Refined partial Hasse invariants and the canonical filtration

Stéphane Bijakowski

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Abstract

Let G be a p-divisible group over a scheme of characteristic p, and assume that it is endowed with an action of the ring of integers of a finite unramified extension F of $\mathbb{Q}_p$. Let us fix the type μ of this action on the sheaf of differentials ω_G. V. Hernandez, following a construction of Goldring and Nicole, defined partial Hasse invariants for G. These are sections of invertible sheaves. The product of these invariants is the μ-ordinary Hasse invariant, and it is non-zero if and only if the p-divisible group is μ-ordinary (i.e. the Newton polygon is minimal given the type of the action).

We show that, if one assumes the existence of a certain filtration refining the Hodge filtration, each of these partial Hasse invariants can be expressed as a product of other sections, the refined partial Hasse invariants. Over a Shimura variety, the condition is satisfied if one considers an explicit closed subscheme of a certain flag variety.

As an application, we relate these refined partial Hasse invariants to the partial degrees of the canonical filtration (if it exists).

Contents

Introduction

Let E be an elliptic curve over an algebraic closed field k of characteristic p. There are two possibilities for the number of k-points of the p-torsion of E : it is either p or 1. In the first case, we say that the elliptic curve is ordinary; otherwise we say that it is supersingular. This condition can also be seen on the group structure of the p-torsion of E : if it is a product of a multiplicative group and an étale one, then the elliptic curve is ordinary. An equivalent condition is the fact that the Eisenstein series E_{p-1} is non-zero at E (if $p \geq 5$).

More generally, if G is a p-divisible group over k, we say that G is ordinary if its p-torsion is the product of a multiplicative part by an étale part. One can associate to G several invariants. The first one is the Newton polygon, and G is ordinary if and only if this polygon has slopes 0 and 1. The second one is the Hasse invariant $ha(G)$; it is a section of the sheaf $(\det \omega_G)^{p-1}$, where ω_G is the sheaf of differentials of G. The section $ha(G)$ is induced by the map $V : \omega_G \to \omega_G^{(p)}$, where V is the Verschiebung, and the superscript denotes a twist by the Frobenius. Then G is ordinary if and only if the Hasse invariant is non-zero.

Assume now that F is an unramified extension of degree f of $\mathbb{Q}_p$, and that G has an action of O_F, the ring of integers of F. The sheaf ω_G thus decomposes into $\omega_G = \oplus_{i=1}^{f} \omega_{G,i}$, where $\omega_{G,i}$ is the subsheaf of ω_G where O_F acts by σ^i, where σ is the Frobenius. Let d_i be the dimension of $\omega_{G,i}$ for all $1 \leq i \leq f$. If there exists an integer d with $d_i = d$ for all d, then the Hasse invariant is the product of partial Hasse invariants $ha_i(G)$. The element $ha_i(G)$ is a section of the invertible sheaf $(\det \omega_{G,i-1})^p (\det \omega_{G,i})^{-1}$, and is induced by the Verschiebung, which decomposes into $V_i : \omega_{G,i} \to \omega_{G,i-1}^{(p)}$. We will refer to this case as the ordinary case.

The general case is more involved. Indeed, if the previous hypothesis is not satisfied, the Hasse invariant $ha(G)$ is always 0 (because at least one of the V_i can never be an isomorphism). There is then an obstruction for the p-divisible group G to be ordinary. Looking at the Newton polygon of G, one can see that this polygon lies always above a certain polygon depending on the collection of integers $\mu := (d_i)_i$, the Hodge polygon. If it is an equality, one say that the p-divisible group is μ-ordinary. It roughly states that the structure of the p-divisible group is the best possible given the constraints of the action of the ring O_F. The construction of a Hasse invariant in this situation, i.e. a section of an invertible sheaf such that its non-vanishing is equivalent to the fact that the p-divisible group is μ-ordinary, has been initiated by Goldring and Nicole ([GN]). Actually, they construct this invariant over a Shimura variety. A local construction has then been done by Hernandez ([He]).

Let us recall the main idea of this construction. The iterated of the Verschiebung V^f induces a map $\omega_{G,i} \to \omega_{G,i}^{(p^f)}$, for all $1 \leq i \leq f$. If d_i is the minimum of the $(d_j)_j$, then there is no obstruction for this map to be an isomorphism, and taking the determinant gives a section $Ha_i(G)$ of $(\det \omega_{G,i})^{p^f-1}$. If it is not the case, the determinant of this map is always 0. However, using crystalline cohomology, one can lift this map to a ring where p is not a zero divisor. Then one can show that the determinant of the map is divisible by an explicit power of p. Making the adequate division gives a section $Ha_i(G)$ of $(\det \omega_{G,i})^{p^f-1}$ for any integer i. The product of these sections is then the μ-ordinary Hasse invariant. This construction is actually valid over an arbitrary scheme of characteristic p, not just the spectrum of an algebraically closed field. Note that the construction in [GN] is valid over the special fiber of Shimura varieties of PEL type, and that Koskivirta and Wedhorn ([KW]) constructed μ-ordinary Hasse invariants for Shimura varieties of Hodge type.

In the ordinary case, the section $Ha_i(G)$ is equal to a product of powers of the sections $ha_j(G)$, $1 \leq j \leq f$. This suggests that the situation is not optimal, and that one should be able to define analogues of the $ha_j(G)$ in general. This is indeed possible if one assumes moreover the existence of a certain filtration on the (contravariant) Dieudonné crystal $\mathcal{E}$ of G evaluated at k. Recall that $\mathcal{E}$ is a k-vector space of dimension the height of G, and that it decomposes into $\mathcal{E} = \oplus_{i=1}^f \mathcal{E}_i$.

Theorem. *Let S be a scheme of characteristic p, and let G be a p-divisible group over S. Assume that G has an action of O_F, and fix the type of this action. Let $\mathcal{E}$ be the Dieudonné crystal of G evaluated at S, and assume the existence of adequate filtrations on the sheaves $\mathcal{E}_i$. Then there exist sections $h_k^{[i]}(G)$ of invertible sheaves for $1 \leq k \leq f$ and $1 \leq i \leq f$, such that*

$$Ha_i(G) = h_i^{[i]} \cdot (h_{i-1}^{[i]})^p \cdot \dots \cdot (h_{i+1}^{[i]})^{p^{f-1}}.$$

We refer to Hypotheses 1.1.1 and 1.1.2 for the precise definition of the adequate filtrations. Let us just make explicit the case $f = 2$. The Hodge filtration give subssheaves $\mathcal{F}_1 \subset \mathcal{E}_1$ and $\mathcal{F}_2 \subset \mathcal{E}_2$, locally free of rank respectively d_1 and d_2, and assume for example that $d_1 < d_2$. The existence of adequate filtrations on $\mathcal{E}_1$ and $\mathcal{E}_2$ amounts to the existence of $\mathcal{F}_1 \subset \mathcal{F}_1^{[2]} \subset \mathcal{E}_1$ and $\mathcal{F}_2^{[1]} \subset \mathcal{F}_2$ such that $\mathcal{F}_1^{[2]}$ is locally a direct factor of rank $h + d_1 - d_2$ containing the intersection of $\mathcal{E}_1$ with the image of the Frobenius F, and $\mathcal{F}_2^{[1]}$ is locally a direct factor of rank $d_2 - d_1$ included in the kernel of the Verschiebung V. The sections $h_1^{[1]}$ and $h_2^{[1]}$ are then induced respectively by the determinant of the maps

$$V : \mathcal{F}_1 \to (\mathcal{F}_2/\mathcal{F}_2^{[1]})^{(p)} \qquad V : \mathcal{F}_2/\mathcal{F}_2^{[1]} \to \mathcal{F}_1^{(p)}.$$

The sections $h_1^{[2]}$ and $h_2^{[2]}$ are then induced respectively by the determinant of the maps

$$F : (\mathcal{E}_2/\mathcal{F}_2)^{(p)} \to \mathcal{F}_1^{[2]}/\mathcal{F}_1 \qquad F : (\mathcal{F}_1^{[2]}/\mathcal{F}_1)^{(p)} \to \mathcal{E}_2/\mathcal{F}_2.$$

If one considers the usual special fiber of a Shimura variety of type (A), then the existence of adequate filtrations is in general not satisfied. Indeed, Hernandez proved in [He] that the elements $Ha_i(G)$ are irreducible in the generic case. But if one considers a certain closed subscheme of a flag variety (see [EV]), then the hypotheses are satisfied and the sections $Ha_i(G)$ are no longer irreducible. Note that the full flag variety for some Shimura varieties, stratifications on this space, and the construction of generalized Hasse invariants on strata, have recently been studied by Goldring and Koskivirta in [GK].

We have thus constructed f^2 refined partial Hasse invariants. Actually, there may be fewer of them : the section $h_k^{[i]}$ depends only on k and the integer d_i. In the ordinary case, we just get the usual partial Hasse invariants. On the other hand, if the elements d_i are pairwise distinct, there are f^2 distinct refined partial Hasse invariants.

The Hasse invariant plays a central role in the theory of the canonical subgroup. Indeed, let K be a finite extension of $\mathbb{Q}_p$ with ring of integers O_K, and let G be a p-divisible group defined over O_K. One can look at the Hasse invariant of $G \times_{O_K} O_K/p$, and taking its (truncated) valuation gives a well defined rational between 0 and 1. Fargues ([Fa2]) proved that if this valuation is small enough (and $p \geq 3$), then there exists a canonical subgroup C in the p-torsion of G. Moreover, one can relate the degree of C to the Hasse invariant.

If the p-divisible group has an action of O_F, in the ordinary case, there is no obstruction for the p-divisible group to be ordinary and to have a canonical subgroup. One can then define the partial degrees $(\deg_i C)_{1 \leq i \leq f}$ of C, and relate them to the partial Hasse invariants $ha_i(G)$ (see

[Bi]). In the general case, Hernandez proved in [He2] (under some assumptions on p) that if the valuation of the μ-ordinary Hasse invariant of G is small enough, then the p-torsion of G admits a canonical filtration. There are thus several canonical subgroups (C_i), each of them being of height fd_i. Actually, Hernandez proved that if the valuation of $Ha_i(G)$ is small enough, then there exists a canonical subgroup C_i of height fd_i. He also relates the valuation of $Ha_i(G)$ to a certain linear combination of the partial degrees of C_i, but is unable to compute each of these partial degrees. They are in fact related to the refined partial Hasse invariants.

Theorem. *Let G be a p-divisible group over O_K with an action of O_F, and assume that there exist adequate filtrations for $G \times_{O_K} O_K/p$. Let $1 \leq i \leq f$ be an integer, and assume that there exists a canonical subgroup C of height fd_i (in the strong sense of Definition 3.2.1). Then we have for $1 \leq k \leq f$*

$$\deg_k C^D = \max(d_i - d_k, 0) + v(h_k^{[i]}).$$

We need the existence of adequate filtrations for $G \times_{O_K} O_K/p$, so that the refined partial Hasse invariants over O_K/p can be defined. We prove that such filtrations always exist, and that the valuations of the refined partial Hasse invariants obtained do not depend on any choice, if the valuation of the μ-ordinary Hasse invariant is small enough.

Let us now talk about the organization of the paper. In the first section, we define adequate filtrations and the refined partial Hasse invariants. We also prove their compatibility with duality, and relate them to the invariants constructed by Hernandez. In the second section, we prove the existence of such filtrations for p-divisible groups over a valuation ring, and prove an uniqueness result. In the third section, we relate these invariants to the partial degrees of the canonical filtration.

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1 Refined partial Hasse invariants

1.1 Definition

Let F be a finite unramified extension of $\mathbb{Q}_p$ of degree f, O_F its ring of integers and $k = \mathbb{F}_p$ the residue field. Let $\mathcal{T}$ be the set of embeddings of F into $\overline{\mathbb{Q}_p}$; it is a cyclic group of order f generated by the Frobenius σ. We will thus identify $\mathcal{T}$ and $\mathbb{Z}/f\mathbb{Z}$. Let S be a k-scheme, and G a p-divisible group over S of height h_0. Let $\mathcal{E}$ be the evaluation of the contravariant Dieudonné crystal of G at S (see [BBM] section 3.3), it is a locally free sheaf over S of rank h_0. The Frobenius and Verschiebung induce morphisms

$$V : \mathcal{E} \to \mathcal{E}^{(p)} \qquad\qquad F : \mathcal{E}^{(p)} \to \mathcal{E}$$

where the superscript denotes a twist by the Frobenius. Let $\mathcal{F} \subset \mathcal{E}$ be the Hodge filtration; it is a locally free subsheaf of $\mathcal{E}$, and induces the exact sequence (see [BBM] Corollary 3.3.5)

$$0 \to \omega_G \to \mathcal{E} \to \omega_{G^D}^\vee \to 0$$

where ω_G is the sheaf of differentials of G, G^D is the Cartier dual of G, and $\omega_{G^D}^\vee$ is the dual of the sheaf ω_{G^D}.

Assume now that G has an action of O_F; the sheaf $\mathcal{E}$ thus decomposes in $\mathcal{E} = \oplus_{i=1}^{f} \mathcal{E}_i$, with O_F acting on $\mathcal{E}_i$ by σ^i. The morphisms F and V decompose in

$$V_i : \mathcal{E}_i \to \mathcal{E}_{i-1}^{(p)} \qquad\qquad F_i : \mathcal{E}_{i-1}^{(p)} \to \mathcal{E}_i$$

for $1 \leq i \leq f$, with $\mathcal{E}_0$ being identified with $\mathcal{E}_f$. The height of G in this case must be a multiple of f; let us note $h_0 = fh$. Let i be an integer between 1 and f; the sheaf $\mathcal{E}_i$ is locally free of rank h over S. Let $\mathcal{F}_i = \mathcal{F} \cap \mathcal{E}_i$, it is a locally free sheaf over S. Let us assume that the rank of this sheaf is constant over S, equal to an integer d_i. The dimension of G is thus $d = \sum_{i=1}^{f} d_i$. The subsheaf $\mathcal{F}_i$ induces an exact sequence

$$0 \to \omega_{G,i} \to \mathcal{E}_i \to \omega_{G^D,i}^{\vee} \to 0$$

where $\omega_{G,i}$ is the subsheaf of ω_G on which O_F acts by σ^i.
Let $\widetilde{\mathcal{F}}_i = \operatorname{Ker} V_i$; it is a locally free subsheaf of $\mathcal{E}_i$ and is also equal to the image of F_i. Recall the equality $\mathcal{F}_{i-1}^{(p)} = \operatorname{Im} V_i = \operatorname{Ker} F_i$ (see [EV] section 3.1). The applications F_i and V_i thus induce isomorphisms

$$V_i : \mathcal{E}_i / \widetilde{\mathcal{F}}_i \simeq \mathcal{F}_{i-1}^{(p)} \qquad F_i : (\mathcal{E}_{i-1}/\mathcal{F}_{i-1})^{(p)} \simeq \widetilde{\mathcal{F}}_i$$

The subsheaf $\widetilde{\mathcal{F}}_i$ will be called the conjugate filtration, and induces the exact sequence

$$0 \to (\omega_{G^D,i-1}^{\vee})^{(p)} \to \mathcal{E}_i \to \omega_{G,i-1}^{(p)} \to 0$$

If $\mathcal{G}$ is a subsheaf of $\mathcal{F}_{i-1}$, then $V_i^{-1}(\mathcal{G}^{(p)})$ is a subsheaf of $\mathcal{E}_i$ containing $\widetilde{\mathcal{F}}_i$. Similarly, if $\mathcal{G}$ is a subsheaf of $\mathcal{E}_{i-1}$ containing $\mathcal{F}_{i-1}$, then $F_i(\mathcal{G}^{(p)})$ is a subsheaf of $\widetilde{\mathcal{F}}_i$. One can then see that a filtration on $\mathcal{E}_{i-1}$ refining the Hodge filtration gives a filtration on $\mathcal{E}_i$ refining the conjugate filtration.

Let r be the cardinality of the set $\{d_i, 1 \leq i \leq f\} \cap [1, h-1]$, and let us write $\delta_1 < \dots < \delta_r$ the different elements of this set. Define also $\delta_0 = 0$, $\delta_{r+1} = h$. For any $1 \leq i \leq f$, there exists a unique integer $0 \leq s(i) \leq r+1$ such that $d_i = \delta_{s(i)}$.
We will now make the following hypothesis.

Hypothesis 1.1.1. *For each $1 \leq i \leq f$, there exists a filtration*

$$0 \subset \mathcal{F}_i^{[s(i)-1]} \subset \dots \subset \mathcal{F}_i^{[1]} \subset \mathcal{F}_i \subset \mathcal{F}_i^{[r]} \subset \dots \subset \mathcal{F}_i^{[s(i)+1]} \subset \mathcal{E}_i$$

such that

- *the sheaf $\mathcal{F}_i^{[j]}$ is locally a direct factor of $\mathcal{E}_i$, and is locally free of rank $d_i - \delta_j$ for $1 \leq j \leq s(i)-1$.*
- *the sheaf $\mathcal{F}_i^{[j]}$ is locally a direct factor of $\mathcal{E}_i$, and is locally free of rank $h + d_i - \delta_j$ for $s(i)+1 \leq j \leq r$.*

Let i be an integer between 1 and f. We will also set $\mathcal{F}_i^{[s(i)]} := 0$ and $\mathcal{F}_i^{[0]} := \mathcal{F}_i^{[r+1]} := \mathcal{F}_i$. Note that the sheaves $\mathcal{F}_i / \mathcal{F}_i^{[j]}$ are locally free of rank δ_j for $0 \leq j \leq s(i)$, and the sheaves $\mathcal{F}_i^{[j]} / \mathcal{F}_i$ are locally free of rank $h - \delta_j$ for $s(i)+1 \leq j \leq r+1$. From our previous remark, these filtrations refining the Hodge filtration induce filtrations refining the conjugate filtration :

$$0 \subset \widetilde{\mathcal{F}}_i^{[r]} \subset \dots \subset \widetilde{\mathcal{F}}_i^{[s(i-1)+1]} \subset \widetilde{\mathcal{F}}_i \subset \widetilde{\mathcal{F}}_i^{[s(i-1)-1]} \subset \dots \subset \widetilde{\mathcal{F}}_i^{[1]} \subset \mathcal{E}_i$$

More precisely, we define $\widetilde{\mathcal{F}}_i^{[j]} := V_i^{-1}((\mathcal{F}_{i-1}^{[j]})^{(p)})$ for $1 \leq j \leq s(i-1)$, and $\widetilde{\mathcal{F}}_i^{[j]} := F_i((\mathcal{F}_{i-1}^{[j]})^{(p)})$ for $s(i-1)+1 \leq j \leq r$. We will set $\widetilde{\mathcal{F}}_i^{[0]} := \mathcal{E}_i$ and $\widetilde{\mathcal{F}}_i^{[r+1]} := 0$. Note that $\widetilde{\mathcal{F}}_i^{[s(i-1)]} = \widetilde{\mathcal{F}}_i$. For each $0 \leq j \leq r+1$, the sheaf $\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}$ is locally free of rank δ_j.
We will also make the following hypothesis.

Hypothesis 1.1.2. *For each integer i between 1 and f, we have $\mathcal{F}_i^{[j]} \subset \widetilde{\mathcal{F}}_i^{[j]}$ if $1 \leq j \leq s(i)-1$, and $\widetilde{\mathcal{F}}_i^{[j]} \subset \mathcal{F}_i^{[j]}$ if $s(i)+1 \leq j \leq r$.*

The filtrations satisfying these two hypotheses will be called adequate. Before going any further, let us make some remarks.

Remark 1.1.3. If there exists an integer $0 < d < h$, with $d_i = d$ for all $1 \leq i \leq f$, then $s(i) = r = 1$ for all $1 \leq i \leq f$ and the two previous hypotheses are empty. We will refer to this case as the ordinary case.

Remark 1.1.4. Assume that $f = 2$, and suppose that $0 < d_1 < d_2 < h$. The first hypothesis is the existence of filtrations

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_1^{[2]} \subset \mathcal{E}_1 \qquad 0 \subset \mathcal{F}_2^{[1]} \subset \mathcal{F}_2 \subset \mathcal{E}_2$$

with $\mathcal{F}_1^{[2]}$ locally free of rank $h - d_2 + d_1$ and $\mathcal{F}_2^{[1]}$ locally free of rank $d_2 - d_1$. The conditions in the second hypothesis are $\mathcal{F}_1^{[2]} \supset \widetilde{\mathcal{F}}_1 = \operatorname{Im} F_1$ and $\mathcal{F}_2^{[1]} \subset \widetilde{\mathcal{F}}_2 = \operatorname{Ker} V_2$.

Remark 1.1.5. Let $1 \leq i \leq f$, and $1 \leq j \leq s(i)-1$; the condition $(\mathcal{F}_i^{[j]})^{(p)} \subset (\widetilde{\mathcal{F}}_i^{[j]})^{(p)}$ is equivalent to the fact that the Verschiebung V_{i+1} sends $\widetilde{\mathcal{F}}_{i+1}^{[j]}$ into $(\widetilde{\mathcal{F}}_i^{[j]})^{(p)}$. Similarly, if $s(i)+1 \leq j \leq r$, the condition $(\widetilde{\mathcal{F}}_i^{[j]})^{(p)} \subset (\mathcal{F}_i^{[j]})^{(p)}$ is equivalent to the fact that the Frobenius F_{i+1} sends $(\widetilde{\mathcal{F}}_i^{[j]})^{(p)}$ into $\widetilde{\mathcal{F}}_{i+1}^{[j]}$.
Adequate filtrations thus induce refinements of the conjugate filtration stable by the Frobenius and Verschiebung.

Definition 1.1.6. Let $1 \leq i \leq f$. If $1 \leq j \leq s(i)$, we define $\mathcal{L}_i^{[j]} := \det(\mathcal{F}_i/\mathcal{F}_i^{[j]})$; if $s(i)+1 \leq j \leq r$, we define $\mathcal{L}_i^{[j]} := \det(\mathcal{E}_i/\mathcal{F}_i^{[j]}) \otimes \det(\mathcal{F}_i)$.

We have thus defined an invertible sheaf $\mathcal{L}_i^{[j]}$ for all $1 \leq i \leq f$ and $1 \leq j \leq r$. Note that $\mathcal{L}_i^{[s(i)]} = \det \mathcal{F}_i$.

Proposition 1.1.7. *Let $1 \leq i \leq f$ and $1 \leq j \leq r$. We have*

$$\det(\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}) \simeq (\mathcal{L}_{i-1}^{[j]})^p$$

Proof. Assume that $j \leq s(i-1)$. Then we have $\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]} \simeq (\mathcal{F}_{i-1}/\mathcal{F}_{i-1}^{[j]})^{(p)}$, hence the result. Suppose now that $j > s(i-1)$. We have

$$\det(\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}) \simeq \det(\mathcal{E}_i/\widetilde{\mathcal{F}}_i) \otimes \det(\widetilde{\mathcal{F}}_i/\widetilde{\mathcal{F}}_i^{[j]}) \simeq \det(\mathcal{F}_{i-1})^p \otimes \det(\mathcal{E}_{i-1}/\mathcal{F}_{i-1}^{[j]})^p = (\mathcal{L}_{i-1}^{[j]})^p$$

□

Definition 1.1.8. Let $1 \leq i \leq f$; if $1 \leq j \leq s(i)$ we define the application $H_i^{[j]} : \mathcal{F}_i/\mathcal{F}_i^{[j]} \to \mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}$. If $s(i) < j \leq r$, we define $H_i^{[j]} : \widetilde{\mathcal{F}}_i^{[j]} \to \mathcal{F}_i^{[j]}/\mathcal{F}_i$.

Note that these applications are well defined thanks to Hypothesis 1.1.2 (recall that $\mathcal{F}_i^{[s(i)]} = 0$).

Proposition 1.1.9. *Let $1 \leq i \leq f$, and let $1 \leq j \leq r$. The determinant of $H_i^{[j]}$ gives a section $h_i^{[j]} \in H^0(S, (\mathcal{L}_{i-1}^{[j]})^p(\mathcal{L}_i^{[j]})^{-1})$.*

Proof. The result is clear if $j \leq s(i)$. Suppose that $s(i) < j \leq r$; the determinant of $H_i^{[j]}$ gives a section of the invertible sheaf

$$\det(\mathcal{F}_i^{[j]}/\mathcal{F}_i) \otimes \det(\widetilde{\mathcal{F}}_i^{[j]})^{-1} \simeq \det(\mathcal{E}_i) \otimes \det(\mathcal{E}_i/\mathcal{F}_i^{[j]})^{-1} \otimes \det(\mathcal{F}_i)^{-1} \otimes \det(\widetilde{\mathcal{F}}_i^{[j]})^{-1} \simeq \det(\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}) \otimes (\mathcal{L}_i^{[j]})^{-1}$$

and this sheaf is isomorphic to $(\mathcal{L}_{i-1}^{[j]})^p(\mathcal{L}_i^{[j]})^{-1}$. □

We will call the elements $(h_i^{[j]})_{i,j}$ the refined partial Hasse invariants. Actually, the section $h_i^{[j]}$ might be expressed as a product of other sections.

Definition 1.1.10. Let $1 \leq i \leq f$ and $1 \leq j \leq r$. If $j \leq s(i)$, we define $\mathcal{M}_i^{[j]} := \det(\widetilde{\mathcal{F}}_i^{[j-1]}/\widetilde{\mathcal{F}}_i^{[j]}) \otimes \det(\mathcal{F}_i^{[j-1]}/\mathcal{F}_i^{[j]})^{-1}$, and $m_i^{[j]}$ the section of this sheaf induced by the determinant of the map

$$\mathcal{F}_i^{[j-1]}/\mathcal{F}_i^{[j]} \to \widetilde{\mathcal{F}}_i^{[j-1]}/\widetilde{\mathcal{F}}_i^{[j]}$$

If $j > s(i)$, we define $\mathcal{N}_i^{[j]} := \det(\mathcal{F}_i^{[j]}/\mathcal{F}_i^{[j+1]}) \otimes \det(\widetilde{\mathcal{F}}_i^{[j]}/\widetilde{\mathcal{F}}_i^{[j+1]})^{-1}$, and $n_i^{[j]}$ the section of this sheaf induced by the determinant of the map

$$\widetilde{\mathcal{F}}_i^{[j]}/\widetilde{\mathcal{F}}_i^{[j+1]} \to \mathcal{F}_i^{[j]}/\mathcal{F}_i^{[j+1]}$$

We also define $\mathcal{N}_i^{[s(i)]} := \det(\mathcal{E}_i/\mathcal{F}_i^{[s(i)+1]}) \otimes \det(\widetilde{\mathcal{F}}_i^{[s(i)]}/\widetilde{\mathcal{F}}_i^{[s(i)+1]})^{-1}$, and $n_i^{[s(i)]}$ the section of this sheaf induced by the determinant of the map

$$\widetilde{\mathcal{F}}_i^{[s(i)]}/\widetilde{\mathcal{F}}_i^{[s(i)+1]} \to \mathcal{E}_i/\mathcal{F}_i^{[s(i)+1]}$$

The next proposition is immediate.

Proposition 1.1.11. *Let $1 \leq i \leq f$ and $1 \leq j \leq r$. If $j \leq s(i)$, we have*

$$h_i^{[j]} = \prod_{k=1}^{j} m_i^{[k]}$$

If $j > s(i)$, we have

$$h_i^{[j]} = \prod_{k=j}^{r} n_i^{[k]}$$

The definition of the partial Hasse invariants, using a refinement of the Hodge filtration, might seem odd. It is in fact more natural to use the conjugate filtration, but one only gets these invariants to the power p.

Proposition 1.1.12. *Let $1 \leq i \leq f$ be an integer, and let $1 \leq j \leq r$. If $j \leq s(i)$, then the determinant of the map*

$$V_{i+1} : \mathcal{E}_{i+1}/\widetilde{\mathcal{F}}_{i+1}^{[j]} \to (\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]})^{(p)}$$

gives the section $(h_i^{[j]})^p$. If $j > s(i)$, then the determinant of the map

$$F_{i+1} : (\widetilde{\mathcal{F}}_i^{[j]})^{(p)} \to \widetilde{\mathcal{F}}_{i+1}^{[j]}$$

gives the section $(h_i^{[j]})^p$.

Proof. Assume that $j \leq s(i)$. The section $h_i^{[j]}$ is induced by the determinant of the map $\mathcal{F}_i/\mathcal{F}_i^{[j]} \to \mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}$. The section $(h_i^{[j]})^p$ is thus induced by the determinant of the map

$$(\mathcal{F}_i/\mathcal{F}_i^{[j]})^{(p)} \to (\mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]})^{(p)}$$

But the Verschiebung induces an isomorphism

$$V_{i+1} : \mathcal{E}_{i+1}/\widetilde{\mathcal{F}}_{i+1}^{[j]} \simeq (\mathcal{F}_i/\mathcal{F}_i^{[j]})^{(p)}$$

hence the result. The case $j > s(i)$ is similar. □

1.2 Compatibility with duality

The goal of this section is to prove the compatibility of the sections $h_i^{[j]}$ with duality. The contravariant Dieudonné crystal of G^D evaluated at S is $\mathcal{E}^\vee$ (see [BBM] section 5.3). We have a decomposition $\mathcal{E}^\vee = \oplus_{i=1}^f \mathcal{E}_i^\vee$; the Verschiebung and Frobenius on $\mathcal{E}^\vee$ are given respectively by $F_i^\vee : \mathcal{E}_i^\vee \to (\mathcal{E}_{i-1}^\vee)^{(p)}$ and $V_i^\vee : (\mathcal{E}_{i-1}^\vee)^{(p)} \to \mathcal{E}_i^\vee$ for $1 \leq i \leq f$. Let i be an integer between 1 and f. The Hodge filtration is given by $\mathcal{F}_i^\perp \subset \mathcal{E}_i^\vee$, and the conjugate filtration by $\widetilde{\mathcal{F}}_i^\perp \subset \mathcal{E}_i^\vee$. These two sheaves are locally free of rank respectively $h - d_i =: d_i'$ and d_{i-1}. The filtrations on $\mathcal{E}_i$ will induce filtrations on $\mathcal{E}_i^\vee$. Let us start with a lemma.

Lemma 1.2.1. *Let $1 \leq i \leq f$ be an integer. Let $\mathcal{A} \subset \mathcal{F}_{i-1}$ be a locally free sheaf, and let $\widetilde{\mathcal{A}} := V_i^{-1}(\mathcal{A}^{(p)})$. Then $(\widetilde{\mathcal{A}})^\perp = V_i^\vee((\mathcal{A}^\perp)^{(p)})$.*
Let $\mathcal{B} \supset \mathcal{F}_{i-1}$ be a locally free sheaf, and let $\widetilde{\mathcal{B}} := F_i(\mathcal{B}^{(p)})$. Then $(\widetilde{\mathcal{B}})^\perp = (F_i^\vee)^{-1}((\mathcal{B}^\perp)^{(p)})$.

Proof. One can work locally and assume that $S = \operatorname{Spec} R$ for some ring R, and that all the sheaves are free R-modules. Let $f \in \mathcal{E}_i^\vee$. Then $f \in (\widetilde{\mathcal{A}})^\perp$ if and only if $f = 0$ when restricted to $\widetilde{\mathcal{A}} = V_i^{-1}(\mathcal{A}^{(p)})$. This is equivalent to the fact that $f = g \circ V_i$, for some $g \in (\mathcal{E}_{i-1}^\vee)^{(p)}$ with $g = 0$ when restricted to $\mathcal{A}^{(p)}$. This last condition is equivalent to $g \in (\mathcal{A}^\perp)^{(p)}$. Thus

$$f \in (\widetilde{\mathcal{A}})^\perp \Leftrightarrow f = V_i^\vee(g) \text{ for some } g \in (\mathcal{A}^\perp)^{(p)}$$

The proof of the second assertion is similar. □

The adequate filtrations give a filtration on $\mathcal{E}_i^\vee$ for each $1 \leq i \leq f$:

$$0 \subset (\mathcal{F}_i^{[s(i)+1]})^\perp \subset \cdots \subset (\mathcal{F}_i^{[r]})^\perp \subset \mathcal{F}_i^\perp \subset (\mathcal{F}_i^{[1]})^\perp \subset \cdots \subset (\mathcal{F}_i^{[s(i)-1]})^\perp \subset \mathcal{E}_i^\vee$$

We also have the filtration

$$0 \subset (\widetilde{\mathcal{F}}_i^{[1]})^\perp \subset \cdots \subset (\widetilde{\mathcal{F}}_i^{[r]})^\perp \subset \mathcal{E}_i^\vee$$

From the previous lemma, these two filtrations are compatible in the sense that $(\widetilde{\mathcal{F}}_i^{[j]})^\perp = (F_i^\vee)^{-1}((\mathcal{F}_{i-1}^{[j]})^{\perp^{(p)}})$ for $1 \leq j \leq s(i-1)$, and $(\widetilde{\mathcal{F}}_i^{[j]})^\perp = V_i^\vee((\mathcal{F}_{i-1}^{[j]})^{\perp^{(p)}})$ for $s(i-1)+1 \leq j \leq r$.

Proposition 1.2.2. *The filtrations on $\mathcal{E}_i^\vee$, $1 \leq i \leq f$, are adequate.*

Proof. Note that the set $\{d'_i, 1 \leq i \leq f\} \cap [1, h-1]$ consists in $h - \delta_r < \cdots < h - \delta_1$. Let us denote $h - \delta_j$ by δ'_j. Let $1 \leq i \leq f$, and let $s(i)+1 \leq j \leq r$ be an integer. The sheaf $(\mathcal{F}_i^{[j]})^\perp$ is locally free of rank $\delta_j - d_i = d'_i - \delta'_j$. If $1 \leq j \leq s(i)-1$, then the sheaf $(\mathcal{F}_i^{[j]})^\perp$ is locally free of rank $h - d_i + \delta_j = h + d'_i - \delta'_j$. This proves that the first hypothesis is satisfied.
If $1 \leq j \leq s(i)-1$, the inclusion $\mathcal{F}_i^{[j]} \subset \widetilde{\mathcal{F}}_i^{[j]}$ implies that $(\widetilde{\mathcal{F}}_i^{[j]})^\perp \subset (\mathcal{F}_i^{[j]})^\perp$. Similarly, one has $(\mathcal{F}_i^{[j]})^\perp \subset (\widetilde{\mathcal{F}}_i^{[j]})^\perp$ for $s(i)+1 \leq j \leq r$. This allows us to check the second hypothesis. □

For each $1 \leq i \leq f$ and $1 \leq j \leq r$, we thus have invertible sheaves $\mathcal{L}_{G^D,i}^{[j]}$, and sections $h_{G^D,i}^{[j]}$ of $(\mathcal{L}_{G^D,i-1}^{[j]})^p(\mathcal{L}_{G^D,i}^{[j]})^{-1}$.

Theorem 1.2.3. *Let $1 \leq i \leq f$, and $1 \leq j \leq r$ be integers. We have an isomorphism*

$$(\mathcal{L}_{G^D,i-1}^{[r+1-j]})^p(\mathcal{L}_{G^D,i}^{[r+1-j]})^{-1} \simeq (\mathcal{L}_{i-1}^{[j]})^p(\mathcal{L}_i^{[j]})^{-1}$$

Using this isomorphism, one has the equality $h_{G^D,i}^{[r+1-j]} = h_i^{[j]}$.

Proof. First assume that $j < s(i)$. The application $H_i^{[j]}$ is the natural map $\mathcal{F}_i/\mathcal{F}_i^{[j]} \to \mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]}$. The application $H_{G^D,i}^{[r+1-j]}$ is the natural map $(\widetilde{\mathcal{F}}_i^{[j]})^\perp \to (\mathcal{F}_i^{[j]})^\perp/\mathcal{F}_i^\perp$ and is thus equal to the dual of $H_i^{[j]}$. Since a map and its dual induce the same section of the same sheaf, the result follows.
The case $j > s(i)$ is similar (or can be treated by duality). We are left with the case $j = s(i)$. The map $H_i^{[s(i)]}$ is the natural map $\mathcal{F}_i \to \mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[s(i)]}$, and the dual of $H_{G^D,i}^{[r+1-s(i)]}$ is the natural map $\widetilde{\mathcal{F}}_i^{[s(i)]} \to \mathcal{E}_i/\mathcal{F}_i$. The result then follows from [Bi3] Proposition 1.3. □

We also have the following compatibility for the sections $(m_i^{[j]})_{i,j}$ and $(n_i^{[j]})_{i,j}$. For $1 \leq i \leq f$ and $1 \leq j \leq r+1-s(i)$, we have an invertible sheaf $\mathcal{M}_{G^D,i}^{[j]}$, and a section $m_{G^D,i}^{[j]}$ of this sheaf. If $1 \leq i \leq f$ and $r+1-s(i) \leq j \leq r$, we have an invertible sheaf $\mathcal{N}_{G^D,i}^{[j]}$, and a section $N_{G^D,i}^{[j]}$ of this sheaf.

Proposition 1.2.4. *Let $1 \leq i \leq f$, and $1 \leq j \leq r$ be integers. If $j \leq s(i)$, we have an isomorphism $\mathcal{M}_i^{[j]} \simeq \mathcal{N}_{G^D,i}^{[r+1-j]}$; under this isomorphism, one has $m_i^{[j]} = n_{G^D,i}^{[r+1-j]}$. If $j \geq s(i)$, we have an isomorphism $\mathcal{N}_i^{[j]} \simeq \mathcal{M}_{G^D,i}^{[r+1-j]}$; under this isomorphism, one has $n_i^{[j]} = m_{G^D,i}^{[r+1-j]}$.*

Proof. Let $1 \leq i \leq f$, and $1 \leq j \leq s(i)$. The section $m_i^{[j]}$ is induced by the determinant of the map

$$\mathcal{F}_i^{[j-1]}/\mathcal{F}_i^{[j]} \to \widetilde{\mathcal{F}}_i^{[j-1]}/\widetilde{\mathcal{F}}_i^{[j]}$$

The section $n_{G^D,i}^{[r+1-j]}$ is induced by the determinant of the map

$$(\widetilde{\mathcal{F}}_i^{[j]})^\perp/(\widetilde{\mathcal{F}}_i^{[j-1]})^\perp \to (\mathcal{F}_i^{[j]})^\perp/(\mathcal{F}_i^{[j-1]})^\perp$$

But the dual of this map is precisely the map $\mathcal{F}_i^{[j-1]}/\mathcal{F}_i^{[j]} \to \widetilde{\mathcal{F}}_i^{[j-1]}/\widetilde{\mathcal{F}}_i^{[j]}$. Since a map and its dual induce the same section of the same sheaf, the first part of the proposition follows. The second part is similar. □

1.3 Relation with the μ-ordinary partial Hasse invariants

Recall that we have a decomposition

$$\omega_G = \bigoplus_{i=1}^{f} \omega_{G,i}$$

where $\omega_{G,i}$ is locally free of rank d_i, for each $1 \leq i \leq f$. Let us call $\mu = (d_i)_{1\leq i\leq f}$; the μ-ordinary partial Hasse invariant attached to i, $Ha_i(G)$, is constructed in [He] and is an element in $H^0(S, (\det \omega_{G,i})^{p^f-1})$ for each $1 \leq i \leq f$ with $d_i \neq 0$. The product of these invariants is the total μ-ordinary Hasse invariant

$$Ha(G) \in H^0(S, (\det \omega_G)^{p^f-1})$$

Note that this last invariant was constructed for some Shimura varieties in [GN]. We will first recall the definition of the elements $Ha_i(G)$; actually, we will use the construction in [BH]. In this article, the authors gave a simpler construction of these invariants, and extended it to the ramified case for p-divisible groups with Pappas-Rapoport condition. We will then relate these elements to the sections $h_i^{[j]}$. More precisely, we will show that $Ha_i(G)$ is equal to the product of some powers of $h_k^{[s(i)]}$, for $1 \leq k \leq f$ (where $1 \leq i \leq f$ is an element with $d_i \notin \{0, h\}$).

Proposition 1.3.1. *Let $1 \leq i \leq f$, and let $d_{i-1} < d \leq h$ be an integer. Then the map*

$$\bigwedge^{d_{i-1}}(\mathcal{E}_i) \otimes \bigwedge^{d-d_{i-1}} \widetilde{\mathcal{F}}_i \to \bigwedge^{d} \mathcal{E}_{i-1}^{(p)}$$
$$(x_1 \wedge \cdots \wedge x_{d_{i-1}}) \otimes (F_i y_1 \wedge \cdots \wedge F_i y_{d-d_{i-1}}) \to (V_i x_1 \wedge \cdots \wedge V_i x_{d_{i-1}}) \wedge (y_1 \wedge \cdots \wedge y_{d-d_{i-1}})$$

is well defined, and factors through the natural surjection $\bigwedge^{d_{i-1}}(\mathcal{E}_i) \otimes \bigwedge^{d-d_{i-1}} \widetilde{\mathcal{F}}_i \twoheadrightarrow \bigwedge^d \mathcal{E}_i$.

This proposition follows from [BH] prop. 2.2.10. In that case, we will call $f_i^d : \bigwedge^d \mathcal{E}_i \to \bigwedge^d \mathcal{E}_{i-1}^{(p)}$ the induced map. If $d \leq d_{i-1}$, we will define f_i^d to be $\bigwedge^d V_i$.

Definition 1.3.2. Let $1 \leq i \leq f$ with $d_i \neq 0$. The μ-ordinary partial Hasse invariant attached to i, $Ha_i(G)$, is the section induced by the map

$$\bigwedge^{d_i} \mathcal{F}_i \hookrightarrow \bigwedge^{d_i} \mathcal{E}_i \xrightarrow{f_i^{d_i}} \bigwedge^{d_i} \mathcal{E}_{i-1}^{(p)} \to \qquad \dots \xrightarrow{(f_{i+2}^{d_i})^{(p^f-2)}} \bigwedge^{d_i} \mathcal{E}_{i+1}^{(p^f-1)} \xrightarrow{\bigwedge^{d_i} V_{i+1}^{(p^f-1)}} \bigwedge^{d_i} \mathcal{F}_i^{(p^f)}$$

The μ-ordinary partial Hasse invariant $Ha_i(G)$, is thus a section of $(\det \mathcal{F}_i)^{p^f-1}$.

Theorem 1.3.3. *Let $1 \leq i \leq f$ with $d_i \notin \{0, h\}$. Then we have*

$$Ha_i(G) = h_i^{[s(i)]} \cdot (h_{i-1}^{[s(i)]})^p \cdot \cdots \cdot (h_{i+1}^{[s(i)]})^{p^{f-1}}$$

Proof. We will prove this statement by giving an alternative description of the maps $h_k^{[j]}$. Let $1 \leq k \leq f$, and $1 \leq j \leq r$ be integers. First assume that $j \leq s(k)$. Then the element $h_k^{[j]}$ is induced by the natural map $\det(\mathcal{F}_k/\mathcal{F}_k^{[j]}) \to \det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]})$. Suppose now that $j > s(k)$. We have isomorphisms of sheaves

$$\begin{aligned}\det(\mathcal{F}_k^{[j]}/\mathcal{F}_k) \otimes \det(\widetilde{\mathcal{F}}_k^{[j]})^{-1} &\simeq \det(\mathcal{E}_k) \otimes \det(\mathcal{E}_k/\mathcal{F}_k^{[j]})^{-1} \otimes \det(\mathcal{F}_k)^{-1} \otimes \det(\widetilde{\mathcal{F}}_k^{[j]})^{-1} \\ &\simeq \det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]}) \otimes \det(\mathcal{E}_k/\mathcal{F}_k^{[j]})^{-1} \otimes \det(\mathcal{F}_k)^{-1}\end{aligned}$$

The element $h_k^{[j]}$ is obtained by taking the determinant of the natural map $\widetilde{\mathcal{F}}_k^{[j]} \to \mathcal{F}_k^{[j]}/\mathcal{F}_k$. Using the previous isomorphism, it is also induced by the natural map

$$\det(\mathcal{E}_k/\mathcal{F}_k^{[j]}) \otimes \det(\mathcal{F}_k) \to \det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]})$$

We are left to unravel the isomorphism $\det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]}) \simeq (\mathcal{L}_{k-1}^{[j]})^p$ from Proposition 1.1.7. Assume that $j \leq s(k-1)$. The isomorphism $\det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]}) \simeq \det((\mathcal{F}_{k-1}/\mathcal{F}_{k-1}^{[j]})^{(p)})$ is given by $\wedge^{\delta_j} V_k$.
Now let us suppose that $j > s(k-1)$. There are isomorphisms

$$\det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]}) \simeq \det(\mathcal{E}_k/\widetilde{\mathcal{F}}_k) \otimes \det(\widetilde{\mathcal{F}}_k/\widetilde{\mathcal{F}}_k^{[j]}) \simeq \det(\mathcal{F}_{k-1}^{(p)}) \otimes \det((\mathcal{E}_{k-1}/\mathcal{F}_{k-1}^{[j]})^{(p)})$$

Assume that we work locally, and let $e_1, \ldots, e_{\delta_j}$ be a basis of $\mathcal{E}_k/\widetilde{\mathcal{F}}_k^{[j]}$. This can be done such that $e_{d_{k-1}+1}, \ldots, e_{\delta_j}$ are in $\widetilde{\mathcal{F}}_k$. Since this space is the image of F_k, we have $e_l = F_k x_l$ for some $x_l \in \mathcal{E}_{k-1}^{(p)}$, for all $d_{k-1} + 1 \leq l \leq \delta_j$. The image of $e_1 \wedge \cdots \wedge e_{\delta_j}$ by the previous isomorphism is then

$$(V_k(e_1) \wedge \cdots \wedge V_k(e_{d_{k-1}})) \otimes (x_{d_{k-1}+1} \wedge \cdots \wedge x_{\delta_j})$$

This concludes the proof. ∎

Remark 1.3.4. If $d_i = h$, then the sheaf $(\det \omega_{G,i})^{p^f-1}$ is trivial, and the element $Ha_i(G)$ is a non-zero section of this trivial sheaf. Indeed, the map f_i^h induces an isomorphism between $\det \mathcal{E}_i$ and $(\det \mathcal{E}_{i-1})^p$ for all $1 \leq i \leq f$.

2 Properties of adequate filtrations

2.1 Existence

In this section, we prove that locally, adequate filtrations always exist. Let $S = \operatorname{Spec} R$, where R is a ring of characteristic p such that if $x, y \in R$, then x divides y or y divides x (for example, R can be a valuation ring of characteristic p). In particular, R is local. Let G be a p-divisible group over S, endowed with an action of O_F as in the previous section. The module ω_G thus decomposes into $\omega_G = \oplus_{i=1}^f \omega_{G,i}$, and let d_i be the rank of $\omega_{G,i}$ for all $1 \leq i \leq f$. We will keep the same notations as in the previous section.

Theorem 2.1.1. *There exist adequate filtrations on the spaces $(\mathcal{E}_i)_{1\leq i\leq f}$.*

Proof. We will explicitly construct adequate filtrations on the spaces $\mathcal{E}_i$. We will construct by induction the spaces $(\mathcal{F}_i^{[j]})_{1\leq i\leq f}$, for $1 \leq j \leq r$. Let j be an integer between 1 and r, and assume we have constructed the spaces $\mathcal{F}_i^{[k]}$ for $1 \leq i \leq f$ and $1 \leq k \leq j-1$. We have to construct the space $\mathcal{F}_i^{[j]}$ for $1 \leq i \leq f$. Note that if $s(i) = j$, there is nothing to be done (as $\mathcal{F}_i^{[s(i)]}$ is defined to be 0). This proves that there is at least one element i for which the space $\mathcal{F}_i^{[j]}$ is constructed. Suppose that the space $\mathcal{F}_{i-1}^{[j]}$ is constructed, and we will construct the space $\mathcal{F}_i^{[j]}$. This will conclude the construction.

If $s(i) = j$, there is nothing to be done. Assume that $j < s(i)$. We look for a direct factor $\mathcal{F}_i^{[j]}$ of $\mathcal{E}_i$ of rank $d_i - \delta_j$ such that $\mathcal{F}_i^{[j]} \subset \mathcal{F}_i^{[j-1]} \cap \widetilde{\mathcal{F}}_i^{[j]}$, with $\mathcal{F}_i^{[0]} = \mathcal{F}_i$ by convention. But $\mathcal{F}_i^{[j-1]}$ is free of rank $d_i - \delta_{j-1}$, and is included in $\widetilde{\mathcal{F}}_i^{[j-1]}$. Since $\widetilde{\mathcal{F}}_i^{[j]}$ is of corank $\delta_j - \delta_{j-1}$ in $\widetilde{\mathcal{F}}_i^{[j-1]}$, the space $\mathcal{F}_i^{[j-1]} \cap \widetilde{\mathcal{F}}_i^{[j]}$ contains a subspace which is free of rank $d_i - \delta_j$. We have thus constructed $\mathcal{F}_i^{[j]}$ in this case.

Suppose now that $j > s(i)$. Let $\mathcal{G} := \mathcal{F}_i^{[j-1]}$ if $j > s(i)+1$, and $\mathcal{G} := \mathcal{E}_i$ if $j = s(i)+1$. We look for a direct factor $\mathcal{F}_i^{[j]}$ of $\mathcal{E}_i$ of rank $h + d_i - \delta_j$ such that $\mathcal{F}_i \subset \mathcal{F}_i^{[j]} \subset \mathcal{G}$ and $\widetilde{\mathcal{F}}_i^{[j]} \subset \mathcal{F}_i^{[j]}$. Note that we have $\widetilde{\mathcal{F}}_i^{[j]} \subset \widetilde{\mathcal{F}}_i^{[j-1]} \subset \mathcal{G}$. The condition on $\mathcal{F}_i^{[j]}$ is then

$$\mathcal{F}_i + \widetilde{\mathcal{F}}_i^{[j]} \subset \mathcal{F}_i^{[j]} \subset \mathcal{G}$$

But inside $\mathcal{G}$, there is a free space of rank $d_i + h - d_j$ containing both $\mathcal{F}_i$ and $\widetilde{\mathcal{F}}_i^{[j]}$. This allows us to construct the space $\mathcal{F}_i^{[j]}$, and concludes the proof. $\square$

2.2 Uniqueness

We will now prove that the reduction of the adequate filtrations modulo a certain ideal is unique.

Let K be a valuation field, which is an extension of $\mathbb{Q}_p$, v be the valuation (normalized by $v(p) = 1$), and let O_K be the ring of integers. For all real $w > 0$, let us define $\mathfrak{m}_w := \{x \in O_K, v(x) \geq w\}$ and $O_{K,\{w\}} := O_K/\mathfrak{m}_w$. In particular $O_{K,\{1\}} = O_K/pO_K$. If M is a O_K-module, and $w > 0$ is a real, then we will note $M_{\{w\}} := M \otimes_{O_K} O_{K,\{w\}}$.

Let G be a p-divisible group defined over $O_{K,\{1\}}$, and assume that G has an action of O_F as before. We will keep the same notations as in the previous sections. Assume the existence of adequate filtrations $(\mathcal{F}_i^{[\bullet]})_{1\leq i\leq f}$ on the spaces $(\mathcal{E}_i)_{1\leq i\leq f}$. We define

$$w_i^{[j]} := v(h_i^{[j]}) \in [0,1]$$

for $1 \leq i \leq f$ and $1 \leq j \leq r$. More precisely, since $h_i^{[j]}$ is the section of an invertible sheaf on $\operatorname{Spec} O_{K,\{1\}}$, a choice of a trivialization of this sheaf allows us to see $h_i^{[j]}$ as an element of $O_{K,\{1\}}$. The valuation of this element is then independent of the choice made. We will also define

$$w = \sum_{i=1}^{f} \sum_{j=1}^{r} w_i^{[j]}$$

Let us start with a lemma.

Lemma 2.2.1. *Let* $(\mathcal{F}_i^{[\bullet]'})_{1\le i\le f}$ *be other adequate filtrations on the spaces* $(\mathcal{E}_i)_{1\le i\le f}$*. Let* $1 \le i \le f$ *and* $1 \le j \le r$ *be integers, and let* $w_i^{[j]} < \alpha \le 1$ *be a real. Assume that* ${\mathcal{F}_{k,\{\alpha\}}^{[l]}}' = \mathcal{F}_{k,\{\alpha\}}^{[l]}$ *for* $1 \le k \le i$ *and* $1 \le l \le j-1$*, and that* ${\mathcal{F}_{i-1,\{\alpha\}}^{[j]}}' = \mathcal{F}_{i-1,\{\alpha\}}^{[j]}$*. Then*

$$ {\mathcal{F}_{i,\{\alpha-w_i^{[j]}\}}^{[j]}}' = \mathcal{F}_{i,\{\alpha-w_i^{[j]}\}}^{[j]} $$

Proof. If $s(i) = j$, the result is obvious. Assume that $j < s(i)$. The space ${\mathcal{F}_{i,\{\alpha\}}^{[j]}}'$ lies inside the kernel of the map

$$ \phi : \mathcal{F}_{i,\{\alpha\}}^{[j-1]} \to \widetilde{\mathcal{F}}_{i,\{\alpha\}}^{[j-1]} / \widetilde{\mathcal{F}}_{i,\{\alpha\}}^{[j]} $$

The matrix of this map is of the form

$$ \begin{pmatrix} 0 & M \end{pmatrix} $$

where this matrix is written with respect to the decomposition $\mathcal{F}_{i,\{\alpha\}}^{[j]} \subset \mathcal{F}_{i,\{\alpha\}}^{[j-1]}$. The matrix M is associated to the map $\mathcal{F}_{i,\{\alpha\}}^{[j-1]} / \mathcal{F}_{i,\{\alpha\}}^{[j]} \to \widetilde{\mathcal{F}}_{i,\{\alpha\}}^{[j-1]} / \widetilde{\mathcal{F}}_{i,\{\alpha\}}^{[j]}$. In particular $v(\det M) \le w_i^{[j]}$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be in the kernel of ϕ. Then $Mx_2 = 0$ so the coordinates of x_2 are of valuation greater than $\alpha - w_i^{[j]}$. This implies that

$$ {\mathcal{F}_{i,\{\alpha-w_i^{[j]}\}}^{[j]}}' = \mathcal{F}_{i,\{\alpha-w_i^{[j]}\}}^{[j]} $$

Suppose now that $j > s(i)$. We keep the notations from the previous section. The space ${\mathcal{F}_{i,\{\alpha\}}^{[j]}}' / \mathcal{F}_{i,\{\alpha\}}$ contains the image of the map

$$ \psi : \widetilde{\mathcal{F}}_{i,\{\alpha\}}^{[j]} \to \mathcal{G}_{\{\alpha\}} / \mathcal{F}_{i,\{\alpha\}} $$

The matrix of this map is of the form

$$ \begin{pmatrix} N \\ 0 \end{pmatrix} $$

where this matrix is written with respect to the decomposition $\mathcal{F}_{i,\{\alpha\}}^{[j]} / \mathcal{F}_{i,\{\alpha\}} \subset \mathcal{G}_{\{\alpha\}} / \mathcal{F}_{i,\{\alpha\}}$. The matrix N is associated to the map $\widetilde{\mathcal{F}}_{i,\{\alpha\}}^{[j]} \to \mathcal{F}_{i,\{\alpha\}}^{[j]} / \mathcal{F}_{i,\{\alpha\}}$. In particular $v(\det N) = w_i^{[j]}$. Now let $(X_1 X_2)$ be the matrix of a basis of ${\mathcal{F}_{i,\{\alpha\}}^{[j]}}'$. Since this space contains the image of ψ, there exists a matrix Z with $N = X_1 Z$ and $0 = X_2 Z$. This implies that $v(\det Z) \le w_i^{[j]}$ and that the coefficients of X_2 are of valuation greater than $\alpha - w_i^{[j]}$. This implies that

$$ {\mathcal{F}_{i,\{\alpha-w_i^{[j]}\}}^{[j]}}' = \mathcal{F}_{i,\{\alpha-w_i^{[j]}\}}^{[j]} $$

□

Proposition 2.2.2. *Assume that* $w < 1$*, and let* $(\mathcal{F}_i^{[\bullet]'})_{1\le i\le f}$ *be other adequate filtrations on the spaces* $(\mathcal{E}_i)_{1\le i\le f}$*. Then we have*

$$ {\mathcal{F}_{i,\{1-w\}}^{[j]}}' = \mathcal{F}_{i,\{1-w\}}^{[j]} $$

for $1 \le i \le f$ *and* $1 \le j \le r$*.*

Proof. We use successively the previous lemma. More precisely, we will prove by induction on j that

$$\mathcal{F}^{[j]}_{i,\{1-\alpha_j\}}{}' = \mathcal{F}^{[j]}_{i,\{1-\alpha_j\}}$$

with $\alpha_j = \sum_i \sum_{k\leq j} w_i^{[k]}$ for all $1 \leq i \leq f$ and $1 \leq j \leq r$. Let $1 \leq k \leq r$, assume that the previous relation is true for $1 \leq i \leq f$ and $1 \leq j \leq k-1$. Let i be an element with $s(i) = k$; then the space $\mathcal{F}_i^{[k]}$ needs not to be defined. From the previous proposition, we have

$$\mathcal{F}^{[k]}_{i+1,\{1-\alpha_{k-1}-w^{[k]}_{i+1}\}}{}' = \mathcal{F}^{[k]}_{i+1,\{1-\alpha_{k-1}-w^{[k]}_{i+1}\}}$$

Applying successively the previous result, one gets

$$\mathcal{F}^{[k]}_{i+l,\{1-\alpha_{k-1}-w^{[k]}_{i+1}-\cdots-w^{[k]}_{i+l}\}}{}' = \mathcal{F}^{[k]}_{i+l,\{1-\alpha_{k-1}-w^{[k]}_{i+1}-\cdots-w^{[k]}_{i+l}\}}$$

for all $1 \leq l \leq f-1$. Hence the result. □

Remark 2.2.3. We could have replaced w by

$$\sum_{i=1}^{f} \sum_{j\neq s(i)} w_i^{[j]}$$

Of course the proposition is also valid if we replace w by the valuation of the total μ-ordinary Hasse invariant.

Corollary 2.2.4. *Assume that $w < 1/2$. Then the elements $(w_i^{[j]})_{1\leq i\leq f, 1\leq j\leq r}$ do not depend on the choice of adequate filtrations, and are thus invariants of the p-divisible group G.*

Proof. Let $(\mathcal{F}_i^{[\bullet]'})_{1\leq i\leq f}$ be other adequate filtrations on the spaces $(\mathcal{E}_i)_{1\leq i\leq f}$, and let $(h_i^{[j]'})_{i,j}$ be the sections computed with these filtrations. Let $1 \leq i \leq f$ and $1 \leq j \leq r$ be integers, and assume that $j \leq s(i)$, the other case being similar (or can be treated by duality). Let $w_i^{[j]'}$ be the valuation of $h_i^{[j]'}$. The elements $w_i^{[j]}$ and $w_i^{[j]'}$ are respectively the valuations of the determinants of

$$\mathcal{F}_i/\mathcal{F}_i^{[j]} \to \mathcal{E}_i/\widetilde{\mathcal{F}}_i^{[j]} \qquad \mathcal{F}_i/\mathcal{F}_i^{[j]'} \to \mathcal{E}_i/(\widetilde{\mathcal{F}}_i^{[j]})'$$

From the previous proposition, the reduction modulo $1-w$ of these maps are equal. Thus

$$\min(w_i^{[j]}, 1-w) = \min(w_i^{[j]'}, 1-w)$$

Since $w_i^{[j]} \leq w < 1-w$, we have $w_i^{[j]'} = w_i^{[j]}$. □

3 The canonical filtration

Let K be a valuation field, which is an extension of $\mathbb{Q}_p$. We will keep the same notations as in the previous section, and we will consider a p-divisible group G over O_K. We will also assume that there are adequate filtrations associated to the group $G \times_{O_K} O_K/p$ (the existence of adequate filtrations follows from the previous section).

3.1 Raynaud group schemes

In this section, we recall some results concerning finite flat group schemes over O_K endowed with an action of O_F. If H is a finite flat group scheme over O_K, its degree has been defined in [Fa] and will be denoted by $\deg H$. When H has moreover an action of O_F, one can define the partial degrees of H, noted $\deg_i H$, for $1 \leq i \leq f$. We refer to [Bi] section 2 for the definition and properties of the partial degrees.
We now recall the structure theorem from Raynaud ([Ra]) concerning finite flat group schemes of height f over O_K, of p-torsion and with an action of O_F.

Proposition 3.1.1. *Let H be a finite flat group scheme of height f over O_K, of p-torsion and with an action of O_F. Then there exist elements $(a_i, b_i)_{1 \leq i \leq f}$ of O_K such that $a_i b_i = pu$ for all $1 \leq i \leq f$ (where u is a fixed p-adic unit), with H isomorphic to the spectrum of*

$$O_K[X_1, \dots, X_f]/(X_i^p - a_{i+1}X_{i+1})$$

where we identify X_{f+1} and X_1. The dual of the group with parameters $(a_i, b_i)_{1 \leq i \leq f}$ is the one with parameter $(b_i, a_i)_{1 \leq i \leq f}$. Moreover, we have $\omega_{H,i} = O_K/a_i$, and therefore $\deg_i H = v(a_i)$, $\deg_i H^D = v(b_i)$ for all $1 \leq i \leq f$.

We will refer to such group schemes as Raynaud group schemes. We will also need a description of the Dieudonné crystal of these group schemes.

Proposition 3.1.2. *Let G be a p-divisible group over O_K with an action of O_F, and let $H \subset G[p]$ be a O_F-stable Raynaud group scheme with parameter $(a_i, b_i)_{1 \leq i \leq f}$. Let $\mathcal{E}_H$ be the (contravariant) Dieudonné crystal of $H \times_{O_K} O_K/p$ evaluated at O_K/p; it decomposes into $\mathcal{E}_H = \oplus_{i=1}^f \mathcal{E}_{H,i}$, with each $\mathcal{E}_{H,i}$ a free O_K/p-module of rank 1. Let i be an integer between 1 and f. The Verschiebung $V_i : \mathcal{E}_{H,i} \to \mathcal{E}_{H,i-1}^{(p)}$ sends a generator to an element of valuation $pv(b_{i-1})$; the Frobenius $F_i : \mathcal{E}_{H,i-1}^{(p)} \to \mathcal{E}_{H,i}$ sends a generator to an element of valuation $pv(a_{i-1})$.*

Proof. Let $\mathcal{E} = \oplus_{i=1}^f \mathcal{E}_i$ be the Dieudonné crystal of $G \times_{O_K} O_K/p$ evaluated at O_K/p. Let $\widetilde{\mathcal{F}}_i \subset \mathcal{E}_i$ and $\widetilde{\mathcal{F}}_{H,i} \subset \mathcal{E}_{H,i}$ be the images of the Frobenius. We have $\mathcal{E}_i/\widetilde{\mathcal{F}}_i \simeq \omega_{G,i-1}^{(p)}$ and have surjective maps

$$\mathcal{E}_i \to \mathcal{E}_{H,i} \qquad \omega_{G,i-1}^{(p)} \to \omega_{H,i-1}^{(p)}$$

Thus

$$\mathcal{E}_{H,i}/\widetilde{\mathcal{F}}_{H,i} \simeq \omega_{H,i-1}^{(p)} \simeq O_{K,\{1\}}/a_{i-1}^p$$

This gives the result for the Frobenius. The result for the Verschiebung can be obtained by duality. □

3.2 Definition of the canonical filtration

We will now recall some definitions and properties of the canonical filtration. Recall that G is a p-divisible group over O_K with an action of O_F. The module ω_G thus decomposes into $\omega_G = \oplus_{i=1}^f \omega_{G,i}$, where $\omega_{G,i}$ is a free O_K-module for all i between 1 and f. Let d_i be the rank of $\omega_{G,i}$ for all $1 \leq i \leq f$. We keep the notations from the section 1; in particular, one has an integer r, and integers $\delta_1, \dots, \delta_r$.

Definition 3.2.1. Let $1 \leq j \leq r$, and let $C \subset G[p]$ be a O_F-stable finite flat subgroup of height $f\delta_j$. We say that C is canonical (of height $f\delta_j$) if

$$\deg C^D < \sum_{i=1}^{f} \max(\delta_j - d_i, 0) + \frac{1}{2}$$

It is a strong canonical subgroup if we have

$$\deg C^D < \sum_{i=1}^{f} \max(\delta_j - d_i, 0) + \frac{1}{p+1}$$

We say that G admits a (strong) canonical filtration if there exist (strong) canonical subgroups of height $f\delta_j$, for all $1 \leq j \leq r$.

Let $C \subset G[p]$ be any O_F-stable finite flat subgroup of height $f\delta_j$, where j is an integer between 1 and r. Note that since $\deg_i C \leq \deg_i G[p] = d_i$, we have $\deg_i C^D \geq \delta_j - d_i$ for all $1 \leq i \leq f$. The degree of C^D is thus always greater or equal than $\sum_{i=1}^{f} \max(\delta_j - d_i, 0)$; the subgroup is canonical if this degree is close to that value.
In the ordinary case (i.e. if there exists an integer $0 < d < h$ with $d_i = d$ for all $1 \leq i \leq f$), this definition agrees with the definition of the canonical subgroup given in [Bi] section 3.1.

Proposition 3.2.2. *Let j be an integer between 1 and r; there exists at most one canonical subgroup of height $f\delta_j$. If C is a canonical subgroup of height $f\delta_j$, then $C^\perp \subset G^D[p]$ is canonical of height $f(h - \delta_j)$.*
Let $1 \leq j \leq k \leq r$ be integers, and assume that C_l is a canonical subgroup of height $f\delta_l$, for each $l \in \{j, k\}$. Then $C_j \subset C_k$.

Proof. This is a local analogue of [Bi2] Proposition 1.24 and 1.25, and the proof is similar. □

In particular, if G admits a canonical filtration, one has

$$0 \subset C_1 \subset \cdots \subset C_r \subset G[p]$$

where C_j is the canonical subgroup of height $f\delta_j$, for $1 \leq j \leq r$.
The main result of [He2] is the existence of a canonical subgroup (of height fd_i) if the valuation of $Ha_i(G)$ is sufficiently small. We recall this theorem here.

Theorem 3.2.3 ([He2] Théorème 6.10)**.** *Let $1 \leq i \leq f$ be an integer with $d_i \notin \{0, h\}$. Assume that $p > 4h$, and that the valuation of $Ha_i(G)$ is strictly less than $1/2$. Then there exists a canonical subgroup C_i of height fd_i. Moreover, one has*

$$\sum_{k=0}^{f-1} p^k (\deg_{i-k} C^D - \max(d_i - d_{i-k}, 0)) = v(Ha_i(G))$$

Remark 3.2.4. This theorem is actually valid under weaker conditions, see [He2] Théorème 6.10 for the precise statement.

3.3 The partial degrees of the canonical filtration

In this section, we will relate the refined partial Hasse invariants constructed previously to the partial degrees of the canonical subgroups (if they exist). Recall that we assumed the existence of adequate filtrations for $G \times_{O_K} O_K/p$, and we noted $w_i^{[j]}$ the valuation of the section $h_i^{[j]}$, for $1 \leq i \leq f$ and $1 \leq j \leq r$.
First let us start with two lemmas.

Lemma 3.3.1. *Let $1 \leq j \leq r$, and assume that C is a canonical subgroup of height $f\delta_j$. Let $\alpha = \deg C^D - \sum_{i=1}^f \max(\delta_j - d_i, 0)$. Then for all $1 \leq i \leq f$, we have*

$$\deg_i C^D \leq \max(\delta_j - d_i, 0) + \alpha$$

Proof. Let $\varepsilon_i := \deg_i C^D - \max(\delta_j - d_i, 0)$ for $1 \leq i \leq f$; this is a non-negative real. Moreover, we have

$$\sum_{i=1}^{f} \varepsilon_i = \alpha$$

We conclude that $\varepsilon_i \leq \alpha$ for all $1 \leq i \leq f$. □

Lemma 3.3.2. *Let $1 \leq j \leq r$, and assume that C is a canonical subgroup of height $f\delta_j$. Let $\alpha = \deg C^D - \sum_{i=1}^f \max(\delta_j - d_i, 0)$, and let $1 \leq i \leq f$ be an integer. If $j \leq s(i)$, then $\omega_{C,i,\{1-\alpha\}}$ is a quotient of $\omega_{G,i,\{1-\alpha\}}$, which is free of rank δ_j over $O_{K,\{1-\alpha\}}$.*
If $j > s(i)$, then $\omega_{C^\perp,i,\{1-\alpha\}}$ is a quotient of $\omega_{G^D,i,\{1-\alpha\}}$, which is free of rank $h-\delta_j$ over $O_{K,\{1-\alpha\}}$.

Proof. By duality, one can assume that $j \leq s(i)$. From the previous lemma, we have

$$\deg_i C \geq \delta_j - \alpha$$

Since $\omega_{C,i}$ is generated by δ_j elements, by the elementary divisors theorem, there exists elements $x_1, \dots, x_{\delta_j}$ in O_K with valuation less than 1 such that

$$\omega_{C,i} \simeq \bigoplus_{k=1}^{\delta_j} O_K/x_k$$

We thus have $\sum_{k=1}^{\delta_j} v(x_k) = \deg_i C$, so $v(x_k) \geq 1 - \alpha$ for all $1 \leq k \leq \delta_j$. This implies that

$$\omega_{C,i,\{1-\alpha\}} \simeq (O_{K,\{1-\alpha\}})^{\delta_j}$$

□

In particular, the existence of a canonical filtration for G implies that both modules $\omega_{G,\{1-\beta\}}$ and $\omega_{G^D,\{1-\beta\}}$ are filtered by free quotients, for some real β depending on the canonical subgroups. This gives adequate filtrations on $\mathcal{E}_{i,\{1-\beta\}}$, $1 \leq i \leq f$.
We can now state the main theorem of this section, which relates the partial degrees of a canonical subgroup to the refined partial invariants. Note that in the ordinary case, the partial degrees of the canonical subgroup have been computed in [Bi] section 3.2.

Theorem 3.3.3. *Let $1 \leq j \leq r$, and assume that C is a strong canonical subgroup of height $f\delta_j$. Then for all $1 \leq i \leq f$, we have*

$$\deg_i C^D = \max(\delta_j - d_i, 0) + w_i^{[j]}$$

Proof. Let $\alpha = \deg C^D - \sum_{i=1}^f \max(\delta_j - d_i, 0)$, and we keep the notations from the previous sections. Let i be an integer between 1 and f, and consider the surjective morphism

$$\mathcal{E}_{i,\{1-\alpha\}} \to \mathcal{E}_{C,i,\{1-\alpha\}}$$

Let $\widetilde{\mathcal{G}}_{i,\{1-\alpha\}}$ be the kernel of this map. This is a free $O_{K,\{1-\alpha\}}$-module of rank $h - \delta_j$.
From the previous lemma $\mathcal{G}_{i,\{1-\alpha\}} := \widetilde{\mathcal{G}}_{i,\{1-\alpha\}} \cap \mathcal{F}_{i,\{1-\alpha\}}$ is a free $O_{K,\{1-\alpha\}}$-module of rank $d_i - \delta_j$ if $j \leq s(i)$. Thus $V_{i+1}^{-1}(\mathcal{G}_{i,\{1-\alpha\}})^{(p)}$ is a free $O_{K,\{1-\alpha\}}$-module of rank $h - d_i + d_i - \delta_j = h - \delta_j$ containing $\widetilde{\mathcal{G}}_{i+1,\{1-\alpha\}}$. Since this last module also free of rank $h - \delta_j$, this is an equality. Note that one has $\mathcal{G}_{i,\{1-\alpha\}} = \mathcal{F}_{i,\{1-\alpha\}}$ and $\widetilde{\mathcal{G}}_{i+1,\{1-\alpha\}} = \widetilde{\mathcal{F}}_{i+1,\{1-\alpha\}}$ if $s(i) = j$.
By duality, if $j > s(i)$, then $\mathcal{G}_{i,\{1-\alpha\}} := \widetilde{\mathcal{G}}_{i,\{1-\alpha\}} + \mathcal{F}_{i,\{1-\alpha\}}$ is a free $O_{K,\{1-\alpha\}}$-module of rank $h - \delta_j + d_i$, and $F_{i+1}(\mathcal{G}_{i,\{1-\alpha\}})^{(p)} = \widetilde{\mathcal{G}}_{i+1,\{1-\alpha\}}$.
Assume now that $j \leq s(i)$. Then the map

$$V_{i+1} : \mathcal{E}_{i+1,\{1-\alpha\}}/\widetilde{\mathcal{G}}_{i+1,\{1-\alpha\}} \to (\mathcal{E}_{i,\{1-\alpha\}}/\widetilde{\mathcal{G}}_{i,\{1-\alpha\}})^{(p)}$$

can thus be identified with the map

$$V_{i+1} : \mathcal{E}_{C,i+1,\{1-\alpha\}} \to (\mathcal{E}_{C,i,\{1-\alpha\}})^{(p)}$$

Moreover, the determinant of this map has a valuation equal to $p \deg_i C^D \leq p\alpha$. This can be proved by filtering the group C by Raynaud group schemes and using Proposition 3.1.2. But this map can also be identified with the natural map

$$(\mathcal{F}_{i,\{1-\alpha\}}/\mathcal{G}_{i,\{1-\alpha\}})^{(p)} \to (\mathcal{E}_{i,\{1-\alpha\}}/\widetilde{\mathcal{G}}_{i,\{1-\alpha\}})^{(p)}$$

Since $p\alpha < 1 - \alpha$, one deduces that the determinant of the natural map

$$\mathcal{F}_{i,\{1-\alpha\}}/\mathcal{G}_{i,\{1-\alpha\}} \to \mathcal{E}_{i,\{1-\alpha\}}/\widetilde{\mathcal{G}}_{i,\{1-\alpha\}}$$

has a determinant of valuation $\deg_i C^D$.
By duality, if $j > s(i)$, the determinant of the map

$$F_{i+1} : (\widetilde{\mathcal{G}}_{i,\{1-\alpha\}})^{(p)} \to \widetilde{\mathcal{G}}_{i+1,\{1-\alpha\}}$$

has a valuation equal to $p \deg_i (C^\perp)^D = p(d_i - \delta_j + \deg_i C^D) \leq p\alpha$. This map can be identified with the natural map

$$(\widetilde{\mathcal{G}}_{i,\{1-\alpha\}})^{(p)} \to (\mathcal{G}_{i,\{1-\alpha\}}/\mathcal{F}_{i,\{1-\alpha\}})^{(p)}$$

Since $p\alpha < 1 - \alpha$, the determinant of the natural map

$$\widetilde{\mathcal{G}}_{i,\{1-\alpha\}} \to \mathcal{G}_{i,\{1-\alpha\}}/\mathcal{F}_{i,\{1-\alpha\}}$$

has valuation equal to $d_i - \delta_j + \deg_i C^D$.
We would be able to conclude if the filtrations $(\mathcal{G}_{i,\{1-\alpha\}})_{1\leq i\leq f}$ and $(\mathcal{F}^{[j]}_{i,\{1-\alpha\}})_{1\leq i\leq f}$ were equal. In the general case, using the same proof as in Proposition 2.2.2, one proves that

$$\mathcal{F}^{[j]}_{i,\{1-2\alpha\}} = \mathcal{G}_{i,\{1-2\alpha\}}$$

for all $1 \leq i \leq f$. Let i be an integer with $s(i) \leq j$. The map $H_i^{[j]}$ agrees with the map

$$\phi_i : \mathcal{F}_{i,\{1-\alpha\}}/\mathcal{G}_{i,\{1-\alpha\}} \to \mathcal{E}_{i,\{1-\alpha\}}/\widetilde{\mathcal{G}}_{i,\{1-\alpha\}}$$

after reduction modulo $\mathfrak{m}_{1-2\alpha}$. But we have seen that the valuation of the determinant of ϕ_i is $\deg_i C^D$. The reduction modulo $\mathfrak{m}_{1-2\alpha}$ of $H_i^{[j]}$ has thus a determinant of valuation $\deg_i C^D$. Since $\deg_i C^D \leq \alpha < 1 - 2\alpha$, one gets

$$\deg_i C^D = w_i^{[j]}$$

The case with $s(i) > j$ is similar or can be obtained by duality. □

Note that the relation in Theorem 3.2.3 due to Hernandez then follows from this last result together with Theorem 1.3.3.

Corollary 3.3.4. *Assume the existence of a strong canonical filtration*

$$0 \subset C_1 \subset \cdots \subset C_r \subset G[p]$$

Then for all $1 \leq i \leq f$ *and* $1 \leq j \leq r$*, we have*

$$\deg_i C_j^D = \max(\delta_j - d_i, 0) + w_i^{[j]}$$

In particular, the elements $(w_i^{[j]})_{1\leq i\leq f, 1\leq j\leq r}$ *are well defined.*

Remark 3.3.5. There is no assumption on p in this corollary (and in our definition of the canonical subgroups), unlike the result of Hernandez (Theorem 3.2.3).

Remark 3.3.6. The partial degrees of the graded parts C_k/C_{k-1} are related to the valuations of the invariants $m_i^{[k]}$ and $n_i^{[k]}$. More precisely, one has

$$\deg_i(C_k/C_{k-1})^D = v(m_i^{[k]})$$

for $1 \leq i \leq f$ and $1 \leq k \leq s(i)$ (with $C_0 := 0$), and

$$\deg_i(C_{k+1}/C_k) = v(n_i^{[k]})$$

for $1 \leq i \leq f$ and $s(i) \leq k \leq r$ (with $C_{r+1} := G[p]$).

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