Model reduction for linear differential inclusions: robustness and time-variance

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Abstract— **This paper deals with the problem of model reduction by moment matching for linear differential inclusions. The problem is formally formulated and the notions of moment-set, perturbed moment trajectory, approximate reduced order model and robust reduced order model are introduced. Two sets of results are presented. The first part of the paper deals with robustness of the reduced order models with respect to input perturbations. Exploiting this result an enhanced model reduction scheme for linear differential equations is presented. In the second part of the paper we focus on the problem of model reduction by moment matching for time-varying systems driven by time-varying signal generators. Finally, these two sets of results are used to solve the problem of model reduction for linear differential inclusions. The results are illustrated by means of numerical examples.**

I. Introduction

Differential inclusions are a generalization of differential equations having multi-valued right-hand side that arise in a multitude of applications [1]. For instance, they provide a framework to study the solutions of differential equations with discontinuous right-hand side [2]. They are a fundamental mathematical tool to prove the existence of control laws in optimal control [3]. They are also useful to model systems with uncertain right-hand side, such as systems in which some of the parameters are known to belong to a certain set but cannot be determined precisely [2]. Recently, differential inclusions have also being used in the literature of hybrid systems and stochastic hybrid systems to model non-unique solutions [4], [5].

Computing the solutions of differential inclusions is numerically challenging since each initial condition can produce a different set of dense solutions. For this reason some numerical methods for the approximation of these solutions have been proposed, see *e.g.* [6], [7]. A possible systematic approach to approximate differential inclusions is represented by model order reduction techniques. The objective would be to determine a reduced order differential inclusion that approximates, in a sense to be specified, the behavior of the higher order differential inclusion. Model reduction for differential *equations* is a mature field. Several approaches have been proposed for linear differential equations [8]–[17], nonlinear differential equations [18]–[23], and more general classes of differential equations [24]–[29], just to cite a few. For differential inclusions the problem of model reduction is almost completely unexplored, with few possible exceptions. For instance in [30] differential inclusions are approximated with *higher-order* differential equations, whereas in [31] a multi-scale approach is used to reduce the complexity of the models. One of the reasons behind this scarcity of results for the problem of model reduction of differential inclusions is probably the additional difficulty which arises when dealing with multi-valued functions. It becomes difficult then to define a notion of reduced order model and to provide numerical tools that can efficiently determine such reduced order models. This paper tries to shed some light on the problem of model reduction for linear differential inclusions exploiting the notion of steady-state. We introduce a series of new concepts, namely the notions of "moment-set", "approximate reduced order model", "perturbed moment trajectory" and "robust reduced order model", which help to precisely formulate the problem (Section II). The problem is solved in two steps. We first consider the problem of model reduction for linear differential equations with additive disturbance (Section III). The solution to this specific problem suggests an enhanced model reduction scheme for unperturbed systems which improves the performance of the model reduction technique firstly introduced in [32] (Section IV). We then consider the problem of model reduction of linear time-varying systems driven by time-varying signal generators (Section V). We finally combine these results to provide a solution to the problem of model reduction for linear differential inclusions (Section VI). The results of the paper are illustrated by means of two numerical examples.

Notation. We use standard notation. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers; $\mathbb{C}_{<0}$ denotes the set of complex numbers with negative real part; $\mathbb{C}_0$ denotes the set of complex numbers with zero real part. The sum of a set and a vector denotes the corresponding set shift. The symbol $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. Given some finite data points set $X = \{x_i \in \mathbb{R} : i = 1, \ldots, m\}$, the discrete ℓ_p norm of a function f is defined as $||f||_{\ell_p,X} := (\sum_{i=0}^{m} |f(x_i)|^p)^{\frac{1}{p}}$, with $1 \leq p < \infty$.

II. Problem formulation

In this section we introduce the class of linear differential inclusions which are studied in this paper. We recall the classical definition of steady-state response and we then define the concepts of "moment-set", "perturbed moment trajectory"

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and "robust reduced order model".

Let $\mathcal{A} \subset \mathbb{R}^{n\times n}$ be a nonempty, closed set of real matrices. For $x \in \mathbb{R}^n$ we say that $v \in \mathcal{A}x$, if there exists a matrix $A \in \mathcal{A}$ such that $v = Ax$. Let $\mathfrak{D}$ be a set of bounded piecewise continuous signals $\delta : \mathbb{R} \to \mathbb{R}$. With this notation, we define a linear differential inclusion of order n as

$$\dot{x} \in \mathcal{A}x + \mathcal{B}_u u + \mathcal{B}_d \delta, \qquad y = Cx, \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\delta(\cdot) \in \mathfrak{D}$, $\mathcal{B}_u \subset \mathbb{R}^{n\times 1}$, $\mathcal{B}_d \subset \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. Given an initial condition $x_0 = x(0)$, we call a continuous function $x : \mathbb{R} \to \mathbb{R}^n$ a solution of (1) if and only if $\dot{x}(t) \in \mathcal{A}x(t) + \mathcal{B}_u u(t) + \mathcal{B}_d \delta(t)$ for almost all $t \in \mathbb{R}$. We define the set of solutions with initial conditions in $X_0 \subset \mathbb{R}^n$ as the set $\mathcal{X}(X_0) := \{x$ is a solution of (1) : $x_0 \in X_0\}$.

Definition 1: [33] Let $\mathbb{B} \subset \mathbb{R}^n$ and suppose $x(\cdot)$ is defined for all $t \geq 0$ and for all $x_0 \in \mathbb{B}$. The *omega limit set of the set* $\mathbb{B}$, denoted $\Omega(\mathbb{B})$, is the set of all points $x \in \mathbb{R}^n$ for which there exists a sequence $\{x_k\}$ of solutions $x_k \in \mathcal{X}(\mathbb{B})$ and a sequence $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that $\lim_{k\to\infty} x_k(t_k) = x$.

Definition 2: [34] Suppose the solutions of system (1), with initial conditions in a closed and positively invariant set $\mathcal{X}$, are ultimately bounded with ultimate bounded subset $\mathbb{B}$. A *steady-state response* is any solution with initial condition $x_0 \in \Omega(\mathbb{B})$.

Consider an autonomous linear differential inclusion described by

$$\dot{\omega} \in \mathcal{S}\omega, \qquad u = L\omega, \tag{2}$$

with $\omega(t) \in \mathbb{R}^\nu$, $\mathcal{S} \subset \mathbb{R}^{\nu\times\nu}$, $L \in \mathbb{R}^{1\times\nu}$. Assume that $\mathcal{S}$ is non-empty, closed and all the matrices $S \in \mathcal{S}$ are non-derogatory[1]. Moreover, we consider only initial conditions $\omega(0)$ for which the triple $(L, S, \omega(0))$ is minimal[2].

Definition 3: Consider the differential inclusion (1), an input u described by (2) and let $\mathfrak{D} = \emptyset$. Suppose that for each solution x in the set of steady-state responses $\mathcal{X}(\Omega(\mathbb{B}))$ there exists a map $\Pi : \mathbb{R} \to \mathbb{R}^{n\times\nu} : t \mapsto \Pi(t)$ such that $x = \Pi\omega$ and let $\mathcal{P} := \{\Pi(\cdot) : x = \Pi\omega$ for all $x \in \mathcal{X}(\Omega(\mathbb{B}))\}$. The *moment-set* of system (1) at $(L, \mathcal{S})$ is defined as the set $C\mathcal{P}$ of the mappings $C\Pi(\cdot)$, with $\Pi(\cdot) \in \mathcal{P}$.

We are now ready to provide two formulations of the problem of model reduction for differential inclusions, a full problem and an approximate problem (in brackets).

Problem 1: Consider the differential inclusion (1) equipped with a moment-set $C\mathcal{P}$ at $(L, \mathcal{S})$. The problem of *(approximate) model reduction by moment matching* consists in determining a differential inclusion of order $\nu < n$ equipped with a moment-set $\mathcal{HP}$ at $(L, \mathcal{S})$ such that for each steady-state output mapping $C\Pi(\cdot)$ belonging to (a subset of) $C\mathcal{P}$, there exists a steady-state output mapping $H(\cdot)P(\cdot)$ belonging to (a subset of) $\mathcal{HP}$ such that $C\Pi(t) = H(t)P(t)$ for all $t \in \mathbb{R}$.

[1]A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

[2]Details on the meaning of the minimality assumption can be found in [35], see also [36].

Definition 4: Let $\mathfrak{D} = \emptyset$. A differential inclusion solving Problem 1 is called an *(approximate) reduced order model* of (1) at $(L, \mathcal{S})$.

While we provide a solution to the non-approximate version of Problem 1, from a practical point of view it is impossible to determine the moment-set of a differential inclusion in closed form because the moment-set is itself the solution of another differential inclusion. Since model reduction is motivated by practical applications, it is imperative to formulate an approximate version of the problem which can be numerically solved. Before explaining the motivation behind the particular chosen formulation, we recall an important fact about model reduction by moment matching.

Remark 1: A reduced order model by moment matching of a linear differential *equation* for a class of inputs u generated by (2) when $\mathcal{S}$ is *single-valued* has the property of having the same steady-state output response of the original system to this specific class of inputs. However, in general there is no guarantee regarding the approximation error provided for inputs that do not belong to the given class.

In the case of differential inclusions, it seems sensible to extend this idea of "zero steady-state error" for the interpolated signals also to the class of systems to be reduced. In particular, we want that the error between the steady-state output response of the reduced differential inclusion and the steady-state output response of a subset of systems belonging to (1) is zero. In other words, we are looking for reduced order models which provide "zero steady-state error" for the "interpolation" signals *and* the "interpolation" systems. We now consider the case in which $\mathfrak{D} \neq \emptyset$ and we formulate a robust version of the problem.

Definition 5: Consider the differential inclusion (1), an input u described by (2) and let $\mathfrak{D} \neq \emptyset$. A *perturbed moment trajectory* of system (1) at $(L, \mathcal{S})$ is defined as a steady-state output response of system (1).

Definition 5 is motivated by the expectation that when $\mathfrak{D} \neq \emptyset$ the steady-state solution is equal to the sum of $\Pi\omega$ and a perturbation term. Hence, we can define a perturbed trajectory but not a perturbed moment.

Problem 2: Consider an (approximate) reduced order model of (1) at $(L, \mathcal{S})$. The problem of *robust (approximate) model reduction by moment matching* consists in minimizing (with respect to some norm) the error between the perturbed moment trajectories of the two systems.

Definition 6: A differential inclusion solving Problem 2 is called a *robust (approximate) reduced order model* of (1) at $(L, \mathcal{S})$.

Exploiting linearity we can first solve the problem of robust model reduction for single-valued systems and then extend the results to differential inclusions. Thus, in the following we first focus on linear systems described by differential *equations* subject to an unknown additive disturbance and then we study the problem of model reduction for unperturbed differential inclusions. Finally, we combine the results to provide a solution to Problems 1 and 2. Note that this approach, in addition to simplifying the presentation, provides individual contributions, such as robust reduced order models and time-varying reduced order models, which are of interest *per se*.

III. Moment matching with additive disturbances

In this section we solve the problem of model reduction by moment matching for the differential inclusion (1) when $\mathcal{A}$, $\mathcal{B}_u$, $\mathcal{B}_d$ and $\mathcal{S}$ are single-valued. In this case, the solution of the differential inclusion is equivalent to the solutions of the differential equations

$$\dot{x} = Ax + B_u u + B_d \delta, \qquad y = Cx, \tag{3}$$

for all $\delta(\cdot) \in \mathfrak{D}$, whereas the signal generator is described by the equations

$$\dot{\omega} = S\omega, \qquad u = L\omega. \tag{4}$$

We begin providing a characterization of the steady-state response of the interconnection of system (3) and the signal generator (4). The result, which combines concepts already known in the literature, is not new. We provide a proof for completeness.

Lemma 1: Consider the interconnection of system (3) and the signal generator (4). Assume $\sigma(A) \subset \mathbb{C}_{<0}$ and $\sigma(S) \subset \mathbb{C}_0$. The steady-state response of the output of such interconnection is

$$y(t) = C\Pi\omega(t) + \int_{-\infty}^{t} Ce^{A(t-\tau)} B_d \delta(\tau) d\tau,$$

for all $t \in \mathbb{R}$, where Π is the unique solution of the Sylvester equation

$$A\Pi + B_u L = \Pi S. \tag{5}$$

Proof: Since $\sigma(S) \cap \sigma(A) = \emptyset$, equation (5) has a unique solution Π, see [37]. Define the variable $z := x - \Pi\omega$ and compute its derivative with respect to time, namely

$$\begin{aligned} \dot{z} = \dot{x} - \Pi\dot{\omega} &= Ax + B_u u + B_d \delta - \Pi S\omega \\ &= A(x - \Pi\omega) + (A\Pi + B_u L - \Pi S)\omega + B_d \delta \\ &= Az + B_d \delta. \end{aligned}$$

The solution of this differential equation is $z(t) = e^{At}z(0) + \int_0^t e^{A(t-\tau)} B_d \delta(\tau) d\tau$, which yields

$$x(t) = \Pi\omega(t) + e^{At}(x(0) - \Pi\omega(0)) + \int_0^t e^{A(t-\tau)} B_d \delta(\tau) d\tau.$$

Note now that the term $e^{At}(x(0) - \Pi\omega(0))$ is a transient response that converges to zero exponentially, whereas the term $\int_0^t e^{A(t-\tau)} B_d \delta(\tau) d\tau$ is a forced response. By definition we can compute the steady-state of this last term starting the integration from $-\infty$ (in fact, in this way the transient will be zero at $t = 0$), obtaining $\int_{-\infty}^t e^{A(t-\tau)} B_d \delta(\tau) d\tau$. This concludes the proof. ■

We can now provide a first family of robust reduced order models.

Lemma 2: Consider the interconnection of system (3) and the signal generator (4). Assume $\sigma(A) \subset \mathbb{C}_{<0}$ and $\sigma(S) \subset \mathbb{C}_0$. The system described by the equations

$$\dot{\xi} = F\xi + Gu + M\delta, \qquad \psi = H\xi, \tag{6}$$

with $\xi(t) \in \mathbb{R}^{\nu}$, $\psi(t) \in \mathbb{R}$, $F \in \mathbb{R}^{\nu\times\nu}$ $G \in \mathbb{R}^{\nu\times 1}$, $M \in \mathbb{R}^{n\times 1}$, $H \in \mathbb{R}^{1\times n}$, is a robust reduced order model of system (3) at (L, S) if $\nu < n$, there exists a unique solution P of the equation

$$FP + GL = PS, \tag{7}$$

such that

$$HP = C\Pi, \tag{8}$$

where Π is the unique solution of (5), and in addition

$$M := \arg\min_{\hat{M}} \left|\left| He^{Ft}\hat{M} - Ce^{At}B_d \right|\right|^2_{\ell_2, \mathbb{R}_{\geq 0}}. \tag{9}$$

Proof: The claim is a direct consequence of Definition 4 and Lemma 1. In fact note that equation (7) defines the moments HP of system (6), whereas equation (8) is the matching condition between the moments of system (3) and the moments of system (6). The resulting model solves Problem 1 when $\mathfrak{D} = \emptyset$. Given such a model, condition (9) provides the matrix M which minimizes the error between the respective perturbed moment trajectories (note that the squared ℓ_2 norm is strictly convex and thus M is unique). ■

Before presenting the next result, we introduce some auxiliary definitions. Define the time-snapshots $Y_p \in \mathbb{R}^{p\times 1}$ and $\Psi_p \in \mathbb{R}^{p\times\nu}$ as

$$Y_p = \begin{bmatrix} CB_d & Ce^{At_1}B_d & \cdots & Ce^{At_p}B_d \end{bmatrix}^{\top},$$

and

$$\Psi_p = \begin{bmatrix} H & He^{Ft_1} & \cdots & He^{Ft_p} \end{bmatrix}^{\top},$$

where $0 < t_1 < \cdots < t_p$, $p \geq \nu$, with $\lim_{p\to\infty} t_p = \infty$. If the matrix Ψ_p is full-rank, let

$$M := \lim_{p\to\infty} \left(\Psi_p^{\top}\Psi_p\right)^{-1} \Psi_p^{\top} Y_p. \tag{10}$$

We now simplify the conditions in Lemma 2, providing a family of parametrized robust reduced order achieving moment matching.

Proposition 1: Consider system (3) and the signal generator (4). Assume $\sigma(A) \subset \mathbb{C}_{<0}$ and $\sigma(S) \subset \mathbb{C}_0$. Then the system described by the equations

$$\dot{\xi} = (S - GL)\xi + Gu + M\delta, \qquad \psi = C\Pi\xi, \tag{11}$$

where Π is the unique solution of (5) and M is given by equation (10), is a robust reduced order model of system (5) at (L, S) for any fixed G such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$ and the pair $(S - GL, C\Pi)$ is observable.

Proof: For model (6), F, G, H and M are free parameters which must be used to solve the conditions (7), (8), (9). To this end note that selecting $P = I$, yields $F = S - GL$ from equation (7) and $H = C\Pi$ from equation (8). Observe that any other selection $\widetilde{P} \neq I$ (invertible) yields a change of coordinates and that the systems expressed in the new coordinates would still have matrices of the form given in (11) (see Remark 4). Problem 1 is then solved for any G such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$. Consider now one of these matrices G solving Problem 1. If G is such that the pair $(S - GL, C\Pi)$ is observable, then it is possible to determine a set of sampling times t_i such that the matrix Ψ_p evaluated at

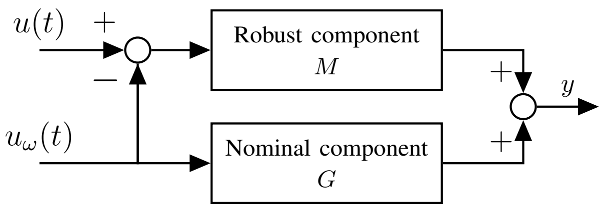

Fig. 1. A schematic overview of the enhanced method.

these t_i has full-rank for any $p \geq \nu$. Hence, we can compute M as in (10), which is the solution of the linear least-squares problem (8). ∎

Remark 2: We have deliberately chosen to exclude G, which is a free parameter, from the minimization of the error between the perturbed moment trajectories. In fact, note that we still want to use G to impose other properties upon the reduced order model. For instance G can be used to set the eigenvalues or the zeros of the reduced order model, see [32]. Thus Problems 1 and 2 have been formulated bearing in mind a two steps procedure: first we determine an unperturbed reduced order model; we then augment such a model by means of M, which we use as a parameter to minimize the error. This provides the greatest design flexibility since G can or can not be used to minimize the error between the perturbed moment trajectories.

Remark 3: The determination of an approximation of M computed truncating the matrices Ψ_p and Y_p to a finite value t_p has a computational complexity that is independent of the order of the system to be reduced. In fact, it depends on the order $\nu < n$ of the reduced order model and on the number of sample times p.

IV. An Enhanced Model Reduction Approach

The results of the previous section suggest an enhanced model reduction scheme for linear, unperturbed, systems. Consider the linear system described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{12}$$

and the problem of obtaining a reduced order model that matches the moment of this system. We can rewrite system (12) as

$$\dot{x} = Ax + Bu_\omega + B(u - u_\omega), \qquad y = Cx, \tag{13}$$

where $u_\omega = L\omega$. System (13) has the form of system (3) with $\delta = u - u_\omega$, $B_u = B$ and $B_d = B$. We can then apply the result of Proposition 1, obtaining the robust reduced order model

$$\dot{\xi} = (S - GL)\xi + Gu_\omega + M(u - u_\omega), \qquad \psi = C\Pi\xi. \tag{14}$$

The resulting reduced order model possesses the following two properties.

Property 1: If the input of the reduced order model (14) is the signal generated by (4), then the steady-state error between y and ψ is zero.

In fact note that in this case $u = u_\omega$ and the model (14) reduces to $\dot{\xi} = (S - GL)\xi + Gu_\omega$, $\psi = C\Pi\xi$.

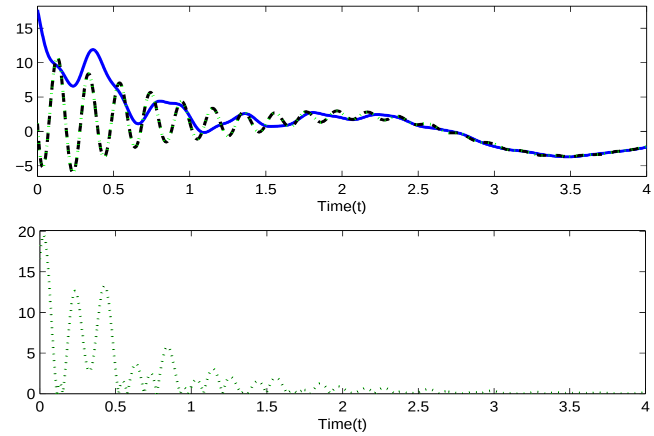

Fig. 2. Case $u = L\omega$. Top: time history of the output of system (12) (solid/blue), time history of the output of the robust reduced order model (14) dashed/black, time history of the output of the nominal reduced order model (dotted/green). Bottom: time history of the corresponding absolute errors.

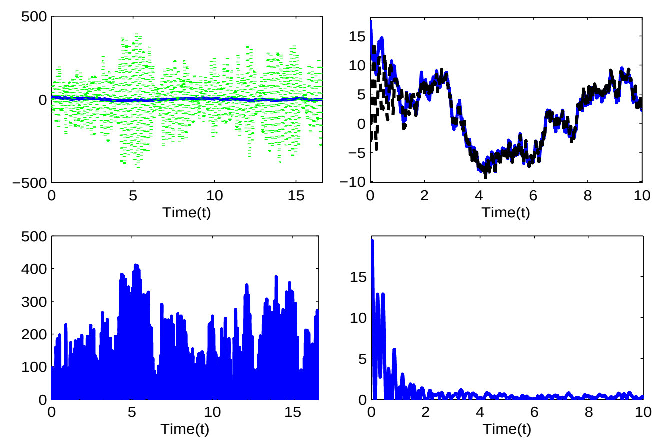

Fig. 3. Case $u \neq L\omega$. Top left: time history of the output of system (12) (solid/blue) and time history of the output of the nominal reduced order model (dotted/green). Bottom left: time history of the corresponding absolute error. Top right: time history of the output of system (12) (solid/blue) and time history of the output of the robust reduced order model (14) (dashed/black). Bottom right: time history of the corresponding absolute error.

Property 2: If the input of the reduced order model (14) is not the interpolated signal generated by (4), then the steady-state error between y and ψ is minimized in the sense of Proposition 1.

In fact note that the resulting model is a robust reduced order model with disturbance $\delta = u - u_\omega$. A schematic interpretation of the reduced model (14) is shown in Fig. 1.

Example 1: We illustrate the enhanced approach with a numerical example. All the quantities in this example have been randomly generated. The results, which are consistent for each random iteration, can be reproduced as follows (obviously obtaining different figures). We have generated a random stable system of order $n = 25$ with the function *rss* of MATLAB. We have generated random vectors L, $\omega(0)$ and $x(0)$ with the function *rand*. We have set $\nu = 5$ and we have selected a matrix S with an eigenvalue at 0 and two pairs of randomly generated imaginary eigenvalues. We have computed a nominal reduced order model assigning the eigenvalues of $S-GL$ using G. Two eigenvalues are selected as the slowest eigenvalues of the matrix A. Two eigenvalues are selected as the eigenvalues with the largest imaginary part of A. The last eigenvalue is a randomly selected real eigenvalue. The matrix M has been computed approximating (10) with $t_p = 20$ and $p = 2000$ (evenly spaced). Fig. 2 shows the results of a simulation in which $u = L\omega$. In the top graph, the solid/blue line is the time history of the output of system (12), the dashed/black line is the time history of the output of the robust reduced order model (14) and the dotted/green is the time history of the output of the nominal reduced order model *i.e.* $\dot{\xi} = (S-GL)\xi + Gu$, $\psi = C\Pi\xi$. The bottom graph displays the corresponding absolute errors. As expected the robust and nominal reduced order model have the same output responses to $u = L\omega$ and both are good asymptotic approximation of the system.

We now consider the input $u(t) = \sum_{i=1}^{4} a_i \sin(b_i t + c_i) + a_5 + 20\,\theta(t)$, where the a_i, b_i and c_i are randomly generated scalars and $\theta(t)$ is a normally distributed noise. Note that the noise is large, *i.e.* comparable with the natural response of the system. Fig. 3 shows the same quantities as Fig. 2 for the new input u for the nominal reduced order model on the left and for the robust reduced order model on the right. We see that the robust reduced order model approximates very well the response of the system (note the small steady-state error in the bottom graph). The behavior of the nominal reduced order model instead is unsatisfactory. The output response undergoes oscillations with intensities ten times greater than the expected value.

V. Model reduction for linear time-varying systems

We now address the problem of model reduction of unperturbed differential inclusions described by

$$\dot{x} \in \mathcal{A}x + \mathcal{B}u, \qquad y = Cx. \tag{15}$$

To solve this problem we first study a related, simpler, problem. Defining a suitable selection criterion, see [38], the linear differential inclusion (15) can be seen as a way of simultaneously considering all time-varying matrices $A : \mathbb{R} \to \mathcal{A}$ and $B : \mathbb{R} \to \mathcal{B}$, and the associated time-varying linear system

$$\dot{x} = A(t)x + B(t)u, \qquad y = Cx. \tag{16}$$

Similarly, the linear differential inclusion (4) can be seen as a way of simultaneously considering all the solutions of the time-varying system

$$\dot{\omega} = S(t)\omega, \qquad u = L\omega. \tag{17}$$

with $S : \mathbb{R} \to \mathcal{S}$. We also assume that $A(\cdot)$, $B(\cdot)$ and $S(\cdot)$ are piecewise continuous functions of t and that $A(\cdot)$ and $S(\cdot)$ are bounded on any finite time interval. Let $(t, t_0) \mapsto \Lambda(t, t_0)$ and $(t, t_0) \mapsto \Sigma(t, t_0)$ be the state transition matrices, see [39], of systems (16) and (17), respectively. We recall the definition of exponential stability for linear time-varying systems.

Definition 7: [40] System (16), or simply A, is exponentially stable on $[t_0, \infty)$ if there exists $\alpha \in \mathbb{R}_{>0}$ and $\gamma \in \mathbb{R}_{>0}$ such that

$$||\Lambda(t,s)|| < \gamma e^{-\alpha(t-s)},$$

for any $t_0 \leq s \leq t < \infty$.

The next result provides a description of the steady-state response of the interconnection of system (16) and the signal generator (17).

Lemma 3: Consider the interconnection of system (16) and the signal generator (17). Assume system (16) is exponentially stable and the signal u is bounded backward and forward in time. The steady-state response of the output of such interconnection is

$$y(t) = C\Pi(t)\omega(t),$$

with

$$\Pi(t) = \int_{-\infty}^{t} \Lambda(t,\tau)B(\tau)L\Sigma(\tau - t)d\tau, \tag{18}$$

which is the unique steady-state solution of the differential Sylvester equation

$$\dot{\Pi}(t) = A(t)\Pi(t) - \Pi(t)S(t) + B(t)L. \tag{19}$$

Proof: We begin proving that $\Pi(\cdot)$ is the unique steady-state solution of (19). First of all note that (18) is the limiting motion of

$$\Pi(t) = \left(\Lambda(t,t_0)\Pi(t_0) + \int_{t_0}^{t} \Lambda(t,\tau)B(\tau)L\Sigma(\tau,t_0)d\tau\right)\Sigma(-t,t_0),$$

when $t_0 = -\infty$ (*i.e.* it is its steady-state solution). Note that this last expression of $\Pi(\cdot)$ solves the differential equation (19). In fact, differentiating the quantity $\Pi(t)\Sigma(t,t_0)$ with respect to time yields

$$\begin{aligned}&\dot{\Pi}(t)\Sigma(t,t_0) + \Pi(t)S(t)\Sigma(t,t_0) = A(t)\Lambda(t,t_0)\Pi(t_0)\\ &\quad + \frac{d}{dt}\int_{t_0}^{t} \Lambda(t,\tau)B(\tau)L\Sigma(\tau,t_0)d\tau = A(t)\Lambda(t,t_0)\Pi(t_0)\\ &\quad + A(t)\int_{t_0}^{t} \Lambda(t,\tau)B(\tau)L\Sigma(\tau)d\tau + \Lambda(t,t)B(t)L\Sigma(t),\end{aligned}$$

where the last expression is obtained using the "differentiation under the integral sign" formula. As a result

$$\dot{\Pi}(t)\Sigma(t,t_0) + \Pi(t)S(t)\Sigma(t,t_0) = A(t)\Pi(t) + B(t)L\Sigma(t,t_0),$$

which yields (19) once we multiply by $\Sigma(t,t_0)^{-1}$. Note that the solution of the differential equation (19) is unique if $A(\cdot)$, $S(\cdot)$ and $B(\cdot)$ are piecewise continuous functions of t and $A(\cdot)$ and $S(\cdot)$ are bounded on any finite time interval, see [41, Theorem 3.2]. Moreover, the steady-state solution is well-defined since system (16) is exponentially stable and the signal u is bounded backward and forward in time, see [34]. We now define the variable $z(t) := x(t) - \Pi(t)\omega(t)$ and compute its derivative with respect to time, namely

$$\begin{aligned}\dot{z} &= \dot{x} - \dot{\Pi}\omega - \Pi\dot{\omega} = Ax + Bu - \dot{\Pi}\omega - \Pi S\omega =\\ &= A(x - \Pi\omega) + (A\Pi + BL - \dot{\Pi} - \Pi S)\omega =\\ &= A(x - \Pi\omega).\end{aligned}$$

Hence,

$$x(t) = \Pi(t)\omega(t) + \Lambda(t,t_0)(x(t_0) - \Pi(t_0)\omega(t_0)).$$

The claim follows since $\Lambda(t,t_0)(x(t_0) - \Pi(t_0)\omega(t_0))$ is the transient response that converges to zero exponentially. ■

We provide now a family of time-varying reduced order models of system (16).

Lemma 4: Consider the interconnection of system (16) and the signal generator (17). Assume system (16) is exponentially stable and the signal u is bounded backward and forward in time. The system described by the equations

$$\dot{\xi} = F(t)\xi + G(t)u, \qquad \psi = H(t)\xi, \tag{20}$$

with $F : \mathbb{R} \to \mathbb{R}^{\nu\times\nu}$ $G : \mathbb{R} \to \mathbb{R}^{\nu\times 1}$, $H : \mathbb{R} \to \mathbb{R}^{1\times n}$, is a reduced order model of system (16) at (L, S) if $\nu < n$ and there exists a unique steady-state solution $P(\cdot)$ of the equation

$$\dot{P}(t) = F(t)P(t) + G(t)L - P(t)S, \tag{21}$$

such that

$$H(t)P(t) = C\Pi(t), \tag{22}$$

where Π is given by (18).

Proof: The claim is a direct consequence of Definition 4 and Lemma 3. ■

We now simplify the conditions in Lemma 4, providing a simple family of reduced order models.

Proposition 2: Consider system (16) and the signal generator (17). Assume system (16) is exponentially stable and the signal u is bounded backward and forward in time. Then the system described by the equations

$$\dot{\xi} = (S(t) - G(t)L)\xi + G(t)u, \qquad \psi = C\Pi(t)\xi, \tag{23}$$

where Π is given by (18), is a reduced order model of system (16) at (L, S) for any G such that $S(t) - G(t)L$ is exponentially stable.

Proof: Similarly to the proof of Proposition 1, we select $P(t) = I$ which yields $F(t) = S(t) - G(t)L$ from equation (21) and $H(t) = C\Pi(t)$ from equation (22). As a result, for any selection of $G(t)$ such that $S(t) - G(t)L$ is exponentially stable, the model (23) has the same moments of system (16) at (L, S). ■

Remark 4: There is no loss of generality in selecting $P(t) = I$. In fact, assume that the equation

$$\dot{\widetilde{P}}(t) = \widetilde{F}(t)\widetilde{P}(t) + \widetilde{G}(t)L - \widetilde{P}(t)S, \tag{24}$$

has a unique solution $\tilde{P}(t) \neq I$ which is invertible for all $t \in \mathbb{R}$ and consider the model

$$\dot{\xi} = \widetilde{F}(t)\xi + \widetilde{G}(t)u, \qquad \psi = \widetilde{H}(t)\xi. \tag{25}$$

Define the new variable $z(t) := P(t)^{-1}\xi(t)$ and note that

$$\dot{\xi} = \dot{\widetilde{P}}(t)z + \widetilde{P}(t)\dot{z} = \widetilde{F}(t)\xi + \widetilde{G}(t)u = \widetilde{F}(t)\widetilde{P}(t)z + \widetilde{G}(t)u.$$

Solving this equation with respect to $\dot{z}$ yields

$$\dot{z} = \widetilde{P}(t)^{-1}(\widetilde{F}(t)\widetilde{P}(t) - \dot{\widetilde{P}}(t))z + \widetilde{P}(t)^{-1}\widetilde{G}(t)u.$$

Substituting equation (24) in the last expression we obtain

$$\begin{aligned}\dot{z} &= \widetilde{P}(t)^{-1}(\widetilde{P}(t)S(t) - \widetilde{G}(t)L)z + \widetilde{P}(t)^{-1}\widetilde{G}(t)u \\ &= (S(t) - G(t)L)z + G(t)u,\end{aligned}$$

where $G = \widetilde{P}^{-1}\widetilde{G}$, and note that as a result $F = \widetilde{P}^{-1}(\widetilde{F}\widetilde{P} - \dot{\widetilde{P}})$. Finally note that the matching condition for system (25) is

$$\widetilde{H}(t)\widetilde{P}(t) = C\Pi(t),$$

which yields $\widetilde{H}(t) = C\Pi(t)\widetilde{P}(t)^{-1}$ and $\psi = \widetilde{H}(t)\xi = C\Pi(t)z$.

We consider now the problem of selecting $G(t)$ such that the reduced order model (23) is exponentially stable. The solution of this problem is given in the next result.

Theorem 1: Let $\Phi(t,t_0)$ be the state transition matrix of system (23). The following statements are equivalent.

(ES1) The matrix $G(t)$ is such that there exists a bounded non-negative definite matrix $Z(t)$ which satisfies

$$\dot{Z}(t) = (S(t) - G(t)L)Z - Z(S(t) - G(t)L)^\top + G(t)G(t)^\top. \tag{26}$$

(ES2) The matrix $G(t)$ is such that

$$Z(t) = \int_{t_0}^{t} \Phi(t,\tau)G(\tau)G(\tau)^\top \Phi(t,\tau)^\top d\tau \tag{27}$$

is bounded and non-negative.

(ES3) System (23) is exponentially stable.

Proof: (ES1) $\iff$ (ES2). Computing the derivative of (27) with respect to time we see that (27) is the solution of equation (26).

(ES1) $\iff$ (ES3). Note that the standing assumption that all the triples $(L, S, \omega(0))$ are minimal implies that all the pairs (L, S) that we consider are observable. This implies that there exists a bounded piecewise continuous matrix $G(t)$ such that $S(t) - G(t)L$ is exponentially stable, see [40, Definition 2.6 (b)]. The result follows from [40, Proposition 2.5 (b)]. ■

VI. Model reduction of linear differential inclusions

We can now summarize the main result of the paper in the next statement which solves Problem 1 and provides a family of reduced order models for the differential inclusion (1). We omit the solution of Problem 2 for reasons of space.

Proposition 3: Consider the differential inclusion (15) and the signal generator (2). Assume that all matrices $A : \mathbb{R} \to \mathcal{A}$ are exponentially stable and the signal u is bounded backward and forward in time. Then the differential inclusion described by

$$\dot{\xi} \in \mathcal{F}\xi + \mathcal{G}u, \qquad \psi = \mathcal{H}\xi, \tag{28}$$

is a reduced order model of the differential inclusion (15) at $(L, \mathcal{S})$ if

- $\mathcal{G} := \{G : \mathbb{R} \to \mathbb{R}^{\nu\times 1} : S(t) - G(t)L$ is exponentially stable, $\forall\, S : \mathbb{R} \to \mathcal{S}\}$.
- $\mathcal{F} := \{F : \mathbb{R} \to \mathbb{R}^{\nu\times\nu} : F(t) = S(t) - G(t)L, \forall\, S : \mathbb{R} \to \mathcal{S}, \forall\, G : \mathbb{R} \to \mathcal{G}\}$.

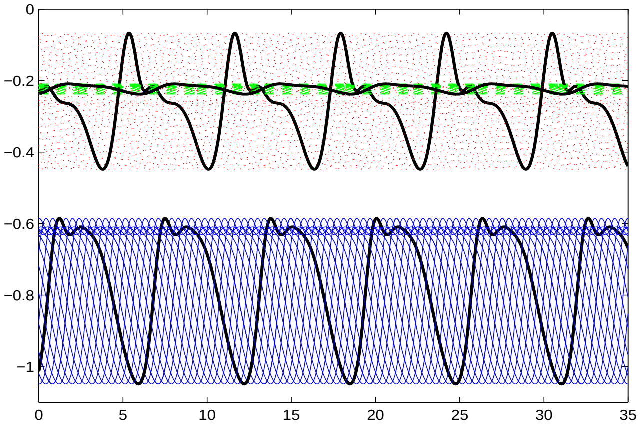

Fig. 4. Time history of the three components of the vector $C\Pi(t)$ for all the time-varying systems in $\bar{\mathcal{A}}$, $\bar{\mathcal{S}}$. One particular solution is highlighted with a solid/black line.

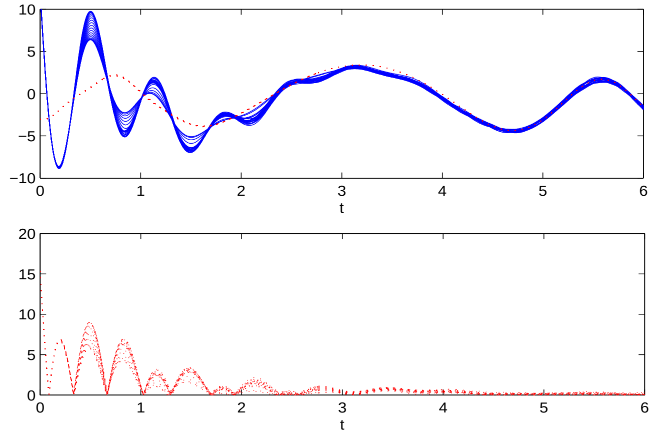

Fig. 6. Top: time history of the output of the differential inclusion (15) (solid/blue) and of the output of the approximate reduced order model (4) (dotted/red). Bottom: time history of the corresponding absolute error.

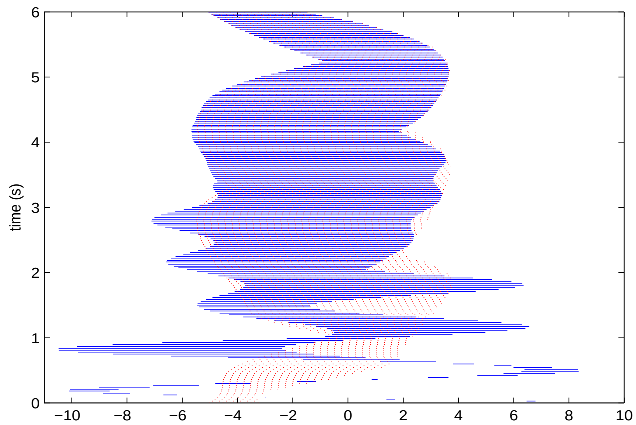

Fig. 5. Set of the outputs of the differential inclusion (15) (solid/blue) and set of the outputs of the approximate reduced order model (4) (dotted/red). Each horizontal line is constant in time.

- $\mathcal{P} := \{\Pi : \mathbb{R} \to \mathbb{R}^{n\times\nu} : \dot{\Pi} \in \mathcal{A}\Pi + \mathcal{B}L - \Pi\mathcal{S}\} = \{\Pi : \mathbb{R} \to \mathbb{R}^{n\times\nu} : \dot{\Pi}(t) = A(t)\Pi(t) + B(t)L - \Pi(t)S(t), \forall A : \mathbb{R} \to \mathcal{A}, \forall B : \mathbb{R} \to \mathcal{B}, \forall S : \mathbb{R} \to \mathcal{S}\}$.
- $\mathcal{H} := \{H : \mathbb{R} \to \mathbb{R}^{1\times\nu} : H(t) = C\Pi(t), \forall \Pi : \mathbb{R} \to \mathcal{P}\}$.

The differential inclusion (28) is an approximate reduced order model of the differential inclusion (15) at $(L, \bar{\mathcal{S}}, \bar{\mathcal{A}}, \bar{\mathcal{B}})$ if

$$\mathcal{P} := \{\Pi : \mathbb{R} \to \mathbb{R}^{n\times\nu} : \dot{\Pi} \in \bar{\mathcal{A}}\Pi + \bar{\mathcal{B}}L - \Pi\bar{\mathcal{S}}\},$$

where $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ and $\bar{\mathcal{S}}$ are subsets of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$, respectively.

Proof: The proposition is a direct consequence of the results proved in the previous sections. ■

Example 2: We illustrate Proposition 3 with a numerical example. Consider the class of linear differential inclusions (15) and of signal generators (4). The sets have been generated as follows. For graphical clarity we select $n = 10$, but similar results have been obtained with $n > 100$. Let A^{n} be a randomly generated stable matrix and let $A^{\mathrm{p}} = \mathrm{diag}(A^{\mathrm{p}}_{11}, \ldots, A^{\mathrm{p}}_{nn})$, $A^{\mathrm{p}}_{ii} > 0$, be a randomly generated diagonal matrix such that $\max A^{\mathrm{p}}_{ii} \leq \max \mathrm{real}\,\mathrm{eig}(A^{\mathrm{n}})$. The set $\mathcal{A}$ has been defined as $\mathcal{A} := \{A : A^{\mathrm{n}}_{ij} - A^{\mathrm{p}}_{ij} \leq A_{ij} \leq A^{\mathrm{n}}_{ij} + A^{\mathrm{p}}_{ij}, i, j = 1, \ldots n\}$. Let $\mathcal{S} := \{S = [0,\, S_{12},\, 0;\, -S_{12},\, 0,\, 0;\, 0,\, 0,\, 0],\, 1.5 \leq S_{12} \leq 4.5\}$. The matrix L, C and the single-valued $\mathcal{B}$ have been randomly generated. The set $\mathcal{A}$ is approximated by the finite set $\bar{\mathcal{A}}$ containing k time varying matrices $A^k(t)$ generated as $A^k_{ij}(t) = A^{\mathrm{n}}_{ij} + A^{\mathrm{p}}_{ij} cos(t + \phi_k)$, with $\phi_k = -\pi, \ldots, \pi$. The set $\mathcal{S}$ is approximated in a similar fashion. The steady-state solution of equation (19) is computed for each time-varying system associated to the differential inclusion. Fig. 4 shows the time history of the three components of the vector $C\Pi(t)$ for all the time-varying systems in $\bar{\mathcal{A}}$, $\bar{\mathcal{S}}$. For each component, the figure shows one particular solution with a solid/black line.

The approximate reduced order model (28) has been constructed as follows. The matrix G^{n} has been computed assigning the eigenvalues of the matrix $F^{\mathrm{n}} = S^{\mathrm{n}} - G^{\mathrm{n}}L$. The time-varying matrix $G(t)$ is generated as time-varying perturbation of G^{n}, namely $G^k_i(t) = G^{\mathrm{n}}_i + G^{\mathrm{p}}_i \cos(t + \phi_k)$, $G^{\mathrm{p}}_i \in \mathbb{R}$, for all $i = 1, \ldots, \nu$, such that the resulting $S(t) - G(t)L$ are exponentially stable. Thus the set $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ have been constructed as in Proposition 3. Fig. 5 shows the set of the outputs of the differential inclusion (15) as it evolves in time in solid/blue line and the set of the outputs of of the approximate reduced order model (4) as it evolves in time in dotted/red line for one randomly generated initial condition. The two sets converge asymptotically to each other as expected.

We now consider a matrix $\tilde{A}(t) \in \mathcal{A} \setminus \bar{\mathcal{A}}$, to test the approximate reduced order model for systems that we have not interpolated by means of an associated time-varying system. Fig. 6 shows the time history of some selected output of the differential inclusion (15) in solid/blue and of some selected output of the approximate reduced order model (4) in dotted/red line. The bottom graph shows the corresponding absolute error. The matrix Π used in the simulation is the one obtained with the closest matrix to $\tilde{A}$ in $\mathcal{A}$. We see that the error does not converge to zero but the approximation is close.

VII. Conclusion

Introducing the notions of moment-set, perturbed moment trajectory, approximate reduced order model and robust reduced order model we have formulated and solved the problem of model reduction by moment matching for linear differential inclusions. We have also provided robust reduced order models with respect to input perturbations, an enhanced model reduction scheme for linear differential equations and reduced order models for time-varying systems driven by time-varying signal generators. Many open problems need to be addressed by further research. For instance the selection of the interpolating subsets $\bar{\mathcal{A}} \subset \mathcal{A}$, $\bar{\mathcal{B}} \subset \mathcal{B}$, $\bar{\mathcal{S}} \subset \mathcal{S}$ is of paramount importance to achieve a satisfactory approximation for the whole sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$. Other research directions include the extension of these results to other classes of inclusions, such as difference inclusions, nonlinear inclusions and hybrid inclusions.

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