Lax-Darboux Schemes and Miura-type Transformations

George Berkeley

Department of Applied Mathematics

University of Leeds

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

- Elements of Part I are based on G. Berkeley, A. V. Mikhailov and P. Xenitidis, "Darboux transformations with tetrahedral reduction group and related integrable systems", preprint arXiv:1603.03289v1.
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Abstract

This thesis is concerned with the study of integrable differential difference and partial difference equations. The thesis splits into two parts.

In part I, we begin by studying Darboux transformations of Lax operators related to the tetrahedral reduction group. We construct a general Darboux matrix of a specified type, then use first integrals to derive interesting novel subcases. In the process, we derive Bäcklund transformations of an associated system of partial differential equations. Subsequently, we utilise the derived Darboux matrices to construct integrable discrete systems, their generalised symmetries and conservation laws. Furthermore, we consider integrable reductions and potentiations of the obtained systems, as well as Miura-type transformations. Moreover, we demonstrate how one can use the Bäcklund transformations to derive explicit solutions to the associated continuous system. Finally, we present results regarding the octahedral reduction group. This is the first time that semi and fully discrete systems have been associated to $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sl}_4(\mathbb{C})$-based automorphic Lie algebras.

In part II, we show that for differential difference equations which possess a Lax pair of a particular type, one can construct Miura-type transformations by considering invariants of associated algebraic structures. We begin by introducing the objects required by the construction and discussing the general theory. Subsequently, we demonstrate the efficacy of the construction by deriving Miura-type transformations related to the Narita-Itoh-Bogoyavlensky lattice and the discrete Sawada-Kotera equation, some of which appear to be new. Furthermore, we discuss how the construction can be applied to partial difference equations and systems, providing examples of its successful application. In the case of systems, the derived Miura-type transformations also appear to be new.

Contents

Chapter 0

Introductory concepts

There are many faces to integrability, some of which we shall meet as this thesis progresses. We take this chapter to introduce the basic topics within the area which are most relevant to the remainder of the text. Our aim is to give a general overview of the concepts involved, as well as to provide examples and references for further study.

The thesis proper, is composed of two parts. Part I is concerned with the study of integrable systems related to finite reduction groups. Part II focuses on the construction of Miura-type transformations, primarily for differential difference equations, although partial difference equations are also discussed. At the beginning of each part we provide specific background material.

Lax pairs play a crucial role in the theory of integrability, and will form a fundamental part of the story in later chapters, thus we now discuss their historical connections to integrable systems.

0.1 Lax pairs

The first partial differential equation (PDE) shown to possess what is now known as a Lax pair, was the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x + u_{xxx}.$$

This equation has multiple applications of physical significance, but was first derived to model waves on shallow water surfaces [40]. In 1967, Gardner, Greene, Kruskal and Miura (GGKM) demonstrated [32] that the initial value problem for the KdV equation, with suitable initial data, can be solved by considering an associated linear problem.

To see this, consider the linear system

$$\Psi_{xx} + u\Psi = \lambda\Psi, \tag{0.1a}$$

$$\Psi_t = 4\Psi_{xxx} + 6u\Psi_x + 3u_x\Psi, \tag{0.1b}$$

where Ψ is a scalar function of x and t. If one computes the compatibility condition, $\Psi_{xxt} = \Psi_{txx}$, where we assume Ψ is not identically zero, then modulo (0.1) one finds

$$\Psi_{xxt} = \Psi_{txx} \qquad \text{if and only if} \qquad \lambda_t - u_t + 6uu_x + u_{xxx} = 0.$$

Hence, if system (0.1) is compatible, then u satisfies the KdV equation if and only if $\lambda_t = 0$. Why is this significant? Well, suppose in (0.1) u satisfies the KdV equation and that we wish to solve its initial value problem with initial data $u(x, 0) = f(x)$. Then at time $t = 0$, (0.1a) becomes

$$\Psi_{xx} + (f(x) - \lambda)\Psi = 0.$$

This equation defines an eigenvalue problem, well known from quantum mechanics. Given $f(x)$, often called the potential, we wish to find the solution $\Psi(x, \lambda)$ as $x \to \pm\infty$. This is known as the direct scattering problem. The solution $\Psi(x, \lambda)$ is asymptotically described by some data, known as the scattering data. Thus, given a potential $f(x)$, i.e.

an initial condition, we associate to this potential scattering data. The inverse scattering problem is to reconstruct $f(x)$, given the scattering data. But how does this help solve the KdV equation? Well $\lambda_t = 0$, thus given the scattering data at $t = 0$, we may use (0.1b), along with this fact, to compute its time evolution. Indeed, this reduces to solving simple ordinary differential equations. But now we have scattering data dependent upon t, which we may view as a parameter. If the inverse scattering problem can be solved, then one can reconstruct the potential $u(x, t)$, i.e. the solution to the initial value problem. It can be shown that to solve the inverse problem, one must solve a linear integral equation

$$K(x, y; t) + F(x + y; t) + \int_x^\infty K(x, z; t)F(z + y; t)dz = 0,$$

for the function K, where F is determined by the scattering data. This equation is known as the Gelfand-Levitan-Marchenko equation. Having K, one can recover the potential from the formula

$$u(x, t) = 2\frac{\partial}{\partial x}K(x, x; t).$$

There it is, a solution to the initial value problem for the KdV equation! Of course, in practice, constructing the scattering data and solving the integral equation are non trivial tasks. Moreover, there are technicalities regarding the initial data which we have skipped over. However, in principle, we have solved the KdV equation. For so-called reflectionless potentials, each step can be carried out explicitly and special solutions called solitons can be derived [33], whose full behaviour had previously only been captured numerically.

Although it was shown that the KdV equation could be treated in this way, it was not clear whether the technique could be applied to other equations. Subsequently, Peter Lax, whom Lax pairs are named after, showed [41] that if one has a linear system

$$L\Psi = \lambda\Psi, \quad \Psi_t = A\Psi,$$

where L and A are differential operators, then if the operator equation

$$L_t - [A, L] = 0,$$

is equivalent to a PDE, it necessarily follows that $\lambda_t = 0$. Thus, in principle, the equation will succumb to the same analysis as the KdV equation. We shall call the pair of operators, L and A, a Lax pair, and at times will use the phrase Lax representation.

Lax opened the door to the prospect of solving other nonlinear PDEs of physical interest. Indeed, many equations then fell to this method. In 1972, Zakharov and Shabat [77] showed that the non-linear Schrödinger equation could be solved via inverse scattering. Later that year, Wadati proved [73] the same was true for the modified KdV equation. In 1973, the sine-Gordon equation was solved [2]. The method became known as the inverse scattering transform (IST). Many nonlinear systems have since been tackled in this way, the key difficulty being the discovery, it if exists, of an associated linear problem.

The existence of the linear problem is essential. If one possesses a Lax pair, then, in principle, IST techniques can be used to solve the corresponding PDE. Although, we acknowledge that the procedure is by no means algorithmic. Furthermore, alternative methods of solution generation follow from a Lax pair, such as the dressing method [78]. For this reason, we consider the definition of integrability to be the existence of a Lax pair. As a technical point, we note that one has to be a little bit careful here as not all Lax pairs are suitable, cf. [25] and [23]. We will not consider the IST itself any further; instead, our focus will solely fall upon the Lax pairs. Moreover, we will consider them in a slightly different form, as we describe below. For a thorough account of the IST, see [1, 29].

As a working definition, we say that a PDE possesses a Lax representation if it is equivalent to the compatibility condition of a linear system

$$\partial_x(\mathbf{\Psi}) = U\mathbf{\Psi}, \ \ \partial_t(\mathbf{\Psi}) = V\mathbf{\Psi},$$

where U and V are $d \times d$ matrices. This system is compatible if and only if

$$U_t - V_x + [U, V] = 0. \tag{0.2}$$

Hence, we require (0.2) to be equivalent to the PDE in question. Equation (0.2) is often referred to as the zero curvature condition. The Lax pair (0.1) can be cast in this form in the following way. Introduce the variables $\Psi^1 = \Psi$ and $\Psi^2 = \Psi_x$, then (0.1a) can be expressed as

$$\partial_x(\Psi^1) = \Psi^2,$$
$$\partial_x(\Psi^2) = (\lambda - u)\Psi^1.$$

Of course the same can be done for the t-derivatives, then (0.1) is written in matrix form as

$$\partial_x(\mathbf{\Psi}) = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \mathbf{\Psi},$$
$$\partial_t(\mathbf{\Psi}) = \begin{pmatrix} -u_x & 2(u+2\lambda) \\ -2u^2 - 2\lambda u + 4\lambda^2 - u_{xx} & u_x \end{pmatrix} \mathbf{\Psi}.$$

Note, the spectral parameter λ occurs polynomially. In general, there could be a rational or even elliptic dependence.

So far we have only mentioned Lax pairs in relation to PDEs. An analogous concept exists for differential difference as well as partial difference equations. In each case, the aim is to rewrite the given nonlinear equation as a compatibility condition for a linear system. We will encounter and construct semi and fully discrete Lax pairs in subsequent sections of this thesis, thus we fix some notation and provide a simple example of each.

0.1.1 Semi-discrete Lax pair

By differential difference equation (D∆E) we mean an equation relating derivatives and shifts of a function of a discrete variable, n, and a continuous variable, t. Associated to the discrete variable is a shift operator $\mathcal{S}$. We take this to act in the following way, $\mathcal{S}^\alpha(u(n,t)) = u(n+\alpha,t) = u_\alpha$ for all $\alpha \in \mathbb{Z}$. A Lax pair for a D∆E of this form is a linear system

$$\mathcal{S}(\mathbf{\Psi}) = M\mathbf{\Psi}, \quad \partial_t \mathbf{\Psi} = U\mathbf{\Psi},$$

such that the compatibility condition

$$M_t + MU - \mathcal{S}(U)M = 0,$$

is equivalent to the D∆E in question. A simple example is the provided by the pair

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}0 & u\\ -1 & \lambda\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$
$$D_t\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}u & \lambda u_{-1}\\ -\lambda & \lambda^2 + u_{-1}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$

where numerical subscripts denote shifts. The compatibility condition is equivalent to the Volterra equation

$$u_t = u(u_1 - u_{-1}). \tag{0.3}$$

We will meet this equation again later. For an in-depth analysis of (0.3) and its related structures, consult Chapter 4 of [72]. For information regarding the IST for D∆Es, see [3, 4].

0.1.2 Discrete Lax pair

By partial difference equation (P∆E) we understand an equation defined on a $\mathbb{Z}^2$ lattice. That is, we consider functions which depend on two discrete variables n and m, such that n, $m \in \mathbb{Z}$. The equation itself relates the values of this function at different lattice points. Subscripts are used to denote shifts. For instance, $u(n+\alpha, m+\beta) = u_{\alpha,\beta}$. We associate to the discrete variable n the shift operator $\mathcal{S}$, and to m, the shift operator $\mathcal{T}$. Consequently, the Lax pair takes the form

$$\mathcal{S}(\mathbf{\Psi}) = M\mathbf{\Psi}, \quad \mathcal{T}(\mathbf{\Psi}) = N\mathbf{\Psi},$$

with compatibility condition

$$\mathcal{S}(N)M - \mathcal{T}(M)N = 0.$$

A simple example is given by

$$\mathcal{S}(\mathbf{\Psi}) = \begin{pmatrix} -u_{0,0} & p^2 + \lambda^2 + u_{0,0}u_{1,0} \\ -1 & u_{1,0} \end{pmatrix} \mathbf{\Psi},$$
$$\mathcal{T}(\mathbf{\Psi}) = \begin{pmatrix} -u_{0,0} & q^2 + \lambda^2 + u_{0,0}u_{0,1} \\ -1 & u_{0,1} \end{pmatrix} \mathbf{\Psi},$$

whose compatibility condition is equivalent to

$$(u_{0,1} - u_{1,0})\,(u_{0,0} - u_{1,1}) = p^2 - q^2, \tag{0.4}$$

the lattice potential KdV (lpKdV). As an aside, we note that (0.4) has the special property that it is multidimensionally consistent, cf. [63]. One can use this property to derive the above Lax pair, a fact true for all such equations [64]. Equations possessing this property are classified in [6], with the resulting set of equations acquiring the name, the ABS list, after the names of the authors, Adler, Bobenko and Suris. Equation (0.4) corresponds to H1 in their list. There exist discrete equations which have Lax pairs but do not possess this property in its original form. To include such equations, the multidimensional consistency approach was extended in [7, 20]. For examples of the discrete IST, see [24] and [22]. In the former, (0.4) is considered with precisely the above Lax pair, and soliton solutions derived. Although, we note that these were previously known via other techniques. For an alternative approach, see [14].

0.2 Symmetries

We take as a definition of integrability the existence of a Lax pair, but another key feature of integrability is the existence of generalised symmetries. In [41], Lax introduced the formal framework for Lax pairs, however he also highlighted that the KdV equation was but one of an infinite hierarchy of equations related to (0.1a). Let L denote the

Schrödinger operator associated to (0.1a). Lax showed that for each $k \in \mathbb{N}$ there exists an operator A_k, such that the operator equation

$$L_t - [A_k, L] = 0,$$

is equivalent to a PDE of order $2k - 1$. The initial equations in the hierarchy are given by

$$u_{t_1} = u_x,$$

$$u_{t_2} = u_{xxx} + 6uu_x,$$

$$u_{t_3} = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x.$$

All equations are compatible, since, $u_{t_i t_j} = u_{t_j t_i}$ for all i and j. Note, when performing this computation, the equality follows modulo the equations in question, thus one must remove all t_i and t_j derivatives. Equations compatible in this manner are said to be symmetries of one another. Thus, the KdV equation has an infinite hierarchy of symmetries of increasing order. It is not alone, cf. [65]. Indeed, this property is a key feature of integrable equations, which turns out to be incredibly useful when considering classification problems.

At first this may not seem to be the case. Given an equation, it appears a difficult task to confirm if such a hierarchy exists. The concept of a recursion operator is of use here. Observe,

$$u_{t_2} = (D_x^2 + 4u + 2u_x D_x^{-1})(u_x),$$

$$u_{t_3} = (D_x^2 + 4u + 2u_x D_x^{-1})(u_{xxx} + 6uu_x).$$

In fact it turns out that for all k, one can generate the right hand side of $u_{t_{k+1}}$ by applying the operator $R = D_x^2 + 4u + 2u_x D_x^{-1}$ to the right hand side of u_{t_k}. We say that R is a recursion operator for the KdV equation. One can generate the whole hierarchy by repeated application of R. Note, it is not obvious that the action of R is well defined, since it depends upon the inverse of D_x, which only makes sense to apply to total derivatives. It can be shown (cf. Theorem 4.7 [66]) that $R^k(u_x) \in \mathrm{Im}(D_x)$ for all $k \in \mathbb{N}$ and thus in this case there is not an issue. But how does this help in classification problems? We explain the general idea without getting into details.

If an equation possesses an infinite hierarchy of symmetries with increasing order, then it can be shown to possess a formal recursion operator. This is an operator $\tilde{R} = a_1 D_x + a_0 + a_{-1} D_x^{-1} + a_{-2} D_x^{-2} + \ldots$, which satisfies a particular relation, which we omit.

It can be shown that Formal recursion operators possess the following two useful properties. The first is that the functions

$$\rho_i = \mathrm{res}(\tilde{R}^i), \quad i = -1, 1, 2, \ldots, \quad \rho_0 = \mathrm{res}\log(\tilde{R}),$$

where $\mathrm{res}(R^k)$ is the coefficient a_{-1} occurring in R^k, define densities of local conservation laws. That is, $D_t(\rho_i) \in \mathrm{Im}(D_x)$. The functions ρ_i are known as canonical densities. The second useful feature is that the coefficients a_i, can be expressed in terms of derivatives of the corresponding PDE. These two properties combined can be used to generate integrability criteria.

The logic is as follows. If an equation is integrable, then it should possess an infinite hierarchy of symmetries. Thus, it should possess a formal recursion operator, and consequently the ρ_i, which we can express in terms of the equation itself, should be densities of local conservation laws. An example is in order. Let us consider the equation

$$u_t = u_{xxx} + f(u, u_x). \tag{0.5}$$

It can be shown that ρ_{-1} and ρ_0 are constant, hence are trivially densities of conservation laws. The first nontrivial condition is provided by ρ_1, which is found to be

$$\rho_1 = \frac{1}{3}\left(\frac{\partial f}{\partial u_1}\right).$$

If (0.5) is integrable, this must be a density of a conservation law. To check this, we make use of the Euler operator. Indeed,

$$\frac{\delta}{\delta u}\left(D_t(\rho_1)\right) = \sum_k (-1)^k D_x^k \left(\frac{\partial D_t(\rho_1)}{\partial u_k}\right) = 0,$$

if and only if $D_t(\rho_1) \in \text{Im}(D_x)$. Using (0.5) to remove time derivatives occurring in the latter relation, the result must be satisfied identically. The x-derivatives of u are independent, hence by setting their coefficients to zero we obtain relations which f must satisfy. Once these relations are satisfied, we move on to ρ_2, then ρ_3, and so on until f is determined. Each ρ_i filters the form of f further. For equations of the type (0.5), one need only consider ρ_1 and ρ_2 before a complete classification is provided. One can check the final equations found against alternative integrability criteria. In the case we have considered, the found equations all possess Lax pairs, hence are integrable in this sense. Variations of this method have been used successfully in many classification problems. What we have described is generally known as the symmetry approach to the classification of integrable systems. It has the advantage of allowing one to test the integrability of a given equation algorithmically, and to obtain exhaustive lists of integrable equations of a particular form. For information regarding recursion operators for PDEs, we refer the reader to [70], and references therein. For a detailed account of the symmetry approach, including solved classification problems, one can consult [11, 54, 56], and references therein.

One can also consider symmetries of DΔEs and PΔEs. In both cases, by symmetry we understand a compatible evolutionary DΔE. For DΔEs, requiring the existence of higher symmetries also constitutes a useful classification tool. For instance, all integrable DΔEs of the form

$$u_t = f(u_{-1}, u, u_1),$$

among other classes of equation have been classified using this approach. We refer the reader to [76] for a review of the results in this area. For PΔEs, the approach becomes harder to apply, although progress has been made, cf. [42, 44, 57, 58]. Moreover, all quad equations found as integrable using multidimensional consistency, i.e. the ABS list, have also been found to possess infinite hierarchies of symmetries, cf. [75].

0.3 Miura and Bäcklund transformations

A remarkable feature displayed by integrable equations is that there often exist relations among their solutions. For instance, suppose one has a solution, v, of the equation

$$v_t = v_{xxx} - 6v^2 v_x. \tag{0.6}$$

Then, if we define u by the formula

$$u = -(v^2 + v_x), \tag{0.7}$$

it follows that u satisfies

$$u_t = u_{xxx} + 6uu_x,$$

the KdV equation. This can be seen by observing that

$$u_t - (u_{xxx} + 6uu_x) = -(2v + \partial_x)(v_t - (v_{xxx} - 6v^2 v_x)).$$

Equation (0.6) is known as the modified KdV (mKdV) equation. This transformation, in a slightly different form, was first presented by Miura [60]. Thus, we shall name differential substitutions such as (0.7), Miura-type transformations (MT). A slight generalisation of (0.7) was used to show that the KdV equation has an infinite number of local conservation laws [61]. Also, analysis of (0.7) led to the discovery of the link between the KdV equation and the Schrödinger operator, a key development on the path to the IST. Hence, MTs can be a useful tool in the analysis of integrable systems. Another example of their use can be found when considering classification problems. Often one finds a core number of integrable equations, with others related to this core few via MTs. For example, if one wishes to classify integrable equations of the type

$$u_t = u_3 + f(x, u, u_x, u_{xx}),$$

then up to MTs, one finds only the KdV and Krichever-Novikov equation, [56].

As usual, there are DΔE and PΔE equivalents. We consider these in detail in later chapters. At the moment, we turn our attention to another important type of transformation, the Bäcklund transformation.

In the case of a MT, one equation can be seen to be sitting above the other, since the MT is not invertible. Whereas for Bäcklund transformations the equations are on an equal footing. Let us consider a simple example. The equation

$$u_t = 3u_x^2 + u_{xxx}, \tag{0.8}$$

is known as the potential KdV (pKdV) equation. It can be shown that if u is a solution of (0.8), then $\tilde{u}$, defined by the relations

$$\tilde{u}_x = \frac{1}{2}p^2 - u_x - \frac{1}{2}(u - \tilde{u})^2, \tag{0.9a}$$

$$\tilde{u}_t = -u_t - (u - \tilde{u})(u_{xx} - \tilde{u}_{xx}) + 2(u_x^2 + u_x\tilde{u}_x + \tilde{u}_x^2), \tag{0.9b}$$

is also a solution. Conversely, if $\tilde{u}$ is a solution of (0.8), by defining u by the above relations, we obtain a second solution of (0.8). Another way to see this is the following. If one eliminates u from (0.9), one finds $\tilde{u}$ satisfies (0.8). Similarly, if one eliminates $\tilde{u}$ from (0.9), one finds u satisfies (0.8). Hence, we can view relations (0.9) as defining an invertible transformation between the solutions u and $\tilde{u}$, which we refer to as a Bäcklund transformation. As u and $\tilde{u}$ both satisfy the same equation, (0.9) defines an auto-Bäcklund transformation. In general, the relations may relate different equations, in this case we simply use the term Bäcklund transformation. One can derive relations (0.9) directly by making an ansatz for the form of the relation and solving the resulting compatibility conditions, cf. [69]. We will present an alternative derivation in a later section using a Lax-Darboux scheme.

A key strength of Bäcklund transformations is their use in generating solutions. Often one starts from a trivial, possibly constant solution, then by solving the Bäcklund relations, derives a nontrivial solution. Moreover, given multiple Bäcklund transformations, one can generate further solutions algebraically. Given a solution u, let us denote the solution obtained using (0.9) as $\tilde{u} = \mathbb{B}(u,p)$, where we observe that the solution obtained is dependent upon p, the parameter occurring in (0.9). By changing the parameter from p to q, we may derive a different solution $\hat{u} = \mathbb{B}(u,q)$. Furthermore, starting with initial solution $\tilde{u}$ and Bäcklund parameter q, we obtain a solution $\hat{\tilde{u}} = \mathbb{B}(\tilde{u},q)$. Similarly, starting with initial solution $\hat{u}$ and parameter p, we obtain $\tilde{\hat{u}} = \mathbb{B}(\hat{u},p)$. If we assume that a solution exists such that $\tilde{\hat{u}} = \hat{\tilde{u}}$, then using the Bäcklund relations the following algebraic relation can be derived

$$(\hat{\tilde{u}} - u)(\tilde{u} - \hat{u}) = p^2 - q^2. \tag{0.10}$$

One can check directly that $\hat{\tilde{u}}$ defined by this relation is a solution provided u, $\tilde{u}$ and $\hat{u}$ are solutions. Hence, starting from a trivial solution u, we can find two solutions $\tilde{u}$ and $\hat{u}$ by solving the Bäcklund relations, then generate a further solution algebraically. In fact, one can generate an infinite number of solutions algebraically by introducing further Bäcklund parameters and repeatedly applying the previous arguments. For instance, the special soliton solutions we mentioned with regards to the IST and reflectionless potentials can be generated in this manner. For more information, consult [69]. This procedure, with regard to the pKdV equation, was first described by Wahlquist and Estabrook, [74].

Relation (0.10) is known as a nonlinear superposition principle. Observe, relabelling $u = u_{0,0}$, $\tilde{u} = u_{1,0}$, $\hat{u} = u_{0,1}$ and $\hat{\tilde{u}} = u_{1,1}$, we obtain (0.4), the lpKdV. Indeed, by interpreting $\tilde{}$ and $\hat{}$ as being shifts on a lattice, we view (0.10) as a PΔE. Many integrable PΔE admit this interpretation, for example Q_4, the master equation in the ABS classification [12], can be interpreted as a nonlinear superposition principle for the Krichever-Novikov equation [5]. Later, we will derive discrete integrable systems by implicitly considering the compatibility of Bäcklund transformations.

Part I

Darboux Transformations, Reduction Groups and Automorphic Lie Algebras

Chapter 1

Introduction

In this part of the thesis, we study integrable systems related to finite reduction groups. In particular, we consider systems with tetrahedral and octahedral symmetry. Continuous systems related to these groups are known, we focus on semi and fully discrete systems. It is always difficult in the study of integrable systems to be certain whether a particular equation has been previously discovered. The myriad of different approaches and the different forms in which an equation may be presented, make certainty a difficult state to reach. That said, we believe many of the systems we find are new, moreover, if they are not, the Lax representations we provide certainly are, and the interplay between the various systems is laid bare. The principal tool that we use is a Lax-Darboux scheme, the raw materials are automorphic Lie algebras. These concepts and topics will be thoroughly explained. As topics, they are independent, so we first discuss the Lax-Darboux scheme via a simple example, then introduce the necessary background material regarding automorphic Lie algebras. Once the stage is set, we tackle the tetrahedral group. We derive associated differential difference and partial difference systems. We discuss their interrelations and make use of fruit of the scheme to construct exact solutions of the corresponding partial differential equations. After this, we finish our study of automorphic Lie algebras by presenting results regarding the octahedral group. Many of the results regarding the tetrahedral reduction group can be found in the preprint [18].

1.1 Lax-Darboux scheme

The Lax-Darboux scheme is a method for deriving integrable DΔE and PΔE by constructing Darboux transformations of a continuous Lax pair. We illustrate the process with a simple example.

Consider the linear system

$$\partial_x(\mathbf{\Psi}) = \begin{pmatrix} 0 & 1 \\ -\lambda - w_x & 0 \end{pmatrix} \mathbf{\Psi}, \tag{1.1a}$$

$$\partial_t(\mathbf{\Psi}) = \begin{pmatrix} -w_{xx} & 2\left(w_x - 2\lambda\right) \\ 4\lambda^2 + 2w_x\lambda - 2w_x^2 - w_{xxx} & w_{xx} \end{pmatrix} \mathbf{\Psi}. \tag{1.1b}$$

This has the form

$$\partial_x(\mathbf{\Psi}) = U\mathbf{\Psi}, \quad \partial_t(\mathbf{\Psi}) = V\mathbf{\Psi}, \tag{1.2}$$

thus, computing the zero curvature condition, cf. (0.2), one finds

$$U_t - V_x + [U, V] = \begin{pmatrix} 0 & 0 \\ -w_{xt} + 6w_x w_{xx} + w_{xxxx} & 0 \end{pmatrix}.$$

Hence,

$$(\mathbf{\Psi}_x)_t = (\mathbf{\Psi}_t)_x \qquad \text{if and only if} \qquad w_{xt} = 6w_x w_{xx} + w_{xxxx}.$$

Thus assuming the system is compatible, w_x must satisfy the KdV equation. Integrating once, w satisfies

$$w_t = 3w_x^2 + w_{xxx}, \tag{1.3}$$

the pKdV equation.

This is the starting point, a continuous Lax pair. Now, suppose we make the change of variables

$$\tilde{\mathbf{\Psi}} = M\mathbf{\Psi},$$

where M is an invertible matrix which may depend on x, t and λ. Then, (1.1) transforms to

$$\partial_x(\tilde{\mathbf{\Psi}}) = (M_x M^{-1} + MUM^{-1})\tilde{\mathbf{\Psi}}, \quad \partial_t(\tilde{\mathbf{\Psi}}) = (M_t M^{-1} + MUM^{-1})\tilde{\mathbf{\Psi}}. \tag{1.4}$$

In general, this system will not have the same form as (1.1), however, they are equivalent. In fact, Lax representations related in this way are said to be gauge equivalent. Suppose that (1.4) has exactly the same form as (1.1), but with w replaced by an "updated" function $\tilde{w}$. Then, as (1.4) is of the form

$$\partial_x(\tilde{\mathbf{\Psi}}) = U(\tilde{w})\tilde{\mathbf{\Psi}}, \quad \partial_t(\tilde{\mathbf{\Psi}}) = V(\tilde{w})\tilde{\mathbf{\Psi}}, \tag{1.5}$$

it follows that

$$(\tilde{\Psi}_x)_t = (\tilde{\Psi}_t)_x \qquad \text{if and only if} \qquad \tilde{w}_{xt} = 6\tilde{w}_x\tilde{w}_{xx} + \tilde{w}_{xxxx},$$

thus, $\tilde{w}$ also satisfies the pKdV equation.

Requiring that the gauged system has the same form as (1.1) is equivalent to imposing the following conditions upon M

$$M_x M^{-1} + MUM^{-1} = \tilde{U}, \quad M_t M^{-1} + MVM^{-1} = \tilde{V}, \tag{1.6}$$

where $\tilde{U} = U(\tilde{w})$ and $\tilde{V} = V(\tilde{w})$. We shall call M which satisfies these conditions a Darboux matrix. This is due to its similarity with classical Darboux transformations, cf. [26, 48, 68], i.e. transformations which leave a linear system covariant. However, we will not consider solutions of the linear system, as in the classical case. Instead, we find M which satisfies conditions (1.6) provided certain differential relations linking w and $\tilde{w}$ hold.

Let us concentrate on the first relation in (1.6), this may be rewritten as

$$M_x + MU = \tilde{U}M. \tag{1.7}$$

By taking the ansatz

$$M = M_0 + \lambda M_1,$$

substituting into (1.7), and using the resulting relations at each power of λ to determine M_0 and M_1, one can find the Darboux matrix

$$M_p = \begin{pmatrix} \frac{1}{2}(w - \tilde{w}) & 1 \\ \frac{1}{4}(-p^2 + w^2 + \tilde{w}^2 - 4\lambda - 2w\tilde{w}) & \frac{1}{2}(w - \tilde{w}) \end{pmatrix}, \tag{1.8}$$

where here the subscript p alludes to the dependence of the Darboux matrix on the parameter p. Taking $M = M_p$ in (1.7), one finds the relation is satisfied provided

$$\tilde{w} = \frac{1}{2}p^2 - w_x - \frac{1}{2}(w - \tilde{w})^2, \tag{1.9}$$

which is nothing but the Bäcklund relation (0.9a). The t part of the Bäcklund relation follows from the second equation in (1.6). Hence, by considering Darboux transformations of the Lax pair, we naturally arrive at Bäcklund transformations for the underlying PDE. Moreover, we can view " ˜ " as defining a shift on a lattice, so can interpret (1.9) as a D∆E. Indeed, if we make the change of variables

$$\tilde{\tilde{\mathbf{\Psi}}} = M_p(\tilde{w}, \tilde{\tilde{w}})\tilde{\mathbf{\Psi}},$$

in (1.5), then we obtain (1.9) with $(w,\tilde{w})$ replaced by $(\tilde{w},\tilde{\tilde{w}})$. This process can be repeated, allowing the derivation of a chain of relations. We encode this set of relations as the D∆E

$$W_{n+1,x} = \frac{1}{2}\alpha^2 - W_{n,x} - \frac{1}{2}\left(W_n - W_{n+1}\right)^2,$$

where $w = W(x,t,0) = W_0$, $\tilde{w} = W(x,t,1) = W_1$, $\tilde{\tilde{w}} = W(x,t,2) = W_2$, etc. By construction this has semi-discrete Lax pair

$$\mathcal{S}(\mathbf{\Psi}) = M_p(W_n, W_{n+1})\mathbf{\Psi}, \quad \partial_x(\mathbf{\Psi}) = U(W_n)\mathbf{\Psi}.$$

We shall often refer to a semi-discrete Lax pair as a Darboux-Lax representation (DLR), due to the interpretation of M as a Darboux matrix.

The above equation is not evolutionary, but nonetheless it has a DLR, hence is integrable. In a later section, having obtained a Darboux matrix, we will vary the equivalent of U from (1.2) to derive integrable evolutionary D∆Es. Thus, even if one is interested in evolutionary equations, possessing a Darboux matrix is invaluable.

One can also use Darboux matrices to derive fully discrete systems. Note that the Darboux matrix (1.8) depends upon a parameter p, one could equally consider a Darboux matrix with parameter q, replacing $\tilde{w}$ with $\hat{w}$. Using M_q to denote this Darboux matrix, we impose that M_p and M_q are compatible. That is, we assume there exists a $\mathbf{\Psi}$ such that $\tilde{\hat{\mathbf{\Psi}}} = \hat{\tilde{\mathbf{\Psi}}}$, in which case, $\tilde{\hat{w}} = \hat{\tilde{w}}$. At the level of the Darboux matrices, we require

$$M_p(\hat{w}, \hat{\tilde{w}})M_q(w, \hat{w}) = M_q(\tilde{w}, \tilde{\hat{w}})M_p(w, \tilde{w}).$$

Performing the computation, one finds this is equivalent to the equation

$$(\hat{\tilde{w}} - w)(\tilde{w} - \hat{w}) = p^2 - q^2,$$

which is the nonlinear superposition formula (0.10). By associating $\mathcal{S}$ to the ˜-shift and $\mathcal{T}$ to the ˆ-shift, we obtain a P∆E. Indeed, the relation becomes

$$(W_{n+1,m+1} - W_{n,m})(W_{n+1,m} - W_{n,m+1}) = p^2 - q^2,$$

where $w = W(x, t, 0, 0) = W_{0,0}$, $\tilde{w} = W(x, t, 1, 0) = W_{1,0}$, $\hat{w} = W(x, t, 0, 1) = W_{0,1}$, $\tilde{\hat{w}} = W(x, t, 1, 1) = W_{1,1}$, etc, and we have started with initial solution $W_{n,m}$. The Lax pair is given by

$$\mathcal{S}(\mathbf{\Psi}) = M_p(W_{n,m}, W_{n+1,m})\mathbf{\Psi}, \quad \mathcal{T}(\mathbf{\Psi}) = M_q(W_{n,m}, W_{n,m+1})\mathbf{\Psi}.$$

At times we may refer to a discrete Lax pair as a Darboux pair.

In this section we have seen that given a Lax pair for a PDE, by considering its Darboux transformations one can naturally derive related semi and fully discrete systems, complete with Lax pairs. Moreover, the relation among the found systems is clear from the construction. A question naturally arises concerning how one derives the initial Lax pair. For a given PDE this is a highly non trivial task, many are constructed using intuition and assuming a particular ansatz. The most direct method belongs to Wahlquist and Estabrook, a review of which can be found in [36] and [28]. In the next section we take a slightly different approach.

1.2 Reduction groups and automorphic Lie algebras

As was mentioned in the previous section, given a nonlinear PDE, it is a non trivial task to ascertain whether it possesses a Lax representation. For this reason, it may be profitable to approach the question from the opposite direction. We may search for integrable equations by studying the structure of admissible Lax representations, instead of directly searching for the equations themselves. In view of this approach, we must first decide which building blocks should be used. This will govern the general form of the elements of the pair. In this thesis, the initial building blocks will be elements of a generalised loop algebra. In particular, we will consider elements of

$$\mathfrak{U}_\lambda(\Gamma) = R_\lambda(\Gamma) \otimes_{\mathbb{C}} \mathfrak{sl}(3, \mathbb{C}),$$

where $R_\lambda(\Gamma)$ is the algebra of rational functions in λ with poles in the set Γ, while $\mathfrak{sl}(3, \mathbb{C})$ is the Lie algebra of 3×3 traceless matrices. Having such a general starting form, we run into the issue that the compatibility condition for a given pair is underdetermined. To rectify this issue, we impose that the elements of the pair belong to a subalgebra of $\mathfrak{U}_\lambda(\Gamma)$ which is invariant with respect to a finite group of automorphisms. In this way we obtain a Lax pair with a high degree of symmetry and whose compatibility condition produces a more applicable PDE. We call the finite group of automorphisms a reduction group.

One can find the origin of the theory of reductions in the papers [50–52]. Subsequently in [45], a solid theoretical foundation is provided, while in [21] explicit irreducible representations for all $\mathfrak{sl}(n, \mathbb{C})$ targeted finite reduction groups are obtained. In this thesis, we mainly focus on Lax operators invariant with respect to the tetrahedral reduction group $\mathbb{T}$, although the octahedral reduction group will also be examined. This is a group of automorphisms of $\mathfrak{U}_\lambda(\Gamma)$ which is isomorphic to the symmetry group of a tetrahedron. Moreover, we aim to derive semi and fully discrete systems, as opposed to investigating continuous systems. In what follows, we give a review of the basic concepts regarding reduction groups and the results so far in this area, then we proceed to apply a Lax-Darboux scheme to tetrahedral invariant operators.

We shall build our Lax pairs from a generalised loop algebra. To be precise, we have the following definition.

Definition 1.2.1 *The algebra* $\mathfrak{U}_\lambda(\Gamma) = R_\lambda(\Gamma) \otimes_{\mathbb{C}} \mathfrak{A}$ *is called the generalised loop algebra with poles in* Γ*, where* $\mathfrak{A}$ *is a finite dimensional Lie algebra, and* $R_\lambda(\Gamma)$ *is the algebra of rational functions in* λ *with poles in the set* Γ.

This is an infinite dimensional Lie algebra with Lie product given by

$$(f \otimes \mathbf{a})(g \otimes \mathbf{b}) = fg \otimes [\mathbf{a}, \mathbf{b}].$$

In practice, the case which is most applicable to the study of integrable systems is where $\mathfrak{A}$ is simple. In [21, 45], the case $\mathfrak{A} = \mathfrak{sl}(n, \mathbb{C})$ is extensively studied with regard to PDEs and algebraic structure. We will also study this case, however with the aim of producing discrete equations.

Next, we give the general definition for an automorphic Lie algebra and reduction group.

Definition 1.2.2 *Let* G *be a subgroup of the group of automorphisms of* $\mathfrak{U}_\lambda(\Gamma)$*, then* $\mathfrak{U}_\lambda^G(\Gamma) = \{\mathbf{a} \in \mathfrak{U}_\lambda(\Gamma) \mid g(\mathbf{a}) = \mathbf{a}, \forall g \in G\}$ *is called the automorphic Lie algebra corresponding to the reduction group* G *with poles in* Γ.

For our purposes, all encountered reduction groups will be finite, however we note that infinite reduction groups do exist and have been used in applications, for instance in the construction of N-soliton solutions for the Landau-Lifschitz equation [53].

As given in the above definition, the reduction group G is a subgroup of the group of automorphisms of $\mathfrak{U}_\lambda(\Gamma)$. One can construct automorphisms of $\mathfrak{U}_\lambda(\Gamma)$ by considering simultaneous automorphisms of its constituent components. The Lie algebra $\mathfrak{U}_\lambda(\Gamma)$ is composed of two objects, a finite dimensional Lie algebra and the algebra of rational functions with poles in Γ. The automorphisms of both objects are well known. In particular, finite subgroups of automorphisms of $R_\lambda(\Gamma)$ were classified by Felix Klein [38], they correspond to

1. The cyclic groups $\mathbb{Z}_n$
2. The dihedral groups $\mathbb{D}_n$
3. The tetrahedral group $\mathbb{T}$
4. The octahedral group $\mathbb{O}$
5. The icosahedral group $\mathbb{I}$

To see this, note that we may view $R_\lambda(\Gamma)$ as being contained within $\mathbb{C}(\lambda)$, the rational function field, automorphisms of which are fractional linear transformations (FLT), i.e. maps $f(\lambda)$ of the form

$$f : \lambda \mapsto \frac{a\lambda + b}{c\lambda + d}, \tag{1.10}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$, with the latter condition ensuring f is invertible. This condition is often called the determinant, and for good reason. To each FLT, (1.10), we may associate a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.11}$$

The condition $ad - bc \neq 0$ simply expresses that the determinant of this matrix is non zero. One can check directly that composition of FLTs corresponds to multiplication of their respective associated matrices. However, note that one has some flexibility in the way in which an FLT is presented. For instance,

$$f(\lambda) = \frac{a\lambda + b}{c\lambda + d} = \frac{-a\lambda - b}{-c\lambda - d} = \frac{2a\lambda + 2b}{2c\lambda + 2d}.$$

Each choice of representation will lead to a different associated matrix. To remove this ambiguity we normalise the determinant of (1.11). We notice that since division of the numerator and denominator of a FLT by a constant α results in the same FLT, we may divide the numerator and denominator by $ad - bc$, which is necessarily nonzero, to obtain a FLT whose corresponding matrix, say A, has unit determinant. However, we could have legitimately divided the numerator and denominator by $-(ad - bc)$, the resulting associated matrix, B, would also have unit determinant. The relation between A and B is $B = -\mathrm{I}A$, where I is the 2×2 identity matrix. Thus, we see that we may uniquely associate a FLT to an element of the group $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm \mathrm{I}\}$. One can show that this association is in fact a group isomorphism, with function composition corresponding matrix multiplication. Hence, if we wish to determine finite subgroups of FLTs, one can equivalently search for finite subgroups of $PSL(2, \mathbb{C})$. Elements of $PSL(2, \mathbb{C})$ can be classified according to the value of their squared trace. In particular, they fall into 4 categories: elliptic, parabolic, hyperbolic and loxodromic. These correspond to the cases $0 \leq \mathrm{tr}^2 < 4$, $\mathrm{tr}^2 = 4$, $\mathrm{tr}^2 > 4$ and $\mathrm{tr}^2 \in \mathbb{C} \setminus [0, 4]$ respectively, where tr is the trace of the element. One can show that if a non identity element, A, is of finite order, i.e. $A^n = I$, then A is necessarily elliptic. Moreover, subgroups formed of elliptic elements, along with the identity, are given by subgroups of $PSU(2, \mathbb{C})$, the group

$$SU(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\},$$

quotiented by $\{\pm \mathrm{I}\}$. There is a group isomorphism $PSU(2, \mathbb{C}) \to SO(3)$, where $SO(3)$ is the group of rotations of $\mathbb{R}^3$. Hence, finite subgroups of FLTs correspond to finite subgroups of $SO(3)$. These are exactly the groups found by Klein. One can find further discussion of these ideas in [13, 16].

Now let H correspond to one of these finite subgroups, then given an injective group homomorphism

$$\psi : H \to \mathrm{Aut}(\mathfrak{A}),$$

we can construct an automorphism of $\mathfrak{U}_\lambda(\Gamma)$ via the map

$$h(f \otimes A) \mapsto h(f) \otimes \psi(h)(\mathbf{a}), \tag{1.12}$$

where $h \in H$, $h(f)$ is the fractional linear transformation (corresponding to the group element h) applied to f, and $\psi(h)(\mathbf{a})$ is the image of h in Aut($\mathfrak{A}$) applied to $\mathbf{a}$. Clearly, due to the injectivity of ψ, we obtain a group of automorphisms of $\mathfrak{U}_\lambda(\Gamma)$ isomorphic to H. Indeed, ψ is nothing more than a faithful representation of H. Automorphisms of $\mathfrak{A} = \mathfrak{sl}_n(\mathbb{C})$ occur as either inner automorphisms or outer automorphisms. Inner automorphisms are of the form

$$\mathbf{a} \mapsto Q\mathbf{a}Q^{-1},$$

for some invertible matrix Q. Outer automorphisms exist for $n > 2$, however, in what follows, we only consider inner automorphisms. For an example of a reduction group built using outer automorphisms, we refer the reader to [55], where Dihedral reduction groups using outer automorphisms are studied. It is important to note that taking $\tilde{Q} = (\alpha \mathrm{I})Q$ with $\alpha \in \mathbb{C}$ and I the identity matrix, results in the same automorphism as taking Q. Thus, we must consider projective faithful representations of H.

For the purposes of classification it is convenient to consider irreducible projective representations. It is known, cf. [27], that finite groups possess a finite number of inequivalent projective irreducible representations when the representation vector space is over an algebraically closed field, for instance $\mathbb{C}$. In fact, one can compute irreducible projective representations of a group by first computing irreducible representations of a Schur cover of the group, before determining which are projectively equivalent. In this thesis, we only consider the tetrahedral and octahedral reduction groups. One can take as Schur covers the binary tetrahedral and binary octahedral groups respectively. Details of an algorithm to compute complex irreducible representations of a finite group can be found in [15]. In practice, one can use the computer algebra package GAP [31], which implements such an algorithm. For information regarding computational representation theory in general, one can consult [47].

Projective irreducible representations of the finite reduction groups occur only for the dimensions $n = 1, 2, 3, 4, 5$ and 6. One can find explicit representations for each case in [21]. One finds that all irreducible representations of cyclic groups $\mathbb{Z}_n$ are 1 dimensional. In the literature, for instance [39], the representations for $\mathbb{Z}_n$ reduction groups used are reducible. For $n = 2$, one finds irreducible representations for the Dihedral groups D_n, the tetrahedral group $\mathbb{T}$, the octahedral group $\mathbb{O}$, and the icosahedral group $\mathbb{I}$. For $n = 3$, the dihedral groups are lost, leaving irreducible representations for $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$. For $n = 4$, only $\mathbb{O}$ and $\mathbb{I}$ remain. Finally, for $n = 5$ and $n = 6$, there only exist irreducible representations for $\mathbb{I}$. Thus we have arrived at a collection of reduction groups.

Once one has chosen a reduction group, the Lie algebra of elements fixed by the group, i.e. an automorphic Lie algebra, can be considered. This is dependent upon which set Γ is taken, as by definition $\mathfrak{U}_\lambda^G(\Gamma)$ is fixed by G, hence it follows that Γ must consist of orbits of the fractional linear transformations which go toward defining G. Otherwise the action of the group would cause poles not in Γ to appear, violating that the algebra is invariant. In this thesis, Γ will be taken to be a single orbit. For a single orbit, there are two cases which we must separate. The case when the orbit is degenerated and the case when the orbit is generic. In the former case, the pole, say $z_0 \in \mathbb{C}$, is a fixed point of some fractional linear transformation σ_g, with g not the identity element. This implies that the cardinality of the orbit set will be less than that of G. In the generic case the orbit will be full, thus will consist of exactly $|G|$ elements. Hence a degenerated orbit is one which has cardinality less than $|G|$.

In the case $n = 2$, i.e. automorphic Lie algebras built from a base Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, it has been shown [21, 46] that there exist 5 distinct classes of algebras. These correspond to a trivial reduction group (no reduction), $\mathbb{Z}_2$ reduction with Γ a degenerated orbit, $\mathbb{Z}_2$ reduction with Γ a generic orbit, $\mathbb{D}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ reduction with Γ a degenerate orbit and $\mathbb{D}_2$ with Γ a generic orbit. In [39], the authors construct Darboux transformations for operators built from each of these algebras, along with the corresponding D∆Es and P∆Es.

In the case $n = 3$, it is shown in [21] that the continuous systems one derives using the automorphic Lie algebras corresponding to the reduction groups $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$, with the degenerated orbit of ∞, are all related by point transformation. Hence, one can consider the tetrahedral reduction group to be representative of finite reduction groups where Γ is the degenerated orbit of ∞. A similar situation occurs in the case $n = 4$. For this degenerated orbit, all PDE systems obtained are related by point transformation. However for $n = 4$, we no longer have an irreducible representation of the tetrahedral reduction group, thus instead the representative is the octahedral group. We consider these two cases in the following sections.

Chapter 2

Tetrahedral group

Lax operators with tetrahedral symmetry were first studied in [51, 59]. This research focused on the associated PDEs. By using a Lax-Darboux scheme, we derive DΔEs and PΔEs associated to these operators. First, we introduce the tetrahedral group.

The tetrahedral group is given abstractly by the group presentation

$$\mathbb{T} =< r, s | r^2 = s^3 = (rs)^3 = e > . \tag{2.1}$$

The group corresponds to the rotational symmetries of a tetrahedron. We have two types of axes of symmetry. One axis allows a rotation of $\frac{2\pi}{3}$. The second allows rotations of π. Alternatively, one can view the tetrahedral group as being given by A_4, the alternating group on 4 objects. This is the group of even permutations of 4 objects. This can be seen by labelling the vertices of the tetrahedron $(1, 2, 3, 4)$, then recognising that A_4 is generated by the cycles $s = (123)$ and $r = (12)(34)$.

As was mentioned in the previous section, we consider simultaneous automorphisms of the algebras $R_\lambda(\Gamma)$ and $\mathfrak{A}$. One can directly verify that the fractional linear transformations

$$\sigma_s(\lambda) = \omega\lambda, \quad \sigma_r(\lambda) = \frac{\lambda + 2}{\lambda - 1}, \tag{2.2}$$

satisfy the group presentation, and therefore generate a group isomorphic to $\mathbb{T}$. Similarly, one can verify that the maps

$$\rho_s : \mathbf{a} \mapsto Q_s \mathbf{a} Q_s^{-1}, \quad \rho_r : \mathbf{a} \mapsto Q_r \mathbf{a} Q_r^{-1}, \qquad \mathbf{a} \in \mathfrak{sl}_3(\mathbb{C}),$$

where

$$\mathbf{Q}_s = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{Q}_r = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix},$$

generate a group of automorphisms of $\mathfrak{sl}_3(\mathbb{C})$ isomorphic to $\mathbb{T}$. Combining these, we obtain automorphisms of $\mathfrak{U}_\lambda(\Gamma)$:

$$g_s : \mathbf{a}(\lambda) \mapsto Q_s \mathbf{a}(\sigma_s^{-1}(\lambda)) Q_s^{-1}, \quad g_r : \mathbf{a}(\lambda) \mapsto Q_r \mathbf{a}(\sigma_r^{-1}(\lambda)) Q_r^{-1}, \quad \mathbf{a}(\lambda) \in \mathfrak{sl}_3(\mathbb{C}(\lambda)).$$

This representation can be found in [21]. Using the above generators one can construct the subalgebra $\mathfrak{U}_\lambda^{\mathbb{T}}(\Gamma)$ of $\mathfrak{U}_\lambda(\Gamma)$ invariant under the action of the group. This is done using the group average operator, given by

$$< a(\lambda) >_{\mathbb{T}} = \frac{1}{|\mathbb{T}|} \sum_{g \in \mathbb{T}} g(a(\lambda)). \tag{2.3}$$

Each g is an automorphism of the Lie algebra, thus is a linear map. Hence, applying an element g to the right hand side of (2.3) simply results in a reshuffling of the summation, as the automorphisms form a group thus they are closed with respect to composition.

Recall that we have some choice with regards to the set Γ. We take it to be a single orbit. From the expressions for σ_s and σ_r found in (2.2), we see that we have 3 degenerate orbits. Two corresponding to the fixed points ∞ and 0 of σ_s

$$\mathbb{T}(\infty) = \{\infty, 1, \omega, \omega^2\}, \quad \mathbb{T}(0) = \{0, -2, -2\omega, -2\omega^2\},$$

and one corresponding to the fixed point $1 + \sqrt{3}$ of σ_r

$$\mathbb{T}(1 + \sqrt{3}) = \{\lambda \in \mathbb{C} | \lambda^6 - 20\lambda^3 - 8 = 0\}.$$

We take $\Gamma = \mathbb{T}(\infty)$, then the subalgebra $\mathfrak{U}_\lambda^{\mathbb{T}}(\Gamma)$ is generated by the elements

$$\mathbf{e}_1 =< \lambda \mathbf{E}_{13} >_{\mathbb{T}}, \ \ \mathbf{e}_2 =< \lambda \mathbf{E}_{21} >_{\mathbb{T}}, \ \ \mathbf{e}_3 =< \lambda \mathbf{E}_{32} >_{\mathbb{T}},$$

where $\mathbf{E}_{ij}$ is the matrix with 1 in the (i, j) position and 0 elsewhere. These are given explicitly as

$$\mathbf{e}_1 = \begin{pmatrix} -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} \\ \frac{\lambda}{\lambda^3-1} & \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} \\ \frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix},$$

$$\mathbf{e}_2 = \begin{pmatrix} -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} \\ \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} \\ -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} & \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix},$$

$$\mathbf{e}_3 = \begin{pmatrix} \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} \\ \frac{\lambda}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda^2}{\lambda^3-1} \\ -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}.$$

In fact, $\mathfrak{U}_\lambda^{\mathbb{T}}(\Gamma)$ possesses the structure $\mathfrak{U}_\lambda^{\mathbb{T}}(\Gamma) = \bigoplus_{k=0}^{\infty} A^k$, where $A^k = \{J^k e_1, J^k e_2, \ldots, J^k e_8\}$, with J an automorphic function of λ with poles on the orbit of ∞ and

$$\mathbf{e}_4 = [\mathbf{e}_1, \mathbf{e}_3], \ \mathbf{e}_5 = [\mathbf{e}_2, \mathbf{e}_1], \ \mathbf{e}_6 = [\mathbf{e}_3, \mathbf{e}_2],$$

$$\mathbf{e}_7 = [[\mathbf{e}_1, \mathbf{e}_3], \mathbf{e}_2], \ \mathbf{e}_8 = [[\mathbf{e}_2, \mathbf{e}_1], \mathbf{e}_3].$$

By automorphic function of λ, we understand that J is invariant with respect to the fractional linear transformations of the group. One can construct such a function by averaging over the group. The theory of automorphic functions is well developed, indeed the authors of [45] cite this theory to be the motivation for the term automorphic Lie algebra. One thing to note is that there is some freedom when choosing J. One can add any constant to J with the result remaining invariant. We can use this property to change its zeros. For instance, as $J = J(\lambda)$, we may consider $\tilde{J} = J(\lambda) - J(c)$. The new automorphic function, $\tilde{J}$, has zeros on the orbit of c. Another thing to note is that $\mathbf{e}_i$, for

$i = 1, 2, 3$ has 1st order poles, while for $i = 4, 5, 6$ has 2nd order poles, and for $i = 7, 8$ has 3rd order poles.

Now that we have an automorphic Lie algebra, we demonstrate the link to PDEs. Let us use $\mathfrak{U}_\lambda^{\mathbb{T}}(\Gamma)$ to construct a Lax pair. To do so, we take operators

$$L = \partial_x + \sum_{i=1}^{3} u_i(x,t)\mathbf{e}_i, \quad A = \partial_t + \sum_{i=1}^{3} v_i(x,t)\mathbf{e}_i + \sum_{i=1}^{3} w_i(x,t)\mathbf{e}_{i+3},$$

where the u_i, v_i and w_i functions are yet to be determined. We use L_1 and L_2 to define a linear system

$$L\Psi = 0, \quad A\Psi = 0, \tag{2.4}$$

for some matrix function Ψ. Its compatibility condition is given by

$$[L, A] = 0. \tag{2.5}$$

As L has first order poles and A has up to second order poles, it follows that the commutator $[L, A]$ involves up to third order poles, and thus is spanned by the elements $\mathbf{e}_i$, for $i = 1, \ldots 8$. Hence, condition (2.5) is equivalent to 8 equations, i.e. one sets the coefficient of each $\mathbf{e}_i$ equal to zero. Solving 6 of these equations allows the v_i and w_i functions to be determined in terms of the u_i functions. Subsequently, setting $u_1 = u$, $u_2 = v$ and $u_3 = w$, the compatibility condition becomes equivalent to the system

$$u_t = \partial_x \left(\frac{u}{3} \left(-u + 2v + 2w + \frac{u_x}{u} + 2\frac{w_x}{w} \right) \right), \tag{2.6a}$$
$$v_t = \partial_x \left(\frac{v}{3} \left(-v + 2w + 2u + \frac{v_x}{v} + 2\frac{u_x}{u} \right) \right), \tag{2.6b}$$
$$w_t = \partial_x \left(\frac{w}{3} \left(-w + 2u + 2v + \frac{w_x}{w} + 2\frac{v_x}{v} \right) \right), \tag{2.6c}$$

with $uvw = 1$. System (2.6) and its Lax pair will be the starting point of the Lax-Darboux scheme. For further details regarding the derivation of the Lax pair and PDE system, one can consult [21]. It can be shown that with an appropriate change of variables, (2.6) is equivalent to equation (u3) in [59].

2.1 Derivation of Darboux transformations

We now construct Darboux transformations for system (2.4). To be explicit,

$$L = \partial_x + u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3, \tag{2.7}$$

with $uvw = 1$. The second operator A is given by

$$A = \partial_t + p\mathbf{e}_1 + q\mathbf{e}_2 + r\mathbf{e}_3 + uw\mathbf{e}_4 + vu\mathbf{e}_5 + wv\mathbf{e}_6, \tag{2.8}$$

where

$$\begin{aligned} p &= \frac{u}{3}\left(-u + 2v + 2w + \frac{u_x}{u} + 2\frac{w_x}{w}\right), \\ q &= \frac{v}{3}\left(-v + 2w + 2u + \frac{v_x}{v} + 2\frac{u_x}{u}\right), \\ r &= \frac{w}{3}\left(-w + 2u + 2v + \frac{w_x}{w} + 2\frac{v_x}{v}\right). \end{aligned}$$

The compatibility condition, $[L, A] = 0$, yields the integrable system (2.6). Note, due to the relation $uvw = 1$, only two of the equations are independent. However, in 3 variables the system retains a symmetric form, thus we do not eliminate w.

We search for Darboux transformations of L. Recall, when considering the x part of the Lax pair, the corresponding Darboux matrix M must satisfy an equation of the form

$$M_x M^{-1} + MUM^{-1} = \tilde{U}, \tag{2.10}$$

where

$$\partial_x(\Psi) = U\Psi,$$

and $\tilde{U}$ has the same form as U, but with the "updated" functions $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$. For notational convenience, we use a subscript to denote an updated function, so $u_1 = \tilde{u}$, $v_1 = \tilde{v}$ and $w_1 = \tilde{w}$. For L, one has $U = -(u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3)$, and therefore after rearrangement, (2.10) becomes

$$M_x = M\left(u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3\right) - \left(u_1\mathbf{e}_1 + v_1\mathbf{e}_2 + w_1\mathbf{e}_3\right)M, \tag{2.11}$$

for M invertible. By inspection, it is clear that (2.6) is invariant with respect to cyclic permutations of the functions u, v and w. Hence, given a solution (u, v, w), one can generate a new solution by simple permutation. This property is expressed at the level of the Lax structure by the existence of the following Darboux matrix

$$M_{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

A short computation demonstrates that

$$M_{(0)}\mathbf{e}_1 M_{(0)}^{-1} = \mathbf{e}_2, \quad M_{(0)}\mathbf{e}_2 M_{(0)}^{-1} = \mathbf{e}_3, \quad M_{(0)}\mathbf{e}_3 M_{(0)}^{-1} = \mathbf{e}_1.$$

Thus, application of $M_{(0)}$ results in the updated functions

$$u_1 = v, \quad v_1 = w, \quad w_1 = u,$$

while applying $M_{(0)}^2$ results in the updated functions

$$u_1 = w, \quad v_1 = u, \quad w_1 = v.$$

Our main interest is in non constant Darboux transformations. As a simplifying assumption, it is natural to consider the Darboux matrix M to be invariant with respect to the reduction group. The simplest such M will have simple poles which belong to a degenerated orbit. In particular, we choose these poles to belong to the orbit of infinity. Hence, we take M of the form

$$M = f\mathrm{I} + a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3. \tag{2.12}$$

Starting from this ansatz, we solve equation (2.11). We make use of the following lemma.

Lemma 2.1.1 *The basis elements* $\mathbf{e}_i$*,* $i = 1, 2, 3$ *satisfy the product relations,*

$$\begin{aligned}
\mathbf{e}_i\mathbf{e}_i &= \frac{2}{9}\mathrm{I} - \frac{1}{3}\mathbf{e}_i, \\
\mathbf{e}_1\mathbf{e}_2 &= -\frac{1}{9}\mathrm{I} + \frac{1}{3}\left(\mathbf{e}_1 + \mathbf{e}_2\right), \\
\mathbf{e}_2\mathbf{e}_3 &= -\frac{1}{9}\mathrm{I} + \frac{1}{3}\left(\mathbf{e}_2 + \mathbf{e}_3\right), \\
\mathbf{e}_3\mathbf{e}_1 &= -\frac{1}{9}\mathrm{I} + \frac{1}{3}\left(\mathbf{e}_3 + \mathbf{e}_1\right).
\end{aligned}$$

Proof

This can be confirmed directly.□

The above relations allow the products $\mathbf{e}_i\mathbf{e}_j$, for $i,j = 1, 2, 3$, to be determined, as recall $\mathbf{e}_4 = [\mathbf{e}_1, \mathbf{e}_3]$, $\mathbf{e}_5 = [\mathbf{e}_2, \mathbf{e}_1]$ and $\mathbf{e}_6 = [\mathbf{e}_3, \mathbf{e}_2]$.

Using the lemma, our strategy will be to expand the compatibility condition in terms of the basis elements $\mathbf{e}_i$ along with the identity matrix. One can then use the fact that the elements are linearly independent to obtain a system of equations which are equivalent to the compatibility condition.

Theorem 2.1.2 *Suppose* M*, as given in (2.12), is a Darboux matrix for* L*, then the functions* a*,* b*,* c *and* f *satisfy the relations*

$$av_1 - bu = 0, \;\; bw_1 - cv = 0, \;\; cu_1 - aw = 0, \tag{2.13}$$

$$a_x + \frac{1}{3}\left((b + c - a + 3f)(u_1 - u) + a(v_1 + w_1 - v - w)\right) = 0, \tag{2.14a}$$

$$b_x + \frac{1}{3}\left((c + a - b + 3f)(v_1 - v) + b(w_1 + u_1 - w - u)\right) = 0, \tag{2.14b}$$

$$c_x + \frac{1}{3}\left((a + b - c + 3f)(w_1 - w) + c(u_1 + v_1 - u - v)\right) = 0. \tag{2.14c}$$

and

$$f_x + \frac{1}{9}\left[(2a-b-c)(u_1-u)+(2b-a-c)(v_1-v)+(2c-a-b)(w_1-w)\right]=0. \tag{2.15}$$

Proof

The compatibility condition implies M satisfies

$$M_x = M\left(u\mathbf{e}_1+v\mathbf{e}_2+w\mathbf{e}_3\right)-\left(u_1\mathbf{e}_1+v_1\mathbf{e}_2+w_1\mathbf{e}_3\right)M.$$

Upon substitution of the ansatz (2.12) and an application of lemma (2.1.1), we arrive at the stated equations. From the coefficients of $\mathbf{e}_4$, $\mathbf{e}_5$ and $\mathbf{e}_6$, one obtains the system of equations (2.13). The coefficients of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ provide the equations (2.14). Finally, the coefficient of I yields (2.15). $\square$

In what follows, we consider reductions of the above general system. In particular, we will make use of first integrals of the system to reduce it to two components. Towards this aim, we prove the following lemma.

Lemma 2.1.3 *For M as given in (2.12), the determinant of M is constant with respect to x and is an automorphic function of λ.*

Proof

For any invertible matrix M, the following identity holds due to Jacobi's formula

$$\partial_x \ln\det(M) \;=\; \mathrm{Tr}\left(M^{-1}M_x\right).$$

Moreover, as M is a Darboux matrix for L, it follows that

$$M^{-1}M_x = \left(u\mathbf{e}_1+v\mathbf{e}_2+w\mathbf{e}_3\right)-M^{-1}\left(u_1\mathbf{e}_1+v_1\mathbf{e}_2+w_1\mathbf{e}_3\right)M.$$

The matrices $\mathbf{e}_i$ are traceless. Since conjugation preserves trace, the right hand side of the above consists of the difference of traceless matrices, and thus is necessarily traceless. Combining the above two relations, we conclude

$$\partial_x \det(M) = 0.$$

The fact that $\det(M)$ is an automorphic function of λ is a consequence of its form. To see this, observe that by construction M is invariant with respect to $\mathbb{T}$. That is, for all $g \in \mathbb{T}$

$$Q_g M(\sigma_g(\lambda)) Q_g^{-1} = M(\lambda),$$

where Q_g and σ_g are the automorphism of $\mathfrak{sl}_3(\mathbb{C})$ and fractional linear transformation respectively, corresponding to the group element g. Taking the determinant of this relation provides the equality

$$\det(M(\sigma_g(\lambda))) = \det(M(\lambda)).$$

This simply states that $\det(M(\lambda))$ is invariant with respect to the fractional linear transformation corresponding to g. As g is an arbitrary element of $\mathbb{T}$, it follows that $\det(M)$ is an automorphic function of λ. □

In light of the above lemma, we express the determinant of M in terms of an automorphic function. This will allow a number of the unknown functions f, a, b and c to be expressed in terms of the remaining functions. The Darboux matrix M is constructed using the automorphic Lie algebra generators $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, so it has poles at the $\mathbb{T}(\infty)$-orbit. This means that we can expect $\det(M)$ to have poles on this particular orbit also. In what follows, we write $\det(M)$ in terms of the following automorphic function

$$J(\lambda) = \frac{\lambda^3 \left(\lambda^3 + 8\right)^3}{4 \left(\lambda^3 - 1\right)^3}, \tag{2.16}$$

since $J(\lambda)$ has the correct pole structure. Using this function we arrive at the following lemma.

Lemma 2.1.4 *The determinant of M, as given in (2.12), can be expressed in the form*

$$\det M = F_1 J(\lambda) + F_2 \,,$$

for F_1 and F_2 constant with respect to x and given by

$$F_1 = \frac{abc}{16}\,, \tag{2.17a}$$

$$F_2 = \frac{1}{27}(3f - 2a + b + c)(3f + a - 2b + c)(3f + a + b - 2c)\,. \tag{2.17b}$$

Proof

The result follows by direct computation and the use of Lemma 2.1.3. □

As F_1 and F_2 are constant, they may be used to determine two of the unknown functions f, a, b and c in terms of the remaining functions. Towards this aim, we may view F_1 and F_2 as elements of the ring $\mathbb{C}[a, b, c, f]$, then determining (a, b, c, f) is equivalent to finding elements of the algebraic variety $V(F_1 - \alpha, F_2 - \beta) = V(F_1 - \alpha) \cap V(F_2 - \beta)$, where α and β are constants in $\mathbb{C}$. The variety $V(F_1 - \alpha)$ is irreducible for $\alpha \neq 0$. The case $\alpha = 0$ implies at least one of a, b or c is zero. From (2.13), this implies $a = b = c = 0$, leading to trivial M. Thus we require $\alpha \neq 0$. For simplicity, we may take $\alpha = \frac{1}{16}$, which implies

$$c = \frac{1}{ab}.$$

For general β, one finds $V(F_1 - \beta)$ is irreducible. Hence, to determine f in terms of a, b and c, one would have to solve a cubic equation. However, it can be seen directly that for $\beta = 0$ the variety $V(F_1 - \beta)$ becomes the union of three irreducible components

$$V(F_1) = V(3f - 2a + b + c) \cup V(3f + a - 2b + c) \cup V(3f + a + b - 2c). \tag{2.18}$$

Thus, three distinct Darboux transformations can be obtained choosing $f = f_1, f_2, f_3$ such that $(a, b, c, f_i) \in V_i$, where the V_i are the irreducible components given above. Since the components are linear, f is expressed rationally in terms of a, b and c.

For $\beta = -abc = -1$, one finds

$$V(F_1-\beta) = V(3f+a+b+c)\cup V(9f^2-3f(a+b+c)-2(a^2+b^2+c^2)+5(ab+bc+ca)). \tag{2.19}$$

By choosing f such that $(a,b,c,f) \in V(3f+a+b+c)$, we obtain a further way to express f rationally in terms of a, b and c. Let J_2 be the automorphic function which has poles in the orbit of infinity and zeros in the orbit of $1+\sqrt{3}$,

$$J_2 = J(\lambda) - J(1+\sqrt{3}) = \frac{\left(\lambda^6 - 20\lambda^3 - 8\right)^2}{4\left(\lambda^3-1\right)^3}.$$

Then we note that expressing $\det(M)$ in terms of J_2, one finds $\det(M) = (F_1+abc) + F_2 J_2$. Setting $(F_1+abc) = 0$, leads to (2.19).

Focusing on the irreducible components of (2.18), provides

$$\begin{aligned} M_{(1)} &= \frac{2a-b-c}{3}\mathrm{I} + \mathrm{a}\mathbf{e}_1 + \mathrm{b}\mathbf{e}_2 + \mathrm{c}\mathbf{e}_3, \\ M_{(2)} &= \frac{2b-a-c}{3}\mathrm{I} + \mathrm{a}\mathbf{e}_1 + \mathrm{b}\mathbf{e}_2 + \mathrm{c}\mathbf{e}_3, \\ M_{(3)} &= \frac{2c-a-b}{3}\mathrm{I} + \mathrm{a}\mathbf{e}_1 + \mathrm{b}\mathbf{e}_2 + \mathrm{c}\mathbf{e}_3, \end{aligned}$$

for $f = f_1$, f_2 and f_3 respectively, while $c = 1/ab$ and a and b are determined as in Theorem 2.1.2. However, it becomes clear from (2.17b) that the permutation

$$\pi : (a,b,c) \mapsto (b,c,a), \quad \pi^3 = \mathrm{id},$$

maps one factor of F_2 into another. Letting T denote the Lie algebra automorphism,

$$T : A \mapsto M_{(0)} A M_{(0)}^{-1},$$

it is easy to see that

$$M_{(2)} = \pi\left(T(M_{(1)})\right), \quad M_{(3)} = \pi\left(T(M_{(2)})\right), \quad M_{(1)} = \pi\left(T(M_{(3)})\right). \tag{2.20}$$

Thus $M_{(1)}$, $M_{(2)}$ and $M_{(3)}$ are related by conjugation by $M_{(0)}$ and a cyclic relabelling of the functions a, b and c.

Turning attention to the irreducible components in (2.19), we see that each component is invariant with respect to π, so in particular, denoting

$$M_{(4)} = -\frac{a+b+c}{3}\mathrm{I} + \mathrm{a}\mathbf{e}_1 + \mathrm{b}\mathbf{e}_2 + \mathrm{c}\mathbf{e}_3, \tag{2.21}$$

it follows that

$$\pi\left(T(M_{(4)})\right) = M_{(4)}.$$

By expressing f and c in terms of the remaining functions using F_1 and F_2 as described above, their presence can be removed from Theorem 2.1.2. The reduction is compatible with the original system, since f and c were determined using first integrals. In order to separate the different cases, we use different indices for the updated functions corresponding to each Darboux transformation. In particular, u_i and v_i denote the updated functions corresponding to $M_{(i)}$, $i = 1, \ldots, 4$. This separation is also convenient for later use when we discuss fully discrete systems.

Theorem 2.1.5 *$M_{(i)}$ for $i = 1, \ldots, 4$ is a Darboux transformation for L provided*

$$a = cuu_iv, \quad b = cu_ivv_i, \quad c^3uu_i^2v^2v_i = 16, \tag{2.22}$$

and the following relations hold.

$$\frac{v_{1,x}}{v_1} - \frac{u_x}{u} - \frac{uv}{v_1} + u + v - v_1 = 0, \tag{2.23a}$$

$$\frac{v_x}{v} + \frac{u_{1,x}}{u_1} + \frac{u_x}{u} - \frac{1}{uv} - \frac{uv}{v_1} + \frac{1}{u_1v_1} + u_1 = 0, \tag{2.23b}$$

$$\frac{v_{2,x}}{v_2} - \frac{u_x}{u} - \frac{u_2v_2}{u} - u + u_2 + v_2 = 0, \tag{2.24a}$$

$$\frac{v_x}{v} + \frac{u_{2,x}}{u_2} + \frac{u_x}{u} - \frac{2u_2v_2}{u} + \frac{1}{u_2v_2} - \frac{1}{uv} - v_2 - v + u - u_2 = 0, \tag{2.24b}$$

$$\frac{v_{3,x}}{v_3} - \frac{u_x}{u} - \frac{1}{uv} + \frac{2}{u_3v} - \frac{1}{u_3v_3} - u + u_3 + v - v_3 = 0, \tag{2.25a}$$

$$\frac{v_x}{v} + \frac{u_{3,x}}{u_3} + \frac{u_x}{u} + \frac{1}{uv} - \frac{1}{u_3v} + u - u_3 = 0, \tag{2.25b}$$

$$\frac{v_{4,x}}{v_4} - \frac{u_x}{u} - u + u_4 + v - v_4 = 0, \tag{2.26a}$$

$$\frac{v_x}{v} + \frac{u_{4,x}}{u_4} + \frac{u_x}{u} - \frac{1}{uv} + \frac{1}{u_4 v_4} + u - u_4 = 0. \tag{2.26b}$$

Proof

The theorem follows from Theorem 2.1.2 by making the previously described reductions. □

As previously stated, the derived Darboux transformations are related via (2.20). The latter relation can be seen at the level of the systems above. For instance, the systems corresponding to $M_{(1)}$ and $M_{(2)}$ are related by the point transformation

$$(u, v) \mapsto (v, u^{-1}v^{-1}), \tag{2.27}$$

and a relabelling of index. The same holds for the pairs of systems corresponding to $M_{(2)}$ and $M_{(3)}$, also for $M_{(3)}$ and $M_{(1)}$.

2.2 Darboux Lax representation for local symmetries

The derived Darboux matrices may be regarded as defining shifts on a lattice. To make it clear, we consider the functions u and v to depend on a set of discrete variables k_i, $i = 1, \ldots, 4$, and interpret the updated functions u_i and v_i as shifts in the corresponding lattice direction. For example, u_1 corresponds to u shifted in the direction corresponding to $M_{(1)}$, while u_2 corresponds to a shift of u in the direction corresponding to $M_{(2)}$. In this interpretation, the Lax operator and its Darboux transformations define a DLR of the corresponding systems found in Theorem 2.1.5. Subsequently, the systems are viewed as DΔEs. Let $\mathcal{S}_i$ denote the shift operator in the i-th direction. Then the differential difference relations corresponding to $M_{(i)}$ are equivalent to the compatibility condition of the following system

$$\mathcal{S}_i(\Psi) = M_{(i)}\Psi, \quad \Psi_x = -\left(u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3\right)\Psi.$$

From this viewpoint, the equations in Theorem 2.1.5 are integrable. However, they are not evolutionary.

We now turn our attention to the derivation of evolutionary D∆Es associated with each $M_{(i)}$. The problem then becomes to find continuous flows $\Psi_{\tau^i} = \Omega_{(i)}\Psi$, such that the compatibility condition

$$\partial_{\tau^i} M_{(i)} = \mathcal{S}_i\left(\Omega_{(i)}\right) M_{(i)} - M_{(i)}\Omega_{(i)}\,, \tag{2.28}$$

is equivalent to an evolutionary system of differential difference equations. We tackle this by taking an ansatz for $\Omega_{(i)}$ of the following form

$$\Omega_{(i)} = M_{(i)}^{-1}(g\mathrm{I} + p\mathbf{e}_1 + q\mathbf{e}_2 + r\mathbf{e}_3). \tag{2.29}$$

This ansatz is based upon observations in [75] that operators producing evolutionary equations are often of this form.

To reduce the number of unknown functions occurring in $\Omega_{(i)}$, we note that due to the choice of F_1 and F_2 taken when deriving each $M_{(i)}$, it follows that $\det\left(M_{(i)}\right)$ is constant with respect to τ_i. This fact, along with Jacobi's identity, yields $\left(\mathcal{S}_i - \mathrm{I}\right)\left(\mathrm{Tr}(\Omega_{(i)})\right) = 0$. Hence, $\mathrm{Tr}(\Omega_{(i)})$ is constant with respect to k_i. Now, consider the transformation

$$\tilde{\Omega}_{(i)} \mapsto \Omega_{(i)} - \frac{(\mathrm{Tr}(\Omega_{(i)}) - \alpha)}{3}\mathrm{I},$$

where α is constant with respect to τ^i and the shift operator $\mathcal{S}_i$. If $\Omega_{(i)}$ satisfies (2.28), then as $\mathrm{Tr}(\Omega_{(i)})$ is constant with respect to $\mathcal{S}_i$, it follows that $\tilde{\Omega}_{(i)}$ also satisfies (2.28). Therefore, we may replace $\Omega_{(i)}$ by $\tilde{\Omega}_{(i)}$ to obtain an operator with constant trace.

Moreover, for constant k,

$$\begin{aligned}\Omega_{(i)} - k\mathrm{I} &= M_{(i)}^{-1}(g\mathrm{I} + p\mathbf{e}_1 + q\mathbf{e}_2 + r\mathbf{e}_3) - k\mathrm{I} \\ &= M_{(i)}^{-1}(g\mathrm{I} + p\mathbf{e}_1 + q\mathbf{e}_2 + r\mathbf{e}_3) - kM_{(i)}^{-1}M_{(i)} \\ &= M_{(i)}^{-1}((g - fk)\mathrm{I} + (p - ka)\mathbf{e}_1 + (q - kb)\mathbf{e}_2 + (r - kc)\mathbf{e}_2),\end{aligned}$$

where a, b, c and f are as in Theorem 2.1.5. Thus by relabelling appropriately $\tilde{\Omega}_{(i)}$ is of the form (2.29), thus we may assume from the start that $\tilde{\Omega}_{(i)}$ has constant trace. We may now proceed to solve the compatibility condition for $\Omega_{(1)}$. This leads to the following result. Note that negative indices denote backward shifts, i.e. $u_{-1} = u(k_1 - 1)$, $v_{-1} = v(k_1 - 1)$.

Theorem 2.2.1 *The matrix $\Omega_{(1)}$ is determined by the relations*

$$\begin{aligned}r &= -\frac{p}{uu_1v} - \frac{q}{u_1vv_1}, \quad g = \frac{uu_1vv_1}{(uu_1v-1)(u-v_1)} + \frac{2p-q-r}{3}, \\ p &= \frac{1}{3}\frac{uu_1v}{u-v_1}\left(\frac{2v_1}{uu_1v-1} - \frac{uv+(u_{-1}-2v)v_1}{(u_{-1}uv_{-1}-1)(u_{-1}-v)}\right), \\ q &= \frac{1}{3}\frac{u_1vv_1}{u-v_1}\left(\frac{2uv-(u_{-1}+v)v_1}{(u_{-1}uv_{-1}-1)(u_{-1}-v)} - \frac{v_1}{uu_1v-1}\right).\end{aligned}$$

The corresponding differential difference system is given by

$$u_{\tau^1} = u(\mathcal{S}_1 - \mathrm{I})\frac{v}{(v-u_{-1})(uu_{-1}v_{-1}-1)}, \tag{2.31a}$$

$$v_{\tau^1} = \frac{v}{uu_{-1}v_{-1}-1}(\mathcal{S}_1 - \mathrm{I})\frac{u_{-1}}{u_{-1}-v}. \tag{2.31b}$$

Proof

The trace of $\Omega_{(1)}$ is constant. Moreover, from the form of $\Omega_{(1)}$ it follows that its trace is an automorphic function of λ. Bearing this in mind, we find that $\mathrm{Tr}(\Omega_{(1)}) = \epsilon_1\frac{1}{J(\lambda)} + \epsilon_2$ where $J(\lambda)$ is given in (2.16), while ϵ_1 and ϵ_2 are as follows

$$\begin{aligned}\epsilon_1 &= \frac{16\,(uu_1v-1)\,(u-v_1)\,(3g-2p+q+r)}{3uu_1vv_1}, \\ \epsilon_2 &= \frac{p}{uu_1v} + \frac{q}{u_1vv_1} + r.\end{aligned}$$

The above relations allow g and r to determined. Moving on to the compatibility condition, we find that u and v must satisfy the differential difference equations

$$\frac{u_{1,\tau^1}}{u_1}+\frac{u_{\tau^1}}{u}+\frac{v_{\tau^1}}{v}+\frac{p}{uu_1v}+\frac{p_1}{u_1u_2v_1}-\frac{q_1}{u_2v_1v_2}-r=0,$$
$$\frac{v_{1,\tau^1}}{v_1}-\frac{u_{\tau^1}}{u}-\frac{2p}{uu_1v}+\frac{p_1}{u_1u_2v_1}+\frac{2q_1}{u_2v_1v_2}-r=0.$$

This system implies the following condition on u

$$(\mathrm{I}+\mathcal{S}_1+\mathcal{S}_1^2)\frac{u_{\tau^1}}{u}=\frac{2q_1}{u_2v_1v_2}+\frac{q_2}{u_3v_2v_3}-\frac{2p}{uu_1v}-\frac{p_2}{u_2u_3v_2}-r+r_1,$$
$$(\mathrm{I}+\mathcal{S}_1+\mathcal{S}_1^2)\frac{u_{\tau^1}}{u}=(\mathrm{I}+\mathcal{S}_1+\mathcal{S}_1^2)\left(\frac{pv_1-qu}{uu_1vv_1}\right).$$

The operator $(\mathrm{I}+\mathcal{S}_1+\mathcal{S}_1^2)$ has kernel ω^{k_1}, where ω is a cube root of unity. Thus, in general, from the above relation we may determine u_{τ^1} up to the addition of a constant multiple of $\omega^{k_1}u$. We shall omit the kernel, as it will not add anything interesting to our subsequent analysis. Inverting the operator $(\mathrm{I}+\mathcal{S}_1+\mathcal{S}_1^2)$, we obtain the equation

$$u_{\tau^1}=\frac{pv_1-qu}{u_1vv_1}.$$

The corresponding equation for v is given by

$$v_{\tau^1}=v\epsilon_2-\frac{p}{uu_1}-\frac{2q}{u_1v_1}.$$

Returning to the compatibility condition, one can solve the remaining equations to determine p and q. Doing so, we find that

$$p=\frac{uu_1v\epsilon_1\left(uv+u_{-1}v_1-2v_1v\right)}{48\left(u_{-1}uv_{-1}-1\right)\left(u_{-1}-v\right)\left(u-v_1\right)}+\frac{uu_1v\left(8\epsilon_2\left(uu_1v-1\right)\left(u-v_1\right)-v_1\epsilon_1\right)}{24\left(uu_1v-1\right)\left(u-v_1\right)},$$
$$q=\frac{-u_1vv_1\epsilon_1\left(2uv-u_{-1}v_1-v_1v\right)}{48\left(u_{-1}uv_{-1}-1\right)\left(u_{-1}-v\right)\left(u-v_1\right)}+\frac{u_1vv_1\left(16\epsilon_2\left(uu_1v-1\right)\left(u-v_1\right)+v_1\epsilon_1\right)}{48\left(uu_1v-1\right)\left(u-v_1\right)}.$$

Finally, eliminating p and q from the differential difference equations above, we arrive at

$$u_{\tau^1}=\frac{-\epsilon_1u}{16}(\mathcal{S}_1-\mathrm{I})\frac{v}{\left(v-u_{-1}\right)\left(uu_{-1}v_{-1}-1\right)},\quad v_{\tau^1}=\frac{-\epsilon_1v}{16\left(uu_{-1}v_{-1}-1\right)}(\mathcal{S}_1-\mathrm{I})\frac{u_{-1}}{\left(u_{-1}-v\right)}.$$

The compatibility condition is now completely satisfied. Note, ϵ_1 can be an arbitrary non zero constant, in particular take $\epsilon_1 = -16$. Additionally, ϵ_2 can be set to zero without affecting the computed flow. Making these choices leads to the expressions in the theorem. □

Corollary 2.2.2 *The DΔEs corresponding to $\Omega_{(2)}$ and $\Omega_{(3)}$ for $i = 2, 3$ are given by*

$$\begin{aligned} u_{\tau^2} &= \frac{u}{(u_2 v v_2 - 1)}(\mathcal{S}_2 - \mathrm{I})\frac{v}{(v - u_{-2})}, \\ v_{\tau^2} &= v(\mathcal{S}_2 - \mathrm{I})\frac{u_{-2}}{(u v_{-2} v - 1)\,(u_{-2} - v)}, \end{aligned}$$

and

$$\begin{aligned} u_{\tau^3} &= \frac{u^2 v_{-3} v}{u v_{-3} v - 1}(\mathcal{S}_3 - \mathrm{I})\frac{1}{u_{-3} u v_{-3} - 1}, \\ v_{\tau^3} &= \frac{u u_3 v^2}{u u_3 v - 1}(\mathcal{S}_3 - \mathrm{I})\frac{1}{u v_{-3} v - 1}. \end{aligned}$$

Proof

The result follows from (2.31), using the transformation $(u, v) \mapsto (v, \frac{1}{uv})$ and relabelling indices. □

Following the same procedure as in the proof of Theorem 2.2.1, we derive the corresponding result for $M_{(4)}$.

Theorem 2.2.3 $\Omega_{(4)}$ *is determined by the relations*

$$\begin{aligned} r &= -\frac{p}{u u_4 v} - \frac{q}{u_4 v v_4}, \quad g = \frac{u u_4 v v_4}{u u_4 v v_4 + u + v_4} - \frac{p + q + r}{3}, \\ p &= \frac{u u_4 v}{3}(\mathcal{S}_4 - \mathrm{I})\frac{u_{-4} + 2v}{u_{-4} u v_{-4} v + u_{-4} + v}, \\ q &= \frac{u_4 v v_4}{3}(\mathcal{S}_4 - \mathrm{I})\frac{u_{-4} - v}{u_{-4} u v_{-4} v + u_{-4} + v}. \end{aligned}$$

The corresponding differential difference system is given by

$$u_{\tau^4} = u(\mathcal{S}_4 - \mathrm{I})\frac{v}{u_{-4}uvv_{-4} + u_{-4} + v}, \tag{2.33a}$$

$$v_{\tau^4} = v(\mathcal{S}_4 - \mathrm{I})\frac{u_{-4}}{u_{-4}uvv_{-4} + u_{-4} + v}. \tag{2.33b}$$

These systems appear to be new.

2.3 Darboux pairs and fully discrete systems

Viewing Darboux transformations as shifts on a lattice allowed DΔEs to be derived. We now derive fully discrete systems as the compatibility condition of two separate Darboux transformations. For system

$$\mathcal{S}_i\,(\Psi) \;=\; M_{(i)}\Psi\,, \quad \mathcal{S}_j\,(\Psi) \;=\; M_{(j)}\Psi\,, \quad i \neq j,$$

the compatibility condition is given by

$$\mathcal{S}_j(M_{(i)})M_{(j)} \;-\; \mathcal{S}_i(M_{(j)})M_{(i)} \;=\; 0\,.$$

Requiring that the Darboux transformations derived in the previous section are compatible leads to the following theorem. We use the notation $Q_{(i,j)}$ to denote the system which corresponds to the compatibility condition of $M_{(i)}$ and $M_{(j)}$. We only consider systems $Q_{(1,2)}$ and $Q_{(1,4)}$. This is because we may employ the point transformation (2.27) to map system $Q_{(1,2)}$ to systems $Q_{(2,3)}$ and $Q_{(3,1)}$, respectively. In the same fashion, system $Q_{(1,4)}$ can be mapped to systems $Q_{(2,4)}$ and $Q_{(3,4)}$, respectively.

Theorem 2.3.1 *The Darboux pairs* $(M_{(1)}, M_{(2)})$ *and* $(M_{(1)}, M_{(4)})$ *are compatible provided the following systems hold.*

$$u_2v_2\left[u_1v\left(v_1 - u\right) + u_{12}v_1\left(u_1 - v_{12}\right)\right] + u_2v_2 - u_1v_1 = 0, \tag{2.34a}$$

$$uv_{12} - u_2v_2 = 0, \tag{2.34b}$$

$$uv_{14} - u_4v_4 + v_{14}v_4 - v_1v_{14} = 0, \tag{2.35a}$$

$$\frac{uv}{v_1} - \frac{1}{u_1v_1} + \frac{1}{u_4v_4} - u_{14} = 0. \tag{2.35b}$$

Proof

The theorem follows by direct computation. □

Note the notation u_{ij} corresponds to a single shift of u in the i direction and a single shift of u in the j direction, similarly for v_{ij}. The above systems appear to be new. They are integrable as they follow from the compatibility condition of a Darboux pair. Another manifestation of their integrability is their multidimensional consistency. If we consider any three systems $Q_{(i,j)}$, $Q_{(j,k)}$ and $Q_{(k,i)}$, where $i \neq j \neq k \neq i$, then u_{ijk} and v_{ijk} can be computed in three different ways, provided by the corresponding systems. All of which yield the same result modulo the systems. In order to understand the connection between the fully discrete and semi discrete systems we have the following lemma.

Lemma 2.3.2 *Suppose M, N, where $\Psi_i = M\Psi$ and $\Psi_j = N\Psi$, are Darboux transformations for the operator $L = \partial_\tau - A$, then*

$$\frac{d}{d\tau}(M_jN - N_iM) = 0, \tag{2.36}$$

when restricted to $M_jN - N_iM = 0$.

Proof

Since M and N are Darboux transformations for L, it follows that

$$M_\tau = A_iM - MA,$$

$$N_\tau = A_jN - NA.$$

Hence,

$$\begin{aligned}\frac{d}{d\tau}(M_jN - N_iM) &= (M_j)_\tau N + M_jN_\tau - (N_i)_\tau M - N_iM_\tau \\ &= A_{ji}(M_jN - N_iM) - (M_jN - N_iM)A \\ &= 0 \mod (M_jN - N_iM).\end{aligned}$$

□

Corollary 2.3.3 *The D∆Es derived in (2.1.5) define non-local symmetries of the corresponding discrete systems* $Q_{(i,j)}$.

One can confirm directly that the τ^i-flow and τ^j-flow define local generalised symmetries in the i-th and j-th lattice direction respectively, for system $Q_{(i,j)}$. We next demonstrate how one can use the found Darboux pairs to construct conservation laws.

2.4 Conservation laws

In the previous section, integrable discrete systems were derived by constructing a Darboux pair of the form

$$\mathcal{S}_i\left(\mathbf{\Psi}\right) \,=\, M_{(i)}\Psi\,, \quad \mathcal{S}_j\left(\mathbf{\Psi}\right) \,=\, M_{(j)}\mathbf{\Psi}\,, \quad i \neq j. \tag{2.37}$$

By considering a particular gauge transformation of this system one may derive conservation laws for the underlying discrete system. The general ideas in the section can be found in [49].

Let the notation $f(\{u_{ij}\})$ denote that f is a function of a finite number of the lattice variables u_{ij}. Then we have the following definition.

Definition 2.4.1 *Let*

$$Q(\{\mathbf{u}_{ij}\}) = 0, \tag{2.38}$$

define a partial difference system. A conservation law is an expression of the form

$$(\mathcal{S}_i - \mathrm{I})f(\{u_{ij}\}) = (\mathcal{S}_j - \mathrm{I})g(\{u_{ij}\}),$$

which holds modulo (2.38). The conservation law is said to be trivial if it holds identically, i.e. not only on solutions of 2.38.

Let $\Phi = W\Psi$, where W is an $n \times n$ invertible matrix which may depend upon the same variables as $M_{(i)}$ and $M_{(j)}$. In the cases that we consider, W is a 3×3 matrix. In terms of Φ, (2.37) is expressed as

$$\mathcal{S}_i(\Phi) = \mathcal{S}_i(W)M_{(i)}W^{-1}\Phi \quad \mathcal{S}_j(\Phi) = \mathcal{S}_j(W)M_{(j)}W^{-1}\Phi.$$

Thus, the effect of the transformation $\Psi \mapsto \Phi$ at the level of the Darboux matrices is the following

$$M_{(i)} \mapsto N_{(i)} = \mathcal{S}_i(W)M_{(i)}W^{-1}, \quad M_{(j)} \mapsto N_{(j)} = \mathcal{S}_j(W)M_{(j)}W^{-1}. \tag{2.39}$$

It is not difficult to show that

$$\mathcal{S}_j(N_{(i)})N_{(j)} - \mathcal{S}_i(N_{(j)})N_{(i)} = \mathcal{S}_i\mathcal{S}_j(W)(\mathcal{S}_j(M_{(i)})M_{(j)} - \mathcal{S}_i(M_{(j)})M_{(i)})W^{-1}.$$

The left hand side is the compatibility condition of system (2.39). From the right hand side one can see that (2.39) is compatible if and only if (2.37) is compatible. This follows as W is invertible. Hence, the system

$$\mathcal{S}_i(\Phi) = N_{(i)}\Phi, \quad \mathcal{S}_j(\Phi) = N_{(j)}\Phi, \tag{2.40}$$

constitutes an equivalent Darboux pair for the underlying discrete system. It is this freedom in choosing a Darboux representation that we shall exploit to derive conservation laws. Suppose one can find W such that $N_{(i)}$ and $N_{(j)}$ are block diagonal, in particular, of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{2.41}$$

Now, the projection of a 3×3 matrix to its $(3, 3)$ element, i.e. the map

$$\pi : A \mapsto [A]_{3,3},$$

is multiplicative on such matrices. Thus, if

$$\mathcal{S}_j(N_{(i)})N_{(j)} = \mathcal{S}_i(N_{(j)})N_{(i)}, \tag{2.42}$$

then

$$\pi[\mathcal{S}_j(N_{(i)})]\pi[N_{(j)}] = \pi[\mathcal{S}_i(N_{(j)})]\pi[N_{(i)}].$$

Which implies

$$\mathcal{S}_j(n_{(i)})n_{(j)} = \mathcal{S}_i(n_{(j)})n_{(i)}, \tag{2.43}$$

where $n_{(i)} = [N_{(i)}]_{3,3}$, and similarly for $n_{(j)}$. Expression (2.43) is in the form of a multiplicative conservation law. By taking logarithms one finds

$$(\mathcal{S}_j - \mathrm{I}) \log(n_{(i)}) = (\mathcal{S}_i - \mathrm{I}) \log(n_{(j)}). \tag{2.44}$$

The equalities (2.42) and (2.43) hold modulo the underlying discrete system. Moreover, these equalities hold for all values of the spectral parameter, which is λ for the Darboux pairs we consider. Hence, expanding (2.44) as a power series in λ, we obtain a sequence of conservation laws. The method relies upon the ability to find a W such that $M_{(i)}$ and $M_{(j)}$ are simultaneously block diagonalised. We find such a W by the process of formal diagonalisation.

Consider $M_{(1)}$, this may be formally expanded as a Taylor series in λ to obtain

$$M_{(1)} = M_0 + \lambda M_1 + \lambda^2 M_2 + \dots.$$

We take the following ansatz for W

$$W = (\mathrm{I} + P_1\lambda + P_2\lambda^2 + \cdots)P,$$

where P_i is off block diagonal for all i, while P is an invertible constant matrix. Explicitly, P_i takes the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}.$$

We take P_i of this form as the set of such matrices is closed under multiplication by matrices of the form (2.41). As shall be shown, this property will allow the subsequent determining relations to be naturally split onto the block diagonal and off block diagonal linear subspaces of the space of 3×3 matrices. We allow P to be an arbitrary constant invertible matrix, as this simply corresponds to first applying a constant gauge transformation to the initial Darboux pair.

Now, assuming $N = N_0 + \lambda N_1 + \lambda^2 N_2 + \ldots$, with each N_i of the form (2.41), we have

$$\mathcal{S}_1(W)M_{(1)}W^{-1} = N,$$

if and only if

$$\mathcal{S}_1(W)M_{(1)} = NW,$$

which is if and only if

$$\begin{aligned} &(\mathrm{I} + \mathcal{S}_1(P_1)\lambda + \mathcal{S}_1(P_2)\lambda^2 + \ldots)P(M_0 + \lambda M_1 + \lambda^2 M_2 + \ldots) \\ &= (N_0 + \lambda N_1 + \lambda^2 N_2 + \ldots)(\mathrm{I} + P_1\lambda + P_2\lambda^2 + \ldots)P. \end{aligned}$$

Equating coefficients at powers of λ, one finds for each n we require

$$\sum_{i+j=n} \mathcal{S}_1(P_i)PM_j = \sum_{i+j=n} N_i P_j P, \tag{2.45}$$

where P_0 is taken to be the identity matrix.

Suppose we are able to solve this condition for each $n \leq k$, and hence have determined N_i and P_i for $i \leq k$. Then, considering the coefficient of λ^{k+1}, we obtain after rearrangement the expression

$$\begin{aligned}\mathcal{S}_1(P_{k+1})N_0 - N_0P_{k+1} - N_{k+1} &= \sum_{i=1}^{k} N_iP_{n-i} - \sum_{i=0}^{k} \mathcal{S}_1(P_i)PM_{n-i}P^{-1} \\ &= R_k,\end{aligned}$$

where the right hand side has been relabelled R_k for convenience. By assumption, R_k is completely determined. As mentioned, the product of an off block diagonal and block diagonal matrix is off block diagonal, hence N_{k+1} and P_{k+1} satisfy the following equations.

$$N_{k+1} = \pi_{BD}(-R_k), \tag{2.46}$$

$$\mathcal{S}_1(P_{k+1})N_0 - N_0P_{k+1} = \pi_{OBD}(R_k), \tag{2.47}$$

where π_{BD} and π_{ODB} are the projections onto the block diagonal and off block diagonal components respectively. From (2.46), N_{k+1} can be determined immediately. Let us concentrate on (2.47). The matrix N_0 is determined by the relation

$$PM_0 = N_0P. \tag{2.48}$$

In general, N_0 may take the form

$$\begin{pmatrix} n_{11}^0 & n_{12}^0 & 0 \\ n_{21}^0 & n_{22}^0 & 0 \\ 0 & 0 & n_{33}^0 \end{pmatrix},$$

since we only require it to be block diagonal. In which case, (2.47) is equivalent to the system of equations

$$n_{33}^0\mathcal{S}(p_{13}^{k+1}) - n_{11}^0p_{13}^{k+1} - n_{12}^0p_{23}^{k+1} = r_{13}^k,$$

$$n_{33}^0\mathcal{S}(p_{23}^{k+1}) - n_{21}^0p_{13}^{k+1} - n_{22}^0p_{23}^{k+1} = r_{23}^k,$$

$$n_{11}^0\mathcal{S}(p_{31}^{k+1})+n_{21}^0\mathcal{S}(p_{32}^{k+1})-n_{33}^0p_{31}^{k+1}=r_{31}^k,$$

$$n_{12}^0\mathcal{S}(p_{31}^{k+1})+n_{22}^0\mathcal{S}(p_{32}^{k+1})-n_{33}^0p_{32}^{k+1}=r_{32}^k,$$

where $p_{ij}^{k+1}=[P_{k+1}]_{ij}$ and similarly $r_{ij}^k=[R_k]_{ij}$. This system separates into the two systems

$$n_{33}^0\begin{pmatrix}\mathcal{S}(p_{13}^{k+1})\\ \mathcal{S}(p_{23}^{k+1})\end{pmatrix}=\begin{pmatrix}r_{13}^k\\ r_{23}^k\end{pmatrix}+\begin{pmatrix}n_{11}^0 & n_{12}^0\\ n_{21}^0 & n_{22}^0\end{pmatrix}\begin{pmatrix}p_{13}^{k+1}\\ p_{23}^{k+1}\end{pmatrix}$$

and

$$n_{33}^0\begin{pmatrix}p_{31}^{k+1}\\ p_{32}^{k+1}\end{pmatrix}=\begin{pmatrix}n_{11}^0 & n_{21}^0\\ n_{12}^0 & n_{22}^0\end{pmatrix}\begin{pmatrix}\mathcal{S}(p_{31}^{k+1})\\ \mathcal{S}(p_{32}^{k+1})\end{pmatrix}-\begin{pmatrix}r_{31}^k\\ r_{32}^k\end{pmatrix}.$$

To ensure that we may determine P_{k+1} locally from these systems for arbitrary k, we require either $n_{33}^0=0$ with $n_{11}^0n_{22}^0-n_{12}^0n_{21}^0\neq 0$, or $n_{11}^0=n_{12}^0=n_{21}^0=n_{22}^0=0$ with $n_{33}^0\neq 0$, so that the systems degenerate from difference equations to purely algebraic relations, which can be solved. Recall, N_0 is given by the relation (2.48), thus we may use the freedom of choice of P to fix N_0. That is, we may find a matrix similar to M_0 which satisfies either of the found conditions.

One can directly compute that

$$M_0=\begin{pmatrix}c\,(uu_1v-1) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & cu_1v\,(u-v_1)\end{pmatrix},$$

where c is as given in Theorem 2.1.5. Clearly, M_0 is not similar to a block diagonal matrix N_0 for which $n_{11}^0=n_{12}^0=n_{21}^0=n_{22}^0=0$, as such an N_0 possesses different eigenvalues to M_0. However, M_0 is similar to

$$N_0=\begin{pmatrix}c\,(uu_1v-1) & 0 & 0\\ 0 & cu_1v\,(u-v_1) & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad (2.49)$$

indeed one can take P to be the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.50}$$

With this choice of P we ensure that one can determine P_k for all $k > 0$ without the need to solve any difference equations. Note this choice is not unique, one can multiply (2.50) by any matrix

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that $ad - bc \neq 0$, to obtain an equally suitable P. Taking P as in (2.50), it follows that

$$P_{k+1} = \begin{pmatrix} 0 & 0 & -\frac{r_{13}^k}{c(uu_1v-1)} \\ 0 & 0 & -\frac{r_{23}^k}{cu_1v(u-v_1)} \\ \mathcal{S}^{-1}\left(\frac{r_{31}^k}{c(uu_1v-1)}\right) & \mathcal{S}^{-1}\left(\frac{r_{32}^k}{cu_1v(u-v_1)}\right) & 0 \end{pmatrix}.$$

Using the above formula, along with (2.46) and the computations for N_0 and P, one can compute W and N up to an arbitrary power of λ. One can verify that modulo the partial difference system (2.35), $M_{(4)}$ is also block diagonalised by W. Thus, making use of W, we compute $N_{(1)}$ and $N_{(4)}$, then as described previously we may use (2.44) to derive conservation laws. We present the first two non trivial conservation laws for systems (2.35) and (2.34).

Theorem 2.4.2 *System (2.35) possesses the conservation laws*

$$(\mathcal{S}_4 - \mathrm{I})\rho_i = (\mathcal{S}_1 - \mathrm{I})\sigma_i, \quad i = 1, 2,$$

with

$$\rho_1 = \log\left(\frac{u^2u_1vv_1^2}{(uu_1v-1)^3(u-v_1)^3}\right),$$
$$\sigma_1 = \log\left(\frac{u^2u_4v}{v_4}\right),$$

$$\rho_2 = \frac{3\left(9uu_1v-1\right)}{8\left(uu_1v-1\right)} + \frac{4\left(uv_{11}+uu_1\right)}{\left(uu_1v-1\right)\left(v_1-u\right)\left(u_1-v_{11}\right)}$$
$$+ \frac{uu_1v\left(u_1-5v_{11}\right)-u_1-3v_{11}}{\left(uu_1v-1\right)\left(u_1-v_{11}\right)\left(u_1v_1v_{11}-1\right)},$$
$$\sigma_2 = \frac{u_1v_1\left(u-v_1+4v_4\right)-u_4v_4\left(u-5v_1\right)}{u_4v_4\left(uu_1v-1\right)\left(u-v_1\right)}.$$

Theorem 2.4.3 *System (2.34) has conservation laws*

$$(\mathcal{S}_2-\mathrm{I})\rho_i = (\mathcal{S}_1-\mathrm{I})\phi_i, \quad i=1,2,$$

with

$$\rho_1 = \log\left(\frac{u^2u_1vv_1^2}{\left(uu_1v-1\right)^3\left(u-v_1\right)^3}\right),$$
$$\phi_1 = \log\left(\frac{u_2v\left(v_2-u\right)^3}{uv_2}\right),$$

$$\rho_2 = \frac{3\left(9uu_1v-1\right)}{8\left(uu_1v-1\right)} + \frac{4\left(uv_{11}+uu_1\right)}{\left(uu_1v-1\right)\left(v_1-u\right)\left(u_1-v_{11}\right)},$$
$$\phi_2 = \frac{u\left(u_2v_2^2+\left(3u_1+4u_2\right)v_1v_2+u\left(u_1v_1-u_2v_2\right)\right)}{u_2v_2\left(uu_1v-1\right)\left(u-v_1\right)\left(u-v_2\right)}.$$

Chapter 3

Related integrable systems & PDE solutions

3.1 Related integrable systems

In this section we present integrable systems related to the previously derived systems. In particular, we consider reductions, potentiations and Miura-type transformations (MTs). We start with MTs of the local symmetries. At present, by MT we simply understand a difference substitution. In part II we study MTs in detail, but for now we simply present transformations which are derived by inspection.

Theorem 3.1.1 *The system*

$$
\begin{aligned}
u_{\tau^1} &= u(\mathcal{S}_1 - \mathrm{I})\frac{v}{(v - u_{-1})\,(uu_{-1}v_{-1} - 1)}\,,\\
v_{\tau^1} &= \frac{v}{uu_{-1}v_{-1} - 1}(\mathcal{S}_1 - \mathrm{I})\frac{u_{-1}}{u_{-1} - v}\,,
\end{aligned}
$$

from Theorem 2.2.1 is linked via the MT

$$\begin{aligned} p &:= \frac{v}{(u_{-1}uv_{-1}-1)(v-u_{-1})}, \\ q &:= \frac{u_{-1}}{u_{-1}-v}, \end{aligned}$$

to the system

$$\begin{aligned} p_{\tau^1} &= \frac{p\left(p_1\left(p-q+1\right)-pq\left(q_1-1\right)\right)}{q-1} - \frac{pp_{-1}\left(p-qq_{-1}+1\right)}{q_{-1}-1}, \\ q_{\tau^1} &= q\left(p_{-1}q-pq_1-p_{-1}+p\right). \end{aligned}$$

Proof

One may use the expression for q to express v in terms of q and u_{-1}. Then, eliminating v from the expression for p, one obtains an expression involving shifts of u, p and q. Explicitly, we have the following

$$v = \frac{(q-1)\,u_{-1}}{q}, \;\; u_{-2} = \frac{q_{-1}\left(p-q+1\right)}{p\left(q_{-1}-1\right)u_{-1}u}.$$

By the definition of p and q, it follows from (2.31) that

$$\begin{aligned} u_{\tau^1} &= u(\mathcal{S}_1-\mathrm{I})p, \\ v_{\tau^1} &= \frac{v}{(uu_{-1}v_{-1}-1)}(\mathcal{S}_1-\mathrm{I})q. \end{aligned}$$

Using this fact, along with the expressions above, a straightforward computation provides the following D∆Es for p and q

$$\begin{aligned} p_{\tau^1} &= \frac{p\left(p_1\left(p-q+1\right)-pq\left(q_1-1\right)\right)}{q-1} - \frac{pp_{-1}\left(p-qq_{-1}+1\right)}{q_{-1}-1}, \\ q_{\tau^1} &= q\left(p_{-1}q-pq_1-p_{-1}+p\right). \end{aligned}$$

□

One can play a similar game with the τ^4 symmetries.

Theorem 3.1.2 *The system*

$$u_{\tau^4} = u(\mathcal{S}_4 - \mathrm{I})\frac{v}{u_{-4}uvv_{-4} + u_{-4} + v}, \tag{3.1a}$$

$$v_{\tau^4} = v(\mathcal{S}_4 - \mathrm{I})\frac{u_{-4}}{u_{-4}uvv_{-4} + u_{-4} + v}, \tag{3.1b}$$

from Theorem 2.2.3 is linked via the MT

$$p := \frac{v}{u_{-4}\left(uv_{-4}v + 1\right) + v},$$
$$q := \frac{u_{-4}}{u_{-4}\left(uv_{-4}v + 1\right) + v},$$

to the system

$$p_{\tau^4} = \frac{1}{3}p\left(q\left(p_4 - q_{-4} + q_4\right) - p\left(p_{-4} - p_4 + q_{-4}\right) + p_{-4} - p_4 + q_{-4} - q\right),$$
$$q_{\tau^4} = \frac{1}{3}q\left(q\left(p_4 - q_{-4} + q_4\right) - p\left(p_{-4} - p_4 + q_{-4}\right) + p - p_4 + q_{-4} - q_4\right).$$

Proof

Introducing the variables

$$p := \frac{v}{u_{-4}\left(uv_{-4}v + 1\right) + v},$$
$$q := \frac{u_{-4}}{u_{-4}\left(uv_{-4}v + 1\right) + v},$$

it is straightforward to show that

$$v = \frac{pu_{-4}}{q}, \quad u = \frac{q_{-4}(p + q - 1)}{p_{-4}pu_{-44}u_{-4}}.$$

Using (3.1), along with the definition of p and q, we find

$$u_{\tau^4} = u(\mathcal{S}_4 - \mathrm{I})p,$$

$$v_{\tau^4} = v(\mathcal{S}_4 - \mathrm{I})q.$$

Computing p_{τ^4}, q_{τ^4}, it follows from the above relations that p and q satisfy the DΔEs

$$p_{\tau^4} = \frac{1}{3}p\left(q\left(p_4 - q_{-4} + q_4\right) - p\left(p_{-4} - p_4 + q_{-4}\right) + p_{-4} - p_4 + q_{-4} - q\right),$$
$$q_{\tau^4} = \frac{1}{3}q\left(q\left(p_4 - q_{-4} + q_4\right) - p\left(p_{-4} - p_4 + q_{-4}\right) + p - p_4 + q_{-4} - q_4\right).$$

□

As an aside, it is worth noting a simple manipulation of the above equations provides the conservation law

$$\partial_{\tau^4} \log\left(\frac{p}{q}\right) = (\mathcal{S}_4 - \mathrm{I})\frac{1}{3}(q - p_{-4}).$$

System $Q_{(1,2)}$ admits a potentiation.

Theorem 3.1.3 *The system $Q_{(1,2)}$, as given in (2.34), is transformed via the potentiation*

$$u = \frac{\phi_1}{\phi}, \;\; v = \frac{\phi}{\phi_{-2}},$$

to the six-point equation

$$\phi\left(\phi_1\phi_{122} - \phi_2\phi_{112}\right) + \phi_{122}\phi_{211}\left(\phi_2 - \phi_1\right) + \phi\phi_{1122}\left(\phi_{112} - \phi_{221}\right) = 0, \tag{3.2}$$

with corresponding symmetries

$$\phi_{\tau^1} = \frac{\phi_{-1-2}\phi_{-1}\phi}{(\phi_{-2} - \phi_{-1})(\phi_{-1-2} - \phi_1)},$$
$$\phi_{\tau^2} = \frac{\phi_{-1-2}\phi_{-2}\phi}{(\phi_{-1} - \phi_{-2})(\phi_{-1-2} - \phi_2)}.$$

Proof

The second equation from $Q_{(1,2)}$, (2.34), can be rewritten in the form

$$uv_{12} = u_2v_2.$$

This is in the form of a multiplicative conservation law. Thus, taking logarithms one obtains

$$(\mathcal{S}_1 - \mathrm{I})\log(v_2) = (\mathcal{S}_2 - \mathrm{I})\log(u).$$

One may force this relation to be trivially satisfied by introducing a potential ϕ such that

$$\log(v_2) = (\mathcal{S}_2 - \mathrm{I})\log(\phi),$$
$$\log(u) = (\mathcal{S}_1 - \mathrm{I})\log(\phi).$$

In particular, one can take

$$u = \frac{\phi_1}{\phi}, \quad v = \frac{\phi}{\phi_{-2}}.$$

Substituting these expressions into the first equation of $Q_{(1,2)}$, we obtain (3.2). Similarly, we find the τ^1 symmetry by applying the potentiation to (2.31). To arrive at the τ^2 symmetry we observe that (3.2) is invariant under the interchange of indices 1 and 2. □

System $Q_{(1,4)}$ can be reduced to a 1-component 6-point equation.

Theorem 3.1.4 *The system $Q_{(1,4)}$, as given in (2.35), is transformed to the six-point equation*

$$\frac{1}{\psi_{114}} + \psi_1\psi_{11} = \psi_4\psi_{14} + \frac{1}{\psi}, \tag{3.3}$$

under the reduction $v_1 = u = \frac{1}{\psi}$. Local symmetries of (3.3) are given by

$$\psi_{\tau^4} = \psi \left(\mathrm{I} - \mathcal{S}_4^{-1}\right) \frac{\psi\psi_1\psi_{14}}{\psi\psi_{14}(\psi_1+\psi_4)+1}, \tag{3.4a}$$

$$\psi_{s^1} = \psi \left(\mathrm{I} - \mathcal{S}_1^{-1}\right) \frac{1}{(\psi_{11}\psi_1\psi - 1)(\psi_1\psi\psi_{-1} - 1)}. \tag{3.4b}$$

Proof

Focusing on the first equation of $Q_{(1,4)}$, it can be easily verified that it turns into an identity under the reduction $u = 1/\psi_1$ and $v = 1/\psi$ (equivalently $u = v_1$), while the second equation of the same system becomes the six-point equation (3.3). Applying the reduction to (2.33) produces the τ^4 symmetry given above. The symmetry corresponding to the τ^1 direction is not compatible with this reduction. To obtain the s^1 symmetry, it is profitable to note that the six-point equation may be rewritten in the following form

$$\left(\psi\mathcal{S}_1 - \psi_{114}\right)\left(\psi\psi_{14}(\psi_1+\psi_4)+1\right) = 0. \tag{3.5}$$

The quadrilateral equation

$$\psi\psi_{14}(\psi_1+\psi_4)+1 = 0, \tag{3.6}$$

defines particular solutions of equation (3.5), and was derived in [58] in the context of second order integrability conditions for difference equations. Knowing from [58] that equation (3.6) admits a second order symmetry in both directions, one can easily show that

$$\psi_{s^1} = \psi \left(\mathrm{I} - \mathcal{S}_1^{-1}\right) \frac{1}{(\psi_{11}\psi_1\psi - 1)(\psi_1\psi\psi_{-1} - 1)},$$

also defines a symmetry for the six-point equation (3.3). It is worth noting that the denominator of the τ^4 symmetry is the defining polynomial of equation (3.6). □

Equation (3.3) can be viewed as a discrete analogue of a third order hyperbolic equation [10]. The equation appears to be new.

3.2 Solutions of continuous system

Our main goal in the study of the tetrahedral group was to derive integrable semi and fully discrete equations. However, along the way, via the Lax-Darboux scheme, we obtained Bäcklund transformations of the initial PDE under consideration. In this section, we demonstrate the utility of the found Bäcklund transformations by deriving exact solutions to the system

$$u_t = \partial_x \left(\frac{u}{3}\left(-u + 2v + 2w + \frac{u_x}{u} + 2\frac{w_x}{w}\right)\right), \tag{3.7a}$$

$$v_t = \partial_x \left(\frac{v}{3}\left(-v + 2w + 2u + \frac{v_x}{v} + 2\frac{u_x}{u}\right)\right), \tag{3.7b}$$

$$w_t = \partial_x \left(\frac{w}{3}\left(-w + 2u + 2v + \frac{w_x}{w} + 2\frac{v_x}{v}\right)\right), \tag{3.7c}$$

with $uvw = 1$.

3.2.1 $M_{(1)}$ Bäcklunds

Recall, the derivation of $M_{(1)}$ produced the following Bäcklund relations

$$\frac{v_{1,x}}{v_1} - \frac{u_{,x}}{u} - \frac{uv}{v_1} + u + v - v_1 = 0, \tag{3.8a}$$

$$\frac{v_{,x}}{v} + \frac{u_{1,x}}{u_1} + \frac{u_{,x}}{u} - \frac{1}{uv} - \frac{uv}{v_1} + \frac{1}{u_1 v_1} + u_1 = 0. \tag{3.8b}$$

This system provides a relation between a solution (u, v) and an updated solution (u_1, v_1). It is easily verified that $(u, v) = (a, b)$, for $a, b \in \mathbb{C}$ with $0 \neq a, b$, is a solution of (3.7). Taking this as an initial solution, one may solve the Bäcklund relations to obtain the x-dependence of the updated solution. Subsequently, one may use the continuous system itself to determine the solutions t-dependence. Note, the t-dependence can also be derived through consideration of the Bäcklund transformations that follow from the compatibility condition of $M_{(1)}$ with the t part of the Lax pair, A, as given in (2.8).

We first concentrate on (3.8a).

Lemma 3.2.1 *Take initial solution $(u, v) = (a, b)$, with $a, b \neq 0$. If $a \neq b$, (3.8a) has the solution*

$$v_1 = a + \frac{b - a}{1 + (b - a)c_1 e^{(b-a)x}}. \tag{3.9}$$

If $a = b$, the solution is given by

$$v_1 = \frac{1}{c_1 - x} + a. \tag{3.10}$$

Proof

Substituting $u = a$ and $v = b$ into (3.8a), we are left with the equation

$$\frac{v_{1,x}}{v_1} - \frac{ab}{v_1} + a + b - v_1 = 0$$

which implies

$$v_{1,x} - ab + (a + b)v_1 - v_1^2 = 0.$$

This is a Ricatti equation. By inspection it is clear that $v_1 = a$ is a particular solution. Hence, following the general procedure for solving Ricatti equations, we reduce the equation to Bernoulli type via the change of variables

$$v_1 = p + a.$$

Doing so we find

$$p_x - ab + (a+b)(p+a) - (p+a)^2 = 0,$$

which implies

$$p_x + (b-a)p = p^2. \tag{3.11}$$

We reduce this Bernoulli type equation to a linear equation by introducing the change of variables $q = p^{-1}$. Then $p = q^{-1}$, $p_x = -q^{-2}q_x$ and thus (3.11) yields

$$q_x + (a-b)q = -1.$$

Now we have the two cases, $a = b$ and $a \neq b$. If $a = b$, then

$$q_x = -1,$$

which implies

$$q = -x + c_1,$$

and thus

$$v_1 = p + a = q^{-1} + a = \frac{1}{c_1 - x} + a, \tag{3.12}$$

where c_1 is constant with respect to x. If $a \neq b$, we multiply by the integrating factor $e^{(a-b)x}$ to obtain

$$\frac{\partial}{\partial x}\left(e^{(a-b)x}q\right) = -e^{(a-b)x},$$

and thus

$$e^{(a-b)x}q = \frac{1}{b-a}e^{(a-b)x} + c_1,$$

which yields

$$q = \frac{1}{b-a} + c_1 e^{-(a-b)x},$$

with c_1 constant with respect to x.

Hence, as $v_1 = p + a = 1/q + a$, it follows that

$$v_1 = a + \frac{b-a}{1+(b-a)c_1 e^{(b-a)x}}. \tag{3.13}$$

□

Now, for equation (3.8b), we consider the two cases, $a = b$ and $a \neq b$, separately.

Lemma 3.2.2 *For $a \neq b$, and v_1 given in (3.9), the equation (3.8b) is solved by*

$$u_1 = \frac{1}{ab} + \frac{\left(a^2b-1\right)\left(ab^2-1\right)\left(ac_1(a-b)e^{x(b-a)}-b\right)}{\left(ab^2-1\right)\left(c_2\left(a^2b-1\right)e^{\frac{x(1-a^2b)}{ab}}-ab^2\right)+a^2bc_1(a-b)\left(a^2b-1\right)e^{x(b-a)}}.$$

Proof

After the substitutions $u = a$ and $v = b$ in (3.8b), we arrive to

$$\begin{aligned} &\frac{u_{1,x}}{u_1} - \frac{1}{ab} - \frac{ab}{v_1} + \frac{1}{u_1 v_1} + u_1 = 0, \\ \implies &u_{1,x} - \left(\frac{1}{ab} + \frac{ab}{v_1}\right)u_1 + \frac{1}{v_1} + u_1^2 = 0. \end{aligned}$$

Again, we have a Ricatti equation. One can verify that $u_1 = \frac{1}{ab}$ is a particular solution. Thus, we make the transformation $u_1 = p + \frac{1}{ab}$, which results in

$$p_x - \left(\frac{1}{ab} + \frac{ab}{v_1}\right)\left(p + \frac{1}{ab}\right) + \frac{1}{v_1} + \left(p + \frac{1}{ab}\right)^2 = 0,$$

leading to

$$p_x + \left(\frac{1}{ab} - \frac{ab}{v_1}\right)p = -p^2.$$

To linearize this Bernoulli type equation, we use the transformation $p = q^{-1}$. This results in

$$q_x + \left(\frac{ab}{v_1} - \frac{1}{ab}\right)q = 1. \tag{3.14}$$

To proceed, we multiply by the integrating factor $e^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx}$, however, first it is convenient to compute the expression inside the integral. As $a \neq b$,

$$\begin{aligned}\frac{ab}{v_1}-\frac{1}{ab} &= \frac{ab}{\frac{b-a}{c_1(b-a)e^{x(b-a)}+1}+a}-\frac{1}{ab},\\ &= \frac{abc_1(a-b)e^{bx}-abe^{ax}}{ac_1(a-b)e^{bx}-be^{ax}}-\frac{1}{ab}.\end{aligned}$$

As a consequence,

$$\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx = \log[ac_1(a-b)e^{bx}-be^{ax}]-\frac{1}{ab}x,$$

and thus

$$\begin{aligned}e^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx} &= e^{-\frac{x}{ab}}\left(ac_1(a-b)e^{bx}-be^{ax}\right),\\ &= ac_1(a-b)e^{(b-\frac{1}{ab})x}-be^{(a-\frac{1}{ab})x}.\end{aligned}$$

Now, returning to (3.14), it follows that

$$\left(qe^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx}\right)' = e^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx},$$

which implies

$$q = \frac{\frac{ac_1(a-b)e^{(b-\frac{1}{ab})x}}{b-\frac{1}{ab}}-\frac{be^{(a-\frac{1}{ab})x}}{a-\frac{1}{ab}}+c_2}{ac_1(a-b)e^{(b-\frac{1}{ab})x}-be^{(a-\frac{1}{ab})x}},$$

where c_2 is constant with respect to x. Thus, as $u_1 = p+\frac{1}{ab} = q^{-1}+\frac{1}{ab}$, we find that

$$u_1 = \frac{1}{ab}+\frac{(a^2b-1)(ab^2-1)\left(ac_1(a-b)e^{x(b-a)}-b\right)}{(ab^2-1)\left(c_2(a^2b-1)e^{\frac{x(1-a^2b)}{ab}}-ab^2\right)+a^2bc_1(a-b)(a^2b-1)e^{x(b-a)}}. \tag{3.15}$$

□

Now, we tackle the case $a = b$.

Lemma 3.2.3 *For $a = b$, and v_1 given in (3.9), the equation (3.8b) has the following solutions.*

For $a^3 \neq 1$

$$u_1 = \frac{(a^3-1)^2\left(a\left(x-c_1\right)-1\right)}{\left(a^3-1\right)a^3c_1-\left(1-2a^3\right)a^2-\left(a^3-1\right)a^3x+\left(a^3-1\right)^2c_2e^{\frac{(1-a^3)x}{a^2}}}.$$

For $a^3 = 1$, the solution is given by

$$u_1 = \frac{1}{a^2} + \frac{a(c_1-x)+1}{x(ac_1+1)-\frac{ax^2}{2}+c_2}.$$

Proof

For the case $a = b$, v_1 is given by (3.12). Following the same sequence of substitutions as in Lemma 3.2.2, we again arrive at equation (3.14). In this case,

$$\begin{aligned}\frac{ab}{v_1}-\frac{1}{ab} &= \frac{a^2(c_1-x)}{1+a(c_1-x)}-\frac{1}{a^2}\\ &= \frac{a^2c_1}{1+a(c_1-x)}-\frac{a^2x}{1+a(c_1-x)}-\frac{1}{a^2}\\ &= \frac{a^2c_1}{1+a(c_1-x)}+a-\frac{a(1+ac_1)}{1+a(c_1-x)}-\frac{1}{a^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx &= \int\left(\frac{a^2c_1}{1+a(c_1-x)}+a-\frac{a(1+ac_1)}{1+a(c_1-x)}-\frac{1}{a^2}\right)dx\\ &= -ac_1\log[1+a(c_1-x)]+ax+(1+ac_1)\log[1+a(c_1-x)]-\frac{x}{a^2}\\ &= ax+\log[1+a(c_1-x)]-\frac{x}{a^2}.\end{aligned}$$

Thus, the correct integrating factor is

$$e^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx} = e^{(a-\frac{1}{a^2})x}(1+a(c_1-x)).$$

From (3.14), it follows that

$$\left(qe^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx}\right)' = e^{\int\left(\frac{ab}{v_1}-\frac{1}{ab}\right)dx},$$

which implies

$$qe^{(a-\frac{1}{a^2})x}(1+a(c_1-x)) = \int e^{(a-\frac{1}{a^2})x}(1+a(c_1-x))dx.$$

Hence, we have two cases, $a^3-1=0$ and $a^3-1\neq 0$. In the former case, the exponential terms are equal to 1, and therefore

$$q(1+a(c_1-x)) = \int (1+a(c_1-x))dx,$$

which yields

$$q = \frac{-\frac{ax^2}{2}+(1+ac_1)x+c_2}{1+a(c_1-x)}.$$

Thus,

$$u_1 = q^{-1}+\frac{1}{a^2} = \frac{1}{a^2}+\frac{a(c_1-x)+1}{x(ac_1+1)-\frac{ax^2}{2}+c_2}.$$

In the latter case,

$$\begin{aligned} qe^{(a-\frac{1}{a^2})x}(1+a(c_1-x)) &= \int e^{(a-\frac{1}{a^2})x}(1+a(c_1-x))dx \\ &= \int e^{(a-\frac{1}{a^2})x}(1+ac_1)dx - a\int xe^{(a-\frac{1}{a^2})x}dx \\ &= \frac{1}{\gamma}e^{\gamma x}(1+ac_1) - a\frac{x}{\gamma}e^{\gamma x}+\frac{a}{\gamma^2}e^{\gamma x} \\ &= \frac{1}{\gamma}(1+a(c_1-x)+\frac{a}{\gamma})e^{\gamma x}+c_2, \end{aligned}$$

where $\gamma = a-\frac{1}{a^2}$.

Consequently,

$$q = \frac{\frac{1}{\gamma}(1+a(c_1-x)+\frac{a}{\gamma})e^{\gamma x}+c_2}{e^{\gamma x}(1+a(c_1-x))},$$

and hence

$$u_1 = q^{-1}+\frac{1}{a^2} = \frac{\gamma^2 e^{\gamma x}\left(a\left(x-c_1\right)-1\right)}{ae^{\gamma x}\left(\gamma\left(x-c_1\right)-1\right)-\gamma\left(\gamma c_2+e^{\gamma x}\right)}+\frac{1}{a^2}.$$

□

Now, u_1 and v_1 satisfy (3.8) for $u = a$ and $v = b$. However, (3.8) only defines the x dependence of the Bäcklund transformation, not the t dependence. Hence, we still must determine the dependence of u_1 and v_1 upon t. As previously mentioned, in principle we could use the Bäcklund relations obtained from the compatibility of $M_{(1)}$ with the t part of the continuous Lax pair. However, this is not necessary, one can instead use the continuous system itself. Substituting u_1 and v_1 into the continuous system (3.7), we find the following relations must hold.

For $a = b$ with $a^3 - 1 = 0$,

$$c_1' = 0, \quad c_2' = a,$$

and thus

$$c_1 = \alpha, \quad c_2 = at + \beta. \tag{3.16}$$

For $a = b$ with $a^3 - 1 \neq 0$,

$$c_1' = \frac{2}{3}\frac{a^3 - 1}{a^2}, \quad c_2' = -\frac{\left(a^3 - 1\right)^2 c_2}{3a^4},$$

therefore

$$c_1 = \frac{2}{3}\frac{a^3 - 1}{a^2}t + \alpha, \quad c_2 = \beta \exp\left(-\frac{\left(a^3 - 1\right)^2 t}{3a^4}\right). \tag{3.17}$$

For $a \neq b$,

$$c_1' = \frac{(a-b)(ab(a+b)-2)}{3ab}c_1, \quad c_2' = \frac{\left(a^2 b - 1\right)\left(ab(a-2b)+1\right)}{3a^2b^2}c_2,$$

leading to

$$c_1 = \alpha \exp\left(\frac{(a-b)(ab(a+b)-2)}{3ab}t\right), \quad c_2 = \beta \exp\left(\frac{\left(a^2 b - 1\right)\left(ab(a-2b)+1\right)}{3a^2b^2}t\right). \tag{3.18}$$

Thus, we obtain the following result.

Theorem 3.2.4 *System (3.7) possesses the following solutions.*

For $a^3 = 1$,

$$u = \frac{1}{a^2} + \frac{a(\alpha - x) + 1}{x(a\alpha + 1) - a\frac{x^2}{2} + at + \beta},$$
$$v = \frac{1}{\alpha - x} + a.$$

For $a^3 \neq 1$, *with* $\gamma = a - \frac{1}{a^2}$,

$$u = \frac{1}{a^2} + \frac{\gamma^2 \left(a\left(x - (\alpha + \frac{2\gamma t}{3})\right) - 1\right)}{a\left(\gamma\left(x - \left(\alpha + \frac{2\gamma t}{3}\right)\right) - 1\right) - \gamma\left(\beta\gamma e^{-\frac{1}{3}\gamma^2 t - \gamma x} + 1\right)},$$
$$v = a + \frac{1}{\alpha + \frac{2\gamma t}{3} - x}.$$

For $a \neq b$, *with* $\gamma_1 = \frac{(a-b)(ab(a+b)-2)}{3ab}$ *and* $\gamma_2 = \frac{\left(a^2b-1\right)(ab(a-2b)+1)}{3a^2b^2}$,

$$u = \frac{1}{ab} + \frac{\left(a^2b - 1\right)\left(ab^2 - 1\right)\left(a(a-b)\alpha e^{\gamma_1 t + (b-a)x} - b\right)}{\left(ab^2 - 1\right)\left(\left(a^2b - 1\right)\beta e^{\gamma_2 t + (\frac{1}{ab} - a)x} - ab^2\right) + a^2b(a-b)\left(a^2b - 1\right)\alpha e^{\gamma_1 t + (b-a)x}},$$
$$v = a + \frac{b - a}{\alpha(b-a)e^{\gamma_1 t + (b-a)x} + 1}.$$

Proof

Result follows from lemmas 3.2.1, 3.2.2 and 3.2.3, along with the expressions (3.16), (3.17) and (3.18). □

In a rather straightforward manner we have derived rational, rational-exponential and exponential solutions. For appropriate choices of a, b, α and β, the latter solution corresponds to a 2-kink soliton for the function u, and a 1-kink soliton for the function v.

3.2.2 $M_{(4)}$ Bäcklunds

We now consider the Bäcklund relations provided by $M_{(4)}$. Recall, these are given by

$$\frac{v_{4,x}}{v_4} - \frac{u_x}{u} - u + u_4 + v - v_4 = 0, \tag{3.22a}$$
$$\frac{v_x}{v} + \frac{u_{4,x}}{u_4} + \frac{u_{,x}}{u} - \frac{1}{uv} + \frac{1}{u_4 v_4} + u - u_4 = 0. \tag{3.22b}$$

Starting with the initial solution $(u, v) = (a, b)$, with $0 \neq a, b \in \mathbb{C}$, the above system becomes

$$\frac{v_{4,x}}{v_4} - a + u_4 + b - v_4 = 0, \tag{3.23a}$$

$$\frac{u_{4,x}}{u_4} - \frac{1}{ab} + \frac{1}{u_4 v_4} + a - u_4 = 0. \tag{3.23b}$$

Rearranging the first equation for u_4 and substituting into the second equation, we obtain a second order ordinary differential equation for v_4 given by

$$abv_{4,xx} - [3abv_4 + ab(a-2b) + 1]\, v_{4,x} = ab + (a-b)\left(ab^2 - 1\right) v_4 - (ab(a-2b)+1)v_4^2 - abv_4^3. \tag{3.24}$$

This equation does not depend explicitly on x, hence we may reduce its order by introducing the variable $p = v_{4,x}$, so that

$$v_{4,xx} = \frac{\partial p}{\partial x} = \frac{\partial p}{\partial v_4} v_{4,x} = p'p,$$

where $p' = p_{v_4}$. The independent variable is now v_4. We set $v_4 = \chi$. In terms of the new variables, (3.24) becomes the first order equation

$$pp' = \frac{(ab(a-2b+3\chi)+1)}{ab} p - \frac{(\chi - b)(ab\chi(a-b+\chi) + a + \chi)}{ab}. \tag{3.25}$$

This is an Abel equation of the second kind. Following [67], we transform the above to canonical form by introducing a new independent variable, z. For an equation of the form $pp' = F_1(\chi)p + F_0(\chi)$, we introduce

$$z = \int F_1(\chi) d\chi.$$

Then, $p' = p_\chi = \frac{dz}{d\chi} p_z = F_1(\chi) p_z$, and thus we obtain the equation

$$pp_z - p = R(z), \tag{3.26}$$

where $R(z)$ is given parametrically by the relations

$$R = \frac{F_0(\chi)}{F_1(\chi)}, \quad z = \int F_1(\chi) d\chi.$$

In the case considered,

$$R = -\frac{(\chi - b)(ab\chi(a - b + \chi) + a + \chi)}{ab(a - 2b + 3\chi) + 1}, \quad z = \frac{\chi}{ab} + a\chi - 2b\chi + \frac{3\chi^2}{2}.$$

The latter is a quadratic equation for χ. This has solutions

$$\chi = \alpha \pm \sqrt{\beta + \gamma z},$$

with α, β and γ given by

$$\alpha = -\frac{ab(a - 2b) + 1}{3ab}, \quad \beta = \frac{(ab(a - 2b) + 1)^2}{9a^2b^2}, \quad \gamma = \frac{2}{3}.$$

Substituting the positive root into the expression for R, we obtain

$$R = g_1 - \frac{2}{9}z + g_2(\beta + \gamma z)^{-\frac{1}{2}},$$

with

$$g_1 = \frac{ab(ab(2a - b)(a + b) - 5a + b) + 2}{27a^2b^2},$$
$$g_2 = \frac{-a^3b^3(a - 2b)(2a - b)(a + b) + 3a^2b^2\left(a^2 + 5ab + b^2\right) + 3ab(a + b) - 2}{81a^3b^3}.$$

A list of Abel's equations in canonical form, for which explicit general solutions are known, is provided by [67]. Surveying this list, one finds no equations containing terms of the form $(\beta + \gamma z)^{-\frac{1}{2}}$. However, there are equations which contain terms of the form $cz^{\pm\frac{1}{2}}$. Thus, we choose a such that $\beta = 0$, i.e. choose a such that $ab(a - 2b) + 1 = 0$. In turn this implies $\alpha = 0$. The equation is quadratic in a, taking either root leads to R taking the form

$$R = \frac{b^3 - 1}{3b} - \frac{2z}{9} + \frac{1}{\sqrt{6}}z^{-\frac{1}{2}}.$$

The solution to equation (3.26), with R taking this form, depends upon the value of $A = \frac{b^3-1}{3b}$. For simplicity, we consider the case $A = 0$. Then, parametrically, the solution is given by

$$z = \frac{3\left(c_1\left(\sqrt{3}\cos(\tau) - \sin(\tau)\right) + 2e^{\sqrt{3}\tau}\right)^2}{8\left(c_1\sin(\tau) + e^{\sqrt{3}\tau}\right)^2}, \tag{3.27}$$

$$p = \frac{3c_1\left(c_1 + 2e^{\sqrt{3}\tau}\left(\sqrt{3}\cos(\tau) - \sin(\tau)\right)\right)}{4\left(c_1\sin(\tau) + e^{\sqrt{3}\tau}\right)^2}, \tag{3.28}$$

cf. equation 1.3.1.26, [67]. Recall, $\chi = \alpha + \sqrt{\beta + \gamma z} = \sqrt{\frac{2}{3}z}$. Moreover,

$$p = \frac{\partial \chi}{\partial x} = \frac{\partial \chi}{\partial \tau}\frac{\partial \tau}{\partial x},$$

which implies

$$\frac{\partial x}{\partial \tau} = \frac{1}{p}\frac{\partial \chi}{\partial \tau}.$$

Performing the computation, one finds

$$\frac{\partial x}{\partial \tau} = -\frac{2}{\sqrt{3}},$$

which yields

$$\tau = -\frac{\sqrt{3}}{2}x + c_2. \tag{3.29}$$

Finally, recall $\chi = v_4$, thus $v_4 = \sqrt{\frac{2}{3}z}$. Replacing z with expression (3.27) and τ with the expression (3.29) yields

$$v_4 = \frac{2e^{\sqrt{3}c_2 - \frac{3x}{2}} + c_1\left(\sin\left(\frac{\sqrt{3}x}{2} - c_2\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}x}{2} - c_2\right)\right)}{2\left(e^{\sqrt{3}c_2 - \frac{3x}{2}} - c_1\sin\left(\frac{\sqrt{3}x}{2} - c_2\right)\right)}. \tag{3.30}$$

Note, to arrive at the preceding case, we first set $\beta = 0$, i.e. $ab(a - 2b) + 1 = 0$, and then also set $A = 0$, i.e. $b^3 - 1 = 0$. Solving these equations simultaneously, we find that $a = b$ with $b^3 - 1 = 0$. To see this, consider the equation $ab(a-2b)+1 = 0$. Multiplying by b^2 and using the relation $b^3 = 1$ produces the equation $a^2 - 2ab + b^2 = (a - b)^2 = 0$, hence $a = b$. Making this choice for a and b, (3.24) becomes

$$v_{4,xx} - 3v_4v_{4,x} + v_4^3 - 1 = 0.$$

One can check directly that expression (3.30) satisfies this. From (3.23), one finds $u_4 = v_4 - \frac{v_{4,x}}{v_4}$, and thus explicitly,

$$u_4 = \frac{2e^{\sqrt{3}c_2} - c_1e^{\frac{3x}{2}}\left(\sqrt{3}\cos\left(\frac{\sqrt{3}x}{2} - c_2\right) - \sin\left(\frac{\sqrt{3}x}{2} - c_2\right)\right)}{c_1e^{\frac{3x}{2}}\left(\sin\left(\frac{\sqrt{3}x}{2} - c_2\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}x}{2} - c_2\right)\right) + 2e^{\sqrt{3}c_2}}.$$

Next, we determine the t-dependence of the constants c_1 and c_2. Substituting u_4 and v_4 into the continuous system (3.7), one finds c_1 and c_2 must satisfy the following simple relations,

$$c_1' = 3c_1, \quad c_2' = \frac{\sqrt{3}}{2}.$$

Hence,

$$c_1 = \alpha e^{3t}, \quad c_2 = \frac{\sqrt{3}}{2}t + \beta.$$

Thus, we have the following result.

Theorem 3.2.5 *For initial solution* $(u, v) = (a, b)$ *with* $a = b$ *and* $a^3 - 1 = 0$*, (3.7) has solution*

$$u = \frac{2e^{\sqrt{3}\beta} - \alpha e^{\frac{3(t+x)}{2}}\left(\sin\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right) + \sqrt{3}\cos\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right)\right)}{2e^{\sqrt{3}\beta} + \alpha e^{\frac{3(t+x)}{2}}\left(\sqrt{3}\cos\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right) - \sin\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right)\right)},$$

$$v = \frac{2e^{\sqrt{3}\beta} - \alpha e^{\frac{3(t+x)}{2}}\sin\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right) + \sqrt{3}\alpha e^{\frac{3(t+x)}{2}}\cos\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right)}{2\left(e^{\sqrt{3}\beta} + \alpha e^{\frac{3(t+x)}{2}}\sin\left(\beta + \frac{\sqrt{3}t}{2} - \frac{\sqrt{3}x}{2}\right)\right)}.$$

Proof

Given above. □

In this case we are led to an exponential-trigonometric solution.

Chapter 4

Octahedral group

In this section we use a Lax-Darboux scheme to derive integrable systems related to the octahedral reduction group. The computations involved are in the same spirit as those in the previous chapters, albeit more involved. Consequently, we present results and comment on significant details only.

The Octahedral group is given abstractly by the group presentation

$$\mathbb{O} =< r, s \mid s^4 = r^2 = (rs)^3 = e > .$$

The group corresponds to the rotational symmetries of an octahedron. It is isomorphic to S_4, the symmetric group on 4 objects. One can verify directly that the automorphisms

$$g_s : \mathbf{a}(\lambda) \mapsto \mathbf{Q}_s \mathbf{a}(\sigma_s^{-1}(\lambda))\mathbf{Q}_s^{-1},$$

$$g_r : \mathbf{a}(\lambda) \mapsto \mathbf{Q}_r \mathbf{a}(\sigma_r^{-1}(\lambda))\mathbf{Q}_r^{-1},$$

where

$$\sigma_s(\lambda) = i\lambda, \quad \sigma_r(\lambda) = \frac{\lambda+1}{\lambda-1},$$

and

$$\mathbf{Q}_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \mathbf{Q}_r = \frac{1}{16}\begin{pmatrix} -4 & 8 & 24 & 48 \\ 2 & 4 & -12 & 24 \\ 2 & -4 & 4 & 8 \\ 1 & 2 & 2 & -4 \end{pmatrix},$$

satisfy the group relations. Setting $\Gamma = \mathbb{O}(\infty)$, i.e. the $\mathbb{O}$-orbit of ∞, the subalgebra $\mathfrak{U}_\lambda^{\mathbb{O}}(\Gamma)$ is generated by the elements

$$\mathbf{e}_1 =< \lambda\mathbf{E}_{14} >_{\mathbb{O}}, \quad \mathbf{e}_2 =< \lambda\mathbf{E}_{21} >_{\mathbb{O}}, \quad \mathbf{e}_3 =< \lambda\mathbf{E}_{32} >_{\mathbb{O}}, \quad \mathbf{e}_4 =< \lambda\mathbf{E}_{43} >, \qquad (4.1)$$

where $< . >_{\mathbb{O}}$ is the group average operator corresponding to the octahedral reduction group. The initial Lax operator in this case is taken as

$$L = \partial_x + \sum_{i=1}^{4} u^i \mathbf{e}_i, \qquad (4.2)$$

with $u^1u^2u^3u^4 = 1$. The representation used can be found in [21]. Moreover, it is shown that a T operator exists such that $[L, T] = 0$ is equivalent to the system

$$\begin{aligned}
u_t^1 &= \partial_x \left(u^1 \left(-\frac{u_x^1}{2u^1} - \frac{u_x^3}{2u^3} - \frac{u_x^2}{u^2} \right) + u^1 \left(-\frac{u^1}{6} - \frac{u^2}{2} + \frac{u^3}{2} + \frac{u^4}{2} \right) \right), \\
u_t^2 &= \partial_x \left(u^2 \left(\frac{u_x^1}{2u^1} - \frac{u_x^3}{2u^3} \right) + u^2 \left(\frac{u^1}{6} + \frac{u^3}{2} - \frac{u^4}{2} \right) \right) - \frac{u^2u^4u_x^1}{u^1}, \\
u_t^3 &= \partial_x \left(u^3 \left(\frac{u_x^1}{2u^1} + \frac{u_x^3}{2u^3} + \frac{u_x^2}{u^2} \right) + u^3 \left(\frac{u^1}{6} + \frac{u^2}{2} - \frac{u^3}{2} - \frac{u^4}{2} \right) \right), \\
u_t^4 &= \partial_x \left(u^4 \left(-\frac{u_x^1}{2u^1} + \frac{u_x^3}{2u^3} \right) + u^4 \left(\frac{u^1}{6} + \frac{u^2}{2} + \frac{u^3}{2} \right) \right) + \frac{u^2u^4u_x^1}{u^1}.
\end{aligned}$$

Searching for Darboux transformations of the form

$$M = f\mathrm{I} + \sum_{i=1}^{4} m^i \mathbf{e}_i, \qquad (4.3)$$

one arrives at the following result.

Theorem 4.0.6 *Suppose (4.3) provides a Darboux transformation for (4.2), then the following relations must hold*

$$m^1u^2_1 - m^2u^1 = 0, \quad m^2u^3_1 - m^3u^2 = 0, \quad m^3u^4_1 - m^4u^3 = 0, \quad m^4u^1_1 - m^1u^4 = 0,$$

$$\begin{aligned}
m^1_x+&\frac{1}{24}m^1(\mathcal{S}-\mathrm{I})(12(u^2+u^4)+6u^3-u^1)\\
&+\frac{1}{4}(4f+2(m^2+m^4)+m^3)(\mathcal{S}-\mathrm{I})u^1 = 0,\\
m^2_x+&\frac{1}{48}m^2[(\mathcal{S}-\mathrm{I})(u^1+12u^3)-(\mathcal{S}+\mathrm{I})u_4]\\
&+\frac{1}{48}(48f+m^1+12m^3)(\mathcal{S}-\mathrm{I})u^2+\frac{1}{2}m^4(\mathcal{S}+\mathrm{I})u^2 = 0,\\
m^3_x+&\frac{1}{48}m^3(\mathcal{S}-\mathrm{I})(u^1-24(u^2+u^3+u^4))\\
&+\frac{1}{48}(48f-24(m^2+m^4)+m^1)(\mathcal{S}-\mathrm{I})u^3 = 0,\\
m^4_x+&\frac{1}{48}m^4[(\mathcal{S}-\mathrm{I})(u^1+12u^3)+(\mathcal{S}+\mathrm{I})24u_2]\\
&+\frac{1}{48}(48f+m^1+12m^3)(\mathcal{S}-\mathrm{I})u^4+\frac{1}{2}m^2(\mathcal{S}+\mathrm{I})u^4 = 0,
\end{aligned}$$

and

$$\begin{aligned}
f_x+&\frac{m^1}{768}(\mathcal{S}-\mathrm{I})(u^1-8(u^2+u^4)-4u^3)\\
&+\frac{m^2}{96}(\mathcal{S}-\mathrm{I})(12u^3+24(u^2-u^4)-u^1)\\
&+\frac{m^3}{196}(\mathcal{S}-\mathrm{I})(36u^3+24(u^2+u^4)-u^1)\\
&+\frac{m^3}{96}(\mathcal{S}-\mathrm{I})(12u^3-24(u^2-u^4)-u^1) = 0,
\end{aligned}$$

where $\mathcal{S}$ is the shift operator associated to M.

Proof

Proof is analogous to that of Theorem 2.1.2. □

As in the case of the tetrahedral group, we may use the fact that the determinant of M is constant to derive two first integrals. Subsequently, we utilise these first integrals to determine the unknown function f, and eliminate a further unknown function.

Expressing the determinant in terms of the automorphic function

$$j = \frac{\left(\lambda^8 + 14\lambda^4 + 1\right)^3}{\lambda^4\left(\lambda^4 - 1\right)^4},$$

one finds

$$\det(M) = F_1 j + F_2,$$

where

$$F_1 = -\frac{m^1 m^2 m^3 m^4}{1296}$$

and

$$F_2 = (48f + m^1 + 24m^2 + 12m^3 - 24m^4)(48f + m^1 - 24m^2 + 12m^3 + 24m^4)q(f),$$

with $q(f)$ a quadratic polynomial in f which does not factor into two rational linear factors. By setting $F_2 = 0$, we fix f in the two distinct ways

$$f_1 = \frac{1}{48}\left(-m^1 - 24m^2 - 12m^3 + 24m^4\right)$$

and

$$f_2 = \frac{1}{48}\left(-m^1 + 24m^2 - 12m^3 - 24m^4\right).$$

Setting $F_1 = -\frac{1}{1296}$, we may fix m^4 using the relation $m^1 m^2 m^3 m^4 = 1$. Let $M_{(1)}$ and $M_{(2)}$ be Darboux transformations obtained by fixing $f = f_1$ and $f = f_2$ respectively. Remembering that $u^1 u^2 u^3 u^4 = 1$, and using the expressions for m^4 and f, the system found in Theorem 4.0.6 can be reduced to a system of three equations, which we may reinterpret as DΔEs for the functions u^1, u^2 and u^3. The Darboux and Lax operator then form a DLR of this system. As it is somewhat unwieldy, we do not present it here. However, we note that just like the systems found in Theorem 2.1.5, it will become a non local symmetry of the upcoming discrete system.

To ease presentation, we make the change of notation $u = u^1$, $v = u^2$ and $w = u^3$. Moreover, let $u_{n,m}$ denote u shifted by n in the direction corresponding to $M_{(1)}$ and m in the direction corresponding to $M_{(2)}$. Similarly for $v_{n,m}$ and $w_{n,m}$. Then, considering the compatibility of $M_{(1)}$ and $M_{(2)}$, we arrive at the following result.

Theorem 4.0.7 *$M_{(1)}$ and $M_{(2)}$ are compatible provided*

$$u_{1,1} = \frac{12uvw\left(u_{1,0}vv_{0,1}ww_{0,1} - w_{1,0}\right)}{v_{0,1}w_{0,1}w_{1,0}\left(u\left(u_{1,0} - u_{0,1}\right)vw + 12\left(u_{0,1}vv_{0,1}w - 1\right)\right)}, \tag{4.4a}$$

$$v_{1,1} = \frac{u_{1,0}vw}{w_{1,0}u_{1,1}}, \tag{4.4b}$$

$$w_{1,1} = \frac{u_{0,1}v_{0,1}\left(u_{1,0}vv_{0,1}ww_{0,1} - w_{1,0}\right)}{u_{1,0}v\left(u_{0,1}v_{0,1}w\left(v - w_{0,1} + w_{1,0}\right) - 1\right)}. \tag{4.4c}$$

Proof

The proof follows from direct computation. □

Following the same sequence of steps as in Theorem 2.2.1, we may derive local symmetries of this system. In particular, we search for a continuous operator Ω_i whose compatibility condition with $M_{(i)}$ produces an evolutionary differential difference system. Taking the ansatz

$$\Omega_{(i)} = M_{(i)}^{-1}\left(g\mathrm{I} + \sum_{\mathrm{i}=0}^{4} \alpha^{\mathrm{i}}\mathbf{e}_{\mathrm{i}}\right),$$

one derives the following result.

Theorem 4.0.8 *The system (4.4) possesses the local symmetries*

$$
\begin{aligned}
u_{\tau^1} &= \frac{u^2 v_{-1,0} v w_{-1,0}}{9\left(u v_{-1,0} v w_{-1,0}-1\right)} (\mathcal{S}_1 - \mathrm{I}) \frac{w}{u_{-1,0} u v_{-1,0} w_{-1,0}\left(v_{-1,0}+w\right)-12 w}, \\
v_{\tau^1} &= \frac{u v w_{1,0}}{108\left(u_{1,0} v v_{1,0} w-1\right)\left(u\left(v+w_{1,0}\right)-12 v_{1,0} w_{1,0}\right)} \\
&\quad - \frac{v}{9\left(u v_{-1,0} v w_{-1,0}-1\right)}\left(\frac{w\left(u_{-1,0} u v_{-1,0} w_{-1,0}-12\right)}{12\left(u_{-1,0} u v_{-1,0} w_{-1,0}\left(v_{-1,0}+w\right)-12 w\right)}+\right. \\
&\quad \left.\frac{w_{1,0}\left(u v\left(u v_{-1,0} w_{-1,0} w_{1,0}+1\right)-12 v_{1,0} w_{1,0}\right)}{\left(u\left(v+w_{1,0}\right)-12 v_{1,0} w_{1,0}\right)\left(u u_{1,0} v w\left(v+w_{1,0}\right)-12 w_{1,0}\right)}\right), \\
w_{\tau^1} &= \frac{w}{108\left(u v_{-1,0} v w_{-1,0}-1\right)} (\mathcal{S}_1 - \mathrm{I}) \frac{u_{-1,0} u v_{-1,0}^2 w_{-1,0}}{u_{-1,0} u v_{-1,0} w_{-1,0}\left(v_{-1,0}+w\right)-12 w},
\end{aligned}
$$

and

$$
\begin{aligned}
u_{\tau^2} &= \frac{u}{9\left(1-u_{0,1} v v_{0,1} w\right)} (\mathcal{S}_2 - \mathrm{I}) \frac{u v^2 w_{0,-1} w}{u v w_{0,-1} w\left(u_{0,-1}-12 v\right)+u_{0,-1}}, \\
v_{\tau^2} &= \frac{u v\left(u_{0,1} v_{0,1} w w_{0,1}+1\right)}{108\left(u_{0,1} v v_{0,1} w-1\right)\left(u_{0,1} v_{0,1} w w_{0,1}\left(u-12 v_{0,1}\right)+u\right)} \\
&\quad - \frac{u_{0,-1} v}{108\left(u v w_{0,-1} w\left(u_{0,-1}-12 v\right)+u_{0,-1}\right)}\left(\frac{u_{0,-1} w}{u_{0,-1} v_{0,-1}+u_{0,-1} w-12 v w}\right. \\
&\quad \left.+\frac{1}{u_{0,1} v v_{0,1} w-1}\right) \\
&\quad - \frac{u_{0,-1} v w}{108\left(u v_{0,-1} v w_{0,-1}-1\right)\left(u_{0,-1} v_{0,-1}+u_{0,-1} w-12 v w\right)}, \\
w_{\tau^3} &= \frac{u_{0,1} v v_{0,1} w^2}{108\left(1-u_{0,1} v v_{0,1} w\right)} (\mathcal{S}_2 - \mathrm{I}) \frac{u_{0,-1}}{u v w_{0,-1} w\left(u_{0,-1}-12 v\right)+u_{0,-1}}.
\end{aligned}
$$

Proof

Can be verified by direct computation. □

The discrete system and symmetries appear to be new.

Chapter 5

Conclusions

We have reached the end of part I. We conclude by summarising the results and suggesting possible directions for future work.

5.1 Summary of results and future work

In the first part of this thesis, we studied integrable systems related to tetrahedral and octahedral reduction groups. Due to classification results mentioned, the cases considered are representative of $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sl}_4(\mathbb{C})$-based automorphic Lie algebras with the most degenerated orbit and inner automorphisms. During this study, various faces of integrability were shown, including symmetries, Bäcklund and Miura transformations, Lax pairs, conservation laws and explicit solutions.

Some of the found systems appear to be new, their integrability is assured by the existence of the presented Lax pairs. As a Lax-Darboux scheme was employed, we made visible the interrelations between the various systems. For instance, we were able to interpret the DΔEs, evolutionary and non-evolutionary, as symmetries of the PΔEs, and explicitly see that the PΔEs are non linear superposition principles for a PDE system. These relations may not be immediately obvious if the systems are found via other means. Moreover, the role of the reduction group may not be visible.

We have not discussed in this thesis the icosahedral reduction group. In the literature, continuous systems have been constructed using icosahedral invariant operators, but at present no discrete systems have been presented. We have some results in this area, however we leave this for future work. Also, we have not considered reducible representations of the reduction groups. One can also construct operators in this case, and thus investigating the associated Darboux transformations and systems is a route of study which also needs to be considered.

In the case of $\mathfrak{sl}_2(\mathbb{C})$-based automorphic Lie algebras, Darboux transformations of the invariant operators have been used to construct Yang-Baxter maps. It would be interesting to see if one can do the same in the $\mathfrak{sl}_3(\mathbb{C})$-based case.

Another direction of research concerns λ-dependent reduction groups. One may consider the conjugating matrices to depend on λ, instead of being constant. A reduction group of this type isomorphic to a Dihedral group has been studied, but at present tetrahedral λ-dependent reduction groups have not.

Finally, we have not derived solutions to the found discrete and semi discrete systems, this is a challenging problem that we leave for future work.

Part II

Miura-type Transformations

Chapter 6

Introduction

This part of the thesis concerns Miura-type transformations (MTs). As has been shown in our consideration of the tetrahedral group, MTs can sometimes be derived by inspection of a given system. Of course, a transformation derived in such a way is no less special than one derived in a systematic fashion, however, this process does not increase our understanding of the underlying structures which makes the existence of MTs possible. To stare at an equation until an MT presents itself is a difficult task for all bar the experienced. We show that for a given semi or fully discrete equation, one can systematically derive MTs by considering algebraic structures associated to the Lax structure of the equation. We initially apply the construction to DΔEs. We demonstrate its efficacy by constructing MTs for the Narita-Itoh-Bogoyavlensky lattice and the discrete Sawada-Kotera equation, some of which appear to be new. To apply the construction, we assume the Lax structure has a particular form. In the differential difference case, this is not so restrictive. In the partial difference case, it is more so. However, we prove that the construction can also be applied to PΔEs, if a suitable Lax pair can be found. This is demonstrated by the construction of MTs of the Levi-Yamilov equation. Finally, we show how the method can be extended to systems of equations, and provide its application to the Toda lattice in Manakov-Flaschka variables, as well as a system presented by Adler and Postnikov.

The main idea behind the construction is based upon the works [30, 34]. In [30], the authors construct MTs solely for the KdV equation by considering functions on particular Lie groups. The paper is very short, a presentation primarily of results, with no proofs. In [34], general (1+1)-dimensional scalar evolutionary PDEs and a small class of multicomponent systems are considered, with proofs provided. In this thesis, we concentrate solely on differential difference and partial difference equations, for which the construction is quite different. We describe the construction in detail, before applying it to derive many MTs. Some of these results can be found in the preprint [17].

6.0.1 The group G and subgroups $(\mathbb{H}_k)_{k\in\mathbb{Z}_{\geq 0}}$

In this section we introduce a key player in the construction. But first, we fix notation and provide the basic definitions.

To start, we consider a single evolutionary DΔE

$$u_t = F(u_a, u_{a+1}, \dots, u_{a+p}). \tag{6.1}$$

As usual, we use the notation $u_\alpha := u(n+\alpha, t)$, for $\alpha \in \mathbb{Z}$, to denote the shift of the discrete variable n by α. We construct MTs of (6.1).

Definition 6.0.1 *Consider the DΔE*

$$v_t = P(v_b, v_{b+1}, \dots, v_{b+q}). \tag{6.2}$$

An MT from (6.2) to (6.1) is a transformation

$$u = f(v_c, \dots, v_{c+r}), \tag{6.3}$$

such that

$$D_t(f(v_c,\dots,v_{c+r})) = F(\mathcal{S}^a f(v_c,\dots,v_{c+r}), \mathcal{S}^{a+1} f(v_c,\dots,v_{c+r}), \dots, \mathcal{S}^{a+p} f(v_c,\dots,v_{c+r})), \tag{6.4}$$

is an identity modulo (6.2).

The above definition expresses the fact that given a solution v of (6.2), one can construct a solution of (6.1) by simply evaluating (6.3). This is completely analogous to the relation between the mKdV and KdV equation mentioned previously. The order of the MT is the difference between the highest and lowest shift, hence (6.3) is of order r.

In what follows, we construct MTs for equations of the form (6.1) which possess a DLR of the following type

$$\mathcal{S}(\mathbf{\Psi}) = M(u,\lambda)\mathbf{\Psi}, \tag{6.5a}$$

$$\partial_t(\mathbf{\Psi}) = \mathcal{U}(u_\alpha, u_{\alpha+1}, \ldots, u_\beta, \lambda)\mathbf{\Psi}, \tag{6.5b}$$

where we assume M and $\mathcal{U}$ are $d \times d$ matrices. As in previous chapters, M is invertible and (6.5) is compatible modulo (6.1). Equivalently,

$$D_t(M) = \mathcal{S}(\mathcal{U})M - M\mathcal{U}, \tag{6.6}$$

modulo (6.1). We have restricted our attention to M which depends only on u and λ, M does not depend on shifts of u. This is important in the subsequent analysis. However, if M depends on only one shift, for example $M = M(u_a, \lambda)$, then by relabelling $u_a = v$, we obtain an equivalent system of the correct form. In a later section, we do just this.

Having provided the basic definitions, we now introduce algebraic structures which will be vital in the construction. Let $\boldsymbol{G}$ be the group of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ. We may view M as an element of $\boldsymbol{G}$. To see this, observe M is a $d \times d$ invertible matrix which depends upon u and λ, by considering u fixed in $\mathbb{C}$, we interpret M as the function

$$M : \lambda \mapsto M(u,\lambda) \in \mathrm{GL}_d(\mathbb{C}).$$

There may be exceptional values (u,λ) for which M is not invertible, we remove these from our consideration.

Now that we have introduced the group $\boldsymbol{G}$, the home in which M lives, we define a sequence of subgroups $(\mathbb{H}_k)_{k\in\mathbb{N}}$ contained in $\boldsymbol{G}$ which will play a pivotal role. To do so, we set $\mathbb{H}_0 = \{\mathrm{I}\}$, the trivial group formed only of the identity matrix. Then, $\mathbb{H}_1$ is generated by elements of the form

$$M(\tilde{u}, \lambda) \cdot M(u, \lambda)^{-1} \in \boldsymbol{G},$$

where u and $\tilde{u}$ take all admissible values, i.e. complex numbers for which M is invertible. Now, given the subgroup $\mathbb{H}_k$ for $k \geq 1$, the subgroup $\mathbb{H}_{k+1}$ is generated by the elements of $\mathbb{H}_k$, along with elements of the form

$$M(u, \lambda) \cdot h_k \cdot M(u, \lambda)^{-1} \in \boldsymbol{G},$$

for $h_k \in \mathbb{H}_k$. By construction we have the inclusions

$$\mathbb{H}_0 \subset \mathbb{H}_1 \subset \mathbb{H}_2 \subset \ldots \subset \boldsymbol{G}. \tag{6.7}$$

Hence, for a given DLR of the form considered, one may associate a sequence of subgroups $(\mathbb{H}_k)_{k\in\mathbb{N}}$. It is by considering invariants of these subgroups that MTs will be constructed. However, we have not yet described in what sense invariance is meant. First, we construct another player in this story.

6.0.2 The groups $\mathcal{G}(c, d, k)$

We now introduce the building blocks for the base Lie group we shall construct. Let $\mathcal{G}(d, c, k)$ be the group of invertible $d \times d$ matrices with entries in the quotient algebra $A_{c,k} = \mathbb{C}[\lambda]/((\lambda - c)^k)$. Using polynomial division, any polynomial $p(\lambda)$ can be expressed in the form

$$p(\lambda) = a(\lambda)(\lambda - c)^k + b(\lambda),$$

with $0 \leq \deg(b(\lambda)) < k$. Thus, modulo the ideal, all polynomials are of degree less than k. Hence, it is clear that $A_{c,k}$ is spanned by the elements $(\lambda - c)^l + I$, for $l = 0, 1, \ldots, k-1$, where I is the ideal $((\lambda - c)^k)$. Therefore, dropping reference to I, elements of $\mathcal{G}(d, c, k)$

are of the form

$$W = \sum_{q=0}^{k-1} (\lambda - c)^q W^q, \quad W^0 \in \mathrm{GL}_d(\mathbb{C}), \quad W^1, \ldots, W^{k-1} \in \mathfrak{gl}_d(\mathbb{C}). \tag{6.8}$$

It is important to highlight that we require $W^0 \in \mathrm{GL}_d(\mathbb{C})$ to ensure that W is invertible. To see this, consider the following simple example. Take $c = 0$, $k = 2$ and consider the elements

$$A = A^0 + \lambda A^1, \quad B = B^0 + \lambda B^1.$$

Then,

$$AB = A^0 B^0 + \lambda(A^0 B^1 + A^1 B^0),$$

where we have used the fact that $\lambda^2 = 0$ in the quotient algebra. It therefore stands that B is an inverse of A if and only if

$$A^0 B^0 = \mathrm{I}, \quad A^0 B^1 + A^1 B^0 = 0.$$

Therefore, we require A^0 to be invertible with inverse B^0, then, B^1 is determined as

$$B^1 = -(A^0)^{-1} A^1 B^0.$$

The same holds for arbitrary c and k. To invert A, as long as A^0 is invertible, we first find B^0 then sequentially solve the relations at each power of $(\lambda - c)^i$ to find B^i, for each $i = 1, \ldots, k-1$.

From the expression (6.8) for W, we see that each element of $\mathcal{G}(d, c, k)$ is determined by k matrices. Since these are $d \times d$ matrices, given an element $g \in \mathcal{G}(d, c, k)$, we may associate g to a point in $\mathbb{C}^{k \times d \times d}$. The group multiplication and taking inverses are smooth, and thus $\mathcal{G}(d, c, k)$ is a kd^2-dimensional Lie group. Now, taking a collection

$$c_1, c_2, \ldots, c_m \in \mathbb{C}, \quad k_1, k_2, \ldots, k_m \in \mathbb{N},$$

for some fixed $m \in \mathbb{N}$, we may consider a cartesian product of such groups

$$G = \mathcal{G}(d, c_1, k_1) \times \mathcal{G}(d, c_2, k_2) \times \cdots \times \mathcal{G}(d, c_m, k_m). \tag{6.9}$$

The group G is of central importance in our construction. An element of G is an m-tuple of elements of the form (6.8), where c and k are replaced appropriately. Thus, if $g \in G$, then $g = (g^1, g^2, \dots g^m)$, with $g^p \in \mathcal{G}(d, c_p, k_p)$, $p = 1, \dots m$. Each g^p is expressed as

$$g^p = \sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q}, \quad W^{p,0} \in \mathrm{GL}_d(\mathbb{C}), \quad W^{p,1}, \dots, W^{p,k_p-1} \in \mathfrak{gl}_d(\mathbb{C}).$$

Defining functions w_{ij}^{pq} such that $w_{ij}^{pq}(g) = (W^{pq})_{ij}$, we obtain coordinate functions on G. We extend the operators $\mathcal{S}$ and D_t to the $w_{i,j}^{pq}$ coordinate functions using the DLR. This is achieved via the formulae

$$\sum_{q}^{k_p-1} (\lambda - c_p)^q \mathcal{S}(W^{p,q}) = M(u, \lambda) \left(\sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q} \right), \quad (\lambda - c_p)^{k_p} \equiv 0, \tag{6.10a}$$

$$\sum_{q}^{k_p-1} (\lambda - c_p)^q D_t(W^{p,q}) = \mathcal{U}(u_\alpha, \dots, u_\beta, \lambda) \left(\sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q} \right), \quad (\lambda - c_p)^{k_p} \equiv 0. \tag{6.10b}$$

We expand the right hand sides of the above as a Taylor series in λ at c_p, then truncate at $(\lambda - c_p)^{k_p-1}$. By equating expressions at the appropriate powers of $(\lambda - c_p)$, we determine $\mathcal{S}(w_{ij}^{pq})$ and $D_t(w_{ij}^{pq})$ as functions of the w_{ij}^{pq}, as well as potentially u and its shifts.

To aid digestion of the formulae, for the case $m = 1, k = 2$ and $d = 2$, they take the form

$$\begin{pmatrix} \mathcal{S}(w_{11}^{10}) & \mathcal{S}(w_{12}^{10}) \\ \mathcal{S}(w_{21}^{10}) & \mathcal{S}(w_{22}^{10}) \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} \mathcal{S}(w_{11}^{11}) & \mathcal{S}(w_{12}^{11}) \\ \mathcal{S}(w_{21}^{11}) & \mathcal{S}(w_{22}^{11}) \end{pmatrix}$$
$$= M(u, \lambda) \left(\begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix} \right), \quad (\lambda - c_1)^2 \equiv 0,$$

$$\begin{pmatrix} D_t(w_{11}^{10}) & D_t(w_{12}^{10}) \\ D_t(w_{21}^{10}) & D_t(w_{22}^{10}) \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} D_t(w_{11}^{11}) & D_t(w_{12}^{11}) \\ D_t(w_{21}^{11}) & D_t(w_{22}^{11}) \end{pmatrix}$$
$$= \mathcal{U}(u_\alpha, \dots, u_\beta, \lambda) \left(\begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix} \right), \quad (\lambda - c_1)^2 \equiv 0.$$

Thus, after Taylor series expansion and truncation of the right hand side, we can simply read off the action of $\mathcal{S}$ and D_t on the coordinate functions w_{ij}^{pq}.

We define the action of $\mathcal{S}$ and D_t on functions $f = f(w_{ij}^{pq}, u_\alpha, \ldots)$ in the natural way, namely

$$\mathcal{S}(f) = f(\mathcal{S}(w_{ij}^{pq}), \mathcal{S}(u_\alpha), \ldots), \tag{6.11a}$$

$$D_t(f) = \sum_{i,j,p,q} \frac{\partial f}{\partial w_{ij}^{pq}} D_t(w_{ij}^{pq}) + \sum_{\alpha} \frac{\partial f}{\partial u_\alpha} D_t(u_\alpha), \tag{6.11b}$$

where $\mathcal{S}(u_\alpha) = u_{\alpha+1}, D_t(u_\alpha) = u_{\alpha,t}$ and $\mathcal{S}(w_{ij}^{pq})$ along with $D_t(w_{ij}^{pq})$ are determined from (6.10). From the above definitions it can be seen that D_t is a derivation of the algebra of functions of the form $f(w_{ij}^{pq}, u_\alpha, \ldots)$, i.e. $D_t(fg) = D_t(f)g + fD_t(g)$. Also, it follows that $\mathcal{S}$ is an automorphism of this algebra. Indeed, using the fact that M is invertible, the action of the inverse shift operator is given by

$$\sum_{q}^{k_p-1} (\lambda - c_p)^q \mathcal{S}^{-1}(W^{p,q}) = M(u_{-1}, \lambda)^{-1} \left(\sum_{q=0}^{k_p-1} (\lambda - c_p)^q W^{p,q} \right), \quad (\lambda - c_p)^{k_p} = 0,$$

$$\mathcal{S}^{-1}(f) = f(\mathcal{S}^{-1}(w_{ij}^{pq}), \mathcal{S}^{-1}(u_\alpha), \ldots).$$

Of course, $\mathcal{S}^{-1}u_\alpha = u_{\alpha-1}$.

Next, we prove the following lemma regarding the interaction of $\mathcal{S}$ and D_t.

Lemma 6.0.2 *$\mathcal{S}$ and D_t commute.*

Proof

It is sufficient to prove that $\mathcal{S}^{-1} \circ D_t \circ \mathcal{S} = D_t$. As $\mathcal{S}$ is an automorphism and D_t is a derivation, it follows that $\mathcal{S}^{-1} \circ D_t \circ \mathcal{S}$ is a derivation. Hence, to prove the required equality it is enough to show that D_t and $\mathcal{S}^{-1} \circ D_t \circ \mathcal{S}$ agree on the functions u_α and w_{ij}^{pq}. That is, we require

$$\mathcal{S}^{-1} \circ D_t \circ \mathcal{S}(u_\alpha) = D_t(u_\alpha) \quad \forall \alpha,$$

$$\mathcal{S}^{-1} \circ D_t \circ \mathcal{S}(w_{ij}^{pq}) = D_t(w_{ij}^{pq}), \quad \forall p, q, i, j. \tag{6.13}$$

The first equation follows trivially, as from the definitions

$$\mathcal{S}^{-1} \circ D_t \circ \mathcal{S}(u_\alpha) = \mathcal{S}^{-1} \circ D_t(u_{\alpha+1}) = \mathcal{S}^{-1}(u_{\alpha+1,t}) = u_{\alpha,t} = D_t(u_\alpha).$$

The second equation is a consequence of (6.6). Indeed, by applying the operator $\mathcal{S}$ to (6.13), we may prove the equivalent relation $D_t \circ \mathcal{S}(w_{ij}^{pq}) = \mathcal{S} \circ D_t(w_{ij}^{pq})$. Applying D_t to (6.10a) and $\mathcal{S}$ to (6.10b), we find after using (6.6) that $D_t \circ \mathcal{S}(w_{ij}^{pq}) = \mathcal{S} \circ D_t(w_{ij}^{pq})$. □

For the above lemma to hold, it is essential that a DLR is used to define the action of the shift and derivative operators on G.

6.0.3 Functions on quotients of G

We shall construct MTs by considering functions on quotients of G. As will be shown, this essentially reduces to considering functions of the w_{ij}^{pq} which satisfy certain properties.

Let $H \subset G$ be a closed connected Lie subgroup of G. Then, G/H is a manifold and the quotient map

$$\pi : G \to G/H, \quad \pi(g) = gH \in G/H, \quad g \in G, \tag{6.14}$$

is an analytic submersion. Given a function f on an open subset $\mathbb{U} \subset G/H$, one can use π to construct a function on $\pi^{-1}(\mathbb{U}) \subset G$. Indeed, we define the function $\pi^*(f)$, where

$$\pi^*(f)(g) = f(\pi(g)) \quad \forall g \in \pi^{-1}(\mathbb{U}). \tag{6.15}$$

A natural question arises concerning how to characterise functions on $\pi^{-1}(\mathbb{U})$ which can be expressed in the form (6.15), i.e. functions y on $\pi^{-1}(\mathbb{U})$ for which there exists an f such that $y = \pi^*(f)$. The answer to this question requires the introduction of the concept of $\mathcal{H}$-right invariance. Let $\mathcal{H} \subset G$ be a subgroup, $U \subset G$ an open subset, and y a function on U. Then, if $h \in \mathcal{H}$ and $g \in U$ are such that $gh \in U$, we say that y is $\mathcal{H}$-right invariant if $y(gh) = y(g)$. Similarly, we say that y is $\mathcal{H}$-left invariant if $y(hg) = y(g)$, assuming now that $hg \in U$, i.e. that it is still in the domain of y.

It is not too difficult to see that a function y is of the form (6.15), i.e. factors through G/H, if and only if y is H-right invariant. To see this note that if $y = \pi^*(f)$, then using the preceding notation

$$y(gh) = \pi^*(f)(gh) = f(\pi(gh)) = f(ghH) = f(gH) = f(\pi(g)) = \pi^*(f)(g) = y(g).$$

Moreover, if y is H-right invariant then one can define the function f via $f(gH) = y(g)$. This is well defined as suppose $g_1 H = g_2 H$, then as H is a subgroup, it contains the identity element, which implies that necessarily $g_1 = g_2 h$ for some $h \in H$, but then $y(g_1) = y(g_2 h) = y(g_2)$. Hence, f is constant on cosets.

Due to the previous analysis, it is seen that when considering functions on open sets of the Lie group G/H, one can equivalently consider H-right invariant functions on open sets of G. In particular, a local system of coordinates on G/H is obtained by finding r functionally independent H-right invariant functions on G, where $r = \dim(G/H)$. For our purposes, we consider systems of local coordinates which exist almost everywhere. We say that a system of local coordinates $f^1, \ldots, f^r$ exists almost everywhere on G/H, if there exists an open dense subset $\mathbb{U} \subset G/H$, such that $f^1, \ldots, f^r$ form a system of local coordinates on a neighbourhood of each point $p \in \mathbb{U}$. This will be necessary as at times rational functions of the w_{ij}^{pq} will be considered.

To ease the exposition we refer to functions on open subsets of G/H and G, as functions on G/H and G respectively. Now that we have described functions on G/H, we consider how the operators $\mathcal{S}$ and D_t interact with the corresponding functions on G.

Lemma 6.0.3 *Let y be a function on G and let H be a closed connected Lie subgroup of G. If y is H-right invariant, then $\mathcal{S}(y)$ and $D_t(y)$ are also H-right invariant.*

Proof

Note $y = y(w_{ij}^{pq})$, so that $y(g) = y(w_{ij}^{pq}(g))$. Moreover, $\mathcal{S}(y)(g) = y(\mathcal{S}(w_{ij}^{pq})(g))$. According to (6.10), it follows that $\mathcal{S}(w_{ij}^{pq})(g) = w_{ij}^{pq}(M(u, \lambda) \cdot g)$, where the multiplication is performed as described. Hence $\mathcal{S}(y)(g) = y(M(u,\lambda)\cdot g)$. Thus for $g \in G$ and $h \in H$,

$$\mathcal{S}(y)(gh) = y(M(u,\lambda)\cdot gh).$$

But $M(u,\lambda)\cdot g \in G$ and consequently as y is H-right invariant, it follows that

$$\mathcal{S}(y)(gh) = y(M(u,\lambda)\cdot gh) = y(M(u,\lambda)\cdot g) = \mathcal{S}(y)(g).$$

A similar argument applies to $D_t(y)$ □

The Lemma provides that the set of functions on G/H is closed with respect to $\mathcal{S}$ and D_t. This fact will be used in the construction of the MTs.

6.0.4 Computing MTs

Recall, $\boldsymbol{G}$ is the group of $\mathrm{GL}_d(\mathbb{C})$-valued functions of λ. The subgroups $\mathbb{H}_k$ live in $\boldsymbol{G}$. We wish to define an action of $\boldsymbol{G}$ on G and its quotient groups G/H, where G is given in (6.9). By power series expansion and truncation we obtain a homomorphism $\rho : \boldsymbol{G} \to G$. The homomorphism ρ provides the link we require between the two groups. Using the homomorphism ρ, we may define an action of $\boldsymbol{G}$ on G and G/H, by first taking elements of $\boldsymbol{G}$ to G, before subsequently applying the standard actions of left multiplication. Note, we may interpret the definition (6.10a) of $\mathcal{S}$, as $\mathcal{S}(w_{ij}^{pq})(g) = w_{ij}^{pq}(\rho(M(u,\lambda))g)$, similarly for D_t.

Using the action of $\boldsymbol{G}$ on G and G/H, we obtain an action of subgroups of $\boldsymbol{G}$ on G and G/H. This allows the notion of $\mathcal{H}$-left invariant functions on either G or G/H to be considered, where $\mathcal{H}$ is now a subgroup of $\boldsymbol{G}$. Hence, we can naturally consider the action of the subgroups $\mathbb{H}_k$ on G and G/H. Explicitly, we shall consider a function y to be $\mathbb{H}_k$-left invariant if y is $\rho(\mathbb{H}_k)$-left invariant, that is, if $y(\rho(h)\hat{g}) = y(\hat{g})$, for all $h \in \mathbb{H}_k$, where $\hat{g}$ belongs to G or G/H. Why would we want to consider such functions? Well, we shall see that such functions interact with the $\mathcal{S}$ operator in a useful manner. Given a function y on G, and hence a function of the w_{ij}^{pq} coordinate functions, one would expect $\mathcal{S}(y) = y(\mathcal{S}(w_{ij}^{pq}))$ to depend on u, due to the definition of the action of $\mathcal{S}$ on the w_{ij}^{pq} coordinates. However, it turns out that if y is $\mathbb{H}_k$-left invariant, for some $k > 0$, where $\mathbb{H}_k$ is one of the subgroups from (6.7). Then, $\mathcal{S}(y)$ does not depend on u. Additionally, it is $\mathbb{H}_{k-1}$-left invariant. We prove this in the following lemma.

Lemma 6.0.4 *Let y be a function on an open dense subset of G, moreover, suppose y is $\mathbb{H}_k$-left invariant, for some $k > 0$. Then, $\mathcal{S}(y)$ does not depend on u and is $\mathbb{H}_{k-1}$-left invariant.*

Proof

The subgroup $\mathbb{H}_k$, for $k > 0$, necessarily contains the elements of $\mathbb{H}_1$. Since $\mathbb{H}_1$ contains all elements of the form

$$M(\tilde{u}, \lambda) \cdot M(u, \lambda)^{-1} \in \boldsymbol{G},$$

it follows from the $\mathbb{H}_k$-left invariance of y that

$$y(\rho(M(\tilde{u}, \lambda) \cdot M(u, \lambda)^{-1})\hat{g}) = y(\hat{g}), \tag{6.16}$$

where u, $\tilde{u}$ are fixed values, and $\tilde{g}$ belongs to an open set in G such that (6.16) is well defined. It is possible to find such a $\tilde{g}$ as y is defined on a dense open set of G. Taking $g = \rho(M(u, \lambda)^{-1})\hat{g}$ and using the fact that ρ is a homomorphism, (6.16) can be rewritten as

$$y(\rho(M(\tilde{u}, \lambda))g) = y(\rho(M(u, \lambda))g). \tag{6.17}$$

But,

$$\mathcal{S}(y)(g) = y(\rho(M(u, \lambda))g), \quad g \in G, \tag{6.18}$$

and so (6.17) implies $\mathcal{S}(y)$ is independent of u. To see that y is $\mathbb{H}_{k-1}$-left invariant, recall that $\mathbb{H}_k$ contains all elements of the form

$$M(u, \lambda) \cdot h \cdot M(u, \lambda)^{-1},$$

where $h \in \mathbb{H}_{k-1}$. Hence, due to the $\mathbb{H}_k$-left invariance of y, it follows that

$$y(\rho(M(u,\lambda) \cdot h \cdot M(u,\lambda)^{-1})\hat{g}) = y(\hat{g}).$$

Taking $g = \rho(M(u,\lambda)^{-1})\hat{g}$, we find that

$$y(\rho(M(u,\lambda))hg) = y(\rho(M(u,\lambda))g),$$

which implies using (6.18) that $\mathcal{S}(y)(hg) = \mathcal{S}(y)(g)$. As h is an arbitrary element of $\mathbb{H}_{k-1}$, it follows that $\mathcal{S}(y)$ is $\mathbb{H}_{k-1}$-left invariant. □

Lemmas 6.0.3 and 6.0.4 allow the following theorem to be proved, which is invaluable in our quest to construct MTs.

Theorem 6.0.5 *Let $H \subset G$ be a closed connected Lie Subgroup of G. Let z be a H-right invariant function on an open dense subset of G. Suppose z is $\mathbb{H}_\alpha$-left invariant for some positive integer $\alpha > 0$, then the functions*

$$\mathcal{S}^k(z) \quad k = 1, \ldots, \alpha,$$

are H-right invariant and do not depend on u_l, for any $l \in \mathbb{Z}$.

Proof

By Lemma 6.0.3, H-right invariance is preserved by the $\mathcal{S}$ operator. Hence, as z is H-right invariant, it follows that $\mathcal{S}(z)$ is also H-right invariant. By repeated application of this argument, it follows that $\mathcal{S}^k(z)$ is H-right invariant for all positive integers k. By assumption, z is $\mathbb{H}_\alpha$-left invariant, hence by Lemma 6.0.4 it follows that $\mathcal{S}(z)$ does not depend on u and is $\mathbb{H}_{\alpha-1}$-left invariant. If $\alpha - 1 \geq 1$, then we may repeat the argument to find that $\mathcal{S}^2(z)$ is $\mathbb{H}_{\alpha-2}$-left invariant and independent of u. Continuing in this manner we obtain the result. Note, at each stage the function is independent of u and so as u is not introduced, the functions are necessarily independent of its shifts. □

Due to the equivalence of H-right invariant functions on G and functions on G/H, we may interpret Theorem 6.0.5 in the context of functions of G/H. This is important as then according to the theorem, if we can find a function z on G/H which is $\mathbb{H}_{k-1}$-left invariant, then we necessarily obtain a collection of functions on G/H

$$\{z, \mathcal{S}(z), \ldots, \mathcal{S}^{k-1}(z)\}, \tag{6.19}$$

which do not depend on u. The shift $\mathcal{S}^k(z)$ is also a function on G/H, but may depend on u. Suppose, $\mathcal{S}^k(z)$ does depend on u. Now, if the dimension of G/H is equal to k, then provided our collection is functionally independent, so forms a system of local coordinates on G/H, it follows that for fixed u, $\mathcal{S}^k(z)$ can be expressed in terms of the collection (6.19) and the fixed value u. Rearranging for u we obtain a MT, where z will play the part of v in definition 6.0.1. We now prove this in detail.

Theorem 6.0.6 *Let z be a $\mathbb{H}_{r-1}$-left invariant function on G/H, equivalently, a $\mathbb{H}_{r-1}$-left invariant and H-right invariant function on G. Suppose the functions*

$$z,\ \mathcal{S}(z),\ \mathcal{S}^2(z), \ldots,\ \mathcal{S}^{r-1}(z), \tag{6.20}$$

form a system of local coordinates almost everywhere on G/H and

$$\frac{\partial}{\partial u}(\mathcal{S}^r(z)) \neq 0. \tag{6.21}$$

Then equation (6.1) possesses a MT.

Proof

Given the assumptions, we shall construct a MT. Set $\mathcal{S}^k(z) = z_k$, where k is a positive integer. From Theorem 6.0.5, it is clear that $z_0, z_1, \ldots, z_{r-1}$ do not depend on u. In general $z_r = \mathcal{S}^r(z)$ can depend on u, and due to the assumption (6.21), it indeed does. This assumption can be seen as a non degeneracy condition. Hence, for fixed u

$$z_r = G(z_0, z_1, \ldots, z_{r-1}, u), \tag{6.22}$$

for some function G, as by assumption, $z_0,\ldots,z_{r-1}$ forms a system of local coordinates almost everywhere on G/H. Moreover, from the definition of the D_t operator, it follows that

$$D_t(z) = Q(z_0, z_1, \ldots, z_{r-1}, u_\alpha, u_{\alpha+1}, \ldots, u_\beta), \tag{6.23}$$

for some function Q. By the implicit function theorem, the expression (6.22) can locally, in a neighbourhood of a point for which $\frac{\partial G}{\partial u} \neq 0$, be rewritten in the form

$$u = f(z_0, z_1, \ldots, z_r), \tag{6.24}$$

for some function f. Introducing the variables v_l for $l \in \mathbb{Z}$, we write the expression

$$u = f(v_0, v_1, \ldots, v_r), \tag{6.25}$$

and will consider the shifts of u, u_l to satisfy

$$u_l = f(v_l, v_{l+1}, \ldots, v_{l+r}).$$

Then, by replacing z_l by v_l and u_l by $f(v_l, \ldots)$ in (6.23), we obtain an expression

$$v_t = Q(v, v_1, \ldots, v_{r-1}, f(v_\alpha, \ldots, v_{r+\alpha}), \ldots, f(v_\beta, \ldots, v_{\beta+r})) = P(v_\alpha, \ldots, v_{\beta+r}), \tag{6.26}$$

for some function P, assuming $\alpha < 0$. Viewing (6.26) as a differential difference equation, we claim (6.25) is a MT from (6.26) to (6.1). To prove this we must demonstrate that

$$D_t(f(v_0, \ldots, v_r)) = F(\mathcal{S}^a(f(v_0, \ldots, v_r)), \mathcal{S}^{a+1}(f(v_0, \ldots, v_r)), \ldots, \mathcal{S}^{a+p}(f(v_0, \ldots, v_r)),$$

is an identity modulo (6.26), i.e. that

$$\sum_{i=0}^{r} \frac{\partial f(v_0, \ldots, v_r)}{\partial v_i} \mathcal{S}^i(P(v_\alpha, \ldots, v_{\beta+r})) = F(f(v_a, \ldots, v_{a+r}), \ldots, f(v_b, \ldots, v_{b+r})), \tag{6.27}$$

holds exactly. To show that this is the case we will require the following lemma.

Lemma 6.0.7 *The functions z_k for $k \in \mathbb{Z}$ are functionally independent.*

Proof

Suppose towards a contradiction that the functions z_k are not functionally independent, then there exists a non trivial relation among them. Applying the appropriate power of $\mathcal{S}^{-1}$, we obtain a relation of the form

$$R(z_0, z_1, \ldots, z_q) = 0. \tag{6.28}$$

By assumption, the set $\{z, z_1, \ldots, z_{r-1}\}$ forms a local system of coordinates on G/H, thus, is necessarily functionally independent, hence we deduce $q > r - 1$. Applying the appropriate power of $\mathcal{S}$ to (6.22), we may replace each z_l for $l \geq r$ by a corresponding function of $z, z_1, \ldots, z_{r-1}, u, u_1, \ldots, u_{l-r}$. Doing so in (6.28), we obtain a non trivial relation among the functions

$$z, z_1, \ldots, z_{r-1}, u, u_1, \ldots, u_{q-r} \tag{6.29}$$

this is a contradiction as the collection (6.29) is functionally independent. □

Now, we finish the proof of the theorem. Applying D_t to (6.24) and using (6.1) to replace u_t, we obtain the expression

$$\sum_{i=0}^{r} \frac{\partial f(z_0, \ldots, z_r)}{\partial z_i} D_t(z_i) = F(u_a, \ldots, u_{a+p}).$$

Recall, $z_i = \mathcal{S}^i(z)$, thus $D_t(z_i) = D_t(\mathcal{S}^i(z)) = \mathcal{S}^i(D_t(z))$ using Lemma 6.0.2. Hence, using (6.23) and (6.24) the above expression becomes

$$\sum_{i=0}^{r} \frac{\partial f(z_0, \ldots, z_r)}{\partial z_i} \mathcal{S}^i(P(z_\alpha, \ldots, z_{\beta+r})) = F(f(z_a, \ldots, z_{a+r}), \ldots, f(z_{a+p}, \ldots, z_{a+p+r})),$$

where $P(z_\alpha, \ldots, z_{\beta+r}) = Q(z, z_1, \ldots, z_{r-1}, f(z_\alpha, \ldots, z_{r+\alpha}), \ldots, f(z_\beta, \ldots, z_{\beta+r}))$. As the functions z_l, $l \in \mathbb{Z}$, are functionally independent, the above expression remains true when z_l is replaced by v_l, for each $l \in \mathbb{Z}$. Doing so, we obtain (6.27) as required. □

6.0.5 Summary

We summarise the general procedure:

1. Choose group G.

2. Choose subgroup H of G.

3. From DLR, generate subgroup $\mathbb{H}_{r-1}$, where $r = \text{dim}(G/H)$.

4. Find function z on G/H which is $\mathbb{H}_{r-1}$-left invariant.

5. Compute MT.

Recall, $G = \mathcal{G}(d, c_1, k_1) \times \mathcal{G}(d, c_2, k_2) \times \ldots \times \mathcal{G}(d, c_r, k_r)$, therefore step 1 amounts to choosing the constants c_i and k_i. As will be observed, the constants c_i will explicitly appear in the computed MT. The constants k_i will not. The DLR determines the value of d. The choice of H taken in step 2 will determine the order of the MT obtained, as the MT has order equal to the codimension of H. Step 3 requires the straight forward application of the relevant definitions. Step 4 presents the main difficulty, however in many situations is achievable. When such a z can be found, the non degeneracy conditions are often satisfied. Note that step 4 is equivalent to finding a function z which is $\mathbb{H}_{r-1}$-left invariant and H-right invariant. In the following sections we shall use Theorem 6.0.6 to derive many MTs.

Chapter 7

Differential difference equations

7.1 Volterra examples

As an example of the general procedure we derive MTs for the Volterra equation

$$u_t = u(u_1 - u_{-1}), \tag{7.1}$$

which has the following DLR

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}0 & u\\ -1 & \lambda\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}, \tag{7.2a}$$

$$D_t\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}u & \lambda u_{-1}\\ -\lambda & \lambda^2 + u_{-1}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}. \tag{7.2b}$$

7.1.1 Example 1

Take $m = 1$ and $k = 2$ in (6.9), then G is the group

$$G = \left\{\begin{pmatrix}w_{11}^{10} & w_{12}^{10}\\ w_{21}^{10} & w_{22}^{10}\end{pmatrix} + (\lambda - c_1)\begin{pmatrix}w_{11}^{11} & w_{12}^{11}\\ w_{21}^{11} & w_{22}^{11}\end{pmatrix} \mid (\lambda - c_1)^2 = 0, w_{11}^{10}w_{22}^{10} - w_{12}^{10}w_{21}^{10} \neq 0\right\}.$$

Take the closed connected Lie subgroup

$$H = \left\{ \begin{pmatrix} w_{11}^{10} & w_{12}^{10} \\ 0 & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ 0 & w_{22}^{11} \end{pmatrix} \mid (\lambda - c_1)^2 = 0, w_{11}^{10} w_{22}^{10} \neq 0 \right\}.$$

The subgroup H has codimension 2, we obtain local coordinates on G/H.

Lemma 7.1.1 *The following functions are H-right invariant*

$$W^1 = \frac{w_{11}^{10}}{w_{21}^{10}}, \qquad W^2 = \frac{w_{21}^{10} w_{11}^{11} - w_{11}^{10} w_{21}^{11}}{(w_{21}^{10})^2}.$$

Proof

Consider the product

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix} = \begin{pmatrix} g_{11}h_{11} & g_{11}h_{12} + g_{12}h_{22} \\ g_{21}h_{11} & g_{21}h_{12} + g_{22}h_{22} \end{pmatrix}.$$

Thus, right multiplication by an upper triangular matrix leaves the ratio of the $(1,1)$ and $(2,1)$ elements invariant. Hence, right multiplication by H will leave the following expression invariant

$$\frac{w_{11}^{10} + w_{11}^{11} (\lambda - c_1)}{w_{21}^{10} + w_{21}^{11} (\lambda - c_1)} = \frac{w_{11}^{10}}{w_{21}^{10}} + \frac{w_{21}^{10} w_{11}^{11} - w_{11}^{10} w_{21}^{11}}{(w_{21}^{10})^2} (\lambda - c_1) + \dots.$$

The expression is invariant for arbitrary λ, thus the coefficients of the powers of $(\lambda - c_1)$ are invariant. W^1 corresponds to the constant term, whereas W^2 corresponds to the linear term. □

Functions W^1 and W^2 are functionally independent, thus can be regarded as local coordinates on G/H. Our aim is to a find a function of W^1 and W^2 which is invariant with respect to the group $\mathbb{H}_1$. Of course before we may do so, it is necessary to compute $\mathbb{H}_1$.

The matrix $M(u, \lambda)$ corresponds to the discrete part of the DLR, so in this case it is given by

$$M(u, \lambda) = \begin{pmatrix} 0 & u \\ -1 & \lambda \end{pmatrix}.$$

Computing $M(u,\lambda)M(\tilde{u},\lambda)^{-1}$ yields

$$M(u,\lambda)M(\tilde{u},\lambda)^{-1} = \begin{pmatrix} \frac{u}{\tilde{u}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $\mathbb{H}_1$ is given by

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid 0 \neq a \in \mathbb{C} \right\}.$$

Let us investigate the action of $\mathbb{H}_1$ on G. Let $h \in \mathbb{H}_1$ and $g \in G$, then

$$hg = \begin{pmatrix} aw_{11}^{10} & aw_{12}^{10} \\ w_{21}^{10} & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} aw_{11}^{11} & aw_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix}. \tag{7.3}$$

So we see

$$W^1 = \frac{w_{11}^{10}}{w_{21}^{10}} \mapsto aW^1, \qquad W^2 = \frac{w_{21}^{10}w_{11}^{11} - w_{11}^{10}w_{21}^{11}}{(w_{21}^{10})^2} \mapsto aW^2.$$

Hence, the ratio

$$z = \frac{W^1}{W^2},$$

is invariant with respect to the action of $\mathbb{H}_1$. Now, to determine the action of $\mathcal{S}$ on G, we must use formula (6.10a). For $g \in G$,

$$\begin{aligned} M(u,\lambda)\cdot g = &\begin{pmatrix} uw_{21}^{10} & uw_{22}^{10} \\ c_1w_{21}^{10} - w_{11}^{10} & c_1w_{22}^{10} - w_{12}^{10} \end{pmatrix} \\ &+ (\lambda - c_1) \begin{pmatrix} uw_{21}^{11} & uw_{22}^{11} \\ c_1w_{21}^{11} - w_{11}^{11} + w_{21}^{10} & c_1w_{22}^{11} - w_{12}^{11} + w_{22}^{10} \end{pmatrix}, \end{aligned} \tag{7.4}$$

where the condition $(\lambda - c_1)^2 \equiv 0$ is made use of. The above expression defines the action of $\mathcal{S}$ on the coordinates w_{ij}^{pq}, and therefore as a consequence defines an action on W^1 and

W^2. For example, we observe $\mathcal{S}(w_{11}^{10}) = uw_{21}^{10}$. Computing the action on W^1, one finds

$$\begin{aligned}\mathcal{S}(W^1) &= \mathcal{S}\left(\frac{w_{11}^{10}}{w_{21}^{10}}\right)\\ &= \frac{\mathcal{S}(w_{11}^{10})}{\mathcal{S}(w_{21}^{10})}\\ &= \frac{uw_{21}^{10}}{c_1 w_{21}^{10} - w_{11}^{10}}\\ &= \frac{u}{c_1 - W^1}.\end{aligned}$$

For W^2,

$$\begin{aligned}\mathcal{S}(W^2) &= \mathcal{S}\left(\frac{w_{21}^{10}w_{11}^{11} - w_{11}^{10}w_{21}^{11}}{(w_{21}^{10})^2}\right)\\ &= \frac{\mathcal{S}(w_{21}^{10})\mathcal{S}(w_{11}^{11}) - \mathcal{S}(w_{11}^{10})\mathcal{S}(w_{21}^{11})}{\mathcal{S}(w_{21}^{10})^2}\\ &= \frac{(c_1 w_{21}^{10} - w_{11}^{10})(uw_{21}^{11}) - (uw_{21}^{10})(c_1 w_{21}^{11} - w_{11}^{11} + w_{21}^{10})}{(c_1 w_{21}^{10} - w_{11}^{10})^2}\\ &= \frac{u(w_{21}^{10}w_{11}^{11} - w_{11}^{10}w_{21}^{11} - w_{21}^{10}w_{21}^{10})}{(c_1 w_{21}^{10} - w_{11}^{10})^2}\\ &= \frac{u\,(W^2 - 1)}{(c_1 - W^1)^{\,2}}.\end{aligned}$$

Armed with the above expressions, z_1 is computed as follows

$$\begin{aligned}z_1 &= \mathcal{S}(z)\\ &= \frac{\mathcal{S}(W^1)}{\mathcal{S}(W^2)}\\ &= \frac{c_1 - W^1}{W^2 - 1}.\end{aligned}$$

As expected, due to the fact that z is $\mathbb{H}_1$-left invariant, $z_1 = \mathcal{S}(z)$ does not depend on u_l for any $l \in \mathbb{Z}$. Taking a shift,

$$\mathcal{S}(z_1) = -\frac{(c_1 - W^1)(c_1(c_1 - W^1) - u)}{(c_1 - W^1)^2 - (W^2 - 1)u}. \tag{7.5}$$

The functions z and z_1 form a system of local coordinates on G/H, indeed W^1 and W^2 can be expressed in terms of z and z_1 in the following way.

$$W^1 = \frac{z\,(c_1 + z_1)}{z + z_1}, \qquad W^2 = \frac{c_1 + z_1}{z + z_1}. \tag{7.6}$$

Inserting the above expressions into (7.5) produces

$$\mathcal{S}(z_1) = -\frac{z_1\left(z_1\left(u - c_1^2\right) + z\left(c_1 z_1 + u\right)\right)}{z_1\left(u - c_1 z_1\right) + z\left(u + z_1^2\right)},$$

which implies

$$u = \frac{z_1^2\left(c_1 - z\right)\left(c_1 + \mathcal{S}(z_1)\right)}{\left(z + z_1\right)\left(\mathcal{S}(z_1) + z_1\right)}.$$

Thus, making the identification $\mathcal{S}^i(z) = v_i$ yields after rearrangement for u, the following MT

$$u = \frac{v_1^2(c_1 - v)(c_1 + v_2)}{(v + v_1)(v_1 + v_2)}. \tag{7.7}$$

To compute the Miura related system one must compute z_t. In terms of W^1 and W^2, this is given by

$$z_t = \frac{W_t^1 W^2 - W_t^2 W^1}{(W^2)^2}.$$

But now W_t^1 and W_t^2 must be determined. Both W^1 and W^2 are functions of the w_{ij}^{pq} coordinates, thus their derivatives can be written in terms of the derivatives of the w_{ij}^{pq}. Explicitly,

$$W_t^1 = \frac{d}{dt}\left(\frac{w_{11}^{10}}{w_{21}^{10}}\right) = \frac{(w_{11}^{10})_t w_{21}^{10} - (w_{21}^{10})_t w_{11}^{10}}{(w_{21}^{10})^2} \tag{7.8}$$

and

$$\begin{aligned} W_t^2 &= \frac{d}{dt}\left(\frac{w_{21}^{10}w_{11}^{11} - w_{11}^{10}w_{21}^{11}}{(w_{21}^{10})^2}\right) \\ &= \frac{(w_{21}^{10})^2(w_{11}^{11})_t - w_{11}^{11}w_{21}^{10}(w_{21}^{10})_t - w_{11}^{10}w_{21}^{10}(w_{21}^{11})_t - w_{21}^{10}w_{21}^{11}(w_{11}^{10})_t + 2w_{11}^{10}w_{21}^{11}(w_{21}^{10})_t}{(w_{21}^{10})^3}. \end{aligned} \tag{7.9}$$

The derivatives of the w_{ij}^{pq} are determined by the action of the t part of the Lax pair. Setting

$$\mathcal{U} = \begin{pmatrix} u & \lambda u_{-1} \\ -\lambda & \lambda^2 + u_{-1} \end{pmatrix}.$$

It follows that for $g \in G$,

$$\mathcal{U}.g = A_0 + (\lambda - c_1)A_1, \tag{7.10}$$

with

$$A_0 = \begin{pmatrix} uw_{11}^{10} + c_1 u_{-1} w_{21}^{10} & uw_{12}^{10} + c_1 u_{-1} w_{22}^{10} \\ w_{21}^{10} c_1^2 - c_1 w_{11}^{10} + u_{-1} w_{21}^{10} & w_{22}^{10} c_1^2 - c_1 w_{12}^{10} + u_{-1} w_{22}^{10} \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} uw_{11}^{11} + u_{-1} w_{21}^{10} + c_1 u_{-1} w_{21}^{11} & uw_{12}^{11} + u_{-1} w_{22}^{10} + c_1 u_{-1} w_{22}^{11} \\ w_{21}^{11} c_1^2 - w_{11}^{11} c_1 + 2 w_{21}^{10} c_1 - w_{11}^{10} + u_{-1} w_{21}^{11} & w_{22}^{11} c_1^2 - w_{12}^{11} c_1 + 2 w_{22}^{10} c_1 - w_{12}^{10} + u_{-1} w_{22}^{11} \end{pmatrix}.$$

Using the above one can compute the derivatives of the w_{ij}^{pq} coordinate functions. For example,

$$(w_{11}^{10})_t = uw_{11}^{10} + c_1 u_{-1} w_{21}^{10}.$$

Hence, replacing the expressions for the $(w_{ij}^{pq})_t$ in (7.8) and (7.9) one finds

$$\begin{aligned} W_t^1 &= \frac{w_{21}^{10}\left(c_1 u_{-1} w_{21}^{10} + uw_{11}^{10}\right) - w_{11}^{10}\left(c_1^2 w_{21}^{10} - c_1 w_{11}^{10} + u_{-1} w_{21}^{10}\right)}{(w_{21}^{10})^2} \\ &= c_1 u_{-1} + W^1\left(u - u_{-1} - c_1^2\right) + c_1 (W^1)^2, \end{aligned}$$

and

$$\begin{aligned} W_t^2 = &\left[w_{21}^{10}\left(2c_1 w_{11}^{10} - w_{21}^{10} w_{11}^{11}\left(c_1^2 - u + u_{-1}\right) + w_{11}^{10}\left(2c_1 w_{11}^{11} + w_{11}^{10}\right) + u_{-1}(w_{21}^{10})^2\right)\right. \\ &\left. + w_{11}^{10} w_{21}^{11}\left(w_{21}^{10}\left(c_1^2 - u + u_{-1}\right) - 2c_1 w_{11}^{10}\right)\right] \frac{1}{(w_{21}^{10})^3} \\ = &\, W^2\left(2c_1 W^1 - c_1^2 + u - u_{-1}\right) - 2c_1 W^1 + u_{-1} + (W^1)^2, \end{aligned}$$

where the final expressions follow from expressing the result of the derivative in terms of the local coordinates W^1 and W^2. Note, this is possible as W_t^2 is necessarily a function on G/H, while W^1 and W^2 form a system of local coordinates on G/H. Hence,

$$\begin{aligned} z_t &= \frac{W_t^1 W^2 - W_t^2 W^1}{(W^2)^2} \\ &= \frac{u_{-1}\left(c_1 W^2 - W^1\right) - (W^1)^2\left(c_1\left(W^2 - 2\right) + W^1\right)}{(W^2)^2}. \end{aligned}$$

However, using (7.6) we may express the W^1 and W^2 in terms of the z and z_1 functions. Doing so yields

$$z_t = -\frac{\left(c_1 - z\right)\left(c_1^2 (z)^2 - u_{-1}\left(z + z_1\right)^2 - (z_1)^2 (z)^2\right)}{\left(z + z_1\right)\left(c_1 + z_1\right)}.$$

Finally, making the identification $\mathcal{S}_u^i(z) = v_i$ and using (7.7) to eliminate u, we arrive at the Miura related system

$$v_t = \frac{v^2\,(v_{-1} - v_1)\,(v - c_1)\,(c_1 + v)}{(v + v_{-1})\,(v + v_1)}.$$

Hence, in summary, to compute $v_t = z_t$, we express z_t in terms of the w_{ij}^{pq}, use the action corresponding to the t part of the DLR to determine the t-derivatives of the w_{ij}^{pq} coordinates, re-express the result in terms of W^1 and W^2, then finally the expressions for W^1 and W^2 in terms of z and z_1 are used to compute the result in terms of z, and so subsequently v. Thus, we obtain the following result.

Theorem 7.1.2 *The equation*

$$v_t = \frac{v^2\,(v_{-1} - v_1)\,(v - c_1)\,(c_1 + v)}{(v + v_{-1})\,(v + v_1)},$$

is related by the MT

$$u = \frac{v_1^2(c_1 - v)(c_1 + v_2)}{(v + v_1)(v_1 + v_2)}, \tag{7.11}$$

to the equation

$$u_t = u(u_1 - u_{-1}).$$

The Miura related system can also be found in [76] as a particular case of the equation V2 in the list of Volterra-type equations. One must take the arbitrary function $p(u^2) = u^2(u^2 - c_1^2)$ and relabel $u = v$. The MT to the Volterra equation is also provided.

7.1.2 Example 2

Consider the subgroup of G given by

$$\tilde{H} = \left\{ \begin{pmatrix} 1 & w_{12}^{10} \\ 0 & w_{22}^{10} \end{pmatrix} + (\lambda - c_1) \begin{pmatrix} w_{11}^{11} & w_{12}^{11} \\ w_{21}^{11} & w_{22}^{11} \end{pmatrix} \mid (\lambda - c_1)^2 = 0, w_{22}^{10} \neq 0 \right\}.$$

One can show that right-multiplication by $\tilde{H}$ on G leaves

$$W^1 = w_{21}^{10}, \qquad W^2 = w_{11}^{10},$$

invariant. Hence we may take W^1 and W^2 as local coordinates on G/H. From (7.3), it can be seen that w_{21}^{10} is also left-$\mathbb{H}_1$ invariant. Hence, we take $z = W^1 = w_{21}^{10}$. Then,

$$z_1 = \mathcal{S}(W^1) = \mathcal{S}(w_{21}^{10}) = c_1 w_{21}^{10} - w_{11}^{10} = c_1 W^1 - W^2,$$

using (7.4). Also,

$$\mathcal{S}(z_1) = c_1 \left(c_1 w_{21}^{10} - w_{11}^{10} \right) - w_{21}^{10} u = (c_1^2 - u) W^1 - c_1 W^2.$$

Using that

$$W^1 = z \qquad W^2 = c_1 z - z_1,$$

rearranging for u and making the identification $\mathcal{S}^i(z) = v_i$, we arrive at the MT

$$u = \frac{c_1 v_1 - v_2}{v}.$$

In this case computing the Miura related system is straightforward, as

$$v_t = z_t = (w_{21}^{10})_t = w_{21}^{10} c_1^2 - c_1 w_{11}^{10} + u_{-1} w_{21}^{10},$$

using (7.10). Expressing this relation fully in terms of v yields the equation

$$v_t = \frac{c_1 v^2 + c_1 v_{-1} v_1 - v_1 v}{v_{-1}}.$$

Hence, we arrive at the following result.

Theorem 7.1.3 *The equation*

$$v_t = \frac{c_1 v^2 + c_1 v_{-1} v_1 - v_1 v}{v_{-1}},$$

is linked via the MT

$$u = \frac{c_1 v_1 - v_2}{v},$$

to the equation

$$u_t = u(u_1 - u_{-1}).$$

If one makes the change of variables

$$v = e^{\tilde{v}}, \quad t = -\tilde{t},$$

the Miura related system becomes

$$\begin{aligned}\tilde{v}_{\tilde{t}} &= e^{\tilde{v}_1 - \tilde{v}_{-1}} - c_1 e^{\tilde{v} - \tilde{v}_{-1}} - c_1 e^{\tilde{v}_1 - \tilde{v}} \\ &= (e^{\tilde{v}_1 - \tilde{v}} - c_1)(e^{\tilde{v} - \tilde{v}_{-1}} - c_1) - c_1^2.\end{aligned}$$

After relabelling $\tilde{v} = u$, this is a particular case of equation V6 with $p(y) = y^2$ in the list of Volterra-type equations in [76]. An equivalent MT is also provided.

7.2 Special case

It can be somewhat unwieldy dealing directly with G and H. The situation can be simplified in the following way. Suppose we take

$$G = \mathcal{G}(d, c_1, 1) \times \mathcal{G}(d, c_2, 1) \times \cdots \times \mathcal{G}(d, c_r, 1). \tag{7.12}$$

Then, $\mathcal{G}(d, c_i, 1)$ is isomorphic to $\mathrm{GL}_d(\mathbb{C})$ for each i, so G is isomorphic to the Cartesian product of r copies of $\mathrm{GL}_d(\mathbb{C})$. It is well known that $\mathrm{GL}_d(\mathbb{C})$ acts transitively on $\mathbb{C}^d \setminus \{0\}$ via left multiplication. The projective space, $\mathbb{CP}^{d-1}$, is nothing but equivalence classes of $\mathbb{C}^d \setminus \{0\}$ under the relation $p_1 \sim p_2$ if and only if $p_1 = \alpha p_2$, for some complex $\alpha \neq 0$. Hence, since $\mathrm{GL}_d(\mathbb{C})$ acts transitively on $\mathbb{C}^d \setminus \{0\}$, the same holds for $\mathbb{CP}^{d-1}$. For $i = 1 \ldots r$, let M_i be one of $\mathbb{C}^d \setminus \{0\}$ and $\mathbb{CP}^{d-1}$. Then, it follows that G acts transitively on $M = M_1 \times M_2 \times \ldots M_r$. Let x be an element of M and fix $H \subset G$ to be the stabilizer of x, ie $H = \{g \in G | gx = x\}$. Then G/H is diffeomorphic to M via the map

$$G/H \to M, \quad gH \to gx.$$

Hence, if we consider G of this type, we may replace G/H by M in our previous analysis, thus may instead speak of functions on dense open subsets of M. This will simplify our considerations. It has the advantage of allowing one to immediately consider functions of the coordinates on M. In the following sections, we often compute MTs by considering G/H to be a cartesian product of $\mathbb{C}^d \setminus \{0\}$ and/or projective spaces. Of course, this does not only apply to these spaces. Whenever we discover a transitive action of G on a manifold, we may replace G/H by this manifold.

7.3 Narita-Itoh-Bogoyavlensky Lattice

In this section we construct MTs for the Narita-Itoh-Bogoyalensky lattice. This lattice has been studied extensively [19, 35, 62]. For fixed p this is given by the equation

$$u_t = u\left(\sum_{k=1}^{p} u_k - \sum_{k=1}^{p} u_{-k}\right), \tag{7.13}$$

which possesses the following operator semi discrete lax pair

$$L\chi = \lambda\chi,$$

$$\chi_t = A\chi, \tag{7.14}$$

with

$$L = \mathcal{S} + u\mathcal{S}^{-p} \tag{7.15}$$

and

$$A = (L^{(p+1)})_{\geq 0}. \tag{7.16}$$

In this form, (7.13) is equivalent to the equation

$$L_t = [A, L].$$

One can find this Lax pair in [37]. Note, the case $p = 1$ corresponds to the Volterra equation. First we shall consider the case $p = 2$.

7.3.1 p=2

For $p = 2$, we obtain the DΔE

$$u_t = u(u_2 + u_1 - u_{-1} - u_{-2}). \tag{7.17}$$

In this case, $L = \mathcal{S} + u\mathcal{S}^{-2}$ and thus

$$\begin{aligned} A &= (L^3)_{\geq 0} \\ &= \left[\left(\mathcal{S}^2 + (u_1 + u)\mathcal{S}^{-1} + uu_{-2}\mathcal{S}^{-4}\right)\left(\mathcal{S} + u\mathcal{S}^{-2}\right)\right]_{\geq 0} \\ &= \mathcal{S}^3 + u + u_1 + u_2. \end{aligned}$$

In order to apply the general theory, the Lax Pair must be expressed in matrix form. Considering the L part, we have the expression

$$\mathcal{S}(\chi) = \lambda\chi - u\mathcal{S}^{-2}(\chi). \tag{7.18}$$

Thus, setting

$$\begin{aligned} \psi^1 &= \mathcal{S}^{-2}\chi, \\ \psi^2 &= \mathcal{S}^{-1}\chi, \\ \psi^3 &= \chi, \end{aligned}$$

we find using (7.18), the following expression for the discrete part of the DLR.

$$\mathcal{S}\begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -u & 0 & \lambda \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}. \tag{7.19}$$

Having obtained the matrix $M(u, \lambda)$, we compute the subgroups $\mathbb{H}_k$. In what follows, we consider manifolds G/H of dimension 2, 3 and 4. Hence, we must consider invariants of the subgroups $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$. We compute these in the next section.

$\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$

Recall, $\mathbb{H}_1$ is the group generated by elements of the form $M(u,\lambda)M(\tilde{u},\lambda)^{-1}$, where $u, \tilde{u}$ are complex numbers such that M is invertible. $\mathbb{H}_2$ is generated by the elements of $\mathbb{H}_1$, along with elements of the form $M(u,\lambda)h_1M(u,\lambda)^{-1}$, where $h_1 \in \mathbb{H}_1$. Similarly, $\mathbb{H}_3$ is generated by elements of $\mathbb{H}_2$, along with elements of the form $M(u,\lambda)h_2M(u,\lambda)^{-1}$, where $h_2 \in \mathbb{H}_2$. Computing $M(u,\lambda)M(\tilde{u},\lambda)^{-1}$, we find

$$M(u,\lambda)M(\tilde{u},\lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda - \frac{u\lambda}{\tilde{u}} & \frac{u}{\tilde{u}} \end{pmatrix}.$$

Hence, elements of $\mathbb{H}_1$ are of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda - a\lambda & a \end{pmatrix},$$

for some $0 \neq a \in \mathbb{C}$. One can check that the set of such elements is indeed a group. Thus, it follows that

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & (1-a)\lambda & a \end{pmatrix} \mid a \in \mathbb{C}, a \neq 0 \right\}$$

To compute $\mathbb{H}_2$, one must consider elements of the form $M(u,\lambda)h_1M(u,\lambda)^{-1}$, for $h_1 \in \mathbb{H}_1$. For an arbitrary element $h_1 \in \mathbb{H}_1$, we find

$$M(u,\lambda)h_1M(u,\lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ (1-a)\lambda & a & 0 \\ (1-a)\lambda^2 & -(1-a)\lambda & 1 \end{pmatrix}.$$

The subgroup $\mathbb{H}_2$ is also required to include $\mathbb{H}_1$, for this reason, and to ensure closure of the subgroup, we must consider elements of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ -(a-1)\lambda & a & 0 \\ d\lambda^2 & b\lambda & c \end{pmatrix}.$$

It is clear that the subgroups are becoming quite complicated. Indeed, computing $M(u,\lambda)h_2M(u,\lambda)^{-1}$ produces

$$M(u,\lambda)h_1M(u,\lambda)^{-1} = \begin{pmatrix} a & -\frac{(a-1)\lambda^2}{u} & \frac{(a-1)\lambda}{u} \\ b\lambda & \frac{d\lambda^3+cu}{u} & -\frac{d\lambda^2}{u} \\ b\lambda^2 & \frac{\lambda\left(d\lambda^3+cu-u\right)}{u} & -\frac{d\lambda^3-u}{u} \end{pmatrix}.$$

To obtain the subgroup $\mathbb{H}_3$ one is required to include elements of the above form, as well as all elements of $\mathbb{H}_2$. It will be a tough task to find an invariant of such a group, thus we shall change tack and consider an equivalent DLR obtained by conjugation. One for which the subgroups take a more manageable form. Note, conjugation by a constant matrix will not introduce shifts of u into M.

Consider the conjugate matrix $\tilde{M}(u,\lambda)$ given by

$$\begin{aligned}\tilde{M}(u,\lambda) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 1 \\ -\lambda & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -u & 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 1 \\ -\lambda & 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \lambda & 0 & 1 \\ -u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.\end{aligned}$$

Computing $\tilde{M}(u,\lambda)\tilde{M}(\tilde{u},\lambda)^{-1}$ provides

$$\tilde{M}(u,\lambda)\tilde{M}(\tilde{u},\lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{u}{\tilde{u}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence,

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid 0 \neq a \in \mathbb{C} \right\}.$$

Computing $\tilde{M}(u,\lambda)h_1\tilde{M}(u,\lambda)^{-1}$ yields

$$\tilde{M}(u,\lambda)h_1\tilde{M}(u,\lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Thus, ensuring $\mathbb{H}_2$ contains all elements of $\mathbb{H}_1$, we find

$$\mathbb{H}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}, ab \neq 0 \right\}.$$

Finally, computing $\tilde{M}(u,\lambda)h_2\tilde{M}(u,\lambda)^{-1}$ provides

$$\tilde{M}(u,\lambda)h_2\tilde{M}(u,\lambda)^{-1} = \begin{pmatrix} b & \frac{b\lambda}{u} - \frac{\lambda}{u} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Hence, subgroup $\mathbb{H}_3$ is given by

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} a & d\lambda & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C}, abc \neq 0 \right\}.$$

It is clear that considering an equivalent DLR obtained by conjugation can aid greatly in increasing the simplicity of the obtained subgroups. Having computed $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$ we proceed to compute MTs.

Standard action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{C}^3 \setminus \{0\}$

Consider the case $G = \mathcal{G}(3, c_1, 1)$, then $G \cong \mathrm{GL}_3(\mathbb{C})$. Since $\mathrm{GL}_3(\mathbb{C})$ acts transitively on the manifold $\mathbb{C}^3 \setminus \{0\}$ via the standard action of left multiplication, it follows that $\mathbb{C}^3 \setminus \{0\}$

is diffeomorphic to G/H for some H. Hence, we may apply the general theory replacing G/H by $\mathbb{C}^3 \setminus \{0\}$. In this case G/H has dimension 3, thus we must find a function of the coordinates on $\mathbb{C}^3 \setminus \{0\}$ which is invariant with respect to the action of $\mathbb{H}_2$. The map $\rho : \boldsymbol{G} \to G$ is simply evaluation, $\lambda \mapsto c_1$. Recall, we considered a gauge equivalent DLR obtained by conjugating by c,

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 1 \\ -\lambda & 1 & 0 \end{pmatrix}.$$

Setting $\boldsymbol{\varphi} = c\boldsymbol{\psi}$, the discrete part of the gauge equivalent DLR takes the form

$$\mathcal{S}\begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 1 \\ -u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{pmatrix}.$$

Let ϕ^1, ϕ^2, ϕ^3 be coordinates on $G/H = \mathbb{C}^3 \setminus \{0\}$. The operator $\mathcal{S}$ acts in the following way

$$\mathcal{S}\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 1 \\ -u & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}. \tag{7.20}$$

We highlight that $\tilde{M}(u, \lambda)$ is mapped into G using ρ (evaluated at c_1), then the standard action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{C}^3$ (left multiplication) is used to define $\mathcal{S}$ on $\mathbb{C}^3 \setminus \{0\}$. Of course, the above expression is almost identical to the discrete part of the DLR, however we should be clear regarding the connection to the general theory. The above determines the action of $\mathcal{S}$ on the manifold G/H. It is not part of an abstract DLR. Observe that although G no longer depends explicitly upon the parameter c_1, as we truncate to the constant term, c_1 still makes an appearance through the $\mathcal{S}$ and D_t operators.

Now, we must find a function of ϕ^1, ϕ^2, ϕ^3 which is left invariant with respect to the action of $\mathbb{H}_2$. Recall,

$$\mathbb{H}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}, ab \neq 0 \right\}.$$

Clearly, ϕ^1 is invariant with respect to the action of $\mathbb{H}_2$. Thus, we set $z = \phi^1$ the invariant. Then, following the general theory the z_1 and z_2 coordinates are given by

$$\begin{aligned} z_1 &= \mathcal{S}(\phi) \\ &= c_1\phi^1 + \phi^3, \\ z_2 &= \mathcal{S}(z_1) \\ &= c_1\mathcal{S}(\phi^1) + \mathcal{S}(\phi^3) \\ &= c_1^2\phi^1 + \phi^2 + c_1\phi^3. \end{aligned}$$

Since z is $\mathbb{H}_2$-left invariant, it follows that, as expected, z_1 and z_2 are independent of u_l for any $l \in \mathbb{Z}$. By definition, $\mathcal{S}(z) = z_1$ and $\mathcal{S}(z_1) = z_2$. Now, computing $\mathcal{S}(z_2)$ we find

$$\begin{aligned} \mathcal{S}(z_2) &= c_1^2\mathcal{S}(\phi^1) + \mathcal{S}(\phi^2) + c_1\mathcal{S}(\phi^3) \\ &= -\left(u - c_1^3\right)\phi^1 + c_1^2\phi^3 + c_1\phi^2. \end{aligned}$$

This depends non-trivially on u, thus a degeneracy has not occurred. Moreover, z, z_1 and z_2 form a system of local coordinates on G/H, and hence we are able to find the following expressions $\phi^1 = z$, $\phi^2 = z_2 - c_1z_1$ and $\phi^3 = z_1 - c_1z$. Thus,

$$\mathcal{S}(z_2) = -uz + c_1z_2. \tag{7.21}$$

The corresponding MT is arrived at by setting $z = v$, $z_1 = \mathcal{S}(v) = v_1$, $z_2 = \mathcal{S}(\mathcal{S}(v)) = v_2$ and rearranging to obtain

$$u = \frac{c_1v_2 - v_3}{v}.$$

To compute the Miura related system, we must compute $D_t(z) = D_t(\phi^1)$. To this end, we first compute the matrix form of the t part of the DLR using the operator Lax pair.

For $p = 2$, (7.16) becomes $A = \mathcal{S}^3 + u + u_1 + u_2$. Then, $\chi_t = \mathcal{S}^3(\chi) + (u + u_1 + u_2)\chi$. From our earlier identification, it follows that

$$\begin{aligned}\psi^3_t &= \mathcal{S}^3(\psi^3) + (u + u_1 + u_2)\psi^3 \\ &= -\lambda^2 u\psi^1 - \lambda u_1\psi^2 + (\lambda^3 + u + u_1)\psi^3,\end{aligned}$$

where above we have used (7.19) to compute $\mathcal{S}^3(\psi^3)$. Hence,

$$\begin{aligned}\psi^2_t &= \mathcal{S}^{-1}\psi^3_t \\ &= -\lambda u\psi^1 + (u_{-1} + u)\psi^2 + \lambda^2\psi^3, \\ \psi^1_t &= \mathcal{S}^{-1}\psi^2_t \\ &= (u_{-1} + u_{-2})\psi^1 + \lambda\psi^3,\end{aligned}$$

where we have used the shifted inverse of (7.19) to obtain $\mathcal{S}^{-1}\psi^1$. Thus,

$$D_t\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix} u_{-2} + u_{-1} & 0 & \lambda \\ -\lambda u & u_{-1} + u & \lambda^2 \\ -u\lambda^2 & -\lambda u_1 & \lambda^3 + u_1 + u\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}.$$

Finally, conjugating by c we obtain the continuous part of the gauged DLR

$$D_t\begin{pmatrix}\varphi^1\\ \varphi^2\\ \varphi^3\end{pmatrix} = \begin{pmatrix} \lambda^3 + u_{-2} + u_{-1} & \lambda & \lambda^2 \\ -\lambda^2 u_{-1} & u + u_1 & -\lambda u_{-1} \\ -\lambda u_{-2} & 0 & u_{-1} + u\end{pmatrix}\begin{pmatrix}\varphi^1\\ \varphi^2\\ \varphi^3\end{pmatrix}. \tag{7.22}$$

Consequently, the action of D_t on G/H is given by

$$D_t\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix} = \begin{pmatrix} c_1^3 + u_{-2} + u_{-1} & c_1 & c_1^2 \\ -c_1^2 u_{-1} & u + u_1 & -c_1 u_{-1} \\ -c_1 u_{-2} & 0 & u_{-1} + u\end{pmatrix}\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix}. \tag{7.23}$$

Now we may compute the Miura related system. As

$$\begin{aligned}z_t &= \phi^1_t \\ &= \left(c_1^3 + u_{-2} + u_{-1}\right)\phi^1 + c_1\phi^2 + c_1^2\phi^3 \\ &= (u_{-2} + u_{-1})z + c_1 z_2,\end{aligned}$$

it follows that

$$\begin{aligned} v_t &= z_t \\ &= (u_{-2} + u_{-1})\, z + c_1 z_2 \\ &= \left(\frac{c_1 v - v_1}{v_{-2}} + \frac{c_1 v_1 - v_2}{v_{-1}} \right) v + c_1 v_2. \end{aligned}$$

Thus, by considering the standard action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{C}^3 \setminus \{0\}$, the following result is obtained.

Theorem 7.3.1 *The equation*

$$v_t = \left(\frac{c_1 v - v_1}{v_{-2}} + \frac{c_1 v_1 - v_2}{v_{-1}} \right) v + c_1 v_2,$$

is linked via the MT

$$u = \frac{c_1 v_2 - v_3}{v}, \tag{7.24}$$

to the equation

$$u_t = u(u_2 + u_1 - u_{-1} - u_{-2}). \tag{7.25}$$

The above MT can also be obtained directly from the operator Lax pair. Indeed, as $L\phi = \lambda\phi$, it follows that

$$\mathcal{S}(\phi) + u\mathcal{S}^{-2}\phi = \lambda\phi.$$

Thus, rearranging for u and setting $v = \mathcal{S}^{-2}(\phi)$, we obtain (7.24). To arrive at the Miura related system we simply use the time part of the operator Lax pair. This is quite a common tactic in the derivation of MTs. As a further rather famous example, in the continuous case one can construct the MT from the modified KdV equation to the KdV equation by simple rearrangement of the KdV equations Lax pair [68]. An issue with this approach is that one is limited in terms of the order of the resulting MT that can be obtained. This limit will be the number of shifts which occur in the Lax pair. This is not the case with the method which we describe.

Action of $\mathrm{GL}_3(\mathbb{C})$ **on** $\mathbb{CP}^2$

Consider the case $G = \mathcal{G}(3, c_1, 1)$, then $G \cong \mathrm{GL}_3(\mathbb{C})$. The group G acts transitively on the 2-dimensional complex projective space, $\mathbb{CP}^2$. Hence, we may find H such that $G/H \cong \mathbb{CP}^2$. Then, applying the general theory, we must find a function z on an open dense subset of $\mathbb{CP}^2$ which is $\mathbb{H}_1$-left invariant. In the previous section we considered ϕ^1, ϕ^2 and ϕ^3 to be coordinates on $\mathbb{C}^3$. We now view them as homogeneous coordinates on $\mathbb{CP}^2$. Then,

$$W^1 = \frac{\phi^2}{\phi^1}, \quad W^2 = \frac{\phi^3}{\phi^1},$$

are affine coordinates on the open dense subset

$$\mathbb{V} = \{(\phi^1 : \phi^2 : \phi^3) \in \mathbb{CP}^2 | \phi^1 \neq 0\} \subset \mathbb{CP}^2.$$

We compute the action of $\mathcal{S}$ and D_t on the affine coordinates using (7.20) and (7.23) as follows

$$\begin{aligned}
\mathcal{S}(W^1) &= \frac{\mathcal{S}(\phi^2)}{\mathcal{S}(\phi^1)} \\
&= -\frac{u\phi^1}{c_1\phi^1 + \phi^3} \\
&= -\frac{u}{c_1 + W^2},
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(W^2) &= \frac{\mathcal{S}(\phi^3)}{\mathcal{S}(\phi^1)} \\
&= \frac{\phi^2}{c_1\phi^1 + \phi^3} \\
&= \frac{W^1}{c_1 + W^2}.
\end{aligned}$$

$$\begin{aligned}
W^1_t &= (\frac{\phi^2}{\phi^1})_t \\
&= \frac{\phi^2_t\phi^1 - \phi^2\phi^1_t}{(\phi^1)^2} \\
&= -c_1^2 u_{-1} - W^1 \left(c_1^3 + u_{-2} + u_{-1} - u - u_1\right) - c_1 u_{-1} W^2 - c_1^2 W^1 W^2 - c_1 (W^1)^2,
\end{aligned}$$

$$\begin{aligned}
W_t^2 &= (\frac{\phi^3}{\phi^1})_t \\
&= \frac{\phi_t^3\phi^1 - \phi^3\phi_t^1}{(\phi^1)^2} \\
&= -c_1 u_{-2} - u_{-2}W^2 + uW^2 - c_1^3 W^2 - c_1^2 (W^2)^2 - c_1 W^1 W^2.
\end{aligned}$$

Recall,

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{C}, a \neq 0 \right\}.$$

The action of $\mathbb{H}_1$ on $\mathbb{C}^3$ leaves ϕ^1 and ϕ^3 invariant, hence the induced action on $\mathbb{CP}^2$ leaves W^2 invariant. Thus, we set $z = W^2$. Then, z_1 is given by $z_1 = \mathcal{S}(W^2) = \frac{W^1}{c_1 + W^2}$, as computed above. Again, we observe that as z is $\mathbb{H}_1$-left invariant, z and $\mathcal{S}(z)$ are independent of u_l for any $l \in \mathbb{Z}$.

Now,

$$\begin{aligned}
\mathcal{S}(z_1) &= \frac{\mathcal{S}(W^1)}{c_1 + \mathcal{S}(W^2)} \\
&= -\frac{u}{c_1\,(c_1 + W^2) + W^1} \\
&= -\frac{u}{(c_1 + z)\,(c_1 + z_1)}.
\end{aligned}$$

Since $\mathcal{S}(z_1)$ depends on u, a degeneracy has not occurred. We set $z = v$, $z_1 = \mathcal{S}(z) = \mathcal{S}(v) = v_1$, $\mathcal{S}(z_1) = v_2$, then

$$u = -v_2(c_1 + v_1)(c_1 + v).$$

Using that $W^1 = c_1 z_1 + z z_1$ and $W^2 = z$, we obtain the Miura related system

$$\begin{aligned}
v_t &= Z_t^1 \\
&= (W^2)_t \\
&= -c_1 u_{-2} - u_{-2}W^2 + uW^2 - c_1^3 W^2 - c_1^2 (W^2)^2 - c_1 W^1 W^2 \\
&= v\,(c_1 + v)\left[(c_1 + v_{-2})(c_1 + v_{-1}) - (c_1 + v_1)(c_1 + v_2)\right].
\end{aligned}$$

Thus, by considering the action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2$, we arrive at the following result.

Theorem 7.3.2 *The equation*

$$v_t = v\,(c_1+v)\left[(c_1+v_{-2})(c_1+v_{-1})-(c_1+v_1)(c_1+v_2)\right], \qquad (7.26)$$

is linked via the MT

$$u = -v_2(c_1+v_1)(c_1+v),$$

to the equation

$$u_t = u(u_2+u_1-u_{-1}-u_{-2}).$$

An MT and equation equivalent to the above can be found in [9].

Action of $\mathrm{GL}_3(\mathbb{C})$ **on** $\mathbb{CP}^2\times\mathbb{CP}^2$

Consider the case $d=3$, $m=2$, $k_1=k_2=1$, then $G=\mathcal{G}(3,c_1,1)\times\mathcal{G}(3,c_2,1)\cong \mathrm{GL}_3(\mathbb{C})\times\mathrm{GL}_3(\mathbb{C})$, with $c_1,c_2\in\mathbb{C}$. Again, using the transitive action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2$, we obtain a transitive action of G on $\mathbb{CP}^2\times\mathbb{CP}^2$. Hence, we may find H such that $G/H\cong\mathbb{CP}^2\times\mathbb{CP}^2$. In this case, $r=\dim(G/H)=4$, thus we must find an invariant of $\mathbb{H}_3$. However, beforehand, we must define the $\mathcal{S}$ and D_t operators on G, as well as describe the action of $\boldsymbol{G}$ on $\mathbb{CP}^2\times\mathbb{CP}^2$, through which, we define the action of $\mathbb{H}_3$.

Let $A(\lambda)$ be a $\mathrm{GL}_3(\mathbb{C})$ valued function of λ, then we may define an action of A on $\mathbb{CP}^2\times\mathbb{CP}^2$ via

$$A(\lambda)\cdot\left(\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix},\begin{pmatrix}\rho^1\\ \rho^2\\ \rho^3\end{pmatrix}\right)=\left(A(c_1)\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix},A(c_2)\begin{pmatrix}\rho^1\\ \rho^2\\ \rho^3\end{pmatrix}\right).$$

By replacing A with M and $\mathcal{U}$, we may compute the action of the $\mathcal{S}$ and D_t operators on the coordinates, ϕ^i and ρ^i. Moreover, for $h \in \mathbb{H}_3$, it can be seen that

$$h(\lambda) \cdot \left(\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}, \begin{pmatrix} \rho^1 \\ \rho^2 \\ \rho^3 \end{pmatrix} \right) = \left(\begin{pmatrix} a\phi^1 + dc_1\phi^2 \\ b\phi^2 \\ c\phi^3 \end{pmatrix}, \begin{pmatrix} a\rho^1 + dc_2\rho^2 \\ b\rho^2 \\ c\rho^3 \end{pmatrix} \right).$$

Thus,

$$\begin{aligned} h\left(\frac{\phi^2}{\phi^3}\frac{\rho^3}{\rho^2}\right) &= \frac{b\phi^2}{c\phi^3}\frac{c\rho^3}{b\rho^2} \\ &= \frac{\phi^2}{\phi^3}\frac{\rho^3}{\rho^2}, \end{aligned}$$

providing $\frac{\phi^2}{\phi^3}\frac{\rho^3}{\rho^2}$ as an invariant of $\mathbb{H}_3$.

We consider the dense open subset of $\mathbb{CP}^2 \times \mathbb{CP}^2$ given by $\phi^3, \rho^3 \neq 0$. We take as local coordinates, $W_\phi^1 = \frac{\phi^1}{\phi^3}$, $W_\phi^2 = \frac{\phi^2}{\phi^3}$, $W_\rho^1 = \frac{\rho^1}{\rho^3}$ and $W_\rho^2 = \frac{\rho^2}{\rho^3}$. Hence, in terms of the local coordinates, the invariant is expressed as $\frac{W_\phi^2}{W_\rho^2}$. Thus, we set

$$z = \frac{W_\phi^2}{W_\rho^2}. \tag{7.27}$$

Then,

$$\begin{aligned} z_1 &= \mathcal{S}(z) \\ &= \frac{W_\rho^2 W_\phi^1}{W_\rho^1 W_\phi^2}, \\ z_2 &= \mathcal{S}(z_1) \\ &= \frac{W_\rho^1\left(c_1 W_\phi^1 + 1\right)}{W_\phi^1\left(c_2 W_\rho^1 + 1\right)}, \\ z_3 &= \mathcal{S}(z_2) \\ &= \frac{\left(c_2 W_\rho^1 + 1\right)\left(c_1 + c_1^2 W_\phi^1 + W_\phi^2\right)}{\left(c_1 W_\phi^1 + 1\right)\left(c_2 + c_2^2 W_\rho^1 + W_\rho^2\right)}, \\ \mathcal{S}(z_3) &= -\frac{\left(c_2 + c_2^2 W_\rho^1 + W_\rho^2\right)\left(W_\phi^1\left(c_1^3 - u\right) + c_1\left(c_1 + W_\phi^2\right)\right)}{\left(c_1 + c_1^2 W_\phi^1 + W_\phi^2\right)\left(W_\rho^1\left(u - c_2^3\right) - c_2\left(c_2 + W_\rho^2\right)\right)}. \end{aligned} \tag{7.28}$$

These relations may be inverted to obtain

$$\begin{aligned}
W^1_\phi &= \frac{1 - z z_1 z_2}{c_2 z_2 - c_1},\\
W^2_\phi &= \frac{z_2\left(c_1 z z_1 - c_2\right)\left(c_1 - c_2 z_3\right)}{\left(z_1 z_2 z_3 - 1\right)\left(c_1 - c_2 z_2\right)},\\
W^1_\rho &= -\frac{1 - z z_1 z_2}{z z_1\left(c_1 - c_2 z_2\right)},\\
W^2_\rho &= \frac{z_2\left(c_1 z z_1 - c_2\right)\left(c_1 - c_2 z_3\right)}{z\left(z_1 z_2 z_3 - 1\right)\left(c_1 - c_2 z_2\right)}.
\end{aligned}$$

Substituting into (7.28) and rearranging for u, we find after making the identification $z = v, z_1 = v_1, z_2 = v_2, z_3 = v_3$ and $\mathcal{S}(z_3) = v_4$ that

$$u = \frac{v_2 v_3\left(c_2 v_4 - c_1\right)\left(c_1 v v_1 - c_2\right)\left(c_1 v_1 v_2 - c_2\right)}{\left(v v_1 v_2 - 1\right)\left(v_1 v_2 v_3 - 1\right)\left(v_2 v_3 v_4 - 1\right)}.$$

Then,

$$\begin{aligned}
v_t &= z_t\\
&= \left(\frac{W^2_\phi}{W^2_\rho}\right)_t\\
&= \frac{(W^2_\phi)_t W^2_\rho - W^2_\phi (W^2_\rho)_t}{(W^2_\rho)^2}.
\end{aligned}$$

If one recalls the definitions of W^i_ϕ and W^i_ρ, the action of the derivative operator on $\mathbb{C}^3$ (7.22), and the action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2 \times \mathbb{CP}^2$, one finds

$$v_t = \frac{v\left(c_1 - c_2 v\right)\left(v_{-2} v_{-1} - v_1 v_2\right)\left(c_1 v v_{-1} - c_2\right)\left(c_1 v v_1 - c_2\right)}{\left(v v_{-2} v_{-1} - 1\right)\left(v v_{-1} v_1 - 1\right)\left(v v_1 v_2 - 1\right)}.$$

Thus, by considering the action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2 \times \mathbb{CP}^2$, we obtain the following result.

Theorem 7.3.3 *The equation*

$$v_t = \frac{v\left(c_1 - c_2 v\right)\left(v_{-2} v_{-1} - v_1 v_2\right)\left(c_1 v v_{-1} - c_2\right)\left(c_1 v v_1 - c_2\right)}{\left(v v_{-2} v_{-1} - 1\right)\left(v v_{-1} v_1 - 1\right)\left(v v_1 v_2 - 1\right)},$$

is linked via the MT

$$u = \frac{v_2 v_3 \left(c_2 v_4 - c_1\right)\left(c_1 v v_1 - c_2\right)\left(c_1 v_1 v_2 - c_2\right)}{\left(v v_1 v_2 - 1\right)\left(v_1 v_2 v_3 - 1\right)\left(v_2 v_3 v_4 - 1\right)},$$

to the equation

$$u_t = u(u_2 + u_1 - u_{-1} - u_{-2}).$$

The above MT and Miura related equation, with arbitrary c_1 and c_2, appear to be new. Particular cases corresponding to $c_2 = 0$ and $c_1 = c_2 \neq 0$, can be found in [72] and [71] respectively. The case $c_2 = 0$ is equivalent, up to scaling, to equation (17.8.24) in [72]. Whereas the case $c_1 = c_2 \neq 0$ is equivalent up, to scaling in t and a fractional linear transformation, to equation (3.4) in [71]. For instance, in the case $c_1 = c_2 = 1$, the fractional linear transformation is given by

$$t \to \frac{v}{1-v}.$$

7.3.2 Arbitrary p

For arbitrary p, as usual, the discrete part of the DLR has the form $\mathcal{S}(\mathbf{\Psi}) = M(u,\lambda)\mathbf{\Psi}$, where $M(u,\lambda)$, as derived from the operator Lax pair, takes the form

$$M(u,\lambda) = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathrm{I}_{p-1\times p-1} & \\ \vdots & & & \\ 0 & & & \\ \hline -u & 0 & \dots & \lambda \end{array}\right). \tag{7.29}$$

Conjugation by the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & \iddots & \iddots & 0 \\ -\lambda & 1 & 0 & 0 \end{pmatrix}, \tag{7.30}$$

results in $\tilde{M}(u,\lambda) = CM(u,\lambda)C^{-1}$, with

$$\tilde{M}(u,\lambda) = \left(\begin{array}{c|c|c} \lambda & \multirow{2}{*}{\Large 0} & 1 \\ -u & & 0 \\ \hline 0 & & 0 \\ \vdots & \mathrm{I}_{p-1\times p-1} & \vdots \\ 0 & & 0 \end{array}\right).$$

Computing $\tilde{M}(u,\lambda)\tilde{M}^{-1}(v,\lambda)$ yields

$$\begin{aligned} \tilde{M}(u,\lambda)\tilde{M}^{-1}(v,\lambda) &= \left(\begin{array}{c|c|c} \lambda & \multirow{2}{*}{\Large 0} & 1 \\ -u & & 0 \\ \hline 0 & & 0 \\ \vdots & \mathrm{I}_{p-1\times p-1} & \vdots \\ 0 & & 0 \end{array}\right) \left(\begin{array}{cc|cc} 0 & -\frac{1}{v} & 0 & \dots \\ \hline 0 & 0 & & \\ \vdots & \vdots & \mathrm{I}_{p-1\times p-1} & \\ 0 & 0 & & \\ \hline 1 & \frac{\lambda}{v} & 0 & \dots \end{array}\right) \\ &= \mathrm{Diag}(1, \frac{u}{v}, 1, \dots, 1). \end{aligned}$$

Thus, $\mathbb{H}_1$ consists of matrices of the form

$$\mathrm{Diag}(1, a, 1, \dots, 1),$$

for arbitrary $0 \neq a \in \mathbb{C}$. Now, consider the product $\tilde{M}(u,\lambda)D\tilde{M}(u,\lambda)^{-1}$, where D is a diagonal matrix. For $D = \mathrm{Diag}(d_{1,1}, d_{2,2}, \dots, d_{p+1,p+1})$, this yields

$$\tilde{M}(u,\lambda)D\tilde{M}(u,\lambda)^{-1} = \left(\begin{array}{cc|ccc} d_{p+1,p+1} & \frac{\lambda(d_{p+1,p+1}-d_{1,1})}{u} & & & \\ 0 & d_{1,1} & & & \\ \hline & & d_{2,2} & & \\ & & & \ddots & \\ & & & & d_{p,p} \end{array}\right),$$

where only nonzero blocks are displayed. Hence, conjugation by $\tilde{M}(u,\lambda)$ cyclically permutes the diagonal elements and adds an element in the (1,2) position, provided

$d_{p+1,p+1} - d_{1,1} \neq 0$. However, for elements of $\mathbb{H}_1$, $d_{1,1} = d_{p+1,p+1} = 1$, and thus conjugating elements of $\mathbb{H}_1$ by $\tilde{M}(u,\lambda)$ will return diagonal matrices of the form $\text{Diag}(1,1,a,1,\dots,1)$, which implies $\mathbb{H}_2 = \{\text{Diag}(1,a,b,1,\dots,1)|a,b \in \mathbb{C}\}$. Repeating this process one finds $\mathbb{H}_3 = \{\text{Diag}(1,a,b,c,1,\dots,1)|a,b,c \in \mathbb{C}\}$, and indeed

$$\mathbb{H}_{p-1} = \{\text{Diag}(1,a_{1,1},a_{2,2},\dots,a_{p-1,p-1},1)|a_{i,i} \in \mathbb{C}\}, \tag{7.31}$$

$$\mathbb{H}_p = \{\text{Diag}(1,a_{1,1},a_{2,2},\dots,a_{p,p})|a_{i,i} \in \mathbb{C}\}. \tag{7.32}$$

Set $G = \mathcal{G}(p+1,c_1,1) \cong \text{GL}_{p+1}(\mathbb{C})$. Since G acts transitively on $\mathbb{C}^{p+1} \setminus \{0\}$, we may replace G/H by $\mathbb{C}^{p+1} \setminus \{0\}$ in the general theory. Let $\phi^1, \phi^2, \dots, \phi^{p+1}$ be coordinates on $\mathbb{C}^{p+1} \setminus \{0\}$, then considering the standard action of $\text{GL}_{p+1}(\mathbb{C})$ on $\mathbb{C}^{p+1}$, we may take as an invariant of $\mathbb{H}_p$ the coordinate ϕ^1. Using $\tilde{M}(u,\lambda)$ and the map $\rho : \boldsymbol{G} \to G$ to compute shifts, it follows that

$$\begin{aligned}
z &= \phi^1,\\
z_1 &= \mathcal{S}(z) = c_1\phi^1 + \phi^{p+1} = c_1 z + \phi^{p+1},\\
z_2 &= \mathcal{S}(z_1) = c_1\mathcal{S}(z) + \mathcal{S}(\phi^{p+1}) = c_1 z_1 + \phi^p,\\
z_3 &= \mathcal{S}(z_2) = c_1\mathcal{S}(z_1) + \mathcal{S}(\phi^p) = c_1 z_2 + \phi^{p-1},\\
&\vdots\\
z_p &= \mathcal{S}(z_{p-1}) = c_1\mathcal{S}(z_{p-2}) + \mathcal{S}(\phi^3) = c_1 z_{p-1} + \phi^2,\\
\mathcal{S}(z_p) &= c_1\mathcal{S}(z_{p-1}) + \mathcal{S}(\phi^2) = z_p - u\phi^1,\\
&= c_1 z_p - uz.
\end{aligned}$$

Hence, setting $z_i = v_i$ and rearranging for u, the last equality becomes

$$u = \frac{c_1 v_p - v_{p+1}}{v}. \tag{7.33}$$

To find the corresponding differential difference equation for v, one must compute $v_t = z_t = \phi^1_t$. In order to do so, we make use of the time dependent part of the scalar lax pair.

Recall,

$$\chi_t = A\chi, \quad A = (L^{(p+1)})_{\geq 0}, \quad L = \mathcal{S} + u\mathcal{S}^{-p}.$$

When deriving $M(u,\lambda)$ from the scalar Lax pair, we identify $\mathcal{S}^{-p}\chi = \psi^1$. We have conjugated $M(u,\lambda)$ by C, which at the level of the DLR results in the relation $\boldsymbol{\varphi} = C\boldsymbol{\psi}$. However, due to the form of C, it follows that $\varphi^1 = \psi^1$, which implies that φ^1 is identified with $\mathcal{S}^{-p}\chi$. This in turn requires the association of $\mathcal{S}^{-p}\chi$ with the ϕ^1 coordinate. Hence,

$$\begin{aligned}
v_t = z_t = \phi^1_t & \\
&= (\mathcal{S}^{-p}\chi)_t \\
&= \mathcal{S}^{-p}\chi_t \\
&= \mathcal{S}^{-p}(A\chi) \\
&= \mathcal{S}^{-p}(\mathcal{S}^{p+1} + u + u_1 + u_2 + \ldots + u_p)\chi \\
&= \mathcal{S}^{p+1}(\mathcal{S}^{-p}\chi) + (u_{-p} + u_{-p+1} + \ldots + u)\mathcal{S}^{-p}\chi \\
&= \mathcal{S}^{p+1}z + \sum_{i=0}^{p} u_{-p+i}z \\
&= c_1 z_p - uz + \sum_{i=0}^{p} u_{-p+i}z \\
&= c_1 v_p + \sum_{i=0}^{p-1} u_{-p+i}v.
\end{aligned}$$

Replacing occurrences of u using (7.33), we arrive at the following theorem.

Theorem 7.3.4 *The equation*

$$v_t = \left[\sum_{i=0}^{p-1}\left(\frac{c_1 v_i - v_{i+1}}{v_{-p+i}}\right)\right] v + c_1 v_p,$$

is linked via the MT

$$u = \frac{c_1 v_p - v_{p+1}}{v},$$

to the equation

$$u_t = u\left(\sum_{i=1}^{p} u_i - \sum_{i=1}^{p} u_{-i}\right).$$

Let us now consider the action of $\mathrm{GL}_{p+1}(\mathbb{C})$ on $\mathbb{CP}^p$. One must find an invariant of $\mathbb{H}_{p-1}$ to construct an MT. From the form of (7.31), observe that we may take $z = \frac{\phi^{p+1}}{\phi^1} = W^p$ as an invariant. Setting $z = W^p$ and using the notation $W^i = \frac{\phi^{i+1}}{\phi^1}$, one finds

$$
\begin{aligned}
z_1 &= \mathcal{S}(z) = \frac{W^{p-1}}{(c_1 + z)},\\
z_2 &= \mathcal{S}(z_1) = \frac{W^{p-2}}{(c_1 + z)(c_1 + z_1)},\\
z_3 &= \mathcal{S}(z_2) = \frac{W^{p-3}}{(c_1 + z)(c_1 + z_1)(c_1 + z_2)},\\
&\vdots\\
z_{p-1} &= \mathcal{S}(z_{p-2}) = \frac{W^1}{(c_1 + z)(c_1 + z_1)\dots(c_1 + z_{p-2})},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathcal{S}(z_{p-1}) &= \frac{\mathcal{S}(W^1)}{(c_1 + z_1)(c_1 + z_2)\dots(c_1 + z_{p-1})},\\
&= \frac{-u}{(c_1 + z)(c_1 + z_1)\dots(c_1 + z_{p-1})}.
\end{aligned}
$$

Rearranging, we find

$$
u = -\mathcal{S}(z_{p-1})\prod_{i=0}^{p-1}(c_1 + z_i).
$$

Thus, identifying $z_i = v_i$, we arrive at the MT

$$
u = -v_p \prod_{i=0}^{p-1}(c_1 + v_i).
$$

To derive the corresponding differential difference equation for v, we must compute $v_t = z_t$. Doing so,

$$
\begin{aligned}
v_t &= \left(\frac{\phi^{p+1}}{\phi^1}\right)_t\\
&= \frac{\phi^{p+1}_t}{\phi^1} - \frac{\phi^1_t}{\phi^1}\frac{\phi^{p+1}}{\phi^1}\\
&= \frac{\phi^{p+1}_t}{\phi^1} - \frac{\phi^1_t}{\phi^1}z.
\end{aligned} \tag{7.34}
$$

In terms of the ψ^i coordinates, we find $\phi^1 = \psi^1$ and $\phi^{p+1} = -c_1\psi^1 + \psi^2$. Recall, $\psi^1 = \mathcal{S}^{-p}\chi$ and $\psi^2 = \mathcal{S}(\psi^1)$. Hence,

$$\begin{aligned}\psi^1_t &= \mathcal{S}^{-p}\chi_t \\ &= \mathcal{S}^{-p}[(\mathcal{S}^{p+1} + u + u_1 + u_2 + \ldots + u_p)]\chi \\ &= \mathcal{S}(\psi^{p+1}) + \sum_{i=0}^{p} u_{-p+i}\psi^1 \\ &= -u\psi^1 + c_1\psi^{p+1} + \sum_{i=0}^{p} u_{-p+i}\psi^1 \\ &= c_1\psi^{p+1} + \sum_{i=0}^{p-1} u_{-p+i}\psi^1,\end{aligned}$$

whereas,

$$\begin{aligned}\psi^2_t &= \mathcal{S}(\psi^1_t) \\ &= -c_1u\psi^1 + c_1^2\psi^{p+1} + \sum_{i=0}^{p-1} u_{-p+i+1}\psi^2.\end{aligned}$$

Now, rewriting (7.34) in terms of the ψ^i coordinates leads to

$$v_t = \frac{-(c_1+z)\psi^1_t + \psi^2_t}{\psi^1}.$$

Using the above expressions for ψ^1_t and ψ^2_t, we arrive after some simplification to

$$v_t = -zc_1\frac{\psi^{p+1}}{\psi^1} - (c_1+z)\sum_{i=0}^{p-1} u_{-p+i} - c_1u + \sum_{i=0}^{p-1} u_{-p+i+1}\frac{\psi^2}{\psi^1}.$$

Making use of

$$\frac{\psi^2}{\psi^1} = \frac{c_1\phi^1 + \phi^{p+1}}{\phi^1} = c_1 + z$$

and

$$\frac{\psi^{p+1}}{\psi^1} = \frac{c_1^p\phi^1 + \sum_{i=0}^{p-1} c_1^i\phi^{i+2}}{\phi^1} = c_1^p + \sum_{i=0}^{p-1} c_1^i W^{i+1},$$

we find after some simplification that

$$v_t = -zc_1^{p+1} - z\sum_{i=0}^{p-1} c_1^{i+1}W^{i+1} - c_1u + (c_1+z)(u - u_{-p}). \qquad (7.35)$$

One can express the W^i coordinates in terms of the z_i coordinates using

$$W^i = z_{p-i}\prod_{i=1}^{p-i}(c_1 + z_{i-1}).$$

Eliminating the W^i in (7.35) using the above relations, as well as replacing occurrences of u using the MT, we find, after relabelling $z_i = v_i$, the following expression

$$v_t = v\left[-c_1^{p+1} - \sum_{i=0}^{p-1} c_1^{i+1} v_{p-i-1} \prod_{j=1}^{p-1-i}(c_1 + v_{j-1}) + \prod_{i=0}^{p}(c_1 + v_{i-p}) - v_p\prod_{i=0}^{p-1}(c_1 + v_i)\right].$$

Finally, this can be simplified to

$$v_t = v\,(c_1 + v)\left[\prod_{i=1}^{p}(c_1 + v_{-i}) - \prod_{i=1}^{p}(c_1 + v_i)\right].$$

Their equivalence can be confirmed by expanding the products in terms of elementary symmetric polynomials. Hence we arrive at the theorem.

Theorem 7.3.5 *The equation*

$$v_t = v\,(c_1 + v)\left[\prod_{i=1}^{p}(c_1 + v_{-i}) - \prod_{i=1}^{p}(c_1 + v_i)\right],$$

is linked via the MT

$$u = -v_p\prod_{i=0}^{p-1}(c_1 + v_i)\,,$$

to the equation

$$u_t = u\left(\sum_{i=1}^{p} u_i - \sum_{i=1}^{p} u_{-i}\right).$$

The MTs and systems found in Theorem 7.3.4 and Theorem 7.3.5 are generalisations of those found in Theorem 7.3.1 and Theorem 7.3.2 respectively. In order to generalise Theorem 7.3.3, one needs to consider the action of $\mathrm{GL}_{p+1}(\mathbb{C})$ on $\mathbb{CP}_p \times \mathbb{CP}_p$. It then becomes necessary to find an invariant of $\mathbb{H}_{2p-1}$. From the structure of the $\mathbb{H}_k$ subgroups, one finds that such an invariant is given by the function

$$z = \frac{W_\phi^p}{W_\rho^p},$$

where $W_\phi^p = \frac{\phi^p}{\phi^{p+1}}$ and $W_\rho^p = \frac{\rho^p}{\rho^{p+1}}$, with ϕ^i and ρ^i homogeneous coordinates on the two patches of $\mathbb{CP}^p$. Using the relations

$$\mathcal{S}(W_\phi^1) = \frac{c_1 W_\phi^1 + 1}{W_\phi^p}, \quad \mathcal{S}(W_\phi^2) = -u\frac{W_\phi^1}{W_\phi^p}, \quad \mathcal{S}(W_\phi^a) = \frac{W_\phi^{a-1}}{W_\phi^p}, \quad a = 3, \dots p,$$

as well as the corresponding relations for W_ρ, it is possible to show that z_i does not depend on u, for $i = 0, \dots, 2p-1$. Indeed, for $a = 0, \dots p-1$,

$$\mathcal{S}^a(z) = \frac{W_\phi^{p-a}}{W_\rho^{p-a}} \frac{1}{\prod_{i=0}^{a-1} z_i} \tag{7.36}$$

and

$$\mathcal{S}^{p+a}(z) = \frac{c_1^{a+1} W_\phi^1 + c_1^a + \sum_{i=1}^a c_1^{a-i} W_\phi^{p-i+1}}{c_2^{a+1} W_\rho^1 + c_2^a + \sum_{i=1}^a c_2^{a-i} W_\rho^{p-i+1}} \frac{1}{\prod_{i=0}^{p+a-1} z_i}. \tag{7.37}$$

From the form of (7.36), it is clear that one can express the W_ϕ^i coordinates in terms of the z_i and W_ρ^i coordinates. Then, using equations (7.37), the W_ρ^i coordinates can be expressed solely in terms of the z_i coordinates. Taking a shift of $\mathcal{S}^{2p-1}(z)$ introduces u, due to its dependence on the terms W_ϕ^2 and W_ρ^2, which occur in the summations. Rearranging for u, we arrive at

$$u = \frac{A - BC}{W_\phi^1 - CW_\rho^1}, \tag{7.38}$$

with

$$A = \sum_{i=0}^{p-2} c_1^{-i+p-1} W_\phi^{p-i} + c_1^p (c_1 W_\phi^1 + 1),$$

$$B = \sum_{i=0}^{p-2} c_2^{-i+p-1} W_\rho^{p-i} + c_2^p (c_2 W_\rho^1 + 1),$$

and

$$C = z_{2p} \left(\prod_{i=0}^{2p-1} z_i \right).$$

Following the procedure for $p = 3$ provides the MT

$$u = \frac{v_3 v_4 v_5 (c_2 v_6 - c_1)(c_1 v v_1 v_2 - c_2)(c_1 v_1 v_2 v_3 - c_2)(c_1 v_2 v_3 v_4 - c_2)}{(v v_1 v_2 v_3 - 1)(v_1 v_2 v_3 v_4 - 1)(v_2 v_3 v_4 v_5 - 1)(v_3 v_4 v_5 v_6 - 1)}.$$

For $p = 4$

$$u = \frac{v_4 v_5 v_6 v_7 (c_2 v_8 - c_1)(c_1 v v_1 v_2 v_3 - c_2)(c_1 v_1 v_2 v_3 v_4 - c_2)(c_1 v_2 v_3 v_4 v_5 - c_2)(c_1 v_3 v_4 v_5 v_6 - c_2)}{(v v_1 v_2 v_3 v_4 - 1)(v_1 v_2 v_3 v_4 v_5 - 1)(v_2 v_3 v_4 v_5 v_6 - 1)(v_3 v_4 v_5 v_6 v_7 - 1)(v_4 v_5 v_6 v_7 v_8 - 1)}.$$

The Miura related systems are found to be

$$v_t = \frac{v\left(c_1 - c_2 v\right)\left(v_{-3}v_{-2}v_{-1} - v_1 v_2 v_3\right)\left(c_1 v v_{-2} v_{-1} - c_2\right)\left(c_1 v v_{-1} v_1 - c_2\right)\left(c_1 v v_1 v_2 - c_2\right)}{\left(v v_{-3} v_{-2} v_{-1} - 1\right)\left(v v_{-2} v_{-1} v_1 - 1\right)\left(v v_{-1} v_1 v_2 - 1\right)\left(v v_1 v_2 v_3 - 1\right)}$$

and

$$\begin{aligned} v_t =& \frac{v\left(c_1 - c_2 v\right)\left(v_{-4}v_{-3}v_{-2}v_{-1} - v_1 v_2 v_3 v_4\right)\left(c_1 v v_{-3} v_{-2} v_{-1} - c_2\right)}{\left(c_1 v v_{-2} v_{-1} v_1 - c_2\right)\left(v v_{-4} v_{-3} v_{-2} v_{-1} - 1\right)\left(v v_{-3} v_{-2} v_{-1} v_1 - 1\right)\left(v v_{-2} v_{-1} v_1 v_2 - 1\right)} \\ &\times \frac{\left(c_1 v v_{-1} v_1 v_2 - c_2\right)\left(c_1 v v_1 v_2 v_3 - c_2\right)}{\left(v v_{-1} v_1 v_2 v_3 - 1\right)\left(v v_1 v_2 v_3 v_4 - 1\right)}, \end{aligned}$$

respectively.

Hence, Theorem 7.3.3 generalises as follows.

Theorem 7.3.6 *The equation*

$$v_t = \frac{v\left(c_1 - c_2 v\right)\left(\prod_{i=1}^{p} v_{-i} - \prod_{i=1}^{p} v_i\right)\left(\prod_{i=0}^{p-1}\left(c_1\left(\prod_{j=i-p+1}^{i} v_j\right) - c_2\right)\right)}{\prod_{i=0}^{p}\left(\prod_{j=i-p}^{i} v_j - 1\right)},$$

is linked via the MT

$$u = \frac{\left(\prod_{j=p}^{2p-1} v_j\right)\left(c_2 v_{2p} - c_1\right)\left(\prod_{i=0}^{p-1}\left(c_1\left(\prod_{j=i}^{i+p-1} v_j\right) - c_2\right)\right)}{\prod_{i=0}^{p}\left(\prod_{j=i}^{i+p} v_j - 1\right)}, \tag{7.39}$$

to the equation

$$u_t = u\left(\sum_{i=1}^{p} u_i - \sum_{i=1}^{p} u_{-i}\right).$$

This MT and Miura related system, for arbitrary c_1 and c_2, appear to be new. Having been shown these formulae, Yu. B. Suris suggested that this can be rewritten in a more symmetric form as follows. The equation

$$v_t = v(\alpha + \beta v)\left(\prod_{i=1}^{p} v_i - \prod_{i=1}^{p} v_{-i}\right)\frac{\prod_{j=1}^{p}(1 + \alpha \prod_{i=1}^{p} v_{j-i})}{\prod_{j=0}^{p}(1 - \beta \prod_{i=0}^{p} v_{j-i})}, \tag{7.40}$$

is linked via the MT

$$w = v\frac{1 + \alpha \prod_{i=1}^{p} v_{-i}}{1 - \beta \prod_{i=0}^{p} v_{-i}}, \tag{7.41}$$

to the equation

$$w_t = w(\alpha + \beta w)(\prod_{i=1}^{p} w_i - \prod_{i=1}^{p} w_{-i}).$$

The latter equation is linked to the Narita-Itoh-Bogoyalensky lattice via the MT

$$u = w(\alpha + \beta w_p)\prod_{i=0}^{p-1} w_i.$$

Equation (7.39) can be obtained, after suitable scaling, by composing the above two MTs. For arbitrary α and β, (7.40) and (7.41) do not appear to have been known.

7.4 Discrete Sawada-Kotera

In this section we construct MTs for the equation

$$u_t = u^2(u_1u_2 - u_{-1}u_{-2}) - u(u_1 - u_{-1}), \tag{7.42}$$

which can be viewed as a discretisation of the fully continuous Sawada-Kotera PDE. A curious feature of this equation is that it is a linear combination of two integrable equations, a modified Narita-Itoh-Bogoyavlensky lattice equivalent to (7.26), and the Volterra equation. These equations belong to different hierarchies, and thus one would not expect their linear combination to remain integrable. This fact is commented upon in [8], where an operator Lax pair is also presented. The authors consider Lax equations of the form

$$P\chi = \lambda Q\chi, \;\; \chi_t = A_k\chi, \tag{7.43}$$

where

$$P = (u\mathcal{S}^m + 1)\mathcal{S}^l, \;\; Q = \mathcal{S}^m + u.$$

They show that for fixed l and m, there exists a sequence of difference operators A_k of increasing order, which define a commutative hierarchy of integrable DΔEs. Equation (7.42) corresponds to the case $m = 2$ and $l = 1$, with

$$A_1 = u_{-1}u_{-2}(\mathcal{S}^{-2} - \mathcal{S}^2).$$

In order to construct MTs, we first derive a DLR.

Writing out explicitly the discrete part of scalar Lax pair leads to

$$(u\mathcal{S}^2+1)\mathcal{S}\chi = \lambda(\mathcal{S}^2+u)\chi,$$

which implies

$$u\chi_3+\chi_1 = \lambda\chi_2+\lambda u\chi,$$

where $\chi_i = \mathcal{S}^i(\chi)$, and thus

$$\chi_3 = \lambda\chi - \frac{1}{u}\chi_1 + \frac{\lambda}{u}\chi_2.$$

Hence, making the identification $\chi = \psi^1$, $\chi_1 = \psi^2$ and $\chi_2 = \psi^3$, we may recast the discrete part of the operator Lax pair in matrix form as follows

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \lambda & -\frac{1}{u} & \frac{\lambda}{u}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}. \tag{7.44}$$

Using equation $\chi_t = A\chi$, and shifting as appropriate, we find the continuous part to be given by

$$D_t\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda^2}-u_{-1} & \frac{u_{-2}u_{-1}-1}{\lambda} & \left(\frac{1}{\lambda^2}-u_{-2}\right)u_{-1}\\ \left(\frac{1}{\lambda}-\lambda u_{-1}\right)u & u_{-1}-u & \frac{u_{-1}\left(u-\lambda^2\right)}{\lambda}\\ u\left(u_1-\lambda^2\right) & \lambda-\lambda u u_1 & u-\lambda^2\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}. \tag{7.45}$$

As required, the discrete part of the Lax pair has been rewritten in the form $\mathcal{S}(\mathbf{\Psi}) = M(u,\lambda)\mathbf{\Psi}$, with

$$M(u,\lambda) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \lambda & -\frac{1}{u} & \frac{\lambda}{u}\end{pmatrix}. \tag{7.46}$$

We proceed to compute the subgroups $\mathbb{H}_1$ and $\mathbb{H}_2$. Performing the multiplication $M(u,\lambda)M(v,\lambda)^{-1}$ provides

$$M(u,\lambda)M(v,\lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{v} - \frac{1}{u} & \left(\frac{1}{u} - \frac{1}{v}\right)\lambda & 1 \end{pmatrix}.$$

Hence, $\mathbb{H}_1$ has the following form

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & -a\lambda & 1 \end{pmatrix} \mid a \in \mathbb{C} \right\}.$$

Computing $M(u,\lambda)h_1M(u,\lambda)^{-1}$, for $h_1 \in \mathbb{H}_1$, produces

$$\begin{pmatrix} 1 & 0 & 0 \\ a\left(\frac{1}{u\lambda} - \lambda\right) & 1 - \frac{a}{u} & \frac{a}{\lambda} \\ \frac{a - au\lambda^2}{u^2} & -\frac{a\lambda}{u^2} & \frac{a+u}{u} \end{pmatrix}.$$

Hence, elements of the form $M(u,\lambda)\mathbb{H}_1M(u,\lambda)^{-1}$ are described by matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{b}{\lambda} - a\lambda & 1 - b & \frac{a}{\lambda} \\ c - d\lambda^2 & -c\lambda & d + 1 \end{pmatrix}.$$

The set of such matrices is not closed under multiplication and thus does not form a group. However, it is clear that the set

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \frac{a_1}{\lambda} + \lambda a_2 & a_3 & \frac{a_4}{\lambda} \\ a_6\lambda^2 + a_5 & a_7\lambda & a_8 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \mid a_i \in \mathbb{C} \right\} \tag{7.47}$$

contains such elements, as well as all elements of $\mathbb{H}_1$. One can show that S is a group, therefore we deduce $\mathbb{H}_2$ is a subgroup of S. Whether $\mathbb{H}_2$ is equal to S is not particularly important for our purposes. What is important is whether we can find an invariant of $\mathbb{H}_2$. In the next section we show that this is indeed possible, and use such an invariant to construct a MT.

Standard Action of $\mathrm{GL}_3(\mathbb{C})$ **on** $\mathbb{C}^3 \setminus \{0\}$

As in the case of the Narita-Itoh-Bogoyavlensky lattice, we set $d = 3$, $m = 1$ and $k_1 = 1$ with $c_1 \in \mathbb{C}$. Then, we may find H such that $G/H \cong \mathbb{C}^3 \setminus \{0\}$. As before, we take ϕ^1, ϕ^2 and ϕ^3 as coordinates on $\mathbb{C}^3 \setminus \{0\}$. As $\dim(G/H) = 3$, by the general theory we must find a function of $\{\phi^1, \phi^2, \phi^3\}$ invariant with respect to the action of $\mathbb{H}_2$. The action of $\mathbb{H}_2$ on $\mathbb{C}^3 \setminus \{0\}$ is defined via the standard action of left multiplication of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{C}^3$, where first we use the map $\rho : \boldsymbol{G} \to \mathrm{GL}_3(\mathbb{C})$ to obtain an element of $\mathrm{GL}_3(\mathbb{C})$. From the form of (7.47), we see that one can take ϕ^1 as the invariant. Thus, set $z = \phi^1$. Then,

$$\begin{aligned} z_1 &= \mathcal{S}(z) \\ &= \phi^2, \\ z_2 &= \mathcal{S}(z_1) \\ &= \phi^3, \\ \mathcal{S}(z_2) &= \frac{c_1\psi^3 + c_1 u\psi^1 - \psi^2}{u} \\ &= \frac{c_1 z_2 + c_1 u z - z_1}{u}. \end{aligned} \tag{7.48}$$

Relabelling $z_i = v_i$, we find (7.48) provides, after some rearrangement, the MT

$$u = \frac{v_1 - c_1 v_2}{c_1 v - v_3}.$$

To obtain the corresponding v equation, we compute $v_t = z_t = \phi^1_t$.

$$\begin{aligned} v_t &= \phi^1_t \\ &= \frac{c_1^2\left(-u_{-1}\right)\left(u_{-2}v_2 + v\right) + c_1\left(u_{-2}u_{-1} - 1\right)v_1 + u_{-1}v_2 + v}{c_1^2}. \end{aligned}$$

Above, we have used (7.45) and the map ρ to determine the action of D_t on the coordinates of $\mathbb{C} \setminus \{0\}$. Finally, using the MT to eliminate occurrences of u, we arrive at the equation

$$v_t = \frac{\left(v_{-2} - v_2\right)\left(v_{-1} - c_1 v\right)\left(v - c_1 v_1\right)}{\left(c_1 v_{-2} - v_1\right)\left(c_1 v_{-1} - v_2\right)}. \tag{7.49}$$

It is noted in [8] that a system related to (7.42) via an MT possesses a discrete symmetry, one can directly check that the solution set of the above equation possesses the same symmetry. It is invariant with respect to the transformation $v \to 1/v$, $c_1 \to 1/c_1$. Thus we obtain a second MT by composition with this discrete symmetry. Hence we have the following theorem.

Theorem 7.4.1 *The equation*

$$v_t = \frac{(v_{-2} - v_2)(v_{-1} - c_1 v)(v - c_1 v_1)}{(v_1 - c_1 v_{-2})(v_2 - c_1 v_{-1})}, \tag{7.50}$$

is linked via the MT

$$u = \frac{v_1 - c_1 v_2}{c_1 v - v_3}, \quad u = \frac{v v_3 (v_1 - c_1 v_2)}{v_1 v_2 (c_1 v - v_3)},$$

to the equation

$$u_t = u^2(u_1 u_2 - u_{-1} u_{-2}) - u(u_1 - u_{-1}). \tag{7.51}$$

Above, two MTs link (7.50) with (7.51), hence implicitly we obtain an auto-Bäcklund transformation for (7.42). To see this, let $u = M_1(\{v_i\})$ and $u = M_2(\{v_i\})$ be two MTs linking the equation $E_1 : v_t = A(\{v_i, c\})$ with $E_2 : u_t = B(\{u_i\})$. Then, given a solution v of E_1, we may obtain via the MTs two solutions of E_2. However, given a solution u^0 of E_2, we can consider its preimage $v^0 = M_1^{-1}(u, c_0)$. Subsequently, making use of M_2, we obtain a second solution $u^1 = M_2(M_1^{-1}(u^0), c_0)$ of the equation for u. Of course, in practice, computing v^0 involves solving an ordinary difference equation, as well as an ordinary differential equation, which may be difficult. However, in some cases this is possible. We give such an example to demonstrate the usefulness of MTs with regard to solution generation.

It is easy to verify that the constant solution, $u = \alpha$, satisfies (7.42). In order to obtain the preimage of α under the first MT, we must solve the equation

$$\alpha v_3 - c_1 v_2 + v_1 - c_1 \alpha v = 0.$$

For α non zero, this is a 3rd order linear difference equation with constant coefficients. As such, it has general solution

$$v = a(r^{(1)})^n + b(r^{(2)})^n + c(r^{(3)})^n,$$

where a, b and c are constants with respect to shifts, while $r^{(1)}$, $r^{(2)}$, $r^{(3)}$ are roots of the polynomial

$$\alpha x^3 - c_1 x^2 + x - c_1 \alpha = 0.$$

The roots $r^{(i)}$, $i = 1, 2, 3$, depend explicitly upon the initial u solution, α, as well as the Miura parameter, c_1. To ease computations and aid in exposition, we take $\alpha = 2/3$ and $c_1 = i$ the imaginary unit. In which case,

$$v = a\left(\frac{i}{2}\right)^n + b(2i)^n + c(-i)^n.$$

Substituting into (7.50), with $c_1 = i$, the resulting expression splits into a system of 3 equations. These allow one to determine the t-dependence of a, b and c. We arrive at a solution of (7.50), for $\lambda = i$, given by

$$v = A\left(\frac{i}{2}\right)^n e^{-\frac{5t}{3}} + B(2i)^n e^{\frac{5t}{3}} + C(-i)^n,$$

for constant A, B and C. Having obtained a solution of the equation for v, we may take advantage of the second MT with $\lambda = i$ to construct a second solution of (7.42). Doing so, we obtain

$$u = \frac{2\left(A + 4^n e^{\frac{10t}{3}} B + (-2)^n e^{\frac{5t}{3}} C\right)\left(A + 4^{n+3} e^{\frac{10t}{3}} B + (-2)^{n+3} e^{\frac{5t}{3}} C\right)}{3\left(A + 4^{n+1} e^{\frac{10t}{3}} B + (-2)^{n+1} e^{\frac{5t}{3}} C\right)\left(A + 4^{n+2} e^{\frac{10t}{3}} B + (-2)^{n+2} e^{\frac{5t}{3}} C\right)}.$$

Hence, starting from a trivial solution $u = 2/3$ we obtain a highly non trivial solution.

Action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2$

Setting $G = \mathcal{G}(3, c_1, 1)$ and making use of the transitive action of $\mathrm{GL}(\mathbb{C})$ on $\mathbb{CP}^2$, we obtain another MT. As $\mathbb{CP}^2$ has dimension 2, following the general theory we must find an invariant of $\mathbb{H}_1$. As usual, we consider the coordinates $\{\phi^1, \phi^2, \phi^3\}$ on $\mathbb{C}^3 \setminus \{0\}$ to be homogeneous coordinates on $\mathbb{CP}^2$. Elements of the subgroup $\mathbb{H}_1$ leave ϕ^1 and ϕ^2 unaffected, hence the local coordinate $W^1 = \frac{\phi^2}{\phi^1}$ on $\mathbb{CP}^2$ is an invariant of $\mathbb{H}_1$. Therefore, set $z = W^1$. Then, $z_1 = \mathcal{S}(z) = \frac{\phi^3}{\phi^2} = W^2/W^1$, where W^2 is the local coordinate ϕ^3/ϕ^1. Shifting z_1, we find

$$\begin{aligned}\mathcal{S}(z_1) &= \frac{c_1\,(u + W_2) - W_1}{uW_2} \\ &= \frac{c_1 u + z\,(c_1 z_1 - 1)}{uzz_1}.\end{aligned}$$

Relabelling $v = z$, $v_1 = z_1$ and rearranging, one is lead to the MT

$$u = \frac{v(1 - c_1 v_1)}{c_1 - v v_1 v_2}.$$

To obtain the corresponding equation for v, one must compute $(z)_t = (W^1)_t = (\frac{\phi^2}{\phi^1})_t$. As in the previous case, the Miura related equation is invariant under the transformation $v \to \frac{1}{v}$, $c_1 \to 1/c_1$, thus, we automatically obtain a second MT

$$u = \frac{v_2(1 - c_1 v_1)}{c_1 - v v_1 v_2}.$$

Theorem 7.4.2 *The equation*

$$v_t = \frac{v\,(c_1 v - 1)}{(c_1 - v_{-1} v v_1)} \left(\frac{v\,(v_{-2} v_{-1} - v_1 v_2)\,(c_1 v_{-1} - 1)\,(c_1 v_1 - 1)}{(c_1 - v_{-2} v_{-1} v)\,(c_1 - v v_1 v_2)} + v_1 - v_{-1} \right),$$

is linked via the MT

$$u = \frac{(1 - c_1 v_1) v}{c_1 - v v_1 v_2}, \quad u = \frac{(1 - c_1 v_1) v_2}{c_1 - v v_1 v_2},$$

to the equation

$$u_t = u^2 (u_1 u_2 - u_{-1} u_{-2}) - u(u_1 - u_{-1}).$$

This MT is also found in [8].

Chapter 8

Partial difference equations

The construction described in previous sections has been wholly framed in the context of DΔEs. However, if one examines the procedure carefully, it is seen that the continuous part of the DLR is not required until one wishes to derive the Miura related system. Moreover, at this stage, the only requirement is that $D_t(z)$ is a function on G/H, ie is H-right invariant. Given a fully discrete system with Darboux pair of the form

$$\mathcal{S}(\mathbf{\Psi}) = M(u, \lambda)\mathbf{\Psi}, \quad \mathcal{T}(\mathbf{\Psi}) = N(\{u_i\}, \lambda)\mathbf{\Psi},$$

one may use N to define the action of $\mathcal{T}$ on G, just as M is used to define the action of $\mathcal{S}$ on G. One can then extend $\mathcal{T}$ to functions on G, and subsequently to functions on G/H. Following the same reasoning as in the $\mathcal{S}$ case, one can show that if z is H-right invariant, then $\mathcal{T}(z)$ is also H-right invariant. Hence, after deriving the MT, provided a degeneracy has not occurred, we may derive a fully discrete Miura related system.

In the fully discrete case, the restriction on the form of M becomes more apparent. The fact that M cannot depend on shifts of u, causes many well known discrete systems to be unsuitable for the method, at least not with their standard Darboux pairs. That said, when an equation does possess a Darboux pair of the correct form, the method can still be effective. We demonstrate this with the following example.

8.0.1 Levi-Yamilov equation

The P∆E given by

$$(u_{0,0}-1)(u_{1,0}+1)-(u_{0,1}+1)(u_{1,1}-1)=0, \tag{8.1}$$

is known as the Levi-Yamilov equation, it has Darboux pair

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}-\frac{1}{\lambda} & u_{0,0}\\ -u_{0,0} & \lambda\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$
$$\mathcal{T}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}\lambda w_{0,0}-\frac{1}{\lambda} & -(w_{0,0}+1)\\ w_{0,0}+1 & \lambda-\frac{w_{0,0}}{\lambda}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$

where

$$w_{n,m}=\frac{u_{n,m}+1}{u_{n,m+1}-1}.$$

One can find this equation and Darboux pair in [43], where (8.1) is presented as an example of an integrable discrete system which is not consistent around the cube. Its construction follows from considering MTs of the Volterra equation.

As usual, we take

$$M(u,\lambda)=\begin{pmatrix}-\frac{1}{\lambda} & u\\ -u & \lambda\end{pmatrix},$$

where we have dropped reference to the shift subscripts. Then, computing $M(u,\lambda)M(v,\lambda)^{-1}$ yields

$$M(u,\lambda)M(v,\lambda)^{-1}=\begin{pmatrix}\frac{uv-1}{v^2-1} & \frac{u-v}{\lambda-v^2\lambda}\\ \frac{(v-u)\lambda}{v^2-1} & \frac{uv-1}{v^2-1}\end{pmatrix}.$$

In order to simplify the above form we conjugate by the matrix

$$C=\begin{pmatrix}-\lambda & 1\\ \lambda & 1\end{pmatrix}.$$

Doing so results in

$$\begin{pmatrix} \frac{u-1}{v-1} & 0 \\ 0 & \frac{u+1}{v+1} \end{pmatrix}.$$

Hence, we ascertain $\mathbb{H}_1$ takes the form

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}, ab \neq 0 \right\},$$

for the equivalent conjugated Darboux pair

$$\mathcal{S} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} -\frac{\left(\lambda^2-1\right)(u_{0,0}-1)}{2\lambda} & -\frac{\left(\lambda^2+1\right)(u_{0,0}-1)}{2\lambda} \\ \frac{\left(\lambda^2+1\right)(u_{0,0}+1)}{2\lambda} & \frac{\left(\lambda^2-1\right)(u_{0,0}+1)}{2\lambda} \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \tag{8.2a}$$

$$\mathcal{T} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} \frac{\left(\lambda^2-1\right)(u_{0,0}+u_{0,1})}{\lambda(u_{0,1}-1)} & \lambda + \frac{1}{\lambda} \\ -\frac{\left(\lambda^2+1\right)(u_{0,0}+1)}{\lambda(u_{0,1}-1)} & 0 \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \tag{8.2b}$$

where $\boldsymbol{\varphi} = C\boldsymbol{\psi}$.

Action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{CP}^1$

Let $G = \mathcal{G}(2, c_1, 1) \cong \mathrm{GL}_2(\mathbb{C})$ and take homogeneous coordinates ϕ^1, ϕ^2 on $\mathbb{CP}^1$. The space $\mathbb{CP}^1$ is 1-dimensional, thus we must find an invariant of H_0, the trivial group. Hence, one may take $z = W^1 = \frac{\phi^1}{\phi^2}$. Then,

$$\begin{aligned} \mathcal{S}(z) &= \frac{\mathcal{S}(\phi^1)}{\mathcal{S}(\phi^2)} \\ &= -\frac{(u-1)\left(\left(c_1^2-1\right)\phi^1 + \left(c_1^2+1\right)\phi^2\right)}{(u+1)\left(\left(c_1^2+1\right)\phi^1 + \left(c_1^2-1\right)\phi^2\right)} \\ &= -\frac{(u-1)\left(c_1^2 + \left(c_1^2-1\right)z + 1\right)}{(u+1)\left(c_1^2 + \left(c_1^2+1\right)z - 1\right)}. \end{aligned}$$

Rearranging for u, relabelling $z = v$ and $\mathcal{S}(z) = v_1$, we obtain

$$u_{0,0} = \frac{c_1^2\left(v_{0,0}+1\right) - v_{1,0}\left(c_1^2\left(v_{0,0}+1\right) + v_{0,0} - 1\right) - v_{0,0} + 1}{c_1^2\left(v_{0,0}+1\right) + v_{1,0}\left(c_1^2\left(v_{0,0}+1\right) + v_{0,0} - 1\right) - v_{0,0} + 1}. \tag{8.3}$$

To find the Miura related system we compute $\mathcal{T}(z)$ using the $\mathcal{T}$ part of the Darboux pair.

$$
\begin{aligned}
\mathcal{T}(z) &= \frac{\mathcal{T}(\phi^1)}{\mathcal{T}(\phi^2)} \\
&= \frac{u_{0,0}\left(\phi_1 - c_1^2\phi_1\right) - u_{0,1}\left(c_1^2\phi_2 + \left(c_1^2 - 1\right)\phi_1 + \phi_2\right) + \left(c_1^2 + 1\right)\phi_2}{\left(c_1^2 + 1\right)\phi_1\left(u_{0,0} + 1\right)} \\
&= \frac{u_{0,0}\left(z - c_1^2 z\right) - u_{0,1}\left(c_1^2(z+1) - z + 1\right) + c_1^2 + 1}{\left(c_1^2 + 1\right) z \left(u_{0,0} + 1\right)}.
\end{aligned}
$$

Using the MT to replace occurrences of $u_{n,m}$, and relabelling $\mathcal{T}(z) = v_{0,1}$, $\mathcal{S}^i(z) = v_{i,0}$, we derive a system

$$E_1 E_2 = 0,$$

with E_1 the trivial equation

$$E_1 = v_{0,1} - \frac{1 - c_1^2}{1 + c_1^2}$$

and

$$E_2 = (1 + c_1^2)\left(v_{0,0} - v_{1,1} + v_{0,0}v_{1,1}(v_{0,1} - v_{1,0})\right) + (1 - c_1^2)(v_{1,0}v_{1,1} - v_{0,0}v_{0,1}).$$

One can check directly that (8.3) provides a MT from $E_2 = 0$ to (8.1). Thus we arrive at the following theorem.

Theorem 8.0.3 *The equation*

$$(1 + c_1^2)\left(v_{0,0} - v_{1,1} + v_{0,0}v_{1,1}(v_{0,1} - v_{1,0})\right) + (1 - c_1^2)(v_{1,0}v_{1,1} - v_{0,0}v_{0,1}) = 0,$$

is linked via the MT

$$u_{0,0} = \frac{c_1^2\left(v_{0,0} + 1\right) - v_{1,0}\left(c_1^2\left(v_{0,0} + 1\right) + v_{0,0} - 1\right) - v_{0,0} + 1}{c_1^2\left(v_{0,0} + 1\right) + v_{1,0}\left(c_1^2\left(v_{0,0} + 1\right) + v_{0,0} - 1\right) - v_{0,0} + 1},$$

to the equation

$$\left(u_{0,0} - 1\right)\left(u_{1,0} + 1\right) - \left(u_{0,1} + 1\right)\left(u_{1,1} - 1\right) = 0.$$

Action of $\mathrm{GL}_2(\mathbb{C})$ **on** $\mathbb{CP}^1 \times \mathbb{CP}^1$

Letting $G = \mathcal{G}(2, c_1, 1) \times \mathcal{G}(2, c_1, 1) \cong \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, we consider the standard action on $\mathbb{CP}^1 \times \mathbb{CP}^1$. This is a 2 dimensional manifold, thus, by the general theory we must find an invariant of the subgroup $\mathbb{H}_1$. Recall, for a $\mathrm{GL}_2(\mathbb{C})$-valued function of λ, $A(\lambda)$, the action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is as follows

$$A(\lambda) \cdot \left(\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}, \begin{pmatrix} \rho^1 \\ \rho^2 \end{pmatrix} \right) = \left(A(c_1) \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}, A(c_2) \begin{pmatrix} \rho^1 \\ \rho^2 \end{pmatrix} \right), \tag{8.4}$$

where ϕ^1, ϕ^2 and ρ^1, ρ^2 are homogeneous coordinates on each component $\mathbb{CP}^1$. The group $\mathbb{H}_1$ is the set of diagonal matrices, therefore $\frac{\phi^1}{\phi^2}\frac{\rho^2}{\rho^1}$ is an invariant. Considering the open dense subset

$$\mathbb{V} = \{(\phi^1 : \phi^2, \rho^1 : \rho^2) \mid \phi^2 \neq 0, \rho^2 \neq 0\},$$

we take local coordinates $W^1_\phi = \frac{\phi^1}{\phi^2}$ and $W^1_\rho = \frac{\rho^1}{\rho^2}$. Then,

$$z = \frac{W^1_\phi}{W^1_\rho}, \tag{8.5}$$

$$\begin{aligned} z_1 &= \mathcal{S}(z) \\ &= \frac{\left(c_2^2 + c_2^2 W^1_\rho + W^1_\rho - 1\right)\left(c_1^2 + \left(c_1^2 - 1\right) W^1_\phi + 1\right)}{\left(c_1^2 + c_1^2 W^1_\phi + W^1_\phi - 1\right)\left(c_2^2 + \left(c_2^2 - 1\right) W^1_\rho + 1\right)}, \end{aligned} \tag{8.6}$$

$$\mathcal{S}(z_1) = \frac{\left(c_2^4 - 2c_2^2 u_{0,0} + (c_2^4 - 1) W^1_\rho + 1\right)\left(c_1^4 + W^1_\phi \left(c_1^4 + 2c_1^2 u_{0,0} + 1\right) - 1\right)}{\left(c_1^4 - 2c_1^2 u_{0,0} + \left(c_1^4 - 1\right) W^1_\phi + 1\right)\left(c_2^4 + W^1_\rho \left(c_2^4 + 2c_2^2 u_{0,0} + 1\right) - 1\right)}, \tag{8.7}$$

finally, we find

$$\mathcal{T}(z) = \frac{\left(c_2^2 + 1\right) W^1_\rho \left(\left(c_1^2 - 1\right) W^1_\phi u_{0,0} + \left(c_1^2 (W^1_\phi + 1) - W^1_\phi + 1\right) u_{0,1} - c_1^2 - 1\right)}{\left(c_1^2 + 1\right) W^1_\phi \left(\left(c_2^2 - 1\right) W^1_\rho u_{0,0} + \left(c_2^2 (W^1_\rho + 1) - W^1_\rho + 1\right) u_{0,1} - c_2^2 - 1\right)}. \tag{8.8}$$

Setting $\mathcal{T}(z) = v_{0,1}$, $z = v_{0,0}$, $z_1 = v_{1,0}$ we obtain an MT in implicit form. We may use (8.5) and (8.6) to express W^1_ϕ and W^1_ρ in terms of z and z_1, and hence in terms of $v_{0,0}$ and $v_{1,0}$. Then, we may use (8.7) to express $u_{0,0}$ in terms of $v_{0,0}$, $v_{1,0}$ and $v_{1,0}$ providing the MT, before finally using (8.8) to recover the related partial difference equation for v.

The resulting MT and related equation is omitted, as it is quite messy and contains square roots. We include this example to show that at times inversion of the relations may not be so straightforward, although in principle possible.

Chapter 9

Systems

We have applied the construction to D∆Es and have shown that in some cases one may also derive MTs for P∆Es. We now show that with minor modification the construction may be extended to systems. Consider the system with dependent variables u^i, given by

$$\begin{aligned} u^1_t &= F^1(\{u^i_j\}), \\ u^2_t &= F^2(\{u^i_j\}), \\ &\vdots \\ u^d_t &= F^d(\{u^i_j\}), \end{aligned} \tag{9.1}$$

where the notation $F^k(\{u^i_j\})$ denotes that F^k is a function of the dependent variables u^i and a finite number of their shifts. Suppose this system has DLR

$$\mathcal{S}(\mathbf{\Psi}) = M(u^1, u^2, \ldots, u^d, \lambda)\mathbf{\Psi}, \quad D_t(\mathbf{\Psi}) = \mathcal{U}(\{u^i_j\}, \lambda)\mathbf{\Psi}. \tag{9.2}$$

We define the $\mathbb{H}_k$ subgroups in the following way. Set $\mathbb{H}_0 = \{\mathrm{I}\}$, then $\mathbb{H}_1$ is the subgroup of $\boldsymbol{G}$ generated by the elements

$$M(u^1, u^2, \ldots, u^d, \lambda)M(\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^d, \lambda)^{-1},$$

for all admissible values u^i and $\tilde{u}^i$. For $k > 1$, $\mathbb{H}_k$ is generated by the elements of $\mathbb{H}_{k-1}$, along with the elements

$$M(u^1, u^2, \dots, u^d, \lambda) h M(u^1, u^2, \dots, u^d, \lambda)^{-1}, \quad h \in \mathbb{H}_{k-1}.$$

The definition of the Lie group G does not require any alteration. For a closed connected Lie subgroup H, following the same logic as in Theorem 6.0.5, one can show that if z is $\mathbb{H}_k$-left invariant and H-right invariant, then each element of the set

$$\{z, \mathcal{S}(z), \dots, \mathcal{S}^k(z)\},$$

is H-right invariant and independent of u^i_l, for all i and l. Now, suppose G/H has dimension r, and moreover we can find d functions z^i, such that z^i is H-right invariant and $\mathbb{H}_{r_i-1}$-left invariant, where $r_i \in \mathbb{N}$. Then each element of the set

$$\{z^1, \mathcal{S}(z^1), \dots, \mathcal{S}^{r_1-1}(z^1), z^2, \mathcal{S}(z^2), \dots, \mathcal{S}^{r_2-1}(z^2), \dots, z^d, \mathcal{S}(z^d), \dots, \mathcal{S}^{r_d-1}(z^d)\}, \tag{9.3}$$

is H-right invariant and independent of u^i_l. The H-right invariance implies that one may view each function as a function on G/H. Suppose further $r_1 + r_2 + \ldots + r_d = r$. Then, one has r functions on a manifold of dimension r. If set (9.3) forms a system of local coordinates on G/H, then for fixed u^i, taking a subsequent shift results in

$$\begin{aligned} \mathcal{S}^{r_1}(z^1) &= P^1(u^1, u^2, \dots, u^d, \{z^i_j\}), \\ \mathcal{S}^{r_2}(z^2) &= P^2(u^1, u^2, \dots, u^d, \{z^i_j\}), \\ &\vdots \\ \mathcal{S}^{r_d}(z^d) &= P^d(u^1, u^2, \dots, u^d, \{z^i_j\}), \end{aligned}$$

for some functions P^i, $i = 1, \dots d$, where $z^i_j = \mathcal{S}^j(z^i)$. Hence, there are d equations for the d functions u^i. If the determinant of the Jacobian matrix, which has $(i$,$j)$ entry

$$\frac{\partial}{\partial u^j}(\mathcal{S}^{r_i}(z^i)),$$

is not identically zero, then using the inverse function theorem we may on a neighbourhood of a point where the determinant is non zero, invert the relations to express the u^i functions in terms of the z^i functions and their shifts. This will provide the MT. To arrive at the Miura related system, one may use the t part of the DLR. We summarise this discussion as the following theorem.

Theorem 9.0.4 *Let H be a closed connected Lie subgroup of G, such that the dimension of G/H is $r \in \mathbb{N}$. For a system of the form (9.1), with DLR (9.2), we associate the collection of subgroups $\mathbb{H}_k$ for $k \in \mathbb{N}$. Suppose there exists d functions $z^1, z^2, \dots z^d$, defined on an open dense subset of G/H, and d natural numbers $r_1, r_2, \dots r_d$, such that each z^i is $\mathbb{H}_{r_i-1}$-left invariant, while $r_1 + r_2 + \dots + r_d = r$. Provided the functions*

$$\mathcal{S}^k(z^i), \quad i = 1, \dots, d, \quad k = 0, \dots r_i - 1,$$

form a system of local coordinates almost everywhere on G/H, and the determinant of the matrix with (i,j) entries

$$\frac{\partial}{\partial u^j}(\mathcal{S}^{r_i}(z^i)),$$

is not identically zero, then (9.1) possesses an MT.

Proof

Proof is analogous to Theorem 6.0.6. □

We now demonstrate the procedure for systems with two examples.

9.0.2 Toda Lattice

The Toda Lattice in Manakov-Flashka coordinates is given by

$$u_t = u(v_1 - v),$$

$$v_t = u - u_{-1}.$$

The following DLR can be found in [37]

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}\lambda+v_1 & u\\ -1 & 0\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$

$$D_t\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}0 & -u\\ 1 & \lambda+v\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}.$$

This is not of the correct type. However, with the change of variables $v := v_1$ and $u := u$, we obtain a suitable DLR

$$\mathcal{S}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}\lambda+v & u\\ -1 & 0\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$

$$D_t\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}=\begin{pmatrix}0 & -u\\ 1 & \lambda+v_{-1}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix},$$

for the equivalent system

$$u_t = u(v - v_{-1}),$$

$$v_t = u_1 - u.$$

We compute a MT by considering the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathrm{GL}_2(\mathbb{C})$. Take $G = \mathcal{G}(2, c_1, 1) \cong \mathrm{GL}_2(\mathbb{C})$, then, taking H to the be the trivial group, G acts on itself via left multiplication. The dimension of G/H is 4, thus according to the general theory, we may find MTs of the form

$$u = A^1(p, p_1, \ldots, p_a, q, q_1, \ldots, q_b),$$

$$v = A^2(p, p_1, \ldots, p_a, q, q_1, \ldots, q_b),$$

where $a+b = 4$, and p, q are the dependent variables of the Miura related system. Suppose $a = b = 2$, then we must find two invariants of the subgroup $\mathbb{H}_1$.

Computing $M(u_1, v_1, \lambda)M(u_2, v_2, \lambda)^{-1}$ yields

$$\begin{pmatrix}\lambda+v_1 & u_1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\lambda+v_2 & u_2\\ -1 & 0\end{pmatrix}^{-1}=\begin{pmatrix}\frac{u_1}{u_2} & -v_1+\frac{u_1v_2}{u_2}+\frac{u_1\lambda}{u_2}-\lambda\\ 0 & 1\end{pmatrix},$$

which may be simplified by conjugating by

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Doing so yields

$$\begin{pmatrix} \frac{u_1}{u_2} & \frac{u_1 v_2}{u_2} - v_1 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\mathbb{H}_1$ corresponding to the conjugated DLR is given by

$$\mathbb{H}_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{C}, a \neq 0 \right\}.$$

The conjugated DLR is given by

$$\mathcal{S} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} v & u - v\lambda \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix},$$
$$D_t \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} \lambda & \lambda v_{-1} - u \\ 1 & v_{-1} \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}.$$

The manifold $\mathrm{GL}_2(\mathbb{C})$ has coordinates functions a^{11}, a^{12}, a^{21} and a^{22}, which evaluate to the entries of the particular element of $\mathrm{GL}_2(\mathbb{C})$. It is clear that left multiplication of $g \in \mathrm{GL}_2(\mathbb{C})$ by an element $h \in \mathbb{H}_1$ will not affect the a^{21} or a^{22} component of g. Hence, it follows that a^{21} and a^{22} are invariants of $\mathbb{H}_1$.

Setting $z = a^{21}$ and $w = a^{22}$, we find

$$\begin{aligned}
z_1 &= \mathcal{S}(z) \\
&= \mathcal{S}(a^{21}) \\
&= a^{21}c_1 - a^{11}, \\
\mathcal{S}(z_1) &= a^{21}(c_1(c_1+v) - u) - a^{11}(c_1+v), \qquad (9.4) \\
w_1 &= \mathcal{S}(w) \\
&= \mathcal{S}(a^{22}) \\
&= a^{22}c_1 - a^{12}, \\
\mathcal{S}(w_2) &= a^{22}(c_1(c_1+v) - u) - a^{12}(c_1+v). \qquad (9.5)
\end{aligned}$$

Rewriting the standard coordinates on $\mathrm{GL}_2(\mathbb{C})$ in terms of the new coordinates z, z_1, w, w_1, one finds the expressions

$$\begin{aligned}
a^{11} &= zc_1 - z_1, \\
a^{12} &= wc_1 - w_1, \\
a^{21} &= z, \\
a^{22} &= w.
\end{aligned}$$

Using these, along with (9.4) and (9.5), one obtains after relabelling $z = p$, $z_1 = p_1$, $w = q$ and $w_1 = q_1$, the MT

$$\begin{aligned}
u &= \frac{p_2q_1 - p_1q_2}{p_1q - pq_1}, \\
v &= \frac{pq_2 - p_2q}{pq_1 - p_1q} - c_1.
\end{aligned}$$

The corresponding p, q system is obtained using the continuous part of the DLR. One must compute $D_t(a^{21})$ and $D_t(a^{22})$. Doing so, we acquire the system

$$\begin{aligned}
p_t &= \frac{p_{-1}\left(pq_1 - p_1q\right)}{p_{-1}q - pq_{-1}}, \\
q_t &= \frac{q_{-1}\left(pq_1 - p_1q\right)}{p_{-1}q - pq_{-1}}.
\end{aligned}$$

Hence, we have the following result.

Theorem 9.0.5 *The system*

$$p_t = \frac{p_{-1}\,(pq_1 - p_1 q)}{p_{-1}q - pq_{-1}},$$
$$q_t = \frac{q_{-1}\,(pq_1 - p_1 q)}{p_{-1}q - pq_{-1}},$$

is linked via the MT

$$u = \frac{p_2 q_1 - p_1 q_2}{p_1 q - pq_1},$$
$$v = \frac{pq_2 - p_2 q}{pq_1 - p_1 q} - c_1,$$

to the system

$$u_t = u(v - v_{-1}),$$
$$v_t = u_1 - u.$$

We cannot locate this MT and related system in the literature, so it may be new.

9.0.3 Adler-Postnikov system

In [8], the authors consider a two component generalisation of the Lax equations (7.43). In particular, they consider the system

$$P\psi = \lambda Q\psi, \quad \psi_t = A_k\psi,$$

with

$$P = (u\mathcal{S}^2 + I)\mathcal{S}, \;\; Q = \mathcal{S}^2 + v.$$

A sequence of operators A_k and B_k define a hierarchy, and are constructed from either positive or negative powers of $\mathcal{S}^2$. The simplest are given by

$$A^- = v_{-2}v_{-1}\mathcal{S}^{-2} + f_{-3} + f_{-2}, \quad B^- = v_{-1}v\mathcal{S}^{-2} + f_{-1} + f, \quad f := uv_1v_2$$

and

$$A^+ = u_{-2}u_{-1}\mathcal{S}^2 + g_{-1} + g, \;\; B^+ = uu_1\mathcal{S}^2 + g + g_1, \quad g := u_{-2}u_{-1}v,$$

which produce the systems

$$\begin{aligned} u_{,t^-} &= u(f_{-1} - f_1), \\ v_{,t^-} &= v(f + f_{-1} - f_{-2} - f_{-3} - v_1 + v_{-1}), \end{aligned}$$

$$\begin{aligned} u_{,t^+} &= u(g + g_1 - g_2 - g_3 - u_{-1} + u_1), \\ v_{,t^+} &= v(g_1 - g_{-1}). \end{aligned}$$

For further details regarding systems produced by Lax operators of this form, we refer the reader to [8]. We merely use these systems to demonstrate the procedure for generating MTs for systems. To this end, it is necessary to write down the discrete part of the Lax pair in matrix form. Following the procedure described in previous sections we find

$$\mathcal{S}\begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda\frac{v}{u} & -\frac{1}{u} & \frac{\lambda}{u} \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}. \tag{9.6}$$

We use the relation

$$\psi_t = A_k\psi$$

to compute the continuous part of the Lax pair.

For A^-,

$$\psi^1_{t_-} = \psi_{t_-} = v_{-2}v_{-1}\mathcal{S}^{-2}(\psi) + (f_{-3} + f_{-2})\psi.$$

Then

$$\psi^2_{t_-} = \mathcal{S}(\psi_{t_-}) = v_{-1}v\mathcal{S}^{-1}(\psi) + (f_{-2} + f_{-1})\mathcal{S}(\psi)$$

and

$$\psi^3_{t_-} = \mathcal{S}^2(\psi_{t_-}) = vv_1\psi + (f_{-1} + f)\mathcal{S}^2(\psi).$$

To determine the inverse shifts of ψ, we use relations

$$\mathcal{S}^{-1}(\mathbf{\Psi}) = (\mathcal{S}^{-1}(M(u,v,\lambda)))^{-1}\mathbf{\Psi}, \quad \mathcal{S}^{-2}(\mathbf{\Psi}) = (\mathcal{S}^{-2}(M(u,v,\lambda)))^{-1}(\mathcal{S}^{-1}(M(u,v,\lambda)))^{-1}\mathbf{\Psi}.$$

Explicitly,

$$\mathcal{S}^{-1}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda v_{-1}} & -\frac{1}{v_{-1}} & \frac{u_{-1}}{\lambda v_{-1}}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}$$

and

$$\mathcal{S}^{-2}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda^2 v_{-2}v_{-1}} - \frac{1}{v_{-2}} & \frac{u_{-2}}{\lambda v_{-2}} - \frac{1}{\lambda v_{-2}v_{-1}} & \frac{u_{-1}}{\lambda^2 v_{-2}v_{-1}}\\ \frac{1}{\lambda v_{-1}} & -\frac{1}{v_{-1}} & \frac{u_{-1}}{\lambda v_{-1}}\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}.$$

Thus, using the above to compute $\mathcal{S}^{-2}(\psi) = \mathcal{S}^{-2}(\psi^1)$ and $\mathcal{S}^{-1}(\psi) = \mathcal{S}^{-1}(\psi^1)$, we find

$$\begin{pmatrix}\psi^1_{t_-}\\ \psi^2_{t_-}\\ \psi^3_{t_-}\end{pmatrix} = \begin{pmatrix} f_{-3} + f_{-2} - v_{-1} + \frac{1}{\lambda^2} & \frac{u_{-2}v_{-1}-1}{\lambda} & \frac{u_{-1}}{\lambda^2}\\ \frac{v}{\lambda} & -v + f_{-2} + f_{-1} & \frac{vu_{-1}}{\lambda}\\ vv_1 & 0 & f + f_{-1}\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}.$$

For A^+,

$$\begin{aligned}
\psi^1_{t_+} &= u_{-2}u_{-1}\mathcal{S}^2(\psi^1) + (g_{-1} + g)\psi^1,\\
\psi^2_{t_+} &= u_{-1}u\mathcal{S}^3(\psi^1) + (g + g_1)\mathcal{S}(\psi^1),\\
\psi^3_{t_+} &= uu_1\mathcal{S}^4(\psi^1) + (g_1 + g_2)\mathcal{S}^2(\psi^1),
\end{aligned}$$

and therefore

$$\begin{pmatrix}\psi^1_{t_+}\\ \psi^2_{t_+}\\ \psi^3_{t_+}\end{pmatrix} = \begin{pmatrix} g + g_{-1} & 0 & u_{-2}u_{-1}\\ v\lambda u_{-1} & g + g_1 - u_{-1} & \lambda u_{-1}\\ v\lambda^2 & \lambda\,(uv_1 - 1) & \lambda^2 - u + g_1 + g_2\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\\ \psi^3\end{pmatrix}.$$

Thus, we have derived DLRs for the t_+ and t_- systems. Now, we compute MTs for these systems by considering the action of $\mathrm{GL}_3(\mathbb{C})$ on $(\mathbb{C}^3 \setminus \{0\}) \times \mathbb{CP}^2$.

Take $G = \mathcal{G}(3, c_1, 1) \times \mathcal{G}(3, c_2, 1) \cong \mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$, then G acts transitively on $\mathbb{M} = (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{CP}^2$. Since $\mathbb{M}$ has dimension 5, we may find MTs of the form

$$u = A^1(p, p_1, \ldots, p_a, q, q_1, \ldots, q_b),$$
$$v = A^2(p, p_1, \ldots, p_a, q, q_1, \ldots, q_b),$$

where $a+b = 5$ and p, q are the dependent variables of the Miura related system. Suppose $a = 3$ and $b = 2$, then we must find an invariant of $\mathbb{H}_2$ and $\mathbb{H}_1$ respectively. We compute

$$M(u_1, v_1, \lambda)M(u_2, v_2, \lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{v_1 - v_2}{u_1 v_2} & \frac{\lambda(v_2 - v_1)}{u_1 v_2} & \frac{u_2 v_1}{u_1 v_2} \end{pmatrix}.$$

One can check that the set of elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & -a\lambda & b \end{pmatrix},$$

for $a, b \in \mathbb{C}$, forms a group, and hence $\mathbb{H}_1$ is given by such elements. Computing $M(u, v, \lambda)h_1 M(u, v, \lambda)^{-1}$, for $h_1 \in \mathbb{H}_1$, produces

$$M(u, v, \lambda)h_1 M(u, v, \lambda)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a\left(\frac{1}{v\lambda} - \lambda\right) & b - \frac{a}{v} & \frac{au}{v\lambda} \\ \frac{a - av\lambda^2}{uv} & -\frac{(a - bv + v)\lambda}{uv} & \frac{a+v}{v} \end{pmatrix}.$$

The set of such elements, along with elements of $\mathbb{H}_1$, are contained within the group

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \frac{a_1}{\lambda} + \lambda a_2 & a_3 & \frac{a_4}{\lambda} \\ a_6\lambda^2 + a_5 & a_7\lambda & a_8 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \mid a_i \in \mathbb{C} \right\}.$$

Since $\mathbb{H}_2$ is a subgroup of S, any invariant of S is also an invariant of $\mathbb{H}_2$.

Let ϕ^1, ϕ^2, ϕ^3 be coordinates on $\mathbb{C}^3 \setminus \{0\}$ and ρ^1, ρ^2, ρ^3 be homogeneous coordinates on $\mathbb{CP}^2$, then consider the dense open subset

$$\mathbb{V} = \{((\phi^1, \phi^2, \phi^3), (\rho^1 : \rho^2 : \rho^3)) \mid \rho_2 \neq 0\}.$$

Take $W_\rho^1 = \rho^1/\rho^2$ and $W_\rho^2 = \rho^3/\rho^2$ as local coordinates on the projective component of $\mathbb{V}$. From the form of $\mathbb{H}_1$ and S, it becomes clear that W_ρ^1 and ϕ^1 are invariants of $\mathbb{H}_1$ and $\mathbb{H}_2$ respectively. Thus, following the general theory, we may set $z = \phi^1$ and $w = W_\rho^1$. Doing so, one finds $z_1 = \phi^2$, $z_2 = \phi^3$ and $w_1 = 1/W_\rho^2$.

Hence,

$$\mathcal{S}(z_2) = c_1 \frac{v}{u} z - \frac{1}{u} z_1 + \frac{c_1}{u} z_2,$$
$$\mathcal{S}(w_1) = \frac{1}{w_1 \left(\frac{c_2 v w}{u} + \frac{c_2}{u w_1} - \frac{1}{u} \right)}.$$

Relabelling $z = p$, $w = q$, and noting that $z_1 = \mathcal{S}(p) = p_1$, $z_2 = \mathcal{S}^2(p) = p_2$ and $w_1 = \mathcal{S}(w) = q_1$, we obtain, on rearrangement of the above equations, the MT

$$u = \frac{q_2 \left(c_2 q q_1 \left(p_1 - c_1 p_2\right) + c_1 p \left(c_2 - q_1\right)\right)}{c_1 p - c_2 p_3 q q_1 q_2},$$
$$v = \frac{p_3 q_2 \left(c_2 - q_1\right) + p_1 - c_1 p_2}{c_1 p - c_2 p_3 q q_1 q_2}.$$

The corresponding Miura related system is derived using the DLRs computed previously. Hence, we have the following results, where to reduce the complexity of the found equations we do not substitute the expressions for u and v.

Theorem 9.0.6 *The systems*

$$p_{,t^+} = \frac{c_1 p_1 \left(u_{-2} v_{-1} - 1\right) + p \left(c_1^2 v_{-1} \left(u_{-2} v + u_{-3} v_{-2} - 1\right) + 1\right) + p_2 u_{-1}}{c_1^2},$$
$$q_{,t^+} = \frac{q \left(c_2^2 v_{-1} \left(u_{-3} v_{-2} - 1\right) + c_2^2 v \left(1 - u_{-1} v_1\right) + 1\right) - c_2 q^2 v}{c_2^2} + \frac{q_1 c_2 \left(u_{-2} v_{-1} - 1\right) + u_{-1} \left(1 - c_2 q v\right)}{c_2^2 q_1},$$

$$p_{,t^-} = p \left(u_{-2} u_{-1} v + u_{-3} u_{-2} v_{-1}\right) + p_2 u_{-2} u_{-1},$$
$$q_{,t^-} = \frac{q \left(q_1 u_{-3} u_{-2} v_{-1} - u_{-1} \left(c_2 + q_1 \left(u v_1 - 1\right)\right)\right) + u_{-2} u_{-1} - c_2 q_1 q^2 u_{-1} v}{q_1},$$

are linked via the MT

$$u = \frac{c_2 q q_1 q_2 \left(p_1 - c_1 p_2\right) + c_1 p q_2 \left(c_2 - q_1\right)}{c_1 p - c_2 p_3 q q_1 q_2},$$
$$v = \frac{p_3 q_2 \left(c_2 - q_1\right) + p_1 - c_1 p_2}{c_1 p - c_2 p_3 q q_1 q_2},$$

to the systems

$$u_{,t^-} = u(f_{-1} - f_1),$$
$$v_{,t^-} = v(f + f_{-1} - f_{-2} - f_{-3} - v_1 + v_{-1}),$$

$$u_{,t^+} = u(g + g_1 - g_2 - g_3 - u_{-1} + u_1),$$
$$v_{,t^+} = v(g_1 - g_{-1}).$$

The above MT appears to be new. We make a link to the MTs calculated for the discrete Sawada-Kotera equation. It is given in [8] that the flows ∂_{t^+} and ∂_{t^-} commute. Moreover, the flow $\partial_t = \partial_{t^-} - \partial_{t^+}$ admits the reduction $v = u$, to the discrete Sawada-Kotera equation (7.42). The Miura related system for the flow ∂_t can be easily obtained from the Miura related systems for the flows ∂_{t^+} and ∂_{t^-}. Using the MT, we can express u and v in terms of shifts of p and q. Hence, the reduction $v = u$ imposes a corresponding condition on p and q. Solutions to this condition will provide a reduction of the Miura related system and the accompanying reduced MT. The condition on p and q is given by

$$q_2 \left(c_1 p - p_3\right) \left(c_2 - q_1\right) + \left(p_1 - c_1 p_2\right) \left(c_2 q q_1 q_2 - 1\right) = 0.$$

A solution of the equation is given by $q = c_2 = 1$. Applying this reduction we obtain the MT described in Theorem 7.50, corresponding to the standard action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{C}^3 \setminus \{0\}$. Another solution is given by setting $p = (-1)^n$ and $c_1 = -1$. From this reduction we recover the MT described in Theorem 7.4.2 corresponding to the action of $\mathrm{GL}_3(\mathbb{C})$ on $\mathbb{CP}^2$.

Chapter 10

Conclusions

We have come to the end of the story. We conclude by summarising the results of part II and suggest possible directions for future work.

10.1 Summary of results and future work

In the second part of this thesis we introduced a method to derive MTs for DΔEs possessing a DLR of a certain type. We later extended the method to discrete equations and systems of differential difference equations. We applied the procedure to various well known systems.

In our study we derived many MTs and Miura related equations. Some of which were known, but interestingly some were not. It is surprising that even in the case of the Narita-Itoh-Bogoyavlensky lattice, which is well studied, we were able to discover new MTs and Miura related equations. This highlights how one can quite naturally arrive at relatively complex MTs by considering quite simple and natural Lie group actions. Even in the cases where the MTs were known, we show that they can be associated to a particular Lie group action and are intimately linked to the DLR.

It is not clear whether all MTs admit this interpretation. An interesting problem for future work is to classify for a given equation, all MTs which can found using the method described. The results of such a study are likely to provide insights to the theory of DLRs, as well as potentially provide new integrable equations. In the continuous case, it is sometimes possible to determine that an equation cannot have a MT above a particular order. One could study whether similar results are able to be achieved in the semi discrete case.

Finally, an issue encountered with regards to discrete equations is that we require M to only depend on u. It may be the case that this can be overcome by altering the set-up. For instance via an alternate definition of the $\mathbb{H}_k$ subgroups. This requires further investigation.

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