Studies on Modal Logics of Time and Space

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Abstract

This dissertation presents original results in Temporal Logic and Spatial Logic. Part I concerns Branching-Time Logic. Since Prior 1967, two main semantics for Branching-Time Logic have been devised: Peircean and Ockhamist semantics. Zanardo 1998 proposed a general semantics, called Indistinguishability semantics, of which Peircean and Ockhamist semantics are limit cases. We provide a finite axiomatization of the Indistinguishability logic of upward endless bundled trees using a non-standard inference rule, and prove that this logic is strongly complete.

In Part II, we study the temporal logic given by the tense operators F for future and P for past together with the derivative operator $\langle d \rangle$, interpreted on the real numbers. We prove that this logic is neither strongly nor Kripke complete, it is PSPACE-complete, and it is finitely axiomatizable.

In Part III, we study the spatial logic given by the derivative operator $\langle d \rangle$ and the graded modalities $\{\Diamond_n \mid n \in \mathbf{N}\}$. We prove that this language, call it L, is as expressive as the first-order language L_t of Flum and Ziegler 1980 when interpreted on T_3 topological spaces. Then, we give a general definition of modal operator: essentially, a modal operator will be defined by a formula of L_t with at most one free variable. If a modal operator is defined by a formula predicating only over points, then it is called point-sort operator. We prove that L, even if enriched with all point-sort operators, however enriched with finitely many modal operators predicating also on open sets, cannot express L_t on T_2 spaces. Finally, we axiomatize the logic of any class between all T_1 and all T_3 spaces and prove that it is PSPACE-complete.

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Declaration of Originality

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Part I of this dissertation is based on and reproduces parts of:

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Chapter 1

Introduction

This dissertation presents original results in Modal Logic. Modal Logic has been an active area of research over the last century (seminal is the work of Lewis [68]) and its roots go back to Ancient Greek Philosophy. Modal Logic has started as the study of the notions of 'necessity' and 'possibility'. These notions have been denoted by symbols, $\Box$ for necessity and $\Diamond$ for possibility, called *modal operators*, and added to Propositional Logic, obtaining formulas of the like:

$$\Box(p \to q) \to (\Box p \to \Box q),$$

meaning: (necessarily (p implies q)) implies ((necessarily p) implies (necessarily q)).

Since the late '50s, this language traditionally has been interpreted over *Kripke models* (seminal is the work of Kripke [59, 60, 61]). Kripke models are structures (W, R, V) such that W is a nonempty set, R is a binary relation on W, and V is a map that associates every propositional variable to a set of elements of W. Intuitively, W represents a set of possible worlds; wRv represents that a world v is accessible from a world w; and, for every propositional variable p, $V(p)$ represents the set of worlds at which p is true. Then, the semantics of 'necessarily' ($\Box$) and 'possibly' ($\Diamond$) have been defined as follows:

- 'Necessarily φ' ($\Box\varphi$) is true at a world w provided φ is true at all worlds accessible from w (that is, all worlds v such that wRv).
- 'Possibly φ' ($\Diamond\varphi$) is true at a world w provided φ is true at some possible world accessible from w (that is, some world v such that wRv).

Observe that $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$.

Kripke semantics has been very successful: the expressivity of modal languages on Kripke models, the definability of particular classes of Kripke models, the axiomatizability of the modal logics of several classes of Kripke models, the decidability and the complexity of these logics have been widely studied (refer to [13, 14] for further reading).

Also, Kripke semantics has been very robust in terms of applications. Modal logicians noticed that modal languages on Kripke models not only could represent the notions of 'necessity' and 'possibility', but also those of 'knowledge', 'belief', 'preference', 'provability', 'in the future', 'after the execution of a program', etc. Thus, Modal Logic found applications in a plethora of fields, such as Artificial Intelligence, Computer Science, Economics, Linguistics, Mathematics, etc. (refer to [14, 95] for further reading).

Another reason for the success of Modal Logic in Kripke semantics is its good computational behavior. In Kripke semantics, the modal operators quantify only over the set of accessible worlds. This feature of locality makes many modal logics decidable with relatively low complexity (refer to [13, 14] for further reading).

1.1 Modal logics of time

Part I of this dissertation is about the temporal interpretation of the modal operators. Since the work of Prior in the '60s [77], it has been noticed that if we take (W, R) as a linear order, then (W, R, V) can be seen as a timeline: the elements of W represent instants; wRv represents 'v being in the future of w'; and every instant w has a set of propositional variables true at w determined by V. Then, $\Box\varphi$ means: 'φ will always happen'; and $\Diamond\varphi$: 'φ will eventually happen'. In this framework, $\Box$ is generally denoted by G, and $\Diamond$ is generally denoted by F. Similarly, past operators H, meaning 'always in the past', and P, meaning 'sometime in the past', can be defined.

Also, it has been noticed that if we take (W, R) to be a tree instead of a linear order, then each maximal linear subset of the tree can be seen as a possible history in a non-deterministic representation of time. Following this intuition, many languages and semantics have been devised. For example, we may evaluate $G\varphi$ at an instant w and decide that $G\varphi$ is true at w provided 'φ will always be true in every possible future of w' (that is, in every v such that wRv); or we may evaluate $G\varphi$ at an instant w with respect to an history h passing through w, and decide that $G\varphi$ is true at (w, h) provided 'φ will always be true in the future of t in h' (that is, in every instant v such that $v \in h$ and wRv); we can add an operator g, evaluate $g\varphi$ at an instant w, and decide that $g\varphi$ is true at w provided 'φ will always be true in some possible future of w' (that is, there is an history h passing through w such that φ is true at every instant $v \in h$ such that wRv); or we can add a modal operator N, and decide that $N\varphi$ is true at (w, h) provided 'φ is true at (w, k) for every history k passing through w'.

Among the most important proposals in this direction, are the so called Peircean and Ockhamist Logics. In [102] Zanardo offered a logic, call it Indistinguishability Logic, which generalizes Peircean and Ockhamist Logics. We devote Part I of this dissertation to the axiomatization of Indistinguishability Logic, presenting an axiomatization that essentially blends the axioms for Peircean Logic presented in [20] with those for Ockhamist Logic presented in [42, Section 7.7, pp. 299–306]. Also the proof that the axioms are complete blends original elements with elements taken from [20] and [42, Section 7.7, pp. 299–306].

1.2 Modal logics of space

Kripke semantics has not been the only semantics for modal languages. Since the '40s, so predating Kripke semantics, a topological semantics has also been suggested (seminal is the work of McKinsey and Tarski [71]). Languages are interpreted over *topological models*. Topological models are structures (A, τ, V) such that (A, τ) is a topological space, and V a map that associates every propositional variable to a subset of A. The semantics for $\Box$ and $\Diamond$ are defined as follows:

- $\Box\varphi$ is true at a point a provided there is a neighborhood U of a such that φ is true at all points of U.
- $\Diamond\varphi$ is true at a point a provided, for every neighborhood U of a, φ is true at some point of U.

Observe that $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$. Also, under this semantics, the $\Box$ and $\Diamond$ operators corresponds respectively to the topological interior and closure operators. For, the set of points at which $\Box\varphi$ is true is equal to the interior of the set of points at which φ is true; while the set of points at which $\Diamond\varphi$ is true is equal to the closure of the set of points at which φ is true.

Other operators have also been considered, for example, the coderivative operator $[d]$ and the derivative operator $\langle d\rangle$ (already suggested in [71]), of which the semantics are defined as follows:

- $[d]\varphi$ is true at a point a provided there is a neighborhood U of a such that φ is true at all points of $U \setminus \{a\}$.
- $\langle d\rangle\varphi$ is true at a point a provided, for every neighborhood U of a, φ is true at some point of $U \setminus \{a\}$.

Observe that $\langle d\rangle\varphi$ is equivalent to $\neg[d]\neg\varphi$. In this case, $[d]$ and $\langle d\rangle$ correspond respectively to the topological coderivative and derivative operators.

Although, since its introduction, Kripke semantics has been more widely studied than topological semantics, topological semantics has been continuously studied since the '70s, in particular by the 'Georgian School', which mostly focused on the operators $[d]$ and $\langle d\rangle$, obtaining results about expressivity, definability, axiomatizability, decidability and complexity, and interest in topological semantics has increased since the

2000s (see [4] and Section 8.2.1 for further reading, and [5, 7, 8, 9, 10, 11, 32, 45, 46, 48, 52, 57, 58, 62, 64, 70, 72, 88, 91] for examples of recent work).

Also, it is interesting to observe that temporal and spatial semantics can be brought together by the following intuition: given a linear order (W, R), the set of open intervals constitutes a base for a topology over W, called *interval topology*. Then, as Scott proposed in [83], together with temporal operators, we can have the following 'present progressive' operator $\Box$: $\Box\varphi$ is true at an instant w provided there is an open interval containing w such that all instants in that interval satisfy φ. This is nothing else than the topological semantics with respect to the interval topology.

This suggestion was undertaken by Shehtman, who studied the $G, H, \Box$-logic of the rationals [86], and by Hodkinson, who studied the $G, H, \Box$-logic of the reals [53]. In Part II we follow this direction and study the $G, H, [d]$-logic of the reals: we axiomatize this logic, prove that it is PSPACE-complete, and show that it is neither strongly complete with respect to topological semantics nor Kripke complete, employing techniques which owe much to both Shehtman's and Hodkinson's results.

Traditionally, the expressivity of modal languages on Kripke models have been compared to the expressivity of first-order languages. This suggests to do the same for modal languages on topological models. In [40], the first-order language L_t to talk about topological spaces has been presented. L_t is a two-sorted language: we have first-sort variables, constants, etc., that are assigned to points, and second-sort variables, constants, etc., that are assigned to open sets. Moreover, second-sort quantification can occur only in certain forms. If we remove these restrictions on second sort quantification, we obtain the more general language L_2. However, L_2, when interpreted on topological spaces, does not enjoy the Compactness, Löwenheim Skolem and Lindström Theorems, whereas L_t does. Furthermore, L_t can express a number of interesting properties of topological spaces, namely every first-order property in L_2 that is invariant under changing base (see Section 17.1).

In Part III, we compare the expressivity of modal languages interpreted on topological spaces with that of L_t. We consider the language L with coderivative operator $[d]$ and graded operators $\{\Diamond_n \mid n \in \mathbf{N}\}$. The semantics of graded operators is defined as follows:

- For every $n \in \mathbf{N}$, $\Diamond_n\varphi$ is true (at a point a) provided there are more than n points at which φ is true.

We prove that L can express L_t on the class of all T_3 topological spaces, and that there is a computable translation from L to L_t. Then, we look at ways to increase the expressivity of L by adding new modal operators. Following Gabbay, Hodkinson and Reynolds [42, Chapter 6], we define a modal operator as a symbol # of which the semantics is given by a formula of L_t with at most one free first-sort variable and no free second-sort variables. Call point-sort those operators of which the semantics is defined by a formula in which only first-sort variables occur. We show that even if we enrich L with all the point-sort modal operators, call L^+ the resulting language, however we add finitely many modal operators to L^+, we cannot express L_t on the class of all T_2 topological models. This result, although stemming from a similar result by Hodkinson in Kripke semantics [51], uses a substantial amount of original techniques. Finally, we study the logic in L of any class of topological models between the class of all T_3 and the class of all T_1 topological models, axiomatizing this logic, and proving that its complexity is PSPACE-complete.

1.3 Plan of the dissertation

- In Part I, we consider Indistinguishability Logic. More precisely, we axiomatize the indistinguishability logic of bundled trees, that is trees of which only a given selection of their histories is considered. This axiomatization uses a non-standard rule, called 'IRR rule'. We conclude this part, discussing ideas to study the decidability and complexity of this logic, ideas to remove the IRR rule, and ideas to axiomatize the unbundled version of this logic.

- In Part II, we consider the language with modal operators G, H and $[d]$ interpreted over the reals. We axiomatize the arising logic, study its complexity proving that it is PSPACE-complete, and we prove that this logic is neither strongly nor Kripke complete (and hence it does not have the finite model property either). We conclude this part, presenting some ideas for future work. In particular, we mention the study of this language interpreted on the rationals, and the splitting of the derivative operator $\langle d \rangle$ in its future component K^+ and past component K^-, suggested in [42, §6.1], of which the semantics is defined as follows:

  - $\mathcal{R}, x \models K^+\varphi$ provided, for every $y > x$, there is $z \in (x, y)$ such that $\mathcal{R}, z \models \varphi$.
  - $\mathcal{R}, x \models K^-\varphi$ provided, for every $y < x$, there is $z \in (y, x)$ such that $\mathcal{R}, z \models \varphi$.

- In Part III, we consider the language L with modal operators $[d]$ and $\{\Diamond_n \mid n \in \mathbf{N}\}$ interpreted over topological models. We compare its expressivity with the expressivity of the two-sorted first-order language L_t of [40]. We prove that L can express L_t on the class of all T_3 topological models, and that there is a computable translation from L to L_t. Then, in order to increase the expressivity of L_t we generalize the definition of modal operator. Following Gabbay, Hodkinson and Reynolds [42, Chapter 6], we define a modal operator as a symbol # the semantics of which is defined by a formula of L_t with at most one free first-sort variable and no free second-sort variables. If a modal operator has its semantics defined by a formula with no second-sort variables occurring, then it is called a point-sort modal operator. Call L^+ the language that we obtain from L by adding all the point-sort modal operators. We prove that, however we add finitely many modal operators to L^+, the resulting language cannot express L_t on the class of all T_2 topological spaces. Then, we study the logic in L of every class $\mathfrak{X}$ of topological models in between the class of all T_1 and the class of all T_3 topological models. We prove that for every such $\mathfrak{X}$, this logic is always the same, in particular the logic in L of all T_1 topological models is equal to the logic of all T_2 topological models and to the logic all T_3 topological models. We axiomatize this logic, and prove that it is PSPACE-complete. We conclude this part, presenting some questions for future work. In particular, questions regarding the study of the logic in L of particular classes of topological models, the enrichment of L with fixed-point operators, the application of Spatial Logic to real-world situations, the study of modal operators intended as per the general definition of modal operator given in this part, and the study of these two modal operators:

- $U(\varphi, \psi)$ holds at x provided there is a neighborhood U of x such that φ holds in the boundary of U and ψ in U.
- $\langle d \rangle_n \varphi$ holds at a point a provided for every neighborhood U of a there are at least n points $b \in U \setminus \{a\}$ such that φ holds at b.

1.4 Final remarks

This dissertation is the result of three and a half years of research activity. On one hand, we present results in some of the traditional lines of research in the field. For example, we have results in axiomatizability, expressivity, decidability and complexity for a number of modal languages and resulting modal logics. On the other hand, we hope that our results can contribute to the development of Modal Logic.

Indeed, consider for example the completeness proofs presented in every part, from Part I to Part III. If, on one hand, they are built on the usual Lemmon-Scott construction of a model satisfying a given formula or set of formulas, on the other hand, a number of innovative elements are present, due to the particular classes of models on which our languages are interpreted.

The same holds for the proof in Part III that the logic in L of any class $\mathfrak{X}$ of topological models between the class of all T_1 and the class of all T_3 topological models is decidable: on one hand, we use the well-known notion of quasi model, on the other hand, we adapt its use to topological semantics. Moreover, the definition of quasi-models of Part III must take into account that we are focusing on a particular class of topological models, namely those between T_1 and T_3 topological models. Finally, also the algorithm, which gives a PSPACE upper bound, is designed so that it can exploit quasi models (in fact, a variation called optimal quasi models).

More generally, we believe that the more mature contributions can be found in Part III. Consider, for example the result according to which our modal language L can express the first-order language L_t on the class of all T_3 topological models. The proof uses a game *à la* Ehrenfeucht-Fraïssé. Both because of the peculiar rules of the game, and because of the particularities of T_3 topological spaces, a number of original elements are present in the proof. Moreover, this result can be seen as analogous to the van Benthem and Kamp Theorems in topological settings. For, analogously to the van Benthem Theorem, it states that on T_3 topological models, L corresponds to the fragment invariant under changing base of the first-order language L_2; analogously to the Kamp Theorem, it states that, on T_3 topological models, L is as expressive as the first-order language L_t.

Furthermore, this result opened the question if L could express L_t on the class of all T_2 topological models. We gave a negative answer, and consequently looked for ways to increase the expressive power of L in order to express L_t on the class of all T_2 topological models. This led to the aforementioned more general definition of modal operators. The result according to which L together with all point-sort modal operators, however enriched with finitely many modal operators, cannot express L_t on T_2 topological spaces perhaps represents the most mature contribution of this dissertation, for at least two reasons. First, the proof, although stemming from a result by Hodkinson [51], required major original contributions. Second, the proof, due to its generality and intricacy, seems to confirm, once again, that there is room in Modal Logic for results of mathematical interest.

Talking about the mathematical potentiality of Modal Logic, it is also worth making the following two points. The first point concerns the more general definition of modal operators given in Part III. This definition seems to suggest a more general theory of Modal Logic. Essentially, if we allow modal operators to be first-order formulas in a given signature with at most one free variable, then we have at our disposal a plethora of new modal operators, and structures on which to interpret them. Programs to study Modal Logic in this generality, like, for example, a classification of modal operators according to validity, expressivity and complexity may have not only an applicative but also a theoretical and mathematical potentiality.

A second point concerns the interaction between Modal Logic and Geometry. Achieving the results of Part III required reasoning about topological spaces according to logical criteria. Also the formulas φ_n of Section 18.2.1 or the basoid models $\mathcal{M}_n$ of Section 18.2.2 are fruits of this paradigm. For, both the formulas and the models were designed having a result in Modal Logic in mind. Nonetheless, the formulas are nothing else than topological properties, and the basoid models $\mathcal{M}_n$ are nothing else than topological spaces satisfying these properties. So, we see that objects of potential interest in Mathematics may arise from logical situations. Also the notion of basoid space (roughly, a couple $(A, \mathcal{B})$ where A is a set and $\mathcal{B}$ a base for a topology on A) comes from the same paradigm. Essentially, basoid spaces have been introduced because of the necessity of having a first-order counterpart of topological models (recall that being topological space is not first-order definable, whereas being a topological base is). This seems to suggest that from Modal Logic we may get contributions in Geometry, obtaining what we may call *Logical Geometry*.

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Chapter 2

Modal Logic

This chapter is about classical definitions and results in Modal Logic. The aim of this chapter is to fix notation and concepts that will be used throughout the rest of this document. As this dissertation treats advanced topics in Logic, we assume the reader familiar with these definitions and results. Therefore, we do not contextualize or discuss them, and we may use them later without mentioning. We refer to [13, Chapter 3]).

2.1 Sets and relations

Call a *countable set* every set A such that the cardinality of A is less or equal than the cardinality of $\mathbf{N}$. For every set A, subset $B \subseteq A$, element $a \in A$ and binary relation R on A, define:

- $\mathcal{P}(A) := \{C \mid C \subseteq A\}$.
- $R(a) := \{b \mid aRb\}$.
- $R{\restriction}B := R \cap (B \times B)$.

For every two sets A, B, we call B an *R-generated* subset of A provided for every element $b \in B$ and $a \in A$, if bRa then $a \in B$.

2.2 Modal languages

Let Φ be a nonempty countable set. The elements of Φ are called *propositional variables* and denoted by small Latin letters $p, q, ...$ Let $\tau = \{\Box_i\}_{i \in I}$ be a non empty countable set. The elements of τ are called *modal operators*. We define the *modal language* $L_{\Phi,\tau}$ as the smallest set such that:

- $\Phi \cup \{\top\} \subseteq L_{\Phi,\tau}$.
- If $\varphi \in L_{\Phi,\tau}$, then $\neg\varphi \in L_{\Phi,\tau}$.
- If $\varphi, \psi \in L_{\Phi,\tau}$, then $(\varphi \wedge \psi) \in L_{\Phi,\tau}$.
- If $\varphi \in L_{\Phi,\tau}$ and $i \in I$, then $\Box_i\varphi \in L_{\Phi,\tau}$.

The elements of $L_{\Phi,\tau}$ are called *formulas* (*in* $L_{\Phi,\tau}$). To avoid proliferation of parentheses, the usual precedence rules among operators are assumed. As abbreviations define $\bot := \neg\top$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and, for every $i \in I$, $\Diamond_i\varphi := \neg\Box_i\neg\varphi$. For every finite set of formulas $\Gamma := \{\gamma_0, ..., \gamma_{n-1}\}$, define:

$$\bigwedge \Gamma := \begin{cases} \top, \text{ if } \Gamma = \emptyset, \\ \gamma_0 \wedge .... \wedge \gamma_{n-1} \text{ otherwise;} \end{cases}$$

and:

$$\bigvee \Gamma := \neg \bigwedge \{\neg\gamma \mid \gamma \in \Gamma\}.$$

2.3 Kripke semantics

Call a *Kripke model* (*for* $L_{\Phi,\tau}$) every tuple $(W, \{R_i\}_{i\in I}, V)$ such that W is a non empty set, for every $i \in I$, R_i is a binary relation on W, and V is a map $V : \Phi \to \mathcal{P}(W)$ called *evaluation*. A Kripke model $(W, \{R_i\}_{\in I}, V)$ is said to be *finite* provided W is finite.

Call a *Kripke frame* (*for* $L_{\Phi,\tau}$) every tuple $(W, \{R_i\}_{i\in I})$ such that W is a nonempty set, and, for every $i \in I$, R_i is a binary relation on W. A Kripke frame $(W, \{R_i\}_{\in I})$ is said to be *finite* provided W is finite.

For every Kripke model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$, $w \in W$, and $\varphi \in L_{\Phi,\tau}$, define $\mathcal{M}, w \models \varphi$ recursively as follows:

- $\mathcal{M}, w \models \top$ always.
- For every $p \in \Phi$, $\mathcal{M}, w \models p$ provided $w \in V(p)$.
- $\mathcal{M}, w \models \neg\varphi$ provided not $\mathcal{M}, w \models \varphi$.
- $\mathcal{M}, w \models \varphi \wedge \psi$ provided $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.
- For every $i \in I$, $\mathcal{M}, w \models \Box_i\varphi$ provided, for every $v \in R_i(w)$, we have $\mathcal{M}, v \models \varphi$.

For every Kripke model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$, and $\varphi \in L_{\Phi,\tau}$, we say that $\mathcal{M}$ *satisfies* φ provided there is $w \in W$ such that $\mathcal{M}, w \models \varphi$.

For every Kripke model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$, and $\varphi \in L_{\Phi,\tau}$, we say that $\mathcal{M}$ *validates* φ provided for every $w \in W$, we have $\mathcal{M}, w \models \varphi$.

For every Kripke frame $\mathcal{F} = (W, \{R_i\}_{i\in I})$, and $\varphi \in L_{\Phi,\tau}$, we say that $\mathcal{F}$ *satisfies* φ provided there is an evaluation $V : \Phi \to \mathcal{P}(W)$, and $w \in W$ such that $(W, \{R_i\}_{i\in I}, V), w \models \varphi$.

For every Kripke frame $\mathcal{F} = (W, \{R_i\}_{i\in I})$, and $\varphi \in L_{\Phi,\tau}$, we say that $\mathcal{F}$ *validates* φ provided, for every evaluation $V : \Phi \to \mathcal{P}(W)$, and $w \in W$, we have $(W, \{R_i\}_{i\in I}, V), w \models \varphi$.

2.3.1 Subframes and submodels

For every two Kripke frames $\mathcal{F} = (W, \{R_i\}_{\in I})$ and $\mathcal{F}' = (W', \{R'_i\}_{i\in I})$, $\mathcal{F}$ is called a *subframe of* $\mathcal{F}'$, notation $\mathcal{F} \subseteq \mathcal{F}'$, provided $W \subseteq W'$, and, for every $i \in I$, $R_i = R'_i \restriction W$.

For every two Kripke models $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathcal{M}' = (W', \{R'_i\}_{i\in I}, V')$, $\mathcal{M}$ is called a *submodel of* $\mathcal{M}'$, notation $\mathcal{M} \subseteq \mathcal{M}'$, provided $(W, \{R_i\}_{\in I}) \subseteq (W', \{R'_i\}_{i\in I})$, and, for every $p \in \Phi$, $V(p) = V'(p) \cap W$.

For every two Kripke frames $\mathcal{F} = (W, \{R_i\}_{\in I})$ and $\mathcal{F}' = (W', \{R'_i\}_{\in I})$ such that $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{F}$ is called a *generated subframe of* $\mathcal{F}'$ provided for every $i \in I$, W is an R'_i-generated subset of W'.

For every two Kripke models $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathcal{M}' = (W', \{R'_i\}_{i\in I}, V')$ such that $\mathcal{M} \subseteq \mathcal{M}'$, $\mathcal{M}$ is called a *generated submodel of* $\mathcal{M}'$ provided $(W, \{R_i\}_{\in I})$ is a generated subframe of $(W', \{R'_i\}_{i\in I})$.

Proposition 2.1.

1. For every two Kripke frames $\mathcal{F}$ and $\mathcal{F}'$ for $L_{\Phi,\tau}$ such that $\mathcal{F}$ is a generated subframe of $\mathcal{F}'$, and $\varphi \in L_{\Phi,\tau}$, if φ is valid on $\mathcal{F}'$ then φ is valid on $\mathcal{F}$.
2. For every two Kripke models $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ and $\mathcal{M}' = (W', \{R'_i\}_{i\in I}, V')$ such that $\mathcal{M}$ is a generated submodel of $\mathcal{M}'$, $\varphi \in L_{\Phi,\tau}$, and $w \in W$, $\mathcal{M}', w \models \varphi$ if and only if $\mathcal{M}, w \models \varphi$.

Proof. See [13, Chapter 3]. □

For every Kripke frame $\mathcal{F} = (W, \{R_i\}_{\in I})$, and nonempty subset X of W, the smallest generated subframe of $\mathcal{F}$ including X (plainly existing) is called the *subframe of* $\mathcal{F}$ *generated by* X.

For every Kripke model $\mathcal{M} = (W, \{R_i\}_{i\in I}, V)$ and nonempty subset X of W, the smallest generated submodel of $\mathcal{M}$ containing X (plainly existing) is called the *submodel of* $\mathcal{M}$ *generated by* X.

2.4 Logics

Call a *modal logic* (*in* $L_{\Phi,\tau}$) every subset L of $L_{\Phi,\tau}$ such that:

- L contains all the propositional tautologies in $L_{\Phi,\tau}$.

- L is closed under *modus ponens* (i.e. if $\varphi \in \mathsf{L}$ and $\varphi \to \psi \in \mathsf{L}$, then $\psi \in \mathsf{L}$).
- L is closed under *uniform substitution* (i.e. if $\varphi \in \mathsf{L}$, then L contains all the substitution instances of φ).

For every modal logic L in $L_{\Phi,\tau}$, and $\varphi \in L_{\Phi,\tau}$, we say that φ *is deducible in* L, notation $\vdash_{\mathsf{L}} \varphi$, provided $\varphi \in \mathsf{L}$. The formula φ is said to be L-*consistent* provided $\nvdash_{\mathsf{L}} \neg\varphi$.

For every modal logic L in $L_{\Phi,\tau}$, and $\Gamma \cup \{\varphi\} \subseteq L_{\Phi,\tau}$, we say that φ *is deducible in* L *from* Γ, notation $\Gamma \vdash_{\mathsf{L}} \varphi$, provided there are $\gamma_0, \gamma_1, ..., \gamma_{m-1} \in \Gamma$ such that $\vdash_{\mathsf{L}} \bigwedge_{i \in m} \gamma_i \to \varphi$. The set $\Gamma \subseteq L_{\Phi,\tau}$ is called L-*consistent* provided $\Gamma \nvdash_{\mathsf{L}} \bot$. Observe that this is equivalent to: for every $\gamma_0, \gamma_1, ..., \gamma_{m-1} \in \Gamma$, $\nvdash_{\mathsf{L}} \neg \bigwedge_{i \in m} \gamma_i$.

Proposition 2.2. For every modal logic L in $L_{\Phi,\tau}$, Γ maximal L-consistent subset of $L_{\Phi,\tau}$, and $\varphi, \psi \in L_{\Phi,\tau}$, the following facts hold:

- $\top \in \Gamma$
- $\bot \notin \Gamma$
- $\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.
- $\varphi \wedge \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ and $\psi \in \Gamma$.
- $\varphi \vee \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ or $\psi \in \Gamma$.
- $\varphi \to \psi \in \Gamma$ if and only if ($\varphi \in \Gamma$ implies $\psi \in \Gamma$).
- If $\varphi, \varphi \to \psi \in \Gamma$, then $\psi \in \Gamma$.
- $\mathsf{L} \subseteq \Gamma$.

Proof. See [13, Chapter 4]. □

Call a *normal modal logic* (*in* $L_{\Phi,\tau}$) every modal logic L in $L_{\Phi,\tau}$ such that:

- *Normality*: for every $i \in I$, for every $p, q \in \Phi$, L contains $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$.
- L is closed under *generalization* (that is, for every $i \in I$, if $\varphi \in \mathsf{L}$, then $\Box_i\varphi \in \mathsf{L}$).

It is well known that:

Proposition 2.3. For every normal modal logic L in $L_{\Phi,\tau}$, $i \in I$, $p, q \in \Phi$, and $\varphi, \psi \in L$, the following facts hold:

1. $\vdash_{\mathsf{L}} \varphi \to \psi$ implies $\vdash_{\mathsf{L}} \Box_i\varphi \to \Box_i\psi$.
2. $\vdash_{\mathsf{L}} \varphi \to \psi$ implies $\vdash_{\mathsf{L}} \Diamond_i\varphi \to \Diamond_i\psi$.

3. $\vdash_{\mathsf{L}} \Box_i(p \wedge q) \leftrightarrow \Box_i p \wedge \Box_i q$.

4. $\vdash_{\mathsf{L}} \Diamond_i(p \vee q) \leftrightarrow \Diamond_i p \vee \Diamond_i q$.

5. $\vdash_{\mathsf{L}} \Box_i p \vee \Box_i q \to \Box_i(p \vee q)$.

6. $\vdash_{\mathsf{L}} \Diamond_i(p \wedge q) \to \Diamond_i p \wedge \Diamond_i q$.

7. $\vdash_{\mathsf{L}} \Box_i p \wedge \Diamond_i q \to \Diamond_i(p \wedge q)$.

Proof's sketch. As an example we prove 7. By Propositional Logic:

$$\vdash_{\mathsf{L}} p \to (\neg(p \wedge q) \to \neg q).$$

Then, by generalization:

$$\vdash_{\mathsf{L}} \Box_i(p \to (\neg(p \wedge q) \to \neg q)).$$

Then, by normality, modus ponens, and Propositional Logic:

$$\vdash_{\mathsf{L}} \Box_i p \to (\Box_i \neg(p \wedge q) \to \Box_i \neg q).$$

Then, by Propositional Logic:

$$\vdash_{\mathsf{L}} \Box_i p \to (\neg\Box_i \neg q \to \neg\Box_i \neg(p \wedge q)).$$

Then, by Propositional Logic and abbreviations:

$$\vdash_{\mathsf{L}} \Box_i p \wedge \Diamond_i\, q \to \Diamond_i(p \wedge q).$$ □

2.5 Canonical models

For every normal modal logic L in $L_{\Phi,\tau}$, define the *canonical model for* L (*in* $L_{\Phi,\tau}$) as the Kripke model $\mathcal{M}^c = (W^c, \{R_i^c\}_{i \in I}, V^c)$ such that:

- W^c is the set of all maximal L-consistent subsets of $L_{\Phi,\tau}$.
- For every $i \in I$, R_i^c is defined by: for every $\Gamma, \Delta \in W^c$, $\Gamma R_i^c \Delta$ provided, for every $\varphi \in L_{\Phi,\tau}$, $\Box_i \varphi \in \Gamma$ implies $\varphi \in \Delta$.
- $V^c : \Phi \to \mathcal{P}(W^c)$ is defined by: for every $p \in \Phi$, $\Gamma \in V^c(p)$ provided $p \in \Gamma$.

Define the *canonical frame for* L (*in* $L_{\Phi,\tau}$) as the frame underlying the canonical model for L. A formula $\varphi \in L_{\Phi,\tau}$ is called *canonical* if, for every normal modal logic L in $L_{\Phi,\tau}$, $\varphi \in \mathsf{L}$ implies φ is valid on the canonical frame for L.

Proposition 2.4. For every $w, v \in W^c$, and $i \in I$, the following conditions are equivalent:

- wR_iv.
- For every $\varphi \in L_{\Phi,\tau}$, $\Box_i\varphi \in w$ implies $\varphi \in v$.
- For every $\varphi \in L_{\Phi,\tau}$, $\varphi \in v$ implies $\Diamond_i\varphi \in w$.

Proof. See [13, Chapter 4]. □

2.6 Sahlqvist formulas

For every propositional variable $p \in \Phi$, and formula $\varphi \in L_{\Phi,\tau}$, we say that an occurrence of p in φ is *positive* provided it is within the scope of an even number of negation signs, and that it is *negative* provided it is within the scope of an odd number of negation signs. We say that φ *is positive in* p provided every occurence of p in φ is postive, and that φ *is negative in* p provided every occurence of p in φ is negative. We say that φ *is positive* provided it is positive in every propositional variable occuring in φ, and that φ *is negative* provided it is negative in every propositional variable occurring in φ.

Call a *boxed atom* (*in* $L_{\Phi,\tau}$) every formula of $L_{\Phi,\tau}$ of the form $\Box_{i_0}\Box_{i_1}...\Box_{i_{k-1}}p$ where, for every $j \in k$, we have that $\Box_{i_j} \in \tau$, $p \in \Phi$, and, if $k = 0$, then $\Box_{i_0}\Box_{i_1}...\Box_{i_{k-1}}p$ is p. In words, a boxed atom is a propositional atom preceded by a number (possibly zero) of operators $\Box_i \in \tau$.

A *Sahlqvist antecedent* (*in* $L_{\Phi,\tau}$) is a formula built up from $\top$, $\bot$, boxed atoms and negative formulas, using $\wedge$, $\vee$ and, for every $i \in I$, operators $\Diamond_i$. A *Sahlqvist implication* (*in* $L_{\Phi,\tau}$) is a formula of the form $\varphi \to \psi$ in which ψ is positive and φ is a Sahlqvist antecedent.

A *Sahlqvist formula* (*in* $L_{\Phi,\tau}$) is a formula that is obtained from Sahlqvist implications by freely applying, for every $i \in I$, operators $\Box_i$ and $\wedge$, and by applying $\vee$ only between formulas that do not share any propositional variables.

2.7 Properties of the canonical model

Proposition 2.5. The following facts hold:

1. (Truth Lemma) For every $\varphi \in L_{\Phi,\tau}$, $\mathcal{M}^c, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$.
2. (Lindenbaum Lemma) For every L-consistent $\Gamma' \subseteq L_{\Phi,\tau}$, there is $\Gamma \in W^c$ such that $\Gamma' \subseteq \Gamma$.
3. (Existence Lemma) For every $\varphi \in L_{\Phi,\tau}$, $\Gamma \in W^c$ and $i \in I$, if $\Diamond_i\varphi \in \Gamma$, then there is $\Delta \in W^c$ such that $\Gamma R_i^c \Delta$ and $\varphi \in \Delta$.

4. (Sahlqvist Completeness Theorem) Every Sahlqvist formula is canonical.

Proof. See [13, Chapters 3, 4 and 5]. □

Part I

On Branching-Time Logic with Indistinguishability

Chapter 3

Introduction

Branching-time logics, that is, logics on tree-like representations of time, traditionally have played a major role in modeling non-deterministic notions of time. Since [77], two main branching-time logics have been considered. Prior called these logics Peircean and Ockhamist. The essential difference between them is the interpretation of the future operator F.

In Peircean Logic, $F\varphi$ is true at an instant t provided 'φ eventually will be true, in every possible future of t'. Peircean language also has a future operator G, of which the semantics is: $G\varphi$ is true at an instant t provided 'φ always will be true, in every possible future of t'.

Instead, the Ockhamist interpretation of $F\varphi$ is relative to instants t with respect to histories h passing through t, and $F\varphi$ is true at (t, h) provided 'φ eventually will be true in the future of t in h'. The Ockhamist language counterpart of the branching aspect of time is a modal operator N that quantifies over the set of histories passing through the moment under consideration: $N\varphi$ is true at (t, h) provided 'φ is true at (t, k) for every history k passing through t'.

Various works on logics of agency considered a partition of the set of histories passing through the instant under consideration into undividedness classes, for example [6]: two histories are *undivided* at t whenever their intersection contains a moment in the future of t.

In [102], a generalization of the notion of undividedness is considered: for every instant t, an equivalence relation I_t (*indistinguishability at t*) between the histories passing through t is given. The only property of the indistinguishability relations is that, if two histories are indistinguishable at an instant t, they are also indistinguishable at every instant in the past of t. This implies that undividedness is a particular case of indistinguishability.

Trees with indistinguishability relations (*I-trees*) provide a semantics for a temporal language with Peircean operators F and G and Ockhamist operator N. In this semantics, truth is relative to couple (t, π), where t is an instant and π is an indistinguishability class at t. The operators F and G have a Peircean reading, but the quantification over histories is restricted to the indistinguishability class under consideration. The operator N has an Ockhamist reading, but the quantification is over indistinguishability classes.

As it is pointed out in [102], Ockhamist and Peircean semantics correspond to the limit cases of the I-tree semantics in which every (respectively, no) history passing through t is distinguishable at t from any other.

3.1 Content of Part I

In Part I, we give a finite axiomatization, with a non standard rule, of the logic of bundled I-trees (see below). Some preliminary definitions are presented in Chapter 4. Syntax and semantics are presented in Chapter 5. The language has the aforementioned operators F, G, and N, plus a past operator H: $H\varphi$ is true at an instant t provided 'φ always happened in the past of t in the class under consideration".

The semantics is given with respect to the class of *bundled* I-trees. A bundle on a tree is a selection of histories of the tree such that every moment occurs in at least one history of the bundle. Moreover, trees are required to be upward endless. The indistinguishability relations and the quantification of the operators are restricted to the histories selected by the bundle.

There are two main reasons why we consider bundled I-trees instead of I-trees. A technical one: completeness is achieved by a particular construction in which only histories of a particular kind are desired and the bundle lets us select these histories. And a conceptual one: it has been argued, for example in [73, 74], that bundled validity is a more correct formalization of human intuition about time and possibility (nonetheless, we cannot avoid to report that, for example in [79], doubts about preferring bundled validity have been raised).

In Chapter 5, we also present a set of axioms and inference rules. These axioms and rules are a mix of the axioms for Peircean semantics presented in [20] and the axioms for Ockhamist semantics presented in [42, Section 7.7, pp. 299–306].

Among the deduction rules of the system, an IRR rule occurs. IRR rules have been introduced in [41], where it is shown that they can characterize irreflexivity, a property that, as it is well known, cannot be characterized by modal axioms. A version of this rule is employed here because it yields important properties of the structure built to prove completeness (see Section 6.1).

Completeness is achieved in Chapter 6 by a construction adapted from [20], mostly for what concerns the Peircean aspects of the logic, and from [42, Section 7.7, pp. 299–306], mostly for what concerns the Ockhamist aspects of the logic.

Part I is based on the following paper:

[43] Alberto Gatto. Axiomatization of a Branching Time Logic with Indistinguishability Relations. *Journal of Philosophical Logic*, 45(2):155–182, 2015.

3.2 Related Works

Peircean and Ockhamist semantics have been introduced in [77]. Peircean Logic has been axiomatized in [20] with a form of the IRR rule and in [101] without. In [20] it has also been proved that Peircean Logic is decidable. The bundled version of Ockhamist logic has been axiomatized in [100] without IRR rule (an axiomatization with a form of the IRR rule due to Gabbay is cited in [92]). An axiomatization of the unbundled Ockhamist logic and a brief sketch of the completeness proof have been presented in [80]. The decidability of bundled Ockhamist logic has been proved in [19, 49].

As for the notion of 'indistinguishability', the semantics with indistinguishability relations has been defined in [102] as a generalization of the semantics with undividedness relations of [6].

Many other logics of branching-time have been introduced. All of them are some kind of variation of either Peircean or Ockhamist logic. For example, in [100] and [101], the temporal operators G and H are replaced by the more expressive Since and Until operators from [55]. Moreover, branching-time logics have often been used in computer science. In that case, since time simulates the steps of a computation, time is assumed discrete. Example of such logics are the Peircean logic CTL of [28], the Ockhamist version CTL* of [29] and the P-extension of CTL*, PCTL*, of [67, 103].

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Chapter 4

Preliminaries

In this chapter we give some preliminary definitions, in particular we define what bundled I-trees are. Bundled I-trees will be the structures on which we will interpret the language studied on this part.

Definition 4.1. For every set A, and binary relation R on A:

1. Let R^r denote the reflexive closure of R.

2. R is said to be *total* provided, for every $a, b \in A$, $aR^r b$ or bRa.

3. R is said to be *linear* provided R is irreflexive, transitive and total.

4. R is called *downward linear* provided, for every $a, b, c \in A$ such that bRa and cRa, we have $bR^r c$ or cRb.

5. R is called *serial* provided, for every $a \in A$, there is $b \in A$ such that aRb.

Definition 4.2. Call a *tree* every tuple $(T, <)$, where T is a nonempty set, and $<$ is an irreflexive, transitive and downward linear binary relation on T.

Example 4.3. Consider the *infinite binary tree* $\mathcal{T}^2 := ({}^{<\mathbf{N}}2, \subset)$. As usual, ${}^{<\mathbf{N}}2$ denotes the set of all functions $f : n \to 2$ for some $n \in \mathbf{N}$. Then, ${}^{<\mathbf{N}}2$ can be identified with the set of all sequences $(a_0, a_1, ..., a_{n-1})$ such that $n \in \mathbf{N}$ and, for every $i \in n$, $a_i \in 2$. Let $\subset$ be a binary relation on ${}^{<\mathbf{N}}2$ defined by: for every $a, b \in {}^{<\mathbf{N}}2$ we have $a \subset b$ provided the domain of a is a proper subset of the domain of b and b restricted to the domain of a is equal to a. The set ${}^{<\mathbf{N}}2$ ordered by $\subset$ is a tree.

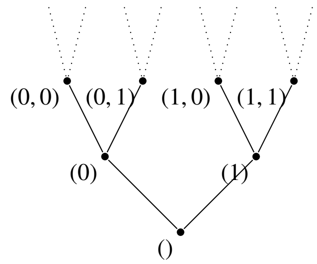

Figure 4.1: infinite binary tree $\mathcal{T}^2$

Definition 4.4. For every tree $\mathcal{T} = (T, <)$, we call a *history* (*of* $\mathcal{T}$) every maximal linear subset of T. Let $H_{\mathcal{T}}$ denote the set of all histories of $\mathcal{T}$. For every $t \in T$, let $H_{\mathcal{T},t}$ denote the set of all histories h of $\mathcal{T}$ such that $t \in h$ - that is, passing through t. When $\mathcal{T}$ is understood, $H_{\mathcal{T},t}$ may be denoted by H_t, that is we may forget to mention $\mathcal{T}$.

Example 4.5. Consider the set ${}^{\mathbf{N}}2$. As usual, for every two sets A and B, ${}^{A}B$ denotes the set of all functions $f : A \to B$. Then, ${}^{\mathbf{N}}2$ can be identified with the set of all countably infinite sequences $(a_0, a_1, ...)$ such that, for every $i \in \mathbf{N}$, we have $a_i \in 2$. For every sequence $a \in {}^{\mathbf{N}}2$, and $i \in \mathbf{N}$, let a_i denote the ith component $a(i)$ of a. One can easily prove that the set $H_{\mathcal{T}^2}$ of all histories of $\mathcal{T}^2$ is the set $\{\{(a_0, a_1, ..., a_i) \mid i \in \mathbf{N}\} \mid a \in {}^{\mathbf{N}}2\}$.

Remark 4.6. For every tree $\mathcal{T} = (T, <)$, and nonempty linear subset $S \subseteq T$, by Zorn Lemma, it is possible to extend S to a history of $\mathcal{T}$.

Definition 4.7. For every tree $\mathcal{T} = (T, <)$, call a *bundle* (*on* $\mathcal{T}$) every subset $B \subseteq H_{\mathcal{T}}$ such that $T = \bigcup B$. Call a *bundled tree* every tuple $(T, <, B)$ such that $(T, <)$ is a tree, and B is a bundle on $(T, <)$. For every bundled tree $(T, <, B)$, and $t \in T$, let $B_{\mathcal{T},t}$ denote the set of all histories $h \in B$ such that $t \in h$. When $\mathcal{T}$ is understood $B_{\mathcal{T},t}$ may be denoted by B_t.

Example 4.8. By Example 4.5, $h \mapsto \bigcup h$ is a bijection from $H_{\mathcal{T}^2}$ to ${}^{\mathbf{N}}2$. Then, we can identify $H_{\mathcal{T}^2}$ with ${}^{\mathbf{N}}2$. Define B as the set of all sequences $a \in {}^{\mathbf{N}}2$ such that there is $i \in \mathbf{N}$ and $b \in \{0, 1\}$ with, for every $j \geq i$, $a_j = b$ - that is, the set of all sequences that become constant from a certain component onward. Let $\subset$ denote proper inclusion. One can easily prove that $B \subset H_{\mathcal{T}^2}$ is a countable bundle on $\mathcal{T}^2$.

Definition 4.9. For every bundled tree $\mathcal{T} = (T, <, B)$, call an *indistinguishability function* (*on* $\mathcal{T}$) every function $I : T \to \mathcal{P}(B \times B)$ such that:

1. For every $t \in T$, $I(t)$ is an equivalence relation on B_t, denoted by I_t.
2. For every $h, k \in B$, and $t, s \in h \cap k$ such that $t < s$, if hI_sk, then hI_tk.

For every $t \in T$, and $h \in H_{\mathcal{T},t}$, $[h]_{I_t}$ denotes the equivalence class of h with respect to I_t. Call a *bundled I-tree* every tuple $\mathcal{T} = (T, <, B, I)$ such that $(T, <, B)$ is a tree and I is an indistinguishability function. For every bundled I-tree $\mathcal{T} = (T, <, B, I)$, and $t \in T$, let $\Pi_{\mathcal{T},t}$ denote the set of all equivalence classes of I_t. When $\mathcal{T}$ is understood $\Pi_{\mathcal{T},t}$ may be denoted by Π_t.

Example 4.10. Suppose that an agent A, for every $a \in {}^{<\mathbf{N}}2$, can distinguish histories in $H_{\mathcal{T}^2,a}$ if and only if they split in less than n steps from a for a fixed $n > 1$. That is, let $len(a)$ denote the length of a, and let $h_1, h_2 \in {}^{\mathbf{N}}2$ be two histories passing through a. We set $h_1 I_a h_2$ provided for all $i \in len(a) + n$, the ith component of h_1 is equal to the ith component of h_2. This defines an indistinguishability function on $\mathcal{T}^2$, which might distinguish undivided histories.

Chapter 5

Language, semantics, axioms and IRR rule

In this Chapter we define the language, the semantics, and we give a system of axioms and inference rules.

5.1 Language

Let Φ be a countably infinite set, and $\tau := \{G, F, H, N\}$. Throughout Part I, we will work with the modal language $L := L_{\Phi,\tau}$. Let $f := \neg G \neg$, $g := \neg F \neg$, $P := \neg H \neg$, and $M := \neg N \neg$.

5.2 Semantics

In this section we define the semantics of L. We explain how the language L will be interpreted on upward endless bundled I-trees.

Definition 5.1. Call a *frame* every bundled I-tree $(T, <, B, I)$ such that $<$ is serial. Call a *model* every tuple $\mathcal{M} = (T, <, B, I, V)$ such that $\mathcal{T} = (T, <, B, I)$ is a frame and $V : \Phi \to \mathcal{P}(\bigcup_{t \in T}(\{t\} \times \Pi_{\mathcal{T},t}))$ is a function called *evaluation*. Let $\mathfrak{M}$ be the class of all models.

$\bigcup_{t \in T}(\{t\} \times \Pi_{\mathcal{T},t})$ is the set of all couples (t, π) such that $t \in T$ and $\pi \in \Pi_{\mathcal{T},t}$. Then, an evaluation maps propositional variables to sets of couples (t, π) where t is a point of $\mathcal{T}$ and π is a class of indistinguishable histories at t. Let $(t, \pi) \in \mathcal{M}$ denote $(t, \pi) \in \bigcup_{t \in T}(\{t\} \times \Pi_{\mathcal{T},t})$.

Definition 5.2. For every model $\mathcal{M} = (T, <, B, I, V)$, $(t, \pi) \in \mathcal{M}$, and $\varphi \in L$, define $\mathcal{M}, (t, \pi) \models \varphi$ recursively as follows:

- $\mathcal{M}, (t, \pi) \models \top$ always.
- For every $p \in \Phi$, $\mathcal{M}, (t, \pi) \models p$ provided $(t, \pi) \in V(p)$.
- $\mathcal{M}, (t, \pi) \models \neg\varphi$ provided $\mathcal{M}, (t, \pi) \not\models \varphi$.
- $\mathcal{M}, (t, \pi) \models \varphi \wedge \psi$ provided $\mathcal{M}, (t, \pi) \models \varphi$ and $\mathcal{M}, (t, \pi) \models \psi$.
- $\mathcal{M}, (t, \pi) \models G\varphi$ provided, for every $h \in \pi$, and $s \in h$ such that $t < s$, we have $\mathcal{M}, (s, [h]_{I_s}) \models \varphi$.
- $\mathcal{M}, (t, \pi) \models F\varphi$ provided, for every $h \in \pi$, there is $s \in h$ with $t < s$ and $\mathcal{M}, (s, [h]_{I_s}) \models \varphi$.
- $\mathcal{M}, (t, \pi) \models H\varphi$ provided, for every $h \in \pi$, and $s \in h$ such that $s < t$, we have $\mathcal{M}, (s, [h]_{I_s}) \models \varphi$.
- $\mathcal{M}, (t, \pi) \models N\varphi$ provided, for every $\rho \in \Pi_{\mathcal{T},t}$, we have $\mathcal{M}, (t, \rho) \models \varphi$.

Analogously to Section 2.3, for every model $\mathcal{M}$, and $\varphi \in L$, we say that $\mathcal{M}$ *satisfies* φ provided there is $(t, \pi) \in \mathcal{M}$ such that $\mathcal{M}, (t, \pi) \models \varphi$. We say that φ *is satisfiable* provided there is a model satisfying φ, and that φ *is valid* provided $\neg\varphi$ is not satisfiable. Let $\mathsf{L}^{\mathfrak{M}}$ be the set of all valid formulas. Plainly, $\mathsf{L}^{\mathfrak{M}}$ is a logic (see Section 2.4).

For every model $\mathcal{M}$, and $\Gamma \subseteq L$, we say that $\mathcal{M}$ *satisfies* Γ provided there is $(t, \pi) \in \mathcal{M}$ such that, for every $\gamma \in \Gamma$, we have $\mathcal{M}, (t, \pi) \models \gamma$. We say that Γ *is satisfiable* provided there is a model satisfying Γ.

For $\Gamma \cup \{\varphi\} \subseteq L$, we say that Γ *semantically entails* φ, notation $\Gamma \models \varphi$, provided, for every model $\mathcal{M}$, and $(t, \pi) \in \mathcal{M}$, if, for every $\gamma \in \Gamma$, $\mathcal{M}, (t, \pi) \models \gamma$, then $\mathcal{M}, (t, \pi) \models \varphi$.

5.3 Axioms and IRR rule

In this section we give a finite list of axioms Λ, and define the smallest 'normal modal logic L (with respect to G, H, and N)' containing Λ and closed under an additional condition, called IRR rule. In Chapter 6, we prove that L *is sound* (that is, $\mathsf{L} \subseteq \mathsf{L}^{\mathfrak{M}}$) and *complete* (that is, $\mathsf{L}^{\mathfrak{M}} \subseteq \mathsf{L}$) *with respect to* $\mathsf{L}^{\mathfrak{M}}$. Then, $\mathsf{L}^{\mathfrak{M}}$ *is axiomatized by* Λ (that is, $\mathsf{L}^{\mathfrak{M}} = \mathsf{L}$). In fact, we also prove that $\mathsf{L}^{\mathfrak{M}}$ *is strongly complete* (that is, for every $\Gamma \cup \{\varphi\} \subseteq L$, $\Gamma \models \varphi$ implies $\Gamma \vdash_{\mathsf{L}^{\mathfrak{M}}} \varphi$).

Λ is a mix of the axioms for Peircean semantics given in [20] and the axioms for Ockhamist semantics given in [42, Section 7.7, pp. 299–306]. In particular, Axioms 2a - 2m come from [20]; Axioms 2n - 2q are S5 for N, exactly as in [42, Section 7.7, pp. 299–306]; and Axioms 2r and 2s are adapted from [42, Section 7.7, pp. 299–306]. Closure under some variation of the IRR rule is required both in [20] and in [42, Section 7.7, pp. 299–306].

The reason why we may think of L as normal in $\{G, H, N\}$ is because of the normality axioms for G, H, and N in Λ (Axioms 2b, 2a, and 2n, respectively), and because L will be closed under generalization for G, H, and N.

On the other hand, the normality axiom for F, i.e. $F(p \to q) \to (Fp \to Fq)$, would not be sound with respect to $\mathsf{L}^{\mathfrak{M}}$, that is, $F(p \to q) \to (Fp \to Fq) \notin \mathsf{L}^{\mathfrak{M}}$. Instead, we use axiom 2c $G(p \to q) \to (Fp \to Fq)$, which is somehow reminiscent of the normality axiom for F. Generalization for F, although preserving validities over models, is not necessary.

The IRR rule will let us use IRR maximal L-consistent sets (see Definition 6.1). For every L-consistent set of formulas Γ, IRR maximal L-consistent sets will let us build a model satisfying Γ. By classical arguments, this yields the completeness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, and, together with the soundness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, also yields the strong completeness of $\mathsf{L}^{\mathfrak{M}}$. Further comments and details will be given later.

Definition 5.3. Define Λ as the following set:

1. All tautologies of propositional logic.
2. Let p, q be any two distinct propositional variables:
   (a) $H(p \to q) \to (Hp \to Hq)$.
   (b) $G(p \to q) \to (Gp \to Gq)$.
   (c) $G(p \to q) \to (Fp \to Fq)$.
   (d) $p \to Hfp$.
   (e) $p \to GPp$.
   (f) $Hp \to HHp$.
   (g) $Gp \to GGp$.
   (h) $FFp \to Fp$
   (i) $Gp \to Fp$.
   (j) $Gp \to gp$.
   (k) $Hp \wedge p \wedge Gp \to GHp$.
   (l) $Hp \wedge p \wedge gp \to gHp$.
   (m) $FGp \to GFp$.
   (n) $N(p \to q) \to (Np \to Nq)$.
   (o) $Np \to NNp$.
   (p) $Np \to p$.
   (q) $p \to NMp$.
   (r) $MPp \to PMp$.

(s) $Hp \wedge N\neg p \wedge Nq \to GNH(M(Hp \wedge N\neg p) \to q)$.

Let L be the smallest subset of L containing Λ and closed under the following inference rules:

- *Modus ponens*: if $\varphi \in \mathsf{L}$ and $\varphi \to \psi \in \mathsf{L}$, then $\psi \in \mathsf{L}$.
- *Uniform substitution*: if $\varphi \in \mathsf{L}$, then L contains all the substitution instances of φ.
- *Generalization*: if $\varphi \in \mathsf{L}$, then $\{G\varphi, H\varphi, N\varphi\} \subseteq \mathsf{L}$.
- *IRR rule*: if $Hp \wedge N\neg p \to \varphi \in \mathsf{L}$ with $p \in \Phi$ not occurring in φ, then $\varphi \in \mathsf{L}$.

We denote $\vdash_{\mathsf{L}}$ by $\vdash$ and refer to L-consistency by saying consistency. Plainly, for every $\varphi \in L$, we have $\vdash \varphi$ if and only if there is $n \in \mathbf{N}$ and $\varphi_0, ...\varphi_n \in L$ such that $\varphi_n = \varphi$ and, for every $i \leq n$, φ_i is either an axiom or is obtained from $\varphi_0, ..., \varphi_{i-1}$ by the application of an inference rule. Such a sequence $\varphi_0, ..., \varphi_n$ is called a *derivation of* φ.

5.4 Properties of L

In this section we prove a number of properties of L. These results will be used to study IRR sets in Sections 6.1 and 6.2.

Proposition 5.4. For every $p, q \in \Phi$, $\varphi, \psi \in L$, and $\Box \in \{G, H, N\}$ (and $\Diamond$ abbreviating $\neg\Box\neg$), the following facts hold:

1. $\vdash \varphi \to \psi$ implies $\vdash \Box\varphi \to \Box\psi$.
2. $\vdash \varphi \to \psi$ implies $\vdash \Diamond\varphi \to \Diamond\psi$.
3. $\vdash \Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$.
4. $\vdash \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$.
5. $\vdash \Box p \vee \Box q \to \Box(p \vee q)$.
6. $\vdash \Diamond(p \wedge q) \to \Diamond p \wedge \Diamond q$.
7. $\vdash \Box p \wedge \Diamond q \to \Diamond(p \wedge q)$.

Proof's sketch. Since L is normal in $\{G, H, N\}$, Proposition 2.3 yields the result. □

Proposition 5.5. For every $\varphi, \psi, \alpha, \beta \in L$, if $\vdash \varphi \leftrightarrow \psi$ and β is like α except for having ψ in one or more places where α has φ, then $\vdash \alpha \leftrightarrow \beta$.

Proof's sketch. The proof is an easy induction on the complexity of α. The propositional and boolean cases are plain. The temporal cases (G, F, H) all follow a similar pattern. We show the case in which α is $F\gamma$. For every $\eta, \vartheta \in L$, write $R(\eta, \vartheta)$ provided ϑ is like η except for having ψ in one or more places where ϑ has φ. By hypothesis, we have $R(\alpha,\beta)$ and β is of the form $F\delta$. Then, plainly, we have $R(\gamma, \delta)$. Then, by inductive hypothesis, we have:

$$\vdash \gamma \to \delta,$$
$$\vdash \delta \to \gamma.$$

Then, by generalization for G, we have:

$$\vdash G(\gamma \to \delta),$$
$$\vdash G(\delta \to \gamma).$$

Then, by Axiom 2c ($G(p \to q) \to (Fp \to Fq)$), uniform substitution, and modus ponens, we have:

$$\vdash F\gamma \to F\delta,$$
$$\vdash F\delta \to F\gamma.$$

□

Remark 5.6. Propositions 5.4 and 5.5 are common in Modal Logic. Therefore, with the exception of this chapter, we will use them without mentioning.

Proposition 5.7. For every $p, q \in \Phi$, and $\varphi, \psi \in L$, the following facts hold:

1. If $\vdash \varphi \to \psi$ then $\vdash F\varphi \to F\psi$.
2. If $\vdash \varphi \to \psi$ then $\vdash g\varphi \to g\psi$.
3. $\vdash Gp \to fp$.
4. $\vdash fgp \to fp$.
5. $\vdash Gp \wedge Fq \to F(p \wedge q)$.
6. $\vdash Fp \wedge gq \to f(p \wedge q)$.
7. $\vdash gHp \to p$.
8. $\vdash Hp \wedge p \wedge gp \to Hgp$.
9. $\vdash Hp \wedge gq \to gH(P\neg p \to q)$.

Proof. 1. Straightforward from hypothesis, generalization for G, Axiom 2c ($G(p \to q) \to (Fp \to Fq)$), uniform substitution, and modus ponens.

2. By hypothesis and propositional logic, we have:

$$\vdash \neg\psi \to \neg\varphi.$$

Then, by 1, we have:

$$\vdash F\neg\psi \to F\neg\varphi.$$

Then, by propositional logic, we have:

$$\vdash \neg F\neg\varphi \to \neg F\neg\psi.$$

3. By Axiom 2j ($Gp \to gp$), we have:

$$Gp \to \neg F\neg p.$$

Moreover, by Axiom 2i ($Gp \to Fp$), uniform substitution, and propositional logic, we have:

$$\neg F\neg p \to \neg G\neg p.$$

Then, by propositional logic, we have:

$$Gp \to \neg G\neg p.$$

4. By Axiom 2i ($Gp \to Fp$), uniform substitution, and propositional logic, we have:

$$\vdash \neg F\neg p \to \neg G\neg p.$$

Then, by Proposition 5.4(2), we have:

$$\vdash \neg G\neg\neg F\neg p \to \neg G\neg\neg G\neg p. \tag{5.1}$$

Moreover, by propositional logic, we have:

$$\vdash G\neg p \to \neg\neg G\neg p.$$

Then, by Proposition 5.4(1), we have:

$$\vdash GG\neg p \to G\neg\neg G\neg p.$$

Then, by propositional logic, we have:

$$\vdash \neg G\neg\neg G\neg p \to \neg GG\neg p. \tag{5.2}$$

Finally, by Axiom 2g ($Gp \to GGp$), uniform substitution, and propositional logic, we have:

$$\vdash \neg GG\neg p \to \neg G\neg p. \tag{5.3}$$

Then, by (5.1), (5.2), and (5.3), we have:

$$\vdash \neg G\neg\neg F\neg p \to \neg G\neg p.$$

5. By propositional logic, we have:

$$\vdash p \to (q \to p \wedge q).$$

Then, by Proposition 5.4(1), we have:

$$\vdash Gp \to G(q \to p \wedge q).$$

Moreover, by Axiom 2c ($G(p \to q) \to (Fp \to Fq)$), and uniform substitution, we have:

$$\vdash G(q \to p \wedge q) \to (Fq \to F(p \wedge q)).$$

Then, by propositional logic, we have:

$$\vdash Gp \to (Fq \to F(p \wedge q)).$$

Then, by propositional logic, we have:

$$\vdash Gp \wedge Fq \to F(p \wedge q).$$

6. By propositional logic, we have:

$$\vdash \neg(p \wedge q) \to (p \to \neg q).$$

Then, by Proposition 5.4(1), we have:

$$\vdash G\neg(p \wedge q) \to G(p \to \neg q).$$

Moreover, by Axiom 2c ($G(p \to q) \to (Fp \to Fq)$), and uniform substitution, we have:

$$\vdash G(p \to \neg q) \to (Fp \to F\neg q).$$

Then, by propositional logic, we have:

$$\vdash G\neg(p \land q) \to (Fp \to F\neg q).$$

Then, by propositional logic, we have:

$$\vdash Fp \land \neg F\neg q \to \neg G\neg(p \land q).$$

7. By Axiom 2e ($p \to GPp$), and uniform substitution, we have:

$$\vdash \neg p \to G\neg H\neg\neg p.$$

Moreover, by Axiom 2i ($Gp \to Fp$), and uniform substitution, we have:

$$\vdash G\neg H\neg\neg p \to F\neg H\neg\neg p.$$

Then, by propositional logic, we have:

$$\vdash \neg p \to F\neg H\neg\neg p.$$

Then, by propositional logic, we have:

$$\vdash \neg F\neg H\neg\neg p \to \neg\neg p. \tag{5.4}$$

Then, by Proposition 5.5, we have:

$$\vdash \neg F\neg Hp \to p.$$

8. By Axiom 2e ($p \to GPp$), and uniform substitution, we have:

$$\vdash \neg p \to GP\neg p.$$

Then, by 1 and 5.4(2), we have:

$$\vdash PF\neg p \to PFGP\neg p. \tag{5.5}$$

By Axiom 2m ($FGp \to GFp$), uniform substitution, and Proposition 5.4(2), we have:

$$\vdash PFGP\neg p \to PGFP\neg p. \tag{5.6}$$

Then, by (5.5), (5.6), and propositional logic, we have:

$$\vdash PF\neg p \to PGFP\neg p. \tag{5.7}$$

Moreover, by Axiom 2d ($p \to Hfp$), and uniform substitution, we have:

$$\vdash \neg p \to H\neg G\neg\neg p.$$

Then, by propositional logic, we have:

$$\vdash \neg H\neg G\neg\neg p \to \neg\neg p.$$

Then, by Proposition 5.5, we have:

$$\vdash \neg H\neg Gp \to p.$$

Then, by uniform substitution, we have:

$$\vdash \neg H\neg GFP\neg p \to FP\neg p. \tag{5.8}$$

Then, by (5.7), (5.8), and propositional logic, we have:

$$\vdash \neg H\neg F\neg p \to F\neg H\neg\neg p.$$

Then, by propositional logic, we have:

$$\vdash \neg F\neg H\neg\neg p \to \neg\neg H\neg F\neg p.$$

Then, by Proposition 5.5, we have:

$$\vdash gHp \to Hgp.$$

Then, by Axiom 2l ($Hp \wedge p \wedge gp \to gHp$), and propositional logic, we have:

$$\vdash Hp \wedge p \wedge gp \to Hgp.$$

9. By propositional logic, we have:

$$\vdash Hp \to \neg P\neg p \vee q. \tag{5.9}$$

Then, by Proposition 5.4(1), we have:

$$\vdash HHp \to H(\neg P\neg p \vee q).$$

Then, by Axiom 2f ($Hp \to HHp$), and propositional logic, we have:

$$\vdash Hp \to H(\neg P\neg p \vee q). \tag{5.10}$$

Moreover, by propositional logic, we have:

$$\vdash q \to \neg P\neg p \vee q.$$

Then, by 2, we have:

$$\vdash gq \to g(\neg P\neg p \vee q). \tag{5.11}$$

Then, by (5.9), (5.10), (5.11), and propositional logic, we have:

$$Hp \wedge gq \to H(\neg P\neg p \vee q) \wedge (\neg P\neg p \vee q) \wedge g(\neg P\neg p \vee q).$$

Then, by Axiom 2l ($Hp \wedge p \wedge gp \to gHp$), uniform substitution, and propositional logic, we have:

$$\vdash Hp \wedge gq \to gH(\neg P\neg p \vee q).$$ □

Proposition 5.8. $\vdash P(p \wedge Fq) \to P(Pp \wedge q) \vee q \vee Fq$.

Proof. Let φ stand for $Pp \wedge q$. By Axiom 2e ($p \to GPp$), propositional logic, and Proposition 5.4(2), we have:

$$\vdash P(p \wedge Fq) \to P(GPp \wedge Fq).$$

Also, by Proposition 5.7(5), uniform substitution, and Proposition 5.4(2) we have:

$$\vdash P(GPp \wedge Fq) \to PF(Pp \wedge q).$$

Then, by propositional logic, we have:

$$\vdash P(p \wedge Fq) \to PF\varphi. \tag{5.12}$$

Moreover, by Proposition 5.5 for the first line, propositional logic for the second, and propositional logic and Proposition 5.7(1) for the third, we have:

$$\vdash \neg P(Pp \wedge q) \to H\neg(Pp \wedge q),$$
$$\vdash \neg q \to \neg(Pp \wedge q),$$
$$\vdash \neg Fq \to g\neg(Pp \wedge q).$$

Then, by propositional logic, we have:

$$\vdash \neg P(Pp \wedge q) \wedge \neg q \wedge \neg Fq \to H\neg\varphi \wedge \neg\varphi \wedge g\neg\varphi.$$

Then, by Proposition 5.7(8), and uniform substitution, we have:

$$\vdash \neg P(Pp \wedge q) \wedge \neg q \wedge \neg Fq \to Hg\neg\varphi. \tag{5.13}$$

Combining (5.12) and (5.13), we obtain:

$$\vdash P(p \wedge Fq) \wedge \neg P(Pp \wedge q) \wedge \neg q \wedge \neg Fq \to \neg H\neg F\varphi \wedge H\neg F\neg\neg\varphi.$$

Then, by Proposition 5.5, we have:

$$\vdash P(p \wedge Fq) \wedge \neg P(Pp \wedge q) \wedge \neg q \wedge \neg Fq \to \neg H\neg F\varphi \wedge H\neg F\varphi.$$

Then, by propositional logic, we have:

$$\vdash \neg(P(p \wedge Fq) \wedge \neg P(Pp \wedge q) \wedge \neg q \wedge \neg Fq). \qquad \square$$

Proposition 5.9. For every:

- $n \in \mathbf{N}$;
- $\varphi, \psi, \chi_1, \chi_2, ..., \chi_n$ in L;
- and $\Box_0, \Box_1, ..., \Box_n \in \{G, H, N\}$;

if we let:

- G^- be H, H^- be G, and N^- be N;
- and, for every $i \in n+1$, $\Diamond_i$ be $\neg\Box_i\neg$;

then:

$$\vdash \varphi \to \Box_0(\chi_1 \to \Box_1(\chi_2 \to ... \to \Box_{n-1}(\chi_n \to \Box_n\neg\psi))...) \tag{5.14}$$

implies:

$$\vdash \psi \to \Box_n^-(\chi_n \to \Box_{n-1}^-(\chi_{n-1} \to ... \to \Box_1^-(\chi_1 \to \Box_0^-\neg\varphi))...). \tag{5.15}$$

Proof. Let us assume (5.14). First we prove by induction on $j \in n+1$ that:

$$\begin{aligned}&\vdash \Diamond_j(\chi_{j+1} \wedge \Diamond_{j+1}(\chi_{j+2} \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...)\\ &\quad \to (\chi_j \to \Box_{j-1}^-(\chi_{j-1} \to \Box_{j-2}^-(\chi_{j-2} \to ... \to \Box_1^-(\chi_1 \to \Box_0^-\neg\varphi))...).\end{aligned} \tag{5.16}$$

Indeed, by (5.14), propositional logic, and Proposition 5.5, we have:

$$\vdash \Diamond_0(\chi_1 \wedge \Diamond_1(\chi_2 \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \rightarrow \neg\varphi.$$

Then, by Proposition 5.4(1), we have:

$$\vdash \Box_0^- \Diamond_0(\chi_1 \wedge \Diamond_1(\chi_2 \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \rightarrow \Box_0^- \neg\varphi.$$

Then, by Axioms 2e ($p \rightarrow GPp$), 2d ($p \rightarrow Hfp$), 2q ($p \rightarrow NMp$), uniform substitution, and propositional logic, we have:

$$\vdash (\chi_1 \wedge \Diamond_1(\chi_2 \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \rightarrow \Box_0^- \neg\varphi.$$

This is (5.16) for $j = 0$.

Consider a natural number m such that $0 \leq m < n$, and assume (5.16) for $j = m$. That is:

$$\begin{aligned} &\vdash \Diamond_m(\chi_{m+1} \wedge \Diamond_{m+1}(\chi_{m+2} \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \\ &\quad \rightarrow (\chi_m \rightarrow \Box_{m-1}^-(\chi_{m-1} \rightarrow \Box_{m-2}^-(\chi_{m-2} \rightarrow ... \rightarrow \Box_1^-(\chi_1 \rightarrow \Box_0^- \neg\varphi))...). \end{aligned}$$

We want to prove (5.16) for $j = m + 1$. By Proposition 5.4(1), we have:

$$\begin{aligned} &\vdash \Box_m^- \Diamond_m(\chi_{m+1} \wedge \Diamond_{m+1}(\chi_{m+2} \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \\ &\quad \rightarrow \Box_m^-(\chi_m \rightarrow \Box_{m-1}^-(\chi_{m-1} \rightarrow \Box_{m-2}^-(\chi_{m-2} \rightarrow ... \rightarrow \Box_1^-(\chi_1 \rightarrow \Box_0^- \neg\varphi))...). \end{aligned}$$

Then, by Axioms 2e ($p \rightarrow GPp$), 2d ($p \rightarrow Hfp$), 2q ($p \rightarrow NMp$), uniform substitution, and propositional logic, we have:

$$\begin{aligned} &\vdash \chi_{m+1} \wedge \Diamond_{m+1}(\chi_{m+2} \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \\ &\quad \rightarrow \Box_m^-(\chi_m \rightarrow \Box_{m-1}^-(\chi_{m-1} \rightarrow ... \rightarrow \Box_1^-(\chi_1 \rightarrow \Box_0^- \neg\varphi))...). \end{aligned}$$

Then, by propositional logic, we have:

$$\begin{aligned} &\vdash \Diamond_{m+1}(\chi_{m+2} \wedge ... \wedge \Diamond_{n-1}(\chi_n \wedge \Diamond_n\psi))...) \\ &\quad \rightarrow (\chi_{m+1} \rightarrow \Box_m^-(\chi_m \rightarrow \Box_{m-1}^-(\chi_{m-1} \rightarrow ... \rightarrow \Box_1^-(\chi_1 \rightarrow \Box_0^- \neg\varphi))...). \end{aligned}$$

That is (5.16) for $j = m + 1$ as desired:

Take (5.16) for $j = n$:

$$\vdash \Diamond_n\psi \rightarrow (\chi_n \rightarrow \Box_{n-1}^- ... \Box_1^-(\chi_1 \rightarrow \Box_0^- \neg\varphi))...).$$

Then, by Proposition 5.4(1), we have:

$$\vdash \Box_n^- \Diamond_n \psi \to \Box_n^-(\chi_n \to \Box_{n-1}^- ... \Box_1^-(\chi_1 \to \Box_0^- \neg\varphi))...).$$

Then, by Axioms 2e ($p \to GPp$), 2d ($p \to Hfp$), 2q ($p \to NMp$), uniform substitution, and propositional logic, we have:

$$\vdash \psi \to \Box_n^-(\chi_n \to \Box_{n-1}^- ... \Box_1^-(\chi_1 \to \Box_0^- \neg\varphi))...).$$

That is (5.15) as desired. □

Chapter 6

Completeness

In this chapter, we prove that L is sound ($\mathsf{L} \subseteq \mathsf{L}^{\mathfrak{M}}$) and complete ($\mathsf{L}^{\mathfrak{M}} \subseteq \mathsf{L}$) with respect to $\mathsf{L}^{\mathfrak{M}}$. That is, $\mathsf{L}^{\mathfrak{M}}$ is axiomatized by Λ ($\mathsf{L}^{\mathfrak{M}} = \mathsf{L}$). In fact, we also prove that $\mathsf{L}^{\mathfrak{M}}$ is strongly complete (for every $\Gamma \cup \{\varphi\} \subseteq L$, $\Gamma \models \varphi$ implies $\Gamma \vdash_{\mathsf{L}^{\mathfrak{M}}} \varphi$).

In order to prove the completeness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, and the strong completeness of $\mathsf{L}^{\mathfrak{M}}$, it is sufficient to prove that:

$$\text{Every consistent set is satisfiable.} \tag{6.1}$$

For, suppose that (6.1) holds, and for some $\Gamma \cup \{\varphi\} \subseteq L$, we have $\Gamma \nvdash \varphi$. Then, $\Gamma \cup \{\neg\varphi\}$ is consistent. Then, by (6.1), $\Gamma \cup \{\neg\varphi\}$ is satisfiable. Then, $\Gamma \not\models \varphi$. Then:

$$\text{For every } \Gamma \cup \{\varphi\} \subseteq L, \text{ we have that } \Gamma \models \varphi \text{ implies } \Gamma \vdash \varphi. \tag{6.2}$$

This yields the completeness of L with respect to $\mathsf{L}^{\mathfrak{M}}$ (just take $\Gamma = \emptyset$). Moreover, together with the soundness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, it also yields the strong completeness of $\mathsf{L}^{\mathfrak{M}}$. For, suppose that $\Gamma \models \varphi$. Then, by (6.2), $\Gamma \vdash \varphi$. Then, by definition, there are $n \in \mathbf{N}$ and $\gamma_0, ..., \gamma_{n-1} \in \Gamma$ such that $\vdash \bigwedge_{i \in n} \gamma_i \to \varphi$. Then, by soundness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, we have $\vdash_{\mathsf{L}^{\mathfrak{M}}} \bigwedge_{i \in n} \gamma \to \varphi$. Then, by definition, $\Gamma \vdash_{\mathsf{L}^{\mathfrak{M}}} \varphi$.

We prove (6.1) in Sections 6.1-6.4. In Section 6.1 we define and study IRR sets. In Section 6.2, for every IRR maximal consistent set Γ, we define a Kripke frame associated to Γ. This Kripke frame is obtained through a construction adapted from [20]. In Section 6.3, given an arbitrary IRR maximal consistent set Γ, the associated Kripke frame is turned into a model satisfying Γ. This model is obtained through a construction adapted from [42]. These results are then used in Section 6.4 to prove that every consistent set is satisfiable on some model.

6.1 IRR maximal consistent sets

In this section we define IRR sets, and relations $\prec$ and $\sim$ among IRR maximal consistent sets like those in canonical models. More precisely, $\prec$ is the relation associated to G, $\succ$ that associated to H, and $\sim$ that associated to N.

The peculiarity of IRR sets is that they contain a 'name' (a formula of the form $Hp \wedge N\neg p$, for some propositional variable p) for themselves and for every IRR set finitely reachable by a sequence of '$(\prec \cup \succ \cup \sim)$-steps'.

This fact will allow us to prove:

- Proposition 6.10, which says that there are no two maximal IRR consistent sets simultaneously $\prec$-related and $\sim$-related.
- Proposition 6.11 (using Axiom 2s), which says that, for every IRR maximal consistent sets $\Gamma \prec \Delta \sim \Delta'$, there is an IRR maximal consistent set Γ' completing the square ($\Gamma \sim \Gamma' \prec \Delta'$).
- Proposition 6.15, which says that if a formula φ is eventually satisfied in every possible future of an IRR maximal consistent set Γ ($F\varphi \in \Gamma$), and there is an IRR maximal consistent set Δ in the future of Γ ($\Gamma \prec \Delta$) such that φ is not satisfied neither at Δ nor eventually in every possible future of Δ ($\varphi, F\varphi \notin \Delta$), then there is an IRR maximal consistent set Θ between Γ and Δ ($\Gamma \prec \Theta \prec \Delta$) such that φ is satisfied at Θ ($\varphi \in \Theta$).

These propositions, in turn, will let us prove important properties of the Kripke frame that will be defined in Section 6.2. In particular, points 3, 4 and 8 of Proposition 6.22 are consequences of respectively Propositions 6.10, 6.11 and 6.15:

- Point 3 is a consequence of Proposition 6.10 and repeats the content of Proposition 6.10 for the Kripke frame.
- Point 4 is a consequence of Proposition 6.11 and repeats the content of Proposition 6.11 for the Kripke frame.
- Point 8 takes care of the operator g stating that for every formula $g\varphi$ satisfied at an instant w, there is one possible future of w in which φ is always satisfied.

These properties, in turn, will let us define the model of Section 6.3.

The use of IRR maximal consistent sets is allowed by Propositions 6.3, 6.4 and 6.9. Observe that Propositions 6.4 and 6.9 are the adaption of respectively the Lindenbaum Lemma (Proposition 2.5(2)) and Existence Lemma (Proposition 2.5(3)) to IRR maximal consistent sets. In proving Propositions 6.3 and 6.4, the IRR rule is crucial.

Let us define what IRR sets are:

Definition 6.1. Call *IRR* every $\Gamma \subseteq L$ such that:

1. For some propositional variable p, $Hp \wedge N\neg p \in \Gamma$.

2. For every formula $\varphi \in \Gamma$, and $n \in \mathbf{N}$, if φ can be read as $\diamond_0(\psi_0 \wedge \diamond_1(\psi_1 \wedge ... \wedge \diamond_{n-1}\psi_{n-1}))...)$, where, for every $i \in n$, $\diamond_i \in \{f, P, M\}$, then, for some q not occurring in φ, $\diamond_0(\psi_0 \wedge \diamond_1(\psi_1 \wedge ... \wedge \diamond_{n-1}(Hq \wedge N\neg q \wedge \psi_{n-1}))...) \in \Gamma$.

Remark 6.2. For every $n \in \mathbf{N}$, one can easily prove by induction on n that if a formula φ can be read as $\diamond_0(\psi_0 \wedge \diamond_1(\psi_1 \wedge ... \wedge \diamond_{n-1}\psi_{n-1}))...)$ and as $\diamond'_0(\psi'_0 \wedge \diamond'_1(\psi'_1 \wedge ... \wedge \diamond'_{n-1}\psi'_{n-1}))...)$, then, for all $i \in n$, $\diamond_i = \diamond'_i$ and $\psi_i = \psi'_i$.

The following two propositions, together with Proposition 6.9, will let us use IRR maximal consistent sets:

Proposition 6.3. For every $\Gamma \subseteq L$, and $p \in \Phi$, if Γ is consistent and, for every $\varphi \in \Gamma$, p does not occur in φ, then $\Gamma \cup \{Hp \wedge N\neg p\}$ is consistent.

Proof. Suppose $\Gamma \cup \{Hp \wedge N\neg p\}$ is inconsistent. Then, there is $n \in \mathbf{N}$ and $\gamma_0, \gamma_1, ..., \gamma_{n-1} \in \Gamma$ such that, if $\gamma := \bigwedge_{i \in n} \gamma_i$, then:

$$\vdash Hp \wedge N\neg p \wedge \gamma \to \bot.$$

Then, by propositional logic, we have:

$$\vdash Hp \wedge N\neg p \to (\gamma \to \bot).$$

Then, by IRR rule, we have:

$$\vdash \gamma \to \bot,$$

which is against the consistency of Γ. □

Proposition 6.4. (Lindenbaum Lemma for IRR maximal consistent sets [42, Lemma 7.7.6].) For every consistent $\Gamma' \subseteq L$, if the cardinality of:

$$\{p \in \Phi \mid \text{for every } \varphi \in \Gamma', p \text{ does not occur in } \varphi\}$$

is infinite, then there is an IRR maximal consistent set Γ such that $\Gamma' \subseteq \Gamma$.

Proof. Assume that the cardinality of:

$$\{p \in \Phi \mid \text{for every } \varphi \in \Gamma', p \text{ does not occur in } \varphi\}$$

is infinite. Define $\Gamma_0 := \Gamma' \cup \{Hp \wedge N\neg p\}$, for some propositional variable p such that, for every $\varphi \in \Gamma$, p does not occur in φ (p is plainly existing by our assumption on the cardinality of $\{p \in \Phi \mid \text{for every } \varphi \in \Gamma, p \text{ does not occur in } \varphi\}$). Then, by Proposition 6.3, Γ_0 is consistent.

Let $(\psi_0, m_0), (\psi_1, m_1), ...$ be an enumeration of all pairs (ψ_i, m_i) such that ψ_i is a formula, and, if $m_i = 0$, then there is no $n \in \mathbf{N}$ such that ψ_i can be read as $\Diamond_1(\psi_1 \wedge \Diamond_2(\psi_2 \wedge ... \wedge \Diamond_n \psi_n))...)$, while if $m_i \neq 0$, then ψ_i can be read as $\Diamond_1(\chi_1 \wedge \Diamond_2(\chi_2 \wedge ... \wedge \Diamond_{m_i} \chi_{m_i}))...)$ (recall Remark 6.2). Moreover, we can assume that ψ_i is readable as $\Diamond_1(\chi_1 \wedge \Diamond_2(\chi_2 \wedge ... \wedge \Diamond_n \chi_n))...)$ if and only if i is odd.

Assume that Γ_n was defined and that the cardinality of:

$$\{p \in \Phi \,|\, \text{for every } \varphi \in \Gamma_n, \, p \text{ does not occur in } \varphi\}$$

is infinite. Then, either $\Gamma_n \cup \{\psi_n\}$ or $\Gamma_n \cup \{\neg\psi_n\}$ is consistent. If $\Gamma_n \cup \{\neg\psi_n\}$ is consistent, set $\Gamma_{n+1} := \Gamma_n \cup \{\neg\psi_n\}$. Otherwise, if n is even, set $\Gamma_{n+1} := \Gamma_n \cup \{\psi_n\}$. If n is odd, ψ_n can be read as $\Diamond_1(\chi_1 \wedge \Diamond_2(\chi_2 \wedge ... \wedge \Diamond_{m_n} \chi_{m_n}))...)$. Since the cardinality of:

$$\{p \in \Phi \,|\, \text{for every } \varphi \in \Gamma_n, \, p \text{ does not occur in } \varphi\}$$

is infinite, there is $q \in \Phi$ such that, for every $\varphi \in \Gamma_n \cup \{\psi_n\}$, q does not occur in φ. Denote $\Diamond_1(\chi_1 \wedge \Diamond_2(\chi_2 \wedge ... \wedge \Diamond_{m_n}(Hq \wedge N\neg q \wedge \chi_{m_n}))...)$ by $\psi_n(q)$. We show that $\Gamma_n \cup \{\psi_n(q)\}$ is consistent. For, suppose not. Then, there is $m \in \mathbf{N}$ and $\gamma_0, \gamma_1, ..., \gamma_{m-1} \in \Gamma_n$ such that, if we call γ the formula $\bigwedge_{i \in m} \gamma_i$, then:

$$\vdash \neg(\gamma \wedge \psi_n(q)).$$

Then, by propositional logic, we have:

$$\vdash \gamma \to \Box_1(\chi_1 \to \Box_2(\chi_2 \to ... \to \Box_{m_n} \neg(Hq \wedge N\neg q \wedge \chi_{m_n}))...).$$

Then, by Proposition 5.9, we have:

$$\vdash Hq \wedge N\neg q \wedge \chi_{m_n} \to \Box^-_{m_n}(\chi_{m_n - 1} \to \Box^-_{m_n - 1}(\chi_{m_n - 2} \to ... \to \Box^-_1 \neg\gamma))...).$$

Then, by propositional logic, we have:

$$\vdash Hq \wedge N\neg q \to (\chi_{m_n} \to \Box^-_{m_n}(\chi_{m_n - 1} \to \Box^-_{m_n - 1}(\chi_{m_n - 2} \to ... \to \Box^-_1 \neg\gamma))...).$$

Then, by IRR rule, we have:

$$\vdash \chi_{m_n} \to \Box^-_{m_n}(\chi_{m_n - 1} \to \Box^-_{m_n - 1}(\chi_{m_n - 2} \to ... \to \Box^-_1 \neg\gamma))...).$$

Then, by Proposition 5.9, we have:

$$\vdash \gamma \to \Box_1(\chi_1 \to \Box_2(\chi_2 \to ... \to \Box_{m_n} \neg\chi_{m_n}))...).$$

Then, by propositional logic, we have:

$$\vdash \neg(\gamma \wedge \psi_n),$$

which is against the consistency of $\Gamma_n \cup \{\psi_n\}$. Then, $\Gamma_n \cup \{\psi_n(q)\}$ is consistent as desired. Also, one can easily prove that $\vdash \psi_n(q) \to \psi_n$. Then, $\Gamma_n \cup \{\psi_n(q)\} \cup \{\psi_n\}$ is consistent as well. Define $\Gamma_{n+1} := \Gamma_n \cup \{\psi_n(q)\} \cup \{\psi_n\}$. Define $\Gamma := \bigcup_{n \in \mathbf{N}} \Gamma_n$.

Plainly, Γ is a maximal consistent set. As for being IRR, by construction, there is $p \in \Phi$ such that $Hp \wedge N\neg p \in \Gamma_0 \subseteq \Gamma$. Then, Condition 1 of Definition 6.1 is satisfied. Moreover, let φ be a formula of Γ such that φ can be read as $\Diamond_0(\psi_0 \wedge \Diamond_1(\psi_1 \wedge ... \wedge \Diamond_m \psi_m))...)$ for some $m \in \mathbf{N}$. Then, φ occurs in the enumeration as (ψ_n, m) for some odd $n \in \mathbf{N}$. Then, $\Gamma_n \cup \{\neg\psi_n\}$ is not consistent. Otherwise, by construction, $\neg\psi_n \in \Gamma_{n+1} \subseteq \Gamma$. Then, $\psi_n \wedge \neg\psi_n \in \Gamma$, a contradiction of the consistency of Γ. Then, by construction, $\psi_n \wedge \psi_n(q) \in \Gamma_{n+1}$ for some $q \in \Phi$ not occurring in ψ_n. Then, Condition 2 of Definition 6.1 is satisfied. □

Let us define relations between IRR maximal consistent sets, like those of canonical models (see Section 2.5), for G, H, and N respectively:

Definition 6.5. For every two IRR maximal consistent sets Γ, Δ, define:

- $\Gamma \prec \Delta$ provided, for every $\varphi \in L$, $G\varphi \in \Gamma$ implies $\varphi \in \Delta$.
- $\Gamma \succ \Delta$ provided, for every $\varphi \in L$, $H\varphi \in \Gamma$ implies $\varphi \in \Delta$.
- $\Gamma \sim \Delta$ provided, for every $\varphi \in L$, $N\varphi \in \Gamma$ implies $\varphi \in \Delta$.

As L is normal in $\{G, H, N\}$, a straightforward adaption of Proposition 2.2 yields:

Proposition 6.6. For every two maximal consistent sets Γ, Δ, and $\Box \in \{G, H, N\}$ (and $\Diamond$ abbreviating $\neg\Box\neg$), the following conditions are equivalent:

- For every $\varphi \in L$, $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$.
- For every $\varphi \in L$, $\varphi \in \Delta$ implies $\Diamond\varphi \in \Gamma$.

Moreover, by Axioms 2d ($p \to Hfp$) and 2e ($p \to GPp$), one can easily prove that $\succ$ is the inverse $\prec^{-1} := \{(a, b) \,|\, (b, a) \in \prec\}$ of $\prec$:

Proposition 6.7. For every two maximal consistent sets Γ, Δ, the following conditions are equivalent:

1. For every $\varphi \in L$, $G\varphi \in \Gamma$ implies $\varphi \in \Delta$.
2. For every $\varphi \in L$, $H\varphi \in \Delta$ implies $\varphi \in \Gamma$.

Remark 6.8. Definitions like Definition 6.5, and propositions like Propositions 6.6 and 6.7 are common in Modal Logic, especially in completeness proofs when building (variations of) the canonical model. Therefore, we will use Definition 6.5 and Propositions 6.6 and 6.7 without mentioning.

We now prove the adaption of the Existence Lemma (Proposition 2.5(3)) to IRR sets:

Proposition 6.9. (Existence Lemma for IRR maximal consistent sets [42, Lemma 7.7.6].) For every $\varphi \in L$, and IRR maximal consistent set Γ, if $f\varphi \in \Gamma$ ($P\varphi \in \Gamma$, $M\varphi \in \Gamma$, respectively), then there is an IRR maximal consistent set Δ such that $\Gamma \prec \Delta$ ($\Gamma \succ \Delta$, $\Gamma \sim \Delta$, resp.) and $\varphi \in \Delta$.

Proof. By definition of IRR set, there is $p \in \Phi$ such that:

$$f(Hp \wedge N\neg p \wedge \varphi) \in \Gamma. \tag{6.3}$$

We show that $\{Hp \wedge N\neg p \wedge \varphi\} \cup \{\psi \mid G\psi \in \Gamma\}$ is consistent. Indeed, suppose it is not. Then, since $\{\psi \mid G\psi \in \Gamma\}$ is closed under conjunction, there is $G\psi \in \Gamma$ such that:

$$\vdash \neg(Hp \wedge N\neg p \wedge \varphi \wedge \psi).$$

Then, by generalization for G, we have:

$$\vdash G\neg(Hp \wedge N\neg p \wedge \varphi \wedge \psi). \tag{6.4}$$

However, by (6.3), and choice of ψ, we have:

$$f(Hp \wedge N\neg p \wedge \varphi \wedge \psi) \in \Gamma. \tag{6.5}$$

Facts (6.4) and (6.5) are against the consistency of Γ. Then, $\{Hp \wedge N\neg p \wedge \varphi\} \cup \{\psi \mid G\psi \in \Gamma\}$ is consistent as desired. Define $\Delta_0 := \{Hp \wedge N\neg p \wedge \varphi\} \cup \{\psi \mid G\psi \in \Gamma\}$.

For the $n+1$ case, we proceed as in Proposition 6.4, but, if $\Delta_n \cup \{\neg\psi_n\}$ is not consistent and n is odd, we need to make sure that there is a propositional variable q such that q does not occur in ψ_n and $\Delta_n \cup \{\psi_n(q)\}$ is consistent. But first let us prove that:

$$f(Hp \wedge N\neg p \wedge \varphi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n) \in \Gamma. \tag{6.6}$$

For, suppose not. Then, we have:

$$G\neg(Hp \wedge N\neg p \wedge \varphi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n) \in \Gamma.$$

Then:

$$\neg(Hp \wedge N\neg p \wedge \varphi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n) \in \Delta_0 \subseteq \Delta_n.$$

This is against the consistency of $\Delta_n \cup \{\psi_n\}$ and (6.6) is proved.

We are now ready to prove that there is a propositional variable q such that q does not occur in ψ_n and $\Delta_n \cup \{\psi_n(q)\}$ is consistent. By (6.6), since Γ is IRR, we have:

$$f(Hp \wedge N\neg p \wedge \varphi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n(q)) \in \Gamma \tag{6.7}$$

for some q not occurring in:

$$f(Hp \wedge N\neg p \wedge \varphi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n).$$

Then, q is not occurring in ψ_n. Suppose $\Delta_n \cup \{\psi_n(q)\}$ is inconsistent. Then, there is $G\psi \in \Gamma$ such that:

$$\vdash \neg(Hp \wedge N\neg p \wedge \varphi \wedge \psi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n(q)).$$

Then, by generalization for G, we have:

$$\neg f(Hp \wedge N\neg p \wedge \varphi \wedge \psi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n(q)) \in \Gamma. \tag{6.8}$$

However, by (6.7) and $G\psi \in \Gamma$, we have:

$$f(Hp \wedge N\neg p \wedge \varphi \wedge \psi \wedge \bigwedge(\Delta_n - \Delta_0) \wedge \psi_n(q)) \in \Gamma. \tag{6.9}$$

Facts (6.8) and (6.9) are a contradiction. Then, $\Delta_n \cup \{\psi_n(q)\}$ is consistent and q as desired. Define $\Delta := \bigcup_{n \in \mathbf{N}} \Delta_n$. One can easily show that Δ is as desired.

The cases in which $P\varphi \in \Gamma$ or $M\varphi \in \Gamma$ are similar. □

We now prove some propositions about the structure of the set of all IRR maximal consistent sets, equipped with the relations $\prec$ and $\sim$ defined above. These propositions are important to prove Proposition 6.22 about the structure of the Kripke frame that will be defined in Section 6.2. In proving Propositions 6.10, 6.11, and 6.15, the fact that IRR maximal consistent sets contain names is crucial.

Proposition 6.10. ([42, Lemma 7.7.7].) $\prec \cap \sim\ = \emptyset$. That is, there are no two IRR maximal consistent sets that are at the same time $\prec$-related and $\sim$-related.

Proof. Suppose $\prec \cap \sim\ \neq \emptyset$. Then, there are two IRR maximal consistent sets Γ, Δ such that $\Gamma \prec \Delta$ and $\Gamma \sim \Delta$. However, since Δ is an IRR set, there is $p \in \Phi$ such that $Hp \wedge N\neg p \in \Delta$. Then, by definition of $\prec$ and $\sim$, we have $p \in \Gamma$ and $\neg p \in \Gamma$, against the consistency of Γ. □

Proposition 6.11. ([42, Lemma 7.7.9].) For every IRR maximal consistent set Γ, Δ, and Δ', if $\Gamma \prec \Delta$ and $\Delta \sim \Delta'$, then there is an IRR maximal consistent set Γ' such that $\Gamma \sim \Gamma'$ and $\Gamma' \prec \Delta'$.

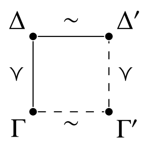

Figure 6.1: There is Γ' such that $\Gamma \sim \Gamma'$ and $\Gamma' \prec \Delta'$

Proof. Since Γ is an IRR maximal consistent set, there is $p \in \Phi$ such that $Hp \wedge N\neg p \in \Gamma$. Then, since $\Gamma \prec \Delta$ and $\Delta \sim \Delta'$, we have $MP(Hp \wedge N\neg p) \in \Delta'$. Then, by Axiom 2r ($MPp \to PMp$), we have $PM(Hp \wedge N\neg p) \in \Delta'$. Then, by Proposition 6.9, there is an IRR maximal consistent set $\Gamma' \prec \Delta'$ such that $M(Hp \wedge N\neg p) \in \Gamma'$. We prove that $\Gamma \sim \Gamma'$. Indeed, suppose $N\varphi \in \Gamma$. Then, by Axiom 2s ($Hp \wedge N\neg p \wedge Nq \to GNH(M(Hp \wedge N\neg p) \to q)$), we have $GNH(M(Hp \wedge N\neg p) \to \varphi) \in \Gamma$. Then, since $\Gamma \prec \Delta$ and $\Delta \sim \Delta'$, we have $M(Hp \wedge N\neg p) \to \varphi \in \Gamma'$. Then, since $M(Hp \wedge N\neg p) \in \Gamma'$, we have $\varphi \in \Gamma'$. □

Proposition 6.12. ([42, Lemma 7.7.8].) $\prec$ is a transitive, serial and downward linear relation on the set of IRR maximal consistent sets.

Proof. Transitivity follows, by classical work, from Axiom 2g ($Gp \to GGp$). Seriality follows from Proposition 5.7(3) ($\vdash Gp \to fp$) (which is obtained combining Axioms 2i ($Gp \to Fp$) and 2j ($Gp \to gp$), and yields $\vdash f\top$) and Proposition 6.9.

Let us show that $\prec$ is downward linear. For, suppose not. Then, there are IRR maximal consistent sets Γ, Δ_1 and Δ_2 such that $\Delta_1, \Delta_2 \prec \Gamma$, $\Delta_1 \neq \Delta_2$, $\Delta_1 \not\prec \Delta_2$ and $\Delta_2 \not\prec \Delta_1$. Then, there are χ_1, χ_2 and χ_3 such that $\chi_1 \in \Delta_1, \chi_1 \notin \Delta_2, \chi_2 \in \Delta_1, f\chi_2 \notin \Delta_2$, $\chi_3 \in \Delta_1$ and $P\chi_3 \notin \Delta_2$. Let φ be $\bigwedge_{i=1}^{3} \chi_i$. Then, $\varphi \in \Delta_1$. Then, we have $fP\varphi \in \Delta_2$. Then, by Axiom 2k ($Hp \wedge p \wedge Gp \to GHp$), we have $P\varphi \vee \varphi \vee f\varphi \in \Delta_2$. If $P\varphi \in \Delta_2$, then we have $P\chi_3 \in \Delta_2$, a contradiction. If $\varphi \in \Delta_2$, then we have $\chi_1 \in \Delta_2$, a contradiction. If $f\varphi \in \Delta_2$, then we have $f\chi_2 \in \Delta_2$, a contradiction. □

Proposition 6.13. ([42, Lemma 7.7.8].) $\sim$ is an equivalence relation over the set of IRR maximal consistent sets.

Proof. By classical work, using Axiom 2p ($Np \to p$) for reflexivity, 2q ($p \to NMp$) for symmetry, 2o ($Np \to NNp$) for transitivity. □

Remark 6.14. Propositions 6.10 and 6.13 imply that $\prec$ is irreflexive.

Proposition 6.15. (Cf. [20, Lemma 3.7].) For every $\varphi \in L$, and IRR maximal consistent sets Γ, Δ, if $\Gamma \prec \Delta$, $F\varphi \in \Gamma$, and $\neg(\varphi \vee F\varphi) \in \Delta$, then there exists an IRR maximal consistent set Θ such that $\Gamma \prec \Theta \prec \Delta$ and $\varphi \in \Theta$.

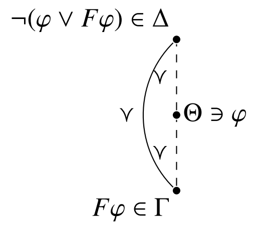

Figure 6.2: There is Θ such that $\Gamma \prec \Theta \prec \Delta$ and $\varphi \in \Theta$

Proof. Since Γ is an IRR maximal consistent set, there is $p \in \Phi$, such that $Hp \wedge N\neg p \in \Gamma$. Then, by Axiom 2p, we have $Hp \wedge \neg p \in \Gamma$. Then, since $\Gamma \prec \Delta$, we have $P(Hp \wedge \neg p \wedge F\varphi) \in \Delta$. Then, by Proposition 5.8 ($\vdash P(p \wedge Fq) \rightarrow P(Pp \wedge q) \vee q \vee Fq$) we have $P(P(Hp \wedge \neg p) \wedge \varphi) \in \Delta$. Then, by Proposition 6.9, there is an IRR maximal consistent set Θ such that $\Theta \prec \Delta$ and $P(Hp \wedge \neg p) \wedge \varphi \in \Theta$. Then, by Proposition 6.12, we have $\Gamma \prec \Theta$ or $\Gamma = \Theta$ or $\Gamma \succ \Theta$. If $\Theta = \Gamma$, then, since $Hp \in \Gamma$ and $P\neg p \in \Theta$, we obtain a contradiction. If $\Theta \prec \Gamma$, then, since $P\neg p \in \Theta$, by Proposition 6.9, there is an IRR maximal consistent set Ξ such that $\Xi \prec \Theta$ and $\neg p \in \Xi$. Then, since $\Xi \prec \Theta \prec \Gamma$ and $\prec$ is transitive (Proposition 6.12), we have $\neg p \in \Xi \prec \Gamma$, against $Hp \in \Gamma$. Then $\Gamma \prec \Theta$.

□

6.2 Kripke Frame

Recall that a Kripke frame for L is a tuple (W, R_0, R_1, R_2) such that W is a nonempty set and R_0, R_1 and R_2 are (possibly empty) binary relations on W associated, respectively, to the operators G, H and N. If we present a Kripke frame for L as triple (W, R, S) we are implicitly assuming that R is the relation associated to G, R^{-1} is the relation associated to H, and S is the relation associated to N.

In this section, for every IRR maximal consistent set Γ, we define a Kripke frame for L associated to Γ. This Kripke frame has, as domain, a set of arbitrary points and, as relations, a relation R associated to G (and implicitly R^{-1} associated to H) and a relation S associated to N. It is related to the set of all IRR maximal consistent sets by means of a function C, named, as in [20], 'chronicle'. More precisely, C associates each point to an IRR maximal consistent set, and for every couple of points x, y, if xRy (respectively, xSy) then $C(x) \prec C(y)$ (respectively, $C(x) \sim C(y)$).

The reason why we consider this Kripke frame, instead of the Kripke frame given by the set of all IRR maximal consistent sets together with $\prec$ and $\sim$, is that the structure needed to perform the construction of Section 6.3, which will produce a model for Γ, is slightly different from the structure of the frame given by the set of all IRR maximal consistent sets together with $\prec$ and $\sim$.

In particular, we want that whenever some $g\varphi$ belongs to $C(x)$ for some point x, then there is an history h passing through x such that, for every $y \in h$ with xRy, we have $\varphi \in C(y)$ (Proposition 6.22(8)). This property will be important to prove the g-case of Proposition 6.41, which is a variant of the Truth Lemma (Proposition 2.5(1)) adapted to the present context. Also, we want no history to be completely S-related to any other history (Corollary 6.31). This last property will be important to define the indistinguishability function of the model that will be built in Section 6.3.

We build the Kripke frame of this section following the construction in [20]. There, Peircean Logic is considered. Here, we also have the Ockhamist operator N. Therefore, in addition to treating the G, F and H-cases (Propositions 6.18, 6.19 and 6.20), treated also in [20, Lemmas 3.9, 3.10, 3.12], we also have to treat the N-case (Proposition 6.21).

Definition 6.16. A *chronicle* on a Kripke frame $\mathcal{F} = (W, R, S)$ is a function C assigning to each $w \in W$ an IRR maximal consistent set in such a way that, for every $w, v \in W$, wRv implies $C(w) \prec C(v)$, and wSv implies $C(w) \sim C(v)$.

Definition 6.17. Consider the following conditions over a chronicle C on a linear order $\mathcal{I} = (I, R)$ (a linear order (I, R) can be viewed as a Kripke frame $(I, R, \emptyset)$):

1. For every formula ψ, and every $i, j \in I$ such that iRj, $P\psi \in C(j)$ and $\psi, P\psi \notin C(i)$, there is $k \in I$ such that iRk, kRj and $\psi \in C(k)$.

2. For every formula ψ, and every $i, j \in I$ such that iRj, $F\psi \in C(i)$ and $\psi, F\psi \notin C(j)$, there is $k \in I$ such that iRk, kRj and $\psi \in C(k)$.

3. For every formula ψ, and every $i \in I$ such that $P\psi \in C(i)$, there is $j \in I$ such that jRi and $\psi \in C(j)$.

4. For every formula ψ, and every $i \in I$ such that $F\psi \in C(i)$, there is $j \in I$ such that iRj and $\psi \in C(j)$.

If C satisfies $1, 2, 3$, it is called *historic*. If C satisfies $1, 2, 4$, it is called *prophetic*. If C satisfies $1, 2, 3, 4$, it is called *perfect*.

The following are intermediate propositions important to prove Proposition 6.22. All of them, from Proposition 6.18 to Proposition 6.21, follow a similar pattern. A linear order $\mathcal{I}$ and a chronicle C on $\mathcal{I}$ satisfying some of the requirements of Definition 6.17 must be defined. Note that these requirements are existential. With the exception of Proposition 6.18, in which a simpler construction suffices, for all the other propositions, $\mathcal{I}$ and C are defined inductively. At step 0, a linear order $\mathcal{I}_0$ and a chronicle C_0 on $\mathcal{I}_0$ are defined. At step $n + 1$, the satisfaction by $\mathcal{I}_n$ and C_n of a particular instance of the desired requirements is checked. If such an instance is not satisfied, the missing elements are added and the chronicle on these new elements defined. The resulting linear order and chronicle define $\mathcal{I}_{n+1}$ and C_{n+1}. $\mathcal{I}$ is then defined as the union of all the $\mathcal{I}_n$ and C as the union of all the C_n.

Proposition 6.18. (Cf. [20, Lemma 3.9].) For every IRR maximal consistent set Γ, there is a linear order $\mathcal{I} = (I, R)$, and a historic chronicle C on $\mathcal{I}$ such that $\mathcal{I}$ has a greatest element, say i, with $C(i) = \Gamma$.

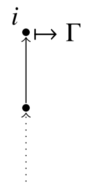

Figure 6.3: $\mathcal{I}$ has a greatest element i such that $C(i) = \Gamma$

Proof. Define:

$$
\begin{aligned}
I &:= \{\Delta \mid \Delta \text{ IRR maximal consistent set} \wedge \Delta \leq \Gamma\},\\
R &:= \prec\restriction I,\\
\mathcal{I} &:= (I, R),\\
C &:= id_I.
\end{aligned}
$$

$\mathcal{I}$ is a linear order by definition, Remark 6.14 and Proposition 6.12. C is plainly a chronicle. Points 1 and 3 of Definition 6.17 follow from definition and Proposition 6.9. Point 2 of Definition 6.17 follows from definition and Proposition 6.15. □

Proposition 6.19. ([20, Lemma 3.10].) For every formula ψ, and IRR maximal consistent set Γ such that $f\psi \in \Gamma$, there is a linear order $\mathcal{I} = (I, R)$ and a prophetic chronicle C on $\mathcal{I}$ such that $\mathcal{I}$ has a smallest element, say i, with $C(i) = \Gamma$, and there is $j \in I$ such that iRj and $\psi \in C(j)$.

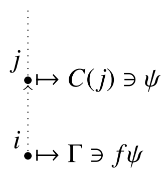

Figure 6.4: $\mathcal{I}$ has a smallest element i st $C(i) = \Gamma$ and there is $j \in I$ st iRj and $\psi \in C(j)$

Proof. Step 0. By Proposition 6.9, there is Δ such that $\Gamma \prec \Delta$ and $\psi \in \Delta$. Consider new objects i, j and define $\mathcal{I}_0 := (I_0, R_0)$, where $I_0 := \{i, j\}$, $R_0 = \{(i, j)\}$, and $C_0 := \{(i, \Gamma), (j, \Delta)\}$.

Let $\varphi_1, \varphi_2, ...$ be an enumeration of all the formulas of the form $P\psi$ or $F\psi$ such that every φ in the list occurs infinitely many times in the list.

Step n+1. Suppose φ_{n+1} is $F\psi$. List the elements of I_n in R_n order: $(i =)i_1, i_2, ..., i_M$. Consider i_1. If $F\psi \notin C_n(i_1)$ or $F\psi \vee \psi \in C_n(i_2)$, go to i_2. Otherwise, by Proposition 6.15, there is an IRR maximal consistent set Θ, with $C_n(i_1) \prec \Theta \prec C_n(i_2)$ and $\psi \in \Theta$. Add an element i'_1 between i_1 and i_2 to I_n, and (i'_1, Θ) to C_n. Go to i_2. Proceed in this way till i_M is reached. If $F\psi \notin C_n(i_M)$, stop. Otherwise, by Axiom 2j ($Gp \to gp$) and Proposition 6.9, there is an IRR maximal consistent set Δ such that $C_n(i_M) \prec \Delta$ and $\psi \in \Delta$. Add an element i'_M after i_M to I_n, and (i'_M, Δ) to C_n. Call the resulting structure $\mathcal{I}_{n+1} = (I_{n+1}, R_{n+1})$, and the resulting chronicle C_{n+1}.

Suppose φ_{n+1} is $P\psi$. List the elements of I_n in R_n order: $(i =)i_1, i_2, ..., i_M$. Consider i_M. If $P\psi \notin C_n(i_M)$ or $P\psi \vee \psi \in C_n(i_{M-1})$, go to i_{M-1}. Otherwise, by Proposition 6.9 and downward linearity of $\prec$, there is an IRR maximal consistent set Δ such that $C_n(i_{M-1}) \prec \Delta \prec C_n(i_M)$ and $\psi \in \Delta$. Add an element i'_M between i_M and i_{M-1} to I_n, and (i'_M, Δ) to C_n. Go to i_{M-1}. Proceed in this way till i_1 is reached and stop, without adding any new object after i_1. Call the resulting structure $\mathcal{I}_{n+1} = (I_{n+1}, R_{n+1})$, and the resulting chronicle C_{n+1}.

Define $\mathcal{I} = (I, R)$ as the union of all the $\mathcal{I}_n$, and C as the union of all the C_n. Plainly $\mathcal{I}$ is linear and C is a chronicle on $\mathcal{I}$.

Let us check that C is prophetic. Let us begin with Condition 1 of Definition 6.17. Suppose that there is a formula ψ and elements $i, j \in I$ such that iRj, $P\psi \in C(j)$ and $\psi, P\psi \notin C(i)$. Then, there is $n \in \mathbf{N}$ such that ψ is φ_n and there are elements $i', j' \in I_n$ such that $iR^r i'$, $i'Rj'$, and $j'R^r j$ (recall that R^r denotes the reflexive closure of R), $\{k \in I_n \mid i'R_n kR_n j'\} = \emptyset$, $P\psi \in C_n(j')$ and $\psi, P\psi \notin C_n(i')$. Then, by construction, there is $k \in I_{n+1}$ such that $i'R_{n+1}k$, $kR_{n+1}j'$ and $\psi \in C_{n+1}(k)$. Then, by construction, $k \in I$, iRk, kRj and $\psi \in C(k)$. With a specular argument, we can prove Condition 2 of Definition 6.17.

Let us check Condition 4 of Definition 6.17. Suppose that there is a formula ψ and an element $i \in I$ such that $F\psi \in C(i)$. Then, there is $n \in \mathbf{N}$ such that ψ is φ_n and $i \in I_n$. If there is $j \in I_n$ such that $iR_n j$ and $\psi, F\psi \notin C_n(j)$ then, by construction, $j \in I$, iRj and $\psi, F\psi \notin C(j)$. Then, by condition 2 of Definition 6.17, there is $k \in I$ such that iRk and $\psi \in k$. Otherwise, $F\psi$ belongs to the greates element i_M of I_n. Then, by construction, there is $j \in I_{n+1}$ such that $iR^r_{n+1} i_M R_{n+1} j$ and $\psi \in j$. Then, by construction, $j \in I$, iRj and $\psi \in j$. Then C is a prophetic chronicle on $\mathcal{I}$ as desired.

All other properties required on $\mathcal{I}$ and C are plain. □

Proposition 6.20. ([20, Lemma 3.12].) For every $\psi \in L$, and IRR maximal consistent set Γ such that $g\psi \in \Gamma$, there exists a linear order $\mathcal{I} = (I, R)$ and a prophetic chronicle C on $\mathcal{I}$ such that $\mathcal{I}$ has a smallest element, say i, with $C(i) = \Gamma$, and for all j, with iRj, $\psi \in C(j)$.

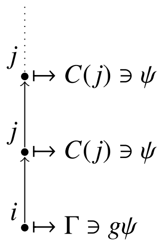

Figure 6.5: $\mathcal{I}$ has a smallest element i st $C(i) = \Gamma$ and for all j with iRj, $\psi \in C(j)$

Proof. Consider the case in which ψ is $H\chi$, for some formula χ. Proceed as in Proposition 6.19, with only the following modifications:

1. At step 0, use Axioms 2h ($FFp \to Fp$) and 2i ($Gp \to Fp$) to find an IRR maximal consistent set Δ, with $\Gamma \prec \Delta$ and $gH\chi \in \Delta$.

2. At each step $n > 0$, if $z_{M,n}$ is the greatest element of $\mathcal{I}_n$, assure that $gH\chi \in C_n(z_{M,n})$, proceeding as follows:

   - According to the proof of Proposition 6.19, the greatest element of $\mathcal{I}_{n+1}$ is different from the greatest element of $\mathcal{I}_n$ only when φ_n is $F\vartheta$, for some formula ϑ, and $F\vartheta \in C_n(z_{M,n})$. Now, if $F\vartheta \wedge gH\chi \in C_n(z_{M,n})$, by Axiom 2$h$ ($FFp \to Fp$) and Proposition 5.7(6) ($\vdash Fp \wedge gq \to f(p \wedge q)$), we have $f(\vartheta \wedge gH\chi) \in C_n(z_{M,n})$. Then, by Proposition 6.9, there is Δ such that $C_n(z_{M,n}) \prec \Delta$ and $\vartheta \wedge gH\chi \in \Delta$. Then, if we define $C_{n+1}(z_{M,n+1}) := \Delta$, we have $gH\chi \in C_{n+1}(z_{M,n+1})$ as desired.

In the end, by Axiom 2i ($Gp \to Fp$), for every $i \in I$, there is $j \in I$, with iRj and $gH\chi \in C(j)$. Then, $fgH\chi \in C(i)$. Applying Proposition 5.7.4 ($\vdash fgp \to fp$) and Axiom 2e ($p \to GPp$), we have $\chi \in C(i)$.

Suppose ψ is an arbitrary formula. Since Γ is an IRR maximal consistent set, there is $p \in \Phi$ such that $Hp \wedge \neg p \in \Gamma$. Then, using Proposition 5.7(9) ($\vdash Hp \wedge gq \to gH(P\neg p \to q)$), we have $gH(P\neg p \to \psi) \in \Gamma$. Then, there exists a linear order $\mathcal{I} = (I, R)$ and a prophetic chronicle C on $\mathcal{I}$ such that $\mathcal{I}$ has a smallest element, say i, with $C(i) = \Gamma$, and, for every element $j \in I$, $P\neg p \to \psi \in C(j)$. Therefore, as $\neg p \in C(i)$, for every $j \in I$ such that iRj, we have $\psi \in C(j)$. □

Proposition 6.21. For every $\psi \in L$, linear order $\mathcal{I} = (I, R)$ with a greatest element, say i, and historic chronicle C on $\mathcal{I}$ such that $M\psi \in C(i)$, there exists a linear order $\mathcal{I}' = (I', R')$, with a greatest element, say i', and a historic chronicle C' on $\mathcal{I}'$ such that $\psi \in C'(i')$ and, for all $j \in I$ there is $j' \in I'$, with $C(j) \sim C'(j')$, and for all $j' \in I'$ there is $j \in I$, with $C'(j') \sim C(j)$.

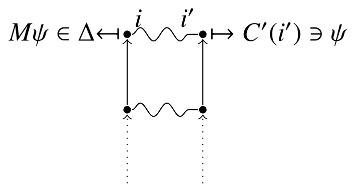

Figure 6.6: Proposition 6.21

Proof. By Proposition 6.9, there is an IRR maximal consistent set Γ such that $C(i) \sim \Gamma$ and $\psi \in \Gamma$. Consider a new object i' and define $I' = \{i'\}$ and $C'(i') = \Gamma$. For every $j \in I$ such that jRi, by Proposition 6.11, there is Γ_j such that $C(j) \sim \Gamma_j$ and $\Gamma_j \prec \Gamma$. Then, for every $j \in I$ such that jRi, add a new object j' to I' and define $C'(j') = \Gamma_j$. Finally, for every $j', k' \in I'$, define $j'R'k'$ if and only if $C'(j') \prec C'(k')$.

Define $\mathcal{I}' := (I', R')$. By definition of R' and irreflexivity, transitivity and downward linearity of $\prec$, we have that $\mathcal{I}'$ is linear. Moreover, C' is a chronicle on $\mathcal{I}'$. Let us prove that C' is a historic chronicle. Consider an arbitrary formula ψ and an arbitrary $j' \in I'$ such that $P\psi \in C'(j')$. Then, by Proposition 6.9, there is Γ' such that $\Gamma' \prec C'(j')$ and $\psi \in \Gamma'$. Then, by Proposition 6.11, there is Γ such that $\Gamma \prec C(j)$ and $\Gamma \sim \Gamma'$. Since Γ is IRR, we have $Hp \wedge N\neg p \in \Gamma$ for some propositional variable p. Then, $P(Hp \wedge N\neg p) \in C(j)$. Then, there is $k \in I$ such that kRj and $Hp \wedge N\neg p \in C(k)$. Then, by downward linearity of $\prec$, we have $\Gamma = C(k)$. Then, there is $k' \in I'$ such that $\Gamma \sim C(k')$. Then, by downward linearity of $\prec$ and $\prec \cap \sim = \emptyset$, we have $\Gamma' = C'(k')$.

Similarly, given an arbitrary ψ and arbitrary $j', l' \in I'$ such that $l'R'j'$, $P\psi \in C'(j')$ and $\psi, P\psi \notin C'(l')$, there is $k' \in I'$, with $l'R'k'$, $k'R'j'$ and $\psi \in C'(k')$; given an arbitrary formula ψ and arbitrary $j', l' \in I'$, with $j'R'l'$, $F\psi \in C'(j')$ and $\psi, F\psi \notin C'(l')$, there is $k' \in I'$, with $j'R'k'$, $k'R'l'$ and $\psi \in C'(k')$. □

The following proposition shows how to build, for every IRR maximal consistent set Γ, the desired Kripke frame. Again, some existential requirements must be satisfied, an inductive construction will be done, and, at each step, the missing part of the frame and the associated part of the chronicle will be added by means of Propositions 6.18-6.21.

Proposition 6.22. For every IRR maximal consistent set Γ, there is a Kripke frame $\mathcal{F} = (W, R, S)$ and a chronicle C on $\mathcal{F}$ such that:

1. R is transitive, downward linear and serial.
2. S is an equivalence relation.
3. $R \cap S = \emptyset$.
4. For every $w, v, v' \in W$, if wRv and vSv', then there is w' such that wSw' and $w'Rv'$.

5. There is $w \in W$ such that $C(w) = \Gamma$.

6. For every $w \in W$, there is $h \in H_w$ such that $C\restriction h$ is a perfect chronicle ($C\restriction h$ denotes C restricted to h).

7. For every $w \in W$, and $\psi \in L$, if $f\psi \in C(w)$, then there is $h \in H_w$ such that $C\restriction h$ is a perfect chronicle, and there is $v \in h$ such that wRv and $\psi \in C(v)$.

8. For every $w \in W$, and $\psi \in L$, if $g\psi \in C(w)$, then there is $h \in H_w$ such that $C\restriction h$ is a perfect chronicle, and, for every $v \in h$ such that wRv, we have $\psi \in C(v)$.

9. For every $w \in W$, and $\psi \in L$, if $M\psi \in C(w)$, then there is $v \in W$ such that wSv and $\psi \in C(v)$.

Proof. Let Γ be an arbitrary IRR maximal consistent set.

Step 0. By Proposition 6.18, there exists a Kripke frame $\mathcal{F} = (I, R, S)$ and a historic chronicle C on $\mathcal{F}$ such that $S = \emptyset$, (I, R) is a linear order with a greatest element, say i, and $C(i) = \Gamma$. Define $\mathcal{F}_0 := \mathcal{F}$ and $C_0 := C$.

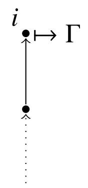

Figure 6.7: Step 0

Step n+1. Consider every pair (w, φ) such that $w \in W_n - W_{n-1}$ and φ is a formula in $C_n(w)$ of the form $f\psi$, $g\psi$ or $M\psi$.

Suppose φ is $f\psi$. By Proposition 6.19, there exists a Kripke frame $\mathcal{F}_{(w,\varphi)} = (I, R, S)$ and a prophetic chronicle $C_{(w,\varphi)}$ on $\mathcal{F}_{(w,\varphi)}$ such that $S = \emptyset$, (I, R) is a linear order with a smallest element, say i, $C_{(w,\varphi)}(i) = C_n(w)$, and there is $j \in I$ with iRj and $\psi \in C_{(w,\varphi)}(j)$. We can assume $i = w$ and $I - \{i\}$ disjoint from W_n.

Suppose φ is $g\psi$. By Proposition 6.20, there exists a Kripke frame $\mathcal{F}_{(w,\varphi)} = (I, R, S)$ and a prophetic chronicle $C_{(w,\varphi)}$ on $\mathcal{F}_{(w,\varphi)}$ such that $S = \emptyset$, (I, R) is a linear order with a smallest element, say i, $C_{(w,\varphi)}(i) = C_n(w)$, and for every j with iRj, $\psi \in C_{(w,\varphi)}(j)$. We can assume $i = w$ and $I - \{i\}$ disjoint from W_n.

Suppose φ is $M\psi$. Assume that, if I_w denote $\{v \in W_n \mid vR_n^r w\}$ (recall that R_n^r denotes the reflexive closure of R_n), then $C_n\restriction I_w$ is historic on I_w (*). Then, by Proposition 6.21, there exists a Kripke frame $\mathcal{F}_{(w,\varphi)} = (I, R, S)$ and a historic chronicle $C_{(w,\varphi)}$ on $\mathcal{F}_{(w,\varphi)}$ such that $S = \emptyset$, (I, R) is a linear order with a greatest element, say i, such that $\psi \in C_{(w,\varphi)}(i)$, and, for every $v \in I_w$, there is $j \in I$ with $C_n(v) \sim C_{(w,\varphi)}(j)$ and, for every $j \in I$, there is $v \in I_w$ with $C_n(v) \sim C_{(w,\varphi)}(j)$. We can assume I disjoint from W_n.

All the $\mathcal{F}_{(w,\varphi)} - \{w\}$ can be assumed disjoint. Let $\mathcal{F}^0_{n+1} := (W_{n+1}, R^0_{n+1}, S_n)$ be the union of $\mathcal{F}_n$ and all the $\mathcal{F}_{(w,\varphi)}$, and C_{n+1} the union of C_n and all the $C_{(w,\varphi)}$. For every $\mathcal{F}_{(w,\varphi)} = (I, R, S)$, with φ of the form $M\psi$, for every $i \in I$, for every $i' \in I_w$ with $C_{(w,\varphi)}(i') \sim C_n(i)$, add (i', i) and (i, i') to S_n, obtaining S^0_{n+1}. (See Figure 6.8.)

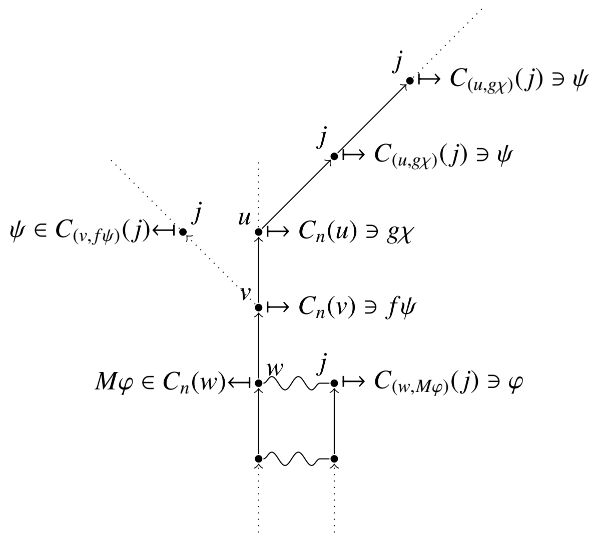

Figure 6.8: Step $n + 1$

Close R^0_{n+1} under transitivity and S^0_{n+1} under reflexivity and transitivity, obtaining R_{n+1} and S_{n+1}. Let $\mathcal{F}$ be the union of all the $\mathcal{F}_n$ and C the union of all the C_n. It is easy to prove that assumption $(*)$ holds at step 0 and that, if assumption $(*)$ holds at step n, then it holds at step $n + 1$. Properties 1-8 easily follow.

A bit of commentary about Properties 6, 7 and 8. To obtain Properties 6, 7 and 8, we "concatenate" linear orders $\mathcal{I}$ such that $\mathcal{I}$ has a greatest element and a historic chronicle defined on it together with linear orders $\mathcal{I}'$ such that $\mathcal{I}'$ has a smallest element and a prophetic chronicle defined on it, identifying the greatest element of $\mathcal{I}$ with the smallest element of $\mathcal{I}'$. A bit of thought shows that by doing this we obtain linear orders with prefect chronicles defined on them. □

6.3 Model

In the previous section, for every IRR maximal consistent set Γ, we defined a Kripke frame associated to Γ. In this section, given an arbitrary IRR maximal consistent set Γ, following [42, Section 7.7, pp. 299–306], we turn the associated Kripke frame $\mathcal{F} = (W, R, S)$ into a model for Γ.

This is done by quotienting over S (S is an equivalence relation). The idea underlying this construction is the following. Points of the Kripke frame $\mathcal{F}$ represent indistinguishability equivalence classes, and they are S-related only if they refer to the same point in time. Then, we define a bundle, an indistinguishability function, and an evaluation on the resulting frame, and we prove that the resulting model satisfies Γ.

Let Γ be an arbitrary IRR maximal consistent set, and $\mathcal{F} = (W, R, S)$ and C the Kripke frame and the chronicle associated to Γ provided by Proposition 6.22.

Definition 6.23. For every $w \in W$, define $\mathbf{w} := \{v \in W \mid wSv\}$, and $\mathbf{W} := \{\mathbf{w} \mid w \in W\}$. That is $\mathbf{w}$ is the equivalence class of w with respect to S and $\mathbf{W}$ is the set of the equivalence classes with respect to S of all the elements of W. For every $\mathbf{w}, \mathbf{v} \in \mathbf{W}$, define $\mathbf{wRv}$ if and only if there are $w' \in \mathbf{w}$ and $v' \in \mathbf{v}$ such that $w'Rv'$. From now on, if we invoke an element $\mathbf{w} \in \mathbf{W}$, we are implicitly stating the existence of an element $w \in W$ such that $\mathbf{w} := \{v \in W \mid wSv\}$.

Remark 6.24. By Proposition 6.22(4), given $\mathbf{w}, \mathbf{v} \in \mathbf{W}$, with $\mathbf{wRv}$, there is $w' \in \mathbf{w}$ such that $w'Rv$.

Proposition 6.25. $\mathbf{R}$ is an irreflexive, transitive, downward linear and serial binary relation over $\mathbf{W}$.

Proof. Given any $\mathbf{w} \in \mathbf{W}$, suppose $\mathbf{wRw}$. Then, by Remark 6.24, there is $w' \in \mathbf{w}$ with $w'Rw$. This contradicts $R \cap S = \emptyset$. Then, $\mathbf{R}$ is irreflexive. Transitivity and downward linearity easily follow from Remark 6.24 and the same properties for R. Seriality for $\mathbf{R}$ easily follows from seriality for R. □

Definition 6.26. For every $h \in H_{\mathcal{F}}$, define $\mathbf{h} := \{\mathbf{w} \mid w \in h\}$. Plainly $\mathbf{h}$ is a history of $(\mathbf{W}, \mathbf{R})$.

Now we prove that there is no history $\mathbf{h}$ such that all its points are S-related to some other history (Corollary 6.31). This property will be important to define the indistinguishability function I (Definition 6.34).

Definition 6.27. Given $w \in W$, define the *date of birth* of w, $dob(w)$, as the natural number n such that w has been introduced at step n in the construction of $\mathcal{F}$. Given $w, w' \in W$ such that wSw', define the *date of birth* of the relation S between w and w', $dob(S, w, w')$ as the natural number n such that w and w' have been S-related at step n.

Proposition 6.28. For every distinct $w, w' \in W$, if wSw' and $dob(S, w, w') = n$, then:

1. Either, for every $v \in W$ such that $vR^r w$, we have $dob(v) = n$ (recall that R^r denotes the reflexive closure of R);

2. Or, for every $v' \in W$ such that $v'R^r w'$, we have $dob(v') = n$.

Proof. According to the construction, there are three ways in which w and w' may have been S-related:

1. There is v such that: $wR^r v$, $dob(v) = n - 1$, and $M\psi \in C(v)$ for some formula ψ. Then, a new element v', together with w', has been introduced at step n to answer $(v, M\psi)$. In this case, the thesis plainly follows.

2. There is v' such that: $w'R^r v'$, $dob(v') = n - 1$, and $M\psi \in C(v')$ for some formula ψ. Then, a new element v, together with w, has been introduced at step n to answer $(v', M\psi)$. In this case, thesis plainly follows.

3. w and w' have been S-related by doing the transitive closure of S^0_n. Then, by definition of transitive closure, there is $m \in \mathbf{N}$ and $x_1, x_2, ..., x_m \in W_n$ such that $w = x_0 S^0_n x_1 S^0_n ... S^0_n x_{m+1} = w'$ and, for every distinct $i, j \in \{0, 1, ..., m + 1\}$, we have $x_i \neq x_j$. Observe that $S^0_n \setminus S_{n-1}$ always relates an element whose dob is $n - 1$ to an element whose dob is n. Then, there is $i \in \{0, 1, ..., m + 1\}$ such that $dob(x_i) = n$. Otherwise, since S_{n-1} is transitive, we have $wS_{n-1}w'$, a contradiction of $dob(S, w, w') = n$. Observe that, by construction, such an x_i has been S^0_n-related to one and only one element v. Then, either w or w' must be x_i. Otherwise, we have $S^0_n(x_i) = \{x_{i-1}, x_{i+1}\}$, a contradiction of $S^0_n(x_i)$ being a singleton. Assume, without loss of generality, that w is x_i. Then, w has been S-related to an object different than w, namely w', at the same step in which w has been introduced. Then, by construction, there must be $v \in W_n$ such that $wR^r v$ and v has been introduced at step n to answer $(v', M\psi)$ for some $v' \in W_{n-1}$ and formula ψ. Then, thesis plainly follows. □

Corollary 6.29. For every distinct $w, w', v \in W$ such that wSw', $dob(S, w, w') = n$, and wRv, if there is $v' \in W$ such that $w'Rv'$ and vSv', then we have $dob(v) \leq n$.

Proof. Assume $dob(v) > n$. Then, $dob(S, v, v') > n$. Then, by Proposition 6.28, either $dob(w) > n$ or $dob(w') > n$, a contradiction of $dob(S, w, w') = n$. □

Proposition 6.30. For every $n \in \mathbf{N}$, and upward endless $h_1 \in H_{\mathcal{F}_n}$, there is no $h_2 \in H_{\mathcal{F}} \setminus \{h_1\}$ such that, for every $w_1 \in h_1$, there is $w_2 \in h_2$ with $w_1 S_n w_2$.

Proof. By induction on $n \in \mathbf{N}$. If $n = 0$, it is easy since there is no upward endless $h_1 \in H_{\mathcal{F}_0}$. Consider $n > 0$ and assume the thesis for every $k < n$. Consider an upward endless $h_1 \in H_{\mathcal{F}_n}$. If $h_1 \in H_{\mathcal{F}_{n-1}}$, then the thesis follows by inductive hypothesis. If $h_1 \in H_{\mathcal{F}_n} \setminus H_{\mathcal{F}_{n-1}}$, then, since h_1 is upward endless, there is $w_1 \in h_1$ such that $\{v_1 \in h_1 \mid w_1 R^r v_1\}$ has been introduced at step n answering (w_1, φ) for some formula φ of the form either $f\psi$ or $g\psi$. Then, the thesis follows by the fact that, by construction, for every $v \in \{v_1 \in h_1 \mid w_1 R^r v_1\}$, we have $S_n(v) = \{v\}$. □

Corollary 6.31. For every $h_1 \in H_{\mathcal{F}}$, there is no $h_2 \in H_{\mathcal{F}} \setminus \{h_1\}$ such that, for every $w_1 \in h_1$, there is $w_2 \in h_2$ with $w_1 S w_2$. Then, $h \mapsto \mathbf{h}$, for $h \in H_{\mathcal{F}}$, is injective.

Proof. Suppose not. Consider $h_1 \in H_{\mathcal{F}}$ and $h_2 \in H_{\mathcal{F}} \setminus \{h_1\}$ such that for every $w_1 \in h_1$ there is $w_2 \in h_2$, with $w_1 S w_2$. Consider $w_1 \in h_1$ and $w_2 \in h_2$ such that $w_1 S w_2$. Let $dob(S, w_1, w_2) = n$. Consider an arbitrary $v_1 \in h_1$ with $w_1 R v_1$. By assumptions, there is $v_2 \in h_2$ such that $w_2 R v_2$ and $v_1 S v_2$. Then, by Corollary 6.29, we have $dob(v_1), dob(v_2)$ and $dob(S, v_1, v_2) \leq n$. Then, $h_1 \in H_{\mathcal{F}_n}$ and for all $w_1 \in h_1$ there is $w_2 \in h_2$ such that $w_1 S_n w_2$. This contradicts Proposition 6.30. □

Now, we define the bundle B over $(\mathbf{W}, \mathbf{R})$:

Definition 6.32. Define B as the set of all $\mathbf{h}$ such that $C \restriction h$ is perfect.

Proposition 6.33. B is a bundle over $(\mathbf{W}, \mathbf{R})$.

Proof. It follows from Proposition 6.22(6). □

Now, we define the indistinguishability function I over $(\mathbf{W}, \mathbf{R}, B)$:

Definition 6.34. Given $\mathbf{w} \in \mathbf{W}$, $\mathbf{h_1}, \mathbf{h_2} \in B_{\mathbf{w}}$, define $\mathbf{h_1} I_{\mathbf{w}} \mathbf{h_2}$ if and only if $h_1 \cap \mathbf{w} = h_2 \cap \mathbf{w}$. Define $I(\mathbf{w}) := I_{\mathbf{w}}$.

Observe that this is a good definition by Corollary 6.31.

Proposition 6.35. I is an indistinguishability function over $(\mathbf{W}, \mathbf{R}, B)$.

Proof. Plainly, I is a function, and, for every $\mathbf{w} \in \mathbf{W}$, $I_{\mathbf{w}}$ is an equivalence relation on $B_{\mathbf{w}}$. Consider arbitrary $\mathbf{v}, \mathbf{w} \in \mathbf{W}$ such that $\mathbf{vRw}$, and arbitrary $\mathbf{h_1}, \mathbf{h_2} \in B_{\mathbf{w}}$ such that $\mathbf{h_1} I_{\mathbf{w}} \mathbf{h_2}$. Consider an arbitrary $w' \in h_1 \cap \mathbf{w}$, and an arbitrary $v' \in h_1 \cap \mathbf{v}$. Then, since $\mathbf{h_1} I_{\mathbf{w}} \mathbf{h_2}$, we have $w' \in h_2$. Moreover, by linearity of h_1, v' and w' must be comparable. If $w' R^r v'$, possibly applying Proposition 6.22(4), we would contradict $R \cap S = \emptyset$. Then $v' R w'$. Then, since $w' \in h_2$, R is downward linear and h_2 is an history, we have $v' \in h_2$. Then, by arbitrariness of v', we have $h_1 \cap \mathbf{v} \subseteq h_2 \cap \mathbf{v}$. In the same way, we have $h_2 \cap \mathbf{v} \subseteq h_1 \cap \mathbf{v}$. Then, $h_1 \cap \mathbf{v} = h_2 \cap \mathbf{v}$. Then, by definition of $I_{\mathbf{v}}$, we have $\mathbf{h_1} I_{\mathbf{v}} \mathbf{h_2}$. □

Now, we define an evaluation V' over $(\mathbf{W}, \mathbf{R}, B, I)$ (Definition 6.38).

Definition 6.36. For every $w \in W$, define $\pi_w := \{\mathbf{h} \in B \mid h \cap \mathbf{w} = \{w\}\}$.

Proposition 6.37. For every $\mathbf{w} \in \mathbf{W}$, for every $\pi \in \Pi_{\mathbf{w}}$, there is a unique $w' \in \mathbf{w}$ such that $\pi = \pi_{w'}$.

Proof. Since $R \cap S = \emptyset$, for every $\mathbf{w} \in \mathbf{W}$, for every $\mathbf{h} \in B_{\mathbf{w}}$, there is (a unique) $w' \in \mathbf{w}$ such that $h \cap \mathbf{w} = \{w'\}$. Then, for every $\mathbf{w} \in \mathbf{W}$, and $\pi \in \Pi_{\mathbf{w}}$, by definition of $I_{\mathbf{w}}$, there is a unique $w' \in \mathbf{w}$ such that $\pi = \pi_{w'}$. □

Then, we can define V' as follows:

Definition 6.38. For every $p \in \Phi$, define:

$$V'(p) := \{(\mathbf{w}, \pi_{w'}) \in \mathcal{P}(\bigcup_{\mathbf{w} \in \mathbf{W}}(\{\mathbf{w}\} \times \Pi_{\mathbf{w}})) \mid p \in C(w')\}.$$

Now, we define the model $\mathcal{N}$:

Definition 6.39. Define $\mathcal{N} := (\mathbf{W}, \mathbf{R}, B, I, V')$.

Proposition 6.40. $\mathcal{N} = (\mathbf{W}, \mathbf{R}, B, I, V')$ is a model.

Proof. It follows from the previous results. □

We now prove that $\mathcal{N}$ is as desired:

Proposition 6.41. For every $\varphi \in L$, and $w \in W$, we have $\varphi \in C(w)$ if and only if $\mathcal{N}, (\mathbf{w}, \pi_w) \models \varphi$. In particular, $\mathcal{N}$ satisfies Γ.

Proof. By induction on the complexity of φ. Easy if φ is $\top$, p, $\neg\psi$ or $\psi \wedge \chi$ for some propositional variable p and formulas ψ and χ.

Suppose φ is $H\psi$ and $H\psi \in C(w)$. Take an arbitrary $\mathbf{h} \in \pi_w$, and an arbitrary $\mathbf{v} \in \mathbf{h}$ such that $\mathbf{vRw}$. Then, there is $v' \in \mathbf{v}$ such that $v' \in h$ and $v'Rw$. Then, since C is a chronicle, we have $C(v') \prec C(w)$. Then, we have $\psi \in C(v')$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{v}, \pi_{v'}) \models \psi$. Then, since $\pi_{v'} = [\mathbf{h}]_{I_\mathbf{v}}$, we have $\mathcal{N}, (\mathbf{v}, [\mathbf{h}]_{I_\mathbf{v}}) \models \psi$. Then, by arbitrariness of $\mathbf{h}$ and $\mathbf{v}$, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \models H\psi$. Suppose $H\psi \notin C(w)$. Then, by Proposition 6.22(6), there is $h \in H_w$ such that $C \restriction h$ is perfect. Then, there is $v \in h$ such that vRw and $\psi \notin C(v)$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{v}, \pi_v) \not\models \psi$. Then, since $\pi_v = [\mathbf{h}]_{I_\mathbf{v}}$, we have $\mathcal{N}, (\mathbf{v}, [\mathbf{h}]_{I_\mathbf{v}}) \not\models \psi$. Then, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \not\models H\psi$.

Suppose φ is $G\psi$ and $G\psi \in C(w)$. Take an arbitrary $\mathbf{h} \in \pi_w$, and an arbitrary $\mathbf{v} \in \mathbf{h}$ such that $\mathbf{wRv}$. Then, there is $v' \in \mathbf{v}$ such that $v' \in h$ and wRv'. Then, since C is a chronicle, we have $C(w) \prec C(v')$. Then, we have $\psi \in C(v')$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{v}, \pi_{v'}) \models \psi$. Then, since $\pi_{v'} = [\mathbf{h}]_{I_\mathbf{v}}$, we have $\mathcal{N}, (\mathbf{v}, [\mathbf{h}]_{I_\mathbf{v}}) \models \psi$. Then, by arbitrariness of $\mathbf{h}$ and $\mathbf{v}$, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \models G\psi$. Suppose $G\psi \notin C(w)$. Then, by Proposition 6.22(7), there is $h \in H_w$ such that $C \restriction h$ is perfect and there is $v \in h$ such that wRv and $\psi \notin C(v)$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{v}, \pi_v) \not\models \psi$. Then, since $\pi_v = [\mathbf{h}]_{I_\mathbf{v}}$, we have $\mathcal{N}, (\mathbf{v}, [\mathbf{h}]_{I_\mathbf{v}}) \not\models \psi$. Then, $\mathcal{N}, (\mathbf{w}, \pi_w) \not\models G\psi$.

Suppose φ is $F\psi$ and $F\psi \in C(w)$. Consider an arbitrary $\mathbf{h} \in \pi_w$. Then, by definition, $C \restriction h$ is perfect. Then, there is $v \in h$ such that wRv and $\psi \in C(v)$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{v}, \pi_v) \models \psi$. Then, since $\pi_v = [\mathbf{h}]_{I_\mathbf{v}}$, we have $\mathcal{N}, (\mathbf{v}, [\mathbf{h}]_{I_\mathbf{v}}) \models \psi$. Then, by arbitrariness of $\mathbf{h}$, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \models F\psi$. Suppose $F\psi \notin C(w)$. Then, by Proposition 6.22(8), there is $h \in H_w$ such that $C \restriction h$ is perfect and, for all $v \in h$ with wRv, we have $\psi \notin C(v)$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{v}, \pi_v) \not\models \psi$. Then, since $\pi_v = [\mathbf{h}]_{I_v}$, we have $\mathcal{N}, (\mathbf{v}, [\mathbf{h}]_{I_\mathbf{v}}) \not\models \psi$. Then, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \not\models F\psi$.

Suppose φ is $N\psi$ and $N\psi \in C(w)$. Take an arbitrary $\pi \in \Pi_{\mathbf{w}}$. Then, by Proposition 6.37, we have $\pi = \pi_{w'}$ for some $w' \in \mathbf{w}$. Then, since C is a chronicle, we have $C(w) \sim C(w')$. Then, we have $\psi \in C(w')$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{w}, \pi_{w'}) \models \psi$. Then, we have $\mathcal{N}, (\mathbf{w}, \pi) \models \psi$. Then, by arbitrariness of π, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \models N\psi$. Suppose $N\psi \notin C(w)$. Then, by Proposition 6.22(9), there is $w' \in \mathbf{w}$ such that $\psi \notin C(w')$. Then, by inductive hypothesis, we have $\mathcal{N}, (\mathbf{w}, \pi_{w'}) \not\models \psi$. Then, we have $\mathcal{N}, (\mathbf{w}, \pi_w) \not\models N\psi$.

Then, since by Proposition 6.22, there is $w \in \mathcal{F}$ such that $C(w) = \Gamma$, we have that $\mathcal{N}$ satisfies Γ as desired. □

6.4 Completeness

At this point, for every IRR maximal consistent set, we are able to produce a model that satisfies it. Nonetheless, to prove the completeness of L with respect of $\mathsf{L}^{\mathfrak{M}}$ and the strong completeness of $\mathsf{L}^{\mathfrak{M}}$, we want to be able to do it for every consistent set. The problem is that not every consistent set can be extended to an IRR maximal consistent set. Therefore, we cannot apply the construction introduced above.

A consistent set Γ can be extended to an IRR maximal consistent set only if the number of propositional variables not occurring among the formulas in Γ is infinite. However, if we index the set Φ of propositional variables and we double the indexes of the propositional variables $p_i \in \Gamma$ (that is, $p_i \mapsto p_{2i}$), the resulting set $\#\Gamma$ is such that the number of propositional variables not occurring among the formulas in $\#\Gamma$ is infinite. Moreover, it can be proved that, if Γ is consistent, then $\#\Gamma$ is consistent as well. Therefore, $\#\Gamma$ can be extended to an IRR maximal consistent set Γ^*. Then, we are able to build a model for Γ^*. This model, by a slight modification of the evaluation function, can be turned into a model for Γ. And then, the completeness of L with respect of $\mathsf{L}^{\mathfrak{M}}$ and the strong completeness of $\mathsf{L}^{\mathfrak{M}}$ follow.

Definition 6.42. Let $p_1, p_2, ...$ be an enumeration without repetition of Φ. Given a formula φ, let $\#\varphi$ denote the formula obtained from φ by replacing every occurrence of every propositional variable p_i with p_{2i}. Given a subset $\Gamma \subseteq L$, let $\#\Gamma$ be the set obtained from Γ by replacing every $\varphi \in \Gamma$ with $\#\varphi$. Given a subset $\Gamma \subseteq L$, call Γ^{-1} the set obtained from Γ as follows: first discard every $\varphi \in \Gamma$ wherein some propositional variable indexed with an odd number occurs; then replace every remaining $\#\varphi$ with φ.

Proposition 6.43. Given a subset $\Gamma \subseteq L$, if Γ is consistent, then $\#\Gamma$ is consistent. Then, by Proposition 6.1, there is an IRR maximal consistent set Γ^*, with $\#\Gamma \subseteq \Gamma^*$.

Proof. Observe that, since uniform substitution is among the inference rules, the definition of derivation given in Definition 5.3 is equivalent to the following: for every $\varphi \in L$, call a *derivation of* φ every sequence $\varphi_0, ..., \varphi_n$ such that $n \in \mathbf{N}$, $\varphi_n = \varphi$, and, for every $i \leq n$, φ is either the *substitution instance of an axiom* or is obtained from $\varphi_0, ..., \varphi_{i-1}$ by the application of an inference rule. In the reset of the proof we use this definition of derivation.

Suppose $\#\Gamma$ is inconsistent. Then, there are $\#\gamma_1, \#\gamma_2, ..., \#\gamma_n \in \#\Gamma$ such that $\vdash \neg \bigwedge_{i=1}^{n} \#\gamma_i$. Consider any derivation of $\neg \bigwedge_{i=1}^{n} \#\gamma_i$. Since such a derivation is finite, there is a bijective function $s : \Phi \to \Phi$ such that, for every propositional variable p with odd index occurring in the derivation, $s(p)$ has an even index and it does not occur in the derivation. Consider the sequence of formulas obtained from the derivation of $\neg \bigwedge_{i=1}^{n} \#\gamma_i$ by replacing every propositional variable p with odd index by $s(p)$. Then, this sequence is of the form $\#\varphi_1, \#\varphi_2, ..., \#\varphi_n$, and $\#\varphi_1, \#\varphi_2, ..., \#\varphi_n$ is a derivation of $\neg \bigwedge_{i=1}^{n} \#\gamma_i$. Then, $\varphi_1, \varphi_2, ..., \varphi_n$ is a derivation of $\neg \bigwedge_{i=1}^{n} \gamma_i$. Then, since $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$, we have that Γ is inconsistent, a contradiction. □

Fix an arbitrary consistent set Γ. Let Γ be fixed from Definition 6.44 to Proposition 6.46.

Definition 6.44. Let $\mathcal{N} = (\mathbf{W}, \mathbf{R}, B, I, V')$ be the model of Definition 6.23 for Γ^*. For every $p \in \Phi$, define

$$V(p) := \{(\mathbf{w}, \pi_{w'}) \in \mathcal{P}(\bigcup_{\mathbf{w} \in \mathbf{W}} (\{\mathbf{w}\} \times \Pi_{\mathbf{w}})) \mid p \in C(w')^{-1}\}.$$

Define the model $\mathcal{M} = (\mathbf{W}, \mathbf{R}, B, I, V)$ and call it *model for* Γ.

Proposition 6.45. For every $\varphi \in L$, $\mathbf{w} \in \mathbf{W}$, and $\pi \in \Pi_{\mathbf{w}}$, we have $\mathcal{M}, (\mathbf{w}, \pi) \models \varphi$ if and only if $\mathcal{N}, (\mathbf{w}, \pi) \models \#\varphi$.

Proof. Easy induction on the complexity of φ. □

Proposition 6.46. For every $\varphi \in L$, and $w \in W$, if $\varphi \in C(w)^{-1}$, then $\mathcal{M}, (\mathbf{w}, \pi_w) \models \varphi$.

Proof. Consider an arbitrary $\varphi \in C(w)^{-1}$. Then, $\#\varphi \in C(w)$. Then, Propositions 6.41 and 6.45 imply $\mathcal{M}, (\mathbf{w}, \pi_w) \models \varphi$. □

Proposition 6.47. For every consistent set Γ, there is a model $\mathcal{T} = (T, <, B, I, V)$ and a pair $(t, \pi) \in \bigcup_{t \in T} (\{t\} \times \Pi_t)$ such that $\mathcal{T}, (t, \pi) \models \Gamma$.

Proof. Consider the model $\mathcal{M} = (\mathbf{W}, \mathbf{R}, B, I, V)$ for Γ. According to the construction, there is $\mathbf{w} \in \mathbf{W}$ such that $\Gamma \subseteq C(w)^{-1}$. Therefore, Proposition 6.46 yields $\mathcal{M}, (\mathbf{w}, \pi_w) \models \Gamma$. □

Theorem 6.48. L is sound and complete with respect to $\mathsf{L}^{\mathfrak{M}}$. Moreover, $\mathsf{L}^{\mathfrak{M}}$ is strongly complete.

Proof. Soundness of L *with respect to* $\mathsf{L}^{\mathfrak{M}}$. As usual, it suffices to prove that every axiom in Λ is valid and that the deductive rules preserve validities. This is easily doable. In particular, we use seriality to prove the validity of Axiom 2i ($Gp \to Fp$), the indistinguishability condition to prove the validity of Axioms 2f ($Hp \to HHp$), 2g ($Gp \to GGp$), 2k ($Hp \wedge p \wedge Gp \to GHp$) and 2l ($Hp \wedge p \wedge gp \to gHp$), both seriality and the indistinguishability condition to prove the validity of Axiom 2m ($FGp \to GFp$), and irreflexivity to prove that the IRR rule preserves validity.

Completeness of L *with respect to* $\mathsf{L}^{\mathfrak{M}}$*, and strong completeness of* $\mathsf{L}^{\mathfrak{M}}$. As mentioned at the beginning of Chapter 6, Proposition 6.47 together with the soundness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, yields the completeness of L with respect to $\mathsf{L}^{\mathfrak{M}}$, and the strong completeness of $\mathsf{L}^{\mathfrak{M}}$. □

Chapter 7

Future works

In this chapter we present a number of open problems:

1. Is there an axiomatization of $\mathsf{L}^{\mathfrak{M}}$ without a form of the IRR rule?

   As highlighted in Section 6.1, the IRR rule is used to get structural properties of the Kripke Frames of Section 6.2, namely Propositions 6.10 and 6.11. These properties seem unavoidable, as they guarantee that what we obtain, by quotienting over S in Section 6.3, is a tree. Then, an axiomatization without a form of the IRR rule seems to need additional axioms to get these properties. Since the axiomatization of this paper is a mix of the axiomatizations with IRR rule of the Peircean and bundled Ockhamist logics, would it be useful to mix the axiomatizations without IRR rule for Peircean and bundled Ockhamist logics of, respectively, [101] and [100]?

2. Is there an axiomatization of the logic of unbundled I-trees?

   First consider the following example showing the crucial role of bundles in our construction. Consider the formula $\gamma = NG(Np \rightarrow MfNp) \rightarrow Mg(Np \rightarrow fNp)$. With a bit of work, one can show that $\gamma \notin \mathsf{L}^{\mathfrak{M}}$. Consider a formula equivalent to $\neg\gamma$ such as $NG(Np \rightarrow MfNp) \wedge NF(Np \wedge GM\neg p)$. Since we proved that L is sound and complete with respect to $\mathsf{L}^{\mathfrak{M}}$, we know that such a formula is consistent. Then, there is an IRR complete consistent theory Γ containing $NG(Np \rightarrow MfNp) \wedge NF(Np \wedge GM\neg p)$.

   Then, the construction of Section 6.2 produces the following situation, where straight lines denote R and waved lines S:

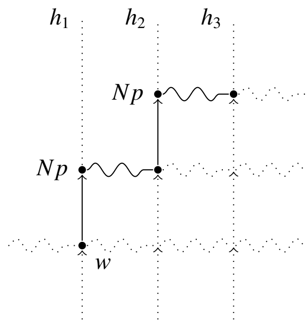

Figure 7.1: Construction of Section 6.2 for the formula $\neg\gamma$

with $C(w) = \Gamma$. When quotienting over S, we obtain a new history, say $\mathbb{h}_\infty$ (straight lines denoting $\mathbf{R}$):

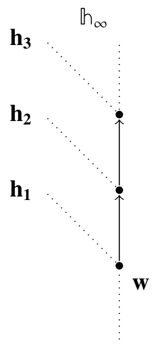

Figure 7.2: When quotienting over S we obtain a new history $\mathbb{h}_\infty$

Without a bundle we would have to consider $\mathbb{h}_\infty$ as well. Surely, for all $\mathbf{v}$ and $\mathbb{h} \in H_{\mathbf{v}}$, if $Np \in v$, then we would define $(\mathbf{v}, [\mathbb{h}]_{\mathbf{v}}) \in V(p)$. Then, $\mathcal{N}, (\mathbf{w}, [\mathbb{h}_\infty]) \models g(Np \rightarrow fNp)$.

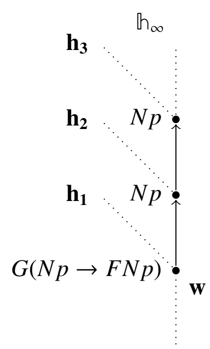

Figure 7.3: $\mathcal{N}, (\mathbf{w}, [\mathbb{h}_\infty]) \models g(Np \to fNp)$

Then $\mathcal{N}, (\mathbf{w}, \pi_w) \models Mg(Np \to fNp)$. However, since $NF(Np \wedge GM\neg p) \in C(w)$, we would contradict Proposition 6.41.

Now, let V_T be the set of all validities of the class of all unbundled I-trees. With a bit of work one can show that $\gamma \in V_T$. Moreover, since an unbundled I-tree can be seen as a bundled I-tree where the bundle select all the histories, $\mathsf{L}^{\mathfrak{M}} \subseteq V_T$. Therefore, $\mathsf{L}^{\mathfrak{M}} \subset V_T$ (recall that $\subset$ denotes proper inclusion). This suggests the need of new axioms and rules to get the missing validities (for example γ). But what are these new axioms and rules? In [79, 80], formulas similar to γ have been used to highlight similar issues concerning the axiomatization of the Ockhamist Logic of unbundled trees. In [80], adding an axiom schema to obtain certain validities on the flavor of γ yields an axiomatization of the Ockhamist Logic of unbundled trees. This suggests to try adding a similar axiom schema.

3. What is the complexity of $\mathsf{L}^{\mathfrak{M}}$?

   In [44], the mosaic technique is used to improve the complexity upper bound for bundled Ockhamist Logic, obtaining 2EXPTIME. We believe that this proof can be adapted to the present context, obtaining 3EXPTIME. Is this also a lower bound?

Part II

On Temporal Logic with derivative operator

Chapter 8

Introduction

In the seminal [71] by McKinsey and Tarski, the authors implicitly defined a topological semantics for the modal operator $\Box$: $\Box\varphi$ is satisfied at a point x provided φ is satisfied at all $y \in O$ for some open set O containing x. They proved that under this interpretation the modal logic of any separable dense-in-itself metric space is **S4**.

In [83] Scott proposed the following 'present progressive' semantics for the operator $\Box$ interpreted on linear orders: $\Box\varphi$ is satisfied at a point x provided φ is satisfied at all $y \in I$ for some open interval I containing x. This is nothing else than the topological semantics with respect to the interval topology (of which the basic open sets are the open intervals).

Following Scott's suggestion, Shehtman put together the temporal operators G, H and the operator $\Box$ and finitely axiomatized the logic of this language interpreted on $\mathbf{Q}$ [86]. Hodkinson finitely axiomatized the logic of this language interpreted on $\mathbf{R}$ [53].

8.1 Content of Part II

In Part II, we consider the coderivative operator $[d]$ instead of the operator $\Box$, together with the temporal operators G and H. The semantics of $[d]$ is as follows: $[d]\varphi$ is satisfied at a point x provided φ is satisfied at all $y \in I \setminus \{x\}$ for some open interval I containing x. We show that the logic of this language interpreted on $\mathbf{R}$ is PSPACE-complete, it is neither strongly nor Kripke complete (and consequently that it fails the finite model property), and it is finitely axiomatizable.

The proofs of these results are mainly adapted from [53] with some modifications. We will use a notation close to that used in [53]. We strongly advise to read [53]. Moreover, when showing the finite axiomatizability of our logic, we will borrow techniques also from [85] and [52]. We will highlight every borrowing when occurring.

In Chapter 9 a number of preliminary definitions and results are given. In Chapter 10 the modal language $G, H, [d]$ interpreted on $\mathbf{R}$ is considered. In Section 11.1 we study the complexity of the validity problem for this language interpreted on **R**. In Section 11.2 we prove that the logic of this language interpreted on **R** is not strongly complete. In Section 11.3 we provide a finite set of axioms for this logic. In Section 11.4 we prove that the axioms are sound. In Section 11.5 we prove that this logic is not Kripke complete (and consequently that it fails the finite model property). Finally, in Chapter 12 we prove that the axioms are complete.

The content of Part II is based on the following paper:

[46] Alberto Gatto. On the Priorean temporal logic with $[d]$ on the real line. *Journal of Logic and Computation*, 26(3):1019–1041, 2016.

8.2 Related Works

In this section we illustrate some related works.

8.2.1 Coderivative operator $[d]$

The semantics of the coderivative operator $[d]$ on topological spaces is generalized to the following definition: a formula $[d]\varphi$ is true at a point x provided φ is satisfied at all $y \in O \setminus \{x\}$ for some open set O containing x.

The coderivative operator is strictly more expressive than the $\Box$ operator and, for every formula φ, $\Box\varphi$ is equivalent to $[d]\varphi \wedge \varphi$. Modal logics with only the coderivative operator (call them d-logics) have long been of interest. Let us list a number of significant results on d-logics.

Logics with the coderivative operator, although already suggested by McKinsey and Tarski in [71, 1944], have been studied firstly by Esakia and the 'Georgian School' from the 1970s. In [30, 1976] and [32, 2001], Esakia proved that **wK4** $:=$ **K** $+$ $\langle d\rangle\langle d\rangle p \to p \vee \langle d\rangle p$ is the d-logic of the class of all topological spaces. In [30, 1976] and [32, 2001], Esakia proved that **K4** is the d-logic of the class of all T_d spaces. In [31, 1981], Esakia proved that the d-logic of the class of all scattered spaces is **GL**. We could not access [30, 1976], [32, 2001], or [31, 1981], but their results are cited, for example, by Esakia himself in [33], and by Kudinov and Shehtman in [64].

In [3, 1987], [1, 1988] and [15, 1990], Abashidze and Blass gave a complete description of the d-logics of ordinals with the interval topology. We have that **K** $+ [d]\bot$ is the d-logic of any finite ordinal with the interval topology, that **GL** $+ [d]^n\bot$ is the logic of any ordinal $\omega^{n-1} \leq \alpha < \omega^n$ for $n \leq 1$, and that **GL** is the d-logic of any ordinal $\alpha \geq \omega^\omega$. Again, we could not access [2, 3, 1987] or [1, 1988], but their results are cited by Kudinov and Shehtman in [64].

In [85, 1990], Shehtman proved that the d-logic of any dense-in-itself 0-dimensional separable metric space, for example **Q**, is **KD4**. Another proof of this result for **Q** was given by Lucero-Bryan in [69, 2011]. In [85, 1990], Shehtman proved that, for all $n \geq 2$, the d-logic of $\mathbf{R}^n$ is $\mathbf{KD4G_1}$:=$\mathbf{KD4}$ + $[d](\blacksquare p \vee \blacksquare\neg p) \to [d]p \vee [d]\neg p$ (with $\blacksquare\varphi$ abbreviating $\varphi \wedge [d]\varphi$). Another proof of this result is given by Kudinov and Shehtman in [64, 2014]. In [85, 1990] Shehtman conjectured that the d-logic of $\mathbf{R}$ is $\mathbf{KD4G_2}$:= $\mathbf{KD4}$ + $[d](\bigvee_{i\in 3} \blacksquare\varphi_i) \to \bigvee_{i\in 3}[d]\neg\varphi_i$ (where φ_i is $p_i \wedge \bigwedge_{j\in 3\setminus\{i\}} \neg p_j$, with p_0, p_1, p_2 distinct propositional variables). Shehtman then proved this conjecture in [88, 2000]. Another proof of this was given by Lucero-Bryan in [70, 2011]. In [88, 2000], Shehtman proved that the d-logic of the class of all 0-dimensional separable metric spaces is $\mathbf{K4}$, and that the d-logic of any dense-in itself separable metric space is $\mathbf{KD4}$. Again, we could not access [88, 2000] but its results are cited by Kudinov and Shehtman in [64].

In [8, 2010], G. Bezhanishvili, Esakia and Gabelaia proved that the d-logic of the class of all Stone spaces is $\mathbf{K4}$, that so is the d-logic of the class of all compact Hausdorff spaces, that the d-logic of the class of all weakly scattered Stone spaces is $\mathbf{K4G} = \mathbf{K4} + \neg[d]\bot \to \langle d\rangle[d]\bot$, and that so is the d-logic of the class of all weakly scattered compact Hausdorff spaces.

In [11, 2011], Bezhanishvili, Esakia and Gabelaia showed $p \wedge \langle d\rangle(q \wedge \langle d\rangle p) \to \langle d\rangle p \vee \langle d\rangle(q \wedge \langle d\rangle q)$ d-defines T_0, and that the d-logic of the class of all T_0-spaces is $\mathbf{wK4} + p \wedge \langle d\rangle(q \wedge \langle d\rangle p) \to \langle d\rangle p \vee \langle d\rangle(q \wedge \langle d\rangle q)$, strictly between $\mathbf{wK4}$ and $\mathbf{K4}$; they showed that this logic is also the logic of the class of all spectral spaces, and enjoy the finite model property; finally they showed that the separation axioms T_1, T_2, T_3 and T_4 are not d-definable.

As mentioned, in [88, 2000], Shehtman showed that the d-logic of $\mathbf{R}$ is $\mathbf{KD4G_2}$. Another proof of this result has been given by Lucero-Bryan in [70, 2011]. In this latter paper, $\langle d\rangle$ is also considered together the universal $\forall$ modality. $\forall\varphi$ holds provided φ holds everywhere. It is shown that $\mathbf{KD4G_n.UC}$ has the finite model property, and that $\mathbf{KD4G_2.UC}$ is the resulting logic of the real line. Axioms $\mathbf{G_1}$, $\mathbf{G_2}$ and, more generally, for all natural numbers $n \geq 2$, axioms:

$$\mathbf{G_n} := [d]\Big(\bigvee_{i\in(n+1)} \blacksquare\varphi_i^n\Big) \to \bigvee_{i\in(n+1)} [d]\neg\varphi_i^n,$$

where φ_i^n is $p_i \wedge \bigwedge_{j\in(n+1)\setminus\{i\}} \neg p_j$, with $p_0, p_1 ..., p_n$ distinct propositional variables, have been introduced by Shehtman in [85, 1990] where he showed that a transitive Kripke frame validates $\mathbf{G}_1$ if and only if it is locally 1-connected. Again in [70, 2011], Lucero-Bryan showed that, for all natural numbers $n \geq 2$, a transitive Kripke frame validates $\mathbf{G}_n$ if and only if it is locally n-connected (see §9.2 for the definition of n-connectedness). In this part, we will use Axiom $\mathbf{G}_2$. Axiom $\mathbf{C} := \forall(\blacksquare p \vee \blacksquare\neg p) \to \forall p \vee \forall\neg p$ defines connected spaces (observe that $\mathbf{C}$ can be expressed by $\Box$ and $\forall$ already, and see [87] for a proof with $[d]$ replaced by $\Box$).

In [10, 2012], G. Bezhanishvili and Lucero-Bryan proved that every extension of

K4 by a set of variable-free axioms is a d-logic of some subspace of **Q**. This gives a continuum of d-logics of countable metric spaces.

In [62, 2013], Kudinov augmented the language with the difference operator $[\neq]$. For all formulas φ, $[\neq]\varphi$ holds at x provided for all $y \neq x$ we have that φ holds at y. He showed that the logic of the resulting language interpreted on **R** has the finite model property but no finite axiomatization. Note that the G and H operators considered in this paper can be seen as a 'splitting' of the difference operator, with, for all formulas φ, $G\varphi$ meaning 'φ holds everywhere after' and $H\varphi$ meaning 'φ holds everywhere before', so that $[\neq]\varphi$ is equivalent to $H\varphi \wedge G\varphi$.

In [64, 2014], Kudinov and Shehtman axiomatized the logic with $\langle d\rangle$ and $[\neq]$ of the class of all spaces, of the class of all T_1 spaces, of the class of all dense-in-themselves spaces, of any 0-dimensional dense-in-itself separable metric space, and of $\mathbf{R}^n$ for $n \geq 2$. They also proved that the d-logic of the class of all trivial spaces is **DL** := **wK4** + $\langle d\rangle[d]p \to p$, and that **DL** is not the logic of any trivial space. In [63, 2010] Kudinov and Shapirovsky proved that the d-logic of any infinite trivial space is not finitely axiomatizable. We could not access [63, 2010], but its results are cited by Kudinov and Shehtman in [64, 2014].

The d-logics of special types of spaces are also studied in [7, 2005] by G. Bezhanishvili, Esakia and Gabelaia, and in [69, 2011] by Lucero-Bryan. They include submaximal, perfectly disconnected, maximal, weakly scattered and some others.

8.2.2 G, H on the real numbers

Interest in the logic of the temporal language G, H interpreted on **R** has been shown by Bull in [18, 1968] in which an axiomatization is provided. To the best of our knowledge, this seems to be the first axiomatization of this logic. In [84, 1970], Segerberg gave a different proof.

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Chapter 9

Preliminaries

In this chapter we give some preliminary definitions and results. They will be used throughout the rest of Part II.

9.1 Clusters

In this section we introduce the mathematical structures called 'clusters'. They are substructures of relational structures and they satisfy certain properties. They will play a pervasive role in Chapter 12.

For every tuple $\mathcal{A} = (A, R)$, where A is a set and R a binary relation on A, an *R-cluster* (*of* $\mathcal{A}$) is a non-empty subset of A, $\subseteq$-maximal among those subsets C of A such that for every two $a, b \in C$ we have aRb. An element $a \in A$ is said to be *R-q-maximal* provided, for every $b \in A$, aRb implies bRa. An R-cluster is said to be *maximal* provided every $a \in C$ is R-q-maximal. It is straightforward to check that:

Proposition 9.1.

1. If R is transitive and $a \in A$ is R-reflexive, then $C(a) := \{b \in A \mid aRbRa\}$ is the only R-cluster of $\mathcal{A}$ containing a.
2. If $a \in A$ is also R-q-maximal then $C(a)$ is a maximal R-cluster of $\mathcal{A}$.

$\mathcal{A}$ has the Zorn Property (*with respect to R*) provided for every $a \in A$, there is $b \in A$ with aRb and b is R-q-maximal.

9.2 Connectedness

In this section we introduce the notions of 'connectedness' and, for every $n \in \mathbf{N}$, of 'local n-connectedness'. These properties apply to relational structures, in particular to clusters. Clusters that are connected and locally 2-connected will be important in

Chapter 12, in particular in the construction of Section 12.3.

Consider a tuple $\mathcal{A} = (A, R)$, where A is a set and R a binary relation on A. Consider two elements $a, b \in A$. Call an *R-path* (*between a, b*) every finite sequence $a_0, a_1, ..., a_n$ of elements of A such that $a = a_0$, $b = a_n$ and, for every $i = 0, 1, ..., n$, we have $a_i R^r a_{i+1}$ (recall that R^r denotes the reflexive closure of R). Define $\widetilde{R} := R \cup R^{-1}$. $\mathcal{A}$ is *connected* provided for every two elements $a, b \in A$, there is an $\widetilde{R}$-path between a and b. A subset of A is called a *connected component* of $\mathcal{A}$ provided it is $\subseteq$-maximal among the subsets B of A such that $(B, R{\restriction}B)$ is connected. Observe that a subset of A is a connected component if and only if it is connected and $\widetilde{R}$-generated. For all natural numbers $n \geq 2$, $\mathcal{A}$ is said to be *locally n-connected* provided for every $a \in A$, $(R(a), R{\restriction}R(a))$ has at most n connected components.

Example 9.2. Consider the infinite binary tree $\mathcal{T}^2 := ({}^{<\mathbf{N}}2, \subset)$ of Example 4.3. $\mathcal{T}^2$ is connected and, for every $n > 1$, locally n-connected. However, $\mathcal{T}^2$ is not locally 1-connected.

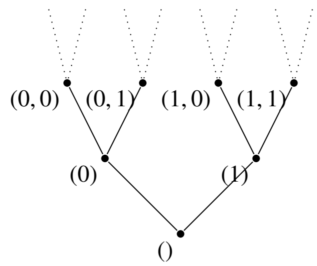

Figure 9.1: Infinite binary tree $\mathcal{T}^2$

9.3 Filtrations

In this section, we introduce the notion of 'filtration'. This notion will be used to define the model $\mathcal{M}_2$ in Section 12.2.3.

Definition 9.3. For every Kripke model $\mathcal{M} = (W, \{R_i\}_{i \in I}, V)$, $w, v \in W$, and $\Psi \subseteq L_{\Phi,\tau}$, define:

1. $w \equiv_{\Psi}^{\mathcal{M}} v$ provided, for every $\varphi \in \Psi$, we have $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}, v \models \varphi$.
2. For every $i \in I$, $w R_{i\Psi} v$ provided, for every $\Box_i \varphi \in \Psi$, if $\mathcal{M}, w \models \Box_i \varphi$ then $\mathcal{M}, v \models \varphi$.

Definition 9.4. For every two Kripke models $\mathcal{M} = (W, \{R_i\}_{i \in I}, V)$ and $\mathcal{M}' = (W', \{R'_i\}_{i \in I}, V')$, $\Psi \subseteq L_{\Phi,\tau}$ closed under taking subformulas, and $h : W \to W'$ a surjective map, $\mathcal{M}'$

is called a *filtration of* $\mathcal{M}$ *through* (Ψ, h) provided, for every $w, v \in W$, and $i \in I$, the following conditions hold:

1. For every $p \in \Phi \cap \Psi$, we have $V'(p) = h(V(p))$.

2. $h(w) = h(v)$ implies $w \equiv_{\Psi}^{\mathcal{M}} v$.

3i. $wR_i v$ implies $h(w)R'_i h(v)$.

4i. $h(w)R'_i h(v)$ implies $wR_{i\Psi} v$.

Proposition 9.5. (Filtration Lemma.) If $\mathcal{M}' = (W', \{R'_i\}_{i \in I}, V')$ is a filtration of $\mathcal{M} = (W, \{R_i\}_{i \in I}, V)$ through (Ψ, h), then, for every $\varphi \in \Psi$, and $w \in W$, we have $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', h(w) \models \varphi$.

Proof. See [86, Lemma 3.2]. □

Chapter 10

Language and semantics

Let Φ be a countably infinite set, and $\tau := \{G, H, [d]\}$. Throughout Part II, we will work with the language $L := L_{\Phi,\tau}$. Let $F := \neg G \neg$, $P := \neg H \neg$, and $\langle d \rangle := \neg [d] \neg$. For every $\varphi \in L$, let $\blacksquare \varphi := \varphi \wedge [d]\varphi$, and $\blacklozenge \varphi := \neg \blacksquare \neg \varphi$.

10.1 Semantics

Let $(\mathbf{R}, <)$ denote the real line with the usual order. We may refer to $(\mathbf{R}, <)$ by just saying $\mathbf{R}$. In this section we define how L will be interpreted on $\mathbf{R}$. Call a *model* every tuple $(\mathbf{R}, V)$ such that $V : \Phi \to \mathcal{P}(\mathbf{R})$ is a function called *evaluation*. Let $\mathfrak{R}$ be the class of all models. For every $\mathcal{R} = (\mathbf{R}, V) \in \mathfrak{R}$, $x \in \mathbf{R}$, and $\varphi \in L_{\Phi,\tau}$, define $\mathcal{R}, x \models \varphi$ recursively as follows:

- $\mathcal{R}, x \models \top$ always.
- $\mathcal{R}, x \models p$ provided $p \in V(x)$.
- $\mathcal{R}, x \models \neg\varphi$ provided not $\mathcal{R}, x \models \varphi$.
- $\mathcal{R}, x \models \varphi \wedge \psi$ provided $\mathcal{R}, x \models \varphi$ and $\mathcal{R}, x \models \psi$.
- $\mathcal{R}, x \models G\varphi$ provided, for every $y > x$, $\mathcal{R}, y \models \varphi$.
- $\mathcal{R}, x \models H\varphi$ provided, for every $y < x$, $\mathcal{R}, y \models \varphi$.
- $\mathcal{R}, x \models [d]\varphi$ provided there are $y, z \in \mathbf{R}$ such that $y < x < z$ and, for every $t \in (y, z) \setminus \{x\}$, $\mathcal{R}, t \models \varphi$.

Additionally, we report the semantics of the modal operator $\Box$ of [53] as follows:

- $\mathcal{R}, x \models \Box\varphi$ provided there are $y, z \in \mathbf{R}$ such that $y < x < z$ and for every $t \in (y, z)$, $\mathcal{R}, t \models \varphi$.

Observe that $\Box$ differs from $[d]$ inasmuch as $\Box$ quantifies also on the point at which the formula is evaluated, whereas $[d]$ ignores such a point. Observe that, for every $\varphi \in L$, $\mathcal{R} \in \mathfrak{R}$, and $x \in \mathbf{R}$, we have $\mathcal{R}, x \models \varphi \wedge [d]\varphi$ if and only if $\mathcal{R}, x \models \blacksquare\varphi$ if and only if $\mathcal{R}, x \models \Box\varphi$. In fact, L is strictly more expressive than the language of [53].

Analogously to Section 2.3, for every $\mathcal{R} \in \mathfrak{R}$, and $\varphi \in L$, we say that $\mathcal{R}$ *satisfies* φ provided there is $x \in \mathbf{R}$ such that $\mathcal{R}, x \models \varphi$.

We say that φ *is satisfiable* provided there is a model satisfying φ, and that φ *is valid* provided $\neg\varphi$ is not satisfiable. Let $\mathsf{L}^{\mathfrak{R}}$ be the set of all valid formulas. $\mathsf{L}^{\mathfrak{R}}$ is a normal modal logic (see Section 2.4).

For every model $\mathcal{R}$, and $\Gamma \subseteq L$, we say that $\mathcal{R}$ *satisfies* Γ provided there is $x \in \mathbf{R}$ such that, for every $\gamma \in \Gamma$, we have $\mathcal{R}, x \models \gamma$. We say that Γ *is satisfiable* provided there is a model satisfying Γ.

For every $\Gamma \cup \{\varphi\} \subseteq L$, we say that Γ *semantically entails* φ, notation $\Gamma \models \varphi$, provided, for every model $\mathcal{R}$, and $x \in \mathbf{R}$, if, for every $\gamma \in \Gamma$, $\mathcal{R}, x \models \gamma$, then $\mathcal{R}, x \models \varphi$.

Chapter 11

Study of $\mathsf{L}^{\mathfrak{R}}$

In this and the following chapter, we study $\mathsf{L}^{\mathfrak{R}}$. In Section 11.1, we study the complexity of the problem of deciding membership to $\mathsf{L}^{\mathfrak{R}}$ - that is, the validity problem for L on $\mathfrak{R}$. In Section 11.2, we prove that $\mathsf{L}^{\mathfrak{R}}$ lacks strong completeness. In Section 11.3, we give a finite set Λ_d of axioms. In Section 11.4, we prove that the smallest normal modal logic L_d containing Λ is sound with respect to $\mathsf{L}^{\mathfrak{R}}$ - that is, $\mathsf{L}_d \subseteq \mathsf{L}^{\mathfrak{R}}$. In Section 11.5, we prove that $\mathsf{L}^{\mathfrak{R}}$ lacks Kripke completeness. In Chapter 12, we prove that L_d is complete with respect to $\mathsf{L}^{\mathfrak{R}}$ - i.e. $\mathsf{L}_d \supseteq \mathsf{L}^{\mathfrak{R}}$. And then, that Λ_d axiomatizes L_d - that is, $\mathsf{L}_d = \mathsf{L}^{\mathfrak{R}}$.

11.1 Complexity of $\mathsf{L}^{\mathfrak{R}}$

In this section we study the complexity of the problem of deciding membership to $\mathsf{L}^{\mathfrak{R}}$ - that is, the validity problem for L on $\mathfrak{R}$. More precisely, we prove that:

Theorem 11.1. (Cf. [53, Theorem 4.1].) The validity problem for L on $\mathfrak{R}$ is PSPACE-complete.

We follow the proof of [53, Theorem 4.1]. Let us introduce the operators U (Until) and S (Since) of [55]. For every $\mathcal{R} = (\mathbf{R}, V) \in \mathfrak{R}$, $x \in \mathbf{R}$, and $\varphi, \psi \in L_{\Phi,\tau}$, define:

- $\mathcal{R}, x \models U(\varphi, \psi)$ provided there is y such that $x < y$, $\mathcal{R}, y \models \varphi$, and, for every $z \in (x, y)$, $\mathcal{R}, z \models \psi$.
- $\mathcal{R}, x \models S(\varphi, \psi)$ provided there is y such that $x > y$, $\mathcal{R}, y \models \varphi$, and, for every $z \in (y, x)$, $\mathcal{R}, z \models \psi$.

In [53, Theorem 4.1], they use the fact that $\Box\psi$ can be expressed by $S(\top, \psi) \wedge \psi \wedge U(\top, \psi)$, whereas here we use the fact that $[d]\psi$ can be expressed by $S(\top, \psi) \wedge U(\top, \psi)$. Also, for every $\varphi \in L$, let $\forall\varphi$ be the abbreviation for $\varphi \wedge \neg U(\neg\varphi, \top) \wedge \neg S(\neg\varphi, \top)$. Observe that, for every $\psi \in L$, every model $\mathcal{R}$, and every $x \in \mathbf{R}$, we have that $\mathcal{R}, x \models \forall\psi$ if and only if, for every $y \in \mathbf{R}$, $\mathcal{R}, y \models \psi$.

Fix $\varphi \in L$. For every $\psi \leq \varphi$, introduce a new propositional atom q_ψ and define the formula $\widehat{\psi}$ recursively as follows:

1. $\widehat{\top} := \forall q_\top$;
2. $\widehat{p} := \forall(q_p \leftrightarrow p)$;
3. $\widehat{\neg\chi} := \forall(q_{\neg\chi} \leftrightarrow \neg q_\chi)$;
4. $\widehat{\chi \wedge \zeta} := \forall(q_{\chi\wedge\zeta} \leftrightarrow q_\chi \wedge q_\zeta)$;
5. $\widehat{F\chi} := \forall(q_{F\chi} \leftrightarrow U(q_\chi, \top))$;
6. $\widehat{P\chi} := \forall(q_{P\chi} \leftrightarrow S(q_\chi, \top))$;
7. $\widehat{[d]\chi} := \forall(q_{[d]\chi} \leftrightarrow U(\top, q_\chi) \wedge S(\top, q_\chi))$.

For every $\psi \leq \varphi$, define:

$$\psi^* := \bigwedge_{\chi \leq \psi} \widehat{\chi} \wedge q_\psi. \tag{11.1}$$

In (11.1) we translate a formula $\psi \in L$ into a brand new propositional atom q_ψ conjuncted with the recursive definition of the representation q_χ of every subformula $\chi \leq \psi$. This recursive definition set q_χ to hold exactly where χ holds. Because of this, we can prove by induction that:

Proposition 11.2. φ is satisfiable on $\mathfrak{R}$ if and only if φ^* is satisfiable on $\mathfrak{R}$.

Proof. First a preliminary lemma:

Lemma 11.3. For every evaluation function $V : \Phi \cup \{q_\psi \mid \psi \leq \varphi\} \to \mathcal{P}(\mathbf{R})$, the following statements are equivalent:

1. For every $\psi \leq \varphi$, and every $x \in \mathbf{R}$, we have $(\mathbf{R}, V), x \models q_\psi$ if and only if $(\mathbf{R}, V), x \models \psi$.
2. For every $\psi \leq \varphi$, and every $x \in \mathbf{R}$, we have $(\mathbf{R}, V), x \models \widehat{\psi}$.

Proof. We begin by proving that 1 implies 2. We report the most interesting cases. Assume that ψ is $p \in \Phi$. By 1, for every $x \in \mathbf{R}$, we have $(\mathbf{R}, V), x \models \forall(q_p \leftrightarrow p)$. We conclude by observing that $\widehat{p}$ is $\forall(q_p \leftrightarrow p)$.

Assume that ψ is $\neg\chi$. Consider an arbitrary x. Suppose that $(\mathbf{R}, V), x \models q_{\neg\chi}$. By 1, this is equivalent to $(\mathbf{R}, V), x \models \neg q_\chi$. Then, $(\mathbf{R}, V), x \models q_{\neg\chi} \leftrightarrow \neg q_\chi$. Then, by arbitrariness of x, for every $x \in \mathbf{R}$, we have $(\mathbf{R}, V), x \models \forall(q_{\neg\chi} \leftrightarrow \neg q_\chi)$. We conclude by observing that $\widehat{\neg\chi}$ is $\forall(q_{\neg\chi} \leftrightarrow \neg q_\chi)$.

Assume that ψ is $[d]\chi$. Consider an arbitrary x. Suppose that $(\mathbf{R}, V), x \models q_{[d]\chi}$. By 1, this is equivalent to $(\mathbf{R}, V), x \models [d]\chi$. By semantics, this is equivalent to $(\mathbf{R}, V), x \models$

$U(\top,\chi) \wedge S(\top,\chi)$. By 1, this is equivalent to $(\mathbf{R}, V), x \models U(\top, q_\chi) \wedge S(\top, q_\chi)$. Then, $(\mathbf{R}, V), x \models q_{[d]\chi} \leftrightarrow U(\top, q_\chi) \wedge S(\top, q_\chi)$. Then, by arbitrariness of x, for every $x \in \mathbf{R}$, we have $(\mathbf{R}, V), x \models \forall(q_{[d]\chi} \leftrightarrow U(\top, q_\chi) \wedge S(\top, q_\chi))$. We conclude by observing that $\widehat{[d]\chi}$ is $\forall(q_{[d]\chi} \leftrightarrow U(\top, q_\chi) \wedge S(\top, q_\chi))$.

We now prove, by induction on the complexity of ψ, that 2 implies 1. We report the most interesting cases. Assume that ψ is $p \in \Phi$. Consider an arbitrary x. By 2, $(\mathbf{R}, V), x \models q_p$ is equivalent to $(\mathbf{R}, V), x \models p$.

Assume that ψ is $\neg\chi$. Consider an arbitrary x. By 2, $(\mathbf{R}, V), x \models q_{\neg\chi}$ is equivalent to $(\mathbf{R}, V), x \models \neg q_\chi$. And, by inductive hypothesis, $(\mathbf{R}, V), x \models \neg q_\chi$ is equivalent to $(\mathbf{R}, V), x \models \neg\chi$.

Assume that ψ is $[d]\chi$. Consider an arbitrary x. By 2, $(\mathbf{R}, V), x \models q_{[d]\chi}$ is equivalent to $(\mathbf{R}, V), x \models U(\top, q_\chi) \wedge S(\top, q_\chi)$. Now, by inductive hypothesis, $(\mathbf{R}, V), x \models U(\top, q_\chi) \wedge S(\top, q_\chi)$ is equivalent to $(\mathbf{R}, V), x \models U(\top,\chi) \wedge S(\top,\chi)$. And, by semantics, $(\mathbf{R}, V), x \models U(\top,\chi) \wedge S(\top,\chi)$ is equivalent to $(\mathbf{R}, V), x \models [d]\chi$. □

Now, let us prove that φ is satisfiable on $\mathfrak{R}$ if and only if φ^* is satisfiable on $\mathfrak{R}$. Assume that φ is satisfiable on $\mathfrak{R}$. Then there is an evaluation function $V : \Phi \to \mathcal{P}(\mathbf{R})$ and $x \in \mathbf{R}$ such that $(\mathbf{R}, V), x \models \varphi$. Define an evaluation function $V' : \Phi \cup \{q_\psi \mid \psi \leq \varphi\} \to \mathcal{P}(\mathbf{R})$ as follows:

1. For every $p \in \Phi$, define $V'(p) := V(p)$.
2. For every $q_\chi \in \{q_\psi \mid \psi \leq \varphi\}$, define $V'(q_\chi) := \{y \in \mathbf{R} \mid (\mathbf{R}, V), y \models \chi\}$.

Observe that V' satisfies Condition 1 of Lemma 11.3. Then, by Lemma 11.3, $(\mathbf{R}, V'), x \models \bigwedge_{\psi\leq\varphi} \widehat{\psi}$. Moreover, by definition of V', we have $(\mathbf{R}, V'), x \models q_\varphi$. We conclude by observing that φ^* is $\bigwedge_{\psi\leq\varphi} \widehat{\psi} \wedge q_\varphi$.

Assume that φ^* is satisfiable on $\mathfrak{R}$. Then there is an evaluation function $V : \Phi \cup \{q_\psi \mid \psi \leq \chi\} \to \mathcal{P}(\mathbf{R})$ and $x \in \mathbf{R}$ such that $(\mathbf{R}, V), x \models \varphi^*$. By definition of φ^*, we have that $(\mathbf{R}, V), x \models q_\varphi$ and V satisfies Condition 2 of Lemma 11.3. Then, by Lemma 11.3, $(\mathbf{R}, V), x \models \varphi$. Then, $(\mathbf{R}, V \restriction \Phi), x \models \varphi$. □

Observe that L could be translated into $L_{\{U,S\}}$ in a perhaps more intuitive way - that is, noting that any $F\chi \in L$ can be expressed by $U(\chi, \top) \in L_{\{U,S\}}$, and any $[d]\chi \in L$ can be expressed by $U(\top,\chi) \wedge S(\top,\chi) \in L_{\{U,S\}}$. The reason why we use the translation given by (11.1) is that we want to prove that:

Proposition 11.4. φ^* is constructible from φ in polynomial time in the length of φ.

Observe that, if instead of using the translation given by (11.1), we translated L into $L_{\{U,S\}}$ expressing, for example, any $[d]\psi \in L$ by $U(\top, \psi) \wedge S(\top, \psi)$, we would have that, for certain formulas in L, the translation would run in more than polynomial time. For example, consider:

$$\psi = \underbrace{[d]...[d]}_{n \text{ times}} p$$

for some $p \in \Phi$. A bit of thought shows that, by expressing every $[d]\psi \in L$ by $U(\top, \psi) \wedge S(\top, \psi)$, we would obtain a formula with 2^n occurrences of p.

Sketch of the proof of Proposition 11.4. Let n be the length of φ. By classical arguments, we can represent φ in a convenient Turing machine as a string of length polynomial in n. For every subformula ψ of φ, write a string representing $\widehat{\psi}$. *By definition of $\widehat{\psi}$, $\widehat{\psi}$ has length polynomial in the length of φ.* By classical arguments, we can represent $\widehat{\psi}$ as a string with length polynomial in n. Then, we can write such a representation in polynomial time in n. The number of subformulas of φ is also polynomial in n. Then, we can write the representation of all $\widehat{\psi}$ in polynomial time in n. Now, it is just a matter of representing the conjunction of all the $\widehat{\psi}$ and q_φ. Again, this can be done in polynomial time in n. We conclude that φ^* can be written in polynomial time in n. □

Now, by Propositions 11.2 and 11.4, the satisfiability problem for L on $\mathfrak{R}$ reduces to the satisfiability problem for $L_{\{U,S\}}$ on $\mathfrak{R}$. The latter has been shown to be PSPACE in [81]. Then, the satisfiability problem for L on $\mathfrak{R}$ is PSPACE. Then, the validity problem for L on $\mathfrak{R}$ is PSPACE as well.

Similarly, given $\varphi \in L_{\Phi,\{\Box\}}$, we can define a formula $\varphi^* \in L$ such that φ is satisfiable on $\mathfrak{R}$ if and only if φ^* is satisfiable on $\mathfrak{R}$, and φ^* is constructible from φ in polynomial time. More precisely, introduce a new propositional atom q_ψ for each subformula ψ of φ and define the formula $\widehat{\psi}$ as follows, where $\forall\psi$ is the abbreviation for $\psi \wedge G\psi \wedge H\psi$:

1. $\widehat{\top} := \forall q_\top$;
2. $\widehat{p} := \forall(q_p \leftrightarrow p)$ for every $p \in \Phi$;
3. $\widehat{\neg\psi} := \forall(q_{\neg\psi} \leftrightarrow \neg q_\psi)$;
4. $\widehat{\psi \wedge \chi} := \forall(q_{\psi\wedge\chi} \leftrightarrow q_\psi \wedge q_\chi)$;
5. $\widehat{\Box\psi} := \forall(q_{\Box\psi} \leftrightarrow q_\psi \wedge [d]q_\psi)$.

Then, the satisfiability problem for $L_{\Phi,\{\Box\}}$ on $\mathfrak{R}$ is reducible to the satisfiability problem for L on $\mathfrak{R}$. Then, the validity problem for $L_{\Phi,\{\Box\}}$ on $\mathfrak{R}$ reduces to the validity problem for L on $\mathfrak{R}$. Then, since the former is PSPACE-hard (for, the set of all valid formulas of $L_{\Phi,\{\Box\}}$ is **S4** [71] and **S4** is PSPACE-hard [13, Chapter 6]), the latter is PSPACE-hard as well.

In conclusion, the validity problem for L on $\mathfrak{R}$ is PSPACE-complete, proving Theorem 11.1

11.2 Lack of strong completeness for $\mathsf{L}^{\mathfrak{R}}$

In this section we study the strong completeness of $\mathsf{L}^{\mathfrak{R}}$. We say that $\mathsf{L}^{\mathfrak{R}}$ is *strongly complete* provided for every $\Gamma \cup \{\varphi\} \subseteq L$, we have that $\Gamma \models \varphi$ implies $\Gamma \vdash_{\mathsf{L}^{\mathfrak{R}}} \varphi$ (see Section 2.4 for the definition of $\vdash_{\mathsf{L}^{\mathfrak{R}}}$). More precisely, we prove that:

Theorem 11.5. (Cf. [53, Theorem 4.2].) $\mathsf{L}^{\mathfrak{R}}$ is not strongly complete.

Proof. We follow the proof of [53, Theorem 4.2]. Consider the following set of formulas:

$$\begin{aligned}\Sigma := \{&\blacksquare p,\\ &F(r \wedge G\neg r),\\ &G(r \vee Fr \to \blacksquare p \vee \blacksquare q),\\ &F(\neg p \wedge F(\neg q \wedge F(\neg p \wedge F(... \wedge Fr \underbrace{))...)}_{n \text{ brackets}} \mid \text{for each natural } n \geq 1\}.\end{aligned}$$

It can easily be proved that every finite subset of Σ is satisfiable on $\mathfrak{R}$. We show that Σ is not satisfiable on $\mathfrak{R}$. Suppose by contradiction that Σ is satisfiable in some model $(\mathbf{R}, V)$. Then, without loss of generality, we can assume that Σ is satisfiable at 0 and $r \wedge G\neg r$ at 1. Then, by the first and third formulas, for every $x \in [0, 1]$, there is an open interval I_x containing x and such that $I_x \subseteq V(p)$ or $I_x \subseteq V(q)$. By the Heine-Borel Theorem, $[0, 1]$ is compact. Then, there exists $n \in \mathbf{N}$ and $x_0, ..., x_{n-1} \in [0, 1]$ such that $[0, 1] \subseteq \bigcup_{i \in n} I_{x_i}$. By the last set of formulas, there are $y_0, ..., y_n \in [0, 1]$ such that $y_0 < y_1 < ... < y_n$ and $y_i \notin V(p)$ if i is even and $y_i \notin V(q)$ if i is odd. Then, by the pigeonhole principle, there exist $i, j \in n$ such that $y_i, y_{i+1} \in I_{x_j}$. Now, $I_{x_j} \subseteq V(p)$ or $I_{x_j} \subseteq V(q)$, a contradiction.

Assume by contradiction that $\mathsf{L}^{\mathfrak{R}}$ is strongly complete. The non satisfiability of Σ on $\mathfrak{R}$ yields $\Sigma \models \bot$. Then, by strong completeness of $\mathsf{L}^{\mathfrak{R}}$, we get $\Sigma \vdash_{\mathsf{L}^{\mathfrak{R}}} \bot$. Then, by definition, there are $\sigma_0, ...\sigma_{n-1} \in \Sigma$ such that $\vdash_{\mathsf{L}^{\mathfrak{R}}} \bigwedge_{i \in n} \sigma_i \to \bot$. Then, by definition of $\mathsf{L}^{\mathfrak{R}}$, we have that $\bigwedge_{i \in n} \sigma_i \to \bot$ is valid on $\mathfrak{R}$. But this is against the finite satisfiability of Σ on $\mathfrak{R}$. $\square$

11.3 Axioms for $\mathsf{L}^{\mathfrak{R}}$

In this section we present a finite set Λ_d of axioms. In Section 11.4, we prove that the smallest normal modal logic L_d containing Λ_d is *sound with respect to* $\mathsf{L}^{\mathfrak{R}}$ (that is, $\mathsf{L}_d \subseteq \mathsf{L}^{\mathfrak{R}}$). In Chapter 12, we prove that L_d is *complete with respect to* $\mathsf{L}^{\mathfrak{R}}$ (that is, $\mathsf{L}_d \supseteq \mathsf{L}^{\mathfrak{R}}$). Then, we obtain that Λ_d *axiomatizes* L_d (that is, $\mathsf{L}_d = \mathsf{L}^{\mathfrak{R}}$).

Let p, q be distinct propositional variables, p_0, p_1, p_2 be distinct propositional variables, and, for every $i \in 3$, φ_i be $p_i \wedge \bigwedge_{j \in \{0,1,2\} \setminus \{i\}} \neg p_j$. Consider the following set Λ_d of formulas.

1. All propositional tautologies.
2. Axioms for G and H (dense linear order without endpoints):
   (a) $G(p \to q) \to (Gp \to Gq)$ — normality;
   (b) $Gp \to GGp$ — transitivity: $\forall xyz(x \sqsubset y \wedge y \sqsubset z \to x \sqsubset z)$;
   (c) $GGp \to Gp$ — density: $\forall xy(x \sqsubset y \to \exists z(x \sqsubset z \sqsubset y))$;
   (d) $p \to GPp$ — $\forall xy(x \sqsubset y \to y \sqsupset x)$;
   (e) $FPp \to p \vee Fp \vee Pp$ — $\forall xyz(x \sqsubset y \wedge y \sqsupset z \to x = z \vee x \sqsubset z \vee x \sqsupset z)$.
3. Axioms for $[d]$:
   (a) $[d](p \to q) \to ([d]p \to [d]q)$ — normality;
   (b) $[d]p \to [d][d]p$ — transitivity: $\forall xyz(xRy \wedge yRz \to xRz)$.
4. Shehtman's 'special axioms' adapted for $[d]$ (cf. [86, pp. 257-258]).
   (a) $Hp \wedge Gp \to [d]p$ — $\forall xy(xRy \to x \sqsupset y \vee x \sqsubset y)$;
   (b) $Gp \to G[d]p$ — $\forall xyz(x \sqsubset y \wedge yRz \to x \sqsubset z)$;
   (c) $p \wedge Gp \wedge [d]p \to [d]Gp$ — $\forall xyz(xRy \wedge y \sqsubset z \to x \sqsubseteq z \vee xRz)$;
   (d) $[d]p \to Fp$ — seriality: $\forall x \exists y(x \sqsubset y \wedge xRy)$.
5. $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\blacklozenge p \wedge \blacklozenge \neg p \wedge Fq)$.
6. Shehtman's Axiom $\mathbf{G_2}$: $[d](\bigvee_{i \in 3} \blacksquare \varphi_i) \to \bigvee_{i \in 3}[d]\neg\varphi_i$.
7. All mirror images of the above axioms (swap G with H and F with P; and $\sqsubset$ with $\sqsupset$ in the first order formulas on the right of the axioms).

The first order formulas on the right of the axioms are their Sahlqvist correspondents. Given a formula φ with a Sahlqvist correspondent ψ, a Kripke frame validates φ if and only if it validates ψ (see [13, Chapter 3]).

Comparison with axioms in [53, §3.1]: our axioms for G and H are the same as in [53, §3.1]. And so are our first two axioms for $[d]$ (normality and transitivity) with respect to [53, §3.1]'s axioms for $\Box$. However, unlike in [53, §3.1], we do not have a reflexivity axiom for $[d]$. Indeed, because of the semantics of $[d]$, such an axiom would not be sound on $\mathfrak{R}$. Also, our Shehtman's Axioms 4a and 4c are slightly different from the respective Shehtman's axioms in [53, §3.1]. Again, this difference are due to the different semantics of $[d]$ and $\Box$ on $\mathfrak{R}$. Then, we have Axiom 5, which is equivalent to the respective axiom $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\Diamond p \wedge \Diamond\neg p \wedge Fq)$ in [53, §3.1]. Again, observe that, if we took $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\langle d\rangle p \wedge \langle d\rangle\neg p \wedge Fq)$ as Axiom 5, we would obtain an axiom which is not sound on $\mathfrak{R}$. Finally, unlike in [53, §3.1], we also have Axiom 6. Axiom 6 will be used in Section 12.2.3.

Let L_d be the smallest normal logic containing Λ_d (plainly existing). We will show that L_d is sound ($\mathsf{L}_d \subseteq \mathsf{L}^{\mathfrak{R}}$) and complete ($\mathsf{L}_d \supseteq \mathsf{L}^{\mathfrak{R}}$) with respect to $\mathsf{L}^{\mathfrak{R}}$. Then, $\mathsf{L}^{\mathfrak{R}}$ is axiomatized by Λ_d ($\mathsf{L}^{\mathfrak{R}} = \mathsf{L}_d$). But first observe:

Proposition 11.6. If we replace $\Box$ with $\blacksquare$, the axioms of [53, §3.1] belong to L_d and the inference rules of [53, §3.1] preserve membership of L_d.

Proof. The axioms of [53] in which $\Box$ does not occur belong to Λ_d. Axiom 5 of [53] belongs to Λ_d with $\Box$ replaced by $\blacksquare$. The inference rules of [53], except for the $\Box$ case of generalization, plainly preserve membership to L_d. As for the remaining axioms of [53], consider, for example, $\Box p \to \Box\Box p$. We want to prove that $\blacksquare p \to \blacksquare\blacksquare p \in \mathsf{L}_d$ - that is:

$$p \wedge [d]p \to p \wedge [d]p \wedge [d](p \wedge [d]p) \in \mathsf{L}_d.$$

By our Axiom 3b, $[d]p \to [d][d]p \in \mathsf{L}_d$. Then, $p \wedge [d]p \to p \wedge [d]p \wedge [d][d]p \in \mathsf{L}_d$. Moreover, since L is a normal modal logic, $[d]\varphi \wedge [d]\psi \to [d](\varphi \wedge \psi) \in \mathsf{L}_d$. Then, $p \wedge [d]p \to p \wedge [d]p \wedge [d](p \wedge [d]p) \in \mathsf{L}_d$, as desired. Similar arguments yield the remaining axioms. As for the $\Box$ case of generalization, suppose that $\varphi \in \mathsf{L}_d$. Then, by our generalization, $[d]\varphi \in \mathsf{L}_d$. Then $\varphi \wedge [d]\varphi \in \mathsf{L}_d$, as desired. □

Corollary 11.7. The following formulas belong to L_d:

1. $F\top$ and $P\top$.
2. $G\neg p \wedge HFp \to \blacklozenge p$.
3. The Prior Axiom $Fq \wedge FG\neg q \to F(G\neg q \wedge HFq)$, and its mirror image.

Proof. The formulas $F\top$, $P\top$, $G\neg p \wedge HFp \to \blacklozenge p$, the Prior Axiom, and its mirror image belong to the logic L of [53]. Then, by Proposition 11.6, they belong to L_d. In proving that the Prior Axiom belongs to L, Axiom 5, with $\blacklozenge$ replaced by $\Diamond$, is used. □

11.4 Soundness of L_d

Proposition 11.8. L_d is sound with respect to **R**.

Proof. Except perhaps for Axiom 6, it can easily be proved that all other axioms of Λ_d are valid on $\mathfrak{R}$. Let us prove that Axiom 6 is valid on $\mathfrak{R}$. We proceed as in [85, Lemma 31]. Suppose by contradiction that it is not. Then, its negation: $[d](\bigvee_{i\in 2}\blacksquare\varphi_i)\wedge\bigwedge_{i\in 2}[d]\varphi_i$ is satisfiable on $\mathfrak{R}$. Then, there is a model $\mathcal{R}$, and $x\in\mathbf{R}$ such that:

$$\mathcal{R}, x \models [d](\bigvee_{i\in 3}\blacksquare\varphi_i), \tag{11.2}$$

$$\mathcal{R}, x \models \bigwedge_{i\in 3}\langle d\rangle\varphi_i. \tag{11.3}$$

For every $\varphi\in L$, let $\|\varphi\| := \{y\in\mathbf{R}\,|\,(\mathbf{R},V),y\models\varphi\}$. For every $A\subseteq\mathbf{R}$, call an *interior point of A* every $a\in A$ such that there is an open interval I (that is, an interval of the form (x,y) for some reals $x<y$) such that $a\in I$ and $I\subseteq A$. Let $int(A) := \{a\in A\,|\,a$ interior point of $A\}$. Observe that, by semantics, for every $\varphi\in L$, we have $\|\blacksquare\varphi\| = int(\|\varphi\|)$. Then, by (11.2), there is an open interval I such that $x\in I$ and $I\setminus\{x\}\subseteq\bigcup_{i\in 3} int(\|\varphi_i\|)$. Moreover, by (11.3), for every $i\in 3$, we have $(I\setminus\{x\})\cap\|\varphi_i\|\neq\emptyset$. Finally, by definition of the formulas φ_i, for every pair of distinct $i,j\in 3$, we have $int(\|\varphi_i\|)\cap int(\|\varphi_j\|)=\emptyset$. Then, $\{int(\|\varphi_i\|)\cap(I\setminus\{x\})\,|\,i\in 3\}$ is a partition of $I\setminus\{x\}$ into three non empty open sets. But this is a contradiction, since, for every open interval I of $\mathbf{R}$, and $x\in I$, $I\setminus\{x\}$ can be partitioned in at most two non empty open sets (this is nothing else than local 2-connectedness in topological settings).

In conclusion, all axioms of Λ_d are valid on $\mathfrak{R}$. Moreover, observe that modus ponens, uniform substitution and generalization preserve validity on $\mathfrak{R}$. Soundness, then, easily follows. □

11.5 Lack of Kripke completeness for $\mathsf{L}^{\mathfrak{R}}$

In this section we prove that $\mathsf{L}^{\mathfrak{R}}$ is not Kripke complete. Recall that a modal logic L is *Kripke complete* (respectively, *has the finite model property*) provided there is a class $\mathfrak{K}$ of (resp. finite) Kripke frames such that L is equal to the set of all formulas that are valid in every Kripke frame in $\mathfrak{K}$.

A Kripke frame (for L) is of the form $(W,\sqsubset,\sqsupset,R)$ (see Section 2.3). $\sqsubset$ interprets G, $\sqsupset$ interprets H, and R interprets $[d]$. When we present a Kripke frame as $(W,\sqsubset,R)$, that is, omitting $\sqsupset$, we tacitly assume $\sqsupset$ defined as $\sqsubset^{-1}$. For example, if we consider a Kripke frame $\mathcal{F}=(W,\sqsubset,\sqsupset,R)$ that validates all the axioms of Λ_d, then by Axiom 2d, we have that $\sqsupset$ is $\sqsubset^{-1}$, so it is convenient to present $\mathcal{F}$ as $(W,\sqsubset,R)$. Moreover, by Axiom 2b, we have that $\sqsubset$ is transitive. However, unlike in [53], $\sqsubset$ is not necessarily reflexive. Given a binary relation $\sqsubset$ on a set W, define $\sqsubseteq$ as the reflexive closure of $\sqsubset$. Let us begin with the following preliminary proposition:

Proposition 11.9. (Cf. [53, Lemma 4.3].) Let $\mathcal{F}=(W,\sqsubset,R)$ be a Kripke frame that validates all the axioms of Λ_d. Let $w,u,x\in W$ be such that $w\sqsubset u$, $w\sqsubset x$, and $\neg(x\sqsubset u)$.

Then, there exists $y \in W$ such that $w \sqsubset y$, $\neg(y \sqsubset u)$, and $yR^r u$.

Proof. We follow the proof of [53, Lemma 4.3]. Consider an evaluation V on $\mathcal{F}$ such that $h(p) = \{u\}$ and let $\mathcal{M} = (W, \sqsubset, R, V)$. Since $\neg(x \sqsubset u)$, we have $\mathcal{M}, x \models G\neg p$. Then, since $w \sqsubset \{u, x\}$, we have $\mathcal{M}, w \models Fp \wedge FG\neg p$. Now, $\mathcal{F}$ validates all the axioms of Λ_d, and frame-validities are preserved by modus ponens, generalization, and uniform substitution. Then, since, by Corollary 11.7(3), the Prior Axiom belongs to L_d, we have that the Prior Axiom is valid on $\mathcal{F}$. Then, $\mathcal{M}, w \models F(G\neg p \wedge HFp)$. Then, there exists $y \in W$ such that:

$$\mathcal{M}, y \models G\neg p \wedge HFp. \tag{11.4}$$

Then, $\neg(y \sqsubset u)$. Moreover, by (11.4) and Corollary 11.7(2), we have $\mathcal{M}, y \models \blacklozenge p$. Then, by definition of V, we have $yR^r u$. □

We are now ready to prove:

Proposition 11.10. (Cf. [53, Lemma 4.4].) $\mathsf{L}^{\mathfrak{R}}$ is not Kripke complete, hence it does not have the finite model property either.

Proof. We follow the proof of [53, Lemma 4.4]. As the reader may confirm, it is sufficient to find a formula satisfiable on $\mathfrak{R}$ and not satisfiable on any Kripke model such that its underlying frame validates all axioms of Λ_d. Consider the following formula, where a and b are propositional variables:

$$\vartheta = H\neg a \wedge H\neg b \wedge \neg a \wedge \neg b \wedge \blacklozenge a \wedge \blacklozenge b \wedge G\neg(\blacklozenge a \wedge \blacklozenge b) \wedge FG\neg a.$$

Consider an arbitrary evaluation function $V : \Phi \to \mathbf{R}$ such that $V(a) = \{1/2^n \mid n \in \mathbf{N}\}$ and $V(b) = \{2/3^n \mid n \in \mathbf{N}\}$. Plainly $(\mathbf{R}, V), 0 \models \vartheta$. Then, ϑ is satisfiable on $\mathfrak{R}$.

Consider an arbitrary Kripke model $\mathcal{M} = (W, \sqsubset, R, V)$ such that its underlying frame validates the axioms in Λ_d. Suppose by contradiction that there is $w \in W$ such that $\mathcal{M}, w \models \vartheta$. Then, $\mathcal{M}, w \models \blacklozenge a \wedge \blacklozenge b$. Then, there exist $u, v \in R^r(w)$ such that $\mathcal{M}, u \models a$ and $\mathcal{M}, v \models b$. Then, by Axiom 4a if $w \neq u$, we have $u \sqsupset w$ or $u \sqsubseteq w$. Now, since $\mathcal{M}, w \models H\neg a \wedge \neg a$, we have $\neg(u \sqsubseteq w)$. Then, we have $u \sqsupset w$. Then, since $wR^r v$, by Axiom 4b if $w \neq v$, we have $u \sqsupset v$. Similarly, we have $u \sqsubset v$. Since $\mathcal{M}, w \models FG\neg a$, there exists $x \in W$ such that $w \sqsubset x$ and $\mathcal{M}, x \models G\neg a$. Then, since $\mathcal{M}, u \models a$, we have $\neg(x \sqsubset u)$. Then, by Proposition 11.9, there exists $y \in W$ such that $w \sqsubset y$, $\neg(y \sqsubset u)$ and $yR^r u$. Now, since $u \sqsubset v$, by Axiom 4c, we have $yR^r v \vee y \sqsubset v$.

If $yR^r v$, then $u, v \in R^r(y)$. Then $\mathcal{M}, y \models \blacklozenge a \wedge \blacklozenge b$. Then, since $w \sqsubset y$, we have $\mathcal{M}, w \models F(\blacklozenge a \wedge \blacklozenge b)$, a contradiction because $\mathcal{M}, w \models \vartheta$ and $G\neg(\blacklozenge a \wedge \blacklozenge b)$ is a conjunct of ϑ. If instead we have $y \sqsubset v$, then, since $v \sqsubset u$, we have $y \sqsubset u$, again a contradiction since $\neg(y \sqsubset u)$. □

Chapter 12

Completeness of L_d

In this chapter we prove that L_d is complete with respect to $\mathsf{L}^{\mathfrak{R}}$ ($\mathsf{L}^{\mathfrak{R}} \subseteq \mathsf{L}_d$). To prove the completeness of L_d, it is sufficient to show that:

$$\text{Every } \mathsf{L}_d\text{-consistent formula } \varphi \text{ is satisfiable on } \mathfrak{R}. \tag{12.1}$$

For, consider an arbitrary formula φ such that $\varphi \notin \mathsf{L}_d$. Then, $\neg\varphi$ is L_d-consistent. Then, by (12.1), $\neg\varphi$ is satisfiable on $\mathfrak{R}$. Then, $\varphi \notin \mathsf{L}^{\mathfrak{R}}$. Then, by contraposition, $\mathsf{L}^{\mathfrak{R}} \subseteq \mathsf{L}_d$ as desired.

Given an L_d-consistent formula φ we might try to produce a finite Kripke model satisfying φ whose frame $(W, \sqsubset, R)$ is of the following form:

$$C_0 \sqsubset u_0 \sqsubset C_1 \sqsubset u_1 \sqsubset ... \sqsubset u_{k-1} \sqsubset C_k,$$

with C_i $\sqsubset$-clusters, u_i $\sqsubset$-irreflexive points and $R(u_i) = C_i \cup C_{i+1}$.

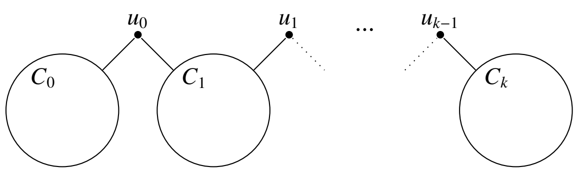

Figure 12.1: $C_0 \sqsubset u_0 \sqsubset C_1 \sqsubset u_1 \sqsubset ... \sqsubset u_{k-1} \sqsubset C_k$

If we require that the $\sqsubset$-clusters are connected and locally 2-connected (with respect to R) then it would be ‘fairly simple’ to build a model on $\mathfrak{R}$ satisfying φ (cf. §12.3).

However, for certain formulas, such a model does not exist. Consider for example ϑ of the previous section, and suppose that ϑ holds at some world w of such a model. Since $\blacklozenge a \wedge \blacklozenge b \wedge G\neg(\blacklozenge a \wedge \blacklozenge b)$ is a conjunct of ϑ, then w is $\sqsubset$-irreflexive. Then, $w = u_i$ for some $i < k$. Then, since $H\neg a \wedge H\neg b \wedge \neg a \wedge \neg b \wedge \blacklozenge a \wedge \blacklozenge b$ is a conjunct of ϑ, we have that a and b are true at some points in C_{i+1}. Then, since $FG\neg a$ is a conjunct of ϑ, we have $C_{i+1} \neq C_k$. Then, $i+1 < k$ and u_{i+1} is defined. Then, $\blacklozenge a \wedge \blacklozenge b$ is satisfied at u_{i+1}. This is a contradiction, since ϑ holds at $w = u_i$, and $G\neg(\blacklozenge a \wedge \blacklozenge b)$ is a conjunct of ϑ.

Then, we consider Kripke models as described above, but we allow $\sqsubset$-clusters with no $\sqsubset$-irreflexive points in between, and we require that every two such clusters contain 'similar' points (see Definition 12.2(5)). For the moment, call these models 'well-behaved' (see Definition 12.2).

We show that every L_d-consistent formula φ is satisfied in a 'well-behaved' model. Then we show that every 'well-behaved' model can be turned into a model on $\mathfrak{R}$ satisfying φ. The latter result is achieved by using 'similar' points to fulfil the 'gap' between $\sqsubset$-clusters with no $\sqsubset$-irreflexive points in between.

12.1 Ψ-linked models

Let us define what 'well-behaved' means for a Kripke model (Definition 12.2). But first let us give the following preliminary definition:

Definition 12.1.

1. For every set A, and binary relation R on A, R is said to be *prelinear* provided, for every $a, b \in A$, we have $aR^r b$ or bRa.
2. For every set A, subset $B \subseteq A$, $c \in A$, and binary relation R on A, define cRB provided, for every $b \in B$, we have cRb; define BRc provided, for every $b \in B$, we have bRc.
3. For every Kripke frame $\mathcal{F} = (W, \sqsubset, R)$, an ordered pair (C, D) of $\sqsubset$-clusters of $\mathcal{F}$ is called *adjacent* provided $C \neq D$ and $\{u \in W \mid C \sqsubset u \sqsubset D\} = C \cup D$.
4. For every set of formulas Ψ, define $\mathcal{B}\Psi$ as the set of formulas in Ψ of the form $G\psi$, $H\psi$ or $[d]\psi$.

Definition 12.2. Let Ψ be a set of formulas. We say that a Kripke model $\mathcal{M} = (W, \sqsubset, R, V)$ is Ψ-*linked* provided:

1. W is finite.
2. The frame of $\mathcal{M}$ validates all the axioms of Λ_d except possibly Axioms 5 and 6.
3. $\sqsubset$ is prelinear.
4. For every $\sqsubset$-cluster $C \subseteq W$, $(C, R\restriction C)$ is connected and locally 2-connected.
5. For every pair of adjacent $\sqsubset$-clusters (C, D), there are R-reflexive points $c \in C$ and $d \in D$ such that $c \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} d$ (recall that $\equiv^{\mathcal{M}}_{\mathcal{B}\Psi}$ was defined in Definition 9.3(2)).

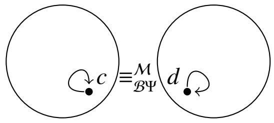

Figure 12.2: Definition 12.2(5)

Ψ-linked models have their frame validating Shehtman's axioms. This implies a number of properties of their structure:

Proposition 12.3. (Cf. [53, Lemma 5.5].) Let $\mathcal{F} = (W, \sqsubset, R)$ be a Kripke frame that validates all the axioms of Λ_d except possibly Axioms 5 and 6. Let $w \in W$. Then:

1. Every $\sqsubset$-cluster in $\mathcal{F}$ is an R-generated subset of W.

2. If w is $\sqsubset$-reflexive, then $R(w)$ is a subset of the $\sqsubset$-cluster $\{v \in W \mid w \sqsubset v \sqsubset w\}$.

3. If w is $\sqsubset$-irreflexive, then there are disjoint $\sqsubset$-clusters $\lambda(w), \rho(w)$ such that:

   (a) $\lambda(w) \sqsubset w \sqsubset \rho(w)$.

   (b) $R(w) = \lambda(w) \cup \rho(w)$.

   (c) For every $v \in W$, we have $v \sqsubset w$ if and only if $v \sqsubset \lambda(w)$, and $w \sqsubset v$ if and only if $\rho(w) \sqsubset v$.

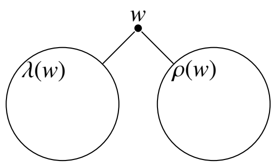

Figure 12.3: Definition 12.3(3)

Proof. 1. Let $C \subseteq W$ be a $\sqsubset$-cluster and let $w \in C$. Then, by Proposition 9.1(1), we have $C = \{v \in W \mid w \sqsubset v \sqsubset w\}$. Then, by Axiom 4b, C is R-generated.

2. Assume that w is $\sqsubset$-reflexive. Then, by Proposition 9.1(1), w belongs to the $\sqsubset$-cluster $C = \{v \in W \mid w \sqsubset v \sqsubset w\}$. By 1, C is R-generated. Then, $R(w) \subseteq C$.

3. Suppose that w is $\sqsubset$-irreflexive. Let:

$$\lambda(w) := R(w) \cap \{u \in W \mid u \sqsubset w\},$$
$$\rho(w) := R(w) \cap \{u \in W \mid w \sqsubset u\}.$$

By Axiom 4d and its mirror image, we have $\lambda(w) \neq \emptyset$ and $\rho(w) \neq \emptyset$. As w is $\sqsubset$-irreflexive and, by Axiom 3b, $\sqsubset$ is transitive, we also have that $\lambda(w) \cap \rho(w) = \emptyset$.

We show that $\lambda(w)$ is a $\sqsubset$-cluster. Consider $t \in W$ and $u \in \lambda(w)$. Then, by Axiom 4b and transitivity of $\sqsubset$, we have:

$$t \sqsubset w \text{ if and only if } t \sqsubset u. \tag{12.2}$$

Since $\lambda(w) \neq \emptyset$, there exists $u \in \lambda(w)$. By (12.2), u is $\sqsubset$-reflexive (just take $t = u$ in (12.2)). Then, by Proposition 9.1(1), the set $U := \{v \in W \mid u \sqsubset v \sqsubset u\}$ is a $\sqsubset$-cluster. We show that $\lambda(w) = U$. By (12.2), for every $t, v \in \lambda(w)$, we have $t \sqsubset v$. Then $\lambda(w) \subseteq U$. Let us prove that $U \subseteq \lambda(w)$. Let $t \in U$ be arbitrary. Since $u \sqsubset w$, by transitivity of $\sqsubset$, we have $t \sqsubset w$. We also have wRu and $u \sqsubset t$. Then, by Axiom 4c, we have $w \sqsubseteq t$ or wRt. If $w \sqsubset t$, then, since $t \sqsubset w$ and $\sqsubset$ is transitive, we have $w \sqsubset w$: against the $\sqsubset$-irreflexivity of w. If $w = t$, then, since $t \sqsubset w$, we have $w \sqsubset w$: again against the assumption that w is $\sqsubset$-irreflexive. Then, wRt. We already have $t \sqsubset w$, so $t \in \lambda(w)$. Then, since t was arbitrary, we have $U \subseteq \lambda(w)$ from which $\lambda(w) = U$ and $\lambda(w)$ is a $\sqsubset$-cluster. Similarly, we can show that $\rho(w)$ is a cluster as well.

Point (a) follows from the definition of $\lambda(w)$ and $\rho(w)$. As for point (b), by Axiom 4a, we have $R(w) \setminus \{w\} = \lambda(w) \cup \rho(w)$. Again by Axiom 4a, if w were R-reflexive then w would be $\sqsubset$-reflexive, against the assumption that w is $\sqsubset$-irreflexive. Then w is not R-reflexive and $R(w) = \lambda(w) \cup \rho(w)$ as desired. Point (c) for $\lambda(w)$ follows from (12.2). Similarly, we can prove point (c) for $\rho(w)$. ◻

Comparison with the proof of [53, Lemma 5.5]: The proof is as in [53, Lemma 5.5] with the following two slight differences in the proof of point (3):

(a) In [53, Lemma 5.5], their axiom 4a ($Hp \wedge p \wedge Gp \to \Box p$) together with the reflexivity of their R is invoked to obtain $R(w) = \lambda(w) \cup \{w\} \cup \rho(w)$. In our proof, since our R might be not reflexive, our axiom 4a yields $R(w) \setminus \{w\} = \lambda(w) \cup \rho(w)$.

(b) In [53, Lemma 5.5], their axiom 4c ($Gp \wedge \Box p \to \Box Gp$) is invoked to obtain that, for a certain point $t \sqsubset w$: $w \sqsubset t$ or wRt. Our axiom 4c yields the additional case in which $w = t$.

We can think of $\langle d \rangle \varphi$ as expressing the existence of points satisfying φ arbitrarily near:

Remark 12.4. If $\mathcal{F} = (W, \sqsubset, R)$ is a Kripke frame validating all the axioms of Λ_d except possibly Axioms 5 and 6, then, as the reader may confirm, by Proposition 12.3, $\mathcal{F}$ validates the following formula:

$$\neg p \wedge G\neg p \to (HFp \leftrightarrow \langle d \rangle p),$$

and its mirror image.

Proposition 12.3 tells us that the frame $(W, \sqsubset, R)$ of a Ψ-linked model $\mathcal{M}$ is the union of finitely many $\sqsubset$-clusters $C_0 \sqsubset C_1 \sqsubset ... \sqsubset C_k$ and irreflexive points u_i, for $i \in I \subseteq \{0, 1, ..., k-1\}$, such that $C_i = \lambda(u_i) \sqsubset u_i \sqsubset \rho(u_i) = C_{i+1}$. There are no consecutive $\sqsubset$-irreflexive points. However we may have two consecutive $\sqsubset$-clusters with no $\sqsubset$-irreflexive points in between. When this happens the two $\sqsubset$-clusters are linked by two R-reflexive $\equiv^{\mathcal{M}}_{\mathcal{B}\Psi}$-equivalent points.

12.2 Satisfiability on a Ψ-linked model

Given an L_d-consistent formula φ_0, let us show that φ_0 is satisfiable on a Ψ-linked model, for Ψ an arbitrary finite set containing $P\varphi_0$, closed under negation and closed under subformulas. The construction, which follows closely that of [53, §6], is in five steps, delivering five Kripke models $\mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_4$ with $\mathcal{M}_4$ a Ψ-linked model.

With the first three models $\mathcal{M}_0$-$\mathcal{M}_2$, we build a (potentially infinite) model ($\mathcal{M}_2$) such that for every of its $\sqsubset$-clusters C we have that $(C, R\restriction C)$ is connected and locally 2-connected (plus some other properties).

However, as we are looking for a Ψ-linked model, we would need the model itself to be finite, not only its $\sqsubset$-clusters, plus some other properties (for example, the presence of 'similar' points when needed). We achieve this by following [53, §6.4] and turning $\mathcal{M}_2$ first into $\mathcal{M}_3$ and then into $\mathcal{M}_4$: the desired Ψ-linked model.

The main differences between the construction of [53, §6] and ours are the definition of $\mathcal{M}_0$ and the filtration by which $\mathcal{M}_2$ is obtained. In [53, §6.1], $\mathcal{M}_0$ is the canonical model for the smallest normal modal logic containing the axioms of [53, §3.1]. Here, we replace Φ (which is countably infinite) with one of its finite subsets with enough propositional variables to write the axioms of Λ_d and φ_0. Then, we define $\mathcal{M}_0$ as the canonical model for the smallest normal modal logic in the resulting language containing Λ_d.

We do this because we follow the construction of [85, §3]. There, it is shown that $\mathbf{KD4G_1}$ has the finite model property. The result is achieved by replacing the set of propositional variables with a suitable one of its finite subsets and then by filtrating the resulting canonical model into a finite model whose frame validates $\mathbf{KD4G_1}$.

We follow this construction because what we are trying to get is similar: we want to turn the canonical model into a model ($\mathcal{M}_2$) whose $\sqsubset$-clusters are finite and locally 2-connected (with respect to R which is already serial and transitive) – that is, a model whose $\sqsubset$-clusters are finite and validating $\mathbf{KD4G_2}$.

As it is shown in [70, §5.1], the filtration of [85, §3] can straightforwardly be adapted to prove that also $\mathbf{KD4G_2}$ has the finite model property. Then, we adapt our filtration (which then becomes different from that of [53, §6.3.1]) in the same way, with the only further requirement that two points are related by the filtration *only if* they lie in the same $\sqsubset$-cluster. This is because, unlike in [85, §3] and [70, §5.1], we have to take into account also the operators G and H, which we do not want to lose in the process of filtration. As a result, instead of obtaining a finite model with its frame validating $\mathbf{KD4G_2}$ we obtain a model ($\mathcal{M}_2$) whose $\sqsubset$-clusters are finite and validating $\mathbf{KD4G_2}$.

12.2.1 Model $\mathcal{M}_0$

Let $\{p_i \mid i \in \mathbf{N}\}$ be an enumeration of Φ without repetition. For every $k \in \mathbf{N}$, let $\Phi^k := \{p_i \mid i \in k\}$, $L^k := L_{\Phi^k,\tau}$ and $\mathsf{L}^k_d := \mathsf{L}_d \cap L^k$. L^k_d is a normal modal logic in L^k.

Consider $k \in \mathbf{N}$ such that $\Lambda_d \cup \{\varphi_0\} \subseteq L^k$. Define $\mathcal{M}_0 = (W_0, \sqsubset_0, R_0, V_0)$ as the canonical model for L^k_d in L^k.

Since φ_0 is L_d-consistent, φ_0 is L^k_d-consistent as well. Then, there is $\Gamma_0 \in W_0$ with $\varphi_0 \in \Gamma_0$. Moreover:

Proposition 12.5. 1. The frame underlying $\mathcal{M}_0$ validates all the axioms of Λ_d, except possibly Axioms 5 and 6.

2. $\mathcal{M}_0$ enjoys the Zorn Property with respect to R_0 (cf. [85, Lemma 5]).

Proof. 1. By Sahlqvist Completeness Theorem. 2. We follow [85, Lemma 5]. For every $\Gamma, \Delta \in W_0$, define:

$$\Gamma \preceq \Delta \text{ if and only if } (\Gamma R_0 \Delta \wedge \neg(\Delta R_0 \Gamma)) \vee \Gamma = \Delta.$$

By Proposition 12.5(1), R_0 is transitive. Then, for all $\Gamma \in W_0$, we have that $(R_0(\Gamma), \preceq\restriction R_0(\Gamma))$ is a partial order.

Since we want to apply the Zorn Lemma, we show that, for every $\Gamma \in W_0$, every linear subset of $(R_0(\Gamma), \preceq\restriction R_0(\Gamma))$ has an upper bound in $R_0(\Gamma)$. Fix an arbitrary $\Gamma \in W_0$ and an arbitrary linear subset Z of $(R_0(\Gamma), \preceq\restriction R_0(\Gamma))$. If Z has a maximum, say Δ, then Δ is an upper bound for Z in $R_0(\Gamma)$. Suppose that Z has no maximum, and define:

$$S := \bigcup_{\Delta \in Z} \{\varphi \mid [d]\varphi \in \Delta\}.$$

We show that S is L^k_d-consistent. For, consider $\varphi_0, ..., \varphi_{n-1} \in S$. Then, for every $i = 0, ..., n-1$, $[d]\varphi \in \Delta_i$ for some $\Delta_i \in Z$. Since Z is linear, we can suppose, without loss of generality, that $\Delta_0 \preceq ... \preceq \Delta_{n-1}$. Since Z has no maximum, we have $R_0(\Delta_{n-1}) \neq \emptyset$. For, otherwise, for every $\Delta_i \in Z$, we would have $\Delta_i \preceq \Delta_{n-1}$, against the fact that Z has no maximum in $R_0(\Gamma)$. Then, we can pick $\Delta \in R_0(\Delta_{n-1})$. Due to the transitivity of R_0, for every $i = 0, ..., n-1$, we have $\Delta_i R_0 \Delta$. Then, by definition of R_0, for every $i = 0, ..., n-1$, we have $\varphi_i \in \Delta$. Then, by L_d-consistency of Δ, we have not $\vdash_{\mathsf{L}_d} \bigwedge_{i=0}^{n-1} \varphi_i \to \bot$. Then, by definition of L^k_d, we have not $\vdash_{\mathsf{L}^k_d} \bigwedge_{i=0}^{n-1} \varphi_i \to \bot$. Then, S is L^k_d-consistent as desired. Then, by the Lindenbaum Lemma, $S \subseteq \Theta$ for some $\Theta \in W_0$. We show that Θ is an upper bound for Z in $R_0(\Gamma)$. For, by definition of R_0 and Θ, we have $\Delta R_0 \Theta$ for every

$\Delta \in Z$. Then, by transitivity of R_0, we have $\Theta \in R_0(\Gamma)$. Moreover, suppose that $\Theta R_0 \Delta_0$ for some $\Delta_0 \in Z$. Then, by transitivity of R_0, for every $\Delta \in Z$, we have $\Delta R_0 \Delta_0$, against the fact that Z has no maximum. Then, for every $\Delta \in Z$, we have $\Delta R_0 \Theta$ and $\neg(\Theta R_0 \Delta)$. Then, by definition of $\preceq$, for every $\Delta \in Z$, we have $\Delta \preceq \Theta$. Therefore, Θ is an upper bound for Z in $(R_0(\Gamma), \preceq\restriction R_0(\Gamma))$.

By Zorn Lemma, $(R_0(\Gamma), \preceq\restriction R_0(\Gamma))$ admits a maximal element, say Δ. We show that Δ is R_0-q-maximal. For, suppose that $\Delta R_0 \Theta$ for some $\Theta \in W_0$ with $\Delta \neq \Theta$. By transitivity of R_0, we have $\Theta \in R_0(\Gamma)$. Then, as Δ is maximal in $(R_0(\Gamma), \preceq\restriction R_0(\Gamma))$, we have $\neg(\Delta \preceq \Theta)$. Then, by definition of $\preceq$, we have $\Theta R_0 \Delta$, as desired. □

12.2.2 Model $\mathcal{M}_1$

Let $\mathcal{M}_1 = (W_1, \sqsubset_1, R_1, V_1)$ be the submodel of $\mathcal{M}_0$ generated by $\{\Gamma_0\}$.

Proposition 12.6.
1. The frame of $\mathcal{M}_1$ validates all the axioms of Λ_d except possibly 5 and 6.
2. $\sqsubset_1$ is prelinear.
3. The frame of $\mathcal{M}_1$ enjoys the Zorn Property with respect to R_1.
4. $\mathcal{M}_1, \Gamma_0 \models \varphi_0$.
5. There are $\sqsubset_1$-clusters $C_{+\infty}, C_{-\infty} \subseteq W_1$ such that for every $\Gamma \in W_1$, $C_{-\infty} \sqsubset_1 \Gamma \sqsubset_1 C_{+\infty}$ (cf. [53, Lemma 6.4]).

Proof. 1 comes from Proposition 2.1(1). As for 2, consider $\Gamma, \Delta \in W_1$. Since $\mathcal{M}_1$ is the submodel of $\mathcal{M}_0$ generated by Γ_0, there is a $(\sqsubset_0 \cup \sqsupset_0 \cup R_0)$-path from Γ_0 to Γ, and from Γ_0 to Δ. Then, by transitivity of $\sqsubset_0$ and possibly using Axioms 2e and 4a, we obtain that $\Gamma \sqsubseteq_0 \Delta \vee \Delta \sqsubset_0 \Gamma$. Then, by definition of $\sqsubset_1$, we have $\Gamma \sqsubseteq_1 \Delta \vee \Delta \sqsubset_1 \Gamma$. 3 comes from Proposition 12.5(2) and definition of $\mathcal{M}_1$. 4 comes from Proposition 2.1(2).

As for 5, we follow [53, Lemma 6.4]. Define $\Lambda_0 := \{P\varphi \,|\, \varphi \text{ is satisfied in } \mathcal{M}_1\}$. We show that Λ_0 is consistent. For, suppose that there is $n \in \mathbf{N}$ and formulas $\varphi_0, ..., \varphi_{n-1}$ satisfied in $\mathcal{M}_1$ at $\Delta_0, ..., \Delta_{n-1}$ respectively. By prelinearity and transitivity of $\sqsubset_1$, there is $i \in n$ such that, for all $j \in n$, we have $\Delta_j \sqsubseteq_1 \Delta_i$. Now, by Corollary 11.7(1), $F\top \in \Delta_i$. Then, there is $\Gamma \in \mathcal{M}_1$ with $\Delta_i \sqsubset_1 \Gamma$. Then, $\Delta_0, ..., \Delta_{n-1} \sqsubset_1 \Gamma$. Then, $P\varphi_0, ..., P\varphi_{n-1} \in \Gamma$, proving the consistency of Λ_0. By Lindenbaum Lemma, there is $\Lambda \in W_0$ with $\Lambda_0 \subseteq \Lambda$. Consider an arbitrary $\Gamma \in W_1$. Then, by definition of Λ_0, we have that if $\varphi \in \Gamma$, then $P\varphi \in \Lambda_0 \subseteq \Lambda$. Then, $\Gamma \sqsubset_0 \Lambda$. Then, by arbitrariness of Γ and definition of $\mathcal{M}_1$, we have $\Lambda \in W_1$ and, for every $\Gamma \in W_1$, $\Gamma \sqsubset_1 \Lambda$. In particular, $\Lambda \sqsubset_1 \Lambda$. Then, by Proposition 9.1(1), Λ lies in a $\sqsubset_1$-cluster, say $C_{+\infty}$, of $\mathcal{M}_1$. By definition of $C_{+\infty}$ and transitivity of $\sqsubset_1$, we have $W_1 \sqsubset C_{+\infty}$. By a similar argument we obtain $C_{-\infty}$. □

Clusters of $\mathcal{M}_1$

Let us now state some results that come from [85, §2, 3]. These results aim to produce a formula ξ that says, given two points lying in the same $\sqsubset_1$-cluster, that they see the same R_1-q-maximal points. To obtain ξ, we use the finitary methods of [85, §2, 3]. The use of finitary methods is made possible by the restriction of the infinite Φ to the finite Φ^k. The formula ξ will have a relevant role in the study of the $\sqsubset_2$-clusters of $\mathcal{M}_2$, in particular in proving their connectedness and local 2-connectedness. But first let us prove a preliminary proposition:

Proposition 12.7. For every R_1-q-maximal point $\Gamma \in W_1$:

1. Γ is R_1-reflexive.
2. Γ is contained in a maximal R_1-cluster C.
3. $R_1(\Gamma) = R_1^r(\Gamma) = C$.

Proof. 1. Let Γ be as per thesis. Since, by Axiom 4d, R_1 is serial, and, by Axiom 3b, R_1 is transitive, the definition of R_1-q-maximality yields that Γ is R_1-reflexive. 2. By 1 and Proposition 9.1(1,2), Γ is contained in a maximal R_1-cluster C. 3. Plainly $C \subseteq R_1(\Gamma)$. Consider $\Delta \in R_1(\Gamma)$. Then, by definition of R_1-q-maximality, we have $\Delta R_1 \Gamma$. Then $\Delta \in C$. Then, $R_1(\Gamma) \subseteq C$. Then, we have $R_1(\Gamma) = C$. Then, by 1, we also have $R_1^r(\Gamma) = C$. □

For every $\Gamma \in W_1$ and $C \subseteq W_1$, define:

$$M(\Gamma) := \{D \subseteq R_1(\Gamma) \mid D \text{ is a maximal } R_1\text{-cluster of } \mathcal{M}_1\},$$

$$\varepsilon(\Gamma) := \{p \in \Phi^k \mid p \in \Gamma\},$$

$$\delta(C) := \{\varepsilon(\Gamma) \mid \Gamma \in C\},$$

$$\gamma(\Gamma) := \{\delta(D) \mid D \in M(\Gamma)\}.$$

Proposition 12.8. (Cf. [85, Lemma 7].) For every $\sqsubset_1$-cluster C, and maximal R_1-clusters $D, E \subseteq C$, if $\delta(D) = \delta(E)$, then $D = E$.

Proof. We follow [85, Lemma 7]. However, we have to remember that our language, unlike the language in [85, Lemma 7], has also temporal operators G and H.

We begin by showing that, for all $\Gamma \in D$ and $\Delta \in E$, if $\varepsilon(\Gamma) = \varepsilon(\Delta)$, then $\Gamma = \Delta$. For, we can prove by induction on $\varphi \in L^k$ that $\varphi \in \Gamma$ if and only if $\varphi \in \Delta$. Indeed, the propositional case is given by $\varepsilon(\Gamma) = \varepsilon(\Delta)$. The temporal cases (G, H), not present in [85, Lemma 7], follow from the fact that, since $\Gamma, \Delta \in C$, then, for every $\Theta \in W_1$, we have $\Gamma \sqsubset_1 \Theta$ if and only if $\Delta \sqsubset_1 \Theta$, and $\Gamma \sqsupset_1 \Theta$ if and only if $\Delta \sqsupset_1 \Theta$. As for the $[d]$-case, suppose that $[d]\varphi \in \Gamma$. Then, for every $\Gamma' \in D$ we have $\varphi \in \Gamma'$ $(*)$. Consider

$\Delta' \in R_0(\Delta)$. Since $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$, we have $\Delta' \in R_1(\Delta)$. Since E is a maximal R_1-cluster, by Proposition 12.7(3), we have $\Delta' \in E$. By $\delta(D) = \delta(E)$, there is $\Gamma' \in D$ with $\varepsilon(\Gamma') = \varepsilon(\Delta')$. By $(*)$, we have $\varphi \in \Gamma'$. By inductive hypothesis, we have $\varphi \in \Delta'$. By arbitrariness of $\Delta' \in R_0(\Delta)$, we have $[d]\varphi \in \Delta$. We can similarly prove that $[d]\varphi \in \Delta$ implies $[d]\varphi \in \Gamma$.

We now prove that $D = E$. Consider $\Gamma \in D$. By $\delta(D) = \delta(E)$, there is $\Delta \in E$ with $\varepsilon(\Gamma) = \varepsilon(\Delta)$. By our claim, $\Gamma = \Delta$. So $\Gamma \in E$, proving $D \subseteq E$. We can similarly prove that $E \subseteq D$, obtaining $D = E$ as desired. □

For every $t \subseteq \Phi^k$, let:

$$q(t) := \bigwedge_{p \in t} p \wedge \bigwedge_{p \in \Phi^k \setminus t} \neg p,$$

and, for every $\Gamma \in W_1$, let $q(\Gamma) := q(\varepsilon(\Gamma))$. For every $T \subseteq \mathcal{P}(\Phi^k)$, let:

$$\alpha(T) := \bigwedge_{t \in T} \blacklozenge q(t) \wedge \bigwedge_{t \in \mathcal{P}(\Phi^k) \setminus T} \neg \blacklozenge q(t),$$

and, for every maximal R_1-cluster C of $\mathcal{M}_1$, let $\alpha(C) := \alpha(\delta(C))$. For every $\mathbf{T} \subseteq \mathcal{P}(\mathcal{P}(\Phi^k))$, let:

$$\xi(\mathbf{T}) := \bigwedge_{T \in \mathbf{T}} \blacklozenge \blacksquare \alpha(T) \wedge \bigwedge_{T \in \mathcal{P}(\mathcal{P}(\Phi^k)) \setminus \mathbf{T}} \neg \blacklozenge \blacksquare \alpha(T),$$

and, for every $\Gamma \in W_1$, let $\xi(\Gamma) := \xi(\gamma(\Gamma))$. Since Φ^k is finite, $q(t)$, $\alpha(T)$ and $\xi(\mathbf{T})$ are all formulas in L^k. From Proposition 12.8 we obtain:

Proposition 12.9. 1. For every $\sqsubset_1$-cluster C, the set of all maximal R_1-clusters included in C is finite (cf. [85, Lemma 8]).

2. For every $\sqsubset_1$-cluster C, maximal R_1-cluster $D \subseteq C$ and R_1-q-maximal $\Gamma \in C$, we have that $\mathcal{M}_1, \Gamma \models \alpha(D)$ if and only if $\Gamma \in D$ (cf. [85, Lemma 9]).

3. For every $\sqsubset_1$-cluster C, maximal R_1-cluster $D \subseteq C$ and $\Gamma \in C$, we have that $\mathcal{M}_1, \Gamma \models \blacklozenge\blacksquare\alpha(D)$ if and only if $D \in M(\Gamma)$ (cf. [85, Lemma 10]).

4. For every $\sqsubset_1$-cluster C, and $\Gamma, \Delta \in C$, we have that $\mathcal{M}_1, \Gamma \models \xi(\Delta)$ if and only if $M(\Gamma) = M(\Delta)$.

Proof. 1. Since Φ^k is finite, given a $\sqsubset_1$-cluster C, we have that $\{\varepsilon(\Gamma) \mid \Gamma \in C\}$ is finite. Then, $\{\delta(D) \mid D \subseteq C \wedge D \text{ a maximal } R_1\text{-cluster}\}$ is finite as well. Thesis follows from Proposition 12.8.

2. Let C, D and Γ be as per thesis. By Proposition 12.7(2), Γ belongs to a maximal R_1-clusters, say E. Assume $\mathcal{M}_1, \Gamma \models \alpha(D)$. Then, by definition of α, we have $\delta(R_1^r(\Gamma)) = \delta(D)$. Now, by Proposition 12.7(3), we have $R_1^r(\Gamma) = E$. Then, we have $\delta(D) = \delta(E)$. Then, by Proposition 12.8, we have $D = E$. Then, $\Gamma \in D$. Conversely, assume that $\Gamma \in D$. Then, since D is an R_1-cluster, by Proposition 9.1(1), we have

$D = E$. Then, by Proposition 12.7(3), $R_1^r(\Gamma) = D$. Then, by definition of α, we have $\mathcal{M}_1, \Gamma \models \alpha(D)$.

3. Let C, D and Γ be as per thesis (observe that, unlike in 2, we are not requiring Γ to be R_1-q-maximal). Assume that $\mathcal{M}_1, \Gamma \models \blacklozenge\blacksquare\alpha(D)$. Then, there is $\Delta \in R_1^r(\Gamma)$ such that $\mathcal{M}_1, \Delta \models \blacksquare\alpha(D)$. Now, by Proposition 12.6(4), there is a R_1-q-maximal $\Theta \in R_1(\Delta)$. Then, as $\mathcal{M}_1, \Delta \models \blacksquare\alpha(D)$, we have $\mathcal{M}_1, \Theta \models \alpha(D)$. Now, by transitivity of R_1, we have $\Theta \in R_1(\Gamma)$. Then, by Proposition 12.3(1), we have $\Theta \in C$. Then, by 2, we have $\Theta \in D$. Therefore $D \in M(\Gamma)$. Conversely, assume that $D \in M(\Gamma)$. Then, there is $\Delta \in D \cap R_1^r(\Gamma)$. Now, by Proposition 12.7(3), we have $R_1^r(\Delta) = D$. Then, by 2, we have $\mathcal{M}_1, \Delta \models \blacksquare\alpha(D)$. Then, as $\Delta \in R_1^r(\Gamma)$, we have $\mathcal{M}_1, \Delta \models \blacklozenge\blacksquare\alpha(D)$. As for 4, it plainly follows from 3. □

12.2.3 Model $\mathcal{M}_2$

Following [85, §3], we define $\mathcal{M}_2$ by filtrating $\mathcal{M}_1$. The filtration is the same as in [85, §3, formula 36], with the exception that ours is relativized to $\sqsubset_1$-clusters. More formally, define the following equivalence relation on $\mathcal{M}_1$:

$$\Gamma \sim \Delta \text{ provided } \Gamma = \Delta \vee (\Gamma \equiv_{\Psi}^{\mathcal{M}_1} \Delta \wedge \Gamma \sqsubset_1 \Delta \wedge \Gamma \sqsupset_1 \Delta \wedge M(\Gamma) = M(\Delta)).$$

Observe that $\Gamma \sim \Delta$ provided either Γ and Δ are equal or they agree on Ψ, belong to the same $\sqsubset_1$-cluster and see the same maximal R_1-clusters through R_1. Define:

1. W_2 as the set of all $\sim$-equivalence classes, with h the projection function.

2. For every $\Gamma, \Delta \in W_1$, $h(\Gamma) \sqsubset_2 h(\Delta)$ provided $\Gamma \sqsubset_1 \Delta$.

3. For every $\Gamma, \Delta \in W_1$, $h(\Gamma) R_2' h(\Delta)$ provided there is $\Gamma' \in h(\Gamma)$ and $\Delta' \in h(\Delta)$ such that $\Gamma' R_1 \Delta'$.

4. R_2 as the transitive closure of R_2'.

5. For every $p \in \Phi^k$, $V_2(p) := \begin{cases} h(V_1(p)) & \text{if } p \in \Phi^k \cap \Psi, \\ \emptyset & \text{otherwise.} \end{cases}$

6. $\mathcal{M}_2 := (W_2, R_2, \sqsubset_2, V_2)$.

Observe that $\sqsubset_2$ is well-defined. For, suppose that $h(\Gamma) \sqsubset_2 h(\Delta)$ and $\Gamma' \in h(\Gamma)$ and $\Delta' \in h(\Delta)$. By definition of $\sqsubset_2$, we have $\Gamma \sqsubset_1 \Delta$, and, by definition of $\sim$, we have $\Gamma' \sqsubseteq_1 \Gamma$ and $\Delta \sqsubseteq_1 \Delta'$. Then, since $\sqsubset_1$ is transitive, we have $\Gamma' \sqsubset_1 \Delta'$.

Remark 12.10. Given a subset C of $\mathcal{M}_1$, define $C/{\sim} := \{h(\Gamma) \mid \Gamma \in C\}$. The definition of $\sqsubset_2$ yields that: for every $\sqsubset_2$-cluster C of $\mathcal{M}_2$, $\bigcup C$ is a $\sqsubset_1$-cluster of $\mathcal{M}_1$; and for every $\sqsubset_1$-cluster C of $\mathcal{M}_1$, $C/{\sim}$ is a $\sqsubset_2$-cluster of $\mathcal{M}_2$.

Proposition 12.11. $\mathcal{M}_2$ is a filtration of $\mathcal{M}_1$ through (Ψ, h) (cf. [86, Lemma 3.3]). Then, by Filtration Lemma 9.5, we have $\mathcal{M}_2, h(\Gamma_0) \models \varphi_0$.

Proof. We need to prove points 1, 2, 3 (cases F, P and $[d]$), and 4 (cases F, P and $[d]$) of Definition 9.4. We follow the proof of [86, Lemma 3.3] with the only exception of point 4 (case $[d]$). This point requires a slight modification due to the fact that our R_1 might be not reflexive.

Consider $\Gamma, \Delta \in W_1$. Point 1 plainly follows from the definition of V_2. Point 2 plainly follows from the definition of $\sim$. Point 3 (cases F, P and $[d]$) plainly follows from the definitions of $\sqsubset_2$ (for cases F, P) and of $\sim$ (for case $[d]$). Point 4 (cases F, P) plainly follows from the definition of $\sqsubset_2$.

The most interesting case is Point 4 (case $[d]$). Consider $\Gamma, \Delta \in W_1$. Assume $h(\Gamma) R_2 h(\Delta)$. We would like $\Gamma R_{1\Psi} \Delta$. Consider $[d]\varphi \in \Psi$ and suppose $\mathcal{M}_1, \Gamma \models [d]\varphi$. By definition of R_2 as the transitive closure of R'_2, there are $\Gamma_0 = \Gamma, \Gamma_1, ..., \Gamma_n = \Delta$ such that $h(\Gamma_i) R'_2 h(\Gamma_{i+1})$ for all $i = 0, 1, ..., n-1$. Suppose $\mathcal{M}_1, \Gamma_i \models [d]\varphi$ for some $i < n$. Now, since $h(\Gamma_i) R'_2 h(\Gamma_{i+1})$, we have $\Gamma'_i R_1 \Gamma'_{i+1}$ for some $\Gamma'_i \in h(\Gamma_i)$ and $\Gamma'_{i+1} \in h(\Gamma_{i+1})$. Then, since, by definition of $\sim$, $\Gamma_i \equiv^{\mathcal{M}_1}_{\Psi} \Gamma'_i$, we have $\mathcal{M}_1, \Gamma'_i \models [d]\varphi$. Then, since $\Gamma'_i R_1 \Gamma'_{i+1}$, and by transitivity of $\sqsubset_1$, we have $\mathcal{M}_1, \Gamma'_{i+1} \models \varphi \wedge [d]\varphi$. Then, since, by definition of $\sim$, $\Gamma_{i+1} \equiv^{\mathcal{M}_1}_{\Psi} \Gamma'_{i+1}$, we have $\mathcal{M}_1, \Gamma_{i+1} \models \varphi \wedge [d]\varphi$. Therefore, by induction, we have $\mathcal{M}_1, \Gamma_n \models \varphi$. Therefore, since $\Gamma_n = \Delta$, we obtain $\Gamma R_{1\Psi} \Delta$ as desired. □

Remark 12.12. Similar arguments show that the frame of $\mathcal{M}_2$ validates Shehtman's special axioms of group 4 (cf. [86, Lemma 3.3]). Transitivity of $\sqsubset_1$ implies transitivity of $\sqsubset_2$. Similarly for the other axioms of group 2. R_2 being a transitive closure is obviously transitive. Therefore the frame of $\mathcal{M}_2$ validates all axioms of Λ_d except possibly 5 and 6.

Remark 12.13. By definition of $\sqsubset_2$ and Filtration Lemma 9.5, a simple induction shows that for every formula φ formed from formulas in Ψ using only boolean and temporal operators and every $\Gamma \in W_1$, $\mathcal{M}_1, \Gamma \models \varphi$ if and only if $\mathcal{M}_2, h(\Gamma) \models \varphi$.

Clusters of $\mathcal{M}_2$

Let us focus on the $\sqsubset_2$-clusters of $\mathcal{M}_2$ and prove that they are finite, connected and locally 2-connected (with respect to R_2). Consider $\Gamma \in W_1$ and define:

$$\chi_\Gamma = \bigwedge_{\varphi \in \Psi \cap \Gamma} \varphi \wedge \bigwedge_{\varphi \in \Psi \setminus \Gamma} \neg\varphi \wedge \xi(\Gamma).$$

Remark 12.14. For every $\sqsubset_1$-cluster C of $\mathcal{M}_1$ and $\Gamma, \Delta \in C$, observe that Δ satisfies the first two conjuncts of χ_Γ if and only if Γ and Δ satisfy the same formulas in Ψ. By Proposition 12.9(4), Δ satisfies the third conjunct of χ_Γ if and only if Δ sees the same maximal R_1-clusters as Γ. Then, by definition of $\sim$, Δ satisfies χ_Γ if and only if $\Delta \sim \Gamma$.

For every $w \in W_2$, define χ_w as χ_Γ for an arbitrary $\Gamma \in w$. The formula χ will let us use Axiom 5 to prove connectedness, and Axiom 6 (**G2**) to prove local 2-connectedness of the $\sqsubset_2$-clusters of $\mathcal{M}_2$:

Proposition 12.15. For every $\sqsubset_2$-cluster C of $\mathcal{M}_2$, the following facts hold:

1. C is finite (cf. [53, Lemma 6.9]).

2. $(C, R_2 \upharpoonright C)$ is connected (cf. [53, Lemma 6.12]).

3. $(C, R_2 \upharpoonright C)$ is locally 2-connected (cf. [85, Theorem 15] and [70, Lemma 5.20]).

Proof. 1. This is a straightforward adaptation of the proof of [53, Lemma 6.9]. Observe that for all $\sqsubset_2$-clusters C of $\mathcal{M}_2$, for all $\Gamma, \Delta \in \bigcup C$, we have that $\Gamma \sim \Delta$ provided $\Psi \cap \Gamma = \Psi \cap \Delta$ and $M(\Gamma) = M(\Delta)$. Then thesis follows by the fact that both Ψ and, by Proposition 12.9(1), the number of maximal R_1-clusters in $\bigcup C$ are finite.

2. This is an adaption of the proof of [53, Lemma 6.12]. Suppose that $C = X \cup Y$ for some X, Y non-empty, disjoint, R_2-generated subsets of C. Let:

$$\alpha := \bigvee_{w \in X} \chi_w.$$

By Remark 12.14, for every $\Gamma \in \bigcup X$, we have:

$$\alpha \in \Gamma \text{ if and only if } \Gamma \in \bigcup X.$$

Choose an arbitrary $\Gamma \in \bigcup C$ and let:

$$\Delta_0 := \{\blacklozenge\alpha, \blacklozenge\neg\alpha\} \cup \{F\gamma, P\gamma \mid \gamma \in \Gamma\}.$$

We show that Δ_0 is consistent. Since Γ is closed under conjunction, it suffices to show that, for an arbitrary $\gamma \in \Gamma$, we have that $\delta := \blacklozenge\alpha \wedge \blacklozenge\neg\alpha \wedge F\gamma \wedge P\gamma$ is consistent. Choose any $\Gamma_X \in \bigcup X$ and $\Gamma_Y \in \bigcup Y$. By Remark 12.10, $\bigcup C$ is a $\sqsubset_1$ cluster of $\mathcal{M}_1$. Then, we have $\Gamma \sqsubset_1 \Gamma_X \sqsubset_1 \Gamma$ and $\Gamma \sqsubset_1 \Gamma_Y \sqsubset_1 \Gamma$. Then, $\alpha \wedge F\gamma \in \Gamma_X$ and $\neg\alpha \wedge F\gamma \in \Gamma_Y$. Then, $F(\alpha \wedge F\gamma) \wedge F(\neg\alpha \wedge F\gamma) \in \Gamma$. Then, by Axiom 5, we have $F(\blacklozenge\alpha \wedge \neg\blacklozenge\alpha \wedge F\gamma) \in \Gamma$. Now, by Axiom 2d, we also have $GP\gamma \in \Gamma$. Then, $F\delta \in \Gamma$. If δ is L_d^k-inconsistent, then $\neg\delta, G\neg\delta \in \mathsf{L}_d^k$ from which $G\neg\delta \in \Gamma$, against the consistency of Γ.

Since Δ_0 is consistent, by Lindendaum Lemma, there is $\Delta \in W_0$ such that $\Delta_0 \subseteq \Delta$. By Definition of Δ_0, we have $\Gamma \sqsubset_0 \Delta \sqsubset_0 \Gamma$. Then, since $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$, we have $\Delta \in \mathcal{M}_1$ and $\Gamma \sqsubset_1 \Delta \sqsubset_1 \Gamma$. Then, by Propositions 9.1(1) and 12.10, we have $\Delta \in \bigcup C$. Since $\blacklozenge\alpha, \blacklozenge\neg\alpha \in \Delta$, we can find $\Delta_X, \Delta_Y \in R_0^r(\Delta)$ with $\alpha \in \Delta_X$ and $\neg\alpha \in \Delta_Y$. Since $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$, we have $\Delta_X, \Delta_Y \in R_1^r(\Delta)$. Since, by Proposition 12.3(1), $\bigcup C$ is R_1-generated, we have $\Delta_X, \Delta_Y \in \bigcup C$. Then, since $\alpha \in \Delta_X$ and $\neg\alpha \in \Delta_Y$, we have $h(\Delta_X) \in X$ and $h(\Delta_Y) \in Y$. Then, there is a $\widetilde{R_2^r}$-path between $h(\Delta_X) \in X$ and $h(\Delta_Y) \in Y$, contradicting the fact that X and Y are disjoint and R_2-generated.

Comparison with the proof of [53, Lemma 6.12]: Since our filtration is partially different from that in [53, Lemma 6.12], we use our formulas χ_w to define $\alpha := \bigvee_{w\in X}\chi_w$. Moreover, when in [53, Lemma 6.12] they use their connectedness axiom $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\Diamond p \wedge \Diamond\neg p \wedge Fq)$, we use our connectedness axiom, Axiom 5, $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\blacklozenge p \wedge \blacklozenge\neg p \wedge Fq)$. Indeed, recall that $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\langle d\rangle p \wedge \langle d\rangle\neg p \wedge Fq)$ is not sound on $\mathfrak{R}$. Then, when in [53, Lemma 6.12] they define Δ_0 as $\{\Diamond\alpha, \Diamond\neg\alpha\} \cup \{F\gamma, P\gamma \,|\, \gamma \in \Gamma\}$, we need to define Δ_0 as $\{\blacklozenge\alpha, \blacklozenge\neg\alpha\} \cup \{F\gamma, P\gamma \,|\, \gamma \in \Gamma\}$. Because of this, when in [53, Lemma 6.12] they obtain a $\widetilde{R_2}$-path between X and Y, we instead obtain an $\widetilde{R_2^r}$-path between X and Y.

3. This is an adaption of the proof of [85, Theorem 15] (the proof of [70, Lemma 5.20] is a similar adaption). In [85, Theorem 15] they prove that a certain filtration enjoys Axiom **G1**, whereas we prove that the $\sqsubset_2$-clusters of our filtration $\mathcal{M}_2$ enjoys Axiom **G2** (that is our Axiom 6). The idea behind the two proofs is the same.

Consider an arbitrary $h(\Gamma) \in C$. Observe that, by Proposition 12.3(1), we have $R_2(h(\Gamma)) \subseteq C$. Suppose for contradiction that $R_2(h(\Gamma))$ can be partitioned into three non empty $\widetilde{R_2}$-generated sets X_1, X_2 and X_3. For $i = 1, 2, 3$ define:

$$\beta_i := \bigvee_{w\in X_i}\chi_w \text{ and } \eta_i := \beta_i \wedge \bigwedge_{j\in\{1,2,3\}\setminus\{i\}}\neg\beta_j.$$

By Remark 12.14, for each $\Delta \in \bigcup C$, we have $\beta_i \in \Delta$ if and only if $h(\Delta) \in X_i$, from which we have $\eta_i \in \Delta$ if and only if $h(\Delta) \in X_i$. Consider any $\Delta \in R_0(\Gamma)$. Since $\mathcal{M}_1$ is a generated subset of $\mathcal{M}_0$, we have $\Delta \in R_1(\Gamma)$. Then, $h(\Delta) \in R_2(h(\Gamma))$. Then, there is $i \in \{1, 2, 3\}$ such that $h(\Delta) \in X_i$. Then, $\eta_i \in \Delta$. Moreover, consider any $\Theta \in R_0(\Delta)$. Since $\mathcal{M}_1$ is a generated subset of $\mathcal{M}_0$, we have $\Theta \in R_1(\Delta)$. Then, $h(\Theta) \in R_2(h(\Delta))$. Since X_i is $\widetilde{R_2}$-generated $h(\Theta) \in X_i$. Then, $\eta_i \in \Theta$. Then, $\blacksquare\eta_i \in \Delta$. Then, $\bigvee_{i=1,2,3}\blacksquare\eta_i \in \Delta$. Then, $[d](\bigvee_{i=1,2,3}\blacksquare\eta_i) \in \Gamma$. Then, by Axiom 6, $\bigvee_{i=1,2,3}[d]\neg\eta_i \in \Gamma$.

Consider any $i \in \{1, 2, 3\}$, $w \in X_i$ and $\Delta \in w$. By Proposition 12.6.3 there is Θ R-q-maximal such that $\Delta R_1 \Theta$. Then, $h(\Delta) R_2 h(\Theta)$. Since X_i is $\widetilde{R_2}$-generated $h(\Theta) \in X_i$. Then, $\eta_i \in \Theta$. Moreover $h(\Gamma) R_2 h(\Theta)$.

Lemma 12.16. $\Gamma R_1 \Theta$.

Proof. Since $h(\Gamma) R_2 h(\Theta)$, there are $n \in \mathbf{N}$ and $v_0, v_1, ..., v_n \in W_2$ with $v_0 = h(\Gamma)$, $v_n = h(\Theta)$ and $v_i R_2' v_{i+1}$. Consider $\Gamma_n \in v_n$. Then, $\Gamma_n \sim \Theta$. Then, $M(\Gamma_n) = M(\Theta)$. Now, since Θ is R_1-q-maximal, by Proposition 12.7(1), $\Theta R_1 \Theta$. Then, since $M(\Gamma_n) = M(\Theta)$, we have $\Gamma_n R_1 \Theta$. Consider $\Gamma_i \in v_i$ and assume $\Gamma_{i+1} R_1 \Theta$ for some $\Gamma_{i+1} \in v_{i+1}$. Since $v_i R_2' v_{i+1}$, we have $\Gamma_i' R_1 \Gamma_{i+1}'$ for some $\Gamma_i' \in v_i$ and $\Gamma_{i+1}' \in v_{i+1}$. Then, $\Gamma_i \sim \Gamma_i'$ and $\Gamma_{i+1} \sim \Gamma_{i+1}'$. Then, $M(\Gamma_i) = M(\Gamma_i')$ and $M(\Gamma_{i+1}) = M(\Gamma_{i+1}')$. Then, again since Θ is R_1-q-maximal and $\Gamma_{i+1} R_1 \Theta$, we have $\Gamma_{i+1}' R_1 \Theta$. Then, since $\Gamma_i' R_1 \Gamma_{i+1}'$, we have $\Gamma_i' R_1 \Theta$, from which, again by R_1-q-maximality of Θ, we have $\Gamma_i R_1 \Theta$. Therefore, by induction, we have $\Gamma R_1 \Theta$. $\square$

Then $\langle d\rangle\eta_i \in \Gamma$. Therefore by arbitrariness of $i \in \{1, 2, 3\}$, we have $\bigwedge_{i=1,2,3}\langle d\rangle\eta_i \in \Gamma$, against $\bigvee_{i=1,2,3}[d]\neg\eta_i \in \Gamma$. $\square$

'Similar' points in $\mathcal{M}_2$

Following [53, §6.3.5] we turn $\mathcal{M}_2$ into a Ψ-linked model $\mathcal{M}_4$ (passing through the model $\mathcal{M}_3$). Let us introduce some preliminary definitions:

Definition 12.17. 1. For $B \subseteq \mathcal{B}\Psi$, define

$$\beta_B := \bigwedge_{\varphi\in B} \varphi \wedge \bigwedge_{\varphi\in\mathcal{B}\Psi\setminus B} \neg\varphi.$$

2. For every Kripke model $\mathcal{M}$ and $w \in \mathcal{M}$, define $\tau_{\mathcal{M}}(w) := \{\varphi \in \mathcal{B}\Psi \mid \mathcal{M}, w \models \varphi\}$ (we drop the subscript where $\mathcal{M}$ is understood).

Given an element w of a Kripke model $\mathcal{M}$, we think of $\tau_{\mathcal{M}}(w)$ as the 'type' of w in $\mathcal{M}$. Plainly:

Proposition 12.18. (Cf. [53, Lemma 6.14].)

1. For every Kripke model $\mathcal{M}$ and $w \in \mathcal{M}$, we have that $\tau(w)$ is the unique subset B of $\mathcal{B}\Psi$ such that $\mathcal{M}, w \models \beta_B$.

2. For every $\Gamma \in W_1$, $\tau_{\mathcal{M}_1}(\Gamma)$ is the unique subset B of $\mathcal{B}\Psi$ such that $\beta_B \in \Gamma$. Moreover, $\tau_{\mathcal{M}_1}(\Gamma) = \Gamma \cap \mathcal{B}\Psi$.

3. For every Kripke model $\mathcal{M}$ and $c, d \in \mathcal{M}$, we have that $c \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} d$ if and only if $\tau_{\mathcal{M}}(c) = \tau_{\mathcal{M}}(d)$.

Then, as observed in [53, Section 6.3.5], to show that two $\sqsubset$-clusters C, D contain similar points it suffices to find R_2-reflexive points $c \in C$ and $d \in D$ with the same type - that is, satisfying the same formula β_B for some $B \subseteq \mathcal{B}\Psi$. We use the Prior Axiom and the Zorn Property to do this. However, the Prior Axiom delivers a $\sqsubset_2$-irreflexive point satisfying β_B, whereas we want a R_2-reflexive point in a $\sqsubset_2$-cluster. So instead of β_B we use $\blacksquare\beta_B$. Then, the Prior Axiom will deliver a $\sqsubset_2$-irreflexive point satisfying $\blacksquare\beta_B$, and we will use the Zorn Property to find a $\sqsubset_2$-reflexive point satisfying $\blacksquare\beta_B$. Let us call 'links' the R_2-reflexive points in $\mathcal{M}_2$ that satisfy $\blacksquare\beta_B$.

There is a second complication. We do not know that the $\blacksquare\beta_B$-instance of the Prior Axiom is valid in $\mathcal{M}_2$. Moreover, the point should also lie in a certain temporal range. To apply the Prior Axiom to obtain a point satisfying $\blacksquare\beta_B$ in the right temporal range we will work in $\mathcal{M}_1$ where the Prior Axiom is valid and $\sqsubset_1$ is defined by formulas.

There is a final complication. Since the Filtration Lemma 9.5 does not apply to $\blacksquare\beta_B$, even if we find a point $\Gamma \in \mathcal{M}_1$ satisfying $\blacksquare\beta_B$, we do not know if the $\sim$-representative of Γ in $\mathcal{M}_2$, namely $h(\Gamma) \in \mathcal{M}_2$, satisfies $\blacksquare\beta_B$. Then, with this definition of 'links' we cannot guarantee that adjacent clusters contain 'similar' points. Then, in the formal definition of 'link', we work directly in $\mathcal{M}_1$:

Definition 12.19. An element $w \in W_2$ is said to be a *link* provided it is R_2-reflexive and there is $\Gamma \in w$ with $\blacksquare\beta_B \in \Gamma$ for some $B \subseteq \mathcal{B}\Psi$.

Proposition 12.20. (Cf. [53, Lemma 6.17].) Let $B \subseteq \mathcal{B}\Psi$ and w an R_2-reflexive point in W_2. Then, $\blacksquare\beta_B \in \bigcup w$ if and only if w is a link and $\tau(w) = B$.

Proof. Let B and w be as per thesis. Suppose that $\blacksquare\beta_B \in \Gamma \in w$. Then, by definition, w is a link. Moreover, $\beta_B \in \Gamma$. Then, by Proposition 12.11, $\mathcal{M}_2, w \models \beta_B$. Then, by definition, $\tau(w) = B$. Conversely, suppose that w is a link and $\tau(w) = B$. Then, by definition, $\blacksquare\beta_{B'} \in \bigcup w$ for some $B' \subseteq \mathcal{B}\Psi$. Then, by the first part, $\tau(w) = B'$. Then, $B' = B$. Then, $\blacksquare\beta_B \in \bigcup w$. □

Links are rather common:

Proposition 12.21. Every $\sqsubset_2$-cluster C of $\mathcal{M}_2$ contains a link.

Proof. By Zorn Property $\bigcup C$ contains an R_1-q-maximal Γ. By Proposition 12.7(1), Γ is R_1-reflexive. Then, $h(\Gamma)$ is R_2-reflexive. For every $\Delta \in R_1(\Gamma)$, the fact that both Γ and Δ belong to the $\sqsubset_1$-cluster C yields that $\sqsubset_1(\Gamma)$ is equal to $\sqsubset_1(\Delta)$ and that $\sqsupset_1(\Gamma)$ is equal to $\sqsupset_1(\Delta)$, whereas R_1-q-maximality of Γ yields that $R_1(\Gamma)$ is equal to $R_1(\Delta)$. Then, $\tau(\Gamma) = \tau(\Delta)$. Then, $\beta_{\tau(\Gamma)} \in \Delta$ and $\blacksquare\beta_{\tau(\Gamma)} \in \Gamma$. Then, $h(\Gamma)$ is a link. □

Let us count the number of 'types' of links between two $\sqsubset_2$-clusters.

Definition 12.22. For C, D two $\sqsubset_2$-clusters of $\mathcal{M}_2$ with $C \sqsubset_2 D$, define:

$$\#(C, D) := |\{\tau(w) \mid w \in W_2 \text{ is a link and } C \sqsubset_2 w \sqsubset_2 D\}|.$$

As observed in [53, §6.3.5], given two adjacent $\sqsubset_2$-clusters C, D of $\mathcal{M}_2$, if $\#(C, D) = 1$, since by Proposition 12.21 both C and D admit links $c \in C$ and $d \in D$, we have $\tau(c) = \tau(d)$ - that is, we have found 'similar' points. This suggests to use induction on $\#(C, D)$ to find 'similar' points in more general situations. We will do so in Proposition 12.25 using the following proposition in the inductive step.

Proposition 12.23. (Cf. [53, Lemma 6.20].) For every $\sqsubset_2$-cluster C of $\mathcal{M}_2$, and $w \in W_2$ such that $C \sqsubset_2 w \notin C$, there is a $\sqsubset_2$-irreflexive $u \in \mathcal{M}_2$ such that:

1. $C \sqsubset_2 u \sqsubseteq_2 w$.
2. There are R_2-reflexive points $c \in C$ and $d \in \lambda(u)$ with $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$.

3. If w is $\sqsubset_2$-irreflexive and $u \sqsubset_2 w$, then $\#(\rho(u), \lambda(w)) < \#(C, \lambda(w))$.

The mirror image holds as well.

Proof. 1. By Proposition 12.21, there exists a link $c \in C$. Define $B := \tau(c)$. Then, by Proposition 12.20, there is $\Gamma \in c$ with $\blacksquare\beta_B \in \Gamma$. Consider an arbitrary $\Delta \in w$. Then, since $C \sqsubset_2 w \notin C$, we have $\Gamma \sqsubset_1 \Delta$ and $\neg(\Delta \sqsubset_1 \Gamma)$. Then, since $\neg(\Delta \sqsubset_1 \Gamma)$, there exists a formula γ such that:

$$\gamma \in \Gamma \text{ and } G\neg\gamma \in \Delta. \tag{12.3}$$

Then, we have $\gamma \wedge \blacksquare\beta_B \in \Gamma$ and $G\neg(\gamma \wedge \blacksquare\beta_B) \in \Delta$.

Now, by Proposition 12.10, Γ is in the $\sqsubset_1$-cluster $\bigcup C$. Then, we have $\Gamma \sqsubset_1 \{\Gamma, \Delta\}$. Then, we have $F(\gamma \wedge \blacksquare\beta_B) \wedge FG\neg(\gamma \wedge \blacksquare\beta_B) \in \Gamma$. Moreover, by Corollary 11.7(3), the Prior Axiom is in L^k_d. Then, $F(G\neg(\gamma \wedge \blacksquare\beta_B) \wedge HF(\gamma \wedge \blacksquare\beta_B))$ is in Γ. Then, there exists $\Theta \in W_1$ such that $\Theta \sqsupset_1 \Gamma$ and:

$$G\neg(\gamma \wedge \blacksquare\beta_B) \wedge HF(\gamma \wedge \blacksquare\beta_B) \in \Theta. \tag{12.4}$$

Then, by (12.4), we have that Θ is $\sqsubset_1$-irreflexive. Define $u := h(\Theta)$. Then, by definition of $\sqsubset_2$, we have that u is $\sqsubset_2$-irreflexive. Moreover, since $\Gamma \sqsubset_1 \Theta$, again by definition of $\sqsubset_2$, we have $C \sqsubset_2 u$. Furthermore, since $HF\gamma \in \Theta$ and $G\neg\gamma \in \Delta$, we have $\neg(\Delta \sqsubset_1 \Theta)$. Then, by prelinearity of $\sqsubseteq_1$, we have $\Theta \sqsubseteq_1 \Delta$. Then, again by definition of $\sqsubset_2$, we have $u \sqsubseteq_2 w$.

2. By (12.4) and Corollary 11.7(2), we have $\blacklozenge(\gamma \wedge \blacksquare\beta_B) \in \Theta$. Then, there exists $\Theta' \in W_1$ such that $\Theta' \in R_1^r(\Theta)$ and $\gamma \wedge \blacksquare\beta_B \in \Theta'$. Two cases are given:

(a) $\Theta = \Theta'$. By the mirror image of Axiom 4d, there exists $\Xi \in W_1$ such that $\Xi \in R_1(\Theta)$ and $\Xi \sqsubset_1 \Theta$. Then, by Axiom 3b, since $\blacksquare\beta_B \in \Theta$, we have $\blacksquare\beta_B \in \Xi$. (Observe that we need $\blacksquare\beta_B$ rather than β_B to get $\blacksquare\beta_B \in \Xi$ from the fact that $\blacksquare\beta_B$ is in the $\sqsubset_1$-irreflexive Θ.)

(b) $\Theta \neq \Theta'$. By (12.4), we have $\neg(\Theta \sqsubset_1 \Theta')$. Then, by prelinearity of $\sqsubset_1$, we have $\Theta' \sqsubset_1 \Theta$. Define $\Xi := \Theta'$. Then, for the same reasons as above, we have $\blacksquare\beta_B \in \Xi$.

In both cases we obtain $\Xi \in W_1$ such that $\Xi \in \lambda(\Theta)$ with $\blacksquare\beta_B \in \Xi$. By Zorn Property, there is an R_1-q-maximal $\Xi' \in R_1(\Xi)$. Then, by remark 12.7, Ξ' is R_1-reflexive. Moreover, by Axiom 3b, since $\blacksquare\beta_B \in \Xi$, we have $\blacksquare\beta_B \in \Xi'$ as well. (Observe that we need $\blacksquare\beta_B$ rather than β_B to get $\blacksquare\beta_B$ in the R_1-reflexive Ξ' from the fact that $\blacksquare\beta_B$ is in the possibly R_1-irreflexive Ξ.) Finally, by Proposition 12.3(1), we have $\Xi' \in \lambda(\Theta)$. Define $d := h(\Xi')$. Then, by definition of $\sqsubset_2$ and R_2, we have that d is R_2-reflexive and

$d \in \lambda(u)$. Now, since $\blacksquare\beta_B \in \Xi'$, by Proposition 12.20, $\tau(d) = B$. Then, since $B = \tau(c)$, we have $\tau(c) = \tau(d)$. Then, $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$.

3. Suppose that w is $\sqsubset_2$-irreflexive and $u \sqsubset_2 w$. Then, by definition of $\sqsubset_2$, Δ is $\sqsubset_1$-irreflexive. Then, we can also assume $H\gamma \in \Delta$. Indeed, since Δ is $\sqsubset_1$-irreflexive, there is δ such that $\neg\delta \wedge H\delta \in \Delta$. Define $\gamma := H\delta$. We want:

- $H\gamma \in \Delta$. Now, by definition of δ, we have $H\delta \in \Delta$. Then, by the mirror image of Axiom 2b, we have $HH\delta \in \Delta$. That is, we have $H\gamma \in \Delta$ as desired.

- $\gamma \in \Gamma$. Now, by the previous item, we have $H\gamma \in \Delta$. Moreover, we have $\Gamma \sqsubset_1 \Delta$. Then, we have $\gamma \in \Gamma$, as desired.

- $G\neg\gamma \in \Delta$. Now, by definition of δ, we have $\neg\delta \in \Delta$. Then, by Axiom 2d, we have $GP\neg\delta \in \Delta$. Then, we have $G\neg H\delta \in \Delta$. That is, we have $G\neg\gamma \in \Delta$ as desired.

Moreover, by (12.4), we have $G\neg(\gamma \wedge \blacksquare\beta_B) \in \Theta$. Then, for every $\Sigma \in W_1$ with $\Theta \sqsubset_1 \Sigma \sqsubset_1 \Delta$, we have $\blacksquare\beta_B \notin \Sigma$. Then, since $h(\Theta) = u$ and $h(\Delta) = w$, by Proposition 12.20, there is no link v with $\tau(v) = B$ and $u \sqsubset_2 v \sqsubset_2 w$. Then, we have:

$$\{\tau(v) \mid v \in W_2 \text{ is a link}, \rho(u) \sqsubset_2 v \sqsubset_2 \lambda(w)\} \subseteq$$
$$\{\tau(v) \mid v \in W_2 \text{ is a link}, C \sqsubset_2 v \sqsubset_2 \lambda(w)\} \backslash \{B\}.$$

Then, since there is a link v of type B with $C \sqsubset_2 v \sqsubset \lambda(w)$, for example $v = c$, we have $\#(\rho(u), \lambda(w)) < \#(C, \lambda(w))$. □

Comparison with the proof of [53, Lemma 6.20]: The two proofs are identical except for the fact that we want an R_2-*reflexive* point d in $\lambda(u)$. The R_2-reflexivity of d is obtained by using the fact that $\mathcal{M}_1$ enjoys the Zorn Property.

Definition 12.24. For every submodel $\mathcal{M} = (W, \sqsubset, R, V)$ of $\mathcal{M}_2$, we say that:

1. $\mathcal{M}$ is *good* provided it is finite, R_2-generated and every $\sqsubset$-cluster of $\mathcal{M}$ is a $\sqsubset_2$-cluster of $\mathcal{M}_2$.
2. $\mathcal{M}$ is *perfect* if it is good and every pair (C, D) of adjacent $\sqsubset$-clusters contains R-reflexive points $c \in C$ and $d \in D$ with $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$ (observe that the equivalence between c and d is with respect to $\mathcal{M}_2$).

Proposition 12.25. (Cf. [53, Lemma 6.22].) For every good submodel $\mathcal{M}$ of $\mathcal{M}_2$, there exists a perfect submodel $\mathcal{M}^*$ of $\mathcal{M}_2$ such that $\mathcal{M} \subseteq \mathcal{M}^*$.

Proof. For every good submodel $\mathcal{M} = (W, \sqsubset, R, V)$ of $\mathcal{M}_2$, define a *defect* as a pair of adjacent $\sqsubset$-clusters (C, D) with no R-reflexive points $c \in C$ and $d \in D$ such that $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$. Let:

$$d(\mathcal{M}) := \Sigma\{\#(C, D) \mid (C, D) \text{ a defect of } \mathcal{M}\}.$$

Observe that $d(\mathcal{M}) \geq 0$. Moreover, since $\mathcal{M}$ is finite, we have that $d(\mathcal{M})$ is finite as well.

Consider an arbitrary good submodel $\mathcal{M} = (W, \sqsubset, R, V)$ of $\mathcal{M}_2$. Among the good submodels $\mathcal{M}^* = (W^*, \sqsubset^*, R^*, V^*)$ such that $\mathcal{M} \subseteq \mathcal{M}^* \subseteq \mathcal{M}_2$, consider a model $\mathcal{M}^*$ such that $d(\mathcal{M}^*)$ is as small as possible.

Now, by Proposition 12.21, if (C, D) is a defect in $\mathcal{M}^*$ then $d(\mathcal{M}^*) > 0$. Then, if $d(\mathcal{M}^*) = 0$, we would have that $\mathcal{M}^*$ has no defect. Then, $\mathcal{M}^*$ would be perfect as desired. We show that $d(\mathcal{M}^*) = 0$.

Assume, by contradiction, that $d(\mathcal{M}^*) > 0$. Then, there exists a defect (C, D) in $\mathcal{M}^*$. Consider an arbitrary $w \in D$. Then, since $\mathcal{M}^* \subseteq \mathcal{M}_2$, we have $C \sqsubseteq_2 w$ and $\neg(w \sqsubset_2 C)$. Then, we can apply Proposition 12.23 to (C, w). Let $u \in \mathcal{M}_2$ be as provided by Proposition 12.23 applied to (C, w). Observe that since C and D are adjacent in $\mathcal{M}^*$, then $u \notin \mathcal{M}^*$. Let $\mathcal{N}$ be the submodel of $\mathcal{M}_2$ given by $\mathcal{M}^*$ together with $R^r(u)$. Let $\sqsubset$ denote $\sqsubset_2 \restriction \mathcal{N}$. Then, we have:

$$\{v \in \mathcal{N} \mid C \sqsubset v \sqsubset D\} = C \cup \lambda(u) \cup \{u\} \cup \rho(u) \cup D. \tag{12.5}$$

(Shown in $\sqsubset$-order, reading from left to right.) Now, $\mathcal{N}$ is plainly good. Moreover, $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_2$. Then, by definition of $\mathcal{M}^*$, we have $d(\mathcal{N}) \geq d(\mathcal{M}^*)$. Now, outside the range between C and D shown in (12.5), $\mathcal{M}^*$ and $\mathcal{N}$ shares the same defects and their #-values. (Recall that # is calculated with respect to $\mathcal{M}_2$). Then, by (12.5), the remaining potential defects in $\mathcal{N}$ are $(C, \lambda(u))$ and $(\rho(u), D)$. Now, by Proposition 12.23, the pair $(C, \lambda(u))$ is not a defect. Then, since $d(\mathcal{N}) \geq d(\mathcal{M}^*)$, we have that $(\rho(u), D)$ is a defect and $\#(\rho(u), D) \geq \#(C, D)$. Now, since $C \sqsubset_2 \rho(u)$, for every $t \in \mathcal{M}_2$ such that $\rho(u) \sqsubset_2 t \sqsubset_2 D$, we have $C \sqsubset_2 t \sqsubset_2 D$. Then, we have $\#(\rho(u), D) \leq \#(C, D)$ as well. Then, $\#(\rho(u), D) = \#(C, D)$. Then $d(\mathcal{M}^*) = d(\mathcal{N})$.

Now, u is irreflexive. Then, $u \sqsubset_2 D$ and $\neg(D \sqsubseteq_2 u)$. Then, we can apply the mirror image of Proposition 12.23 to (u, D). Let $v \in \mathcal{M}_2$ be as provided by Proposition 12.23 applied to (u, D). Then, v is $\sqsubset_2$-irreflexive, and such that $u \sqsubseteq_2 v \sqsubset_2 D$ and $\rho(v)$ and D contains R_2-reflexive $\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$-equivalent points. Now, since $(\rho(u), D)$ is a defect in $\mathcal{N}$, $\rho(u)$ and D do not contain R_2-reflexive $\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$-equivalent points. Then, $u \neq v$ and $u \sqsubset_2 v \sqsubset_2 D$. Let $\mathcal{N}'$ be the submodel of $\mathcal{M}_2$ consisting of $\mathcal{N}$ together with $R^r_2(v)$. Let $\sqsubset$ denote $\sqsubset_2 \restriction \mathcal{N}'$. Then, we have:

$$\{v \in \mathcal{N} \mid \rho(u) \sqsubset v \sqsubset D\} = \rho(u) \cup \lambda(v) \cup \{v\} \cup \rho(v) \cup D. \tag{12.6}$$

(Again shown in $\sqsubset$-order, reading from left to right.) Now, outside the range $\rho(u) - D$ shown in (12.6), $\mathcal{N}'$ and $\mathcal{N}$ have the same defects. Then, the possible defects in $\mathcal{N}'$ are within the range between $\rho(u)$ and D. Then, they are $(\rho(u), \lambda(v))$ and $(\rho(v), D)$. Now, by Proposition 12.23, we have that $(\rho(v), D)$ is not a defect in $\mathcal{N}'$ and that $\#(\rho(u), \lambda(v)) <$

$\#(\rho(u), D)$. Then, we have $d(\mathcal{N}') < d(\mathcal{N}) = d(\mathcal{M}^*)$. Then, since $\mathcal{N}'$ is plainly good and $\mathcal{M} \subseteq \mathcal{N}' \subseteq \mathcal{M}_2$, we obtain a contradiction of the minimality of $\mathcal{M}^*$. □

12.2.4 Models $\mathcal{M}_3$ and $\mathcal{M}_4$

Proceeding as in [53, §6.4], we extract, through a selective filtration, a good submodel $\mathcal{M}_3$ of $\mathcal{M}_2$ as follows:

1. Add $C_{+\infty}/{\sim}$ and $C_{-\infty}/{\sim}$ to $\mathcal{M}_3$.

2. For every $\psi \in \Psi$ such that $F\neg\psi \wedge FG\psi$ is satisfied in $\mathcal{M}_2$, by validity of Prior Axiom in $\mathcal{M}_1$ and remark 12.13, there is $w \in \mathcal{M}_2$ satisfying $G\psi \wedge HF\neg\psi$. Add $R_2^r(w) = \lambda(w) \cup \{w\} \cup \rho(w)$ to $\mathcal{M}_3$.

3. Perform mirror image of 2.

$\mathcal{M}_3$ is plainly a nonempty good submodel of $\mathcal{M}_2$. Then, by Proposition 12.25, there exists a perfect submodel $\mathcal{M}_4 = (W_4, \sqsubset_4, R_4, V_4)$ such that $\mathcal{M}_3 \subseteq \mathcal{M}_4$. We prove that the satisfiability of formulas in Ψ is preserved between $\mathcal{M}_2$ and $\mathcal{M}_4$:

Proposition 12.26. (Cf. [53, Lemma 6.23].) For every $\varphi \in \Psi$ and $w \in W_4$, we have $\mathcal{M}_4, w \models \varphi$ if and only if $\mathcal{M}_2, w \models \varphi$.

Proof. By induction on the complexity of ψ. If ψ is atomic, thesis follows by the fact that $\mathcal{M}_4 \subseteq \mathcal{M}_2$. The boolean cases follow by induction and the closure of Ψ under taking subformulas. The case $[d]\psi$ follows by the fact that $\mathcal{M}_4$ is a R_2-generated submodel of $\mathcal{M}_2$.

The most interesting cases are $G\psi$ and $H\psi$. Let us consider $G\psi$. That $\mathcal{M}_2, w \models G\psi$ implies $\mathcal{M}_4, w \models G\psi$ follows from $\mathcal{M}_4 \subseteq \mathcal{M}_2$. Conversely, assume that $\mathcal{M}_2, w \models \neg G\psi$. Then $\mathcal{M}_2, w \models F\neg\psi$. Two cases are given:

1. $\mathcal{M}_2, w \models GF\neg\psi$. Now, by Proposition 12.6(5), for every $u \in C_{+\infty}/\sim$, we have $u \sqsupset_2 w$ Then, $\mathcal{M}_2, u \models F\neg\psi$. Then, there is $v \in W_2$ such that $v \sqsupset_2 u$ and $\mathcal{M}_2, v \models \neg\psi$. Again by Proposition 12.6(5), $v \in C_{+\infty}/\sim$ as well. Then, by definition of $\mathcal{M}_3$, we have $v \in W_3$. Moreover, by transitivity of $\sqsubset_2$, we have $w \sqsubset_2 v$.

2. $\mathcal{M}_2, w \models F\neg\psi \wedge FG\psi$. Then, by definition of $\mathcal{M}_3$, there exists $u \in W_2$ such that $\mathcal{M}_2, u \models G\psi \wedge HF\neg\psi$ and $R_2^r(u) \subseteq W_3$. Now, if $u \sqsubseteq_2 w$, then $\mathcal{M}_2, w \models G\psi$: a contradiction of $\mathcal{M}_2, w \models F\neg\psi$. Then, by prelinearity of $\sqsubset_2$, we have $w \sqsubset_2 u$. Now, by Remark 12.4, $G\psi \wedge HF\neg\psi \to \blacklozenge\neg\psi$ is valid in $\mathcal{M}_2$. Then, $\mathcal{M}_2, u \models \blacklozenge\neg\psi$. Then, there exists $v \in R_2^r(u)$ such that $\mathcal{M}_2, v \models \neg\psi$. Now, by definition of $\mathcal{M}_3$, we have $R_2^r(u) \subseteq W_3$. Then, we have $v \in W_3$. Moreover, as $w \sqsubset_2 uR_2^r v$, by Axiom 4b if $u \neq v$, we have $w \sqsubset_2 v$.

In both cases, we obtain $v \in W_3$ such that $w \sqsubset_2 v$ and $\mathcal{M}_2, v \models \neg\psi$. Then, since $W_3 \subseteq W_4$, we have $v \in W_4$. Moreover, since $\mathcal{M}_4 \subseteq \mathcal{M}_2$, we have $w \sqsubset_4 v$. Finally, by inductive hypothesis, we have $\mathcal{M}_4, v \models \neg\psi$. Then, $\mathcal{M}_4, w \models \neg G\psi$. The case $H\psi$ can be proved similarly. □

Comparison with the proof of [53, Lemma 6.23]: The proof is the same as that of [53, Lemma 6.23] with the only slight differences that we use Remark 12.4 instead of [53, Lemma 3.3], and that when, for a point $w \in W_2$, they refer to $R_2(w)$, we refer to $R_2^r(w)$ instead. This is due to the fact that our R_2 is generally not reflexive.

We now prove that $\mathcal{M}_4$ is as desired:

Proposition 12.27. (Cf. [53, Lemma 6.24].) $\mathcal{M}_4$ is a Ψ-linked model satisfying φ_0.

Proof. We follow the proof of [53, Lemma 6.24]. We show that conditions 1-5 of Definition 12.2 hold.

1. W_4 is plainly finite. 2. We show that the frame $(W_4, \sqsubset_4, R_4)$ validates all the axioms in Λ_d, excluding Axioms 5 and 6. Now, all the axioms in Λ_d, excluding Axioms 5 and 6, are valid in the frame of $\mathcal{M}_2$. Then, their first-order correspondents are true in the frame of $\mathcal{M}_2$. Now, the correspondents of these axioms, except 2c and 4d, are universal first-order sentences. Then, since the frame of $\mathcal{M}_4$ is a substructure of the frame of $\mathcal{M}_2$, their truth is preserved in the frame of $\mathcal{M}_4$. Moreover, the correspondent of Axiom 4d is preserved by R_2-generated subframes of $\mathcal{M}_2$. Then, it remains true in the frame of $\mathcal{M}_4$. Finally, the correspondent of Axiom 2c is density for $\sqsubset$. Now, the frame of $\mathcal{M}_4$ consists of $\sqsubset_4$-clusters with possibly single $\sqsubset_4$-irreflexive points between pairs of consecutive $\sqsubset_4$-clusters. Then, $\sqsubset_4$ is plainly dense.

3. Since $\mathcal{M}_2$ is prelinear and $\mathcal{M}_4 \subseteq \mathcal{M}_2$, we have that $\sqsubset_4$ is prelinear as well. 4. Let C be a $\sqsubset_4$-cluster of $\mathcal{M}_4$. Then, since $\mathcal{M}_4$ is good, C is a $\sqsubset_2$-cluster of $\mathcal{M}_2$ as well. Then, by Proposition 12.15, $(C, R_2{\restriction}C)$ is connected and locally 2-connected. Now, since $\mathcal{M}_4 \subseteq \mathcal{M}_2$, we have $(C, R_2{\restriction}C) = (C, R_4{\restriction}C)$. Then, $(C, R_4{\restriction}C)$ is connected and locally 2-connected as well.

5. Since $\mathcal{M}_4$ is perfect, for every pair (C, D) of adjacent $\sqsubset_4$-clusters, there are R_4-reflexive $c \in C$ and $d \in D$ such that $c \equiv_{\mathcal{B}\Psi}^{\mathcal{M}_2} d$. Now, by Proposition 12.26, $c \equiv_{\mathcal{B}\Psi}^{\mathcal{M}_4} d$. Then, $\mathcal{M}_4$ is Ψ-linked.

We show that $\mathfrak{M}_4$ satisfies φ_0. By Proposition 12.11, $\mathcal{M}_2$ satisfies φ_0. Consider an arbitrary $w \in C_{+\infty}/\sim$. Then, by Proposition 12.6(5) and definition of $\sqsubset_2$, $\mathcal{M}_2, w \models P\varphi_0$. Now, by definition of Ψ, we have $P\varphi_0 \in \Psi$. Then, since $w \in W_4$, by Proposition 12.26, we have $\mathcal{M}_4, w \models P\varphi_0$. Then, φ_0 is satisfied in $\mathcal{M}_4$. □

In fact, V_4 has Φ^k as domain, but it is harmless to extend V_4 to Φ by defining $V_4(p) = \emptyset$ for every $p \in \Phi \setminus \Phi^k$.

12.3 Satisfiability on $\mathfrak{R}$

In this section we show that if a formula is satisfiable on a Ψ-linked model, then it is satisfiable on $\mathfrak{R}$ (cf. [53, §7]).

Proposition 12.28. (Cf. [53, Theorem 7.11].) Given a set of formulas Ψ closed under taking subformulas and such that $\varphi \in \Psi$, if φ is satisfiable on a Ψ-linked model, then φ is satisfiable on $\mathfrak{R}$.

To prove Proposition 12.28 we closely follow [53, §7] with the only exception that we use [52, Lemma 6.1] instead of [53, Lemma 7.10] (observe that the latter is in turn an adaption of [52, Definition 4.1 and Lemma 4.2]).

We report and prove [52, Lemma 6.1] as Proposition 12.38. To prove it we need to introduce the notions of 'lexicographic sums of linear orders', 'lexicographic sums of functions on linear orders', and most importantly of 'shuffles'. We do so in Sections 12.3.1, 12.3.2, 12.3.3.

12.3.1 Lexicographic sums of linear orders

In this section we define 'lexicographic sums of linear orders'.

Given a tuple $\mathcal{A} = (A, R)$, if R is linear (see Chapter 4, Definition 4.1(3)) we say that $\mathcal{A}$ is a *linear order*. An *interval* of $\mathcal{A}$ is a non-empty subset B of A such that for every $a, b, c \in A$, if $a, c \in B$ and $aRbRc$, then $b \in B$. Given $a, b \in \mathcal{A}$, define $(a, b) := \{c \in A \,|\, aRcRb\}$, $[a, b) := \{c \in A \,|\, aR^r cRb\}$, $(a, b] := \{c \in A \,|\, aRcR^r b\}$, $[a, b] := \{c \in A \,|\, aR^r cR^r b\}$, $(-\infty, a) := \{c \in A \,|\, cRa\}$ and $(a, +\infty) := \{c \in A \,|\, aRc\}$.

Let $\mathcal{J} = (J, <_{\mathcal{J}})$ be a linear order and, for every $j \in J$, let $\mathcal{I}_j = (I_j, <_{\mathcal{I}_j})$ be a linear order as well. Define:

$$\sum_{j \in J} \mathcal{I}_j := (\sum_{j \in J} I_j, <)$$

where $\sum_{j \in J} I_j := \{(i, j) \,|\, j \in J, i \in I_j\}$ and $<$ is a binary relation on $\sum_{j \in J} I_j$ defined as follows: $(i, j) < (i', j')$ provided $j <_{\mathcal{J}} j'$ or ($j = j'$ and $i <_{\mathcal{I}_j} i'$). $\sum_{j \in J} \mathcal{I}_j$ is a linear order. If $(J, <_{\mathcal{J}}) = (\{0, 1, ..., n\}, <)$, we may write the sum as $\mathcal{I}_0 + \mathcal{I}_1 + ... + \mathcal{I}_n$.

Example 12.29. Let $\mathcal{J}$ be $0, 1$, $\mathcal{I}_0$ be i, ii, and $\mathcal{I}_1$ be ii, iii. Then, $\sum_{j \in J} \mathcal{I}_i = \mathcal{I}_0 + \mathcal{I}_1$ is

$$(i, 0), (ii, 0), (ii, 1), (iii, 1)$$

(shown in order, reading from left to right).

Suppose that for each $j \in J$, I_j is an interval of $\mathbf{R}$. Then:

Proposition 12.30. Suppose one of the following conditions holds:

1. $J = \{0, 1, ..., n\}$ for some $n \in \mathbf{N}$. $\mathcal{I}_0$ has no least and no greatest element, and, for every $j > 0$, $\mathcal{I}_j$ has a least and no greatest element.

2. $(J, <_{\mathcal{J}}) = (\mathbf{Z}, <)$ and, for every $j \in J$, $\mathcal{I}_j$ has no least element and has a greatest element.

3. $(J, <_{\mathcal{J}}) = (\mathbf{R}, <)$ and, for every $j \in J$, $\mathcal{I}_j$ has a least and a greatest element, and, for every irrational j, $\mathcal{I}_j$ is a singleton.

Then, $\sum_{j \in J} \mathcal{I}_j$ is isomorphic to $(\mathbf{R}, <)$, notation $\sum_{j \in J} \mathcal{I}_j \cong \mathbf{R}$.

Sketch of the proof. It is well known that a linear order $\mathcal{I}$ is isomorphic to $(\mathbf{R}, <)$ if and only if the following conditions hold:

- $\mathcal{I}$ is dense.
- $\mathcal{I}$ has no endpoints.
- $\mathcal{I}$ has a countable dense subset.
- $\mathcal{I}$ is Dedekind complete.

It is easy and we leave to the reader to check that the linear orders given by Points 1-3 satisfy these conditions (see [82] and [21] for further reading). □

When $\sum_{j \in J} \mathcal{I}_j \cong (\mathbf{R}, <)$, we may identify the two structures.

12.3.2 Lexicographic sums of functions on linear orders

In this section we define 'lexicographic sums of functions on linear orders'.

Let A be a non empty set and, for every $j \in J$, $f_j : I_j \to A$ a function. Define

$$\sum_{j \in J} f_j : \sum_{j \in J} I_j \to A$$

by $(i, j) \mapsto f_j(i)$. If $(J, <_{\mathcal{J}}) = (\{0, 1, ..., n\}, <)$, we may write the sum as $f_0 + f_1 + ... + f_n$. If I_j is a singleton $\{x\}$ and $f_j(x) = a$, we may write the function f_j as a. If $g = \sum_{j \in J} f_j$, define $\mathrm{dom}_g(f_j) := \{(i, j) \mid i \in I_j\}$. In what follows, it may be convenient to identify f_j with $g \restriction \mathrm{dom}_g(f_j)$.

Example 12.31. Let $\mathcal{J}$, $\mathcal{I}_0$ and $\mathcal{I}_1$ be as in Example 12.29. Consider a set $A = \{a, b\}$ and functions $f_0 : I_0 \to A$ defined by:

$$i \mapsto a,$$
$$ii \mapsto b,$$

and $f_1 : I_1 \to A$ defined by:

$$ii \mapsto a,$$
$$iii \mapsto a.$$

Then $\sum_{j\in J} f_j = f_0 + f_1$ is:

$$(i,0) \mapsto a,\ (ii,0) \mapsto b,\ (ii,1) \mapsto a,\ (iii,1) \mapsto a.$$

12.3.3 Shuffles

In this section we define a special case of lexicographic sum of functions on linear orders, called 'Shuffle'.

Let A be a non-empty set, and $\mathcal{G}$ a countable (possibly empty) set of functions $g : K_g \to A$ such that K_g is an interval of $\mathbf{R}$ with a least and a greatest element. Let $g_0 : K_{g_0} \to A$ be a function such that K_{g_0} is a singleton interval of $\mathbf{R}$. Let $\vartheta : \mathbf{R} \to \mathcal{G} \cup \{g_0\}$ be a function such that:

- $\vartheta^{-1}(g_0) = \mathbf{R} \setminus \mathbf{Q}$.
- For every $g \in \mathcal{G}$, $\vartheta^{-1}(g)$ is a dense subset of $\mathbf{Q}$.

It is not difficult to prove that such a ϑ exists. Then, for every $g \in \mathcal{G} \cup \{g_0\}$, we have that $\vartheta^{-1}(g)$ is a dense subset of $\mathbf{R}$. Now, for every $j \in \mathbf{R}$, define $I_j = K_{\vartheta_j}$ and:

$$I := \sum_{j\in\mathbf{R}} I_j.$$
$$\sigma : \sum_{j\in\mathbf{R}} \vartheta_j : I \to A.$$

Then, $\sigma((i, j)) = (\vartheta(j))(i) \in A$.

We call *σ-endpoint* any element $x \in I$ such that $x = (i, j)$ with $j \in \mathbf{R}$ and i the least or greatest element of I_j.

Proposition 12.32. (Cf. [53, Lemma 7.3].) Let I and σ be defined as above. Then, for every $x, y, z \in \mathbf{R}$ such that $y < x < z$ and x a σ-endpoint, we have $\mathrm{rng}\sigma = \sigma((y, z) \setminus \{x\})$.

Proof. We follow the proof of [53, Lemma 7.3]. That $\sigma((y, z) \setminus \{x\}) \subseteq \mathrm{rng}\sigma$ is trivial. We show that $\mathrm{rng}\sigma = \sigma((y, z) \setminus \{x\})$. Consider an arbitrary element $a \in \mathrm{rng}\sigma$. Then, there exist $g \in \mathcal{G} \cup \{g_0\}$ and $k \in I_g$ such that $g(k) = a$. Suppose that $x = (i, j)$ and that i is the least element of I_j. Now, y is of the form (i', j') for some $j' \in \mathbf{R}$ and $i' \in I_{j'}$. Then, since $y < x$ and x is the least element of I_j, we have $j' < j$. Now, $\vartheta^{-1}(g)$ is dense in $\mathbf{R}$. Then, there exists $j^* \in \vartheta^{-1}(g)$ such that $j' < j^* < j$. Then, $x \neq (k, j^*)$, $y < (k, j^*) < z$, and $a = \sigma((k, j^*)) \in \sigma((y, z) \setminus \{x\})$. □

As observed in [53, §7.3], by Proposition 12.30(3), we have $(I, <) \cong (\mathbf{R}, <)$. Then, by choosing a suitable isomorphism, we car regard σ as a function $\sigma : \mathbf{R} \to W$. This function depends on the choice of ϑ and of the isomorphism between $(I, <)$ and $(\mathbf{R}, <)$. However, for every choice of ϑ and of the isomorphism between $(I, <)$ and $(\mathbf{R}, <)$, we obtain the same result, modulo an order preserving automorphism of $\mathbf{R}$. Then, we let $\mathrm{Shuffle}(\mathcal{G}; g_0)$ denote a function $\sigma : \mathbf{R} \to W$ for an arbitrary choice of ϑ and of the isomorphism between $(I, <)$ and $(\mathbf{R}, <)$. The elements of $\mathcal{G} \cup \{g_0\}$ are called the *ingredients* of the shuffle.

Example 12.33. If $W = \{w, v, u\}$, then $\mathrm{Shuffle}\{\{w, v\}; u\}$ will be a map $\sigma : \mathbf{R} \to W$ such that $\sigma^{-1}(w)$ and $\sigma^{-1}(v)$ are two dense subsets of $\mathbf{Q}$ and they form a partition of $\mathbf{Q}$, whereas $\sigma^{-1}(u) = \mathbf{R} \setminus \mathbf{Q}$.

More generally, given two linear orders $\mathcal{I}_1 = (I_1, <_1)$ and $\mathcal{I}_2 = (I_2, <_2)$, a set A, and two functions $f_1 : I_1 \to A$ and $f_2 : I_2 \to B$, if there is an isomorphism i from $\mathcal{I}_1$ to $\mathcal{I}_2$, and for every $x \in I_1$ we have $f_1(x) = f_2(i(x))$, then we say that f_1 *is isomorphic to* f_2, write $f_1 \cong f_2$, and we may identify f_1 with $f_2 \circ i^{-1}$.

12.3.4 [d]: from Kripke models to R

In this section, we report and prove [52, Lemma 6.1] as Proposition 12.38. This is a substantial proposition, closely related to Shehtman's result that the d-logic of $\mathbf{R}$ is $\mathbf{KD4G_2}$ [88]. Essentially, it enables to translate the semantics of the coderivative operator from Kripke models to models on the reals. This proposition will be crucial in proving the main Proposition 12.28, according to which if a formula is satisfiable on a Ψ-linked model, then it is satisfiable on $\mathfrak{R}$, in Section 12.3.5. But first let us give some preliminary definitions.

Definition 12.34. Given a tuple $\mathcal{F} = (W, R)$ such that W is a set and R is a binary relation on W, a linear order $(I, <)$, a function $g : I \to W$, and an element $x \in I$, we say that x is *g-fair* (*with respect to* $\mathcal{F}$) provided there are $y, z \in I$ such that $y < x < z$ and, for all y', z' with $y \le y' < x < z' \le z$, we have $g((y', z') \setminus \{x\}) = R(g(x))$.

Our definition of fairness slightly differs from [53, Definition 7.7]: in our definition the point to which the property of fairness refers is excluded, whereas in [53, Definition 7.7.] it is not. This reflects the difference between the semantics of [d] and of $\Box$.

Remark 12.35. (Cf. [53, Remark 7.8].) If $\mathcal{F}$ is an R-generated subframe of a $\mathbf{K4}$-frame $\mathcal{G} = (W, R)$, $(I, <)$ a linear order, and g a function from I to $\mathcal{F}$, then a point in I is g-fair with respect to $\mathcal{F}$ if and only if it is g-fair with respect to $\mathcal{G}$.

Remark 12.36. (Cf. [53, Remark 7.9].) Fairness is a 'local' property: if $g = \sum_{j \in J} f_j$, $j \in J$, and $x \in \mathrm{dom}_g(f_j)$ is in the interior of $\mathrm{dom}_g(f_j)$ (that is, if x is not the least or greatest point of $\mathrm{dom}_g(f_j)$), then x is g-fair if and only if it is f_j-fair. (Recall that we identify f_j with $g{\restriction}\mathrm{dom}_g(f_j)$.)

Definition 12.37. Consider a tuple $\mathcal{F} = (W, R)$ such that W is a finite set, R a transitive binary relation on W, and $\mathcal{F}$ connected and locally 2-connected. For every $w \in W$, define $R^\bullet(w) := \{v \in W \mid wRv \wedge \neg vRw\}$. For every $w \in W$, we say that w is a *leaf* provided $R^\bullet(w) = \emptyset$. For every $w \in W$, define $W_w := R^r(w)$ and $\mathcal{F}_w := (W_w, R{\restriction}W_w)$. Observe that $\mathcal{F}_w$ is a generated, connected and locally 2-connected subframe of $\mathcal{F}$.

Proposition 12.38. (Cf. [53, Theorem 7.10] and [52, Lemma 6.1].) For every frame $\mathcal{F} = (W, R)$ such that W is a finite set, R a serial and transitive binary relation on W, and $\mathcal{F}$ is connected and locally 2-connected, there is a function $g_{\mathcal{F}} : \mathbf{R} \to W$ such that, for every $x \in \mathbf{R}$ and $v \in W$:

1. x is $g_{\mathcal{F}}$-fair.

2. $g_{\mathcal{F}}(-\infty, x) = g_{\mathcal{F}}(x, +\infty) = W$.

3. If v is a leaf, then $g_{\mathcal{F}_v}$ is defined inductively and there are non-empty functions g_A, g_B such that $g_{\mathcal{F}} \cong g_A + g_{\mathcal{F}_v} + g_B$.

Figure 12.4: $g_{\mathcal{F}} \cong g_A + g_{\mathcal{F}_v} + g_B$

Proof. By induction on $|\mathcal{F}|$. Two cases are given:

(a) Suppose that there exists $w \in W$ such that w is reflexive and $\mathcal{F} = \mathcal{F}_w$. Define:

$$g_{\mathcal{F}} := \mathrm{Shuffle}(\{w + g_{\mathcal{F}_u} + w \mid u \in R^\bullet(w)\} \cup \{u \mid u \in C(w)\}; w). \tag{12.7}$$

Now, if $u \in R^\bullet(w)$ then $|\mathcal{F}_u| < |\mathcal{F}_w|$. Then, by inductive hypothesis $g_{\mathcal{F}_u}$ is well defined. Then, $g_{\mathcal{F}}$ is well defined. We show that $g_{\mathcal{F}}$ satisfies Points 1-3.

1. Suppose that $x \in g_{\mathcal{F}}^{-1}(u)$ and $u \in R^\bullet(w)$. Then 1 holds by inductive hypothesis and Remark 12.36. Otherwise, $x \in g_{\mathcal{F}}^{-1}(u)$ and $u \in C(w)$ (since w is reflexive, $w \in C(w)$ and the case $x \in g_{\mathcal{F}}^{-1}(w)$ is also covered). Then, x is a $g_{\mathcal{F}}$-endpoint. Then, by Proposition 12.32, for every $y, z \in \mathbf{R}$ such that $y < x < z$, we have:

$$g_{\mathcal{F}}((y, z) \setminus \{x\}) = \mathrm{rng}(g_{\mathcal{F}}) = R(w) = R(u).$$

2. Consider an arbitrary $x \in \text{dom} g_{\mathcal{F}}$ and $v \in W$. Then, since $W = W_w$, we have that either $v = u$ for some $u \in R^\bullet(w)$ or $v = u \in C(w)$. By definition of shuffle and equation (12.7) defining $g_{\mathcal{F}}$, in both cases we find $y, z \in \text{dom}(g_{\mathcal{F}})$ such that $g_{\mathcal{F}}(y) = g_{\mathcal{F}}(z) = v$.

3. Suppose that $v \in W = W_w$ is a leaf. If $v \in R^\bullet(w)$, then 3 holds for v because $g_{\mathcal{F}_v}$ is an ingredient of the shuffle in (12.7) defining $g_{\mathcal{F}}$. Otherwise, $\neg w R^\bullet v$. Then, $v \in C(w)$. Then, $\mathcal{F}_v = \mathcal{F}_w = \mathcal{F}$. Now, since $g_{\mathcal{F}}$ is a shuffle, as the reader may confirm, we have $g_{\mathcal{F}} \cong g_A + g_{\mathcal{F}} + g_B$ for some g_A and g_B.

(b) Suppose the assumption of Case (a) does not hold. Then, since $\mathcal{F}$ is connected and locally 2-connected, a bit of work shows that there is a sequence

$$...u_{-1}d_0u_0d_1u_1d_2...$$

of elements of W such that:

- For every $i \in \mathbf{Z}$, we have u_iRd_i and u_iRd_{i+1}.
- For every $i \in \mathbf{Z}$, we have $W = \bigcup_{j<i} W_{u_j}$ and $W = \bigcup_{j>i} W_{u_j}$.
- $\{d_i \mid i \in \mathbf{Z}\}$ is the set of all leaves of $\mathcal{F}$.
- For every $i \in \mathbf{Z}$, if C_i and D_i are the connected components of the frame $(R(u_i), R{\restriction}R(u_i))$ that contain d_i and d_{i+1} respectively, then $R(u_i) = C_i \cup D_i$.

Let us sketch how we can find such a sequence. Since $\mathcal{F}$ is connected, there is a cycle $d_0u_0d_1...d_{n-1}u_nd_n$ such that $d_0 = d_n$ and:

- For every $i \in n$, we have u_iRd_i and u_iRd_{i+1}.
- $W = \bigcup_{i\in n} W_{u_i}$.
- The leaves of $\mathcal{F}$ are among $d_0, d_1, ..., d_n$.

If d_i is not a leaf, then there is d_i' such that d_i' is a leaf and d_iRd_i'. Then, we can replace d_i with d_i'. Then, we can assume that all the d_i are leaves. Consider u_i for an arbitrary $i \leq n$. Since $\mathcal{F}$ is locally 2-connected, $(R(u_i), R{\restriction}R(u_i))$ can be partitioned in two (possibly equal) connected components C, D. If $d_i \in C$ and $d_{i+1} \notin D$, choose a leaf d in D, and replace the subsequence $d_iu_id_{i+1}$ with $d_iu_idu_id_{i+1}$. Do this for every $i \leq n$. At the end, we obtain a cycle $d_0u_0d_1...d_{m-1}u_{m-1}d_m$ such that $d_0 = d_m$. The sequence:

$$...d_mu_0d_1...d_{m-1}u_{m-1}d_mu_0d_1...d_{m-1}u_{m-1}d_mu_0...$$

is as desired.

Now, for every $i \in \mathbf{Z}$, if we define $C_i := (C_i, R{\restriction}C_i)$, we obtain a frame satisfying the hypotheses of this proposition. We can define a frame $\mathcal{D}_i$ on D_i similarly. If

$C_i = \mathcal{F}$, we would have $\mathcal{F} = C_i \subseteq R(u_i) \subseteq \mathcal{F}$. Then, $\mathcal{F} = R(u_i)$. Then, since $u_i \in \mathcal{F}$, we would have $u_i \in R(u_i)$. Then, u_i would be reflexive and $\mathcal{F} = \mathcal{F}_{u_i}$, a contradiction of the assumptions of Case (b). Then, we have $|C_i| < |\mathcal{F}|$ and similarly for $\mathcal{D}_i$. Then, we can apply the inductive hypothesis on C_i and $\mathcal{D}_i$ obtaining functions g_{C_i} and $g_{\mathcal{D}_i}$ respectively. Now, since $d_i \in C_i$ and $d_{i+1} \in D_i$, and d_i and d_{i+1} are leaves, then, by the inductive hypothesis, there are functions g_{A_i}, g_{B_i}, $g_{A'_i}$, and $g_{B'_i}$ such that:

$$
\begin{aligned}
g_{C_i} &\cong g_{A_i} + g_{\mathcal{F}_{d_i}} + g_{B_i}, \\
g_{\mathcal{D}_i} &\cong g_{A'_i} + g_{\mathcal{F}_{d_{i+1}}} + g_{B'_i}.
\end{aligned}
$$

Then, since $\mathrm{dom}(g_{C_i}) = \mathrm{dom}(g_{\mathcal{F}_{d_i}}) = \mathrm{dom}(g_{\mathcal{D}_i}) = \mathrm{dom}(g_{\mathcal{F}_{d_{i+1}}}) = \mathbf{R}$, we have that $\mathrm{dom}(g_{A_i})$ has a greatest element and no least element, $\mathrm{dom}(g_{B_i})$ has a least element and no greatest element, and similarly for $\mathrm{dom}(g_{A'_i})$ and $\mathrm{dom}(g_{B'_i})$. Define:

$$
g_{\mathcal{F}} := \sum_{i \in \mathbf{Z}} (g_{\mathcal{F}_{d_i}} + g_{B_i} + u_i + g_{A'_i}). \tag{12.8}
$$

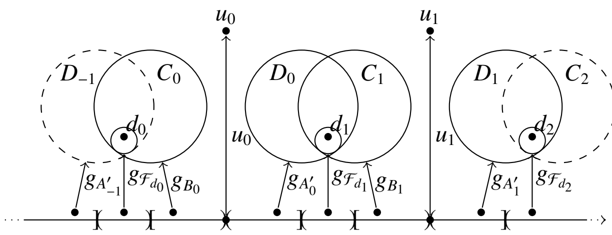

Figure 12.5: $g_{\mathcal{F}} := \sum_{i \in \mathbf{Z}} (g_{\mathcal{F}_{d_i}} + g_{B_i} + u_i + g_{A'_i})$

Now, for each $i \in \mathbf{Z}$, $\mathrm{dom}(g_{\mathcal{F}_{d_i}} + g_{B_i} + u_i + g_{A'_i})$ is plainly isomorphic to an interval of $\mathbf{R}$ with no least element and with greatest element. Then, by Proposition 12.30(2), we can assume $\mathrm{dom}(g_{\mathcal{F}}) = \mathbf{R}$. We show that $g_{\mathcal{F}}$ satisfies $1 - 3$.

1. Consider an arbitrary $x \in \mathrm{dom}(g_{\mathcal{F}})$. Suppose that $x \in \mathrm{dom}(u_i)$ for some $i \in \mathbf{Z}$. Then, by (12.8), x lies just after $\mathrm{dom}(g_{B_i})$. Now, $\mathrm{dom}(g_{B_i})$ is isomorphic to a final part of $\mathrm{dom}(g_{C_i})$. Consider an arbitrary $y \in \mathrm{dom}(g_{B_i})$ such that $y < x$. Then, by 2 for C_i, we have $g_{\mathcal{F}}((y, x)) = C_i$. Then, there exists $y \in \mathbf{R}$ such that $y < x$ and, for every $y' \in \mathbf{R}$ such that $y \le y' < x$, we have $g_{\mathcal{F}}((y', x)) = C_i$. Similarly, there is $z \in \mathbf{R}$ such that $x < z$ and, for every $z' \in \mathbf{R}$ such that $x < z' \le z$, we have $g_{\mathcal{F}}((x, z')) = D_i$. Then, there are $y, z \in \mathbf{R}$ such that $y < x < z$ and, for every $y', z' \in \mathbf{R}$ such that $y \le y' < x < z' \le z$, we have:

$$
g_{\mathcal{F}}((y', z') \setminus \{x\}) = C_i \cup D_i = R(u_j) = R(g_{\mathcal{F}}(x)).
$$

Then, x is $g_{\mathcal{F}}$-fair.

Otherwise, if $x \notin \mathrm{dom}(u_i)$ for every $i \in \mathbf{Z}$, then x lies in $\mathrm{dom}(g_{C_i})$ or $\mathrm{dom}(g_{\mathcal{D}_i})$ for some $i \in \mathbf{Z}$. Then, by inductive hypothesis and Remark 12.36, 1 holds for such an x.

2. Consider arbitrary $x \in \mathbf{R}$ and $v \in \mathcal{F}$. Suppose that x is in $\mathrm{dom}(g_{\mathcal{F}_{d_i}} + g_{B_i} + u_i + g_{A'_i})$ for some $i \in \mathbf{Z}$. Then, by assumption on the u_i, there are $j, k \in \mathbf{Z}$ such that $j < i < k$ and $v \in W_{u_j} \cap W_{u_k}$. Consider $y \in \mathrm{dom} u_j$. Then, $y < x$. If $v = u_j$, then plainly $g_{\mathcal{F}}(y) = v$. If $v \neq u_j$, then $v \in R(u_j)$. Then, by 1, there are $z, t \in \mathbf{R}$ such that $z < y < t < x$ and $g_{\mathcal{F}}((z, t) \setminus \{y\}) = R(u_j)$. Then, in both cases, we find $y \in \mathbf{R}$ such that $y < x$ and $g_{\mathcal{F}}(y) = v$. We similarly find $y \in \mathbf{R}$ such that $x < y$ and $g_{\mathcal{F}}(y) = v$.

3. By assumption on the d_i, every leaf v of $\mathcal{F}$ is equal to some d_j. Then, we conclude by observing that, by (12.8), $\mathrm{dom}(g_{\mathcal{F}_{d_i}})$ occurs as an interval in $\mathrm{dom}(g_{\mathcal{F}})$. □

Comparison with [52, Lemma 6.1]: Our proof follows exactly that of [52, Lemma 6.1]. However the thesis is written in a different form. In [52, Lemma 6.1] the thesis is written according to the following observation: given a Kripke frame $\mathcal{F} = (W, R)$, and a function $g : \mathbf{R} \to W$, if we take W as the set of propositional variables and $\overline{g} : W \to \mathcal{P}(\mathbf{R})$ is defined as $\overline{g}(w) = g^{-1}(w)$, then $(\mathbf{R}, \overline{g})$ is a model.

12.3.5 $G, H, [d]$: from Ψ-linked models to $\mathbf{R}$

We are ready to prove Proposition 12.28, according to which if a formula is satisfiable on a Ψ-linked model, then it is satisfiable on $\mathfrak{R}$.

Let $\mathcal{M} = (W, \sqsubset, R, V)$ be a Ψ-linked model. Then, for every $\sqsubset$-cluster C of $\mathcal{M}$, $(C, R{\restriction}C)$ satisfies the hypotheses of Proposition 12.38. Moreover, prelinearity of $\sqsubset$ yields an enumeration without repetition $\{C_i \mid i = 0, 1, ..., k\}$ of the $\sqsubset$-clusters of $\mathcal{M}$ with $C_0 \sqsubset C_1 \sqsubset ... \sqsubset C_k$ (for k some natural number). As observed, there are two cases:

1. C_i and C_{i+1} are not adjacent. Then, there is a unique $\sqsubset$-irreflexive u_i with $C_i \sqsubset u_i \sqsubset C_{i+1}$, $C_i = \lambda(u_i)$ and $C_{i+1} = \rho(u_i)$. Let us say that i is *open*, C_i *right-open* and C_{i+1} *left-open*.

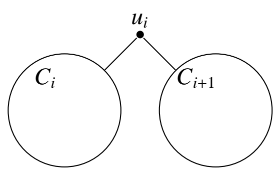

Figure 12.6: C_i and C_{i+1} are not adjacent

2. C_i and C_{i+1} are adjacent. Then, there are R-reflexive points $d_i \in C_i$ and $s_{i+1} \in C_{i+1}$ with $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1}$. Let us say that i is *closed*, C_i *right-closed* and C_{i+1} *left-closed*.

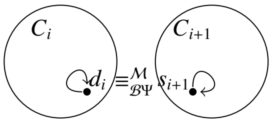

Figure 12.7: C_i and C_{i+1} are adjacent

Moreover we say that C_0 is left-open and C_k is right-open.

Definition 12.39. Consider a sequence of elements of $\mathbf{R} \cup \{-\infty, +\infty\}$:

$$-\infty = l_0 < r_0 \leq l_1 < r_1 \leq l_2 < ... \leq l_k < r_k = +\infty$$

such that $r_i = l_{i+1}$ if and only if i is open. For every $i \leq k$, define f_i^* as follows:

1. If C_i is left-open and right-open, let $f_i^* : (l_i, r_i) \to C_i$ be a function given by Proposition 12.38.

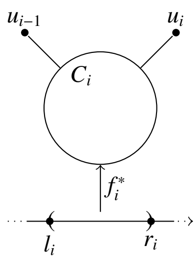

Figure 12.8: Definition 12.39(1)

2. If C_i is left-open and right-closed, let $f_i^* : (l_i, +\infty) \to C_i$ be a function given by Proposition 12.38. Since f_i^* is surjective, there is $z \in \mathbf{R}$ such that $f_i^*(z) = d_i$. Then, the function

$$x \mapsto f_i^*\left(l_i + \frac{(x - l_i)(z - l_i)}{r_i - l_i}\right) : (l_i, +\infty) \mapsto C_i$$

has the properties provided by Proposition 12.38 and additionally maps r_i to d_i. Then, by replacing f_i^* by this scaled version, we can suppose, without loss of generality, that $f_i^*(r_i) = d_i$.

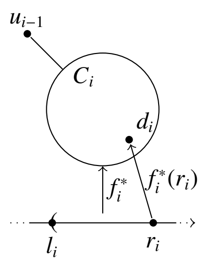

Figure 12.9: Definition 12.39(2)

3. If C_i is left-closed and right-open, by a specular argument, let $f_i^* : (-\infty, r_i) \to C_i$ be a function given by Proposition 12.38. By a scaling similar to that of Point 2, we can suppose, without loss of generality, that $f_i^*(l_i) = s_i$.

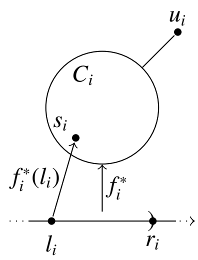

Figure 12.10: Definition 12.39(3)

4. If C_i is left-closed and right-closed, let $f_i^* : \mathbf{R} \to C_i$ be a function given by Proposition 12.38. Then, there are $x < y$ in $\mathbf{R}$ with $f_i^*(x) = s_i$, $f_i^*(y) = d_i$ and, since C_i is finite, $f_i^*((x, y)) = C_i$. By a scaling similar to that of Point 2, we can assume $x = l_i$ and $y = r_i$.

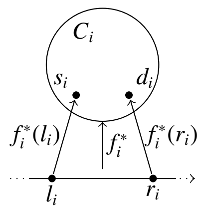

Figure 12.11: Definition 12.39(4)

For every $i \leq k$, define f_i as $f_i^* \restriction (l_i, r_i)$.

Reasoning as in [53, Lemmas 7.13-15], we prove:

Proposition 12.40. (Cf. [53, Lemmas 7.13].) For every $i \leq k$ and $x \in (l_i, r_i)$, we have:

1. If C_i is left-open, then $f((l_i, x)) = C_i$.

2. If C_i is right-open, then $f((x, r_i)) = C_i$.

Proof. Suppose that C_i is left-open. Then, $f_i((l_i, x)) \subseteq \mathrm{rng}(f_i^*) = C_i$. Conversely, consider an arbitrary $w \in C_i$. Now, since C_i is left-open, we have $\mathrm{dom}(f_i^*) = (l_i, z)$ for $z = r_i$ or $z = +\infty$. Moreover, by Proposition 12.38(2), there is $y \in (l_i, x)$ with $f_i^*(y) = w$. Then, since $f_i = f^* \restriction (l_i, r_i)$, we have $f_i(y) = w$. Then, $w \in f_i((l_i, x))$. Then, since w was arbitrary, we have $C_i \subseteq f_i((l_i, x))$. Therefore, $C_i = f_i((l_i, x))$ as desired. This proves 1. We can prove 2 by a specular argument. □

Proposition 12.41. (Cf. [53, Lemma 7.14].) For every $i \leq k$, the function $f_i : (l_i, r_i) \to C_i$ is surjective.

Proof. If C_i is both left- and right-closed, then the thesis follows from the definition of f_i^*. Otherwise, the thesis follows from Proposition 12.40. □

Proposition 12.42. (Cf. [53, Lemma 7.15].) For every $i < k$, if i is closed, then:

1. There is $y \in (l_i, r_i)$ such that $f_i((y, r_i)) \subseteq R(d_i)$.

2. There is $z \in (l_{i+1}, r_{i+1})$ such that $f_i((l_{i+1}, z)) \subseteq R(s_{i+1})$.

Proof. If i is closed then C_i is right-closed, C_{i+1} is left-closed and d_i, s_{i+1} are defined. Now, by Proposition 12.38(2), r_i is f_i^*-fair. Then, there are $y, z \in \mathrm{dom} f_i^* = (l_i, +\infty)$ such that $y < r_i < z$ and $f_i^*((y, z) \setminus \{r_i\}) = R(f^*(r_i)) = R(d_i)$. Then, $f_i((y, r_i)) = f_i^*((y, r_i)) \subseteq f_i^*((y, z) \setminus \{r_i\}) = R(d_i)$. We can find z by a specular argument. □

In the following definition, our condition on d_i to be R-reflexive is used. In [53, Definition 7.16], there is no need of this condition because R is reflexive.

Definition 12.43. For each $i < k$, define functions $f_i' : [r_i, l_{i+1}] \to W$ as follows:

1. If i is open, then $r_i = l_{i+1}$. Define $f'(r_i) = u_i$.

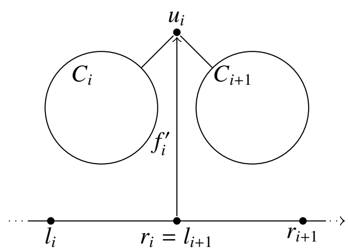

Figure 12.12: Definition 12.43(1)

2. If i is closed, then C_i is right-closed, C_{i+1} is left-closed and d_i and s_{i+1} are defined. Now, $(R(d_i), R\restriction R(d_i))$ is a finite, transitive frame. Moreover, since d_i is R-reflexive, $(R(d_i), R\restriction R(d_i))$ is connected (here is where we use the R-reflexivity

of d_i). Furthermore, since $(C_i, R{\upharpoonright}C_i)$ is locally 2-connected, $(R(d_i), R{\upharpoonright}R(d_i))$ is locally 2-connected as well. Finally, $(r_i, l_{i+1}) \cong (\mathbf{R}, <)$. So we can apply Proposition 12.38. Consider a function $f'_i : (r_i, l_{i+1}) \to R(d_i)$ given by Proposition 12.38. Moreover, define $f'_i(r_i) := d_i$ and $f'_i(l_{i+1}) := s_{i+1}$.

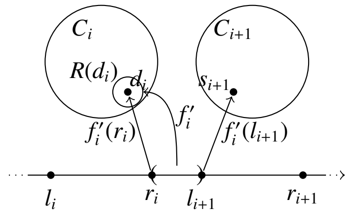

Figure 12.13: Definition 12.43(2)

Reasoning as in [53, Lemma 7.17] we show that:

Proposition 12.44. (Cf. [53, Lemma 7.17].) If $i < k$ is closed and $x \in (r_i, l_{i+1})$, then $f'_i((r_i, x)) = f'_i((x, l_{i+1})) = R(d_i)$.

Proof. The thesis is an immediate consequence of Proposition 12.38(2). □

Definition 12.45. Define $g := \bigcup_{i \le k}(f_i \cup f'_i) : \mathbf{R} \to W$ - that is, for every $x \in \mathbf{R}$, we have:

$$g(x) := \begin{cases} f_i(x) & \text{if } x \in (l_i, r_i) \text{ for some } i \le k, \\ f'_i(x) & \text{if } x \in [r_i, l_{i+1}] \text{ for some } i < k. \end{cases}$$

Example 12.46. Let $k = 2$ and 0 be open and 1 be closed. Then, the definition of g yields the following picture:

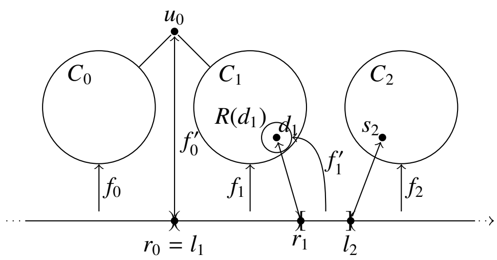

Figure 12.14: Example 12.46

Following the proofs of [53, Lemmas 7.21-24] it can be shown that:

Proposition 12.47. (Cf. [53, Lemmas 7.21-24].) The following facts hold:

1. g is surjective and order-preserving.

2. Every $x \in \mathbf{R} \setminus \{r_i, l_{i+1} \mid i < k\}$ is g-fair with respect to the frame of $\mathcal{M}$.

3. For every $i \leq k$ and $x \in (l_i, r_i)$, we have:

   (a) If C_i is left-open, then $g((l_i, x)) = C_i$.

   (b) If C_i is right-open, then $g((x, r_i)) = C_i$.

4. For every $i \leq k$ closed, we have:

   (a) $g((y, r_i)) \subseteq R(d_i)$ for some $y < r_i$.

   (b) $g((r_i, x)) = g((x, l_{i+1})) = R(d_i)$ for every $x \in (r_i, l_{i+1})$.

   (c) $g((l_{i+1}, z)) \subseteq R(s_{i+1})$ for some $z > l_{i+1}$.

Proof. 1. Surjectivity follows from Proposition 12.41 and the fact that, for every i such that u_i is defined, we have $g(r_i) = u_i$. That g is order-preserving plainly follows from the definition of g.

2. Let $x \in \mathbf{R} \setminus \{r_i, l_{i+1} \mid i < k\}$. Then, two cases are given:

(a) x is in dom(f_i) for some $i \leq k$. Then, by Proposition 12.38(1), we have that x is f_i-fair with respect to $(C_i, R{\restriction}C_i)$.

(b) x is in the interior of dom(f_i) for some $i < k$. Then, again by Proposition 12.38(1), we have that x is f_i-fair with respect to $(R(d_i), R{\restriction}R(d_i))$.

Then, in both cases by Remarks 12.35 and 12.36, we have that x is g-fair with respect to the frame of $\mathcal{M}$.

3. Points (3a) and (3b) follow from points (1) and (2), respectively, of Proposition 12.40 and the fact that $g{\restriction}(l_i, r_i) = f_i$.

4. Points (4a) and (4c) follow from points 1 and 2, respectively, of Proposition 12.42 and the fact that $g{\restriction}(l_i, r_i) = f_i$ and that $g{\restriction}(l_{i+1}, r_{i+1}) = f_i$, respectively. Point (4b) follows from Proposition 12.44 and the fact that $g{\restriction}(r_i, l_{i+1}) = f'_i$. □

Let $V' : \Phi \to \mathcal{P}(\mathbf{R})$ be a function defined by, for every propositional variable p, $V'(p) = g^{-1}(V(p))$. Define $\mathcal{R} := (\mathbf{R}, <, V')$.

Following [53, Lemma 7.25], we prove that:

Proposition 12.48. (Cf. [53, Lemma 7.25].) For every $\varphi \in \Psi$ and $x \in \mathbf{R}$, we have $\mathcal{R}, x \models \varphi$ if and only if $\mathcal{M}, g(x) \models \varphi$.

Proof. By induction on φ. If φ is a propositional variable, then the thesis follows immediately from the definition of V'. The boolean cases are easy and left to the reader. The most interesting cases are $G\psi$, $H\psi$ and $[d]\psi$. Now, since Ψ is closed under taking subformulas, then, by inductive hypothesis, we have that:

$$\text{For every } x \in \mathbf{R}, \mathcal{R}, x \models \psi \text{ if and only if } \mathcal{M}, g(x) \models \psi. \quad (12.9)$$

Assume that $\mathcal{M}, g(x) \models G\psi$ and consider an arbitrary $y \in \mathbf{R}$ such that $y > x$. Then, by Proposition 12.47(1), we have $g(x) \sqsubset g(y)$. Then, we have $\mathcal{M}, g(y) \models \psi$. Then, by (12.9), we have $\mathcal{R}, y \models \psi$. Then, as y was arbitrary, we have $\mathcal{R}, x \models G\psi$.

Conversely, suppose that $\mathcal{M}, g(x) \models \neg G\psi$. Then, since W is finite and $\sqsubset$ transitive and prelinear, there is $w \in W$ such that $\mathcal{M}, w \models \neg\psi$ and, for every $u \in W$ with $\mathcal{M}, u \models \neg\psi$, we have $w \sqsupseteq u$. That is, w is a witness to $\neg\psi$ in the last possible cluster. Then, $g(x) \sqsubset w$. We show that:

$$\text{There is } y > x \text{ in } \mathbf{R} \text{ with } g(y) = w. \tag{12.10}$$

There are two cases:

1. $w = u_i$ for some $i < k$. Then, since $g(x) \sqsubset w = u_i$, we have $g(x) < \lambda(u_i) = C_i$. Then, by definition of g, we have $x \in (-\infty, \lambda_{i+1})$. Then, we have $x < l_{i+1}$ and, by definition of g, $g(l_{i+1}) = u_i = w$, proving (12.10) in this case.

2. $w \in C_i$ for some $i \leq k$. We show that C_i is right-open. Suppose, by contradiction, that C_i is right-closed. Then, we have $i < k$, the points d_i and s_{i+1} are defined, and w and d_i are in the same $\sqsubset$-cluster C_i. Then, we have $d_i \sqsubset w$. Then, we have $\mathcal{M}, d_i \models \neg G\psi$. Now, $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1}$ and $G\psi \in \Psi$. Then, we have $\mathcal{M}, s_{i+1} \models \neg G\psi$ as well. Then, there is $u \in W$ such that $s_{i+1} \sqsubset u$ and $\mathcal{M}, u \models \neg\psi$. Then, by choice of w, we have $u \sqsubset w$. Then, we have $s_{i+1} \sqsubset u \sqsubseteq w \sqsubset d_i$. Then, by transitivity of $\sqsubset$, we have $s_{i+1} \sqsubset d_i$. Moreover, since $d_i \in C_i$ and $s_{i+1} \in C_{i+1}$, we have $\neg(s_{i+1} \sqsubset d_i)$, obtaining a contradiction. Then, C_i is right-open. Then, if $i < k$, we have $r_i = l_{i+1}$. Moreover, we have $r_k = +\infty$. Then, by construction, we have that $\{y \in \mathbf{R} \mid g(y) \sqsubset C_i\} = (-\infty, r_i)$. Then, since $g(x) \sqsubset w \in C_i$, we have $x \in (-\infty, r_i)$. Then, since C_i is right-open, by Proposition 12.47(3b), we have $C_i \subseteq g((x, r_i))$. Then, there is $y > x$ with $g(y) = w$, proving (12.10) in this case as well.

Then, we can take y as in (12.10). Then, by (12.9), we have $\mathcal{R}, y \models \neg\psi$. Then, we have $\mathcal{R}, x \models \neg G\psi$. The case $H\psi$, although not completely specular, can be proved similarly.

The only remaining case is $[d]\psi$. First, an intermediate result:

Lemma 12.49. If x is g-fair, then $\mathcal{R}, x \models [d]\psi$ if and only if $\mathcal{M}, g(x) \models [d]\psi$.

Proof. Assume that x is g-fair. Suppose that $\mathcal{M}, g(x) \models [d]\psi$. Then, by g-fairness of x, there exists $y, z \in \mathbf{R}$ such that $y < x < z$ and $g((y, z) \setminus \{x\}) \subseteq R(g(x))$. Consider an arbitrary $t \in (y, z) \setminus \{x\}$. Then, we have $g(t) \in R(g(x))$. Then, by semantics, we have $\mathcal{M}, g(t) \models \psi$. Then, by (12.9), we have $\mathcal{R}, t \models \psi$. Then, by arbitrariness of t, we have $\mathcal{R}, x \models [d]\psi$.

Conversely, suppose that $\mathcal{R}, x \models [d]\psi$. Then, by semantics, there exist $y, z \in \mathbf{R}$ such that $y < x < z$ and, for every $t \in (y, z) \setminus \{x\}$, we have $\mathcal{R}, t \models \psi$. Consider an arbitrary

$w \in R(g(x))$. Now, by g-fairness of x, we have $R(g(x)) \subseteq g((y,z) \setminus \{x\})$. Then, there is $t \in (y,z) \setminus \{x\}$ such that $g(t) = w$. Then, we have $\mathcal{R}, t \models \psi$. Then, by (12.9), we have $\mathcal{M}, w \models \psi$. Therefore, by arbitrariness of w, we have $\mathcal{M}, g(x) \models [d]\psi$. $\square$

Consider an arbitrary $x \in \mathbf{R}$. The are four cases:

1. $x \in \mathbf{R} \setminus \{r_i, l_{i+1}\}$. Then, by Lemma 12.47(2), we have that x is g-fair. Then, by Lemma 12.49, we have $\mathcal{R}, x \models [d]\psi$ if and only if $\mathcal{M}, g(x) \models [d]\psi$, as desired.

2. $x = r_i = l_{i+1}$ for some $i < k$. We show that x is g-fair. Since $r_i = l_{i+1}$, we have that C_i is right-open, C_{i+1} is left-open, and $g(x) = u_i$. Consider arbitrary $y, z \in \mathbf{R}$ such that $l_i < y < x < z < r_{i+1}$. Then, since $x = r_i = l_{i+1}$, by Proposition 12.47(3a,3b), we have $g((y,x)) = C_i$ and $g((x,z)) = C_{i+1}$. Then, we have:

$$g((y,z) \setminus \{x\}) = g((y,x)) \cup g((x,z)) = C_i \cup C_{i+1} = R(u_i).$$

   Then, x is g-fair. Therefore, by Lemma 12.49, we have $\mathcal{R}, x \models [d]\psi$ if and only if $\mathcal{M}, g(x) \models [d]\psi$, as desired.

3. $x = r_i < l_{i+1}$ for some $i < k$. Then, i is closed and $g(x) = d_i$. Then, by Proposition 12.47(4a), for every large enough $y < x$, we have $g((y,x)) \subseteq R(d_i)$. Moreover, if $z \in (x, l_{i+1})$, then, by Proposition 12.47(4b), we have $g((x,z)) = R(d_i)$. Then, for all large enough y and small enough z such that $y < x < z$, we have:

$$g((y,z) \setminus \{x\}) = g((y,x)) \cup g((x,z)) = R(d_i).$$

   Then, x is g-fair. Then, by Lemma 12.49, we have $\mathcal{R}, x \models [d]\psi$ if and only if $\mathcal{M}, g(x) \models [d]\psi$, as desired.

4. $r_i < l_{i+1} = x$ for some $i < k$. Then, the claim does not apply, as x is not g-fair. Indeed, $g((r_i, x))$ is disjoint from $R(g(x))$. Now, since $r_i < l_{i+1}$, we have that C_i is right-closed, C_{i+1} is left-closed, d_i and s_{i+1} are defined, $g(x) = s_{i+1}$, and $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1}$.

   Suppose that $\mathcal{R}, x \models [d]\psi$. We show that $\mathcal{M}, d_i \models [d]\psi$. Consider an arbitrary $y \in (r_i, x)$ such that, for every $t \in (y,x)$, we have $\mathcal{R}, t \models \psi$. Then, by Lemma 12.47(4b), we have $g((y,x)) = R(d_i)$. Consider an arbitrary $w \in R(d_i)$. Then, since $g((y,x)) = R(d_i)$, there is $t \in (y,x)$ such that $g(t) = w$. Then, by (12.9), we have $\mathcal{M}, w \models \psi$. Then, since w was arbitrary, we have $\mathcal{M}, d_i \models [d]\psi$. Moreover, we have $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1} = g(x)$ and $[d]\psi \in \mathcal{B}\Psi$. Then, we have $\mathcal{M}, g(x) \models [d]\psi$.

   Conversely, suppose that $\mathcal{M}, g(x) \models [d]\psi$. Then, since $g(x) = s_{i+1} \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} d_i$ and $[d]\psi \in \mathcal{B}\Psi$, we have $\mathcal{M}, s_{i+1} \models [d]\psi$ and $\mathcal{M}, d_i \models [d]\psi$. Then, for every $w \in R(d_i) \cup R(s_{i+1})$, we have $\mathcal{M}, w \models \psi$. Let $y := r_i < x$. By Proposition

12.47(4b), we have $g((y, x)) = R(d_i)$. Moreover, by Proposition 12.47(4c), there is $z \in \mathbf{R}$ such that $z > x$ and $g((x, z)) \subseteq R(s_{i+1})$. Then:

$$g((y, z) \setminus \{x\}) = g((y, x)) \cup g((x, z)) \subseteq R(d_i) \cup R(s_{i+1}). \tag{12.11}$$

Consider an arbitrary $t \in (y, z) \setminus \{x\}$. Then, by (12.11), we have $g(t) \in R(d_i) \cup R(s_{i+1})$. Then, we have $\mathcal{M}, g(t) \models \psi$. Then, by (12.9), we have $\mathcal{R}, t \models \psi$. Therefore, since t was arbitrary, we obtain $\mathcal{R}, x \models [d]\psi$. □

Now, pick $w \in W$ such that $\mathcal{M}, w \models \varphi$. By Proposition 12.47.1, g is surjective. Then, there is $x \in \mathbf{R}$ with $g(x) = w$. Then, since $\varphi \in \Psi$, by Proposition 12.48, we have $\mathcal{R}, x \models \varphi$. This proves Proposition 12.28.

12.4 Completeness of L_d

As stated in section 12, to prove the completeness of L_d with respect to $\mathfrak{R}$ it suffices to show that every L_d-consistent formula φ is satisfiable on $\mathfrak{R}$. In section 12.2 we showed that for every L_d-consistent formula φ there is a 'well-behaved' model satisfying φ. In section 12.3, we showed that every L_d-consistent formula φ satisfiable on a 'well-behaved' model is satisfiable on $\mathfrak{R}$. Then, every L_d-consistent formula is satisfiable on $\mathfrak{R}$, as desired.

Chapter 13

Future works

In this chapter we present a number of open problems:

1. Study the logic of L on $\mathbf{Q}$. In [86], Shehtman axiomatized the $G, H, \Box$-logic of $\mathbf{Q}$. This might be a starting point.

2. As mentioned, G and H can be seen as a splitting of the difference operator $[\neq]$, with G meaning 'everywhere after' and H meaning 'everywhere before'. In [42, §6.1] the $\langle d \rangle$ operator has been split into its future and past counterparts, K^+ and K^- respectively, with the following semantics:

   - $\mathcal{R}, x \models K^+\varphi$ provided for every $y > x$ there is $z \in (x, y)$ such that $\mathcal{R}, z \models \varphi$.
   - $\mathcal{R}, x \models K^-\varphi$ provided for every $y < x$ there is $z \in (y, x)$ such that $\mathcal{R}, z \models \varphi$.

   Can we axiomatize K^+, K^- over classes of linear orders? In our completeness proof for $G, H, [d]$ on the reals, and in that of [53] for $G, H, \Box$ on the reals, a crucial role is played by the Shuffle construction. To use Shuffle for G, H, K^+, K^-, we may have to modify Shuffle and/or add other constructions in a way that makes them sensitive to being after and to being before a point.

I would like to thank Valentin Shehtman for suggesting the following two open questions:

3. Study the logic of G only together with $[d]$. Observe that, without the operator H, we lose the Prior Axiom.

4. Study the logic of L interpreted on a continuous circle rather than on $\mathbf{R}$.

Part III

On Spatial Logic with derivative and graded operators

Chapter 14

Introduction

Among other semantics, modal languages have been interpreted also on topological spaces. Topological semantics has a long history (for example, see the seminal [71]) and there is ongoing interest in the field (see [4] and Section 8.2.1 for further reading, and [5, 7, 8, 9, 10, 11, 32, 45, 46, 48, 52, 57, 58, 62, 64, 70, 72, 88, 91] for examples of recent work). The main idea is to associate propositional variables to sets of points of a topological space and give a topologically flavored semantics to modal operators. For example:

- *Derivative operator*: $[d]\varphi$ holds at a point a provided there is a neighborhood U of a such that, for every $b \in U \setminus \{a\}$, we have that φ holds at b.

Together with these operators, we may have other operators more focused on points than on open sets. For example:

- *Graded, or counting, operators*: for every $n \in \mathbf{N}$, $\Diamond_n\varphi$ holds (at a point a) provided there are more than n points b such that φ holds at b.

In this part we will study the expressivity of the modal language L with modal operators $\{[d], \Diamond_n \mid n \in \mathbf{N}\}$ interpreted on topological spaces. Traditionally, the expressivity of modal languages have been compared to that of first-order languages. For example:

- The *van Benthem Characterization Theorem* states (roughly) that on Kripke models, the basic modal language is equivalent to the bisimulation invariant fragment of the first-order language associated to Kripke models [94].

- The *Kamp Theorem* states (roughly) that, on the naturals or the reals, the linear temporal language with 'Until' and 'Since' is equivalent to the first order language associated to the naturals or the reals [55].

We would like to follow this path. Then, we must find a suitable first-order language to describe topological spaces.

If one wants to describe topological spaces in first order terms, the following language L_2 is probably one of the most 'natural'. L_2 is a two-sorted first order language: we have first-sort variables $x, y, ...$, that are assigned to points, and second-sort variables $X, Y, ...$, that are assigned to open sets. L_2 may be defined over the desired signature of relation, function, and constant symbols. But we always have a symbol $=$, that is interpreted as the equality relation, and a symbol ε, that is interpreted as the set membership relation. L_2 has the 'usual' boolean connectives, and quantifiers $\forall x$, $\exists x$ for first-sort variables and quantifiers $\forall X$, $\exists X$ for second-sort variables with the 'usual' meaning. As we would like to characterize (a fragment of) L_2 in modal terms, we restrict the signature of L_2 to a countably infinite set of unary relation symbols.

Unfortunately, L_2 on topological spaces fails to have two important properties 'characterizing' First-order Logic[1]: Compactness and Löwenheim-Skolem Theorems. However, if we add some restrictions to L_2, we obtain the first-order language L_t of [40, Part 1 §2]. L_t is just as L_2 except for the definition of second-sort quantification. For L_t, second-sort quantification is defined by:

- If φ is positive[2] in X, $\forall X(x \varepsilon X \to \varphi)$ is a formula of L_t;
- If φ is negative in X, $\exists X(x \varepsilon X \wedge \varphi)$ is a formula of L_t.

The language L_t, unlike L_2, interpreted on topological spaces enjoys the Compactness and Löwenheim-Skolem Theorem [40, Part 1 §2, 3]. In fact, there is no language (under certain conditions) for describing topological spaces that is more expressive than L_t and enjoys compactness and Löwenheim-Skolem Theorem [40, Part 1 §8].

Moreover L_t can express 'non-trivial' topological properties: for examples, T_0, T_1, T_2 and T_3 axioms, triviality, discreteness, continuity, etc. (However, L_t cannot express normality, connectedness and compactness.) (See [40, Part 1 §3].)

Furthermore the L_t-theory of all T_3 topological spaces is decidable. (However, for $i = 0, 1, 2$, the L_t theory of all T_i topological spaces is undecidable, even without unary relations.) (See [40, Part 2 §1].)

Finally, L_t is equivalent over topological spaces to the base-invariant fragment of L_2 [40, Part 1, Theorem 4.19], where 'base-invariance' is defined as follows. Call a *basoid model* every structure $(A, \mathcal{B})$ where A is a set and $\mathcal{B}$ is a base for a topology over A. Let $\widehat{\mathcal{B}}$ denote the topology generated by $\mathcal{B}$. Let us interpret L_2 over basoid models by interpreting second-sort variables as elements of $\mathcal{B}$. A formula $\varphi[x_1, ..., x_n, X_1, ..., X_m]$ of L_2 is said to be *base-invariant* provided for every basoid model $(A, \mathcal{B})$, $a_1, ..., a_n \in A$ and $U_1, ..., U_m \in \mathcal{B}$,

$$(A, \mathcal{B}) \models \varphi[a_1, ..., a_n, U_1, ..., U_m] \text{ iff } (A, \widehat{\mathcal{B}}) \models \varphi[a_1, ..., a_n, U_1, ..., U_m].$$

[1] Recall that, according to the Lindström Theorem, First-order Logic is (roughly) 'the strongest logic (satisfying certain conditions) that enjoys compactness and satisfies the Löwenheim-Skolem Theorem'.

[2] An L_2 formula is *positive* (*negative*) *in a second-sort variable* X provided all the free occurrences of X are under an even (odd) number of negation signs.

Because of its 'nice' properties, we will take L_t as our first-order language to describe topological spaces, and we will compare the expressivity of L to that of L_t.

14.1 Content of Part III

In Chapter 15, we fix preliminary notation and results. In Chapter 16, we introduce the language L with modal operators $\{[d], \Diamond_n \mid n \in \mathbf{N}\}$. In Chapter 17, we prove that on the class of all T_3 topological spaces L_t and L are equivalent:

- For all formulas $\varphi[x] \in L_t$ there is a formula $\psi \in L$ such that, for every T_3 model $\mathcal{M}$, and point $a \in \mathcal{M}$, we have:

$$\mathcal{M} \models \varphi[a] \text{ if and only if } \mathcal{M}, a \models \psi.$$

- For all formulas $\varphi \in L$ there is a formula $\psi[x] \in L_t$ such that, for every T_3 model $\mathcal{M}$, and point $a \in \mathcal{M}$, we have:

$$\mathcal{M}, a \models \varphi \text{ if and only if } \mathcal{M} \models \psi[a].$$

There are at least two interpretations of this result that are worth mentioning. We can read this result as a van Benthem characterization theorem: on the class of all T_3 topological spaces L is the base invariant fragment of L_2. We can read this result also as a Kamp theorem: on the class of all T_3 topological spaces, L 'captures' L_t.

In Section 17.2.3, we prove that there is a computable translation from L_t to L. In Section 17.2.4 we prove that the equivalence between L_t and L fails over the class of all T_2 spaces. In Chapter 18 we try to answer the question if it is possible to extend L to regain the equivalence between an extension of L and L_t on the class of all T_2 topological spaces.

Traditionally, one way to extend a modal language is to add new modal operators. This leads to the questions of what is a modal operator, and of whether we can add modal operators to L to regain the equivalence between the extension of L and L_t on the class of all T_2 topological spaces.

As an answer, in Section 18.1 we give a more general definition of modal operator. Following Gabbay, Hodkinson and Reynolds [42, Chapter 6], the semantics of modal operators will be defined by a formula of L_t with at most one free first-sort variable and no free second-sort variables. We will call *point-sort modal operator* every modal operator whose semantics is defined by an L_t formula without occurrences of second-sort variables. In Section 18.2, we show that even if we enrich L with all the point-sort modal operators, call L^+ the resulting language, however we add finitely many modal operators to L^+, we cannot express L_t on T_2 topological models.

Finally, in Chapter 19, we study the logic of L on any class X between the class of all T_1 topological spaces and the class of all T_3 topological spaces. We prove that this logic is decidable and PSPACE-complete. Moreover, we provide an axiomatization for this logic, and prove that this logic is strongly complete.

14.2 Related works

14.2.1 Coderivative operator

See Section 8.2.1.

14.2.2 Graded modalities

Graded modalities have been introduced by Fine [38, 1969], [39, 1972], Goble [47, 1970], and Kaplan [56, 1970]. Fine gave axiomatizations for the class of all Kripke models, all **T** (reflexive), all **B** (symmetric), all **S5** (reflexive, transitive, symmetric), and other classes of Kripke models. Their completeness is proved with a construction *à la* Lemmon and Scott. Goble interpreted graded modalities as different grades of possibility on a non-Kripke semantics, and provided a complete axiomatization. We could not access Kaplan's work.

In [36, 1985], Fattorosi-Barnaba and de Caro also considered graded modalities on the class of all **T** Kripke models, provided a set of axioms, and proved the completeness and decidability of the resulting logic. As Fattorosi-Barnaba later stated in [34, 1999], he was not aware of Fine's and Kaplan's work when he and de Caro wrote [36, 1985]. In [26, 1988], de Caro extended [36, 1985] providing a more general completeness proof, *à la* Lemmon and Scott, and complete axiomatizations for the class of all Kripke models, all **K4** (transitive), all **T**, and all **S5**. In [35, 1988], Fattorosi-Barnaba and Cerrato provided a complete axiomatization for the class of all **S4** (reflexive and transitive) Kripke models. In [22, 1990] Cerrato proved that the construction of [26, 1988] does not work for **KB** (symmetric), **KBD** (symmetric and serial), and **KBT** (reflexive and symmetric) Kripke models, and provided a new, more general, construction and complete axiomatizations for these classes of Kripke models. In [96, 1992], van der Hoek studied the expressivity, and definability of graded modalities. Also, he proved the finite model property and decidability for the graded logic of the main classes of Kripke models between **K** and **S5**. In [23, 1994], Cerrato also obtained decidability for the logics of the main classes of Kripke models between **K** and **S5**, by means of a filtration. In [98, 1995], de Rijke and van der Hoek proved that the satisfiability problem for graded modalities is PSPACE-complete (if the grades are coded in unary) and EXPTIME (if the grades are coded in binary) on the class of all Kripke models, and NP-complete (if the grades are coded in unary) and PSPACE (if the grades are coded in binary) on the class of all **S5** Kripke models. In [93, 1999], Tobies proved that the satisfiability problem for graded modalities is PSPACE-complete (even if the grades are coded in binary). In [27, 2000], de Rijke offered a notion of bisimulation for graded modalities, and from this he proved the finite model property, and some definability and invariance results.

In [65, 2002], Kupferman, Sattler and Vardi investigated the graded μ-calculus, proving that its satisfiability problem is EXPTIME-complete (both if the grades are coded in unary and binary). In [17, 2004], Bonatti and Peron proved that μ-calculus with inverse programs, graded modalities, and nominals is undecidable (full μ-calculus). In [16, 2006], Bonatti, Lutz, Murano and Vardi investigated the complexity of the satisfiability problem for the fragments of full μ-calculus obtained dropping one among inverse programs, graded modalities, and nominals. They obtained EXPTIME-completeness in all cases (both if the grades are coded in unary and binary).

In [37, 1995], Fattorosi-Barnaba and Grassotti studied the infinitary version of Graded Modal Logic. They noticed that in this framework, a finite modality stating the existence of a finite number of elements, could be defined. They also provided a complete axiomatization of this logic. Following this path, in [34, 1999], Fattorosi-Barnaba and Balestrini considered the (finitary) modal logic with a modality stating the existence of a finite number of elements, and they provided a complete axiomatization. Similar to this, in [75, 2004] Pacuit and Salame introduced a majority modality quantifying over more than half of the accessible worlds, and provided a complete axiomatization for the resulting logic. And earlier in [97, 1996], van der Hoek introduced a modality $\geq$ defined as $\varphi \geq \psi$ provided φ holds in at least as many accessible worlds as ψ.

Graded modalities have also been interpreted according to other semantics. We already mentioned the work of Goble [47, 1970]. In [90, 1997], in the context of Fuzzy Logic, Suzuki considered grades of modality ranging continuously from 0 to 1. In [50, 2007] Hershfeld and Rabinovich, in the context of Metric Temporal Logic, interpreted graded modalities as stating the existence of at least n points in the next unit of time, and proved some results on expressivity, which are somehow related to our results in Chapter 18. In [78, 2010] Rabinovich proved that the satisfiability problem for Until, Since and this interpretation of graded modalities on the real line is PSPACE-complete if the index is coded in unary and EXPSPACE-complete if the index is coded in binary. In [12, 2012], in the context of Computational Tree Logic (CTL), Bianco, Mogavero and Murano enriched CTL with graded modalities stating the existence of at least n (equivalence classes of) paths, proved results on expressivity, and that the satisfiability problem is EXPTIME, both if the grades are coded in unary and binary.

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Chapter 15

Preliminaries

15.1 Topology

In this section we introduce some basic aspects of Topology. Topology is a well established branch of Mathematics that studies topological spaces. Topological spaces are a general representation of 'space', that allow us to define and study notions such as 'continuity', 'connectedness', and 'convergence'. Other representations of spaces in Mathematics, for example metric spaces, rely on topological spaces enriched with additional structure. For further reading see [99].

15.1.1 Topological spaces

A *topological space* is a tuple (A, τ) such that A is a set, $\tau \subseteq \mathcal{P}(A)$, and:

1. $\emptyset, A \in \tau$.
2. τ is closed under unions and finite intersections - that is, for every set I, and $\{U_i\}_{i \in I} \subseteq \tau$, we have $\bigcup_{i \in I} U_i \in \tau$, and, for every $U_1, U_2 \in \tau$, we have $U_1 \cap U_2 \in \tau$.

The set τ is called a *topology on A*. The elements of A are called *points*. The elements of τ are called *open sets*. For every $a \in A$, we call a *neighborhood of a* every $U \in \tau$ such that $a \in U$. For every $C \subseteq A$, if $A \setminus C \in \tau$, then we call C a *closed set*. For every open U, if U is also closed, then we call U a *clopen set*.

Example 15.1. We give some examples of topological spaces $\mathcal{A} = (A, \tau)$:

1. $\tau = \mathcal{P}(A)$. Plainly, τ is a topology on A, called the *discrete topology on A*.
2. $\tau = \{\emptyset, A\}$. Plainly, τ is a topology on A, called the *trivial topology on A*.
3. $A = \{a, b\}$, and $\tau = \{\emptyset, \{a\}, A\}$. Plainly, τ is a topology on A.
4. Further examples will be given later.

15.1.2 Topological bases

For every topological space (A, τ), we call a *base for* τ every set $\mathcal{B} \subseteq \tau$ such that $\tau = \{\bigcup B \mid B \subseteq \mathcal{B}\}$. If $\mathcal{B}$ is a base for τ, we say that $\mathcal{B}$ *generates* τ.

Proposition 15.2. For every set A, and $\mathcal{B} \subseteq \mathcal{P}(A)$, the following facts are equivalent:

1. There is a topology τ on A generated by $\mathcal{B}$.
2. $\mathcal{B}$ enjoys:
   (a) $A = \bigcup \mathcal{B}$.
   (b) For every $B, B' \in \mathcal{B}$, and $a \in B \cap B'$, there is $B'' \in \mathcal{B}$ such that $a \in B'' \subseteq B \cap B'$.

Proof. See [99, Theorem 5.3]. □

Example 15.3. For every $x, y \in \mathbf{R}$ such that $x < y$, the set $(x, y) = \{z \in \mathbf{R} \mid x < z < y\}$ is called an *open interval of* $\mathbf{R}$. The set of all open intervals of $\mathbf{R}$ is a base for a topology over $\mathbf{R}$, called the *usual topology* and denoted by ε.

A topological space (A, τ) such that there is a base $\mathcal{B}$ for τ of which every element is a clopen set of τ is called a 0*-dimensional topological space*.

15.1.3 Interior and closure

Definition 15.4. Consider a topological space (A, τ), and a subset $B \subseteq A$. Define the *interior* of B, notation $int(B)$, as the biggest open set included in B:

$$int(B) := \bigcup \{U \in \tau \mid U \subseteq B\}.$$

Define the *closure* of B, notation $cl(B)$, as the smallest closed set including B:

$$cl(B) := \bigcap \{U \mid U \text{ is closed} \wedge B \subseteq U\}.$$

Observe that, since topologies are closed under arbitrary unions, the notions of interior and closure are well defined.

Example 15.5. For every $a, b \in \mathbf{R}$ such that $a < b$, recall that $(a, b] = \{c \in \mathbf{R} \mid a < c \leq b\}$, $[a, b) = \{c \in \mathbf{R} \mid a \leq c < b\}$, and $[a, b] = \{c \in \mathbf{R} \mid a \leq c \leq b\}$. Consider $(\mathbf{R}, \varepsilon)$. As the reader may confirm, we have:

$$int((a, b]) = (a, b),$$
$$cl((a, b]) = [a, b].$$

15.1.4 T_i topological spaces ($i \in \{1, 2, 3\}$)

In this section we define T_1, T_2, and T_3 topological spaces. These definitions provide a classification of topological spaces according to 'how much their topologies separate', with T_1 topologies separating the least, and T_3 topologies separating the most[1].

Definition 15.6. We say that:

1. A topological space (A, τ) is a *T_1 topological space* provided, for every distinct $a, a' \in A$, there is $U \in \tau$ such that $a \in U$ and $a' \notin U$.

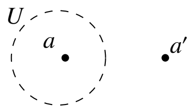

Figure 15.1: There is $U \in \tau$ such that $a \in U$ and $a' \notin U$

2. A topological space (A, τ) is a *T_2 topological space* provided, for every distinct $a, a' \in A$, there are $U, U' \in \tau$ such that $a \in U$, $a' \in U'$, and $U \cap U' = \emptyset$.

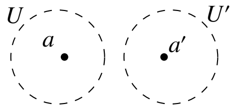

Figure 15.2: There are $U, U' \in \tau$ such that $a \in U$, $a' \in U'$, and $U \cap U' = \emptyset$

3. A topological space (A, τ) is a *regular topological space* provided, for every $a \in A$, and closed set C such that $a \notin C$, there are $U, U' \in \tau$ such that $a \in U$, $C \subseteq U'$, and $U \cap U' = \emptyset$.

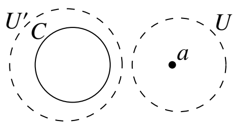

Figure 15.3: There are $U, U' \in \tau$ such that $a \in U$, $C \subseteq U'$, and $U \cap U' = \emptyset$

4. A topological space (A, τ) is a *T_3 topological space* provided it is T_2 and regular.

For every $i \in \{1, 2, 3\}$, let $\mathfrak{T}_i$ be the class of all T_i topological spaces. Plainly, for every $i \in \{1, 2\}$, $\mathfrak{T}_{i+1} \subseteq \mathfrak{T}_i$.

Example 15.7. Consider a topological space $\mathcal{A} = (A, \tau)$ such that A is an infinite set, and τ is the set of all cofinite subsets of A (recall that a subset is *cofinite* provided its complement is finite). Plainly $\mathcal{A}$ is T_1, and not T_2.

[1]There are also other classes. For example, a topological space (A, τ) is a *T_0 topological space* provided, for every distinct $a, a' \in A$, there is $U \in \tau$ such that ($a \in U$ and $a' \notin U$) or ($a \notin U$ and $a' \in U$).

Example 15.8. Let $\mathcal{B}_1 := \{(a,b) \mid a,b \in \mathbf{R} \wedge a < b\}$, $\mathcal{B}_2 := \{(a,b) \cap \mathbf{Q} \mid a,b \in \mathbf{R} \wedge a < b\}$, and $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$. Plainly, $\mathcal{B}$ is a base for a topology on $\mathbf{R}$, say τ. To continue our example, we show that:

Proposition 15.9. $(\mathbf{R}, \tau)$ is T_2 and not T_3.

Proof. Observe that $\mathcal{B}_1$ generates the usual topology ε over $\mathbf{R}$. As the reader may confirm, $(\mathbf{R}, \varepsilon)$ is T_2. Then, since $\mathcal{B}_1 \subseteq \mathcal{B}$ and $(\mathbf{R}, \varepsilon)$ is T_2, we have that $(\mathbf{R}, \tau)$ is T_2 as well.

We show that $(\mathbf{R}, \tau)$ is not T_3. Since T_3 means T_2 and regular, showing that $(\mathbf{R}, \tau)$ is not regular suffices. Since $\mathbf{Q} = \bigcup_{n \in \mathbf{N}}((-n,n) \cap \mathbf{Q})$, we have $\mathbf{Q} \in \tau$. Then, $\overline{\mathbf{Q}} := \mathbf{R} \setminus \mathbf{Q}$ is closed in $(\mathbf{R}, \tau)$. Consider a point $a \in \mathbf{R} \setminus \overline{\mathbf{Q}}$. Suppose by contradiction that $(\mathbf{R}, \tau)$ is regular. Then, there are open sets $B_a \in \mathcal{B}$ and $O_{\overline{\mathbf{Q}}} \in \tau$ such that $a \in B_a$, $\overline{\mathbf{Q}} \subseteq O_{\overline{\mathbf{Q}}}$, and $B_a \cap O_{\overline{\mathbf{Q}}} = \emptyset$.

Now, since $B_a \in \mathcal{B}$, and $B_a \cap \overline{\mathbf{Q}} = \emptyset$, we have $B_a \in \mathcal{B}_2$. Then, $B_a = (b,c) \cap \mathbf{Q}$ for some $b, c \in \mathbf{R}$ such that $b < c$. Then, since $\overline{\mathbf{Q}}$ is dense in $(\mathbf{R}, \varepsilon)$, there is $d \in (b,c) \cap \overline{\mathbf{Q}}$. Consider an arbitrary set B in $\mathcal{B}$ such that $d \in B$. Then, since $d \in \overline{\mathbf{Q}}$, we have that $B \in \mathcal{B}_1$. Moreover, since $d \in (b,c)$ and $d \in B$, we have that $(b,c) \cap B \neq \emptyset$. Then, since $\mathbf{Q}$ is dense in $(\mathbf{R}, \varepsilon)$, we have that $((b,c) \cap B) \cap \mathbf{Q} \neq \emptyset$. Then, since $B_a = (b,c) \cap \mathbf{Q}$, we have that $B_a \cap B \neq \emptyset$. Then, by arbitrariness of B, we have that $d \in cl(B_a)$. Then:

$$cl(B_a) \cap \overline{\mathbf{Q}} \neq \emptyset. \tag{15.1}$$

However, the closure $cl(B_a)$ of B_a in $(\mathbf{R}, \tau)$ is the smallest closed set in $(\mathbf{R}, \tau)$ including B_a. Then, its complement $\overline{cl(B_a)}$ is the biggest open set in $(\mathbf{R}, \tau)$ disjoint from B_a. Then, since $O_{\overline{\mathbf{Q}}}$ is an open set in $(\mathbf{R}, \tau)$ disjoint from B_a, we have $O_{\overline{\mathbf{Q}}} \subseteq \overline{cl(B_a)}$. Then, as $\overline{\mathbf{Q}} \subseteq O_{\overline{\mathbf{Q}}}$, we have $\overline{\mathbf{Q}} \subseteq \overline{cl(B_a)}$, a contradiction of (15.1). □

Recall that $\subset$ denotes proper inclusion. Then, for every $i \in \{1,2\}$, we have $\mathfrak{T}_{i+1} \subset \mathfrak{T}_i$.

Example 15.10. As the reader may confirm, $(\mathbf{R}, \varepsilon)$ is T_3.

15.2 First-order Logic

The aim of this section is to fix notation and concepts of First-order Logic that will be used throughout the rest of this document. Since this dissertation treats advanced topics in Logic, we assume the reader familiar with these definitions and results. Then, we do not contextualize or discuss them, and we may use them later without mentioning. We refer to [24] for a deeper treatment.

15.2.1 First-order languages

Consider a countably infinite set *Var*, whose elements are called *variables* and are denoted by small latin letters $x, y, ...$

Call a *signature* every tuple $(\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a})$. The elements of $\{f_i\}_{i\in I}$ are called *function symbols*, the elements of $\{R_j\}_{j\in J}$ are called *relation symbols*, and the elements of $\{c_k\}_{k\in K}$ are called *constant symbols*. $\mathfrak{a}$ is a function $\mathfrak{a} : \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J} \to \mathbf{N}$, called *arity*. For every $\xi \in \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J}$, $\mathfrak{a}(\xi)$ is called the *arity of* ξ.

For every signature $\sigma = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a})$, define the *terms* (*in* σ) recursively as follows:

- Every variable is a term.
- Every constant symbol is a term.
- If f is a function symbol, and $t_0, ..., t_{\mathfrak{a}(f)-1}$ are terms, then $f(t_0, ..., t_{\mathfrak{a}(f)-1})$ is a term.

Define the *atomic formulas* (in σ) as follows:

- $\top$ is an atomic formula.
- For every two terms t_0, t_1, $t_0 = t_1$ is an atomic formula.
- For every relation symbol R, and terms $t_0, ..., t_{\mathfrak{a}(R)-1}$, $R(t_0, ..., t_{\mathfrak{a}(R)-1})$ is an atomic formula.

Then, 'equality' is always available. Define the *first-order language* (*in* σ), denoted by L_σ, as the smallest set such that:

- All atomic formulas are in L_σ.
- For every $\varphi \in L_\sigma$, we have $\neg\varphi \in L_\sigma$.
- For every $\varphi, \psi \in L_\sigma$, we have $(\varphi \wedge \psi) \in L_\sigma$.
- For every $x \in Var$, and $\varphi \in L_\sigma$, we have $\forall x\, \varphi \in L_\sigma$.

Call a *formula* (*of* L_σ) every element of L_σ. To avoid proliferation of parentheses, the usual precedence rules among operators are assumed. $\perp$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, and $\exists x\, \varphi$ are defined as the usual abbreviations.

For every $x \in Var$, and $\varphi \in L_\sigma$, we say that x *occurs free in* φ provided:

- If φ is an atomic formula, x occurs in φ.
- If φ is $\neg\psi$, x is free in ψ.
- If φ is $\psi \wedge \chi$, x is free in ψ or χ.

- If φ is $\forall y\, \psi$, x is free in ψ and x is different from y.

For every pairwise distinct $x_0, ..., x_{n-1} \in Var$, and $\varphi \in L_\sigma$, if we write $\varphi[x_0, ..., x_{n-1}]$, then the variables that occur free in φ are among $x_0, ..., x_{n-1}$. An array of variables $x_0, ..., x_{n-1}$ may be denoted by $\overline{x}$. Call a *sentence* of L_σ every formula of L_σ with no occurring free variables. For every $\varphi \in L_\sigma$, $x \in Var$, and term t, define $\varphi[{}^t/_x]$ as the formula that we obtain by replacing every free occurrence of x in φ with t.

The quantifier depth of a first-order formula is the maximal nesting of quantifiers. More precisely, for every $\varphi \in L_\sigma$, define the notion of *quantifier depth of* φ, denoted by $qd(\varphi)$, recursively as follows:

- If φ is an atomic formula, then $qd(\varphi) = 0$.
- If φ is $\neg\psi$, then $qd(\varphi) = qd(\psi)$.
- If φ is $\psi \wedge \chi$, then $qd(\varphi) = max\{qd(\psi), qd(\chi)\}$.
- If φ is $\forall x\, \psi$, then $qd(\varphi) = 1 + qd(\psi)$.

For every $n \in \mathbf{N}$, let $L_\sigma^n := \{\varphi \in L_\sigma \,|\, qd(\varphi) \leq n\}$ be the set of formulas of L_σ with quantifier depth at most n.

15.2.2 Semantics

For every signature $\sigma = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a})$, call a *model* (*for* σ) every tuple:

$$\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$$

such that:

- A is a non-empty set, called the *domain* (*of* $\mathcal{M}$).
- For every $i \in I$, $f_i^{\mathcal{M}} : A^{\mathfrak{a}(f_i)} \to A$.
- For every $j \in J$, $R_j^{\mathcal{M}} \subseteq A^{\mathfrak{a}(R_j)}$.
- For every $k \in K$, $c_k^{\mathcal{M}} \in A$.

For every $\xi \in \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J} \cup \{c_k\}_{k\in K}$, $\xi^{\mathcal{M}}$ is called the *interpretation of* ξ (*in* $\mathcal{M}$).

For every model $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$, call an *assignment* (*in* $\mathcal{M}$) every function $g : Var \to A$. For every assignment g, $x \in Var$, and $a \in A$, define $g[{}^a/_x]$ as the function that associates x to a and on every other variable behaves like g.

For every model $\mathcal{M}$, term t, and assignment g, define $g(t)$ recursively as follows:

- If t is a variable x, then $g(t) := g(x)$.
- If t is a constant symbol c, then $g(t) := c^{\mathcal{M}}$.

- If t is of the form $f(t_0, ..., t_{\mathfrak{a}(f)-1})$ (f a function symbol, and $t_0, ..., t_{\mathfrak{a}(f)-1}$ terms), then $g(t) := f^{\mathcal{M}}(g(t_0), ..., g(t_{\mathfrak{a}(f)-1}))$.

For every model $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$, $\varphi \in L_\sigma$, and assignment g, define $\mathcal{M} \models \varphi(g)$ recursively as follows:

- If φ is $\top$, then $\mathcal{M} \models \varphi(g)$ always.
- If φ is $t_0 = t_1$ (t_0, t_1 terms), then $\mathcal{M} \models \varphi(g)$ provided $g(t_0) = g(t_1)$.
- If φ is $R(t_0, ..., t_{\mathfrak{a}(R)-1})$ (R a relation symbol, and $t_0, ..., t_{\mathfrak{a}(R)-1}$ terms), then $\mathcal{M} \models \varphi(g)$ provided $(g(t_0), ..., g(t_{\mathfrak{a}(R)-1})) \in R^{\mathcal{M}}$.
- If φ is $\neg\psi$ ($\psi \in L_\sigma$), then $\mathcal{M} \models \varphi(g)$ provided not $\mathcal{M} \models \psi(g)$.
- If φ is $\psi \wedge \chi$ ($\psi, \chi \in L_\sigma$), then $\mathcal{M} \models \varphi(g)$ provided $\mathcal{M} \models \psi(g)$ and $\mathcal{M} \models \chi(g)$.
- If φ is $\forall x \psi$ ($x \in Var$, $\psi \in L_\sigma$), then $\mathcal{M} \models \varphi(g)$ provided, for every $a \in A$, we have $\mathcal{M} \models \psi(g[^a/_x])$.

If $\mathcal{M} \models \varphi(g)$, then we say that $\mathcal{M}$ *satisfies* φ *under the assignment* g. It is easy to see that for every model $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$, formula $\varphi[x_0, ..., x_{n-1}] \in L_\sigma$, and assignment g, whether $\mathcal{M} \models \varphi(g)$ depends only on the value of g on $x_0, ..., x_{n-1}$ - that is, for every two assignments g and g', if:

$$g \upharpoonright \{x_i \mid i \in n\} = g' \upharpoonright \{x_i \mid i \in n\},$$

then:

$$\mathcal{M} \models \varphi(g) \text{ if and only if } \mathcal{M} \models \varphi(g').$$

For every pairwise distinct $a_0, ..., a_{n-1} \in A$, let $\mathcal{M} \models \varphi[a_0, ..., a_{n-1}]$ denote that there is an assignment g such that $g(x_i) = a_i$ ($i \in n$) and $\mathcal{M} \models \varphi(g)$. By what we said, $\mathcal{M} \models \varphi[a_0, ..., a_{n-1}]$ is equivalent to say that for every assignment g such that $g(x_i) = a_i$ ($i \in n$), we have $\mathcal{M} \models \varphi(g)$.

As a consequence, for every sentence φ, and model $\mathcal{M}$, whether $\mathcal{M} \models \varphi(g)$ does not depend on the assignment g - that is, for every assignment g, g':

$$\mathcal{M} \models \varphi(g) \text{ if and only if } \mathcal{M} \models \varphi(g').$$

For every sentence φ, let $\mathcal{M} \models \varphi$ denote that there is an assignment g such that $\mathcal{M} \models \varphi(g)$. Then, $\mathcal{M} \models \varphi$ is equivalent to the fact that, for every assignment g, we have $\mathcal{M} \models \varphi(g)$. If $\mathcal{M} \models \varphi$, then we say that $\mathcal{M}$ *satisfies* φ.

For every two models $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ *and* $\mathcal{N}$ *are elementarily equivalent*, notation $\mathcal{M} \equiv_{L_\sigma} \mathcal{N}$, provided, for every sentence $\varphi \in L_\sigma$, we have:

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{N} \models \varphi.$$

For every set of sentences $\Gamma \subseteq L_\sigma$, we say that Γ *has a model* provided there is a model $\mathcal{M}$ such that, for every $\gamma \in \Gamma$, we have $\mathcal{M} \models \gamma$. For every model $\mathcal{M}$, if $\mathcal{M}$ is a model of Γ, then we write $\mathcal{M} \models \Gamma$.

15.2.3 Isomorphisms

For every two models for σ $\mathcal{M} = (A, ...)$ and $\mathcal{N} = (B, ...)$, call an *isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ every function $g : A \to B$ such that:

1. g is a bijection.
2. For every relation symbol R, and $a_0, ..., a_{\mathfrak{a}(R)-1} \in A$, we have $(a_0, ..., a_{\mathfrak{a}(R)-1}) \in R^{\mathcal{M}}$ if and only if $(g(a_0), ..., g(a_{\mathfrak{a}(R)-1})) \in R^{\mathcal{N}}$.
3. For every function symbol f, and $a_0, ..., a_{\mathfrak{a}(f)-1} \in A$, we have $g(f^{\mathcal{M}}(a_0, ..., a_{\mathfrak{a}(f)-1})) = f^{\mathcal{N}}(g(a_0), ..., g(a_{\mathfrak{a}(f)-1}))$.
4. For every constant symbol c, we have $g(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

If $\mathcal{M} = \mathcal{N}$, then g is called *automorphism*. If there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ then we say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic*. If two models are isomorphic, then they are elementarily equivalent. For further reading see ([24, Chapter 1]).

15.2.4 Properties

A vast number of results in First-order Logic lies on the following two properties of first-order languages.

Theorem 15.11. Let σ be a signature. Then, the following facts hold:

- *Compactness Theorem.* For every set of sentences $\Gamma \subseteq L_\sigma$, if every finite subset of Γ has a model, then Γ has a model.
- *Löwenheim-Skolem Theorem.* For every countable set of sentences $\Gamma \subseteq L_\sigma$, if Γ has an infinite model, then Γ has a countably infinite model. (This is the Löwenheim-Skolem Theorem as stated in [40]. More general versions of the Löwenheim-Skolem Theorem exist - see [24, Chapter 2].)

Not only are these properties important, they also characterize First-order Logic in the sense of the *Lindström Theorem*. Roughly, the Lindström Theorem states that there is no language (satisfying certain properties) that is more expressive than first-order languages and enjoys Compactness and Löwenheim-Skolem Theorem simultaneously. A more formal statement of the Lindström Theorem would require a detour which is outside the scope of this thesis (for further reading see [24, Chapter 2]).

15.2.5 n-sorted first-order languages ($n \in \mathbf{N}$)

Suppose that we want to talk about mathematical structures whose domain is made of objects of n different types ($n \in \mathbf{N}$). Then, we may want to use the so-called *n-sorted first-order languages*.

For every sort $i = 1, ..., n$ we have a countably infinite set of variables Var_i, whose elements are called *ith sort variables*.

Signatures have to specify information about sorts. Call a *signature* every tuple $(\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$. As before, the elements of $\{f_i\}_{i\in I}$ are called function symbols, the elements of $\{R_j\}_{j\in J}$ are called relation symbols, the elements of $\{c_k\}_{k\in K}$ are called constant symbols, and $\mathfrak{a} : \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J} \to \mathbf{N}$ is called arity. $\mathfrak{b}$ is a function $\mathfrak{b} : \{f_i\}_{i\in I} \cup \{c_k\}_{k\in K} \to \{1, ..., n\}$. For every $\xi \in \{f_i\}_{i\in I} \cup \{c_k\}_{k\in K}$, $\mathfrak{b}(\xi)$ is called the *sort* of ξ. $\mathfrak{c}$ is a function that associates every $\xi \in \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J}$ to an element of ${}^{\mathfrak{a}(\xi)}\{1, ..., n\}$ - that is, to an $\mathfrak{a}(\xi)$-tuple $(n_0, ..., n_{\mathfrak{a}(\xi)-1})$ such that, for every $i \in \{0, ..., \mathfrak{a}(\xi) - 1\}$, we have $n_i \in \{1, ..., n\}$. For every $\xi \in \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J}$, the $\mathfrak{a}(\xi)$-tuple $\mathfrak{c}(\xi) = (n_0, ..., n_{\mathfrak{a}(\xi)-1})$ is called the *template of* ξ.

Terms, atomic formulas and formulas are defined taking into account the sort of terms and the template of function and relation symbols. For every signature $\sigma = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, define the *terms* (*in* σ) and their *sort* recursively as follows:

- For every $i = 1, ..., n$, every ith sort variable is a term, whose sort is i.
- Every constant symbol c is a term, whose sort is $\mathfrak{b}(c)$.
- If f is a function symbol such that $\mathfrak{c}(f) = (n_0, ..., n_{\mathfrak{a}(f)-1})$, and $t_0, ..., t_{\mathfrak{a}(f)-1}$ are terms such that, for every $i \in \mathfrak{a}(f)$, the sort of t_i is n_i, then $f(t_0, ..., t_{\mathfrak{a}(f)-1})$ is a term, whose sort is $\mathfrak{b}(f)$.

Define *atomic formulas* (*in* σ) as follows:

- $\top$ is an atomic formula.
- For every $i \in \{1, ..., n\}$, and terms t_1, t_2 with the same sort, $t_1 = t_2$ is an atomic formula.
- If R is a relation symbol such that $\mathfrak{c}(R) = (n_0, ..., n_{\mathfrak{a}(R)-1})$, and $t_0, ..., t_{\mathfrak{a}(R)-1}$ are terms such that, for every $i \in \mathfrak{a}(R)$, the sort of t_i is n_i, then $R(t_0, ..., t_{\mathfrak{a}(R)-1})$ is an atomic formula.

Define the *n-sorted first-order language* (*in* σ), denoted by L_σ, as the smallest set such that:

- All atomic formulas are in L_σ.
- For every $\varphi \in L_\sigma$, we have $\neg\varphi \in L_\sigma$.

- For every $\varphi, \psi \in L_\sigma$, we have $\varphi \wedge \psi \in L_\sigma$.
- For every $i = 1, ..., n$, $x \in Var_i$, and $\varphi \in L_\sigma$, we have $\forall x \, \varphi \in L_\sigma$.

15.2.6 Semantics

For every signature $\sigma = (\{f_i\}_{i \in I}, \{R_j\}_{j \in J}, \{c_k\}_{k \in K}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, call a *model* (*for* L_σ) every tuple:

$$\mathcal{M} = (A_1, ..., A_n, \{f_i^{\mathcal{M}}\}_{i \in I}, \{R_j^{\mathcal{M}}\}_{j \in J}, \{c_k^{\mathcal{M}}\}_{k \in K})$$

such that:

- $A_1, ..., A_n$ are pairwise disjoint non-empty sets;
- For every $i = 1, ..., n$, A_i is called the *ith-sort domain* of $\mathcal{M}$.
- The domain of the interpretation of a function symbol is defined by $\mathfrak{c}$, while the range by $\mathfrak{b}$ - that is, if f is a function symbol such that $\mathfrak{c}(f) = (n_0, ..., n_{\mathfrak{a}(f)-1})$, then:
$$f_i^{\mathcal{M}} : A_{n_0} \times ... \times A_{n_{\mathfrak{a}(f)-1}} \to A_{\mathfrak{b}(f)}.$$
- The domain of the interpretation of a relation symbol is defined by $\mathfrak{c}$ - that is, if R is a relation symbol such that $\mathfrak{c}(R) = (n_0, ..., n_{\mathfrak{a}(R)-1})$, then:
$$R_i^{\mathcal{M}} \subseteq A_{n_0} \times ... \times A_{n_{\mathfrak{a}(R)-1}}.$$
- The domain of the interpretation of a constant symbol is defined by $\mathfrak{b}$ - that is, if c is a constant symbol, then:
$$c^{\mathcal{M}} \in A_{\mathfrak{b}(c)}.$$

For every $\xi \in \{f_i\}_{i \in I} \cup \{R_j\}_{j \in J} \cup \{c_k\}_{k \in K}$, $\xi^{\mathcal{M}}$ is called the *interpretation of* ξ (*in* $\mathcal{M}$).

For every model $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i \in I}, \{R_j^{\mathcal{M}}\}_{j \in J}, \{c_k^{\mathcal{M}}\}_{k \in K})$, call an *assignment* (*in* $\mathcal{M}$) every tuple $g = (g_1, ..., g_n)$ of functions such that, for every $i \in \{1, ..., n\}$, $g_i : Var_i \to A_i$. For every g_i, $x \in Var_i$, and $a \in A_i$, define $g_i[a/x]$ as the function that associates x to a and on every other ith-sort variable behaves like g_i.

For every model $\mathcal{M}$, term t, and assignment $g = (g_1, ..., g_n)$, define $g(t)$ recursively as follows:

- If t is a ith-sort variable x, then $g(t) := g_i(x)$.
- If t is a constant symbol c, then $g(t) := c^{\mathcal{M}}$.
- If t is of the form $f(t_0, ..., t_{\mathfrak{a}(f)-1})$ (f a function symbol and $t_0, ..., t_{\mathfrak{a}(f)-1}$ terms), then $g(t) := f^{\mathcal{M}}(g(t_0), ..., g(t_{\mathfrak{a}(f)-1}))$.

For every model $\mathcal{M} = (A_1, ..., A_n, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$, $\varphi \in L_\sigma$, and assignment g, define $\mathcal{M} \models \varphi(g)$ recursively as follows:

- If φ is $\top$, then $\mathcal{M} \models \varphi(g)$ always.
- If φ is $x = y$ ($i \in \{1, ..., n\}$, and $x, y \in Var_i$), then $\mathcal{M} \models \varphi(g)$ provided $g_i(x) = g_i(y)$.
- If φ is $R(t_0, ..., t_{\mathfrak{a}(R)-1})$ (R a relation symbol and $t_0, ..., t_{\mathfrak{a}(R)-1}$ terms), then $\mathcal{M} \models \varphi(g)$ provided $(g(t_0), ..., g(t_{\mathfrak{a}(R)-1})) \in R^{\mathcal{M}}$.
- If φ is $\neg\psi$ ($\psi \in L_\sigma$), then $\mathcal{M} \models \varphi(g)$ provided not $\mathcal{M} \models \psi(g)$.
- If φ is $\psi \wedge \chi$ ($\psi, \chi \in L_\sigma$), then $\mathcal{M} \models \varphi(g)$ provided $\mathcal{M} \models \psi(g)$ and $\mathcal{M} \models \chi(g)$.
- If φ is $\forall x\, \psi$ ($i \in \{1, ..., n\}$, $x \in Var_i$, and $\psi \in L_\sigma$), then $\mathcal{M} \models \varphi(g)$ provided, for every $a \in A_i$, we have $\mathcal{M} \models \psi(g_0, ..., g_i[a/x], ..., g_n)$.

15.2.7 Properties

Plainly, first-order languages can be seen as 1-sorted first-order languages. Conversely, every n-sorted first-order language can be translated into a first-order language. Indeed, for every sort $i \in \{1, ..., n\}$, introduce a relation symbol P_i with arity 1. To have n many domains, one for every sort, require that the domain is partitioned into n non-empty parts. This can be expressed by a first-order sentence using the $P_i s$:

$$\bigwedge_{i\in\{1,...,n\}} \exists x\, P_i(x) \wedge \forall x \Big(\bigvee_{i\in\{1,...,n\}} P_i(x) \wedge \bigwedge_{\substack{i,j\in\{1,...,n\} \\ i\neq j}} \neg(P_i(x) \wedge P_j(x)) \Big).$$

To quantify over sort i, use the P_is to restrict the quantification to the sort i:

$$\forall x\, (P_i(x) \to \varphi).$$

For every $n \in \mathbf{N}$, to say that an n-ary function symbol f has sort i, use the P_is in the formula:

$$\forall x_0 ... \forall x_{n-1} P_i(f(x_0, ..., x_{n-1})).$$

To say that a constant symbol c has sort i, use the P_is in the formula:

$$P_i(c).$$

Therefore, the Compactness, Löwenheim-Skolem and Lindström Theorems hold for n-sorted first-order languages as well.

15.3 Monadic Second-order Logic

Monadic second-order languages extend first-order languages. Together with a countably infinite set of variables Var_1, which will be assigned to elements of the domain and whose elements are called *first-order variables*, we have a second countably infinite set Var_2 of variables, which will be assigned to subsets of the domain and whose elements are called *second-order variables*. Moreover, we can say that a (first-order) term belongs to a second order variable: we just write $X(t)$, where X is a second-order variable and t a (first-order) term. Finally, quantification is allowed over both set of variables.

Formally, for every signature $\sigma = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a})$, we have the additional atomic formulas in σ given by:

- For every (first-order) term t and $X \in Var_2$, $X(t)$ is an atomic formula.

Finally, we have the additional formulas given by:

- For every $X \in Var_2$, $\varphi \in L_\sigma$, we have $\forall X\, \varphi \in L_\sigma$.

Observe that monadic second-order languages can be translated into 2-sort first-order languages by interpreting the latter over models whose second-sort domain is the powerset of the first-sort domain and is assumed disjoint from the first-sort domain.

15.3.1 Semantics

For every signature $\sigma = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a})$, models for L_σ are, as before, tuples $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$. For every model $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$, call an *assignment* (*in* $\mathcal{M}$) every tuple $g = (g_1, g_2)$ of functions $g_1 : Var_1 \to A$ and $g_2 : Var_2 \to \mathcal{P}(A)$. For every model $\mathcal{M} = (A, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K})$, formula φ, and assignment $g = (g_1, g_2)$, the definition of $\mathcal{M} \models \varphi(g)$ requires the following additional conditions:

- If φ is $X(t)$ (t a (first-order) term, and $X \in Var_2$), then $\mathcal{M} \models \varphi(g)$ provided $g_1(t) \in g_2(X)$.
- If φ is $\forall X\, \psi$ ($X \in Var_2$), then $\mathcal{M} \models \varphi(g)$ provided, for every $B \in \mathcal{P}(A)$, we have $\mathcal{M} \models \psi(g_1, g_2[B/X])$.

15.3.2 Properties

Monadic second-order languages are strictly more expressive than first-order languages. For example, when interpreted over $(\mathbf{R}, <)$, the corresponding monadic second-order language can express the Completeness Axiom - that is, the sentence stating that every non-empty, bounded subset of $\mathbf{R}$ has a least upper bound. Indeed:

- Let $\varphi_{\mathrm{ne}(X)} := \exists x\, X(x)$. Then $\varphi_{\mathrm{ne}(X)}$ states that X is non-empty.
- Let $\varphi_{\mathrm{ub}(x,X)} := \forall y\,(X(y) \to y \leq x)$. Then $\varphi_{\mathrm{ub}(X)}$ states that x is an upper bound of X.
- Let $\varphi := \forall X(\varphi_{\mathrm{ne}}(X) \wedge \exists x\, \varphi_{\mathrm{ub}}(x, X) \to \exists x\,(\varphi_{\mathrm{ub}(x,X)} \wedge \forall z\,(\varphi_{\mathrm{ub}}(z, X) \to x \leq z))$.

Then, φ is the desired formula.

On the other hand, the corresponding first-order language cannot express the Completeness Axiom. Indeed, the real numbers are the unique (up to isomorphism) complete ordered field [89]. Now, being an ordered field is expressible in first-order languages. Then, if φ were expressible in first order-languages, by the Löwenheim-Skolem Theorem, we would have a countable complete ordered field, say $\mathcal{M}$. Then, the reals would be isomorphic to $\mathcal{M}$. Then the reals would be countable, a contradiction.

This example also shows that monadic second-order languages do not satisfy the Löwenheim-Skolem Theorem. For, suppose they did. Then, since being a complete order field is expressible in second-order languages, by the Löwenheim-Skolem Theorem, we would have a countable complete ordered field, obtaining the same contradiction.

Finally second-order languages do not enjoy the Compactness Theorem either. For, consider the following set Γ of sentences:

1. $<$ is linear.
2. For every $n \in \mathbf{N}$, there are at least n distinct points.
3. For every set X, if X is non-empty, then X has a greatest element or a least element.

Plainly, the sentences of Γ are expressible in second-order languages. Moreover, each finite subset of Γ has a model. Now, suppose by contradiction that the Compactness Theorem held for Monadic Second-order Logic. Then, Γ would have a model, say $\mathcal{M}$. Then, by 2, $\mathcal{M}$ would be an infinite linear order. However, as the reader may confirm, each infinite linear order admits an infinite ascending sequence or an infinite descending sequence. Then, $\mathcal{M}$ would not satisfy 3, a contradiction.

Chapter 16

Language and semantics

Let Φ be a countable set, and $\tau := \{[d], \Diamond_n \mid n \in \mathbf{N}\}$. Throughout Part III, we will work with the language $L := L_{\Phi,\tau}$. Let $\langle d \rangle := \neg[d]\neg$ and, for every $n \in \mathbf{N}$, $\Box_n := \neg\Diamond_n\neg$.

The modal depth of a formula is the maximal nesting of modal operators and is defined as:

Definition 16.1. For every $\varphi \in L$, define the *modal depth of* φ, denoted by $md(\varphi)$, recursively as follows:

- If φ is $\top$ or a propositional variable, then $md(\varphi) := 0$.
- If φ is $\neg\psi$, then $md(\varphi) := md(\psi)$.
- If φ is $\psi \wedge \chi$, then $md(\varphi) := max\{md(\psi), md(\chi)\}$.
- If φ is $[d]\psi$, then $md(\varphi) := 1 + md(\psi)$.
- If φ is $\Diamond_n\psi$, then $md(\varphi) := (n + 1) + md(\psi)$.

Cf. with the notion of quantifier depth defined in Section 15.2.1. For every $n \in \mathbf{N}$, let $L^n := \{\varphi \in L \mid md(\varphi) \leq n\}$ be the set of formulas of L with modal depth at most n.

16.1 Semantics

Call a *topological model* (*for L*) every tuple (A, τ, V) such that (A, τ) is a topological space and $V : \Phi \to \mathcal{P}(A)$ is a function, called *evaluation*. Given a topological model $\mathcal{M} = (A, \tau, V)$, if we write $a \in \mathcal{M}$, we mean $a \in A$.

For every topological model $\mathcal{M} = (A, \tau, V)$, every $a \in A$, every $p \in \Phi$ and every $\varphi, \psi \in L$, define:

- $\mathcal{M}, a \models \top$ always.

- $\mathcal{M}, a \models p$ provided $a \in V(p)$.
- $\mathcal{M}, a \models \neg\varphi$ provided $\mathcal{M}, a \not\models \varphi$.
- $\mathcal{M}, a \models \varphi \wedge \psi$ provided $\mathcal{M}, a \models \varphi$ and $\mathcal{M}, a \models \psi$.
- $\mathcal{M}, a \models [d]\varphi$ provided there is a neighborhood U of a such that, for every $a' \in U \setminus \{a\}$, we have $\mathcal{M}, a' \models \varphi$.
- For every $n \in \mathbf{N}$, $\mathcal{M}, a \models \Diamond_n\varphi$ provided there are more than n distinct points $a' \in A$ such that $\mathcal{M}, a' \models \varphi$ - that is:

$$|\{a' \in A \mid \mathcal{M}, a' \models \varphi\}| > n.$$

Observe that then, for every $n \in \mathbf{N}$, $\mathcal{M}, a \models \Box_n\varphi$ provided there are at most n distinct points $a' \in A$ such that $\mathcal{M}, a' \not\models \varphi$ - that is:

$$|\{a' \in A \mid \mathcal{M}, a' \not\models \varphi\}| \leq n.$$

Moreover, the satisfiability of formulas of the form $\Diamond_n\varphi$ and $\Box_n\varphi$ ($n \in \mathbf{N}$) is invariant under changing the point of evaluation - that is, for every $a, a' \in A$ we have:

$$\mathcal{M}, a \models \Diamond_n\varphi \text{ if and only if } \mathcal{M}, a' \models \Diamond_n\varphi,$$

$$\mathcal{M}, a \models \Box_n\varphi \text{ if and only if } \mathcal{M}, a' \models \Box_n\varphi.$$

For this reason, we call *sentences of* L the formulas in L which are a boolean combination of formulas of the form $\Diamond_n\varphi$ and $\Box_n\varphi$. For every sentence $\varphi \in L$, for every topological model $\mathcal{M}$, if we write $\mathcal{M} \models \varphi$, we mean $\mathcal{M}, a \models \varphi$ for some (every) $a \in \mathcal{M}$.

Analogously to Section 2.3, for every topological model $\mathcal{M}$, and $\varphi \in L$ ($\Gamma \subseteq L$, respectively), we say that $\mathcal{M}$ *satisfies* φ ($\mathcal{M}$ *satisfies* Γ, resp.) provided there is $a \in \mathcal{M}$ such that $\mathcal{M}, a \models \varphi$ (such that $\mathcal{M}, a \models \delta$ for every $\gamma \in \Gamma$, resp.).

For every two topological models $\mathcal{M}$ and $\mathcal{N}$, a point $a \in \mathcal{M}$ and a point $b \in \mathcal{N}$ are said to be *modally equivalent* (*up to modal depth n* resp.), denoted $\mathcal{M}, a \equiv_L \mathcal{N}, b$ ($\mathcal{M}, a \equiv_L^n \mathcal{N}, b$ resp.), provided, for every formula $\varphi \in L$ ($\varphi \in L^n$ resp.), we have $\mathcal{M}, a \models \varphi$ if and only if $\mathcal{N}, b \models \varphi$. If the models are understood, we simply write $a \equiv_L b$ ($a \equiv_L^n b$ resp.).

For every two topological models $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}$ are said to be *modally equivalent* (*up to modal depth n* resp.), denoted $\mathcal{M} \equiv_L \mathcal{N}$ ($\mathcal{M} \equiv_L^n \mathcal{N}$ resp.), provided, for every sentence $\varphi \in L$ ($\varphi \in L^n$ resp.), we have $\mathcal{M} \models \varphi$ if and only if $\mathcal{N} \models \varphi$.

For every $i \in \{1, 2, 3\}$, and topological model $\mathcal{M} = (A, \tau, V)$, if (A, τ) is a T_i topological space, then we say that $\mathcal{M}$ is a *T_i topological model*. We say that φ *is satisfiable*

(Γ *is satisfiable*, resp.) *on* $\mathfrak{T}_i$ provided there is a T_i topological model satisfying φ (satisfying Γ, resp.), and that φ *is valid on* $\mathfrak{T}_i$ provided $\neg\varphi$ is not satisfiable on $\mathfrak{T}_i$. Let $\mathsf{L}^{\mathfrak{T}_i}$ be the set of all valid formulas on $\mathfrak{T}_i$. Observe that $\mathsf{L}^{\mathfrak{T}_i}$ is a logic. For $\Gamma \cup \{\varphi\} \subseteq L$, we say that Γ *semantically entails* φ *on* $\mathfrak{T}_i$, notation $\Gamma \models^{\mathfrak{T}_i} \varphi$, provided, for every T_i topological model $\mathcal{M}$, and $a \in \mathcal{M}$, if $\mathcal{M}, a \models \gamma$ for every $\gamma \in \Gamma$, then $\mathcal{M}, a \models \varphi$.

Remark 16.2. The semantics for $[d]$ defined in this section generalizes the semantics for $[d]$ defined in Section 10.1. Indeed, given a linear order $\mathcal{I} = (I, <)$, define the *order topology on* $\mathcal{I}$ as the topology $\tau_<$ generated by the base $\mathcal{B}$ consisting of all the *open intervals of* $\mathcal{I}$ - that is, $\mathcal{B} = \{(a, b) \,|\, a, b \in I \wedge a < b\}$. It is easy to prove that, for every evaluation function $V : \Phi \to \mathcal{I}$, formula $\varphi \in L_{\Phi,\{[d]\}}$, and $a \in I$, we have that the following two statements are equivalent:

1. There are $b, c \in I$ such that $b < a < c$ and, for every $d \in (b, c) \setminus \{a\}$, we have $(I, <, V), d \models \varphi$. (This is the semantics for $[d]$ defined in Section 10.1.)

2. There is a neighborhood U of a such that for every $b \in U \setminus \{a\}$ we have $(I, \tau_<, V), b \models \varphi$. (This is the semantics for $[d]$ defined in this section.)

Chapter 17

Expressivity

In this chapter we will study the expressivity of L. One way to study the expressivity of a language is to compare its expressivity with the expressivity of another language. Traditionally, the expressivity of modal languages has been compared to the expressivity of first-order languages. Famous examples are the van Benthem Characterization Theorem [94] and the Kamp Theorem [55]. We will follow this tradition and compare the expressivity of L to that of the first-order language L_t of Section 17.1.3. (For a formal definition of what this means see Definitions 17.4 and 17.5.)

17.1 First-order languages

If one wants to talk about topological spaces in first-order terms, the 'most natural' choice is probably the language L_2 of [40, Part 1, §1].

L_2 is a two-sorted first-order language. We have a countably infinite set Var_1 of first-sort variables, which will be assigned to points, and will be denoted by small Latin letters $x, y, ...$; and we have a countably infinite set Var_2 of second-sort variables, which will be assigned to open sets, and will be denoted by capital Latin letters $X, Y, ...$.

L_2 can be defined over the desired signature $\sigma = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, with the only requirement that among the relation symbols $\{R_j\}_{j\in J}$ of σ there is a relation symbol ε that takes as arguments couples whose first component is a first-sort variable and second component is a second-sort variable - that is, $\mathfrak{c}(\varepsilon) = (1, 2)$. The relation symbol ε will be interpreted as the set-membership relation $\in$.

We will interpret L_2 over topological models:

Definition 17.1. Call a *topological model* (*for* L_2) every model for σ:

$$\mathcal{M} = (A, \tau, \{f_i^{\mathcal{M}}\}_{i\in I}, \{R_j^{\mathcal{M}}\}_{j\in J}, \{c_k^{\mathcal{M}}\}_{k\in K}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}),$$

such that (A, τ) is a topological space, and $\varepsilon^{\mathcal{M}}$ is the set-membership relation $\in$. We say that a topological model $(A, \tau, ...)$ is *countable* provided A is countable and τ is generated by a countable base[1].

Let $\forall X_x\, \varphi$ abbreviate $\forall X\, (x\varepsilon X \to \varphi)$, and $\exists X_x\, \varphi$ abbreviate $\exists X\, (x\varepsilon X \wedge \varphi)$. Moreover, when, as in the case of topological models, Var_1 is assigned to elements of a set, and Var_2 is assigned to subsets of this set, $X = Y$ can, and will, be taken as an abbreviation for $\forall x(x\varepsilon X \leftrightarrow x\varepsilon Y)$.

17.1.1 Properties of L_2

On one hand, L_2 can express a number of interesting topological properties [40, Part 1]:

- L_2 can say that a topology is trivial:

$$\forall X\, (\exists x\, x\varepsilon X \to \forall x\, x\varepsilon X).$$

- That a topology is discrete:

$$\varphi_{\mathrm{disc}} := \forall x\, \exists X\, \forall y\, (y\varepsilon X \leftrightarrow y = x).$$

- That a topological space is T_1:

$$\forall x\, \forall y\, \exists X_x\, (\neg x = y \to \neg y\varepsilon X).$$

- That a topological space is T_2:

$$\varphi_{T_2} := \forall x\, \forall y\, (\neg x = y \to \exists X\, \exists Y\, (x\varepsilon X \wedge y\varepsilon Y \wedge \forall z\, \neg(z\varepsilon X \wedge z\varepsilon Y))).$$

- That a topological space is regular:

$$\varphi_{\mathrm{reg}} := \forall x\, \forall X_x\, \exists Y_x\, \forall y\, (\neg y\varepsilon X \to \exists W_y\, \forall z\, \neg(z\varepsilon W \wedge z\varepsilon Y)).$$

  Observe that this is a somehow ‘unusual’ definition of ‘regularity’. A more usual one would be: for every closed set X, for every point $x \notin X$, there are disjoint open sets Y and Z such that $X \subseteq Y$ and $x \in Z$. This definition is expressible in L_2 and it is equivalent to the former definition of ‘regularity’. Nonetheless, the former definition of ‘regularity’ is the one given in [40], and important formulas in Chapter 18 are inspired by it. Therefore, we think it it useful to present this definition of ‘regularity’ and define φ_{reg} as we did.

[1] In the literature, a topological space satisfying this condition is usually called *second countable*. However, we follow [40, p. 8] and say ‘countable’.

- That a topological space is T_3:

$$\varphi_{T_2} \wedge \varphi_{\text{reg}}.$$

- That a topological space is Alexandroff:

$$\forall x\, \exists X_x\, \forall Y_x\, \forall y\, (y \varepsilon X \to y \varepsilon Y).$$

- That a topological space is dense-in-itself:

$$\forall x\, \forall X_x\, \exists y\, (\neg y = x \wedge y \varepsilon X).$$

- That a topological space is connected:

$$\varphi_{\text{con}} := \forall X\, \forall Y\, (\forall x\, (x \varepsilon X \leftrightarrow \neg x \varepsilon Y) \to \forall x\, x \varepsilon X \vee \forall x\, \neg x \varepsilon X).$$

- That a topological space is normal:

$$\varphi_{\text{nor}} := \forall X\, \forall Y\, (\overline{X} \cap \overline{Y} = \emptyset \to \exists Z\, \exists W\, (Z \cap W = \emptyset \wedge \overline{X} \subseteq Z \wedge \overline{Y} \subseteq W)),$$

  with, $\overline{X} \cap \overline{Y} = \emptyset$ being an abbreviation for $\forall x\, (x \varepsilon X \vee x \varepsilon Y)$, $X \cap Y = \emptyset$ for $\forall x\, \neg(x \varepsilon X \wedge x \varepsilon Y)$, and $\overline{X} \subseteq Y$ for $\forall x\, (\neg x \varepsilon X \to x \varepsilon Y)$.

- And, for every n-ary function symbol f, that f is continuous:

$$\begin{aligned} &\forall x_1 \ldots \forall x_n\, \forall Y_{f(x_1, \ldots, x_n)}\, \exists X_{1\, x_1} \ldots \exists X_{n\, x_n}\, \forall y_1 \ldots \forall y_n \\ &\quad (y_1 \varepsilon X_1 \wedge \ldots \wedge y_n \varepsilon X_n \to f(y_1, \ldots, y_n) \in Y). \end{aligned}$$

On the other hand, as stated in Section 15.2.4, the Compactness and Löwenheim-Skolem Theorems are important properties of first-order languages. Moreover, they characterize first-order languages in the sense of the Lindström Theorem. Therefore, we would like L_2 to enjoy these theorems.

Since L_2 is a 2-sorted first-order language, as mentioned in Section 15.2.7, L_2 enjoys the Compactness and Löwenheim-Skolem Theorems when interpreted on the class of all models for L_2. But what about when interpreted on the class of all topological models? Do we have the following versions of the Compactness and Löwenheim-Skolem Theorems for topological models?

1. *Compactness Theorem.* For every set of sentences $\Gamma \subseteq L_2$, if every finite subset of Γ has a topological model, then Γ has a topological model.

2. *Löwenheim-Skolem Theorem.* For every countable set of sentences $\Gamma \subseteq L_2$, if Γ has an infinite topological model, then Γ has a countable topological model.

As observed in [40, Part 1, §1], the answer is no. Indeed, for every topological model $\mathcal{M} = (A, \tau, ...)$, we have:

$$\mathcal{M} \models \varphi_{\text{disc}} \text{ if and only if } \mathcal{M} \text{ has the discrete topology } \tau = \mathcal{P}(A).$$

Then, monadic second-order languages can be expressed in L_2 interpreted on the class of all topological models. And, as mentioned in Section 15.3, the Compactness and Löwenheim-Skolem Theorems do not hold for monadic second-order languages.

17.1.2 A solution

If being a topological space were expressible in L_2, then it would be easy to prove the Compactness and Löwenheim-Skolem Theorems for L_2 interpreted on the class of all topological models. Indeed, suppose that being a topological space were expressible by a countable set of sentences $\Gamma \subseteq L_2$. Then, for every set of sentences $\Delta \subseteq L_2$, applying the Compactness and Löwenheim-Skolem Theorems to $\Gamma \cup \Delta$, we would obtain models for $\Gamma \cup \Delta$. Then, these models would be topological models for Δ, as desired.

Although being a topological space is not expressible in L_2, being a base for a topology is - namely by the conjunction φ_{bas} of the following formulas:

- $\forall x \, \exists X \, x \varepsilon X$.
- $\forall x \, \forall X \, \forall Y \, (x \varepsilon X \wedge x \varepsilon Y \rightarrow \exists Z \, (x \varepsilon Z \wedge \forall y \, (y \varepsilon Z \rightarrow y \varepsilon X \wedge y \varepsilon Y)))$.

Moreover, call a *basoid model* every model:

$$\mathcal{M} = (A, \mathcal{B}, ...),$$

such that $\mathcal{B}$ is a base for a topology over A. Then, by reasoning as above, we obtain the following versions of the Compactness and Löwenheim-Skolem Theorems:

1. *Compactness Theorem.* For every set of sentences $\Gamma \subseteq L_2$, if every finite subset of Γ has a basoid model, then Γ has a basoid model.

2. *Löwenheim-Skolem Theorem.* For every countable set of sentences $\Gamma \subseteq L_2$, if Γ has an infinite basoid model, then Γ has a countable basoid model.

Suppose that Γ is *invariant under changing base* - that is, for every two basoid models $(A, \mathcal{B}, ...)$ and $(A, \mathcal{B}', ...)$ such that $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology, we have:

$$(A, \mathcal{B}, ...) \models \Gamma \text{ if and only if } (A, \mathcal{B}', ...) \models \Gamma.$$

Then, since every topology is a base for itself, we obtain:

Theorem 17.2. The invariant under changing base fragment of L_2 enjoys the Compactness and Löwenheim-Skolem Theorems:

1. *Compactness Theorem.* For every set $\Gamma \subseteq L_2$ of sentences invariant under changing base, if every finite subset of Γ has a topological model, then Γ has a topological model.

2. *Löwenheim-Skolem Theorem.* For every countable set $\Gamma \subseteq L_2$ of sentences invariant under changing base, if Γ has an infinite topological model, then Γ has a countable topological model.

Moreover, they characterize this fragment in the sense of the Lindström Theorem [40, Part 1 §8]. Roughly, there is no language (satisfying certain properties) that is more expressive than the invariant under changing base fragment of L_2 and enjoys the Compactness and Löwenheim-Skolem Theorems simultaneously.

Recall that countable topological model means a model $(A, \tau, ...)$ such that A is countable and τ is generated by a countable base. This is different from the statement of the Löwenheim-Skolem Theorem for n-sorted first-order languages. There, we required the union of the n domains to be countable, while here τ may be uncountable. However, since we are considering the invariant under changing base fragment of L_2, requiring A to be countable and τ to be generated by a countable base, let us work with a basoid model $(A, \mathcal{B})$ such that A is countable and $\mathcal{B}$ is a countable base for τ.

Now, one may ask:

- Can this fragment express interesting topological properties?

- Can we characterize this fragment in a 'less obscure' way than as the invariant under changing base fragment of L_2?

We try to answer these questions, starting from the latter, presenting the language L_t of [40, Part 1, §2]. In [40, Part 1, Corollary 2.4 and Theorem 4.19], L_t is shown to be equivalent to the invariant under changing base fragment of L_2. We will then show that L_t can express some interesting topological properties, although not as many as L_2.

17.1.3 Language L_t

In this section we present the language L_t, introduced in [40, Part 1, §2]. For every $\varphi \in L_2$, and $X \in Var_2$, say that φ *is positive* (*negative*, respectively) *in* X provided every free occurrence of X in φ is within the scope of an even (odd, resp.) number of negation symbols. L_t is the fragment of L_2 obtained by allowing second-sort quantification only in the following form:

- If s is a first-sort term, $X \in Var_2$, and $\varphi \in L_t$ is positive in X, then $\forall X_s\, \varphi \in L_t$.

- If s is a first-sort term, $X \in Var_2$, and $\varphi \in L_t$ is negative in X, then $\exists X_s\, \varphi \in L_t$.

17.1.4 Properties of L_t

In this section we present some properties of L_t coming from [40, Part 1]. L_t can express some interesting topological properties:

- L_t can say that a topology is trivial:

$$\forall x\, \forall X_x \forall y\, y\varepsilon X.$$

- That a topology is discrete:

$$\forall x\, \exists X_x\, \forall y\, (y\varepsilon X \rightarrow y = x).$$

- That a topological space is T_1:

$$\forall x\, \forall y\, (x = y \vee \exists X_x \neg y\varepsilon X).$$

- That a topological space is T_2:

$$\varphi_{T_2} := \forall x\, \forall y\, (x = y \vee \exists X_x\, \exists Y_y\, \forall z\, \neg(z\varepsilon X \wedge z\varepsilon Y)).$$

- That a topological space is regular:

$$\varphi_{\mathrm{reg}} := \forall x\, \forall X_x\, \exists Y_x\, \forall y\, (y\varepsilon X \vee \exists W_y\, \forall z\, \neg(z\varepsilon W \wedge z\varepsilon Y)).$$

- That a topological space is T_3:

$$\varphi_{T_3} := \varphi_{T_2} \wedge \varphi_{\mathrm{reg}}.$$

- That a topological space is Alexandroff:

$$\forall x\, \exists X_x\, \forall Y_x\, (\neg y\varepsilon X \vee y\varepsilon Y).$$

- That a space is dense-in-itself:

$$\forall x\, \forall X_x\, \exists y\, (\neg y = x \wedge y\varepsilon X).$$

- And, for every n-ary function symbol f, that f is continuous:

$$\forall x_1 \ldots \forall x_n\, \forall Y_{f(x_1,\ldots,x_n)}\, \exists X_{1\,x_1} \ldots \exists X_{n\,x_n}\, \forall y_1 \ldots \forall y_n$$
$$(y_1\varepsilon X_1 \wedge \ldots \wedge y_n\varepsilon X_n \rightarrow f(y_1, \ldots, y_n) \in Y).$$

Moreover:

Theorem 17.3. The following facts hold:

1. L_t is invariant under changing base [40, Part 1, Corollary 2.4].

Therefore:

2. *Compactness Theorem*. For every set of sentences $\Gamma \subseteq L_t$, if every finite subset of Γ has a topological model, then Γ has a topological model [40, Part 1, Theorem 3.2].

3. *Löwenheim-Skolem Theorem*. For every countable set of sentences $\Gamma \subseteq L_t$, if Γ has an infinite topological model, then Γ has a countable topological model [40, Part 1, Theorem 3.4].

Furthermore:

4. For every invariant under changing base sentence $\varphi \in L_2$, there is a sentence $\psi \in L_t$ such that, for every topological model $\mathcal{M}$, we have:

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{M} \models \psi.$$

   [40, Part 1, Theorem 4.19].

Points 1 and 4 together state that L_t is equivalent on topological models to the invariant under changing base fragment of L_2. Finally:

5. *Lindström Theorem*. Roughly, there is no language (satisfying certain properties) that is more expressive than L_t and enjoys the Compactness and Löwenheim-Skolem Theorem simultaneously. (For a more formal statement see [40, Part 1, Theorem 8.1].)

Since L_t can express a number of interesting topological properties, enjoys the Compactness and Löwenheim-Skolem Theorems, by the Lindström Theorem is maximal with respect to the property of enjoying these theorems simultaneously, and is equivalent to the invariant under changing base fragment of L_2, we believe that L_t is an excellent language to talk about topological spaces in first-order terms, and, since L translate into L_t (Theorem 17.9), we will compare the expressive power of L to that of L_t.

Nonetheless, there are interesting topological properties that L_t cannot express [40, Part 1§3]. For example connectedness, expressible in L_2 by the formula φ_{con} of Section 17.1.1. Indeed, every connected and ordered topological field is isomorphic to the field of the real numbers, Then uncountable. However, being an ordered topological field is expressible in L_t. If connectedness were expressible in L_t too, then, by the Löwenheim-Skolem Theorem, there would be a countable connected and ordered topological field, a contradiction [40, Part 1, Proof of Corollary 3.6].

17.2 Expressive power of L

In this section we compare the expressive power of L to the expressive power of L_t, establishing that the two languages are equivalent on T_3 topological models (see Definitions 17.4 and 17.5, and Theorem 17.13).

More formally, define L_t on the 'same signature' of L. That is, a signature σ consisting only of the relation symbol ε and the propositional variables of L, intended as unary relation symbols taking as arguments first-sort variables. An example of formula in L_t is:

$$\forall X_x \, \exists y \, (\neg y = x \wedge y \varepsilon X \wedge p(x)).$$

Observe that this is nothing else than the semantics of $\langle d \rangle$.

Incidentally, one may ask whether L_2, defined over this signature and interpreted over topological models, enjoys the Compactness and Löwenheim-Skolem Theorems simultaneously. The answer is no (even if the signature consists of the only relation symbol ε). Indeed, suppose by contradiction that L_2, defined over this signature and interpreted over topological models, enjoyed the Löwenheim-Skolem Theorem. Then, as mentioned in [40, Part 1, Corollary 3.5], since L_2 can express that a topological space is regular and not normal, namely by the sentence $\varphi_{\text{reg}} \wedge \neg\varphi_{\text{nor}}$, there would be a regular and not normal topological space with the topology generated by a countable base. However, regular topological spaces with the topology generated by a countable base are normal (see [99, Example 15.3(c), and Theorem 23.1]), and we would obtain a contradiction.

Since the interpretation of ε is fixed to set-membership, from now on we will omit mentioning ε when presenting topological models for L_t.

Moreover, observe that, for every topological model $\mathcal{M} = (A, \tau, V)$ for L, and $p \in \Phi$, if we intend $V(p)$ as the interpretation of p, then $\mathcal{M}$ can be seen as a topological model for L_t. Conversely, for every topological model $\mathcal{M} = (A, \tau, \{p^{\mathcal{M}} \mid p \in \Phi\})$ for L, and $p \in \Phi$, if we intend $p^{\mathcal{M}}$ as the evaluation of p, then $\mathcal{M}$ can be seen as a topological model for L. Then, we will confuse the two notions and simply call topological models these objects.

Finally:

Definition 17.4. Given a class $\mathfrak{M}$ of topological models, we say that:

1. *L_t can express L sententially on* $\mathfrak{M}$ provided, for every sentence $\varphi \in L$, there is a sentence $\psi \in L_t$ such that *φ and ψ are equivalent on* $\mathfrak{M}$ - that is, for every topological model $\mathcal{M} \in \mathfrak{M}$, we have:

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{M} \models \psi.$$

2. *L can express L_t sententially on* $\mathfrak{M}$ provided, for every sentence $\varphi \in L_t$, there is

a sentence $\psi \in L$ such that φ and ψ are equivalent on $\mathfrak{M}$.

3. If both 1 and 2 hold, then we say that *L_t and L are sententially equivalent on* $\mathfrak{M}$.

Definition 17.5. Given a class $\mathfrak{M}$ of topological models, we say that:

1. *L_t can express L on* $\mathfrak{M}$ provided, for every formula $\varphi \in L$, there is a formula $\psi[x] \in L_t$ such that *φ and ψ are equivalent on* $\mathfrak{M}$ - that is, for every topological model $\mathcal{M} \in \mathfrak{M}$, and $a \in \mathcal{M}$, we have:

$$\mathcal{M}, a \models \varphi \text{ if and only if } \mathcal{M} \models \psi[a].$$

2. *L can express L_t on* $\mathfrak{M}$ provided, for every formula $\varphi[x] \in L_t$, there is a formula $\psi \in L$ such that φ and ψ are equivalent on $\mathfrak{M}$.

3. If both 1 and 2 hold, then we say that *L_t and L are equivalent on* $\mathfrak{M}$.

Definition 17.4 is motivated by the fact that sentences are evaluated on whole topological models. Definition 17.5 is motivated by the fact that modal formulas are evaluated at points of topological models. Observe that:

Proposition 17.6. For every class $\mathfrak{M}$ of topological models, if L_t can express L on $\mathfrak{M}$, then L_t can express L sententially on $\mathfrak{M}$.

Proof. Suppose that L_t can express L on $\mathfrak{M}$. Consider a sentence $\varphi \in L$. Then, φ is a formula in L. Then, since, by hypothesis, L_t can express L on $\mathfrak{M}$, there is a formula $\psi[x] \in L_t$ such that φ and ψ are equivalent on $\mathfrak{M}$. Then, plainly, φ and $\forall x \psi$ are equivalent on $\mathfrak{M}$. Then, thesis follows by observing that $\forall x \psi$ is a sentence of L_t. $\square$

Proposition 17.7. For every class $\mathfrak{M}$ of topological models, if L can express L_t on $\mathfrak{M}$, then L can express L_t sententially on $\mathfrak{M}$.

Proof. Suppose that L can express L_t on $\mathfrak{M}$. Consider a sentence $\varphi \in L_t$. Then, φ is a formula in L_t with at most one free first-sort variable. Then, since, by hypothesis, L can express L_t on $\mathfrak{M}$, there is a formula $\psi \in L$ such that φ and ψ are equivalent on $\mathfrak{M}$. Then, plainly, φ and $\Box_0\psi$ are equivalent on $\mathfrak{M}$. Then, thesis follows by observing that $\Box_0\psi$ is a sentence of L. $\square$

Corollary 17.8. For every class $\mathfrak{M}$ of topological models, if L is equivalent to L_t on $\mathfrak{M}$, then L is sententially equivalent to L_t on $\mathfrak{M}$.

In the next two sections we will prove that L and L_t are equivalent on the class of all T_3 topological spaces.

17.2.1 L_t can express L on topological models

L_t can 'always' express L:

Theorem 17.9. The following facts hold:

1. L_t can express L sententially on the class of all topological models.

2. L_t can express L on the class of all topological models.

To prove this, we define the 'standard translation' from L to L_t.

Definition 17.10. For every formula $\varphi \in L$, and $x \in Var_1$, define the *standard translation $ST_x(\varphi)$ of φ (from L to L_t)* recursively as follows:

- If φ is $\top$, then $ST_x(\varphi)$ is $\top$.

- If φ is $p \in \Phi$, then $ST_x(\varphi)$ is $p(x)$.

- If φ is $\neg\psi$, then $ST_x(\varphi)$ is $\neg ST_x(\psi)$.

- If φ is $\psi \wedge \chi$ then $ST_x(\varphi)$ is $ST_x(\psi) \wedge ST_x(\chi)$.

- For every $n \in \mathbf{N}$, if φ is $\Diamond_n\psi$, then $ST_x(\varphi)$ is:

$$\exists x_0 \ldots \exists x_n \, (\bigwedge_{\substack{i,j \in n+1 \\ i \neq j}} \neg x_i = x_j \wedge \bigwedge_{i \in n+1} ST_{x_i}(\psi)),$$

  with $x_0, \ldots, x_n$ pairwise distinct first-sort variables.

- If φ is $[d]\psi$, then $ST_x(\varphi)$ is:

$$\exists X_x \, \forall y \, (\neg x = y \wedge y \varepsilon X \rightarrow ST_y(\psi)),$$

  with $y \in Var_1 \setminus \{x\}$.

Remark 17.11. Plainly, for every formula $\varphi \in L$, we have that $ST_x(\varphi)$ is a formula in L_t with at most one free first-sort variable, namely x. Moreover, if φ is a sentence in L, then $ST_x(\varphi)$ is a sentence in L_t.

Proof of Theorem 17.9. For every $\varphi \in L$, take $ST_x(\varphi)$. A simple induction shows that:

Lemma 17.12. For every topological model $\mathcal{M}$, and $a \in \mathcal{M}$, we have:

$$\mathcal{M}, a \models \varphi \text{ if and only if } \mathcal{M} \models ST_x(\varphi)[a].$$

Proof. The only interesting cases are $\Diamond_n\psi$ and $\langle d \rangle\psi$.

Suppose that φ is $\Diamond_n\psi$. Consider an arbitrary topological model $\mathcal{M}$, and an arbitrary $a \in \mathcal{M}$. Suppose that $\mathcal{M}, a \models \varphi$. Then, by semantics, this is equivalent to the existence of pairwise distinct $a_0, ..., a_n \in \mathcal{M}$ such that $\mathcal{M}, a_i \models \psi$ $(i \in n)$. Then, by inductive hypothesis, this is equivalent to $\mathcal{M} \models ST_{x_i}(\psi)[a_i]$ ($i \in n$, and $x_0, ..., x_n \in Var_1$ pairwise distinct). Then, by semantics, this is equivalent to $\mathcal{M} \models ST_x(\varphi)[a]$.

Suppose that φ is $\langle d \rangle \psi$. Consider an arbitrary topological model $\mathcal{M}$, and an arbitrary $a \in \mathcal{M}$. Suppose that $\mathcal{M}, a \models \varphi$. Then, by semantics, this is equivalent to: for every neighborhood of a, there is a point $b \in U \setminus \{a\}$ such that $\mathcal{M}, b \models \psi$. Then, by inductive hypothesis, this is equivalent to $\mathcal{M} \models ST_y(\psi)[b]$ $(y \in Var_1 \setminus \{x\})$. Then, by semantics, this is equivalent to $\mathcal{M} \models ST_x(\varphi)[a]$. □

From Lemma 17.12 and Remark 17.11, points 1 and 2 easily follow. □

17.2.2 L_t and L are equivalent on T_3 topological models

In this section we prove that L is equivalent to L_t on the class of all T_3 topological models. More precisely, we prove that:

Theorem 17.13. The following facts hold:

1. L can express L_t sententially on the class of all T_3 topological models. Moreover, for every sentence $\varphi \in L_t$, there is an equivalent sentence $\psi \in L$ such that $qd(\varphi) = md(\psi)$.

2. L can express L_t on the class of all T_3 topological models. Moreover, for every formula $\varphi[x] \in L_t$, there is an equivalent formula $\psi \in L$ such that $qd(\varphi) = md(\psi)$.

This, together with Theorem 17.9 yields the equivalence between L and L_t on T_3 topological models. Observe that, here, we are also proving that we can go from L_t to L by preserving the quantifier/modal depth of formulas. If we were not proving this preservation property, by Proposition 17.7, we could get Point 1 from Point 2. However, proving the preservation of quantifier/modal depth in Point 1 requires a direct proof of Point 1.

To prove Theorem 17.13, we introduce (Definition 17.14) the Ehrenfeucht-Fraïssé game $G_n(\mathcal{M}, \mathcal{N})$ ($n \in \mathbf{N}$ and $\mathcal{M}, \mathcal{N}$ topological models) of [40, Part 1, §4]. To prove 1, we prove that: if Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$, then $\mathcal{M}$ and $\mathcal{N}$ agree on the sentences of L_t with quantifier depth at most n (Corollary 17.16); and if $\mathcal{M}$ and $\mathcal{N}$ ($\mathcal{M}, \mathcal{N}$ T_3 0-dimensional topological models) agree on the same sentences of L with modal depth at most n, then Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ (Proposition 17.18).

To prove 2, we prove that: if Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ ($\mathcal{M}, \mathcal{N}$ topological models) after $(a, b) \in \mathcal{M} \times \mathcal{N}$ has been played, then $\mathcal{M}$ and $\mathcal{N}$ agree on the formulas of L_t with quantifier depth at most $n - 1$ and at most one free variable assigned to a and b respectively (Corollary 17.17); and if $\mathcal{M}, a$ and $\mathcal{N}, b$ ($\mathcal{M}, \mathcal{N}$ T_3 0-dimensional topological models, and $(a, b) \in \mathcal{M} \times \mathcal{N}$) agree on the same formulas of

L with modal depth at most $n-1$, then Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ after $(a, b) \in \mathcal{M} \times \mathcal{N}$ has been played (Proposition 17.21).

Definition 17.14. For every $n \in \mathbf{N}$, and two distinct[2] models $\mathcal{M}$ and $\mathcal{N}$, define the game $G_n(\mathcal{M}, \mathcal{N})$ between Player I and Player II as follows. n rounds are played. At every round of the game, Player I first chooses a model $C \in \{\mathcal{M}, \mathcal{N}\}$, then Player I chooses either to play a point $c \in C$ or to play a couple (c, O), where c is a point in C already played either by Player I or by Player II and O is a neighborhood of c. Then, if Player I chose a point $c \in C$, Player II must choose a point d in $\mathcal{D} \in \{\mathcal{M}, \mathcal{N}\} \setminus \{C\}$ while, if Player I chose a couple (c, O), Player II must choose a couple (d, P) such that d is the point played in the same round as c, and P is an open in $\mathcal{D} \in \{\mathcal{M}, \mathcal{N}\} \setminus \{C\}$ containing d. Player II wins if, at the end of the game:

1. If points $a \in \mathcal{M}$ and $b \in \mathcal{N}$ were played in the same round, and points $a' \in \mathcal{M}$ and $b' \in \mathcal{N}$ were played in the same round, then $a = a'$ if and only if $b = b'$, and, for every $p \in \Phi$, $p^{\mathcal{M}}(a)$ if and only if $p^{\mathcal{N}}(b)$.
2. For every model $C \in \{\mathcal{M}, \mathcal{N}\}$, and point $c \in C$ played either by Player I or by Player II, if c is contained in an open O played by Player II, then the point d played in the same round as c is contained in the open played by Player I in the same round as O.

From now on, if we present a play in $G_n(\mathcal{M}, \mathcal{N})$ as (a, b), then we mean that $a \in \mathcal{M}$ and $b \in \mathcal{N}$; similarly, if we present a play in $G_n(\mathcal{M}, \mathcal{N})$ as $((a, U), (b, V))$, then we mean that $a \in \mathcal{M}$ and U is a neighborhood of a, and $b \in \mathcal{N}$ and V is a neighborhood of b.

The game G_n offers a criterium to check whether two models are indistinguishable by sentences in L_t of quantifier depth at most n (Corollary 17.16), and whether two points in two models are indistinguishable by formulas in L_t with at most one free variable and quantifier depth at most $n-1$ (Corollary 17.17). To prove this, for every $\varphi \in L_t$, define φ to be in *negation normal form* provided, treating $\vee$ and $\exists$ as additional primitive operators, negation signs in φ occur only in front of atomic formulas. Then, prove the following Proposition (cf. [40, Part 1, Lemma 4.6]):

Proposition 17.15. Let n be an arbitrary natural number. Let φ be an arbitrary formula of $\in L_t^n$ such that:

- φ is in negation normal form;
- Each of φ's free second-sort variables is either positive or negative - say φ is $\varphi[\overline{x}, \overline{X^+}, \overline{X^-}]$, with $\overline{x} = (x_1, ..., x_{m_1})$ denoting the array of the first-sort variables occurring free in φ, $\overline{X^+} = (X^+_{m_1+1}, ..., X^+_{m_2})$ denoting the array of the positive

[2] Assuming the models distinct will simplify the notation. This assumption is harmless because we can always assume the two models distinct, for example by conveniently renaming the elements of one of them.

second-sort variables occurring free in φ, and $\overline{X^-} = (X^-_{m_2+1}, ..., X^-_m)$ denoting the array of the negative second-sort variables occurring free in φ;

- $qd(\varphi) + m \leq n$.

Assume that:

- Plays and only plays $\{(a_i, b_i)\}_{i=1}^{m_1}$, $\{((a_i, U_i^+), (b_i, V_i^+))\}_{i=m_1+1}^{m_2}$, and $\{((a_i, U_i^-), (b_i, V_i^-))\}_{i=m_2+1}^{m}$ have been played in some order;

- Player I played all the (a, U^-)s;

- Player II played all the (a, U^+)s;

- Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ at this point of the game.

Then:

- $\mathcal{M} \models \varphi[\overline{a}, \overline{U^+}, \overline{U^-}]$ implies $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

Proof. By induction on φ. Since φ is in negation normal form, we treat the following cases: $\top$, $x = y$, $x \varepsilon X$, $p(x)$, $\neg\top$, $\neg x = y$, $\neg x \varepsilon X$, $\neg p(x)$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall X_x \psi$, and $\exists X_x \psi$.

The $\top$ case is plain. Assume that φ is $x = y$. Then φ is $x_i = x_j$ for some $i, j \in \mathbf{N}$. Assume $\mathcal{M} \models x_i = x_j[\overline{a}, \overline{U^+}, \overline{U^-}]$. Then, $a_i = a_j$. Then, since Player II has a winning strategy at this point of the game, by winning condition 1, we have $b_i = b_j$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

Assume that φ is $x \varepsilon X$. Then, X occurs positively in φ. Then, φ is $x_i \varepsilon X_j^+$ for some $i, j \in \mathbf{N}$. Assume $\mathcal{M} \models x_i \varepsilon X_j[\overline{a}, \overline{U^+}, \overline{U^-}]$. Then, $a_i \in U_j^+$. Now, by hypothesis, U_j^+ has been played by Player II. Then, since Player II has a winning strategy at this point of the game, by winning condition 2, we have $b_i \in V_j^+$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

Assume that φ is $p(x)$. Then, φ is $p(x_i)$ for some $i \in \mathbf{N}$. Assume $\mathcal{M} \models p(x_i)[\overline{a}, \overline{U^+}, \overline{U^-}]$. Then, $p^{\mathcal{M}}(a_i)$. Then, since Player II has a winning strategy at this point of the game, by winning condition 1, we have $p^{\mathcal{N}}(b_i)$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

The $\neg\top$ case is plain. Assume that φ is $\neg(x = y)$. Then, φ is $\neg(x_i = x_j)$ for some $i, j \in \mathbf{N}$. Assume $\mathcal{M} \models x_i = x_j[\overline{a}, \overline{U^+}, \overline{U^-}]$. Then, $a_i \neq a_j$. Then, since Player II has a winning strategy at this point of the game, by winning condition 1, we have $b_i \neq b_j$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

Assume that φ is $\neg(x \varepsilon X)$. Then, X occurs negatively in φ. Then, φ is $\neg(x_i \varepsilon X_j^-)$ for some $i, j \in \mathbf{N}$. Assume $\mathcal{M} \models \neg(x_i \varepsilon X_j)[\overline{a}, \overline{U^+}, \overline{U^-}]$. Then, $a_i \notin U_j^-$. Now, by hypothesis, V_j^- has been played by Player II. Then, since Player II has a winning strategy at this point of the game, by winning condition 2, we have $b_i \notin V_j^-$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

Assume that φ is $\neg p(x)$. Then, φ is $\neg p(x_i)$ for some $i \in \mathbf{N}$. Assume $\mathcal{M} \models \neg p(x_i)[\overline{a}, \overline{U^+}, \overline{U^-}]$. Then, not $p^{\mathcal{M}}(a_i)$. Then, since Player II has a winning strategy at this point of the game, by winning condition 1, we have not $p^{\mathcal{N}}(b_i)$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

The $\wedge$, $\vee$ cases are plain. Assume that φ is $\exists X_x\, \psi$. Then, x is x_i for some $i \in \mathbf{N}$ and, by definition of L_t, X is negative in $\psi[\overline{x}, \overline{X^+}, \overline{X^-}X]$. Assume:

$$\mathcal{M} \models \exists X_{x_i}\, \psi[\overline{a}, \overline{U^+}, \overline{U^-}].$$

Then, there is a neighborhood U of a_i such that $\mathcal{M} \models \psi[\overline{a}, \overline{U^+}, \overline{U^-}U]$. Let Player I play (a_i, U). Observe that in this case $qd(\varphi) > 0$. Then, $m < n$. Then, Player II can still make a move according to its winning strategy. Let Player II play so choosing (b_i, V). By definition of the game, V is a neighborhood of b_i. Observe that ψ and the plays $\{(a_i, b_i)\}_{i=1}^{m_1}$, $\{((a_i, U_i^+), (b_i, V_i^+))\}_{i=m_1+1}^{m_2}$, $\{((a_i, U_i^-), (b_i V_i^-))\}_{i=m_2+1}^{m}$, and $((a_i, U), (b_i, V))$ satisfy the inductive hypothesis. Then, since $\mathcal{M} \models \psi[\overline{a}, \overline{U^+}, \overline{U^-}U]$, by inductive hypothesis, we have $\mathcal{N} \models \psi[\overline{b}, \overline{V^+}, \overline{V^-}V]$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$.

Assume that φ is $\forall X_x\, \psi$. Then, x is x_i for some $i \in \mathbf{N}$ and, by definition of L_t, X is positive in $\psi[\overline{x}, \overline{X^+}X, \overline{X^-}]$. Assume:

$$\mathcal{M} \models \forall X_{x_i}\, \psi[\overline{a}, \overline{U^+}, \overline{U^-}].$$

Then, for every neighborhood U of a_i, we have $\mathcal{M} \models \psi[\overline{a}, \overline{U^+}U, \overline{U^-}]$. Consider an arbitrary neighborhood V of b_i and let Player I play (b_i, V). Observe that in this case $qd(\varphi) > 0$. Then, $m < n$. Then, Player II can still make a move according to its winning strategy. Let Player II play so choosing (a_i, U). By definition of the game, U is a neighborhood of a_i. Observe that ψ and the plays $\{(a_i, b_i)\}_{i=1}^{m_1}$, $\{((a_i, U_i^+), (b_i, V_i^+))\}_{i=m_1+1}^{m_2}$, $\{((a_i, U_i^-), (b_i V_i^-))\}_{i=m_2+1}^{m}$, and $((a_i, U), (b_i, V))$ satisfy the inductive hypothesis. Then, since $\mathcal{M} \models \psi[\overline{a}, \overline{U^+}U, \overline{U^-}]$, by inductive hypothesis, we have $\mathcal{N} \models \psi[\overline{b}, \overline{V^+}V, \overline{V^-}]$. Then, $\mathcal{N} \models \varphi[\overline{b}, \overline{V^+}, \overline{V^-}]$. □

Now, observe that every formula in L_t^n admits an equivalent formula in L_t^n in negation normal form. Then, by Proposition 17.15, the following corollaries follow:

Corollary 17.16. (Cf. [40, Lemma 4.6].) For every $n \in \mathbf{N}$, and two models $\mathcal{M}$ and $\mathcal{N}$, if Player II has a winning strategy for $G_n(\mathcal{M}, \mathcal{N})$, then, for every sentence $\varphi \in L_t^n$, we have:

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{N} \models \varphi.$$

Corollary 17.17. For every $n \in \mathbf{N}$, two models $\mathcal{M}$ and $\mathcal{N}$, and two points $a \in \mathcal{M}$ and $b \in \mathcal{N}$, if Player II has a winning strategy for $G_n(\mathcal{M}, \mathcal{N})$ after (a, b) has been played, then, for every formula $\varphi[x]$ of L_t^{n-1}, we have:

$$\mathcal{M} \models \varphi[a] \text{ if and only if } \mathcal{N} \models \varphi[b].$$

We now prove that:

Proposition 17.18. Assume that Φ is finite. Then, for every $n \in \mathbf{N}$, for every two T_3 0-dimensional topological models $\mathcal{M}$ and $\mathcal{N}$, if $\mathcal{M} \equiv_L^n \mathcal{N}$, then Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$.

Observe that, being a T_0 0-dimensional topological model implies T_3ness. So we could just as well have replaced T_3ness with T_0ness in the hypothesis. We kept T_3ness for sake of simplicity. Proposition 17.18 and its Corollary 17.21, together with Corollaries 17.16 and 17.17, will let us prove Theorem 17.13.

To prove Proposition 17.18, let us fix some notation:

Definition 17.19. 1. For every $n \in \mathbf{N}$, topological model $\mathcal{M}$, and point $a \in \mathcal{M}$, define $type^n(\mathcal{M}, a) := \{\varphi \in L^n \mid \mathcal{M}, a \models \varphi\}$. If the model $\mathcal{M}$ is understood, then $type^n(\mathcal{M}, a)$ may be referred as $type^n(a)$.

2. For every two formulas $\varphi, \psi \in L$, φ and ψ are said to be *equivalent* provided for every topological model $\mathcal{M}$, and point $a \in \mathcal{M}$, we have:

$$\mathcal{M}, a \models \varphi \text{ if and only if } \mathcal{M}, a \models \psi.$$

3. For every subset $\Gamma \subseteq L$, call a *representative of* Γ every subset $R \subseteq \Gamma$ of in-equivalent formulas such that every formula in Γ is equivalent to a formula in R.

4. For every $n \in \mathbf{N}$, topological model $\mathcal{M} = (A, \tau, V)$, and $B \subseteq A$, call a *representative with respect to equivalence up to modal depth n of B* every subset $R \subseteq B$ such that, for every point $a \in B$, there is a point $a' \in R$ with $a \equiv_L^n a'$ and, for every pair a, a' of points in R, we have $a \not\equiv_L^n a'$.

Proof of Proposition 17.18. First let us make the following remark:

Remark 17.20. Observe that, by winning conditions 1 and 2 for Player II in $G_n(\mathcal{M}, \mathcal{N})$, for every natural number $m \leq n$, if Player I chooses (c, O) and Player II has a winning strategy, then Player II would have a winning strategy also if Player I had chosen (c, O') with O a subset of O'. Therefore, there is no harm in assuming that Player I plays as small opens as desired.

Now, assume that m rounds, with $m < n$, have been played, that Player II is still in the game and that:

1. For every natural number $i \leq m$, if (a_i, b_i) were played at round i, then $a_i \equiv_L^{n-i} b_i$;

2. For every natural number $i \leq m$, and (c, O) played in $C \in \{\mathcal{M}, \mathcal{N}\}$ either by Player I or by Player II at round i:

    (a) O is a clopen;

(b) O is included in or disjoint from every other open played either by Player I or by Player II earlier than the ith round. That is, if O' has been played by Player I or by Player II earlier than the ith round then $O \subseteq O'$ or $O \cap O' = \emptyset$;

(c) O contains c and does not contain any other point played either by Player I or by Player II earlier than the ith round;

(d) For every point c' in $O \setminus \{c\}$, and every neighborhood O' of c, there is $c'' \in O' \setminus \{c\}$ with $c'' \equiv_L^{n-(i+1)} c'$;

(e) For every point $c' \in O \setminus \{c\}$ there are at least $n - i$ points:

   i. Modally equivalent up to modal depth $n - (i + 1)$ to c';

   ii. Different from every point played either by Player I or by Player II earlier than the ith round;

   iii. Not contained in O nor in any open played either by Player I or by Player II earlier than the ith round disjoint from O;

   iv. Contained in the intersection of all opens played either by Player I or by Player II earlier than the ith round and including O (if any).

Consider the $m + 1$st round and assume that Player I has played in $\mathcal{M}$. Let us show that Player II can play without losing $G_n(\mathcal{M}, \mathcal{N})$ and that Assumptions 1 and 2 remain satisfied (with m replaced by $m + 1$ in their statement). Let us consider the possible moves of Player I:

1. Player I chose a point $a \in \mathcal{M}$. If a is a point a' already played either by Player I or by Player II, then Player II chooses the point $b \in \mathcal{N}$ played at the same round as a'. By Assumption 2c, Player II is still in the game. Otherwise, a is different from every point played either by Player I or by Player II. Let us consider two cases:

   (a) a does not belong to any open played by Player II. Consider, among the points played either by Player I or by Player II earlier than a, those that are modally equivalent up to modal depth $n - (m + 1)$ to a. Let l be their number. Let R be a *representative* of $type^{n-(m+1)}(a)$. Since Φ is finite, we can assume R finite as well. Then, $\mathcal{M} \models \Diamond^l \bigwedge R$. Observe that $l \leq m$. Then $(l + 1) + n - (m + 1) \leq n$. Now, by definition of modal depth, $md(\Diamond_l \bigwedge R) = (l + 1) + n - (m + 1)$. Then, $md(\Diamond^l \bigwedge R) \leq n$. Then, since $\mathcal{M} \equiv_L^n \mathcal{N}$, we have $\mathcal{N} \models \Diamond^l \bigwedge R$. So, by semantics, there are at least $l + 1$ points in B modally equivalent up to modal depth $n - (m + 1)$ to a.

   By Assumption 1, l is also the number of points played either by Player I or by Player II in $\mathcal{N}$ modally equivalent up to modal depth $n - (m + 1)$ to a. Therefore, there is a point $b \in B$, modally equivalent up to modal depth $n - (m + 1)$ to a that has not been played either by Player I or by Player II.

Now, two cases are given:

i. If b is not contained in any open played by Player II, then Player II chooses b.

ii. Otherwise, b is contained in some open played by Player II. Consider the latest open $V \in B$ played by Player II that at the end of round m is not contained in an open played by Player II (other than itself) and that contains a point $b' \equiv_L^{n-(m+1)} b$.

Assume that i is the round in which V was played. By Assumption 2e and choice of V, at the end of round i there were at least $n - i$ points different from every point played either by Player I or by Player II earlier than round i, not contained in V nor in any open played by Player II earlier than the ith round, and modally equivalent up to modal depth $n - (i + 1)$ to b.

By Assumption 2b and choice of V, every open played by Player II in B later than the ith round is either included in V or included in some open disjoint from V played by Player II earlier than the ith round or does not contain points b' modally equivalent up to modal depth $n - (m + 1)$ to b.

Then, at round m, there are still at least $n - i$ points not contained in any open played by Player II modally equivalent up to modal depth $n - (i + 1)$ to b'. Observe that, since $i \leq m$, we have $n - (i + 1) \geq n - (m + 1)$. Then, at round m, there are at least $n - i$ points not contained in any open played by Player II modally equivalent up to modal depth $n - (m + 1)$ to b.

After round i until the end of round m, at most $m - i$ among these points have been played. So, at least $(n - i) - (m - i) = n - m$ points among these points remain to be played. Observe that, since $m < n$, $n - m > 0$. Let Player II choose one of these remaining points. It is obvious that Player II, by choosing a point in this way, chooses a point modally equivalent up to modal depth $n - (m + 1)$ to a, different from every point played either by Player I or by Player II, and not contained in any open played by Player II.

(b) Otherwise a belongs to some opens played by Player II, say, in the order in which they were played, $U_1, U_2, ..., U_l$. Let a' be the point played with U_l and (b', V_l) be the couple played by Player I at the same round as (a', U_l).

By definition of $G_n(\mathcal{M}, \mathcal{N})$, $a' \in U_l$. Since $\mathcal{N}$ is T_3 and hence T_1, for every point b different than b' played either by Player I or by Player II, there is a neighborhood V_b of b' not containing b. Since the number of points b played by either Player I or Player II different from b' is finite, if V denotes the intersection of the V_bs, then $V \cap V_l$ is a neighborhood of b' included in V_l, not containing any of the points different from b' played either by Player I or by Player II.

As before, we can find a finite representative R of $type^{n-(m+1)}(a)$. Since U_l was played with a' and $a \in U_l \setminus \{a'\}$, by Assumption 2d, we have $\mathcal{M}, a' \models \langle d \rangle \bigwedge R$. Then, since, by Assumption 1, $a' \equiv_{L^n}^{n-m} b'$ and, by definition of modal depth, $md(\langle d \rangle \bigwedge R) = n - m$, we obtain $\mathcal{N}, b' \models \langle d \rangle \bigwedge R$.

Since $\mathcal{N}, b' \models \langle d \rangle \bigwedge R$, we can find a point b in $V \cap \bigcap_{i=1}^{l} V_i$, then different from any other point played either by Player I or by Player II, modally equivalent up to modal depth $n - (m + 1)$ to a. Now, two cases are given:

i. If b does not belong to any open played by Player II, then Player II chooses b;

ii. Otherwise b belongs to some open played by Player II, say V. Observe that, by assumption 2b, if V is included in V_l, then V has been played later than V_l.

Then, by arguments similar to those used in case 1(a)ii, there is a point b'' different from every point played either by Player I or by Player II, contained in V_l, not contained in any open played by Player II included in V_l, and modally equivalent up to modal depth $n - (m + 1)$ to a.

Then, by assumption 2b, if b'' is contained in some open V played by Player II, then V was played before than V_l. Moreover, as $b'' \in V_l$, V contains V_l.

Then, as $b' \in V_l$, $b' \in V$. Then, if U is the open played at the same round as V, by winning condition 2, $a' \in U$. Then, as $a' \in U_l$ and U has been played earlier than U_l, by Assumption 2b, U includes U_l. So, as $a \in U_l$, $a \in U$. Therefore, if Player II chooses b'', Player II is still in the game and Assumptions 1 and 2 remain satisfied.

2. Player I chooses (a, U). Let us define a neighborhood of a included in U and satisfying Assumption 2. By Remark 17.20, U can then be assumed to be this open. In order to do this, we define a sequence of neighborhoods of a, $U_1 \supseteq U_2 \supseteq ... \supseteq U_5$ with U_5 as desired.

(U_1) Let $\mathcal{X}$ be the set of all opens played either by Player I or by Player II until the end of round m containing a, and $\mathcal{X}'$ be the set of all opens played either by Player I or by Player II until the end of round m not containing a. Observe that, by Assumption 2a, for every $U' \in \mathcal{X}'$, we have that U' is clopen and so that $A \setminus U'$ is open. Let $U'' := \bigcap_{U' \in \mathcal{X}} U' \cap \bigcap_{U' \in \mathcal{X}'} (A \setminus U')$. Since $\mathcal{X} \cup \mathcal{X}'$ is finite, the set $U_1 := U \cap U''$ is a neighborhood of a, included in U and satisfying Assumption 2b.

(U_2) Consider all the points different from a played either by Player I or by Player II. With arguments similar to those above, we can find a neighborhood U_2 of a contained in U_1, and, then, still satisfying Assumption 2b, satisfying Assumption 2c as well.

(U_3) Let us focus on the points $a' \in U_2 \setminus \{a\}$ such that there is a neighborhood $U'_{a'}$ of a with, for every $a'' \in U'_{a'} \setminus \{a\}$, $a'' \not\equiv^{n-(m+1)}_{L^n} a'$. Let P be the set of these points.

Consider a representative with respect to equivalence up to modal depth $n - (m + 1)$ R of P. Since Φ is finite, we can assume R finite as well. Let $U' := \bigcap_{a' \in R} U_{a'}$ and $U_3 := U_2 \cap U'$. U_3 satisfies Assumptions 2b, 2c and 2d.

(U_4) Consider a representative with respect to equivalence up to modal depth $n-(m+1)$ R of $U_3 \setminus \{a\}$. Again, and for the same reasons as above, we can assume R finite.

For every $a' \in R$, since A is T_3 and hence T_1, we can find a neighborhood U' of a such that $a' \notin U'$. If we define $U'_4 := U_3 \cap U'$, we obtain a neighborhood of a still satisfying 2b, 2c and 2d and such that the number of points modally equivalent up to modal depth $n - (m + 1)$ to a' outside U'_4 is one more than the number of points modally equivalent up to modal depth $n - (m + 1)$ to a' outside U_3.

Observe that, since U'_4 satisfies Assumption 2d, there is a point $a'' \in U'_4$ with $a'' \equiv^{n-(m+1)}_{L^n} a'$ and the same operation can be repeated. Plainly, by repeating this operation as finitely many times as necessary and for every $a' \in R$, we can define a neighborhood U_4 of a still satisfying 2b, 2c and 2d and satisfying 2e as well.

(U_5) Finally, since by assumption $\mathcal{M}$ is 0-dimensional, there is a clopen neighborhood U_5 of a included in U_4. U_5 satisfies 2a, 2b, 2c, 2d and 2e, as desired.

By Remark 17.20, U can be assumed to be U_5. Let us consider the point b in B played at the same round as a. By reasoning as above, Player II can choose a neighborhood of b satisfying 2.

3. If Player I plays in $\mathcal{N}$, Player II proceeds symmetrically

Clearly, by definition of $G_n(\mathcal{M}, \mathcal{N})$ this strategy is a winning strategy for Player II in $G_n(\mathcal{M}, \mathcal{N})$. □

Corollary 17.21. Assume that Φ is finite. Then, for every $n \in \mathbf{N}$, two T_3 0-dimensional topological models $\mathcal{M}$ and $\mathcal{N}$, and points $a \in \mathcal{M}$ and $b \in \mathcal{N}$, if $\mathcal{M}, a \equiv^{n-1}_{L} \mathcal{N}, b$, then Player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ after (a, b) has been played.

Proof of Theorem 17.13. 1. Consider a sentence $\varphi \in L_t$. Let n be its quantifier depth and restrict Φ to the propositional variables occurring in φ. Then, Φ is finite. Then, the set of sentences in L^n admits a finite representative R. For every $\Gamma \subseteq R$, consider the sentence:

$$\psi_\Gamma := \bigwedge_{\psi \in \Gamma} \psi \wedge \bigwedge_{\psi \in R \setminus \Gamma} \neg\psi.$$

The sentence ψ_Γ says that, among the sentences in R, those contained in Γ are satisfied while the remainings are not. Then, since R is a representative of the set of all sentences of L^n, we have that two topological models $\mathcal{M}$ and $\mathcal{N}$ agree on the same sentences in L^n if and only if, for some $\Gamma \subseteq R$, they both satisfy ψ_Γ. Consider the formula:

$$\chi := \bigvee \{\psi_\Gamma \mid \exists \text{ a } \mathrm{T}_3 \text{ topological model } \mathcal{M} \text{ s.t. } \mathcal{M} \models ST_x(\psi_\Gamma) \wedge \varphi\}.$$

By Corollary 17.16 and Proposition 17.18, the following lemma holds:

Lemma 17.22. For every T_3 topological model $\mathcal{N}$, $\mathcal{N} \models \chi$ if and only $\mathcal{N} \models \varphi$.

Proof. From right to left is trivial. Let us prove the direction from left to right. Assume that $\mathcal{N} \models \chi$. Then, for some $\psi_\Gamma \in L^n$ such that there is a T_3 topological model $\mathcal{M}$ such that $\mathcal{M} \models ST_x(\psi_\Gamma) \wedge \varphi$, we have $\mathcal{N} \models \psi_\Gamma$. Then, as observed, since both $\mathcal{M}$ and $\mathcal{N}$ satisfy $\psi_\Gamma \in L^n$, we have $\mathcal{M} \equiv^n_L \mathcal{N}$. Now, by [40, Part 2, Theorem 1.17, p. 88], there are T_3 0-dimensional topological models $\mathcal{M}'$ and $\mathcal{N}'$ such that $\mathcal{M} \equiv_{L_t} \mathcal{M}'$ and $\mathcal{N} \equiv_{L_t} \mathcal{N}'$. Then, since $\mathcal{M} \equiv^n_L \mathcal{N}$, $\mathcal{M} \equiv_{L_t} \mathcal{M}'$, and $\mathcal{N} \equiv_{L_t} \mathcal{N}'$, by Theorem 17.9, we have $\mathcal{M}' \equiv^n_L \mathcal{N}'$. Then, by Proposition 17.18, Player II has a winning strategy in $G_n(\mathcal{M}', \mathcal{N}')$. Then, by Proposition 17.16, $\mathcal{M}'$ and $\mathcal{N}'$ agree on the sentences of L^n_t. Then, $\mathcal{M}$ and $\mathcal{N}$ agree on the sentences of L^n_t. Then, since φ is a sentence of L^n_t and $\mathcal{M} \models \varphi$, we have that $\mathcal{N} \models \varphi$ as desired. □

Then, χ is the desired sentence.

2. Consider an arbitrary formula $\varphi[x] \in L_t$. Let n be its quantifier depth and restrict Φ to the propositional variables occurring in φ. Then, Φ is finite and L^n admits a finite representative R. For every $\Gamma \subseteq R$, consider the formula:

$$\psi_\Gamma := \bigwedge_{\psi \in \Gamma} \psi \wedge \bigwedge_{\psi \in R \setminus \Gamma} \neg\psi.$$

For every two topological models $\mathcal{M}$ and $\mathcal{N}$, and points $a \in \mathcal{M}$ and $b \in \mathcal{N}$, we have that $\mathcal{M}, a \equiv^n_L \mathcal{N}, b$ if and only if, for some $\Gamma \subseteq L^n$, $\mathcal{M}, a \models \psi_\Gamma$ and $\mathcal{N}, b \models \psi_\Gamma$. Consider the formula:

$$\chi := \bigvee \{\psi_\Gamma \mid \exists \text{ a } \mathrm{T}_3\ \mathcal{M} \text{ and } a \in \mathcal{M} \text{ s.t. } \mathcal{M} \models (ST_x(\psi_\Gamma) \wedge \varphi)[a]\}.$$

By Corollaries 17.17 and 17.21, the following lemma holds:

Lemma 17.23. For every T_3 topological model $\mathcal{N}$ and point $b \in \mathcal{N}$ we have:

$$\mathcal{N}, b \models \chi \text{ if and only if } \mathcal{N} \models \varphi[b].$$

Proof. From right to left is trivial. Let us prove the direction from left to right. Assume that $\mathcal{N}, b \models \chi$. Then, for some $\psi_\Gamma \in L^n$ such that there is a T_3 topological model $\mathcal{M}$ and a point $a \in \mathcal{M}$ with $\mathcal{M} \models (ST_x(\psi_\Gamma) \wedge \varphi)[a]$, we have $\mathcal{N}, b \models \psi_\Gamma$. Then, since both $\mathcal{M}, a \models \psi_\Gamma$ and $\mathcal{N}, b \models \psi_\Gamma$, we have $\mathcal{M}, a \equiv^n_L \mathcal{N}, b$.

Consider a new constant symbol c. Let $L_{t\{c\}}$ denote the language defined over the signature of L_t augmented with c. For every topological model $\mathcal{D}$ for L_t and point $d \in \mathcal{D}$, let $\mathcal{D}_{\{d\}}$ denote the topological model for $L_{t\{c\}}$ obtained from $\mathcal{D}$ by adding d as the interpretation of c. If we present a topological model for $L_{t\{c\}}$ as $\mathcal{D}_{\{d\}}$, then we mean that $\mathcal{D}$ is a topological model for L_t and d is the interpretation of c in $\mathcal{D}_{\{d\}}$.

By [40, Part 2, Theorem 1.17, p. 88], there are T_3 0-dimensional topological models $\mathcal{M}'_{\{a'\}}$ and $\mathcal{N}'_{\{b'\}}$ of $L_{t\{c\}}$ and such that $\mathcal{M}_{\{a\}} \equiv_{L_{t\{c\}}} \mathcal{M}'_{\{a'\}}$ and $\mathcal{N}_{\{b\}} \equiv_{L_{t\{c\}}} \mathcal{N}'_{\{b'\}}$.

We prove that $\mathcal{M}', a' \equiv^n_L \mathcal{N}', b'$. For, consider $\psi \in L^n$. Then, by Lemma 17.12, we have:

$$\mathcal{M}', a' \models \psi \text{ if and only if } \mathcal{M}' \models ST_x(\psi)[a'].$$

And, since a' is the interpretation of c in $\mathcal{M}'_{a'}$, we have:

$$\mathcal{M}' \models ST_x(\psi)[a'] \text{ if and only if } \mathcal{M}'_{\{a'\}} \models ST_x(\psi)[c/x].$$

And, since $\mathcal{M}_{\{a\}} \equiv_{L_{t\{c\}}} \mathcal{M}'_{\{a'\}}$, we have:

$$\mathcal{M}'_{\{a'\}} \models ST_x(\psi)[c/x] \text{ if and only if } \mathcal{M}_{\{a\}} \models ST_x(\psi)[c/x].$$

And, since a is the interpretation of c in $\mathcal{M}_a$, we have:

$$\mathcal{M}_{\{a\}} \models ST_x(\psi)[c/x] \text{ if and only if } \mathcal{M} \models ST_x(\psi)[a].$$

And, by Lemma 17.12, we have:

$$\mathcal{M} \models ST_x(\psi)[a] \text{ if and only if } \mathcal{M}, a \models \psi.$$

And, since $\mathcal{M}, a \equiv^n_L \mathcal{N}, b$, we have:

$$\mathcal{M}, a \models \psi \text{ if and only if } \mathcal{N}, b \models \psi.$$

And, similarly, we have:

$$\mathcal{N}, b \models \psi \text{ if and only if } \mathcal{N}', b' \models \psi.$$

Then, $\mathcal{M}', a' \equiv_L^n \mathcal{N}', b'$, as desired.

Then, by Proposition 17.21, we have that Player II has a winning strategy in $G_{n+1}(\mathcal{M}', \mathcal{N}')$ after that (a', b') has been played. Then, by Corollary 17.17, for every formula $\varphi(x)$ of L_t^n, we have $\mathcal{M}' \models \varphi[a']$ if and only if $\mathcal{N}' \models \varphi[b']$. Now, $\mathcal{M} \models \varphi[a]$. Then, with arguments similar to those above, we can show that $\mathcal{N} \models \varphi[b]$ as desired. □

Then, χ is the desired sentence. □

17.2.3 The translation is computable

Proposition 17.24. 1. For every sentence $\varphi \in L_t$, there is an algorithm that computes a sentence $\psi \in L$ such that φ and ψ are equivalent on $\mathfrak{T}_3$, and $qd(\varphi) = md(\psi)$.

2. For every formula $\varphi[x] \in L_t$, there is an algorithm that computes a formula $\psi \in L$ such that $\varphi[x]$ and ψ are equivalent on $\mathfrak{T}_3$, and $qd(\varphi) = md(\psi)$.

Proof. 1. Consider a sentence $\varphi \in L_t$. Compute its quantifier depth, say $n \in \mathbf{N}$. Restrict Φ to the set of propositional variables occurring in φ. By Lemma 17.22, there is a subset of sentences $\Sigma \subseteq L^n$ such that $\bigvee \Sigma$ is equivalent to φ on $\mathfrak{T}_3$. Since Φ is finite, we can compute a finite representative R of the set of all sentences in L^n. For every subset $\Sigma \subseteq R$, check whether $ST_x(\bigvee \Sigma) \leftrightarrow \varphi$ is valid on $\mathfrak{T}_3$. By [40, Part 2, Corollary 1.25, p. 91] there is an algorithm that can do it. By Lemma 17.22, the algorithm will answer positively for some $\overline{\Sigma}$. Since the number of possible Σ is finite, the algorithm will compute $\overline{\Sigma}$ in a finite number of steps. $\bigvee \overline{\Sigma}$ is the desired sentence.

2. Consider a formula $\varphi[x] \in L_t$. Compute its quantifier depth, say $n \in \mathbf{N}$. Restrict Φ to the set of propositional variables occurring in φ. By Lemma 17.23, there is a set $\Sigma \subseteq L^n$ such that $ST_x(\bigvee \Sigma)$ is equivalent to φ on $\mathfrak{T}_3$. Since the Φ is finite, we can compute a finite representative R of L^n. For every subset $\Sigma \subseteq R$, check whether $\forall x(ST_x(\bigvee \Sigma) \leftrightarrow \varphi)$ is valid on $\mathfrak{T}_3$ structures. By [40, Part 2, Corollary 1.25, p. 91] there is an algorithm that can do it. By Lemma 17.23, the algorithm will answer positively for some $\overline{\Sigma}$. Since the number of possible Σ is finite, the algorithm will compute $\overline{\Sigma}$ in a finite number of steps. $\bigvee \overline{\Sigma}$ is the desired formula. □

17.2.4 The result fails for T_2 spaces

In this section, we prove that L cannot express L_t sententially on $\mathfrak{T}_2$ (Theorem 17.28). Then, by Proposition 17.7, L cannot express L_t on $\mathfrak{T}_2$ either.

Observe that Theorem 18.3 in Chapter 18 implies Theorem 17.28. Nonetheless, a direct proof of Theorem 17.28 seems instructive because it allows us to introduce a notion of bisimulation for L. Moreover, this proof essentially lies on the topological spaces of Example 15.8 and does not consume much space.

More precisely, to prove Theorem 17.28, we define a notion of bisimulation for L (Definition 17.25), and prove that if two points in two topological models are bisimilar, then they are modally equivalent (Proposition 17.26).

Then, we consider two topological models, one of them T_2 but not T_3, and the other T_3, and prove that they have two bisimilar points. Then, since L_t can express that a topological model is T_3, namely by the sentence φ_{T_3} of Section 17.1.4, if L could express L_t on $\mathfrak{T}_2$, there would be a sentence $\psi \in L$ equivalent to φ on $\mathfrak{T}_2$. Then, since the two models have bisimilar points, they both would satisfy ψ. Then, they both would satisfy φ_{T_3}. Therefore, they both would be T_3, a contradiction.

Definition 17.25. For every two topological models $\mathcal{M} = (A, ...)$ and $\mathcal{N} = (B, ...)$ define a *bisimulation between* $\mathcal{M}$ *and* $\mathcal{N}$ as a relation $Z \subseteq A \times B$ such that for every two points $a \in \mathcal{M}$ and $b \in \mathcal{N}$ if aZb then the following conditions hold:

PV. For every $p \in \Phi$, we have $\mathcal{M}, a \models p$ if and only if $\mathcal{N}, b \models \varphi$;

$[d]$F. For every neighborhood U of a there is a neighborhood V of b such that for every point $b' \in V \backslash \{b\}$ there is a point $a' \in U \backslash \{a\}$ with $a'Zb'$;

$[d]$B. For every neighborhood V of b there is a neighborhood U of a such that for every point $a' \in U \backslash \{a\}$ there is a point $b' \in V \backslash \{b\}$ with $a'Zb'$;

- For every $n \in \mathbf{N}$:

  $\Diamond_n$F. For every $n+1$ distinct points $a_0, a_1, ..., a_n \in \mathcal{M}$ there are $n+1$ distinct points $b_0, b_1, ..., b_n \in \mathcal{N}$ such that for every $i \in n$ we have $a_i Z b_i$;

  $\Diamond_n$B. For every $n+1$ distinct points $b_0, b_1, ..., b_n \in \mathcal{N}$ there are $n+1$ distinct points $a_0, a_1, ..., a_n \in \mathcal{M}$ such that for every $i \in n$ we have $a_i Z b_i$.

For every two points $a \in \mathcal{M}$ and $b \in \mathcal{N}$ we say that $\mathcal{M}, a$ and $\mathcal{N}, b$ are *bisimilar*, notation $\mathcal{M}, a \leftrightarrows \mathcal{N}, b$, provided there is a bisimulation Z between $\mathcal{M}$ and $\mathcal{N}$ such that aZb. If the topological models are understood, we just say that a and b are *bisimilar*, notation $a \leftrightarrows b$.

Proposition 17.26. For every two topological models $\mathcal{M}$ and $\mathcal{N}$, for every two points $a \in \mathcal{M}$ and $b \in \mathcal{N}$, if $a \leftrightarrows b$ then $a \equiv_L b$.

Proof. We prove it by induction. Consider an arbitrary formula $\varphi \in L$. The only interesting cases are when φ is $[d]\psi$, and $\Diamond_n \psi$ ($n \in \mathbf{N}$).

Suppose that φ is $[d]\psi$. Suppose that $\mathcal{M}, a \models [d]\psi$. Then, there is a neighborhood U of a such that for every $a' \in U \setminus \{a\}$, we have $\mathcal{M}, a' \models \psi$. Then, by $[d]$F, there is a neighborhood V of b such that, for every $b' \in V \setminus \{b\}$, there is $a' \in U \setminus \{a\}$ with $a'Zb'$. Then, by inductive hypothesis, there is a neighborhood V of b such that, for every $b' \in V \setminus \{b\}$, we have $\mathcal{M}, b' \models \psi$. Then, $\mathcal{M}, b \models \varphi$, as desired.

Suppose that $\mathcal{N}, b \models [d]\psi$. Then, we can prove that $\mathcal{M}, a \models \varphi$ proceeding as before but using $[d]$B instead of $[d]$F.

Suppose that φ is $\Diamond_n \psi$ ($n \in \mathbf{N}$). Suppose that $\mathcal{M}, a \models \Diamond_n \psi$. Then, there are $n+1$ distinct points $a_0, a_1, ..., a_n \in \mathcal{M}$ such that, for every $i \in n+1$, we have $\mathcal{M}, a_i \models \psi$. Then, by $\Diamond_n$F, there are $n+1$ distinct points $b_0, b_1, ..., b_n \in \mathcal{B}$ such that, for every $i \in n+1$, we have $a_i Z b_i$. Then, by inductive hypothesis, there are $n+1$ distinct points $b_0, b_1, ..., b_n \in \mathcal{B}$ such that, for every $i \in n+1$, we have $\mathcal{N}, b_i \models \psi$. Then, $\mathcal{M}, b \models \varphi$, as desired.

Suppose that $\mathcal{N}, b \models \Diamond_n \psi$. Then, we can prove that $\mathcal{M}, a \models \varphi$ proceeding as before but using $\Diamond_n$B instead of $\Diamond_n$F. □

Let $(\mathbf{R}, \varepsilon)$ be the real numbers with the usual topology. Let $(\mathbf{R}, \tau)$ be the real numbers with the topology defined in Example 15.8. As mentioned, the former is a T_3 topological space, whereas the latter is a T_2 but not T_3 topological space. Let $V : \Phi \to \mathcal{P}(\mathbf{R})$ be the evaluation defined by, for every $p \in \Phi$, $V(p) := \emptyset$. Let $\mathcal{R}_1 := (\mathbf{R}, \varepsilon, V)$ and $\mathcal{R}_2 := (\mathbf{R}, \tau, V)$. Plainly:

Proposition 17.27. $\mathbf{R} \times \mathbf{R}$ is a bisimulation between $\mathcal{R}_1$ and $\mathcal{R}_2$.

Proof. The only interesting cases are $[d]$F, $[d]$B, $\Diamond_n$F, and $\Diamond_n$B. The former two follows by the fact that both $(\mathbf{R}, \varepsilon)$ and $(\mathbf{R}, \tau)$ are *dense-in-itself* - that is, for every $a \in \mathbf{R}$ and neighborhood U of a, there is $b \in U \setminus \{a\}$ - whereas the latter two follows from the fact that $\mathbf{R}$ is infinite. □

Then, reasoning as above, we can prove that:

Theorem 17.28. L cannot express L_t sententially on $\mathfrak{T}_2$. Then, L cannot express L_t on $\mathfrak{T}_2$ either.

Observe that, in the proof of Proposition 17.27, we only use that the two spaces are infinite and dense-in-themselves. Then:

Corollary 17.29. Let $\Phi = \emptyset$. For every two infinite and dense-in-themselves topological models $\mathcal{M}$ and $\mathcal{N}$, for every $a \in \mathcal{M}$ and $b \in \mathcal{N}$:

- a and b are bisimilar.
- a and b satisfies the same formulas of L.

Chapter 18

On increasing the expressive power of L

In Section 17.2.4 we proved that L cannot express L_t on T_2 topological models. Can we increase the expressive power of L so that the resulting language can express L_t on T_2 topological models? Traditionally, one way to increase the expressive power of a modal language is to add new modal operators. But what is a modal operator? And are there modal operators that added to L give a modal language expressing L_t on T_2 topological models?

In Section 18.1 we give a more general definition of modal operator. Following Gabbay, Hodkinson and Reynolds [42, Chapter 6], the semantics of modal operators will be defined by a formula of L_t with at most one free first-sort variable and no free second-sort variables. We will call point-sort modal operator every modal operator whose semantics is defined by an L_t formula without occurring second-sort variables. In Section 18.2, we show that even if we enrich L with all the point-sort modal operators, call L^+ the resulting language, however we add finitely many modal operators to L^+, we cannot express L_t on T_2 topological models.

18.1 Spatial languages

In this section we give a more general definition of modal operator. Following Gabbay, Hodkinson and Reynolds [42, Chapter 6], we define the semantics of modal operators by a first-order formula with at most one free variable and no free second-sort variables (Definition 18.1(1)). Given a set of modal operators, we define the resulting modal language recursively as usual (Definition 18.1(2)).

For every topological model $\mathcal{M} = (A, \tau, V)$, $A_0, ..., A_{n-1} \subseteq A$, and $p_0, ..., p_{n-1} \in \Phi$, define $\mathcal{M}[{}^{A_0}/_{p_0}, ..., {}^{A_{n-1}}/_{p_{n-1}}]$ as the model obtained from $\mathcal{M}$ by replacing, for every $i \in n$,

$p_i^{\mathcal{M}}$ with A_i.

Definition 18.1. 1. A *modal operator* is a tuple $(\#, \varphi[x], p_0, ..., p_{n-1})$ where # is an identifier, $x \in Var_1$, n is a natural number, $p_0, ..., p_{n-1} \in \Phi$, and φ is a formula of L_t with at most one occurring free variable, namely x, and such that the propositional atoms occurring in φ are among $p_0, ..., p_{n-1}$.

2. For every set of modal operators τ, define the *spatial language* $L_{\Phi,\tau}$ as the smallest set such that:

   - $\Phi \cup \{\top\} \subseteq L_{\Phi,\tau}$.
   - If $\varphi \in L_{\Phi,\tau}$ then $\neg\varphi \in L_{\Phi,\tau}$.
   - If $\varphi, \psi \in L_{\Phi,\tau}$ then $(\varphi \wedge \psi) \in L_{\Phi,\tau}$.
   - If $(\#, \varphi[x], p_0, ..., p_{n-1}) \in \tau$ and $\varphi_0, ..., \varphi_{n-1} \in L_{\Phi,\tau}$, then $\#(\varphi_0, ..., \varphi_{n-1}) \in L_{\Phi,\tau}$.

3. For every spatial language $L_{\Phi,\tau}$, formula $\varphi \in L_{\Phi,\tau}$, topological model $\mathcal{M}$, and $a \in \mathcal{M}$, define $\mathcal{M}, a \models \varphi$, recursively as follows:

   - $\mathcal{M}, a \models \top$ always.
   - For every $p \in \Phi$, $\mathcal{M}, a \models p$ provided $a \in p^{\mathcal{M}}$.
   - $\mathcal{M}, a \models \neg\varphi$ provided not $\mathcal{M}, a \models \varphi$.
   - $\mathcal{M}, a \models \varphi \wedge \psi$ provided $\mathcal{M}, a \models \varphi$ and $\mathcal{M}, a \models \psi$.
   - For every $(\#, \varphi[x], p_0, ..., p_{n-1}) \in \tau$, and $\#(\varphi_0, ..., \varphi_{n-1}) \in L_{\Phi,\tau}$, $\mathcal{M}, a \models \#(\varphi_0, ..., \varphi_{n-1})$ provided $\mathcal{M}[\varphi_0^{\mathcal{M}}/p_0, ..., \varphi_{n-1}^{\mathcal{M}}/p_{n-1}] \models \varphi[a]$ where, for every $i \in n$, $\varphi_i^{\mathcal{M}} := \{b \in \mathcal{M} \mid \mathcal{M}, b \models \varphi_i\}$.

4. Call a *point-sort formula* every $\varphi \in L_t$ such that there are no second-sort variables occurring in φ. Call a *point-sort modal operator* every modal operator $(\#, \varphi[x], p_0, ..., p_{n-1})$ such that φ is a point-sort formula. Let ρ be the set of all point-sort modal operators.

Example 18.2. For every $n \in \mathbf{N}$, the modal operator $\Diamond_n$ of L, defined as per Definition 18.1(1), is:

$$(\Diamond_n, \exists x_0 ... \exists x_n \Big(\bigwedge_{\substack{i,j \in n+1 \\ i \neq j}} x_i \neq x_j \wedge \bigwedge_{i \in n+1} p(x_i)\Big), p).$$

Observe that $\Diamond_n$ is a point-sort modal operator. The modal operator $[d]$ of L, defined as per Definition 18.1(1), is:

$$([d], \exists X_x \, \forall y \, (y \varepsilon X \wedge y \neq x \rightarrow p(y)), p).$$

We prove that:

Theorem 18.3. There is no finite set σ of modal operators such that, if $\tau := \rho \cup \sigma$, then $L_{\Phi,\tau}$ can express L_t on $\mathfrak{T}_2$.

Then, since $\{\Diamond_n \mid n \in \mathbf{N}\} \subseteq \rho$, we have:

Corollary 18.4. There is no finite set σ of modal operators such that, if $\tau := \{[d], \Diamond_n \mid n \in \mathbf{N}\} \cup \sigma$, then $L_{\Phi,\tau}$ can express L_t on $\mathfrak{T}_2$.

18.2 Proof of Theorem 18.3

The proof stems from a result by Hodkinson [51] in Kripke semantics. However, substantial original contributions are required. The proof is as follows. For every $n \in \mathbf{N}$, we introduce a formula φ_n. Then, we define sets A_1, A_2 and $A := A_1 \cup A_2$. Then, consider two propositional variables $p, q \in \Phi$. We define an evaluation function V such that $V(p) = A_1$, $V(q) = A_2$ and, for every $r \in \Phi \setminus \{p, q\}$, $V(r) = \emptyset$. Then, for every natural number $n > 0$, we define a basoid model $\mathcal{M}_n = (A, \mathcal{B}_n, V)$ (Section 18.2.2). Then, we prove that:

1. For every $n \in \mathbf{N}$, $\mathcal{M}_n$ has A as first-sort domain (by definition).
2. $V(p) = A_1$, $V(q) = A_2$ and, for every $r \in \Phi \setminus \{p, q\}$, $V(r) = \emptyset$ (by definition).
3. For every $n \in \mathbf{N}$, $\mathcal{M}_n$ is indeed a basoid model (Proposition 18.7).
4. For every $n \in \mathbf{N}$, $\mathcal{M}_n \models \varphi_{\mathrm{T}_2}$ (Proposition 18.9).
5. For every $n \in \mathbf{N}$, $\varphi[x] \in L_t$, $i \in \{1, 2\}$, and $a, a' \in A_i$, we have $\mathcal{M}_n \models \varphi[a]$ if and only if $\mathcal{M}_n \models \varphi[a']$ (Corollary 18.16).
6. There is a point $a \in A$ such that, for every $n \in \mathbf{N}$, $\mathcal{M}_n \models \varphi_n[a]$ and, for every natural $m > n$, $\mathcal{M}_n \not\models \varphi_m[a]$.

Then, consider an arbitrary finite set σ of modal operators, and two distinct propositional variables $\{p, q\}$. Then, by 5, and since $\{p, q\}$ and σ are finite, there are $n, m \in \mathbf{N}$ such that $n < m$ and, for every $a \in A$, and formula φ of the form $\#(\varphi_0, ..., \varphi_{k-1})$ with $\# \in \sigma$ and, for every $i \in k$, $\varphi_i \in \{\top, \bot, p, q\}$, we have:

$$\mathcal{M}_n, a \models \#(\varphi_0, ..., \varphi_{k-1}) \text{ if and only if } \mathcal{M}_m, a \models \#(\varphi_0, ..., \varphi_{k-1}). \tag{18.1}$$

Then, for every formula $\varphi \in L_{\Phi,\tau}$ with no occurring propositional variables, and $a \in A$, we have:

$$\mathcal{M}_n, a \models \varphi \text{ if and only if } \mathcal{M}_m, a \models \varphi. \tag{18.2}$$

This is Lemma 18.22 and is proved by induction on the formula, Point 1 above is used when $\# \in \rho$, and Point 2 above and (18.1) when $\# \in \sigma$.

Suppose, by contradiction, that $L_{\Phi,\tau}$ can express L_t on $\mathfrak{T}_2$. Then, since $\varphi_m \in L_t$, there is $\varphi \in L_{\Phi,\tau}$ equivalent to φ_m on $\mathfrak{T}_2$. Since φ is equivalent to φ_m, and there are no propositional variables occurring in φ_m, we can assume that there are no propositional variables occurring in φ.

Recall that L_t is invariant under changing bases, and so, since the semantics of modal operators is determined by an L_t formula, also $L_{\Phi,\tau}$ is invariant under changing bases. Let a be the point given by Point 6 above. Then, $\mathcal{M}_m \models \varphi_m[a]$. Then, since φ_m and φ are equivalent on $\mathfrak{T}_2$, by Point 3 above, we have $\mathcal{M}_m \models \varphi[a]$. Then, since there are no propositional variables occurring in φ, by (18.2), we have $\mathcal{M}_n \models \varphi[a]$. Then, since φ_m and φ are equivalent on $\mathfrak{T}_2$, we have $\mathcal{M}_n \models \varphi_m[a]$, against Point 5 above.

18.2.1 Formulas

In this section we introduce the formulas φ_n. For every $0 < n \in \mathbf{N}$, define:

$$\begin{aligned}\varphi_n :=& \exists X_{0_{x_0}} \forall Y_{0_{x_0}} \exists x_1 (\neg x_1 \varepsilon X_0 \wedge x_1 \varepsilon cl(Y_0) \wedge \\ & \exists X_{1_{x_1}} \forall Y_{1_{x_1}} \exists x_2 (\neg x_2 \varepsilon X_1 \wedge x_2 \varepsilon cl(Y_1) \wedge \\ & \ldots \\ & \exists X_{n-1_{x_{n-1}}} \forall Y_{n-1_{x_{n-1}}} \exists x_n (\neg x_n \varepsilon X_{n-1} \wedge x_n \varepsilon cl(Y_{n-1}) \wedge x_n \varepsilon cl(Y_0) \underbrace{))...)}_{n \text{ brackets}},\end{aligned}$$

where, for every $x \in Var_1$, and $X \in Var_2$, we define $x\varepsilon cl(X)$ as $\forall Y_x \exists y (y \varepsilon Y \wedge y \varepsilon X)$ for some fresh $y \in Var_1$, and $Y \in Var_2$. Plainly, $\varphi_n \in L_t$. And, if $n = 1$, then φ_1 is:

$$\varphi_1 := \exists X_x \forall Z_x \exists y (\neg y \varepsilon X \wedge \forall X_y \exists x (x \varepsilon Z \wedge x \varepsilon X)), \text{ that is } \neg\varphi_{\text{reg}}.$$

Observe that, for every $0 < n \in \mathbf{N}$, φ_n can be written in L_t using only 5 variables. For example, if $n = 3$, φ_3 can be written as:

$$\begin{aligned}\varphi_3 :=& \exists X_x \forall Z_x \exists y (\neg y \varepsilon X \wedge \forall X_y \exists x (x \varepsilon Z \wedge x \varepsilon X) \wedge \\ & \exists X_y \forall Y_y \exists x (\neg x \varepsilon X \wedge \forall X_x \exists x (x \varepsilon Y \wedge x \varepsilon X) \wedge \\ & \exists X_x \forall Y_x \exists y (\neg y \varepsilon X \wedge \forall X_y \exists x (x \varepsilon Y \wedge x \varepsilon X) \wedge \forall X_y \exists x (x \varepsilon Z \wedge x \varepsilon X)))).\end{aligned}$$

18.2.2 Basoid Models

In this section we define the basoid models $\mathcal{M}_n$. They are somewhat technical and we need a bit of preliminaries in order to define them and study their properties:

Preliminaries

For every two sets A, B, let ${}^{A}B$ denote the set of all functions from A to B. For every $n \in \mathbf{Z}$, define:

$$[0, n) := \{i \in \mathbf{Z} \,|\, 0 \leq i < n\},$$
$$(-\infty, n) := \{i \in \mathbf{Z} \,|\, i < n\}.$$

Then, ${}^{[0,0)}\mathbf{Z}$ is a singleton, namely the set containing the empty function $\emptyset : \emptyset \to \mathbf{Z}$. Define:

$$\mathcal{N} = \bigcup_{n \in \mathbf{N}} {}^{[0,n)}\mathbf{Z}.$$

The elements of $\mathcal{N}$ will be denoted by bold small latin letters $\mathbf{a}, \mathbf{b}, \ldots$. Plainly, for every $\mathbf{a} \in \mathcal{N}$, there is a unique $n \in \mathbf{N}$ such that $\mathbf{a} \in {}^{[0,n)}\mathbf{Z}$. Define:

$$|\mathbf{a}| := n - 1.$$

As a special case, the elements of ${}^{[0,1)}\mathbf{Z}$ will be denoted by small latin letters $b, c \ldots$. For every element $\mathbf{a}$ of $\mathcal{N}$, and $b \in \mathbf{Z}$, define $\mathbf{a}b \in {}^{[0,|a|+2)}\mathbf{Z} \subseteq \mathcal{N}$ as:

$$(\mathbf{a}b)(i) = \mathbf{a}(i) \text{ for every integer } 0 \leq i < |\mathbf{a}| + 1,$$
$$(\mathbf{a}b)(|\mathbf{a}| + 1) = b.$$

Define:

$$\mathcal{Z} := \bigcup_{n \in \mathbf{Z}} {}^{(-\infty,n)}\mathbf{Z}.$$

The elements of $\mathcal{Z}$ will be denoted by small greek letters $\nu, \mu, \ldots$. Plainly, for every $\nu \in \mathcal{Z}$, there is a unique $n \in \mathbf{Z}$ such that $\nu \in {}^{(-\infty,n)}\mathbf{Z}$. Define:

$$|\nu| := n - 1.$$

For every element $\nu \in \mathcal{Z}$, and $\mathbf{a} \in \mathcal{N}$, define $\nu\mathbf{a} \in {}^{(-\infty,|\nu|+|\mathbf{a}|+2)}\mathbf{Z} \subseteq \mathcal{Z}$ as:

$$(\nu\mathbf{a})(i) = \nu(i) \text{ for every integer } i < |\nu| + 1, \tag{18.3}$$
$$(\nu\mathbf{a})(i) = \mathbf{a}(i - (|\nu| + 1)) \text{ for every integer } |\nu| + 1 \leq i < |\nu| + |\mathbf{a}| + 2. \tag{18.4}$$

For every element $\nu \in \mathcal{Z}$, define:

$$\nu - 1 := \nu\restriction{}^{(-\infty,|\nu|)}\mathbf{Z}.$$

Plainly:

Remark 18.5. For every two $A, B \subseteq \mathbf{Z}$, and translation $t : \mathbf{Z} \to \mathbf{Z}$, if A is cofinite in $\mathbf{Z}$ and $t[A] \subseteq B$, then B is cofinite in $\mathbf{Z}$. (Recall that a *translation* (*from* $\mathbf{Z}$ *to* $\mathbf{Z}$) is any function $t : \mathbf{Z} \to \mathbf{Z}$ such that there is $m \in \mathbf{Z}$ such that, for every n, we have $t(n) = n + m$.)

Basoid Models

We are now ready to define the basoid models $\mathcal{M}_n$. We begin by defining their first-sort domain A:

$$A_1 := \mathcal{Z},$$
$$A_2 := \{(v, n, m) \mid v \in A_1 \wedge n, m \in \mathbf{Z}\},$$
$$A := A_1 \cup A_2.$$

A bit of commentary. A_1 is $\mathcal{Z}$. $\mathcal{Z}$ is the set of all functions from any interval $(-\infty, n]$ for $n \in \mathbf{Z}$ to $\mathbf{Z}$. Recall that for every two functions f, g, we have that $f \subset g$ denotes that the domain of f is a proper subset of the domain of g and g restricted to the domain of f is equal to f. Then, if we order A_1 by $\subset$, we obtain a tree that is downward endless and such that each point branches into $\mathbf{Z}$-many new points.

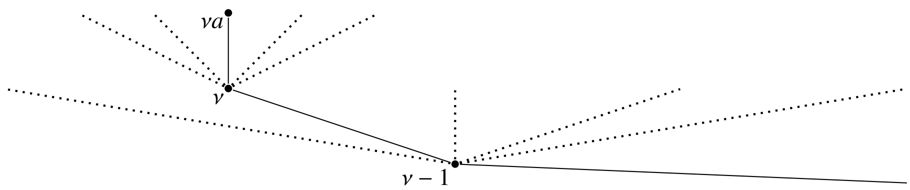

Figure 18.1: $(A_1, \subset)$, $v \in \mathcal{Z}$ and $a \in \mathbf{Z}$

As for A_2, we may think of it as some ‘dust’ around the points of A_1. For every $v \in A_1$, we may imagine the dust around v organized in $\mathbf{Z}$-many copies of $\mathbf{Z}$. More precisely, consider $(v, n, m) \in A_2$. Then, the first component v tells us that (v, n, m) is a particle of dust around v; the second component $n \in \mathbf{Z}$ tells us that (v, n, m) belongs to the nth copy of $\mathbf{Z}$ around v; finally, the third component $m \in \mathbf{Z}$ distinguishes (v, n, m) from the other elements $(v, n, ...)$ of the nth copy of $\mathbf{Z}$ around v.

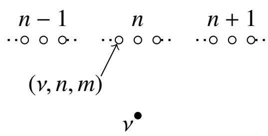

Figure 18.2: A particle of dust $(v, n, m) \in A_2$ around $v \in A_1$

Since v branches into $\mathbf{Z}$-many new points, and there are $\mathbf{Z}$-many copies of $\mathbf{Z}$ around v, we can bijectively associate every new point with a copy of $\mathbf{Z}$.

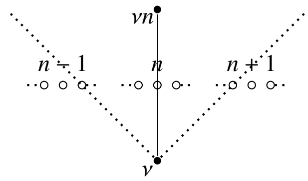

Figure 18.3: We can bijectively associate every new point with a copy of **Z**

Now, we define the second-sort domain of $\mathcal{M}_n$. That is, a topological base $\mathcal{B}_n$. We begin by defining the preliminary notion of n-selection. For every $n \in \mathbf{N}$, define:

$$^{<n+1}\mathcal{N} := \bigcup_{0 \leq i < n+1} {}^{[0,i)}\mathbf{Z}.$$

For every integer $n > 0$, call a *n-selection* every tuple $(C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in C \rangle)$ such that:

1. $C \subseteq {}^{<n+1}\mathcal{N}$.
2. $C \neq \emptyset$.
3. If $\mathbf{c}b \in C$, then $\mathbf{c} \in C$.
4. For every $\mathbf{c} \in C \cap {}^{<n}\mathcal{N}$, $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C\}$ is cofinite in $\mathbf{Z}$.
5. For every $\mathbf{c} \in C$, $D_{\mathbf{c}}$ is a cofinite subset of $\mathbf{Z}$.

Point 1 says that C is a set of functions from $[0, i)$ for any $0 \leq i < n + 1$ to $\mathbf{Z}$. Point 2 that C is non-empty. Point 3 that C is closed under taking subsequences. Recall that an element of C can be seen as a sequence:

$$(\mathbf{c}(0), \mathbf{c}(1), ..., \mathbf{c}(|\mathbf{c}|)).$$

Then, given an element $\nu \in A_1$, we can imagine C as a set of extensions of ν: if $\mathbf{c} \in C$, then $\nu\mathbf{c}$ is an extension of ν. Moreover, we can imagine $\mathbf{c}$ identifying a path starting at the $\mathbf{c}(0)$th copy of $\mathbf{Z}$ around ν, followed by the $\mathbf{c}(1)$th copy of $\mathbf{Z}$ around $\nu(\mathbf{c}(0))$, and so on until the $\mathbf{c}(|\mathbf{c}|)$th copy of $\mathbf{Z}$ around $\nu\mathbf{c} - 1$ (see Figure 18.4 for $\mathbf{c} = (0, 1, -1)$).

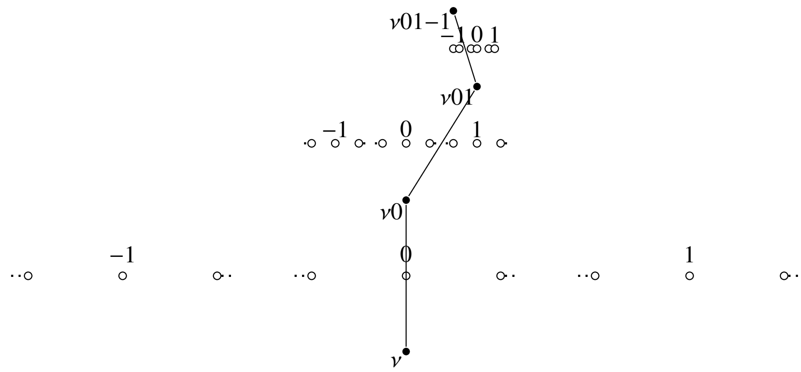

Figure 18.4: We can imagine C as a set of extensions of ν

For every $\mathbf{c} \in C \cap {}^{<n}\mathcal{N}$, Point 4 says that the set of copies of $\mathbf{Z}$ around $\nu\mathbf{c}$ identifiable by an extension $\mathbf{c}b \in C$ of $\mathbf{c}$ is cofinite in $\mathbf{Z}$. Moreover, for every $\mathbf{c} \in C$, a set $D_{\mathbf{c}}$ is associated to $\mathbf{c}$. We may think of $D_{\mathbf{c}}$ as a subset of the $\mathbf{c}(|\mathbf{c}|)$th copy of $\mathbf{Z}$ around $\nu\mathbf{c} - 1$. Point 5 says that such subset $D_{\mathbf{c}}$ is cofinite in $\mathbf{Z}$. Cofiniteness required by Point 4 and 5 will be important to prove that $\mathcal{M}_n$ are T_2 (Proposition 18.9).

For every $\nu \in A_1$, and $S = (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in C\rangle)$ an n-selection, define:

$$B(\nu, S) := \{\nu\} \cup \{(\nu\mathbf{c} - 1, (\nu\mathbf{c})(|\nu\mathbf{c}|), d) \,|\, \mathbf{c} \in C \wedge d \in D_{\mathbf{c}}\}.$$

Observe that $B(\nu, S) \subseteq A$. Sets $B(\nu, S)$ will be some of the basics opens of $\mathcal{B}_n$. They are made of a point $\nu \in A_1$ plus the dust in A_2 given by the n-selection S. More precisely, together with ν we have dust particles of the form $(\nu\mathbf{c} - 1, (\nu\mathbf{c})(|\nu\mathbf{c}|), d)$ for $\mathbf{c} \in C$ and $d \in D_{\mathbf{c}}$. That is, for every $\mathbf{c} \in C$ and $d \in D_{\mathbf{c}}$, the particle identified by d in the $\mathbf{c}(|\mathbf{c}|)$th copy of $\mathbf{Z}$ around $\nu\mathbf{c} - 1$ is in $B(\nu, S)$. Observe that, if $\mathbf{c}$ is $\emptyset$, then we obtain dust particles of the form $(\nu - 1, (\nu)(|\nu|), d)$. That is, dust particles in the $(\nu)(|\nu|)$th copy of $\mathbf{Z}$ around $\nu - 1$.

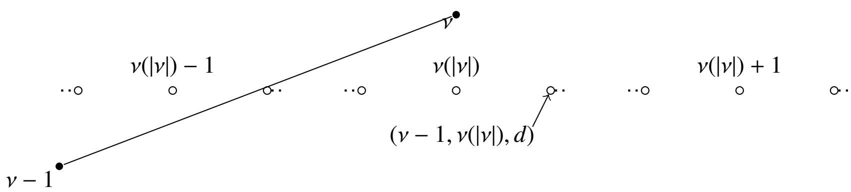

Figure 18.5: A dust particle $(\nu - 1, (\nu)(|\nu|), d)$ in the $(\nu)(|\nu|)$th copy of $\mathbf{Z}$ around $\nu - 1$

Define:

$$\mathcal{B}_n := \{B(\nu, S) \,|\, \nu \in A_1 \wedge S \text{ an } n\text{-selection}\} \cup \{\{a\} \,|\, a \in A_2\}.$$

Then, $\mathcal{B}_n$ is made of the sets $B(\nu, S)$ together with the singleton of every element of A_2.

Now, we introduce the evaluation function. We would like to distinguish A_1 from A_2 by means of propositional variable. This will be used in multiple places in Section 18.2.3. Then, we fix two distinct propositional variables $p, q \in \Phi$, and, for every $r \in \Phi$, we define:

$$V(r) := \begin{cases} A_1 & \text{provided } r = p, \\ A_2 & \text{provided } r = q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Define:

$$\mathcal{M}_n := (A, \mathcal{B}_n, V).$$

Remark 18.6. Observe that, since A and V are the same for every $\mathcal{M}_n$ ($n \in \mathbf{N}$), then, for every $n, m \in \mathbf{N}$, point-sort formulas $\varphi \in L_t$ and assignment g, we have:

$$\mathcal{M}_n \models \varphi[g] \text{ if and only if } \mathcal{M}_m \models \varphi[g].$$

Properties of $\mathcal{M}_n$

We now prove the aforementioned properties of $\mathcal{M}_n$.

Proposition 18.7. $\mathcal{M}_n \models \varphi_{\text{bas}}$ - that is, $\mathcal{M}_n$ is a basoid model.

Proof. Plainly, $A \subseteq \bigcup \mathcal{B}_n$. Consider $B_1, B_2 \in \mathcal{B}_n$. Three cases are given:

(a) $B_1 = \{a_1\}$, and $B_2 = \{a_2\}$ for some $a_1, a_2 \in A_2$. Then, either $B_1 \cap B_2 = \emptyset$ or $B_1 = B_2$. Then, for every $a \in B_1 \cap B_2$, there is $B \in \mathcal{B}_n$, namely $B_1 = B_2$, such that $a \in B \subseteq B_1 \cap B_2$.

(b) $B_1 = \{a_1\}$, and $B_2 = B(v, S)$ for some $v \in A_1$, and S n-selection. Then, either $B_1 \cap B_2 = \emptyset$ or $B_1 \cap B_2 = B_1$. Then, for every $a \in B_1 \cap B_2$, there is $B \in \mathcal{B}_n$, namely B_1, such that $a \in B \subseteq B_1 \cap B_2$.

(c) $B_1 = (v_1, S_1)$, and $B_2 = B(v_2, S_2)$ for some $v_1, v_2 \in A_1$, and S_1, S_2 selections for n. Two cases are given:

   i. $v_1 \neq v_2$. Then, by definition, $B_1 \cap B_2 \subseteq A_2$. Then, for every $a \in B_1 \cap B_2$, there is $B \in \mathcal{B}_n$, namely $\{a\}$, such that $a \in B \subseteq B_1 \cap B_2$.

   ii. $v_1 = v_2$. Let $S_1 = (C_1, \langle D_{1\mathbf{c}} \,|\, \mathbf{c} \in C_1 \rangle)$, and $S_2 = (C_2, \langle D_{2\mathbf{c}} \,|\, \mathbf{c} \in C_2 \rangle)$. Define $C := C_1 \cap C_2$, and, for every $\mathbf{c} \in C$, $D_{\mathbf{c}} := D_{1\mathbf{c}} \cap D_{2\mathbf{c}}$. Define $S := (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in D \rangle)$.

   Lemma 18.8. $S := (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in D \rangle)$ is an n-selection.

   Proof. 1. By point 1 of the definition of n-selection, we have $C_1, C_2 \subseteq {}^{<n+1}\mathcal{N}$. Then, by definition of C, we have $C \subseteq {}^{<n+1}\mathcal{N}$.

2. By point 2 of the definition of n-selection, we have $\emptyset \in C_1$ and C_2. Then, by definition of C, we have $\emptyset \in C$. Then $C \neq \emptyset$.

3. Consider $\mathbf{c}b \in C$. Then, by definition of C, $\mathbf{c}b \in C_1$ and $\mathbf{c}b \in C_2$. Then, by point 3 of the definition of n-selection, we have $\mathbf{c} \in C_1$ and $\mathbf{c} \in C_2$. Then, by definition of C, we have $\mathbf{c} \in C$.

4. Consider $\mathbf{c} \in C \cap {}^{<n}\mathcal{N}$. Then, by definition of C, we have:

$$\begin{aligned}\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C\} &= \{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C_1 \cap C_2\} \\ &= \{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C_1\} \cap \{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C_2\}.\end{aligned}$$

Now, by point 4 of the definition of n-selection, both $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C_1\}$ and $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in C_2\}$ are cofinite in $\mathbf{Z}$. Then, their intersection C is cofinite in $\mathbf{Z}$ as well.

5. Consider $\mathbf{c} \in C$, and $D_{\mathbf{c}}$. Then, by definition of $D_{\mathbf{c}}$, we have $D_{\mathbf{c}} = D_{1\mathbf{c}} \cap D_{2\mathbf{c}}$. Then, by point 5 of the definition of n-selection, both $D_{1\mathbf{c}}$ and $D_{2\mathbf{c}}$ are cofinite subsets of $\mathbf{Z}$. Then, their intersection $D_{\mathbf{c}}$ is a cofinite subset of $\mathbf{Z}$. ☐

Then $B(v_1, S) \in \mathcal{B}_n$. We show that $B(v_1, S) = B_1 \cap B_2$. Consider $a \in B(v_1, S)$. If $a = v_1$, then, by definition (and recall that $v_1 = v_2$), we have $a \in B_1 \cap B_2$. If $a \neq v_1$, then $a = (v_1\mathbf{c} - 1, v\mathbf{c}(|v\mathbf{c}|), d)$ for some $\mathbf{c} \in C$, and $d \in D_{\mathbf{c}}$. Then, $a = (v_1\mathbf{c} - 1, v\mathbf{c}(|v\mathbf{c}|), d)$ for some $\mathbf{c} \in C_1 \cap C_2$, and $d \in D_{1\mathbf{c}} \cap D_{2\mathbf{c}}$. Then, by definition, $a \in B_1 \cap B_2$. Then $B(v_1, S) \subseteq B_1 \cap B_2$. Conversely, consider $a \in B_1 \cap B_2$. If $a = v_1$, then, by definition, $a \in B(v_1, S)$. If $a \neq v_1$, then, $a = (v_1\mathbf{c} - 1, v\mathbf{c}(|v\mathbf{c}|), d)$ for some $\mathbf{c} \in C_1 \cap C_2$, and $d \in D_{1\mathbf{c}} \cap D_{2\mathbf{c}}$. Then, $a = (v_1\mathbf{c} - 1, v\mathbf{c}(|v\mathbf{c}|), d)$ for some $\mathbf{c} \in C$, and $d \in D_{\mathbf{c}}$. Then, by definition, $a \in B(v_1, S)$. ☐

Proposition 18.9. $\mathcal{M}_n \models \varphi_{\mathrm{T}_2}$ - that is, since $\varphi_{\mathrm{T}_2} \in L_t$ and L_t is invariant under changing base, the topological space generated by $(A, \mathcal{B}_n)$ is a T_2 topological space.

Proof. Consider $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Three cases are given:

(a) $a_1, a_2 \in A_2$. Then, $a_1 \in \{a_1\} \in \mathcal{B}_n$, $a_2 \in \{a_2\} \in \mathcal{B}_n$, and $\{a_1\} \cap \{a_2\} = \emptyset$.

(b) $a_1 \in A_1$ and $a_2 \in A_2$. Then $a_1 = v_1$ and $a_2 = (v_2, n, m)$ for some $v_1, v_2 \in \mathcal{Z}$, and $n, m \in \mathbf{Z}$. Consider $B \in \mathcal{B}_n$ such that $v_1 \in B$. Then, by definition of $\mathcal{B}_n$, we have $B = B(v_1, S)$ for some n-selection $S = (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in D_{\mathbf{c}}\rangle)$. Suppose $(v_2, n, m) \notin B(v_1, S)$. Then, by definition, we have $v_2 \in \{(v_2, n, m)\} \in \mathcal{B}_n$, and, by hypothesis, we have $B(v_1, S) \cap \{(v_2, n, m)\} = \emptyset$. Suppose that $(v_2, n, m) \in B(v_1, S)$. Then, $(v_2, n, m) = (v_1\mathbf{c} - 1, v_1\mathbf{c}(|v_1\mathbf{c}|), d)$ for some $\mathbf{c} \in C$ and $d \in D_{\mathbf{c}}$. Consider $D'_{\mathbf{c}} := D_{\mathbf{c}} \setminus \{d\}$. Since $D_{\mathbf{c}}$ is cofinite in $\mathbf{Z}$, we have that $D'_{\mathbf{c}}$ is cofinite in $\mathbf{Z}$. Then, the object S' that we obtain from S by replacing D with D' is an n-selection such that $(\nu_2, n, m) \notin B(\nu_1, S')$. Then $\nu_1 \in B(\nu_1, S') \in \mathcal{B}_n$, $(\nu_2, n, m) \in \{(\nu_2, n, m)\} \in \mathcal{B}_n$, and $B(\nu_1, S') \cap \{(\nu_2, n, m)\} = \emptyset$.

(c) $a_1, a_2 \in A_1$. Then, $a_1 = \nu_1$, and $a_2 = \nu_2$ for some $\nu_1, \nu_2 \in \mathcal{Z}$. Consider $B(\nu_1, S_1)$ and $B(\nu_2, S_2)$ for some selections for n $S_1 = (C_1, \langle D_{1\mathbf{c_1}} \mid \mathbf{c_1} \in C_1\rangle)$ and $S_2 = (C_2, \langle D_{2\mathbf{c_2}} \mid \mathbf{c_2} \in C_2\rangle)$. Suppose that $B(\nu_1, S_1) \cap B(\nu_2, S_2) \neq \emptyset$, then we have:

$$(\nu_1\mathbf{c_1} - 1, \nu_1\mathbf{c_1}(|\nu_1\mathbf{c_1}|), d_1) = (\nu_2\mathbf{c_2} - 1, \nu_2\mathbf{c_2}(|\nu_2\mathbf{c_2}|), d_2) \tag{18.5}$$

for some $\mathbf{c_1} \in C_1$ and $d_1 \in D_{1\mathbf{c_1}}$, and for some $\mathbf{c_2} \in C_2$ and $d_2 \in D_{2\mathbf{c_2}}$. Then, we have $\nu_1\mathbf{c_1} = \nu_2\mathbf{c_2}$. Then, we have $\nu_1 \subset \nu_2$ or $\nu_1 \supset \nu_2$ (recall that $\subset$ denotes proper inclusion). Suppose, without loss of generality that $\nu_1 \subset \nu_2$. Define $C'_1 = \{\mathbf{c} \in C_1 \mid \nu_1\mathbf{c} \not\supseteq \nu_2\}$, and $S' := (C'_1, \langle D_\mathbf{c} \mid \mathbf{c} \in C'_1\rangle)$.

Lemma 18.10. S'_1 is an n-selection.

Proof. 1. Plainly, $C'_1 \subseteq C_1$. Then, by point of 1 of the definition of n-selection, we have $C'_1 \subseteq {}^{<n+1}\mathcal{N}$. 2. By point 2 of the definition of n-selection, we have $\emptyset \in C_1$. Moreover, since $\nu_1 \subset \nu_2$, we have $\nu_1\emptyset \not\supseteq \nu_2$. Then, by definition of C'_1, we have $\emptyset \in C'_1$. Then, $C'_1 \neq \emptyset$. 3. Consider $\mathbf{c}b \in C'_1$. Then, by definition of C'_1, we have $\mathbf{c}b \in C_1$ and $\nu_1\mathbf{c}b \not\supseteq \nu_2$. Then, by point 3 of the definition of n-selection and since $\mathbf{c} \supseteq \nu_2$ implies $\nu_1\mathbf{c}b \supseteq \nu_2$, we have $\mathbf{c} \in C_1$ and $\mathbf{c} \not\supseteq \nu_2$. Then, by definition of C'_1, we have $\mathbf{c} \in C'_1$. 4. Consider $\mathbf{c} \in C'_1 \cap {}^{<n}\mathcal{N}$. Then, by definition of C'_1, we have:

$$\{b \in \mathbf{Z} \mid \mathbf{c}b \in C'_1\} = \{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1 \wedge \nu_1\mathbf{c}b \not\supseteq \nu_2\}. \tag{18.6}$$

Now, since $\mathbf{c} \in C'_1$, by definition of C'_1, we have $\nu_1\mathbf{c} \not\supseteq \nu_2$. Then, two cases are given:

i. $\nu_1\mathbf{c} \subset \nu_2$. Then:

$$\{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1 \wedge \nu_1\mathbf{c}b \not\supseteq \nu_2\} \supseteq \{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1\} \setminus \{\nu_2(|\nu_2|)\}.$$

ii. There is $i < |\nu_2| + 1$, such that $(\nu_1\mathbf{c})(i) \neq \nu_2(i)$. Then:

$$\{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1 \wedge \nu_1\mathbf{c}b \not\supseteq \nu_2\} = \{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1\}.$$

In both cases:

$$\{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1 \wedge \nu_1\mathbf{c}b \not\supseteq \nu_2\} \supseteq \{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1\} \setminus \{\nu_2(|\nu_2|)\}. \tag{18.7}$$

Now, by point 4 of the definition of n-selection, we have that $\{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1\}$ is cofinite in $\mathbf{Z}$. Then $\{b \in \mathbf{Z} \mid \mathbf{c}b \in C_1\} \setminus \{\nu_2(|\nu_2|)\}$ is cofinite in $\mathbf{Z}$. Then, by (18.6) and (18.7), we have that $\{b \in \mathbf{Z} \mid \mathbf{c}b \in C'_1\}$ is cofinite in $\mathbf{Z}$.

4. Consider $\mathbf{c} \in C'_1$. Then, by definition of C'_1, we have $\mathbf{c} \in C_1$. Then, since S_1 is an n-selection, we have $D_{\mathbf{c}}$ is a cofinite subset of $\mathbf{Z}$. □

Plainly, $\nu_1 \in B(\nu_1, S'_1) \in \mathcal{B}_n$ and $\nu_2 \in B(\nu_2, S_2) \in \mathcal{B}_n$. Suppose that $B(\nu_1, S'_1) \cap B(\nu_2, S_2) \neq \emptyset$. Then, we have (18.5) for some $\mathbf{c_1} \in C'_1$ and $d_1 \in D_{1\mathbf{c_1}}$, and for some $\mathbf{c_2} \in C_2$ and $d_2 \in D_{2\mathbf{c_2}}$. Then, we have $\nu_1\mathbf{c_1} = \nu_2\mathbf{c_2}$. Then, we have $\nu_1\mathbf{c_1} \supseteq \nu_2$, against $\mathbf{c_1} \in C'_1$. □

The points of A_1 are pairwise automorphic, and the same holds for the points of A_2 (Proposition 18.11). Then, the points of A_1 agree on the same formulas of L_2, and the same holds for the points of A_2 (Corollary 18.16).

Proposition 18.11. 1. For every $a_1, a_2 \in A_1$, there is an automorphism $f : \mathcal{M}_n \to \mathcal{M}_n$ such that $f(a_1) = a_2$.

2. For every $a_1, a_2 \in A_2$, there is an automorphism $f : \mathcal{M}_n \to \mathcal{M}_n$ such that $f(a_1) = a_2$.

Overview: we define a transformation, some kind of translation, depending on some parameters, on A_1 (18.8). This transformation induces a transformation, again depending on a parameter, on A_2 (18.9), and a map $\mathcal{B}_n \to \mathcal{B}_n$ (18.10-18.14). Then, we take the union of these maps and prove that the resulting function f is an automorphism on $\mathcal{M}_n$ (Lemma 18.14). Finally, we prove that, for every $i \in \{1, 2\}$, and $a_1, a_2 \in A_i$, there exist values of the parameters such that we have $f(a_1) = a_2$.

Proof. Let $\beta \in \mathbf{Z}$, for every $i \in \mathbf{Z}$, let $\gamma(i) \in \mathbf{Z}$, and $\delta \in \mathbf{Z}$. For every $\nu \in A_1$, define $f(\nu) \in {}^{(+\infty, |\nu|+\beta+1)}\mathbf{Z}$ by, for every integer $i < |\nu| + \beta + 1$:

$$f(\nu)(i) := \nu(i - \beta) + \gamma(i). \tag{18.8}$$

For every $(\nu, m, n) \in A_2$, define $f((\nu, n, m)) \in A_2$ by:

$$f((\nu, n, m)) := (f(\nu n) - 1, f(\nu n)(|f(\nu n)|), m + \delta). \tag{18.9}$$

For every $\{a\} \in \mathcal{B}_n$, define $f(\{a\}) \in \mathcal{B}_n$ by:

$$f(\{a\}) := \{f(a)\}. \tag{18.10}$$

For every $\nu \in A_1$, n-selection $S = (C, \langle D_{\mathbf{c}} \mid \mathbf{c} \in D\rangle)$, and $\mathbf{c} \in C$, define $f_\nu(\mathbf{c}) \in {}^{[0,|\mathbf{c}|+1)}\mathbf{Z}$

by, for every integer $0 \leq i < |\mathbf{c}| + 1$:

$$f_\nu(\mathbf{c})(i) := f(\nu\mathbf{c})(|f(\nu)| + 1 + i). \tag{18.11}$$

Define:

$$f_\nu(C) := \{f_\nu(\mathbf{c}) \mid \mathbf{c} \in C\}. \tag{18.12}$$

Let us study f_ν:

Lemma 18.12. The following facts hold:

1. For every $\mathbf{c} \in C$, and every integer $0 \leq i < |f_\nu(\mathbf{c})| + 1$, we have:

$$f_\nu(\mathbf{c})(i) = \mathbf{c}(i) + \gamma(|f(\nu)| + 1 + i).$$

2. f_ν is a bijection between C and $f_\nu(C)$.

3. For every $\mathbf{c} \in f_\nu(C)$, for every $0 \leq i < |f_\nu^{-1}(\mathbf{c})| + 1$, we have:

$$f_\nu^{-1}(\mathbf{c})(i) = \mathbf{c}(i) - \gamma(|f(\nu)| + 1 + i).$$

4. For every $\mathbf{c} \in C$, we have:

$$f(\nu)f_\nu(\mathbf{c}) = f(\nu\mathbf{c}).$$

Proof. 1. For every integer $0 \leq i < |f_\nu(\mathbf{c})| + 1$, we have:

$$\begin{aligned}
f_\nu(\mathbf{c})(i) &= f(\nu\mathbf{c})(|f(\nu)| + 1 + i) \text{ (by definition of } f_\nu(\mathbf{c})\text{) (18.11)} \\
&= (\nu\mathbf{c})(|f(\nu)| + 1 + i - \beta) + \gamma(|f(\nu)| + 1 + i) \text{ (by definition of } f \text{ (18.8))} \\
&= (\nu\mathbf{c})(|\nu| + \beta + 1 + i - \beta) + \gamma(|f(\nu)| + 1 + i) \text{ (by definition of } f(\nu) \text{ and of } |_|) \\
&= (\nu\mathbf{c})(|\nu| + 1 + i) + \gamma(|f(\nu)| + 1 + i) \text{ (by arithmetic)} \\
&= \mathbf{c}(i) + \gamma(|f(\nu)| + 1 + i) \text{ (by definition of } \nu\mathbf{c})
\end{aligned}$$

as desired.

2. Consider $\mathbf{c}_1, \mathbf{c}_2$ such that

$$f_\nu(\mathbf{c}_1) = f_\nu(\mathbf{c}_2).$$

Then, by definition of f_v and of $|\_|$, we have:

$$|\mathbf{c}_1| = |f_v(\mathbf{c}_1)|,$$
$$|\mathbf{c}_2| = |f_v(\mathbf{c}_2)|.$$

Then, since $f_v(\mathbf{c}_1) = f_v(\mathbf{c}_2)$, we have:

$$|\mathbf{c}_1| = |\mathbf{c}_2|.$$

Moreover, by Point 1, for every $i < |\mathbf{c}_1| + 1 = |\mathbf{c}_2| + 1$, we have:

$$f_v(\mathbf{c}_1)(i) = \mathbf{c}_1(i) + \gamma(|f(v)| + 1 + i),$$
$$f_v(\mathbf{c}_2)(i) = \mathbf{c}_2(i) + \gamma(|f(v)| + 1 + i).$$

Then, since $f_v(\mathbf{c}_1) = f_v(\mathbf{c}_2)$, for every $i < |\mathbf{c}_1| + 1 = |\mathbf{c}_2| + 1$, we have:

$$\mathbf{c}_1(i) + \gamma(|f(v)| + 1 + i) = \mathbf{c}_2(i) + \gamma(|f(v)| + 1 + i).$$

Then, by arithmetic, for every $i < |\mathbf{c}_1| + 1 = |\mathbf{c}_2| + 1$, we have:

$$\mathbf{c}_1(i) = \mathbf{c}_2(i).$$

Then, $\mathbf{c}_1 = \mathbf{c}_2$ as desired. Surjectivity plainly follows by definition of $f_v(C)$.

3. For every integer $0 \leq i < |\mathbf{c}|$, we have:

$$\begin{aligned}\mathbf{c}(i) &= f_v(f_v^{-1}(\mathbf{c}))(i) \text{ (by Point 2)}\\ &= f_v^{-1}(\mathbf{c})(i) + \gamma(|f(v)| + 1 + i) \text{ (by Point 1).}\end{aligned}$$

From this, thesis plainly follows.

4. We have that:

$$\begin{aligned}|f(v)f_v(\mathbf{c})| &= |f(v)| + |f_v(\mathbf{c})| + 1 \text{ (by definition of } f(v)f_v(\mathbf{c}) \text{ and of } |\_|)\\ &= |v| + \beta + |f_v(\mathbf{c})| + 1 \text{ (by definition of } f(v) \text{ and of } |\_|)\\ &= |v| + \beta + |\mathbf{c}| + 1 \text{ (by definition of } f_v(\mathbf{c}) \text{ and of } |\_|)\\ &= |v\mathbf{c}| + \beta \text{ (by definition of } v\mathbf{c} \text{ and of } |\_|)\\ &= |f(v\mathbf{c})| \text{ (by definition of } f(v\mathbf{c}) \text{ and of } |\_|).\end{aligned}$$

Moreover, for every integer $i < |f(\nu)| + 1$, we have:

$$\begin{aligned} f(\nu)f_\nu(\mathbf{c})(i) &= f(\nu)(i) \text{ (by definition of } f(\nu)f_\nu(\mathbf{c}) \text{ (18.3))} \\ &= \nu(i-\beta) + \gamma(i) \text{ (by definition of } f \text{ (18.8))} \\ &= (\nu\mathbf{c})(i-\beta) + \gamma(i) \text{ (by definition of } \nu\mathbf{c} \text{ (18.3))} \\ &= f(\nu\mathbf{c})(i) \text{ (by definition of } f \text{ (18.8)).} \end{aligned}$$

And, for every integer $|f(\nu)| + 1 \leq i < |f(\nu)| + |f_\nu(\mathbf{c})| + 2$, we have:

$$\begin{aligned} f(\nu)f_\nu(\mathbf{c})(i) &= f_\nu(\mathbf{c})(i - (|f(\nu)| + 1)) \text{ (by definition of } f(\nu)f_\nu(\mathbf{c}) \text{ (18.4))} \\ &= f(\nu\mathbf{c})(|f(\nu)| + 1 + i - (|f(\nu)| + 1)) \text{ (by definition of } f_\nu \text{ (18.11))} \\ &= f(\nu\mathbf{c})(i) \text{ (by arithmetic).} \end{aligned}$$

Then, $f(\nu)f_\nu(\mathbf{c}) = f(\nu\mathbf{c})$ as desired. □

For every $\mathbf{c} \in f_\nu(C)$, define:

$$D_\mathbf{c} := \{d + \delta \mid d \in D_{f_\nu^{-1}(\mathbf{c})}\}. \tag{18.13}$$

By Lemma 18.12(2), this is a good definition. Define:

$$f_\nu(S) := (f_\nu(C), \langle D_\mathbf{c} \mid \mathbf{c} \in f_\nu(C)\rangle). \tag{18.14}$$

Lemma 18.13. $f_\nu(S)$ is an n-selection.

Proof. 1. Consider $\mathbf{c} \in f_\nu(C)$. Then, by definition of $f_\nu(C)$, there is $\mathbf{c}' \in C$ such that $f_\nu(\mathbf{c}') = \mathbf{c}$. By point 1 of the definition of n-selection, we have $|\mathbf{c}'| < n$. Moreover, by definition of f_ν, we have $|f_\nu(\mathbf{c}')| = |\mathbf{c}'|$. Then, since $\mathbf{c} = f_\nu(\mathbf{c}')$, we have $|\mathbf{c}| < n$. Then, by arbitrariness of $\mathbf{c} \in f_\nu(C)$, we have $f_\nu(C) \subseteq {}^{<n+1}\mathcal{N}$.

2. By point 2 of definition of n-selection, there is $\mathbf{c} \in C$. Then, by definition of $f_\nu(C)$, we have $f_\nu(\mathbf{c}) \in f_\nu(C)$. Then, $f_\nu(C) \neq \emptyset$.

3. Consider $\mathbf{c}b \in f_\nu(C)$. Then, by definition of $f_\nu(C)$, there is $\mathbf{c}' \in C$ such that $f_\nu(\mathbf{c}') = \mathbf{c}b$. We prove that $f_\nu(\mathbf{c}' - 1) = \mathbf{c}$. Indeed, for every $0 \leq i < |\mathbf{c}' - 1| + 1$, we have:

$$\begin{aligned} f_\nu(\mathbf{c}' - 1)(i) &= (\mathbf{c}' - 1)(i) + \gamma(|\nu| + \beta + 1 + i) \text{ (by Lemma 18.12(1))} \\ &= \mathbf{c}'(i) + \gamma(|\nu| + \beta + 1 + i) \text{ (by definition of } \mathbf{c}' - 1) \\ &= f_\nu(\mathbf{c}')(i) \text{ (by Lemma 18.12(1))} \\ &= \mathbf{c}b(i) \text{ (by choice of } \mathbf{c}') \\ &= \mathbf{c}(i) \text{ (by definition of } \mathbf{c}b). \end{aligned}$$

Moreover, by definition of f_ν, we have $|f_\nu(\mathbf{c}')| = |\mathbf{c}'|$. Then, since $f_\nu(\mathbf{c}') = \mathbf{c}b$, we have

$|\mathbf{c}'| = |\mathbf{c}b| \geq 0$. Then, by point 3 of definition of n-selection, we have $\mathbf{c}' - 1 \in C$. Then, by definition of $f_\nu(C)$, we have $f_\nu(\mathbf{c}' - 1) \in f_\nu(C)$. Then, since $f_\nu(\mathbf{c}' - 1) = \mathbf{c}$, we have $\mathbf{c} \in f_\nu(C)$.

4. Consider $\mathbf{c} \in f_\nu(C) \cap {}^{<n}\mathcal{N}$. Then, we have:

$$\begin{aligned}\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in f_\nu(C)\} &= \{b \in \mathbf{Z} \,|\, f_\nu^{-1}(\mathbf{c}b) \in C\} \text{ (by Lemma 18.12(2))}\\ &= \{b \in \mathbf{Z} \,|\, (f_\nu^{-1}(\mathbf{c}b) - 1)(b - \gamma(|f(\nu)| + 1 + |\mathbf{c}b|) \in C\} \text{ (by Lemma 18.12(3))}.\end{aligned}$$

Define:

$$\begin{aligned}E &:= \{b \in \mathbf{Z} \,|\, (f_\nu^{-1}(\mathbf{c}b) - 1)b \in C\}.\\ F &:= \{b \in \mathbf{Z} \,|\, (f_\nu^{-1}(\mathbf{c}b) - 1)(b - \gamma(|f(\nu)| + 1 + |\mathbf{c}b|)) \in C\},\\ t : \mathbf{Z} \to \mathbf{Z} &:= n \mapsto n + \gamma(|f(\nu)| + 1 + |\mathbf{c}b|).\end{aligned}$$

Then, t is a translation from $\mathbf{Z}$ to $\mathbf{Z}$. Consider $b \in E$. Then, by definition of E, we have $(f_\nu^{-1}(\mathbf{c}b) - 1)b \in C$. Then, by definition of t, we have $(f_\nu^{-1}(\mathbf{c}b) - 1)(t(b) - \gamma(|f(\nu)| + 1 + |\mathbf{c}b|)) \in C$. Then, by definition of F, we have $t(b) \in F$. Then $t[E] \subseteq F$. Moreover, by Lemma 18.12(2), we have $f_\nu^{-1}(\mathbf{c}) \in C$. Then, by point 4 of the definition of n-selection, we have that E is cofinite in $\mathbf{Z}$. Then, by Remark 18.5, F is cofinite in $\mathbf{Z}$. Then, since we showed $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in f_\nu(C)\} = F$, we have that $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in f_\nu(C)\}$ is cofinite in $\mathbf{Z}$.

5. Consider $\mathbf{c} \in f_\nu(C)$. Consider $t : \mathbf{Z} \to \mathbf{Z} := n \mapsto n + \delta$. Observe that t is a translation from $\mathbf{Z}$ to $\mathbf{Z}$. Moreover, by definition of $D_\mathbf{c}$, we have $t[D_{f_\nu^{-1}(\mathbf{c})}] = D_\mathbf{c}$. Moreover, by point 5 of the definition of n-selection, we have that $D_{f_\nu^{-1}(\mathbf{c})}$ is cofinite in $\mathbf{Z}$. Then, by Remark 18.5, we have that $D_\mathbf{c}$ is cofinite in $\mathbf{Z}$. □

Define:

$$f(B(\nu, S)) := B(f(\nu), f_\nu(S)). \tag{18.15}$$

Lemma 18.14. f is an automorphism on $\mathcal{M}_n$.

Proof. 1. *Injectivity (first-sort).* Consider $a_1, a_2 \in A$. Suppose that:

$$f(a_1) = f(a_2). \tag{18.16}$$

Two cases are given:

(a) $a_1, a_2 \in A_1$. Then, $a_1 = \nu_1$ and $a_2 = \nu_2$ for some $\nu_1, \nu_2 \in \mathcal{Z}$. Then, by (18.16), we have:

$$|f(\nu_1)| = |f(\nu_2)|.$$

Then, by definition of f, we have:

$$|\nu_1| + \beta = |\nu_2| + \beta.$$

Then:

$$|\nu_1| = |\nu_2|.$$

Moreover, by definition of f (18.8), for every $i < |\nu_1| + 1 = |\nu_2| + 1$, we have:

$$\nu_1(i) + \gamma(i + \beta) = f(\nu_1)(i + \beta),$$
$$\nu_2(i) + \gamma(i + \beta) = f(\nu_2)(i + \beta).$$

Then, by (18.16), for every $i < |\nu_1| + 1 = |\nu_2| + 1$, we have:

$$\nu_1(i) + \gamma(i + \beta) = \nu_2(i) + \gamma(i + \beta).$$

Then, for every $i < |\nu_1| + 1 = |\nu_2| + 1$, we have:

$$\nu_1(i) = \nu_2(i).$$

Then, $\nu_1 = \nu_2$ as desired.

(b) $a_1, a_2 \in A_2$. Then, $a_1 = (\nu_1, n_1, m_1)$ and $a_2 = (\nu_2, n_2, m_2)$ for some $\nu_1, \nu_2 \in \mathcal{Z}$ and $n_1, n_2, m_1, m_2 \in \mathbf{Z}$. Then, by definition of f (18.9), we have:

$$f(\nu_1, n_1, m_1) = (f(\nu_1 n_1) - 1, f(\nu_1 n_1)(|f(\nu_1 n_1)|), m_1 + \delta),$$
$$f(\nu_2, n_2, m_2) = (f(\nu_2 n_2) - 1, f(\nu_2 n_2)(|f(\nu_2 n_2)|), m_2 + \delta).$$

Then, we have:

$$f(\nu_1 n_1) = f(\nu_2 n_2),$$
$$m_1 + \delta = m_2 + \delta.$$

Then, by point 1(a) for the first equation, we have:

$$\nu_1 n_1 = \nu_2 n_2,$$
$$m_1 = m_2.$$

Then, we have:

$$\nu_1 = \nu_2,$$
$$n_1 = n_2,$$
$$m_1 = m_2.$$

Then, we have $(\nu_1, n_1, m_1) = (\nu_2, n_2, m_2)$ as desired.

2. *Injectivity (second-sort).* Consider arbitrary $B_1, B_2 \in \mathcal{B}_n$. Suppose that:

$$f(B_1) = f(B_2). \tag{18.17}$$

Then, two cases are given:

(a) $B_1 = \{a_1\}$ and $B_2 = \{a_2\}$ for some $a_1, a_2 \in A_2$. Then, by definition of f (18.10) and (18.17), we have:

$$\{f(a_1)\} = \{f(a_2)\}.$$

Then, we have:

$$f(a_1) = f(a_2).$$

Then, by injectivity of f on A, we have $a_1 = a_2$ as desired.

(b) $B_1 = B(\nu_1, S_1)$ and $B_2 = (\nu_2, S_2)$ for some $\nu_1, \nu_2 \in A_1$ and n-selections $S_1 = (C_1, \langle D_{1\mathbf{c}} \mid \mathbf{c} \in C_1\rangle)$ and $S_2 = (C_2, \langle D_{2\mathbf{c}} \mid \mathbf{c} \in C_2\rangle)$. Then, by definition of f (18.15) and (18.17), we have:

$$B(f(\nu_1), f_{\nu_1}(S_1)) = B(f(\nu_2), f_{\nu_2}(S_2)). \tag{18.18}$$

Moreover, by definition (18.14-18.13), we have:

$$f_{\nu_1}(S_1) = (f_{\nu_1}(C_1), \{d + \delta \mid d \in D_{1 f_{\nu_1}^{-1}(\mathbf{c})}\} \mid \mathbf{c} \in f_{\nu_1}(C_1)\}), \tag{18.19}$$
$$f_{\nu_2}(S_2) = (f_{\nu_2}(C_2), \{d + \delta \mid d \in D_{2 f_{\nu_2}^{-1}(\mathbf{c})}\} \mid \mathbf{c} \in f_{\nu_2}(C_2)\}). \tag{18.20}$$

Then, by (18.18), definition of $B(\nu, S)$, and (18.19,18.20), we have:

$$\{f(\nu_1)\} \cup \{(f(\nu_1)\mathbf{c} - 1, f(\nu_1)\mathbf{c}(|f(\nu_1)\mathbf{c}|), d + \delta) \mid \mathbf{c} \in f_{\nu_1}(C_1) \wedge d \in D_{1 f_{\nu_1}^{-1}(\mathbf{c})}\} =$$
$$\{f(\nu_2)\} \cup \{(f(\nu_2)\mathbf{c} - 1, f(\nu_2)\mathbf{c}(|f(\nu_2)\mathbf{c}|), d + \delta) \mid \mathbf{c} \in f_{\nu_2}(C_2) \wedge d \in D_{2 f_{\nu_2}^{-1}(\mathbf{c})}\}. \tag{18.21}$$

Then, we have:

$$f(\nu_1) = f(\nu_2).$$

Then, by injectivity of f on A, we have:

$$\nu_1 = \nu_2. \tag{18.22}$$

Then, again by (18.21) and arithmetic, for every $\mathbf{c_1} \in f_{\nu_1}(C_1)$ and $d_1 \in D_{1 f_{\nu_1}^{-1}(\mathbf{c_1})}$, there is $\mathbf{c_2} \in f_{\nu_1}(C_2)$ and $d \in D_{2 f_{\nu_1}^{-1}(\mathbf{c_2})}$ such that:

$$\begin{aligned}&(f(\nu_1)\mathbf{c_1} - 1, f(\nu_1)\mathbf{c_1}(|f(\nu_1)\mathbf{c_1}|), d_1) = \\ &\quad (f(\nu_1)\mathbf{c_2} - 1, f(\nu_1)\mathbf{c_2}(|f(\nu_1)\mathbf{c_2}|), d_2).\end{aligned}$$

Then, for every $\mathbf{c_1} \in f_{\nu_1}(C_1)$ and $d_1 \in D_{1 f_{\nu_1}^{-1}(\mathbf{c_1})}$, there is $\mathbf{c_2} \in f_{\nu_1}(C_2)$ and $d_2 \in D_{2 f_{\nu_1}^{-1}(\mathbf{c_2})}$ such that:

$$\begin{aligned}\mathbf{c_1} &= \mathbf{c_2}, \\ d_1 &= d_2.\end{aligned}$$

Then, for every $\mathbf{c_1} \in f_{\nu_1}(C_1)$ and $d_1 \in D_{1 f_{\nu_1}^{-1}(\mathbf{c_1})}$, we have $\mathbf{c_1} \in f_{\nu_1}(C_2)$ and $d_1 \in D_{2 f_{\nu_1}^{-1}(\mathbf{c_1})}$. Moreover, similarly, for every $\mathbf{c_2} \in f_{\nu_1}(C_2)$ and $d_2 \in D_{2 f_{\nu_1}^{-1}(\mathbf{c_2})}$, we have $\mathbf{c_2} \in f_{\nu_1}(C_1)$ and $d_2 \in D_{1 f_{\nu_1}^{-1}(\mathbf{c_2})}$. Then, $f_{\nu_1}(C_1) = f_{\nu_1}(C_2)$ and, for every $\mathbf{c} \in f_{\nu_1}(C_1)$, $D_{1 f_{\nu_1}^{-1}(\mathbf{c})} = D_{2 f_{\nu_1}^{-1}(\mathbf{c})}$. Then, by injectivity of f_{ν_1} we have $C_1 = C_2$, and $\langle D_{1\mathbf{c}} \,|\, \mathbf{c} \in C_1 \rangle = \langle D_{2\mathbf{c}} \,|\, \mathbf{c} \in C_2 \rangle$. Then, $B_1 = B_2$ as desired.

3. *Surjectivity (first-sort).* Consider $a \in A$. Then, two cases are given:

   (a) $a \in A_1$. Then, $a = \nu$ for some $\nu \in \mathcal{Z}$. Define $\mu \in {}^{(-\infty,|\nu|-\beta+1)}\mathbf{Z}$ by, for all $i < |\nu| - \beta + 1$:

$$\mu(i) := \nu(i + \beta) - \gamma(i + \beta). \tag{18.23}$$

   Then, we have:

$$\begin{aligned}|f(\mu)| &= |\mu| + \beta \text{ (by definition of } f \text{ and of } |\_|) \\ &= |\nu| - \beta + \beta \text{ (by definition of } \nu \text{ and of } |\_|) \\ &= |\nu| \text{ (by arithmetic).}\end{aligned}$$

Moreover, by definition of f, for all $i < |f(\mu)| + 1 = |\nu| + 1$, we have:

$$\begin{aligned} f(\mu)(i) &= \mu(i-\beta) + \gamma(i) \text{ (by definition of } f \text{ (18.8))} \\ &= \nu(i) - \gamma(i) + \gamma(i) \text{ (by definition of } \mu \text{ (18.23))} \\ &= \nu(i) \text{ (by arithmetic).} \end{aligned}$$

Then, we have $f(\mu) = \nu$ as desired.

(b) $a \in A_2$. Then, $a = (\nu, n, m)$ for some $\nu \in \mathcal{Z}$, and $n, m \in \mathbf{Z}$. By Point 2(a), there is $\mu \in A_1$ such that $f(\mu) = \nu n$. Define, $b := (\mu - 1, \mu(|\mu|), m - \delta)$. Then, we have:

$$\begin{aligned} f((\mu - 1, \mu(|\mu|), m - \delta)) &= (f(\mu) - 1, f(\mu)(|f(\mu)|), m) \text{ (by definition of } f \text{ (18.9))} \\ &= (\nu, n, m) \text{ (by definition of } \mu \text{ (18.23)).} \end{aligned}$$

Then, $f(b) = a$ as desired.

Surjectivity (second-sort). Consider $B \in \mathcal{B}_n$. Two cases are given:

(a) $B = \{a\}$ for some $a \in A_2$. Then, by surjectivity of f on A_2, there is $a' \in A_2$ such that $f(a') = (a)$. Then, by definition of $\mathcal{B}_n$, we have $\{a'\} \in \mathcal{B}_n$. Moreover, we have:

$$\begin{aligned} f(\{a'\}) &= \{f(a')\} \text{ (by definition of } f \text{ (18.10))} \\ &= \{a\} \text{ (by definition of } a'\text{).} \end{aligned}$$

Then, $f(\{a'\}) = B$ as desired.

(b) $B = B(\nu, S)$ for some $\nu \in A_1$ and n-selection $S = (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in C\rangle)$. By surjectivity of f on A_1, there is $\nu' \in A_1$ such that:

$$f(\nu') = \nu. \tag{18.24}$$

Define:

$$f_{\nu'}^{-1}(C) := \{f_{\nu'}^{-1}(\mathbf{c}) \,|\, \mathbf{c} \in C\}.$$

For every $\mathbf{c} \in f_{\nu'}^{-1}(C)$, define:

$$D'_{\mathbf{c}} := \{d - \delta \,|\, d \in D_{f_{\nu'}(\mathbf{c})}\}. \tag{18.25}$$

Define:

$$f_{\nu'}^{-1}(S) := (f_{\nu'}^{-1}(C), \langle D'_{\mathbf{c}} \,|\, \mathbf{c} \in f_{\nu'}^{-1}(C)\rangle).$$

Lemma 18.15. $f_{\nu'}^{-1}(S)$ is a n-selection.

Proof. 1. Consider $\mathbf{c} \in f_{\nu'}^{-1}(C)$. Then, by definition of $f_{\nu'}^{-1}(C)$, there is $\mathbf{c}' \in C$ such that $f_{\nu}^{-1}(\mathbf{c}') = \mathbf{c}$. By point 1 of the definition of n-selection, we have $|\mathbf{c}'| < n$. Moreover, by definition of f_ν, we have $|f_{\nu'}^{-1}(\mathbf{c}')| = |\mathbf{c}'|$. Then, since $\mathbf{c} = f_{\nu'}^{-1}(\mathbf{c}')$, we have $|\mathbf{c}| < n$. Then, by arbitrariness of $\mathbf{c} \in f_\nu(C)$, we have $f_\nu(C) \subseteq {}^{<n+1}\mathcal{N}$.
2. By point 2 of definition of n-selection, there is $\mathbf{c} \in C$. Then, by definition of $f_{\nu'}^{-1}(C)$, we have $f_{\nu'}^{-1}(\mathbf{c}) \in f_{\nu'}^{-1}(C)$. Then, $f_{\nu'}^{-1}(C) \neq \emptyset$.
3. Consider $\mathbf{c}b \in f_{\nu'}^{-1}(C)$. Then, by definition of $f_{\nu'}^{-1}(C)$, there is $\mathbf{c}' \in C$ such that $f_{\nu'}^{-1}(\mathbf{c}') = \mathbf{c}b$. We prove that $f_{\nu'}^{-1}(\mathbf{c}' - 1) = \mathbf{c}$. Indeed, for every $0 \leq i < |\mathbf{c}' - 1| + 1$, we have:

$$\begin{aligned}
f_{\nu'}^{-1}(\mathbf{c}' - 1)(i) &= (\mathbf{c}' - 1)(i) - \gamma(|f(\nu')| + 1 + i) \text{ (by Lemma 18.12(3))}\\
&= \mathbf{c}'(i) - \gamma(|f(\nu')| + 1 + i) \text{ (by definition of } \mathbf{c}' - 1)\\
&= f_{\nu'}^{-1}(\mathbf{c}')(i) \text{ (by Lemma 18.12(3))}\\
&= \mathbf{c}b(i) \text{ (by choice of } \mathbf{c}')\\
&= \mathbf{c}(i) \text{ (by definition of } \mathbf{c}b).
\end{aligned}$$

Moreover, by definition of $f_{\nu'}$, we have $|f_{\nu'}^{-1}(\mathbf{c}')| = |\mathbf{c}'|$. Then, since $f_{\nu'}^{-1}(\mathbf{c}') = \mathbf{c}b$, we have $|\mathbf{c}'| = |\mathbf{c}b| \geq 0$. Then, by point 3 of definition of n-selection, we have $\mathbf{c}'-1 \in C$. Then, by definition of $f_{\nu'}^{-1}(C)$, we have $f_{\nu'}^{-1}(\mathbf{c}'-1) \in f_{\nu'}^{-1}(C)$. Then, since $f_{\nu'}^{-1}(\mathbf{c}' - 1) = \mathbf{c}$, we have $\mathbf{c} \in f_{\nu'}^{-1}(C)$.
4. Consider $\mathbf{c} \in f_{\nu'}^{-1}(C) \cap {}^{<n}\mathcal{N}$. Then, we have:

$$\begin{aligned}
\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in f_{\nu'}^{-1}(C)\} &= \{b \in \mathbf{Z} \,|\, f_{\nu'}(\mathbf{c}b) \in C\} \text{ (by Lemma 18.12(2))}\\
&= \{b \in \mathbf{Z} \,|\, (f_{\nu'}(\mathbf{c}b) - 1)(b + \gamma(|f(\nu)| + 1 + |\mathbf{c}b|) \in C\} \text{ (by Lemma 18.12(1))}.
\end{aligned}$$

Define:

$$\begin{aligned}
E &:= \{b \in \mathbf{Z} \,|\, (f_{\nu'}(\mathbf{c}b) - 1)b \in C\}.\\
F &:= \{b \in \mathbf{Z} \,|\, (f_{\nu'}(\mathbf{c}b) - 1)(b + \gamma(|f(\nu')| + 1 + |\mathbf{c}b|)) \in C\},\\
t : \mathbf{Z} \to \mathbf{Z} &:= n \mapsto n - \gamma(|f(\nu')| + 1 + |\mathbf{c}b|).
\end{aligned}$$

Then, t is a translation from $\mathbf{Z}$ to $\mathbf{Z}$. Consider $b \in E$. Then, by definition of E, we have $(f_{\nu'}(\mathbf{c}b) - 1)b \in C$. Then, by definition of t, we have $(f_{\nu'}(\mathbf{c}b) - 1)(t(b) + \gamma(|f(\nu')| + 1 + |\mathbf{c}b|)) \in C$. Then, by definition of F, we have $t(b) \in F$. Then $t[E] \subseteq F$. Moreover, by definition of $f_{\nu'}^{-1}(C)$, we have $f_{\nu'}(\mathbf{c}) \in C$. Then, by point 4 of the definition of n-selection, we have that E is cofinite in $\mathbf{Z}$. Then, by Remark 18.5, F is cofinite in $\mathbf{Z}$. Then, since we showed $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in f_{\nu'}^{-1}(C)\} = F$, we have that $\{b \in \mathbf{Z} \,|\, \mathbf{c}b \in f_{\nu'}^{-1}(C)\}$ is cofinite in $\mathbf{Z}$.

5. Consider $\mathbf{c} \in f_{\nu'}^{-1}(C)$. Consider $t : \mathbf{Z} \to \mathbf{Z} := n \mapsto n - \delta$. Observe that t is a translation from $\mathbf{Z}$ to $\mathbf{Z}$. Moreover, by definition of $D'_{\mathbf{c}}$, we have $t[D_{f_{\nu'}(\mathbf{c})}] = D'_{\mathbf{c}}$. Moreover, by point 5 of the definition of n-selection, we have that $D_{f_{\nu'}(\mathbf{c})}$ is cofinite in $\mathbf{Z}$. Then, by Remark 18.5, we have that $D'_{\mathbf{c}}$ is cofinite in $\mathbf{Z}$. □

By definition of f (18.14,18.13), we have:

$$f_{\nu'}(f_{\nu'}^{-1}(S)) := (f_{\nu'}(f_{\nu'}^{-1}(C)), \langle\{d + \delta \,|\, d \in D'_{f_{\nu'}^{-1}(\mathbf{c})}\} \,|\, \mathbf{c} \in f_{\nu'}(f_{\nu'}^{-1}(C))\rangle). \tag{18.26}$$

By definition of $f_{\nu'}^{-1}(C)$ and of f (18.12), we have:

$$f_{\nu'}(f_{\nu'}^{-1}(C)) = C.$$

Then, (18.26) simplifies in:

$$f_{\nu'}(f_{\nu'}^{-1}(S)) = (C, \langle\{d + \delta \,|\, d \in D'_{f_{\nu'}^{-1}(\mathbf{c})}\} \,|\, \mathbf{c} \in C\rangle).$$

Then, by definition of $D'_{f_{\nu'}^{-1}(\mathbf{c})}$ (18.25), we have:

$$\begin{aligned} f_{\nu'}(f_{\nu'}^{-1}(S)) &= (C, \langle\{d - \delta + \delta \,|\, d \in D_{f_{\nu'}(f_{\nu'}^{-1}(\mathbf{c}))}\} \,|\, \mathbf{c} \in C\rangle) \\ &= (C, \langle\{d \,|\, d \in D_{f_{\nu'}(f_{\nu'}^{-1}(\mathbf{c}))}\} \,|\, \mathbf{c} \in C\rangle) \text{ (by arithmetic)} \\ &= (C, \langle\{d \,|\, d \in D_{\mathbf{c}}\} \,|\, \mathbf{c} \in C\rangle) \text{ (by Proposition 18.12(2))} \\ &= (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in C\rangle) \\ &= S. \end{aligned} \tag{18.27}$$

By definition of f (18.15), we have:

$$\begin{aligned} f(B(\nu', f_{\nu'}^{-1}(S))) &= B(f(\nu'), f_{\nu'}(f_{\nu'}^{-1}(S))) \\ &= B(\nu, S) \text{ (by (18.24,18.27))}. \end{aligned}$$

Then, $f(B(\nu', f_{\nu'}^{-1}(S))) = B$ as desired.

4. *Relation symbols* $r \in \Phi$. Consider $a \in A$ and $r \in \Phi$. Suppose that $a \in V(r)$. Then, by definition of V, we have that either $r = p$ or $r = q$. Suppose that $r = p$. Then, by definition of V, we have that $a \in A_1$. Then by definition of f, we have $f(a) \in A_1$. Then, by definition of V, we have $f(a) \in V(p)$. Suppose that $r = q$. Then, by definition of V, we have that $a \in A_2$. Then by definition of

f, we have $f(a) \in A_2$. Then, by definition of V, we have $f(a) \in V(q)$. Suppose that $f(a) \in V(r)$. Then, by definition of V, we have that either $r = p$ or $r = q$. Suppose that $r = p$. Then, by definition of V, we have that $f(a) \in A_1$. Then by definition of f, we have $a \in A_1$. Then, by definition of V, we have $a \in V(p)$. Suppose that $r = q$. Then, by definition of V, we have that $f(a) \in A_2$. Then by definition of f, we have $a \in A_2$. Then, by definition of V, we have $a \in V(q)$.

5. *Relation symbol ε.* Consider $a \in A$ and $B \in \mathcal{B}_n$. Then, either $B = \{a'\}$ for some $a' \in A_2$ or $B = B(\nu, S)$ for some $\nu \in A_1$ and n-selection S. Suppose that $B = \{a'\}$ for some $a' \in A_2$. Suppose that $a \in B$. Then, $a = a'$. Then $f(a) \in \{f(a')\}$. Then, since, by definition of f (18.10), $f(B) = \{f(a')\}$, we have $f(a) \in f(B)$ as desired. Suppose that $B = B(\nu, S)$ for some $\nu \in A_1$ and n-selection $S = (C, \langle D_{\mathbf{c}} \,|\, \mathbf{c} \in C\rangle)$. Two cases are given:

   (a) $a \in A_1$. Then, $a = \mu$ for some $\mu \in \mathcal{Z}$. Suppose that:

   $$\mu \in B(\nu, S).$$

   Then, by definition of $B(\nu, S)$, this is equivalent to:

   $$\mu = \nu.$$

   Then, by injectivity of f (Lemma 18.12(2)), this is equivalent to:

   $$f(\mu) = f(\nu).$$

   Then, by definition of $B(f(\nu), f_\nu(S))$, this is equivalent to:

   $$f(\mu) \in B(f(\nu), f_\nu(S)).$$

   Then, by definition of $f(B(\nu, S))$ (18.15), this is equivalent to:

   $$f(\mu) \in f(B(\nu, S)).$$

   (b) $a \in A_2$. Then $a = (\mu, n, m)$ for some $\mu \in \mathcal{Z}$, and $n, m \in \mathbf{Z}$. Suppose that:

   $$(\mu, n, m) \in B(\nu, S).$$

   Then, by definition of $B(\nu, S)$, this is equivalent to:

   $$(\mu, n, m) = (\nu\mathbf{c} - 1, (\nu\mathbf{c})(|\nu\mathbf{c}|), d)$$

   for some $\mathbf{c} \in C$, and $d \in D_{\mathbf{c}}$. Then, by injectivity of f (Lemma 18.12(2)) and a bit of thought on the definition of f, this is equivalent to:

$$f((\mu, n, m)) = (f(\nu\mathbf{c}) - 1, f(\nu\mathbf{c})(|f(\nu\mathbf{c})|), d + \delta)$$

for some $\mathbf{c} \in C$, and $d \in D_{\mathbf{c}}$. Then, by Lemma 18.12(4), this is equivalent to:

$$f((\mu, n, m)) = (f(\nu)f_\nu(\mathbf{c}) - 1, (f(\nu)f_\nu(\mathbf{c}))(|f(\nu)f_\nu(\mathbf{c})|), d + \delta)$$

for some $\mathbf{c} \in C$, and $d \in D_{\mathbf{c}}$. Then, by (18.12) and (18.13), this is equivalent to:

$$f((\mu, n, m)) = (f(\nu)\mathbf{c} - 1, (f(\nu)\mathbf{c})(|f(\nu)\mathbf{c}|), d)$$

for some $\mathbf{c} \in f_\nu(C)$, and $d \in D_{\mathbf{c}}$. Then, by (18.14), this is equivalent to:

$$f((\mu, n, m)) \in B(f(\nu), f_\nu(S)).$$

Then, by (18.15), this is equivalent to:

$$f((\mu, n, m)) \in f(B(\nu, S))$$

as desired. □

We are ready to prove Points 1 and 2. 1. Consider arbitrary $a_1, a_2 \in A_1$. Then $a_1 = \nu_1$ and $a_2 = \nu_2$ for some $\nu_1, \nu_2 \in \mathcal{Z}$. Define:

$$\beta := |\nu_2| - |\nu_1|. \tag{18.28}$$

For every integer $i < |\nu_2| + 1$, define:

$$\gamma(i) := \nu_2(i) - \nu_1(i - \beta). \tag{18.29}$$

For every integer $i \geq |\nu_2| + 1$, let $\gamma(i)$ be arbitrary. Let δ be arbitrary. Let f be the automorphism on $\mathcal{M}_n$ defined by $\beta, \{\gamma(i) \mid i \in \mathbf{Z}\}$, and δ. Then, we have:

$$\begin{aligned} |f(\nu_1)| &= |\nu_1| + \beta \text{ (by definition of } f \text{ and of } |\_|) \\ &= |\nu_1| + |\nu_2| - |\nu_1| \text{ (by definition of } \beta \text{ (18.28))} \\ &= |\nu_2| \text{ (by arithmetic).} \end{aligned}$$

Moreover, for every $i < |f(\nu_1)| + 1 = |\nu_2| + 1$, we have:

$$
\begin{aligned}
f(\nu_1)(i) &= \nu_1(i-\beta) + \gamma(i) \text{ (by definition of } f \text{ (18.8))} \\
&= \nu_1(i-\beta) + \nu_2(i) - \nu_1(i-\beta) \text{ (by definition of } \gamma(i) \text{ (18.29))} \\
&= \nu_2(i) \text{ (by arithmetic).}
\end{aligned}
$$

Then $f(\nu_1) = \nu_2$ as desired.

2. Consider arbitrary $a_1, a_2 \in A_2$. Then $a_1 = (\nu_1, n_1, m_1)$ and $a_2 = (\nu_2, n_2, m_2)$ for some $\nu_1, \nu_2 \in \mathcal{Z}$ and $n_1, m_1, n_2, m_2 \in \mathbf{Z}$. Define an automorphism f on $\mathcal{M}_n$ such that $f(\nu_1 n_1) = \nu_2 n_2$ as per Point 1, taking:

$$\delta = m_2 - m_1. \tag{18.30}$$

Then, we have:

$$
\begin{aligned}
f((\nu_1, n_1, m_1)) &= (f(\nu_1 n_1) - 1, f(\nu_1 n_1)(|f(\nu_1 n_1)|), m_1 + \delta) \text{ (by definition of } f \text{ (18.9))} \\
&= (f(\nu_1 n_1) - 1, f(\nu_1 n_1)(|f(\nu_1 n_1)|), m_1 + m_2 - m_1) \text{ (by definition of } \delta \text{ (18.30))} \\
&= (\nu_2 n_2 - 1, (\nu_2 n_2)(|(\nu_2 n_2)|), m_1 + m_2 - m_1) \text{ (by choice of } f) \\
&= (\nu_2, n_2, m_2) \text{ (by definition of } \nu_2 n_2 - 1, (\nu_2 n_2)(|(\nu_2 n_2)|) \text{, and arithmetic).}
\end{aligned}
$$

Then, $f((\nu_1, n_1, m_1)) = (\nu_2, n_2, m_2)$ as desired. □

By Proposition 18.11, plainly follows:

Corollary 18.16.

For every $a_1, a_2 \in A_1$, and $\varphi[x] \in L_2$, we have:

$$\mathcal{M}_n \models \varphi[a_1] \text{ if and only if } \mathcal{M}_n \models \varphi[a_2].$$

For every $a_1, a_2 \in A_2$, and $\varphi[x] \in L_2$, we have:

$$\mathcal{M}_n \models \varphi[a_1] \text{ if and only if } \mathcal{M}_n \models \varphi[a_2].$$

We now prove that:

Proposition 18.17. For every $a \in A_1$, $\mathcal{M}_n \models \varphi_n[a]$.

Proof. Consider $a \in A_1$. Then $a = \nu_0$ for some $\nu_0 \in \mathcal{Z}$. Let $S_{\mathbf{Z}} = (C_{\mathbf{Z}}, \langle D_{\mathbf{Zc}} \,|\, \mathbf{c} \in C \rangle)$ be the n-selection given by $C_{\mathbf{Z}} := {}^{<n+1}\mathcal{N}$, and, for every $\mathbf{c} \in C_{\mathbf{Z}}$, $D_{\mathbf{Zc}} := \mathbf{Z}$. Consider $B(\nu_0, S_{\mathbf{Z}})$. Consider an arbitrary $B(\nu_0, S_0)$ such that $S_0 = (C_0, \langle D_{0\mathbf{c}} \,|\, \mathbf{c} \in C_0 \rangle)$ is an n-selection.

Lemma 18.18. For every $0 \leq i < n$:

- Let $b_1, ..., b_i$ be an arbitrary element of ${}^{[0,i)}\mathbf{Z}$ (if $i = 0$, then $b_1, ..., b_i = \emptyset \in {}^{[0,0)}\mathbf{Z}$);
- Let $S_i = (C_i, \langle D_{i\mathbf{c}} \,|\, \mathbf{c} \in C_i \rangle)$ be an arbitrary n-selection.

Suppose that:

H1. $b_1...b_i \in C_0$.

Then, there is $b_{i+1} \in C_i$ such that:

T1. $b_1...b_{i+1} \in C_0$.

Proof. Consider an arbitrary $0 \leq i < n$. Suppose that hypothesis H1 holds. Then, by point 4 of the definition of n-selection, we have that $\{b \in \mathbf{Z} \,|\, b_1...b_i b \in C_0\}$ is cofinite in $\mathbf{Z}$. Moreover, by point 2 of the definition of n-selection, we have that $\emptyset \in C_i$. Then, by point 4 of the definition of n-selection, we have $\{b \in \mathbf{Z} \,|\, b \in C_i\}$ is cofinite in $\mathbf{Z}$. Then, there is $b_{i+1} \in \{b \in \mathbf{Z} \,|\, b_1...b_i b \in C_0\} \cap \{b \in \mathbf{Z} \,|\, b \in C_i\}$. □

Then, by Lemma 18.18 ($i = 0$), there is $b_1 \in C_0$ such that $b_1 \in C_0$. (Observe that, if $i = 0$, then $b_1...b_i = \emptyset$, and, by point 2 of the definition of n-selection, we have $\emptyset \in C_0$. Then, H1 ($i = 0$) is satisfied.) Consider $\nu_1 := \nu_0 b_1$. Consider $B(\nu_1, S_\mathbf{Z})$. Consider an arbitrary $B(\nu_1, S_1)$ such that $S_1 = (C_1, \langle D_{1\mathbf{c}} \,|\, \mathbf{c} \in C_1 \rangle)$ is an n-selection. Plainly, H1 ($i = 1$) is satisfied. Repeat until you obtain $b_n \in \mathbf{Z}$ such that $b_1...b_n \in C_0$.

Lemma 18.19. For every $0 < i \leq n$:

1. $\nu_i \notin B(\nu_{i-1}, S_\mathbf{Z})$.
2. $\nu_i \in cl(B(\nu_{i-1}, S_{i-1}))$.
3. $\nu_n \in cl(B(\nu_0, S_0))$.

Proof. 1. By definition, for every $0 < i \leq n$, we have $\nu_i = \nu_{i-1} b_i$. Then, we have $\nu_i \neq \nu_{i-1}$. Then, by definition of $B(\nu_{i-1}, S_\mathbf{Z})$, we have $\nu_i \notin B(\nu_{i-1}, S_\mathbf{Z})$.

2. Consider $0 < i \leq n$. Then, by Lemma 18.18, we have:

$$\nu_i = \nu_{i-1} b_i; \tag{18.31}$$

$$b_i \in C_{i-1}. \tag{18.32}$$

Consider an arbitrary $B(\nu_i, S'_i)$ where $S'_i = (C'_i, \langle D'_{i\mathbf{c}} \,|\, \mathbf{c} \in C'_i \rangle)$ is an n-selection. By point 2 and 3 of the definition of n-selection, we have $\emptyset \in C'_i$. Then, $D'_{i\emptyset}$ is defined. By (18.32), $D_{i-1 b_i}$ is defined. By definition of n-selection, both $D_{i-1 b_i}$ and $D'_{i\emptyset}$ are cofinite in $\mathbf{Z}$. Then, there is $d \in D_{i-1 b_i} \cap D'_{i\emptyset}$. Then, we have:

$$\begin{aligned} B(\nu_{i-1}, S_{i-1}) &\ni (\nu_{i-1} b_i - 1, \nu_{i-1} b_i(|\nu_{i-1} b_i|), d) \text{ (by definition of } B(\nu_{i-1}, S_{i-1})) \\ &= (\nu_i - 1, \nu_i(|\nu_i|), d) \text{ (by (18.31))} \\ &\in B(\nu_i, S'_i) \text{ (by definition of } B(\nu_i, S'_i)). \end{aligned}$$

Then, we have $B(v_i, S'_i) \cap B(v_{i-1}, S_{i-1}) \neq \emptyset$. Then, by arbitrariness of $B(v_i, S'_i)$, we have $v_i \in cl(B(v_{i-1}, S_{i-1}))$.

3. By Lemma 18.18, we have:

$$v_n = v_0 b_1 ... b_n; \tag{18.33}$$

$$b_1 ... b_n \in C_0. \tag{18.34}$$

Consider an arbitrary $B(v_n, S'_n)$ where $S'_n = (C'_n, \langle D'_{n\mathbf{c}} \,|\, \mathbf{c} \in C'_n \rangle)$ is an n-selection. By point 2 of the definition of n-selection, we have $\emptyset \in C'_n$. Then, $D'_{n\emptyset}$ is defined. By (18.34), $D_{0b_1...b_n}$ is defined. By definition of selection in n, both $D_{0b_1...b_n}$ and $D'_{n\emptyset}$ are cofinite in $\mathbf{Z}$. Then, there is $d \in D_{0b_1...b_n} \cap D'_{n\emptyset}$. Then, we have:

$$\begin{aligned} B(v_0, S_0) &\ni (v_0 b_1 ... b_n - 1, v_0 b_1 ... b_n(|v_0 b_1 ... b_n|), d) \text{ (by definition of } B(v_0, S_0)) \\ &= (v_n - 1, v_n(|v_n|), d) \text{ (by (18.33))} \\ &\in B(v_n, S'_n) \text{ (by definition of } B(v_n, S'_n)). \end{aligned}$$

Then, we have $B(v_n, S'_n) \cap B(v_0, S_0) \neq \emptyset$. Then, by arbitrariness of $B(v_n, S'_n)$, we have $v_n \in cl(B(v_0, S_0))$. □

Then, by choice of a and, for every $0 \leq i < n$, of $B(v_i, S_{\mathbf{Z}})$ and $B(v_i, S_i)$, thesis follows. □

Finally:

Proposition 18.20. For every $m > n$, and $a \in A_1$, $\mathcal{M}_n \not\models \varphi_m[a]$.

Proof. We begin with a preliminary lemma:

Lemma 18.21. For every $v_1 \in A_1$, $B(v_1, S_1)$ where $S_1 = (C_1, \langle D_{1\mathbf{c}} \,|\, \mathbf{c} \in C_1 \rangle)$ is an n-selection, and $a \in A$, if $a \notin B(v_1, S_1)$ and $a \in cl(B(v_1, S_1))$, then $a = v_2$ for some $v_2 \in \mathcal{Z}$ and $v_1 \subset v_2$.

Proof. If $a \in A_2$, then, since $a \notin B(v_1, S_1)$ and $\{a\} \in \mathcal{B}_n$, we have that $a \notin cl(B(v_1, S_1))$, a contradiction. Then $a \in A_1$. Then, $a = v_2$ for some $v_2 \in \mathcal{Z}$. Consider $B(v_2, S_2)$ where $S_2 = (C_2, \langle D_{2\mathbf{c}} \,|\, \mathbf{c} \in C_2 \rangle)$ is an n-selection. Since $v_2 \in cl(B(v_1, S_1))$, then $B(v_1, S_1) \cap B(v_2, S_2) \neq \emptyset$. Then, we have:

$$(v_1\mathbf{c_1} - 1, v_1\mathbf{c_1}(|v_1\mathbf{c_1}|), d_1) = (v_2\mathbf{c_2} - 1, v_2\mathbf{c_2}(|v_2\mathbf{c_2}|), d_2) \tag{18.35}$$

for some $\mathbf{c_1} \in C_1$ and $d_1 \in D_{1\mathbf{c_1}}$, and for some $\mathbf{c_2} \in C_2$ and $d_2 \in D_{2\mathbf{c_2}}$. Then, we have $v_1\mathbf{c_1} = v_2\mathbf{c_2}$. Then, we have $v_1 \subset v_2$ or $v_1 = v_2$ or $v_1 \supset v_2$. If $v_1 = v_2$, then $v_2 \in B(v_1, S_1)$, a contradiction.

Suppose $\nu_1 \supset \nu_2$. Define $C'_2 = \{\mathbf{c} \in C_2 \,|\, \nu_2\mathbf{c} \not\supseteq \nu_1\}$, and $S'_2 := (C'_2, \langle D_\mathbf{c} \,|\, \mathbf{c} \in C'_2\rangle)$. By Lemma 18.10, S'_2 is an n-selection. Plainly, $\nu_2 \in B(\nu_2, S'_2) \in \mathcal{B}_n$. Then, since $\nu_2 \in cl(B(\nu_1, S_1))$, we have (18.35) for some $\mathbf{c_1} \in C_1$ and $d_1 \in D_{1\mathbf{c_1}}$, and for some $\mathbf{c_2} \in C'_2$ and $d_2 \in D_{2\mathbf{c_2}}$. Then, we have $\nu_1\mathbf{c_1} = \nu_2\mathbf{c_2}$. Then, we have $\nu_2\mathbf{c_2} \supseteq \nu_1$, against $\mathbf{c_2} \in C'_2$. □

Consider $a \in A_1$. Then, $a = \nu_0$ for some $\nu_0 \in \mathcal{Z}$. Suppose that $\mathcal{M}_n \models \varphi_m[\nu_0]$. Then, there is $B(\nu_0, S_0) \in \mathcal{B}_n$ with $S_0 = (C_0, \langle D_{0\mathbf{c}} \,|\, \mathbf{c} \in D_0\rangle)$ an n-selection, and $a_1 \in A$ such that $a_1 \notin B(\nu_0, S_0)$ and $a_1 \in cl(B(\nu_0, S_0))$. Then, by Lemma 18.21, $a_1 = \nu_1$ for some $\nu_1 \in A_1$ and $\nu_0 \subset \nu_1$. Repeat until you find $\nu_0, \nu_1, ..., \nu_m \in A_1$ such that $\nu_0 \subset ... \subset \nu_m$. Moreover, by definition of φ_m, we can suppose $\nu_m \in cl(B(\nu_0, S_0))$. Consider an arbitrary $B(\nu_m, S_m)$ where $S_m = \{D_{m\mathbf{c}} \,|\, \mathbf{c} \in C_m\}$. Then, since $\nu_m \in cl(B(\nu_0, S_0))$, we have $B(\nu_0, S_0) \cap B(\nu_m, S_m) \neq \emptyset$. Then, we have:

$$(\nu_0\mathbf{c_0} - 1, \nu_0\mathbf{c_0}(|\nu_0\mathbf{c_0}|), d_0) = (\nu_m\mathbf{c_m} - 1, \nu_m\mathbf{c_m}(|\nu_m\mathbf{c_{m-1}}|), d_m) \tag{18.36}$$

for some $\mathbf{c_0} \in C_0$ and $d_0 \in D_{0\mathbf{c_0}}$, and for some $\mathbf{c_m} \in C_m$ and $d_m \in D_{m\mathbf{c_m}}$. Then, since $\nu_0 \subset \nu_m$, there is $\mathbf{c} \subseteq \mathbf{c_0}$, such that $\nu_0\mathbf{c} = \nu_m$. Then, by point 3 of definition of n-selection, we have $\mathbf{c} \in C_0$. Moreover, since $\nu_0 \subset ... \subset \nu_m$, we have $|\mathbf{c}| + 1 \geq m > n$, a contradiction since $\mathbf{c} \in C_0 \subseteq \bigcup_{\text{integer } i<n+1} {}^{[0,i)}\mathbf{Z}$. □

18.2.3 Proof of Theorem 18.3

Proof of Theorem 18.3. Consider an arbitrary finite set σ of modal operators. Then, by Corollary 18.16, and since $\{p, q\}$ and σ are finite, there are distinct $n, m \in \mathbf{N}$ such that $n < m$ and, for every $a \in A$, and formula φ of the form $\#(\varphi_0, ..., \varphi_{k-1})$ with $(\#, \psi[x], p_0, ..., p_{k-1}) \in \sigma$ and, for every $i \in k$, $\varphi_i \in \{\top, \bot, p, q\}$, we have:

$$\mathcal{M}_n, a \models \#(\varphi_0, ..., \varphi_{k-1}) \text{ if and only if } \mathcal{M}_m, a \models \#(\varphi_0, ..., \varphi_{k-1}). \tag{18.37}$$

Then:

Lemma 18.22. For every formula $\varphi \in L_{\Phi,\tau}$ such that all propositional variables occurring in φ are among $\{p, q\}$, and $a \in A$, we have $\mathcal{M}_n, a \models \varphi$ if and only if $\mathcal{M}_m, a \models \varphi$.

Proof. By induction on φ. The case in which φ is $\top$ is trivial. Suppose $\varphi \in \Phi$. Then, by hypothesis, $\varphi \in \{p, q\}$. Then, thesis plainly follows by definition of V. The boolean cases are trivial. Suppose φ is $\#(\varphi_0, ..., \varphi_{k-1})$ for $(\#, \psi[x], p_0, ..., p_{k-1}) \in \tau$. Observe that, since all propositional variables occurring in φ are among $\{p, q\}$, then, for every $i \in k$, all propositional variables occurring in φ_i are among $\{p, q\}$. Then, by inductive hypothesis, we have:

$$\varphi_i^{\mathcal{M}_n} = \varphi_i^{\mathcal{M}_m} \text{ for every } i \in k. \tag{18.38}$$

Suppose that $(\#, \psi[x], p_0, ..., p_{k-1}) \in \rho$. Then, ψ is a point-sort formula. Then, $\mathcal{M}_n, a \models \#(\varphi_0, ..., \varphi_{k-1})$ if and only if, by semantics, $\mathcal{M}_n[\varphi_0^{\mathcal{M}_n}/p_0, ..., \varphi_{k-1}^{\mathcal{M}_n}/p_{k-1}], a \models \psi[a]$ if and only if, by Remark 18.6 and (18.38), $\mathcal{M}_m[\varphi_0^{\mathcal{M}_m}/p_0, ..., \varphi_{k-1}^{\mathcal{M}_m}/p_{k-1}] \models \psi[a]$ if and only if, by semantics, $\mathcal{M}_m, a \models \#(\varphi_0, ..., \varphi_{k-1})$.

Suppose that $(\#, \psi[x], p_0, ..., p_{k-1}) \in \sigma$. Let χ be $\#(\chi_0, ..., \chi_{k-1})$ where, for every $i \in k$:

$$\chi_i := \begin{cases} \top & \text{provided } \varphi_i^{\mathcal{M}_n} = A, \\ p & \text{provided } \varphi_i^{\mathcal{M}_n} = A_1, \\ q & \text{provided } \varphi_i^{\mathcal{M}_n} = A_2, \\ \bot & \text{provided } \varphi_i^{\mathcal{M}_n} = \emptyset. \end{cases}$$

Then, by Corollary 18.16, the formulas χ_i are well defined. Moreover, by definition of V, we have:

$$\varphi_i^{\mathcal{M}_n} = \chi_i^{\mathcal{M}_n} \text{ for every } i \in k, \tag{18.39}$$

and, by choice of n, m, we have:

$$\chi_i^{\mathcal{M}_n} = \chi_i^{\mathcal{M}_m} \text{ for every } i \in k. \tag{18.40}$$

Then, we have $\mathcal{M}_n, a \models \varphi$ if and only if, by semantics, $\mathcal{M}_n[\varphi_0^{\mathcal{M}_n}/p_0, ..., \varphi_{k-1}^{\mathcal{M}_n}/p_{k-1}] \models \psi[a]$ if and only if, by (18.39), $\mathcal{M}_n[\chi_0^{\mathcal{M}_n}/p_0, ..., \chi_{k-1}^{\mathcal{M}_n}/p_{k-1}] \models \psi[a]$ if and only if, by semantics, $\mathcal{M}_n, a \models \chi$. Observe that χ is a formula of $L_{\Phi,\tau}$ such that all propositional variables occurring in χ are among $\{p, q\}$. Then, by choice of $\mathcal{M}_n$ and $\mathcal{M}_m$, we have $\mathcal{M}_n, a \models \chi$ if and only if $\mathcal{M}_m, a \models \chi$ if and only if, by semantics, $\mathcal{M}_m[\chi_0^{\mathcal{M}_m}/p_0, ..., \chi_{k-1}^{\mathcal{M}_m}/p_{k-1}] \models \psi[a]$ if and only if, by (18.38), (18.39) and (18.40), $\mathcal{M}_m[\varphi_0^{\mathcal{M}_m}/p_0, ..., \varphi_{k-1}^{\mathcal{M}_m}/p_{k-1}] \models \psi[a]$ if and only if, by semantics, $\mathcal{M}_m, a \models \varphi$. □

Suppose, by contradiction, that $L_{\Phi,\tau}$ can express L_t on $\mathfrak{T}_2$. Then, since $\varphi_m \in L_t$, there is $\varphi \in L_{\Phi,\tau}$ equivalent to φ_m on $\mathfrak{T}_2$. Since φ is equivalent to φ_m, and there are no propositional variables occurring in φ_m, we can assume that there are no propositional variables occurring in φ. Indeed, let $p_0, ..., p_{l-1} \in \Phi$ be the propositional variables occurring in φ, and $\varphi[\top/p_0, ..., \top/p_{l-1}]$ the formula obtained by replacing, for every $i \in l$, every occurrence of p_i by $\top$. Then, for every T_2 topological model $\mathcal{M} = (A, \tau, V)$, we have $\mathcal{M} \models \varphi_m[a]$ if and only if, since there are no occurring propositional variables in φ_m, $\mathcal{M}[A/p_0, ..., A/p_{l-1}] \models \varphi_m[a]$ if and only if, since φ_m and φ are equivalent on $\mathfrak{T}_2$, $\mathcal{M}[A/p_0, ..., A/p_{l-1}], a \models \varphi$ if and only if, by definition of $\varphi[\top/p_0, ..., \top/p_{l-1}]$, $\mathcal{M}, a \models \varphi[\top/p_0, ..., \top/p_{l-1}]$. Then, φ_m and $\varphi[\top/p_0, ..., \top/p_{l-1}]$ are equivalent on $\mathfrak{T}_2$ and, by definition of $\varphi[\top/p_0, ..., \top/p_{l-1}]$, there are no propositional variables occurring in $\varphi[\top/p_0, ..., \top/p_{l-1}]$. Take $\varphi[\top/p_0, ..., \top/p_{l-1}]$ as φ.

Recall that L_t is invariant under changing bases, and so, since the semantics of modal operators is determined by an L_t formula, also $L_{\Phi,\tau}$ is invariant under changing bases. By Proposition 18.17, for every $a \in A_1$, we have $\mathcal{M}_m \models \varphi_m[a]$. Then, since φ_m and φ are equivalent on $\mathfrak{T}_2$, by Proposition 18.9, we have $\mathcal{M}_m \models \varphi[a]$. Then, since there are no propositional variables occurring in φ, by Lemma 18.22, we have $\mathcal{M}_n \models \varphi[a]$. Then, since φ_m and φ are equivalent on $\mathfrak{T}_2$, we have $\mathcal{M}_n \models \varphi_m[a]$, against Proposition 18.20. □

Chapter 19

Logic of $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$

In this chapter we study the logic $\mathsf{L}_{\mathfrak{X}}$ of any class of topological models $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$. We axiomatize this theory, and prove that it is PSPACE-complete.

More precisely, in Section 19.1 we give a list of axioms Λ (let L_Λ be the modal logic generated by Λ as defined in Section 19.1). In Section 19.2, we prove that L_Λ *is sound with respect to* $\mathsf{L}_{\mathfrak{T}_1}$ (that is, $\mathsf{L}_\Lambda \subseteq \mathsf{L}_{\mathfrak{T}_1}$), and in Section 19.4 that L_Λ *is complete with respect to* $\mathsf{L}_{\mathfrak{T}_3}$ (that is, $\mathsf{L}_{\mathfrak{T}_3} \subseteq \mathsf{L}_\Lambda$). Then, for every $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$, we obtain that $\mathsf{L}_{\mathfrak{X}}$ *is axiomatized by* Λ (that is, $\mathsf{L}_\Lambda = \mathsf{L}_{\mathfrak{X}}$). In particular, $\mathsf{L}_\Lambda = \mathsf{L}_{\mathfrak{T}_1} = \mathsf{L}_{\mathfrak{T}_2} = \mathsf{L}_{\mathfrak{T}_3}$. Moreover, we prove that $\mathsf{L}_{\mathfrak{X}}$ *is strongly complete* (that is, for every $\Gamma \cup \{\varphi\} \subseteq L$, we have that $\Gamma \models \varphi$ implies $\Gamma \vdash_{\mathsf{L}_{\mathfrak{X}}} \varphi$).

In Section 19.3, for every $\varphi \in L$, we define 'φ-quasi models'. They are finite codifications of topological models satisfying φ (Cf. [13, §6.4]). Indeed, we prove that if there is a φ-quasi model, then there is a T_3 topological model satisfying φ (Proposition 19.3), and that if there is a T_1 topological model satisfying φ, then there is a φ-quasi model (Proposition 19.6). From this, since $\mathsf{L}_{\mathfrak{T}_1} = \mathsf{L}_{\mathfrak{X}} = \mathsf{L}_{\mathfrak{T}_3}$, we obtain that deciding if $\varphi \in \mathsf{L}_{\mathfrak{X}}$ is reducible to deciding if there is no φ-quasi models. Finally, in Section 19.5, we prove that deciding if there is a φ-quasi model is PSPACE (Proposition 19.20). Then, deciding if there is no φ-quasi model is PSPACE. Then, deciding if $\varphi \in \mathsf{L}_{\mathfrak{X}}$ is PSPACE. In fact, we prove that the validity problem for $\mathsf{L}_{\mathfrak{X}}$ is PSPACE-complete (Theorem 19.21).

19.1 The logic L_Λ

For every $\varphi \in L$, define $\Diamond_{!0}\varphi$ as an abbreviation for $\neg\Diamond_0\varphi$, and, for every natural number $n > 0$, define $\Diamond_{!n}\varphi$ as an abbreviation for $\Diamond_{n-1}\varphi \wedge \neg\Diamond_n\varphi$. Plainly, $\Diamond_{!n}\varphi$ means that there are exactly n points satisfying φ. Consider the following $\Lambda \subseteq L$:

PL. All propositional tautologies.

1. Axioms for $\Diamond_n$ ($n \in \mathbf{N}$). Let p, q be distinct propositional variables:

   $\Diamond_{\mathbf{N}}1$. $\Diamond_{n+1} p \to \Diamond_n p$ ($n \in \mathbf{N}$).

   $\Diamond_{\mathbf{N}}2$. $\Box_0(p \to q) \to (\Diamond_n p \to \Diamond_n q)$ ($n \in \mathbf{N}$).

   $\Diamond_{\mathbf{N}}3$. $\Diamond_{!0}(p \wedge q) \to ((\Diamond_{!n_1} p \wedge \Diamond_{!n_2} q) \to \Diamond_{!n_1+n_2}(p \vee q))$ ($n_1, n_2 \in \mathbf{N}$).

   $\Diamond_{\mathbf{N}}4$. $\Box_0 p \to p$.

   $\Diamond_{\mathbf{N}}5$. $\Diamond_n p \to \Box_0 \Diamond_n p$ ($n \in \mathbf{N}$).

2. Axioms for $[d]$:

   $[d]1$. $[d](p \to q) \to ([d]p \to [d]q)$.

   $[d]2$. $[d]p \to [d][d]p$.

4. $\langle d \rangle p \to \Diamond_n p$ ($n \in \mathbf{N}$).

Let L_Λ be the smallest subset of L containing Λ and closed under the following inference rules:

- *Modus ponens*: If $\varphi \in \mathsf{L}_\Lambda$ and $\varphi \to \psi \in \mathsf{L}_\Lambda$, then $\psi \in \mathsf{L}_\Lambda$.
- *Uniform substitution*: if $\varphi \in \mathsf{L}_\Lambda$, then L_Λ contains all the substitution instances of φ.
- *Generalization*: if $\varphi \in \mathsf{L}_\Lambda$, then $\Box_0\varphi, [d]\varphi \in \mathsf{L}_\Lambda$.

Plainly, L_Λ is a modal logic. Axioms $\Diamond_{\mathbf{N}}1$-4 come from [36], Axiom $\Diamond_{\mathbf{N}}5$ comes from [26], Axioms $[d]1$,2 are **K4** for $[d]$, Axiom 4 is an original axiom that relates $[d]$ to the $\{\Diamond_n \mid n \in \mathbf{N}\}$.

Observe that L_Λ is not normal in the sense of Section 2.4. Indeed, let $A := \{0, 1\}$, τ be the discrete topology on A, and V any evaluation such that $V(p) = \{0\}$ and $V(q) = \emptyset$. Then, $\mathcal{M} := (A, \tau, V)$ is a T_1 topological model such that:

$$|\{a \in \mathcal{M} \mid \mathcal{M}, a \not\models p \to q\}| \leq 1,$$

$$|\{a \in \mathcal{M} \mid \mathcal{M}, a \not\models p\}| \leq 1,$$

$$|\{a \in \mathcal{M} \mid \mathcal{M}, a \not\models q\}| > 1.$$

Then, by soundness of L_Λ with respect to $\mathfrak{T}_1$ (Proposition 19.1), we have $\Box_1(p \to q) \to (\Box_1 p \to \Box_1 q) \notin \mathsf{L}_\Lambda$. Nonetheless, $\Box_0(p \to q) \to (\Box_0 p \to \Box_0 q)$ is plainly valid on $\mathfrak{T}_3$. Then, by completeness of L_Λ with respect to $\mathfrak{T}_3$ (Theorem 19.8), we have $\Box_0(p \to q) \to (\Box_0 p \to \Box_0 q) \in \mathsf{L}_\Lambda$. This, together with Generalization for $\Box_0$, gives 'normality of L_Λ with respect to $\Box_0$'. Moreover, 'L_Λ is normal with respect to $[d]$' too. Furthermore, Axiom Schema $\Diamond_{\mathbf{N}}2$'s structure is reminiscent of normality axioms.

Observe also that Axiom $\Diamond_{\mathbf{N}}4$ is **T** for $\Box_0$, and Axiom $\Diamond_{\mathbf{N}}4$ for $n = 0$ is **5** for $\Box_0$. Finally, Axiom Schema 4 states that if a point sees other points satisfying p arbitrarily near, then there are infinitely many points satisfying p.

19.2 Soundness of L_Λ with respect to $\mathsf{L}_{\mathfrak{T}_1}$

Theorem 19.1. L_Λ is sound with respect to $\mathsf{L}_{\mathfrak{T}_1}$.

Proof. As usual, to prove the soundness of L_Λ with respect to $\mathsf{L}_{\mathfrak{T}_1}$, we prove that every formula in Λ is valid on $\mathfrak{T}_1$, and that the inference rules preserve validity on $\mathfrak{T}_1$.

Validity on $\mathfrak{T}_1$ of propositional tautologies, $\Diamond_{\mathbf{N}}1$-5, and $[d]1$ is plain. As for Axiom $[d]2$, consider a topological model $\mathcal{M} \in \mathfrak{T}_1$, and a point $a \in \mathcal{M}$, such that $\mathcal{M}, a \models [d]p$. Then, there is a neighborhood U of a such that, for every $a' \in U \setminus \{a\}$, we have $\mathcal{M}, a' \models p$. Consider an arbitrary $a' \in U \setminus \{a\}$. Then, since $\mathcal{M}$ is T_1, there is a neighborhood U' of a' with $a \notin U'$. Then, $U \cap U'$ is a neighborhood of a' such that, for every $a'' \in U'' \setminus \{a'\}$, $\mathcal{M}, a'' \models p$. Then, $\mathcal{M}, a' \models [d]p$. Then, by arbitrariness of $a' \in U \setminus \{a\}$, we have $\mathcal{M}, a \models [d][d]p$. (Cf. [66, p. 198, Axiom IV'] (in French) by Kuratowski, 1922.)

As for Axiom 4, consider an arbitrary $\mathcal{M} \in \mathfrak{T}_1$, and a point $a \in \mathcal{M}$ such that $\mathcal{M}, a \models \langle d \rangle p$. Then, there exists a neighborhood U of a, and a point $a' \in U \setminus \{a\}$ such that $\mathcal{M}, a' \models p$. Then, since $\mathcal{M}$ is T_1, reasoning as before, we can find a neighborhood U' of a such that $U' \subseteq U \setminus \{a'\}$. Moreover, since $\mathcal{M}, a \models \langle d \rangle p$, we can find $a'' \in U' \setminus \{a\}$, and then $a'' \neq a'$, with $\mathcal{M}, a'' \models p$. Iterating, we obtain ω distinct $a' \in \mathcal{M}$ such that $\mathcal{M}, a' \models p$.

Finally, observe that Modus Ponens, Substitution and Generalization preserve validity on $\mathfrak{T}_1$. □

19.3 Quasi models

In this section, for every $\varphi \in L$, we define 'φ-quasi models'. They are finite codifications of topological models satisfying φ (Cf. [13, §6.4]). They are used to study the computability of $\mathsf{L}_{\mathfrak{X}}$ ($\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$).

Fix $\overline{\varphi} \in L$. Let $sub(\overline{\varphi}) := \{\psi \in L \,|\, \psi \leq \varphi\}$ denote the set of subformulas of $\overline{\varphi}$. Call a *Hintikka set for* $\overline{\varphi}$ every set $w \subseteq sub(\overline{\varphi})$ such that:

H1. For every $\neg\varphi \leq \overline{\varphi}$, we have $\neg\varphi \in w$ if and only if $\varphi \notin w$.

H2. For every $\varphi \wedge \psi \leq \overline{\varphi}$, we have $\varphi \wedge \psi \in w$ if and only if $\varphi \in w$ and $\psi \in w$.

Let $Hin(\overline{\varphi})$ be the set of all Hintikka sets for $\overline{\varphi}$.

Hintikka sets encode points and formulas that these points satisfy ($\varphi \in w$ meaning that the point encoded by w satisfies φ).

Definition 19.2. A $\overline{\varphi}$-*quasi model* is a tuple $\mathcal{M} = (W, m)$ with $W \subseteq Hin(\overline{\varphi})$, and $m : W \to \omega + 1$ such that:

Q1. $\overline{\varphi} \in \bigcup W$.

Q2. For every $n \in \mathbf{N}$, if $\Diamond_n \varphi \in \bigcup W$, then $\sum\{m(w) \mid \varphi \in w \in W\} > n$.

Q3. For every $n \in \mathbf{N}$, and $\Diamond_n \varphi \leq \overline{\varphi}$, if $\sum\{m(w) \mid \varphi \in w \in W\} > n$, then $\Diamond_n \varphi \in \bigcap W$.

Q4. For every $w \in W$, and every $[d]\varphi \leq \overline{\varphi}$ such that $[d]\varphi \notin w$, there is $w' \in W$ such that:

1. $\varphi \notin w'$.
2. For every $[d]\psi \in w$, we have $\psi, [d]\psi \in w'$.
3. $m(w') = \omega$.

A $\overline{\varphi}$-quasi model (W, m) encodes a T_3 topological model $\mathcal{M} = (A, \tau, V)$ satisfying $\overline{\varphi}$. (W, m) encodes the domain A of $\mathcal{M}$, with $m(w)$ encoding how many times a point encoded by w occurs in A. Q1 encodes the fact that $\overline{\varphi}$ is satisfied in $\mathcal{M}$. Q2 encodes the fact that, for every $n \in \mathbf{N}$, and $\Diamond_n \varphi \leq \overline{\varphi}$, if a point in $\mathcal{M}$ satisfies $\Diamond_n \varphi$, then there are more than n points in $\mathcal{M}$ satisfying φ. Q3 encodes the fact that, for every $n \in \mathbf{N}$, and $\Diamond_n \varphi \leq \overline{\varphi}$, if there are more than n points in $\mathcal{M}$ satisfying φ, then all points in $\mathcal{M}$ satisfy $\Diamond_n \varphi$. Q4 encodes the fact that for every point $a \in \mathcal{M}$, there is a neighborhood U of a such that:

- For every $[d]\varphi \leq \overline{\varphi}$, if a does not satisfy $[d]\varphi$, then there is $a' \in U \setminus \{a\}$ that does not satisfy φ;
- For every $[d]\varphi \leq \overline{\varphi}$, if a satisfies $[d]\varphi$, then every $a' \in U \setminus \{a\}$ satisfies φ and $[d]\varphi$.
- For every $a' \in U \setminus \{a\}$, there are infinitely many points in $\mathcal{M}$ that satisfy the same formulas $\varphi \leq \overline{\varphi}$ as a.

We prove that, if there is a $\overline{\varphi}$-quasi model, then there is a T_3 topological model satisfying $\overline{\varphi}$ (Proposition 19.3), and that if there is a T_1 topological model satisfying $\overline{\varphi}$, then there is a $\overline{\varphi}$-quasi model (Proposition 19.6).

Proposition 19.3. If there is a $\overline{\varphi}$-quasi model then there is a T_3 topological model satisfying $\overline{\varphi}$.

Proof. Consider an arbitrary $\overline{\varphi}$-quasi model $\mathcal{M} = (W, m)$. Define:

$$\begin{aligned} A_0 &:= \{(w,n) \mid w \in W \wedge n \in m(w)\}, \\ \alpha_0 &:= \emptyset, \\ \lambda_0 : A_0 \to W &:= (w,n) \mapsto w. \end{aligned}$$

Consider an arbitrary $i \in \mathbf{N} \setminus \{0\}$, and suppose that $\mathcal{M}_i = (A_i, \alpha_i, \lambda_i)$ has been defined. For every $a \in A_i$, for every $[d]\varphi \leq \overline{\varphi}$ such that $[d]\varphi \notin \lambda_i(a)$, let $w'_{(\lambda_i(a),[d]\varphi)}$ be as per Q4 applied to $(\lambda_i(a), [d]\varphi)$. Let $U_a := \{w'_{(\lambda_i(a),[d]\varphi)} \mid [d]\varphi \leq \overline{\varphi} \wedge [d]\varphi \notin \lambda_i(a)\}$. Let X_a be a set of new objects and f_a a bijection from X_a to U_a. Assume the sets X_a pairwise disjoint. Define $\mathcal{M}_{i+1} = (A_{i+1}, \alpha_{i+1}, \lambda_{i+1})$ as:

$$\begin{aligned} A_{i+1} &:= A_i \cup \bigcup_{a \in A_i} X_a, \\ \alpha_{i+1} &:= \{\{a\} \cup X_a \mid a \in A_i\} \cup \{U \cup \bigcup_{a \in U} X_a \mid U \in \alpha_i\}, \\ \lambda_{i+1} &:= \lambda_i \cup \bigcup_{a \in A_i} f_a. \end{aligned}$$

In words, we obtain α_{i+1} by adding, for every $a \in A_i$, new elements $\{a\} \cup X_a$ to α_{i+1} (if $\{a\} \notin \alpha_i$ and $X_a \neq \emptyset$) and extending old elements $U \in \alpha_i$ to $U \cup \bigcup_{a \in U} X_a \in \alpha_{i+1}$. For each of the new elements $U \in \alpha_{i+1}$ define (the *date of birth of* U) $dob(U) := i+1$. Each of the extended elements $V \in \alpha_{i+1}$ must be obtained from an element $U \in \alpha_n$ with $dob(U) = n$ (for some $n < i+1$) by subsequent extending. Define $at\_age(U, i+1-n) := V$. Define:

$$\begin{aligned} A &:= \bigcup_{i \in \mathbf{N}} A_i, \\ \alpha &:= \{\bigcup_{i \in \mathbf{N}} at\_age(U, i) \mid n \in \mathbf{N} \wedge U \in \alpha_n \wedge dob(U) = n\}, \\ \lambda &:= \bigcup_{i \in \mathbf{N}} \lambda_i. \end{aligned}$$

Lemma 19.4. The following facts hold:

1. The set α is a base for a topology, say τ, over A.
2. (A, τ) is a T_2 topological space.
3. For every $U \in \alpha$, U is a closed set.
4. (A, τ) is a regular topological space.
5. (A, τ) is a T_3 topological space.

Proof. 1. By construction each $a \in A$ belongs to some $U \in \alpha$. Moreover, by construction, for every $U, U' \in \alpha$ such that $U \cap U'$, we have that $U \subseteq U'$ or $U \supset U'$. Then, for every $U, U' \in \alpha$ and $a \in U \cap U'$, there is $U'' \in \alpha$ with $a \in U'' \subseteq U \cap U'$. Then, by Proposition 15.2, α is a base.

2. Consider $a, a' \in A$ such that $a \neq a'$. Then, by construction there is $i \in \mathbf{N}$ such that $a, a' \in A_i$ and there are $U, U' \in \alpha_i$ such that $a \in U$, $a' \in U'$ and $U \cap U' = \emptyset$. Then, by construction, we have $a \in \bigcup_{i \in \mathbf{N}} at\_age(U, i) \in \alpha$, $a' \in \bigcup_{i \in \mathbf{N}} at\_age(U', i) \in \alpha$, and $\bigcup_{i \in \mathbf{N}} at\_age(U, i) \cap \bigcup_{i \in \mathbf{N}} at\_age(U', i) = \emptyset$.

3. Consider $U \in \alpha$, and $a \in A \setminus U$. Then, by construction there is $i \in \mathbf{N}$ such that $a \in A_i$, $at\_age(U, i) \in \alpha_i$ and there is $V \in \alpha_i$ such that $a \in V$ and $V \cap at\_age(U, i) = \emptyset$. Let $V_a := \bigcup_{i \in \mathbf{N}} at\_age(V, i)$. Then, by construction, $a \in V_a \in \alpha$ and $U \cap V_a = \emptyset$. In conclusion, for every $U \in \alpha$, and $a \in A \setminus U$, there is $V_a \in \alpha$ with $a \in V_a$ and $V_a \cap U = \emptyset$. Then, for every $U \in \alpha$, we have $A \setminus U = \bigcup_{a \in A \setminus U} V_a$. Then, U is a closed set.

4. Consider a closed set C and $a \in A \setminus C$. Then, $A \setminus C$ is open. Then, there is $U \in \alpha$ such that $a \in U \subseteq A \setminus C$. Moreover, by 3, we have that $A \setminus U$ is open. Finally, plainly, $C \subseteq A \setminus U$ and $U \cap (A \setminus U) = \emptyset$.

5. By definition, and points 2 and 4. □

For every $p \in \Phi$, define $V(p) := \{a \in A \mid p \in \lambda(a)\}$, and $\mathcal{M}' := (A, \tau, V)$. We show that:

Lemma 19.5. For every $a \in \mathcal{M}'$ and $\varphi \leq \overline{\varphi}$, we have:

$$\varphi \in \lambda(a) \text{ if and only if } \mathcal{M}', a \models \varphi.$$

Proof. By induction on the complexity of φ. Assume that $\varphi \in \Phi$. Then $\varphi \in \lambda(a)$ if and only if, by definition of V, $a \in V(\varphi)$ if and only if, by semantics, $\mathcal{M}, a \models \varphi$.

Assume that φ is $\neg\psi$. $\neg\psi \in \lambda(a)$ if and only if, by H1, $\psi \notin \lambda(a)$ if and only if, by inductive hypothesis, $\mathcal{M}', a \not\models \psi$ if and only if, by semantics, $\mathcal{M}', a \models \neg\psi$.

Assume that φ is $\psi \wedge \chi$. $\psi \wedge \chi \in \lambda(a)$ if and only if, by H2, $\psi \in \lambda(a)$ and $\chi \in \lambda(a)$ if and only if, by inductive hypothesis, $\mathcal{M}', a \models \psi$ and $\mathcal{M}', a \models \chi$ if and only if, by semantics, $\mathcal{M}', a \models \psi \wedge \chi$.

Assume that φ is of the form $\Diamond_n \psi$. Assume that $\varphi \in \lambda(a)$. By definition of λ, we have $\lambda(a) \in W$. Then, by Q2, we have $\sum\{m(w) \mid \psi \in w \in W\} > n$. Then, by definition of A_0 and λ_0, we have $|\{a' \in A_0 \mid \psi \in \lambda_0(a')\}| > n$. Now, by definition of A and λ, $A_0 \subseteq A$ and $\lambda_0 \subseteq \lambda$. Then, we have $|\{a' \in A \mid \psi \in \lambda(a')\}| > n$. Then, by inductive hypothesis, we have $|\{a' \in A \mid \mathcal{M}, a' \models \psi\}| > n$. Then, by semantics, we have $\mathcal{M}', a \models \Diamond_n \psi$.

Assume:

$$\Diamond_n \psi \notin \lambda(a). \tag{19.1}$$

By definition of λ, we have $\lambda(a) \in W$. Then, by (19.1) and Q3, we have:

$$\sum\{m(w) \mid \psi \in w \in W\} \leq n.$$

Then, by definition of A_0 and λ_0, we have:

$$|\{a' \in A_0 \mid \psi \in \lambda_0(a')\}| \leq n. \tag{19.2}$$

Now, by definition of λ, we have $\lambda_0 \subseteq \lambda$. Then, by (19.2), we have:

$$|\{a' \in A_0 \mid \psi \in \lambda(a')\}| \leq n. \tag{19.3}$$

Consider $a' \in A \setminus A_0$. Then, by construction, a' is obtained by application of Q4. Then, by Q4(3), we have:

$$m(\lambda(a')) = \omega. \tag{19.4}$$

Assume:

$$\psi \in \lambda(a'). \tag{19.5}$$

By definition of λ, we have $\lambda(a') \in W$. Then, by (19.4), (19.5) and Q3, we have $\Diamond_n\psi \in \lambda(a)$: contradiction, since, by (19.1), $\Diamond_n\psi \notin \lambda(a)$. Then, we have:

$$|\{a' \in A \setminus A_0 \mid \psi \in \lambda(a')\}| = 0. \tag{19.6}$$

Then, by (19.3) and (19.6), we have $|\{a' \in A \mid \psi \in \lambda(a')\}| \leq n$. Then, by inductive hypothesis, we have $|\{a' \in A \mid \mathcal{M}, a' \models \psi\}| \leq n$. Then, by semantics, we have $\mathcal{M}, a \not\models \Diamond_n\psi$.

Assume that φ is of the form $[d]\psi$. Assume $\varphi \in \lambda(a)$. Let i be the step at which a has been introduced. Then, by construction, there is $U \in \alpha_{i+1}$ with $a \in U_{i+1}$ and $dob(U) = i + 1$. Then, by construction, $a \in U' := \bigcup_{i \in \mathbf{N}} at\_age(U, i) \in \alpha$. Now, by construction, we have that every element a' of $U' \setminus \{a\}$ have been introduced by some sequence of applications of Q4:

$$(a, [d]\chi) = (a_0, [d]\chi_0), (a_1, [d]\chi_1), ..., (a_n, [d]\chi_n), a_{n+1} = a'$$

(n some natural number) such that, for every $j \in n + 1$, we have $[d]\chi_j \leq \overline{\varphi}$ and $[d]\chi_j \notin \lambda_{i+j}(a_j)$, and, for every $j \in n$, we have a_{j+1} is the element obtained by applying Q4 to $(\lambda_{i+j}(a_j), [d]\chi_j)$. Then, since $[d]\psi \in \lambda_i(a)$, by Q4(2), for every $a' \in U' \setminus \{a\}$ we have $\psi \in \lambda_{n+1}(a')$. Then, since $\lambda_{n+1} \subseteq \lambda$, for every $a' \in U' \setminus \{a\}$, we have $\psi \in \lambda(a')$. Then, by inductive hypothesis, for every $a' \in U' \setminus \{a\}$, we have $\mathcal{M}', a' \models \psi$. Then, by semantics, we have $\mathcal{M}', a \models [d]\psi$.

Assume:

$$[d]\psi \notin \lambda(a). \tag{19.7}$$

Let U be an arbitrary neighborhood of a. Then, since α is a base for τ, there is $U' \in \alpha$ with $a \in U' \subseteq U$. Then, by construction, there is $i \in \mathbf{N}$ and $U'' \in \alpha_i$ such that $dob(U'') = i$ and $U' = \bigcup_{i' \in \mathbf{N}} at\_age(U'', i')$. Then, by construction, there is $i' \in \mathbf{N}$ such that $a \in at\_age(U'', i' - i) \in \alpha_{i'}$. Now, by (19.7), Q4(1), and construction:

$$\text{there is } a' \in at\_age(U'', i' + 1 - i) \setminus \{a\} \text{ with } \psi \notin \lambda_{i'+1-i}(a'). \tag{19.8}$$

Now, by definition of λ and U', we have $\lambda_{i'+1-i} \subseteq \lambda$ and $at\_age(U'', i' + 1 - i) \subseteq U'$. Then, by (19.8), there is $a' \in U' \setminus \{a\}$ with $\psi \notin \lambda(a')$. Then, since $U' \subseteq U$, there is $a' \in U \setminus \{a\}$ with $\psi \notin \lambda(a)$. Then, by inductive hypothesis, there is $a' \in U \setminus \{a\}$ with $\mathcal{M}', a' \not\models \psi$. Then, by arbitrariness of U and semantics, we have $\mathcal{M}', a \not\models [d]\psi$. □

Now, by construction, we have $\lambda_0[A_0] \supseteq W$. Then, since, by definition, $\lambda_0 \subseteq \lambda$, we have $\lambda[A] \supseteq W$. Then, by Q1 and Lemma 19.5, we have that $\overline{\varphi}$ is satisfiable in $\mathcal{M}'$. □

Proposition 19.6. If there is a T_1 topological model satisfying $\overline{\varphi}$, then there is a $\overline{\varphi}$-quasi model (W, m).

Proof. Let $\mathcal{M} = (A, \tau, V)$ be a T_1 topological model satisfying $\overline{\varphi}$. For every $a \in W$, define:

$$type(a) := \{\varphi \leq \overline{\varphi} \mid \mathcal{M}, a \models \varphi\}.$$

Define:

$$W := \{type(a) \mid a \in A\},$$
$$m : W \to \omega + 1 := type(a) \mapsto \begin{cases} |\{a' \in A \mid type(a') = type(a)\}| & \text{if finite,} \\ \omega & \text{otherwise.} \end{cases}$$
$$\mathcal{M}' := (W, m).$$

Lemma 19.7. $\mathcal{M}'$ is a $\overline{\varphi}$-quasi model.

Proof. Q1. Since $\mathcal{M}$ satisfies $\overline{\varphi}$, there is $a \in A$ such that $\mathcal{M}, a \models \overline{\varphi}$. Then, by definition of *type*, we have $\overline{\varphi} \in type(a)$. Then, by definition of W, we have $\overline{\varphi} \in \bigcup W$.

Q2. Assume that $\Diamond_n \varphi \in \bigcup W$. Then, by definition of W, there is $type(a) \in W$ with $\Diamond_n \varphi \in type(a)$. Then, by definition of *type*, we have $\mathcal{M}, a \models \Diamond_n \varphi$. Then, by semantics, we have $|\{a' \in A \mid \mathcal{M}, a' \models \varphi\}| > n$. Then, by definition of *type*, we have $|\{a' \in A \mid \varphi \in type(a')\}| > n$. Then, by definition of W and m, we have $\sum\{m(w) \mid \varphi \in w \in W\} > n$.

Q3. Assume that $\sum\{m(w) \mid \varphi \in w \in W\} > n$. Then, by definition of W and m, we have $|\{a' \in A \mid \varphi \in type(a')\}| > n$. Then, by definition of *type*, $|\{a' \in A \mid \mathcal{M}, a' \models \varphi\}| > n$. Then, by semantics, for every $a \in A$, we have $\mathcal{M}, a \models \Diamond_n \varphi$. Then, by definition of *type*, for every $a \in A$, we have $\Diamond_n \varphi \in type(a)$. Then, by definition of W, we have $\Diamond_n \varphi \in \bigcap W$.

Q4. Consider an arbitrary $w \in W$. Then, there is $a \in A$ such that $w = type(a)$. Consider $S_1 := \{[d]\varphi \mid [d]\varphi \in type(a)\}$. Then, by definition of *type* and semantics, for every $[d]\varphi \in S_1$, there is a neighborhood $U_{[d]\varphi}$ of a such that for every $a' \in U_{[d]\varphi} \setminus \{a\}$ we have:

$$\varphi \in type(a'). \tag{19.9}$$

Moreover, by definition of *type*, Axiom $[d]2$, Proposition 19.1, and semantics, for every $[d]\varphi \in S_1$, there is a neighborhood $U'_{[d]\varphi}$ of a such that for every $a' \in U'_{[d]\varphi} \setminus \{a\}$ we have:

$$[d]\varphi \in type(a'). \tag{19.10}$$

Consider $S_2 := \{type(a') \mid \mathcal{M}, a \not\models \langle d \rangle \bigwedge type(a')\}$. Then, by definition of *type*, and semantics, for every $v \in S_2$, there is a neighborhood U_v of a such that for every $a' \in U_v \setminus \{a\}$ we have:

$$type(a') \neq v. \tag{19.11}$$

Now, since S_1 and S_2 are finite, we have that:

$$U_a := \bigcap_{\varphi \in S_1} U_\varphi \cap \bigcap_{\varphi \in S_1} U'_\varphi \cap \bigcap_{v \in S_2} U_v$$

is a neighborhood of a.

Consider an arbitrary $[d]\varphi \leq \overline{\varphi}$ such that $[d]\varphi \notin w$. Then, by definition of *type* and semantics, there is $a' \in U_a \setminus \{a\}$ such that:

$$\varphi \notin type(a').$$

This is Q4(1).

Moreover, by (19.9) and (19.10), for every $[d]\varphi \in w$, we have $\varphi, [d]\varphi \in type(a')$. This is Q4(2).

Finally, by (19.11), we have $type(a') \notin S_2$. Then, by definition of S_2, we have $\mathcal{M}', a \models \langle d \rangle \bigwedge type(a')$. Then, by Axiom 4, Proposition 19.1, and semantics, we have $\mathcal{M}, a \models \Diamond_n \bigwedge type(a')$ for every $n \in \mathbf{N}$. Then, by semantics, we have $|\{a'' \in A \mid \mathcal{M}, a'' \models \bigwedge type(a')\}|$ is infinite. Then, by definition of *type*, we have, $|\{a'' \in A \mid type(a'') = type(a')\}|$ is infinite. Then, by definition of m, we have $m(type(a')) = \omega$. This is Q4(3). $\square$

This completes the proof. $\square$

19.4 Completeness of L_Λ with respect to $\mathsf{L}_{\mathfrak{T}_3}$

In this section we prove that:

Theorem 19.8. L_Λ is complete with respect to $\mathsf{L}_{\mathfrak{T}_3}$. Moreover, $\mathsf{L}_{\mathfrak{T}_3}$ is strongly complete.

Theorems 19.1 and 19.8, together with $\mathfrak{T}_1 \supseteq \mathfrak{T}_3$, imply that:

Corollary 19.9. For every $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$, we have that $\mathsf{L}_\Lambda = \mathsf{L}_\mathfrak{X}$ and $\mathsf{L}_\mathfrak{X}$ is strongly complete.

By classical arguments, to prove Theorem 19.8 it suffices to show that every maximal consistent $\Gamma \subseteq L$ is satisfiable in $\mathfrak{T}_3$ (see for example [13, Chapter 4]). We use results in [26, 36] (and we strongly advise reading [26, 36]) to build what, with abuse of language, could be described as a 'Γ-quasi model' (Proposition 19.11). Then, the proof of Proposition 19.3 can straightforwardly be adapted to obtain a T_3 topological model satisfying Γ.

Let MC be the set of all maximal consistent $\Gamma \subseteq L$. Fix $\overline{\Gamma} \in MC$. For every $\Gamma, \Gamma' \in MC$, define:

$$m(\Gamma,\Gamma') := \begin{cases} \omega \text{ provided, for every } \varphi \in \Gamma' \text{ and } n \in \mathbf{N}, \text{ we have } \Diamond_n\varphi \in \Gamma, \\ min\{n \in \mathbf{N} \,|\, \exists\varphi \text{ such that } \varphi \in \Gamma' \wedge \Diamond_{!n}\varphi \in \Gamma\} \text{ otherwise,} \end{cases}$$
$$m(\Gamma) := sup\{m(\Gamma',\Gamma) \,|\, \Gamma' \in MC\},$$
$$R := \{(\Gamma,\Gamma') \,|\, m(\Gamma,\Gamma') \neq 0\},$$
$$S := \{(\Gamma,\Gamma') \,|\, \forall[d]\varphi, \text{ if } [d]\varphi \in \Gamma \text{ then } \varphi \in \Gamma'\}.$$

Plainly, the function $m(\_,\_)$ is well-defined. Then, $\Gamma R \Gamma'$ if and only if, by definition of R, $m(\Gamma,\Gamma') \neq 0$ if and only if, by definition of $m(\_,\_)$, for every $\varphi \in \Gamma'$, we have $\Diamond_0\varphi \in \Gamma$. Then, (MC, R, S), together with the 'canonical evaluation', constitutes the canonical model for $L_{\{\Phi,\Box_0,[d]\}}$. However, we have to take into account also the modal operators $\Diamond_n$ for $n > 0$. This is done by $m(\_,\_)$. For, in words, $m(\_,\_)$ determines the multiplicity of the relation $\Gamma R \Gamma'$, and induces the multiplicity $m(\_)$ of Γ (see Proposition 19.10(1).)

Proposition 19.10. The following facts hold:

1. For every $\varphi \in L$, $\Gamma \in MC$ and $n \in \mathbf{N}$ we have:

$$\Diamond_n\varphi \in \Gamma \text{ if and only if } \sum\{m(\Gamma,\Gamma') \,|\, \varphi \in \Gamma' \in MC\} > n.$$

2. For every $\Gamma, \Gamma' \in MC$, if $\Gamma R \Gamma'$ then $m(\Gamma) = m(\Gamma',\Gamma)$.

3. $S \subseteq R$.

Proof. 1 This is [26, Theorem 1]. It says that $\Diamond_n \varphi \in \Gamma$ is equivalent to having more than n maximal consistent sets Γ' (counting their multiplicity) R-related to Γ and containing φ. This is the Kripke semantics of $\Diamond_n$. The proof uses Axioms $\Diamond_{\mathbf{N}}1-3$, which indeed reflect the Kripke semantics for $\Diamond_n$.

2. Assume $\Gamma R \Gamma'$. Then, by definition of R, we have $m(\Gamma, \Gamma') \neq 0$. Then, by the proof of [26, Lemma 9], we have $m(\Gamma) = m(\Gamma', \Gamma)$, as desired. A bit of commentary. Observe that the 0-instances of Axioms $\Diamond_{\mathbf{N}}4, 5$ give **S5**. Then, R is an equivalence relation. Then, this point reflects that within R-equivalence classes, R is the universal relation. And the $\Diamond_n$'s semantics, from restricted to R-successors, becomes universal as defined by us.

3. Consider $\Gamma, \Gamma' \in MC$ such that $\Gamma S \Gamma'$, and $\varphi \in \Gamma'$. Then, by classical work, we have $\langle d \rangle \varphi \in \Gamma$. Then, by Axiom 4, for every $n \in \mathbf{N}$, we get $\Diamond_n \varphi \in \Gamma$. Then, by definition of m, we have $m(\Gamma, \Gamma') = \omega$. Then, by definition of R, we have $\Gamma R \Gamma'$. □

Since within R-equivalence classes, $\Diamond_n$'s semantics, from restricted to R-successors, becomes universal as defined by us, we consider the smallest R-generated subset $W \subseteq MC$ containing $\overline{\Gamma}$ (plainly existing). That is, the R-equivalence class containing $\overline{\Gamma}$. Observe that by Proposition 19.10(3), W is also an S-generated subset of MC.

Proposition 19.11. The following facts hold:

Q'1. $\overline{\Gamma} \in W$.

Q'2. If $\Diamond_n \varphi \in \bigcup W$ then $\sum\{m(\Gamma) \mid \varphi \in \Gamma \in W\} > n$.

Q'3. If $\sum\{m(\Gamma) \mid \varphi \in \Gamma \in W\} > n$ then $\Diamond_n \varphi \in \bigcap W$.

Q'4. For every $\Gamma \in W$, for every $[d]\varphi \notin \Gamma$, there is $\Gamma' \in W$ such that:

1. $\varphi \notin \Gamma'$.
2. For every $[d]\varphi \in \Gamma$, then $\varphi, [d]\varphi \in \Gamma'$.
3. $m(\Gamma') = \omega$.

Proof. Q'1. By definition of W. Q'2 and Q'3 follows by Proposition 19.10(1,2). Q'4. By classical work, there is $\Gamma' \in S(\Gamma)$ such that $\varphi \notin \Gamma'$. Just take Γ'. 1 follows by definition of Γ'. 2 follows by definition of S, Axiom $[d]2$, and classical work. 3 follows by Axiom 4, classical work, and definition of m. □

By Proposition 19.11, the proof of Proposition 19.3 can straightforwardly be adapted to obtain a T_3 topological model satisfying $\overline{\Gamma}$.

19.5 Complexity of $\mathsf{L}_{\mathfrak{X}}$ ($\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$)

In this section, we prove that, for every $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$, the validity problem for $\mathsf{L}_{\mathfrak{X}}$ is PSPACE-complete (Theorem 19.21).

To prove Theorem 19.21, we prove that deciding the existence of a φ-quasi model is PSPACE (Proposition 19.20). To prove Proposition 19.20, we alter the definition of φ-quasi models into that of 'φ-optimal quasi models' (Definition 19.12). Then, we prove that there is a φ-quasi model if and only if there is a φ-optimal quasi model (Propositions 19.13 and 19.15). Finally, we prove that deciding the existence of a φ-optimal quasi model is PSPACE (Proposition 19.16).

Now, by Corollary 19.9 and Propositions 19.3 and 19.6, deciding if $\varphi \in \mathsf{L}_{\mathfrak{X}}$ is reducible to deciding if there is no φ-quasi models. And, by Proposition 19.20, deciding if there is no φ-quasi models is PSPACE. Then, deciding if $\varphi \in \mathsf{L}_{\mathfrak{X}}$ is PSPACE. In fact, we prove that the validity problem for $L_{\mathfrak{X}}$ is PSPACE-complete.

19.5.1 Optimal quasi models

For every formula $\varphi \in L$ define $len(\varphi)$ as:

- If $\varphi \in \Phi$, then $len(\varphi) := 1$.
- If φ is of the form $\neg\psi$, then $len(\varphi) := 1 + len(\psi)$.
- If φ is of the form $\psi \wedge \chi$, then $len(\varphi) := 1 + len(\psi) + len(\chi)$.
- If φ is of the form $\Diamond_n\psi$, then $len(\varphi) := n + 1 + len(\varphi)$.
- If φ is of the form $[d]\psi$, then $len(\varphi) := 1 + len(\psi)$.

Fix $\overline{\varphi} \in L$ and let $\overline{n} := len(\overline{\varphi})$.

Definition 19.12. A *$\overline{\varphi}$-optimal quasi model* is a tuple $\mathcal{M} = (W, W', m)$ such that $W \subseteq Hin(\overline{\varphi})$, $W' \subseteq W$, $m : W' \to \omega + 1$, and:

O0. $|W'| < \overline{n} + 1$.

O1. $\overline{\varphi} \in \bigcup W'$.

O2. For every $n \in \mathbf{N}$, if $\Diamond_n\varphi \in \bigcup W'$, then $\sum\{m(w) \mid \varphi \in w \in W'\} > n$.

O3. For every $n \in \mathbf{N}$, and $[d]\varphi \leq \overline{\varphi}$, if $\sum\{m(w) \mid \varphi \in w \in W'\} > n$ then $\Diamond_n\varphi \in \bigcap W'$.

O4. For every $w \in W$ and $[d]\varphi \leq \overline{\varphi}$, if $[d]\varphi \notin w$ then there is $w' \in W$ such that:

0. For every $n \in \mathbf{N}$, and $\Diamond_n\psi \leq \overline{\varphi}$, $\Diamond_n\psi \in w'$ if and only if $\Diamond_n\psi \in \bigcap W'$.

1. $\varphi \notin w'$.

2. For every $[d]\psi \in w$, we have $\psi, [d]\psi \in w'$.

3. For every $n \in \mathbf{N}$, and $\Diamond_n \psi \leq \overline{\varphi}$, if $\psi \in w'$ then $\Diamond_n \psi \in \bigcup W'$.

Proposition 19.13. If there is a $\overline{\varphi}$-optimal quasi model, then there is a $\overline{\varphi}$-quasi model.

Proof. Suppose that $\mathcal{M} = (W, W', m)$ is a $\overline{\varphi}$-optimal quasi model. Define $V_0 := W'$. Suppose that V_i (for some $i \in \mathbf{N}$) has been defined such that $V_i \subseteq W$. For every $v \in V_i$ and $[d]\varphi \leq \overline{\varphi}$ with $[d]\varphi \notin v$, let $v'_{(v,[d]\varphi)}$ be the point in W provided by O4. Define:

$$V'_{i+1} := \{v'_{(v,[d]\varphi)} \mid v \in V_i \wedge [d]\varphi \leq \overline{\varphi} \wedge [d]\varphi \notin v\},$$
$$V_{i+1} := V_i \cup V'_{i+1}.$$

Plainly, $V_{i+1} \subseteq W$. Define $V := \bigcup_{i \in \mathbf{N}} V_i$. Define V' as $\bigcup_{i \in \mathbf{N} \setminus \{0\}} V'_i$ - that is, as the set of elements of V that have served as a point provided by some application of O4 in the construction of V. Observe that $V \setminus V' \subseteq V_0 = W'$, but we might have $V_0 \cap V' \neq \emptyset$. Define $n : V \to \omega + 1$ as follows: if $v \in V'$, then define $n(v) := \omega$; if $v \in V \setminus V'$, define $n(v) := m(v)$. Define:

$$\mathcal{N} := (V, n).$$

Lemma 19.14. $\mathcal{N}$ is a $\overline{\varphi}$-quasi model.

Proof. Q1 follows by O1. Q2. Assume that $\Diamond_n \varphi \in \bigcup V$. Then, by definition of V and O4(0), $\Diamond_n \varphi \in \bigcup W'$. Then, by O2, $\sum\{m(w) \mid \varphi \in w \in W'\} > n$. Then, by definition of n, $\sum\{n(w) \mid \varphi \in w \in V\} > n$.

Q3. Assume $\sum\{n(w) \mid \varphi \in w \in V\} > n$ and $\Diamond_n \varphi \leq \overline{\varphi}$. Two cases are given:

1. φ, among the elements of V, belongs only to elements in $V_0 \setminus V'$. Then, since $V_0 \setminus V' \subseteq W' \subseteq V$, and definition of n:

$$\begin{aligned}\sum\{m(w) \mid \varphi \in w \in W'\} &= \sum\{m(w) \mid \varphi \in w \in V_0 \setminus V'\} \\ &= \sum\{n(w) \mid \varphi \in w \in V_0 \setminus V'\} \\ &= \sum\{n(w) \mid \varphi \in w \in V\} > n.\end{aligned}$$

2. Otherwise we have that φ belongs to some point $v \in V'$ - that is, φ belongs to some point v provided by the recursive application of O4 in the definition of V. So, by O4(3), we have $\Diamond_n \varphi \in \bigcup W'$. Then, by O2:

$$\sum\{m(w) \mid \varphi \in w \in W'\} > n.$$

In both cases, $\sum\{m(w) \mid \varphi \in w \in W'\} > n$. Then, by O3, $\Diamond_n \varphi \in \bigcap W'$. Then, by O4(0), $\Diamond_n \varphi \in \bigcap V$.

Q4. Plainly, it follows by O4, and definition of n. □

This completes the proof. □

Proposition 19.15. If there is a $\overline{\varphi}$-quasi model then there is a $\overline{\varphi}$-optimal quasi model.

Proof. Let $\mathcal{M} = (W, m)$ be a $\overline{\varphi}$-quasi model. By Q1-Q3 we can find $W' \subseteq W$ and $n : W' \to \omega + 1$ satisfying O0-O3. Indeed, by Q1, there is $\overline{w} \in W$ with $\overline{\varphi} \in \overline{w}$. Moreover, let $\Diamond_{n_0}\varphi_0, ..., \Diamond_{n_{m-1}}\varphi_{m-1}$ be all the formulas in $\bigcup W$ of the form $\Diamond_n\varphi$. By Q2, there are subsets $A_0, ..., A_{m-1}$ of W such that, for every $i \in m$:

1. $\varphi_i \in \bigcap A_i$.
2. $|A_i| \leq n_i + 1$.
3. $\sum\{m(w) \mid w \in A_i\} > n_i$.

Define $W' := \{\overline{w}\} \cup \bigcup_{i \in m} A_i$ and $n := m \upharpoonright W'$. By definition of W, we have $W' \subseteq Hin(\overline{\varphi})$. Since $\Diamond_{n_0}\varphi_0, ..., \Diamond_{n_{m-1}}\varphi_{m-1} \leq \overline{\varphi}$, we have $\sum_{i\in m}(n_i + 1) \leq len(\overline{\varphi})$. Then, by 2, we have $|W'| \leq \overline{n} + 1$, proving O0. By definition of W', $\overline{\varphi} \in \bigcup W'$, proving O1. O2 follows by 1 and 3, and definition of n. As for O3, suppose that $\sum\{m(w) \mid \varphi \in w \in W'\} > n$. Then, since $W' \subseteq W$, $\sum\{m(w) \mid \varphi \in w \in W\} > n$. Then, by Q3, $\Diamond_n\varphi \in \bigcap W$. Then, again as $W' \subseteq W$, $\Diamond_n\varphi \in \bigcap W'$, proving O3.

Plainly, Q4 yields that (W, W', n) satisfies O4(0,1,2): for every $w \in W$ and $[d]\varphi \leq \overline{\varphi}$ such that $[d]\varphi \notin w$, just take $w' \in W$ as per Q4 applied to $(w, [d]\varphi)$. As for O4(3), observe that, by Q4(3), we have $m(w') = \omega$. Then, for every $n \in \mathbf{N}$, and $\Diamond_n\psi \leq \overline{\varphi}$, if $\psi \in w'$, by Q3, we have $\Diamond_n\psi \in \bigcap W \subseteq \bigcup W'$. □

Proposition 19.16. There is an algorithm Alg that given $\varphi \in L$ decides the existence of a φ-optimal quasi model in PSPACE in $len(\varphi)$.

Proof. By Savitch's Theorem, it suffices to show that the problem is NPSPACE in $\overline{n}$ (see for example [76, Theorem 7.5]). Fix $\overline{\varphi} \in L$. Let $\overline{n} := len(\overline{\varphi})$.

Guess $W' \subseteq \mathcal{P}(sub(\overline{\varphi}))$ and $m : W' \to \omega + 1$ satisfying O0-O3. For every $w \in W'$ run $Witness(w)$ where $Witness$ is the following algorithm:

if {
  w is an Hintikka set
  and for every $[d]\varphi \leq \overline{\varphi}$: $[d]\varphi \notin w \Rightarrow$ there is $w' \subseteq sub(\overline{\varphi})$ such that:
    {
    for every $n \in \mathbf{N}$, and $\Diamond_n\psi \leq \overline{\varphi}$: $\Diamond_n\psi \in w' \Leftrightarrow \Diamond_n\psi \in \bigcap W'$
    and $\varphi \notin w'$
    and for every $[d]\psi \in w$: $\psi, [d]\psi \in w'$
    and for every $n \in \mathbf{N}$, and $\Diamond_n\psi \leq \overline{\varphi}$: $\psi \in w' \Rightarrow \Diamond_n\psi \in \bigcup W'$
    and {
      if {

w' is an Hintikka set

and for every $[d]\psi \in w'$: $[d]\psi \in w$

}

then true

else $Witness(w')$

}

}

}

then true

else false

If $Witness(w) =$ **true** for every $w \in W'$, then return **true**, otherwise return **false**.

Lemma 19.17. For every $\overline{\varphi} \in L$, if $Alg(\overline{\varphi}) =$ **true**, then there is a $\overline{\varphi}$-optimal quasi model.

Proof of Lemma 19.17. Assume that $Alg(\overline{\varphi}) =$ **true**. Then, there is $W' \subseteq Hin(\overline{\varphi})$ and $m : W' \to \omega + 1$ satisfying O1-O3 and such that for every $w \in W'$ we have $Witness(w) =$ **true**. Define $W_0 := W'$. Suppose that W_i ($i > 0$) has been defined and that for every $w \in W_i$ we have either:

C1. There is $w' \in W_{i-1}$ such that, for every $[d]\varphi \leq \overline{\varphi}$, we have $[d]\varphi \in w$ if and only if $[d]\varphi \in w'$;

or $Witness(w) =$ **true**. For every $w \in W_i$, if w fails C1, then, for every $[d]\varphi \leq \overline{\varphi}$, if $[d]\varphi \notin w$, let $w'_{(w,[d]\varphi)}$ be the Hintikka set guessed by $Witness$ on $(w, [d]\varphi)$. Define:

$$W_{i+1} := W_i \cup \{w'_{(w,[d]\varphi)} \mid w \in W_i \wedge w \text{ fails C1} \wedge [d]\varphi \leq \overline{\varphi} \wedge [d]\varphi \notin w\}.$$

Observe that, since $Alg(\overline{\varphi}) =$ **true**, then, by definition of W_{i+1} and $Witness$, for every $w \in W_{i+1}$, we have that either w satisfies C1 (with i replaced by $i + 1$) or $Witness(w) =$ **true**. Define:

$$W := \bigcup_{i \in \mathbf{N}} W_i,$$

$$\mathcal{M} := (W, W', m).$$

We show that $\mathcal{M}$ is a $\overline{\varphi}$-optimal quasi model. O0-O3 hold by choice of W' and m. O4. Consider an arbitrary $w \in W$ and an arbitrary $[d]\varphi \leq \overline{\varphi}$ with $[d]\varphi \notin w$. Then, by construction, there is $i \in \mathbf{N}$ such that $w \in W_i$. Denote w by w_i. By construction, there is $0 \leq n \leq i$ such that:

1. For every $0 \leq j \leq n$, there is $w_{i-j} \in W_{i-j}$ with, for every $[d]\psi \leq \overline{\varphi}$: $[d]\psi \in w_i$ if and only if $[d]\psi \in w_{i-j}$;

2. w_{i-n} fails C1. (Observe that, by definition of C1, every element in W_0 fails C1.)

Then, $Witness(w_{i-n}) = \mathbf{true}$. Then, we have $w' \in W_{i-n}$ satisfying O4(0-3) applied to $(w_{i-n}, [d]\varphi)$. Then, since for every $[d]\psi \leq \overline{\varphi}$, we have $[d]\psi \in w_i$ if and only if $[d]\psi \in w_{i-n}$, and w_i is w, w' satisfies O4(0-3) applied to $(w, [d]\varphi)$. □

And, plainly:

Lemma 19.18. For every $\varphi \in L$, if there is a φ-optimal quasi model then $Alg(\varphi) = \mathbf{true}$.

Moreover:

Lemma 19.19. For every $\varphi \in L$, we have that $Witness(\varphi)$ answer in NPSPACE in $len(\varphi)$.

Proof. By O0, the crucial point is whether the recursive calls of *Witness* cause a blow-up in space requirements. Observe that, by definition of *Witness*, if $Witness(w) = \mathbf{true}$ ($w \in Hin(\varphi)$) and w' is given by *Witness* applied to w, then:

$$\{[d]\varphi \mid [d]\varphi \in w'\} \supseteq \{[d]\varphi \mid [d]\varphi \in w\}. \tag{19.12}$$

By definition of *Witness*, we have that *Witness* is recursively applied to w' if and only if

$$\{[d]\varphi \mid [d]\varphi \in w'\} \supset \{[d]\varphi \mid [d]\varphi \in w\}$$

(recall $\subset$ denotes proper inclusion in this dissertation). And, by (19.12), this can recursively happen at most $len(\varphi)$ many times. □

Then *Alg* run in NPSPACE, as desired. □

Propositions 19.13, 19.15 and 19.16 yield:

Proposition 19.20. For every $\varphi \in L$, deciding the existence of a φ-quasi model is PSPACE.

19.5.2 Complexity of $\mathsf{L}_{\mathfrak{X}}$ ($\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$)

By Corollary 19.9 and Propositions 19.3 and 19.6, for every $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$, deciding if $\varphi \in \mathsf{L}_{\mathfrak{X}}$ is reducible to deciding if there is no φ-quasi models. And, by Proposition 19.20, deciding if there is no φ-quasi models is PSPACE. Then, the validity problem for $\mathsf{L}_{\mathfrak{X}}$ is PSPACE.

Moreover, a simple modification of the completeness proof shows that the $[d]$-logic of $\mathfrak{X}$ is **K4**. Now, by Ladner's Theorem (see for example [13, Theorem 6.50]), **K4** is PSPACE-hard. And, the $[d]$-language can be translated into L. Then, the validity problem for $\mathsf{L}_{\mathfrak{X}}$ is PSPACE-hard. Then:

Theorem 19.21. For every $\mathfrak{X}$ such that $\mathfrak{T}_1 \supseteq \mathfrak{X} \supseteq \mathfrak{T}_3$, the validity problem for $\mathsf{L}_{\mathfrak{X}}$ is PSPACE-complete.

Chapter 20

Future works

There are a number of questions, related to the work of Part III, that we would like to investigate in the future:

1. We proved that there is a computable translation from L_t to L on the class of all T_3 topological spaces: what is the complexity of this translation?

2. We would like to study the axiomatizability and complexity of the L-theory of particular topological spaces, or classes of topological spaces. For example, the real line, or $\mathbf{R}^n$ (for $n > 1$); the class of all metric spaces; the class of all T_0 spaces; and the class of all topological spaces. As for T_0, and all topological spaces, it is worth noting that Axiom 2 would not be sound.

3. Is L as expressive as the spatial language with modal operators $\{[d]\} \cup \rho$ (recall that ρ is the set of all point-sort modal operators) on the class of all T_3 topological models?

4. We would like to increase the expressivity of L by adding fixed-point operators. Call L^+ the resulting language. Since L_t cannot express some important topological properties such as connectedness or compactness, we would like to investigate if we can express these properties in L^+.

5. We would like to study the axiomatizability and complexity of L^+ when interpreted on particular classes of topological spaces.

6. The expressivity of modal languages, when interpreted on Kripke models, has traditionally been compared to that of first-order languages. A fundamental tool in this comparison is the notion of *bisimulation*. As it is well known, every bisimulation-invariant property expressible in first-order languages is equivalent to some property expressible in modal languages [94].

In [5], a notion analogous to bisimulation, called *topo-bisimulation*, has been devised to compare the expressivity of modal languages to the expressivity of first-order languages interpreted on topological spaces.

Fixed point modal languages can express properties that are not expressible in first-order languages. So we cannot compare their expressivity to that of first-order languages. However, every property expressible in fixed point modal languages is equivalent to some property expressible in second-order languages. And in [54], it is proved that every bisimulation-invariant property expressible in second-order languages is equivalent to some property expressible in fixed point modal languages.

From these observations, three questions arise:

(a) We would like to study if every topo-bisimulation-invariant properties expressible in L_t is equivalent to some property expressible in L. In [91], a similar result is proved for a language simpler than L.

(b) Given the results in [54] and [91], we would like to investigate if we can define the second-order version of L_t, call it L_t^+, and prove that all properties expressible in L^+ are expressible in L_t^+, and that all topo-bisimulation-invariant properties expressible in L_t^+ are equivalent to some property expressible in L^+.

(c) Since L is as expressive as L_t when used to reason about T_3 topological spaces, we would like to study whether L^+ is as expressive as L_t^+ when used to reason about T_3 topological spaces.

7. In Section 18.1, we gave a more general definition of modal operator. Following Gabbay, Hodkinson and Reynolds [42, Chapter 6], the semantics of a modal operator # is defined by a first-order formula in L_t with at most one free first-sort variable and no free second-sort variables. We would like to study this notion in its generality focusing on questions like:

    - Given a class of models $\mathfrak{M}$, and a formula $\varphi \in L_{\{\#\}}$, which semantics for # makes φ sound on $\mathfrak{M}$? For example, which semantics for # makes axiom **K** $\#(p \to q) \to (\#p \to \#q)$ sound on the class of all Kripke models?

8. Can spatial modal logics be applied to real-world problems? In this respect, it is worth mentioning [25, 2014], where a spatial modal logic similar to ours is used for reasoning about geographical maps. It is worth mentioning their use of an Until operator similar to:

    - $U(\varphi, \psi)$ holds at x provided there is a neighborhood U of x such that φ holds on the boundary of U and ψ in U.

9. We would like to study Until on topological spaces.

10. We would like to blend together the derivative and graded operators and study operators $\langle d \rangle_n$ with semantics:

    - $\langle d \rangle_n \varphi$ holds at a point a provided for every neighborhood U of a there are at least n points $b \in U \setminus \{a\}$ such that φ holds at b.

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