumption and the amenity services that the natural resource stock provides. This utility function is the same as that in Krautkraemer (1985), described above. The backstop is included in two ways;

First it is a sector of the economy that provides output independent of the resource. Therefore the optimal control problem is to:

$$\max \int_0^\infty U\big(C(t), S(t)\big)e^{-\delta t}.dt$$

subject to:

$$\dot{K}(t) = Q + F\big(K(t), R(t)\big) - C(t)$$
$$\dot{S}(t) = -R(t)$$

and

$C(t)$, $R(t)$, $S(t)$, $K(t)$ are non-negative

$K(0) = K_0$ and $S(0) = S_0$

where $K(t)$, $S(t)$ and $C(t)$ denote capital, resource stocks and consumption respectively, $Q$ denotes the production from the backstop technology, $\delta$ is the rate of discount, and $U(C,S)$ and $F(K,R)$ denote the utility function and the production function respectively.

The first differential equation shows that the rate of change of the stock of capital increases with production from the backstop technology and production from resource use, and decreases as the level of consumption increases. The second differential equation shows that the rate of change of the resource stock is negatively related to the extraction rate.

He shows that production from the backstop technology will increase the steady state level of consumption; this will have the effect of reducing the productive value of the non-renewable resource and so the optimal level of permanently preserved environments will increase. Also a lower discount rate increases the present value of the amenity services from the preserved environments. This will increase the demand for capital for production. The higher demand for capital will increase the demand for the resource as an input as higher capital intensity will result in an increase in production. The increase in the extractive demand for the resource will outweigh the increase in demand for preserved environments if the output elasticity with respect to the non-renewable resource input is lower than the output elasticity with respect to capital.

The second way that Krautkraemer models the use of a backstop technology is as a perfect substitute input for the non-renewable resource. The optimal control problem is to:

$$\max \int_0^{\infty} U\big(C(t), S(t)\big)e^{-\delta t}.dt$$

subject to:

$$\dot{K}(t) = F\big(K(t), R(t) + M(t)\big) - \phi(M(t)) - C(t)$$
$$\dot{S}(t) = -R(t)$$

and

$C(t)$, $R(t)$, $S(t)$, $K(t)$ are non-negative

$$K(0) = K_0 \text{ and } S(0) = S_0$$

where $M(t)$ denotes the quantity of the substitute input and $\phi(M(t))$ denotes the physical cost of providing the substitute.

The differential equation for the capital stock shows that the rate of change of the capital stock increases as output increases and decreases as consumption and the physical cost of providing the substitute increases.

He shows that it is optimal to completely exhaust the resource stock if the marginal cost of the backstop is high enough so that it isn't used until after the marginal productivity of capital falls below the rate of discount. The capital stock will then decline, output and consumption will fall to zero and the productive value of the resource input increases without bound. Therefore there is no preservation.

He also shows that some permanent preservation can be optimal. If the marginal cost of the backstop technology is relatively low then it will be used before the marginal productivity of capital falls below the rate of discount. Capital will continue to increase, consumption will rise, the resource price will drop to zero and permanent preservation can be optimal. If the marginal productivity of capital does fall to the rate of discount, then the economy will converge to a steady state with positive values of consumption and environmental preservation.

**Differential Games**

So far the literature reviewed has been dynamic optimisation models with a single decision maker. For example, problems involving a single industry country. These are optimal control problems in which it is a single individuals choice of the control trajectory that changes the state of the system. The problems so far have been formulated so that there is only one decision maker and there has been no allowance for the fact that often decision-making may be carried out by more than one country or individual. There are many situations in which the overall state of the system is determined by more than one individual. Situations in which the joint actions of several individuals, each acting independently, affect a common state variable are modelled as differential games. Non-cooperative games are those in which individuals, referred to as players, do not co-operate in selecting the values of the control variables, and for which the state of the system changes according to one or more state equations. Thus in a differential game the players interact continuously through time.

There has been much literature concerning the theory and application of differential games. See for example, Starr and Ho (1969). This literature has been extended to include environmental concerns and the control strategies of different countries

Dockner and Long (1993) develop a dynamic game model of international pollution control involving two countries. Each country produces goods that, when consumed by domestic houdeholds, generates pollution emissions. Each household's utility is positively related to consumption and negatively related to the stock of pollution. The Governments aim is to maximise the discounted stream of net benefits of a representative consumer. They examine two strategies - cooperative and non-cooperative. Cooperative assumes that both countries have a high degree of commitment to follow an agreed strategy. Non-cooperative scenario is when each country's emission policy is based on self interest, which is based on the other country's pollution emissions.

They find that when the Governments follow a linear strategy, non-cooperative behaviour results in a high level of pollution stock and overall losses for both countries. The steady state pollution stock that results when both countries adopt linear strategies is greater than that under a fully cooperative behaviour and the level of welfare for both countries is lower. A linear strategy will cause a country to adopt a decision rule which implies a negative linear relationship between the permitted emission rate and the level of pollution stock at any time. If country one found that it would be optimal to decrease its pollution emissions, the overall level of pollution stock will fall. The environment will now be cleaner and, as a clean environment is a public good, then country two will benefit from this fall in pollution. According to the linear decision rule, country two can now increase its emissions and so in the long run a higher steady state pollution stock will result.

They find that if the countries adopt non-linear strategies and the discount rate is sufficiently low, then the use of these strategies will enable the two countries to reach a self-enforcing agreement that approximates to the fully cooperative scenario.

## CONCLUSION

This chapter has reviewed economic growth theory involving natural resources and the environment and the bearing this has on sustainable development. In Chapter 1 we saw the basic issues in addressing sustainability and it can been seen that the literature has developed in such a way as to address these concerns. The alternative approach to natural resource scarcity recognises that the environment provides us with services essential to human welfare Here, this is contrasted with the conventional approach to natural resource scarcity which is mainly concerned with the optimal allocation of economically valuable exhaustible resources.

It is clear that there has been a change in the way that the environment has been perceived in economics, not just through past history dating back to Malthus but since the 1970's. The environmental effects of economic activity used to be regarded as a mistake, an externality to the economic process. More recently, however, concerns have been focused on the physical dependency of economic activity and human welfare on the sustainability of crucial natural resource systems and ecological functions. Now the issue of sustainability figures prominently in contemporary discussions of natural resource use and environmental management and economic development.

It has not been the purpose of this chapter to attempt the task of reviewing the methodological critiques of neo-classical growth theory, but to identify the most notable issues in formal economic analysis of sustainability and give a review of the literature that is within the mainstream of neo-classical economics.

This chapter gives an overview of how this literature has developed since the work of Hotelling (1931) and shows that changes over time of economic approaches to environmental issues is linked to the emergence of the concept of sustainable development.

In the following five chapters I have selected the various issues concerning sustainability, i.e. the preservation of land, the generation of pollution, the optimal extraction of non-renewable resources, endogenous population growth and the use of carbon taxes and developed them further. Each chapter contains a far more detailed review of the relevant literature to give further insight into each issue.

# CHAPTER THREE

# METHODOLOGY

## INTRODUCTION

In this and subsequent chapters, the problems posed are problems of dynamic optimisation. The methodology employed is the Maximum Principle of L.S.Pontryagin and his associates (Pontryagin *et al* 1962). This is a modern and much more powerful version of the classical calculus of variations as developed by Euler, Lagrange, Legendre, Hamilton and Jacobi. In Section I, a typical optimal control problem with a finite time horizon will be set out. Section II will detail Pontryagin's Maximum Principle which is used to determine the optimal solution to the problem. Section III will examine the current value Hamiltonian where the value of the variables at time $T$ are discounted back to give their equivalent value at time zero, i.e. their present value. Section IV will look at infinite horizon problems and Section V will examine how one can solve an optimisation problems when the Hamiltonian is linear in the control variables. In Section VI we will show how one can establish whether it is optimal to reach the optimal solution in the minimum amount of time Section VII will look at the steady state solutions to optimisation problems and the stability of the dynamic systems and in Section VIII we will look at how this methodology can take into consideration issues of sustainable development.

## SECTION I

### A Typical Optimal Control Problem

A typical optimal control problem contains the following elements:

1). An objective functional which it is desired to be maximised (or minimised) over some time interval. Typically we may wish to maximise;

$$\int_{t0}^{T} F(x(t), u(t), t).dt \qquad (1).$$

where $F(x(t), u(t), t)$ is a continuously differentiable function, $u$ is a piecewise continuous vector of instruments/control variables $(u^1, ...., u^k)$, $x$ is a continuous piecewise differentiable vector of target/state variables $(x^1, ..., x^n)$ and $t$ represents time. Here the objective functional is to be maximised over a finite time period, $T$. This however, may be infinite as will be discussed later.

2). A set of differential equation constraints (which show how the state variable changes over time);

$$\dot{x} = g(x(t), u(t), t) \qquad (2).$$

together with initial conditions;

$$x(t_0) = x_0 \qquad (3).$$

3). In addition there may be some terminal time requirements on the state variables and both state and control variables may have to satisfy a set of inequality constraints. For example;

$$x^i(T) = x^i{}_T \qquad i = 1, ............, m$$

$$x^i(T) \geq x^i{}_T \qquad i = m+1, ........, q$$

$$x^i(T) \text{ free} \qquad i = q+1, ........., n \qquad (4).$$

4). A control variable restriction

$$u(t) \in \mathcal{U} \qquad \mathcal{U} \text{ given set in } R^r \qquad (5)$$

where $\mathcal{U}$ denotes some bounded control set, (which may be $R^r$). Admissable controls are the class for all piecewise continuous real functions $u(t)$ defined on $0 \leq t \leq T$ and satisfying (5).

The problem essentially consists of finding a feasible piecewise continuous time path for the vector of control variables, $u$ , defined on the time interval $t \in (0,T)$ that maximises the criterion functional. This will generate a time path for the state variables as the solution to the first order differential equations (2).

If a pair $(x(t), u(t))$ satisfies (2) - (5) it is a feasible pair. A feasible pair is an optimal pair if it maximises the integral in (1).

For example, consider the differential equation;

$$\frac{dS}{dt} = -R(t) \tag{6}$$

where $S$ denotes the stock of a natural exhaustible resource, as in the Hotelling model (see Hotelling 1931) with $S(0) = S_0$ and $R(t)$ denotes the rate of resource extraction at time $t$. Equation (6) is known as the state equation, it describes the evolution of the system from its initial state $S_0$ resulting from the application of a given control $R(t)$. $R(t)$ is a control variable because it is something that is subject to our discretionary choice. $R(t)$ is like a steering mechanism, it drives the state variable $S(t)$ to various positions via the state equation at any time $t$.

The aim is to optimise some performance criterion. Suppose society wants to maximise total utility which is derived from using the exhaustible resource over a time period (0, $T$). The problem is to:

$$\max \int_0^T u(R).dt$$

subject to:

$$\frac{dS}{dt} = -R(t)$$

$$S(0) = S_0$$

Note: The path of the control variable over time does not have to be continuous throughout the time period for it to be feasible. However, it does need to be piecewise continuous which means that it is allowed to have discontinuities, i.e. it can jump in value within the time period. See Fig 1 in the appendix to this chapter for the general case where the control variable is denoted by $u$ and the state variable is denoted by $x$. All diagrams relating to this chapter are contained in the Appendix to this chapter.

The path for the state variable over time does have to be continuous but may have a finite number of points where it is not differentiable. For the state path to be feasible it only needs to be piecewise differentiable. See Fig 2:

The non-differentiable points along the state path occur at the same time that the discontinuities occur along the optimal control path. This is easy to explain. Once the control path for the time interval $(0,t_1)$ has been determined, say the curve *ab* in fig (1), the corresponding state path for the time interval $(0,t_1)$ must be determined. Suppose this is given by the curve AB in fig (2), whose initial point is given in the initial condition. Next we need to determine the optimal state path for the next time interval $(t_1,t_2)$, corresponding to the optimal control path curve *cd* in Fig 1. But now $B$ is the starting point of the optimal state path segment. Therefore for the first time interval point $B$ is the end point and for the second time interval $B$ is the initial point for the optimal state path. Therefore at point $B$ there may be a non-differentiable point but there can be no discontinuity.

This can be explained using the previous example of an exhaustible resource. The control which is the rate of extraction $R(t)$ can vary as it is subject to our discretionary choice. Therefore the optimal control path can have discontinuities

- i.e. it is piecewise continuous. As the resource is extracted, it runs down the stock of the resource $S(t)$ - the state variable. For the time interval $(0,t_1)$ the extraction rate of the resource, $R(t)$ is shown by the curve ab in fig (1). In fig (2), $B$ is the terminal point for the first time interval. The extraction rate at time $t_1$ jumps to a higher rate indicated by point $C$, but the stock of the resource must start from point $B$. Therefore the optimal state path must be continuous but the control path need only be piecewise continuous.

# SECTION II

## The Maximum Principle

The Maximum Principle is expressed in terms of the Hamiltonian:

$$\mathrm{H}(x(t),u(t),t:\lambda(t)) = f(x(t),u(t),t) + \lambda(t)g(x(t),u(t),t)$$

where $\lambda(t) = (\lambda_1(t),....,\lambda_n(t))$ is a vector of adjoint or costate variables which are valuation variables measuring the shadow price of an associated state variable. In the same way as the state and control variables, the costates are piecewise continuous functions of time. The first term on the right hand side is the objective functional at time $t$. Using the previous example this is the utility function at time $t$ based on the current resource stock and the current policy decision at $t$. It is the present utility corresponding to policy $R(t)$, (in the general case - $u(t)$). The second term on the right hand side, $g(x(t),u(t),t)$, shows the rate of change of the resource stock ($\dot{S}$), (in the general case ($\dot{x}$)) corresponding to policy $R(t)$. The whole of this expression relates to the future utility effect of policy $R(t)$. Here $\lambda(t)$ converts this expression to a monetary value, it is the imputed value of future utility streams. The Hamiltonian represents overall utility prospects of various policy decisions with both the immediate and future prospects taken into consideration.

It is convenient here to state the Maximum Principle:

For $(x(t), u(t))$ to be a feasible pair it is necessary that there exists a constant $\lambda_0$ and a continuous $n$-vector function $\lambda(t) = (\lambda_1(t),....,\lambda_n(t))$ where for all $t \in (t_0, T)$, $(\lambda_0, \lambda(t)) \neq (0,0)$ and such that;

For any $t \in (t_0, T)$

$$H(\bar{x}(t), u(t), \lambda(t), t) \leq H(\bar{x}(t), \bar{u}(t), \lambda(t), t) \qquad \text{for all } u \in \mathcal{U} \qquad (7)$$

Except at the points of discontinuity of $\bar{u}(t)$,

$$\dot{\lambda}_i(t) = -H'_{xi}(\bar{x}(t), \bar{u}(t), \lambda(t), t) \qquad i = 1,.......,n \qquad (8)$$

Furthermore the following transversality conditions must be satisfied:

$$\lambda_i(T) \text{ no conditions} \qquad i = 1,........,p$$

$$\lambda_i(T) \geq 0 \qquad (= 0 \text{ if } \bar{x}_i(T) > x_{i1}) \qquad i = p+1,....,q$$

$$\lambda_i(T) = 0 \qquad i = q+1,....,n \qquad (9)$$

(Seierstad and Sydsaeter (1977))

Equations (7), (8) and (9) are thus the necessary conditions that must be satisfied by an optimal control. Equation (8) is known as the adjoint equation and simplifies to: (from now on, for notational simplicity we suppress explicit dependence upon time)

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$$

which when written in full is:

$$\dot{\lambda}_i = -\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} \qquad (10)$$

The left hand side of (10) denotes the change in the shadow price over time. The first of the two terms on the right hand side, $\frac{\partial f}{\partial x_i}$, represents the marginal contribution of the state variable to the instantaneous utility. The second term, $\lambda\frac{\partial g}{\partial x}$, represents the value of the marginal contribution of the state variable to the future utility prospects.

The Maximum Principle requires that the shadow price of the state variable, $x(t)$, depreciates at the rate at which the state variable is contributing to the current and future pay offs.

## The transversality conditions

Equations (9) depend on the particular end point constraint dictated by the problem. When no terminal value for the state variable $x(t)$ is specified - i.e. free terminal value, the transversality condition is:

$$\lambda(T) = 0$$

This means that at the terminal time $T$, the shadow price of the state should be driven down to zero. Using the example used earlier, the reason for this is that the benefits of the resource to Society arise solely from its potential for producing utility. Given that the terminal time is fixed, $T$, only the utility derived in that period $(0,T)$ matters and that Society derives no utility from remaining resource stocks. Therefore the shadow price of the resource should be set equal to zero at time $T$.

If there is some minimum acceptable level for the terminal resource stock, say $S_{\min}$, the transversality condition is now:

$$\lambda(T) \geq 0 \qquad \text{and } [S^*_{\min}(T) - S_{\min}]\lambda(T) = 0$$

If the optimal level of resource stock at time $T$, $[S^*(T)]$, is greater than $S_{\min}$, the restriction placed on the terminal resource stock does not hold. This would be the same as in the previous case and the transversality condition $\lambda(T)=0$ still applies. But if the shadow price of the resource at the terminal time, $\lambda(T)$, is optimally positive then the optimal amount of the resource stock left at time $T$ will be exactly the same as the required minimum level $S_{\min}$.

For a problem in which there is a prespecified level of resource stock at the terminal time $T$, but the time horizon may be freely chosen, there is an additional condition and that is that $T$ should be chosen such that the maximised value of the Hamiltonian should be zero at terminal time. This means that at $T$ the sum of the current and future utility levels must be zero. Therefore we should not achieve the prespecified level of resource stock when the sum of the current and future utility levels is positive, i.e. $H_T > 0$. This would imply that there was additional utility to be gained and the full utility potential had not been maximised. We should reach $S(T)$ when no more utility can be derived, i.e. when the sum is zero.

In general terms again, the Maximum Principle requires the maximisation of the Hamiltonian with respect to the controls $u$. Referring back to equation (7), the control set is often of the form: $u_i(t) \in [\alpha_i, \beta_i]$ where i = 1,......,$n$ and $\alpha_i < \beta_i$ ; $\alpha_i$ and $\beta_i$ are either constants or functions of $t$ and/or $x(t)$. If the control set is unrestricted then $u \in (-\infty, \infty)$ and the range of $u_i$ is the real line.

For each $t \in [0, T]$ either:

i). $\frac{\partial H}{\partial u} > 0 \quad \forall u_i \in [\alpha_i, \beta_i]$ then we have a boundary solution and set $u_i = \beta_i$. This is the upper boundary solution (fig 3).

ii). $\frac{\partial H}{\partial u} < 0 \quad \forall u_i \in [\alpha_i, \beta_i]$ then we have a lower boundary solution and set $u_i = \alpha_i$ (fig 4).

iii). $\frac{\partial H}{\partial u} = 0$ and $\forall u_i \in [\alpha_i, \beta_i]$ then $u_i = u^*_i$ and this is called an interior solution (fig 5).

## The Sufficiency Conditions

Suppose an admissible pair $[x(t), u(t)]$ is found that satisfies (7) - (9). Will it be an optimal pair? The above conditions are necessary but not necessarily sufficient. However when certain concavity conditions are satisfied, the necessary conditions stipulated by the Maximum Principle are also sufficient for maximisation.

Let $[x(t), u(t)]$ be a feasible pair satisfying (7) - (9). Then if $H(x, u, \lambda, t)$ is jointly concave in $x$ and $u$, $[x^*(t), u^*(t)]$ is an optimal pair, (Mangasarian 1966).

# SECTION III

## The Current Value Hamiltonian:

With many economic applications of optimal control theory, the pay off function $g$ often contains a discount factor $e^{-\rho t}$. To explain this it is best to look at a small example. If $A$ pounds (£$A$) were invested at an interest rate $r$% per year, after 1 year the amount would grow to £$(1+r)A$, after 2 years £$[(1+r)A + r(1+r)A] =$ £$(1+r)^2A$, and after $T$ years £$(1+r)^TA$.

If the interest were compounded not annually but twice a year, then for a 6 month period the interest rate would be $(r/2)$%. So if £A were invested, after 1 year it would grow to £$(1+r/2)^2A$ and after $T$ years it would be $(1+r/2)^{2T}$. If interest was compounded $m$ times a year then the rate per period would be $(r/m)$%. Then £$A$ would grow to £$(1+r/m)^mA$ after 1 year and £$(1+r/m)^{mT}A$ after $T$ years.

Continuous compounding means letting $m \rightarrow \infty$ and since:

$$\lim_{m \to \infty} (1 + r/m)^{mT} = e^{-rT}$$

This means that £$A$ invested at an annual rate of $r$%, if continuously compounded, grows to £$Ae^{rT}$ in $T$ years time. If we call this amount $B$, then, in other words, £$A$ is worth $Ae^{rT}$ in $T$ years time and $Ae^{rT} = B$. Thus $A = e^{-rT}B$. The term, $e^{-rT}B$, is the present value of £$B$ available $T$ years in the future.

This is called discounting. We are discounting the value of £$B$ to find it's present value, i.e. it's value at time zero.

The same procedure is employed in optimal control theory. The value of the variables at time $T$ is discounted back to give their equivalent value at time zero, i.e. their present value. The optimal control problem is now:

$$\max J = \int_0^T F(x,u)\, e^{-\rho t}.dt$$

subject to $\dot{x} = g(x,u)$ and boundary conditions as before.

where $F(x,u)e^{-\rho t} = f(t,x,u)$ and $\rho$ is the rate of (subjective) time preference.

It is convenient to define the current value costate as:

$$y(t) \equiv \lambda(t)\, e^{\rho t}$$

Thus:

$$\lambda(t) \equiv y(t) e^{-\rho t} \tag{11}$$

and then the Hamiltonian may be written as:

$$\mathrm{H} = e^{-\rho t}\,[F(x,u) + yg(x,u)]$$

Then the adjoint equations are:

$$\frac{d}{dt}(y_i e^{-\rho t}) = -\frac{\partial \mathrm{H}}{\partial x}$$

The remaining necessary conditions are as before. The result of this is to enable us to express all the necessary conditions in a simple non time dependent form.

$\lambda(t)$ represents the marginal valuation of the state variable at time $t$ discounted back to zero. But it is more convenient to use terms of current value - that is value at time $t$ rather than their equivalent at time zero. It is desirable to define a new Hamiltonian called the current value Hamiltonian. This can be written:

$$Hc = \mathrm{H}\, e^{\rho t} = F(x,u) + yg(x,u)$$

$H_c$ is now free of the discount factor.

We need to re-examine all the conditions of the Maximum Principle. The first condition is to maximise $H_c$ with respect to $u$ at every point in time. Because $e^{-\rho t}$ is a constant for any given $t$ and it is always strictly positive, using the current value Hamiltonian the condition is unchanged. The particular $u$ that maximises H will therefore also maximise $H_c$. Therefore equation (7) is unchanged.

Looking at the equation of motion for the costate variable:

$$\dot{\lambda}_i = -\frac{\partial \mathrm{H}}{\partial x_i}{}_i \qquad i = 1,\ldots..,n \qquad (12)$$

we need to transform each side of (12) into expressions involving $y$. By differentiating (11):

$$\dot{\lambda} = \dot{y} e^{-\rho t} - \rho y e^{-\rho t}$$

The right hand side of (12), $-\frac{\partial \mathrm{H}}{\partial x_i}$, can be rewritten as:

$$-\frac{\partial \mathrm{H}}{\partial x_i} = -\frac{\partial Hc}{\partial x_i} e^{-\rho t} \qquad \text{by using } \mathrm{H}=H_c e^{-\rho t}$$

By equating these two results we get:

$$\dot{y} e^{-\rho t} - \rho y e^{-\rho t} = -\frac{\partial Hc}{\partial x_i} e^{-\rho t}$$

Cancelling out the common factor, $e^{-\rho t}$, we get the following revised equation of motion:

$$\dot{y} = -\frac{\partial Hc}{\partial x} + \rho y \tag{13}$$

As compared with the original equation of motion for $\lambda(t)$, the new equation of motion for $y(t)$ now has an extra term , $\rho y$.

Next we need to look at the transversality conditions. For the condition where no terminal value for the state variable is prescribed the transversality condition is:

$$\lambda(T) = 0 \qquad \text{(equation (9)).}$$

Using (11) this implies that:

$$[\, y e^{-\rho t} \,]_{t=T} = 0$$

Therefore:

$$y(T) e^{-\rho T} = 0$$

For the case where there is a minimum acceptable level for the state variable the transversality condition is:

$$\lambda(T) \geq 0$$

Therefore

$$y(T)e^{-\rho T} \geq 0 \quad \Rightarrow y(T) = 0 \text{ since } e^{-\rho T} > 0.$$

For the case where the terminal time is free but the state variable is fixed to some prespecified level, the transversality condition is:

$$H(T) = 0$$

Therefore $$[H_c e^{-\rho T}] = 0 \quad \Rightarrow H_c(T) = 0$$

# SECTION IV

## Infinite Horizon Problems

So far the time horizon considered has been finite. But most economic models have a planning horizon which is infinite ( $T = \infty$ ). But with an infinite horizon two methodological issues need to be addressed. One is a matter of the transversality conditions, the other has to do with the convergence of the objective functional. The convergence problem arises because the objective functional (1) may itself become infinite. In this case there may be difficulties in discriminating between alternative optimal policy options where there are more than one pair $(x(t), u(t))$ that satisfies the optimality conditions.

A simple criterion has been proposed by *Von Weizsäcker* (1965) called the 'overtaking' criterion which deals with the problem of infinite integrals.

By putting $T = \infty$ and assuming that the necessary and sufficient conditions hold, (with the possible exception of the transversality conditions), $(x^*(t), u^*(t))$, suggests itself to be an optimal pair, $(x(t), u(t))$ is any other feasible pair.

Define:

$$\Delta(t) = \int_{t0}^{t} f(x^*(t), u^*(t), t).dt - \int_{t0}^{t} f(x(t), u(t), t).dt$$

If there exists a finite number $t'$ such that $\Delta(t) \geq 0$ for all $t \geq t'$ then we say that $(x^*(t), u^*(t))$ "overtakes" $(x^*(t), u^*(t))$ and $(x^*(t), u^*(t))$ is an optimal pair.

$$M[x(t), \lambda(t)] = \underset{\{u\}}{Sup}\, H[x(t), u(t), t, \lambda(t)]$$

## Transversality Conditions for Infinite Horizon Problems

Halkin (1974) has shown that while the necessary conditions carry over to the case where $T$ is infinite, the transversality conditions may not. The problem arises because the planning horizon is infinite, and the terminal state value may also be free. Arrow (1968) however, has shown that the transversality conditions do carry over for an infinite horizon problem where the sufficiency conditions hold. The transversality condition for a free terminal state is:

$$\lim_{t \to \infty} \lambda(t) = 0$$

Similarly, in the case of the terminal state which is subject to a pre-specified minimum level $x_{min}$ as $t \to \infty$ the infinite horizon transversality condition is:

$$\lim_{t \to \infty} \lambda(t) \geq 0 \qquad \text{and} \qquad \lim_{t \to \infty} \lambda(t)[y(t) - y_{min}] = 0$$

## Sufficiency Conditions Re-examined

The sufficiency conditions for the optimality of a feasible pair $(x^*(t), u^*(t))$ requires that the Hamiltonian function of the optimal control problem is concave with respect to $x$ and $u$ or $H^*(x)$ is concave in $x$, where $H^*(x)$ is the maximised Hamiltonian. But Sorger (1992) presents new sufficiency conditions that do not require the Hamiltonian function to be concave with respect to $x$ and $u$.

The Hamiltonian is defined as before as:

$$H[x(t), t, u(t), \lambda(t)]$$

and let

$$M[x(t),\lambda(t)] = \underset{\{u\}}{Sup}\, H[x(t),u(t),t,\lambda(t)]$$

If $M(x,\lambda)$ turns out to be convex in $(x^*,\lambda^*)$ the usual sufficiency conditions cannot be satisfied. Sorger showed that for a problem with a single state variable a sufficiency condition for a stationary state to be (locally) optimal is that:

i). There exists a stationary state $(x^*,\lambda^*)$and an open neighbourhood $N$ of $(x^*,\lambda^*)$ such that the function $H[x,u,\lambda,t]$ has a unique maximum with respect to feasible values of $u$ for all $(x,\lambda)\in N$ and such that $M(x,\lambda)$ is twice continuously differentiable on $N$.

ii). At the stationary state $(x^*,\lambda^*)$, the Hamiltonian function is strictly convex with respect to $\lambda$ and $M_{\lambda\lambda}> 0$ at $(x^*,\lambda^*)$.

iii). If i). and ii). are satisfied and $M^2{}_{x\lambda} - rM_{x\lambda} - M_{xx}M_{\lambda\lambda} > 0$ at $(x^*,\lambda^*)$, then $(x^*,\lambda^*)$ is a locally stable optimal stationary state.

See Sorger (1992) Corollary 2.1 p150, also see Chappell and Dury (1994) for an example of how these new sufficiency conditions are used.

# SECTION V

## Singular Arc solutions to Optimal Control

The Hamiltonian (6) may sometimes be linear in the control variables and for an unbounded control set the control can take any value. The necessary conditions for optimality that $\frac{\partial H}{\partial u} = 0$ cannot be solved for the optimal value of the controls and therefore the choice of control cannot be determined in the usual way. Under these circumstances, the natural solution to look for is a singular arc solution.

In this thesis, problems of this type do arise from time to time but in a relatively simple form with scalar state and control variables and an infinite time horizon. The most general formulation of such a problem is:

$$\max_{\{u\}} \int_0^\infty e^{-\rho t}[a(x)+b(x)u].dt$$

subject to:

$$\dot{x} = c(x)+d(x)u$$

$$x(0) = x_0 \qquad \text{for scalar } u \text{ and } x.$$

The present value Hamiltonian is defined as:

$$\mathrm{H} = e^{-\rho t}\{a(x)+b(x)u+y[c(x)+d(x)u]\}$$

The current value hamiltonian is therfore:

$$H_c = \mathrm{H}e^{\rho t} = \{a(x)+b(x)u+y[c(x)+d(x)u]\}$$

Note that the control set is unbounded (also for notational simlicity we suppress the dependence on $x$) and:

$$\frac{\partial H_c}{\partial u} = [b+dy] = 0 \tag{1}$$

This does not enable us to solve for the control. However in these circumstances along the singular arc;

$$\frac{\partial H_c}{\partial u} = 0$$

and therefore

$$b+dy = 0 \tag{2}$$

The equation for the costate is:

$$\dot{y} = \rho y - \frac{\partial H_c}{\partial x}$$

and therefore

$$\dot{y} = \rho y - a' - b'u - c'y - d'uy \qquad (3)$$

It follows that along the singular arc:

$$\frac{d}{dt}(\partial H_c / \partial u) = 0$$

therefore

$$\frac{d}{dt}(\partial H_c / \partial u) = d\dot{y} + (b' + d'\dot{y})\dot{x} = 0$$

Substituting in (3) gives:

$$d(\rho y - a' - b'u - c'y - d'u) + (b' + d')(c + du) = 0$$

Simplifying gives:

$$\rho dy - a'd - c'dy + b'c + cd'y = 0 \qquad (4)$$

Note that this expression still does not enable us to solve for the control variable so we must differentiate again. It also follows that along the singular arc

$$\frac{d^2}{dt^2}(\partial H / \partial u) = 0$$

Therefore

$$(\rho d - c'd + cd')\dot{y} + (\rho yd' - a''d - a'd' - c''dy - c'd'y + b''c + b'c' + c'd'y + cd''y)\dot{x} = 0$$

Substituting for $\dot{y}$ and $\dot{x}$ and simplifying gives:

$$
\begin{aligned}
&(\rho d - c'd + cd')(ry - a' - b'u - c'y - d'yu) \\
&+(c+du)(\rho yd' - a''d - a'd' - c''dy + b''c + b'c' + cd''y) = 0
\end{aligned}
\tag{5}
$$

Note that this condition can in fact be solved for $u$ in terms of $x$ and $y$ and, (provided the "coefficient" on $u$ is non-zero), these three equations, (2), (4) and (5), can be solved for the optimal values of $x$, $u$ and $y$ along the singular arc. Denoting optimal values with a * superscript, it follows that from (2):

$$
y^* = \frac{-b(x^*)}{d(x^*)} \tag{6}
$$

Substituting this into (4) and simplifying gives:

$$
\begin{aligned}
&-\rho b(x^*) - a'(x^*)d(x^*) \\
&+b(x^*)c'(x^*) + b'(x^*)c(x^*) - \left[\frac{b(x^*)c(x^*)d'(x^*)}{d(x^*)}\right] = 0
\end{aligned}
\tag{7}
$$

and:

$$
u^* = -\frac{c(x^*)}{d(x^*)} \tag{8}
$$

Clearly multiply solutions are possible. However any candidate solution must satisfy an additional necessary condition.

## Necessary Conditions for Singular Optimal Controls

Pontryagin's Maximum Principle does not yield any information directly on singular controls, so new necessary conditions for optimality are needed.

Kelley (1964) discovered, and Robbins (1967), Tait (1965) and Kelley et al (1967) generalised, a new necessary condition known as the Generalised Legendre Clebsch, (GLC), for the optimality of singular arcs. Along a singular arc Kelley et al, Robbins and Tait prove that an additional necessary condition of optimality is as follows;

$$(-1)^k \frac{\partial}{\partial u}\left[\frac{d^{2k}}{dt^{2k}}(\partial H / \partial u)\right] < 0 \qquad (9)$$

where $u$ is the control variable and the $2kth$ time derivative of $H_u$ is the first to contain explicitly the control $u$. If this condition holds then the singular arc solution is optimal.

# SECTION VI

Next it is necessary to establish whether it is optimal to reach the optimal solution in the minimum amount of time. This property may be established by writing the objective functional as a line integral and applying Green's Theorem, see Kamien and Schwartz (1991).

## Line Integrals

By solving the state equation for the control and substituting into the objective functional, this can then be written as a line integral.

Consider the linear control problem with scalar state and control variables of the form;

$$\max \int_0^{\infty} [f_0(t,x) + g_0(t,x)u].dt \qquad (10)$$

subject to:

$$\dot{x} = f_1(t,x) + g_1(t,x)u \qquad (11)$$

Therefore rearranging (11) gives

$$u = \frac{[\dot{x} - f_1(t,x)]}{g_1(t,x)} \qquad (12)$$

We can then eliminate $u$ from our problem by substituting (12) into (10). The problem then becomes:

$$\max \int_0^\infty [G(t,x) + H(t,x)\dot{x}].dt \tag{13}$$

where $G(t,x) = \{f_0(t,x) - [g_0(t,x)f_1(t,x)]/g_1(t,x)\}, H(t,x) = g_0(t,x)/g_1(t,x)$.

(13) can then be written as a line integral, Kamien and Schwartz (1991):

$$\int_0^\infty [G(t,x) + H(t,x)\dot{x}].dt = \int_\ell G.dx + H.dt \tag{14}$$

where $\ell$ is the curve $x = x(t),\ t \geq 0$. Suppose that the situation is as shown in Fig 6 where the curve $j(t)$ is the asserted optimal trajectory denoted by *mn*. Consider an alternative feasible path, the curve $q(t)$, denoted by *mp* which takes longer to reach $x^*$, the optimal level of the state variable $x$. Let *tp* be the time $q(t)$ gets to $x^*$. Then $j(t)$ and $q(t)$ coincide for $t \geq t_p$, i.e. both paths have $x(t) = x^*$ for $t \geq t_p$. For $j(t)$ to reach $x^*$ quicker than $q(t)$ we need to show that $j(t)$ gives a higher value to the integral from 0 to $t_p$ than $q(t)$ does, i.e:

$$\int_{mn} G.dx + H.dt - \int_{mp} G.dx + H.dt \geq 0 \tag{15}$$

Using Green's Theorem (see Kamien and Schwartz (1991)):

$$\oint_{pnmp} Gdx + Hdt = \iint_R [dH/dt - dG/dx]dxdt \tag{16}$$

where $R$ is the closed and bounded region *pnmp*.

If equation (15) is $\geq 0$, i.e.:

$$\iint_R [\partial H/\partial t - \partial G/\partial x].dxdt \geq 0$$

then the curve $j(t)$ gives a higher value to the integral from 0 - *tp* than $q(t)$ does. This means that it is optimal to reach $x^*$ in the minimum amout of time.

# SECTION VII

## Steady states and their stability

In the long run the optimal solution may converge to an equilibrium, or rest point.

This constant solution of a system of differential equations is known as the steady state or the particular solution.

A typical first order system of differential equations in $R^n$ is:

$$\begin{aligned}\dot{y}_1 &= f_1(y_1,\ldots,y_n)\\ &\vdots \\ \dot{y}_n &= f_n(y_1,\ldots,y_n)\end{aligned}$$

or in vector notation $\dot{y} = F(y)$, where $F \equiv (f_1,\ldots,f_n)$. Since each $\dot{y}(t) = 0$ for a steady state solution to the system, a point $y^* = (y_1{}^*,\ldots,y_2{}^*)$ is a steady state if and only if:

$$\begin{aligned}f_1(y_1{}^*,\ldots,y_n{}^*) &= 0\\ &\vdots \\ f_n(y_1{}^*,\ldots,y_n{}^*) &= 0\end{aligned}$$

and in vector notation, $F(y^*) = 0$. Therefore to find the steady state solutions is a matter of solving $n$ algebaic equations in $n$ variables.

## Stability

If $y^*$ is the equilibrium point for the $n$-dimensional first order system of differential equations, as above, then $y^*$ is locally asymptotically stable if every solution $y(t)$ which starts near $y^*$, converges to $y^*$ as $t \to \infty$.

If every solution to the system of equations $\dot{y} = F(y)$ tends to $y^*$ as $t \to \infty$ for any initial values, $y_0$, the equilibrium is globally asymptotically stable.

If an equilibrium $y^*$ is neither globally or locally asymptotically stable, then the system is unstable and $y(t)$ may not converge to the steady state equilibrium as $t \to \infty$.

To ascertain the stability of the steady states it is necessary to determine the characteristic roots (eigenvalues) of the Jacobian matrix evaluated at the steady state, $DF(y^*)$.

In a system $\dot{y} = F(y)$ with $n$ variables and steady state $y^*$, the Jacobian matrix $DF(y^*)$ is defined as:

$$DF(y^*) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(y^*) & \cdots & \frac{\partial f_1}{\partial y_n}(y^*) \\ \frac{\partial f_2}{\partial y_1}(y^*) & \cdots & \frac{\partial f_2}{\partial y_n}(y^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(y^*) & \cdots & \frac{\partial f_n}{\partial y_n}(y^*) \end{bmatrix}$$

If each eigenvalue of the Jacobian matrix $DF(y^*)$ is negative or has negative real parts, then $y^*$ is a locally asymptotically stable steady state.

If $DF(y^*)$ has at least one positive real eigenvalue or one complex eigenvalue with a positive real part, then $y^*$ is an unstable steady state, possibly a saddle point. This method is not valid if any of the eigenvalues are purely imaginery.

There may be no simple closed-form expression for the eigenvalues of this matrix. For example if $n > 4$ a different method is then required to analyse the stability of the steady state.

Using corollary 2c in Sorger (1989), the equilibrium point of the system $\dot{y} = F(y)$ possess the saddle point property if the 'curvature matrix,'

$$C = \begin{bmatrix} H^*_{ii} & \left(\frac{\delta}{2}\right)I_n \\ \left(\frac{\delta}{2}\right)I_n & -H^*_{jj} \end{bmatrix}$$

is negative definite, where $H^*$ is the maximised Hamiltonian, $i$ represents the state variables, $j$ represents the costate variables of the system, $n$ is the number of equality constraints and $\delta$ is the time rate of preference.

Cass and Shell (1976) show that the Hamiltonian is convex in the costate and concave in the state for problems with a concave objective function and so the matrices $H^*_{ii}$ and $H^*_{jj}$ are negative definite.

Brock and Scheinkman (1976) show that if the matrices $H^*_{ii}$ and $H^*_{jj}$ are negative definite with minimum eigenvalues below zero, the curvature matrix is negative definite with a low rate of time preference.

Sorger (1989) shows that if the curvature matrix $C$ is negative definite then the equilibrium point i.e. the steady state, is globally stable for bounded solutions. This implies that with any initial levels of state variables the optimal path will converge toward a unique steady state.

# SECTION VIII

Dynamic optimisation modelling is used as the analytical tool for the theoretical analysis of sustainable development. Models are formulated that combine the economic and environmental processes from a long term point of view. In this section, the elements of these models that incoporate sustainable development considerations are discussed.

The concept central to sustainable development is the idea that we have a responsibility for the well being of future generations.

'Development is sustainable if it satisfies present needs without compromising the ability of future generations to meet their own needs." (WCED, 1987).

Optimal control modelling can incorporate intergenerational concerns. It can do this through certain elements of the models; the use of social discount rates, social welfare function, time horizon and constraints imposed on variables in the model.

## Discounting

The problems addressed in this thesis are dynamic optimisation problems. These are economic problems in an inter-temporal setting, i.e. where the variables and the objective function are not timeless. They pose the question of what is the optimal magnitude of a choice variable at each point in time in the planning horizon. The solution of these problems will take the form of an optimal time path for each choice variable and will give the optimal value of the variable today, tomorrow and for each point in time in the whole planning period.

Those that may be affected by a decision today may not be involved in the decision making process, i.e. future generations. Thus there is the potential for an intertemporal externality. Today's generation must decide how to allocate, for instance, the natural resources over time based on their considerations of equity.

Discounting is a process by which the costs and benefits of different time periods are compared. A discount rate is applied to future costs and benefits to yield their present value. It can be said that a high discount rate may discriminate against future generations. This is because a high discount rate will favour projects where social costs occur in the long term and social benefits occur in the short term. Costs are therefore shifted to later generations and there are fewer incentives for projects that have a long term pay off. Projects that yield benefits in the long term are less likely to be undertaken with a high discount rate. These are very likely to be environmentally favourable projects, Opschoor (1987). If a high discount rate is applied in natural resource management, renewable resources will tend to be depleted at higher rates as current consumption is favoured. This will lead to a lower level of resource stock or may even lead to exhaustion of the resource before the planning period has ended. It can be argued that there is a case for lowering the discount rate for natural resource management as this will tend to result in a larger stock of natural resources and thereby create more options for current and future generations.

Many environmentalists would prefer a lower discount rate to allow for environmentally sound projects to be undertaken. Pearce (1991) makes the case that if the discount rate is above 1 - 2 %, then global warming considerations are not taken into account seriously enough and that the cost of this would be borne by future generations. Krautkraemer (1988), argued that lower discount rates would worsen environmental degradation because lower discount rates would lower the cost of capital and thus lower the cost of production; so more would be consumed in the short term than if higher discount rates were used.

Some would argue for a zero discount rate to be used so that future generations interest can be protected. But this would also be inequitable. It would discriminate against current generations as it would imply a policy of total current sacrifice and this would be inconsistent with the aim of intergenerational equity. In developing countries there is a great urgency to satisfy immediate wants rather than guaranteeing long term, say food security. Therefore here the rate of time preference is very high.

It can also be argued that a consequence of higher discount rates is a low level of development due to less investment in general and this may benefit environmentally sound projects as there would be less demand for resources and less emissions, Markandya and Pearce (1988). So there seems to be a contradiction here.

Barnett and Morse (1963) argued that:

> 'By devoting itself to improving the lot of the living, therefore, each generation, whether recognising a future orientated obligation to do so or not, transmits a more productive world to those who follow." (Barnett and Morse p 288).

They argue that current generations should use a high rate of discount as future generations will almost certainly be much wealthier. Goodland and Ledec (1987), point out that today's affluence can be attributed to the irreversible consumption and depletion of cheap petroleum and, since the oil price shocks, poorer countries have found it hard to adjust to higher energy prices. Between 1960 and 1982, oil importing countries in sub-Saharan Africa have experienced negative per-capita growth rates, (World Bank (1984)). They therefore argue that it would be prudent for planners to entertain the possibility that because of higher energy prices alone, future generations might live more frugally, rather than become wealthier and have a more lavish lifestyle. They state that if this is a plausible argument, then the equity case for higher discount rates loses its validity. In any case, since the future is uncertain, discounting valuable natural resources at a high rate is not a prudent option in view of the future.

It is not the purpose of this thesis to discuss the ongoing controversy about the appropriate level of the discount rate. However, it can be said that discounting reflects considerations of intergenerational equity. As future generations cannot be here, the current generation must decide on its behalf, therefore we must decide on a fair distribution of costs and benefits between generations. Discounting, then, is a way of incorporating intergenerational equity into optimisation problems.

## Time Horizon

The notion of sustainable development implies a long term planning horizon, although the exact choice of time span is arbitrary. It is argued that if we are to take into consideration the needs of future generations in a meaningful way then the time span should cover at least a period long enough to include the next generation after the current one has disappeared, (van den Bergh (1991)). For sustainable development it is the long run behaviour of economic and ecological systems that is relevant. Individuals may plan for a finite time horizon as even the most farsighted people are likely not to plan very far beyond their expected lifetime. But for society, or even some corporations, it is reasonable to expect or assume that its existence will be permanent. It will therefore be desirable to extend the planning period indefinitely into the future. Dynamic optimisation modelling in an optimal control framework allows for long term planning periods, indeed the time horizon can be finite or infinite. It therefore allows for the consideration of future generations. It is also possible to include the notion of over lapping-generations, (See Fisher (1992)).

## Welfare function

A social welfare function is an ordinal index of society's welfare and is a function that attempts to aggregate the utility functions of all individuals in that society. The form of the welfare function depends on the judgement of the person who is formulating the problem and its arguments depend on what that person deems appropriate to have an effect on human welfare. Therefore welfare functions can include consumption, output, population levels. Pollution stocks and resource stocks can be included then to recognise that the quality of the environment has an effect on the welfare of society. In this way the welfare of society can be maximised with environmental concerns being taken into account. This is essential for the analysis of sustainable development.

## Long Term Uncertainty

Sustainable development implies a long term time horizon and this concerns the issue of uncertainty. Uncertainty surrounds the future behaviour of economic and ecological systems. Uncertainty comes from unforeseen changes that can occur in a system which are caused by exogenous impacts. Timmerman (1986), classifies five patterns of behaviour that a system can take: (1) perfectly stable, (2) resilient, (3) cyclical, (4) switching between multiple stability points, and (5) catastrophic.

If an equilibrium of the system exists, then there is the problem of actually attaining it and whether or not the equilibrium is optimal. With regard to long run uncertainty, it is important to determine the stability of the system. If the time paths of the variables eventually reach equilibrium or steady state, then the underlying dynamic system is stable. A system that is perturbed from a stable steady state will eventually return to it. For an unstable steady state, any perturbation will move the system away from the equilibrium. The steady state is locally stable if the equilibrium is eventually reached with initial values of the variables sufficiently close to the equilibrium point. The steady state is globally stable if the equilibrium is eventually reached for any set of initial values. Global stability implies local stability but local stability does not imply global stability.

Therefore if it can be determined that the system is stable, something can be said about the certainty of the system to reach the optimal equilibrium point. If the initial values places us on a stable path the uncertainty over the future state of the system is reduced.

Long term uncertainty can also be handled with the use of sensitivity analysis to determine how sensitive the model is to parameter values. This will show which parameter values affect the outcome of the model and identify which parameter values to concentrate research on to try and improve their estimates. If the system is highly sensitive to a particular parameter then this will indicate that research could be carried out into the value of this parameter, so that an accurate as possible estimate is obtained.

## Economic-Ecological Interactions.

It is very important to integrate development, the economy and the environment so that insight can be gained into sustainable development. An approach that is frequently used to describe the interactions between the economy and the environment is to use formal models. These models are useful for the study of sustainable development in that they can provide us with the dynamic features of economic-environmental systems and with the long run steady states of such systems.

The impact that the economic productive system has on the environment needs to be considered in terms of resource extraction, waste emissions and disturbing activities. Production can have a positive effect on the environment, for instance environmental protection or abatement activities. The impact that the environment may have on the economy can also be considered, for instance soil quality for agricultural productivity, and also considered here is the material outflow from the environment as resources are extracted and are used as inputs in the production process. It is also possible to consider the negative effects that the environment may have on productivity, for example the quality of the air from pollution activities that may affect workers health. Not only material services or services that can be tagged with a price label can be included in these models, but other services such as amenity services that the environment provides us with, for instance, recreational, aesthetic or scientific . These services can be included in the welfare function.

All these considerations show that it is the dynamic behaviour of the systems that is essential to the study of sustainable development. Dynamic optimisation modelling can allow us to take account of these positive and negative feedbacks from the economy to the environment and vice versa.

Many of the economic activities and the natural processes may be independent of each other. But some processes that yield services to society may result in conflict between the users, for instance the recreational and aesthetic uses of lakes and their commercial exploitation. Thus the multiple use of resources can be dealt with in formal modelling to include the different independent and conflicting uses of economic and environmental resources.

# APPENDIX

## METHODOLOGY DIAGRAMS

Fig 1.

Control path

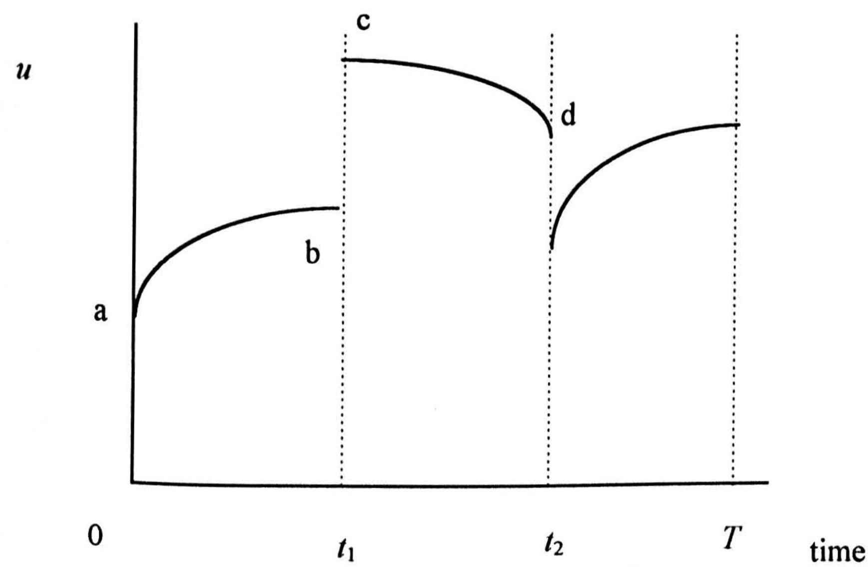
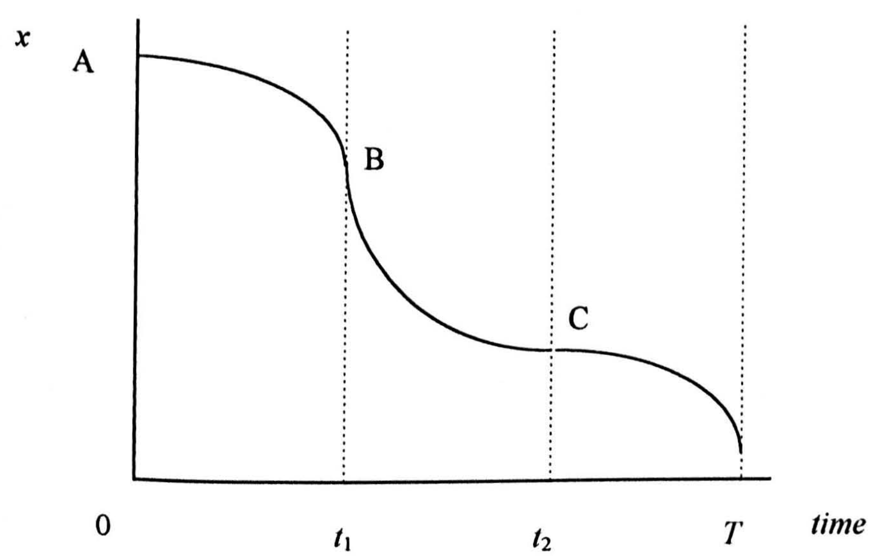


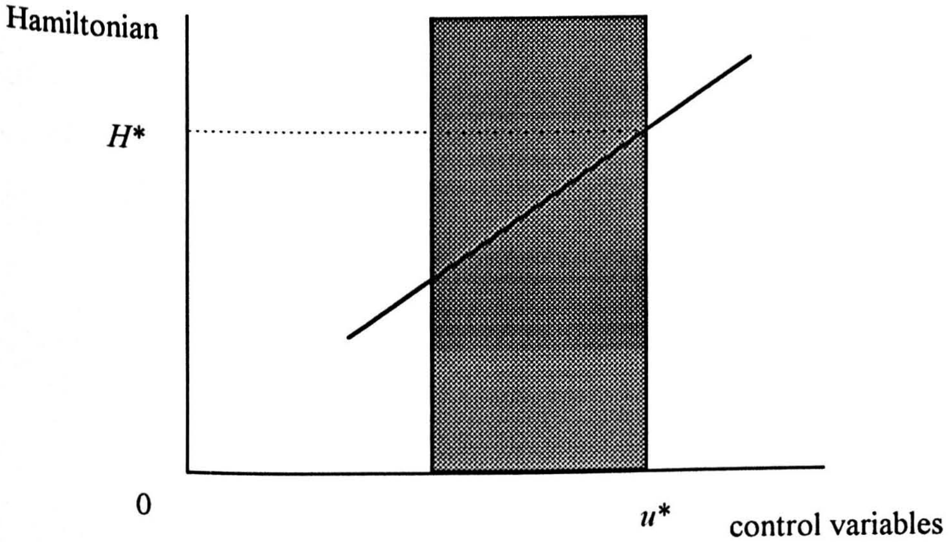

Fig 2

State path



**Upper boundary control**

**Fig 3.**




where the shaded region is the bounded contol set.

**Fig 4.**

**Lower boundary control**

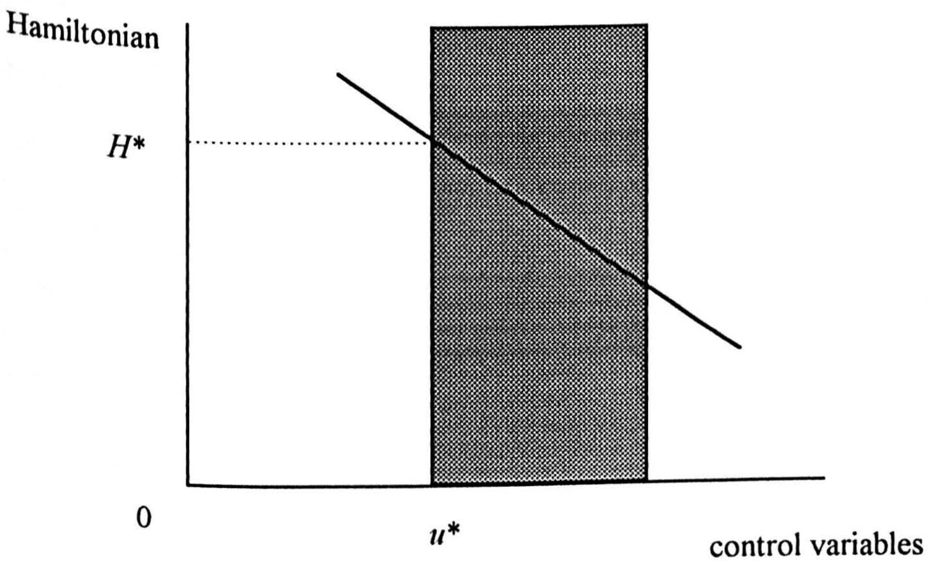


**Fig 5.**

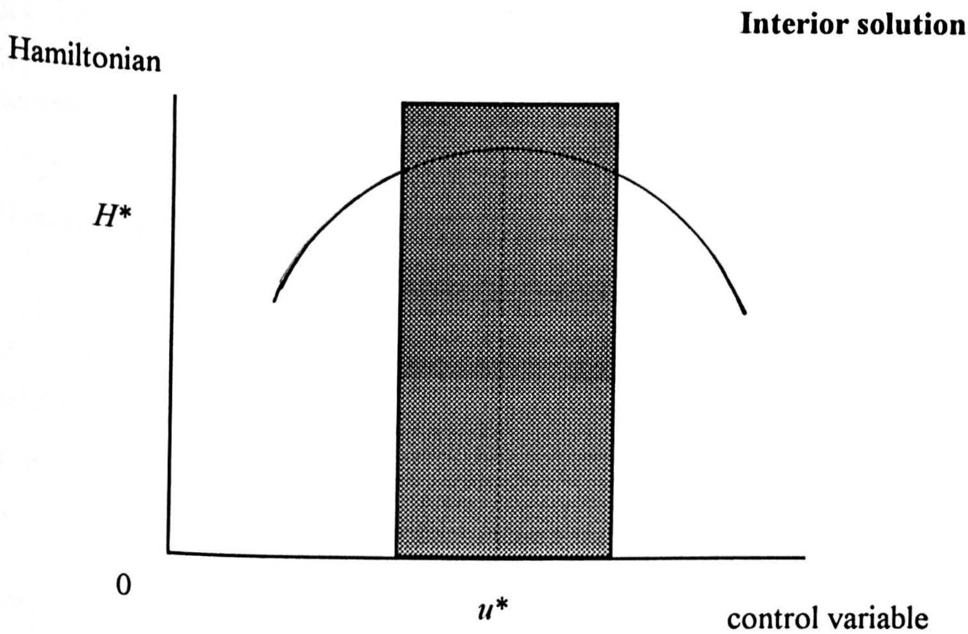
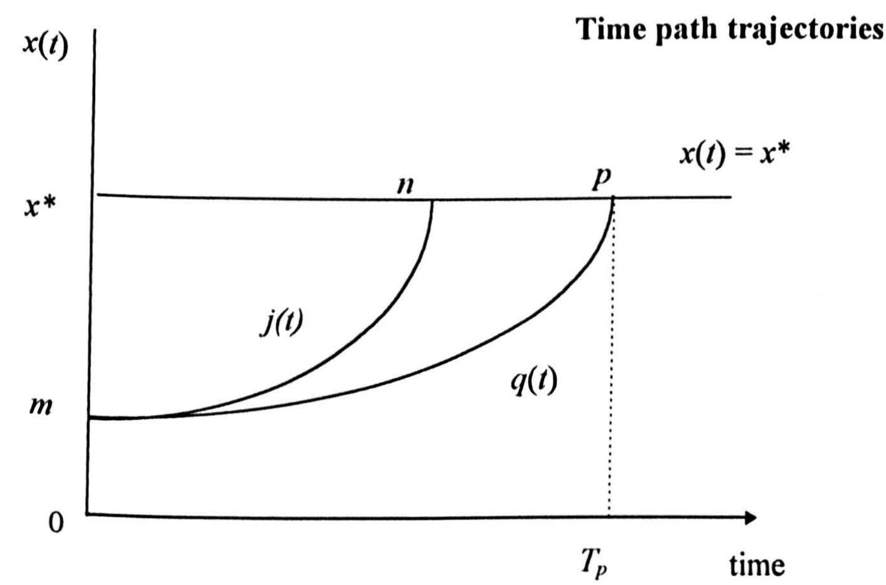

Interior solution
Hamiltonian
H*
0
u*
control variable

**Fig 6.**


Time path trajectories
x(t)
x(t) = x*
x*
n
p
j(t)
q(t)
m
0
T_p
time

# CHAPTER FOUR

# ECONOMICS OF ENVIRONMENTAL PRESERVATION

## Preservation versus Development

Society has a desire for both the preservation and development of natural environments. The natural environment is valued for both the natural resources and the environmental amenities it provides - recreational, educational, scientific, aesthetic as well as regulating the climate and the global atmosphere. The need for preserving our environment is becoming an increasingly more important issue.

However, not only is it necessary to preserve our natural environment but there is also the need to develop land for production and for living. The environment also provides us with the natural resources we require for these purposes. Land does need to be developed, we need to be able to produce commodities, develop land for agriculture and for housing etc., we need all this to live. Developing land however renders these resources more scarce and so puts our future prosperity at risk.

Preservation and development of our natural environment are in direct competition with each other. One of the best examples that illustrates this is the cutting down of the Brazilian tropical rain forests to make way for agricultural production and with the world population growing, there is increasing competition between the two uses. Destroying natural capital invariably means that the environmental losses that occur are irreversible. Fisher and Krutilla (1985), point out that by irreversible conversion of natural areas the future possibilities of obtaining environmental services from these areas are lost. One may argue that technology of the future may make it possible to restore the area back to its original state. It may be possible to clean up pollution that has caused damage to the environment that so far has been viewed as permanent. But what if that pollution has destroyed the natural habitat of a particular plant or animal which subsequently becomes extinct. Once a species is lost it is gone forever; no amount of future technology will give it back to us. The tropical rain forests harbour a diversity of life. Pulling down a forest and using the land for development will destroy the natural habitat of thousands of species. The land will also become fully degraded, soil fertility will be lost and other essential services provided by the forest ecosystem such as watershed protection and regulation will be destroyed. Deforestation will also have serious impacts on global, regional and local climates. Also other areas of land such as desertified land are very difficult to reclaim once development has been undertaken. It must be realised that it is necessary to achieve a balance between preservation and development - only then will it be possible to sustain human life.

## Literature on land use models

Irreversible investments have received a great amount of attention in the environmental economics literature. Fisher, Krutilla and Cicchetti (1972), (hereafter FKC), formulated a model for the allocation of natural environments between, preservation and development. They make the assumption that conversion of the natural area for development purposes represents an irreversible development. Their examples of irreversible development include:

> "transformation and loss of whole environments as could result from clear cutting a redwood forest, or developing a hydroelectric project in the Grand Canyon." (p 605).

Cummings and Norton (1974), (hereafter C and N), criticise FKC for assuming that development of natural environments represents an irreversible investment. They state that it may be impossible to restore a natural area, such as a flooded canyon, to its exact original state and that it is not necessarily desirable to do so. They argue that such exactness is not necessarily a prerequisite for the future generation of environmental benefits from that area and that an area can be restored to some kind of natural environment in the future - even a flooded canyon. C and N agree that investment can be irreversible but that is unlikely, and so irreversible investment should be treated as a special case. The implications from FKC arguments is that their development begins with irreversible investment and they state that:

> "were the transformations reversible, much of the conflict between preservation and development would vanish." (p607).

FKC make it clear from their paper that it is irreversible development that is the issue - so why the criticism from C and N?

C and N recognise that irreversible investment is a special case but also that the cost of reverting back to its original state may be so large relative to the benefits of restoration that investment for development is economically irreversible. C and N formulate a model that incorporates this view and show that FKC's model is a special case of the following optimisation problem:

$$\max \sum_{t=1}^{T} \left[ B_1^t(P^t) + B_2^t(D^t) - I^t - G^t \right] \beta^t \qquad (1)$$

subject to:

$$\begin{aligned} D^{t+1} &= D^t + \sigma I^t - \gamma G^t \\ P^{t+1} &= P^t + \gamma G^t - \sigma I^t \\ P^t + D^t &= L \end{aligned} \qquad (2)$$

$B_1{}^t$ and $B_2{}^t$ are the net benefits at time $t$, from preservation, $P^t$, and development, $D^t$, respectively. $I^t$ represents the investment in developing the land. All correspond to FKC's model. The difference is that $G^t$ is investment in preserving the land, i.e. $G^t$ allows for reversibility, by converting the land back into its original state. The first equation in (2) states that development in period $t$ plus investment in development less the investment in preservation gives the amount of land developed in the next period. The second equation in (2) is a transition equation for the preserved land. It states that preservation in period $t$ plus investment in preservation less investment in development gives the amount of land preserved in the next period. The third equation shows that there is a fixed amount of land and that it can either be developed of preserved. Their model uses the same assumption as FKC in that the benefits of the alternative uses of the land, i.e. preservation and development, change with time in a known way. They view the problem in a much broader perspective and it is a valid extension of the literature. However the main concern in this chapter is irreversible development and it is wrong to criticise FKC for concentrating on such an issue.

Miller *et al* (1981) uses the irreversibility concept but applies it to the preservation of endangered species. The same concept applies in that the problem is the allocation of land between preservation and development, except the area of land is home to a particular endangered species. Therefore a model is developed which analyses the problem of the allocation of land between the production of economic goods (development), and the preservation of the species (preservation). Conversion of the land is assumed to be irreversible - once the land has been developed for the production of goods it is not possible to use the habitat in the future, i.e. the species will become extinct. For example a lake or some body of water may be used for irrigation purposes that results in the extinction of some aquatic species. The arguments of an individual's utility function are the aggregate of economic goods and services, $Q$ and the stock of a species of wildlife, $X$.

$$U^i = U^i(Q_1^i, \ldots, Q_T^i; X_1 \ldots X_T)$$

$X$ is not used in the production of output, but production requires the use of land, $L$, and non-land resources, $R$.

The welfare function for society defined over the utility of $S$ individuals is:

$$V = V[U^1, \ldots, U^s]$$

The problem posed is to maximise the welfare function over a finite time horizon $t = T$ subject to a species growth constraint, constraints on land and non-land resources, the irreversibility condition, an adding up constraint in produced goods, and the production constraint for output. These are respectively:

$$\begin{aligned}
&X_t = G(X_{t-1}, R_t^x, L_t^x) \\
&\overline{L} = L_t^Q + L_t^X \\
&\overline{R_t} = R_t^Q + R_t^X \\
&L_t^X \leq L_{t-1}^X \\
&\sum Q_t^i = Q_t \\
&Q_t = Q(R_t^Q, L_t^Q)
\end{aligned}$$

where:

$Q_t$ = goods produced in time $t$

$X_t$ = species stock in time $t$

$X_{t-1}$ = species stock in $(t - 1)$

$R_t^Q$ = non-land resources used to produce $Q$ in time $t$

$L_t^Q$ = land used to produce $Q$ in time $t$

$R_t^X$ = nonland resources used to produce $X$ in time $t$

$L_t^X$ = land used to produce $X$ in time $t$

Miller's results are comparable to FKC's when applied to an endangered species, in that there is less conversion of the species habitat than would occur in the absence of irreversibility.

Arrow and Fisher (1974) focus on irreversible development, but they ask the question - does the introduction of uncertainty as to the benefits and costs of the proposed development affect the investment decision? They consider a two period model consisting of the first period, the present, followed by all future intervals being compressed into the second period. The second period expectations are conditional on what happens in the first period. Some amount of development is planned at the start of the first period. Information about benefits in the first period accumulate and plans can be revised at the start of the second period, (but only in the direction of more investment). Results show that the natural area is less likely to be developed under these uncertainties.

Henry (1974) makes it clear how the prospect of receiving more information in the future will affect the decisions taken in the initial period. He also incorporates an information structure into his investment decision and shows that a decision maker is led to adopt an irreversible decision more often than he should. Jones and Olstroy (1984) incorporated an information structure that incorporates the amount of learning in the future about future values.

Haspel and Johnson (1982) argue there is a limited number of cases that satisfy the irreversibility condition and argue that "many areas do have physical substitutes and are as a result not unique". They give some examples:

> 'There are some designated wilderness areas in the western United States which are located in close proximity to other designated wilderness areas. The terrain, flora, fauna and climate are extremely similar if not identical. That is the same plant or pond cannot exist in each area", but 'the same species of plant and a similar pond may, making the areas extremely alike". (p. 80).

They treat irreversibility as a special case and develop a model that considers reversibility through man-made technology and natural processes, but goes further than C and N and incorporates the substitution in the demand for natural resources. They show that the availability of substitutes and the reversibility of an area reduces the social cost of using the resource and so there will be an increased level of development. They argue that if substitution is ignored by making the assumption that the asset is unique, then the potential benefits from the preservation of the asset may be overestimated. But the example they give above seems to be a very rare case and maybe this is the special case, not the irreversible situation.

Usategui, (1990) analyses the problem of the allocation of land between preservation and development in a 2 period model with uncertainty about future benefits of those uses, about the irreversibility of development and about the costs of changing the allocation of resources. The decision maker receives information at the end of the first period which completely solves these uncertainties and therefore the decision maker can act accordingly.

Some of the later literature mentioned above, that incorporates uncertainty about irreversibility of investment, may be a feasible extension to the existing literature but in many cases, as argued previously, environmental losses that occur are irreversible; there is no uncertainty. Also some of the above literature incorporates uncertainty about the future benefits of preservation and development. It can be argued that we know how these benefits change over time. If future generations place a higher value on environmental amenities than the current generation does, then it would be desirable to develop less of our irreplaceable resources. It is very likely that there will be an upward movement in the relative price of environmental amenities. Society's wealth is increasing over time and so one would expect increased future demand for environmental quality. Therefore there will be a positive income elasticity of demand. Also, the fact that unique environmental areas are becoming increasingly scarce contributes to this upward shift in values. Thus we can argue that the benefits of alternative uses of land, preservation and development change over time in a known way. For these reasons this chapter concentrates on FKC's paper (1972), where irreversibility of investment decisions is a constraint on the model and changes in the benefits of preservation and development are known.

FKC formulate a model for the optimal allocation of land between preservation and development. They find that development should stop short of the level indicated by current valuations whenever future values indicate that reduced development is desirable. Their analysis presents a strong case for the permanent preservation of some natural environments whose current return to preservation is less than the current return to development, i.e. inter temporal optimisation is required. Preservation, therefore, should possibly be greater than current values would indicate.

Krautkraemer (1985), shows that even if the value of the amenity services rise relative to the value of commodities, then it may be optimal to completely exhaust the resource stock. He argues that technical progress will increase the productivity of the resource, and the productive value of the resource may be rising even though the value of commodities may be falling. Therefore it may be optimal to fully develop natural environments where a productive resource is found.

In Krautkraemer's model the economy seeks to maximise the present value of utility which is a function of consumption and the flow of resource amenities, which in turn is a function of the level of resource stock. Therefore the problem is to :

$$\max \int_0^{\infty} U[C(t), S(t)] e^{-\delta t}$$

subject to:

$$\dot{S}(t) = -R(t)$$

with initial conditions:

$$S(0) = S_0$$
$$R(t) \geq 0$$
$$S(t) \geq 0$$

where $C$ is the level of consumption, $R$ is the level of resource extraction and $S$ is the level of resource stock all at time $t$. The rate of technical progress is given by $\tau$.

He proves that it may be optimal to extract the whole resource stock. However, he does not include the fact that that there are benefits to the stock of developed land.

Barrett (1992), presents a more general model of optimal economic growth and environmental preservation and shows that the results of FKC and Krautkraemer are special cases. He shows that Krautkraemer's sombe result breaks down if developing the environment produces a form of capital from which it is possible to obtain future consumption, and also if technical progress in resource extraction is accompanied by technical progress in the developed sector. Barrett's paper adds weight to the arguement that it is optimal to permanently preserve some natural areas and protect our environment.

In his model, the problem is to:

$$\max \int_0^\infty U[C(t), S(t)]e^{-\delta t}\,dt$$

subject to:

$$\dot{S}(t) = -r(t)$$
$$C = \sigma e^{\gamma t} + f(S(t))e^{\omega t}$$

where $C$ is consumption, $S$ is the resource stock, $\gamma$ and $\omega$ are the rates of technical progress in the extraction and development sectors respectively. $\sigma$ is the constant that changes the rate of extraction of the resource in the initial period into the consumption rate. The production function in the development sector is $F(D(t))$. But in the above problem this is expressed in the original state variable, $S(t)$. It is assumed that the amount of resources is fixed, therefore:

$$D(t) - D_0 = S_0 - S(t)$$

And since $S_0$ and $D_0$ are fixed,

$$F(D(t)) = F(S_0 + D_0 - S(t)) = f(S(t))$$

hence the presence of $f(S(t))$ in the consumption function. Therefore, here, account is being taken of the benefits from development. However Barrett argues that he has shown that FKC's results hold under more general conditions. But, FKC do not fully develop the solution. The conclusion they reach is derived from an incomplete analysis of the model. It is argued here that the solution derived in this chapter is the full correct solution and that from this different results are obtained to that of FKC.

FKC begin their paper with a general model for the allocation of land between preservation and development. Their model is described in Section I . By applying Pontryagin's Maximum Principle, the investment path is chosen so as to maximise the discounted utility subject to the constraint that investment is irreversible ( i.e. $I \geq 0$ ). In their paper, FKC (1972), they show that the optimal development path for a given area of land is given by a sequence of investment intervals. They define periods of investment as free intervals, and periods of no investment as blocked intervals.

But why should investment in developing the land be undertaken in stages. If there is an optimal level of development, surely it would be ideal to proceed with investment as soon as possible and achieve that optimal state.

In section II of their paper they present a different result for the proposed development of a hydroelectric power project in the Grand Canyon. They argue that the benefits from developing the hydro project (i.e. the difference in costs between the most economic alternative source of energy and the hydro project), are decreasing over time because the costs of the best alternative source of energy decrease as new technologies are brought in. They also argue that the benefits from preservation will be increasing over time.

They show that if the benefits from development are decreasing over time relative to the benefits from preservation then the optimal level of development decreases. If $D^*$, the optimal level of development, is monotone decreasing, then there is an infinite blocked interval where investment is zero and the level of development remains constant. They argue that development should be frozen at the initial level or jump to the optimal level $D^*$ at time $t = 0$ and remain there.

But development of a given area of land cannot be completed at $t = 0$, this is impossible. It takes time to construct buildings, build roads, erect factories, it cannot be done instantaneously.

Section II will extend their paper by showing that there is a singular arc solution to the optimal control problem. It will be shown that it is optimal to reach the singular arc in the shortest amount of time possible and then the level of development should be frozen. This means that all investment is concentrated at the beginning of the plan. This is in direct contrast to both the conclusions reached by FKC and is arguably a far more realistic result.

In Section III, specific functional forms are used for the benefit functions of preservation and development. This enables us to fully characterise the solution to the problem.

In Section IV a different dynamic constraint is applied to the model to allow for decreasing returns to investment. This produces a different optimal transition path to the steady state. Section V offers some conclusions.

# SECTION I

## The Model

Suppose there is an area of land which can be divided between two uses - preservation and development. Allocation of the latter is to the highest valued use and the same assumption applies to preservation - optimal use is for recreation. The objective, then, is to develop a model to help us choose the optimum division between these two uses.

The problem is to choose a path for investment, (the cost of transforming preserved areas into developed areas), so as to maximise discounted net social benefits over the whole future, from the quantities of the preserved area and the developed area.

If $L$ is the fixed amount of land then:

$$P + D = L \tag{1}$$

where $P$ is the amount of preserved land and $D$ is the amount of developed land.

The problem then is to maximise:

$$\int_0^{\infty} e^{-\rho t} \; [B^P(P(t),t) + B^D(D(t),t) - I(t)].dt \tag{2}$$

subject to :

$$\dot{D}(t) = \sigma I(t) \tag{3}$$

where $B^P$ is the net social benefit from preserved land, $B^D$ is the net social benefit from developed land, $I$ is total investment and $\sigma$ is a positive constant. The first two terms of the criterion functional (2) show the discounted flow of expected net social benefits (from $P$ and $D$). The third term shows the total investment in developing the area. Equation (3) indicates that the change in the area of developed land is proportional to total investment. This implies constant returns to investment at any point in time.

We assume that returns to increasing preservation and development are positive but diminishing i.e. concave benefit functions:-

$$B^P{}_P \ , B^D{}_D > 0 \quad \text{and} \quad B^P{}_{PP} \ , B^D{}_{DD} < 0$$

The problem posed is an optimal control problem solved by using Pontryagin's ... Maximum Principle [1962]. The present value Hamiltonian is:

$$\mathrm{H} = e^{-\rho t}\left[B^P(P,t) + B^D(D,t) - I(t)\right] + p(t)\sigma I(t) \tag{4}$$

where $p(t)$ is the costate variable, measuring the value of future benefits of development. $I$ is the control variable for which we need to find the optimal path and $D$ is the state variable. Rearranging (4) gives:

$$\mathrm{H} = e^{-\rho t}\left[B^P(P,t) + B^D(D,t)\right] + I(t)[\sigma p(t) - e^{-\rho t}] \tag{5}$$

Let $$q(t) = \sigma p(t) - e^{-\rho t} \tag{6}$$

Then (5) simplifies to:

$$\mathrm{H} = e^{-\rho t}\left[B^P(P,t) + B^D(D,t)\right] + q(t)I(t) \tag{7}$$

Given that $L = P + D$

(i.e. $P = L - D$)

Then (7) becomes:

$$\mathrm{H} = e^{-\rho t}\left[B^P(L-D,t) + B^D(D,t)\right] + q(t)I(t) \tag{8}$$

From (6) $$p(t) = \frac{[\,q(t) + e^{-\rho t}\,]}{\sigma}$$

Therefore $$\dot{p}(t) = \frac{\dot{q}(t) - \rho e^{-\rho t}}{\sigma} \tag{9}$$

The Hamiltonian is linear in the control variable and, for an unbounded control set, the control can take any value. Under these circumstances the natural solution to look for is a singular arc solution. To make economic sense, $I$ must be bounded from below (i.e. $I(t) \geq 0$). In what follows we will derive the singular arc solution in cases when the control set is bounded from above and when it is unbounded from above.

# SECTION II

## The Unbounded Case

For a singular arc solution the necessary conditions which must be satisfied are:

$$\frac{\partial H}{\partial I} = q = 0 \tag{10}$$

$$\dot{p} = -\frac{\partial H}{\partial D} = -e^{-\rho t}\left[-\frac{dB^P}{dP} + \frac{dB^D}{dD}\right] \tag{11}$$

Substituting (11) into (9) gives:

$$-e^{-\rho t}\left[-\frac{dB^P}{dP} + \frac{dB^D}{dD}\right] = \frac{\dot{q}(t) - \rho e^{-\rho t}}{\sigma} \tag{12}$$

Rearranging gives:

$$\dot{q} = e^{-\rho t}\left[\rho + \sigma\frac{dB^P}{dP} - \sigma\frac{dB^D}{dD}\right] \tag{13}$$

It follows that along the singular arc:

$$\frac{d}{dt}\left[\frac{\partial H}{\partial I}\right] = 0$$

i.e. $\dot{q} = 0$

From (13)

$$e^{-\rho t}\left[\rho + \sigma\frac{dB^P}{dP}((L-D),t) - \sigma\frac{dB^D}{dD}(D,t)\right] = 0 \qquad (14)$$

In the same way it follows that:

$$\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right] = 0$$

Therefore:

$$\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right] = -\rho e^{-\rho t}\left[\rho + \sigma\frac{dB^P}{dP}((L-D),t) - \sigma\frac{dB^D}{dD}(D,t)\right] +$$

$$e^{-\rho t}\left[-\sigma\frac{d^2B^P}{dP^2}\dot{D} - \sigma\frac{d^2B^D}{dD^2}\dot{D}\right] = 0$$

Substituting (3) into the above equation gives:

$$\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right] = -\rho e^{-\rho t}\left[\rho + \sigma\frac{dB^P}{dP} - \sigma\frac{dB^D}{dD}\right] +$$

$$e^{-\rho t}\sigma^2\left[-\frac{d^2B^P}{dP^2} - \frac{d^2B^D}{dD^2}\right]I = 0 \qquad (15)$$

The equations (10), (14) and (15) are a set of 3 algebraic equations in 3 unknowns. Solving for $q$, $D$ and $I$ along the singular arc:

$$q^* = 0 \tag{16}$$

$$-\frac{dB^P}{dP}(L - D^*) + \frac{dB^D}{dD}(D^*) = \frac{\rho}{\sigma} \tag{17}$$

Substituting (17) into (15) gives:

$$I^* = 0 \tag{18}$$

This result corresponds to that of FKC. Equation (17) indicates that at the optimal level of investment in development the marginal benefits of development $\frac{dB^D}{dD}$ equals the sum of the marginal costs of development $\frac{dB^P}{dP}$( this is equivalent to the marginal benefits of preservation because the opportunity cost of developing the land is the benefits lost from preservation), and the direct opportunity costs $\frac{\rho}{\sigma}$.

For the singular arc solution to give a maximum there is an additional necessary condition, the Generalised Legendre Clebsch condition (henceforth GLC ), Lewis (1980). This condition is stated as :

$$(-1)^k\left(\frac{\partial}{\partial U}\right)\left[\left(\frac{d^{2k}}{dt^{2k}}\right)\left(\frac{\partial H}{\partial U}\right)\right] < 0 \tag{19}$$

where $2k$ is the order of the first time derivative of $\frac{\partial H}{\partial U}$ that explicitly depends on $U$. Therefore for this problem $k = 1$ and $U = I$;

Using (15)

$$\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right] = -\rho e^{-\rho t}\left[\rho + \sigma\frac{dB^P}{dP} - \sigma\frac{dB^D}{dD}\right] +$$

$$e^{-\rho t}\sigma^2\left[-\frac{d^2B^P}{dP^2}-\frac{d^2B^D}{dD^2}\right]I = 0 \qquad (15)$$

and differentiating with respect to I gives;

$$\frac{\partial}{\partial I}\left[\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right]\right] = e^{-\rho t}\sigma^2\left[-\frac{d^2B^P}{dP^2}-\frac{d^2B^D}{dD^2}\right] > 0$$

Therefore:

$$(-1)\left(\frac{\partial}{\partial I}\right)\left[\left(\frac{d^2}{dt^2}\right)\left(\frac{\partial H}{\partial I}\right)\right] < 0$$

Therefore the GLC condition is satisfied and the singular arc solution is optimal.

Next it is necessary to establish that it is optimal to reach $D^*$ in the minimum time, if $D_0 \neq D^*$. This may be established by writing the objective functional as a line integral and then applying Green's Theorem (Kamien and Schwarz (1991)).

Solving (3) for $I$ and substituting into (2) gives:

$$\int_0^{\infty} e^{-\rho t}[B^P(L-D,t)+B^D(D,t)-\frac{\dot{D}}{\sigma}]\,dt \qquad (20)$$

Writing (20) as a line integral:

$$\int_0^{\infty}[\,F(D,t) \;+\; G(D,t)\,\dot{D}\,]\,dt = \int_{\xi} FdD + Gdt$$

Where $F(D,t) = \left[B^P(L-D,t)+B^D(D,t)\right]e^{-\rho t}$ ; $G(D,t) = \left[\frac{-1}{\sigma}\right]e^{-\rho t}$ and $\xi$ is the curve $D = D(t)$ , $t \geq 0$ (as shown in fig 1):

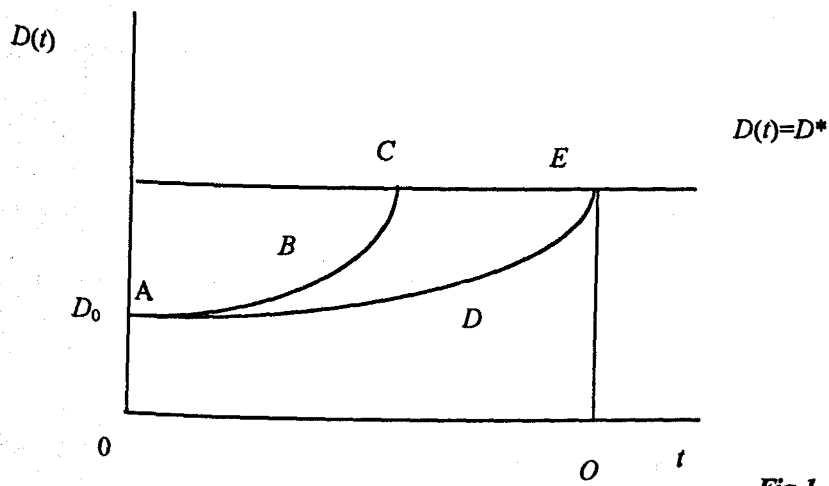



*Fig 1.*

Suppose that $D_0 < D^*$ then the optimal initial control solution is a shift from $D_0$ to $D^*$ as soon as possible. Let ABC be the path that reaches $D^*$ as quickly as possible and let some other feasible path be ADE which takes longer to reach $D^*$. Let $T_S$ be the time it takes for ADE to reach $D^*$. Then both paths coincide for $t > T_S$.

We need to show that :

$$\int_{ABC} F\,dD + G dt - \int_{ADE} F\,dD + G\,dt \geq 0 \qquad (21)$$

i.e.

$$\int_{ABC} F\,dD + G\,dt \geq \int_{ADE} F\,dD + G\,dt \qquad (22)$$

Applying Green's Theorem:

$$\oint_{ECBADE} FdD + Gdt = \iint_R [\frac{dG}{\mathrm{dt}} - \frac{dF}{dD}]dDdt \tag{23}$$

where $R$ is the bounded region $ECBADE$, $\frac{dF}{dD} = e^{-\rho t}\left[-\frac{dB^P}{dP} + \frac{dB^D}{dD}\right]$ and $\frac{dG}{dt} = \left[\frac{\rho}{\sigma}\right]e^{-\rho t}$

Therefore:

$$\iint_R [\frac{\rho}{\sigma} + \frac{\mathrm{dB^P}}{\mathrm{dP}} - \frac{\mathrm{dB^D}}{\mathrm{dD}}]\mathrm{e}^{-\rho t} \geq 0 \tag{24}$$

Since $\frac{dB^P}{dP} - \frac{dB^D}{dD} = \frac{\rho}{\sigma}$ (equation (17)) for $D < D^*$.

Therefore it is optimal to reach $D^*$ in the minimum amount of time. This is in direct contrast to the result of FKC. Here the optimal growth path of developed land over time takes the form of alternating sequences of rising segments and plateaus. Over a free interval where $q(\mathrm{t}) = 0$, $D^*(t)$ is rising. Since $t_0$ is the end of a free interval, $I = 0$ and $q(t_0) < 0$. FKC find that $D(t)$ is constant over the blocked interval $(t_0, t_1)$.

This is illustrated below in fig 2.

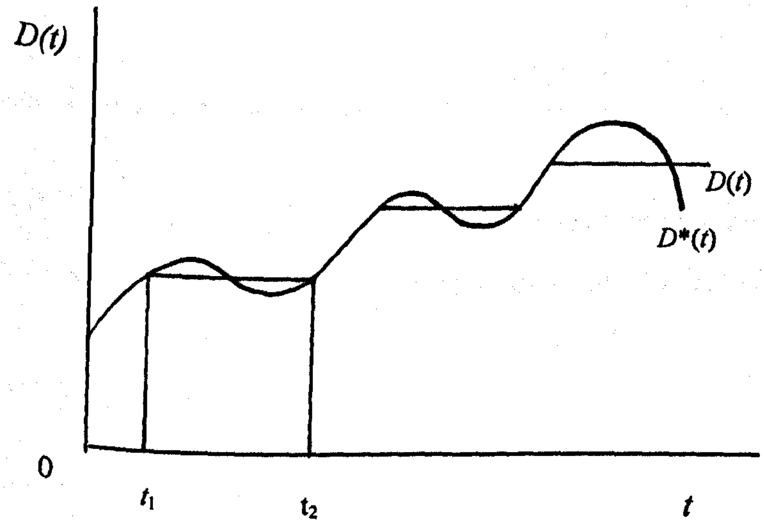



*fig 2.*

Over the full interval ($t_0$, $t_1$) the sum of discounted marginal benefits of development equal the sum of discounted marginal costs. However within this interval FKC specify a myopic or short-sighted path ($D^*(t)$) for development. At some point ($t_0 \leq t \leq t_1$) optimal development begins to fall implying that marginal benefits are less than marginal costs. There has been too much development and we need to disinvest or reverse previous development. But if development of natural environments is irreversible, FKC find that development then should stop short of the level that is indicated by current valuations if in the near future reduced development is desirable. Investment in development should then cease until another free interval ($t_1$). This gives the corrected path for development ($D(t)$). Then the optimal growth path for development would be an alternating sequence of periods of investment in development and periods of no investment where $D(t)$ is constant over the blocked interval.

## The Bounded Case

Looking at the case where the level of investment is bounded , i.e. $I \leq k$:

First of all looking at when $I = k$; the solution of equation (3) is

$$D(t) = D_0 + \sigma k t$$

When the level of development is at the optimum then:

$$D(T) = D_0 + \sigma k T = D^*$$

Thus $D^*$ will be reached at some time $T$ i.e. $t = \dfrac{D^* - D_0}{\sigma k}$

Therefore the optimal control is that the level of investment between $t = 0$ and $t = T$ is $k$, i.e. $I = k$ $t \in [0, T]$. For $t$ greater than $T$ investment is zero as the optimum level has been reached and no more investment is needed.

# SECTION III

## Functional Forms

To throw more light on the type of solution we may expect to get, we now turn to a specific example.

We assume the functional forms of the benefit functions at time $t$ for Preservation, $P$, and Development, $D$, are given by:

$$B^P(P) = C_1 \ln P \tag{1}$$

$$B^D(D) = C_2 \ln D \tag{2}$$

Where $C_1$ and $C_2$ are positive constants

Equations (1) and (2) show us the expected net social benefits to $P$ and $D$ respectfully. The conditions necessary for concavity are satisfied with these functional forms.

$$\frac{dB^P}{dP} = \frac{C_1}{P} > 0 \qquad \text{and} \qquad \frac{d^2B^P}{dP^2} = -\frac{C_1}{P^2} < 0$$

$$\frac{dB^D}{dD} = \frac{C2}{D} > 0 \qquad \text{and} \qquad \frac{d^2B^D}{dD^2} = -\frac{C_2}{D^2} < 0$$

The problem then is to maximise:

$$\int_0^{\infty} e^{-rt}\left[ C_1 \ln (L - D) + C_2 \ln D - I \right] dt \tag{3}$$

Subject to :

$$\dot{D} = \sigma I \qquad\qquad D(0) = D_0 > 0 \tag{4}$$

The Hamiltonian is:

$$\mathrm{H} = e^{-\rho t}\left[C_1 \ln(L-D) + C_2 \ln(D) - I\right] + \rho\sigma I \tag{5}$$

Rearranging gives:

$$\mathrm{H} = e^{-\rho t}\left[C_1 \ln(L-D) + C_2 \ln(D)\right] + I\left[\rho\sigma - e^{-\rho t}\right]$$

Again letting;

$$q = p\sigma - e^{-\rho t} \qquad \text{[ equation (6) in Section I ]}$$

The Hamiltonian becomes:

$$\mathrm{H} = e^{-\rho t}\left[C_1 \ln(L-D) + C_2 \ln(D)\right] + qI \tag{6}$$

The necessary conditions for an interior solution are:

$$\frac{\partial H}{\partial I} = q = 0 \tag{7}$$

$$\dot{p} = -\frac{\partial H}{\partial D} = -e^{-\rho t}\left[-\frac{C_1}{(L-D)} + \frac{C_2}{D}\right] \tag{8}$$

Using equation (9) from page 3;

$$\dot{p} = \left[\frac{\dot{q} - \rho e^{-\rho t}}{\sigma}\right]$$

and substituting into (8) gives;

$$-e^{-\rho t}\left[-\frac{C_1}{(L-D)} + \frac{C_2}{D}\right] = \left[\frac{\dot{q} - \rho e^{-\rho t}}{\sigma}\right]$$

Rearranging gives:

$$\dot{q} = -e^{-\rho t}\left[-\sigma\frac{C_1}{(L-D)} + \sigma\frac{C_2}{D}\right] + \rho e^{-\rho t}$$

Therefore:

$$\dot{q} = e^{-\rho t}\left[\rho + \sigma\frac{C_1}{(L-D)} - \sigma\frac{C_2}{D}\right] \tag{9}$$

It follows that along the singular arc :

$$\frac{d}{dt}\left[\frac{\partial H}{\partial I}\right] = 0 \tag{10}$$

$$\text{i.e. } \dot{q} = 0 \tag{11}$$

Therefore:

$$e^{-\rho t}\left[\rho + \sigma\frac{C_1}{(L-D)} - \sigma\frac{C_2}{D}\right] = 0 \tag{12}$$

It also follows that:

$$\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right] = e^{-\rho t}\left[\sigma\frac{C_1\dot{D}}{(L-D)^2} + \sigma\frac{C_2\dot{D}}{D^2}\right] - \rho e^{-\rho t}\left[\rho + \sigma\frac{C_1}{(L-D)} - \sigma\frac{C_2}{D}\right] = 0 \tag{13}$$

substituting in $\dot{D} = \sigma I$ gives:

$$e^{-\rho t}\left[\sigma^2\frac{C_1}{(L-D)^2} + \sigma^2\frac{C_2}{D^2}\right]I - \rho e^{-\rho t}\left[\rho + \sigma\frac{C_1}{(L-D)} - \sigma\frac{C_2}{D}\right] = 0 \tag{14}$$

To see if this solution is a maximum we must again look at the GLC condition (equation (19) in Section II).

Here $k = 1$

$$\frac{\partial}{\partial I}\left[\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right]\right] = \sigma^2 e^{-\rho t}\left[\frac{C_1}{(L-D)^2}+\frac{C_2}{D^2}\right] > 0$$

given the concavity assumptions of the functional forms.

Therefore:

$$(-1)\frac{\partial}{\partial I}\left[\frac{d^2}{dt^2}\left[\frac{\partial H}{\partial I}\right]\right] = -\sigma^2 e^{-\rho t}\left[\frac{C_1}{(L-D)^2}+\frac{C_2}{D^2}\right] < 0$$

The GLC condition is satisfied and the singular arc solution is optimal.

To find the steady state solution for $q$, $D$ and $I$ (denoted by $q^*$, $D^*$ and $I^*$) we need to solve equations (7), (11) and (14) which are a set of 3 algebraic equations in 3 unknowns. Solving for $q$, $D$ and $I$ along the singular arc:

$$q^* = 0 \tag{16}$$

$$I^* = 0 \tag{17}$$

$$\rho + \sigma\frac{C_1}{(L-D^*)} - \sigma\frac{C_2}{D^*} = 0 \tag{18}$$

Rearranging 18 gives:

$$-\frac{C_1}{(L-D^*)}+\frac{C_2}{D^*} = \frac{\rho}{\sigma} \quad \text{(equivalent to equation 17 in Section II)}$$

This shows that the optimal investment policy equates marginal benefits from development to the sum of the direct and marginal opportunity costs of development at any point in time.

Rearranging again gives a quadratic which can be solved to find $D^*$

i.e.

$$\rho D^{*2} - [\rho L + \sigma(C_1 + C_2)]D^* + \sigma C_2 L = 0$$

Solving the quadratic for the equilibrium point, the steady state of the system gives (see appendix):

$$D^* = D_2 = \left[\rho L + \sigma(C_1 + C_2) - \sqrt{[\rho L - \sigma C_2)]^2 + C_1^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)}\right] / 2\rho$$

$$I^* = 0$$

and $0 < D^* < L$ for $\rho > 0$

If $\rho = 0$ then: $$D^* = \frac{C_2 L}{(C_1 + C_2)}$$

Which also satisfies $0 < D^* < L$

Again using line integrals and applying Green's Theorem we can establish that it is optimal to reach $D^*$ in the minimum amount of time.

Writing (4) as a line integral:

$$\int_x F\,dD + G dt$$

Where $F(D,t) = [\ C_1 \ln (L - D) + C_2 \ln D\ ]\ e^{-\rho t}$, $G(t) = [\ -1/\sigma\ ]\ e^{-\rho t}$ and $\xi$ is the curve $D = D(t)$, $t \geq 0$ (as shown in previous diagram)

As before, we need to show that the value of the integral along the path $ABC$ from 0 to $T_S$ is greater than the value of the integral along the path $ADE$ from 0 to $T_S$.

i.e.

$$\int_{ABC} F\,dD + G dt - \int_{ADE} F\,dD + G dt \geq 0 \tag{20}$$

Applying Green's Theorem:

$$\oint_{ECBADE} F\,dD + G\,dt \geq \iint_R [dG/dt - dF/dD]\,dD\,dt \qquad (21)$$

where $R$ is the bounded region $ECBADE$ and $\dfrac{dG}{dt} = \left[\dfrac{\rho}{\sigma}\right]e^{-\rho t}$ and $\dfrac{dF}{dD} = \left[-\dfrac{C_1}{(L-D^*)} + \dfrac{C_2}{D^*}\right]e^{-\rho t}$

Therefore (21) becomes:

$$= \iint_R [\rho/s + C_1/(L-D^*) - C_2/D^*\,]\,e^{-\rho t}\,.dt \;\geq 0 \quad (22)$$

From (17) $\qquad \dfrac{\rho}{\sigma} = \dfrac{C_1}{(L-D^*)} - \dfrac{C_2}{D^*}$

Therefore it is optimal to reach $D^*$ in the shortest time possible.

The next section will now consider the case when the returns to investment are not constant but in fact decreasing as more marginal land is developed.

# SECTION IV

## Decreasing Returns to Investment

FKC maximise the present value of net social benefits from the area of land subject to the constraint that:

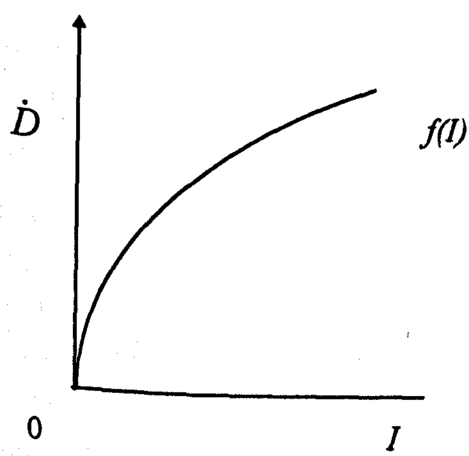
$$\dot{D}(t) = \sigma I(t)$$

where $D(t)$ is the amount of developed land, $I(t)$ is the level of total investment and $\sigma$ is a positive constant. This constraint implies constant returns to increasing investment. But wouldn't it be the case that the easiest land to be developed would be developed first and then from then on the land would become more difficult to develop? Developing this more 'marginal land' would require greater investment. This would imply decreasing returns to increasing investment.

The dynamic constraint would thus change to:

$$\dot{D} = \sigma f(I)$$

The relationship between the change in development and investment is shown graphically below:




At first, a small increase in investment induces a relatively large increase in the amount of developed land. However as investment increases through time, successively larger amounts are required to induce the same initial increase in development. This implies that the more marginal land is developed last as it requires greater investment.

Giving the dynamic equation a specific form say;

$$\dot{D} = \sigma \ln(1+I) \qquad (1)$$

changes the problem to:

$$\max \int_0^{\infty} [C_1 \ln(L-D) + C_2 \ln(D) - I] e^{-\rho t} . dt$$

subject to (1).

The Hamiltonian is therefore:

$$\mathrm{H} = e^{-\rho t}\{C_1 \ln(L-D) + C_2 \ln(D) - I + \lambda\sigma \ln(1+I)\}$$

where $\lambda e^{-\rho t}$ is the costate variable.

The Current Value Hamiltonian is:

$$H = \mathrm{H}e^{\rho t} = \{C_1 \ln(L-D) + C_2 \ln(D) - I + \lambda\sigma \ln(1+I)\}$$

The necessary conditions for an interior solution are:

$$\frac{\partial H}{\partial I} = -1 + \frac{\sigma\lambda}{(1+I)} \tag{1}$$

$$\dot{\lambda} = \rho\lambda - \frac{C_2}{D} + \frac{C_1}{(L-D)} \tag{2}$$

This is no longer a singular arc problem as the Hamiltonian is not linear in the control. From (1):

$$\left.\begin{aligned} \lambda &= \frac{(1+I)}{\sigma} \\ \dot{\lambda} &= \frac{\dot{I}}{\sigma} \end{aligned}\right\} \tag{3}$$

Substituting (3) into (2) and rearranging gives:

$$\dot{I} = \rho(1+I) - \frac{C_2\sigma}{D} + \frac{C_1\sigma}{(L-D)} \tag{4}$$

and $$\dot{D} = \sigma \ln(1+I)$$

At the stationary point:

$$\dot{D} = 0$$

Thus

$$I^* = 0 \tag{5}$$

Where $I^*$ is the steady state level of investment.

Substituting (5) into (4)and rearranging gives:

$$\rho D^{*2} - (\rho L + \sigma(C_1 + C_2))D^* + C_2\sigma L = 0$$

Where $D^*$ is the steady state level of development

Solving gives the equilibrium point, (the steady state of the system), therefore we get:

$$D^* = D_2 = \left[\rho L + \sigma(C_1 + C_2) - \sqrt{[\rho L - \sigma C_2)]^2 + C_1{}^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)}\right] / 2\rho$$

$$I^* = 0$$

and $0 < D^* < L$ is satisfied, (see appendix).

It should be noted here that this is the same equilibrium point as in the original case where $\dot{D} = \sigma I$. We will now look at the transition path to $D^*$.

Analysisg the stability of the system

$$\dot{D} = \sigma \ln(1 + I)$$

$$\dot{I} = \rho(1 + I) - \frac{C_2\sigma}{D} + \frac{C_1\sigma}{(L - D)}$$

Differentiating and evaluating at the steady state gives:

$$\frac{\partial \dot{I}}{\partial I} = \rho \qquad \frac{\partial \dot{I}}{\partial D} = \frac{C_2\sigma}{D^{*2}} + \frac{C_1\sigma}{(L-D^*)^2} = K, \text{ say}$$

$$\frac{\partial \dot{D}}{\partial I} = \frac{\sigma}{(1+I^*)} \qquad \frac{\partial \dot{D}}{\partial D} = 0$$

Putting into matrix form gives:

$$\begin{bmatrix} \dot{I} \\ \dot{D} \end{bmatrix} = \begin{bmatrix} \rho & K \\ \frac{\sigma}{(1+I)} & 0 \end{bmatrix} \begin{bmatrix} I^* \\ D-D^* \end{bmatrix}$$

The eigenvalues of the coefficient matrix satisfy:

$$Z^2 - \rho Z - \frac{\sigma K}{(1+I)} = 0$$

Therefore:

$$Z = \frac{\rho \pm \sqrt{\rho^2 + \frac{4\sigma K}{(1+I)}}}{2}$$

The eigenvalues are real and opposite in sign, therefore ($I^*$, $D^*$) is a saddle point

## Phase Diagram

To construct a phase diagram, we first need to draw the $\dot{I} = 0$ and $\dot{D} = 0$ curves. These curves represent the subset of points in the ($I$, $D$) space where $I$ and $D$ respectively are stationary. Where these two curves intersect determines the equilibrium point - the steady state - of the system, i.e. where $\dot{I} = \dot{D} = 0$ .

Setting $\dot{I} = 0$ and solving for $I$ we get:

$$\dot{I} = \rho(1+I) - \frac{C_2\sigma}{D} + \frac{C_1\sigma}{(L-D)} = 0$$

$$\Rightarrow I = -1 + \frac{\sigma C_2}{\rho D} - \frac{\sigma C_1}{\rho(L-D)}$$

Differentiating with respect to $D$ gives:

$$\left.\frac{dI}{dD}\right|_{\dot{I}=0} = -\frac{\sigma C_2}{\rho D^2} - \frac{\sigma C_1)}{\rho(L-D)^2}$$

The gradient of this curve is negative. As $D \to 0$, $I \to \infty$ and as $D \to L$, $I \to -\infty$.

Solving $\dot{D} = 0$ for $I$ gives:

$$\dot{D} = \sigma \ln(1+I) = 0$$

and so:

$$I = 0$$

Therefore the curve $\dot{D} = 0$ is a horizontal line going through $I = 0$ and is shown below.

The phase diagram below shows the saddle point equilibrium at $D^*$, this occurs where the two curves $\dot{I} = 0$ and $\dot{D} = 0$ intersect. All points on the $\dot{I} = 0$ and $\dot{D} = 0$ are stationary in $I$ and $D$ respectively. Points off these curves are not stationary, they are involved in the dynamic motion and the following analysis explains the dynamics of the system . The arrowheads depict the direction of movement of trajectories from any starting point in the ($I$, $D$) space.

Let us examine the equations:

$$\dot{D} = \sigma \ln(1+I)$$

$$\dot{I} = \rho(1+I) - \frac{C_2\sigma}{D} + \frac{C_1\sigma}{(L-D)}$$

For $I > 0$, $\dot{D} = \sigma \ln(1+I) > 0$ which means that $D$ increases when $I$ is positive. This is shown by rightward pointing arrowheads in quadrants $A$ and $B$.

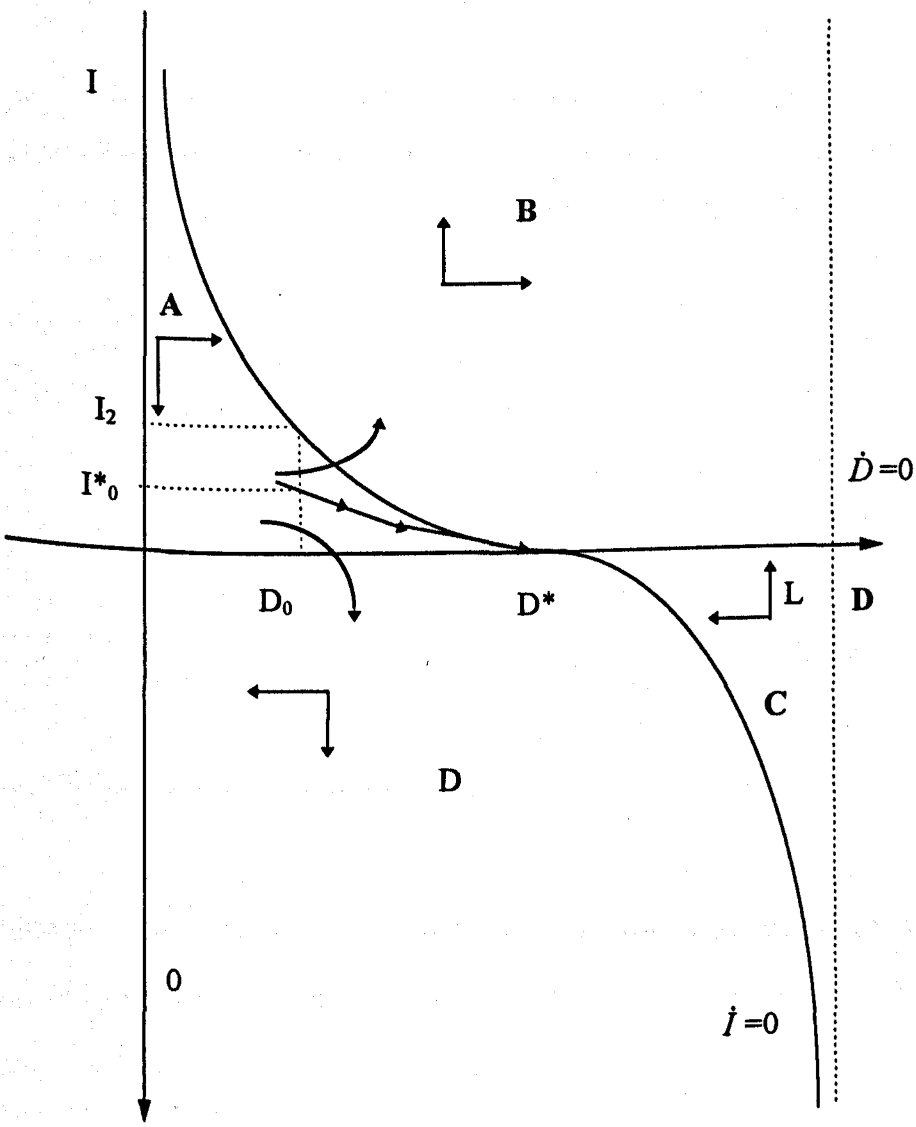
For $I < 0$, $\dot{D} = \sigma \ln(1+I) < 0$ which means that $D$ in decreasing and this is shown by the leftward pointing arrowheads in the $C$ and $D$ quadrant.

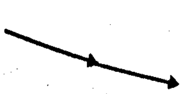
For $D \rightarrow L$, then $\dot{I} \rightarrow \infty$, therefore $I$ is increasing and this is shown by upward pointing arrowheads in the $B$ and $C$ quadrants.

As $D \rightarrow 0$, then $\dot{I} \rightarrow -\infty$, therefore $I$ is decreasing in the $A$ and $D$ quadrant and this is shown by the downward pointing arrowheads.

The equilibrium of the system occurs at point $D^*$ where the two stationary curves intersect.

**Phase Diagram**



depicts a typical optimal trajectory

There are two stable branches leading towards the equilibrium point. All paths starting off the stable branch diverge. It is a saddle-point equilibrium and the only way to reach $D^*$ is to follow one of the stable branches. Given the initial development $D_0$ we must chose the initial level of investment $I^*_0$ as this will ensure that the level of development will reach the optimum level $D^*$. This is indicated by the typical trajectory path in the phase diagram with $D_0 < D^*$. Any initial level of investment greater than $I_2$ will result in excessive investment and development and failure to reach the optimum point $D^*$.

Following this stable path we can see that the level of development is steadily increasing and the gap between $D^*$ and the $D_0$ is gradually falling. With the increase in $D$ there is a steady fall in the level of investment $I$. This is consistent with the earlier negative value for the derivative of $I$ with respect to $D$ - as $D$ increases, $I$ decreases.

Looking at the phase diagram it can be seen that in quadrants $C$ and $D$ the level of investment is negative. This is inadmissable in this problem as investment is irreversible and so can never be negative.

Therefore there is only one meaningful stable path leading to the optimal level of development, $D^*$. There are an infinite number of unstable paths which lead to excessive investment and development. Whether $D^*$ is reached depends on the initial level of investment.

# SECTION V

## Conclusion.

This chapter has demonstrated a different result to that presented by FKC, in that there is a singular arc solution to the optimal control problem and that it is optimal to reach $D^*$ as soon as possible. FKC showed in their theoretical work that investment should be undertaken in intervals whereas in their empirical work they argued that development should jump to $D^*$ at time $t = 0$, i.e. an impulse response. It is argued here that the solution derived in this chapter is a correct solution and that FKC do not fully develop their solution. The conclusion they reach is derived from an incomplete analysis of the model.

The results presented in this chapter are much more realistic. Investment cannot jump to some specified optimal level instantly, nor does it seem optimal to have periods of investment and periods of no investment. Surely if there is some optimal development level then this should be achieved as quickly as possible, and this is what has been shown to be the optimal solution.

A different dynamic constraint was applied to the model to capture the characteristic of decreasing returns to investment instead of the constant returns implied by Fisher et al in their dynamic constraint $\dot{D} = \sigma I$. This gives the same optimal solution as the case where there are constant returns to investment but the transition path is entirely different. The optimal solution is now a saddle point with the optimal investment being undertaken gradually through time. The stability of the system depends on the initial level of investment. If initial investment is too high then the system will become unstable and exhibit excessive investment and development.

# Appendix

Rearranging (18) gives a quadratic which can be solved to find $D^*$

i.e.

$$\rho D^{*2} - [\rho L + \sigma(C_1 + C_2)]D^* + \sigma C_2 L = 0$$

Therefore:

$$D^* = \left[\rho L + \sigma(C_1 + C_2) \pm \sqrt{[\rho L + \sigma(C_1 + C_2)]^2 - 4\rho C_2 \sigma L]}\right] / 2\rho$$

or

$$D^* = \left[\rho L + \sigma(C_1 + C_2) \pm \sqrt{[\rho L - \sigma C_2)]^2 + C_1{}^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)}\right] / 2\rho$$

Both roots are real and positive. We need to check which lie between 0 and $L$, i.e. we need to satisfy the condition that $L - D \geq 0$. It cannot be the case that the amount of developed land is greater than the amount of land that we started with.

There are two alternatives for $D^*$ ($D_1$ and $D_2$);

$$D_1 = \left[\rho L + \sigma(C_1 + C_2) + \sqrt{[\rho L - \sigma C_2)]^2 + C_1{}^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)}\right] / 2\rho$$

$$D_2 = \left[\rho L + \sigma(C_1 + C_2) - \sqrt{[\rho L - \sigma C_2)]^2 + C_1{}^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)}\right] / 2\rho$$

Both roots are real. We now need to satisfy the condition $D^* < L$,

$$\text{i.e. } L - D^* > 0 \qquad (19)$$

First taking $D_1$ and substituting into (19) gives:

$$2\rho L-\rho L-\sigma(C_1+C_2)-\sqrt{[\rho L-\sigma C_2)]^2+C_1{}^2\sigma^2+2\sigma(\rho LC_1+\sigma C_1C_2)}>0$$

rearranging gives:

$$2\rho L-\rho L-\sigma(C_1+C_2)>\sqrt{[\rho L-\sigma C_2)]^2+C_1{}^2\sigma^2+2\sigma(\rho LC_1+\sigma C_1C_2)}>0$$

Squaring both sides gives:

$$\rho^2L^2-2\rho L\sigma C_1-2\rho L\sigma C_2+\sigma^2C_1^2+\sigma^2C_2^2+2\sigma^2C_1C_2>$$

$$\rho^2L^2-2\rho\sigma LC_2+\sigma^2C_2^2+C_1{}^2\sigma^2+2\sigma\rho LC_1+2\sigma^2C_1C_2$$

This cancels to:

$$-2\rho L\sigma C_1>\ 2\sigma\rho LC_1$$

This condition obviously doesn't hold therefore $L - D_1 < 0$ (i.e. $D_1 > L$) which is inadmissable. The area of developed land obviously cannot be bigger that the area of land to start with.

Now looking at $D_2$:

Substituing $D_2$ into (19) and rearranging gives:

$$\sqrt{[\rho L-\sigma C_2)]^2+C_1{}^2\sigma^2+2\sigma(\rho LC_1+\sigma C_1C_2)}>\rho L-2\rho L+\sigma(C_1+C_2)$$

Squaring both sides gives:

$$2\sigma\rho LC_1>-2\rho L\sigma C_1$$

The inequality holds, i.e. $0 < D^* < L$ and so the equilibrium point, the steady state of the system is:

$$D^*=D_2=\left[\rho L+\sigma(C_1+C_2)-\sqrt{[\rho L-\sigma C_2)]^2+C_1{}^2\sigma^2+2\sigma(\rho LC_1+\sigma C_1C_2)}\right]/2\rho$$

$I^* = 0$

and $0 < D^* < L$ for $\rho > 0$

This cancels to:

$$-2\rho L\sigma C_1 > 2\sigma\rho L C_1$$

This condition obviously doesn't hold therefore $L - D_1 < 0$ (i.e. $D_1 > L$) which is inadmissable. The area of developed land obviously cannot be bigger that the area of land to start with.

Now looking at $D_2$:

Substituing $D_2$ into (19) and rearranging gives:

$$\sqrt{[\rho L - \sigma C_2)]^2 + C_1^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)} > \rho L - 2\rho L + \sigma(C_1 + C_2)$$

Squaring both sides gives:

$$2\sigma\rho L C_1 > -2\rho L\sigma C_1$$

The inequality holds, i.e. $0 < D^* < L$ and so the equilibrium point, the steady state of the system is:

$$D^* = D_2 = \left[\rho L + \sigma(C_1 + C_2) - \sqrt{[\rho L - \sigma C_2)]^2 + C_1^2\sigma^2 + 2\sigma(\rho L C_1 + \sigma C_1 C_2)}\right] / 2\rho$$

$I^* = 0$

and $0 < D^* < L$ for $\rho > 0$

# CHAPTER FIVE

# SUSTAINABLE ECONOMIC DEVELOPMENT AND NATURAL RESOURCE USE IN A POLLUTED ENVIRONMENT

## Pollution issues

Society derives utility from the consumption of produced goods. Evidence shows that pollution flows from the production processes in the economic system to a stock in the natural environment. Pollution as a stock in the environment generates disutility - no one likes to see a river or lake polluted with waste products or visit the beach when the sea is foaming with toxic substances. Fumes from industrial works can spoil the countryside, not to mention playing havoc with ones health. The stock of pollution therefore has a negative effect on human welfare.

But there is a dilemma: By reducing present production, and thereby reducing utility, the level of pollution in the future may be reduced and increase future utility. There is therefore a trade-off between producing output for consumption and the level of pollution. Maximising the utility from consumption of produced goods is not the same as maximising society's welfare. The welfare effect that the stock of pollution has on society must be taken into account. But pollution can also have other effects on the environment. Pollution stock can have a negative influence on the regenerative and assimilative capacity of the environment. Renewable resources have a natural regeneration rate and this can be reduced as the level of pollution increases. For example, a forest will regenerate itself, but its growth rate will be reduced as more acid rain is deposited over the trees. The stock of pollution can also affect the assimiliative capacity of the environment. The rate at which the environment can clean itself up can be depressed as the level of pollution stock increases. A small amount of pollution may be cleaned up by of pollution stock increases. A small amount of pollution may be cleaned up by the environment very easily and quickly but as the level of pollution stock increases the environment will find it increasingly difficult to clean itself up. Therefore the assimilative rate will fall. It is these negative effects that pollution has on the regenerative and assimiliative capacity of the environment and the negative effect on human welfare that will be investigated in this chapter.

In section I there follows an overview of the literature concerning pollution. Section II formulates a model where a renewable resource is extracted from the environment and used in the production process along with capital and labour services. Productive activity generates a flow of pollution, which in turn builds up as a stock in the environment. This stock of pollution has a negative impact on the regenerative capacity of the renewable natural resource and also affects the assimilative capacity (the natural self-purification process) of the environment. The problem is to choose a time path for harvesting the resource so as to maximise some objective functional whose arguments are the time path of consumption and the stock of pollution. Thus indicating that social welfare at any point in time depends on the flow of consumption and the quality of the environment. In section III Pontryagin's Maximum Principle (1962) is used to discover the optimal solution and the steady state values. Section IV analyses the stability of the system. Section V considers the effects that changes in the parameters of the model have on the steady state solutions.

# SECTION I

## Pollution Models

In the past, the consequences of pollution that have been considered important have been the direct disutility to consumers and abatement costs to producers, Keeler et al (1972). Pollution is usually treated as a flow that dissipates and does not build up into a stock, Keeler et al (1972), Forster (1980), Gruver (1976) and Forster (1973a), i.e. a flow variable. This assumption is used by Forster (1980), to gain analytical simplicity. Treating pollution as a flow variable and not as a stock reduces the number of differential equations in the problem. Forster presents a model in which there is assumed to be a single energy source, such as a fossil fuel that produces a non-accumulating pollutant when it is used. Consumption creates utility, but the use of the fuel generates a flow of pollution as a by-product which creates disutility. An example of this type of pollution is that which is emitted by automobiles. The level of utility will therefore depend upon the level of consumption and the flow of pollution as society is concerned about the quality of the environment. Withdrawing the natural resource from the environment for energy use reduces the stock of the resource An Energy Board is appointed to plan the optimal path for energy use over a specified time. The dynamic optimisation problem that must be solved is:

$$Max \int_0^T U(C(E), P(E)).dt$$

$$\text{subject to } \dot{S} = -E$$
$$S(0) = S_0$$
$$S(T) \geq 0$$

where consumption, $C$ and the flow of pollution, $P$ are functions of energy use, $E$. $S$ is the stock of fuel, $\dot{S}$ is the change in the resource stock and $E$ is the rate of extraction (energy use) at time $t$.

He shows that when pollution is treated as a flow variable, the rate of use of the resource is constant over time. The level of extraction depends on the length of time of the planning horizon and the initial resource endowment. For long time periods and small resource endowments, environmental considerations are ignored and are not a constraint on extraction. An extraction rate is chosen which exhausts the resource by the end of the time period. When the endowments of resource are large relative to the time period then the level of resource stock at the end of the time period is positive. The extraction rate is chosen which balances the costs and benefits of additional fuel use from an environmental stand point.

However evidence shows that pollution flows into the environment and does build up into a stock. Pollution such as radioactive waste and oil spills, do emerge as a stock and produce lasting effects. The accumulating stock of pollutants has direct negative marginal effects on human welfare.

Forster (1977), also presents a model in which the level of consumption and the stock of pollution, instead of the flow, are arguments in the utility function. Utility is derived from consumption but by consuming output a flow of pollution is generated which in this model builds up into a stock in the environment, causing disutility. Therefore the utility function is:

$$U = U(C, P)$$

where $C$ is the level of consumption and $P$ is the stock of pollution. Forster assumes that a fixed amount of output, $\phi$ is produced each time period and this is allocated to consumption, $C$ and pollution control activities, $E$. Therefore:

$$\phi = C + E$$

However, Forster does not include the production process in the model, nor does he include natural resources. The flow of pollution or the change in the stock of pollution increases with respect to consumption at a constant rate, $g$ where $g>0$ . The environment has the capacity to clean itself up and get rid of the waste at a constant rate $\alpha$ where $\alpha > 0$ , this is known as the assimiliative capacity. Therefore the flow is reduced in a proportionate manner with respect to the level of pollution. The change in the stock of the resource when there is no expenditure on anti-pollution activities is thus given by:

$$\dot{P} = g(C) - \alpha P$$

The amount of pollution cleaned up by pollution control activities is a function of the amount of expenditure on pollution control activities, $E$. Therefore the differential equation relating to the change in the stock of pollution now becomes

$$\dot{P} = g(C) - h(E) - \alpha P$$

where $h$ is the constant rate at which pollution stock is cleaned up and $h>0$.

The problem then is to maximise the discounted flow of utility, i.e.:

$$\int_0^\infty e^{-rt} U(C,P).dt$$

subject to

$$\dot{P} = g(C) - h(E) - \alpha P$$

where future utility is discounted at a constant exponential rate $r$.

Forster shows that if the marginal utility of consumption is greater than the cost of additional consumption then it is not optimal to eliminate the pollution. The equilibrium of the system is characterised by non-zero level of pollution and a consumption level greater than the initial level, $C_0$. If the marginal utility of consumption is less than the marginal cost of consumption then the optimal policy results in a clean environment and once it has been cleaned up consumption is at $C_0$.

Siebert (1982), also treats pollution as a stock variable and, again, his model does not include a capital good in the production process but he does include the extraction of a natural resource. It differs from the basic economic growth model as only naturally produced goods or services are consumed. The economic process he describes is that society gains utility from consumption of extracted amounts of a renewable resource. Consumption and/or extraction of the resource causes accumulation of a stock of pollution, and the stock of pollution has a negative impact on the regenerative capacity of the renewable resource. The resource grows at a natural rate which depends on the amount of resource stock at time $t$. The stock of the resource is affected by the level of pollution. In each time period the level of pollution reduces the resource stock at a constant rate $\alpha$, where α>0. The stock is also reduced by the extraction of the resource from the environment which is then consumed. Therefore the change in the resource stock at time $t$ is given by:

$$\dot{R} = g(R) - aS - C$$

where $R$ is the level of resource stock, $\dot{R}$ is the change in the resource stock, $g$ is the natural growth rate, $S$ is stock of pollution in the environment and $C$ is consumption.

The consumption process generates pollutants at a constant proportion, $\beta$, per unit of resource consumed. The stock of pollution is cleaned up by the environment at a constant rate $\pi$. The change in the stock of pollution increases due to pollution flowing into the environment from consumption and decreases due to the assimilative capacity of the environment. Therefore the change in the stock of pollution is given by the equation:

$$\dot{S} = \beta C - \pi S$$

where $\dot{S}$ is the change in the stock of pollutants.

Formally his model is specified as follows:

$$\max \int_0^\infty u(C)e^{-rt}.dt$$

subject to:

$$\dot{R} = g(R) - aS - C$$
$$\dot{S} = \beta C - \pi S$$

$$C, R, S \geq 0$$
$$N(0) = N_0$$
$$S(0) = S_0$$

Siebert does not take into account that the stock of pollution has a negative effect on social welfare, the welfare function has only the flow of consumption as its argument and there is no productive activity in the economy.

Siebert shows that if the initial levels of resource and pollution stock are lower than their steady state levels, then the system can move towards the optimal steady state by a policy of low extraction rate, thereby slowly increasing the stock of resources and the stock of pollution. If the initial level of resource stock is less than its steady state value and the initial level of pollution stock is greater than its steady state value, then the system will not reach the optimal steady state. The negative effect of pollution will not allow regeneration to occur fast enough and the resource stock will decline.

Forster (1980) formulates a model in the same paper cited previously, where pollution does build up into a stock and he does take into account the negative effects that this pollution stock has on social welfare. Therefore consumption and pollution are arguments in the utility function, showing that society gains utility from consumption and disutility from pollution. But again it is only natural resources that are consumed, there is no productive process in the model. Forster specifies the utility function as follows:

$$U = U(C, P)$$

The stock of the resource can be thought of again as a fossil fuel which is extracted from the environment and makes it possible for goods and services to be produced. These goods and services are consumed by society, which creates utility. Therefore consumption is a function of the energy use. As in his previous model, the use of energy generates a flow of pollution, but here the flow of pollution builds up into a stock in the environment. The change in the stock of pollution, i.e. flow of pollution increases by a constant proportion, $\alpha$ ,of the energy used where $\alpha > 0$,and is also affected by antipollution activities. It is assumed that these activities can reduce the flow of pollution by some proportion

$\beta$, where $\beta > 0$. Furthermore the change in pollution stock is subject to exponential decay at a constant rate $\delta$, where $\delta > 0$. The stock of fuel is reduced by extraction of the resource. Also, as abatement activities requires the use of energy, then this also reduces the stock of the resource.

The dynamic optimisation problem is stated as:

$$\max \int_0^T U(C(E), P).dt$$

subject to:

$$\dot{P} = \alpha E - \beta A - \delta P$$
$$\dot{S} = -A - E$$

$$P(0) = P_0 > 0 \quad P(T) \geq 0$$
$$S(0) = S_0 > 0 \quad S(T) \geq 0$$
$$E(0) = 0 \quad 0 \leq A \leq \tilde{A}$$

where $P$ is the stock of pollution, $C$ is the level of consumption, $E$ is the energy used ($\alpha > 0$), $A$ is the level of anti pollution activities and $\tilde{A}$ is an upper limit on $A$ and $S$ is the level of resource stock.

Forster shows that it may not be optimal to undertake pollution abatement activities which are energy using. Rather in the initial stages the resource should be used more slowely so that the level of environmental damage is lowered. Over the planning period the resource use is increased and by the end, the resource is completely exhausted and there is a positive stock of pollution.

Barbier (1989), presents a similar model in that he does treat pollution as a stock, but in the form of an environmental degradation variable. Barbier's model recognises the fact that as the natural environment supplies more and more resources to society, it is forced to absorb more and more waste products and there may be a point at which ecological stability is threatened. However, again no productive activity is included in the model. His argument is that resources are harvested from the environment to provide for consumption, therefore the flow of emissions and resource extraction are still functions of the economic process but Barbier makes them dependent on the flow of consumption. Pollution is not a product of the productive process. Barbier formulates a model in which the degradation of the environment is increased as a result of waste emissions and renewable and nonrenewable resource extraction, and is decreased by the natural assimiliative capacity of the environment and the natural regeneration of the renewable resource. He explains this in the form of a differential equation for environmental degradation and is defined as:

$$\dot{D} = (W - A) + (R_N - G) + R_s$$

where $\dot{D}$ is the change in environmental degradation, $W$ is the level of waste emissions, $A$ is the level of assimilated waste, $R_N$ is the level of renewable resource extraction, $G$ is the level of natural regeneration and $R_S$ is the level of nonrenewable resource extraction.

As resources are extracted and wastes are emitted by the economic process to provide for consumption then:

$$W = W(C) \quad R_N = R_N(C) \quad R_S = R_S(C)$$

Barbier does include in his model that the regeneration of the resource and the assimiliative capacity of the environment are affected by the quality of the environment. Let $X$ be some measure of environmental quality, which is measured by a stock of environmental goods that yield a flow of services proportional to that stock at each point in time, then:

$$A = A(X) \quad G = G(X)$$

Therefore:

$$\dot{D} = [W(C) + R_N(C) + R_S(C)] - [A(X) + G(X)]$$

i.e.

$$\dot{D} = [N(C) - Q(X)]$$

where $N(C)$ is the increasing environmental degradation resulting from the various resource demands that are being put on the environment. $Q(X)$ is the resilience of the environment. He assumes an inverse relationship between the change in environmental degradation and the change in environmental quality;

i.e.

$$\dot{X} = -a\dot{D}$$

If environmental degradation is increasing over time then environmental quality is falling at a constant proportional rate $a$, where $a$>0.

Therefore:

$$\dot{X} = a[\mathrm{Q}(X) - N(C)]$$

Formally his problem is defined as optimising a social welfare function that has the flow of consumption and the level of environmental quality as its arguments:

i.e.

$$\max \int_0^\infty e^{-rt} U(C, X).dt$$

subject to :

$$\dot{X} = a[\mathrm{Q}(X) - N(C)]$$

$$X(0) = X_0 \quad X_\infty free$$

He examines the optimal conditions that would lead an economy to choose a sustainable or unsustainable economic growth path. He finds that if the initial level of environmental quality is less than the minimum sustainable level - the level at which the flow of waste emitted is equal to the assimilative capacity of the environment - then environmentally unsustainable economic growth may be the optimal strategy. In this case the assimilative capacity of the environment will have been destroyed and the economy will be forced to exhaust existing resource stocks and so the economy will collapse.

If the initial level of environmental quality is equal to the minimum sustainable level, then it is optimal to remain at that growth rate forever. If it is greater, then the economic growth path will end up at a stable equilibrium which represents environmentally sustainable economic growth. In this case and where they are equal, the biophysical constraints are being adhered to - harvesting of the renewable resource is within its regeneration rate, non-renewable resources are being extracted at a rate at which renewables can be substituted for them, and emissions of pollutants are within the assimilative capacity of the environment.

He concludes that it is the initial level of environmental quality and the rate of discount that are significant factors in determining whether a sustainable or unsustainable economic growth path is the optimal strategy. In his analysis a decrease in the discount rate will lead to a unique stable equilibrium and so the optimal strategy is to follow a sustainable growth path regardless of the initial environmental quality. A sufficient increase in the discount rate will lead to a unique unstable equilibrium and so there is only one optimal strategy to follow and that is an unsustainable growth path regardless of the initial level of environmental quality. These conclusions are as one would expect, a high discount rate will favour current consumption and so here it is optimal to deplete resources at higher rates leading to lower levels of resource stock and maybe even total exhaustion. A higher discount rate will favour projects where the benefits occur in the short term and so there are fewer incentives for projects that have a long term pay off. Projects that yield benefits in the long term are less likely to be undertaken with a high discount rate, and these are very likely to be environmentally favourable projects.

However, a more meaningful analysis would include a production function with capital services and natural resources as inputs. Such a model is developed in section II.

Van de Bergh and Nijkamp (1991), present a model, similar to Keeler et al's (1972), but take both the flow and stock of pollution into account. The welfare function includes both of these as well as the flow of consumption. The stock of pollution accumulates as a result of waste generation by the production and consumption process. However they do not include natural resources in their model. The production function $Q$ has inputs of capital, $K_1$ and material input, $M$ and it is assumed that the effects of pollution may harm the production process. Therefore the production function is decreasing in the stock of pollution and is given below:

$$Q(t) = F[K_1(t), P(t), M(t)]$$

Capital is allocated between pollution control activities, $K_2$ and in the productive process where it is used as an input, $K_1$. Therefore:

$$K(t) = K_1(t) + K_2(t)$$

The change in the capital stock , or the flow of capital, increases with respect to the level of output and decreases with respect to the level of expenditure on pollution control activities, the level of consumption and the amount of depreciated capital. Therefore the change in the capital stock is given by the differential equation:

$$\dot{K} = (1-\beta)Q(t) - C(t) - \delta_k K(t)$$

where consumption is denoted by $C$, $\beta$ is a fixed constant where $0<\beta<1$ showing the proportion of total expenditure allocated to pollution control activities and capital depreciates at a constant rate $\delta_K$ where $0<\delta_K<1$.

The flow of pollution increases with respect to emissions into the environment which is generated by the production and the consumption processes. Consumption generates the material outflow, $M$ , into the environment and emissions from the production process is a proportion, $w$ of the amount of capital used in production, where $w$ is constant and $w>0$. Pollution is reduced by pollution control expenditure, at a constant rate $h$ where capital input $K_2$ is allocated to pollution reduction activities. The environment assimilates waste products at a rate $\delta_p$, which may depend on the level of $P$, because pollution may affect the capacity of the environment to cleanse itself. The amount of expenditure allocated to pollution control activities is $\beta Q$ and pollution is reduced at a rate $d$, where $0<d<1$. Formally their model is:

$$\max \int_0^{\infty} e^{-rt}\left[C(t), P(t)\frac{dP(t)}{dt}\right].dt$$

subject to:

$$\begin{aligned}
\dot{K} &= (1-\beta)Q(t) - C(t) - \delta_k K(t) \\
\dot{P} &= w[K_1(t)] + M(t) - h[K_2(t)] - d\beta Q(t) - \delta_p P(t)
\end{aligned}$$

where:

$$\begin{aligned}
&Q(t) = F[K_1(t), P(t), M(t)] \\
&K(t) = K_1(t) + K_2(t) \\
&C(t), K_i(t)(i=1,2), P(t) \geq 0 \\
&K_i(0) = K_{i0}(i=1,2), P(0) = P_0
\end{aligned}$$

This model does include a production function and the fact that pollution can build up into a stock in the environment. It also takes into account that the stock of pollution affects utility and therefore includes this in the utility function. Another interesting aspect of the model is that the rate of assimilated waste, $\delta_p$ may depend on the level of pollution in the environment. This aspect of a non-constant rate of decay will be dealt with later in the chapter. The main critisms of van den Bergh and Nijkamp's work is that the extraction of natural resources is not included in the model and the fact that they do not solve the system, thereby offering no conclusions to their work.

Another important work is that presented by Brock (1977). Brock analyses the problem of growth and stock pollution. He argues that because of the inputs of energy in production, emissions that are generated are closely related to the production level rather than the level of consumption. In his model he presents a production function with capital and emissions as factors of production. He also includes the stock of pollution as an argument in the social welfare function.

The change in the capital stock increases with respect to output and is reduced by the level of consumption, therefore:

$$\dot{K} = P(K,E) - C$$

where $K$ is the stock of capital, $E$ is emissions flow and $C$ is consumption.

The stock of pollution increases as emissions increase and is reduced by the natural decay of the environment. Therefore the evolution of the stock of pollution over time is given by:

$$\dot{Z} = E - \alpha Z$$

where $Z$ is the pollution stock, $E$ is the emissions flow and $\alpha > 0$ is the rate of decay. The problem is to maximise a social welfare function subject to the constraint of capital accumulation and pollution stock. Formally, the problem is to :

$$\max \int_0^\infty U(C,Z)e^{-\delta t}.dt$$

subject to :

$$\dot{K} = P(K,E) - C$$

$$\dot{Z} = E - \alpha Z$$

He shows that an optimal solution does exist and that the steady state is a local saddle point.

Tahvonen and Kuuluvainen (1993), (original version in Tahvonen and Kuuluvainen (1991)), extend Brock's model to include natural resources. Also, they take into account the negative effect pollution has on social welfare by including the stock of pollution in the welfare function. The production function, $Q$ now includes the renewable resource inputs; $Q$ has the stock of capital, $K$, the rate of harvest, $h$ and emissions, $e$ as inputs. The change in the capital stock increases with respect to output and is reduced by the amount of output that is consumed by society, $C$. Therefore:

$$\dot{K} = Q(K, h, e) - C$$

$X$ is the stock of the renewable natural resource which is sensitive to the stock of pollution, $Z$. The regeneration of the resource also depends on the amount of resource in stock at time $t$, $X$ therefore the growth function is given by:

$$F = F(X, Z)$$

The change in the resource stock, or the flow of the resource, increases with respect to the growth function $F$, and decreases as the resource is harvested for production. Therefore:

$$\dot{X} = F(X, Z) - h$$

Emissions accumulate from the productive process into the environment and this therefore increases the stock of pollution, $Z$. The pollution stock also decays naturally at a constant rate, $\alpha$ where $\alpha > 0$. Therefore:

$$\dot{Z} = e - \alpha Z$$

where the change in the stock of pollution at time $t$ is equal to the amount of emissions at time $t$ minus the amount of pollution naturally assimilated by the environment at time $t$, at rate $\alpha$.

Formally, the problem is to choose time paths for consumption, resource harvesting and emissions in an economy where production is based on renewable resources. The problem for a social planner is to:

$$\max \int_0^\infty e^{-rt} U(C,Z).dt$$

subject to:

$$\dot{K} = Q(K,h,e) - C$$
$$\dot{X} = F(X,Z) - h$$
$$\dot{Z} = e - \alpha Z$$

where $r \geq 0$ is the discount rate.

Tahvonen and Kuuluvainen show that the optimal steady state in Brock's model is independent of the discount rate. When they include renewable resources in the model, the steady state also has the saddle point propery but this depends on the discount rate being small.

However, in both Brock and Tahvonen and Kuuluvainens' models, emissions are dealt with as inputs in the production process. In the model to be developed in this chapter, the view is taken that it would be more meaningful to show emissions as an outflow from the production process, not as a necessary factor of production. Brock and Tahvonen and Kuuluvainen also include a decay rate for the stock of pollution but treat this as constant. They also neglect the decay of the capital stock.

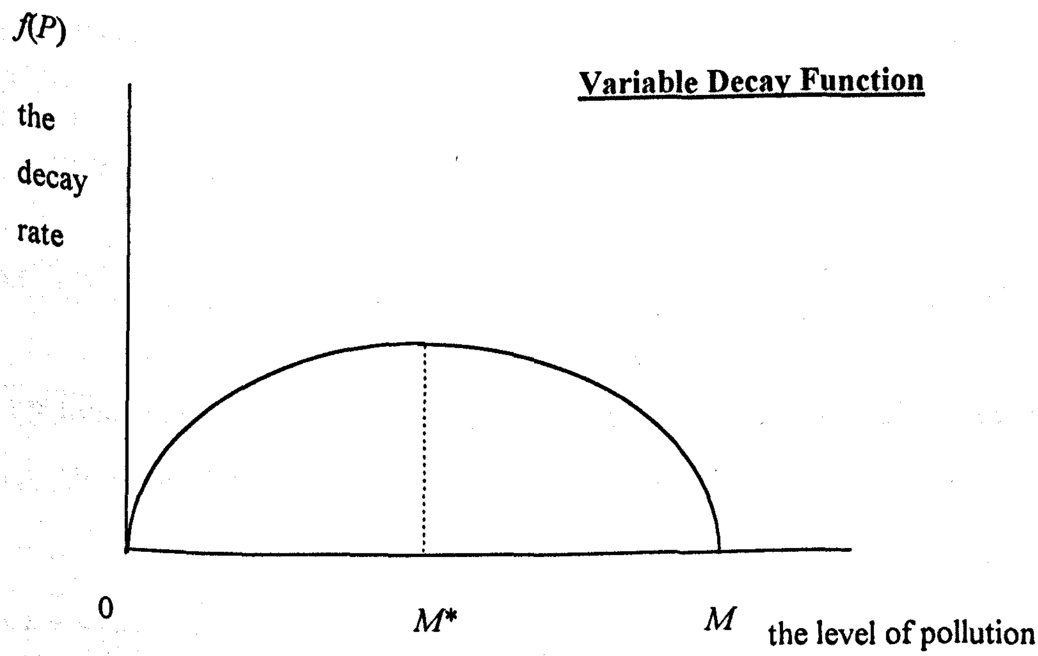
Van den Bergh (1991), identifies a weakness in the literature on the development of ecological-economic models in that they fail to be complete models, he argues that models must include every aspect of interaction between the environment and the economic system. He goes on to present an aggregate economic-ecological model which is consistent with a macroeconomic system. However such a model is very complex and intractable. In section II in this chapter, a comprehensive model is formulated that takes into account all the previous weaknesses in the literature and still allows for mathematical tractability.

Most of the past work on pollution has been concerned with a constant exponential decay rate [see Forster (1980), Siebert (1982), Tahvonen and Kuuluvainen (1993), Forster (1973b), Plourde (1972), D'Arge (1971) and Forster (1977)]. If the decay rate is constant, it is true that for any level of pollution, the stock of pollution would eventually completely decay if there were no new additions. In this situation the pollution does not destroy the natural purification process of the environment. D'Arge (1971), for instance, assumes a constant decay rate that is completely independent of the level of pollution. He assumes that it is the density of waste that is the proper measure of environmental quality. He specifies that the average change in the density of waste, or the average flow of waste density, increases with respect to the flow of waste per unit measure of the natural environment less the flow of waste cleaned up by capital that is invested specifically for waste control minus the natural decay rate. Therefore:

$$\dot{D} = \frac{1}{V}W - hI_k - \delta$$

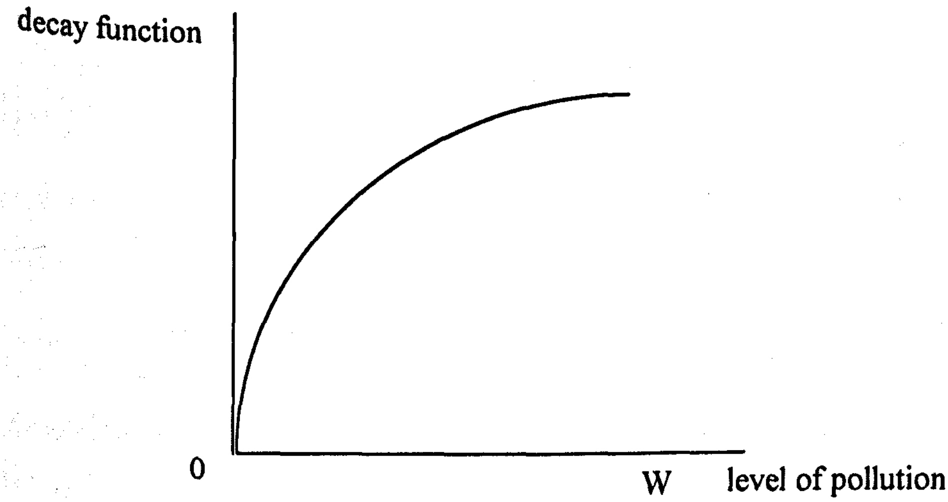
where $W$ is the waste flow, $D$ is the average density of waste, $V$ is a volume measure of the natural environment. The example D'arge gives is the size of the global natural life zone which is assumed fixed. The coefficient on $I_k$ reflects the rate at which capital investment can clean up the environment. The natural decay rate of waste density is $\delta$ and is constant.

The assumption of a constant decay rate is very limiting. Forster (1975), argues that the rate of self-purification may depend upon the amount and nature of the waste load. Certain toxic substances may inhibit the self-purification properties of water by killing the bacteria that is required to break down the organic wastes. For example if the amount of pollution is great enough it may cause a waterway to be biologically dead and unable to cleanse itself. This is the problem associated with lakes such as lake Erie in North America, Ehrlich and Ehrlich (1970). Forster formulates a model to allow for the fact that pollution may depress the rate of self-purification. Social welfare is measured by a utility function with consumption and the stock of pollution as arguments in the utility function. However, he doesn't include production and natural resources in his model. Pollution is therefore a function of consumption. Forster's decay function $f(P)$ can be shown to have the shape shown below:




Over the interval (0, $M^*$), the decay rate is increasing at a decreasing rate; therefore, the more pollution there is the faster it is dissipated. After $M^*$ the level of pollution depresses the natural decay process. When $M$ is reached the pollution has killed off the natural clean up process. But why should it be that over the interval (0, $M^*$) the more pollution, the faster it is cleaned up? This surely doesn't make sense.

Smith (1977), treats pollution as a stock variable and also assumes that the natural decay rate $h(W)$ depends on the level of pollution ($W$), where $h'>0$ and $h''<0$ - i.e. pollution decay increases at a decreasing rate with respect to the stock of pollution. The decay function would have the shape shown below:




But again, why should the rate of assimilation increase as there is more pollution. The difference between Smith and Forster is that Forster assumes that there is an upper limit on the level of pollution at which point the decay rate becomes zero. It is feasible that there will be some point when the environment simply cannot cope with the pollution level and the self-purification powers will have been destroyed. This critism will be dealt with in section III.

# SECTION II

## The Mathematical Model.

Consider a simple closed economy with no government intervention. Therefore output is either consumed or invested;

i.e.

$$Y(t) = C(t) + I(t) \tag{1}$$

where $Y(t)$ represents output, $C(t)$ represents consumption and $I(t)$ represents investment, all at time $t$. Note that for the rest of the chapter dependence on time is not explicitly shown for notational simplicity. Technology in the economy is summarised by a Cobb Douglas production function with inputs of capital services and a renewable natural resource and which exhibits constant returns to scale, hence;

$$Y = AK^{\alpha}U^{1-\alpha} \tag{2}$$

$\alpha$ is some constant where $0 < \alpha < 1$, $K$ is the flow of capital services and $U$ is the flow of services from the renewable natural resource, i.e. the harvest rate since all of the extracted resource is used up in production. It is assumed in this model that there is a constant labour input, $L$ into production, and without loss of generality $L \equiv 1$. The labour input is subsumed within the constant $A$ and the production function is normalised so that $A \equiv 1$.

Let $S$ equal the savings rate (which is assumed constant); then total savings is a constant proportion of output and total savings equals total investment.

$$\begin{aligned} \text{Total Savings} &= \text{Total Investment} \\ &= \dot{K} + \phi K \\ &= \text{(new investment + replacement investment)} \end{aligned}$$

where $\phi > 0$ is the constant rate of depreciation of the capital stock.

Total savings is a constant proportion, $S$ of output, i.e.

$$\text{Total Savings} = SK^{\alpha}U^{1-\alpha} \tag{3}$$

By definition $S = I$

Therefore:

$$\text{Total Investment} = SK^{\alpha}U^{1-\alpha}$$

Using the original income identity $Y = C + I$:

$$K^{\alpha}U^{1-\alpha} = SK^{\alpha}U^{1-\alpha} + C$$

Therefore:

$$C = (1-S)K^{\alpha}U^{1-\alpha} \tag{4}$$

Consumption in this economy is simply the amount of output not saved (i.e. invested). We assume that society wishes to choose a production and extraction plan so as to maximise a discounted linear combination of consumption and the stock of pollution.

i.e.

$$\max \int_0^{\infty} e^{-rt}(w_1 C + w_2 Z).dt$$

where $w_1$ and $w_2$ are weights on consumption, $C$ and pollution stock, $Z$. It is further assumed that $w_1$ is positive and $w_2$ is negative. To simplify, since only relative weights matter, we can normalise the weight on consumption to equal unity and then maximise:

$$\int_0^{\infty} e^{-rt}(C - wZ).dt \tag{5}$$

where $w$ is a positive constant and $0 < w < 1$. Therefore substituting equation (4) into (5) gives:

$$\int_0^{\infty} e^{-rt}\left[(1-S)K^{\alpha}U^{1-\alpha} - wZ\right].dt \tag{6}$$

The instantaneous utility function exhibits positive but diminishing marginal utility from consumption and negative, but constant, marginal disutility from pollution. Future utility is discounted at a constant exponential rate $r$, where $r > 0$.

The change in the capital stock at time $t$ will increase as new capital is invested and will decrease as the stock of capital depreciates. We can obtain a differential equation for the change in the stock of capital by using the condition:

$$\begin{aligned} \text{Total Savings} &= \text{Total Investment} \\ &= \text{new investment} + \text{replacement investment} \\ &= \dot{K} + \phi K \end{aligned}$$

Therefore:

$$SK^{\alpha}U^{1-\alpha} = \dot{K} + \phi K$$

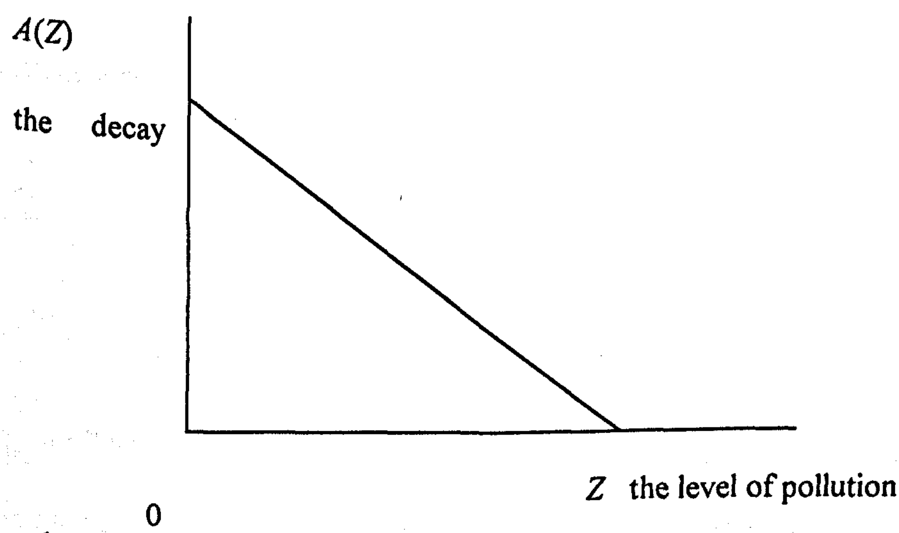
and

$$\dot{K} = SK^{\alpha}U^{1-\alpha} - \phi K$$

Let $X$ be the stock of the renewable resource at time $t$ which is harvested and used for the production of output. Pollution is generated by this production process and builds up into a stock $Z$ at time $t$. The natural regeneration of the resource is given by $g - \beta Z$ where $g$ and $\beta$ are both positive constants. It is assumed that when there is no pollution, $g$ is the natural growth rate. As the level of pollution grows, the rate of regeneration falls at a constant rate, $\beta$. When $Z = g/\beta$ there is no more regeneration of the resource and the pollution has destroyed its natural growth rate. Therefore it follows that the rate of change in the stock of the resource is equal to the rate of regenerated resource minus the harvest rate, $U$; $U$ is also the flow of the resource used in production at time $t$, i.e:

$$\dot{X} = (g - \beta Z)X - U$$

We assume that the rate of change of pollution stock ($\dot{Z}$) into the environment is the difference between the flow of pollution into the environment (which is proportional to the rate of output, where $0 < \gamma < 1$, i.e. $\gamma K^{\alpha} U^{1-\alpha}$), and the rate of assimilative capacity of the environment (i.e. the natural clean up process). Forster (1975) has argued that the rate of assimilated capacity of the environment may depend on the amount of pollution in the environment. Forster's decay function *f*(*P*) is shown in section *I*. But repeating the previous critism of Barbier's function, why should it be that over the interval (0,*M**) the more pollution the faster it is dissipated? A more feasible function would be of the form shown below:




Where the assimilative capacity of the environment is modelled by $A(Z) = \theta - \delta Z$, where, $\theta > 0$, $0 < \delta < 1$ and $Z > 0$. It is assumed that production has taken place in the past therefore there is a stock of pollution already existing in the environment. It shows that as the stock of pollution rises the rate of clean up falls at a constant rate $\delta$. When $Z$ has risen to $\frac{\theta}{\delta}$ the decay rate is zero and the environment is unable to cleanse itself. Therefore if follows that the stock of pollution will evolve over time according to:

$$\dot{Z} = \gamma K^{\alpha} U^{1-\alpha} - (\theta - \delta Z)$$

## SECTION III

### The Formal Problem

Formally, the problem we wish to address is to:

$$\max\int_0^\infty e^{-rt}\left[(1-S)K^\alpha U^{1-\alpha} - wZ\right].dt$$

subject to:

$$\dot{K} = SK^\alpha U^{1-\alpha} - \phi K$$
$$\dot{X} = (g-\beta Z)X - U$$
$$\dot{Z} = \gamma K^\alpha U^{1-\alpha} - (\theta - \delta Z)$$

with initial conditions:

$$K(0) = K_0 > 0$$

$$X(0) = X_0 > 0$$

$$Z(0) = Z_0 > 0$$

The transversality consitions are:

$$\lim_{t\to\infty} e^{-rt}\lambda_1 K = 0, \qquad \lim_{t\to\infty} e^{-rt}\lambda_2 X = 0, \qquad \lim_{t\to\infty} e^{-rt}\lambda_3 Z = 0$$

Where $\lambda_1 e^{-rt}$ ,$\lambda_2 e^{-rt}$ and $\lambda_3 e^{-rt}$ are costate variables which are valuation variables measuring the shadow price of capital, the renewable resource and the stock of pollution respectively.

Therefore the problem posed is a dynamic optimisation problem and the method employed is the Maximum Principle of L.S. Pontryagin et al (1962).

The present value Hamiltonian is defined as:

$$H = e^{-rt}\left[(1-S)K^{\alpha}U^{1-\alpha} - wZ + \lambda_1[K^{\alpha}U^{1-\alpha} - \phi K] + \lambda_2[(g-\beta Z)X - U] + \lambda_3[\gamma K^{\alpha}U^{1-\alpha} + \delta Z - \theta]\right]$$

Assuming an interior solution the the first necessary condition is:

$$\partial H / \partial U = 0 \Rightarrow (1-S)\left(\frac{K}{U}\right)^{\alpha}(1-\alpha) + \lambda_1 S\left(\frac{K}{U}\right)^{\alpha}(1-\alpha)$$

$$-\lambda_2 + \lambda_3\gamma\left(\frac{K}{U}\right)^{\alpha}(1-\alpha) = 0 \qquad (1)$$

From (1):

$$\left(\frac{K}{U}\right) = \left[\frac{\lambda_2}{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}\right]^{\frac{1}{\alpha}} \qquad (2)$$

The second necessary condition is:

$$d/dt\left(\lambda_1 e^{-rt}\right) = -\partial H / \partial K$$

Therefore:

$$\dot{\lambda}_1 = (r+\phi)\lambda_1 - \left(\frac{K}{U}\right)^{\alpha-1}[\alpha(1-S+S\lambda_1+\gamma\lambda_3)] \qquad (3)$$

Substituting (2) into (3) gives:

$$\dot{\lambda}_1 = (r+\phi)\lambda_1 - \left[\frac{\lambda_2}{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}\right]^{\frac{\alpha-1}{\alpha}}[\alpha(1-S+S\lambda_1+\gamma\lambda_3)]$$

Therefore simplifying gives:

$$\dot{\lambda}_1 = (r+\phi)\lambda_1 - \frac{\alpha\lambda_2}{(1-\alpha)}\left[\frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}\right]^{\frac{1}{\alpha}} \tag{4}$$

The third necessary condition is:

$$d/dt\left(\lambda_2 e^{-rt}\right) = -\partial H/\partial X$$

i.e.

$$\dot{\lambda}_2 = r\lambda_2 - \partial H/\partial X = \lambda_2(r-g+\beta Z) \tag{5}$$

And the fourth necessary condition is:

$$d/dt\left(\lambda_3 e^{-rt}\right) = -\partial H/\partial Z$$

i.e.

$$\dot{\lambda}_3 = r\lambda_3 - \partial H/\partial Z = w+(r-\delta)\lambda_3 + \beta\lambda_2 X \tag{6}$$

Substituting (2) into the dynamic constraints gives:

$$\dot{K} = K\left[S\left[\frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}\right]^{\frac{1-\alpha}{\alpha}} - \phi\right] \tag{7}$$

$$\dot{X} = (g-\beta Z)X - K\left[\frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}\right]^{\frac{1}{\alpha}} \tag{8}$$

$$\dot{Z} = \gamma K\left[\frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}\right]^{\frac{1-\alpha}{\alpha}} + \delta Z - \theta \tag{9}$$

Note that these conditions are also sufficient because of the concavity of the objective functional, see Mangasarian (1966) and the appendix A to this chapter.

This implies that if a path is found that converges towards a steady state then it is an optimal path.

In the long run the optimal solution may converge to an equilibrium point. This constant solution of a system of differential equations is known as the steady state. If the steady state exists then the system, given the initial state values, is sustainable and there is an optimal extraction and consumption plan that will lead to this sustainable equilibrium state. The analysis continues by deriving the steady state.

At the steady state $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_3 = \dot{K} = \dot{X} = \dot{Z} = 0$

From (5):

$$Z^* = \frac{g-r}{\beta} \qquad (10)$$

From (7):

$$K\left[S\left[\frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}\right]^{\frac{1-\alpha}{\alpha}} - \phi\right] = 0$$

Let:

$$Q = \frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}$$

Therefore:

$$SQ^{\frac{1-\alpha}{\alpha}} - \phi = 0$$

And

$$Q^{\frac{1-\alpha}{\alpha}} = \frac{\phi}{S} \qquad (11)$$

Therefore:

$$Q^{\frac{1}{\alpha}} = \left(\frac{\phi}{S}\right)^{\frac{1}{1-\alpha}} \tag{12}$$

Substitute (10) and (11) into (9) gives:

$$K^* = \left[\frac{(\beta\theta - \delta(g-r))S}{\beta\gamma\phi}\right] \tag{13}$$

Substitute (10), (12) and (13) into (8) gives:

$$X^* = \frac{[\beta\theta - \delta(g-r)]S}{r\gamma\phi\beta}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}} \tag{14}$$

$Z^*$, $X^*$ and $K^*$ are steady state values for $Z$, $X$ and $K$ respectively. In equations (4), (6) and (12) there are 3 equations with 3 unknowns, $\lambda_1$, $\lambda_2$ and $\lambda_3$, (the shadow prices of capital, stock of the renewable natural resource and the stock of pollution).

Solving for the steady state of these equations gives (see appendix B):

$$\lambda_1^* = \frac{\alpha w(\gamma\alpha - \alpha) - (\partial - r)(1-\alpha)(1-S)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}}{(r+\phi)(1-\alpha)\left[(\delta - r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}} p - \frac{[\beta\theta - \delta(g-r)]S}{r\gamma\phi}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}}(\gamma\alpha - \alpha)\right]}$$

$$\lambda_2^* = \frac{w(\gamma\alpha - \alpha) - (\partial - r)(1-\alpha)(1-S)}{\left[(\delta - r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}} p - \frac{[\beta\theta - \delta(g-r)]S}{r\gamma\phi}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}}(\gamma\alpha - \alpha)\right]}$$

$$\lambda_3{}^* = w + \frac{\beta}{(\delta - r)} \left[ \frac{w(\gamma\alpha - \alpha) - (\partial - r)(1-\alpha)(1-S)}{\left[ (\delta - r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}} P - \frac{[\beta\theta - \delta(g-r)]S}{r\gamma\phi}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}}(\gamma\alpha - \alpha) \right]} \right] \left[\frac{(\beta\theta - \delta(g-r))S}{r\gamma\beta}\right]\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}$$

A simpler way of displaying these results is to use matrix notation (see appendix B):

$$\begin{bmatrix} \lambda_1{}^* \\ \\ \lambda_2{}^* \\ \\ \lambda_3{}^* \end{bmatrix} = \begin{bmatrix} 0 & \frac{[\beta\theta - \delta(g-r)]}{r\gamma}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & (r-\delta) \\ (r+\phi) & \frac{-\alpha}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & 0 \\ S & \frac{-1}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & \gamma \end{bmatrix}^{-1} \begin{bmatrix} -w \\ \\ 0 \\ \\ -(1-S) \end{bmatrix}$$

Using the equation $\dot{K} = SK^{\alpha}U^{1-\alpha} - \phi K = 0$ to solve for the steady state value for $U$ we get:

$$U^* = \left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}} \left[\frac{(\beta\theta - \delta(g-r))S}{\beta\gamma\phi}\right] \qquad (22)$$

Again using the same equation as above we can solve for the steady state value for output, $Q$, consumption, $C$ and savings, $S$.

$$Y^* = K^{\alpha}U^{1-\alpha} = \frac{\phi K^*}{S}$$

Therefore:

$$\left.\begin{aligned} Y^* &= \frac{\phi}{S}\left[\frac{(\beta\theta - \delta(g-r))S}{\beta\gamma\phi}\right] \\ C^* &= (1-S)\frac{\phi}{S}\left[\frac{(\beta\theta - \delta(g-r))S}{\beta\gamma\phi}\right] \end{aligned}\right\} \tag{23}$$

To make economic sense the steady state values of *Y*, *C*, *U*, *X*, *K* and *Z* cannot be negative. Therefore the following restrictions on the parameters are assumed to hold:

$$g > r$$
$$\beta\theta > \delta(g-r) > 0$$

Rearranging the second equation gives:

$$\frac{r}{\beta} > \frac{g}{\beta} - \frac{\theta}{\delta} > 0$$

The first condition demands that the natural growth rate of the resource must be greater than the discount rate. The right hand side of the second condition demands that level of pollution which kills the natural growth process of the resource must be greater than the level of pollution that kills the assimilitive capacity of the environment. The difference between these capacity constraints must be less than the ratio of the discount rate to the rate at which regeneration falls.

# SECTION IV

## Stability

It is now important to analyse the system to determine the stability of the steady state. To do this it is necessary to determine the characteristic roots (eigenvalues) of the Jacobian matrix evaluated at the steady state. If the all the roots have positive real parts then the system is unstable. If the roots have negative real parts then the system is stable. Determining the eigenvalues of the Jacobian evaluated at the steady state in this model becomes a very intractable problem to solve as there are 6 simultaneous non-linear differential equations. The Jacobian is:

$$J = \begin{bmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial X} & \frac{\partial \dot{K}}{\partial Z} & \frac{\partial \dot{K}}{\partial \lambda_1} & \frac{\partial \dot{K}}{\partial \lambda_2} & \frac{\partial \dot{K}}{\partial \lambda_3} \\ \frac{\partial \dot{X}}{\partial K} & \frac{\partial \dot{X}}{\partial X} & \frac{\partial \dot{X}}{\partial Z} & \frac{\partial \dot{X}}{\partial \lambda_1} & \frac{\partial \dot{X}}{\partial \lambda_2} & \frac{\partial \dot{X}}{\partial \lambda_3} \\ \frac{\partial \dot{Z}}{\partial K} & \frac{\partial \dot{Z}}{\partial X} & \frac{\partial \dot{Z}}{\partial Z} & \frac{\partial \dot{Z}}{\partial \lambda_1} & \frac{\partial \dot{Z}}{\partial \lambda_2} & \frac{\partial \dot{Z}}{\partial \lambda_3} \\ \frac{\partial \dot{\lambda}_1}{\partial K} & \frac{\partial \dot{\lambda}_1}{\partial X} & \frac{\partial \dot{\lambda}_1}{\partial Z} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_3} \\ \frac{\partial \dot{\lambda}_2}{\partial K} & \frac{\partial \dot{\lambda}_2}{\partial X} & \frac{\partial \dot{\lambda}_2}{\partial Z} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_3} \\ \frac{\partial \dot{\lambda}_3}{\partial K} & \frac{\partial \dot{\lambda}_3}{\partial X} & \frac{\partial \dot{\lambda}_3}{\partial Z} & \frac{\partial \dot{\lambda}_3}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_3}{\partial \lambda_2} & \frac{\partial \dot{\lambda}_3}{\partial \lambda_3} \end{bmatrix}$$

There is no simple closed-form expression for the eigenvalues of this matrix. Therefore a new method will be used to analyse the stability of the steady state.

Using corollary 2c in Sorger (1989), (see methodology), it can be shown that the steady state is globally asymptotically stable for bounded solutions.

The equilibrium point of this system possess the saddle point property if the 'curvature matrix,'

$$C = \begin{bmatrix} H_{ii}^{*} & \left(\frac{\delta}{2}\right) I_n \\ \left(\frac{\delta}{2}\right) I_n & -H_{jj}^{*} \end{bmatrix}$$

is negative definite, where $H^*$ is the maximised Hamiltonian and $i = K, X, Z$, and $j = \lambda_1, \lambda_2, \lambda_3$ and $n = 3$. Cass and Shell (1976) show that the Hamiltonian is convex in the costate and concave in the state for problems with a concave objective function and so the matrices $H_{ii}^{*}$ and $H_{jj}^{*}$ are negative definite. The objective function in this model is concave (see appendix A), therefore in this model the matrices $H_{ii}^*$ and $H_{jj}^*$ are negative definite. As the matrices $H_{ii}^*$ and $H_{jj}^*$ are negative definite with minimum eigenvalues below zero, the curvature matrix $C$ is negative definite with a low rate of discount, (Brock and Scheinkman (1976)). As the curvature matrix $C$ is negative definite then the equilibrium point i.e. the steady state, is globally stable for bounded solutions, Sorger (1989). This implies that with any initial levels of capital and resource stocks in the environment, the optimal path converges toward a unique optimal steady state.

# SECTION V

## Comparative Statics.

A sensitivity analysis was carried out on the steady state solutions of the variables shown in the following table to determine their dependency on the parameters. The results are given below:

| | $g$ | $\beta$ | $\gamma$ | $\delta$ | $w$ | $r$ | $\alpha$ | $\phi$ | $\theta$ | $S$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $K^*$ | - | + | - | - | 0 | + | 0 | - | + | + |
| $X^*$ | - | + | - | - | 0 | + | $\gtreqless 0$ as $\phi \gtreqless S$ | + | + | - |
| $Z^*$ | + | + | 0 | 0 | 0 | - | 0 | 0 | 0 | 0 |
| $U^*$ | - | + | - | - | 0 | + | $\gtreqless 0$ as $\phi \gtreqless S$ | + | + | - |
| $Y^*$ | - | + | - | - | 0 | + | 0 | 0 | + | 0 |
| $C^*$ | - | + | - | - | 0 | + | 0 | 0 | + | - |

Regarding $w$; it is interesting to see that the results show that if there is any change in the social preference towards the stock of pollution, $w$ then there is no change in any of the steady state solutions, i.e. the steady state does not depend on tastes.

Regarding $g$; if there was an increase in the natural growth rate, $g$, then we would expect that the steady state level of resource would rise along with the harvest rate. This would increase output, consumption and savings and therefore the level of pollution. Therefore we would expect that all the steady state solutions would increase when there was an increase in $g$. As we can see, the comparative static results seem contradictory to a-priori expectations. However, a feasible explanation for some of the solutions is that if there is an increase in the natural growth rate then it would be possible to run down the level of resource stock to a lower level because of its capacity for higher growth. Therefore the steady state level, $X^*$, may be lower. During this time period, the level of harvest would increase, output would increase and due to this, the steady state level of pollution would rise. At the steady state there is now less of the resource to harvest as it has been run down. Therefore output, consumption and the harvest rate will be lower in the steady state; though they may, of course, have been higher along the transition path. Another reason why the steady state level of the resource stock may be lower is because as the stock of pollution builds up it will affect the growth rate of the resource. Therefore the steady state values of output, consumption and resource extraction could be lower.

Regarding $\delta$; an increase in $\delta$,the rate at which the rate of assimilated waste falls, (i.e. the capability of the environment to clean itself up is falling at a faster rate) would be expected to result in an increase in the pollution stock. However the table above gives us contradictory results. If there was a rise in the level of pollution as would be expected, then there would be fall in the level of resource stock as a rise in the pollution stock would affect the growth of the resource. For example, as a lake becomes more and more polluted because the capability to cleanse itself has fallen, the less the fish stock be able to regenerate. Eventually the pollution will start to destroy the growth process and the steady state stock of fish will reduce. The steady state level of harvest will also fall as there is less of the fish to harvest. With $\gamma$ constant, and the level of pollution rising, output would have to fall to maintain a steady state stock of pollution. If output falls then there would be less output to consume therefore consumption and savings would also fall.

Regarding $\beta$; an increase in $\beta$, the rate at which the growth rate falls, means that the growth rate is now falling at a faster rate. If the growth of the resource is now more affected by the level of pollution we would expect that the steady state level of the resource stock to fall. This would imply lower output and consumption. Using the previous example, an increase in $\beta$ would mean that the growth rate of fish stock in a lake is now more affected by the level of pollution and the fish are now regenerating at a slower rate. This would imply that there is less of the resource stock to harvest therefore less output, which in turn would reduce consumption and savings.

The results show otherwise, but there is a possible explanation for this. With an increase in $\beta$, implying a lower growth rate at each pollution level, it is likely that the resource would still be growing. Therefore at the steady state there would be more to harvest, i.e. $U^*$ would be higher and hence $Y^*$ and $C^*$ would increase. Due to the increased output, the steady state level of the pollution stock would rise.

Regarding $\phi$; if there was an increase in the rate of depreciation on capital we would expect the level of capital to fall. If $g = r$ then the results show that capital would fall. To maintain output, i.e. no change in the steady state level of output, more of the resource would be needed in production (assuming that capital and the resource are substitutable), therefore the rate of harvest would increase and the resource stock would decrease. If output remains unchanged, consumption and savings would also remain at their original steady state level. This is shown by the results above. The results imply that there is a maximum level for capital used in production; this is feasible as there is only so much we can invest in new capital. Also there is a maximum level of the stock of the resource; this would also be feasible because, for example, a lake could only hold a certain amount of fish and after some maximum level there would be overcrowding and fish would start to die. However the results imply a minimum level on the extraction rate, this might seem unfeasible but it could be explained by the fact that if there is a maximum level on the use of capital, then to get a certain amount of output there would be a minimum amount of the resource you need to use in the production process to achieve that output. In that case the level of output would not change, neither would consumption and savings.

Regarding $\gamma$, if there was an increase in the proportion of output that turns into emissions that flow into the environment we would expect the level of pollution stock to rise. However the results show that there is no change in the steady state level of pollution. If there was more pollution, the resource stock would fall. This would definitely be the case if $g = r$ ( i.e. we are taking no more out of the environment than can be produced). The resource stock would fall because there is now a higher level of pollution affecting the regeneration capacity. With less resource stock there is less to harvest, therefore $U^*$ would fall. In turn this would mean output, consumption and savings would also fall. If the growth rate was large then the level of resource stock could still rise even though the level of pollution has risen. It seems all the results can be explained except the fact that an increase in $\gamma$ doesn't affect the level of pollution stock.

Regarding r; a counter-intuitive result is the effect of a change in the discount rate, $r$. With a rise in $r$ we would anticipate a rise in output at the beginning of the plan. But as the economy depends so much on the natural resource for production, increased extraction of the resource would run down the stock and there would be less left for future production. Therefore the steady state level of output and consumption would be lower. This is the underlying argument for sustainable development. If we use up our resources unsustainably then the future economy is put at risk. The table shows a different story. In the standard neoclassical optimal growth model, Cass (1965), a higher discount rate results in higher consumption at the start of the plan and lower steady state consumption. Here, the position is not so clear cut. An increase in consumption early in the plan results in lower output and less pollution. The lower pollution increases the growth rate of the resource and therefore it is possible to increase the extraction of the resource so that steady state output and consumption in this model can actually be higher. The steady state stock of the resource could still be higher even though it is possible to increase the extraction rate, as it is feasible that the resource is still growing along the transition path. Of course, at the steady state there is no change in the resource stock.

Regarding $\theta$, represents the natural assimilative capacity of the environment. If this increased then at each level of pollution, the pollution stock would be cleaned up at a faster rate and we would expect the steady state stock of pollution to fall. With less pollution there would be a greater amount of the resource stock as there is less pollution to affect the growth rate of the resource. With a greater amount of resource stock there would be more to harvest therefore there would be a greater amount of output and thereby consumption and savings would increase. The results show that there is an increase in all the steady state values as expected except that there would in fact be no change in the steady state stock of pollution.

Regarding $S$; an increase in $S$ implies that there is an increase in the rate of saving. The comparative static results are as one would expect. They show that there would be an increase in the steady state level of the capital stock and a fall in the level of consumption. With an increase in $S$ and a fall in the steady state level of consumption, then it is feasible that the steady state level of output would not change as the identity $Y = C + I$ is assumed. If more is saved then more capital can be used for future output and the level of the resource stock and the level of resource extraction can be lower in the steady state.

Regarding $\alpha$; an increase in $\alpha$ implies that the share of output accuring to capital has increased and that the marginal productivity of capital has increased. Output is given by:

$$Y = K^{\alpha} U^{1-\alpha}$$

The marginal productivity of capital is thus given by:

$$\frac{\partial Y}{\partial K} = \alpha K^{\alpha-1} U^{1-\alpha}$$

or

$$\frac{\partial Y}{\partial K} = \alpha \frac{U^{1-\alpha}}{K^{1-\alpha}} = \alpha \frac{Y}{K}$$

This shows that an increase in $\alpha$ increases the marginal productivity of capital. In the comparitive statics the result shows that $\alpha$ does not affect the steady state level of the capital stock. In the standard neoclassical optimal growth model, Cass (1965), this is not the case and $\alpha$ does affect the steady state level of capital stock. Here $\alpha$ also does not affect the steady state level of the pollution stock, the level of consumption or the level of output. The only effect that $\alpha$ has is on the steady state level of the resource extraction and the steady state level of the resource stock. There is also no effect on them if the level of capital depreciation is equivalent to the savings rate.

## CONCLUSION

In this chapter a new dynamic optimisation model has been developed in which there is an intertemporal trade off between producing output for consumption and the level of pollution. The model highlights the effect that pollution has on human welfare and on the environment, i.e. the effect on the regeneration rate of a renewable resource and the effect on the assimilative capacity of the environment. It has been recognised that there is a weakness in the literature on environmental-economic models in that they fail to take into account all aspects of their interaction. From various models presented in section I, it can be seen that this is the case and it can be said that models are over simplified to gain analytical simplicity. For instance, pollution is usually treated as a flow that doesn't build up into a stock in the environment, Keeler et al (1972), Gruver (1976), and Forster (1973a). Siebert (1982) and Barbier(1989) do include pollution stock in their models and natural resource use but not capital accumulation, therefore pollution is genererated from consumption. Other models do include production but do not include natural resource use, van den Bergh and Nijkamp (1991) and Brock (1977). Most of these models also treat the natural rate of assimilation as constant, others include, Plourde (1972) and D'Arge (1971). It has been argued in Forster (1975) and Smith (1977) and in this chapter that this is an unrealistic assumption and also serves to allow analyitical simplicity. Here the natural decay rate does depend on the level of pollution stock in the environment and is a decreasing function of that stock.

It is argued here that the current environmental-economic models, do not take account of all the necessary interactions that occur. The closest model is that of Tahvonen and Kuuluvainen, but here there are assumptions that are unrealistic. For instance, pollution emissions are treated as a factor of production and also the natural decay rate is constant. They also ignore capital depreciation.

The model in this chapter combines characteristics of other models not previously brought together into one optimal control model. It is a more realistic version of Tahvonen and Kuuluvainens' model in that takes into account the aspects that they have neglected.

It is found that an optimal and sustainable consumption and resource harvesting policy does exist. Sustainable steady state solutions of the variables are derived and sufficient conditions for the existence of the steady state are given. New theory on the stability of dynamical systems is used and it is shown that when the rate of discount is small enough all bounded solutions converge to a unique optimal steady state. Therefore the same result as Tahvonen and Kuuluvainen is shown to exist under more realistic assumptions. The effect that changes in the parameters of the problem have on the steady states are also examined and the results discussed.

# APPENDICES

## APPENDIX A

### Concavity of the objective function, *W*

$$W = (1-S)K^{\alpha}U^{1-\alpha} - wZ$$

For this to be concave in *K*, *U* and *Z*, then the matrix of second derivatives must have non-positive eigenvalues.

Therefore we need to find the eigenvalues of the matrix:

$$\begin{bmatrix} \frac{\partial W^2}{\partial K \partial K} & \frac{\partial W^2}{\partial K \partial U} & \frac{\partial W^2}{\partial K \partial Z} \\ \frac{\partial W^2}{\partial U \partial K} & \frac{\partial W^2}{\partial U \partial U} & \frac{\partial W^2}{\partial U \partial Z} \\ \frac{\partial W^2}{\partial Z \partial K} & \frac{\partial W^2}{\partial Z \partial U} & \frac{\partial W^2}{\partial Z \partial Z} \end{bmatrix}$$

Therefore the matrix is:

$$\begin{bmatrix} (\alpha-1)\alpha(1-S)K^{\alpha-2}U^{1-\alpha} & (1-\alpha)\alpha(1-S)K^{\alpha-1}U^{-\alpha} & 0 \\ (1-\alpha)\alpha(1-S)K^{\alpha-1}U^{-\alpha} & -(1-\alpha)\alpha(1-S)K^{\alpha}U^{-\alpha-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $B = (1-\alpha)\alpha(1-S)$, therefore:

$$\begin{bmatrix} -BK^{\alpha-2}U^{1-\alpha} & BK^{\alpha-1}U^{-\alpha} & 0 \\ BK^{\alpha-1}U^{-\alpha} & -BK^{\alpha}U^{-\alpha-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The characteristic equation is:

$$-\lambda\left[\left(-BK^{\alpha-2}U^{1-\alpha}-\lambda\right)\left(-BK^{\alpha}U^{-\alpha-1}-\lambda\right)-\left(BK^{\alpha-1}U^{-\alpha}\right)\left(BK^{\alpha-1}U^{-\alpha}\right)\right]=0$$

where λ represents the eigenvalues.

Multiplying out gives:

$$-\lambda\left[B^{2}K^{2\alpha-2}U^{-2\alpha}+\lambda^{2}+\lambda BK^{\alpha}U^{-\alpha-1}+\lambda BK^{\alpha-2}U^{1-\alpha}-B^{2}K^{2\alpha-2}U^{-2\alpha}\right]=0$$

Cancelling out gives:

$$-\lambda\left[\lambda(\lambda+BK^{\alpha}U^{-\alpha-1}+BK^{\alpha-2}U^{1-\alpha})\right]=0$$

The eigenvalues are:

$$\lambda=0,\lambda=0,\lambda=-BK^{\alpha}U^{-\alpha-1}-BK^{\alpha-2}U^{1-\alpha}$$

Therefore all eigenvalues are non-positive and the objective function is therefore concave.

# APPENDIX B

## Steady State Analysis

At the steady state, $\dot{\lambda}_1=\dot{\lambda}_2=\dot{\lambda}_3=0$

Substituting (12) into (4) gives:

$$\lambda_1=\frac{\alpha\lambda_2}{(r+\phi)(1-\alpha)}\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}} \tag{24}$$

Rearranging (12) we get:

$$\lambda_1(S-\alpha S)+(1-\alpha)(1-S+\gamma\lambda_3)=\lambda_2\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}} \tag{25}$$

Substituting (24) into (25) gives:

$$\frac{\alpha\lambda_2}{(r+\phi)(1-\alpha)}\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}(S-\alpha S)+(1-\alpha)(1-S+\gamma\lambda_3)=\lambda_2\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}$$

Therefore rearranging:

$$\lambda_3=\frac{\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}\lambda_2\left[\frac{\alpha(S-\alpha S)-(r+\phi)(1-\alpha)}{(r+\phi)(1-\alpha)}\right]+(1-\alpha)(1-S)}{(\gamma\alpha-\alpha)}$$

Let:

$$P=\frac{\alpha(S-\alpha S)-(r+\phi)(1-\alpha)}{(r+\phi)(1-\alpha)} \tag{26}$$

Therefore $\lambda_3$ becomes:

$$\lambda_3=\frac{\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}\lambda_2 P+(1-\alpha)(1-S)}{(\gamma\alpha-\alpha)}$$

From (6):

$$\lambda_3=\frac{w+\beta\lambda_2 X}{\delta-r} \tag{27}$$

Substituting (27) into (26) gives:

$$\frac{\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}\lambda_2 P+(1-\alpha)(1-S)}{(\gamma\alpha-\alpha)}=\frac{w+\beta\lambda_2 X}{\delta-r}$$

Rearranging gives:

$$\lambda_2\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\beta X(\gamma\alpha-\alpha)\right]=w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)$$

Therefore:

$$\lambda_2{}^*=\frac{w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)}{\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\beta X(\gamma\alpha-\alpha)\right]}$$

$$\lambda_2{}^*=\frac{w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)}{\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\frac{[\beta\theta-\delta(g-r)]S}{r\gamma\phi}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}}(\gamma\alpha-\alpha)\right]} \qquad (28)$$

Substituting (28) into (24) gives:

$$\lambda_1{}^*=\frac{\alpha w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}}{(r+\phi)(1-\alpha)\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\beta X(\gamma\alpha-\alpha)\right]}$$

substituting X* into the above gives:

$$\lambda_1{}^*=\frac{\alpha w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}}{(r+\phi)(1-\alpha)\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\frac{[\beta\theta-\delta(g-r)]S}{r\gamma\phi}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}}(\gamma\alpha-\alpha)\right]} \qquad (29)$$

Substituting (28), and (14) into (27):

$$\lambda_3{}^{*}=w+\frac{\beta}{(\delta-r)}\left[\frac{w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)}{\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\beta X(\gamma\alpha-\alpha)\right]}\right]\left[\frac{(\beta\theta-\delta(g-r))S}{r\gamma\beta}\right]\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}$$

Substituting X* into this gives:

$$\lambda_3{}^{*}=w+\frac{\beta}{(\delta-r)}\left[\frac{w(\gamma\alpha-\alpha)-(\partial-r)(1-\alpha)(1-S)}{\left[(\delta-r)\left[\frac{\phi}{S}\right]^{\frac{\alpha}{1-\alpha}}P-\frac{[\beta\theta-\delta(g-r)]S}{r\phi\gamma}\left[\frac{\phi}{S}\right]^{\frac{1}{1-\alpha}}(\gamma\alpha-\alpha)\right]}\right]\left[\frac{(\beta\theta-\delta(g-r))S}{r\gamma\beta}\right]\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}$$

A simpler way of displaying these results is to use matrix formation:

Therefore rearranging (24) we get:

$$(r+\phi)\lambda_1-\frac{\alpha\lambda_2}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{(1-\alpha)}}=0 \tag{31}$$

Rearranging (27):

$$(\delta-r)\lambda_3-\beta\lambda_2X=w \tag{32}$$

Therefore substituting (14) into (32) gives:

$$(\delta-r)\lambda_3-\lambda_2\left[\frac{\beta\theta-\delta(g-r)}{r\gamma}\right]\left(\frac{\phi}{S}\right)^{\frac{\alpha}{(1-\alpha)}}=w \tag{33}$$

Rearranging (33) gives:

$$\lambda_2\left[\frac{\beta\theta-\delta(g-r)}{r\gamma}\right]\left(\frac{\phi}{S}\right)^{\frac{\alpha}{(1-\alpha)}}+(r-\delta)\lambda_3=-w \tag{34}$$

Using (12)gives:

$$\frac{(1-\alpha)(1-S+S\lambda_1+\gamma\lambda_3)}{\lambda_2}=\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}$$

Rearranging we get:

$$(1-\alpha)(1-S+S\lambda_1)+\gamma(1-\alpha)\lambda_3-\lambda_2\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}=0$$

Dividing through by (1-$\alpha$):

$$(1-S+S\lambda_1)+\gamma\lambda_3-\frac{\lambda_2}{(1-\alpha)}\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}=0$$

Therefore, rearranging:

$$S\lambda_1+\gamma\lambda_3-\frac{\lambda_2}{(1-\alpha)}\left[\frac{\phi}{S}\right]^{\frac{\alpha}{(1-\alpha)}}=-(1-S) \tag{35}$$

Therefore using equations (31), (34) and (35) we can show the steady state solutions in matrix form:

$$\begin{bmatrix} 0 & \frac{[\beta\theta-\delta(g-r)]}{r\gamma}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & (r-\delta) \\ (r+\phi) & \frac{-\alpha}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & 0 \\ S & \frac{-1}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & \gamma \end{bmatrix}\begin{bmatrix} \lambda_1 * \\ \lambda_2 * \\ \lambda_3 * \end{bmatrix} = \begin{bmatrix} -w \\ 0 \\ -(1-S) \end{bmatrix}$$

Rearranging gives:

$$\begin{bmatrix} \lambda_1 * \\ \lambda_2 * \\ \lambda_3 * \end{bmatrix} = \begin{bmatrix} 0 & \frac{[\beta\theta-\delta(g-r)]}{r\gamma}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & (r-\delta) \\ (r+\phi) & \frac{-\alpha}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & 0 \\ S & \frac{-1}{(1-\alpha)}\left(\frac{\phi}{S}\right)^{\frac{\alpha}{1-\alpha}} & \gamma \end{bmatrix}^{-1}\begin{bmatrix} -w \\ 0 \\ -(1-S) \end{bmatrix}$$

# CHAPTER SIX

# ECONOMIC GROWTH AND THE PRESERVATION OF NATURAL ENVIRONMENTS WITH A GROWING POPULATION

## Population Pressures

The cause of the environmental problems that we face today can be attributed to human activity - the more people, the more problems. A growing population puts pressure on both the economic and environmental systems. There is a need for more food, more water, more goods. The consequent rise in economic activity, as well as the direct impact of a greater amount of people, places an increasing burden on the eco-system. The question is, can the environmental and economic systems cope? For a system to be sustainable it must have the capacity to generate enough wealth to provide for investment and the maintenance of the environment as well as for the material requirements of the population.

Rapid population has increased the demand on the environment and natural resource supplies causing increasing degradation to the environment. For example, the Brazilian rain forest is being cut down at an alarming rate to make way for the increased need for agricultural production to support the ever increasing number of people. Population growth is putting increasing pressure on the depletion of our natural resources. The neo-Malthusian's view is that more people will lead to an increased use of resources and this will eventually lead to a shortage. They argue that the loss of natural resources that are essential to life could come as a Holocaust (the bang), or could come slowly and painfully (the whimper) as a result of the consequences of over population. Whatever happens, they say we are doomed. Simon (1990), argues that the increased demand for new supplies. He argues that eventually new sources will be found. But how can a finite world supply us with an infinite amount of resources?

It is true that as resources start to become more scarce their price will rise and this will encourage the owners of the resource to conserve their supply. Shortages may arise because individual countries or regions will be unable to pay for the resources they need. But in the aftermath of the oil shocks in the mid-1970's when there was increasing prices for oil, there were new discoveries of fossil fuels. Known fossil fuel reserves have risen much faster than consumption in the past forty years. The world's reserves of oil and natural gas stood at 30 billion tons of oil equivalent in 1950, and today there are more than 250 billion tons, even though world oil consumption has totalled 100 billion tons in the intervening years, (World Bank (1993)).

But even if the world as a whole has enough, individual countries and regions may be faced with serious shortages. For example, Asian countries do not have enough energy resources, particularly oil and gas, of their own to meet demands.

Thomas Malthus at first wrote about the dangers of population outstripping food supply due to the shortage of good farmland. He was writing at the start of the nineteenth century, when accelerating population and industrial growth were raising demands for food faster than English agriculture could respond, (Brinley (1985)). However his predictions were confounded by the increase in international trade and technological change.

Since Malthus first worried about the possibility of mass starvation, the world's output of food has risen faster than its population. For example, in India between 1950 and 1990, food output increased by 2.7 per cent and its population increased on average by 2.1 per cent. However, although the global averages conceal some countries which are finding food resources increasingly scarce, some of the countries of sub-Saharan Africa, Nepal and Bangladesh are becoming increasingly hungry (World Bank (1984)).

Neo-classical economists emphasise the ability of the market to adapt to resources scarcities over time. If future scarcities are expected then businesses will hold the supplies of the resource off the market anticipating future prices to increase and they will invest in discovering and developing new supplies. Prices can guide market adjustment processes but this presupposes that those resources have a price, i.e. they can be bought and sold. This in turn implies that they are owned. However, many important resources are not privately owned. The sustainability of common property resources requires collective control, through legal and regulatory restraints and by decisions of the leaders in societies. If the amount of the resource is vast then users will assume that the impact of their actions will have little if any effect on it and that any sacrifice by them to preserve it will have an insignificant effect. These resources are treated as free goods and are at risk. Some of these resources are renewable, such as the tropical forests, but some of the exhaustible resources may be more secure than some that are renewable.

The ideas of Malthus have resurfaced in the latter half of this century as concerns about the environment have grown. It has been argued that the world would simply run out of the essential raw materials, such as coal and oil, and so nature would limit growth. In 1972 the Club of Rome, a group of eminent people, produced the Limits To Growth (Meadows et al (1972)). The conclusion is extremely gloomy. They predicted that:

> "If the present growth trends in population, industrialisation, pollution, food production, and resource depletion continue unchanged, the limits to growth on this planet will be reached in some time within the next 100 years", (Meadows et al (1972)).

However, after the publication of the Limits to Growth, the world's reserves of some know fossil fuels have risen faster than consumption, consumption has not outstripped production, as pointed out earlier; the world's reserves of oil and gas have risen. The lesson of the years since this publication is that it is not necessarily the case that exhaustible resources will be exhausted. It is argued that the market will protect natural resources from over-use. But, as mentioned earlier, those environmental resources that are most at risk from over exploitation are not those that the Club of Rome were worried about, but the free environmental goods such as the tropical forests, the fish in the world's seas and the supply of fresh water.

## Literature on Population models

Stimulated by the Limits to Growth and environmental and energy concerns, a vigorous literature has evolved on natural resources and the environment.

There is an extensive literature on economic growth with exhaustible resources as essential inputs in the production process where population is treated as stationary (Dasgupta and Heal (1975), Solow (1974) and Dasgupta and Heal (1979)).

Solow (1974), starts his paper with the simplest of cases - modelling capital accumulation with constant population, no technical progress and no scarce resources, i.e. the production function is:

$$Q = F(K, L)$$

where $K$ is the capital stock and $L$ is the flow of labour services. Since $Q$ is net output to be produced under constant returns to scale.

$$Q = C + \dot{K}$$

where $C$ is consumption and $\dot{K}$ is investment. The dynamic constraint is:

$$\dot{K} = Q[F(K, L)] - C \qquad (1)$$

He then introduces an exponentially growing population and assumes:

$$L = L_0 e^{nt}, \qquad \text{i.e.} \quad \dot{L} = nL \qquad (2)$$

Using:

$$Q = F(K, L) = Lf(k)$$

where $k=K/L$, the capital to labour ratio. equation (1) becomes:

$$\frac{\dot{K}}{L} = f(k) - c \tag{3}$$

where $c=C/L$, the consumption per capita.

Using equation (2) and the capital resource ratio we get:

$$\dot{K} = \dot{k}L + knL$$

Substituting this into (3) gives:

$$\frac{\dot{k}L + knL}{L} = f(k) - c$$

Therefore:

$$\dot{k} = f(k) - nk - c$$

He then extends his model to take into account exhaustible resources; the production function becomes:

$$Q = F(K, L)R^h \qquad 0 < h < 1$$

where $F$ is homogeneous of degree $1\text{-}h^2$, $R$ is the rate of flow of a natural resource and is an essential input in the productive process. He then argues that a model with exponential population growth seems ridiculous.

> "We all know that population cannot grow forever if only for square footage reasons. The convention of exponential population growth makes excellent sense as an approximation so long as population is well below its limit. On a time-scale appropriate to finite resources however, exponential growth of population is an

inappropriate idealisation. But then we might as well treat the population as constant." (See Solow (1974), p 36).

He then treats population growth as zero, i.e. $n = 0$.

Empirical evidence of population growth indicates that the population is not stationary. Given our empirical experience of population growth it is worthwhile to explore the consequences of a growing population in models involving a non-renewable resource (Stiglitz (1974a), Ingham and Simmons (1975), Cigno (1981) and Stiglitz (1974b)).

Ingham and Simmons (1975), examine optimal growth paths for an economy which does possess an exponentially growing labour force and a scarce non-renewable resource but with the particular criterion of intergenerational equity. In their model there is a single non-renewable natural resource that is extracted from the environment. A single composite commodity is produced from capital, labour and natural resource inputs with the assumption of constant returns to scale. Labour is assumed to grow at a constant exponential rate $n$. The problem is to:

$$\max \int_0^T L(t)U(c(t)).dt$$

subject to :

$$\dot{k}(t) = f(k(t), x(t)) - nk(t) - c(t)$$

where $U(c(t))$ is the welfare ascribed to a particular individual of generation $t$, $L$ is the labour force, $k = K/L$ the capital:labour ratio and $x = X/L$ the resource:labour ratio. Also the stock of the resource $X$ must be allocated to production over time so that:

$$\int_0^T L(t)x(t)dt = X$$

Labour grows at a rate n, i.e. $L(t) = e^{nt}$

Production does include the natural resource but they do not take into account the amenity value of the environment. The utility function only contains consumption as its argument. The purpose of their paper is to determine whether there are feasible and optimal growth paths under a variety of technological constraints. They show that the nature of the optimal paths will depend on the properties of the functions $f(k,x)$ and $U(c)$. They found that if the elasticity of substitution is constant i.e. using a CES production function:

$$f(k,x)=\left[\alpha_1 k^{-\beta}+\alpha_2 x^{-\beta}+\alpha_3\right]^{\frac{-1}{\beta}}$$

and is greater than unity and if population does not grow too rapidly in relation to the parameters of the production function, then it is not certain that an infinite horizon optimum exists. They show that if the time horizon is finite then an optimal solution exists.

Stiglitz (1974b), examines the implications of introducing exhaustible natural resources as an essential factor of production in the standard neo-classical growth model and he treats the rate of population growth as constant. In his model output is a function of labour, capital and natural resource inputs and it can be either consumed or invested.

$$Q=F(K,L,R,t)=K^{\alpha_1}L^{\alpha_2}R^{\alpha_3}e^{\lambda_t}=C+\dot{K}$$

where $\alpha_1+\alpha_2+\alpha_3=1$ , $\lambda$ is the rate of technical progress and $C$ is the level of consumption. Labour gross at a constant exponential rate, $n$, such that:

$$\dot{L}=nL$$

Also:

$$R=-\dot{S}$$

where $S$ is the stock of the resource. He shows that if there is a steady state solution, it is a saddle point. Hence introducing exhaustible resources into the model has the effect of making the system highly unstable. Stiglitz (1974a), shows that steadily growing per capita consumption may be feasible forever if a wasting and non-replenishable resource is indispensable to production and if population is growing at a constant exponential rate. He argues that even with a limited amount of resources, the economy need not decline. Technical change and capital accumulation will offset the effects of falling natural resource inputs. Capital accumulation alone could do this if the share of capital in production is greater than that of the natural resource. He argues that we can just use up our resources and that the technical change can offset the effects on output of a slowly declining natural resource input. Stiglitz then shows the optimal growth paths for an economy with exhaustible natural resources with the criterion of maximising per capita consumption. Stiglitz, along with others such as Dasgupta and Heal (1974), Solow (1974) and Kamien and Schwartz (1978), show that it may be optimal to completely exhaust a non-renewable resource if the availability of future technologies and perfect substitutes mean that depletion of the resource is no longer essential for future production.

Mitra (1983) is concerned with what patterns of population growth are consistent with the attainment of some social objectives in the presence of exhaustible resource constraints. He is not concerned with finding an optimal population policy where population or it's growth rate is treated as a control variable, but population is exogenously given and satisfies $L_0 = L, \quad L_{t+1} \geq L_t \quad \text{for } t \geq 0$. The problem he states is to determine precisely what population profiles are consistent (or inconsistent) with economic welfare objectives, first the attainment of a non-trivial "maximum" program, i.e. it is a maximum programme and can maintain a positive per capita consumption level, and secondly the attainment of an "optimal programme". Mitra examines the precise limitations that must be imposed on population growth, in order to attain these welfare objective.

He states that population growth should not be too fast and shows that population growth consistent with the welfare objectives is a "quasi-arithmetic progression", i.e. $L_t = (t+1)^{\lambda}$ for $t \geq 0$ and $\lambda > 0$, rather than a geometric progression.

Papers such as, Koopmans (1974), Lane (1977) and Dasgupta and Mitra (1979), deal with the problem of optimal population policies with exhaustible resource constraints where the population is treated as a control variable.

Cigno (1981) argues that in an economy constrained by exhaustible resources, the assumption of a constant rate of population growth is implausible. He examines the implications of making the population growth rate a function of consumption and capital per capita, therefore the growth rate is endogenously determined. He argues that both natality and mortality rates, the difference of which is the population growth rate, are bound up with a country's standard of living and degree of industrialisation. He postulates that the rate of population growth is positively related to per capita consumption and inversely related to the degree of industrialisation. He argues that this is consistent with empirical observations that at low levels of industrialisation, the rate of population growth tends to move in the same direction as per capita consumption, while at high levels of industrialisation, it tends to move in the opposite direction. Cigno uses Stiglitz's model but makes the minor alteration that in the dynamic equation, $\dot{L} = nL$, $n$ is not constant and he assumes that the rate of growth is:

$$n = \left\{(1-s)\left(\frac{Q}{L}\right)\right\}^{V_1}\left(\frac{K}{L}\right)^{V_2}$$

where $(1-s)\left(\frac{Q}{L}\right)$ represents the standard of living with s as the savings rate assumed constant, and $\frac{K}{L}$ represents the degree of industrialisation. He finds that an economy with exhaustible resources and an endogenously determined population is capable of stable growth. He shows that the stability of the system depends on the choice of savings/income ratio. An economy may be put on a stable path if a policy maker is able to control the choice of savings/income ratio. However, Cigno considers the dynamic model of an economy with a non-renewable resource as an essential factor of production but not in an optimising framework. The model presented in the following sections will be an extension of this model.

In the above articles, natural resources are treated as those environmental resources that provide us with valuable productive services; thus only one function of the natural environment is considered relevant to the aspect of natural resource scarcity and that is as a supplier of the raw material and energy inputs to the economic process.

The literature concerning natural resource depletion, population growth and the amenity values associated with the environment is much sparser.

Krautkraemer (1985), broadens the conventional approach to optimal resource depletion by taking into account the amenity value of preserved environments. The extraction of a non-renewable resource, which is an essential input into the productive process, irreversibly disrupts the natural environment where the resource is found. In his model the economy produces a composite commodity with inputs capital and the natural resource. Utility is a function of the flow of consumption and the amenity services from the resource stock. The commodity can be consumed or invested to increase future production. The problem is to choose the optimal path of extraction and investment so as to maximise the present value of utility.

$$\max \int_0^\infty e^{-\delta t} U(C(t), S(t)).dt$$

subject to:

$$\begin{aligned}
&\dot{S}(t) = -R(t) \\
&\dot{K}(t) = F[K(t), R(t)] - C(t) \\
&C(t), K(t), R(t), S(t) \text{ non-negative} \\
&K(0) = K_0; \quad S(0) = S_0
\end{aligned}$$

where $C$ is consumption, $S$ is the non-renewable resource stock, $R$ is the resource input to production, $K$ is capital input in production.

He finds that the optimal level of preservation will depend upon the initial capital stock.

> "If the elasticity of capital-resource substitution is greater than the inverse of the elasticity of the marginal utility of consumption, then an increase in the initial capital stock will increase the optimal level of preservation.", (p 165).

However, within his model there is the assumption that the population is constant, i.e. there is a stationary population.

Burt and Cummings' paper (1970), is concerned with developing a comprehensive model for simultaneously optimising the rate of resource extraction and investment in natural resource industries in general. They postulate that the model is sufficiently general to be applicable to any specific resource, i.e. non-renewable or renewable. They state that optimisation is viewed from the stand point of society and the level of population is treated as stationary. The social benefit function contains in it's arguments the level of resource stock, and the rate of resource use, at time *t*:

$$B_t(u_t, v_t, x_t, y_t)$$

where $u_t$, $v_t$, $x_t$, $y_t$ represent the rate of use of the resource, capital investment, resource stocks and amount of investment in the natural resource industry. This can be viewed as a welfare function that is taking into consideration the benefits of resource amenities and so the objective then would be to maximise these benefits to society. However there is no mention of environmental concerns in this paper. Their rationale for including the level of resource stocks in the social benefit function is that $x_t$, the resource stock at time $t$, reflects the accessibility of the resource. For example;

> " large stocks of fish permit capture of a given quantity at a lower cost, and large stocks of a mineral are associated with relatively accessible and rich ores." (Burt and Cummings 1970, p 578).

$u_t$ is included because the rate of use of the resource is assumed to have value in economic production or as a direct consumption good. The level of capital invested in the natural resource industry, $y_t$, is included for the reason that higher stocks of capital imply lower costs of production in each period.

The social benefit function is viewed as a profit function and results applying to the behaviour of the firm.

Clark (1976) considers problems of maximising discounted net revenue derived from the exploitation of renewable resources. He extends the theory of optimal exploitation of renewable resources to more complex biological models that involve age structure. However he does not take into account the amenity services provided by natural environments and there is no treatment of population, indicating that he assumes that population is stationary.

Beddington, Watts and Wright (1975) derive optimal time paths for the cropping of self-reproducible natural resources. In their paper they present four models of renewable resources that are inputs in the production process. Population is treated as constant and so doesn't enter into the analysis. In all the models the economic objective is to maximise the present value of profit obtained from the selling of the crop. However they do examine the conditions under which the resource becomes exterminated, but under the strategy of maximising discounted cash flow.

This chapter deals with the scenario of an economy that possess a single renewable resource that is extracted from a pre-existing pool. The resource is self-replenishable. This situation could apply to the cutting down of trees where the forest re-seeds itself. There are no controls on regeneration - no additional inputs, e.g. fertiliser. It is only managed only by cropping. The resource is used in the production of a single composite commodity which is either consumed or invested. The objective is to find the optimal extraction rate so as to maximise per capita consumption. However, here there is the additional constraint to the usual analysis - the population (which is assumed equivalent to the labour force) is growing at an endogenous rate. This chapter is an extension to the work of Cigno (1981) in that the full optimal solution to the problem will be derived. Also, renewable resources will be the essential factor of production rather than an exhaustible resource as in most of the previous literature.

In Section I, the model will be developed and the dynamical system explained. In Section II the formal optimal control problem will be presented and solved for the optimal sustainable steady state solution. In Section III it is shown that for the steady state to exist, there are restrictions on the range which the discount rate can take. In Section IV the stability of the system will be analysed and Section V considers the effects that changes in the paramenters of the model have on the steady state solutions. Section VI will offer some conclusions.

# SECTION I

## The Model

Consider an economy that possesses a single renewable natural resource that is extracted from a pre-existing pool. This resource is used in the production of a single composite commodity which is either consumed or invested. Efficient output possibilities for the commodity are given by the production function:

$$Y(t) = K^{\alpha_1} L^{\alpha_2} R^{\alpha_3} \tag{1}$$

where $K$ is the capital input, $L$ is the labour input and $R$ is the resource input, all at time $t$. The production function has the property that if there is no amount of the resource used in porduction then there will be no output, ie. if $R = 0$ then $Y = 0$. This is described by Dasgupta and Heal (1974) as an essential resource.

"..one regards a resource as being essential if output of final consumption goods is nil in the absense of the resource. Otherwise call the resource inessential." (Dasgupta and Heal (1974) p. 4).

The Production function is twice differentiable and strictly concave, i.e. ;

$$F_K > 0 \qquad F_{KK} < 0$$

$$F_L > 0 \qquad F_{LL} < 0$$

$$F_R > 0 \qquad F_{RR} < 0$$

We also assume the usual neoclassical assumption of constant returns to scale (linear homogeniety), i.e. $\alpha_1 + \alpha_2 + \alpha_3 = 1$

The development over time of the capital stock ($\dot{K}$) can be described by the dynamic equation:

$$\dot{K}(t) = K^{\alpha_1} L^{\alpha_2} R^{\alpha_3} - C \tag{2}$$

where $C$ is the consumption of the composite commodity. Output that is not invested is consumed; therefore output minus consumption, all at time $t$, will equal the change in the stock of capital at time $t$.

Here we are assuming that the costs of extracting the resource are zero and that extraction of the resource causes irreversible degradation to the environment. The change in the resource stock is given by:

$$\dot{S} = gS - R \tag{3}$$

where $S$ denotes the level of remaining resource stock at time $t$, $g$ is the growth rate of the renewable resource at time $t$ and $R$ is the level of resource extraction at time $t$, all of which is used up in production.

The available labour force is considered to be identical to the population and grows at a rate n. Therefore the change in the labour supply ( $\dot{L}$ ) is given by:

$$\dot{L} = nL \tag{4}$$

The growth rate of the population is not assumed to be constant as it is in most of the literature on natural resources. Here it is assumed that the growth rate, $n(t)$, of the population is a function of the country's standard of living and the degree to which the country is industrialised. This is an extension of an idea of Cigno (1981), where he argued that the variables sometime treated as constants in other models are infact actual economic variables themselves. He then let the rate of population growth be positively related to the country's standard of living, which he takes to be the level of *per capita* consumption, and inversely related to the degree of industrialisation. He argues that this is consistent with empirical observations that:

> "at low levels of industrialisation the rate of population growth tends to move in the same direction as per capita consumption, while at higher levels of industrialisation it tends to move in the opposite direction", (Cigno (1980), p 287).

He then lets the rate of population growth be given by:

$$n = \left[(1-w)\left(\frac{Y}{L}\right)\right]^{v_1}\left(\frac{L}{K}\right)^{v_2}$$

where $v_1$ and $v_2$ are positive constants representing the share of population growth that is attributed to per capita consumption and the degree of industrialisation, and $w$ is the savings rate, assumed constant.

Per capita consumtion is :

$$c = \left[(1-w)\left(\frac{Y}{L}\right)\right]$$

Therefore the growth function can be written:

$$n = c^{v_1}\left(\frac{L}{K}\right)^{v_2} \tag{5}$$

Now, writing the stock of capital, stock of resource, output and consumption in per capita terms:

$k = K/L$ $\qquad$ $c = C/L$ $\qquad$ $r = R/L$ $\qquad$ $y = Y/L$

i.e.

$K = kL$ $\qquad$ $C = cL$ $\qquad$ $R = rL$ $\qquad$ $Y = yL$

Where $\quad k$ = capital to labour ratio / the amount of capital per unit of labour.

$c$ = amount of consumption per unit of labour / consumption per capita.

$r$ = amount of resource flow used per unit of labour

$y$ = total output per unit of labour, i.e. average product of labour.

$$\dot{K} = (kL)^{\alpha_1} L^{\alpha_2} (rL)^{\alpha_3} - cL$$

Multiplying out gives:

$$\dot{K} = (L^{\alpha_1} L^{\alpha_2} L^{\alpha_3}) k^{\alpha_1} r^{\alpha_3} - cL$$

But since, $\alpha_1 + \alpha_2 + \alpha_3 = 1$ this can be simplified to:

$$\dot{K} = Lk^{\alpha_1} r^{\alpha_3} - cL \tag{6}$$

Looking at the resource stock equation:

$$\dot{S} = gsL - rL \tag{7}$$

Substituting $K = kL$ into the population growth function gives:

$$n = c^{v_1}\left(\frac{L}{kL}\right)^{v_2} = c^{v_1}k^{-v_2} \tag{8}$$

Therefore the change in the stock of labour/population is:

$$\dot{L} = c^{v_1}k^{-v_2}L \tag{9}$$

We now write the change in the capital stock and resource stock in per capita quantities:

From the above definitions, $K = kL$. Differentiating, we get:

$$\dot{K} = \dot{k}L + k\dot{L}$$

and, using equation (9):

$$\dot{K} = \dot{k}L + kc^{v_1}k^{-v_2}L$$

Equating this with equation (6):

$$\dot{K} = \dot{k}L + kc^{v_1}k^{-v_2}L = Lk^{\alpha_1}r^{\alpha_3} - cL$$

Rearranging gives:

$$\dot{k} = k^{\alpha_1}r^{\alpha_3} - kc^{v_1}k^{-v_2} - c \tag{10}$$

Using $S = sL$ and differentiating gives:

$$\dot{S} = s\dot{L} + \dot{s}L$$

Substituting equation (9) into this gives:

$$\dot{S} = sc^{v_1} k^{-v_2} L + \dot{s}L$$

Equating this with equation (7) gives:

$$\dot{S} = sc^{v_1} k^{-v_2} L + \dot{s}L = gsL - rL$$

and, rearranging:

$$\dot{s} = gs - r - sc^{v_1} k^{-v_2} \tag{11}$$

# SECTION II

## The formal problem

Utility is an increasing function of consumption. Therefore a reasonable objective for a social planner is to maximise per capita utility over time given the constraints on capital accumulation and the renewable resource stock described in equations (10) and (11) above.

Therefore the problem is to:

$$\max \int_0^\infty U(c)e^{-\psi t}.dt$$

subject to :

$$\dot{k} = k^{\alpha_1} r^{\alpha_3} - kc^{v_1} k^{-v_2} - c$$

$$\dot{s} = gs - r - sc^{v_1} k^{-v_2}$$

and the initial conditions:

$$k(0) = k_0 > 0$$

$$S(0) = S_0 > 0$$

where $\psi$ is the social rate of discount and where we assume for simplicity that $U(c) = \ln(c)$. Note that this utility function is strictly concave (i.e. exhibits diminishing marginal utility):

$$Uc > 0 \qquad\qquad Ucc < 0$$

The current value Hamiltonian is:

$$H = \left[\ln c + \lambda_1[k^{\alpha_1} r^{\alpha_3} - kc^{v_1} k^{-v_2} - c] + \lambda_2(gs - r - sc^{v_1} k^{-v_2})\right]$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the current value shadow prices associated with capital and resource stocks respectively.

First order necessary conditions for an interior solution are:

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda_1 - V_1 \lambda_1 k^{1-v_2} c^{v_1 - 1} - V_1 \lambda_2 s k^{-v_2} c^{v_1 - 1} = 0$$

$$\frac{\partial H}{\partial r} = \alpha_3 \lambda_1 k^{\alpha_1} r^{\alpha_3 - 1} - \lambda_2 = 0$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial k} + \psi \lambda_1$$

$$= \psi\lambda_1 - \alpha_1 \lambda_1 k^{\alpha_1 - 1} r^{\alpha_3} + \lambda_1 c^{v_1}(1 - v_2) k^{-v_2} - v_2 \lambda_2 s c^{v_1} k^{-(v_2 + 1)}$$

$$\lambda_2 = -\frac{\partial H}{\partial S} + \psi \lambda_2 = \psi \lambda_2 - \lambda_2 g + \lambda_2 c^{v_1} k^{-v_2}$$

Note that these conditions are also sufficient because of the concavity of the objective functional, see Mangasarian (1966).

In the long run, the optimal solution may "settle down" to an equilibrium point. This particular solution of a system of differential equations is known as the steady state. If the steady state exists, then the system, given the initial state values, is sustainable and there is an optimal extraction and consumption plan that will lead to this sustainable equilibrium state. The analysis continues by deriving the steady state.

The steady state solutions are characterised by:

$$\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{k} = \dot{s} = 0$$

Therefore, from equations (10) and (11):

$$k^{\alpha_1} r^{\alpha_3} - kc^{v_1} k^{-v_2} - c = 0 \tag{1}$$

$$gs - r - sc^{v_1} k^{-v_2} = 0 \tag{2}$$

And from the necessary conditions:

$$\frac{1}{c} - \lambda_1 - V_1 \lambda_1 k^{1-v_2} c^{v_1 - 1} - V_1 \lambda_2 s k^{-v_2} c^{v_1 - 1} = 0 \tag{3}$$

$$\alpha_3 \lambda_1 k^{\alpha_1} r^{\alpha_3 - 1} - \lambda_2 = 0 \tag{4}$$

$$\psi\lambda_1 - \alpha_1 \lambda_1 k^{\alpha_1 - 1} r^{\alpha_3} + \lambda_1 c^{v_1} (1 - v_2) k^{-v_2} - v_2 \lambda_2 s c^{v_1} k^{-(v_2 + 1)} = 0 \tag{5}$$

$$\psi\lambda_2 - \lambda_2 g + \lambda_2 c^{v_1} k^{-v_2} = 0 \tag{6}$$

There are 6 equations in 6 unknowns. These can then be solved for the steady state values of the variables $s$, $k$, $r$, $c$, $\lambda_1$, $\lambda_2$, denoted by $s^*$, $k^*$, $r^*$, $c^*$, $\lambda_1^*$, $\lambda_2^*$, all in terms of $k^*$.

From (6):

$$\lambda_2(\psi - g + c^{*v_1}\, k^{*-v_2}) = 0$$

Solving this for $c^*$, assuming $\lambda_2 \neq 0$:

$$c^* = (g-\psi)^{\frac{1}{v_1}} k^{*\frac{v_2}{v_1}} \qquad (7)$$

Solving for $r^*$:

Substituing (7) into (1) and solving for $r^*$ gives:

$$r^* = \left[ (g-\psi)k^{*1-\alpha_1} + (g-\psi)^{\frac{1}{v_1}} k^{*\frac{1}{v_2} - \alpha_1} \right]^{\frac{1}{\alpha_3}} \qquad (8)$$

Substituting (8) and $c^{*v_1}\, k^{*-v_2} = g - \psi$ into (2)gives:

$$gs^* - \left[ (g-\psi)k^{*1-\alpha_1} + (g-\psi)^{\frac{1}{v_1}} k^{*\frac{v_2}{v_1} - \alpha_1} \right]^{\frac{1}{\alpha_3}} - s^*(g-\psi) = 0$$

Therefore:

$$s^* = \frac{1}{\psi}\left[ (g-\psi)k^{*1-\alpha_1} + (g-\psi)^{\frac{1}{v_1}} k^{*\frac{v_2}{v_1} - \alpha_1} \right]^{\frac{1}{\alpha_3}} \qquad (9)$$

Substituting (8) into (4) gives:

$$\alpha_3\lambda_1{}^*k^{*\alpha_1}\left[(g-\psi)k^{*1-\alpha_1}+(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}-\alpha_1}\right]^{\frac{\alpha_3-1}{\alpha_3}}-\lambda_2{}^*=0$$

Therefore:

$$\lambda_2{}^*=\alpha_3\lambda_1{}^*k^{*\alpha_1}\left[(g-\psi)k^{*1-\alpha_1}+(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}-\alpha_1}\right]^{\frac{\alpha_3-1}{\alpha_3}} \qquad (10)$$

Solving for $\lambda_1$* (see appendix $A$), gives:

$$\lambda_1{}^*=\frac{\psi}{(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}}\left[\psi+V_1(g-\psi)^{\frac{v_1-1}{v_1}}k^{*\frac{(v_1-v_2)}{v_1}}\left[\psi+\alpha_3(g-\psi)\right]+V_1\alpha_3(g-\psi)\right]}=0(11)$$

Solving for $k$* (See appendix $A$), gives:

$$k^*=\left[\frac{\psi^2-\alpha_1\psi(g-\psi)+\psi(g-\psi)(1-v_2)-v_2\alpha_3(g-\psi)^2}{\left[\alpha_1(g-\psi)^{\frac{1}{v_1}}+V_2\alpha_3(g-\psi)^{\frac{1}{v_1}+1}\right]}\right]^{\frac{v_1}{v_2-v_1}} \qquad (12)$$

Equations (7), (8), (9), (10) and (11) are the steady state solutions of $c$, $r$, $s$, $\lambda_1$ and $\lambda_2$. Equation (12) shows the steady state solution of the per capita capital stock , $k$.

Looking at equation (7);

$$c^*=(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}}$$

$g > \psi$ must hold if the solution is to be economically sensible. This condition also ensures that the level of resource extraction, $r$, the level of resource stock, $S$, the shadow price of capital, $\lambda_1$, the shadow price of resource stock, $\lambda_2$, and the level of capital stock are positive. Equation (12) gives an interesting result. The exponent on the inverse of the degree of industrialisation, $v_2$, and the exponent on per capita consumption, $v_1$, must not be equal to each other. If this were the case then the whole system would collapse. This implies that the share of per capita consumption to population growth cannot be equal to the share of industrialisation to population growth.

# SECTION III

## Conditions for the Existence of the Steady State

We know from above that the condition $g > \psi$ must hold. Therefore the denominator of $k^*$ is positive and so for $k^* > 0$, the numerator must be positive. Now let us find the range of values of $\psi$ for which $k^*$ is positive:

Let the numerator of $k^*$ be denoted by $Y$:

$$Y = \psi^2 - \alpha_1\psi(g-\psi) + \psi(g-\psi)(1-v_2) - v_2\alpha_3(g-\psi)^2$$

Mulitplying out and collecting terms gives:

$$Y = [\alpha_1 + v_2 - v_2\alpha_3]\psi^2 + [g(1-\alpha_1 - v_2 + 2v_2\alpha_3)]\psi - v_2\alpha_3 g^2$$

The roots of $Y = 0$ satisfy:

$$[\alpha_1 + v_2 - v_2\alpha_3]\psi^2 + [g(1-\alpha_1 - v_2 + 2v_2\alpha_3)]\psi - v_2\alpha_3 g^2 = 0$$

Let $\text{A} = [\alpha_1 + v_2 - v_2\alpha_3]\psi^2$

$\text{B} = g(1-\alpha_1 - v_2 + 2v_2\alpha_3)$

$$C = v_2 \alpha_3 g^2$$

Then the roots are:

$$\psi = \frac{-B \pm \sqrt{B^2 + 4AC}}{2A}$$

There is one positive root, $\psi^+$, and one negative root, $\psi^-$, since $\sqrt{B^2 + 4AC} > B$

Now consider the parabola $Y = Y(\psi)$:

The turning point satisfies $\frac{dY}{d\psi} = 2A\psi + B = 0$. Therefore:

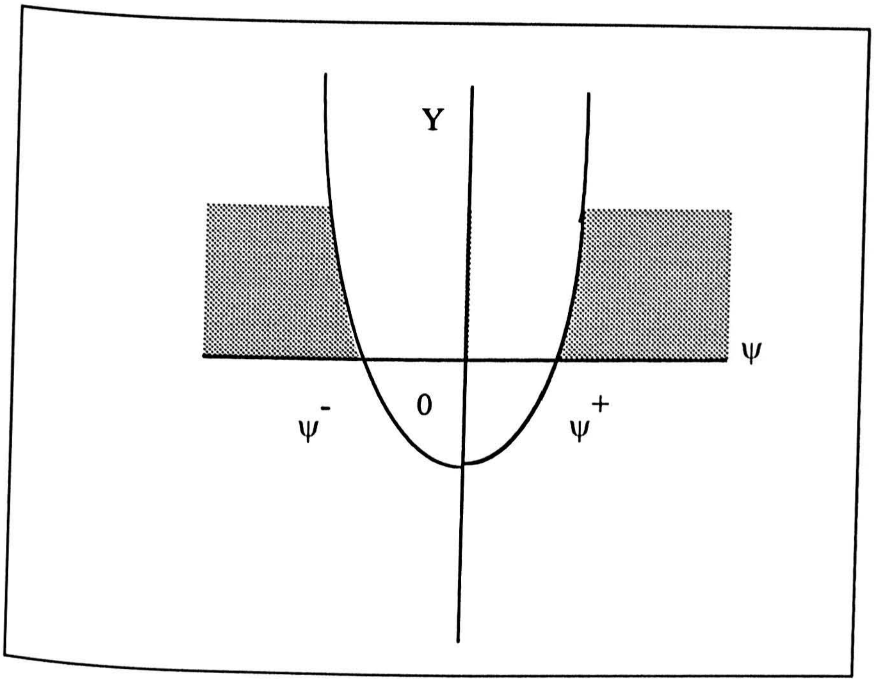
$$\psi = -\frac{B}{2A} \Rightarrow -\frac{[g(1-\alpha_1 + 2v_2\alpha_3 - v_2)]}{2[\alpha_1 + v_2(1-\alpha_3)]}$$

and since the second derivative is positive,

i.e. $\frac{d^2Y}{d\psi^2} = 2A \Rightarrow 2[\alpha_1 + v_2(1-\alpha_3)] > 0,$

then the turning point is a minimum.

The curve Y is presented below:




The shaded regions show the values for $\psi$ for which $Y > 0$. $\psi$ cannot be negative, therefore $\psi$ must be greater than or equal to the positive root ; $\psi^{+} \geq \frac{-B+\sqrt{B^2+4AC}}{2A}$. However we know that g > ψ from (7), and so we also require that:

$$g > \psi^{+} = \frac{-[g(1-\alpha_1-\nu_2+2\nu_2\alpha_3)]+\sqrt{[g(1-\alpha_1-\nu_2+2\nu_2\alpha_3)]^2+4[\alpha_1+\nu_2-\nu_2\alpha_3]\nu_2\alpha_3 g^2}}{2[\alpha_1+\nu_2-\nu_2\alpha_3]}$$

Multiplying by the denominator of $\psi^{+}$:

$$2g[\alpha_1+\nu_2-\nu_2\alpha_3] > -g[1-\alpha_1-\nu_2+2\nu_2\alpha_3]+g\sqrt{[1-\alpha_1-\nu_2+2\nu_2\alpha_3]^2+4[\alpha_1+\nu_2-\nu_2\alpha_3]\nu_2\alpha_3}$$

Rearrange and cancel:

$$1+\alpha_1+\nu_2 > \sqrt{[1-\alpha_1-\nu_2+2\nu_2\alpha_3]^2+4[\alpha_1+\nu_2-\nu_2\alpha_3]\nu_2\alpha_3}$$

Squaring both sides:

$$1+2\alpha_1+2v_2+\alpha_1^2+v_2^2+2\alpha_1 v_2>[1-\alpha_1-v_2+2v_2\alpha_3]^2+4[\alpha_1+v_2-v_2\alpha_3]v_2\alpha_3$$

Rearranging gives:

$$1+2\alpha_1+2v_2+\alpha_1^2+v_2^2+2\alpha_1 v_2-4[\alpha_1+v_2-v_2\alpha_3]v_2\alpha_3>[1-\alpha_1-v_2+2v_2\alpha_3]^2 \qquad (1)$$

expanding the right hand side:

$$[1-\alpha_1-v_2+2v_2\alpha_3][1-\alpha_1-v_2+2v_2\alpha_3]=$$

$$1-\alpha_1-v_2+2v_2\alpha_3-\alpha_1+\alpha_1{}^2+\alpha_1 v_2-2v_2\alpha_1\alpha_3-v_2+\alpha_1 v_2+v_2{}^2-$$

$$2v_2{}^2\alpha_3+2v_2\alpha_3-2v_2\alpha_3\alpha_1-2v_2{}^2\alpha_3+4v_2{}^2\alpha_3{}^2$$

Cancelling out and collecting terms gives:

$$1-2\alpha_1-2v_2+\alpha_1{}^2+4v_2\alpha_3+2\alpha_1 v_2-4v_2\alpha_1\alpha_3+v_2{}^2-4v_2{}^2\alpha_3+4v_2{}^2\alpha_3{}^2$$

Substituting this into equation (1):

$$1+2\alpha_1+2v_2+\alpha_1^2+v_2^2+2\alpha_1 v_2-4\alpha_1 v_2\alpha_3-4v_2^2\alpha_3+4v_2^2\alpha_3^2>$$
$$1-2\alpha_1-2v_2+\alpha_1{}^2+4v_2\alpha_3+2\alpha_1 v_2-4v_2\alpha_1\alpha_3+v_2{}^2-4v_2{}^2\alpha_3+4v_2{}^2\alpha_3{}^2$$

Cancelling terms we get:

$$4\alpha_1+4v_2-4v_2\alpha_3>0$$

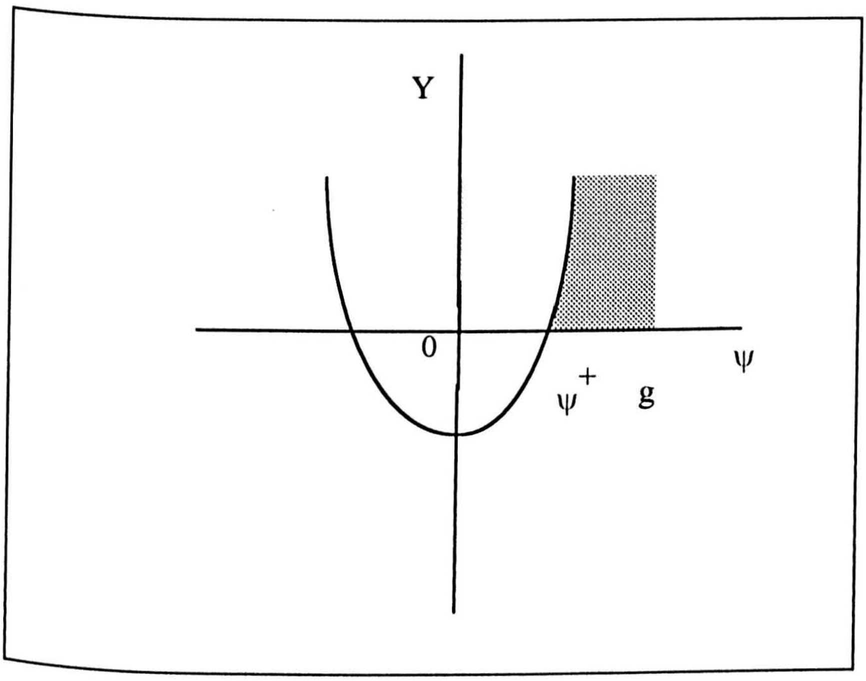
Rearranging:

$$\alpha_1+v_2(1-\alpha_3)>0$$

It is clear that this condition holds, thus it has been proved that $g$ is greater than the positive root, $\psi^{+}$, therefore there is a non-empty set for $\psi^{+}$ where the value of $k^* > 0$. Therefore the discount rate must lie in the range:

$$\frac{-B+\sqrt{B^2+4AC}}{2A} \leq \psi^{+} < g$$

and is shown below, where the shaded region indicates the possible values for $\psi$.




It then follows that the steady states $c^*$, $r^*$, $s^*$, $\lambda_1{}^*$ and $\lambda_2{}^*$ are all positive.

# SECTION IV

## Stability

In this section the system is analysed to determing the stability of the steady state. It is necessary therefore to determine the eigenvalues of the Jacobian matrix evaluated at the steady state. The Jacobian is:

$$
J = \begin{bmatrix}
\frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial r} & \frac{\partial \dot{k}}{\partial s} & \frac{\partial \dot{k}}{\partial \lambda_1} & \frac{\partial \dot{k}}{\partial \lambda_2} & \frac{\partial \dot{k}}{\partial c} \\
\frac{\partial \dot{r}}{\partial k} & \frac{\partial \dot{r}}{\partial r} & \frac{\partial \dot{r}}{\partial s} & \frac{\partial \dot{r}}{\partial \lambda_1} & \frac{\partial \dot{r}}{\partial \lambda_2} & \frac{\partial \dot{r}}{\partial c} \\
\frac{\partial \dot{s}}{\partial k} & \frac{\partial \dot{s}}{\partial r} & \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial \lambda_1} & \frac{\partial \dot{s}}{\partial \lambda_2} & \frac{\partial \dot{s}}{\partial c} \\
\frac{\partial \dot{\lambda}_1}{\partial k} & \frac{\partial \dot{\lambda}_1}{\partial r} & \frac{\partial \dot{\lambda}_1}{\partial s} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} & \frac{\partial \dot{\lambda}_1}{\partial c} \\
\frac{\partial \dot{\lambda}_2}{\partial k} & \frac{\partial \dot{\lambda}_2}{\partial r} & \frac{\partial \dot{\lambda}_2}{\partial s} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} & \frac{\partial \dot{\lambda}_2}{\partial c} \\
\frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial r} & \frac{\partial \dot{c}}{\partial s} & \frac{\partial \dot{c}}{\partial \lambda_1} & \frac{\partial \dot{c}}{\partial \lambda_2} & \frac{\partial \dot{c}}{\partial c}
\end{bmatrix}
$$

There is no simple closed-form expression for the eigenvalues of this matrix. Therefore a new method was used to analyse the stability of the steady state.

Using corollary 2c in Sorger (1989), (see methodology), it can be shown that the steady state is globally asymptotically stable for bounded solutions.

The equilibrium point of this system possess the saddle point property if the 'curvature matrix,'

$$
C = \begin{bmatrix} H_{ii}^{*} & \left(\frac{\delta}{2}\right) I_n \\ \left(\frac{\delta}{2}\right) I_n & H_{jj}^{*} \end{bmatrix}
$$

is negative definite, where $H^*$ is the maximised Hamiltonian and $i = k, s$ and $j = \lambda_1, \lambda_2, \lambda_3$ and $n = 2$. Cass and Shell (1976) show that the Hamiltonian is convex in the costate and concave in the state for optimal control problems with a concave objective function and so the matrices $H_{ii}^{*}$ and $H_{jj}^{*}$ are negative definite. The objective function here is strictly concave and as the matrices $H_{ii}^{*}$ and $H_{jj}^{*}$ are negative definite with minimum eigenvalues below zero the curvature matrix $C$ is negative definite with a low rate of discount, (Brock and Scheinkman (1976)).

As the curvature matrix $C$ is negative definite then the equilibrium point i.e. the steady state, is globally stable for bounded solutions, Sorger (1989). This implies that with any initial levels of capital and resource stocks in the environment the optimal path converges toward a unique steady state.

# SECTION V

## Comparitive Statics

The next step is to carry out a sensitivity analysis on the steady state solutions of the variables to determine their dependency on the parameters. The complex nature of the steady state values makes it difficult to obtain the partial derivatives with respect to each parameter. To simplify matters the assumption that $V_2 = 1$ is assumed throughout this section. Looking back at Section III if $V_2 = 1$, then equation (1) is still positive and so the condition that there is a non-empty set for $\psi^+$ where the value of $k^* > 0$ still holds. So $k^*$, $c^*$, $r^*$, $s^*$, $\lambda_1^*$ and $\lambda_2^*$ are positive.

We know from the steady state expression for $k$ (equation (12)), that $V_1$ cannot be equal to $V_2$, so $V_1$ can take values below or above 1, however $V_1 \neq 1$.

The results of the sensitivity analysis on the steady state solutions of the variables are given below (see appendix B). The following table summarises the results for $v_1 > 1$

The comparitive statics are very complicated and not all the parameters are of great interest. It was chosen to analyse the effects that the parameters $g$ and $\psi$ have on the steady state values of $k$, $c$, $r$ and $s$. An increase in the growth rate of the resource, $g$, gives a counter-intuitive result and an increase in $\psi$ gives the same result as a standard neoclassical growth model.

**Table of results**

| | $g$ | $\psi$ |
|---|---|---|
| $k^*$ | + | - |
| $c^*$ | + | - |
| $r^*$ | + if $\alpha_1 < \frac{1-2\nu_1}{\nu_1(1-\nu_1)}$ | - if $\alpha_1 < \frac{1-2\nu_1}{\nu_1(1-\nu_1)}$ |
| $s^*$ | + if $\alpha_1 < \frac{1-2\nu_1}{\nu_1(1-\nu_1)}$ | - if $\alpha_1 < \frac{1-2\nu_1}{\nu_1(1-\nu_1)}$ |

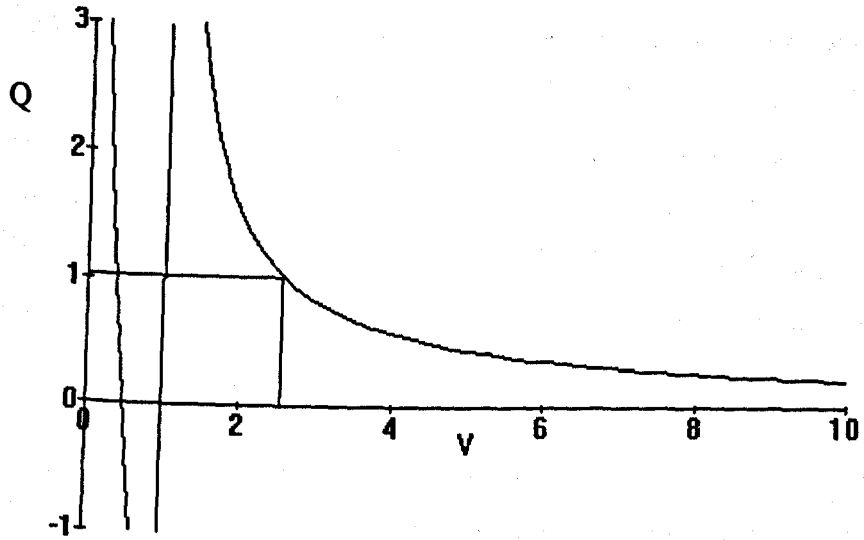
Regarding $g$: We have already assumed that $\nu_1 > 1$, i.e. $\nu_1 > \nu_2$, which implies that the positive effect that consumption has on the population growth rate is greater that the negative effect of industrialisation. The table shows that if there was an increase in the growth rate of the resource, then the steady state level of per capita consumption rises. As population growth rate is increasing, one may expect that per capita consumption would fall. However, the result shows otherwise. A feasible explanation for this is that per capita consumption would rise even though population growth rate is rising, if total consumption was rising faster than population.

With an increase in the growth of the resource, the rate of extraction rises as would be expected, this is only the case if the condition on $\alpha_1$ given in the table is satisfied.

Let:

$$Q = \frac{1-2\nu_1}{\nu_1(1-\nu_1)}$$

We know that $\alpha_1$ must be less than this for the steady state level of resource extraction to rise. The graph of $Q$ against increasing values of $v_1$ is shown below:




It is easy to show that $Q = 1$ when $v_1 = 2.6$ (1dp).

We know that $\alpha_1 < 1$. For values of $2.6 > v_1 > 1$, $Q$ is greater than 1, thus the condition on $\alpha_1$ holds automatically as $\alpha_1$ is assumed to be less than 1 throughout. Hence the condition must hold for values of $v_1$ greater than 2.6 (1dp).

It follows that if $v_1$ is large, i.e. the effect of consumption on population growth rate is large, there is some positive declining number that is less than 1 that must be greater than $\alpha_1$ for the results to hold.

With an increase in the amount of the resource that is extracted at the steady state, then more can be consumed per capita, even though the population growth rate is increasing. The rise in the growth of the resource may still allow the steady state level of the resource to increase even though the amount extracted has risen, if the growth rate increases at a faster rate than the extraction rate increases. This is borne out by the results.

Regarding $\psi$: The results show that a rise in the discount rate, $\psi$, implying an increased preference for consumption in the present, would cause a rise in consumption at the beginning of the plan and lower steady state consumption. This is the same result as in the standard neoclassical growth model (Cass 1965). As the economy depends so much on the natural resource for production, increased extraction of the resource would run down stocks and there would be less left for future production. As the steady state stock is now lower, there is less left to extract and so the steady state extraction rate falls. This is the underlying argument for sustainable development, it we extract too much now and pay less regard for future generations then they will have less resources to use and so their output prospects will be diminished.

These results are even more unfavourable for future generations if the population growth rate is rising. There will be more pressure on the per capita quantities of consumption and resource stocks to fall in the steady state.

# SECTION VI

## Conclusion

In this chapter a model has been formulated of a simple closed economy with a renewable natural resource that is an essential factor of production. This economy also has an endogenously determined rate of population growth. The model was presented in Section I. In Section II the necessary and sufficient conditions for an optimal solution were presented and solved for the steady state. The model highlights the importance of including population growth in dynamic optimisation models. From the various models presented in section I, it can be seen that population is usually treated as constant. For instance, Solow (1974), Krautkraemer (1985) and Dasgupta and Heal (1974 and 1979), formulate models where an exhaustible resource is an essential input in production but population is constant; hence they assume that there is a stationary population. In other models, the theory has been extended to renewable resource extraction, Beddington et al (1975) and Clark (1976).

Other research has explored the consequences of a growing population in models involving a non-renewable resource, Stiglitz(1974a), Ingham and Simmons (1975) and Stiglitz (1974b). In these models the labour force (equivalent to the population) grows at a constant exponential rate. The growth rate in these models is treated as an exogenously given constant.

This chapter is an extension to an earlier paper by Cigno (1980), where the implications of making the population growth rate endogenous were explored. The growth rate is a function of consumption and capital per capita. His model is an extension to Stiglitz's model (1974b). Stiglitz found that introducing exhaustible resources has the effect of making the standard neo-classical growth model unstable. Cigno argues that putting endogenous population growth and exhaustible resources together may result in a stable solution. However Cigno considers the dynamic model of an economy with a non-renewable resource as an essential factor of production but not in an optimising framework. Stiglitz also presents his work in this way. Also Cigno does not include in his model the dynamics of the non-renewable resource, it simply appears in the production function.

In this chapter a version of Cigno's model was presented and solved for the *optimal* sustainable steady state. In this model the economy possesses a single renewable natural resource which is used as an input in production. It is found that an optimal and sustainable consumption and resource harvesting policy does exist. Also the conditions that are necessary for the steady state to exist are shown. New theory on the stability of dynamical systems is used and it is shown that when the rate of discount is small enough all bounded solutions converge to a unique steady state. This is in contrast to Stiglitz's work and agrees with Cigno's results. However our problem is solved in an optimising framework and for a renewable natural resource and is thus more general.

Some of the past work has shown that it is optimal to completely exhaust a non-renewable resource stock if the availability of future technologies and perfect substitutes mean that depletion of the resource is no longer essential for future production. Stiglitz (1974a), argues that even with a limited amount of the resource, the economy need not decline. This is because he assumes that technical change and capital accumulation will offset the effects of falling natural resource inputs and so we should just use up our resources. Other work has also argued this point, Dasgupta and Heal (1974), Solow (1974) and Kamien and Schwartz (1978).

The model in this chapter concerns a renewable resource. Renewable resources tend not to get as much attention as non-renewable resources in the literature as they are not seen as limiting factors for economic growth. But as we are aware, the over-exploitation of these resources can render them extinct and harm the environment and biosphere. We need to pay as much attention to renewable resources as non-renewable. This analysis presented here shows that it is not optimal to completely exhaust the resource, this is in direct contrast to some of the previous literature (See Clark (1976)).

The comparative statics also shows that if there is a greater preference to deplete the resource earlier on i.e. there is a greater preference for current consumption, then future generations will be deprived of some output possibilities. Sustainable development requires that the options of future generations are not diminished. If we use up natural resources at too fast a rate then we are removing an option for future generations.

The World Commission on Environment and Development was established by the General Assembly of the United Nations in 1983. They define Sustainable Development as:

> "Development is sustainable if it satisfies present needs without compromising the ability of future generations to meet their own needs." (WCED, 1987).

Current over-exploitation of the worlds natural resources and destruction of natural environments jeopardises the future possibilities of obtaining environmental services from these areas and so threatens the future world economy. The goal of SD therefore is to achieve the maximum level of economic welfare that can be perpetuated for many generations in the future and to establish a viable path for economic development that is compatible with environmental quality.

# APPENDICES

## APPENDIX A

### Solving for λ*

Substituting (7) into (3) gives:

$$\frac{1}{(g-\psi)^{\frac{1}{v_1}} k*^{\frac{v_2}{v_1}}} - \lambda_1 * - \lambda_1 * k*^{1-v_2} V_1 (g-\psi)^{\frac{v_1-1}{v_1}} k*^{\frac{v_2(v_1-1)}{v_1}} -$$

$$\lambda_2 * s * k*^{-v_2} V_1 (g-\psi)^{\frac{v_1-1}{v_1}} k*^{\frac{v_2(v_1-1)}{v_1}} = 0$$

substituting (9) into (3) gives:

$$\frac{1}{(g-\psi)^{\frac{1}{v_1}} k*^{\frac{v_2}{v_1}}} - \lambda_1 * - \lambda_1 * k*^{1-v_2} V_1 (g-\psi)^{\frac{v_1-1}{v_1}} k*^{\frac{v_2(v_1-1)}{v_1}} -$$

$$\lambda_2 * \frac{1}{\psi}\left[(g-\psi)k*^{1-\alpha_1} + (g-\psi)^{\frac{1}{v_1}} k*^{\frac{v_2}{v_1}-\alpha_1}\right]^{\frac{1}{\alpha_3}} k*^{-v_2} V_1 (g-\psi)^{\frac{v_1-1}{v_1}} k*^{\frac{v_2(v_1-1)}{v_1}} = 0$$

Substituing (10) into (3) gives:

$$\frac{1}{(g-\psi)^{\frac{1}{v_1}} k*^{\frac{v_2}{v_1}}} - \lambda_1 * - \lambda_1 * k*^{1-v_2} V_1 (g-\psi)^{\frac{v_1-1}{v_1}} k*^{\frac{v_2(v_1-1)}{v_1}} -$$

$$\left[\alpha_3\lambda_1{}^*\left[(g-\psi)k^*+(g-\psi)^{\frac{1}{\nu_1}}k^{*\frac{\nu_2}{\nu_1}}\right]^{\frac{\alpha_3-1}{\alpha_3}}\right.$$

$$\left.\left[\frac{1}{\psi}\left[(g-\psi)k^{*1-\alpha_1}+(g-\psi)^{\frac{1}{\nu_1}}k^{*\frac{\nu_2}{\nu_1}-\alpha_1}\right]^{\frac{1}{\alpha_3}}k^{*-\nu_2}V_1(g-\psi)^{\frac{\nu_1-1}{\nu_1}}k^{*\frac{\nu_2(\nu_1-1)}{\nu_1}}\right]\right]=0$$

Collecting terms together:

$$\frac{1}{(g-\psi)^{\frac{1}{\nu_1}}k^{*\frac{\nu_2}{\nu_1}}}-\lambda_1{}^*-\lambda_1{}^*V_1(g-\psi)^{\frac{\nu_1-1}{\nu_1}}k^{*\frac{(\nu_1-\nu_2)}{\nu_1}}-$$

$$\left[\frac{\alpha_3\lambda_1{}^*k^{*-\frac{\nu_2}{\nu_1}}}{\psi}\left[(g-\psi)k^*+(g-\psi)^{\frac{1}{\nu_1}}k^{*\frac{\nu_2}{\nu_1}}\right]\left[V_1(g-\psi)^{\frac{\nu_1-1}{\nu_1}}\right]=0\right]$$

Mulitplying out the last term and simplifying:

$$\frac{1}{(g-\psi)^{\frac{1}{\nu_1}}k^{*\frac{\nu_2}{\nu_1}}}-\lambda_1{}^*-\lambda_1{}^*V_1(g-\psi)^{\frac{\nu_1-1}{\nu_1}}k^{*\frac{(\nu_1-\nu_2)}{\nu_1}}-$$

$$\frac{V_1\alpha_3\lambda_1{}^*}{\psi}\left[(g-\psi)^{\frac{2\nu_1-1}{\nu_1}}k^{*\frac{\nu_1-\nu_2}{\nu_1}}+(g-\psi)\right]=0$$

Collecting $\lambda_1$*'s together:

$$\frac{1}{(g-\psi)^{\frac{1}{\nu_1}}k^{*\frac{\nu_2}{\nu_1}}}-\lambda_1{}^*\left[1+V_1(g-\psi)^{\frac{\nu_1-1}{\nu_1}}k^{*\frac{(\nu_1-\nu_2)}{\nu_1}}+\frac{V_1\alpha_3}{\psi}\left[(g-\psi)^{\frac{2\nu_1-1}{\nu_1}}k^{*\frac{\nu_1-\nu_2}{\nu_1}}+(g-\psi)\right]\right]=0$$

collecting terms together:

$$\frac{1}{(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}}}-\lambda_1{}^*\left[1+V_1(g-\psi)^{\frac{v_1-1}{v_1}}k^{*\frac{(v_1-v_2)}{v_1}}\left[1+\frac{\alpha_3}{\psi}(g-\psi)\right]+\frac{V_1\alpha_3(g-\psi)}{\psi}\right]=0$$

Multiplying through by $\psi$ gives:

$$\frac{\psi}{(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}}}-\lambda_1{}^*\left[\psi+V_1(g-\psi)^{\frac{v_1-1}{v_1}}k^{*\frac{(v_1-v_2)}{v_1}}\left[\psi+\alpha_3(g-\psi)\right]+V_1\alpha_3\right]=0$$

Solving for $\lambda_1{}^*$

$$\lambda_1{}^*=\frac{\psi}{(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}}\left[\psi+V_1(g-\psi)^{\frac{v_1-1}{v_1}}k^{*\frac{(v_1-v_2)}{v_1}}\left[\psi+\alpha_3(g-\psi)\right]+V_1\alpha_3(g-\psi)\right]}=0 \quad (11)$$

## Solving for $k^*$

Substituting (7) into (5):

$$\psi\lambda_1{}^*-\alpha_1\lambda_1{}^*k^{*\alpha_1-1}r^{*\alpha_3}+\lambda_1{}^*(g-\psi)(1-v_2)-v_2\lambda_2{}^*s^*(g-\psi)k^{*-1}=0$$

Substituting (8) into the above equation and collecting the k*'s:

$$\psi\lambda_1{}^*-\alpha_1\lambda_1{}^*\left[(g-\psi)+(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}-1}\right]+\lambda_1{}^*(g-\psi)(1-v_2)-v_2\lambda_2{}^*s^*(g-\psi)k^{*-1}=0$$

Substituting (9) into the above equation gives:

$$\psi\lambda_1{}^*-\alpha_1\lambda_1{}^*\left[(g-\psi)+(g-\psi)^{\frac{1}{v_1}}k^{*\frac{v_2}{v_1}-1}\right]+\lambda_1{}^*(g-\psi)(1-v_2)-$$

$$\nu_2\lambda_2 * \frac{1}{\psi}\left[(g-\psi)k*^{1-\alpha_1} + (g-\psi)^{\frac{1}{\nu_1}} k*^{\frac{\nu_2}{\nu_1}-\alpha_1}\right]^{\frac{1}{\alpha_3}} (g-\psi)k*^{-1} = 0$$

Substituting (10) into the above equation gives:

$$\psi\lambda_1 * -\alpha_1\lambda_1 * \left[(g-\psi) + (g-\psi)^{\frac{1}{\nu_1}} k*^{\frac{\nu_2}{\nu_1}-1}\right] + \lambda_1 * (g-\psi)(1-\nu_2) -$$

$$\nu_2\alpha_3\lambda_1 * k*^{\alpha_1}\left[(g-\psi)^{\frac{1}{\nu_1}} k*^{\frac{\nu_2}{\nu_1}-\alpha_1} + (g-\psi)k*^{1-\alpha_1}\right]^{\frac{\alpha_3-1}{\alpha_3}}$$

$$\frac{1}{\psi}\left[(g-\psi)k*^{1-\alpha_1} + (g-\psi)^{\frac{1}{\nu_1}} k*^{\frac{\nu_2}{\nu_1}-\alpha_1}\right]^{\frac{1}{\alpha_3}} (g-\psi)k*^{-1} = 0$$

Dividing through by $\lambda_1$, times by $\psi$ and simplifying:

$$\psi^2 - \alpha_1\psi\left[(g-\psi) + (g-\psi)^{\frac{1}{\nu_1}} k*^{\frac{\nu_2}{\nu_1}-1}\right] + \psi(g-\psi)(1-\nu_2) -$$

$$\nu_2\alpha_3\left[(g-\psi)^{\frac{1}{\nu_1}+1} k*^{\frac{\nu_2}{\nu_1}-1} + (g-\psi)^2\right] = 0$$

Multiplying out gives:

$$\psi^2 - \alpha_1\psi(g-\psi) - \alpha_1(g-\psi)^{\frac{1}{\nu_1}} k*^{\frac{\nu_2}{\nu_1}-1} + \psi(g-\psi)(1-\nu_2) -$$

$$v_2\alpha_3(g-\psi)^{\frac{1}{v_1}+1}k^{*\frac{v_2}{v_1}-1}-v_2\alpha_3(g-\psi)^2=0$$

Collecting $k^{*\frac{v_2}{v_1}-1}$ terms together:

$$k^{*\frac{v_2}{v_1}-1}\left[-\alpha_1(g-\psi)^{\frac{1}{v_1}}-V_2\alpha_3(g-\psi)^{\frac{1}{v_1}+1}\right]=-\psi^2+\alpha_1\psi(g-\psi)-\psi(g-\psi)(1-v_2)+v_2\alpha_3(g-\psi)^2$$

Multiplying by -1:

$$k^{*\frac{v_2}{v_1}-1}\left[\alpha_1(g-\psi)^{\frac{1}{v_1}}+V_2\alpha_3(g-\psi)^{\frac{1}{v_1}+1}\right]=\psi^2-\alpha_1\psi(g-\psi)+\psi(g-\psi)(1-v_2)-v_2\alpha_3(g-\psi)^2$$

Rearranging:

$$k^{*\frac{v_2}{v_1}-1}=\frac{\psi^2-\alpha_1\psi(g-\psi)+\psi(g-\psi)(1-v_2)-v_2\alpha_3(g-\psi)^2}{\left[\alpha_1(g-\psi)^{\frac{1}{v_1}}+V_2\alpha_3(g-\psi)^{\frac{1}{v_1}+1}\right]}$$

Therefore:

$$k^*=\left[\frac{\psi^2-\alpha_1\psi(g-\psi)+\psi(g-\psi)(1-v_2)-v_2\alpha_3(g-\psi)^2}{\left[\alpha_1(g-\psi)^{\frac{1}{v_1}}+V_2\alpha_3(g-\psi)^{\frac{1}{v_1}+1}\right]}\right]^{\frac{v_1}{v_2-v_1}} \tag{12}$$

# APPENDIX B

First it is necessary to simplify the expression for the steady state values of *k*, *c*, *r* and *s*. Simplifying equation (12) gives:

$$k^* = \left[ (g-\psi)^{1-\frac{1}{v_1}} \right] \left[ \frac{\frac{\psi^2}{(g-\psi)} - \alpha_1\psi - \alpha_3(g-\psi)}{[\alpha_1 + \alpha_3(g-\psi)]} \right]^{\frac{v_1}{1-v_1}}$$

By substituting $k^*$ into equation (7) gives $c^*$:

$$c^* = [(g-\psi)] \left[ \frac{\frac{\psi^2}{(g-\psi)} - \alpha_1\psi - \alpha_3(g-\psi)}{[\alpha_1 + \alpha_3(g-\psi)]} \right]^{\frac{v_1}{1-v_1}}$$

Substituting $k^*$ into equation (8) gives:

$$r^* = \left[ (g-\psi)^{\left[\frac{(v_1-1)(1-\alpha_1)+v_1}{v_1}\right]} \left[ \frac{\frac{\psi^2}{(g-\psi)} - \alpha_1\psi - \alpha_3(g-\psi)}{[\alpha_1 + \alpha_3(g-\psi)]} \right]^{\frac{v_1(1-\alpha_1)}{1-v_1}} \right]^{\frac{1}{\alpha_3}} +$$

$$\left[ (g-\psi)^{\left[\frac{2v_1-1-\alpha_1 v_1^2+\alpha_1 v_1}{v_1^2}\right]} \left[ \frac{\frac{\psi^2}{(g-\psi)} - \alpha_1\psi - \alpha_3(g-\psi)}{[\alpha_1 + \alpha_3(g-\psi)]} \right]^{\frac{v_1}{1-v_1}\left[\frac{1}{v_1}-\alpha_1\right]} \right]^{\frac{1}{\alpha_3}}$$

The steady state value of $s$ is now:

$$s^* = \left[\frac{1}{\psi}\right]\left[(g-\psi)^{\left[\frac{(v_1-1)(1-\alpha_1)+v_1}{v_1}\right]}\left[\frac{\frac{\psi^2}{(g-\psi)}-\alpha_1\psi-\alpha_3(g-\psi)}{\left[\alpha_1+\alpha_3(g-\psi)\right]}\right]^{\frac{v_1(1-\alpha_1)}{1-v_1}}\right]^{\frac{1}{\alpha_3}} +$$

$$\left[\frac{1}{\psi}\right]\left[(g-\psi)^{\left[\frac{2v_1-1-\alpha_1 v_1^2+\alpha_1 v_1}{v_1^2}\right]}\left[\frac{\frac{\psi^2}{(g-\psi)}-\alpha_1\psi-\alpha_3(g-\psi)}{\left[\alpha_1+\alpha_3(g-\psi)\right]}\right]^{\frac{v_1}{1-v_1}\left[\frac{1}{v_1}-\alpha_1\right]}\right]^{\frac{1}{\alpha_3}}$$

The simplified expressions for each variable followed by the derivative with respect to $g$ and $\psi$ are given below. The derivatives have been calculated using Maple V 2.0. Each Greek symbol is assigned a conventional letter as Maple is unable to write Greek symbols.

For the following calculations note that:

$$v_2 = 1$$

$$\alpha_1 = A$$

$$\alpha_3 = F$$

$$\psi = U$$

$$v_1 = X$$

$$k := (g-U)^{\left(1-\frac{1}{X}\right)}\left(\frac{\frac{U^2}{g-U}-A\,U-F\,(g-U)}{A+F\,(g-U)}\right)^{\left(\frac{X}{1-X}\right)}$$

$$c := (g-U)\left(\frac{\frac{U^2}{g-U} - A\,U - F\,(g-U)}{A + F\,(g-U)}\right)^{\left(\frac{X}{1-X}\right)^{-}}$$

$r=:$

$$\left(\left((g-U)^{\left(\frac{2X-1-A\,X^2+A\,X}{X^2}\right)}\left(\frac{\frac{U^2}{g-U} - A\,U - F\,(g-U)}{A + F\,(g-U)}\right)^{\left(\frac{X\left(\frac{1}{X}-A\right)}{1-X}\right)}\right)^{\left(\frac{1}{F}\right)} + (g-U)^{\left(\frac{(X-1)\,(1-A)+X}{X}\right)}\left(\frac{\frac{U^2}{g-U} - A\,U - F\,(g-U)}{A + F\,(g-U)}\right)^{\left(\frac{X\,(1-A)}{1-X}\right)}\right)^{\left(\frac{1}{F}\right)}$$

$$s := \left( (g-U)^{\left(\frac{2X-1-AX^2+AX}{X^2}\right)} \left( \frac{\frac{U^2}{g-U} - A\,U - F\,(g-U)}{A+F\,(g-U)} \right)^{\left(\frac{X\left(\frac{1}{X}-A\right)}{1-X}\right)} + (g-U)^{\left(\frac{(X-1)\,(1-A)+X}{X}\right)} \left( \frac{\frac{U^2}{g-U} - A\,U - F\,(g-U)}{A+F\,(g-U)} \right)^{\left(\frac{X(1-A)}{1-X}\right)} \right)^{\left(\frac{1}{F}\right)} \Bigg/ U$$

The derivatives with respect to $g$ of $k^*$, $c^*$, $r^*$ and $s^*$ respectively are given by:

$$\frac{\partial k^*}{\partial g} :=$$

$$\frac{(g-U)^{\left(1-\frac{1}{X}\right)} \left(1-\frac{1}{X}\right) \left( \frac{\%1}{A+F\,(g-U)} \right)^{\left(\frac{X}{1-X}\right)}}{g-U} + (g-U)^{\left(1-\frac{1}{X}\right)} \left( \frac{\%1}{A+F\,(g-U)} \right)^{\left(\frac{X}{1-X}\right)} X \left( \frac{-\frac{U^2}{(g-U)^2} - F}{A+F\,(g-U)} - \frac{\%1\,F}{(A+F\,(g-U))^2} \right)$$
$$(A+F\,(g-U))/((1-X)\,\%1)$$

$$\%1 := \frac{U^2}{g-U} - A\,U - F\,(g-U)$$

$$\frac{\partial c^*}{\partial g} :=$$

$$\left(\frac{\%1}{A+F(g-U)}\right)^{\left(\frac{X}{1-X}\right)} + (g-U)\left(\frac{\%1}{A+F(g-U)}\right)^{\left(\frac{X}{1-X}\right)} X$$

$$\left(\frac{-\frac{U^2}{(g-U)^2}-F}{A+F(g-U)} - \frac{\%1\,F}{(A+F(g-U))^2}\right)(A+F(g-U))/((1-X)\,\%1)$$

$$\%1 := \frac{U^2}{g-U} - A\,U - F(g-U)$$

$$\frac{\partial r^*}{\partial g} :=$$

$$\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)^{\left(\frac{1}{F}\right)}\left(\frac{(g-U)^{\left(\frac{\%5}{X^2}\right)}\%5\,\%4}{X^2\,(g-U)}+(g-U)^{\left(\frac{\%5}{X^2}\right)}\%4\,X\right.$$

$$\left(\frac{1}{X}-A\right)\left(\frac{-\frac{U^2}{(g-U)^2}-F}{A+F\,(g-U)}-\frac{\%1\,F}{(A+F\,(g-U))^2}\right)(A+F\,(g-U))/((1-X)\,\%1)$$

$$+\frac{\%3\,((X-1)\,(1-A)+X)\,\%2}{X\,(g-U)}$$

$$\left.+\frac{\%3\,\%2\,X\,(1-A)\left(\frac{-\frac{U^2}{(g-U)^2}-F}{A+F\,(g-U)}-\frac{\%1\,F}{(A+F\,(g-U))^2}\right)(A+F\,(g-U))}{(1-X)\,\%1}\right)\Bigg/\left(\vphantom{\frac{1}{1}}\right.$$

$$\left.F\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)\right)$$

$$\%1 := \frac{U^2}{g-U}-A\,U-F\,(g-U)$$

$$\%2 := \left(\frac{\%1}{A+F\,(g-U)}\right)^{\left(\frac{X\,(1-A)}{1-X}\right)}$$

$$\%3 := (g-U)^{\left(\frac{(X-1)\,(1-A)+X}{X}\right)}$$

$$\%4 := \left(\frac{\%1}{A+F\,(g-U)}\right)^{\left(\frac{X\left(\frac{1}{X}-A\right)}{1-X}\right)}$$

$$\%5 := 2\,X-1-A\,X^2+A\,X$$

$$\frac{\partial k^*}{\partial U} :=$$

$$-\frac{(g-U)^{\left(1-\frac{1}{X}\right)}\left(1-\frac{1}{X}\right)\left(\frac{\%1}{A+F(g-U)}\right)^{\left(\frac{X}{1-X}\right)}}{g-U}+(g-U)^{\left(1-\frac{1}{X}\right)}$$

$$\left(\frac{\%1}{A+F(g-U)}\right)^{\left(\frac{X}{1-X}\right)} X \left(\frac{2\frac{U}{g-U}+\frac{U^2}{(g-U)^2}-A+F}{A+F(g-U)}+\frac{\%1\, F}{(A+F(g-U))^2}\right)$$

$$(A+F(g-U))/((1-X)\%1)$$

$$\%1 := \frac{U^2}{g-U}-A\,U-F(g-U)$$

$$\frac{\partial c^*}{\partial U} :=$$

$$-\left(\frac{\%1}{A+F(g-U)}\right)^{\left(\frac{X}{1-X}\right)}+(g-U)\left(\frac{\%1}{A+F(g-U)}\right)^{\left(\frac{X}{1-X}\right)} X$$

$$\left(\frac{2\frac{U}{g-U}+\frac{U^2}{(g-U)^2}-A+F}{A+F(g-U)}+\frac{\%1\, F}{(A+F(g-U))^2}\right)(A+F(g-U))/((1-X)\%1)$$

$$\%1 := \frac{U^2}{g-U}-A\,U-F(g-U)$$

$$\frac{\partial r^*}{\partial U} :=$$

$$\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)^{\left(\frac{1}{F}\right)}\left(-\frac{(g-U)^{\left(\frac{\%5}{X^2}\right)}\%5\,\%4}{X^2\,(g-U)}+(g-U)^{\left(\frac{\%5}{X^2}\right)}\%4\,X\right.$$

$$\left(\frac{1}{X}-A\right)\left(\frac{2\,\frac{U}{g-U}+\frac{U^2}{(g-U)^2}-A+F}{A+F\,(g-U)}+\frac{\%1\,F}{(A+F\,(g-U))^2}\right)(A+F\,(g-U))/($$

$$(1-X)\,\%1)-\frac{\%3\,((X-1)\,(1-A)+X)\,\%2}{X\,(g-U)}+\%3\,\%2\,X\,(1-A)$$

$$\left(\frac{2\,\frac{U}{g-U}+\frac{U^2}{(g-U)^2}-A+F}{A+F\,(g-U)}+\frac{\%1\,F}{(A+F\,(g-U))^2}\right)(A+F\,(g-U))/((1-X)\,\%1)$$

$$\left.\right)\Big/\left(F\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)\right)$$

$$\%1 := \frac{U^2}{g-U}-A\,U-F\,(g-U)$$

$$\%2 := \left(\frac{\%1}{A+F\,(g-U)}\right)^{\left(\frac{X\,(1-A)}{1-X}\right)}$$

$$\%3 := (g-U)^{\left(\frac{(X-1)\,(1-A)+X}{X}\right)}$$

$$\%4 := \left(\frac{\%1}{A+F\,(g-U)}\right)^{\left(\frac{X\left(\frac{1}{X}-A\right)}{1-X}\right)}$$

$$\%5 := 2\,X-1-A\,X^2+A\,X$$

$$\frac{\partial s^*}{\partial U} :=$$

$$-\frac{\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)^{\left(\frac{1}{F}\right)}}{U^2}+\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)^{\left(\frac{1}{F}\right)}\Bigg($$

$$-\frac{(g-U)^{\left(\frac{\%5}{X^2}\right)}\%5\,\%4}{X^2\,(g-U)}+(g-U)^{\left(\frac{\%5}{X^2}\right)}\%4\,X\left(\frac{1}{X}-A\right)$$

$$\left(\frac{2\,\frac{U}{g-U}+\frac{U^2}{(g-U)^2}-A+F}{A+F\,(g-U)}+\frac{\%1\,F}{(A+F\,(g-U))^2}\right)(A+F\,(g-U))/((1-X)\,\%1)$$

$$-\frac{\%3\,((X-1)\,(1-A)+X)\,\%2}{X\,(g-U)}+\%3\,\%2\,X\,(1-A)$$

$$\left(\frac{2\,\frac{U}{g-U}+\frac{U^2}{(g-U)^2}-A+F}{A+F\,(g-U)}+\frac{\%1\,F}{(A+F\,(g-U))^2}\right)(A+F\,(g-U))/((1-X)\,\%1)$$

$$\Bigg)\Bigg/\left(U\,F\left((g-U)^{\left(\frac{\%5}{X^2}\right)}\%4+\%3\,\%2\right)\right)$$

$$\%1 := \frac{U^2}{g-U}-A\,U-F\,(g-U)$$

$$\%2 := \left(\frac{\%1}{A+F\,(g-U)}\right)^{\left(\frac{X\,(1-A)}{1-X}\right)}$$

$$\%3 := (g-U)^{\left(\frac{(X-1)\,(1-A)+X}{X}\right)}$$

$$\%4 := \left(\frac{\%1}{A+F\,(g-U)}\right)^{\left(\frac{X\left(\frac{1}{X}-A\right)}{1-X}\right)}$$

$$\%5 := 2\,X-1-A\,X^2+A\,X$$

The sign of each differential is given in the table in Section V for $X > 1$. However, on some of the results, i.e. $\frac{\partial r}{\partial g}, \frac{\partial s}{\partial g}$, are only positive if $\alpha_1 < \frac{1-2v_1}{v_1(1-v_1)}$. Similarly, $\frac{\partial r}{\partial U}$ and $\frac{\partial s}{\partial U}$ are only negative if this condition holds as well.

Let $Q = \frac{1-2v_1}{v_1(1-v_1)}$, then, we need to find out what values of $v_1$ does $Q = 1$.

Set $\frac{1-2v_1}{v_1(1-v_1)} = 1$, then multiplying out and solving we get:

$v_1 = 2.61$ and $0.38$ (2dp). We also know that $v_1 > 1$, therefore $v_1 = 2.61$ when $Q = 1$. Thus for values of $1 < v_1 \leq 2.6$ the condition $\alpha_1 < \frac{1-2v_1}{v_1(1-v_1)}$ automatically holds as $\alpha_1 < 1$, assumed throughout. It follows that for values of $v_1 > 2.6$ the condition must hold.

# CHAPTER SEVEN

# THE OPTIMAL TIME PATH OF A CARBON TAX

## Pollution and the taxation of carbon emissions

One important by-product of the combustion of fossil fuels is the emission of carbon dioxide into the atmosphere. $CO_2$ is not generally thought of as a pollutant but rather as something that plays an important role in the determination of the global climate. The presence of $CO_2$ in the atmosphere produces a "greenhouse effect". $CO_2$ and other greenhouse gases (GHG) are mostly transparent to sunlight. The sunlight passes through them and warms the earth. The earth radiates heat out into space and it is this infrared radiation that is absorbed by the gases. There is evidence that greenhouse gases are changing the climate but the seriousness of global warming is not known.

The need for public intervention to control environmental pollution arises because of the externalities that are incurred by other members of society, i.e. there are costs involved with pollution. The polluter may have no reason to take these external costs associated with the emission of carbon dioxide into account. Therefore the objective of environmental policy should be to fully internalise these costs. A tax on carbon emissions would have two effects on fuel use. Firstly, it would create an incentive to switch away from the most $CO_2$-intensive fuels to less carbon intensive fuels. Secondly, consumers would be motivated to utiltize energy more efficiently and to reduce energy consumption where possible. It would also encourage investment in energy saving technology development of new, less carbon intensive technologies, products and processes.

Carbon taxes have some drawbacks: a sizeable tax may imply major problems of institutional control and political acceptability. Also a very important concern is for the resulting distributional effects. Energy intensive sectors of the economy, such as the steel and cement industries, may experience detrimental effects from the impostion of a carbon tax, especially with regards to their international competitiveness, output and employment. Also there is the concern that lower income households would not be able to afford at least a minimum level of domestic heating. The advantage of a carbon tax is that revenue would be created and this could be recycled back into the economy to offset these negative effects. For example, if the revenue was used to reduce the level of other more distortionary indirect taxes in the economy. This could also broadly offset the effect that the tax might have on the price level.

Nordhaus (1993a), for instance, estimated that if a US$56 per ton carbon tax was implemented, $CO_2$ emissions in 1995 would be reduced by 20% below 1990 levels, but would result in an annual global loss in GDP of US$762 billion. When burdensome taxes were reduced, however, a tax of US$59 per ton of carbon would be optimal and this would reduce emissions by 20% below the 1990 level resulting in a gain in GDP of US$206 billion..

Similarly, Jorgenson and Wilcoxen (1993) argue that a policy of recycling the revenue as lump sum payments to households is:

> "not the most likely use of the revenue...Using the revenue to reduce distortionary taxes would lower the net cost of a carbon tax by removing inefficiency elsewhere in the economy" (Jorgenson and Wilcoxen (1993))

Indeed they found that a 1.7% GDP loss under a policy lump sum redistribution is converted to a 0.69% loss if labour taxes are reduced and a 1.1% gain if capital taxes are reduced.

It is not, however, the purpose of this chapter to discuss these aspects For a more in depth discussion of the distributional effects see Johnson et al (1990) and Smith (1992).

In 1992 the British Government, along with around 150 other countries, signed the Framework Convention on Climate Change in Rio de Janeiro. The target set was to return emissions of $CO_2$ and other GHGs to their 1990 levels by the year 2000. In response to this, a few countries have introduced a carbon tax. For example, Finland, Norway and Sweden. The European Community proposed to introduce a carbon tax in 1993 at $3 a barrel of oil and this was to increase by $1 annually until it reached $10 in the year 2000. It has been calculated by Barker et al (1993), that in the year 2000 the proposed tax of US$10 would yield approximately £11.5 billion in the UK and would reduce $CO_2$ emissions by 8% compared to the 1990 level. For a detailed report on the European Community's proposals see Pearson and Smith (1991). But since the Rio Earth Summit in 1992 policies to reduce emissions have been pigeonholed.

## The literature on carbon tax models

There has been a substantial literature on pollution externalities, (see chapter 5). Recently the literature has focused on the issue of a carbon tax, but has tended to look at the level of tax that is required to reduce $CO_2$ emissions to some target level by some specified date.

Manne and Richels (1991) analysis is concerned with the level of carbon tax that will achieve a particular target of emission levels by 2020. The analysis is based on the Global 2100 Model which is an analytic framework for estimating the economy wide impacts of rising energy costs. They estimate how emissions are likely to evolve over time when there are no carbon limits and indicate the size of the carbon tax that would be needed to encourage lower dependency on carbon intensive fuels. They explore the impacts of carbon emission limits on five regions - 1). USA 2).other OECD countries (Western Europe, Canada, Japan, Australia and New Zealand 3). USSR 4).China and 5). ROW (Rest Of the World). The reason for this regional categorising is that the solution to the climate problem is likely to require different responses by industrialised countries than by developing countries. Also, $CO_2$ emissions largely result from coal burning and 97% of the world's coal resources are found in the OECD, USSR, Eastern Europe and China (see Cline (1982)). The ROW category is needed to keep a consistent global balance of energy and flows from carbon. They establish a region-by-region time path for carbon taxation, that is required to achieve specific limits on carbon emissions. The time path varies between regions. A high tax is required in the industrialised countries because there is an assumed agreement to reduce emissions by 20% by the year 2020 and China and the ROW are allowed to increase emissions beyond their current levels. Those that find it easier to remove the link between energy consumption and GDP growth will find that the tax rises but are at a lower rate than those who find it harder to adjust to the emission limits

The USSR and Eastern Europe will find it more difficult to adjust to emission limits as they lack the supply of, and the demand for, alternative energy. They will have to maintain their taxes at a higher level to encourage consumers to substitute high carbon-intensive fuels for lower carbon-intensive fuels. Their tax rate will not start to fall significantly until 2070. Other OECD countries are better off in terms of supply and demand for energy alternatives. Therefore they require a reduced tax level, but the tax will still rise due to the 20% reduction commitment by 2020. However, after the year 2020, US and OECD taxation level will fall. China and the ROWs' tax level will rise contiually until 2040 and then stabilise.

From their paper, Manne and Richels give an insight into the fact that the costs of limiting carbon emissions is likely to vary among regions. However, the long run carbon tax for each region that is required to reduce consumers dependency on carbon based fuels converges in 2100 and the long run carbon tax is calculated to be $250 per ton of carbon emissions.

Although this is an interesting and significant piece of work they do not take any account of the damage costs of global warming. In a similar paper, Whalley and Wigle develop a CGE (Computer Generalised Equilibrium) Model and use this to identify the effects of cutting $CO_2$ emissions by 50% by 2005 and the international effects that could result. Again they abstract from the issue of damage from global warming.

There are a few attempts to estimate the monetary quantification of global warming damage. Fankhauser (1994) provides estimates of the marginal social cost of GHG emissions. He estimates the social cost to be \$20 per ton of carbon emission between 1991 - 2000. This then rises over time to about \$28 per ton of carbon between 2021 - 2030. This shows that as emissions are rising, the marginal damage costs are rising. Therefore the damage function is non-linear, implying that a ton of $CO_2$ added to an existing large stock is likely to cause a higher damage than a ton emitted when there is a low concentration level.

Neither of these two papers, or the proposed European carbon tax policy deal with the task of designing an optimal policy response to global warming and the figures with which they provide us with give little indication of the socially optimal carbon tax, let alone the optimal emission and carbon tax trajectories. This requires optimal control modelling.

The pioneering paper paper in this area is Nordhaus (1991), where costs are estimated that could be caused by global warming. Nordhaus investigates the impact of doubling $CO_2$ concentrations in the atmosphere, which he takes as a 3 % rise in mean surface temperature. He estimates the flow damages from this climate change to be 1/4% of GNP for the US and claims that this would hold for the rest of the world. His damage function would incorporate for example, land lost due to the sea level rising and crop yields changing due to climatic variations. To allow for non market impacts, such as damage to natural systems, he raises this value to 1% and estimates that the cost per ton of carbon emission to be \$7.33. Others also agree with Nordhaus and predict damage costs in the order of 1-2% of world GNP (Cline (1992), Titus (1992) and Fankhauser (1992, 1993)). Nordhaus, however, makes a particularly questionable assumption. He assumes that the economy is already in a resource steady state, implying that all physical flows and concentrations of GHGs are also constant, and that the impact of climate has been stabilised. This does not take into account that higher emissions will increase damage costs. For instance, the IPCC (1992), (Intergovernmental Panel on Climate Change (1990)), forecasts an increase in annual $CO_2$ emissions of 9-14 gigatonnes by the year 2025. Nordhaus does not consider the optimal carbon tax that would result from his model.

It can be argued it is the time path that a carbon tax should take is what matters. The major source of $CO_2$ in the atmosphere is the burning of fossil fuels, so to encourage delayed depletion of the resource a falling tax would be the correct policy. Another possible reason for a falling carbon tax is that technology that will reduce emissions is already in existence and it is very possible that technology may be invented that might eliminate GHG emissions and reduce global warming. This is another argument in favour of delaying some fossil fuel burning and a falling carbon tax would encourage this. However as the damage arising from global warming is an increasing function of the level of $CO_2$ emissions then maybe the correct policy would be a rising tax rate. It seems then, that the issue that we should be concerned with is the optimal time path that the carbon tax should take. As mentioned above this requires the application of optimal control theory.

The CETA (Carbon Emission Trajectory Assessment) Model used by Peck and Teisberg (1991) provides an assessment of what the optimal trajectory might be and, consequently, what is the optimal time path that a carbon tax should follow. They experiment with a linear and a non-linear damage function. They show that the carbon tax in either case is likely to be non-decreasing over time. In the case of the linear function the tax rises steadily and in the case of the non-linear function the tax rises sharply over time because they argue that an increase in the stock of the pollution increases the marginal damage caused, thus requiring a higher emission charge. They do not, however, include the damage function as an argument in the utility function or take into account the optimal allocation of natural resources over time. There are energy inputs in the production function but there is no explicit treatment of the dynamics of the stock of the natural resource.

The DICE (Dynamic Integrated Climate Economy) model, Nordhaus (1993b), is also an optimal control model. A dynamic optimisation model is constructed for estimating the optimal path of reductions of $CO_2$. The optimal growth model is extended to include a climate module and a damage sector. These feed back into the economy. He assumes a non-linear damage function and shows that the carbon tax that is required to raise fossil fuel prices sufficiently to induce substitution from carbon intensive goods and services to less carbon intensive uses, increases gradually over time. The model however does not include extraction of non-renewable resources, the very fossil fuels that he is raising the prices of. This work is an extension to the earlier resource steady state work by Nordhaus (1991). In this later model he develops the dynamics of the economy and the climate and argues that the resource steady state approach is unsatisfactory because of the time lags involved in the reaction of the climate and the economy to $CO_2$ emissions. He states that scientific estimates indicate that the GHGs stay in the atmosphere for over 100 years. Also the climate has a lag of several decades behind the changes in GHG stocks because the oceans contain a lot of the heat. For a full analysis of the DICE model see Nordhaus (1992).

In the DICE model the objective is to maximise the discounted sum of the utilities of consumption, summed over the relevant time horizon.

$$\max_{\{c(t)\}} \sum_t U\left[c(t), P(t)(1+\rho)^{-t}\right]$$

where $U$ is the flow of utility or social well-being, $c(t)$ is the flow of consumption at time $t$, $P(t)$ is the level of population at time $t$ and $\rho$ is the pure rate of social time preference. The maximisation is subject to a number of constraints, the first set are the economic constraints while the second is the set of climate-emissions constraints.

## *The Economic Constraints*

First, he takes the definition of total utility to be the product of population $[P(t)]$ and utility of per capita consumption $u[(t)]$. He takes a power function to represent the form of the utility function:

$$U[c(t)] = \frac{P(t)\left\{[c(t)]^{1-\alpha} - 1\right\}}{(1-\alpha)} \tag{1}$$

$\alpha$ is the parameter that measures the social valuation of different levels of consumption, which he calls the "rate of inequality aversion". When $\alpha$ is zero, then there is no social aversion to inequality, as $\alpha$ increases then society becomes more egalitarian, i.e. society holds the principle of equal rights for everyone. In his model, Nordhaus takes $\alpha = 1$, which is the logarithmic or Bernoullian utility function.

The definition of per capita consumption is $c(t)$:

$$c(t) = \frac{C(t)}{P(t)}$$

where $C(t)$ is total consumption and $P(t)$ is total population.

The production function used is the Cobb Douglas function in capital $[K(t)]$, Labour $[P(t)]$, (which is assumed proportional to population), and technology $[A(t)]$.

$$Q(t) = \Omega(t)A(t)K(t)^{\gamma}\,P(t)^{1-\gamma} \tag{2}$$

where $\gamma$ is the elasticity of output with respect to capital. There are constant returns to scale in capital and labour. The term $\Omega(t)$ relates to climate impacts which are described in equation (9).

Output is either consumed or invested, i.e.

$$C(t) = Q(t) - I(t) \tag{3}$$

The capital accumulation equation is:

$$K(t) = (1 - \delta_K)K(t-1) + I(t) \tag{4}$$

where $\delta_K$ is the rate of capital depreciation. Therefore the stock of capital at time t is equal to the stock of capital in the previous period less the depreciated capital in that time plus new investment at time t.

***The climate, emission and damage equations***

The first constant links economic activity with GHG emissions:

$$E(t) = [1 - \mu(t)]\sigma(t)Q(t) \tag{5}$$

$E(t)$ represents emissions at time $t$, $\sigma(t)$ is the ratio of uncontrolled GHG emissions to gross output, $\mu(t)$ is the fractional reduction of emissions relative to an uncontrolled level.

The next constant represents the build up of GHG in the atmosphere.

$$M(t) = \beta E(t) + (1 - \delta_M)M(t-1) \tag{6}$$

$M(t)$ is the stock of $CO_2$ in the atmosphere, $\beta$ is the rate at which the atmosphere retains $CO_2$, $\delta_M$ is the rate of removal of $CO_2$ from the atmosphere. This shows that the stock of $CO_2$ at time t is equal to $CO_2$ retained in the atmosphere from emissions at time t plus the net stock of $CO_2$ in the previous time period.

The impact of climate change on human and natural systems is taken to be a non-linear function of temperature increase:

$$D(t) = 0.0133\left[\frac{T(t)}{3}\right]^2 Q(t) \tag{7}$$

where $D(t)$ is the loss of global output from greenhouse warming and $T(t)$ is the temperature in period $t$. The cost of greenhouse warming of 3% is estimated in a previous paper by Nordhaus (1991) to be 1/4% (0.0025)% of national income. He has adjusted this here to take into account that there are areas that are unquantifiable by increasing the value to 0.0133%.

Total cost of reducing emissions is:

$$\frac{TC(t)}{GDP(t)} = b_1 \mu(t)^{b_2} \tag{8}$$

expressed as a fraction of world output. $\mu(t)$ is the reduction in GHG emissions. This equation shows that the cost curve rises as more costly measures are required.

The damage and cost functions are then combined to form the relationship $\Omega$ in the production function:

$$\Omega(t) = [1 - b_1 \mu(t)^{b_2}] / [1 + d(t)] \tag{9}$$

Nordhaus uses cost and damage for global warming as components of the production function. He does not however, include the damage function as an argument in the utility function. He talks about market impacts from the global warming damage and that one must also take into consideration that there are unquantifiable costs such as damage done to natural systems and everyday life. It would be more appropriate to include this damage cost as a negative argument in the utility function. The model also does not include extraction of non-renewable resources - there is no treatment of the dynamics of natural resource stocks. There is no use of fossil fuels, pollution emissions simply come from production. Various parameters are given estimates and then the model is simulated using the GAMS (Generalised Algorithm Modelling System) computer package and run under several scenarios. He shows the optimal carbon tax should rise gradually over time.

Van der Ploeg and Withagen (1991) show that if the stock of $CO_2$ is below its steady state level then the carbon tax should rise over time.

The problem in their model is to:

$$\max \int_0^\infty [B(Y-A) - D_F(\alpha Y) - D_S(S)]e^{-\theta t} dt$$

subject to:

$$\dot{S} = \alpha Y - \sigma(A)S$$
$$S(0) > S_0$$

where $S$ is the stock of pollutant, $\alpha$ is the emission output ratio, $\sigma(A)$ denotes the rate at which the pollutants are assimilated by the environment. $Y$ is the production of goods and an amount of output is used to clean up the environment, $(A)$. Consumption is thus $Y$-$A$. Net social benefits of consumption are $B(C)$, where $B'(C) > 0$ and $B''(C) < 0$. $D_F(\alpha Y)$ and $D_S(S)$ denote the social damage caused by the flow and the stock of pollution respectively. $\theta > 0$ is the social rate of discount. The stock of pollution and the optimal emission charge evolve over time according to:

$$\dot{S} = \alpha Y(\tau, S) - \gamma A(\tau, S))S$$
$$\dot{\tau} = [\theta + \sigma(A(\tau, S))]\tau - D'_S(S)$$

where $\tau$ denotes the optimal emission charge per unit of pollution, $\alpha Y$.

They show that the optimal emissions charge should rise over time if the stock of pollution is below its optimal steady state. This is because the optimal pollution stock is rising and if this is the case then an increase in the stock of pollution increases the marginal damage done to the environment and therefore requires a rising tax.

They also allow for pollution stock in the classical Ramsey problem. The problem then is to maximise a social welfare function, $W$ subject to constraints on capital accumulation and pollution stock accumulation, i.e. maximise:

$$W = \int_0^{\infty} \left[ B(C(t)) - D_s(S(t)) \right] e^{-\theta t} dt$$

subject to:

$$\dot{K} = f(K) - \delta K - C \qquad K(0) = K_0$$
$$\dot{S} = \alpha f(K) - \sigma S \qquad S(0) = S_0$$

where $B(C)$ is the net benefits from per capita consumption, $C$. $K$ is the per capita capital stock, $\sigma$ is the depreciation of the stock of pollutants.

An interior solution must satisfy:

$$\dot{C} = \left\{ f'(K) \left[ 1 - \left( \frac{\alpha}{B'(C)} \right) \tau \right] - \theta - \delta \right\} \eta(C)$$
$$\dot{\tau} = (\theta + \sigma)\tau - D'_S(S)$$

Again, the optimal time path for the emission charge, $\tau$, is increasing if the stock of pollution is below the steady state level.

Neither of their models include any treatment of natural resource stock extraction. They do provide a brief illustrative example of how to include renewable or non-renewable resources into such models thereby stressing the interrelationship between the economy, the environment and renewable or non-renewable resources, however they do not extend their analysis to include emission charges.

There are many other models that relate stock externalities to the use of natural resources that do not examine the implications of their models for the time path of a carbon tax, (see chapter 5). However more recently there have been a few authors that have looked into this issue.

Ulph and Ulph (1994) argue that a carbon tax should first rise sharply and then fall. This is because after some time the exhaustion constraint will start to bite, i.e. the resource stock is getting lower and the producer price of the non-renewable resource increases and so chokes off demand. The carbon tax is then steadily eliminated and so both the specific and ad valorem tax will fall over time. In their model they assume one very important feature - constant marginal costs of extraction. This therefore implies that each unit costs the same to extract. As mining goes deeper, then a unit of coal extracted costs the same to extract as the previous unit. The work in this chapter does not entirely hold with this view. Costs in coal mining have been kept down to some extent because 'uneconomic' deposits have not been extracted. As each additional unit is extracted the resource stock goes down. Extracting further deposits become more difficult and therefore the cost of extracting more rises. Therefore there are increasing marginal costs of extraction. To argue that each unit costs the same to extract is an unrealistic assumption to make.

In Ulph and Ulph the problem is to:

$$\max \int_0^\infty [B(x) - c(x) - D(M)]e^{-rt}.dt$$

subject to:

$$\dot{S} = -x$$
$$\dot{M} = \gamma x - \delta M$$

where

$$B(x) - c(x) = \alpha x - \frac{\beta}{2}x^2$$
$$D(M) = \frac{\varepsilon}{2}M^2$$

where $S$ is the stock of the resource, $M$ is the stock of $CO_2$ in the atmosphere, $x$ is the extraction rate, $B(x)$ is benefits of resource extraction, $c(x)$ is cost of resource extraction, $D(M)$ is the damage function, $\delta$ is the rate that the $CO_2$ decays in the atmosphere, each unit of the resource extracted adds to the stock of $CO_2$ by and amount $\gamma$, $\alpha$, $\beta$ and $\varepsilon$ are positive constants.

The costs of extraction are constant and the damage cost function from global warming is monotonically increasing in $M$, implying that as the stock of the resource increases, the cost of the damage increases at an increasing rate.

In this paper they assume that the resource is exhausted in finite time although they do not prove this to be the case. They look at the problem in 2 phases - one where there is production , i.e. there is positive extraction of the resource and one where there is no extraction , the terminal phase. Their model is not solved in the general case and they use numerical calibration of the model to determine the optimal time path of the carbon tax.

A falling carbon tax over time is argued by Sinclair (1992). He offers a model where there are two sectors, oil extraction and other production. Oil is extracted from the ground at no cost and used as an input in the production process. However their is no explicit treatment of the dynamics of the stock of carbon in the atmosphere. He assumes that the non-oil production sector is affected by the emissions and incorporates this into the model by introducing a negative link between the rate of technical progress and the level of oil depletion.

Ulph and Ulph in their paper described previously, develop Sinclair's model into an optimal control problem to show that Sinclair's intuition that there should be a falling carbon tax is perfectly correct from his analysis, but that it is the implausible assumption that he makes that drives him to his conclusion.

He assumes that the damage caused by $CO_2$ in the atmosphere affects the productive capacity of the economy, where:

$$K(M) = M^{-\tau} \qquad \tau > 0 \tag{1}$$

$K(M)$ is the technological capabilities of the economy as affected by the stock of $CO_2$. This equation shows that as the stock of $CO_2$ increases the technological capabilities of the economy reduce at an increasing rate.

He then shows that Sinclair assumes that the burning of fossil fuels effects the percentage rate of growth of $CO_2$, rather than the absolute rate that Ulph and Ulph assume. Their constraint on the stock of pollution growth above now changes to :

$$\dot{M} = M(\gamma x - \delta) \tag{2}$$

They then show that it is the percentage reduction in the stock of fossil fuel rather than the absolute, which affects the percentage rate of increase in $CO_2$.

Therefore the percentage rate of fossil fuel reduction is given by $\xi$, where $\xi$ is defined as:

$$\xi = \frac{x}{S} \tag{3}$$

It is $\xi$ which affects the percentage rate of increase in $CO_2$, therefore:

$$\dot{M} = M(\gamma\xi - \delta) \tag{4}$$

This means that if a new source of an exhaustible resource were to be discovered, i.e. $S$ increases, then the rate of accumulation of $CO_2$ in the atmosphere would reduce. But as there is more of the resource available to extract, then more would be extracted and the stock of $CO_2$ would increase.

Differentiating (1) with respect to time gives:

$$\dot{K} = -\tau M^{-\tau-1}\dot{M}$$

Substituting (1) and (4) into this gives and rearranging gives:

$$\dot{K} = (\varsigma - \chi\xi)K \qquad (5)$$

where $\varsigma = \tau\delta$ and $\chi = \gamma\tau$. This is the crucial equation that Sinclair gives showing that it is the burning of the resources that reduces the rate of growth of the technological capabilities of the economy.

Ulph and Ulph then go on to develop Sinclair's model into a full optimal control problem. The change in the resource stock can now be given by substituting equation (3) into the original resource stock constraint:

$$\dot{S} = -\xi S \qquad (6)$$

The optimal extraction problem is then to:

$$\max \int_0^\infty K_t B(\xi_t, S_t)$$

subject to equations (5) and (6).

They show that the social exhaustion price of the fossil fuel (the shadow price of S or the social value of the stock), is falling over time, and that given Sinclair's assumption - it is the percentage reduction in the stock of fossil fuels which determines the rate of growth of $CO_2$ -this implies the following. As the stock of the fossil fuels is reduced then the percentage reduction in the stock that will arise from future extraction will increase. Therefore the amount of damage done by $CO_2$ will increase. They prove using Sinclair's analysis, that the ad valorem tax must be falling in the steady state.

In another paper, Sinclair (1994) proves that the carbon tax should still indeed be falling over time, without making these assumptions. In his paper he formulates a model of endogeneous growth, oil extraction and global warming. First he looks at the decentralised case and introduces a tax on oil. Here the Lagrangian multiplier method is used and the problem is to:

$$\max\ \varphi=\int_0^{\infty}\left[A(t)+\lambda(t)\left(e^{ht}\bar{S}(t)^m K(t)^{\alpha_1} N(t)^{\alpha_2} D(t)^{1-\alpha_1-\alpha_2}+B(t)-C(t)\right)\right]dt$$

where:

$$B(t)=-P(t)D(t)[1+Z(t)]+\dot{S}(t)+y(t)N(t)$$
$$C(t)=c(t)N(t)+jK(t)+\dot{K}(t)$$

subject to the initial conditions:

$$K(0)=K_0>0$$

$$S(0)=K_0>0$$

In the model:

$$A(t)=\frac{e^{-it}c(t)^{1-v}}{[1-v]}$$

$c(t)$ is per capita consumption, $i$ is the discount rate and $v$ is a coefficient of relative risk aversion. $h$ is an exogeneous rate of technical progress, $K$, $N$ and $D$ are the inputs of capital, labour and oil respectively. $\bar{S}(t)$ is the stock of oil underground and so $-\dot{S}(t)$ is the rate of extraction, $P$ is the price of oil, $y$ is the per capita return on oil excise receipts, $j$ is the rate of capital depreciation $Z$ is the ad valorem tax on oil, $m$ is the green house effect parameter representing the elasticity of technology to resource stock, $\alpha_1$ and $\alpha_2$ are positive constants and $\alpha_1+\alpha_2<1$.

He then goes on to formulate a social planners problem and looks at the time path of the tax that will make these two situations equivalent.

The planner's problem is to:

$$\max\ \psi = \int_0^{\infty}\left[A(t) - \mu(t)\left(e^{ht}s(t)^m K(t)^{\alpha}\ \ N(t)^{\alpha_2}\left[-\dot{S}(t)\right]^{1-\alpha_1-\alpha_2} - C(t)\right)\right]dt$$

subject to $K(0)$, and $S(0)$ given. The variables are as before but here there is a greenhouse effect parameter, m. The difference between the two maximisation problems is that the planner recognises that there is benefit from the stock of unburnt oil, whereas individuals ignore it.

He shows that the tax is falling over time as the greenhouse parameter is positive. Although Sinclair includes in the social planner's problem the favourable affect of an unburnt oil stock, (i.e. there is less pollution in the atmosphere), he does not include any treatment of the dynamics of the flow or stock of $CO_2$ in the atmosphere.

In this chapter it is shown that a version of the Ulph and Ulph model can be solved in general for the time path of the carbon tax without having to make assumptions about some of the parameters. Their model is developed further to incorporate increasing marginal costs of extraction and it is shown that the specific tax on carbon emissions should be held constant over time and the ad valorem tax should fall. This agrees with the conclusion of Sinclair (1992), but here it is the absolute reduction in the fossil fuel stock that determines the rate of growth of $CO_2$, not the percentage. This also agrees with the conclusion of Sinclair (1994), but here the dynamics of the $CO_2$ stock is a constraint on the maximisation problem.

It is also shown that the steady state level of the resource stock is positive, and that extraction of the resource is terminated before the resource is totally exhausted; this is in contrast to Ulph and Ulph where the resource stock is completely exhausted. It is also shown that the level of steady state resource stock is greater than that which occurs when environmental considerations are not taken into account.

In Section I the model is formulated. In Section II Pontryagin's Maximum Principle (Pontryagin et al (1962)) is used to derive the first order conditions and the optimal steady state. In section III the optimal time paths of the relevant variables are derived, including the optimal time path of the carbon tax.

# SECTION I

## The Model

In this section a model of the extraction of a non-renewable resource and its consumption over time is constructed. Consumption of the resource generates a flow of pollution which builds up into a stock in the atmosphere. However it has been observed that the increase in the stock of $CO_2$ is about half of what it should be if all the anthropogenic (manmade) emissions that have occurred since the industrial revolution had been added to the pre-industrial stock of $CO_2$. This implies that carbon is being removed from the atmosphere, into the oceans and the biosphere. In this model there is assumed to be a constant depreciation/decay rate for the stock of $CO_2$.

The problem then is to identify a time path of a tax on carbon emissions that will make the decentralised optimum by individual agents equivalent to the socially optimal solution.

A stock $S$ of non-renewable resource is extracted from the ground at a rate $x$. The behaviour of the stock of the resource over time is given by:

$$\dot{S} = -x$$

Consumption of the resource generates a flow of pollution which evolves over time according to:

$$\dot{M} = \gamma x - \delta M$$

where $M$ is the stock of $CO_2$ in the atmosphere at time $t$. Each unit of the resource extracted adds to the stock of $CO_2$ by an amount $\gamma > 0$. The stock of the resource decays at rate $\delta$.

The stock of $CO_2$ causes damage to the environment. Costs associated with global warming can be put into two categories (Nordhaus (1991)). The first is the market effects of global warming; for example the reduced productivity from land loss and the reduced productivity in the climate dependent commercial activities, such as agriculture, forestry and fishing. Also, there are shore protection costs due to rising sea levels. The second category contains the damage to unmanaged natural systems and changes in the amenity value of everyday life. The damage function in this model and in Ulph and Ulph is given by a monotonically increasing damage function, $D(M)$ where $D'(M) > 0$ and $D''(M) \geq 0$. This implies that as the level of pollution rises the damage done by the $CO_2$ increases either at an increasing or constant rate.

The cost of extracting the resource increases as the level of the resource stock goes down. This implies that there is increasing marginal costa of extraction. Therefore the cost function is given by $C(x,S)$ where $C_S(x,S) > 0$ and $C_{SS}(x,S) \geq 0$.

The flow of the resource generates benefits from consumption. This is given by a concave benefit function $B(x)$ where $B_x(x) > 0$ and $B_{xx}(x) < 0$. This implies that as the amount of resouce consumed increases then the benefit increases but at a falling rate, i.e. diminishing marginal utility.

Social welfare, $W$, is given by:

$$W = B(x) - C(x,S) - D(M)$$

# SECTION II

## The formal problem

The task of a social planner then is to maximise discounted social welfare from fossil fuel extraction, subject to the constraints on resource stock and pollution stock accumulation.

Formally the problem is to :

$$\max \int_0^{\infty} [B(x) - C(x,S) - D(M)]e^{-rt}.dt$$

subject to:

$$\dot{S} = -x \tag{1}$$

$$\dot{M} = \gamma x - \delta M \tag{2}$$

with initial conditions:

$$M(0) = M_0 > 0$$
$$S(0) = S_0 > 0$$

where $r$ is the social rate of time preference.

The current value Hamiltonian is:

$$H = B(x) - c(x,S) - D(M) - \mu(x) - \lambda(\gamma x - \delta M)$$

where the shadow price associated with the stock of pollutants corresponds to $-\lambda$, because this is a stock with a negative social value. The shadow price associated with the stock of the resource corresponds to $\mu$, which has positive social value, i.e. $\mu$ and $\lambda$ are costate variables.

Assuming an interior solution, the first order necessary conditions are:

$$\frac{\partial H}{\partial x} = B_x(x) - c_x(x,S) - \mu - \gamma\lambda = 0$$

$$\frac{d}{dt}(\mu e^{-rt}) = -\frac{\partial H}{\partial S} \rightarrow \dot{\mu} = r\mu - c_S(x,S)$$

$$\frac{d}{dt}(\lambda e^{-rt}) = -\frac{\partial H}{\partial M} \rightarrow \dot{\lambda} = (r+\delta)\lambda - D_M(M)$$

The transversality conditions are:

$$\lim_{t \to \infty} e^{-rt} \mu(t) S(t) = 0$$

$$\lim_{t \to \infty} e^{-rt} \lambda(t) M(t) = 0$$

Rearranging the first equation gives:

$$B_X(x) = c_S(x, S) + \mu + \gamma\lambda$$

This equation demands that the marginal benefits from consuming the resource equal the marginal costs (i.e. marginal private costs) plus social value placed on the reduction of an extra unit of the resource and the social value of additional unit of $CO_2$ placed in the atmosphere(i.e. marginal social costs).

Therefore the social optimum can be decentralised by a producer price, $q(t) = c_X(x, S) + \mu$ and a consumer price $p(t) = q(t) + \gamma\lambda$. $\lambda(t)$ is the positive specific tax placed on each unit of $CO_2$ emissions. The behaviour of the specific tax over time can be seen by the differential equation:

$$\dot{\lambda} = (\mathrm{r} + \delta)\lambda - D'(M)$$

The ad valorem tax, $V$(t), which is a tax set as a percentage of the value of the commodity is given by:

$$V(t) = \frac{\lambda(t)}{q(t)}$$

Therefore to specialise further and solve for the time path of the specific and the ad valorem taxes, functional forms for the functions above need to be specified.

Let:

$$B(x) = ax - \frac{\beta}{2}x^2$$
$$c(x,S) = (c - \theta S)x$$
$$D(M) = \varepsilon M$$

From this we see that $B(x)$ is strictly concave in $x$. Also $c(x,S)$ is (weekly) convcex in $x$, and $D(M)$ is linear and increasing in $M$.

A linear damage function is used here to permit the derivation of the general solution. Peck and Teisberg (1991) show that the carbon tax would rise over time whether it is a linear or non-linear function. It is only the magnitude that is different. Since the aim of this chapter is to gain insight into the optimal *time profile* of the tax rather than its magnitude, the damage function used will be a linear function of the increase in the stock of pollution.

The current value Hamiltonian is:

$$H = ax - \frac{\beta}{2}x^2 - (c - \theta S)x - \varepsilon M - \mu x - \lambda(\gamma x - \delta M)$$

The first order necessary conditions are:

$$\frac{\partial H}{\partial x} = a - \beta x - c + \theta S - \mu - \gamma\lambda = 0 \qquad (3)$$

$$\frac{d}{dt}(\mu e^{-rt}) = -\frac{\partial H}{\partial S} \quad \rightarrow \quad \dot{\mu} = r\mu - \theta x \qquad (4)$$

The final condition is derived from:

$$\frac{d}{dt}(\lambda e^{-rt}) = -\frac{\partial H}{\partial M} \quad \rightarrow \quad \dot{\lambda} = (r + \delta)\lambda - \varepsilon \qquad (5)$$

These conditions are also sufficient because of the concavity of the objective function in $x$ and $M$. (See Mangarsarian (1966)).

From (3):

$$x = \frac{a - c + \theta S - \mu - \gamma\lambda}{\beta} \tag{6}$$

Substituting this into the state equations, (1) and (2) and costate equations, (4) and (5) we get:

$$\dot{S} = -\frac{1}{\beta}[a - c + \theta S - \mu - \gamma\lambda] \tag{7}$$

$$\dot{M} = \frac{\gamma}{\beta}[a - c + \theta S - \mu - \gamma\lambda] - \delta M \tag{8}$$

$$\dot{\mu} = r\mu - \frac{\theta}{\beta}[a - c + \theta S - \mu - \gamma\lambda] \tag{9}$$

$$\dot{\lambda} = (r + \delta)\lambda - \varepsilon \tag{10}$$

In the long run the optimal solution may "settle down" or converge to a steady state. If the steady state exists and is optimal, then there is an optimal extraction plan that will lead to this steady state. At the steady state $\dot{S} = \dot{M} = \dot{\lambda} = \dot{\mu} = 0$ therefore:

$$\dot{S} = -x^* = 0$$

$$\dot{M} = \gamma x^* - \delta M^* = 0$$

$$\dot{\mu} = r\mu^* - \theta x^* = 0$$

$$\dot{\lambda} = (r + \delta) - \varepsilon = 0$$

Thus

$$x^* = 0$$

and substituting this into the other equations gives:

$$M^* = 0$$

$$\mu^* = 0$$

$$\lambda^* = \frac{\varepsilon}{(r+\delta)}$$

Rearranging (6) and substituting in $x^* = 0$ gives:

$$S^* = \frac{c - a + \mu^* + \gamma\lambda^*}{\theta}$$

Therefore substituting $\lambda^*$ and $\mu^*$ into $S^*$ gives:

$$S^* = \frac{c-a}{\theta} + \frac{\gamma\varepsilon}{\theta(r+\delta)}$$

Rearranging gives:

$$S^* = \frac{(c-a)(r+\delta) + \gamma\varepsilon}{\theta(r+\delta)} \tag{11}$$

This steady state level of resource stock is greater than in the decentralised case when no environmental considerations are taken into account, this is shown in Chappell and Dury (1994), where the optimisation problem is to:

$$\max \int_0^\infty \left\{ax - \beta x^2 - (c - \theta S)x\right\} e^{-rt}.dt$$

subject to:

$$\dot{S} = -x$$

The steady state level of the resource stock is:

$$S^* = \frac{(c-a)}{\theta}$$

and

$$\frac{(c-a)}{\theta} < \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}$$

Therefore less is extracted in the socially optimum case.

# SECTION III

## The optimum time path

The problem here is to identify a time path for a tax on emissions that makes a decentralised optimum by individual agents equivalent to a socially optimal solution. Equation (3) can be rearranged to give:

$$a - \beta x = c - \theta S + \mu + \gamma\lambda \tag{12}$$

This equation shows that the optimum can be decentralised by a producer price, $q$ and a consumer price, $p$;

$$q = c - \theta S + \mu$$
$$p = q + \gamma\lambda$$

where $\lambda$ is a positive specific tax per unit of $CO_2$ emissions. It represents the optimal emission charge per unit of pollution.

Equation (3) damands that the marginal social benefit of consuming the resource equals the marginal cost plus the marginal social value of the damage done to the environment due to consumption. The pollution externality is fully internalised. In the absence of pollution, i.e. $D(M) = 0$, emission charges are zero, $\lambda = 0$, so that the marginal social benefit equals the marginal social cost of extracting the resource. The market outcome corresponds to the optimal solution.

Since the market does not internalise the externality associated with pollution emissions, the level of production and consumption, i.e. the level of extraction is too high:

In Chappell and Dury:

$$x = \frac{a - c + \theta S - \mu}{\beta}$$

Here, the pollution damage is taken into account and the level of extraction is reduced:

$$x = \frac{a - c + \theta S - \mu - \gamma\lambda}{\beta}$$

The socially optimum outcome is achieved by levying a consumption tax at a rate $\lambda$. The dynamics of the specific tax are given by (5):

$$\dot{\lambda} = (r + \delta)\lambda - \varepsilon$$

The ad valorem tax is set as a percentage of the value of the commodity, therefore:

$$V = \frac{\lambda}{q}$$

where q is the producer price of the extracted resource.

The evolution of the ad valorem tax, $V$, is given by:

$$\frac{\dot{V}}{V} = \frac{\dot{\lambda}}{\lambda} - \frac{\dot{q}}{q} = (r + \delta) - \frac{\varepsilon}{\lambda} + \frac{\theta\dot{S} - \dot{\mu}}{c - \theta S + \mu}$$

Therefore substituting (7) and (9) into the above equation gives:

$$\frac{\dot{V}}{V}=\frac{\dot{\lambda}}{\lambda}-\frac{\dot{q}}{q}=(r+\delta)-\frac{\varepsilon}{\lambda}-\frac{r\mu}{c-\theta S+\mu} \tag{13}$$

To proceed any further and determine the time path for the specific and ad valorem tax, then the system needs to be solved for the time paths of the variables.

Equations (7), (8), (9) and (10) can be expressed in matrix form, $\underline{Y}=A\underline{X}+\underline{B}$:

$$\begin{bmatrix} \dot{S} \\ \dot{M} \\ \dot{\mu} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{-\theta}{\beta} & 0 & \frac{1}{\beta} & \frac{\gamma}{\beta} \\ \frac{\gamma\theta}{\beta} & -\delta & \frac{-\gamma}{\beta} & \frac{-\gamma^2}{\beta} \\ \frac{-\theta^2}{\beta} & 0 & r+\frac{\theta}{\beta} & \frac{\gamma\theta}{\beta} \\ 0 & 0 & 0 & (r+\delta) \end{bmatrix} \begin{bmatrix} S \\ M \\ \mu \\ \lambda \end{bmatrix} + \begin{bmatrix} \frac{(c-a)}{\beta} \\ \frac{\gamma(a-c)}{\beta} \\ \frac{\theta(c-a)}{\beta} \\ -\varepsilon \end{bmatrix}$$

The characteristic equation of $A$ is:

$$(r+\delta-z)(-\delta-z)\left[(\frac{-\theta}{\beta}-z)(r+\frac{\theta}{\beta}-z)+\frac{\theta^2}{\beta}\right]=0$$

Therefore the eigenvalues of the system are (see appendix A):

$$z= \ -\delta \ , (r+\delta) \ , \ \frac{r-\sqrt{Q}}{2} \ , \ \frac{r+\sqrt{Q}}{2}$$

where $Q=r^2+\frac{4\theta r}{\beta}$

there are two positive eigenvalues $z= \ (r+\delta) \ , \ \frac{r+\sqrt{Q}}{2}$

and two negative eigenvalues, $z= \ -\delta \ , \ \frac{r-\sqrt{Q}}{2}$

.as $r-\sqrt{Q}=r-\sqrt{r^2+\frac{4\theta r}{\beta}}<0$

The associated eigenvectors are (see appendix B):

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad \begin{bmatrix}1\\ -\frac{\gamma(r+\delta)}{(r+2\delta)}\\ \frac{\theta(r+\delta)}{\delta}\\ \frac{\delta\beta(r+\delta)-r\theta}{\delta\gamma}\end{bmatrix},\quad \begin{bmatrix}1\\ -\frac{\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})}\\ \beta\left[\frac{\theta}{\beta}+\frac{r-\sqrt{Q}}{2}\right]\\ 0\end{bmatrix},\quad \begin{bmatrix}1\\ -\frac{\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})}\\ \beta\left[\frac{\theta}{\beta}+\frac{r+\sqrt{Q}}{2}\right]\\ 0\end{bmatrix}$$

The particular solution $\tilde{x}$ is equal to the steady state solution proved previously, therefore (see apendix C):

$$\underline{\tilde{x}}=\begin{bmatrix}\tilde{S}\\ \tilde{M}\\ \tilde{\mu}\\ \tilde{\lambda}\end{bmatrix}=\begin{bmatrix}\frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}\\ 0\\ 0\\ \frac{\varepsilon}{(r+\delta)}\end{bmatrix} \qquad (11)$$

The paths of the variables over time are, (see appendix D):

$$\begin{bmatrix}S(t)\\ M(t)\\ \mu(t)\\ \lambda(t)\end{bmatrix}=\begin{bmatrix}0 & e^{(r+\delta)t} & e^{\frac{(r-\sqrt{Q})}{2}t} & e^{\frac{(r+\sqrt{Q})}{2}t}\\ e^{-\delta t} & Ae^{(r+\delta)t} & Be^{\frac{(r-\sqrt{Q})}{2}t} & De^{\frac{(r+\sqrt{Q})}{2}t}\\ 0 & Ee^{(r+\delta)t} & Fe^{\frac{(r-\sqrt{Q})}{2}t} & Ge^{\frac{(r+\sqrt{Q})}{2}t}\\ 0 & He^{(r+\delta)t} & 0 & 0\end{bmatrix}\begin{bmatrix}C_1\\ C_2\\ C_3\\ C_4\end{bmatrix}+\begin{bmatrix}\frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}\\ 0\\ 0\\ \frac{\varepsilon}{(r+\delta)}\end{bmatrix}$$

where

$$A = \frac{-\gamma(r+\delta)}{(r+2\delta)}, \quad B = \frac{-\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})}, \quad D = \frac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})}$$

$$E = \frac{\theta(r+\delta)}{\delta}, \quad F = \beta\left[\frac{\theta}{\beta} + \frac{(r-\sqrt{Q})}{2}\right], \quad G = \beta\left[\frac{\theta}{\beta} + \frac{(r+\sqrt{Q})}{2}\right]$$

$$H = \frac{\delta\beta(r+\delta) - r\theta}{\delta\gamma}$$

Multiplying out gives:

$$S(t) = C_2 e^{(r+\delta)t} + C_3 e^{\frac{(r-\sqrt{Q})}{2}t} + C_4 e^{\frac{(r+\sqrt{Q})}{2}t} + \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta((r+\delta)}$$

$$M(t) = C_1 e^{-\delta t} + C_2 A e^{(r+\delta)t} + C_3 B e^{\frac{(r-\sqrt{Q})}{2}t} + C_4 D e^{\frac{(r+\sqrt{Q})}{2}t}$$

$$\mu(t) = C_2 E e^{(r+\delta)t} + C_3 F e^{\frac{(r-\sqrt{Q})}{2}t} + C_4 G e^{\frac{(r+\sqrt{Q})}{2}t}$$

$$\lambda(t) = C_2 H e^{(r+\delta)t} + \frac{\varepsilon}{(r+\delta)}$$

Solving for $C_1$, $C_2$, $C_3$ and $C_4$ using the Boundary and transversality conditions:

$$M(0) = M_0 > 0$$
$$S(0) = S_0 > 0$$

$$\lim_{t\to\infty} e^{-rt}\mu(t)S(t) = 0$$
$$\lim_{t\to\infty} e^{-rt}\lambda(t)M(t) = 0$$

gives:

$$C_2 = C_4 = 0$$

$$C_1 = \left[ M_0 + \frac{\gamma\theta S_0(r+\delta)(r-\sqrt{Q}) - \gamma(r-\sqrt{Q})(c-a)(r+\delta)}{\theta(r+\delta)(2\delta+(r-\sqrt{Q}))} + \frac{\gamma^2\varepsilon(r-\sqrt{Q})}{\theta(r+\delta)(2\delta+(r-\sqrt{Q}))} \right]$$

$$C_3 = \left[ \frac{S_0\theta(r+\delta) - (c-a)(r+\delta) + \gamma\varepsilon}{\theta(r+\delta)} \right]$$

Thus the optimal time paths are:

$$S(t) = C_3 e^{\frac{(r-\sqrt{Q})}{2}t} + \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}$$

$$M(t) = C_1 e^{-\delta t} + +C_3 B e^{\frac{(r-\sqrt{Q})}{2}t}$$

$$\mu(t) = C_3 F e^{\frac{(r-\sqrt{Q})}{2}t}$$

$$\lambda(t) = \frac{\varepsilon}{(r+\delta)}$$

The constant value for $\lambda(t)$ implies that there is a constant specific tax on emissions of $\frac{\varepsilon}{(r+\delta)}$ over time.

Also as $t \to \infty$, $S(t) \to \left( \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)} \right)$, $M(t) \to 0$ and $\mu(t) \to 0$. These correspond to the steady state values derived above.

From (6), $x = \frac{a-c+\theta S-\mu-\gamma\lambda}{\beta}$ and it follows that:

$$x(t)=\left[\frac{a-c}{\beta}\right]+\frac{\theta}{\beta}S(t)-\frac{1}{\beta}\mu(t)-\frac{\gamma}{\beta}\lambda(t)$$

Substituting $S(t)$, $\mu(t)$ and $\lambda(t)$ into this equation and rearranging gives:

$$x(t)=\left[\frac{a-c}{\beta}\right]-\left[\frac{a-c}{\beta}\right]+\left[\frac{\gamma\varepsilon}{\beta(r+\delta)}\right]-\frac{\gamma}{\beta}\left(\frac{\varepsilon}{(r+\delta}\right)-\frac{1}{\beta}\left[\frac{(r-\sqrt{Q})}{2}\right]\left[C_3e^{\left(\frac{r-\sqrt{Q}}{2}\right)t}\right]$$

As that $r-\sqrt{Q}<0$, then as $t\rightarrow\infty$ the rate of extraction is falling over time and $x(t)\rightarrow 0$. This agrees with the steady state value for $x(t)$ that was derived above.

Looking at the ad valorem tax $V(t)=\dfrac{\lambda(t)}{q(t)}$:

The producer price $q(t)$ is:

$$q(t)=c-\theta S(t)+\mu(t)$$

Using the equations for $S(t)$ and $\mu(t)$ and substituting into the above equation and rearranging gives:

$$q(t)=c-\frac{(c-a)(r+\delta)+\gamma\varepsilon}{(r+\delta)}+C_3e^{\frac{(r-\sqrt{Q})}{2}t}\left[\frac{\beta(r-\sqrt{Q})}{2}\right]$$

As $t\rightarrow\infty$ then $-q(t)\rightarrow 0$. Therefore the producer price is increasing over time.

This is because $F-\theta=\dfrac{\beta(r-\sqrt{Q})}{2}$ is negative, remembering that $r<\sqrt{Q}$

Differentiating with respect to time confirms this:

$$\frac{dq(t)}{dt} = \beta\left[\frac{(r-\sqrt{Q})}{2}\right]^2 C_3 e^{\frac{(r-\sqrt{Q})}{2}t}$$

Since $C_3 = S_0 - S^*$ ,(the amount taken out of the ground which by definition cannot be negative), the above derivative is positive; therefore $q(t)$ is rising over time.

It follows that if $q(t)$ is rising over time then the ad valorem tax , $V(t)$, is falling over time.

Ulph and Ulph argue that the social value of the resource stock,. i.e. the exhaustion rent, increases over time and that it is this that chokes off demand therefore both the specific and the ad valorem tax fall. But here the social value of the resource stock, $\mu$, falls over time and so it is the fact that the producer and therefore the consumer price is rising that chokes off demand. Thus it is optimal for the ad valorem carbon tax to fall.

## CONCLUSION

This model extended the Ulph and Ulph analysis of stock externalities to incorporate increasing marginal costs of extraction and it was shown that the model can be solved in general for the time path of the carbon tax without having to make assumptions about some of the parameters.

The socially optimal time path for the carbon tax, that makes the decentralised optimisation by individual agents equivalent to the socially optimal solution is found. It was shown that the price of output increases over time and it is this that causes the optimal carbon tax to fall. This is in contrast to Ulph and Ulph who argue that the optimal carbon tax trajectory should be one that first rises and then falls. They argue that this is because the exhaustion rent, the social price of the resource stock, rises and this raises fossil fuel prices, thus doing the work of choking off demand. The carbon tax will therefore be gradually eliminated.

The falling tax rate agrees with the models presented by Sinclair (1992), but without making the implausible assumption that it is the percentage reduction in the stock of resource which affects the percentage increase in the stock of $CO_2$. This also agrees with Sinclair (1994) but here the dynamics of the stock of $CO_2$ are incorporated into the model.

It is also shown that the steady state level of the resource stock is positive; hence extraction of the resource is terminated before the resource is totally exhausted, this is in contrast to Ulph and Ulph where the resource stock is completely exhausted in finite time. It is also shown that the level of steady state resource stock is greater than that which occurs when environmental considerations are not taken into account. This can be seen by comparing the level of steady state resource stock with that resulting in the case where environmental considerations are not taken into account.

# APPENDICES

## APPENDIX A

Equations (7), (8), (9) and (10) can be expressed in matrix form, $\underline{Y} = A\underline{X} + \underline{B}$:

$$\begin{bmatrix} \dot{S} \\ \dot{M} \\ \dot{\mu} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{-\theta}{\beta} & 0 & \frac{1}{\beta} & \frac{\gamma}{\beta} \\ \frac{\gamma\theta}{\beta} & -\delta & \frac{-\gamma}{\beta} & \frac{-\gamma^2}{\beta} \\ \frac{-\theta^2}{\beta} & 0 & r+\frac{\theta}{\beta} & \frac{\gamma\theta}{\beta} \\ 0 & 0 & 0 & (r+\delta) \end{bmatrix} \begin{bmatrix} S \\ M \\ \mu \\ \lambda \end{bmatrix} + \begin{bmatrix} \frac{(c-a)}{\beta} \\ \frac{\gamma(a-c)}{\beta} \\ \frac{\theta(c-a)}{\beta} \\ -\varepsilon \end{bmatrix}$$

The eigenvalues can be found by solving $|A - zI| = 0$, where z is an eigenvalue. Therefore solving:

$$\begin{vmatrix} \frac{-\theta}{\beta} - z & 0 & \frac{1}{\beta} & \frac{\gamma}{\beta} \\ \frac{\gamma\theta}{\beta} & -\delta - z & \frac{-\gamma}{\beta} & \frac{-\gamma^2}{\beta} \\ \frac{-\theta^2}{\beta} & 0 & r+\frac{\theta}{\beta} - z & \frac{\gamma\theta}{\beta} \\ 0 & 0 & 0 & (r+\delta) - z \end{vmatrix} = 0$$

gives the characteristic equation:

$$(r+\delta-z)(-\delta-z)\left[(\frac{-\theta}{\beta} - z)(r + \frac{\theta}{\beta} - z) + \frac{\theta^2}{\beta}\right] = 0$$

multiplying and cancelling out:

$$(r+\delta-z)(-\delta-z)\left[z^2 - rz - \frac{\theta^2}{\beta}\right] = 0$$

Therefore the eigenvalues of the system are:

$$z = \; -\delta, \; (r+\delta) \;, \; \frac{r-\sqrt{Q}}{2} \;, \; \frac{r+\sqrt{Q}}{2}$$

where $Q = r^2 + \dfrac{4\theta r}{\beta}$

## Appendix B

The associated eigenvectors can be found by solving $|A - zI|\underline{F} = 0$, where $\underline{F}$ is an eigenvector, therefore:

$$\begin{bmatrix} -\frac{\theta}{\beta} - z & 0 & \frac{1}{\beta} & \frac{\gamma}{\beta} \\ \frac{\gamma\theta}{\beta} & -\delta - z & -\frac{\gamma}{\beta} & -\frac{\gamma^2}{\beta} \\ -\frac{\theta^2}{\beta} & 0 & r + \frac{\theta}{\beta} - z & \frac{\gamma\theta}{\beta} \\ 0 & 0 & 0 & (r+\delta) - z \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

When z = - δ:

$$\left[-\frac{\theta}{\beta} + \delta\right] F_1 + \frac{1}{\beta} F_3 + \frac{\gamma}{\beta} F_4 = 0 \qquad (1a)$$

$$\frac{\gamma\theta}{\beta} F_1 - \frac{\gamma}{\beta} F_3 - \frac{\gamma^2}{\beta} F_4 = 0 \qquad (2a)$$

$$-\frac{\theta^2}{\beta} F_1 + \left[r + \frac{\theta}{\beta} + \delta\right] F_3 + \frac{\theta\gamma}{\beta} F_4 = 0 \qquad (3a)$$

$$[r + 2\delta] F_4 = 0 \qquad (4a)$$

From (4a), $F_4 = 0$. Therefore rearranging (1a);

$$F_3 = (\theta - \beta\delta)F_1$$

and substituting this into (2a) gives:

$$\frac{\gamma\theta}{\beta}F_1 - \frac{\gamma}{\beta}[\theta - \beta\delta]F_1 = 0 \quad \rightarrow \quad F_1[\beta\delta] = 0$$

Therefore:

$$F_1 = F_3 = F_4 = 0$$

and the corresponding eigenvector (which is determined only up to a scale factor) is:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

When $z = (r + \delta)$:

$$\left[-\frac{\theta}{\beta} - (r + \delta)\right]F_1 + \frac{1}{\beta}F_3 + \frac{\gamma}{\beta}F_4 = 0 \qquad (1b)$$

$$\frac{\gamma\theta}{\beta}F_1 - (r + 2\delta)F_2 - \frac{\gamma}{\beta}F_3 - \frac{\gamma^2}{\beta}F_4 = 0 \qquad (2b)$$

$$-\frac{\theta^2}{\beta}F_1 + \left[\frac{\theta}{\beta} - \delta\right]F_3 + \frac{\theta\gamma}{\beta}F_4 = 0 \qquad (3b)$$

Solving (1b) and (3b) in terms of $F_1$ gives:

$$\begin{bmatrix} \frac{1}{\beta} & \frac{\gamma}{\beta} \\ \frac{\theta}{\beta}-\delta & \frac{\theta\gamma}{\beta} \end{bmatrix}\begin{bmatrix} F_3 \\ F_4 \end{bmatrix}=\begin{bmatrix} \frac{\theta}{\beta}+(r+\delta) \\ \frac{\theta^2}{\beta} \end{bmatrix}F_1$$

Therefore:

$$\begin{bmatrix} F_3 \\ F_4 \end{bmatrix}=\frac{\beta}{\delta\gamma}\begin{bmatrix} \frac{\theta\gamma}{\beta} & -\frac{\gamma}{\beta} \\ \delta-\frac{\theta}{\beta} & \frac{1}{\beta} \end{bmatrix}\begin{bmatrix} \frac{\theta}{\beta}+(r+\delta) \\ \frac{\theta^2}{\beta} \end{bmatrix}F_1$$

Multiplying out gives:

$$F_3=\frac{(r+\delta)\theta}{\delta}F_1$$

$$F_4=\frac{\beta}{\delta\gamma}\left[\left(\delta-\frac{\theta}{\beta}\right)\left(\frac{\theta}{\beta}+(r+\delta)\right)+\frac{\theta^2}{\beta^2}\right]F_1$$

This cancels to:

$$F_4=\frac{\delta\beta(r+\delta)-r\theta}{\delta\gamma}F_1$$

Substitute $F_3$ and $F_4$ into (2b)

$$\frac{\gamma\theta}{\beta}F_1-[r+2\delta]F_2-\frac{\gamma(r+\delta)\theta}{\beta\delta}F_1-\frac{\gamma^2}{\beta}\left[\frac{\beta}{\delta\gamma}\left[\delta(r+\delta)-\frac{r\theta}{\beta}\right]\right]F_1=0$$

multiplying out gives:

$$\frac{\gamma\theta}{\beta}F_1-[r+2\delta]F_2-\frac{\gamma r\theta}{\beta\delta}F_1-\frac{\gamma\theta}{\beta}F_1-\gamma(r+\delta)-\frac{\gamma r\theta}{\delta\beta}F_1=0$$

Simplifying gives:

$$F_2 = \left[\frac{-\gamma(r+\delta)}{(r+2\delta)}\right]F_1$$

Therefore the corresponding eigenvector is:

$$\begin{bmatrix} 1 \\ \\ \dfrac{-\gamma(r+\delta)}{(r+2\delta)} \\ \\ \dfrac{\theta(r+\delta)}{\delta} \\ \\ \dfrac{\delta\beta(r+\delta)-r\theta}{\delta\gamma} \end{bmatrix}$$

When $z = \dfrac{r-\sqrt{Q}}{2}$:

$$\left[-\frac{\theta}{\beta}-\frac{(r-\sqrt{Q})}{2}\right]F_1 + \frac{1}{\beta}F_3 + \frac{\gamma}{\beta}F_4 = 0 \tag{1c}$$

$$\frac{\gamma\theta}{\beta}F_1 - \left[\delta + \frac{(r-\sqrt{Q})}{2}\right]F_2 - \frac{\gamma}{\beta}F_3 - \frac{\gamma^2}{\beta}F_4 = 0 \tag{2c}$$

$$-\frac{\theta^2}{\beta}F_1 + \left[r + \frac{\theta}{\beta} - \frac{(r-\sqrt{Q})}{2}\right]F_3 + \frac{\theta\gamma}{\beta}F_4 = 0 \tag{3c}$$

$$\left[r + \delta - \frac{(r-\sqrt{Q})}{2}\right]F_4 = 0 \tag{4c}$$

Therefore $F_4 = 0$

Rearranging (1c) gives:

$$F_3 = \beta\left[\frac{\theta}{\beta} + \left(\frac{(r-\sqrt{Q})}{2}\right)\right]F_1$$

Substituting this into (2c) gives:

$$\frac{\gamma\theta}{\beta}F_1 - \left[\delta + \frac{(r-\sqrt{Q})}{2}\right]F_2 - \frac{\gamma}{\beta}\left[\beta\left(\frac{\theta}{\beta} + \frac{(r-\sqrt{Q})}{2}\right)\right]F_1 = 0$$

Multiplying out:

$$\frac{\gamma\theta}{\beta}F_1 - \frac{\gamma\theta}{\beta}F_1 - \frac{\gamma(r-\sqrt{Q})}{2}F_1 - \left[\delta + \frac{(r-\sqrt{Q})}{2}\right]F_2 = 0$$

Simplifying gives:

$$F_2 = \left[\frac{-\gamma(r-\sqrt{Q})}{2\delta + (r-\sqrt{Q})}\right]F_1$$

Therefore the corresponding eigenvector is:

$$\begin{bmatrix} 1 \\ \dfrac{-\gamma(r-\sqrt{Q})}{2\delta + (r-\sqrt{Q})} \\ \beta\left[\dfrac{\theta}{\beta} + \dfrac{(r-\sqrt{Q})}{2}\right] \\ 0 \end{bmatrix}$$

When $z = \dfrac{r+\sqrt{Q}}{2}$:

$$\left[\frac{-\theta}{\beta} - \frac{(r+\sqrt{Q})}{2}\right]F_1 + \frac{1}{\beta}F_3 + \frac{\gamma}{\beta}F_4 = 0 \quad \text{(1d)}$$

$$\frac{\gamma\theta}{\beta}F_1 - \left[\delta + \frac{(r+\sqrt{Q})}{2}\right]F_2 - \frac{\gamma}{\beta}F_3 - \frac{\gamma^2}{\beta}F_4 = 0 \quad \text{(2d)}$$

$$\frac{-\theta^2}{\beta}F_1 + \left[r + \frac{\theta}{\beta} - \frac{(r+\sqrt{Q})}{2}\right]F_3 + \frac{\theta\gamma}{\beta}F_4 = 0 \quad \text{(3d)}$$

$$\left[r + \delta - \frac{(r+\sqrt{Q})}{2}\right]F_4 = 0 \quad \text{(4c)}$$

Therefore $F_4 = 0$.

From (1d):

$$F_3 = \beta\left[\frac{\theta}{\beta} + \left(\frac{(r+\sqrt{Q})}{2}\right)\right]F_1$$

Substituting this into (2d) gives:

$$\frac{\gamma\theta}{\beta}F_1 - \left[\delta + \frac{(r+\sqrt{Q})}{2}\right]F_2 - \frac{\gamma}{\beta}\left[\beta\left(\frac{\theta}{\beta} + \frac{(r+\sqrt{Q})}{2}\right)\right]F_1 = 0$$

Simplifying gives:

$$F_2 = \left[\frac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})}\right] F_1$$

Therefore the corresponding eigenvector is:

$$\begin{bmatrix} 1 \\ \dfrac{-\gamma(\mathrm{r}+\sqrt{\mathrm{Q}})}{2\delta+(r+\sqrt{Q})} \\ \beta\left[\dfrac{\theta}{\beta}+\dfrac{(r+\sqrt{Q})}{2}\right] \\ 0 \end{bmatrix}$$

## Appendix C

The particular solution (which is of course the steady state) is found by setting, $\underline{Y} = 0$;

$$0 = A\underline{X} + \underline{B} \quad \rightarrow \quad \tilde{X} = -A^{-1}B$$

Thus:

The inverse of the matrix $A$ is (using elementary row operations)

$$\begin{bmatrix} -\left[\dfrac{1}{r}+\dfrac{\beta}{\theta}\right] & 0 & \dfrac{1}{r\theta} & \dfrac{\gamma}{\theta(r+\delta)} \\ -\dfrac{\gamma}{\delta} & -\dfrac{1}{\delta} & 0 & 0 \\ -\dfrac{\theta}{r} & 0 & \dfrac{1}{r} & 0 \\ 0 & 0 & 0 & \dfrac{1}{(r+\delta)} \end{bmatrix}$$

Thus:

$$\underset{\sim}{\tilde{X}} = \begin{bmatrix} \tilde{S} \\ \tilde{M} \\ \tilde{\mu} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} \dfrac{(c-a)(r+\delta)+\gamma\varepsilon}{(r+\delta)} \\ 0 \\ 0 \\ \dfrac{\varepsilon}{(r+\delta)} \end{bmatrix}$$

## Appendix D

As $\dot{\underset{\sim}{X}} = A\underset{\sim}{X} + \underset{\sim}{B}$, and that the particular solution is $\underset{\sim}{\tilde{X}} = -A^{-1}\underset{\sim}{B}$.

let $\underset{\sim}{Y} = \underset{\sim}{X} - \underset{\sim}{\tilde{X}} = \underset{\sim}{X} + A^{-1}\underset{\sim}{B}$.

Then:

$\underset{\sim}{X} = \underset{\sim}{Y} - A^{-1}\underset{\sim}{B}$.

Therefore:

$$\begin{bmatrix} S \\ M \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} Y_S \\ Y_m \\ Y_\mu \\ Y_\lambda \end{bmatrix} + \begin{bmatrix} \dfrac{(c-a)(r+\delta)+\gamma\varepsilon}{(r+\delta)} \\ 0 \\ 0 \\ \dfrac{\varepsilon}{(r+\delta)} \end{bmatrix}$$

Differentiating both sides gives,

$\dot{\underset{\sim}{X}} = \dot{\underset{\sim}{Y}}$.

In other words:

$$\begin{bmatrix} \dot{S} \\ \dot{M} \\ \dot{\mu} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \dot{Y}_s \\ \dot{Y}_m \\ \dot{Y}_\mu \\ \dot{Y}_\lambda \end{bmatrix}$$

Substituting the expressions for $\underline{X}$ and $\underline{\dot{X}}$ for $\underline{\dot{X}} = A\underline{X} + \underline{B}$, results in:

$$\underline{\dot{Y}} = A(\underline{Y} - A^{-1}\underline{B}) + \underline{B} = A\underline{Y}$$

Therefore:

$$\begin{bmatrix} \dot{Y}_s \\ \dot{Y}_m \\ \dot{Y}_\mu \\ \dot{Y}_\lambda \end{bmatrix} = \begin{bmatrix} -\frac{\theta}{\beta} & 0 & \frac{1}{\beta} & \frac{\gamma}{\beta} \\ \frac{\gamma\theta}{\beta} & -\delta & -\frac{\gamma}{\beta} & -\frac{\gamma^2}{\beta} \\ -\frac{\theta^2}{\beta} & 0 & r + \frac{\theta}{\beta} & \frac{\gamma\theta}{\beta} \\ 0 & 0 & 0 & (r+\delta) \end{bmatrix} \begin{bmatrix} Y_s \\ Y_m \\ Y_\mu \\ Y_\lambda \end{bmatrix}$$

Let P be a matrix whose columns are the eigenvectors of

$$\text{A: P} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & \frac{-\gamma(r+\delta)}{(r+2\delta)} & \frac{-\gamma(r-\sqrt{Q})}{2\delta + (r-\sqrt{Q})} & \frac{-\gamma(r+\sqrt{Q})}{2\delta + (r+\sqrt{Q})} \\ 0 & \frac{\theta(r+\delta)}{\delta} & \beta\left(\frac{\theta}{\beta} + \frac{r-\sqrt{Q}}{2}\right) & \beta\left(\frac{\theta}{\beta} + \frac{r+\sqrt{Q}}{2}\right) \\ 0 & \frac{\delta\beta(r+\delta) - r\theta}{\delta\gamma} & 0 & 0 \end{bmatrix}$$

Let $\underline{Y} = P\underline{v}$. Differentiating gives:

$$\underline{\dot{Y}} = P\underline{\dot{v}}$$

Therefore:

$$\begin{bmatrix} \dot{Y}_s \\ \dot{Y}_m \\ \dot{Y}_\mu \\ \dot{Y}_\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & \dfrac{-\gamma(r+\delta)}{(r+2\delta)} & \dfrac{-\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})} & \dfrac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})} \\ 0 & \dfrac{\theta(r+\delta)}{\delta} & \beta\left(\dfrac{\theta}{\beta}+\dfrac{r-\sqrt{Q}}{2}\right) & \beta\left(\dfrac{\theta}{\beta}+\dfrac{r+\sqrt{Q}}{2}\right) \\ 0 & \dfrac{\delta\beta(r+\delta)-r\theta}{\delta\gamma} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_s \\ \dot{v}_m \\ \dot{v}_\mu \\ \dot{v}_\lambda \end{bmatrix}$$

Now substituting $\underline{Y}$ and $\underline{\dot{Y}}$ into $\underline{\dot{Y}} = A\underline{Y}$ gives:

$$P\underline{\dot{v}} = AP\underline{v} \rightarrow \underline{\dot{v}} = P^{-1}AP\underline{v} = D\underline{v}$$

where D is a diagonal matrix with the eigenvalues of A on the principal diagonal and zero's elsewhere.

This results in the system:

$$\begin{bmatrix} \dot{v}_s \\ \dot{v}_m \\ \dot{v}_\mu \\ \dot{v}_\lambda \end{bmatrix} = \begin{bmatrix} -\delta & 0 & 0 & 0 \\ 0 & (r+\delta) & 0 & 0 \\ 0 & 0 & \dfrac{(r-\sqrt{Q})}{2} & 0 \\ 0 & 0 & 0 & \dfrac{(r+\sqrt{Q})}{2} \end{bmatrix} \begin{bmatrix} v_s \\ v_m \\ v_\mu \\ v_\lambda \end{bmatrix}$$

Then it follows that:

$$\underline{\dot{v}}_i = \underline{z}_i \underline{v}_i; \quad i = 1,2,3,4.$$

where $z_i$ represents the eigenvalues of A. Separating these variables and integrating gives:

$$\underline{v}_i = C_i \exp(z_i t); \quad i = 1,2,3,4.$$

where the C's are arbitrary constants. Stacking these solutions to the individual equations in a vector we may write:

$$\underline{v} = \exp(zt)\underline{C}$$

where $\exp(zt)$ is a diagonal matrix with $\exp(z_1t)$, $\exp(z_2t)$,....., $\exp(z_4t)$ on the principal diagonal and zeros elsewhere and $\underline{C}$ is a column vector with elements $C_1$, $C_2$, $C_3$, $C_4$.

Therefore we can write:

$$\begin{bmatrix} v_s \\ v_m \\ v_\mu \\ v_\lambda \end{bmatrix} = \begin{bmatrix} e^{-\delta t} & 0 & 0 & 0 \\ 0 & e^{(r+\delta)t} & 0 & 0 \\ 0 & 0 & e^{\frac{(r-\sqrt{Q})}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{(r+\sqrt{Q})}{2}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

Substituting $\underline{P}$ and $\underline{v}$ into $\underline{Y} = P\underline{v}$ gives:

$$
\underline{P_V}=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & \dfrac{-\gamma(r+\delta)}{(r+2\delta)} & \dfrac{-\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})} & \dfrac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})} \\ 0 & \dfrac{\theta(r+\delta)}{\delta} & \beta\left(\dfrac{\theta}{\beta}+\dfrac{r-\sqrt{Q}}{2}\right) & \beta\left(\dfrac{\theta}{\beta}+\dfrac{r+\sqrt{Q}}{2}\right) \\ 0 & \dfrac{\delta\beta(r+\delta)-r\theta}{\delta\gamma} & 0 & 0 \end{bmatrix}\begin{bmatrix} e^{-\delta t} & 0 & 0 & 0 \\ 0 & e^{(r+\delta)t} & 0 & 0 \\ 0 & 0 & e^{\frac{(r-\sqrt{Q})}{2}t} & 0 \\ 0 & 0 & 0 & e^{\frac{(r+\sqrt{Q})}{2}t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}
$$

$$
\begin{bmatrix} Y_S \\ Y_m \\ Y_\mu \\ Y_\lambda \end{bmatrix}=\begin{bmatrix} 0 & e^{(r+\delta)t} & e^{\frac{(r-\sqrt{Q})}{2}t} & e^{\frac{(r+\sqrt{Q})}{2}t} \\ e^{-\delta t} & \dfrac{-\gamma(r+\delta)}{(r+2\delta)}e^{(r+\delta)t} & \dfrac{-\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})}e^{\frac{(r-\sqrt{Q})}{2}t} & \dfrac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})}e^{\frac{(r+\sqrt{Q})}{2}t} \\ 0 & \dfrac{\theta(r+\delta)}{\delta}e^{(r+\delta)t} & \beta\left(\dfrac{\theta}{\beta}+\dfrac{r-\sqrt{Q}}{2}\right)e^{\frac{(r-\sqrt{Q})}{2}t} & \beta\left(\dfrac{\theta}{\beta}+\dfrac{r+\sqrt{Q}}{2}\right)e^{\frac{(r+\sqrt{Q})}{2}t} \\ 0 & \dfrac{\delta\beta(r+\delta)-r\theta}{\delta\gamma}e^{(r+\delta)t} & 0 & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}
$$

Given that $\underline{X}=\underline{Y}-A^{-1}\underline{B}$, substituting the particular solution and $\underline{Y}$ into this equation gives:

$$
\begin{bmatrix} S(t) \\ M(t) \\ \mu(t) \\ \lambda(t) \end{bmatrix} =
\begin{bmatrix}
0 & e^{(r+\delta)t} & e^{\frac{(r-\sqrt{Q})}{2}t} & e^{\frac{(r+\sqrt{Q})}{2}t} \\
e^{-\delta t} & \frac{-\gamma(r+\delta)}{(r+2\delta)} e^{(r+\delta)t} & \frac{-\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})} e^{\frac{(r-\sqrt{Q})}{2}t} & \frac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})} e^{\frac{(r+\sqrt{Q})}{2}t} \\
0 & \frac{\theta(r+\delta)}{\delta} e^{(r+\delta)t} & \beta\left(\frac{\theta}{\beta}+\frac{r-\sqrt{Q}}{2}\right) e^{\frac{(r-\sqrt{Q})}{2}t} & \beta\left(\frac{\theta}{\beta}+\frac{r+\sqrt{Q}}{2}\right) e^{\frac{(r+\sqrt{Q})}{2}t} \\
0 & \frac{\delta\beta(r+\delta)-r\theta}{\delta\gamma} e^{(r+\delta)t} & 0 & 0
\end{bmatrix}
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)} \\ 0 \\ 0 \\ \frac{\varepsilon}{(r+\delta)} \end{bmatrix}
$$

Simplifying the system gives:

$$\begin{bmatrix} S(t) \\ M(t) \\ \mu(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} 0 & e^{(r+\delta)t} & e^{\frac{(r-\sqrt{Q})}{2}t} & e^{\frac{(r+\sqrt{Q})}{2}t} \\ e^{-\delta t} & Ae^{(r+\delta)t} & Be^{\frac{(r-\sqrt{Q})}{2}t} & De^{\frac{(r+\sqrt{Q})}{2}t} \\ 0 & Ee^{(r+\delta)t} & Fe^{\frac{(r-\sqrt{Q})}{2}t} & Ge^{\frac{(r+\sqrt{Q})}{2}t} \\ 0 & He^{(r+\delta)t} & 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)} \\ 0 \\ 0 \\ \frac{\varepsilon}{(r+\delta)} \end{bmatrix}$$

$$A = \frac{-\gamma(r+\delta)}{(r+2\delta)}, \quad B = \frac{-\gamma(r-\sqrt{Q})}{2\delta+(r-\sqrt{Q})}, \quad D = \frac{-\gamma(r+\sqrt{Q})}{2\delta+(r+\sqrt{Q})}$$

$$E = \frac{\theta(r+\delta)}{\delta}, \quad F = \beta\left[\frac{\theta}{\beta}+\frac{(r-\sqrt{Q})}{2}\right], \quad G = \beta\left[\frac{\theta}{\beta}+\frac{(r+\sqrt{Q})}{2}\right]$$

$$H = \frac{\delta\beta(r+\delta)-r\theta}{\delta\gamma}$$

Therefore multiplying out gives:

$$S(t) = C_2 e^{(r+\delta)t} + C_3 e^{\frac{(r-\sqrt{Q})}{2}t} + C_4 e^{\frac{(r+\sqrt{Q})}{2}t} + \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta((r+\delta)}$$

$$M(t) = C_1 e^{-\delta t} + C_2 Ae^{(r+\delta)t} + C_3 Be^{\frac{(r-\sqrt{Q})}{2}t} + C_4 De^{\frac{(r+\sqrt{Q})}{2}t}$$

$$\mu(t) = C_2 E e^{(r+\delta)t} + C_3 F e^{\frac{(r-\sqrt{Q})}{2}t} + C_4 G e^{\frac{(r+\sqrt{Q})}{2}t}$$

$$\lambda(t) = C_2 H e^{\frac{(r-\sqrt{Q})}{2}t} + \frac{\varepsilon}{(r+\delta)}$$

Using the initial conditions,

$$M(0) = M_0$$
$$S(0) = S_0$$

gives:

$$S_0 = C_2 + C_3 + C_4 + \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}$$

$$M_0 = C_1 + AC_2 + BC_3 + DC_4$$

Next, using the transversality condition,

$$\lim_{t \to \infty} e^{-rt}\mu(t)S(t) = 0$$

we get:

$$e^{-rt}\mu(t)S(t) = \left[ C_2 E e^{\delta t} + C_3 e^{-\left(\frac{r+\sqrt{Q}}{2}\right)t} + C_4 G e^{-\left(\frac{r-\sqrt{Q}}{2}\right)t} \right]$$

$$\left[ C_2 e^{(r+\delta)t} + C_3 e^{\left(\frac{r-\sqrt{Q}}{2}\right)t} + C_4 e^{\left(\frac{r+\sqrt{Q}}{2}\right)t} + \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)} \right]$$

Multiplying out gives:

$$e^{-rt}\mu(t)S(t) = C_2^2 Ee^{(r+2\delta)t} + C_2C_3e^{\left(\frac{r+\sqrt{Q}+2\delta}{2}\right)t} + C_2C_4Ge^{\left(\frac{r-\sqrt{Q}+2\delta}{2}\right)t}$$

$$+C_2C_3Ee^{\left(\frac{r-\sqrt{Q}+2\delta}{2}\right)t} + C_3^2e^{\left(-\sqrt{Q}\right)t} + C_3C_4G + C_2C_4Ee^{\left(\frac{r+\sqrt{Q}+2\delta}{2}\right)t}$$

$$+C_3C_4 + C_4^2Ge^{\left(\sqrt{Q}\right)t} + C_2E\left[\frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}\right]e^{\delta t} + \left[\frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}\right]C_3e^{-\left(\frac{r+\sqrt{Q}}{2}\right)t}$$

$$+\left[\frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}\right]C_4Ge^{-\left(\frac{r-\sqrt{Q}}{2}\right)t}$$

To satisfy the transversality condition $\lim\limits_{t\to\infty} e^{-rt}\mu(t)S(t) = 0$

$C_2 = C_4 = 0$,

Therefore

$$C_3^2e^{\left(-\sqrt{Q}\right)t} + \left[\frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}\right]C_3e^{-\left(\frac{r+\sqrt{Q}}{2}\right)t} \to 0$$

therefore as the exponents are negative, $C_3$ has a value yet to be determined.

Next using the second transversality condition,

$$\lim_{t\to\infty} e^{-rt}\lambda(t)M(t) = 0$$

we get:

$$e^{-rt}\lambda(t)M(t) = \left[C_2He^{\delta t} + \frac{\varepsilon}{(r+\delta)}e^{-rt}\right]$$

$$\left[C_1e^{-\delta t} + C_2Ae^{(r+\delta)t} + C_3Be^{\left(\frac{r-\sqrt{Q}}{2}\right)t} + C_4De^{\left(\frac{r+\sqrt{Q}}{2}\right)t}\right]$$

Multiplying out gives:

$$C_1C_2H + \frac{C_1\varepsilon}{(r+\delta)}e^{-(r+\delta)t} + C_2^2AHe^{(r+2\delta)} + \frac{C_2A\varepsilon}{(r+\delta)}e^{\delta t} + C_2C_3BHe^{\left(\frac{r+2\delta-\sqrt{Q}}{2}\right)t}$$

$$+\frac{C_3B\varepsilon}{(r+\delta)}e^{-\left(\frac{r+\sqrt{Q}}{2}\right)t} + C_2C_4DHe^{\left(\frac{r+2\delta+\sqrt{Q}}{2}\right)t} + \frac{C_4D\varepsilon}{(r+\delta)}e^{-\left(\frac{r-\sqrt{Q}}{2}\right)t}$$

To satisfy the transversality condition, again:

$$C_2 = C_4 = 0$$

Therefore:

$$\frac{C_3B\varepsilon}{(r+\delta)}e^{-\left(\frac{r+\sqrt{Q}}{2}\right)t} + \frac{C_1\varepsilon}{(r+\delta)}e^{-(r+\delta)t} \to 0$$

therefore $C_1$ and $C_3$ have a value yet to be determined. Using this and substituting into

$S(0) = S_0$ and $M(0) = M_0$ equations we get:

$$S_0 = C_3 + \frac{(c-a)(r+\delta)+\gamma\varepsilon}{\theta(r+\delta)}$$

and

$$M_0 = C_1 + BC_3 = C_1 - \frac{\gamma(r-\sqrt{Q})}{2\delta + (r-\sqrt{Q})} C_3$$

Therefore from rearranging the $S_0$ equation:

$$C_3 = \left[ \frac{S_0\theta(r+\delta) - (c-a)(r+\delta) + \gamma\varepsilon}{\theta(r+\delta)} \right]$$

and substituting this into $M_0$ and rearranging gives:

$$C_1 = M_0 + \frac{\gamma(r-\sqrt{Q})}{2\delta + (r-\sqrt{Q})} \left[ \frac{S_0\theta(r+\delta) - (c-a)(r+\delta) + \gamma\varepsilon}{\theta(r+\delta)} \right]$$

Multiplying out gives:

$$C_1 = \left[ M_0 + \frac{\gamma\theta S_0(r+\delta)(r-\sqrt{Q}) - \gamma(r-\sqrt{Q})(c-a)(r+\delta)}{\theta(r+\delta)(2\delta + (r-\sqrt{Q}))} + \frac{\gamma^2\varepsilon(r-\sqrt{Q})}{\theta(r+\delta)(2\delta + (r-\sqrt{Q}))} \right]$$

Therefore:

$$S(t) = C_3 e^{\left(\frac{r-\sqrt{Q}}{2}\right)t} + \frac{(c-a)(r+\delta) + \gamma\varepsilon}{\theta(r+\delta)}$$

$$M(t) = C_1 e^{-\delta t} + C_3 B e^{\left(\frac{r-\sqrt{Q}}{2}\right)t}$$

$$\mu(t) = C_3 F e^{\left(\frac{r-\sqrt{Q}}{2}\right)t}$$

$$\lambda(t) = \frac{\varepsilon}{(r+\delta)}$$

where $C_1$, $C_3$, $B$ and $F$ are as stated above.

# CHAPTER EIGHT

# AN ECONOMIC MODEL OF OPENCAST COAL MINING

## Open Cast Coal Mining

The opencasting of coal involves deeply ploughing an area of land to extract the coal. Coal that is mined by opencasting has less extraneous dirt, less free moisture, better sizing and so better handling qualities than deep mined coal. Opencast coal has a far lower chlorine content than deep mined coal, and deep mines have relied on the supply of opencast coal so that it can be mixed with the deep mined coal to give an overall acceptable level of chlorine, particularly when it is sold to the customer for power generation as this will reduce the amount of chlorine emitted into the atmosphere, (Department of the Environment (1988)). Opencast coal is cheaper to extract than deep mined coal and so it is in the national interest to maximise its production. However opencast coal mining inevitably causes adverse environmental effects in the area concerned; therefore it is necessary to strike a balance between the benefits of development and protecting the environment.

It has been recognised by the Government that the environmental effects of any proposal for opencast coal mining need to be considered along with the scope for mitigating those effects. There are guidelines and strict regulations for any opencast coal operations and any proposal is subject to examination of all the possible effects that may occur: for a brief overview see Department of the Environment (1988). For example, the mineral planning authorities will need to make a judgement on each application for mining, taking into the account the case for development and the environmental effects that may occur. Where there are objections to the development being undertaken, the applicant must show how these detrimental effects can be overcome. Where the application for mining concerns an area of environmental richness, such as a National Park or National Nature Reserve, then the mineral planning authorities must take the advice of the Department of the Environment which instructs any proposal for these workings to be placed under the most rigorous examinations: see Department of the Environment Circular 4/76 (Welsh Office 7/76). The Policy concerning Green Belts is set out in Planning Policy Guideline Notes PPG 2, where the site must be well restored. The Department of the Environment Circular 16/87 (Welsh Office 25/87) emphasises that where an application concerns agricultural land, land quality after restoration as well as environmental effects must be considered. Proposals for the aftercare and restoration of a site must be submitted with any proposal for mining and should be in sufficient detail so that a realistic view can be taken of the aftercare and restoration intentions. These proposals must be agreed, before planning permission is granted to the applicant, with the mineral planning authorities, district councils, land owners, and the local community. The mineral planning authorities will therefore make the decision on whether the proposed site should be mined by comparing the benefits, whether these are economic or not, with the environmental costs.

The after-use of the land must be decided before planning permission is granted. This will affect the course of restoration. Briefly (to be more fully explained later), if the afteruse is for agriculture, then the land must be restored in stages throughout mining. If the afteruse is development, then the restoration can take place at the end of the extraction phase.

The council for the Protection of Rural England has argued that opencast coal mining is one of the most destructive activities being carried out in the UK: see Trade and Industry Select Committee (1993). The main forms of environmental impact include visual impacts, noise, blasting vibrations, water pollution, coal transportation effects, impacts on agriculture, and air pollution. Not only is there the visual impact that occurs from the intrusion of the excavation on the landscape, including the fixed plant and machinery; but topsoil and subsoil mounds are formed close to the site boundary, (these are known as spoil banks). Noise emanates from the plant and machinery on the site, arising from soil stripping, the workings within the site, blasting, and the transportation of the coal and the restoration phase. The Secretary of State for Energy imposes noise levels for sites for day time and night time working. However, the noise produced during soil stripping and restoration are excluded from these conditions. The spoil mounds and the depth of the site will help to reduce the noise level, although their effectiveness may be reduced if housing is higher than the working site or if machinery is higher than the spoil piles. Blasting has three impacts associated with it; ground vibration, air blast waves which can cause vibrating windows, and rock particles that are projected into the atmosphere. The degree to which these cause disturbance to the public depends on the type and quantity of explosive, how far away the nearest houses are, the geology and topography of the site and the weather conditions (foggy, hazy or smoky conditions give rise to increased noise levels).

Water pollution occurs because an opencast site needs to be kept dry and the water that accumulates in the site will need to be pumped away. This outflow of water will contain suspended solids and acidic drainage from the metals and sulphide in the waste rock. These can be quite harmful to water habitats if they find their way into natural waterways.

Apart from the congestion that is caused from the extra traffic on the roads, due to the transportation of coal to the coal washeries or disposal points, there is the problem of dust which is created by the movement of the transportation vehicles. The problem will vary according to the weather conditions. The dust can partly be controlled by watering the site roads and by planting trees and hedges around the site. Transport vehicles that are using public roads will undergo wheel washing before leaving the site.

Another environmental problem that is caused by opencast mining is the affect it has on the landscape and the natural habitats that existed there. Woodlands, trees and hedgerows are removed, wetlands drained and heath, downland and moorland are ploughed up. Although restoring the landscape after mining will involve replanting trees and hedges, it will take a long time for them to re-establish themselves, and it unlikely that a mature woodland will be established. Sometimes the woodland and trees that once existed have been replaced by fences and in some cases where there has been replanting, the trees and hedgerow have failed to mature because of soil problems, see CoEnCo (1980). There is also the problem of the loss of species habitat that is very likely to occur from opencast mining.

One of the main effects though is that on agriculture and the fertility of the soil after restoration has occurred. The soil after it has been mined tends to be poorly structured and there are areas where the soil is very compact which will have the effects of impeding drainage, restrict the growth of roots and make the land more difficult to work. Also, the land will become stonier and this might hinder farm machinery. The top soil will be of a shallower depth. The storage of top soil and sub soil causes deterioration of their biological value and this affects the fertility of the soil. Farmers have argued that their crop yields have been halved and many farmers have had to change from growing crops to raising cattle. They have also argued that opencasted land needs a greater amount of fertiliser inputs, especially that of phosphate and nitrogen, than before the land was disturbed. CoEnCo state that they are concerned that the loss in soil fertility is very long term and they are not convinced that it could be restored back to its original level of productivity. Therefore it is recommended by CoEnCo (1980), that the soil should be replaced in phases so that the deterioration of the soil fertility is minimised.

There is a great debate between environmentalists and the Coal Authority on the environmental consequences of opencast coal mining. Environmentalists argue that there is a long term degradation of the natural environment and a reduction in the agricultural productivity of land that is restored after opencast mining is completed. The Coal Authority argues that the land that is restored after mining is invariably improved and that land used for opencast coal mining is mostly derelict land anyway.

As far back as 1981 the Flowers Commission calculated that of all the land used for opencast mining, only about "14.4% could be attributed to some form of derelict land clearance" Flowers Commission on Energy and the Environment (1981). But as derelict sites have been worked there are increasing applications made to move into land of higher agricultural and scenic value, Beynon et al (1990).

The Government does not make any policy recommendations as to the overall level of opencast coal production;

> "It will be for current or prospective developers in the licensed sector, in the light of their own business plans, to decide the level of output for which they wish to aim." Department of Trade and Industry (1993);

This White Paper states that the market should be allowed to decide how much to produce and there will be fewer restrictions after British coal has been privatised. This liberalisation by the Department of Trade and Industry contradicts the tightening up of rules in the guidance issued by the Department of the Environment. These new guidelines state that while:

> "coal which can be produced economically is an important indigenous energy resource," ....it must be produced in an "environmentally acceptable way and consistent with wider environmental objectives including sustainable development", Department of the Environment (1993).

The underlying concept of sustainable development is to provide for an acceptable standard of living for all, and ensure that all aspects of this development are fulfilled in the long run by the availability of natural resources, ecosystems and life support systems. We have a responsibility to future generations not to jeopardise their needs. Therefore when opencast mining is proposed in a certain area, the future effects must be taken seriously into account. For example, if the afteruse of the area is for agriculture, the restoration must be undertaken in phases during the excavation period so that in the future the deterioration of the soil quality is minimised and future generations' agricultural needs are not jeopardised.

## Open Cast Coal Models

Ever since the work of Hotelling (1931), the problem of the optimal depletion of a non-renewable resource has received a lot of attention in the literature, and has been the focus of much research by economists.

The symposium issue of the Review of Economic Studies (1974) was a major contribution to theoretical modelling in this field. A few of these articles show that if the availability of future technologies and future substitutes mean that the resource is no longer an essential for future production, then it is optimal to completely exhaust the resource: see Dasgupta and Heal (1974), Solow (1974), Stiglitz (1974); also see Kamien and Schwartz (1978).

Later work has looked at different aspects of the problem of non-renewable resource depletion. For instance, Sethi and Sorger (1990) analysed a model where a country possessed a non-renewable resource and it could either consume it or export it. There is a backstop price at which a substitute for the resource can be imported. The problem is to determine the optimal consumption, export and import of the resource so as to maximise consumer's utility and maximise producer profits over a finite horizon. They show that there is an initial phase where none of the substitute is consumed, although the price of the substitute is the same as the price of the resource. The substitute is only consumed in the second phase when the resource is completely exhausted.

In some of the past literature, some of the consequences of pollution have been considered along with the optimal depletion of non-renewable resources, (see chapter 5). For example, Forster (1980) presents a model where a fossil fuel is the single energy resource which when burned, produces a non-accumulating flow of pollution, and this creates disutility. In the same paper he presents a model where a non-renewable resource, which is required in the production of goods and services, is extracted from the environment. Here the flow of pollution builds up into a stock in the environment. He also takes into account the negative effects that this pollution stock has on social welfare and allows for abatement activities, which also require the use of energy and which further reduce the stock of the resource.

Barbier (1989) formulates a model where the problem is to maximise an objective function that has the flow of consumption and the level of environmental quality its arguments, subject to an environmental degradation variable which is increased as a result of waste emissions and renewable and non-renewable resource extraction. He finds that it is the initial level of environmental quality and the level of discount rate that are significant factors in determining whether it is optimal for the economy to follow an unsustainable or sustainable growth path. For more detail see chapter 5.

These examples show that environmental considerations have been taken into account in the literature, (see chapter 5). However it is generally assumed in the literature that the costs that a mining firm will incur from extracting the resource end when extraction itself is completed. We have argued that this assumption is not always realistic. There are often post extraction payments that will be incurred, for instance clean up costs, payments to compensate for mutilation of the environment and there may be costs of storing or abating harmful and toxic materials. Permission for open cast coal mining requires the mining firm to clean-up the environment and restore the land that has been excavated, to its original state. However, as discussed above, the timing of this restoration is affected by the after use of the land. Therefore the post extraction payments should be included in an analysis of opencast coal mining whether they occur as a flow during the extraction period or at the end as an equivalent lump sum payment.

Kemp and Long, (1980) formulated a model of the mining firm that has access to a single deposit of a nonrenewable resource. They take into account post extraction payments and they distinguish between those costs which are incurred at the end of the mining period, when the mine has closed, from those which occur when the mine is temporarily closed. Their model is presented below:

They assume that:

$$\pi(q(t),x(t)) = \begin{cases} \pi^1(q(t),x(t)), & if \quad q(t')>0, \quad for\ some\ t'>t, \\ \phi(x(t)), & if \quad q(t')=0, \quad for\ all\ t'\geq t, \end{cases} \qquad (*)$$

Here, $\pi(0, x(t))$ is the rate at which costs are incurred when production is temporarily stopped, and $\phi(x(t))$ is the rate at which costs accrue after the mine is closed down completely. The rate of flow of net revenue depends on the rate of extraction and on the accumulated extraction. They assume that it costs as much to close down temporarily as it does to shut down the mine permanently, therefore (*) collapses to:

$$\pi(q(t),x(t)) \equiv \pi'(q(t),x(t))$$

The task of the firm is to maximise its flow of net revenue, i.e.

$$\max \int_0^\infty e^{-rt}\pi(q(t),x(t)).dt$$

subject to:

$$\dot{x}(t) = q(t) \geq 0$$

with the boundary conditions:

$$x(t) \leq a$$
$$x(0) = x_0$$

where $a$ is the initial stock of the resource, $\pi(t)$ is the flow of net revenue, $q(t)$ is the rate of extraction, and $x(t)$ is the accumulated extraction, all at time $t$.

Kemp and Long do not fully characterise their model and actually solve for the time paths of the variables, but use phase portrait analysis to examine typical optimal trajectories. They show that, under certain conditions, it is optimal for the firm to completely exhaust the resource in finite time.

They then present a case when the rate at which costs accrue after the mine has shut down to are less than the rate at which costs are incurred when production is temporarily shut down. The objective function of the firm is shown below:

$$\max \int_0^T e^{-rt} \pi^1(q,x)dt + e^{-rT}(\phi(x)/r),$$

They do not, however, give a full solution to the problem and state that it is optimal to incompletely exhaust the resource and to close down after a finite working lifetime, and to extract at a positive terminal rate, i.e. the extraction rate at time $T$ is positive. Again Kemp and Long do not fully characterise the solution to the problem and use phase portrait analysis to examine typical optimal trajectories.

Kemp and Long have not taken into account the fact that the post extraction payments can be incurred at different times. In their analysis, the costs are incurred when there is no production taking place, everything is at a stand still and no resource is being extracted. This does not distinguish fully from the case when costs accrue to the firm at the end of the mining period, again when there is no extraction taking place. Costs that are associated with extraction that occur during the mining period, should be considered as a flow. In the following analysis these costs occur in phases during the mining plan and it is shown that the extraction rate is positive for the whole of the extraction period and there is no temporary closure of the site. This is quite realistic as there is no reason why mining should come to a standstill while another part of the area is being restored.

The purpose of this chapter is to consider the optimal exploitation of a non-renewable natural resource, such as a fossil fuel, under the ownership of a monopolist who faces conditions of increasing marginal costs of extraction and new regularity constraints to protect the environment. This is an extension to an earlier paper by Chappell and Dury (1994) where there are no regulatory constraints on the monopolist to restore the land after it has been mined. This is an infinite horizon problem and will be presented in Section I. The problem will also be solved for a finite time horizon and will be presented in Section II.

In Section III, two models are presented where there are regulatory constraints imposed on the monopolist to infill the site after is has been mined. Two models are considered where different after uses of the site affect the timing of the reclamation of the land. In Section III.i, the land is to be used for development and so infilling is undertaken after the site has been mined. In Section III.ii, the reclaimed land is to be used for agriculture or forestry. In this case the infilling is undertaken in phases to ensure that the subsoil and topsoil are replaced at the earliest opportunity to minimise deterioration of the biological value of the soil during storage.

# SECTION I

## The Model - the infinite time horizon case

Suppose that a profit maximising monopolist has sole extraction rights over the resource stock of coal, $x(t)$. The rate at which the monopolist extracts the coal at time $t$ is denoted by $u(t)$. It is clear then, that $x$ satisfies the following differential equation:

$$\dot{x} = -u \tag{1}$$

With initial condition:

$$x(0) = x_0 > 0$$

Let $p(t)$ denote the selling price per unit of (extracted) coal at time $t$ and suppose that the inverse demand function is linear and given by:

$$p(t) = a - bu(t) \qquad a, b, > 0$$

Suppose that total extraction costs depend on the rate of extraction and on the remaining deposits of coal, i.e. it is assumed:

$$\text{Total costs} = uF(x)$$

Now assume that the function $F(x)$ which is the marginal cost, is decreasing function of $x$ and is, for simplicity, is linear so that:

$$F(x) = c - kx$$

where $k > 0$ and $0 \leq c - a < kx_0$. It is also assumed that $F(0) \geq a > F(x_0)$. This assumption is made to keep the ensuing dynamic problem tractable and to ensure that it has an 'interior' solution. If $F(0) < a$, there would not be an interior solution to the problem; if $F(x_0) \geq a$, i.e. marginal cost at time zero is greater than the marginal price, then the monopolist would not extract any coal.

It is assumed that the monopolist wishes to maximise his discounted profits over the entire future. Total revenue is given by:

$$\text{Total Revenue} = au - bu^2$$

Profits are then equal to total revenue minus total costs, therefore:

$$\text{Profits} = au - bu^2 - (c - kx)u$$

Thus the problem he faces is to choose an optimal extraction plan that maximises:

$$\max \int_0^{\infty} \left[au - bu^2 - (c - kx)u\right] e^{-rt} . dt$$

subject to (1) and where $r > 0$ is his discount rate.

and given the initial condition:

$$x(0) = x_0 > 0$$

This is the problem in Chappell and Dury (1994). Let $\lambda(t)$ denote the costate variable associated with the constraint (1). Thus $\lambda(t)$ is the shadow value of the resource stock at time $t$. The Hamiltonian is defined as:

$$\mathrm{H} = e^{-rt}\left\{(a-c)u - bu^2 + kux - \lambda u\right\}$$

The current value Hamiltonian is:

$$\mathrm{H}e^{rt} = H = \left\{(a-c)u - bu^2 + kux - \lambda u\right\}$$

Assuming an interior solution, the necessary conditions for a maximum are:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow a - c - 2bu + kx - \lambda = 0$$

which gives:

$$u = \frac{a - c + kx - \lambda}{2b} \tag{2}$$

and:

$$\dot{\lambda} = r\lambda - ku \tag{3}$$

and the transversality condition:

$$\lim_{t \to \infty} e^{-rt} \lambda(t) x(t) = 0$$

It is clear that the system has a unique stationary state defined as $x^*$, $\lambda^*$, $u^*$. From (3):

$$\dot{\lambda} = r\lambda - ku = 0$$

Therefore $\lambda^* = \frac{ku^*}{r}$. From (1), $u^* = 0$. Therefore $\lambda^* = 0$. From (2):

$$u^* = \frac{a - c + kx^* - \lambda^*}{2b}$$

Substituting in $u^*$ and $\lambda^*$ gives:

$$x^* = \frac{c - a}{k} > 0$$

The usual sufficiency conditions for the optimality of the steady state requires that the Hamiltonian function is concave with respect to $x$ and $u$ or $M(x)$ is concave in $x$, where $M(x)$ denotes the maximised Hamiltonian. In this optimal control problem the Hamiltonian is convex in $x$ at the steady state, ($x^*$, $\lambda^*$). Therefore the usual sufficiency conditions cannot be satisfied. However Sorger (1992) has shown that for a problem with a single state variable a sufficiency condition for a stationary state to be locally optimal is that:

i). There exists a stationary state ($x^*$, $\lambda^*$) and an open neighbourhood N of ($x^*$, $\lambda^*$) such that $H(x, \lambda, u)$ has a unique maximum with respect to admissible values of u for all $(x, \lambda) \in N$ and such that $M(x, \lambda)$ is twice continuously differentiable on N;

ii). $M_{\lambda\lambda} > 0$ at ($x^*$, $\lambda^*$);

iii). $M^2_{x\lambda} - rM_{x\lambda} - M_{xx} M_{\lambda\lambda} > 0$ at ($x^*$, $\lambda^*$);

(see Sorger (1992), p. 150, Corollary 2.1 and methodology)

Solving for $M$ by substituting (2) into the current value Hamiltonian gives:

$$M(x,\lambda) = (a-c)\left[\frac{a-(c-kx)-\lambda}{2b}\right] - b\left[\frac{a-(c-kx)-\lambda}{2b}\right]^2 +$$

$$k\left[\frac{a-(c-kx)-\lambda}{2b}\right]x - \lambda\left[\frac{a-(c-kx)-\lambda}{2b}\right]$$

Therefore:

$$M(x,\lambda) = \frac{(a-(c-kx)-\lambda)^2}{4b} \tag{4}$$

We now show that conditions (i), (ii) and (iii) on the preceding page are satisfied. Clearly $M$ is twice continuously differentiable to satisfy (i) and:

$$M_\lambda = -\frac{(a-c+kx-\lambda)}{2b} \quad ; \quad M_x = \frac{(a-c+kx-\lambda)k}{2b} \; ;$$

$$M_{\lambda\lambda} = \frac{1}{2b} \quad ; \quad M_{xx} = \frac{k^2}{2b}$$

$$M_{x\lambda} = -\frac{k}{2b}$$

Substituting these into the conditions (I) and (iii) we get:

ii). $M_{\lambda\lambda}$ at $(x^*, \lambda^*)$ is:

$$\frac{1}{2b} > 0$$

iii). $M^2_{x\lambda} - rM_{x\lambda} - M_{xx}M_{\lambda\lambda} \quad = \quad \frac{k^2}{4b^2} + \frac{rk}{2b} - \frac{k^2}{4b^2} \quad = \quad \frac{rk}{2b}$

at the steady state $\left(x^* = \dfrac{c-a}{k}\ ,\ \lambda^* = 0\right)$:

$$= \quad \frac{rk}{2b} > 0$$

Therefore all the sufficiency conditions are satisfied and the steady state:

$$x^* = \frac{c-a}{k},\ u^* = 0, \text{ and } \lambda^* = 0$$

is a locally optimal stationary state. Thus it is not optimal for the monopolist to completely exhaust the stock of coal.

Examining the behaviour of $x$ and $u$ along the transition path to the optimal stationary state.

From (2):

$$\lambda = a - 2bu - c + kx$$

Differentiating with respect to time gives:

$$\dot{\lambda} = -2b\dot{u} - ku \tag{5}$$

Equating equations (3) and (5) gives:

$$-2b\dot{u} - ku = r\lambda - ku$$

Substituting (4) into the above gives:

$$-2b\dot{u} - ku = r[a - c - 2bu + kx] - ku$$

Cancelling out and simplifying gives:

$$\dot{u} = \frac{r[c + 2bu - kx - a]}{2b} \tag{6}$$

and

$$\dot{x} = u \tag{1}$$

Therefore we need to solve the system:

$$\begin{bmatrix} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} r & -\dfrac{rk}{2b} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} -\dfrac{r(a-c)}{2b} \\ 0 \end{bmatrix}$$

The particular solution is $(x^*, u^*) = \left(\dfrac{c-a}{k}, 0\right)$

Solving for the eigenvalues of the system:

$$\begin{bmatrix} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} r & -\dfrac{rk}{2b} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} -\dfrac{r(a-c)}{2b} \\ 0 \end{bmatrix}$$

The characteristic equation is:

$$(r-z)(-z) - \frac{rk}{2b} = 0$$

Therefore the eigenvalues are:

$$z_1 = \frac{r + \sqrt{r^2 + \dfrac{2rk}{b}}}{2} \quad , \quad z_2 = \frac{r - \sqrt{r^2 + \dfrac{2rk}{b}}}{2}$$

Letting:

$$Q = \sqrt{r^2 + \frac{2rk}{b}}$$

The eigenvalues may be written:

$$z = \frac{r+Q}{2} \quad , \quad z = \frac{r-Q}{2}$$

The general solution then is:

$$x(t) = Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t} + \frac{(c-a)}{k} \tag{7}$$

where $A$ and $B$ are arbitrary constants.

Differentiating $x(t)$ with respect to time gives:

$$\dot{x}(t) = \left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} - \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}$$

Substituting this into (1) to derive the time path of the control variable we get:

$$u(t) = -\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t} \tag{8}$$

Using the boundary conditions:

$$x(0)=x_0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-rt}\lambda(t)x(t) = 0$$

we can solve for $A$ and $B$ (see appendix A):

$$A = 0 \quad \text{and} \quad B = \frac{kx_0 + a - c}{k}$$

Differentiating $u(t)$ with respect to time we get:

$$\dot{u} = -\left(\frac{Q+r}{2}\right)^2 Ae^{\left(\frac{Q+r}{2}\right)t} - \left(\frac{Q-r}{2}\right)^2 Be^{-\left(\frac{Q-r}{2}\right)t}$$

$A = 0$ and $B > 0$ therefore $\dot{u} < 0$. This implies that the change in the extraction rate is always negative, and that the extraction rate $u(t)$ is monotonically decreasing over time.

Looking at the price equation:

$$p(t) = a - bu(t) \qquad\qquad a,\ b, > 0$$

It is clear that as $u(t)$ is decreasing, the price of coal will be increasing over time. Therefore a profit maximising monopolist will adopt a policy of increasing price and decreasing production over time and will not completely exhaust the resource.

# SECTION II

## The model - the finite time horizon case

In this section the same optimal control problem will be solved but in this case there is only a limited period in which the monopolist can extract the coal. This is because a developer will only be granted a limited period of time to mine the area - planning permission will be limited to a finite time period. The monopolist's objective then is to maximise his discounted profits over a finite time horizon subject to the same constraints as in the previous section. Thus the problem he faces is to choose an optimal extraction plan that maximises:

$$\max \int_0^T \left[au - bu^2 - (c - kx)u\right] e^{-rt}.dt$$

subject to:

$$\dot{x} = -u \qquad (1)$$

With the initial condition:

$$x(0) = x_0 > 0$$

The current valued Hamiltonian is:

$$H = \left\{(a-c)u - bu^2 + kux - \lambda u\right\}$$

Assuming an interior solution the necessary conditions for a maximum are:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow a - c - 2bu + kx - \lambda = 0 \tag{2}$$

$$\dot{\lambda} = r\lambda - ku \tag{3}$$

The transversality condition is now:

$$\lambda(T) = 0$$

From (2):

$$\lambda = a - c - 2bu + kx \tag{4}$$

Differentiating with respect to time gives:

$$\dot{\lambda} = -2b\dot{u} - ku \tag{5}$$

Equating equations (3) and (5) gives:

$$-2b\dot{u} - ku = r\lambda - ku$$

Substituting (4) into the above gives:

$$-2b\dot{u} - ku = r[a - c - 2bu + kx] - ku$$

Simplifying gives:

$$\dot{u} = \frac{r[c + 2bu - kx - a]}{2b} \tag{6}$$

and

$$\dot{x} = -u \tag{1}$$

Therefore we need to solve the system:

$$\begin{bmatrix} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} r & -\frac{rk}{2b} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} -\frac{r(a-c)}{2b} \\ 0 \end{bmatrix}$$

The eigenvalues are, as in the previous section,:

$$z_1 = \frac{r+Q}{2} \quad , \quad z_2 = \frac{r-Q}{2}$$

where:

$$Q = \sqrt{r^2 + \frac{2rk}{b}}$$

The general solution is:

$$x(t) = Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t} + \frac{(c-a)}{k} \tag{7}$$

where $A$ and $B$ are arbitrary constants (See Appendix B).

Differentiating $x(t)$ with respect to time gives:

$$\dot{x}(t) = \left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} - \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}$$

Substituting this into (1) we get:

$$u(t) = -\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t} \tag{8}$$

Using the boundary conditions:

$$x(0) = x_0 \qquad \text{and} \qquad \lambda(T) = 0$$

we can solve for $A$ and $B$:

$$A = \frac{[kx_0 + a - c][b(Q-r) - k]}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}$$

$$B = \frac{[kx_0 + a - c][b(Q+r) + k]e^{QT}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}$$

Substituting $A$ and $B$ into $u(t)$ and $x(t)$, equations (7) and (8) respectively, gives expressions for the time path of $x$ and $u$:

$$x(t) = \frac{[kx_0 + a - c][b(Q-r) - k]}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{\left(\frac{Q+r}{2}\right)t} +$$

$$\frac{[kx_0 + a - c][b(Q+r) + k]e^{QT}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{-\left(\frac{Q-r}{2}\right)t} + \frac{c-a}{k}$$

$$u(t) = -\frac{(Q+r)[kx_0 + a - c][b(Q-r) - k]}{2k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{\frac{(Q+r)}{2}t}$$

$$+\frac{(Q-r)[kx_0 + a - c][b(Q+r) + k]e^{QT}}{2k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{-\frac{(Q-r)}{2}t}$$

Substituting $A$ and $B$ into $u(T)$ and $x(T)$ gives expressions for the extraction rate and the level of resource left in the ground at the end of the time period:

$$x(T) = e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q-r) - k]}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] +$$

$$e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q+r) + k]e^{QT}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] + \frac{c-a}{k}$$

$$u(T) = -\left(\frac{Q+r}{2}\right)e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q-r) - k]}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] +$$

$$\left(\frac{Q-r}{2}\right)e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q+r) + k]e^{QT}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right]$$

It is important to look at the level of extraction at the end of the time period. We need to establish that the rate of extraction at time $T$ is non-negative and also that this holds at any point in time, i.e. $u(t) \geq 0$. The rate of extraction can never be negative as this would imply putting the resource back in the ground - this would not make economic sense. Firstly we will look at the rate of extraction at time $T$.

Multiply out:

$$u(T) = -\left(\frac{Q+r}{2}\right)\left[\frac{[kx_0 + a - c][b(Q-r) - k]e^{\left(\frac{Q+r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] +$$

$$\left(\frac{Q-r}{2}\right)\left[\frac{[kx_0 + a - c][b(Q+r) + k]e^{\left(\frac{Q+r}{2}\right)}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right]$$

Multiply the numerator and the denominator by $e^{-QT}$ and rearranging gives:

$$u(T) = \frac{[kx_0 + a - c]e^{-\left(\frac{Q-r}{2}\right)T}}{2k\left[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]\right]}\left\{-bQ^2 - rbQ + rbQ + br^2 + kbQ + \right.$$

$$\left. kbr + bQ^2 - rbQ + rbQ - br^2 + kbQ - kbr\right\}$$

$$= \frac{[kx_0 + a - c]e^{-\left(\frac{Q-r}{2}\right)T}}{2k\left[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]\right]}\{2kbQ\} > 0$$

Therefore $u(T)$ is positive - the rate of extraction at time $T$ is positive.

We then will look at the rate of extraction at time $t = 0$:

$$u(0) = \frac{[kx_0 + a - c]\{bQ\}}{[[b(Q+r)+k]]}$$

Therefore the rate of extraction at time $t = 0$ is positive.

We need to check that $u(t) \geq 0$:

Looking back at $A$ and $B$:

$$A = \frac{[kx_0 + a - c][b(Q-r)-k]}{k[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]]}$$

$$B = \frac{[kx_0 + a - c][b(Q+r)+k]e^{QT}}{k[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]]}$$

The denominator of $A$ and $B$ is:

$$k[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]]$$

Rewriting this equation gives:

$$bQ(e^{QT}+1)+(br+k)(e^{QT}-1)$$

$Q = \sqrt{(r^2 + \frac{2rk}{b}} > 0$, and $(e^{QT} - 1) > 0$, therefore:

$$bQ(e^{QT}+1)+(br+k)(e^{QT}-1) > 0$$

Thus $A$ and $B$ have positive denominators. It is easy to see that the numerator of $B$ is positive, $A$ is not so clear. The numerator of $A$ is:

$$\left[kx_0 + a - c\right]\left[b(Q-r) - k\right]$$

The first bracket is positive. Expanding the second gives:

$$bQ - br - k$$

We will use a proof by contradiction to show that this is negative.

Let us assume that it is non-negative:

$$bQ - br - k \geq 0$$

It follows that:

$$bQ \geq br + k$$

Squaring both sides gives:

$$b^2(r^2 + \frac{2rk}{b}) \geq b^2r^2 + 2brk + k^2$$

Multiplying out gives:

$$b^2r^2 + 2brk \geq b^2r^2 + 2brk + k^2$$

This is obviously false, and it follows that $A$ is negative.

The extraction rate at time t is:

$$u(t) = -\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}$$

Therefore, as $A$ is negative and $B$ is positive, $u(t)$ is positive. Thus the extraction rate at any point in time in the finite horizon case is positive. If $A$ and $B$ were both positive then it is clear that the extraction rate would be decreasing over time. i.e. $\dot{u} < 0$.

$$\dot{u}(t) = -\left(\frac{Q+r}{2}\right)^2 Ae^{\left(\frac{Q+r}{2}\right)t} - \left(\frac{Q-r}{2}\right)^2 Be^{-\left(\frac{Q-r}{2}\right)t}$$

However as $A$ is negative then it is unclear whether $u(t)$ is increasing or decreasing over the whole of the planning period.

We can now establish whether or not it is optimal for the monopolist to completely exhaust the resource, or whether it is optimal for him to leave some in the ground, i.e. is $x(T) \geq 0$;

$$x(T) = e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q-r)-k]}{k\left[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]\right]}\right]+$$

$$e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q+r)+k]e^{QT}}{k\left[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]\right]}\right]+\frac{c-a}{k}$$

Simplifying and rearranging gives:

$$x(T) = \left[\frac{e^{\left(\frac{r-Q}{2}\right)T}[kx_0 + a - c]}{k\left[[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]\right]}\right]\left\{[b(Q-r)-k]+[b(Q+r)+k]\right\}+\frac{(c-a)}{k}$$

$$= \left[\frac{e^{\left(\frac{r-Q}{2}\right)T}[kx_0 + a - c]}{k\left[[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]\right]}\right]\left\{bQ - br - k + bQ + br + k\right\}+\frac{(c-a)}{k}$$

$$= \left[\frac{e^{\left(\frac{r-Q}{2}\right)T}[kx_0 + a - c]}{k\left[[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]\right]}\right]\left\{2bQ\right\}+\frac{(c-a)}{k}$$

Therefore $x(T)$ is positive and as one would expect, (since in the infinite horizon case $x^* > 0$), it is not optimal for the monopolist to exhaust the resource totally in the finite time horizon. As one would expect, the stock of coal left in the ground at the time $T$ is greater than in the infinite case when the monopolist is not restricted to a mining the area for a certain length of time.

For completeness, comparing these results with the infinite horizon case. Then as $T \to \infty$:

$$\lim_{T \to \infty} x(T) = \frac{c - a}{k} > 0$$

and:

$$\lim_{T \to \infty} u(T) = 0$$

These steady state solutions agree with the infinite horizon case.

# SECTION III

## Section III.i

### After-use: Development

In this section, the previous model will be extended to take account of the regulations imposed on the monopolist to restore the environment. An open-cast coal developer will only be granted a limited time period for mining and he will always be obliged to restore the land after he has mined the area. In this model the area is to be used for development purposes, possibly housing, after the area has been mined. The planning permission granted by the local Mineral Planning Authority will depend on the proposal. If there were great economic benefits to the local area then planning permission might be granted for a longer period. If there were sufficiently large environmental effects then the time period may be less. Indeed in this case, where the monopolist has to restore the land to its original condition after mining has finished, it would be unrealistic to have an infinite horizon time period. This is because no permission would be granted for a longer period. If there were sufficiently large environmental effects then the time period may be less. Indeed in this case, where the monopolist has to restore the land to its original condition after mining has finished, it would be unrealistic to have an infinite horizon time period. This is because no permission would be granted for mining if the area was to be excavated for very long periods. Therefore it is legitimate to examine the finite time horizon case. The monopolist will now face a different problem. The regulations will affect the extraction plan as additional costs will be imposed on him. The problem then for the monopolist is to maximise his discounted profits subject to the constraints on the resource stock and the regulatory constraints imposed upon him. Formally, the problem is to:

$$\max \int_0^T \left[ au - bu^2 - (c - kx)u \right] e^{-rt}.dt - \gamma e^{-rT} \int_0^T u(t).dt$$

subject to:

$$\dot{x} = -u \tag{1}$$

With the initial condition:

$$x(0) = x_0 > 0$$

where $x(t)$ denotes the remaining reserves of coal at time $t$. The rate of extraction at time $t$ is denoted by $u(t)$, $\gamma > 0$ is a constant of proportionality. For each unit extracted, the monopolist faces restoration costs of $\gamma u$. The first part of the objective functional shows that total profits are maximised over the entire time horizon and so are discounted at each point in time, $t$. The second part shows that the costs of infilling occur at the end of the mining period and so they are discounted to give the current value, (i.e. at $t$ = 0). The total revenue and marginal cost functions are as in *Section I*.

It is assumed that $0 \leq c - a + \gamma < kx_0$. If $a - (c-kx_0) - \gamma \geq 0$, then the monopolist would not extract any coal. This is because if at time $t = 0$, the marginal price of coal, $a$, was less than the marginal cost of extraction, $(c + kx_0)$, plus the marginal cost of infilling, $\gamma$, then none would be extracted. This condition is assumed to hold throughout the subsequent analysis.

The Hamiltonian is:

$$\mathrm{H} = e^{-rt}\left\{(a-c)u - bu^2 + kux - \gamma ue^{-r(T-t)} - \lambda u\right\}$$

The current value Hamiltonian is therefore:

$$\mathrm{H}e^{rt} = H = \left\{(a-c)u - bu^2 + kux - \gamma ue^{-r(T-t)} - \lambda u\right\}$$

Assuming an interior solution, the necessary conditions are:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow a - c - 2bu + kx - \gamma e^{-r(T-t)} - \lambda = 0 \tag{2}$$

$$\dot{\lambda} = r\lambda - ku \tag{3}$$

with the transversality condition:

$$\lambda(T) = 0$$

From (2):

$$\lambda = a - c - 2bu + kx - \gamma e^{-r(T-t)} \tag{4}$$

Differentiating with respect to time gives:

$$\dot{\lambda} = -2b\dot{u} - ku - r\gamma e^{-r(T-t)} \tag{5}$$

Equating equations (3) and (5) we get:

$$-2b\dot{u} - ku - r\gamma e^{-r(T-t)} = r\lambda - ku$$

Substituting (4) into the above equation gives:

$$-2b\dot{u} - ku - r\gamma e^{-r(T-t)} = r\left[a - c - 2bu + kx - \gamma e^{-r(T-t)}\right] - ku$$

Simplifying gives:

$$\dot{u} = \frac{r\left[c + 2bu - kx - a\right]}{2b} \tag{6}$$

and

$$\dot{x} = -u \tag{1}$$

Therefore we need to solve the system:

$$\begin{bmatrix} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} r & -\dfrac{rk}{2b} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} -\dfrac{r(a-c)}{2b} \\ 0 \end{bmatrix}$$

The particular solution of is :

$$\begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{(c-a)}{k} \end{bmatrix}$$

Solving for the eigenvalues of the system:

$$\begin{bmatrix} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} r & -\dfrac{rk}{2b} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} -\dfrac{r(a-c)}{2b} \\ 0 \end{bmatrix}$$

The characteristic equation is:

$$(r - z)(-z) - \frac{rk}{2b} = 0$$

and the eigenvalues are:

$$z_1 = \frac{r + \sqrt{r^2 + \frac{2rk}{b}}}{2} \quad , \quad z_2 = \frac{r - \sqrt{r^2 + \frac{2rk}{b}}}{2}$$

Letting:

$$Q = \sqrt{r^2 + \frac{2rk}{b}}$$

The eigenvalues may be written:

$$z_1 = \frac{r+Q}{2} \quad , \quad z_2 = \frac{r-Q}{2}$$

The general solution is:

$$x(t) = Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t} + \frac{(c-a)}{k} \tag{7}$$

where $A$ and $B$ are arbitrary constants.

Differentiating $x(t)$ with respect to time gives:

$$\dot{x}(t) = \left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} - \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}$$

Therefore substituting this into (1):

$$u(t) = -\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t} \tag{8}$$

Using the boundary conditions:

$$x(0)=x_0 \qquad \text{and} \qquad \lambda(T) = 0$$

we can solve for $A$ and $B$, (see Appendix C):

$$A = \frac{[kx_0 + a - c][b(Q-r)-k] + \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}$$

$$B = \frac{[kx_0 + a - c][b(Q+r)+k]e^{QT} - \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}$$

Substituting $A$ and $B$ into $u(t)$ and $x(t)$ gives expressions for the time path of the rate of extraction and the stock of resource in the ground at time $t$.

$$x(t) = \frac{[kx_0 + a - c][b(Q-r)-k] + \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{\left(\frac{Q+r}{2}\right)t} +$$

$$\frac{[kx_0 + a - c][b(Q+r)+k]e^{QT} - \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{-\left(\frac{Q-r}{2}\right)t} + \frac{(c-a)}{k}$$

$$u(t) = -\frac{(Q+r)[kx_0 + a - c][b(Q-r)-k] + \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{2k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{\left(\frac{Q+r}{2}\right)t} +$$

$$\frac{(Q-r)[kx_0 + a - c][b(Q+r)+k]e^{QT} - \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{2k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]} e^{-\left(\frac{Q-r}{2}\right)t}$$

Substituting $A$ and $B$ into $u(T)$ and $x(T)$ gives expressions for the resource stock and rate of extraction at the end of the time period. It is important to establish whether it is optimal for the monopolist to completely exhaust the resource and also to check that the rate of extraction is always non-negative.

The level of resource at time $T$ is:

$$x(T) = e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q-r)-k] + \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] +$$

$$e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q+r)+k]e^{QT} - \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] + \frac{c-a}{k}$$

And the level of extraction at time $T$ is::

$$u(T) = -\left(\frac{Q+r}{2}\right)e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q-r)-k] + \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] +$$

$$\left(\frac{Q-r}{2}\right)e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q+r)+k]e^{QT} - \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right]$$

Looking at the rate of extraction first we simplify the equation and get:

$$u(T) = -\left(\frac{Q+r}{2k}\right)\left[\frac{[kx_0 + a - c][b(Q-r) - k]e^{-\left(\frac{Q-r}{2}\right)T} + \gamma k}{[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]]}\right] +$$

$$\left(\frac{Q-r}{2k}\right)\left[\frac{[kx_0 + a - c][b(Q+r) + k]e^{-\left(\frac{Q-r}{2}\right)} - \gamma k e^{-QT}}{[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]]}\right]$$

Then as $T \to \infty$ :

$$\lim_{T\to\infty} u(T) = \frac{-\gamma(Q+r)}{2[b(Q+r)+\delta]} < 0$$

This would imply that if the time horizon were very long the monopolist would be putting coal back into the ground. This does not make sense and it was argued at the beginning that in this situation only the finite case is realistic. However it is interesting to see the result if we do let $T \to \infty$. This agrees with Kemp and Long (1980) who state that it is optimal for the mine to be closed down after a finite working life. They do not give a full solution to the problem and this result above shows why it is optimal to shut down the mine - after a finite amount of time the rate of extraction becomes negative, Kemp and Long do not explicitly point this out.

If $T = 0$ we find that:

$$u(0) = \frac{[kx_0 + a - c]\{-(Q+r)[b(Q-r)-k] + (Q-r)[b(Q+r)+k]\} - (Q+r)\gamma k - (Q-r)\gamma k}{2k[[b(Q+r)+k] + [b(Q-r)-k]]}$$

$$= \frac{[kx_0 + a - c - \gamma]}{2b}$$

Therefore for $u(0) > 0$, the following condition must hold:

$$kx_0 + a - c - \gamma > 0$$

But this condition must hold because if the expression was negative none of the resource would ever be extracted. At time $t = 0$, if the marginal price of coal, $a$, was less than the marginal cost of extraction, $(c + kx_0)$, plus the marginal cost of infilling, $\gamma$, then none would be extracted. This condition is assumed to hold throughout the analysis and was stated at the beginning.

It follows that there are some finite time $T$ where $u(T)$ is non-negative. It implies that for each unit extracted from the ground, the cost of infilling is $\gamma u$ where $\gamma$ is a constant fraction.

Looking at the extraction rate at time $T$ and letting:

$$\xi = k\Big[\big[b(Q+r)+k\big]+e^{-QT}\big[b(Q-r)-k\big]\Big]$$

$$u(T) = \frac{1}{2\xi}\Bigg[\big[kx_0 + a - c\big]e^{-\left(\frac{Q-r}{2}\right)T}\Big(-(Q+r)\big[b(Q-r)-k\big]+(Q-r)\big[b(Q+r)+k\big]\Big)$$

$$-(Q+r)\gamma k + (Q-r)\gamma k e^{-QT}\Bigg]$$

$$= \frac{1}{2\xi}\left[\big[kx_0 + a - c\big]e^{-\left(\frac{Q-r}{2}\right)T}(2kQ) - Q\gamma k - r\gamma k + Q\gamma k e^{-QT} - r\gamma k e^{-QT}\right]$$

The sign of this expression cannot be unambiguously determined. We need to look at the boundary condition which is defined as:

$$2bu(T) - kx(T) = a - c - \gamma$$

If $x(T) > 0$ then $u(T) > 0$, since $a - c - \gamma > 0$. It is unclear, however, whether the extraction rate is decreasing or increasing over time.

Looking at the optimal amount of resource stock that is left in the ground after the site has been mined, i.e. $x(T)$ is:.

$$x(T) = e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q-r)-k] + \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]\right]}\right]+$$

$$e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c][b(Q+r)+k]e^{QT} - \gamma k e^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}[b(Q+r)+k]+[b(Q-r)-k]\right]}\right]+\frac{c-a}{k}$$

$$=\left[\frac{[kx_0 + a - c][b(Q-r)-k]e^{\left(\frac{r-Q}{2}\right)T} + \gamma k}{k\left[[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]\right]}\right]+$$

$$\left[\frac{[kx_0 + a - c][b(Q+r)+k]e^{\left(\frac{r-Q}{2}\right)T} - \gamma k e^{-QT}}{k\left[[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]\right]}\right]+\frac{(c-a)e^{-QT}}{ke^{-QT}}$$

since:

$$\xi = k\left[[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]\right]$$

Factorising gives:

$$x(T) = \left[\frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi}\right]\left\{[kx_0 + a - c][b(Q-r)-k+b(Q+r)+k]\right\}+\left\{\frac{\gamma k - \gamma k e^{-QT}}{\xi}\right\}+\frac{c-a}{k}$$

$$= \left[ \frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi} \right] \{[kx_0 + a - c][2bQ]\} + \left\{ \frac{\gamma k - \gamma k e^{-QT}}{\xi} \right\} + \frac{(c-a)}{k}$$

$x(T) > 0$ as $\gamma k - \gamma k e^{-QT} > 0$ since $Q > 0$, $\xi > 0$ and $1 > e^{-QT}$. This means that $u(T)$ is positive. As the extraction rate is positive at time $t = 0$ and at time $t = T$, it will be assumed here that the level of resource extraction is non-negative for $t \leq T$.

Thus at time $T$, the level of coal left in the ground is positive, therefore it is not optimal for the monopolist to completely exhaust the resource. We can compare the level of resource stock at time $T$ with $x(T)$ in the unregulated case in *Section II*

In the unregulated case (*Section II*), the level of resource stock at time T is:

$$x(T) = \left[ \frac{e^{\left(\frac{r-Q}{2}\right)T} [kx_0 + a - c]}{\xi} \right] \{2bQ\} + \frac{(c-a)}{k}$$

In this case, where the monopolist is obliged to restore the environment, the level of resource stock is:

$$x(T) = \left[ \frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi} \right] \{[kx_0 + a - c][2bQ]\} + \left\{ \frac{\gamma k - \gamma k e^{-QT}}{\xi} \right\} + \frac{(c-a)}{k}$$

There is an extra term in the regulated case, $\frac{\gamma k - \gamma k e^{-QT}}{\xi} > 0$. Therefore we can see that the level of coal left in the ground when there is a requirement to restore the environment after excavation, is greater than the level of resource stock when the monopolist is not faced with these extra costs.

## Section III.ii

**After use - Agriculture**

In the previous model the land was to be used for development and so infilling is undertaken after the site has been mined. In this second model, the reclaimed land is to be used for agriculture or forestry. In this case the infilling is undertaken in phases during the excavation period to ensure that the subsoil and topsoil are replaced at the earliest opportunity to minimise deterioration of the biological value of the soil during storage. The model will be solved for a finite time horizon and then the steady states of the variables will be derived so that comparison can be made with the unregulated case where the monopolist has no restrictions on the excavation time and state of the land when mining has finished. This is a valid comparison in this case as it is possible that permission would be granted for excavating the site for long periods if the site was to be restored while the mining was actually taking place.

It is assumed that a profit maximising monopolist wishes to maximise his discounted profits over a finite time horizon. Thus the problem he faces is to choose an optimal extraction plan so as to maximise those profits. However the monopolist faces the new constraint that is he is obliged to fill the site and restore the land in phases while the excavations are taking place. Again this will impose additional costs on the monopolist. The problem then is to:

$$\max \int_0^T \left[ au - bu^2 - (c - kx)u - \gamma u \right] e^{-rt} . dt$$

subject to:

$$\dot{x} = -u \tag{1}$$

With initial condition:

$$x(0) = x_0 > 0$$

where $x(t)$ denotes the remaining reserves of coal at time t. The rate of extraction at time t is denoted by $u(t)$, $\gamma > 0$ is a constant of proportionality. For each unit extracted, the monopolist faces infilling costs of $\gamma u$. The objective function shows that the costs of infilling are incurred during the extraction plan rather than at the end of the mining period. The total revenue and marginal cost functions are as in Section I.

It is assumed that $0 \leq c - a + \gamma < kx_0$. If $a - (c - kx_0) - \gamma \geq 0$, then the monopolist would not extract any coal. At time $t = 0$, if the marginal price of coal, $a$, was less than the marginal cost of extraction, $(c + kx_0)$, plus the marginal cost of infilling, $\gamma$, then none would be extracted. Therefore this condition is assumed to hold throughout the subsequent analysis.

The Hamiltonian is:

$$\mathrm{H} = e^{-rt}\left\{(a-c)u - bu^2 + kux - \gamma u - \lambda u\right\}$$

where $\lambda e^{-rt}$ is the costate variable.

The current value Hamiltonian is, therefore:

$$\mathrm{H}e^{rt} = H = \left\{(a-c)u - bu^2 + kux - \gamma u - \lambda u\right\}$$

Assuming an interior solution the necessary conditions are:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow a - c - 2bu + kx - \gamma - \lambda = 0 \tag{2}$$

$$\dot{\lambda} = r\lambda - ku \tag{3}$$

with the transversality condition:

$$\lambda(T) = 0$$

From (2):

$$\lambda = a - c - 2bu + kx - \gamma \tag{4}$$

Differentiating with respect to time gives:

$$\dot{\lambda} = -2b\dot{u} - ku \tag{5}$$

Equating equations (3) and (5) gives:

$$-2b\dot{u} - ku = r\lambda - ku$$

Substituting (4) into the above gives:

$$-2b\dot{u} - ku = r[a - c - 2bu + kx - \gamma] - ku$$

and it follows that:

$$\dot{u} = \frac{r[c + 2bu - kx - a + \gamma]}{2b} \tag{6}$$

$$\dot{x} = -u$$

Solving the system:

$$\begin{bmatrix} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} r & -\dfrac{rk}{2b} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} -\dfrac{r(a - c + \gamma)}{2b} \\ 0 \end{bmatrix}$$

The particular solution is:

$$\begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{c - a - \gamma}{k} \end{bmatrix}$$

Solving for the eigenvalues of the system, the characteristic equation is:

$$(r - z)(-z) - \frac{rk}{2b} = 0$$

Therefore the eigenvalues are:

$$z_1 = \frac{r + \sqrt{r^2 + \frac{2rk}{b}}}{2} \quad , \quad z_2 = \frac{r - \sqrt{r^2 + \frac{2rk}{b}}}{2}$$

Let

$$Q = \sqrt{r^2 + \frac{2rk}{b}}$$

and the eigenvalues may be written:

$$z_1 = \frac{r+Q}{2} \quad , \quad z_2 = \frac{r-Q}{2}$$

(Note that these are the same as in the previous model).

The general solution is:

$$x(t) = Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t} + \frac{(c-a+\gamma)}{k} \tag{7}$$

where $A$ and $B$ are arbitrary constants.

Differentiating $x(t)$ with respect to time gives:

$$\dot{x}(t) = \left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} - \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}$$

Therefore substituting this into (1) gives:

$$u(t) = -\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t} \tag{8}$$

Using the boundary conditions:

$$x(0)=x_0 \qquad \text{and} \qquad \lambda(T)=0$$

we can solve for $A$ and $B$, (see Appendix D):

$$A=\frac{\left[kx_0+a-c-\gamma\right]\left[b(Q-r)-k\right]}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}$$

$$B=\frac{\left[kx_0+a-c-\gamma\right]\left[b(Q+r)+k\right]e^{QT}}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}$$

where $B$ is positive and $A$ is negative since (see previous model):

$$\left[b(Q-r)-k\right]<0$$

Substituting $A$ and $B$ into $u(T)$ and $x(T)$ gives:

$$x(T)=e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{\left[kx_0+a-c-\gamma\right]\left[b(Q-r)-k\right]}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}\right]+$$

$$e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{\left[kx_0+a-c-\gamma\right]\left[b(Q+r)+k\right]e^{QT}}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}\right]+\frac{c-a+\gamma}{k}$$

And:

$$u(T)=-\left(\frac{Q+r}{2}\right)e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{\left[kx_0+a-c-\gamma\right]\left[b(Q-r)-k\right]}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}\right]+$$

$$\left(\frac{Q-r}{2}\right)e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{\left[kx_0+a-c-\gamma\right]\left[b(Q+r)+k\right]e^{QT}}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}\right]$$

Rearranging and simplifying, it follows that:

$$u(T) = -\left(\frac{Q+r}{2k}\right)\left[\frac{[kx_0 + a - c - \gamma][b(Q-r) - k]e^{-\left(\frac{Q-r}{2}\right)T}}{[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]]}\right] +$$

$$\left(\frac{Q-r}{2k}\right)\left[\frac{[kx_0 + a - c - \gamma][b(Q+r) + k]e^{-\left(\frac{Q-r}{2}\right)}}{[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]]}\right]$$

Suppose $T = 0$, then:

$$u(0) = \frac{[kx_0 + a - c - \gamma][-(Q+r)[b(Q-r)-k] + (Q-r)[b(Q+r)+k]]}{2k[[b(Q+r)+k] + [b(Q-r)-k]]}$$

$$= \frac{[kx_0 + a - c - \gamma][-bQ^2 - rbQ + rbQ + r^2b + kQ + rk + bQ^2 - rbQ + rbQ - r^2b + kQ - rk]}{4bkQ}$$

$$= \frac{[kx_0 + a - c - \gamma][2kQ]}{4bkQ} = \frac{[kx_0 + a - c - \gamma]}{2b}$$

Therefore for $u(0) > 0$, the following condition must hold:

$$kx_0 + a > c + \gamma$$

Or:

$$0 < \gamma < kx_0 + a - c$$

This is the assumption stated at the beginning of the section and is assumed to hold throughout the analysis. If this was not the case then none of the resource would ever be extracted.

Looking at $u(t)$:

$$u(t) = -\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}$$

Since $A$ is negative and $B$ is positive, then $u(t)$ is positive for all $t$. It is unclear, however, whether $u(t)$ is increasing or decreasing over the finite time period.

Looking at the optimal amount of resource stock that is left in the ground after the site has been mined.

$$x(T) = e^{\left(\frac{Q+r}{2}\right)T}\left[\frac{[kx_0 + a - c - \gamma][b(Q-r) - k]}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] +$$

$$e^{-\left(\frac{Q-r}{2}\right)T}\left[\frac{[kx_0 + a - c - \gamma][b(Q+r) + k]e^{QT}}{k\left[e^{QT}[b(Q+r)+k] + [b(Q-r)-k]\right]}\right] + \frac{c-a+\gamma}{k}$$

Rearranging and simplifying, we may write:

$$x(t) = \left[\frac{[kx_0 + a - c - \gamma][b(Q-r) - k]e^{\left(\frac{r-Q}{2}\right)T}}{k\left[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]\right]}\right] +$$

$$\left[\frac{[kx_0 + a - c - \gamma][b(Q+r) + k]e^{\left(\frac{r-Q}{2}\right)T}}{k\left[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]\right]}\right] + \frac{(c-a+\gamma)e^{-QT}}{ke^{-QT}}$$

Let:

$$\xi = k\left[[b(Q+r)+k] + e^{-QT}[b(Q-r)-k]\right]$$

Factorising gives:

$$x(T)=\left[\frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi}\right]\left[\left[kx_0+a-c-\gamma\right]\left[b(Q-r)-k+b(Q+r)+k\right]\right]+\frac{c-a+\gamma}{k}$$

$$=\left[\frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi}\right]\left[\left[kx_0+a-c-\gamma\right]\left[2bQ\right]\right]+\frac{c-a+\gamma}{k}$$

If the monopolist was not operating under any constraints concerning the length of time, and he could mine for as long as he wanted, we can compare this case with the infinite horizon problem in Section I. This is a valid assumption to make as it is feasible that permission may be granted to a monopolist to extract coal over very long periods if he was obliged to restore the land as he went along. It is straight forward to show that:

$$\lim_{T\to\infty} u(T)=0$$

Therefore $u(t)$ tends to zero just as in the infinite horizon case when there are no regulations on the extracting monopolist.

$$\lim_{T\to\infty} x(T)=\frac{c-a+\gamma}{k}>0$$

The optimal level of resource stock left in the ground by the monopolist when no regulatory constraints to clean up the environment, is:

$$x^*=\frac{c-a}{k}$$

It is easy to see that:

$$\frac{c-a+\gamma}{k}>\frac{c-a}{k}$$

Again, it will not be optimal for the monopolist to completely exhaust the resource in the infinite horizon case and in fact there will be more left in the ground than when no restoration constraints are imposed on the monopolist.

Next we will compare the resource stock at finite time $T$ in this section, i.e. when the monopolist is obliged to infill the site as he goes along, with the resource stock when there are no constraints imposed on the monopolist (Section II), and when the monopolist is required to infill at the end of the mining period (Section III.i).

In Section II, the unregulated case, $x(T)$ is:

$$x(T) = \left[ \frac{e^{\left(\frac{r-Q}{2}\right)T} \left[kx_0 + a - c\right]}{\xi} \right] \{2bQ\} + \frac{(c-a)}{k}$$

We will use a proof by contradiction to show that $x(T)$ in this section i.e. $x(T)$ when there are constraints, is greater than in the unregulated case in Section II :

Suppose that $x(T)$ in the unregulated case is greater than in the case where there are constraints imposed on the monopolist:

$$\left[ \frac{e^{\left(\frac{r-Q}{2}\right)T} \left[kx_0 + a - c\right]}{\xi} \right] \{2bQ\} + \frac{(c-a)}{k} >$$

$$\left[ \frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi} \right] \left[\left[kx_0 + a - c - \gamma\right]\left[2bQ\right]\right] + \frac{c-a+\gamma}{k}$$

Simplifying gives:

$$0 > -\frac{2b\gamma Qe^{\left(\frac{r-Q}{2}\right)T}}{\xi} + \frac{\gamma}{k}$$

Substituting for $\xi$ and simplifying we get:

$$0 > -2bQe^{\left(\frac{r-Q}{2}\right)T} + [b(Q+r)+k] + e^{-QT}[b(Q-r)-k]$$

or:

$$2bQe^{\left(\frac{r-Q}{2}\right)T} > [b(Q+r)+k] + e^{-QT}[b(Q-r)-k]$$

If $T$ is large enough the left hand side of the inequality and the second term on the right hand side will be very small. It follows that it is likely that the inequality will not hold; therefore if $T$ is large enough, the resource stock in the unregulated case will be less than in the regulated case.

In Section III.i, the case where infilling is required at the end of the mining period, $x(T)$ is:

$$x(T) = \left[\frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi}\right]\{[kx_0 + a - c][2bQ]\} + \left\{\frac{\gamma k - \gamma k e^{-QT}}{\xi}\right\} + \frac{(c-a)}{k}$$

Assuming that this level of resource stock is greater than $x(T)$ in this section we get:

$$\left[\frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi}\right]\{[kx_0+a-c][2bQ]\}+\left\{\frac{\gamma k-\gamma ke^{-QT}}{\xi}\right\}+\frac{(c-a)}{k}>$$

$$\left[\frac{e^{\left(\frac{r-Q}{2}\right)T}}{\xi}\right][[kx_0+a-c-\gamma][2bQ]]+\frac{c-a+\gamma}{k}$$

Simplifying gives:

$$k[1-e^{-QT}]>-2bQe^{\left(\frac{r-Q}{2}\right)T}+[b(Q+r)+k]+e^{-QT}[b(Q-r)-k]$$

Cancelling and rearranging gives:

$$2Qe^{\left(\frac{r-Q}{2}\right)T}-(Q+r)-e^{-QT}(Q-r)>0$$

When $T=0$ then the left hand side of the inequality is zero, implying that at the beginning of the plan the level of resource in the ground is the same in each case. This doesn't tell us whether one is bigger than the other at time $T$, i.e. if one leaves more in the ground than the other. Next we differentiate the left hand side of the inequality and if the derivative is positive, which would imply that as time increases the left hand side would become positive, then we can say the inequality holds.

Let:

$$\phi=2Qe^{-\left(\frac{Q-r}{2}\right)T}-(Q+r)-e^{-QT}(Q-r)$$

Differentiating $\phi$ with respect $T$ gives:

$$\frac{\partial \phi}{\partial T} = -(Q-r)Qe^{-\left(\frac{Q-r}{2}\right)T} + Qe^{-QT}(Q-r)$$

Multiplying by $\dfrac{e^{QT}}{e^{QT}}$:

$$\left[\frac{e^{QT}}{e^{QT}}\right]\frac{\partial \phi}{\partial T} = -(Q-r)Qe^{\left(\frac{Q+r}{2}\right)T} + Q(Q-r)$$

$$= Q(Q-r)\left[1-e^{\left(\frac{Q+r}{2}\right)}\right]e^{-QT} \; < \; 0$$

Therefore an increase in $T$ results in a fall in $\phi$. Thus the inequality does not hold and the level of resource stock at time $T$ is greater in the case when the monopolist has to infill the site as he goes along. This would make sense because the present value of the costs of infilling would be greater in this case as they are discounted along the time period and not right at the end. Therefore the level of extraction when the monopolist infills as he goes along would be less resulting in a greater level of $x$ at time $T$.

# SECTION IV

## CONCLUSION

In this chapter, two models have been formulated to consider the optimal depletion of a non-renewable resource - open-cast coal, under conditions of increasing marginal costs of extraction. In much of the past literature the cost of extraction come to an end when the site has been mined. It has been discussed earlier that this assumption is unrealistic in the case of open-cast coal mining. The monopolist who is excavating the site wants to maximise his profits but he is required to restore the site. The timing of the restoration is dependent on whether the site is to be used for development or agricultural land. If the site is to be developed then the restoration of the area can be undertaken after mining has finished. However, if the land is to be used for agriculture or forestry, then the infilling must be undertaken continually during the excavation period so as to minimise the deterioration of the soil quality. The post extraction costs that the monopolist will incur are therefore either a lump sum payment at the end of the period or they are a flow during excavation.

This is an extension to an earlier paper by Chappell and Dury (1994), where the problem was to determine the optimal depletion of a non-renewable resource under the ownership of a monopolist who faces increasing marginal costs of extraction, but where the monopolist is not obliged to incur costs of restoring the area once it has been mined. This paper is presented in Section I and it is shown that for the infinite time horizon case, the profit maximising monopolist will adopt a policy of increasing price and decreasing production over time. It is also shown that it is not optimal for the monopolist to completely exhaust the resource in either the infinite time horizon of finite time horizon. In the finite time horizon case, Section II, the monopolist will not extract all the resource, but it is indeterminate whether the extraction rate is decreasing or increasing over time.

In Section III.i the monopolist is obliged to restore the land at the end of the mining period. The monopolist is now faced with a new regulatory constraint which changes the extraction plan and it becomes optimal for the firm to leave more of the resource stock in the ground at time $T$ than in the unregulated case where there are no constraints imposed on the monopolist. It is also unclear whether production is increasing or decreasing over the transition path. In Section III.ii the monopolist faces restoration payments during the mining period and the result is that with an infinite time horizon the steady state resource stock is greater than the unregulated situation. The level of resource stock at the end of the finite time horizon is also greater in the regulated than the unregulated case.

However, it is unclear whether production increases or decreases along the transition path. Nevertheless, it is shown that when the post extraction payments are a flow during the mining by the firm, it is not optimal for the firm to shut down temporarily, therefore production is not zero at any point during the mining period, the extraction rate, $u(t)$, is always positive.

It has also been shown that the level of resource stock when the monopolist is faced with continuous payments to restore the land, is greater than when he has to pay a lump sum at the end of the finite time period. This is as would be expected. The present value of the restoration payments would yield a greater value if they were discounted during the time period rather than if they were discounted at the end of the time period. This is assuming that the infilling costs are the same for both options which is a fair assumption to make. This would make it more expensive to extract the resource and so he would leave more in the ground if he was obliged to infill the land continuously along the time span of the project.

This analysis extends that of Kemp and Long (1980), who formulated a model where the mining firm incurs post extraction payments. However, they do not fully characterise their model and assume that the mine closes down temporarily or that it shuts down permanently, for restoration purposes. Therefore the two possible situations, in their optimal control problem, collapses to just one where the mine is shut down. They do not distinguish fully between the two situations. In this chapter a more realistic analysis is presented and the two different scenarios are fully explored.

# APPENDICES

## APPENDIX A

First using $x(0) = x_0$ (7) becomes:

$$A + B = \frac{kx_0 + a - c}{k} \tag{9}$$

From (2):

$$\lambda = a - c - 2bu + kx$$

Therefore:

$$\lambda x = (a - c)x - 2bux + kx^2$$

Substituting in $u(t)$ and $x(t)$ gives:

$$e^{-rt}\lambda x = (a-c)\left[Ae^{\left(\frac{r+Q}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t}\right]e^{-rt} - 2be^{-rt}\left[-\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)t} + \left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)t}\right]$$

$$\left[Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t}\right] + ke^{-rt}\left[Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t}\right]\left[Ae^{\left(\frac{Q+r}{2}\right)t} + Be^{-\left(\frac{Q-r}{2}\right)t}\right]$$

which simplifies to:

$$e^{-rt}\lambda x=(a-c)Ae^{\left(\frac{Q-r}{2}\right)t}+(a-c)Be^{-\left(\frac{Q+r}{2}\right)t}+2b\left(\frac{Q+r}{2}\right)A^2e^{Qt}+2b\left(\frac{Q+r}{2}\right)AB+$$

$$2b\left(\frac{Q-r}{2}\right)AB+2b\left(\frac{Q-r}{2}\right)B^2e^{-Qt}+kA^2e^{Qt}+kAB+kAB+B^2e^{-Qt}$$

For the boundary condition to hold, i.e.:

$$\lim_{t\to\infty}\lambda(t)x(t)=0$$

$A$ must be equal to zero, $A = 0$. To find $B$, substitute $A = 0$ into equation (9) and:

$$B=\frac{kx_0+a-c}{k}>0$$

## APPENDIX B

First using $x(0) = x_0$ (7) becomes:

$$A+B=\frac{kx_0+a-c}{k} \tag{9}$$

At time $T$, u($T$) and x($T$) are:

$$x(T)=Ae^{\left(\frac{Q+r}{2}\right)T}+Be^{-\left(\frac{Q-r}{2}\right)T}+\frac{(c-a)}{k} \tag{10}$$

$$u(T)=-\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)T}+\left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)T} \tag{11}$$

Substituting $\lambda(T) = 0$ into (2) gives:

$$2bu(T)-kx(T)=a-c$$

Substituting (10) and (11) into the above condition and simplifying gives:

$$Ae^{\left(\frac{Q+r}{2}\right)T}\left[b(Q+r)+k\right]-Be^{-\left(\frac{Q-r}{2}\right)T}\left[b(Q-r)-k\right] \quad (12)$$

Expressing (9) and (12) in matrix form:

$$\begin{bmatrix} e^{\left(\frac{Q+r}{2}\right)T}\left[b(Q+r)+k\right] & -e^{-\left(\frac{Q-r}{2}\right)T}\left[b(Q-r)-k\right] \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} 0 \\ \frac{kx_0+a-c}{k} \end{bmatrix}$$

Using Cramer's Rule and simplifying the resultant expressions gives:

$$A=\frac{\left[kx_0+a-c\right]\left[b(Q-r)-k\right]}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}$$

$$B=\frac{\left[kx_0+a-c\right]\left[b(Q+r)+k\right]e^{QT}}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}$$

## APPENDIX C

First using $x(0)=x_0$ (7) becomes:

$$A+B=\frac{kx_0+a-c}{k} \quad (9)$$

At time $T$ u($T$) and x($T$) are:

$$x(T)=Ae^{\left(\frac{Q+r}{2}\right)T}+Be^{-\left(\frac{Q-r}{2}\right)T}+\frac{(c-a)}{k} \quad (10)$$

$$u(T)=-\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)T}+\left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)T} \quad (11)$$

Substituting $\lambda(T)=0$ into (2) gives:

$$2bu(T)-kx(T)=a-c-\gamma$$

Substituting (10) and (11) into the above condition and simplifying we get:

$$Ae^{\left(\frac{Q+r}{2}\right)T}\left[b(Q+r)+k\right]-Be^{-\left(\frac{Q-r}{2}\right)T}\left[b(Q-r)-k\right]=\gamma \qquad (12)$$

Expressing (9) and (12) in matrix form gives:

$$\begin{bmatrix} e^{\left(\frac{Q+r}{2}\right)T}\left[b(Q+r)+k\right] & -e^{-\left(\frac{Q-r}{2}\right)T}\left[b(Q-r)-k\right] \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} \gamma \\ \frac{kx_0+a-c}{k} \end{bmatrix}$$

Using Cramer's Rule and simplifying the resultant expressions gives:

$$A=\frac{[kx_0+a-c][b(Q-r)-k]+\gamma ke^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}$$

$$B=\frac{[kx_0+a-c][b(Q+r)+k]e^{QT}-\gamma ke^{\left(\frac{Q-r}{2}\right)T}}{k\left[e^{QT}\left[b(Q+r)+k\right]+\left[b(Q-r)-k\right]\right]}$$

## APPENDIX D

First using $x(0)=x_0$ (7) becomes:

$$A+B=\frac{kx_0+a-c-\gamma}{k} \qquad (9)$$

At time $T$, $u(T)$ and $x(T)$ are:

$$x(T)=Ae^{\left(\frac{Q+r}{2}\right)T}+Be^{-\left(\frac{Q-r}{2}\right)T}+\frac{(c-a+\gamma)}{k} \qquad (10)$$

$$u(T)=-\left(\frac{Q+r}{2}\right)Ae^{\left(\frac{Q+r}{2}\right)T}+\left(\frac{Q-r}{2}\right)Be^{-\left(\frac{Q-r}{2}\right)T} \qquad (11)$$

Substituting $\lambda(T) = 0$ into (2) gives:

$$2bu(T) - kx(T) = a - c - \gamma$$

Substituting (10) and (11) into the above condition and simplifying we get:

$$Ae^{\left(\frac{Q+r}{2}\right)T}\left[b(Q+r)+k\right] - Be^{-\left(\frac{Q-r}{2}\right)T}\left[b(Q-r)-k\right] = 0 \qquad (12)$$

Expressing (9) and (12) in matrix form gives:

$$\begin{bmatrix} e^{\left(\frac{Q+r}{2}\right)T}\left[b(Q+r)+k\right] & -e^{-\left(\frac{Q-r}{2}\right)T}\left[b(Q-r)-k\right] \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{kx_0 + a - c - \gamma}{k} \end{bmatrix}$$

Using Cramer's Rule and simplifying the resultant expressions gives:

$$A = \frac{\left[kx_0 + a - c - \gamma\right]\left[b(Q-r)-k\right]}{k\left[e^{QT}\left[b(Q+r)+k\right] + \left[b(Q-r)-k\right]\right]}$$

$$B = \frac{\left[kx_0 + a - c - \gamma\right]\left[b(Q+r)+k\right]e^{QT}}{k\left[e^{QT}\left[b(Q+r)+k\right] + \left[b(Q-r)-k\right]\right]}$$

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