ADAPTIVE CONTROL USING VARIABLE STRUCTURE SYSTEMS

BY

OMAR MOHAMMAD EL-BASHEER EL-GHEZAWI

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TO MY PARENTS

بسم الله الرحمن الرحيم

وَقُلْ رَبِّ ارْحَمْهُمَا كَمَا رَبَّيَانِي صَغِيرًا

In the name of Allah the Beneficent the Merciful

"and say: My Lord bestow thy mercy on them both as they did care for me in childhood"

The Qur'an 17:24.

SUMMARY

Adaptive control is employed in control systems required to operate satisfactorily regardless of parameter variations, external disturbances and changes in the environment. A conceptually simple approach to adaptive control is the model reference approach which yields a nonlinear feedback system. In a model reference control system the system output is made to follow the output of a specified model.

There are numerous approaches to the design of model reference adaptive control systems (MRAC). In this thesis the theory of variable structure systems (VSS) is studied and applied in the design of MRAC systems. VSS are inherently nonlinear feedback systems which exhibit certain adaptive properties including insensitivity to a range of parameter variations and certain external disturbances when operating in the sliding mode.

The application of VSS theory to the problem of adaptive model-following has demonstrated the simplicity of the design. It also ensures the asymptotic stability of the controlled system and provides direct control over the error transient.

The notion of system zeros arises naturally when tackling the problem of output model-following control systems. Certain interrelations between VSS, system zeros and the output model-following problem have suggested a new method for computing the zeros of linear multivariable square systems.

A fundamental operator in VSS is shown to be a projector. The employment of projector theory in the study of VSS provides further insight into their operation. Furthermore new methods for constructing the switching hyperplanes matrix are formulated by utilizing projector theory.

The linear control law ensuring output model-following and the necessary order reduction is shown to be identical to the equivalent control encountered in VSS. The control law also decouples the system, assigns arbitrary poles and possesses certain adaptive properties. The extension of VSS theory to output model-following systems using output information is also discussed.

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CONTENTS

Page No.

CHAPTER ONE.

INTRODUCTION.

The desire and need for improved performance of increasingly complex engineering systems with large uncertainties have given rise to a special class of feedback control systems known as "adaptive control systems". In contrast to the majority of conventional feedback control theory, which deals with the control of dynamical systems whose mathematical representations are completely known, adaptive control theory deals with partially known systems.

Partially known systems encompass those which have time-varying parameters, those which are amenable to extreme changes in the environment and those which are subject to major external disturbances. In all of these cases the adaptive control system should be able to react or adapt itself to meet the newly evolving circumstances. In order that the adaptive system performs satisfactorily, the changes in the system and in the environment should be considered as a set of conditions which have to be taken into account by the designer. Such conditions will obviously include the range of parameter variations, the range of the external disturbances, the nature of the input signals and any other factors to be catered for by the adaptive controller.

An appealing and successful approach to adaptive control is the so-called model reference adaptive control (MRAC). The MRAC approach is well documented (see Chapter 2), is conceptually simple and easily implemented. Continuous time MRAC systems will be the subject of this thesis.

MRAC was first applied to aerospace problems as early as 1958 and since then it has been applied to numerous engineering problems (see Chapter 2).

MRAC systems are inherently nonlinear time-varying feedback systems. Investigating and ensuring the stability of such systems is therefore essential. Global stability is required in order to secure an acceptable performance under hostile working conditions. This has lead to extensive use of the Liapunov and the hyperstability methods in the design of stable MRAC systems as discussed in Chapter 2. A new method for the design of MRAC systems, which besides guaranteeing stability, also offers direct control over the error transient has been proposed by Young (1978b) and is based on the theory of variable structure systems.

Variable structure systems (VSS) are a special class of non-linear systems. They are characterised by discontinuous control which changes structure on reaching a set of switching surfaces (hyperplanes). The change in structure is intentional and is dictated by a preassigned algorithm or law.

In VSS different structures are combined giving a fixed structure which enforces the subsequent motion of the system known as the "sliding motion" or "motion in the sliding mode". During sliding the system acts like a linear system and the discontinuous control effectively acts like a linear continuous equivalent control.

The characteristic advantageous feature of VSS is that during sliding the system has invariance properties yielding motion which is independent of certain system parameter variations and external disturbances. Thus variable structure systems are usefully employed in systems with uncertain and time-varying parameters.

However, variable structure systems are not isolated from linear control systems. Certain unifying aspects exist and the sliding mode is found to provide the link between the two. The motion of the "slow mode" found in high gain systems is identical to sliding motion and the invariance conditions are the same as perfect model-following conditions. Methods for computing the system zeros have been motivated by the links between the switching hyperplanes and the output zeroing problem which is associated with the system zeros. Equivalent control also arises naturally in decoupling theory and in the output model-following problem. The last three topics will be substantiated later in Chapters 5 and 7.

A recent application of VSS has been to adaptive model-following control systems (AMFC). This has been motivated by the equivalence of the perfect model-following conditions and the VSS disturbance invariance conditions. Previous designs of AMFC systems were based on stability theory and as mentioned before although the stability of the overall system is guaranteed no direct control over the error transient was possible. Using VSS and by organizing sliding on the intersection of the switching hyperplanes involving the error states, complete control over the error transient is achieved. In addition the global asymptotic stability of the system is guaranteed. The design of variable structure adaptive model-following control systems (VSMFC) is systematic and involves inequalities which simplify the calculations (see Chapter 4). Three design examples are included to demonstrate the basic concepts of variable structure adaptive model-following control systems.

One inevitably encounters system zeros when studying adaptive output model-following systems. Certain concepts related to system zeros have analogous counterparts in the theory of VSS (see Chapter 5). The similarity between the output zeroing problem and the switching hyperplanes initiated the employment of VSS in the determination of the system zeros. The zeros and the state zero directions are found as eigenvalues and eigenvectors of a matrix which arise naturally in VSS design.

A new algorithm for computing the system zeros is proposed in Chapter 5. The algorithm is conceptually simple and yields insight into the operation of VSS in the sliding mode. The algorithm offers the advantage over known techniques of the ability to calculate the state and the input zero directions independently of each other and without resorting to the determination of the null space of the (n+m)th order system matrix. The generalization of the algorithm to the case where CB is singular is more complex and is discussed in Chapter 5 and Appendix 3. A new design method for zero assignment is given utilizing projector theory.

A new treatment of VSS in the sliding mode is developed using projector theory (see Chapter 6). The basic operator $[I-B(CB)^{-1}C]$ governing VSS during the sliding mode is a projector. This observation provides a neat method of analysing VSS. Using projector theory certain VSS features are explained and others are expanded. A simple explanation of order reduction is given together with a re-examination of the invariance principle of Draženović (1969) which is extended to the case where CB is singular. The physical interpretation of invariance is given. The system zeros are demonstrated to be invariant during the sliding mode and the perfect model-following conditions are also revisited.

Projector theory is further exploited in formulating new methods for the construction of the switching hyperplanes matrix C. Systematic design methods are proposed which offer two design options in addition to the arbitrary assignment of eigenvalues; namely (i) the ability to specify CB arbitrarily and (ii) the ability to excercise partial control over the selection of the closed-loop eigenvectors.

Output model-following with full state feedback is studied in Chapter 7. It is shown that the equivalent control exists naturally and that it decouples the closed-loop system. The importance of system zeros are highlighted when matching the plant outputs with the desired outputs of a lower order model. A particular error equation is synthesized and the resulting control law is found to exhibit certain advantageous features. These include

(i) Automatic order reduction and decoupling.

(ii) Arbitrary eigenvalue assignment to the observable part of the plant.

However, since this control law cancels all of the plant zeros by pole-zero cancellation the plant will be unstable if any zero has a positive real part.

Adaptive VSS output model-following systems using only output information are also tackled and the limitations and difficulties are pointed out. The applicability of variable structure adaptive output model-following seems to be restricted and compromises may be needed. The linear synthesized control, however, is shown to exhibit adaptive properties subject to certain conditions (see Chapter 7).

The thesis is organised as follows. Chapter 2 introduces continuous time model reference adaptive systems. The theory of variable structure systems is introduced in Chapter 3. Variable structure adaptive model-following are introduced and developed in detail in Chapter 4. The determination of system zeros using VSS theory is presented in Chapter 5. The analysis and design of VSS using projector theory is detailed in Chapter 6. Adaptive output model following control systems are discussed in Chapter 7. Chapter 8 concludes this study by briefly reviewing the contents and proposing further areas which need to be studied in the future.

Three appendices are included. Appendix 1 concerns hyperstability theory, Appendix 2 explains some concepts of Linear algebra and generalized inverses of matrices, while Appendix 3 presents the proofs to the theorems presented in Chapter 5.

CHAPTER TWO

MODEL REFERENCE ADAPTIVE CONTROL SYSTEMS.

2.1 Introduction.

Adaptive control continues to attract considerable research interest ever since it was introduced some 25 years ago. This is testified by the amount of literature available on the subject. In 1976, Asher, Andrisani and Dorato compiled more than 700 references on adaptive control and in a recent introduction to the subject by Jacobs (1981) it was indicated that well over 1000 papers have appeared on this topic over the past 10 years. Recently, adaptive control has been the subject of four books by Landau (1979), Narendra and Monopoli (1980), Unbehauen (1980) and Harris and Billings (1981).

Among the various adaptive schemes the model reference approach proves to be the most successful. It is conceptually simple, fast and easily implemented. However, model reference adaptive systems (MRAS) are non-linear and therefore, the stability of such systems is of paramount importance. This has lead to extensive use of stability methods in the design of stable adaptive schemes as indicated by the techniques reviewed in the following sections.

2.2 The Need for Adaptive Controllers.

While conventional feedback systems are oriented towards the elimination of state perturbations, adaptive control systems are oriented towards the elimination of structural perturbations. Such structural perturbations are caused by variations in the dynamic parameters of the controlled system or by variations in the system operational conditions. Three examples will be cited below as potential situations where adaptive control is needed (Landau, 1979).

i) Aeronautical Situations: The dynamic behaviour of an aircraft depends on altitude, speed and configuration of the craft. Some parameters may vary by a multiplication factor of up to fifty.

ii) Nautical Situations: The dynamic characteristics of an oil tanker, because of its large dimensions, vary drastically from deep water to shallow water where the depth and volume of the ship become comparable with those of the water basin in which it navigates. Its dynamic characteristics also vary with the load.

iii) Electromechanical Systems: The dynamic behaviour of a dc motor varies with the moment of inertia and the friction of the load. The maximum to minimum ratio of parameter variations ranges from about 3 to 100. The thyristor bridges which are now used almost exclusively for motor control introduce structural perturbations in the control system when the firing angle varies. This affects both the equivalent gain and the time constant of the process.

Further examples are referenced in Unbehauen (1980).

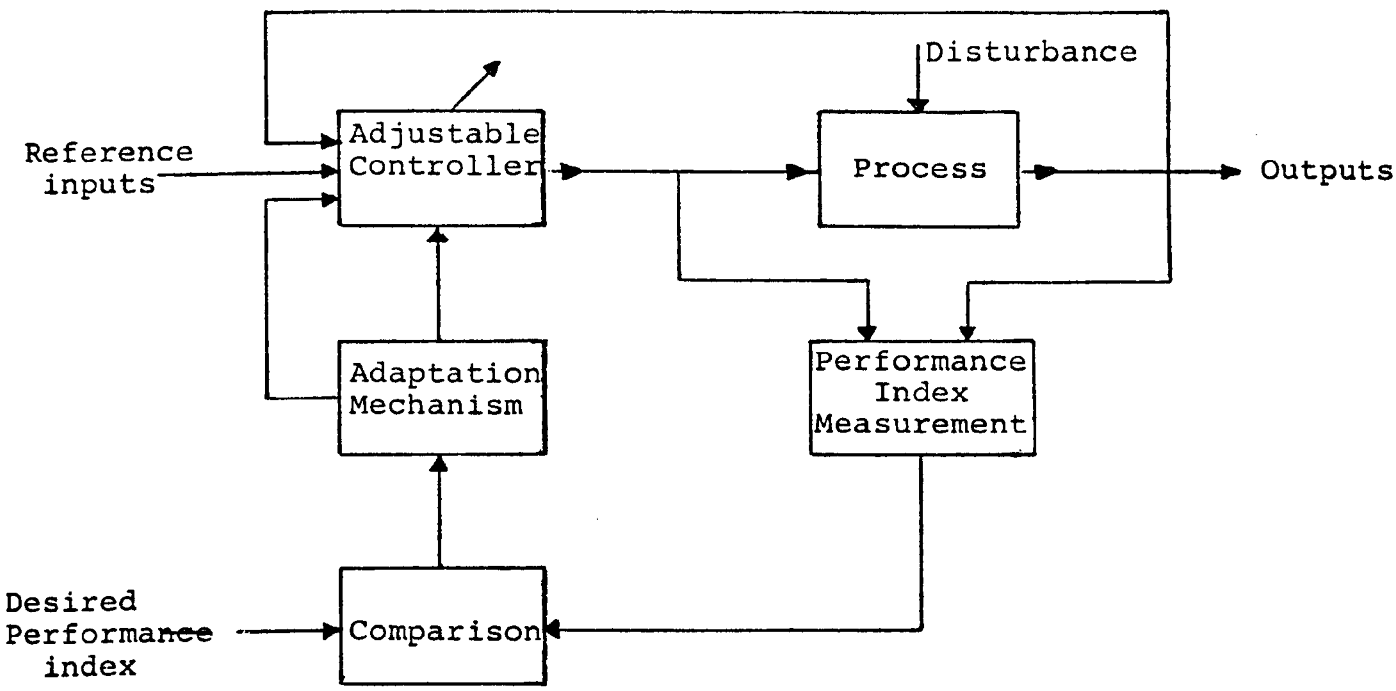
2.3 Formulation of Model Reference Adaptive Control Systems.

Landau (1974, 1979) defines an adaptive system as a system that measures a certain performance index (PI) using the inputs, the states and the outputs of the adjustable system. By comparing the measured performance index with a desired set of indices, the adaptive mechanism modifies the parameters of the adjustable system or generates auxiliary inputs in order to maintain the performance index close to the desired set of indices. This definition is depicted in Fig.2.3.1a.

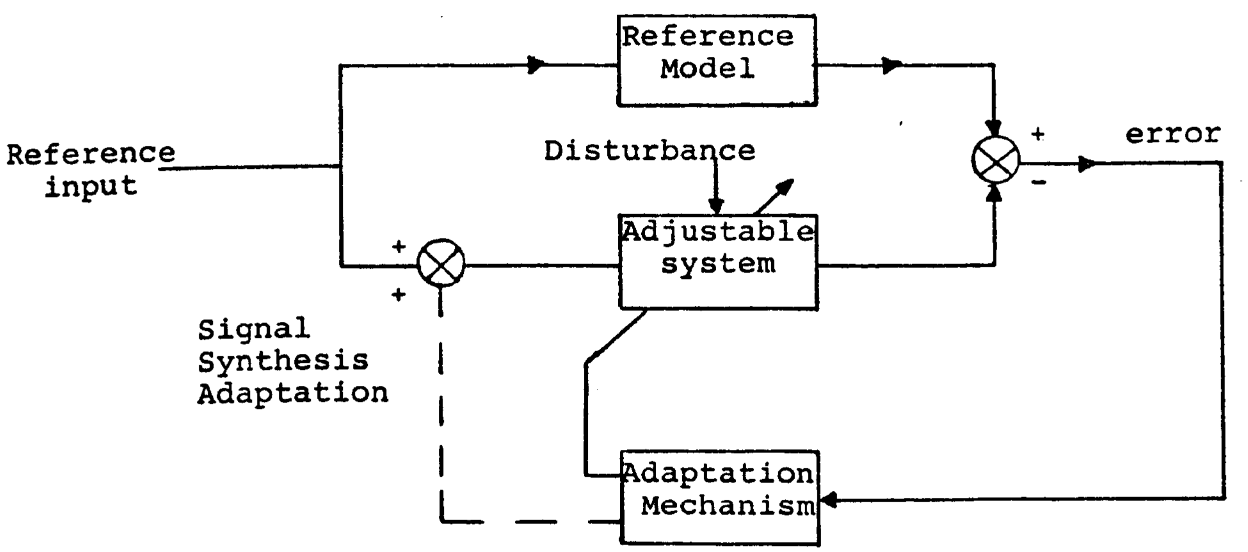
An important type of adaptive system is the model reference adaptive control (MRAC) system. This has the advantage of easy implementation and high speed adaptation. In model reference adaptive systems (MRAC) the set of (PI's) appearing in Fig. 2.3.1a are replaced by one dynamic performance index which becomes the reference (PI). To generate the reference (PI) one uses an auxiliary dynamic system called the reference model which is excited by the same external inputs as the adjustable system. The reference model, therefore, specifies in terms of inputs and model states a given performance index. The comparison between the given (PI) and the measured (PI) in this case is achieved directly by comparing the states (or outputs) of the adjustable system with those of the model using a subtractor. The difference between the two (PI's), i.e., the error, is used by the adaptation mechanism either to modify the parameters of the adjustable system or to generate auxiliary input signals. In either case, the adaptive mechanism chosen should drive the error signal to zero at least in the steady state. A typical parallel adaptive model reference system is shown in Fig.2.3.1b.

An extensive study of model reference adaptive systems has been conducted by Landau (1979). Prior to that Landau presented a survey paper with 253 references on the theory and applications of model reference adaptive techniques (Landau,1974).

An important class of model reference adaptive systems is the adaptive model following control system (AMFC). The treatment and applications of such systems will be deferred until Chapter 4.



Basic configuration of adaptive systems
(a)



A Parallel model reference adaptive control system
(b)

Fig.2.3.1

2.4 Mathematical Description of Model Reference Adaptive Control Systems.

The basic idea of model reference adaptive systems (MRAS) is to synthesize a controller that is capable of driving the error between the model and plant outputs to zero irrespective of parameter variations and external disturbances.

The linear plant and model to be considered are those described by the differential equations.

$$\dot{x}_p = A_p(t)\, x_p + B_p(t)\, u_p \tag{2.4.1}$$

and

$$\dot{x}_m = A_m\, x_m + B_m\, u_m \tag{2.4.2}$$

where x_p, x_m are the plant and the model states and u_p, u_m are the plant and model control inputs, respectively, $x_p \varepsilon R^n$, $x_m \varepsilon R^n$, $u_p \varepsilon R^m$, $u_m \varepsilon R^\ell$. The pair and (A_m, B_m) is assumed to be controllable.

The error between the model and plant states is given by

$$e = x_m - x_p \qquad e \,\varepsilon\, R^n \tag{2.4.3}$$

The derivative of the error ($\dot{e}$) is given by

$$\dot{e} = A_m\, x_m + B_m\, u_m - A_p(t)\, x_p - B_p(t)\, u_p$$

Adding and subtracting $A_m\, x_p$ yields

$$\dot{e} = A_m(x_m - x_p) + (A_m - A_p(t))\, x_p + B_m u_m - B_p(t)\, u_p$$

$$= A_m\, e + (A_m - A_p(t))\, x_p + B_m u_m - B_p(t)\, u_p \tag{2.4.4}$$

In systems with time-varying parameters the matrices $A_p(t)$, $B_p(t)$ can be decomposed into two matrices; one with fixed parameters and the other with parameters dependent on the error and time.

$$A_p(t) = A_p + F(e,t) \qquad (2.4.5)$$

$$B_p(t) = B_p + G(e,t) \qquad (2.4.6)$$

where A_p, B_p are time invariant matrices and F,G are time varying error-dependent matrices of dimension nxn and nxm respectively. The pair (A_p, B_p) is assumed to be controllable

Therefore

$$\dot{e} = A_m e + (A_m - A_p - F(e,t))x_p + B_m u_m - (B_p + G(e,t))\, u_p \qquad (2.4.7)$$

This shows that the error dynamics are described by a time varying nonlinear differential equation.

2.5 Design of Model Reference Adaptive System Using Stability Theory.

Model reference adaptive control systems are inherently non-linear time-varying feedback systems. This is because the error between even a linear plant and model is governed by non-linear time-varying differential equations. Therefore, investigating and ensuring the stability of the whole system is essential. This has been recognised and the extensive use of Liapunov functions in the design of stable adaptive schemes is reviewed in sections 2.5.2 and 2.5.4.

Another stability criterion which lends itself naturally to the design of stable MRAS is the hyperstability method proposed by Popov (1963,1973). This was adopted by Landau (1969) in his design of MRAS. A brief exposition of this method is given in section 2.5.3.

A new design method of asymptotically stable MRAC systems has been proposed by Young (1978b). This method utilizes the theory of variable structure systems and will be detailed in this thesis.

2.5.1 The MIT Rule.

The model reference adaptive control (MRAC) approach was first suggested by Witaker et al. (1958) at the Massachusetts Institute of Technology. The adaptive scheme was applied to aerospace problems and was developed further by Osburn et al. (1961) becoming known as the MIT rule.

The MIT rule is based upon the minimization of the integral of the squared error between the model and the adjustable system outputs. The mechanization of the adaptive law was implemented through sensitivity functions. However, as pointed out by Butchart et al. (1965) and Parks (1966) the stability of adaptive systems based on the MIT rule is not guaranteed even for simple systems and inputs. This drawback limits the applicability of the scheme.

2.5.2 Liapunov Design Methods.

Since 1961 considerable attention has been given to the design of model reference adaptive systems from a stability viewpoint. The early design of adaptive model reference controllers utilized Liapunov's second method almost exclusively. The Liapunov based design philosophy relies on the choice of a controller that will ensure at least a negative semidefinite first order derivative of the Liapunov function chosen. Besides guaranteeing stability for all kinds of inputs, the Liapunov method allows high gains in the adaptive loops, thus speeding up the adaptation and it often leads to considerable simplifications of such loops.

The utilization of Liapunov functions in the design of stable adaptive controllers seems to have been first proposed by Grayson

(1961). Since then many authors have used Liapunov functions in the design of MRAC systems (Hiza et al, 1963), (Butchart et al., 1965), (Grayson, 1965), (Parks, 1966), (Shackcloth,1967), (Winsor et al., 1968), (Phillipson, 1969), (Gilbart et al., 1970), and (Hang, 1974). The difference between the design approaches mentioned above lies in the particular choice of the Liapunov function. In a simulation study, Hang and Parks (1973) compared five different adaptive schemes; the MIT rule; Dressler's (1967); Price's (1970); Monopoli's (1968) and the Liapunov design scheme proposed by Gilbart et al. (1970). The last two schemes based on Liapunov functions were found to exhibit superior performance over the other designs. A survey paper on the use of Liapunov functions in the design of MRAC systems was presented by Lindorff and Carroll (1973). The survey listed 40 references.

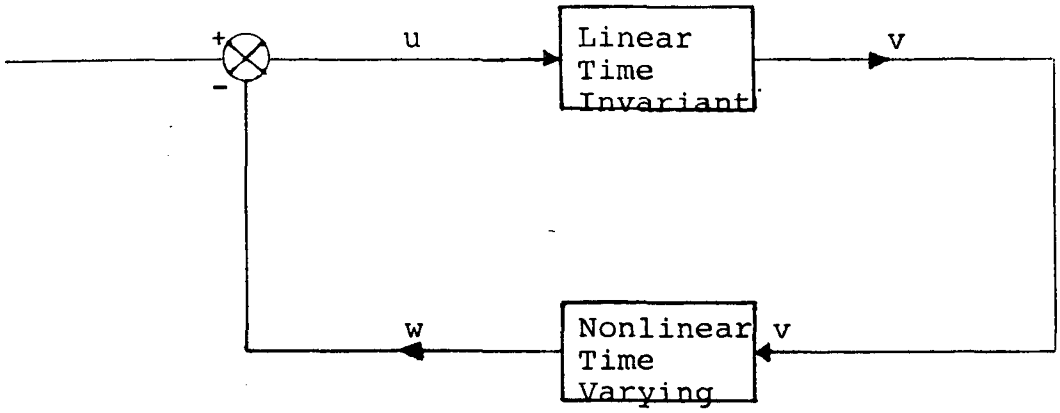
Derivatives of the errors are sometimes required but may be avoided in gain adjustment schemes if the system transfer function is positive real (Parks, 1966). Derivatives of the outputs can also be avoided by employing the augmented error scheme proposed by Monopoli (1974). Another feature of Monopoli's design method is that the plant inputs and outputs are only needed and that the adaptive system is asymptotically stable.

2.5.3 Hyperstability Design Methods.

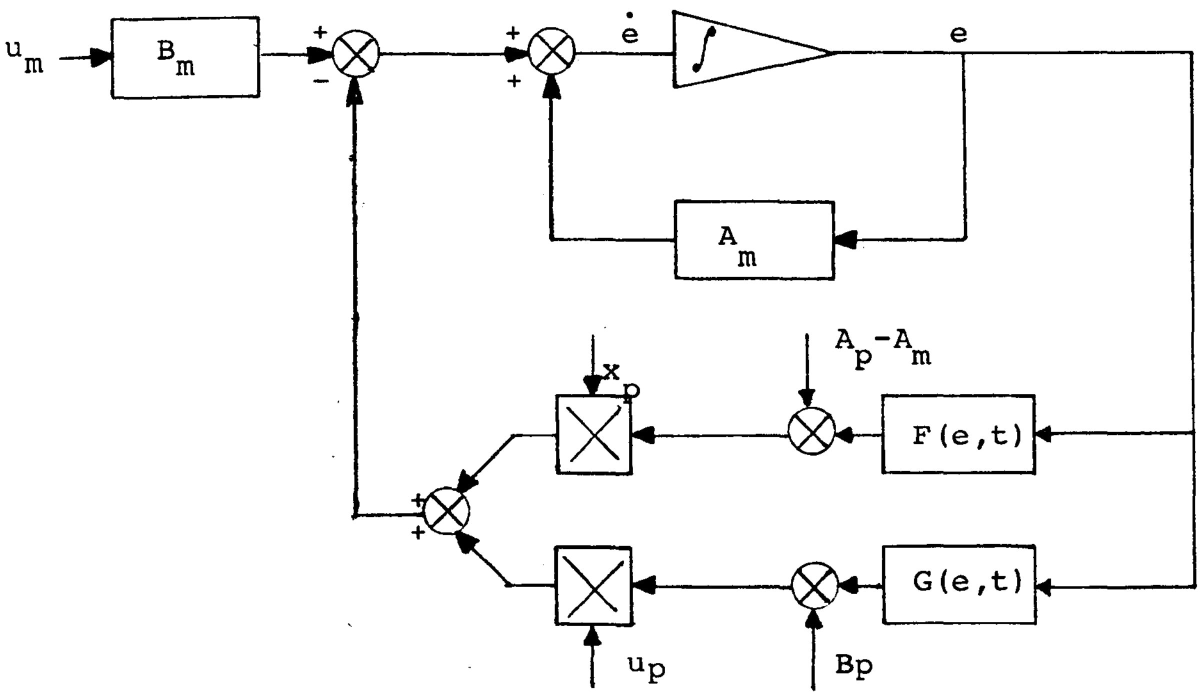
The natural decomposition of the adaptive error equation (2.4.7) into a linear feedforward part and a nonlinear time-varying feedback part lead to the introduction of hyperstability theory in the design of adaptive model reference control systems. The hyperstability concept concerns the stability properties of a class of feedback systems which can be split into two blocks; a linear feedforward block and a nonlinear time-varying feedback block as shown in Fig.2.5.1a. The adaptive error equation (2.4.7) belongs to this class of feedback systems (see Fig. 2.5.1b).

The definitions of hyperstability and asymptotic hyperstability are discussed in Appendix 1. Suffice to mention here that a necessary and sufficient condition for a system to be hyperstable is that the transfer function matrix of the linear feedforward part should be positive real. This seems to be a result common to hyperstability and some Liapunov based design methods. The hyperstability approach, however, offers a more systematic design procedure.

The application of hyperstability theory in the design of model reference adaptive control systems was first described by Landau (1969) and Laundau and Courtiol (1972), (1974). The papers considered the design procedure of adaptive model following control systems using hyperstability theory together with simulations of an aircraft control problem which showed the feasibility of the hyperstable design and the improved performance over the linear model following design. The design of an adaptive model reference electromechanical system utilizing hyperstability theory was reported by Courtiol and Landau (1975). This is discussed in section 2.6, together with the method of Irving et al. (1979) who implemented a hyperstable adaptive controller to power plants. Many papers have appeared on the design and applications of hyperstable adaptive schemes (see Landau, 1979).



Standard nonlinear time-varying feedback system
(a)



Equivalent feedback representation of the adaptive error equation (2.4.7)
(b)

Fig.2.5.1

2.5.4 Narendra's Error Models.

A different approach to the design of adaptive model reference control systems has been undertaken by Narendra. His scheme is centered on matching the plant transfer function with that of the model. Narendra and Kudva (1974) presented a two part paper for the design of stable adaptive controllers based on Liapunov functions. In part I, techniques were developed for the identification and control of unknown plants when the state variables are accessible for measurement. In part II, the results of part I were extended to the case when only the plant outputs are available for measurement, in which case stable adaptive observers were used to synthesize the adaptive controller. Further employment of adaptive observers in the design of stable MRAC systems has been considered by Narendra and Valavani (1976). The global asymptotic uniform stability of the adaptive loop for the case where the plant has n poles and (n-1) or (n-2) zeros has been demonstrated by Narendra and Valavani (1978). A unified approach to direct control, where no identification is involved but a reference model is used, and indirect control where identification of the plant parameters is achieved using an adaptive observer has been given by Narendra and Valavani (1979). Using a very specific choice of observer and controller structure in the indirect control problem the two approaches were shown to be equivalent and the stability questions that arise in the two cases were identical.

In an attempt to unify the two stability approaches to the design of model reference adaptive control systems, Narendra and Valavani (1980) presented a comparative study of the Liapunov and the hyperstability methods. The two authors concluded that when the input to the error model is uniformly bounded stability and asymptotic stability are achieved under exactly the same conditions as hyperstability and asymptotic hyperstability. When the input to the error model cannot be assumed to be uniformly bounded the problem is not completely resolved using either method.

Parks (1981) has illustrated the three design approaches described in sections 2.5.2-2.5.4 by means of simple examples.

2.5.5 In Search of Global Stability.

Although various adaptive schemes have been proposed, no study dedicated to the question of global stability had been attempted till 1978. Globally stable adaptive schemes are important since they are more likely to perform satisfactorily under practical conditions involving noise, nonlinearities and time-varying plant parameters.

Feuer and Morse (1978) and Morse (1980) seem to be the first workers who have tackled the problem of global stability for continuous time model reference adaptive systems. The treatment was restricted to single-input single output systems and although the controller is complex, it is differentiator-free and achieves global stability without making unnecessarily restrictive process -model assumptions. Another proof of global stability has been provided by Narendra, Lin and Valavani (1980) and Egardt (1978). Kreisselmeier (1980b) demonstrated global asymptotic stability by incorporating an adaptive observer followed by an adaptive feed-back matrix. The system is globally stable provided that the input to the plant is sufficiently excited by means of an external command signal. Sufficiently excited means that the command signal is sufficiently rich in frequencies; this is discussed in Kreisselmeier (1980a) For a discussion of stable adaptive schemes reference is also made to Narendra and Lin (1980).

2.6 Practical Applications of Adaptive Controllers.

Although adaptive control has been studied for well over twenty years, relatively few applications have been reported. This is due to

(a) insufficient development of the design methods

(b) a relative complexity of the adaptive schemes in comparison with classical linear time-invariant feedback controllers.

However, the need for high-quality control systems in the presence of large and unknown plant parameters variations, the development of feasible design methods and the advent of economical powerful microprocessors; all open the way for numerous practical implementations of MRAC systems in the near future.

One of the early practical applications of adaptive control to physical systems was presented by Porter and Tatnall in (1970). The adaptive controller was based on a Liapunov stability approach devised by Porter and Tatnall (1969) and was implemented on a specially constructed hydraulic servo-mechanism. The numerous test experiments that have been carried out demonstrated the success of the implementation.

Gilbart and Winston (1974) successfully applied an adaptive model reference controller to an optical tracking telescope. The control law was based on the Liapunov design technique proposed by Gilbart et al. (1970). The authors demonstrated the capability of the adaptive technique to compensate for mounting irregularities such as inertial variations and bearing friction. Results obtained via field tests on a large tracking telescope showed a 6:1 improvement in tracking accuracy for a worst case satellite trajectory.

Courtiol and Landau (1975) implemented a hyperstable based adaptive controller to a 5.3 kW d.c electrical drive. The adaptive system followed the model very closely and maintained good performance in the presence of variations of the moment of inertia (ratio 4:1), variations of the field current and variations of the gain due to the fluctuation of the firing angles of the thyristors used for motor feeding. The adaptive performance was compared with that obtained using a conventional PID controller. A different adaptive controller for the same system may be designed utilizing the theory of adaptive variable structure systems and will be discussed in Chapter 4. For other applications of adaptive control in Germany and France reference is made to Landau and Unbehauen (1974).

Irving, Barret, Charcossey and Monville (1979) implemented a hyperstable model reference adaptive controller in order to improve the steady state stability of power generators and to reduce the unit stress in the generators. Owing to the success of the adaptive scheme it will be applied to all future 1300 mW units to be installed in France.

Adaptive autopilots for ships have also been implemented by Van Amergon and Udink Ten Cate (1975) and Van Amergon (1980). Such autopilots are favoured because of the unpredictable variations in the ship steering characteristics and in order to prevent course instability which may occur in very large ships.

In addition to the practical implementation of the adaptive controllers mentioned above, there have been some simulation studies of other engineering systems. Such simulated systems have involved robotic manipulators (Dubowsky and DesForges, 1979), gas turbine engines (Monopoli, 1981) and coal-fired power plants (Mabius, 1981).

Finally, a comprehensive survey of the applications of adaptive controllers has been carried out by Parks et al.(1980). The authors presented a table listing some 58 known applications drawn from the cement, metallurgy and chemical industries, process control, power systems and electromechanical systems. Other applications appear in Landau (1979), Narendra and Monopoli (1980), Unbehauen (1980) and Harris and Billings (1981).

CHAPTER THREE

INTRODUCTION TO VARIABLE STRUCTURE SYSTEMS.

3.1 Introduction.

In linear control systems the designer is usually confronted with the two opposing requirements; static accuracy (stability, zero steady-state error and noise immunity) on one hand and dynamic accuracy (high speed of response) on the other. The two requirements are usually satisfied by a compromise. However, one way to alleviate such conflict is through the introduction of decision-making or nonlinearities into the system.

A special class of nonlinear systems which overcome such conflicting requirements is known as variable structure systems (VSS). As suggested by the name VSS, the control system undergoes a change in structure when the system state reaches a set of switching hyperplanes in the state space. The change in structure is intentional and in accordance with a preassigned algorithm or law. The times at which these changes occur are determined by the current value of the state and its derivatives. This is a fundamental property of VSS which distinguishes it from programmed controllers.

In variable structure systems different structures (subsystems) are combined giving a fixed structure which has a dynamic behaviour different from the individual structures on which the design of the original VSS was based. The fixed structure is specified by the designer and it enforces the subsequent motion of the system which is known as "sliding motion" or the "sliding mode". During the sliding mode the state point keeps jumping from one structure to the other crossing the switching hyperplane at each transition. Sliding motion is therefore identical to the chatter motion well known in relay control systems. Effectively, during sliding the state point moves along a new trajectory which is different from any of the trajectories of the individual constituent structures.

Variable structure systems resolve the conflict between static and dynamic accuracy by splitting the system transient into two independent stages. The two stages are: a brief motion up to the vicinity of the switching hyperplane known as "hitting" and an unlimited sliding motion of infinite frequency (theoretically) along the switching hyperplane.

The sliding mode is the most important feature of VSS as it offers many advantages which are discussed in section 3.2. VSS design is mainly concerned with satisfying certain conditions which enforce hitting and maintain sliding as discussed in section 3.5. An important fact about VSS is that they behave like linear systems during the sliding mode with an associated linear control law termed the "equivalent control".

Variable structure systems have been developed over the past 25 years almost exclusively in the USSR. Consequently, most of the literature on the subject is in Russian. Nevertheless, translations are available and the subject has attracted a wider interest in recent years. VSS are the subject of three books in English (Barbashin, 1970),(Itkis,1976) and (Utkin, 1978a). A recent survey on the subject is presented by Utkin(1977).

3.2 Advantages of Variable Structure Systems (VSS).

Although changing the structure introduces additional complexity into the system, it also yields a system exhibiting certain advantages unparalleled in systems with fixed structure. The advantageous features of VSS are:

a) High speed of response without loss of stability. This is obtained by increasing the gains of the feedback controller and by a suitable choice of switching hyperplanes.

b) Straightforward synthesis of asymptotically stable systems by combining two or more structures which may be unstable on their own.

c) Simplicity of physical realization.

d) Insensitivity to parameter variations over a wide range.

e) Invariance to a certain class of external disturbances.

3.3 Formulation of VSS.

Variable structure systems are a special class of nonlinear systems which may be considered as relay systems where the output signal of the relay element is proportional to the state and its derivatives. They are characterized by discontinuous control which changes structure on reaching a set of switching surfaces (hyperplanes). For the linear system

$$\dot{x} = Ax + Bu \qquad (3.3.1)$$

where $x \in R^n$, $u \in R^m$, the control u has the form

$$u_i = \begin{cases} u_i^+(x) & s_i(x) > 0 \\ u_i^-(x) & s_i(x) < 0 \end{cases} \qquad (3.3.2)$$

where u_i is the ith component of u and $s_i(x)$ is the ith component of the m switching hyperplanes given by

$$s(x) = C\,x = 0 \qquad (3.3.3)$$

where C is a constant mxn matrix.

The above system with discontinuous control is termed a variable structure system since the effect of the switching hyperplanes is to alter the feedback structure of the system whenever the state x reaches a switching hyperplane $s_i(x) = 0$.

3.4 The Sliding Mode.

Sliding motion occurs if, at a point on a switching hyperplane $s_i(x) = 0$, the directions of motions along the state trajectories on either side of the hyperplane are directed towards the switching hyperplane. The describing point then slides and remains on the switching hyperplane. The condition for sliding motion to occur on the ith hyperplane may be stated as

$$\lim_{s_i \to 0^+} \dot{s}_i < 0 \qquad \text{and} \qquad \lim_{s_i \to 0^-} \dot{s}_i > 0 \tag{3.4.1}$$

or, equivalently,

$$s_i \dot{s}_i < 0 \qquad\qquad s_i \to 0 \tag{3.4.2}$$

In the sliding mode the system satisfies the equations

$$s_i(x) = 0 \qquad \text{and} \qquad \dot{s}_i(x) = 0 \tag{3.4.3}$$

Mathematically, the right hand side of equation (3.3.1) with control (3.3.2) does not satisfy the classical theorems on existence and uniqueness of solutions of differential equations. However, the ambiguity in the system behaviour can be eliminated if various types of nonidealities such as time-delay, small time constants and hysteresis are considered when deriving the sliding mode equations (Utkin, 1971, 1978a). Incorporating such non-idealities will result in the so-called real sliding in a certain small neighbourhood of the discontinuity hyperplane. Ideal sliding results when the nonidealities tend to zero. We shall be dealing with this idealized situation throughout this thesis. A rigorous mathematical description of sliding has been given by Filippov (1960) and is also described in Utkin (1978a). A general approach to the investigation of the dynamics of VSS in the sliding mode was presented by Bermant, Emel'yanov and Taran (1965).

The invariance of VSS to parameter variations during sliding is due to the fact that the system motion is restricted to lie on the switching hyperplane, i.e., the system motion satisfies

$$Cx(t) = 0 \tag{3.3.3}$$

which is independent of the system parameters and depends only on the elements of the switching hyperplane s. Consequently, by a suitable choice of C asymptotic stability can be ensured during sliding. Furthermore, sliding can be made arbitrarily fast by the appropriate selection of C.

The conditions for sliding to exist and to be maintained along the intersection of the switching hyperplanes from any initial condition are the conditions of asymptotic stability in the large. Stability of VSS is therefore ensured if hitting is guaranteed and sliding is sustained. The stability of VSS has been investigated by André and Seibert (1960), Barbashin (1970) and Weissenberger (1966, 1969). Stability of variable structure controllers incorporated in model reference systems has also been demonstrated by Devaud and Caron (1975) and Young (1977).

3.5 The Design of VSS.

The design procedure consists of two independent stages. First, the switching hyperplanes are chosen to satisfy the designer's objectives (e.g. arbitrary eigenvalues, minimizing a quadratic performance index). Secondly, a discontinuous control is sought which guarantees the existence of sliding at every point on the switching hyperplanes and which steers the state to the sliding hyperplanes. The two design stages are systematic and furthermore, the second stage involves inequalities which ease the design problem. A recent method for the selection of the discontinuity hyperplanes has been presented by Utkin and Yang (1978). Other procedures for the construction of the switching hyperplanes are developed and presented in this thesis.

Assuming the appropriate switching hyperplanes that ensure stability of the system during sliding have been chosen, the design of the discontinuous control law which enforces sliding and maintains it will now be described. The design description below is for a scalar system in canonical companion form. The design of general systems follows the same lines. Let the system be described by

$$\begin{aligned} \dot{x}_i &= x_{i+1} \qquad\qquad i=1,\dots,n-1 \\ \dot{x}_n &= -\sum_{i=1}^{n} a_i(t)x_i - b(t)u \end{aligned} \tag{3.5.1}$$

where the scalar control u is given by

$$u = \sum_{i=1}^{n} \psi_i \, x_i \tag{3.5.2}$$

where

$$\psi_i = \begin{cases} \alpha_i & \text{if } x_i s > 0 \\ \beta_i & \text{if } x_i s < 0 \end{cases} \tag{3.5.3}$$

and

$$s = \sum_{i=1}^{n} c_i \, x_i \quad , \quad c_n = 1 \tag{3.5.4}$$

The plant parameters $a_i(t)$ and gain $b(t)$ are bounded piecewise-continuous and are given by

$$a_{i,min} \leqslant a_i(t) \leqslant a_{i,max} \quad ; \quad b_{min} \leqslant b(t) \leqslant b_{max}$$

Invoking the condition $s\dot{s} < 0$ it can be shown that the necessary and sufficient conditions for existence of a sliding mode at each point of the switching hyperplane s are:

$$\alpha_i > \max_t \frac{1}{b(t)} [c_{i-1} - a_i(t)] \tag{3.5.5}$$

$$i=1,\ldots,n$$

$$\beta_i < \min_t \frac{1}{b(t)} [c_{i-1} - a_i(t)] \tag{3.5.6}$$

A more general form of the control u is

$$u = \sum_{i=1}^{n} \psi_i x_i + \delta_u \tag{3.5.7}$$

where ψ_i is the same as in (3.5.3) and δ_u is given by

$$\delta_u = \begin{cases} \delta_o & \text{if } s > 0 \\ -\delta_o & \text{if } s < 0 \end{cases} \tag{3.5.8}$$

The relay component δ_u is included to ensure sliding at the origin in the presence of disturbances. Relay control alone can be used to enforce sliding but does not generally ensure global sliding.

The inequalities in equations (3.5.5) and (3.5.6) provide the designer with some freedom when selecting the control law u. Larger values of α_i and β_i shorten the time of the state transition to the switching hyperplane , thus reducing the effect of disturbances acting upon the system. Sliding can also be ensured by feeding back k states where $k < n$. However, the freedom in choosing arbitrary c_i is lost in this case and sometimes conflicting requirements have to be satisfied (Itkis, 1976). Various methods have been proposed for the systematic selection of the control parameters α_i, β_i for more general systems. These methods are discussed in Utkin (1978a) and include the method of control hierarchy which is also systematically formulated in Young (1978b).

3.6 The Equivalent Control.

A precise technique for finding the equations of ideal sliding is known as the "equivalent control method". In this technique, the time derivative of the switching hyperplanes s(x) is set to zero and the resulting algebraic equation is solved for the control vector u. This equivalent control u (if it exists) is then substituted in the original system. The resulting equations describe the system behaviour during sliding. The equivalent control is a continuous linear control which constrains the state x to remain on the intersection of the discontinuity hyperplanes. The equivalent control is now derived for the system described by (3.3.1) and (3.3.3)

$$\dot{s}(x) = C\dot{x} = 0 \tag{3.6.1}$$

$$= CAx + CBu = 0 \tag{3.6.2}$$

Assuming $|CB| \neq 0$ the equivalent control is the solution of equation 3.6.2, i.e.

$$u = u_{eq} = -(CB)^{-1} CAx \tag{3.6.3}$$

Substituting u_{eq} in the original system described by 3.3.1 yields the equations of ideal sliding

$$\dot{x} = (I - B(CB)^{-1}C)\, Ax \qquad (3.6.4)$$

During sliding the order of the system is reduced by m degrees. This means that m states of the system are given as functions of the remaining n-m states. This is a direct consequence of the solutions of m homogeneous equations ($s(x)=0$) in n unknowns. Further insight into this reduction property is developed in this thesis using projector theory and is presented in Chapter 6.

The equivalent control is unique if $|CB| \neq 0$ and the existence of the sliding mode is ensured in this case. Sliding may also exist if CB is singular ($|CB| = 0$) but this is subject to satisfying certain conditions on CB and CAx (see Utkin 1972, 1978a).

Recent studies of high gain systems have exposed the relation between VSS and linear high gain systems. Young, et al. (1977) and Utkin (1978b) have shown that the motion before and that during sliding is similar to the "fast" and "slow" motion which characterises high gain systems. The equivalent control therefore, yields the slow motion during sliding. This is consistent with the physical interpretation of the equivalent control given by Utkin (1972, 1978a). Utkin demonstrated that the equivalent control is the average value of the discontinuous control.

3.7 The Invariance Principle.

In addition to being insensitive to parameter variations, VSS enjoy another characteristic feature, namely invariance to a certain class of disturbances during the sliding mode. The "invariance" principle was first introduced by Draženović (1969) who showed that during sliding VSS are insensitive to external disturbances which belong to the range space of the gain matrix B, R(B). The disturbances need not be incorporated in the control law as long as sliding is maintained. This is a great advantage especially in the case of unmeasurable disturbances. The effects of the disturbances before the sliding mode is attained can be minimized by increasing the system gain which decreases the time taken for hitting to occur. To obtain the invariance condition consider the system

$$\dot{x} = Ax + Bu + Df \qquad (3.7.1)$$

$$s = Cx \qquad (3.3.3)$$

where f is the disturbance vector, $f \in R^{\ell}$ and D is an $(n \times \ell)$ disturbance gain matrix.

Calculating the equivalent control and substituting back in equation (3.7.1) we get

$$\dot{x} = (I-B(CB)^{-1}C)(Ax+Df) \qquad (3.7.2)$$

If col(D) belongs to the range space of B, i.e. if

$$D = BM \qquad (3.7.3)$$

where M is $m \times \ell$ [illegible] matrix, (3.7.2) then reduces to

$$\dot{x} = (I-B(CB)^{-1}C)\,Ax \qquad (3.7.4)$$

which is independent of the disturbance vector f. Thus the disturbance f has been rejected during the sliding mode.

The invariance principle will be revisited in Chapter 6 where an alternative proof is developed together with a generalization of the principle to include the case where CB is singular.

3.8 An Example.

Much of the previous theory can best be demonstrated by means of a simple example. The objective is to demonstrate certain variable structure concepts such as sliding, discontinuous control, switching hyperplanes, the condition $s\dot{s}<0$, and the equivalent control. Consider an unstable double-integrator system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (3.8.1)$$

where

$$u = -\psi_1 x_1 - \psi_2 x_2 \qquad (3.8.2)$$

$$\psi_i = \begin{cases} \alpha_i & \text{if} \quad x_i s>0 \\ \beta_i & \text{if} \quad x_i s<0 \end{cases} \qquad i=1,2 \qquad (3.8.3)$$

$$s = c_1 x_1 + x_2 \qquad (3.8.4)$$

Satisfying the condition $s\dot{s}<0$ yields the values of α_i and β_i which can be obtained in terms of the inequalities

$$\alpha_1 > 0 \quad , \quad \alpha_2 > c_1 \quad , \quad \beta_1 < 0 \quad , \quad \beta_2 < c_1$$

Large values of these parameters decrease the time taken for hitting to occur since the system gain is increased. During sliding

$$s = c_1 x_1 + x_2 = 0$$

i.e.

$$x_2 = -c_1 x_1 = \dot{x}_1 \qquad (3.8.5)$$

Solving for $x_1(t)$ we get

$$x_1(t) = x_{1s} e^{-c_1(t-t_s)} \qquad (3.8.6)$$

where x_{1s} is the value $x_1(t_s)$ at the time t_s the system begins sliding. A similar expression can be obtained for $x_2(t)$ from (3.8.5)

$$x_2(t) = -c_1 x_{1s} e^{-c_1(t-t_s)}$$

$$= x_{2s} e^{-c_1(t-t_s)} \qquad (3.8.7)$$

One can easily observe that the order of the system is reduced since the system dynamics are totally specified by (3.8.5) which is a first order differential equation. The rate of decay of the states can be seen to increase with c_1.

A simulation study has been conducted with the following values

$c_1 = 1$, $x_1(0)=1$, $x_2(0)=0$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = -1$, $\beta_2 = 0$,

ΔT (simulation time step) = 0.02 sec.

Fig.3.8.1a shows the time response of x_1 and x_2. Fig.3.8.1b demonstrates the sliding motion on the switching hyperplane (a line in this case). The describing point for $0 \leq t \leq t_s$ approaches the switching line and then undergoes small damped oscillations crossing the switching line every time the sign of s changes for $t > t_s$ ($t_s = 0.8$ sec). The amplitude of these oscillations decreases as Δt decreases. As expected x_1 and x_2 both maintain zero values in the steady state. This is a consequence of sliding at the origin since $s = 0$ at every point on the switching hyperplane including the origin.

Fig. 3.8.2a demonstrates how the magnitude of the switching function s decreases and how the describing point keeps jumping from the positive to the negative half and vice versa. The graph also demonstrates that the condition $s\dot{s} < 0$ is always satisfied at every point in the phase plane ($\dot{s}$ is the slope of s and it can be easily seen that $\dot{s} < 0$ when $s > 0$ and $\dot{s} > 0$

when $s < 0$). Fig.3.8.2b shows the discontinuous nature of the variable structure controller once sliding has occurred and how its magnitude approaches zero as the phase point approaches the origin. The continuous nature of the equivalent control is shown in Fig.3.8.2c. The equivalent control is calculated using (3.6.3) and is represented by the graph on the right hand side of the vertical line drawn to indicate the time sliding occurs.

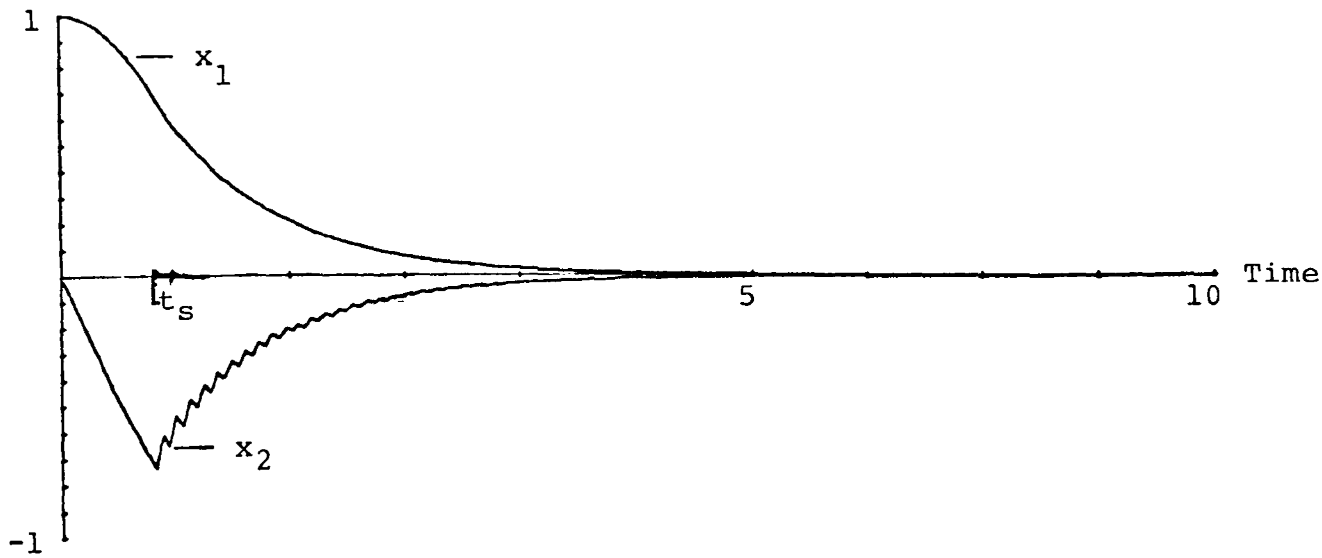
Fig.3.8.3a,b show the effect of increasing the feedback gains (α,β) which result in a decrease in the time taken for sliding to occur (indicated by the vertical lines). Note also the decrease in the slope of s with larger gains (Fig.3.8.3b). With $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = -2$, $\beta_2 = -1$, t_s decreased to 0.38 sec. Δt has been decreased to 0.01 sec so as to expand the graph.

3.9 Applications of VSS.

Variable structure controllers are useful whenever a system is required to perform satisfactorily in a hostile environment which entails large parameter variations, external disturbances or extreme working conditions. VSS perform well in these cases because of their self-adaptive properties. Some practical implementations of VSS controllers are mentioned below.

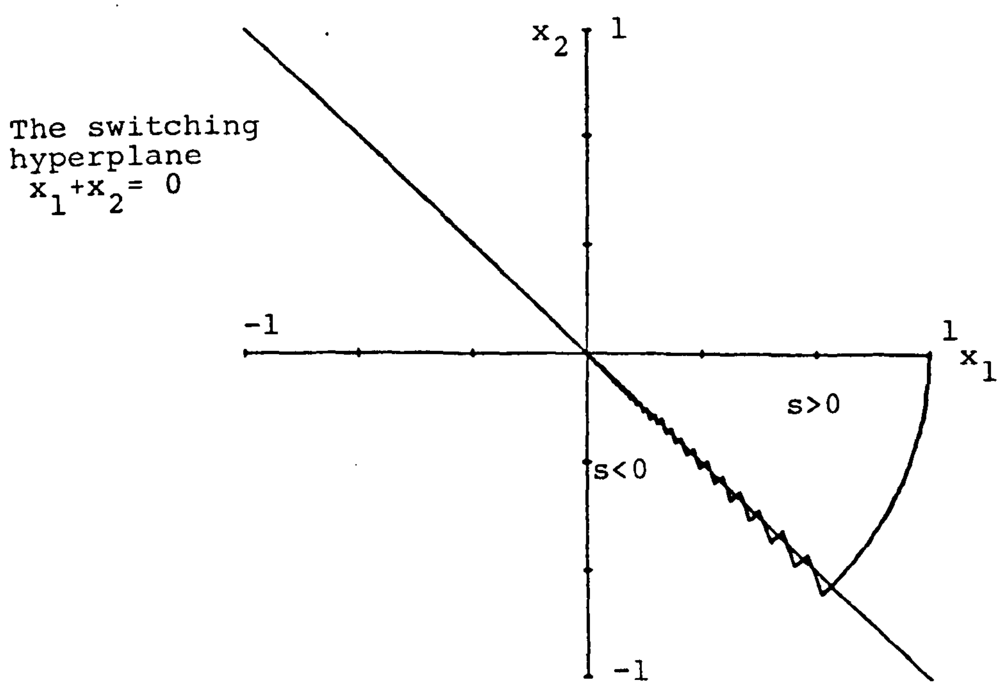
Emel'yanov et al. (1969), applied variable structure theory to the control of strip-thickness in a continuous hot-rolling mill. The industrial implementation of the VSS controller showed a marked improvement in the static accuracy and lead to an almost time optimal system.

The insensitivity of VSS to parameter variations together with their good static precision have given favourable control



(a)

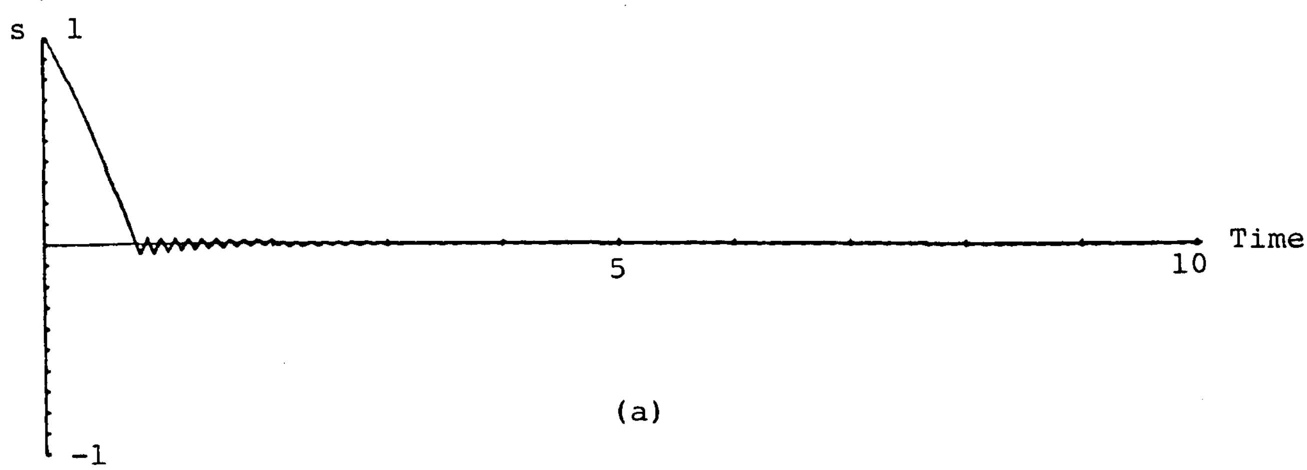
The time response of x_1 and x_2



(b)

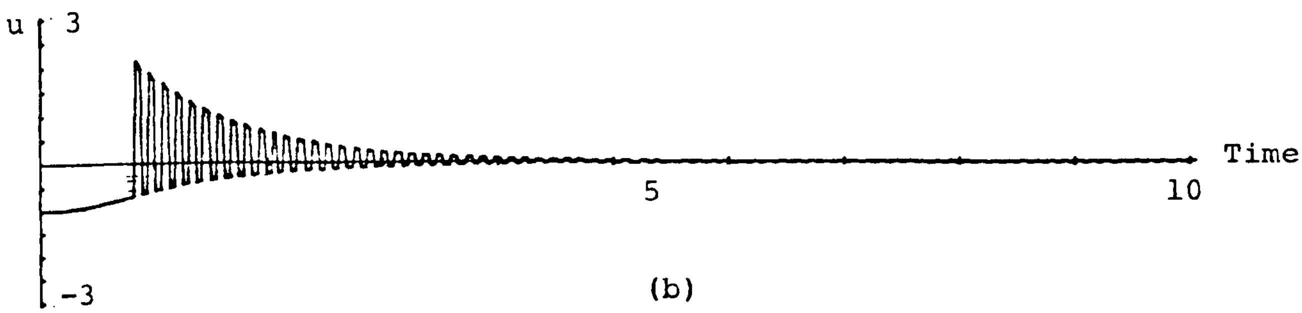
Sliding on the switching hyperplane $s = 0$

Fig.3.8.1



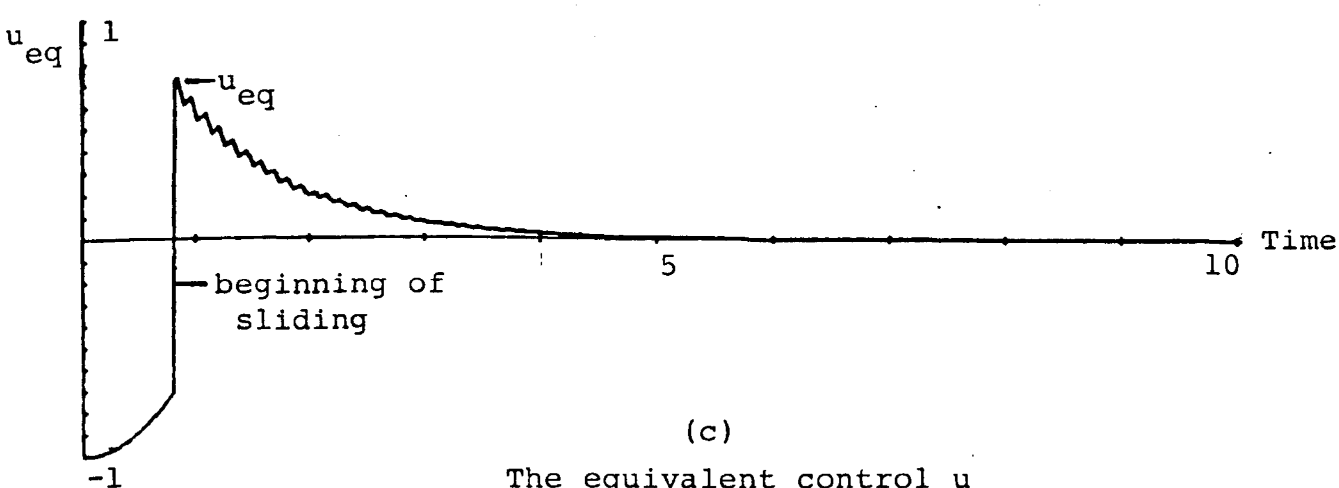
(a)

The time evolution of the switching hyperplane s = 0



(b)

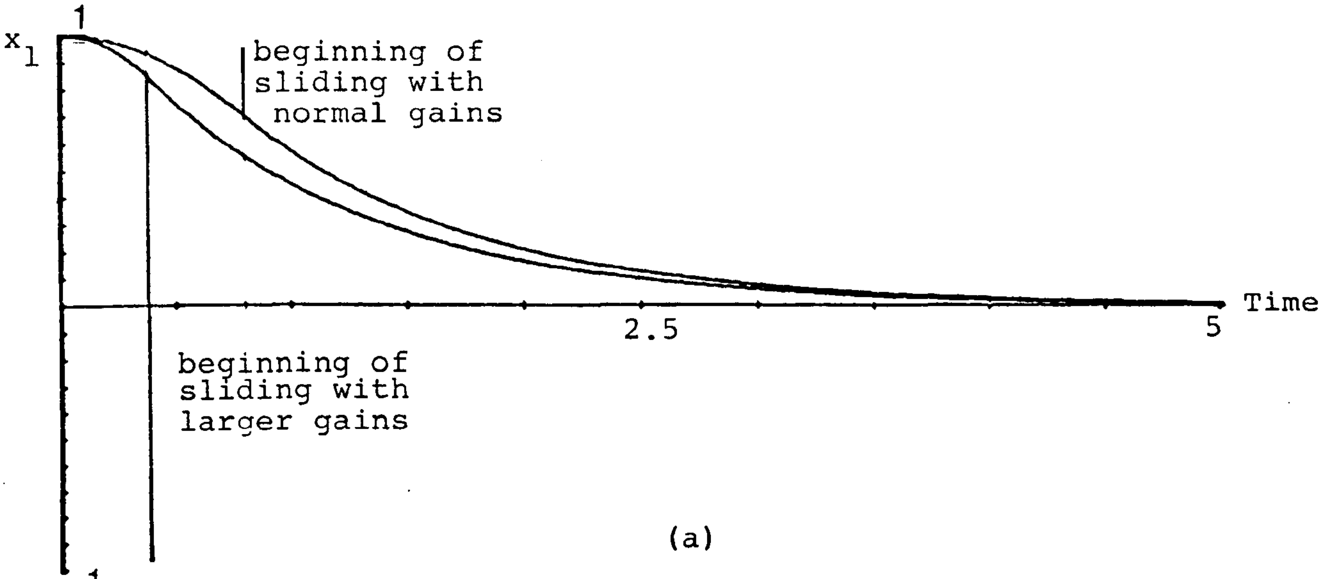
The discontinuous control u



(c)

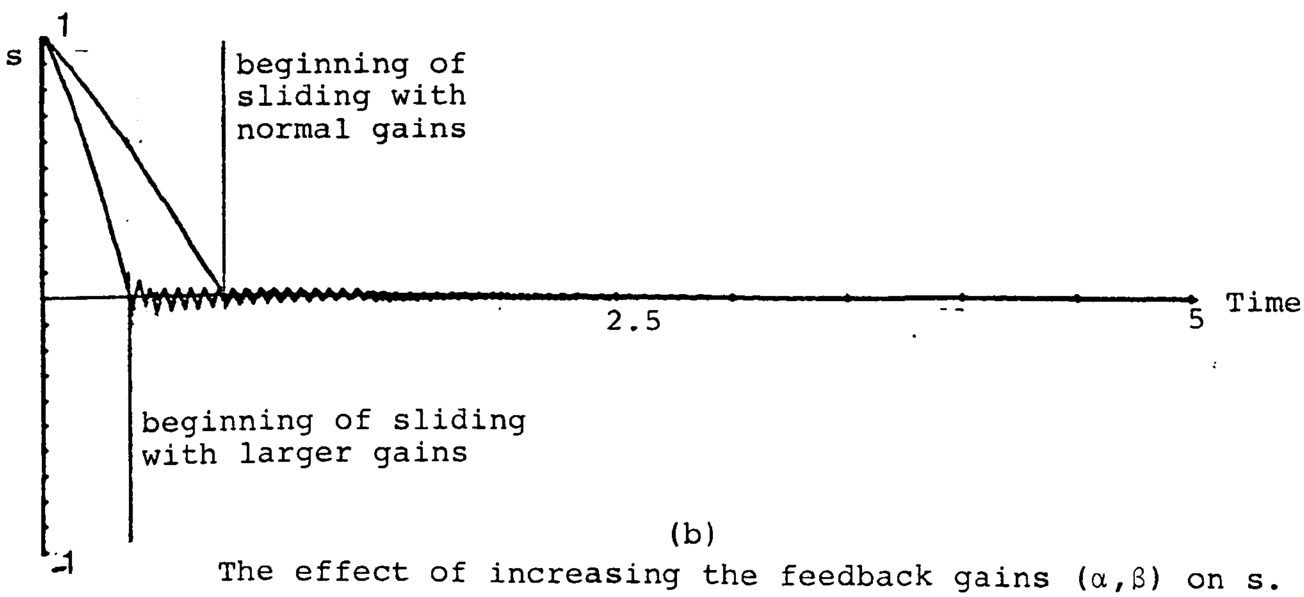
The equivalent control u_{eq}

Fig.3.8.2



(a)

The effect of increasing the feedback gains (α, β) on x_1



(b)

The effect of increasing the feedback gains (α, β) on s.

Fig.3.8.3

of hydroelectric power stations (Erschler et al., 1974). The authors implemented a variable structure controller to the water-gate system of the station and the experiments conducted showed good practical performance.

Application of variable structure theory to the control of position, speed and torque of squirrel cage induction motors has been investigated by Šabanović and Izosimov (1981). Discussion of the practical realization of the implemented controller was presented together with the experimental tests conducted.

Simulation studies have also demonstrated the potential of variable structure theory in the control of many engineering systems. Such simulated systems have involved robotic manipulators (Young, 1978a), subsonic aircrafts (Young, 1978b), overhead cranes (Zinober, 1979) and overspeed protection for large steam turbines (Kwatny and Young, 1981). Other simulation examples will be presented in this thesis; one of which concerns the speed control of dc motors which is also discussed in Zinober, El-Ghezawi and Billings (1981).

3.10 Links Between the Sliding Mode and Linear Systems.

Variable structure systems may at first appear to be an isolated branch of control theory which has nothing in common with conventional linear control systems. This is not the case. As has been demonstrated recently, certain unifying aspects exist between VSS and linear control systems. The sliding mode has been found to provide the link between the two. This is justified since VSS behave like linear systems during sliding.

Young et al. (1977) and Utkin (1978b) have shown that the system motion during sliding is identical to the slow motion which exists in high gain systems. The equations governing the sliding motion have also lead to a method for computing the system zeros (Young et al., 1977), (El-Ghezawi et al., 1982a,1982b). The condition which satisfies the invariance principle and that which guarantees model-following are identical as pointed out by Young (1978b). The equivalent control is confined not only to VSS but is also found to exist naturally in decoupling theory and in output model following. This will be substantiated later in this thesis. Furthermore, a geometric approach provides additional insight into the analysis and design of VSS. This approach is believed to be new and is developed in detail in this thesis.

Finally, it is to be hoped that cross-fertilization between linear and discontinuous control theory will help to enrich our understanding of these diverse concepts.

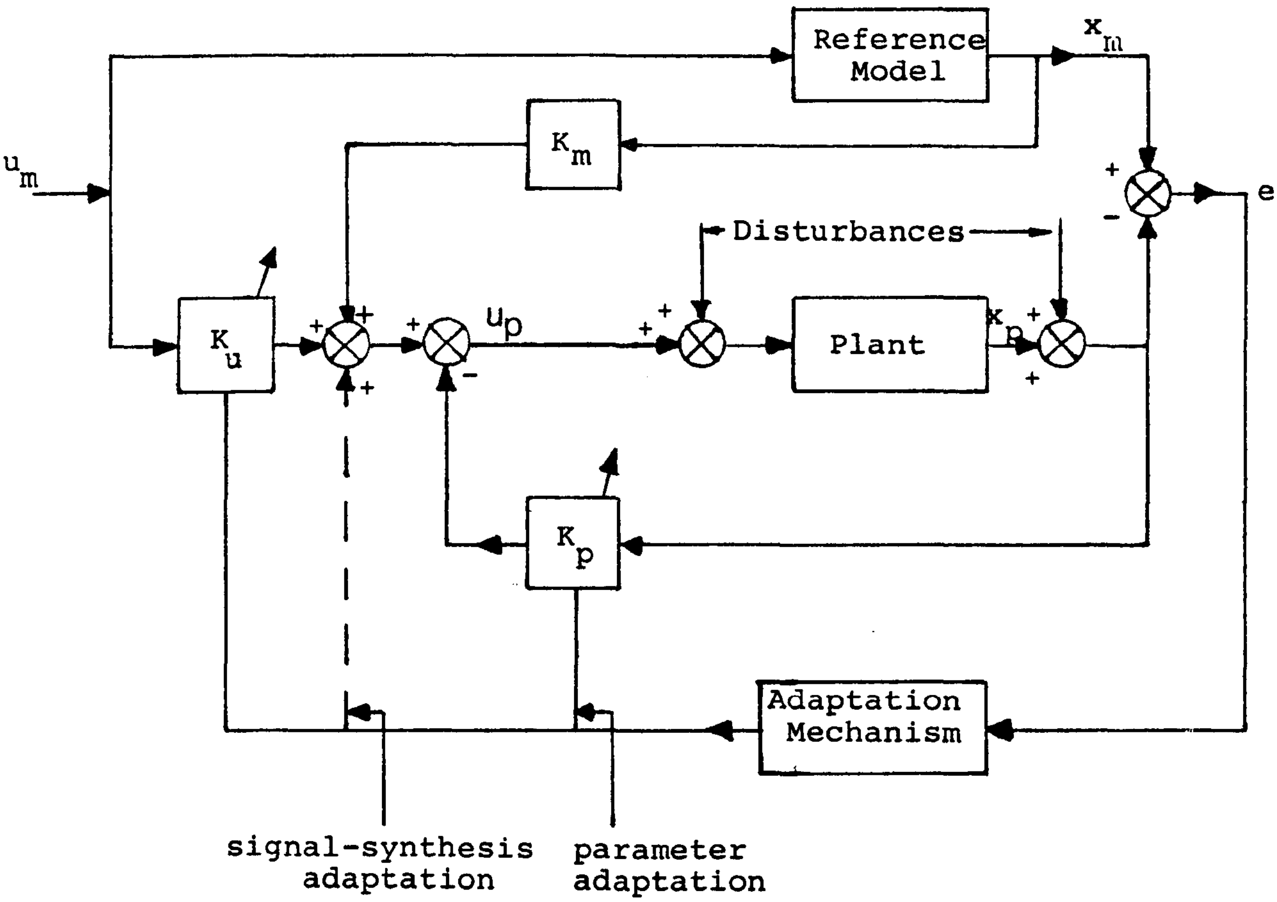
CHAPTER FOUR

VARIABLE STRUCTURE ADAPTIVE MODEL-FOLLOWING CONTROL SYSTEMS (VSMFC).

4.1 Introduction.

The difficulty of selecting weighting matrices for a quadratic performance index when specifying a desired transient characteristic is a well known problem in optimal control theory. An efficient method which avoids such difficulties is known as linear model-following control (LMFC). In LMFC the model specifies the desired performance of the control system and feedforward and feedback matrices or auxiliary inputs are determined to ensure perfect following of the model evolution. However, LMFC systems are inadequate when there are large parameter variations or disturbances. This has lead to the development of so-called adaptive model-following control systems (AMFC) (Landau, 1979). AMFC differs from LMFC by incorporating an adaptation mechanism which has provisions for modifying the parameters of the control matrices. The basic configuration of AMFC systems is illustrated in Fig.4.1.1.

There are two types of model-following systems known as implicit model-following and real model-following. In implicit model-following, the model is not part of the overall system; it is only needed at the design stage when calculating the control law. On the other hand, real model-following systems require the states of the model and therefore, the model is a necessary part of the system (Erzberger, 1968) and (Chan, 1973). Although real model-following is more complex to implement than implicit model following it offers the best



Adaptive model-following control system.

Fig.4.1.1

performance when uncertainties and disturbances are present in the system (Erzberger, 1968). Both implicit and real model-following controllers should force the error between the model and the plant outputs to zero as time tends to infinity, i.e. $\lim_{t \to \infty} e(t) = 0$.

Early designs of LMFC systems have been presented by Tyler (1964), Erzberger (1968), Markland (1970) and Newell et al. (1972). In these cases the controller was obtained by minimizing a quadratic cost function involving the error. Non-optimal controllers have also been utilized in the design of LMFC systems (Chan, 1973), (Morse, 1973), (Kudva et al., 1976) and (Shaked, 1977a). The design of AMFC systems, however, is more complex because of the external disturbances and the variations in the plant parameters. The stability of such systems also warrants more detailed attention. The design of AMFC systems has been approached from a stability viewpoint (Landau et al., 1972, 1974), (Courtiol et al., 1975), (Irving et al, 1979) and (Monopoli, 1981). Although the stability approach guarantees that the error goes to zero as $t \to \infty$ it offers no direct quantitative control over the error transients.

A new design method for AMFC systems has been proposed by Young (1977, 1978b). His approach utilizes the theory of variable structure systems (VSS) in designing asymptotically stable systems. The control is discontinuous on a number of switching hyperplanes. During the sliding mode which exists on the intersection of the hyperplanes the system becomes less sensitive to parameter variations and noise disturbances. The advantage of this method is that it provides a systematic and effective procedure for specifying the transient response of the error.

4.2 Model-following Control Systems.

In real model following systems, the plant is controlled in such a way that its dynamic behaviour approximates that of a specified model. The model is part of the system and it specifies the design objectives. The adaptive controller should force the error between the model and the plant to zero as time tends to infinity i.e., $\lim_{t\to\infty} e(t) = 0$.

The plant and model are described by

$$\dot{x}_p(t) = A_p(t)\, x_p(t) + B_p(t)\, u_p(t) \tag{4.2.1}$$

$$\dot{x}_m(t) = A_m\, x_m(t) + B_m\, u_m(t) \tag{4.2.2}$$

where $x_p, x_m \in R^n$, $u_p \in R^m$, $u_m \in R^\ell$. The error vector is

$$e(t) = x_m(t) - x_p(t) \tag{4.2.3}$$

We shall assume that the pairs (A_p, B_p), (A_m, B_m) and (A_m, B_p) are stabilizable i.e., the uncontrollable modes lie in the left half plane. The matrix A_m is assumed to be stable. The plant matrices $A_p(t)$ and $B_p(t)$ may be uncertain and time varying. The upper and lower bounds of the elements of these matrices are assumed to be known to the designer.

It can be easily shown that

$$\dot{e}(t) = A_m\, e(t) + (A_m - A_p(t))\, x_p(t) + B_m\, u_m(t) - B_p(t)\, u_p(t) \tag{4.2.4}$$

Perfect model-following occurs if, for zero initial conditions, the error vector e(t) is null for any u_m belonging to the class of piecewise-continuous vector functions. Erzberger (1968) was the first to provide the necessary conditions for perfect following. This was followed by Chan (1973) who provided a different version of the same conditions. The perfect following conditions are those of the equation

$$(A_m - A_p)\ x_p + B_m\ u_m - B_p\ u_p = 0 \tag{4.2.5}$$

to have a solution u_p irrespective of x_p and u_m.@ This is a classical problem in linear algebra in which the necessary and sufficients conditions are (as stated by Erzberger, (1968))

$$(I - B_p B_p^{g})\ (A_m - A_p) = (I - B_p B_p^{g})\ B_m = 0 \tag{4.2.6}$$

or, equivalently (Chan, 1973)

$$\text{rank } B_p = \text{rank } (A_m - A_p : B_p) = \text{rank } (B_m : B_p) \tag{4.2.7}$$

where B_p^{g} is any generalized inverse of the matrix B_p.*

We shall assume throughout that the perfect model-following conditions hold. Adaptive model-following design allows the plant parameters to vary and to be uncertain, but not the structure of the plant, unless eqn (4.2.6) or (4.2.7) remains satisfied. For brevity, the conditions (4.2.7) will be referred to as the perfect following conditions.

If conditions (4.2.7) are satisfied then perfect following control u_p is obtained from (4.2.5)

$$u_p = B_p^{+}\ (A_m - A_p)\ x_p + B_p^{+}\ B_m\ u_m \tag{4.2.8}$$

---

*See Appendix 2 for the definition of the generalized inverse.

@ A_p and B_p in (4.2.5) are taken as the nominal values.

where B_p^+ is a special class of generalized inverse called the Penrose pseudoinverse. The use of the pseudoinverse is not necessary as any other generalized inverse will suffice. Its main advantage is that it leads to a minimum Euclidian norm $||u_p||$ (Chan, 1973). Furthermore, the boundedness of the pseudoinverse implies that the control u_p is bounded (Erzberger, 1968). Erzberger (1968) has also shown that for the cases where conditions (4.2.6) fail zero error is still possible by enlarging the class of controls to include delta functions.

So far the discussion of determining the control u_p has been restricted to the problem of non adaptive LMFC, that is, systems which are not subject to parameter variations or external disturbances. Landau and Courtiol (1972) seem to be the first to approach the problem of adaptive model-following (AMFC) using the hyperstability model reference adaptive techniques. The two authors showed that parameter adaptation and signal synthesis adaptation illustrated in Fig.4.1.1 are identical. In parameter adaptation the control is given by

$$u_p = - k_p(e,t)\ x_p + k_u(e,t)\ u_m + k_m\ x_m \qquad (4.2.9)$$

where k_p, k_u are time-varying matrices dependent on the error e. The signal synthesis adaptive controller can be obtained from (4.2.9) by expressing $k_p(e,t)$, $k_u(e,t)$ as

$$k_p(e,t) = k_p + \Delta k_p(e,t) \qquad (4.2.10)$$

$$k_u(e,t) = k_u + \Delta k_u(e,t) \qquad (4.2.11)$$

where k_p, k_u are constant matrices designed for specific plant parameter values chosen to satisfy the perfect model-following conditions. With this decomposition, (4.2.9) can be written as

$$u_p = u_{pl} + \Delta u_{pl} \qquad (4.2.12)$$

where

$$u_{pl} = -k_p x_p + k_u u_m + k_m x_m \qquad (4.2.13)$$

$$\Delta u_{pl} = -\Delta k_p(e,t)\, x_p + \Delta k_u(e,t)\, u_m \qquad (4.2.14)$$

u_{pl} is the familiar controller chosen to ensure perfect model-following while Δu_{pl} is the contribution of the adaptive loop which is applied to the system as an auxiliary input.

In order to ensure strong stability characteristics of the adaptive system, Landau et al. (1972) applied the concepts of hyperstability to generate the adaptive controller. The synthesis of the input Δu_{pl} does not require real time identification. Computations are only needed at the design stage. The controller is implemented using summators, multipliers and integrators.

Although Landau's method guarantees the stability of the adaptive system it offers no direct control over the error transients. An attractive method which offers direct control over the error transients and also guarantees the stability of the system has been presented by Young (1978b). This method is based on the theory of variable structure systems (VSS) and will be discussed in the following section.

4.3 Variable Structure Adaptive Model-Following Control Systems (VSMFC).

The introduction of variable structure systems into the design of adaptive model-following control systems has been motivated by the observation that the perfect model matching conditions (4.2.7) coincide with the invariance condition (3.7.3) during the sliding mode (Young, 1978b). Since the prime objective of adaptive control is to drive the error between the model and the plant to zero, the design of AMFC systems is therefore analogous to that of organizing sliding in the error state space. By maintaining sliding along the intersection of selected switching hyperplanes associated with the error equation described by (4.2.4) the error can be guaranteed to approach zero as $t \to \infty$. The salient feature of variable structure adaptive model following control systems (VSMFC) is that both x_p and u_m are considered as disturbances and therefore are rejected during sliding owing to the invariance principle and the perfect model-following conditions.

The properties of VSMFC systems during the sliding mode when $s = \dot{s} = 0$ will now be examined. The variable structure system in our case is

$$\dot{e} = A_m e + (A_m - A_p) x_p + B_m u_m - B_p u_p \qquad (4.3.1)$$

with a variable structure control given by

$$u_p = \psi_e e + \psi_p x_p + \psi_m u_m \qquad (4.3.2)$$

where ψ_e, ψ_p, ψ_m are discontinuous functions as will be shown later.

Assuming sliding is organized on all of the switching hyperplanes

$$s = C e = 0 \qquad (4.3.3)$$

where C is a full rank mxn constant matrix. The equivalent control can be calculated as described in Chapter 3 and is given by

$$u_{eq} = (CB_p)^{-1} C[A_m e + (A_m - A_p)x_p + B_m u_m] \qquad (4.3.4)$$

The equation governing the error during sliding is therefore

$$\dot{e} = [I - B_p(CB_p)^{-1}C][A_m e + (A_m - A_p)x_p + B_m u_m] \qquad (4.3.5)$$

Due to the coincidence of the perfect model-following conditions and the invariance condition eqn. (4.3.5) reduces to

$$\dot{e} = [I - B_p(CB_p)^{-1}C] A_m e \qquad (4.3.6)$$

Therefore, given A_m, B_p and assuming the pair (A_m, B_p) is stabilizable a matrix C can be found such that eqn (4.3.6) is asymptotically stable, thus ensuring that the error goes to zero as $t \to \infty$. If the pair (A_m, B_p) is controllable, then complete control over the error transient is possible. Fast error decay can be ensured by placing n-m eigenvalues deep in the left-half of the complex plane.

The construction of the appropriate switching hyperplanes is important in order to guarantee the asymptotic stability of the system. Through a comparative study of high-gain systems and VSS Young et al. (1977) have shown that the motion during sliding is identical to that of the 'slow' subsystem of a high-gain system. Furthermore the authors presented a design procedure for constructing the switching hyperplanes. Using a similarity transformation W, eqn (4.3.6) can be transformed into (Young, 1978b)

$$\dot{e}_1' = (A_m^{11} - A_m^{12} C_2^{-1} C_1) e_1' \; ; \; e_1' \in R^{n-m} \qquad (4.3.7)$$

where $$e' = \begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = W e \qquad (4.3.8)$$

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \quad \text{such that } W_1 B_p = 0 \tag{4.3.9}$$

$$W A_m W^{-1} = \begin{bmatrix} A_m^{11} & A_m^{12} \\ A_m^{21} & A_m^{22} \end{bmatrix} \tag{4.3.10}$$

$$C W^{-1} = [\mathcal{C}_1 \quad \mathcal{C}_2]^{@} \tag{4.3.11}$$

The inverse of $\mathcal{C}_2$ always exists since the inverse of CB is assumed to exist. By suitable choice of $\mathcal{C}_1, \mathcal{C}_2$ arbitrary eigenvalues can be assigned to eqn (4.3.7) assuming that (A_m, B_p) is completely controllable. Eqn (4.3.7) is associated with the slow subsystem in high gain systems. It governs the system behaviour during sliding.

The eigenvalues of eqn (4.3.7) are the transmission zeros of the system S(A,B,C) taking the switching hyperplanes as the system output. Once matrices $\mathcal{C}_1$ and $\mathcal{C}_2$ have been selected the switching hyperplanes are obtained from eqn (4.3.11)

$$C = [\mathcal{C}_1 \quad \mathcal{C}_2]\, W \tag{4.3.12}$$

Young (1978b) also studied the effect of parameter perturbation on B_p. He showed that if perturbations ΔB_p satisfy

$$B_p = B_o + \Delta B_p$$

$$\text{and} \qquad \mathcal{R}(\Delta B_p) \subset R(B_o) \tag{4.3.13}$$

then the eigenvalues of (4.3.7) will be unaffected by ΔB_p. However, if $\mathcal{R}(\Delta B_p)$ is independent of $\mathcal{R}(B_o)$, i.e. $\mathcal{R}(\Delta B_p) \not\subset R(B_o)$ then system stability can still be assured by placing the eigenvalues of (4.3.7) deep in the left half plane. The insensitivity to

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@ Equation (24) in Young (1978b) is wrong. Equation (4.3.11) is the correct one.

parameter variations in B_p is considered as another advantageous feature of VSMF system over hyperstability design where variations in B_p may cause problems (Young 1978b).

A different procedure for constructing the switching hyperplanes has been developed by the author and will be discussed in Chapter 6. The new method offers two advantages not enjoyed by Young's et al. (1977) procedure; namely the ability to impose a specification on CB and the ability to exercise partial control over the eigenvectors associated with the n-m assigned eigenvalues.

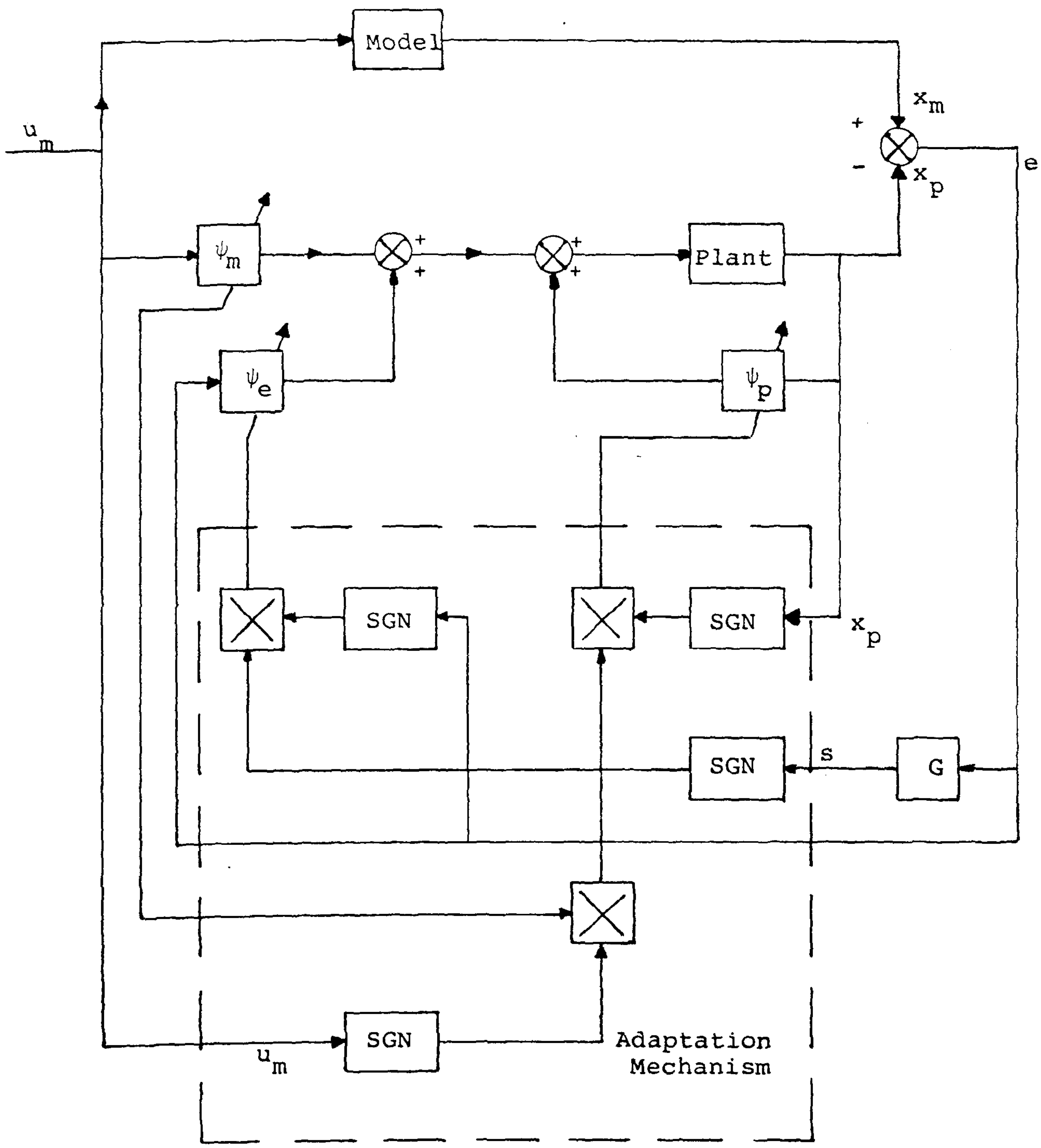
4.4 The Control u_p in VSMFC.

In the previous section it has been shown that VSMFC systems possess adaptive properties similar to that of AMFC systems. The variable structure control law employed (4.3.2) is similar to the"paramater adaptation" control law adopted in AMFC systems except that the gain matrices ψ_e, ψ_p and ψ_m are discontinuous. The adaptation mechanism arising from these discontinuities is governed by the switching hyperplanes. In other words, the discontinuous gain matrices ψ_e, ψ_p, ψ_m may be considered as adaptable gains.

This interpretation of VSMFC systems is illustrated in Fig. 4.4.1.

The gains ψ_e, ψ_p, ψ_m in the control law (4.3.2) are given by

$$\psi^i_{\gamma_j} = \begin{cases} \alpha^i_j & \gamma_j \; s_i(e)>0 \\ \beta^i_j & \gamma_j \; s_i(e)<0 \end{cases} \qquad \begin{array}{l} i=1,\ldots,m \\ j=1,\ldots,2n+\ell \end{array} \quad (4.4.1)$$



Variable Structure Adaptive Model-Following Control System.

Fig.4.4.1

where $\gamma = [e_1,\ldots,e_n,x_{p1},\ldots x_{pn},u_{m1},\ldots u_{ml}]^T$

If a control law of the form ($|z|$ is the absolute value of each component of z).

$$u_p = \psi_e\ |e| + \psi_p\ |x_p| + \psi_m\ |u_m| \qquad (4.4.2)$$

is used then the ψ_{γ_j}'s are given by

$$\psi^i_{\gamma_j} = \begin{cases} \alpha^i_j & s_i(e) > 0 \\ \beta^i_j & s_i(e) < 0 \end{cases} \qquad (4.4.3)$$

Once sliding occurs the error tends to zero asymptotically. Furthermore ψ_e, ψ_p and ψ_m can be considered to be continuous functions which are identical to those making up the equivalent control described by (4.3.4), i.e.

$$\psi_e^{eq} = -(CB)^{-1}C\ A_m \qquad (4.4.4)$$

$$\psi_p^{eq} = -(CB)^{-1}C\ (A_m-A_p) \qquad (4.4.5)$$

$$\psi_m^{eq} = -(CB)^{-1}C\ B_m \qquad (4.4.6)$$

Thus, during sliding the discontinuous control law (4.3.2) yields the effective continuous equivalent control (4.3.4).

A successful method for determining the α_i and β_i values is the method of control hierarchy which has been formulated in detail by Utkin (1978a) and Young (1978b) to ensure sliding motion. The advantage of the method is its ability to allow for parameter variations and external disturbances. There is ample freedom in the choice of α^i_j, β^i_j since their values need to satisfy certain inequalities.

4.5 Design Examples.

The simplicity of the variable structure design technique will be illustrated by applying it to the adaptive model-following control of a second-order dc drive, a fourth-order multivariable system and to a multivariable batch process. A further example is also discussed in Zinober, El-Ghezawi and Billings (1982).

4.5.1 The Examples and the Design of the VSMFC Systems dc Drive.

An adaptive controller has been designed by Courtiol and Landau (1975) for a dc drive and the same system will be considered here. The system equations are

$$\dot{x}_p = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{bmatrix} x_p + \begin{bmatrix} 0 \\ \frac{k_m A}{J\tau} \end{bmatrix} u_p \tag{4.5.1}$$

where x_{p1} and x_{p2} are the speed and acceleration of the motor respectively, u_p is the reference of the armature current control loop, A and τ are respectively the equivalent gain and time constant of the armature current loop, k_m is the torque and back-emf constant and J is the moment of inertia of the load. J, k_m, A and τ are parameters subject to variations. The model is given by

$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -\frac{\alpha\, k_m^o\, A^o}{J^o\, \tau^o} & -\frac{1}{\tau^o} \end{bmatrix} x_m + \begin{bmatrix} 0 \\ \frac{\alpha\, k_m^o\, A^o}{J^o\, \tau^o} \end{bmatrix} u_m \tag{4.5.2}$$

where the superscript "o" denotes the nominal value of the plant parameters. The gain α yields the desired dynamics of the speed control loop. The perfect model-following conditions are satisfied. The load moment of inertia is considered to vary in the range 0.21 to 0.78 Kgm^2. The parameters of the reference model are $\tau^o = 10$ msec, $A^o = 3$ A/V, $k_m{}^o = 2$ Nm/A, $\alpha = 0.83$ and $J^o = 0.5$ Kgm^2.

The switching line and the control u_p are chosen to be

$$s = c\, e_1 + e_2 = 0 \qquad c > 0 \tag{4.5.3}$$

and

$$u_p = -\psi_1\, x_{p1} - \psi_2\, x_{p2} - \psi_3\, e_1 - \psi_4\, e_2 - \psi_5\, u_m \tag{4.5.4}$$

where the ψ_i are given by the expressions

$$\psi_i = \begin{cases} \alpha_i & \text{if} \quad \gamma_i\, s > 0 \\ \beta_i & \text{if} \quad \gamma_i\, s < 0 \end{cases} \qquad i=1,\ldots,5 \tag{4.5.5}$$

with $\gamma_1 = x_{p1}$, $\gamma_2 = x_{p2}$, $\gamma_3 = e_1$, etc. To satisfy the reachability and sliding condition $s\dot{s} < 0$ the α_i's should satisfy the inequalities

$$\alpha_1 < \min \left\{ \alpha\, \frac{\tau}{\tau^o} \cdot \frac{J}{J^o} \cdot \frac{k_m{}^o}{k_m} \cdot \frac{A^o}{A} \right\} \tag{4.5.6}$$

$$\alpha_2 < \min \left\{ \frac{J}{k_m A} \cdot \frac{(\tau - \tau^o)}{\tau^o} \right\} \tag{4.5.7}$$

$$\alpha_3 < \min \left\{ \alpha\, \frac{A}{A^o} \cdot \frac{J}{J^o} \cdot \frac{k_m{}^o}{k_m} \cdot \frac{\tau^o}{\tau} \right\} \tag{4.5.8}$$

$$\alpha_4 < \min \left\{ \frac{J\tau}{k_m\, A\, \tau^o} \cdot (1 - c\tau^o) \right\} \tag{4.5.9}$$

$$\alpha_5 < -\max \left\{ -\alpha\, \frac{A}{A^o} \cdot \frac{J}{J^o} \cdot \frac{k_m{}^o}{k_m} \cdot \frac{\tau^o}{\tau} \right\} \tag{4.5.10}$$

The β_i's satisfy corresponding inequality conditions with the less than signs replaced by greater than signs and the minima by maxima.

Fourth-order System.

Consider the fourth-order system with two controls where the plant is given by:

$$\dot{x}_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -6 & -11 & -10 \end{bmatrix} x_p + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u_p \tag{4.5.11}$$

The model is given by

$$\dot{x}_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -12 & -19 & -8 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} u_m \tag{4.5.12}$$

The switching hyperplanes are chosen to be

$$s = \begin{bmatrix} 0 & 0 & 0 & 1 \\ C_1 & C_2 & 1 & 0 \end{bmatrix} e \tag{4.5.13}$$

The initial conditions are zero except for $x_{p1} = -1$. We now follow the control hierarchy method to ensure the sliding mode. Sliding on the hyperplane $s_1 = e_4 = 0$ is assured at $t = 0$. Thus $\dot{e}_4 = 0$ and substitution yields the 3rd-order system.

$$\dot{e} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u_m - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{p2} \tag{4.5.14}$$

and

$$u_{1eq} = 3x_{p1} - 6x_{p2} - 8x_{p3} + 2x_{p4} - 3e_1 - 12e_2 - 19e_3 - 8e_4 + 3u_m - u_{p2} \tag{4.5.15}$$

Choosing

$$u_{p2} = \sum_{i=1}^{2n+\ell} \psi_i \gamma_i \qquad (4.5.16)$$

where

$$\psi_i = \begin{cases} \alpha_i & \text{if} \quad \gamma_i s_2 > 0 \\ \beta_i & \text{if} \quad \gamma_i s_2 < 0 \end{cases} \qquad (4.5.17)$$

and

$$\gamma_1 = x_{p1}, \gamma_2 = x_{p2}, \ldots, \gamma_5 = e_1, \ldots, \gamma_9 = u_m$$

Sliding is ensured on the second hyperplane $s_2 = 0$ by satisfying the reachability conditions. In the sliding mode $\dot{s} = 0$ and therefore, the error transient may be specified to be those of a desired second-order system.

$$\ddot{e}_1 + c_2 \dot{e}_1 + c_1 e_1 = 0 \qquad (4.5.18)$$

by suitably choosing c_1 and c_2. The inequality constraints yield

$$\alpha_6 < -c_1, \alpha_7 < -c_2, \beta_6 > -c_1, \beta_7 > -c_2$$

and the remaining α's need to be negative. Symmetric results arise for the β's, i.e. the β's are positive.

A Chemical Reactor.

Munro (1972) considered the linearized open-loop unstable chemical reactor given by

$$\dot{x}_p = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} x_p + \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix} u_p$$

(4.5.19)

A model chosen to satisfy the perfect model-following conditions and to yield acceptable dynamics is given by

$$\dot{x}_m = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ 10.1952 & -4.29 & 9.998 & -13.802 \\ 1 & 0 & -5 & 2 \\ 2.2037 & 4.273 & 3.343 & -5 \end{bmatrix} x_m + \begin{bmatrix} 0 & 0 \\ 4.999 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} u_m \tag{4.5.20}$$

The new method developed in this thesis will be used for the selection of the switching hyperplanes (see Chapter 6). The two eigenvalues governing the error dynamics will be placed at -10. Applying the design procedure described in Chapter 6 with CB_p specified as*

$$CB_p = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tag{4.5.21}$$

the switching hyperplanes $s = Cx$ are found to be (the values are to four significant digits)

$$C = \begin{bmatrix} 0.0751 & -0.1128 & 0 & -0.3163 \\ 1.0773 & -0.0197 & 0.6357 & -0.5373 \end{bmatrix} \tag{4.5.22}$$

The control law used is

$$u_p^j = -\sum_{i=1}^{2n+\ell} \psi_i^j \, |\gamma_i| \qquad j=1,m \tag{4.5.23}$$

where

$$\psi_i^j = \begin{cases} K_i^j & \text{if } s_j(e) > 0 \\ -K_i^j & \text{if } s_j(e) < 0 \end{cases} \tag{4.5.24}$$

and

$$\gamma_1 = x_{p1}, \ldots, \gamma_4 = x_{p4}, \quad \gamma_5 = e_1, \ldots\ldots, \gamma_{10} = u_{m2}$$

---

*The determination of C is developed in Chapter 6 (see Example 6.9.3).

Following the method of control hierarchy (Young, 1978b) and by requiring sliding to commence on the first hyperplane and then on the second the values of K_i^j have been chosen as

$$K = \begin{bmatrix} 2 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \end{bmatrix} \qquad (4.5.25)$$

4.5.2 Simulation Results.

Second-order dc drive.

Throughout the simulation (unless otherwise indicated) the following values have been used

$x_{p_1}(0) = -2,\ x_{p_2}(0)=0,\ x_{m_1}(0)=0,\ x_{m_2}(0),\ u_m = \begin{cases} 4 & t \leq 0.6 \text{ sec} \\ 0 & t > 0.6 \text{ sec} \end{cases}$

$c = 50$

$J = J_{max} = 0.78$

$\alpha_1 = -0.2, \alpha_2 = -0.2,\ \alpha_3 = -0.2,\ \alpha_4 = -0.2,\ \alpha_5 = -1.3$

$\beta_1 = 1.3,\ \beta_2 = 0.2,\ \beta_3 = 1.3,\ \beta_4 = 0.2,\ \beta_5 = -0.3$

T (simulation time step) = 0.001

The dynamic behaviour of the plant and model is shown in Fig. 4.5.1a. In order to demonstrate the influence of the switching hyperplane gradient (c) on the error transient two values of c were considered. As expected, the larger value (c = 50) leads to a more rapid decay of the error than the smaller value (c = 20). This is depicted in Fig.4.5.1b. Note also that the error reduces exponentially to zero. At t = 0.6 (where $u_m = 0$) the error vector is zero and sliding commences immediately. The error therefore remains zero and the plant and model responses are identical. This is not the case for the hyperstability design method (see Fig.11 in Courtiol et al., 1975).

Fig.4.5.2a demonstrates the adaptive properties of the system as the load moment of inertia was perturbed about its nominal value ($J^o = 0.5$). The perturbations are step-wise changing randomly every 0.03 sec. The level of these perturbations are such that the load moment of inertia assumes different values including the minimum ($J = 0.21$) and the maximum ($J = 0.78$). As compared with the fixed moment of inertia (see Fig.4.5.1a) no marked difference is seen. This graph also helps to demonstrate the invariance principle since the fluctuation of J does not alter the range space of B_p (i.e. $R(B_p)$). Fig.4.5.2b shows the effect of an added random disturbance ($h(t)$) which does not belong to $R(B_p)$. Satisfactory response can still be obtained with $|h(t)| \leq 5$.

It may be impractical to apply the discontinuous control directly to the plant. The insertion of a 1st-order lowpass filter with output $u_f(t)$ ahead of the plant yields a smoother control signal as shown in Fig.4.5.3c, but during sliding we now have $|s| < \varepsilon$ with ε not infinitesimally small. The resulting response (see Fig.4.5.3a) is close to that of Fig.4.5.1a and the discontinuous control $u_p(t)$ is shown in Fig.4.5.3b. The filter time constant is 20 s^{-1}.

<u>Fourth-order Multivariable system.</u>

The following values have been used in the simulation

$$c_1 = 1, \quad c_2 = 2$$

The values of the α's and β's were -0.4 and 0.4 respectively except for $\alpha_{6,7}$ and $\beta_{6,7}$ where

$$\alpha_6 = -0.4-c_1, \ \alpha_7 = -0.4-c_2, \ \beta_6 = 0.4-c_1, \ \beta_7 = 0.4-c_2$$

All initial conditions are zero except for x_{p1} which is equal to -1

$\Delta T = 0.05$

As shown in Fig.4.5.4a the plant has a highly oscillatory transient response without adaptive control. The choice of the model (4.5.12) and of c_1 and c_2 in (4.5.13) yields a suitable model transient and error transient. A typical response is shown in Fig.4.5.4b. The response from zero error at $t = 25$ yields identical plant and model transients for $t \geq 25$.

The Chemical Reactor

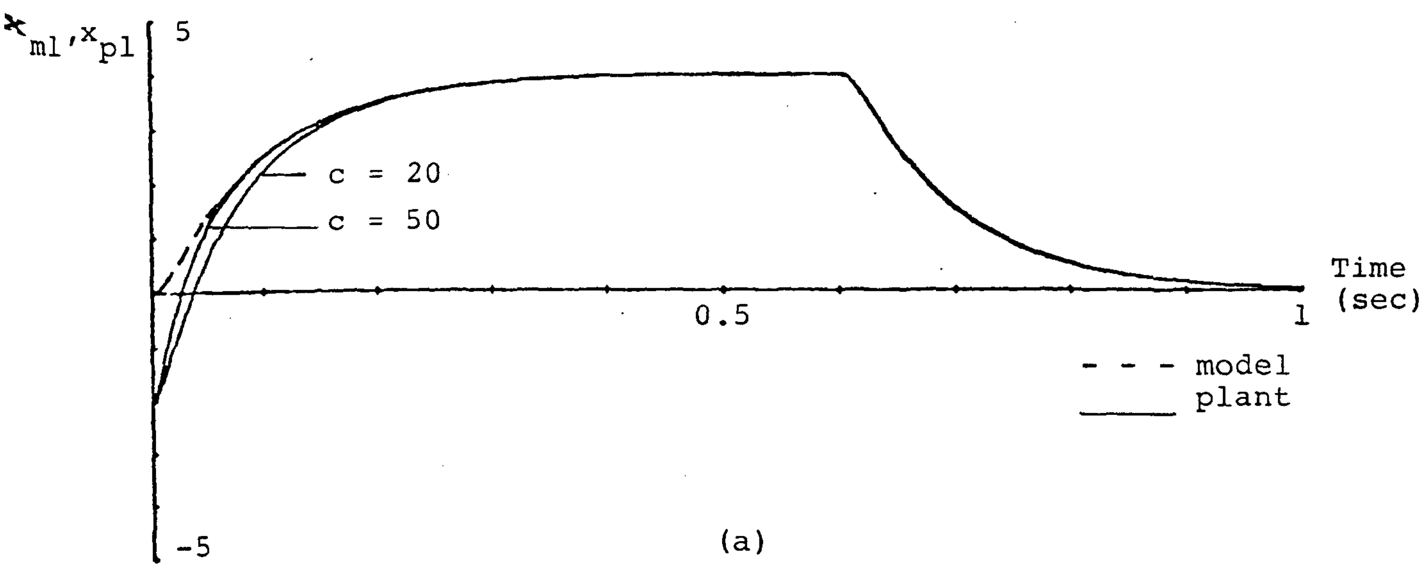
All initial conditions are zero except for x_{p_1}, x_{p_2} where

$$x_{p_1}(0) = 3, \quad x_{p_2}(0) = 3$$

$$\Delta T = 0.02$$

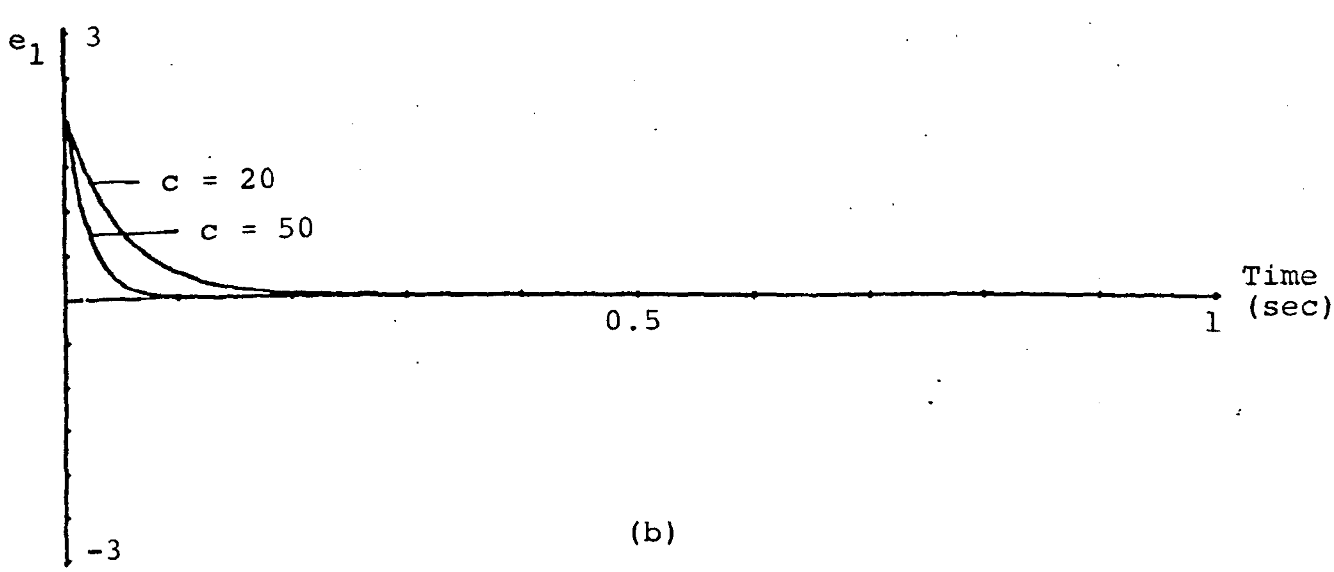
Fig.4.5.5a demonstrates the plant and model behaviour. As can be seen rapid following of the model is achieved. Simulation with more realistic switching hyperplanes where all the elements of c have been rounded to one significant digit have also been carried out and no marked difference is noticed.

Fig.4.5.5b demonstrates sliding on the two hyperplanes. As expected sliding begins on the first switching hyperplane and then on the second. The time step has been decreased to $\Delta T=0.0002$.



(a)

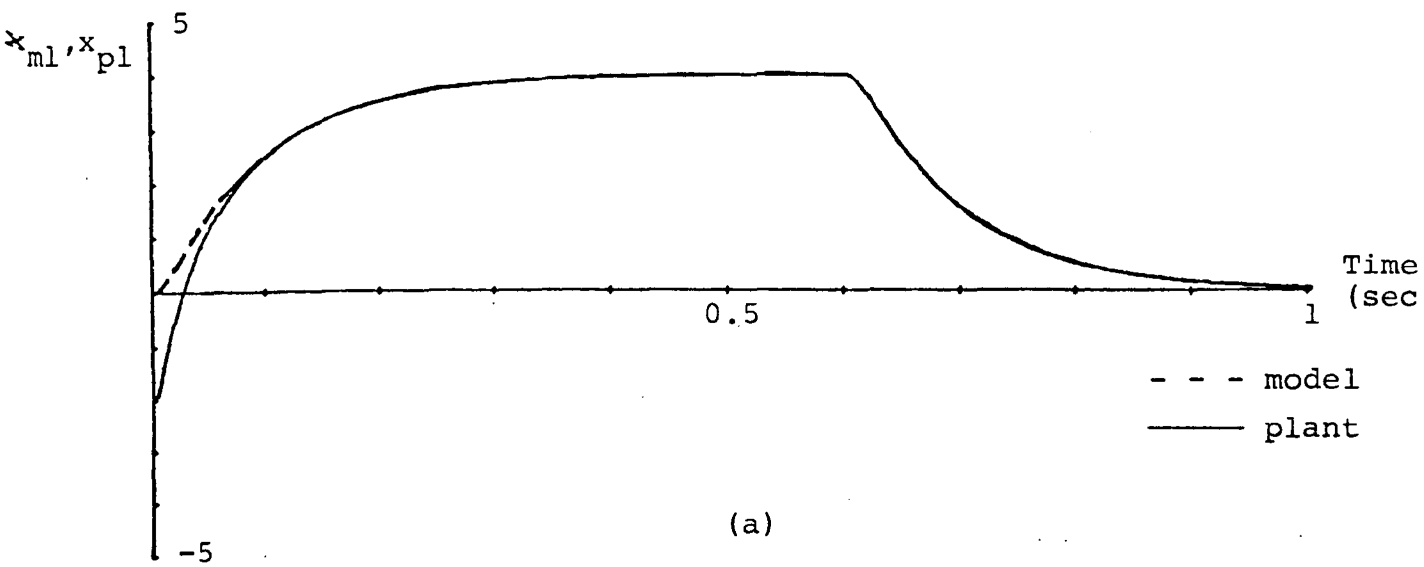
The model and plant response.



(b)

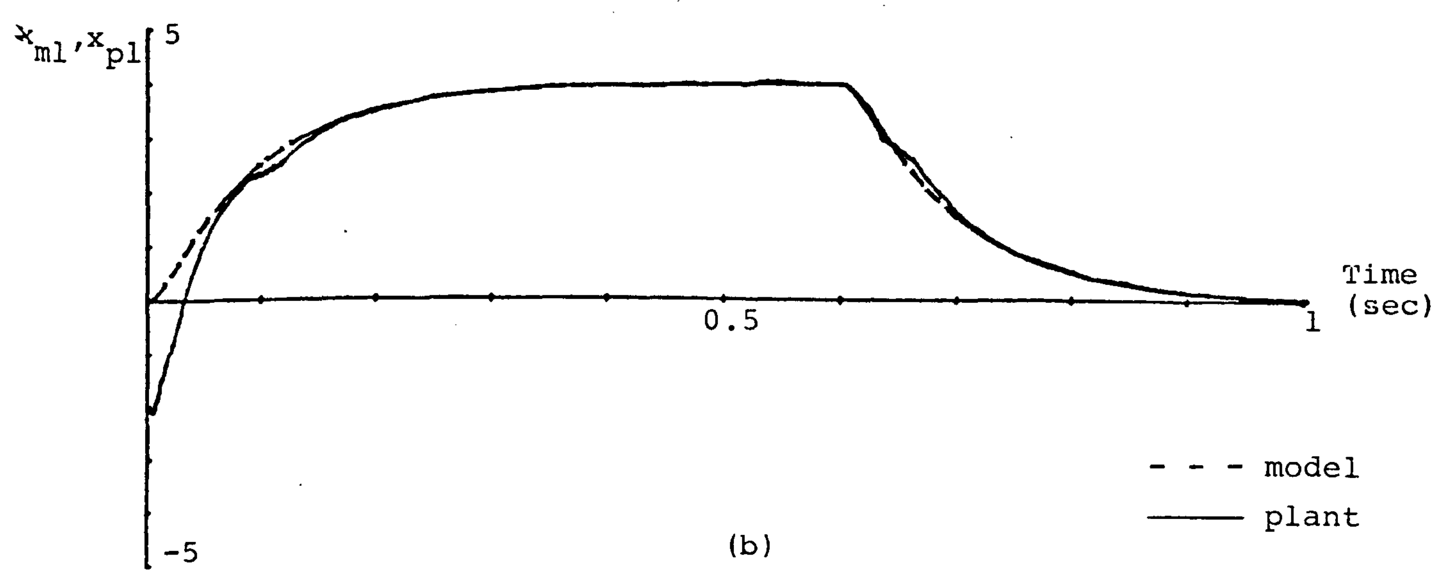
The error between x_{m1} and x_{p1}.

Fig.4.5.1



(a)

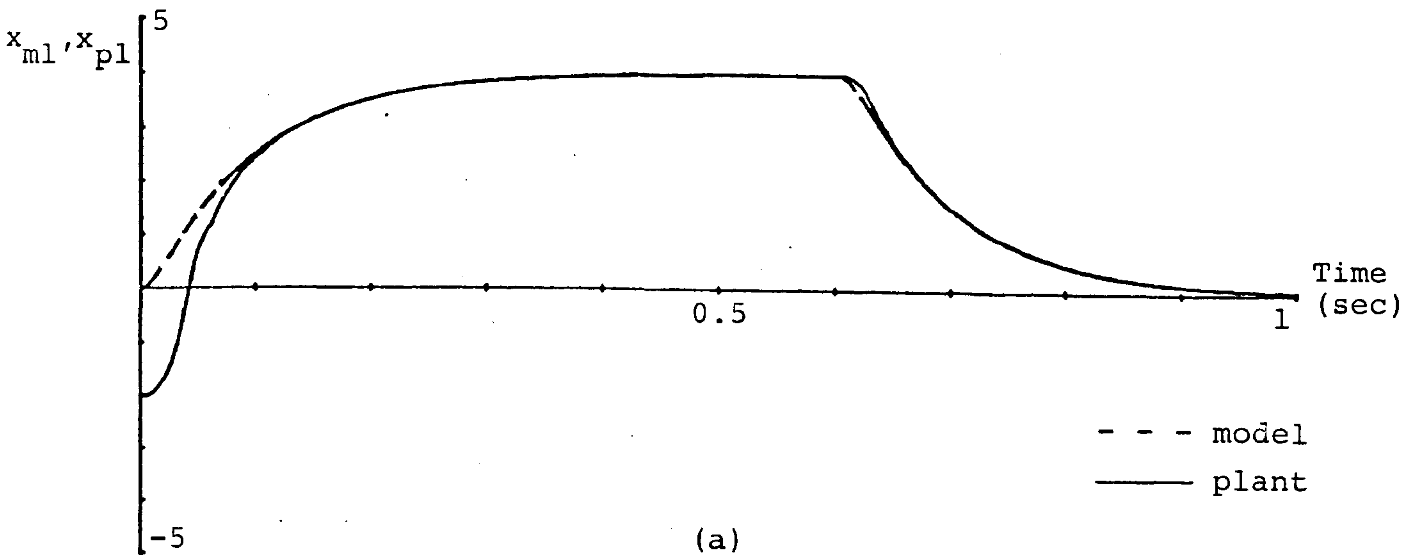
Response of the plant with perturbation in the load moment of inertia.



(b)

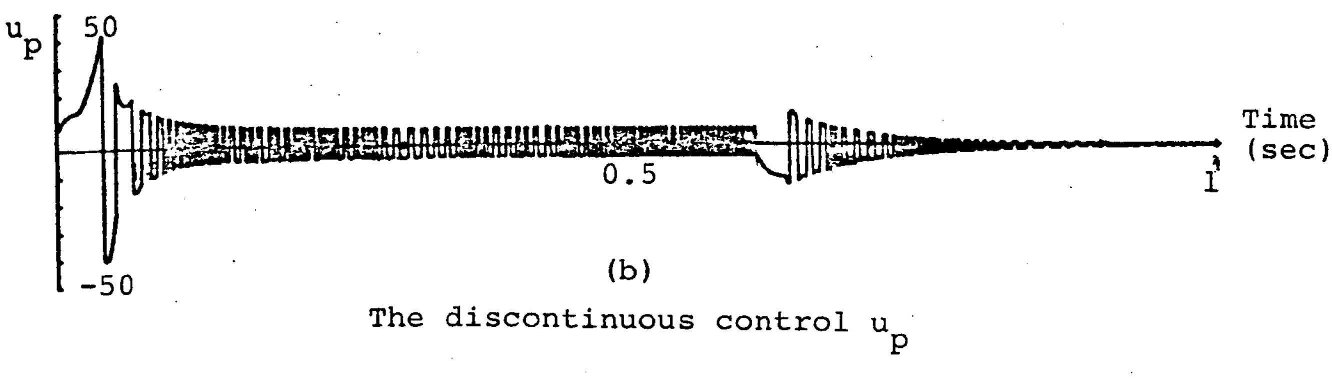
Response of the plant with added disturbance h(t) which does not belong to R(B).

Fig.4.5.2



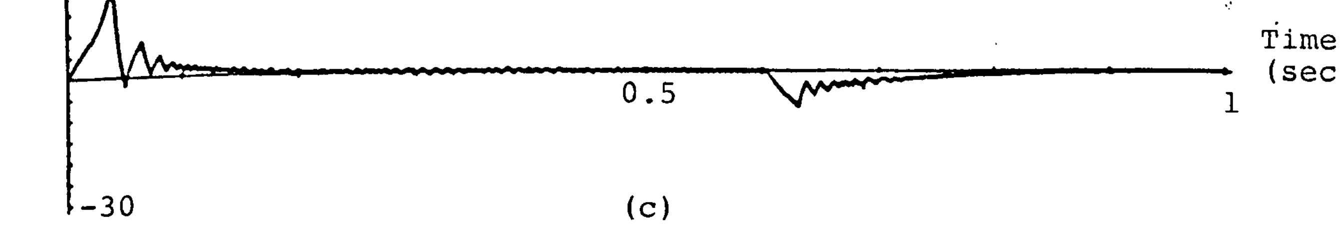
(a)

The plant response with filtered input u_f



(b)

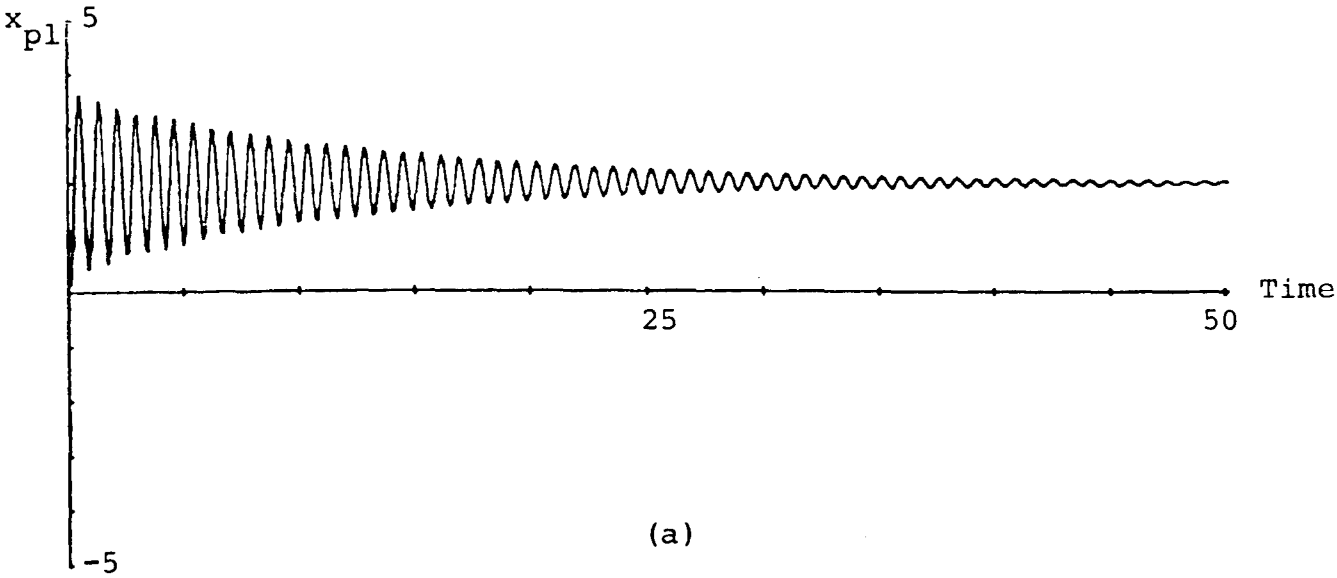
The discontinuous control u_p



(c)

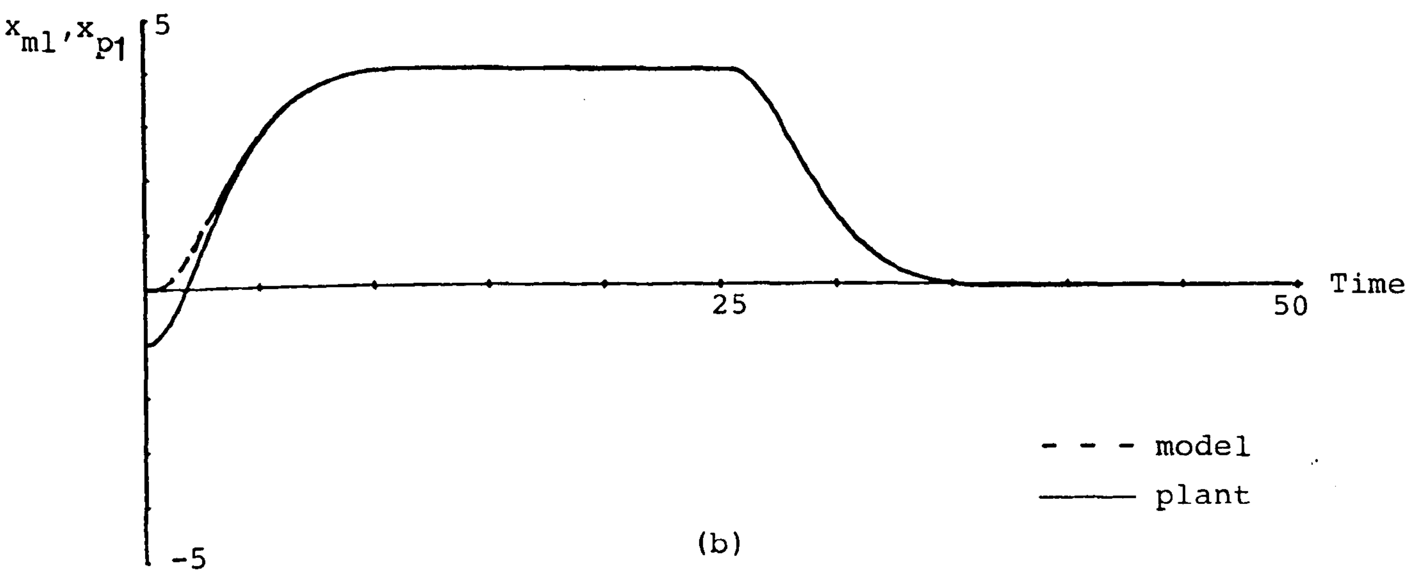
The filtered control u_f

Fig.4.5.3



(a)

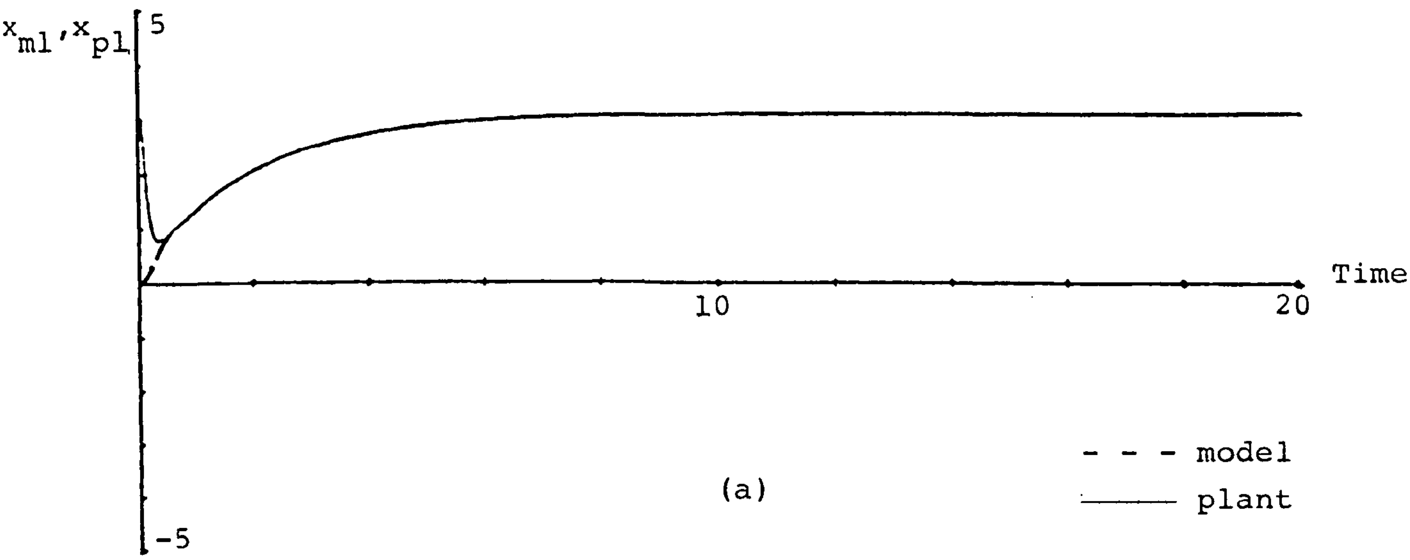
The highly oscillatory uncontrolled plant.



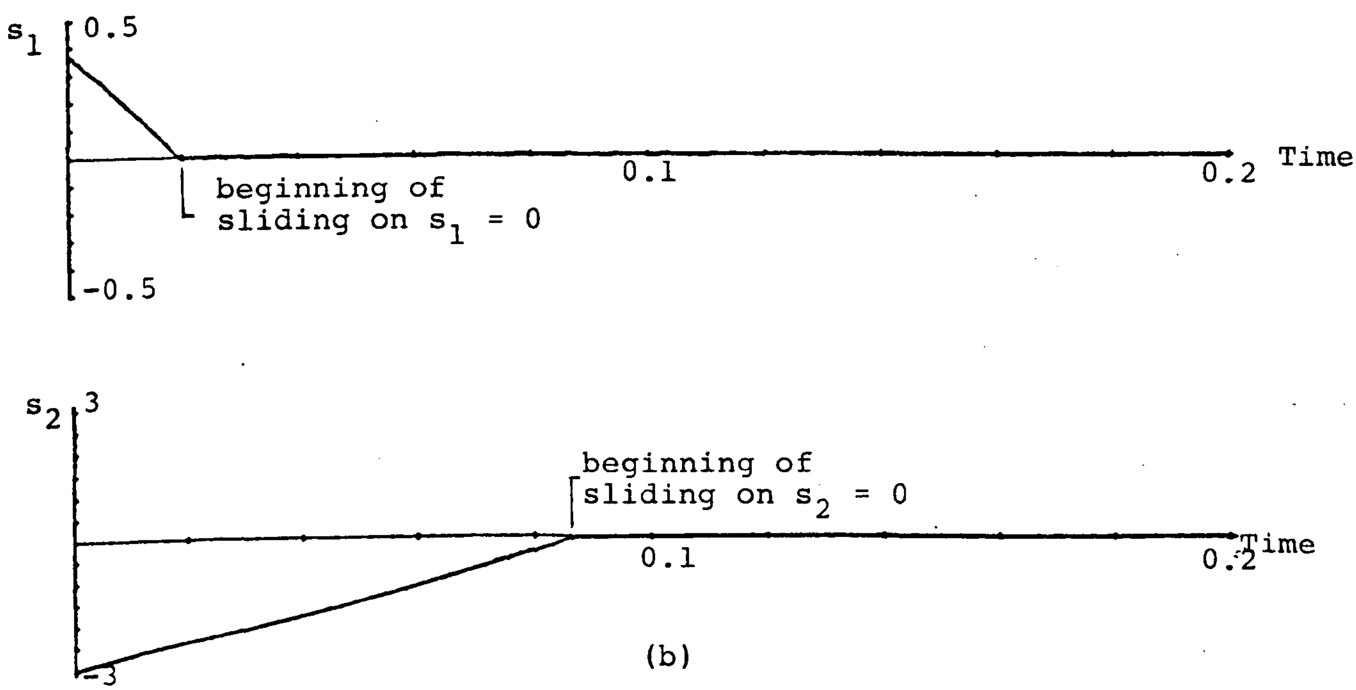
(b)

The model and plant response.

Fig.4.5.4



The model and plant response



Sliding on the switching hyperplanes $s_1 = 0$ and $s_2 = 0$

Fig.4.5.5

CHAPTER FIVE.

THE DETERMINATION OF MULTIVARIABLE SYSTEM ZEROS USING VARIABLE STRUCTURE SYSTEM THEORY.

5.1 Introduction.

Reading through the recent literature dealing with multivariable system zeros, zero directions, the output zeroing problem, unobservable subspaces and output model-following systems, there emerges a close similarity between these concepts and those of variable structure systems (VSS). The output zeroing question which is associated with system zeros and zero directions provides the link between the two.

The output zeroing problem (defined in section 5.2) of the system (S,A,B)

$$\dot{x} = Ax + Bu \qquad , x \in \mathcal{C}^n,\ u \in \mathcal{C}^m \qquad (5.1.1)$$

$$y = Cx \qquad y \in \mathcal{C}^m,\ m<n \qquad (5.1.2)$$

seeks the appropriate choice of state "x_o" and control "u" which makes the m outputs of the system identically zero i.e. $y(t) \equiv 0$ for all $t \geqslant 0$. Associated with this notion are the system zeros and the zero directions (MacFarlane and Karcanias, 1976). The number of system zeros and zero directions is n-m for the case where CB is nonsingular and the state zero directions (x_o) are constrained to lie in the null space of C i.e. $C\,x_o = 0$ These ideas have analogous counterparts in the theory of variable structure systems where

1) The zero output of the system S(A,B,C) can be associated with the zero-valued switching hyperplanes during the sliding

mode. In the VSS case, x_o and u are such that the state is constrained to lie on the intersection of the switching hyperplanes.

2) During the sliding mode the state belongs to the R^{n-m} solution space of $Cx_o = 0$. Consequently n-m independent solutions exist. The subsequent motion of the system is dictated by these n-m modes as is well known in VSS.

In addition to the above observations, a recent approach to model reduction and model-following systems is based on the cancellation of n-m poles of the closed loop system by the n-m zeros of the system (Shaked and Karcanias, 1976), (Shaked, 1977a). This is achieved through forcing n-m modes of the plant to be unobservable. In other words, the corresponding n-m eigenvectors are forced to lie in the null space of the output matrix C. Again, these concepts tie up with the (n-m)-dimensional null space of C in which the state is constrained to lie during the sliding mode. If the switching hyperplanes are considered to be the system outputs then the system is unobservable. The relationship between model-following systems, VSS and systems zeros will be deferred until Chapter 7.

The above observations suggest the possibility of exploiting VSS theory in the sliding mode to yield an algorithm for the determination of system zeros and zero directions. Such an idea is strengthened by the fact that the equivalent control and the control which achieves output model-following of a lower order model (consequently order reduction) are similar. For the moment it suffices to say that knowledge of the plant zeros is necessary in output model-following systems. The new algorithm proposed provides an easy method to do that. Furthermore, the control law upon which this algorithm is based appears as part of the control law which achieves output model-following. This provides a self-sufficient method for tackling the problem of output model-following. These topics will be explored in Chapter 7.

The importance of the proposed algorithm is not confined to the determination of the zeros and the zero directions but goes further to provide a design method for specifying system zeros and state zero directions. Furthermore, having associated the switching hyperplanes with the system output existing methods for zero assignment (Kouvaritakis et al. 1976, Part 2) can be used for the specification of the switching hyperplanes.

The calculation of system zeros has been studied by many authors and numerous methods have been proposed. The concepts underlying such calculations range from the use of high gain output feedback (Davison et al., 1974), (Young et al., 1977) and generalized eigenvalue QZ algorithms (Laub et al., 1977), (Porter, 1979) to geometric formulations (Kouvaritakis et al., 1976), (Owens, 1977) and the use of generalized inverses (Lovass-Nagy et al., 1980). Other algorithms obtain the zeros as the poles of the system minimal order right or left inverse (Patel, 1977) or by pole-zero cancellation techniques (Wolovich, 1977).

In the present study the relationship between variable structure systems (VSS) and the zeros of the square linear time-invariant multivariable system $S(A,B,C)$ described by (5.1.1) and (5.1.2) are investigated. It is assumed for the present that B and C have full rank and $|CB| \neq 0$. The case where CB is singular can be considered with suitable modifications and will be presented in section 5.10 and in Appendix 3.

A new method of computing the zeros of S(A,B,C) is derived by considering the theory of VSS. It will be shown that the system zeros and the state zero directions are eigenvalues and eigenvectors of a matrix which arises naturally in VSS design. The algorithm is computationally simple and yields insight into the operation of VSS in the sliding mode. The algorithm provided offers the advantage over known techniques of the ability to calculate the state and the input zero directions independently of each other and without resorting to the determination of the null space of the (n+m)th order system matrix. The relationship between the VSS algorithm and the NAM algorithm (Kouvaritakis et al., 1976) is discussed together with a new method of zero assignment. The generalization of the algorithm to the case where CB is singular is more complex and will be presented in section 5.10 and Appendix 3.

This chapter is based on two papers by the author (El-Ghezawi et al., 1982a, 1982b).

5.2 System Zeros.

The use of variable structure systems theory in calculating the system zeros is motivated by the observation that variable structure systems in the sliding mode are very closely related to the output zeroing problem.

The output zeroing problem

MacFarlane and Karcanias (1976) state that a necessary and sufficient condition for an input

$$u(t) = g \exp(zt)\, 1(t) \tag{5.2.1}$$

to yield rectilinear motion in the state space of the form

$$x(t) = x_o \exp(zt)\, 1(t) \tag{5.2.2}$$

such that

$$y(t) = 0 \quad \text{for } t \geq 0 \tag{5.2.3}$$

is that

$$\begin{bmatrix} zI-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_o \\ g \end{bmatrix} = 0 = P(z) \begin{bmatrix} x_o \\ g \end{bmatrix} \tag{5.2.4}$$

where z is a system zero, x_o the related state zero direction, g the input zero direction and 1(t) denotes the Heaviside unit step function. At the complex frequency s = z the state (x_o) and input (g) zero directions satisfy

$$\begin{bmatrix} x_o \\ g \end{bmatrix} \varepsilon\ N(P(z)) \tag{5.2.5}$$

where N(P(z)) is the null space or kernel of P(z).

5.3 Zero calculation using VSS theory.

Considering the switching functions s to be the system outputs y, VSS and the output zeroing problem reduce to the selection of a control u and a state vector x such that the output $y(t) = 0$ for $t \geq 0$. Calculation of the system zeros using VSS theory exploits the fact that if $y(t) = 0$ for $t \geq 0$ then $\dot{y}(t) = 0$ for $t \geq 0$. The algorithm which consists of determining the eigenvalues of the matrix A_{eq} which arises when the feedback control yielding $\dot{y}(t) = 0$ is applied to S(A,B,C) is summarised below.

(i) Calculate u using $\dot{y} = C\dot{x} = 0$. This yields

$$C(Ax + Bu) = 0$$

and

$$u = -(CB)^{-1}CAx \tag{5.3.1}$$

which is the equivalent control u_{eq} of VSS theory.

(ii) Substitute in (5.1.1) to yield

$$\dot{x} = [I - B(CB)^{-1}C]Ax = A_{eq}x. \tag{5.3.2}$$

(iii) Determine the eigenvalues and eigenvectors of A_{eq}.

(iv) Any eigenvector x_o^i satisfying $Cx_o^i = 0$ is a state zero direction and the corresponding eigenvalue is a system zero.

(v) The corresponding input zero directions are given by

$$g_i = u_{eq}\Big|_{x=x_o^i} = -(CB)^{-1}CAx_o^i \tag{5.3.3}$$

In practice we need consider only the (n-m) eigenvalues $\lambda_i \in sp(A_{eq}) - \{0\}^m$ in steps (iii) and (iv). This becomes clear in section 5.5 (see equations (5.5.13) and (5.5.15)), since it is evident that only the eigenvectors associated with the (n-m) eigenvalues λ_i of A_{eq} lie in the null space of C and satisfy $Cx_o^i = 0$. (see Appendix 2 for the definition of the null space) The (n-m) eigenvalues λ_i are not necessarily non-zero valued.

5.4 Physical Interpretation.

The (VSS) equivalent control is such that it assigns some of the closed loop eigenvalues and the corresponding eigenvectors to coincide with the system zeros and state zero directions,thus driving the system to be unobservable. To show this consider the observability matrix

$$O(C,A_{eq}) = [\,C^T, (CA_{eq})^T \;, \;\ldots\ldots \;, \;(CA_{eq}^{n-1})^T\,]^T \qquad (5.4.1)$$

Since

$$CA_{eq} = C[I - B(CB)^{-1}C]A = 0 \qquad (5.4.2)$$

it follows that

$$CA_{eq}^{i} = 0 \qquad i = 1, 2, 3, \ldots n-1$$

and

$$\mathrm{rank}[O(C,A_{eq})] = \mathrm{rank}\ (C) = m < n.$$

The action of the equivalent control u_{eq} is therefore to drive the system to be unobservable through the cancellation of system zeros with some closed-loop poles. The system zeros are therefore a subset of the eigenvalues of A_{eq}.

5.5 Decomposition of the state space.

The calculation of the system zeros using the proposed method relies on pole-zero cancellation through appropriate state feedback using the equivalent control u_{eq}. The zeros of the system are given by the eigenvalues of a certain matrix associated with the (n-m) dimensional unobservable subspace. To find this matrix an observability decomposition is employed (Kudva et al., 1976).

Introduce the similarity transformation

$$\bar{x} = Tx \qquad (5.5.1)$$

where

$$T = \begin{bmatrix} C \\ P \end{bmatrix} \begin{matrix} \}m \\ \}n-m \end{matrix} \qquad (5.5.2)$$

C is the output matrix which spans the observable subspace, and P is a matrix chosen to ensure that T is nonsingular. Therefore

$$\dot{\bar{x}} = TA_{eq} T^{-1}\bar{x} \tag{5.5.3}$$

$$y = CT^{-1}\bar{x} . \tag{5.5.4}$$

Define the partitioned inverse of T as

$$T^{-1} = [\underbrace{V}_{m} : \underbrace{W}_{n-m}] \tag{5.5.5}$$

and recall that a generalised inverse S^g of an mxn matrix S of rank m satisfies (Graybill, 1969), (see also Appendix 2).

$$SS^gS = S \text{ and } SS^g = I_m \tag{5.5.6}$$

Using these definitions and exploiting the identify $TT^{-1} = I_n$ it can be readily shown that

$$CV = I_m \tag{5.5.7}$$

$$PW = I_{n-m} \tag{5.5.8}$$

$$PV = 0 \tag{5.5.9}$$

$$CW = 0 \tag{5.5.10}$$

Therefore V and W can be taken to be generalized inverses of C and P such that conditions (5.5.9) and (5.5.10) are satisfied, i.e. $PC^g = 0$ and $CP^g = 0$. From equation (5.5.3)

$$\dot{\bar{x}} = \begin{bmatrix} C \\ P \end{bmatrix} A_{eq}[C^g \; P^g]\bar{x} = \begin{bmatrix} CA_{eq}C^g & CA_{eq}P^g \\ PA_{eq}C^g & PA_{eq}P^g \end{bmatrix} \bar{x} \tag{5.5.11}$$

and

$$y = C[C^g \quad P^g]\bar{x}. \tag{5.5.12}$$

From (5.4.2), $CA_{eq} = 0$ and therefore

$$\dot{\bar{x}} = \begin{bmatrix} 0_m & 0 \\ PA_{eq}C^g & PA_{eq}P^g \end{bmatrix} \bar{x} \tag{5.5.13}$$

$$y = [I_m : 0]\bar{x}. \tag{5.5.14}$$

This is the standard observability decomposition. The zeros are therefore given by the (n-m) eigenvalues of

$$PA_{eq}P^g \tag{5.5.15}$$

Because the eigenvalues are invariant under similarity transformation and because of (5.5.13) and (5.5.15) the n-m zeros can also be obtained as

$$sp(A_{eq}) - \{0\}^m \tag{5.5.16}$$

The calculation of the inverse of T yields V and W. However, if the computation of the matrix inverse is to be avoided, the matrix P should be chosen such that equations (5.5.9) and (5.5.10) are satisfied. Furthermore, since $\mathcal{R}(P^g) \varepsilon N(C)$, a possible choice for P^g is M where $\mathcal{R}(M) = N(C)$. P is then equal to M^g subject to the condition $M^gM = I_{n-m}$. The zeros of the system are therefore equal to the eigenvalues of $M^gA_{eq}M$ where M is a basis matrix for the null space of C.

Since the eigenvectors x_o^i (or state zero directions) lie in the null space of C, they can be expressed as a linear combination of the basis vectors of M,

$$x_o^i = M\alpha_i \qquad \alpha_i \neq 0 \quad i=1,2,\ldots,n-m \tag{5.5.17}$$

Thus

$$A_{eq}x_o^i = A_{eq}M\alpha_i = z_iM\alpha_i$$

where z_i is the zero associated with x_o^i and

$$M^gA_{eq}M\alpha_i = z_iM^gM\alpha_i = z_i\alpha_i \quad . \tag{5.5.18}$$

Therefore α_i is an eigenvector of $M^gA_{eq}M$ corresponding to the zero z_i.

To calculate the state zero directions x_o the eigenvectors α_i of the (n-m) th order matrix $M^gA_{eq}M$ should be determined and substituted in (5.5.17). The input zero directions g_i are given by replacing x_o by $M\alpha_i$ in equation (5.3.3) to yield

$$g = -(CB)^{-1}CAM\alpha_i \quad . \tag{5.5.19}$$

An alternative derivation of (5.5.15) and (5.5.16) can be obtained geometrically (El-Ghezawi et al., 1982b). This is presented in theorem 5.1.

Theorem 5.1.1 For the system S(A,B,C) with $|CB| \neq 0$

(i) the n-m zeros z_i of the system are given by

(a) $\{z_i\} = sp(A_{eq}) - \{0\}^m$

or (b) are the eigenvalues of the (n-m)th order matrix $M^gA_{eq}M$ where M is a basis matrix for N(C) and M^g is a generalized inverse of M.

(ii) the state zero directions x_o associated with the zeros z_i are

(a) the corresponding eigenvectors of the matrix A_{eq} which belong to N(C).

and (b) satisfy $x_o = M\,\alpha_i$ where the α_i are the eigenvectors of $M^gA_{eq}M$.

(iii) the input zero directions are $g_i = -(CB)^{-1}CAx_o = -(CB)^{-1}CAM\alpha_i$.

Proof:

(i) Let ν^* be the maximal (A,B)-invariant subspace in the kernel or null space of C (Wonham, 1979). Then $\nu^* = N(C)$ and since CB is nonsingular

$$R^n = \nu^* \oplus R(B)$$

Since

$$CA_{eq} = CA - (CB)(CB)^{-1}CA = 0 \qquad (5.5.20)$$

it therefore follows that the range space (see Appendix 2) of A_{eq} belongs to N(C) or ν^*. The range of $A_{eq}R^n$ is the same as that of A_{eq}, therefore, $A_{eq}R^n \subset \nu^*$ and in particular

$$A_{eq}\ \nu^* \subset \nu^* \qquad (5.5.21)$$

Since zeros are invariant under state feedback, the n-m zeros of S(A,B,C) and the closed-loop system $S(A_{eq}, B, C)$ are equal and therefore from (5.5.21) the zeros are a subset of the eigenvalues of A_{eq}. The relation $A_{eq}R^n \subset \nu^*$ implies that the other m eigenvalues of A_{eq} are zero-valued. This proves (i)(a).

Furthermore, if z_i is a zero, there exists a non-zero eigenvector $x_o \in N(C)$ such that

$$A_{eq}\ x_o = z_i\ x_o \qquad i = 1,2, \ldots, n-m \qquad (5.5.22)$$

Let M be a basis matrix for N(C) and write $x_o = M\alpha_i$ then

$$A_{eq}M\alpha_i = M\alpha_i z_i \qquad (5.5.23)$$

and

$$M^g A_{eq} M\ \alpha_i = z_i M^g M\alpha_i \qquad (5.5.24)$$

$$= z_i\ \alpha_i \qquad (5.5.25)$$

where M^g is any generalized inverse of M satisfying $M^g M = I_{n-m}$. It follows from (5.5.24) that the zeros of S(A,B,C) are given by the eigenvalues of $M^g A_{eq} M$ which has order n-m. This proves (i)(b).

(ii) This follows from equations (5.5.22) and (5.5.25).

(iii) Substituting $x = x_o = M\alpha_i$ in

$$g_i = -(CB)^{-1}CA\, x_o = -(CB)^{-1}\, CA\, M\, \alpha_i \qquad (5.5.26)$$

which completes the proof of the theorem.

Using section (i)(a) in the theorem yields a technique for determining the (n-m) zeros by finding the eigenvalues of the specified matrix A_{eq}, without the need for calculating the annihilator matrices M and N, c.f. the NAM algorithm of Kouvaritakis and MacFarlane (1976). The method of section (i)(b) has links with the NAM algorithm (see section 5.8). For both cases (a) and (b) the zero-directions are calculated without resorting to the determination of the null space of the (n+m)th order system matrix as defined in Kouvaritakis and MacFarlane (1976).

For the case where n-m is large, method (a) is computationally simpler than (b). For n-m small, i.e. $n \approx m$, method (b) may be a reasonable alternative.

5.6 Classification of the zeros.

Since CB is nonsingular, the system S(A,B,C) has the maximum number of zeros, i.e. n - m, and the zeros are the system zeros (MacFarlane et al., 1976). Since the system is square, all the input, output and input-output decoupling zeros are invariant zeros, and therefore the invariant zeros are the system zeros. The set of transmission zeros, Z_T, is the set of invariant zeros, Z_I, minus the set of decoupling zeros, Z_D.

Thus

$$Z_T = Z_I - Z_D \tag{5.6.1}$$

An output decoupling zero satisfies (MacFarlane et al., 1976)

$$Ap_i = z_i p_i \tag{5.6.2}$$

and

$$Cp_i = 0 \tag{5.6.3}$$

where p_i is a nonzero eigenvector of A associated with the zero z_i. Equation (5.2.4) implies that $g_i = 0$, i.e. the input zero direction is zero.

An input decoupling zero satisfies

$$q_i A = z_i q_i \tag{5.6.4}$$

and

$$q_i B = 0 \tag{5.6.5}$$

where q_i is a nonzero left eigenvector of A associated with the zero z_i. If (5.6.2)-(5.6.5) are satisfied we have an input-output decoupling zero.

Alternatively the output decoupling zeros can be determined by noting that for $g_i = 0$ to hold we need from (5.3.3) $g_i = -(CB)^{-1}CAx_o^i = 0$, and since CB is nonsingular, $CAx_o^i = 0$. The transmission zeros are determined by establishing which zeros do not satisfy (5.6.2)-(5.6.5).

5.7 Zero Assignment.

The problem of zero assignment for a system with given A and B matrices seeks the appropriate selection of an output matrix C which will give rise to desired zeros. The method described below is simple and furthermore, it offers arbitrary specification of CB. This new method relies on finding state zero directions (x_o) associated with the desired zeros. The state zero directions are not completely arbitrary but have to satisfy a certain condition which is formulated in Chapter 6 using projector theory. Some freedom may be possible when selecting x_o and this increases with the number of inputs, or equivalently, with the range of B as will become apparent in Chapter 6. Assuming the x_o are available then the output matrix C can be found as follows (this method is developed in Chapter 6).

Since $\mathrm{col}(x_o) \varepsilon N(C)$ it follows that

$$C = \Gamma x_o^{\perp} \tag{5.7.1}$$

where Γ is an arbitrary nonsingular mxm matrix and $x_o^{\perp}$ is a left matrix annihilator of x_o (i.e. $x_o^{\perp} x_o = 0$). (see Appendix 2).

If no specification of CB is required then C can be taken as

$$C = x_o^{\perp} \tag{5.7.2}$$

If CB is required to assume a certain value S, i.e.

$$CB = S \tag{5.7.3}$$

then Γ is not arbitrary and is found using (5.7.1) and (5.7.3)

$$\Gamma x_o^{\perp} B = S$$

$$\Gamma = S(x_o^{\perp} B)^{-1} \tag{5.7.4}$$

$(x_o^{\perp} B)^{-1}$ always exists since $\mathcal{R}(x_o)$ are always independent of $\mathcal{R}(B)$. Eqns (5.7.1) and (5.7.4) give

$$C = S(x_o^{\perp} B)^{-1} x_o^{\perp} \qquad (5.7.5)$$

The other methods developed in Chapter 6 for the construction of the switching hyperplanes can all be used to obtain C.

5.8 The Relationship with the NAM Algorithm.

The algorithm presented above resembles that of the NAM method of MacFarlane and Kouvaritakis (1976). In both methods the zeros are determined as the eigenvalues of an (n-m)-dimensional matrix. It can be easily shown that the matrix $N_e = M^g[I - B(CB)^{-1}C]$ qualifies for the matrix N in the NAM algorithm. This is because

$$N_e B = 0$$

and

$$N_e M = I_{n-m}$$

where the matrix M is the same in both methods. However, the approach proposed in this paper offers the advantage of calculating the state and input zero directions without resorting to the null space of the (n+m)-dimensional system matrix P(**z**). The above technique involves only certain matrix multiplications in the calculation of the zero directions.

5.9 Computational Aspects.

Two techniques for determining zeros have been described. In Section 5.3 the set of eigenvalues of the nth order matrix A_{eq}, i.e. $sp(A_{eq})$ needs to be determined. The n-m zeros are given by $sp(A_{eq}) - \{0\}^m$. In Section 5.5 the zeros are obtained as the eigenvalues of the (n - m)th order matrix $M^g A_{eq} M$ where M is a basis for the nullspace of C. The choice of method depends upon the values of m and n, and the ease of calculation of the basis M. The method of Section 5.3 is to be favoured for n >> m.

We have assumed throughout the system order to be n and that rank B, rank C and rank (CB) are given. Otherwise they may be determined using suitable algorithms (Davison et al.,1978). Ill-conditioning of (CB), A_{eq} and ker C may cause difficulties. A highly accurate method for determining the zeros such as the QZ-type algorithms for the solution of non-symmetric generalized eigenvalue problems (Laub et al., 1977) (Porter, 1979) should then be used.

5.10 System Zeros for Systems with Singular CB.

The algorithm (Theorem 5.5.1) for calculating the system zeros requires suitable modifications for the case where CB is singular. The zeros can be identified as a subset of the eigenvalues of the matrix

$$A_K = A - B(CA^{K-1}B)^{-1}CA^K \qquad 1 \leq K \leq n \qquad (5.10.1)$$

where K is the smallest integer such that $CA^{K-1}B$ is nonsingular. The choice of (5.10.1) is motivated by the CB nonsingular case. For K=1 eqn (5.10.1) reduces to A_{eq} (see eqn (5.3.2)).

The problem of calculating the zeros and zero directions for the case where CB is singular has been tackled using two different approaches depending upon the form of CB (El-Ghezawi et al., 1982b). The uniform rank case where

$$CA^{i-1}B = 0 \text{ for all } 1 \leq i < K \quad \text{and } |CA^{K-1}B| \neq 0 \qquad (5.10.2)$$

is treated in theorem 5.10.1. The non-uniform rank case is treated in theorem 5.10.2. A necessary condition for the existence of system zeros when CB is singular is given in theorem 5.10.3. Remarks related to theorem 5.10.3 are given in Appendix 3.

Case 1: Uniform Rank.

A system is said to have uniform rank k (Owens, 1979) if

$$CA^{i-1}B = 0 \qquad 1 \leq i < k \tag{5.10.3}$$

and

$$|CA^{k-1}B| \neq 0 . \tag{5.10.4}$$

R^n is then decomposed as the direct sum

$$R^n = R(B) \oplus AR(B) \oplus \ldots \oplus A^{k-1}R(B) \oplus \nu^* \tag{5.10.5}$$

and

$$\nu^* = \bigcap_{i=1}^{k} N(CA^{i-1}) \tag{5.10.6}$$

Theorem 5.10.1: Given the feedback control

$$u = -(CA^{k-1}B)^{-1}CA^kx \tag{5.10.7}$$

and

$$A_k = A - B(CA^{k-1}B)^{-1}CA^k \tag{5.10.8}$$

then

(i) the (n-km) zeros z_i of S(A,B,C) are given by

(a) $\{z_i\} = sp(A_k) - \{0\}^{km}$

or (b) the eigenvalues of the matrix $M_k^g A_k M_k$ where M_k is a basis matrix for ν^*.

(ii) the state zero directions w_i associated with the zeros z_i are

(a) the corresponding eigenvectors of the matrix A_k

and (b) satisfy $w_i = M_k \alpha_i$ where the α_i are eigenvectors of $M_k^g A_k M_k$

(iii) the input zero directions are given by $g_i = -(CA^{k-1}B)^{-1} CA^k M_k \alpha_i$

Proof: see Appendix 3.

Case 2: The Non-Uniform Rank Case.

The uniform rank case is completely resolved in the preceding section. The case of a non-uniform rank system is more complex. The following result does however, identify a condition when Theorems 5.5.1 and 5.10.1 have a natural generalization.

Theorem 5.10.2.

Let $k \geq 2, |CA^{i-1}B| = 0$ for $i=1,2,\ldots k-1, |CA^{k-1}| \neq 0$ and $\nu^* \subset \bigcap_{i=1}^{k} N(CA^{i-1})$. Then, $A_k \; \nu^* \subset \nu^*$ and the zeros of $S(A,B,C)$ are a subset of the eigenvalues of A_k. The zero directions are then calculated as in Theorem 5.10.1.

Proof: see Appendix 3.

A Necessary Condition for the Existence of System Zeros.

Theorem 5.10.3.

A necessary condition for the non-degenerate system $S(A, B, C)$ with $|CB| = 0$ to have zeros is that

$$n > m + d_{max} \qquad (5.10.9)$$

where d_{max} is the maximum of the rank deficiencies d_i of the matrices

$$CA^{i-1}B \qquad i = 1,\ldots,k. \qquad (5.10.10)$$

Proof: see Appendix 3.

5.11 Examples.

Example 1: Consider the system (Kouvaritakis et al., 1976).

$$A = \begin{bmatrix} -1 & 1 & 3 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -3 & -1 \\ 0 & 3 & -1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & -1 & -3 & 2 \end{bmatrix}$$

From (5.3.2)

$$A_{eq} = \frac{1}{3} \begin{bmatrix} -2 & -6 & 6 & 4 \\ 1 & 12 & 42 & -29 \\ 1 & 6 & 12 & -11 \\ 2 & 15 & 39 & -31 \end{bmatrix}$$

From (5.5.16) the zeros are given by $\{0,0,-1,-2\} - \{0,0\} = \{-1,-2\}$

Alternatively

$$P^g = M = \begin{bmatrix} 2 & -6 \\ 2 & 0 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

and from (5.5.18) the zeros are given by the eigenvalues of

$$M^g A_{eq} M = \begin{bmatrix} -0.5 & -1.5 \\ 0.5 & -2.5 \end{bmatrix}$$

i.e., the system has two zeros at -1 and -2.

The eigenvectors α_i of $M^g A_{eq} M$ are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The state zero directions are given by $x_o^i = M\alpha_i$ and hence

$$x_o^1 = [0 \quad 6 \quad 2 \quad 6)^T \text{ and } x_o^2 = [-4 \quad 2 \quad 2 \quad 4]^T$$

The input zero directions (5.5.19) are

$$g_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \quad \text{and } g_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Example 2: Consider the uniform rank system (see Appendix 3).

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 & -4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Since CB is singular the method described in Appendix 3 is used.

$$CB = 0 \text{ and } CAB = \begin{bmatrix} 1 & 0 & -4 & 1 & -4 \\ 0 & 1 & -3 & 1 & -4 \end{bmatrix} B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We therefore use Theorem 5.10.1 with k = 2. Using method (a) the zeros are given by $sp(A_k) - \{0\}^{km} = \{-3,0,0,0,0\} - \{0,0,0,0\}$

$$= -3$$

The algorithm (b) yields

$$\mathcal{R}(M_k) = \nu^* = N(C) \cap N(CA) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \cap \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$$

From (5.10.8)

$$A_k = \begin{bmatrix} 1 & 1 & -9 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ -1 & 3 & -3 & 4 & -16 \\ 0 & 0 & 1 & 1 & -4 \end{bmatrix}$$

$$M_k^g A_k M_k = [0 \quad 1 \quad 0 \quad 0 \quad 0] \quad A_k . \nu^*$$

$$= [1 \quad 0 \quad -5 \quad 0 \quad 0] \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} = -3$$

Therefore the system has a single zero at $z = -3$. The state zero-direction is equal to $w_1 = M_k\alpha_1 = [2 \quad 1 \quad 1 \quad -2 \; -1)^T\alpha_1$ where α_1 is an arbitrary scalar since $n-km=1$. The input zero-direction

$$g_1 = (CAB)^{-1} \, CA^2M_k$$

$$= -(CAB)^{-1}CA.AM_k$$

$$= \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

Example 3: Zero Classification. Consider the system with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [10, \; 7 \; 1] .$$

From (5.3.2)

$$A_{eq} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -7 \end{bmatrix}$$

with eigenvalues $\{0, -2, -5\}$. The zeros are $\{0, -2, -5\} - \{0\} = \{-2, -5\}$ and the corresponding state zero directions are

$$x_o^1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad \text{and } x_o^2 = \begin{bmatrix} 1 \\ -5 \\ 25 \end{bmatrix}$$

Since -5 is not an eigenvalue of A the zero $z_2 = -5$ is a transmission zero. $z_1 = -2$ is an eigenvalue of A. Since $CAx_o^1 = 0$, the input zero direction corresponding to $z_1 = -2$ is zero. So -2 is an output decoupling zero. The output decoupling zero $z_1 = -2$ is not an input decoupling zero since the left eigenvector $q_1 = [3 \quad 4 \quad 1]$ does not satisfy $q_1 B = 0$ (eqn 5.6.5)

<u>Example 4:</u> Zero assignment. Consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \quad C = ?$$

It is required to assign a zero at $z = -3$. Following the method described in Chapter 6 a state zero direction can be found as

$$x_o = [-1 \quad 3 \quad 1]^T$$

i) Case 1: no specification on CB is required. Using (5.7.2)

$$C = x_o^{\perp} = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

ii) Case 2: CB is required to be equal to I_2, i.e. $S = I_2$. Using (5.7.5)

$$C = I_2 (x_o^{\perp} \; B)^{-1} \, x_o^{\perp}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} x_o^{\perp} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

CHAPTER SIX

ANALYSIS AND DESIGN OF VARIABLE STRUCTURE SYSTEMS USING PROJECTOR THEORY.

6.1 Introduction.

Order reduction is a fundamental property of variable structure systems (VSS) in the sliding mode. This is due to the motion of the state which is constrained to lie on the intersection of the m switching hyperplanes. During the sliding mode the order of the system is reduced because the motion of the state is governed by n-m "slow" modes. The remaining m modes are the "fast" modes (see Young et al., 1977).

A new method of analysing VSS in the sliding mode is developed in this chapter. The study has been motivated by the observation that the basic operator $[I-B(CB)^{-1}C]$ associated with A_{eq} qualifies as a projector.

Projector theory provides a neat method for the analysis and design of VSS. Using projector theory certain VSS features are explained and others are expanded. A simple explanation of order reduction is given together with a re-examination of the invariance principle of Draženović (1969). It is shown that Draženović's conditions are a special case of a more general condition although the two conditions are the same when CB is nonsingular. The physical interpretation of invariance is also given. The invariance of the system zeros in the sliding mode is investigated. Perfect model-following is also revisited.

It is found that certain interrelations exist between the n-m closed-loop eigenvectors W (associated with the n-m assigned eigenvalues), A, A_{eq}, B, W^g, B^g and P ($P = B(CB)^{-1}C$). Such interrelations are exploited further when formulating new methods for constructing the matrix C specifying the switching hyperplanes and hence specifying the n-m closed-loop eigenvalues.

Design methods are proposed for the design of the switching hyperplanes. The methods given have the ability to assign the matrix CB arbitrarily. This may be useful as a design option since it has already been established that a diagonally dominant CB matrix ensures the convergence of the fast motion to the switching hyperplanes (Utkin, 1978b). Another advantage is the ability to excercise partial control over the closed loop eigenvectors associated with the n-m assigned eigenvalues. The freedom in selecting these eigenvectors increases with the number of inputs, or equivalently, with increasing range space of B (see El-Ghezawi et al. 1982c).

Throughout this chapter we consider the time-invariant system S(A,B,C).

$$\dot{x} = Ax + Bu \tag{6.1.1}$$

$$s = Cx \tag{6.1.2}$$

where $x \varepsilon R^n$, $u \varepsilon R^m$, $s \varepsilon R^m$. The matrices B and C are assumed to have full rank m and $|CB| \neq 0$. Define

$$P = B(CB)^{-1}C \tag{6.1.3}$$

and then from (3.6.4)

$$A_{eq} = (I-P)A \tag{6.1.4}$$

6.2 Projectors.

6.2.1 Definition: Given a decomposition of space S into subspaces S_1 and S_2 so that for any $x \varepsilon S$

$$x = x_1 + x_2 \quad ; \quad x_1 \varepsilon S_1 \ , \quad x_2 \varepsilon S_2 \qquad (6.2.1)$$

the linear operator P that maps x into x_1 is called a projector on S_1 along S_2, i.e.

$$Px = x_1 \quad ; \quad Px_2 = 0 \qquad (6.2.2)$$

6.2.2 Properties of Projectors.

Some useful properties of projectors are listed below (Pease, 1965):

1) A linear operator P is a projector if and only if it is idempotent, i.e. if

$$P^2 = P \qquad (6.2.3)$$

2) If P is the projector on S_1 along S_2 then (I-P) is the projector on S_2 along S_1 (6.2.4)

3) If P is the projector on R(P) (Range of P) along N(P) (Null space of P) then (I-P) is the projector on N(P) along R(P) (6.2.5)

4) For any $x \varepsilon R(P)$

$$Px = x \qquad (6.2.6)$$

and

$$(I-P)x = 0 \qquad (6.2.7)$$

5) rank (P) = trace (P) (6.2.8)

and

rank (I-P) = n-rank(P) (6.2.9)

6)

$$R(P) = N(I-P) \qquad (6.2.10)$$

and

$$N(P) = R(I-P) \qquad (6.2.11)$$

6.2.3 Relevant examples on Projectors.

It is noted that certain matrix operators encountered in variable structure systems (VSS) are projectors.

1) $B(CB)^{-1}C$ is a projector.

Proof: Since

$$[B(CB)^{-1}C]^2 = B(CB)^{-1}C\ B(CB)^{-1}C = B(CB)^{-1}C$$

$B(CB)^{-1}C$ is therefore idempotent and consequently a projector.

$B(CB)^{-1}C$ projects R^n on $R(B)$ along $N(C)$ because

$$R[B(CB)^{-1}C] = R(B) \qquad (6.2.12)$$

since rank (BK) = rank(B) if B and K are full rank. In our case $K = (CB)^{-1} C$ which is full rank since B and CB are full rank (see Appendix 2). Similarly

$$N[B(CB)^{-1}C] = N(C)$$

Since nullity (HC) = nullity (C) if H and C are full rank where $H = B(CB)^{-1}$ which is full rank.

2) $[I-B(CB)^{-1}C]$ is a projector.

Proof: Either from (6.2.4) or by expanding $[I-B(CB)^{-1}C]^2$ and showing that it is equal to $[I-B(CB)^{-1}C]$

$[I-B(CB)^{-1}C]$ projects R^n on $N(C)$ along $R(B)$.

Since the rank of a matrix is the dimension of its range space then by letting $P = B(CB)^{-1}C$ and from (6.2.12)

$$\text{rank}(P) = \text{rank}(B) = m \qquad (6.2.13)$$

From (6.2.9)

$$\begin{aligned}\text{rank}(I-P) &= n - \text{rank}(P)\\ &= n - m\end{aligned}$$

Therefore any nxn matrix pre-multiplied by $[I-B(CB)^{-1}C]$ will at most have rank n-m.

Both of the above projectors turn out to be of invaluable help in exploring the basic features of Variable Structure Systems.

3) If A is an mxn matrix and A^g is a generalized inverse of A then (see Appendix 2)

$$AA^g,\ A^gA,\ I_m-AA^g,\ I_n-A^gA$$

are all idempotent and therefore projectors (Graybill, 1969). The proof follows immediately from the definition of the generalized inverse of a matrix, i.e.

$$(AA^g)^2 = AA^g AA^g = AA^g$$

AA^g and A^gA are projectors on R(A) and R(A^g) respectively and (I_m-AA^g), (I_n-A^gA) are projectors on N(A^g) and N(A) respectively. Further properties of $B(CB)^{-1}C$ and $[I_n-B(CB)^{-1}C]$ include

a) The matrix $B(CB)^{-1}$ qualifies as a right inverse of C

Proof:

$$\text{Since } C.B(CB)^{-1} = I_m$$

it follows that

$$C^g = B(CB)^{-1}$$

Also $(CB)^{-1}C$ qualifies as a left inverse of B.

b) C^gC projects R^n on R(C^g) or R(B) along N(C)

c) $[I-C^gC]$ projects R^n on N(C) along R(B). In other words the column space of $[I-C^gC]$ is the same as that of N(C).

6.3 Projector Theory and Variable Structure System in the Sliding Mode.

We shall now apply the above theory to VSS.

6.3.1 Order Reduction.

In the sliding mode the equation describing the system is given by

$$\dot{x} = [I-B(CB)^{-1}C]\ Ax = A_{eq}x \tag{6.1.4}$$

Since $[I-B(CB)^{-1}C]$ is a projector it maps all the columns of A on N(C). The order of the system has therefore been reduced because the state vector is now constrained to lie in N(C) which is an (n-m)th dimensional subspace.

6.3.2 The Invariance Principle Revisited.

The invariance principle formulated by Draženović (1969) states that for the system given by

$$\dot{x} = Ax + Bu + Df \tag{6.3.2}$$

$$s = Cx \tag{6.3.3}$$

to be invariant to disturbance $f \varepsilon R^{\ell}$ in the sliding mode; the columns of matrix D should belong to the range space of B i.e. $R(D) \subset R(B)$. This principle will now be re-examined and a more general version derived. This generalization extends the theory to the case where CB is singular (assuming sliding exists).

Theorem 6.3.1: The system given by (6.3.2) and (6.3.3) is invariant with respect to the disturbance f in the sliding mode if

$$\text{col}(D) \varepsilon\ R[B(CB)^{-1}C] \tag{6.3.4}$$

or

$$\text{col}(D) \varepsilon\ R[B(CB)^{g}C] \quad \text{if CB is singular} \tag{6.3.5}$$

where col( ) stands for columns of ( ).

Proof: The system in the sliding mode satisfies

$$\dot{x} = [I-B(CB)^{-1}C]Ax + [I-B(CB)^{-1}C]Df \tag{6.3.6}$$

For the system to be invariant to f, $[I-B(CB)^{-1}C]D$ should be zero. Suppose $|CB| \neq 0$. If

$$\text{col}(D) \varepsilon\ R[B(CB)^{-1}C]$$

then, since $B(CB)^{-1}C$ is a projector, the property (6.2.7) gives

$$[I-B(CB)^{-1}C]D = 0$$

as required.

Conversely, if

$$[I-B(CB)^{-1}C]D = 0$$

then

$$\mathcal{R}(D) \subset N[I-B(CB)^{-1}C]$$

and from (6.2.10)

$$\mathcal{R}(D) \subset R[B(CB)^{-1}C]$$

For $|CB| = 0$ we can replace $(CB)^{-1}$ by $(CB)^g$ in the above proof.

The condition (6.3.4) is identical to that given by Draženović. This is so because

$$R[B(CB)^{-1}C] = R(B)$$

<u>Remark</u> 6.3.1 $B(CB)^gC$ can easily be shown to be a projector.

<u>Remark</u> 6.3.2 When CB is singular the invariancy is weakened since

$$R(B(CB)^gC) \subset R(B)$$

Therefore in this case there will be no rejection in the sliding mode to any disturbance that belongs to R(B) and not to $R(B(CB)^gC)$.

<u>Remark</u> 6.3.3 It is well known that the scalar system

$$\dot{x}_i = x_{i+1} \qquad i = 1, \ldots, n-1$$

$$\dot{x}_n = - \sum_{i=1}^{n} a_i x_i + bu$$

is invariant to parameter variations when it is in the sliding mode. This is because all the parameter variation A_v are in the range of B where $B = [0,0,\ldots,b]^T$.

6.3.3 <u>Physical Interpretation of Disturbance Invariance.</u>

Physical insight into the invariance principle is achieved using projector theory. Let $P = B(CB)^{-1}C$ in the rest of this chapter.

From the previous definitions, the projector P decomposes the state space X into the direct sum

$$X = R(P) \oplus N(P) \tag{6.3.7}$$

or

$$X = R(P) \oplus R(I-P) \tag{6.3.8}$$

Alternatively

$$R(P) \cap R(I-P) = \{0\} \tag{6.3.9}$$

Since $x \in R(I-P)$ during sliding, for x not to be affected by any disturbance f, the disturbance should lie in the complementary subspace of (I-P), i.e. $f \in R(P)$ which is the condition of invariance.

6.3.4 Effect of Sliding on the System Zeros.

Young (1977) has shown for scalar variable structure systems that the system zeros are unaffected by the sliding mode. This is to be expected since the sliding mode results from state feedback, and it is well known that state feedback cannot affect the system zeros (Kouvaritakis et al., 1976). However, it is instructive to demonstrate that this is indeed the case.

Given a system S(A,B,C) we wish to show that the system zeros are not affected by the organization of a sliding mode on the intersection of the hyperplane

$$s = Gx = 0 \tag{6.3.10}$$

We note that sliding results from the application of state feedback

$$u_{eq} = -(GB)^{-1} GAx \tag{6.3.11}$$

which yields the closed loop system

$$\dot{x} = [A-B(GB)^{-1}GA]x = A_G x \tag{6.3.12}$$

Let us prove that the zeros of $S(A_G,B,C)$ are identical to the zeros $S(A,B,C)$. The zeros of $S(A_G,B,C)$ are given by the n-m eigenvalues of (see Chapter 5)

$$M^g[I-B(CB)^{-1}C]\ A_GM \tag{6.3.13}$$

$$= M^g[I-B(CB)^{-1}C]\ [I-B(GB)^{-1}G]\ AM$$

$$= M^g[I-B(GB)^{-1}G-B(CB)^{-1}C + B(CB)^{-1}CB(GB)^{-1}G]AM$$

$$= M^g[I-B(CB)^{-1}C]\ AM \tag{6.3.14}$$

But the eigenvalues of (6.3.14) are the zeros of the systems $S(A,B,C)$ and $S(A_G,B,C)$. Therefore sliding does not alter the system zeros.

6.4 The Perfect Model-Following Conditions Revisited.

In Chapter 4 the perfect model-following conditions were given (see 4.2.7) and it was pointed out that these conditions are the necessary and sufficient conditions for a solution of a system of linear equations to exist. The same conditions can be obtained using projector theory as will be demonstrated below. If the control

$$u_p = B_p^g\ (A_m-A_p)x_p + B_p^g\ B_m\ u_m \tag{6.4.1}$$

were obtained without regard to conditions (4.2.7) and then substituted in the original system given by (4.2.1) to yield

$$\dot{x}_p = A_px_p + B_p(B_p^g\ (A_m-A_p)\ x_p + B_p^g\ B_m\ u_m) \tag{6.4.2}$$

adding and subtracting $A_m\ x_p$

$$\dot{x}_p = A_m\ x_p - (I-B_pB_p^g)(A_m-A_p)x_p + B_pB_p^gB_mu_m \tag{6.4.3}$$

but for model-following to result eqn (6.4.3) should assume the form

$$\dot{x}_p = A_m x_p + B_m u_m \qquad (6.4.4)$$

Comparing (6.4.3) and (6.4.4) yields

$$(I-B_pB_p^g)(A_m-A_p) = 0 \qquad (6.4.5)$$

and

$$B_pB_p^gB_m = B_m \qquad (6.4.6)$$

Since $B_pB_p^g$ is a projector with range $R(B_p)$, it follows that both $\mathcal{R}((A_m-A_p))$ and $\mathcal{R}(B_m)$ should belong to $R(B_p)$ in order to satisfy (6.4.5) and (6.4.6). These are the same conditions as those in (4.2.7).

6.5 Further Insight into Variable Structure Systems.

It is now apparent that projector theory provides a neat method of studying many properties of VSS in the sliding mode. It also exposes the relationships between recurring themes associated with VSS in the sliding mode. Such themes involve the n-m closed-loop eigenvectors W of A_{eq}, the input matrix B and the projector matrix $P(P = B(CB)^{-1}C)$ together with the generalized inverses of W and B.

The relationships obtained in this section will be of invaluable help when formulating the new methods for constructing the switching hyperplanes matrix C.

6.5.1 The Relationship Between B, W and P.

Lemma 6.5.1. The closed-loop eigenvectors W of A_{eq} are independent of the columns of B, i.e.

$$\mathcal{R}(W) \cap \mathcal{R}(B) = \{0\} \qquad (6.5.1)$$

Proof:

The nonsingularity of CB implies that the columns of B are independent of N(C) and since $W \subseteq N(C)$ then (6.5.1) is established.

Q.E.D.

<u>Theorem 6.5.1</u>

In VSS the selected generalized inverses of B and W should satisfy

$$B^g W = 0 \tag{6.5.2}$$

and

$$W^g B = 0 \tag{6.5.3}$$

Proof:

Since $\mathcal{R}(B)$ is the range space of the projector P and $\mathcal{R}(W)$ is the null space of P then (Hohn, 1973),

$$P[B\ W] = [B\ 0] \tag{6.5.4}$$

From (6.5.1) $\mathcal{R}(B) \cap \mathcal{R}(W) = \{0\}$ and therefore the inverse of [B W] always exists. Thus P is given by

$$P = [B\ 0]\ [B\ W]^{-1} \tag{6.5.5}$$

Let
$$T = [B\ W] \tag{6.5.6}$$

and
$$T^{-1} = \begin{bmatrix} F \\ G \end{bmatrix} \tag{6.5.7}$$

then using $T^{-1}T = I_n$ it can be shown that (c.f. Chapter 5)

$$F = B^g \qquad (6.5.8$$

and
$$G = W^g \qquad (6.5.9$$

such that

$$B^g W = 0 \tag{6.5.2}$$

$$W^g B = 0 \tag{6.5.3}$$

Q.E.D.

Substituting F(6.5.8) and G(6.5.9) in (6.5.7) and then in (6.5.5) we get

$$P \;=\; BB^g \tag{6.5.10}$$

If the calculation of the inverse of the matrix T is to be avoided, P should be obtained either from (6.5.10) subject to condition (6.5.2) or from (6.5.4) as the solution of

$$PW \;=\; 0 \tag{6.5.11}$$

The solution of (6.5.11) is (see Appendix 2)

$$P \;=\; H\,(I - WW^g) \tag{6.5.12}$$

where H is an nxn arbitrary matrix and W^g satisfies (6.5.3).

6.5.2 The Relationship Between P and A_{eq}.

Using

$$A_{eq} = (I-P)A \tag{6.1.4}$$

and multiplying both sides by (I-P) gives

$$(I-P)A_{eq} = (I-P)^2A = (I-P)A = A_{eq} \tag{6.5.13}$$

Therefore A_{eq} is (I-P)-invariant or equivalently the columns of A_{eq} belong to R(I-P). This implies that

$$PA_{eq} \;=\; 0 \tag{6.5.14}$$

Eqn (6.5.14) can also be obtained by multiplying both sides of (6.1.4) by P.

From (6.1.4), assuming $|A| \neq 0$,

$$(I-P) = A_{eq}A^{-1} \tag{6.5.15}$$

which establishes that $A_{eq}A^{-1}$ is a projector since (I-P) is a projector. From (6.5.15)

$$P = I - A_{eq}A^{-1} \tag{6.5.16}$$

By multiplying both sides of (6.1.4) by B^g and noting from (6.5.10) that $P = BB^g$ we get

$$B^g A_{eq} = 0 \qquad (6.5.17)$$

6.6 Projector theory and the Design of VSS in the Sliding Mode.

The utilization of projector theory in the design of VSS in the sliding mode appears to be promising as it leads to new methods for constructing the switching hyperplanes.

The problem of selecting the switching hyperplanes with desired design objectives can be easily solved using projector theory. Desired design objectives may encompass

(i) Arbitrary eigenvalue assignment.

(ii) Arbitrary specification of CB.

(iii) The choice (partially) of the eigenvectors of Aeq.

The freedom in choosing the assigned eigenvectors is partial and the degree of freedom increases with increasing R(B).

Existing design methods (Young et al., 1977) and (Utkin et al., 1978) cater for case (i) above. All the proposed methods require the availability of the closed-loop eigenvectors W. The determination of these eigenvectors will be described in the following section.

6.7 The determination of the Eigenvectors W.

The design methods described in section 6.8 for the construction of the switching hyperplanes require the availability of the closed-loop eigenvectors W. A well known fact related to linear feedback systems eigenvalue-eigenvector assignment question is that

$$(A + BK)W = WJ \tag{6.7.1}$$

where K is an mxn feedback matrix chosen to yield the desired closed-loop poles specified by the eigenvalues of J (Sinswat et al., 1977). The matrix J may be diagonal or have Jordan block form. If rank (K) = m then eqn (6.7.1) implies that

$$R(AW - WJ) \subset R(B) \tag{6.7.2}$$

The problem of arbitrary eigenvector assignment has been tackled by Shah et al. (1975) where it has been shown that, in general, it is impossible to specify all components of any one eigenvector arbitrarily using state feedback. In matrix form (6.7.2) is equivalent to

$$AW - WJ = BL \tag{6.7.3}$$

where L is an arbitrary mx(n-m) matrix chosen to provide linear combinations of the columns of B. This influences the solution of W and provides partial control over the n-m eigenvectors W. The eigenvectors should be independent of B, i.e. they satisfy

$$R(W) \cap B = \{\emptyset\} \tag{6.7.4}$$

The author knows of no general algorithm for obtaining W which satisfies both (6.7.3) and (6.7.4). Therefore the solution of (6.7.3) which also satisfies (6.7.4) will be determined algebraically utilizing the structure of the given system. The solution can be obtained by allowing each eigenvector W_i to satisy (6.7.3) separately. Two cases are considered.

Case 1: J is diagonal.

The eigenvector W_i is found as the solution of

$$(A - \lambda_i I) W_i = b_i \tag{6.7.5}$$

where b_i is a linear combination of the columns of B. This is achieved through the matrix L which is specified by the designer. A solution always exists if the assigned eigenvalue λ_i is not an eigenvalue of A. In this case $(A-\lambda_i I)$ is nonsingular and W_i is given by

$$W_i = (A - \lambda_i I)^{-1} b_i \tag{6.7.6}$$

The controllability of the pair (A, B) always guarantees the existence of at least one eigenvector W_i obtained from (6.7.6) which is independent of B. If n-m>m then at most only m eigenvectors independent of B can be obtained. This is to be observed when identical λ_i's are to be assigned. In this case the λ_i's are specified by a Jordan block and the method in Case 2 should be used (see Example 1).

However, if $|A-\lambda_i| = 0$ then λ_i is an eigenvalue of A and a solution to (6.7.5) exists if

$$\text{rank } [(A-\lambda_i I), b_i] = \text{rank } (A-\lambda_i I) \tag{6.7.7}$$

If condition (6.7.7) is satisfied then by letting $A-\lambda_i I = A_s$

$$W_i = A_s^g b_i + (I - A_s^g A_s)h \tag{6.7.8}$$

where h is an arbitrary nx1 vector which may be utilized to ensure $W_i \notin \mathcal{R}(B)$ and to provide additional freedom in the selection of W_i. If condition (6.7.7) is not satisfied for a given b_i then it may be satisfied with another linear combination of the columns of B.

Otherwise b_i can be taken as zero. This is justified since the zero vector trivially belongs to R(B). In this case W_i will be an eigenvector of A corresponding to λ_i. Any control over the selection of W_i is completely lost in this case.

Case 2: J is a Jordan block.

In this case J has the usual Jordan block form. After finding the first eigenvector using the procedure described for Case 1 the generalized eigenvectors are obtained from

$$(A-\lambda_i I)W_{i+1} - W_i = b_K$$

or equivalently

$$(A-\lambda_i I)W_{i+1} = b_K + W_i \qquad i=1,\ldots,d-1 \qquad (6.7.9)$$

where b_K is a suitable linear combination of the columns and B and d is the total number of the repeated eigenvalues of J.

The solution of (6.7.9) now follows as in Case 1.

6.8 The Construction of the Switching Hyperplanes C.

The problem of constructing the switching hyperplanes constitutes a special case in the more general problem of pole-assignment. The switching hyperplanes matrix C is to be chosen such that A_{eq} will always have m zero-valued eigenvalues and n-m eigenvalues specified by the designer. Therefore any eigenvalue assignment method can be used. However, a reduction in the computational effort involved especially in the case when $m \approx n$ can be obtained using properly adapted eigenvalue placement algorithms (Young et al., 1977 and Utkin et al., 1978). Other design methods select the switching hyperplanes which minimize the quadratic functionals (Utkin et al., 1978).

$$I_1 = \int_{t_s}^{\infty} x^T Q x \, dt$$

or

$$I_2 = \int_{t_s}^{\infty} (x^T Q x + u_{eq}^T R u_{eq}) \, dt$$

where Q and R are positive semi-definite symmetric matrices and t_s is the starting time of the sliding mode.

In addition, methods available for zero assignment given matrices A, B can also be used to obtain the matrix C (Kouvaritakis et al., 1976, part 2).

Method I: The "B^g" method.

Let the matrix C satisfy

$$CB = S \tag{6.8.1}$$

where S is an arbitrary mxm nonsingular matrix and

$$CW = 0 \tag{6.8.2}$$

A solution to (6.8.1) always exists, since B is full rank, giving the particular solution

$$C = SB^g \tag{6.8.3}$$

This solution also satisfies (6.8.2) since it is required from (6.5.2) that $B^gW = 0$. A systematic method of finding B^g which will always satisfy $B^gW = 0$ is by constructing the inverse of $[B\ W]^{-1}$. The first m rows of this inverse gives B^g such that $B^gW = 0$ (see (6.5.5)-(6.5.9)).

Remark 6.8.1: It can be shown that the direct calculation of P is not necessary for the determination of C as the solution will always depend on B^g.

Since

$$B(CB)^{-1}C = BS^{-1}C = P \tag{6.8.4}$$

a solution for C always exists since BS^{-1} and P in (5.8.4) have the same range (see (6.2.12)). A particular solution is

$$C = SB^{g}P \tag{6.8.5}$$

From (6.5.10) $P = BB^{g}$ and therefore

$$C = SB^{g}BB^{g} \tag{6.8.6}$$

Since

$$B^{g}B = I_m \tag{6.8.7}$$

it follows that

$$C = SB^{g} \tag{6.8.8}$$

which is independent of P.

Method II: The "W" method.

Here C is determined directly from the nx(n-m) eigenvector matrix W.

Since

$$\mathcal{R}(W) \subseteq N(C) \tag{6.8.9}$$

it follows that

$$C = \Gamma W^{\perp}$$

where Γ is an arbitrary nonsingular mxm matrix and $W^{\perp}$ is the annihilator of W (i.e. $W^{\perp}W = 0$). If the value of CB is immaterial, Γ can be chosen arbitrarily. However, if CB is required to assume a certain value S, Γ must be determined.

$$CB = S = \Gamma W^{\perp}B \tag{6.8.11}$$

$$\Gamma = S(W^{\perp}B)^{-1} \tag{6.8.12}$$

The inverse of $(W^{\perp}B)$ always exists since $\mathcal{R}(W) \cap \mathcal{R}(B) = \{0\}$

Therefore

$$C = S(W^{\perp}B)^{-1}W^{\perp} \tag{6.8.13}$$

The C calculated using this method is also equal to SB^g. This is because $(W^{\perp}B)^{-1}W^{\perp}$ qualifies as a generalized inverse of B and satisfies

$$(W^{\perp}B)^{-1}W^{\perp}\, W = B^gW = 0 \tag{6.8.14}$$

The matrix $(W^{\perp}B)^{-1}W^{\perp}$ will always qualify as a B^g that satisfies $B^gW = 0$ irrespective of the choice of $W^{\perp}$.

6.9 Examples.

In all these examples A and B are given. It is required to find the switching hyperplanes matrix C which will assign the specified eigenvalues of J to A_{eq}.

Example 1.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We wish to assign two repeated eigenvalues at -4. Since λ_i's are identical and n-m>m we should use the Jordan block form

$$J = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}$$

We next calculate W the eigenvector matrix. Invoking condition (6.7.2) for each eigenvector separately,

$$AW_1 + 4W_1 = b\alpha_1$$

$$AW_2 - W_1 + 4W_2 = b\alpha_2$$

where α_1, α_2 are arbitrary scalars. We solve for W_1, taking the arbitrary $\alpha_1 = 1$.

$$W_1 = (A + 4I)^{-1} b$$

$$W_1 = \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ -6 & -8 & -4 \\ 24 & 38 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W_1 = \frac{1}{6}\begin{bmatrix} 1 \\ -4 \\ 16 \end{bmatrix}$$

$$W_2 = (A + 4I)^{-1} (\alpha_2 b + w_1)$$

$$= (A + 4I)^{-1} (\alpha_2 b + w_1)$$

taking α_2 =-16 a generalized eigenvector W_2 can be found as

$$W_2 = \frac{1}{6}\begin{bmatrix} 3 & 2 & 1 \\ -6 & -8 & -4 \\ 24 & 38 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$$

$$W_2 = \frac{1}{6}\begin{bmatrix} -5 \\ 26 \\ -128 \end{bmatrix}$$

Using the W method (method II) with CB = 1 we get

$$C = [W_1 \; W_2]^{\perp} = \frac{1}{6}\begin{bmatrix} 1 & -5 \\ -4 & 26 \\ 16 & -128 \end{bmatrix}^{\perp}$$

$$= [16 \quad 8 \quad 1]$$

As a check, the resulting eigenvalues of A_{eq} are given by

$$sp(A_{eq}) = Sp \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -16 & -8 \end{bmatrix} = \{0, \; -4, \; -4\}$$

Two eigenvectors associated with the two eigenvalues at -4 can be easily checked to be equal to W_1 and W_2.

Example 2.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

It is required to assign a single eigenvalue at $\lambda = -1$ so $J = -1$. Note that A has three eigenvalues at -1

$$(A-\lambda I) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A_s$$

By choosing $\ell_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where ℓ_1 is the first column of L the consistency condition (6.7.7) is satisfied and a solution exists which is given by (6.7.8).

$$W = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^g \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (I_n - A_s^g \, A_s)h$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$\therefore \; W_1 = \begin{bmatrix} h_1 \\ -1 \\ 1 \end{bmatrix}$$

where h_1 is arbitrary. Furthermore h_1 should be non-zero in order to satisfy

$$W_1 \notin R(B)$$

Let $h_1 = 1$

Using method II with $CB = I_2 = S$

$$C = I_2 \, (W_1^{\perp} B)^{-1} \, W_1^{\perp}$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Note that if method I has been used then the B^g which ensures $B^g W = 0$ will be obtained as

$$B^g = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

giving

$$C = SB^g = I_2 B^g = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Example 3.

The matrices A and B are the same as A_m and B_p used in example 3 in Chapter 4.

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ 10.1952 & -4.29 & 9.998 & -13.802 \\ 1 & 0 & -5 & 2 \\ 2.2037 & 4.273 & 3.343 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}$$

The solution of (6.7.3) with

$$J = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

gives $$W = \begin{bmatrix} -0.3235 & 0.3891 \\ 0.3891 & 0.3936 \\ 0.3781 & -0.6878 \\ -0.2155 & -0.0480 \end{bmatrix}$$

Since the two eigenvectors W are independent and $R(B) \cap R(W) = \{0\}$ then using (6.5.5) the matrix P can be determined together with B^g which is given by the first two rows of the inverse of $[B \;\; W]^{-1}$. The matrix CB is assigned the value

$$CB = S = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Using method I, B^g is found as the first two rows of

$$[B \; W]^{-1} \begin{bmatrix} -0.0751 & 0.1128 & 0 & 0.3163 \\ -0.5386 & 0.0098 & -0.3178 & 0.2686 \\ -0.8167 & 0.5016 & 0 & -2.5077 \\ 1.8908 & 0.4171 & 0 & -2.085 \end{bmatrix}$$

$$C = SB^g$$
$$= \begin{bmatrix} 0.0751 & -0.1128 & 0 & -0.3163 \\ 1.0773 & -0.0196 & 0.6357 & -0.5373 \end{bmatrix}$$

CHAPTER SEVEN

ADAPTIVE OUTPUT MODEL-FOLLOWING CONTROL SYSTEMS.

7.1 Introduction.

The adaptive model-following of a plant which has the same order as the model has been treated in detail in Chapter 4. The output adaptive model-following of a plant to a lower order model with full state feedback is more complex and warrants detailed attention. Output model-following has been studied by Tyler (1964) Erzberger (1968), Markland (1970) and Kudva et al (1976). The importance of the system zeros was not recognised in the above papers and in some cases unstable systems resulted, although the perfect model-following conditions were satisfied as was the case in Kudva et al. (1976). This was due to the plant having non-minimum phase zeros (i.e. zeros with positive real parts).

A recent approach to the problem which highlights the importance of the system zeros in model reduction and model-following has been presented by Shaked (1977a). The problem considered is that of output following, i.e. the plant output is required to follow the model output. The plant $S_p(A,B,C)$ is described by

$$\begin{aligned} \dot{x}_p &= A\, x_p + B\, u_p \\ y_p &= C\, x_p \end{aligned} \qquad (7.1.1)$$

where $x_p \in R^n$, $u_p \in R^\ell$ and $y_p \in R^m$. The pairs (A,B) and (C,A) are respectively assumed to be controllable and observable.

The desired behaviour of the plant is provided by the model $S_m(M,N,L)$

$$\dot{x}_m = M x_m + N u_m \tag{7.1.3}$$

$$y_m = L x_m \tag{7.1.4}$$

where $x_m \varepsilon R^g$, $u_m \varepsilon R^r$ and $y_m \varepsilon R^m$. It is assumed that the model is stable and is completely controllable and observable. The matrices B, C, N and L are assumed to have full rank and $g \leqslant n$.

The study conducted by Shaked (1977a) has revealed that perfect model-following is impossible in the case where CB is rank deficient and the system is non-degenerate (A degenerate system has infinite number of zeros). Perfect model-following is achieved by driving the system to be unobservable through pole-zero cancellation. For all of the plant zeros z_i to be cancelled all of the z_i should have negative real parts to ensure stability of the closed loop system. A necessary condition for model-following is that $g \geqslant m + d$ where d is the rank deficiency of CB. Once the plant order has been reduced the problem of model-following becomes that of transfer function admissibility (Shaked, 1977 a,b). The model zeros are required to be a subset of the plant zeros for perfect model-following.

The output model-following problem treated in this chapter is that of driving the plant output to approach the model output using a suitable feedback law. The linear feedback law is based upon specially constructed error equation which has been motivated by the variable structure adaptive error equations of Chapter 4. In the case where CB is square and non-singular the control law exhibits certain structural properties. These properties are

(i) the automatic reduction of the plant order through forcing the plant to be unobservable,

(ii) the straight forward and automatic arbitrary assignment of the eigenvalues associated with the observable part of the plant,

(iii) the resulting closed-loop plant is decoupled.

Since the order reduction is automatic the determination of the plant zeros and the associated state and input zero directions are not needed as they are in Shaked (1977a) and Shaked et al. (1976). We need to know whether the plant is minimum phase or not. If this knowledge is not available, certain terms in the control can be used to determine the plant zeros. This will be clarified later. However, the control law achieves cancellation of all of the plant zeros and cannot allow for selective cancellation as in Shaked (1977a).

A review of decoupling theory is presented in section 7.2 and the equivalent control is shown to provide a special case in decoupling theory. A special error equation is then synthesized and each component of the resulting linear feedback control is examined.@ The stability, order reduction and the decoupled properties of the plant are then demonstrated. The case where CB is rectangular is tackled and the difficulties encountered are pointed out. The above control algorithm employs the information of all the states.

An approach to Output VSS model-following with only output information is proposed and the limitations are exposed. The

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@ The VSS implementation of this linear control ensures perfect pole zero cancellation in the presence of system variation. This is not the case for the linear controller (Shaked, 1977a).

control law employed which is a linear time-invariant is shown to exhibit adaptive properties subject to certain conditions. Design examples are included at the end of the Chapter.

7.2 Review of Decoupling Theory.

7.2.1 Definition.

A linear time invariant system described by

$$\dot{x} = Ax + Bu \tag{7.2.1}$$

$$y = Cx \tag{7.2.2}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$ and $x(0) = 0$ is said to be decoupled or noninteracting if its transfer function matrix $G(s)$ is diagonal and nonsingular (Fortmann et al., 1977)(Falb et al., 1967)(Power et al., 1978). The definition implies that every output is affected only by the corresponding input, i.e.

$$\begin{aligned} y_i(s) &= g_i(s)\, u_j(s) \qquad \text{if } i=j;\ i=1,2,..,m \\ &= 0 \qquad \text{if } i \neq j \end{aligned} \tag{7.2.3}$$

where $g_i(s)$ is the ith diagonal polynomial element of $G(s)$.

7.2.2 Decoupling by State Feedback.

A system given by (7.2.1) and (7.2.2) can be decoupled by incorporating state feedback. In this case a feedback control law of the form

$$u = Kx + Hv \tag{7.2.4}$$

is sought such that the closed loop system is stable and decoupled. Matrices K and H are mxn and mxm respectively and v is the new input (set point). Matrices K and H are found as follows

For $i = 1,2,...m$, we denote the rows of C by C_i and define d_i to be the smallest integer such that $0 \leq d_i \leq n-1$

for which (Falb et al., 1967), (Fortmann, 1977)

$$C_i\, A^{d_i}\, B \neq 0 \qquad (7.2.5)$$

or, if

$$C_i\, A^j\, B = 0 \qquad \text{for all } j = 0,1,\ldots,n-1 \qquad (7.2.6)$$

then d_i is set equal to n-1. We now define the following matrices

$$D = \begin{bmatrix} C_1\, A^{d_1} \\ C_2\, A^{d_2} \\ \vdots \\ C_m\, A^{d_m} \end{bmatrix} ;\; E = DB = \begin{bmatrix} C_1\, A^{d_1}\, B \\ C_2\, A^{d_2}\, B \\ \vdots \\ C_m\, A^{d_m}\, B \end{bmatrix} ;\; F = DA = \begin{bmatrix} C_1\, A^{d_1+1} \\ C_2\, A^{d_2+1} \\ \vdots \\ C_m\, A^{d_m+1} \end{bmatrix} \qquad (7.2.7)$$

The system S(A,B,C) may be decoupled using the linear state variable feedback control law (7.2.4) if and only if the mxm matrix E is nonsingular. If so the choice

$$K = -E^{-1}F \qquad (7.2.8)$$

$$H = E^{-1} \qquad (7.2.9)$$

results in the closed-loop system

$$\dot{x} = (A - BE^{-1}F)x + BE^{-1}\, v \qquad (7.2.10)$$

$$y = Cx \qquad (7.2.2)$$

which is decoupled. The transfer function matrix G(s) will then assume the form

$$G(s) = C[sI-A+BE^{-1}F]^{-1}BE^{-1} = \begin{bmatrix} \frac{1}{s^{d_1+1}} & & O \\ & \frac{1}{s^{d_2+1}} & \\ O & & \ddots \quad \frac{1}{s^{d_m+1}} \end{bmatrix} \qquad (7.2.11)$$

and is said to be in integrator decoupled form. Additional state variable feedback or output feedback can be applied to assign arbitrary poles to the integrator decoupled system while maintaining the decoupled properties of the closed loop system.

Decoupling is achieved through driving the system to be unobservable. The control law selected makes some of the closed loop poles coincide with the system zeros and thus the corresponding modes become unobservable. This is acceptable as long as the system zeros lie in the left half plane, i.e. have negative real parts.

It will now be shown that the VSS linear equivalent control is a special case in decoupling theory.

7.2.3 The Equivalent Control and Decoupling Theory.

Let us synthesize the control having the form (7.2.4) using VSS theory as a motivation. For a unique equivalent control to exist $|CB| \neq 0$. This condition immediately implies that all the d_i are equal to zero since

$$C_i\ B = C_i\ A^0\ B \neq 0 \qquad i=1,\ldots,m$$

(by the definition of the decoupling indices). The trivial case where $A \equiv I$ is excluded because d_i can then take any value between 0 and ∞.

If $d_i = 0$ for all $i = 1,\ldots,m$ then the decoupling matrices E and F (7.2.7) reduce to

$$E = \begin{bmatrix} C_1\ A^0\ B \\ C_2\ A^0\ B \\ \vdots \\ C_m\ A^0\ B \end{bmatrix} = CB$$

and

$$F = \begin{bmatrix} c_1 A^{0+1} \\ c_2 A^{0+1} \\ \vdots \\ c_m A^{0+1} \end{bmatrix} = \begin{bmatrix} c_1 A \\ c_2 A \\ \vdots \\ c_m A \end{bmatrix} = CA$$

The matrices K and H are from (7.2.8) and (7.2.9)

$$K = -E^{-1}F = -(CB)^{-1} CA$$

$$H = (CB)^{-1}$$

The matrix K is the same as that found using the equivalent control method for the system given by (7.2.1) and (7.2.2) i.e.

$$u_{eq} = -(CB)^{-1}CAx = K_{eq}x$$

It remains to be proved that the equivalent control matrix $K_{eq} = -(CB)^{-1}CA$ and the feedforward matrix $H = (CB)^{-1}$ reduce the transfer function matrix to

$$G(s) = \begin{bmatrix} \frac{1}{s^{0+1}} & & \\ & \frac{1}{s^{0+1}} & 0 \\ 0 & & \frac{1}{s^{0+1}} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & & \\ & \frac{1}{s} & 0 \\ 0 & & \frac{1}{s} \end{bmatrix} \quad (7.2.12)$$

In order to avoid repetition, this decoupling property of the equivalent control will be demonstrated in Section 7.

7.3 Formulation of Output Model-Following.

The problem of model-following treated in this chapter is that of output matching. The output of the plant is required to approach that of a lower order model such that the error between them tends to zero as $t \to \infty$. A special error equation is synthesized below. The synthesis has been motivated by the state error equation encountered in Chapter 4.

Consider the plant and model given by (7.1.1)-(7.1.4). The error e between their outputs is defined as

$$e = y_m - y_p \tag{7.3.1}$$

Therefore

$$\begin{aligned} \dot{e} &= \dot{y}_m - \dot{y}_p \\ &= L(Mx_m+Nu_m) - C(Ax_p+Bu_p) \\ &= LMx_m+LNu_m - CAx_p-CBu_p \end{aligned}$$

Adding and subtracting Λy_m and Λy_p gives

$$\begin{aligned} \dot{e} &= \Lambda y_m-\Lambda y_m+\Lambda y_p-\Lambda y_p+LMx_m+LNu_m-CAx_p-CBu_p \\ &= \Lambda(y_m-y_p)-\Lambda Lx_m+\Lambda Cx_p+LMx_m+LNu_m-CAx_pCBu_p \\ &= \Lambda e + (LM-\Lambda L)x_m+LNu_m+(\Lambda C-CA)x_p-CBu_p \end{aligned} \tag{7.3.2}$$

The mxm matrix Λ is chosen to be nonsingular and stable i.e. Re $(\lambda_i) < 0$, $i=1,\ldots,m$ where λ_i are the eigenvalues of Λ.

Perfect model-following is achieved if

$$(LM-\Lambda L)\, x_m + LNu_m + (\Lambda C-CA)x_p - CBu_p = 0$$

or

$$CBu_p = (LM-\Lambda L)x_m + LNu_m + (\Lambda C-CA)x_p \tag{7.3.3}$$

A controller u_p exists if

$$\begin{aligned} \text{rank } (CB) &= \text{rank } [LM-\Lambda L \quad CB] \\ &= \text{rank } [\ LN \quad CB] \\ &= \text{rank } [\Lambda C-CA \quad CB] \end{aligned}$$

The case where CB is rank deficient and Sp(A,B,C) is non-degenerate is ruled out as perfect following is impossible (Shaked, 1977a). Therefore the only cases considered here will be CB square and nonsingular and CB rectangular and full rank.

7.4 Model-following with $|CB| \neq 0$

In this case model-following is always possible as CB is full rank and therefore the rank conditions (7.3.4) are always satisfied irrespective of A, L, M, N and Λ. A controller u_p therefore exists and is given, from (7.3.3), by

$$u_p = (CB)^{-1} [(\Lambda C-CA)x_p + (LM-\Lambda L)x_m + LNu_m] \qquad (7.4.1)$$

The control u_p yields from (7.3.2) the error equation.

$$\dot{e} = \Lambda e \qquad (7.4.2)$$

Since Λ is stable the error e will asymptotically approach zero as $t \to \infty$.

A careful look at the control law (7.4.1) shows that it consists of three parts

$$u_p = -(CB)^{-1}CAx_p + (CB)^{-1}\Lambda Cx_p + (CB)^{-1}[(LM-\Lambda L)x_m + LNu_m] \qquad (7.4.3)$$

The first part $-(CB)^{-1}CAx_p$ has been encountered before and it is the familiar equivalent control u_{eq}. It arises naturally in the output model-following problem owing to the similarity between the error equation and the switching hyperplanes which are required to be zero together with their derivatives. The equivalent control u_{eq} has already been shown to drive the system unobservable thus reducing the system order (see Chapter 5). This is achieved by forcing some of the closed-loop poles to coincide with the system zeros. The order reduction property of

u_{eq} is illustrated further by the observation that another control law which achieves order reduction by deliberately driving the system to be unobservable through pole-zero cancellation has the same structure. This control law is given by Shaked (1977a) and is given by

$$u_s = K\ W^g\ x_p \qquad (7.4.4)$$

where K is a given set of input zero directions and W^g is a generalized inverse of the associated state zero directions W. We will now show that u_{eq} and u_s (7.4.4) have the same structure when $|CB| \neq 0$ and the number of zero directions W and K is n-m.

Theorem 7.4.1 The two control laws

$$u_{eq} = -\ (CB)^{-1}CAx_p = K_{eq}\ x_p \qquad (7.4.5)$$

and

$$u_s = KW^g x_p = L_s\ x_p \qquad (7.4.4)$$

which drive the system given by (7.1.1) and (7.1.2) with $m=\ell$ to be unobservable have the same structure and achieve the same results.

Proof: The control u_s (7.4.4) originates from the particular solution for the matrix L_s (Shaked et al., 1976), Shaked (1977a) where

$$L_s W = K \qquad (7.4.6)$$

since it has been shown in Chapter 5 that the input zero directions K are given by (letting x_o = W)

$$K = -(CB)^{-1}CAW \qquad (5.3.3)$$

then

$$L_s W = -\ (CB)^{-1}CAW \qquad (7.4.7)$$

Therefore from (7.4.7), $-(CB)^{-1}CA$ qualifies as a solution for L_s and therefore K_{eq} and L_s in (7.4.5) and (7.4.4) have the same structure and achieve the same results i.e. order reduction. Note that since W^g is not unique K_{eq} and L_s are generally non-identical. Q.E.D.

It has been established in Chapter 5 that the equivalent control automatically assigns zero values to m eigenvalues of the resulting closed loop A_{eq} matrix where

$$A_{eq} = A - B(CB)^{-1}CA \tag{7.4.8}$$

The second part of the control law (7.4.3) i.e. $(CB)^{-1}\Lambda Cx_p$ reassigns the values of these m eigenvalues to arbitrary values given by the eigenvalues of Λ. This will become clear in Section 7.5.

The third part of u_p $(CB)^{-1}$ $[(LM-\Lambda L)x_m+LNu_m]$ is independent of x_p and provides the new set points of the system. This will be demonstrated when exploring the decoupling of the system in Section 7.7.

The control u_p (7.4.3) therefore automatically drives the plant unobservable thus reducing its order, and also decouples the system if the matrix Λ is diagonal. Previous approaches to the problem rely on the deliberate reduction of the system order (see Shaked, 1977a). Arbitrary pole assignment to the observable part is also automatic. This is achieved through the matrix Λ which is chosen to be stable.

However a drawback to u_p is that it cancels all of the plant zeros and does not allow selective cancellation as in Shaked (1977a).

Therefore, if a plant has a non-minimum phase zero, applying u_p will result in an unstable closed-loop system. u_p also requires all the plant states x_p to be fed back (this is also required by Shaked). Since the plant is assumed to be observable an observer may be employed to reconstruct the states. A dynamic observer adapted for model reduction has been proposed by Shaked and Karcanias (1976).

The equivalent control in (7.4.3) can be implemented as a discontinuous VSS control law. The implementation, however, will require all the plant states. The design of VSS output model-following control systems using only the error information between the plant and model outputs is difficult and the difficulties encountered are discussed in section 7.9.

With the control law (7.4.3) only M, N, L and Λ have to be determined to satisfy the requirements. The zeros of the plant have to be determined to ensure stability and to check the necessary conditions for perfect matching. However, if the plant zeros are known a priori to have negative real parts, the determination of the zeros is not needed. If the zeros need to be determined then as has been shown in Chapter 5, the matrix

$$A_{eq} = A-B(CB)^{-1}CA \tag{7.4.8}$$

can be used to determine the system zeros. As can be seen from (7.4.8) part of A_{eq} namely $-(CB)^{-1}CA$ appears in u_p (7.4.3). This makes the control u_p self-sufficient (in the sense that the computational effort involved is reduced) in tackling the problem of output model-following in the case when $|CB| \neq 0$ and when complete cancellation is required.

The next step is to investigate the stability of the closed-loop system and to show that u_p reduces the order of the plant, decouples it and assigns arbitrary poles to the observable part.

7.5 Stability of the Closed-loop System.

Substituting u_p (7.4.3) into the original system gives

$$\dot{x}_p = Ax_p + B(CB)^{-1} [(\Lambda C-CA)x_p + (LM-\Lambda L)\ x_m + LN\ u_m]$$

$$= [A-B(CB)^{-1}CA + B(CB)^{-1}\Lambda C]\ x_p+B(CB)^{-1}[(LM-\Lambda L)x_m+LNu_m]$$

$$= A_c\ x_p + B(CB)^{-1}\ [(LM-\Lambda L)\ x_m + LNu_m] \qquad (7.5.1)$$

This closed-loop system is stable if the matrix A_c is stable. In order to investigate the stability of A_c a nonsingular similarity transformation T is employed.

Let

$$\bar{x}_p = T\ x_p$$

Substituting in (7.5.1) yields

$$\dot{\bar{x}} = T\ A_c\ T^{-1}\ \bar{x} + TB(CB)^{-1}\ [(LM-\Lambda L)\ x_m + LNu_m]$$

$$y_p = CT^{-1}\ \bar{x}$$

The transformation T is chosen in Chapter 5 to be

$$T = \begin{bmatrix} C \\ P \end{bmatrix}, \quad T^{-1} = [C^g\ P^g] \qquad (7.5.2)$$

where P is chosen such that $|T| \neq 0$.

Recall that

$$A_{eq} = A-B(CB)^{-1}CA \qquad (7.4.8)$$

Then using (7.5.1)

$$A_s = TA_cT^{-1} = \begin{bmatrix} C \\ P \end{bmatrix} A_c \quad [C^g \quad P^g] \qquad (7.5.3)$$

$$A_s = \begin{bmatrix} C \\ P \end{bmatrix} [A_{eq} + B(CB)^{-1}\Lambda C] \quad [C^g \quad P^g] \tag{7.5.4}$$

$$A_s = \left[\begin{array}{c|c} C(A_{eq}+B(CB)^{-1}\Lambda C)C^g & C(A_{eq}+B(CB)^{-1}\Lambda C)P^g \\ \hline P(A_{eq}+B(CB)^{-1}\Lambda C)C^g & P(A_{eq}+B(CB)^{-1}\Lambda C)P^g \end{array}\right] \tag{7.5.5}$$

It has been shown in Chapter 5 that

$$CA_{eq} = C[A-B(CB)^{-1}CA] = 0$$

$$CC^g = I_m$$

$$CP^g = 0$$

$$PA_{eq}P^g = Z$$

Therefore

$$A_s = \left[\begin{array}{c|c} 0+CB(CB)^{-1}\Lambda CC^g & 0+CB(CB)^{-1}\Lambda CP^g \\ \hline P(A_{eq}+B(CB)^{-1}\Lambda C)C^g & PA_{eq}P^g+PB(CB)^{-1}\Lambda CP^g \end{array}\right] \tag{7.5.6}$$

$$= \left[\begin{array}{c|c} \Lambda & 0 \\ \hline P(A_{eq}+B(CB)^{-1}\Lambda C)C^g & PA_{eq}P^g \end{array}\right] \tag{7.5.7}$$

The eigenvalues of the matrix $Z=PA_{eq}P^g$ are the system zeros. The stability is therefore determined by Λ and Z only. Since we have no control over Z through state feedback it is a prerequisite that the zeros (given by the eigenvalues of Z) have negative real parts. It then remains to choose the eigenvalues of Λ to have negative real parts.

7.6 Order Reduction.

The plant order is reduced by n-m degrees. This is because the control law u_p has driven the plant unobservable. It will be shown using two approaches that the plant is unobservable and that the matrix Λ does not affect this property.

1) Decomposition of A_c: The closed-loop system (7.5.1) has been decomposed into (see 7.5.7)

$$\dot{\bar{x}} = \left[\begin{array}{c|c} \Lambda & 0 \\ \hline P(A_{eq}+B(CB)^{-1}\Lambda C)C^g & Z \end{array}\right] \bar{x}_p \tag{7.6.1}$$

and

$$y = CT^{-1}\bar{x} = C[C^g : P^g]\bar{x}_p$$

Since $CC^g = I_m$ and $CP^g = 0$ y reduces to

$$y = [I_m : 0]\bar{x}_p \tag{7.6.2}$$

Equations (7.6.1) and (7.6.2) show that the system is unobservable and therefore the order of the plant is reduced.

2) The observability matrix $O(C, A_c)$.

Consider the observability matrix

$$\mathbf{O}(C,A_c) = [C^T, (C A_c)^T, \ldots, (CA_c^{n-1})^T]^T \tag{7.6.3}$$

$$CA_c = C(A_{eq} + B(CB)^{-1}\Lambda C) = \Lambda C$$

$$CA_c^2 = \Lambda C(A_{eq}+B(CB)^{-1}\Lambda C) = \Lambda^2 C$$

$$\vdots$$

$$CA_c^{n-1} = \Lambda^{n-2}C(A_{eq}+B(CB)^{-1}\Lambda C) = \Lambda^{n-1}C$$

$$\therefore \quad O(C,A_c) = [C^T, (\Lambda C)^T, \ldots (\Lambda^{n-1}C)^T]^T$$

$$O(C,A_c) = \begin{bmatrix} C \\ \Lambda C \\ \vdots \\ \Lambda^{n-1}C \end{bmatrix} = \begin{bmatrix} I_m & & 0 \\ & \Lambda & \\ 0 & & \Lambda^{n-1} \end{bmatrix} \begin{bmatrix} C \\ C \\ \vdots \\ C \end{bmatrix} \tag{7.6.4}$$

$$= \Lambda_d \begin{bmatrix} C \\ C \\ \vdots \\ C \end{bmatrix} \tag{7.6.5}$$

Since Λ is assumed to be nonsingular (see section 7.3) then Λ_d is nonsingular and therefore the rank of $O(C,A_c)$ is that of $[C^T\ C^T \cdot\cdot C^T]^T$ which is equal to the rank of C (equal to m). Therefore the plant is unobservable since $m < n$.

7.7 Decoupled Output Model-Following Control.

As has been pointed out the synthesized control law u_p decouples the plant outputs. Consider the transfer function matrix G(s) of the closed-loop system (7.5.1). Note from (7.5.1) that the equivalent input matrix is $B(CB)^{-1}$. Therefore

$$G(s) = C[sI-A_c]^{-1} B(CB)^{-1} \tag{7.7.1}$$

Since the transfer function matrix G(s) is invariant under state transformation it follows that

$$G(s) = CT^{-1}[sI_n-TA_cT^{-1}]^{-1}TB(CB)^{-1} \tag{7.7.2}$$

where T is the previously used similarity transformation. We substitute $TA_cT^{-1} = A_s$ (see 7.5.7) into (7.7.2) giving (note that $Z = PA_{eq}P^g$)

$$G(s) = C[C^g\ P^g] \begin{bmatrix} sI_m - \Lambda & 0 \\ -PA_cC^g & sI_{n-m}-Z \end{bmatrix}^{-1} \begin{bmatrix} C \\ P \end{bmatrix} B(CB)^{-1}$$

$$= [I_m\ 0] \begin{bmatrix} sI_m-\Lambda & 0 \\ -PA_cC^g & sI_{n-m}-Z \end{bmatrix}^{-1} \begin{bmatrix} I_m \\ PB(CB)^{-1} \end{bmatrix} \tag{7.7.3}$$

because $CC^g = I_m$ and $CP^g = 0$ (see section 7.5).

A standard identity for the inverse of a partitioned matrix (Kailath, 1980) assuming $|F| \neq 0$ and $|G| \neq 0$ is

$$\begin{bmatrix} F & 0 \\ H & G \end{bmatrix}^{-1} = \begin{bmatrix} F^{-1} & 0 \\ -G^{-1}HF^{-1} & G^{-1} \end{bmatrix} \tag{7.7.4}$$

Therefore

$$G(s) = [I_m \ 0] \begin{bmatrix} (sI_m-\Lambda)^{-1} & 0 \\ J & (sI_{n-m}-Z)^{-1} \end{bmatrix} \begin{bmatrix} I_m \\ PB(CB)^{-1} \end{bmatrix} \tag{7.7.5}$$

where $J = (sI_{n-m} - Z)^{-1} PA_c P^g (sI_m-\Lambda)^{-1}$

Simplifying (7.7.5) gives

$$\begin{aligned} G(s) &= [(sI_m-\Lambda)^{-1} \ 0] \begin{bmatrix} I_m \\ PB(CB)^{-1} \end{bmatrix} \\ &= (sI_m - \Lambda)^{-1} \end{aligned} \tag{7.7.6}$$

Therefore the plant is decoupled and G(s) assumes the diagonal form

$$G(s) = \begin{bmatrix} \frac{1}{s-\lambda_i} & O \\ O & \frac{1}{s-\lambda_m} \end{bmatrix}$$

Consider the input matrix to the closed-loop system to be $B(CB)^{-1}$. The new inputs to the plant are therefore given by

$$v = [(LM-\Lambda L)\, x_m + LN\, u_m] \tag{7.7.7}$$

We shall now show that these inputs ensure that $y_p = y_m$.

From (7.7.7)

$$\begin{aligned} v &= [L(M\, x_m + N\, u_m) - \Lambda\, L\, x_m] \\ &= L\, \dot{x}_m - \Lambda\, L\, x_m \\ &= \dot{y}_m - \Lambda\, y_m \end{aligned}$$

Taking the Laplace transform and assuming $y_m(0)$

$$V(s) = (s\,I_m - \Lambda)\; y_m(s) \tag{7.7.8}$$

Recall that the input-output relationship of a system is given by

$$y_p(s) = G(s)\; v(s) \tag{7.7.9}$$

then from (7.7.6) and (7.7.8)

$$\begin{aligned} y_p(s) &= (sI - \Lambda)^{-1}\,(sI - \Lambda)\; y_m(s) \\ &= y_m(s) \end{aligned}$$

<u>Remark 7.7.1.</u> If $\Lambda = 0$ then

$$u_p = -(CB)^{-1}CAx_p + (CB)^{-1}\,[LM\,x_m + LN\,u_m] \tag{7.7.10}$$

and

$$G(s) = (sI_m)^{-1}$$

Since u_p in (7.7.10) involves the equivalent control and feedforward matrix $(CB)^{-1}$ and G(s) is diagonal and of the form (7.2.12) then the equivalent control decouples the system as stated in section 7.2.

7.8 <u>Output Model-Following with Unequal number of Inputs and and Outputs ($m \neq \ell$).</u>

In the previous sections the problem of output model-following for the case CB square has been treated in detail. The control law which ensures perfect following has been shown to possess inherent structural properties which achieve automatic order reduction and arbitrary eigenvalue assignment. The extension of the theory to the case of rectangular CB (i.e. unequal number of inputs and outputs) is rather involved and requires careful attention. In this case, assuming the perfect-model following conditions are satisfied then

$$u_p = (CB)^g \, [(\Lambda C - CA) \, x_p + (LM-\Lambda L) \, x_m + LN \, u_m] \tag{7.8.1}$$

where $(CB)^g$ is the generalized inverse of CB. The resulting closed-loop system is given by

$$\dot{x}_p = [A-B(CB)^g CA+B(CB)^g \Lambda C]x_p+B(CB)^g[(LM-\Lambda L)x_m+LN \, u_m] \tag{7.8.2}$$

$$= A_k \, x_p + B(CB)^g \, [(LM-\Lambda L) \, x_m + LN \, u_m] \tag{7.8.3}$$

The stability and the decoupling properties of the closed-loop system will now be investigated.

<u>Case 1 : The number of outputs is less than the number of inputs</u> $(m < \ell)$

In this case the decoupling property of the system is preserved. The stability of the system, however, is no longer dictated by the system zeros alone but also depends upon the particular choice of $(CB)^g$. Using the previous similarity transformation T (7.5.2) and noting that (see Appendix 2)

$$(CB)(CB)^g = I_m \tag{7.8.4}$$

we have

$$A_t = TA_k T^{-1} = T(A_g+B(CB)^g \Lambda C]T^{-1} \tag{7.8.5}$$

where

$$A_g = A - B(CB)^g CA \tag{7.8.6}$$

Following the same procedure as in (7.5.3)-(7.5.7) noting that $C \, A_g = 0$ we get

$$A_t = \begin{bmatrix} \Lambda & 0 \\ PA_k P^g & PA_g P^g \end{bmatrix} \tag{7.8.7}$$

and

$$y = [I_m \quad 0] \tag{7.8.8}$$

Following the same procedure as in section 7.7 the decoupling property of the system can be shown to be preserved. However, the stability of the system is now governed by the eigenvalues of PA_gP^g giving new zeros in addition to the original zeros of the system. These new zeros have been introduced by the squaring down of the matrix B. The squaring down of B is achieved through $(CB)^g$ and therefore a judicious selection of a generalized inverse should be sought such that the eigenvalues of A_g lie in the left hand half of the complex plane. Whether there are enough degrees of freedom in the choice of $(CB)^g$ to yield desired eigenvalues has yet to be established.

<u>Case 2 : The number of outputs are greater than the number of inputs</u> $(m > \ell)$.

In this case $(CB)(CB)^g \neq I_m$ and therefore (7.8.3) will not assume the simple structure of (7.8.7). Instead A_t (7.8.5) will now be

$$A_t = \begin{bmatrix} CA_kC^g & CA_kP^g \\ PA_kC^g & PA_kP^g \end{bmatrix} \tag{7.8.9}$$

$$y = [\, I_m \quad 0 \,] \tag{7.8.10}$$

Since $CA_kP^g \neq 0$ the unobservability of the closed loop system cannot be assessed from (7.8.9) and (7.8.10). Also the decoupling property of the system can no longer be assessed from (7.8.9) since $CA_kP^g \neq 0$ and $CA_k\, C^g$ is not diagonal. The stability is dependent upon the choice of $(CB)^g$ because new zeros are introduced into the system due to the squaring down of the outputs. Again, whether there are enough degrees of freedom in the choice of $(CB)^g$ to yield desired eigenvalues has yet to be established.

7.9 Variable Structure Adaptive Output Model-Following Control Systems using only the output information. (VSOMF)

As mentioned before the specially synthesized error equation (7.3.3) has been motivated by the error equation related to state adaptive model-following and variable structure adaptive model-following (VSMFC) which have been studied in Chapter 4. The attractive features of VSMFC suggest the possibility of extending the theory to Output model-following (OMF). However, as will be shown below, the usefulness of employing VSS theory to the OMF problem appears to be limited, and rather complex, if at all possible. This is due to the three forms taken by CB which are treated separately.

In VSOMF it is required to organise sliding at the intersection of the switching hyperplanes

$$s = G\,e = 0 \qquad (7.9.1)$$

where G is a pxm full rank matrix. The number p of switching hyperplanes will vary according to the form of CB. Consider the three cases.

Case 1 : CB is rectangular with more outputs than inputs $(m > \ell)$.

In this case we choose $p = \ell$ so that GCB is square of order $\ell x \ell$. GCB must be nonsingular in order to ensure the uniqueness of the equivalent control u_{eq} where from (7.9.1) and (7.3.2)

$$u_{eq} = (GCB)^{-1}G[\Lambda e + (\Lambda C-CA)\,x_p + (LM-\Lambda L)\,x_m + LN\,u_m] \qquad (7.9.2)$$

which yields

$$\dot{e} = [I-CB(GCB)^{-1}G]\ [\Lambda e+(\Lambda C-CA)\ x_p + (LM-\Lambda L)\ x_m + LN\ u_m] \qquad (7.9.3)$$

and

$$\dot{x}_p = [A + B(GCB)^{-1}G(\Lambda C-CA)]x_p+B(GCB)^{-1}G\ [\Lambda e + Q] \qquad (7.9.4)$$

with

$$Q = [(LM-\Lambda L)\ x_m + LN\ u_m] \qquad (7.9.5)$$

The term GC effectively squares the outputs down and therefore new zeros are introduced in addition to the original ones. Care should therefore be exercised when choosing G so that any zeros introduced will lie in the left half of the complex plane. G should at the same time assign m-ℓ arbitrary eigenvalues (assuming the pair (Λ,CB) is controllable) to

$$[I_m - CB(GCB)^{-1}G]\ \Lambda$$

as required by VSS theory. Whether G has enough degrees of freedom to accomplish this and ensuring at the same time that no non-minimum phase zeros are introduced remains to be established. When the above constraints have been satisfied a discontinuous control is sought to enforce sliding and to maintain e on the intersection of the switching hyperplanes s = Ge = 0 as has been described in Chapters 3 and 4.

Case 2 : CB is square ($m = \ell$).

The attractiveness of the insensitivity of VSS to parameter variations and to external disturbances relies on the exploitation of the null space N(G) of the switching hyperplane matrix G. If G is square and nonsingular N(G) will be the zero vector at the origin. Therefore G should be either square and singular (and have a nontrivial null space) or rectangular of order pxm where $p < m$. Cases where G and B are rectangular and GB is singular have been studied by Utkin (1978a). Utkin has shown that sliding may or may not result. The case where G and B are square with G singular has never been studied before. However, one way round this problem will be discussed below. Let CB be partitioned as

$$CB = [\,K \quad F\,] \tag{7.9.6}$$

where K is mxp and F is mx(m-p). The matrix K is chosen such that $|GK| \neq 0$. This can always be achieved by rearranging the columns of CB. The error equation will become

$$\dot{e} = \Lambda e + (\Lambda C - CA)\, x_p + K\, u_K + F\, u_f + Q \tag{7.9.7}$$

where Q is given by (7.9.5) and u_K and u_f are the p and m-p components of u_p. The objective now is to organize sliding on the intersection of the switching hyperplanes using the control u_K only. This gives

$$u_{K_{eq}} = (GK)^{-1}\, [\Lambda e + (\Lambda C - CA)\, x_p + F\, u_f + Q] \tag{7.9.8}$$

and

$$\dot{e} = [I - K(GK)^{-1}G]\, [\Lambda e + (\Lambda C - CA)\, x_p + F\, u_f + Q] \tag{7.9.10}$$

with

$$Ge = 0 \tag{7.9.11}$$

During sliding the error is invariant to disturbances if

$$(\Lambda C - CA)\, x_p + F\, u_f + Q = 0 \tag{7.9.12}$$

The invariance principle and the model following conditions require each of these terms to lie in the range space of K if they are to be rejected. Difficulties may arise in satisfying such conditions if $(\Lambda C-CA)\ x_p$ and Q are full rank. Furthermore, the requirement of CB being full rank implies that the matrix F will always be independent of K and therefore it will not be rejected during the sliding mode. Added control elements are therefore needed to counteract these disturbances,(see Draženović,1969),(Itkis,1976) and (Utkin,1978a). The pair (Λ,K) should be completely controllable for arbitrary eigenvalue assignment.

Case 3 : CB is rectangular with more inputs than outputs ($\ell > m$).

The best partitioning of CB in this case will be to have K of dimension $(m-1)xm$. This means that the dimension of F is $mx(\ell-m+1)$ which will at least have one column which is independent of K. Therefore the invariance principle will never be satisfied. Difficulties may also arise if $(\Lambda C-CA)\ x_p$ and Q do not belong to the range space of K. Added control elements are therefore needed to counteract these disturbances. Also the pair (Λ,K) should be controllable for arbitrary eigenvalue assignment.

7.10 Adaptive Output Model-Following with a Time-Invariant Linear Controller.

In the previous sections 7.5 - 7.7 it has been demonstrated that Λ assigns arbitrary eigenvalues to the observable part of the system. It will now be shown that by a suitable choice of the eigenvalues of Λ output model-following may still be possible in the presence of disturbances using the fixed linear control u_p (7.4.3) designed for disturbance free systems. In this sense the control u_p exhibits adaptive properties subject to the conditions

(i) the parameter variations A_v do not make the system non-minimum phase.

(ii) the eigenvalues of Λ have sufficiently large negative real parts.

Condition (i) is obvious since we have no control over the system zeros using state feedback. However a certain class of disturbances which do not affect the zeros of the system can be identified. Let the new matrix representing the system be

$$A_d = A + A_v \tag{7.10.1}$$

where A is the same as in (7.1.1) and A_v is the parameter variations in A due to the disturbance. The new zeros of the system are given by the (n-m) eigenvalues of (see Chapter 5 and (7.5.7) letting $M = P^g$)

$$M^g\ [I-B(CB)^{-1}C]\ (A + A_v)\ M \tag{7.10.2}$$

where M is a basis for the null space of the output matrix C. Since the original zeros of the system are given by

$$M^g\ [I-B(CB)^{-1}C]\ A\ M$$

then for the zeros of (7.10.1) not to be affected by the parameter variations A_v it is sufficient that

$$[I - B(CB)^{-1}C]\ A_v = 0 \tag{7.10.3}$$

Since $[I-B(CB)^{-1}C]$ is a projector (see Chapter 6) it is then required that

$$\text{col}\ (A_v)\ \varepsilon\ R(B) \tag{7.10.4}$$

Condition (7.10.4) is the same as that provided by the invariance principle (see Chapter 3 section 3.7 and Chapter 6 section 6.3).

The closed-loop system after substituting u_p which is given by (7.4.3) is (see (7.5.1))

$$\dot{x}_p = (A + A_v)\, x_p + B\, u_p$$

$$= A_c\, x_p + A_v\, x_p + B(CB)^{-1}\, [(LM-\Lambda L)x_m + LNu_m] \qquad (7.10.5)$$

To investigate the stability of (7.10.5) we use the same similarity transformation T (see 7.5.2). The stability of the closed-loop system is dictated by the eigenvalues of (c.f. 7.5.7)

$$T\,(A_c + A_v)\,T^{-1} \qquad (7.10.6)$$

Following (7.5.2)-(7.5.7) it can be shown that (note that $Z = PA_cP^g$ in (7.5.7))

$$T(A_c+A_v)T^{-1} = \begin{bmatrix} \Lambda & 0 \\ PA_cC^g & Z \end{bmatrix} + \begin{bmatrix} CA_vC^g & CA_vP^g \\ PA_vC^g & PA_vP^g \end{bmatrix} \qquad (7.10.7)$$

Also from (7.6.2)

$$y = [\, I_m \; : \; 0] \qquad (7.10.8)$$

It can be seen from (7.10.7) that the decoupling property of the closed-loop system (7.10.5) is preserved if

$$C\,A_v\,P^g = C\,A_v\,C^g = 0 \qquad (7.10.9)$$

i.e. if $R(A_v) \subset N(C)$ since $R(A_vP^g) = R(A_vC^g) = R(A_v)$, (see Appendix 2). However, from (7.10.7) and (7.10.8) the system zeros are now given by

$$sp\;\{Z + PA_v\,P^g\} \qquad (7.10.10)$$

i.e. unless the eigenvalues of (7.10.10) have negative real parts the system will be unstable. The variations in the zeros of the system depend on A_v alone and cannot be influenced by Λ.

If $CA_v \neq 0$ the decoupling and unobservability of the system is lost. In this case

$$T(A_c+A_v)T^{-1} = \begin{bmatrix} \Lambda+CA_vC^g & CA_vP^g \\ PA_cC^g+PA_vC^g & Z+PA_vP^g \end{bmatrix} \qquad (7.10.11)$$

$$y = [\, I_m \ : \ 0] \qquad (7.10.8)$$

From (7.10.11) and (7.10.8) the system is no longer decoupled. Equation (7.10.11) however, is influenced by Λ and therefore by choosing Λ with eigenvalues whose real parts are sufficiently negative the eigenvalues of (7.10.11) may be ensured to have negative real parts, thus ensuring the stability of the closed-loop systems provided that A_v does not make the system zeros non-minimum phase.

The above procedure can be extended to study the effect of parameter variations in the input matrix where B can now be represented as

$$B_d = B + B_v \qquad (7.10.12)$$

The closed-loop system after substituting u_p (7.4.3) is given by (see 7.5.1)

$$\dot{x}_p = A_c x_p + B_v(CB)^{-1}(\Lambda C-CA)x_p+B_d(CB)^{-1}[(LM-\Lambda L)x_m+LNu_m] \qquad (7.10.13)$$

To avoid repetition and since the matrix $B_v(CB)^{-1}(\Lambda C-CA)$ is square of order nxn with range belonging to $R(B_v)$ (see Appendix 2) then $B_v(CB)^{-1}(\Lambda C-CA)$ can be represented as

$$A_v^b = [a_1 \ \ldots \ a_n] \qquad (7.10.14)$$

where a_i, $i=1,..,n$ belongs to $R(B)$. The same conditions previously imposed on A_v will now apply to A_v^b and consequently to B_v.

The effect of parameter variations on B has also been studied by Young (1978b) using VSS theory. The system has been shown to be insensitive to parameter variations in B if the eigenvalues corresponding to the "slow" modes are sufficiently deep in the left half of the complex plane (see also Chapter 4 section 4.3).

7.11 Examples.

Two examples are presented to demonstrate output model-following, decoupling and the adaptive property of u_p (4.7.3)

Example 1: This is the same chemical reactor considered in Chapter 4. It is now required to control this unstable plant such that its outputs match those of a lower order model.

Consider the plant

$$\dot{x}_p = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} x_p + \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix} u_p \tag{7.11.1}$$

$$y_p = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_p \tag{7.11.2}$$

The model is given by

$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 6 \end{bmatrix} u_m \tag{7.11.3}$$

$$y_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_m \tag{7.11.4}$$

Since CB is nonsingular the rank deficiency $d=0$ and the necessary condition for model-following i.e. $g \geq m+d$ is satisfied.

Also since it is known a priori that the plant has no non-minimum phase zeros (Munro, 1972); the control law (7.4.3) will be employed without the need to calculate these zeros.

$$u_p = (CB)^{-1}CA\ x_p + (CB)^{-1}\Lambda C\ x_p + (CB)^{-1}[(LM-\Lambda 1)x_m + LNu_m] \qquad (7.4.3)$$

with Λ chosen as

$$\Lambda = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \qquad (7.11.5)$$

To achieve order reduction, there is no need to determine the system zeros, the state and input zero directions as in Shaked (1977a). Order reduction is achieved automatically through the employment of u_p (7.4.3).

The following values have been used in the simulation. all initial conditions are zero except $x_{p_1}(0) = -0.5$ and

$x_{p_2} = -0.5$.

The dynamic behaviour of the plant and model outputs is shown in Fig.7.11.1a and perfect matching is achieved. As shown in Fig.7.11.1a output model-following can be speeded up by increasing the absolute value of the eigenvalues of Λ. Each eigenvalue has been increased to -10. Fig.7.11.1b shows the error decay corresponding to the two sets of eigenvalues of Λ.

The decoupling property of the controlled plant is shown in Fig.7.11.2a-d. Perfect model-following is achieved and the plant and model responses are identical (in this case $x_{p_1}(0) = x_{p_2}(0) = 0$), see Fig. 7.11.2a,c. Fig.7.11.2a shows the response of the plant output y_{p_1} when the input associated with y_{p_1} is excited by y_{m_1} only. The interaction

Δy_{p_2} in the second unexcited output y_{p_2} is depicted in Fig. 7.11.1b and can be seen to be very small. Note that the maximum of $\Delta y_{p_2} = \pm\, 0.0005$. Fig.7.11.2c shows the response of y_{p_2} when the input associated with it is excited by y_{m_2} only. The interaction Δy_{p_1} can be seen to be very small as shown in Fig.7.11.2d The maximum of $\Delta y_{p_1} = \pm\, 0.005$. Therefore from 7.11.2a-c the closed-loop plant is decoupled.

<u>Example 2:</u>

This example demonstrates the adaptive properties (discussed in section 7.10) of the linear control law u_p (7.4.3). Consider the plant

$$\dot{x}_p = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -3 \end{bmatrix} x_p + \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} u_p \qquad (7.11.6)$$

$$y_p = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} x_p \qquad (7.11.7)$$

and the model

$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 9 \end{bmatrix} u_m \qquad (7.11.8)$$

$$y_m = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} x_m \qquad (7.11.9)$$

The necessary condition for model-following is satisfied i.e. $g \geq m+d$ where the rank deficiency $d = 0$. Model-following is possible with this plant since it has a minimum phase zero at -3, see (Patel, 1977), (El-Ghezawi et al., 1982b).

The simulation study of this example demonstrates the good performance of the plant when it is subject to constant perturbations (disturbances) in A and B. The added perturbations in A and B (see section 7.10) are given by

$$A_V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad (7.11.9)$$

The idea is to demonstrate that the control law (7.4.3) calculated with the known values of A, B as in (7.11.6) and then implemented as a feedback control law exhibits adaptive properties. It will now be demonstrated that by choosing the eigenvalues of Λ with sufficiently large negative real parts good model-following is achieved despite the perturbations in A and B.

Fig.7.11.3(a) shows the plant response when affected by the perturbation A_V (7.11.9). Λ is given by

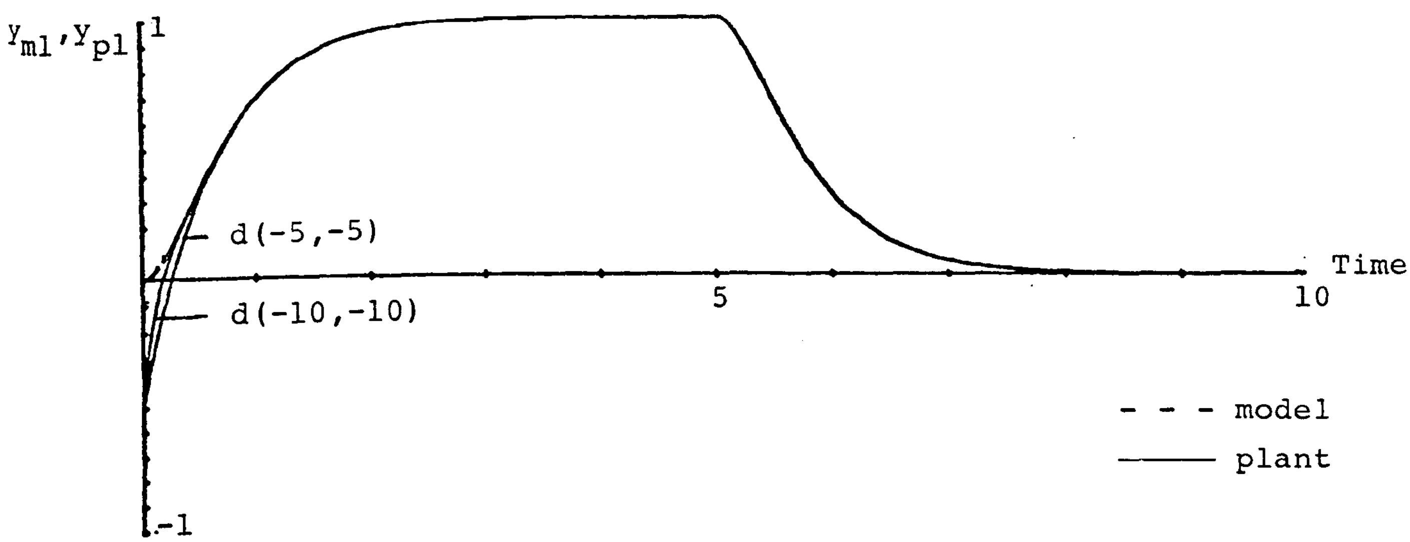
$$\Lambda = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix} \qquad (7.11.10)$$

The closed-loop system is stable but has steady-state error. This error can be reduced as shown in Fig.7.11.3(b) by choosing

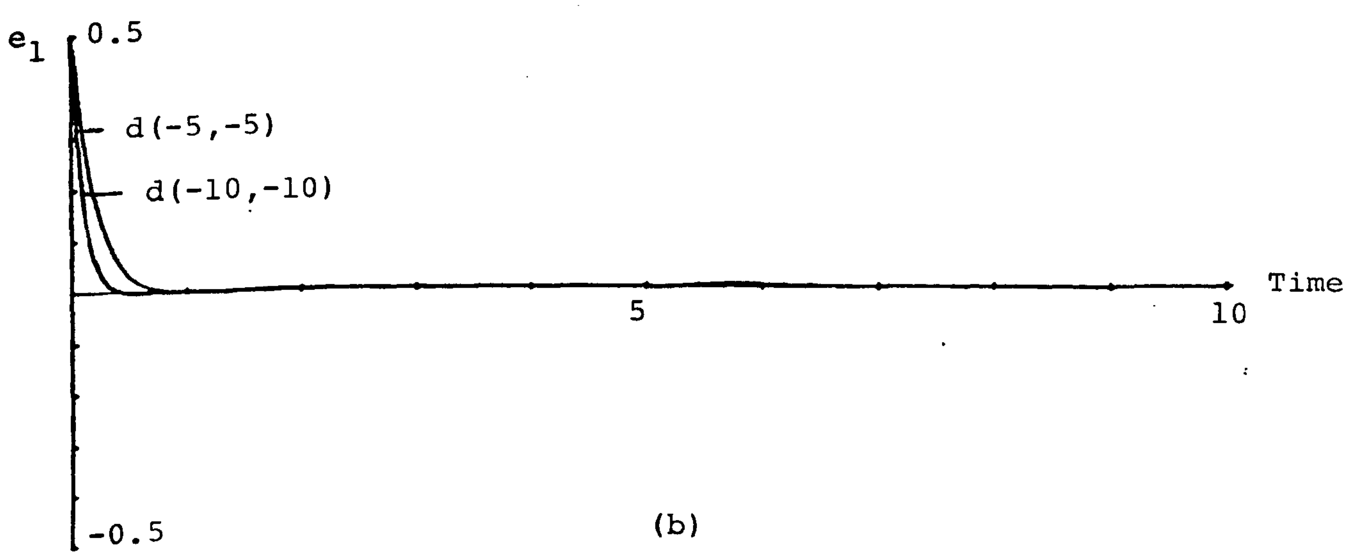
$$\Lambda = \begin{bmatrix} -50 & 0 \\ 0 & -50 \end{bmatrix} \qquad (7.11.11)$$

The stability of the system suggests that the perturbation A_V has not made the zero of the system non-minimum phase.

Fig.7.11.4(a) shows the plant response with perturbation B_V (7.11.9). Good model-following is achieved. The steady state error shown in Fig.7.11.4(a) with Λ given by (7.11.10) is reduced by choosing Λ as in (7.11.11). This is depicted in Fig.7.11.4(b).



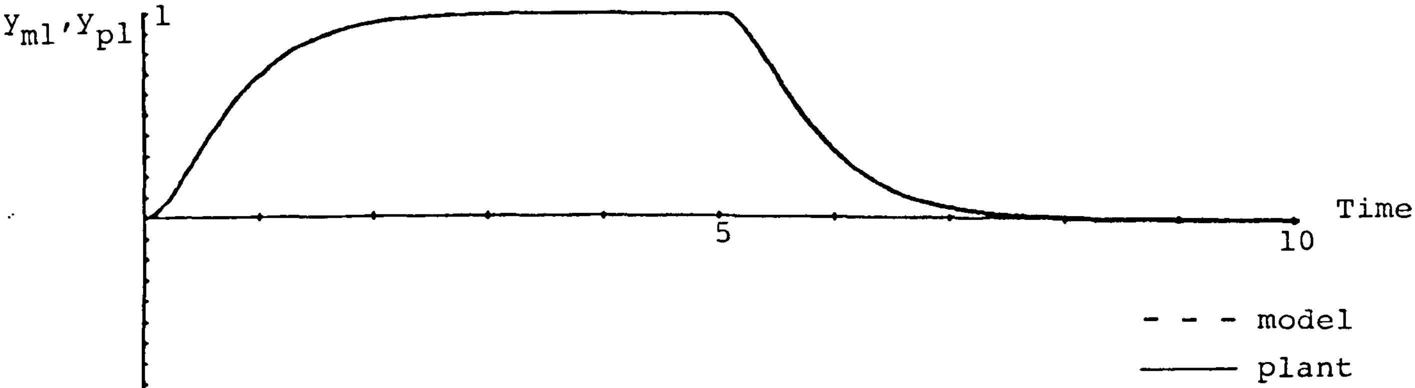
The plant and model response (the diagonal elements of Λ are represented by d(,).



(b)

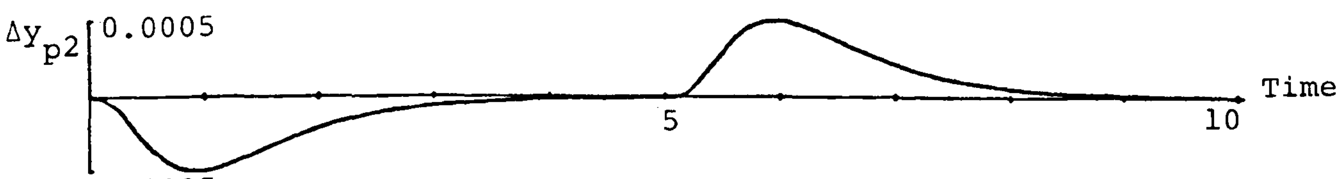
The error between y_{m1} and y_{p1}

Fig.7.11.1



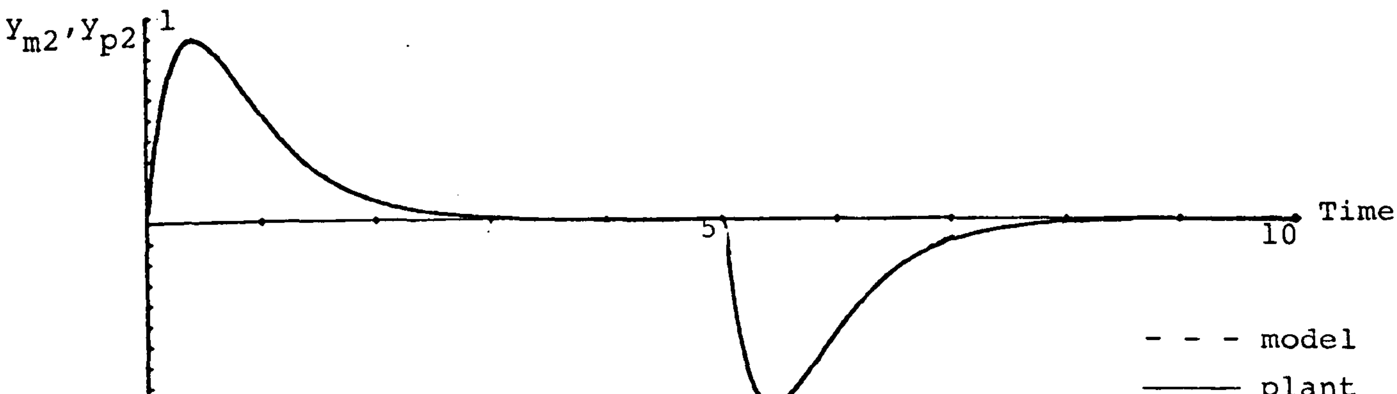
(a)

Identical model and plant response.



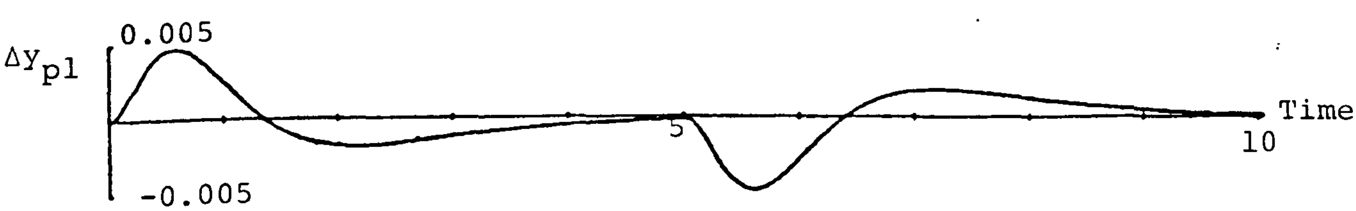
(b)

Interaction in y_{p2}



(c)

Identical model and plant response

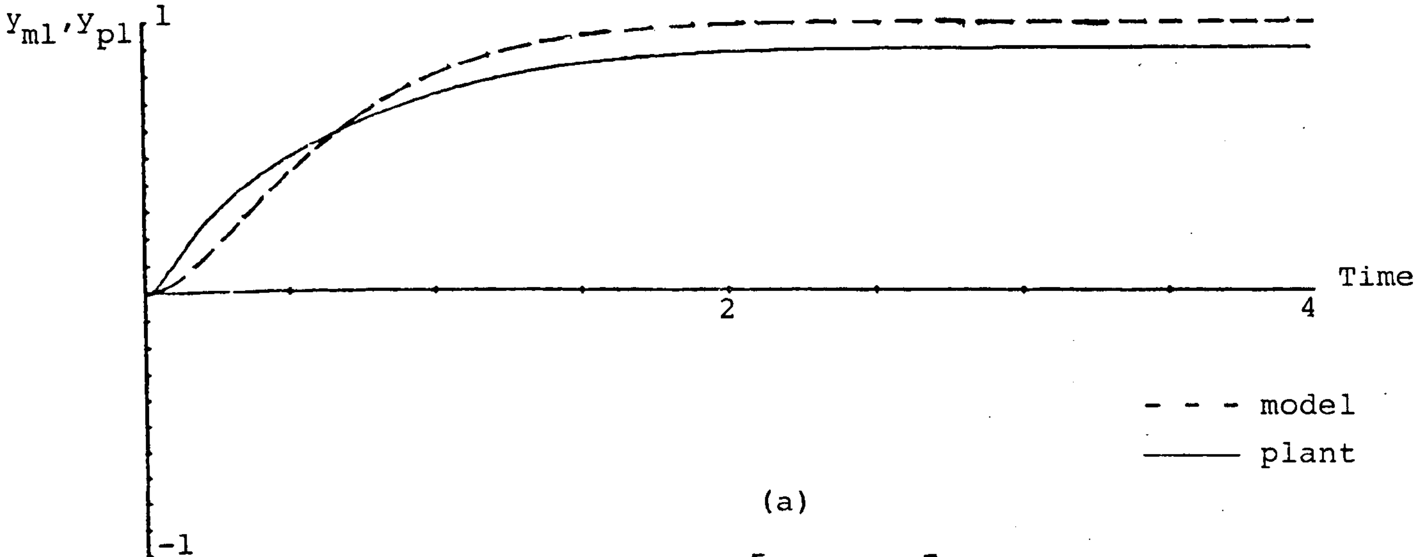


(d)

Interaction in y_{p1}

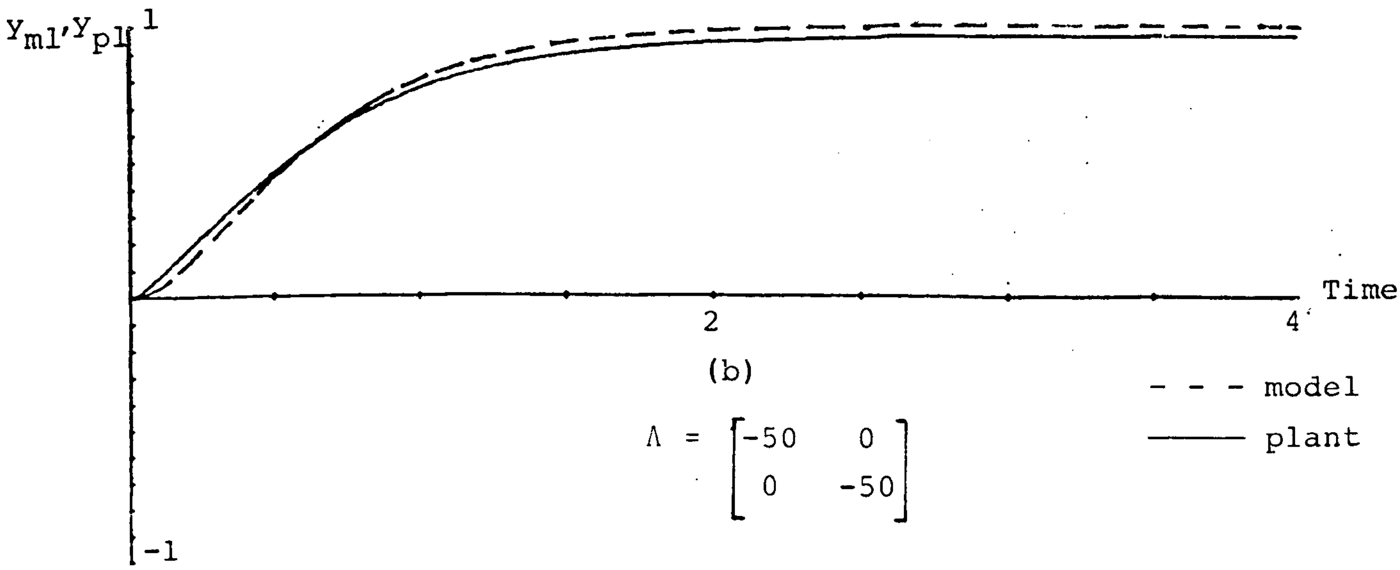
The decoupled plant.

Fig.7.11.2



(a)

$$\Lambda = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix}$$

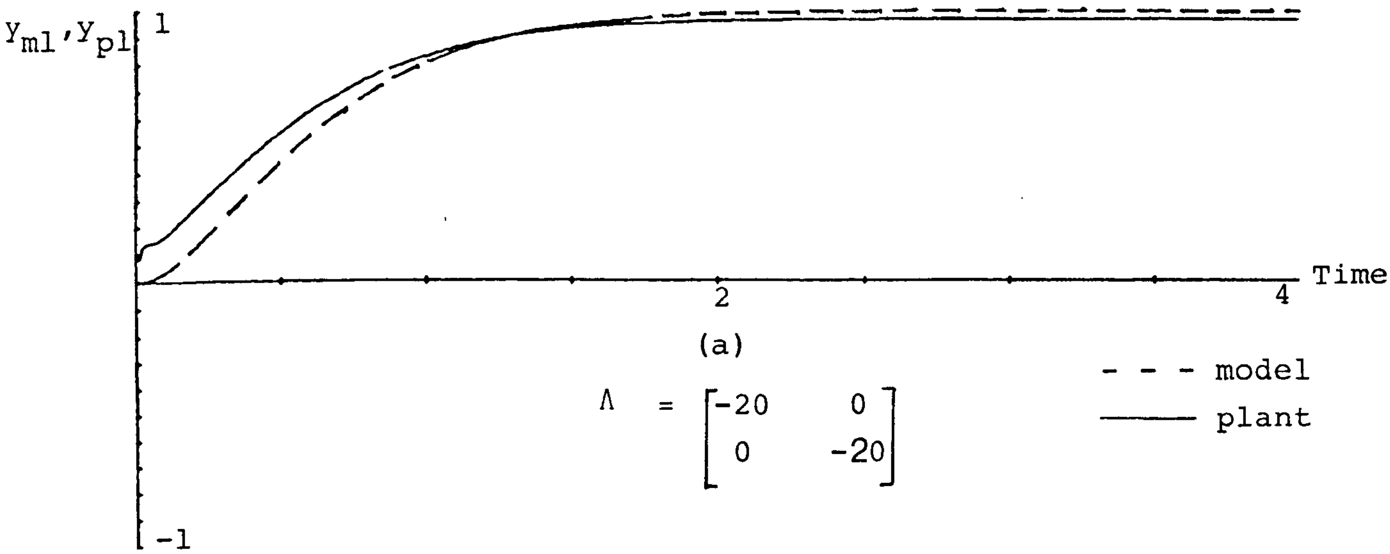


(b)

$$\Lambda = \begin{bmatrix} -50 & 0 \\ 0 & -50 \end{bmatrix}$$

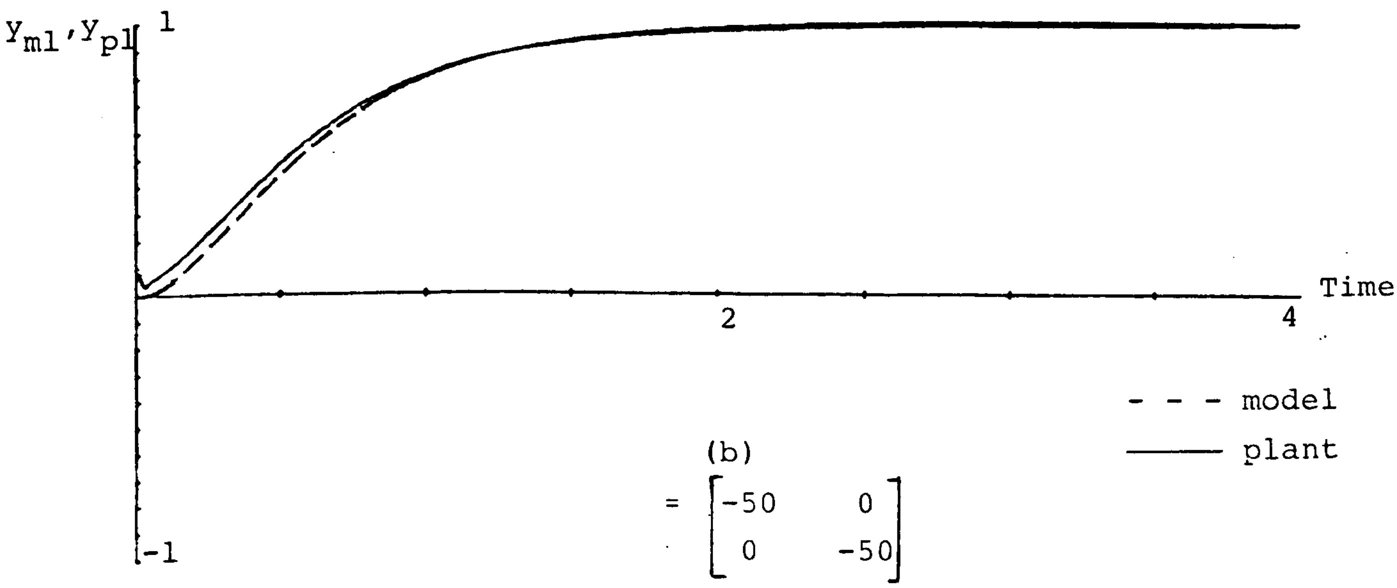
The model and plant response with added disturbance A_v (the value of Λ used is indicated below each figure).

Fig.7.11.3



(a)

$$\Lambda = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix}$$



(b)

$$= \begin{bmatrix} -50 & 0 \\ 0 & -50 \end{bmatrix}$$

The model and plant response with added disturbance B_v (the value of Λ used is indicated below each figure).

Fig.7.11.4

CHAPTER EIGHT

CONCLUSION.

Variable structure control of linear multivariable adaptive model-following systems has been described. The design technique is systematic, straightforward and requires little computional effort. Most designs can be carried out without any computer assistance. By ensuring sliding motion on the switching hyperplanes, insensitivity to parameter variations and disturbances is achieved. Design studies have been conducted and simulation studies of three examples have been carried out. The examples include a dc drive, a fourth order multivariable system and a chemical batch process.

The analogy between variable structure systems (VSS) in the sliding mode, and the output zeroing problem and the output model-following problem have suggested a new method of computing the system zeros and zero directions of linear multivariable square systems. The calculation of the system zeros is straightforward and offers the advantage of the ability to calculate the state and the input zero directions independently of each other and without resorting to the determination of the null space of the (n+m)th order system matrix. A design procedure for zero assignment has been presented and four examples are given.

The employment of projector theory in the study of VSS in the sliding mode has been shown to provide further insight into their operation. Furthermore new methods for constructing the switching hyperplanes have been formulated utilizing projector theory. In addition to solving the eigenvalue placement problem these methods allow CB to be specified arbitrarily and allow partial control over the choice of the closed-loop eigenvectors. The examples included illustrate the feasibility of the proposed methods.

The output model-following problem is shown to be rather complex and the system zeros play a major role in the design of such systems. A specially constructed control law has been shown to reduce the order of the plant, to assign arbitrary poles to the observable part and to decouple the plant. All of the above is achieved automatically. However, this control law is unsuitable if the plant has a non-minimum phase zero. Other techniques may be used in this case. The case where CB is rectangular is shown to be dependent upon the particular generalized inverse chosen. The examples included demonstrate the ease of the design of output model-following control systems using the theory presented. Variable structure adaptive output model-following using output information only is shown to be a difficult problem and the limitations of the results given are pointed out. The linear synthesized control is shown to exhibit adaptive properties subject to certain conditions.

Future research into variable structure systems and adaptive model-following systems should involve

(i) utilizing VSS in the calculation of system zeros in the case of unequal number of inputs and outputs.

(ii) obtaining a general solution W of the system of equations

$$R(AW-WJ) \varepsilon R(B)$$

$$R(W) \cap R(B) = \{0\}$$

where $B \varepsilon R^{n \times m}$, $W \varepsilon R^{n \times (n-m)}$ and A, W, J, B are as defined in Chapter 6.

(iii) investigating the degrees of freedom available in the generalized inverse of CB when assigning the eigenvalues of the matrix

$$A-B(CB)^{g}CA$$

which is encountered in section 7.8

(iv) developing an adaptive variable structure observer to estimate unmeasured states.

(v) investigating model-following variable structure systems with output information only and mismatched plant and model order.

(vi) comparing VSS design with other methods of adaptive control based on Liapunov and hyperstability theory.

(vii) evaluating VSS designs by applying the techniques to industrial processes using microprocessors.

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APPENDIX 1.

HYPERSTABILITY.

The hyperstability concept concerns the stability properties of a class of feedback systems which can be split into two blocks; a linear feedforward block and a nonlinear time-varying feedback block as shown in Fig.2.5.1a. See also (Landau, 1974, 1979), (Landau and Courtiol, 1974).

Consider a non-linear time-varying feedback system consisting of two parts. The linear part is described by

$$\dot{x} = Ax + Bu = Ax - Bw \tag{A.1.1}$$

$$v = Cx + Ju = Cx - Jw \tag{A.1.2}$$

where $x \varepsilon R^n$, $u \varepsilon R^m$, $v \varepsilon R^m$, $w \varepsilon R^m$. The pair (A,B) and (C,A) are assumed to be completely controllable and observable respectively. The nonlinear time varying part is described by

$$u = -w \tag{A.1.3}$$

$$w = f(v(\tau), t) \qquad 0 \leqslant \tau \leqslant t \tag{A.1.4}$$

where f denotes a functional dependence in the interval $0 \leqslant \tau \leqslant t$ between w and v.

Consider the subset of feedback blocks of the type (A.1.4) which satisfies the following integral inequality

$$\eta(0,t) = \int_0^t w^T v \, d\tau \geqslant -\gamma_o^2 - \gamma_o^2 \sup||x(\tau)|| \qquad 0 \leqslant \tau \leqslant t \tag{A.1.5}$$

where $\gamma_o \geqslant 0$

Definition : Hyperstability.

The feedback system (A.1.1)-(A.1.4) is hyperstable if there exists constants $\delta \geqslant 0$ and $\gamma \geqslant 0$ such that all the solutions x(t) of eqn (A.1.1) and (A.1.2) satisfy the inequality

$$||x(t)|| < \delta \; (||x(0)|| + \gamma) \qquad t \geqslant 0 \qquad (A.1.6)$$

for all feedback blocks $w = f(v(\tau), t)$ satisfying the inequality (A.1.5)

<u>Definition : Asymptotic hyperstability.</u>

The feedback system (A.1.1)-(A.1.4) is asymptotically hyperstable if it is hyperstable and in addition

$$\lim_{t \to \infty} x(t) = 0 \qquad (A.1.7)$$

for all feedback blocks $w = f(v(\tau), t)$ satisfying the inequality (A.1.5).

The necessary and sufficient condition for the feedback system described by (A.1.1)-(A.1.4) to be asymptotically hyperstable is that the transfer function matrix

$$Z(s) = C(sI-A)^{-1} B + J \qquad (A.1.8)$$

be strictly positive real, i.e.

i) the poles of $Z(s)$ lie in the left half of the complex plane.

ii) $Z(j\omega) + Z^T(-j\omega)$ should be positive definite hermitian for all real ω.

As an illustration of the hyperstability design of MRAC systems Landau (1974) considered the simple plant-model equations given by

$$\dot{x}_p = A_p(t) x_p + B_p(t)\, u \qquad (A.1.9)$$

$$\dot{x}_m = A_m\, x_m + B_m\, u \qquad (A.1.10)$$

where x_p and x_m are the plant and model states (n dimensional) u is the input vector (m dimensional), A_m B_m are constant matrices and $A_p(t)$, $B_p(t)$ are adjustable time-varying matrices. The error equation e $(e=x_m-x_p)$ can be arranged in the form

$$\dot{e} = A_m e + [A_m - A_p(t)]\ x_p + [B_m - B_p(t)]u \qquad (A.1.11)$$

In the hyperstability approach one requires a linear compensator D which generates the vector v

$$v = D\ e \qquad (A.1.12)$$

where D is a square matrix. The second and third term in (A.1.11) are replaced by

$$[A_m - A_p(t)]\ x_p + [B_m - B_p(t)]\ u = -\ w \qquad (A.1.13)$$

and w is defined as a function of $v(t,\tau)$

$$w = [\int_0^t \phi(v(\tau),t)d\tau - A_m + A_p(0)]x_p + [\int_0^t \psi(v(\tau),t)d\tau - B_m + B_p(0)]\ u \qquad (A.1.14)$$

Equations (A.1.11)-(A.1.14) define a nonlinear time-varying feedback system. The linear part with input -w and output v and transfer function matrix $Z(s) = D(SI-A_m)^{-1}$ is given by (A.1.11) and (A.1.12). The nonlinear time-varying part is given by (A.1.14).

In the hyperstability design ϕ and ψ are chosen such that

$$\int_0^t v^T\ w\ d\tau \geq -\ \gamma_o^2 \qquad t \geq 0, \gamma_o^2 < \infty \qquad (A.1.15)$$

Particular solutions for ϕ and ψ satisfying (A.1.15) are

$$\dot{A}_p(t) = \phi(v(t),t) = R\ v\ x_p^T \qquad (A.1.16)$$

$$\dot{B}_p(t) = \psi(v(t),t) = Q\ v\ u^T \qquad (A.1.17)$$

where R and Q are positive definite matrices. If asymptotic stability is required then D should be chosen to ensure a strictly positive real transfer function matrix $Z(s)$. The hyperstable adaptation law is given by (A.1.16) and (A.1.17).

APPENDIX 2.

SOME CONCEPTS OF LINEAR ALGEBRA AND GENERALIZED INVERSES OF MATRICES.

This appendix contains a collection of definitions of certain matrix algebra concepts used in this thesis together with some properties of these concepts.

i) Range and null spaces (Grossman, 1980), (Graham, 1979)

a) The range space of an mxn matrix A is defined as

$$R(A) = \{y \in R^m : Ax = y \text{ for some } x \in R^n\} \qquad (A.2.1)$$

R(A) is a subspace of R^m. The rank of A is

$$\text{rank}(A) = \dim R(A) \qquad (A.2.2)$$

where dim stands for dimension of. If B is an nxk matrix then

$$R(AB) \subseteq R(A) \qquad (A.2.3)$$

If rank (A) = rank (B) then

$$R(AB) = R(A) \qquad (A.2.4)$$

b) The null space or kernel of an mxn matrix A is defined as

$$N(A) = \{x \in R^n : Ax = 0\} \qquad (A.2.5)$$

N(A) is a subspace of R^n. The nullity ν of A is

$$\nu(A) = \dim N(A) \qquad (A.2.6)$$

If A has full rank m then

$$\nu(A) = n - \text{rank}(A) = n - m \qquad (A.2.7)$$

If a matrix H is kxm then

$$N(A) \subseteq N(HA) \qquad (A.2.8)$$

If $\nu(A) = \nu(H)$ then

$$N(A) = N(HA) \tag{A.2.9}$$

Remark A.2.1 If A is square and nonsingular then

$$\text{rank}(A) = n \text{ and } \nu(A) = 0 \tag{A.2.10}$$

ii) Annihilators (Nering, 1970)

The annihilator $W^{\perp}$ of a subset W of a space V is the set of all linear functionals $\emptyset$ such that $\emptyset\alpha = 0$ for all $\alpha \varepsilon W$.

If W is a subspace of dimension m then $W^{\perp}$ is of dimension n-m and

$$(W^{\perp})^{\perp} = W \tag{A.2.11}$$

A matrix A has a left annihilator $A_L^{\perp}$ if $m > n$, i.e.

$$A_L^{\perp} A = 0 \tag{A.2.12}$$

and a right annihilator if $m < n$ i.e.

$$A A_R^{\perp} = 0 \tag{A.2.13}$$

Note that the annihilators of A are non-unique.

iii) The generalized inverse of matrices (Graybill, 1969), (Rao, 1971).

The generalized inverse of an mxn matrix A is an nxm matrix A^g satisfying

$$AA^gA = A \tag{A.2.14}$$

Some properties of A^g are listed below

i) A^g is non-unique

ii) If A has full rank m and $m < n$ then

$$AA^g = I_m \tag{A.2.15}$$

iii) $AA^g, A^gA, I_m - AA^g, I_n - A^gA$ are idempotent (an idempotent matrix P satisfies $P^2=P$)

iv) The equation

$$Ax = y \tag{A.2.16}$$

where A is mxn and y is an mx1 vector is consistent iff

$$AA^g y = y \tag{A.2.17}$$

If (A.2.17) is satisfied then the solution of (A.2.16) is given by

$$x = A^g y + (I_n - A^g A)\ h \tag{A.2.18}$$

where h is an arbitrary nx1 vector. If x and y are matrices (X,Y) of dimension nxq and mxq respectively then h is an arbitrary nxq matrix (H).

v) If A is square and nonsingular then

$$A^g = A^{-1} \tag{A.2.19}$$

i.e. A^g is the usual inverse.

vi) If A has full rank then A^g may be computed as

$$A^g = A^T\ (AA^T)^{-1} \qquad \text{if } m < n \tag{A.2.20}$$

$$A^g = (A^TA)^{-1}\ A^T \qquad \text{if } m > n \tag{A.2.21}$$

where A^T is the transpose of A.

vii) A particular generalized inverse is the Penrose pseudo-inverse A^+ which satisfies four conditions

$$\begin{aligned} &(a) \quad AA^+A = A \\ &(b) \quad A^+A\,A^+ = A^+ \\ &(c) \quad (AA^+)^T = AA^+ \\ &(d) \quad (A^+A)^T = A^+A \end{aligned} \tag{A.2.22}$$

The Penrose inverse is unique.

APPENDIX 3.

THE PROOF OF THEOREMS 5.10.1-5.10.3

In this appendix the proof of theorems 5.10.1 - 5.10.3 is given.

Theorem 5.10.1

Proof: (i) The zeros of S(A,B,C) are invariant under state feedback and are therefore equal to the zeros of $S(A_k,B,C)$.

Noting that

$$CA^{i-1}A_k = CA^i - CA^{i-1}B(CA^{k-1}B)^{-1}CA^i = \begin{cases} CA^i & 1 \leqslant i < k \\ 0 & i = k \end{cases} \qquad (A.3.1)$$

and therefore, from (A.3.1) and (5.10.6)

$$CA^{i-1}A_k \quad \nu^* = 0 \qquad 1 \leqslant i \leqslant k \qquad (A.3.2)$$

i.e. $A_k\ \nu^*$ belongs to $\bigcap_{i=1}^{k} N(CA^{i-1})$ which is equal to ν^*. Therefore

$$A_k \quad \nu^* \subset \nu^* . \qquad (A.3.3)$$

and hence from (5.10.6) n-km eigenvalues of A_k are the zeros of S(A,B,C). The remaining km eigenvalues will now be shown to be equal to zero. Using a similarity transformation T where

$$\bar{x} = Tx \qquad (A.3.4)$$

the system is now given by

$$\dot{\bar{x}} = T\ A_k\ T^{-1}\ \bar{x} \qquad (A.3.5)$$

$$y = C\ T^{-1}\ \bar{x} \qquad (A.3.6)$$

Let

$$T = \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{k-1} \\ \hline P \end{bmatrix} \quad ; \quad T^{-1} = [C^g:(CA)^g:\ldots:(CA^{k-1})^g:P^g] \qquad (A.3.7)$$

where C^g is the generalized inverse of C.

The condition $TT^{-1} = I_n$ implies that

$$\begin{aligned}
(CA^i)(CA^i)^g &= I_m && i = j;\ i = 0,\dots,K-1 \\
&= 0 && i \neq j \\
(CA^i)P^g &= 0 && i = 0,\dots,K-1 \\
PP^g &= I_{n-km} && \\
P(CA^i)^g &= 0 && i = 0,\dots K-1
\end{aligned}$$

(A.3.8)

$$\dot{\bar{x}} = \begin{bmatrix} C \\ \hdashline CA \\ \hdashline \vdots \\ \hdashline CA^{K-1} \\ \hdashline P \end{bmatrix} [A-B(CA^{K-1}B)^{-1}CA^K]\,[C^g : (CA)^g : \dots : (CA^{K-1})^g : P^g]\bar{x} \tag{A.3.9}$$

$$y = C[C^g : (CA)^g : \dots : (CA^{K-1})^g : P^g] \tag{A.3.10}$$

From (A.3.1)

$$\dot{\bar{x}} = \begin{bmatrix} CA \\ \hdashline CA^2 \\ \hdashline \vdots \\ \hdashline CA^K \\ \hdashline PA_K \end{bmatrix} [C^g : (CA)^g : \dots : (CA^{K-1})^g : P^g]\bar{x}$$

$$= \begin{array}{c} \left[\begin{array}{cccc:c} O_m & I_m & O & \cdots & O_{m,n-Km} \\ \hdashline O_m & O_m & I_m & & \vdots \\ \vdots & \vdots & & I_m & \\ O_m & O_m & O_m & O_m & O_{m,n-Km} \\ \hline PA_K C^g & PA_K(CA)^g & \cdots & & PA_K P^g \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \} \; n-Km \end{array} \\ \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{Km} \; \underbrace{\qquad\qquad}_{n-Km} \end{array} \bar{x} \tag{A.3.11}$$

and $y = [I_m : 0 : 0 : \cdots : 0] \bar{x}$ (A.3.12)

This is the standard observability decomposition. Therefore the n-km eigenvalues of PA_kP^g are the zeros of S(A,B,C). Furthermore, km eigenvalues of the matrix A_k are zero-valued. The system zeros are therefore given by

$$sp\,(A_k) - \{0\}^{km} \quad (A.3.13)$$

This proves (i)(a). For an alternative proof (see El-Ghezawi et al., 1982b).

To prove (i)(b) let M_k be a basis for ν^*. Then

$$w_i = M_k\alpha_i \quad , \quad \alpha_i \neq 0 \quad (A.3.14)$$

$$A_kM_k\alpha_i = M_k\alpha_i z_i$$

and

$$M_k^gA_kM_k\alpha_i = z_iM_k^gM_k\alpha_i$$

$$= z_i\alpha_i \quad (A.3.15)$$

where M_k^g is any matrix satisfying $M_k^gM_k = I_{n-km}$. Therefore, the system zeros are given by the eigenvalues of

$$M_k^gA_kM_k \quad (A.3.16)$$

(ii) From (A.3.13) — (A.3.15) (ii)(a) and (ii)(b) are proved respectively

(iii) Substituting in (5.10.7) with x replaced by $w_i = M_k\,\alpha_i$ yields

$$g_i = -(CA^{k-1}B)^{-1}CA^kM_k\alpha_i \quad (A.3.17)$$

This proves (iii)

<u>Theorem 5.10.2</u>

<u>Proof:</u> Let ν be a basis matrix for ν^* then (Wonham, 1979)

$$A\nu = \nu J + \hat{B} \quad (A.3.18)$$

By the definition of ν^* there exists an m x n matrix F such that

$$\hat{B} = BF\nu. \quad (A.3.19)$$

From (A.3.18)

$$CA^{i}\nu = CA^{i-1}\nu\, J + CA^{i-1}\hat{B} \qquad 1 \leqslant i \leqslant k$$

If $i < k$ then $CA^{i-1}\hat{B} = 0$

If $i = k$ then $CA^{k}\nu = CA^{k-1}\hat{B}$

$$= CA^{k-1}BF\nu$$

i.e. a valid solution for F is

$$F = (CA^{k-1}B)^{-1}CA^{k}.$$

Also,

$$[A - B(CA^{k-1}B)^{-1}CA^{k}]\nu = A\nu - B(CA^{k-1}B)^{-1}CA^{k}\nu$$

$$= A\nu - B(CA^{k-1}B)^{-1}CA^{k-1}(\nu J + \hat{B})$$

$$= A\nu - B(CA^{k-1}B)^{-1}CA^{k-1}BF\nu \; ; \quad \text{since } CA^{k-1}\nu = 0$$

$$= A\nu - BF\nu$$

$$= (A - BF)\nu$$

which has range in ν^{*}. This proves that $A_k\ \nu^{*} \subset \nu^{*}$ and the remainder of the theorem follows in a similar manner to theorem 5.10.1.

This result has a similar interpretation to Theorems 5.5.1 and 5.10.1 but may be difficult to apply in practice as k is not necessarily known and, although the zeros are a subset of the eigenvalues of A_k, we have not identified which subset. In this sense the result is primarily of theoretical interest.

<u>Theorem 5.10.3.</u>

<u>Proof:</u> The defining relations in Theorems 5.10.1 and 5.10.2 are needed to establish the proof. Since the existence of zeros assures the existence of the state zero directions we shall here establish the conditions for the state zero directions to exist.

The state zero directions are given by

$$w \subseteq \bigcap_{i=1}^{k} N(CA^{i-1})$$

$$\subseteq \bigcap_{i=1}^{k-1} N(CA^{i-1}) \bigcap N(CA^{k-1})$$

Since $\bigcap_{i=1}^{k-1} N(CA^{i-1})$ may at most contain d_{max} linear independent vectors that belong to R(B) and since $N(CA^{k-1})$ is always independent of R(B) because of (5.10.4), then for w to be nonempty it is necessary (but not sufficient) that every

$$N(CA^{i-1}) \qquad\qquad i = 1, 2, \ldots, k$$

should be large enough so as to contain vectors which are independent of R(B). This is guaranteed only if

$$\min\left(\dim(N(CA^{i-1}))\right) > d_{max} \qquad i=1, 2, \ldots, k \qquad (A.3.20)$$

or

$$n - m > d_{max}$$

i.e. $$n > m + d_{max}. \qquad (A.3.21)$$

This is only a necessary condition since w may be empty even if

$$n > m + d_{max}.$$

Remark

If $d_1 < d_{max}$ we can use the condition $n > m + d_1$ as a first check whether zeros may exist. (If $n \leqslant m + d_1$ no zeros exist). At successive stages of the algorithms in Theorems 2 and 3 the matrices $CA^{i-1}B$ $(i = 1,2,\ldots k)$ are determined and the check $n > m + d_i$ should be repeated. Failure of this condition to hold at any stage indicates the non-existence of zeros and the algorithm should be terminated.

Even if $n > m + d_i \quad \forall i = 1, 2, \ldots, k$ we still require

$$w \subseteq \bigcap_{i=1}^{k} N(CA^{k-1}) \neq \emptyset$$

for zeros to exist.

Corollary:

A necessary condition for a uniform rank system to have zeros is that

$$n > 2m.$$

Proof:

This follows from Theorem 5.10.3 since in this case

$$d_{max} = m$$

and

$$n > m + m = 2m.$$