

## Comment on “Finite-size scaling of survival probability in branching processes”

Nanxin Wei\* and Gunnar Pruessner†

Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

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R. Garcia-Millan *et al.* [*Phys. Rev. E* **91**, 042122 (2015)] reported a universal finite-size scaling form of the survival probability in discrete time branching processes. In this comment, we generalize the argument to a wide range of continuous time branching processes. Owing to the continuity, the resulting differential (rather than difference) equations can be solved in closed form, rendering some approximations by R. Garcia-Millan *et al.* superfluous, although we work along similar lines. In the case of binary branching, our results are in fact exact. Demonstrating that discrete time and continuous time models have their leading order asymptotics in common, raises the question to what extent corrections are identical.

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In the following we will briefly outline the derivation of the survival probability in continuous time branching processes, many aspects of which are covered by textbooks such as Ref. [1]. In the process considered, each individual node undergoes branching with Poissonian rate  $\lambda$ , replacing it with probability  $p_k$  by  $k \in \mathbb{N}$  nodes. In particular,  $k = 0$  corresponds to spontaneous extinction of an individual and  $k = 1$  to no change. As in Ref. [2], the generating function  $f(s) = \sum_{k=0}^{\infty} p_k s^k$  establishes normalization via  $f(1) = 1$ , the mean as  $m = f'(1)$ , and the variance  $\sigma^2$  via  $f''(1) = \sigma^2 + m^2 - m$ . While  $f(s)$  characterizes the branching, the generating function  $G(s, t)$  is concerned with the process as a whole. If  $N_t$  denotes the number of individuals at time  $t$ , then  $G(s, t)$  is the expectation of  $s^{N_t}$  so that

$$G(s, t) = s e^{-\lambda t} + \int_0^t d\tau \lambda e^{-\lambda \tau} f(G(s, t - \tau)), \quad (1)$$

where the first term accounts for no branching and the second for (the first) branching at time  $\tau$ , leaving the process time  $t - \tau$  to develop further. The probability  $P_0(t)$  of extinction by time  $t$ , which is the probability of  $N_t = 0$ , is given by  $\lim_{s \rightarrow 0} G(s, t)$  and obeys  $\dot{P}_0/\lambda = f(P_0) - P_0$ , by differentiating Eq. (1) with respect to  $t$ . Solving  $P_0$  in closed form is generally dependent on the precise choice of  $f(s)$ . The procedure is particularly easy if all  $p_k$  vanish for  $k > k^*$ . In the following, we will expand  $f(s)$  about its smallest fixed point, recovering exact results for  $k^* = 2$ , i.e., quadratic  $f(s)$ .

The eventual extinction probability  $q = \lim_{t \rightarrow \infty} P_0(t)$  is given by the smallest fixed point of the offspring generating function [3],  $f(q) = q$  with  $q = 1$  for  $m \leq 1$  and  $q < 1$  otherwise.  $P_0$  approaches  $q$  from below as  $t$  increases. Defining  $Q = q - P_0$ , it obeys

$$\dot{Q}(t) = -\lambda Q \left[ 1 - f'(q) + \frac{1}{2} f''(q) Q + \mathcal{O}(Q^2) \right] \quad (2)$$

by expanding  $f(q - Q)$  to second order in small  $Q$ , which is expected for  $t \ll \lambda^{-1}$ . In the *subcritical regime*,  $m < 1$ , it

follows that  $q = 1$ , and therefore  $Q$  is the complement of  $P_0$ , or the survival probability  $P_{\text{surv}} = 1 - P_0$ . Using the expansion in Eq. (2), which is exact for  $k^* = 2$ , we find

$$P_{\text{surv}}(t) \approx \frac{1}{t f''(1)} \frac{2y \exp(y\lambda)}{\exp(y\lambda) - 1 - \frac{2(1-m)}{f''(1)}}, \quad (3)$$

where  $y = t(m - 1)$  (using the notation of Ref. [2]). In large  $t \gg \lambda^{-1}$  with  $y$  fixed,  $f''(1)$  converges to  $\sigma^2$  and  $(1 - m)/f''(1)$  vanishes, recovering Eq. (4) of Ref. [2]. In the *critical regime*,  $m = 1$  and we find

$$P_{\text{surv}}(t) \approx \left( 1 + \frac{1}{2} \lambda \sigma^2 t \right)^{-1}, \quad (4)$$

which has the same asymptote in  $t$  as Eq. (3) for small  $y \ll 1$ . The *supercritical regime*,  $m > 1$ , is somewhat messier, because the smallest fixed point  $q$  of  $f(s)$  is no longer unity, whereas mean and variance of the offspring number derive from properties of  $f(s)$  there. Because  $f(1) = 1$  and  $f(s)$  is strictly convex on  $[0, 1]$ , we have  $f'(s) < 1$  and  $Q = q - 1 + P_{\text{surv}}$  therefore follows the derivation in the *subcritical regime*, resulting in

$$P_{\text{surv}}(t) \approx 1 - q + \frac{2(f'(q) - 1)}{f''(q)} \times \frac{\exp(\lambda(f'(q) - 1)t)}{\exp(\lambda(f'(q) - 1)t) - 1 - \frac{2(1-f'(q))}{f''(q)q}} \quad (5)$$

exact for  $k^* = 2$  and extending the result in Ref. [2] to  $q$  not necessarily close to unity. In Ref. [2],  $f'(q)$  is approximated by  $2 - m$  and  $f''(q)$  by  $\sigma^2$ , after some algebra recovering Eq. (3) above. Noticeably, Eq. (5) does not rely on the introduction of the “critical variance”  $\sigma_c^2$  in Ref. [2]. In summary, we recover for the continuous time branching process the asymptotes determined by Ref. [2]. The approximate signs in Eqs. (3), (4), and (5) become equalities in the case of a branching process producing at most two offspring at a time,  $k^* = 2$ , which is equivalent to a birth-death process with constant birth and extinction rates for each individual.

Given the similarity between the continuous time branching process and the Galton-Watson process analyzed by Ref. [2] and the fact that the latter is embedded in the former [3], the shared asymptotics may not come as a great surprise. However,

\*n.wei14@imperial.ac.uk

†g.pruessner@imperial.ac.uk

not much is known about the universal finite-size scaling of these processes, derivable in the present case in closed form. This is obviously the major advantage of considering

a continuous time process such as the present one. Finding subleading terms, even just in  $1 - q$  for the supercritical case [Eq. (5)], amounts to a great challenge for discrete time models.

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