THE $2A$-MAJORANA REPRESENTATIONS OF THE HARADA-NORTON GROUP

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Abstract. We show that all $2A$-Majorana representations of the Harada-Norton group F_5 have the same shape. If $\mathcal{R}$ is such a representation, we determine, using the theory of association schemes, the dimension and the irreducible constituents of the linear span U of the Majorana axes. Finally, we prove that, if $\mathcal{R}$ is based on the (unique) embedding of F_5 in the Monster, U is closed under the algebra product.

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1. Introduction

Let $(W, \cdot)$ be a real commutative algebra endowed with a scalar product $(\,,\,)_W$ and denote with $Aut(W)$ the group of algebra automorphisms of W that preserve the scalar product. We shall assume that, for every $u, v, w \in W$,

(M1) $(\,,\,)_W$ is *associative* , that is $(u \cdot v, w) = (u, v \cdot w)$,
(M2) the *Norton Inequality*, $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$, holds.

Recall that a *Majorana axis* of W (see [10, Definition 8.6.1] or, equivalently, [9, p.2423]) is a vector $a \in W$ such that

(M3) a has length 1,
(M4) the adjoint endomorphism $ad(a)$, induced by multiplication by a on the $\mathbb{R}$-vector space W, is semisimple with spectrum contained in $\{1, 0, 2^{-2}, 2^{-5}\}$,
(M5) a spans linearly the eigenspace relative to the eigenvalue 1 of $ad(a)$,
(M6) the linear transformation $a^\tau : W \to W$, that inverts the eigenvectors of $ad(a)$ relative to 2^{-5} and centralises the other eigenvectors, preserves the algebra product,
(M7) the linear transformation $a^\sigma : C_W(a^\tau) \to C_W(a^\tau)$, that inverts the eigenvectors of $ad(a)$ relative to 2^{-2} and centralises the other eigenvectors contained in $C_W(a^\tau)$, preserves the restriction to $C_W(a^\tau)$ of the algebra product.

Denote with $\mathcal{A}$ the set of Majorana axes of W. If $a \in \mathcal{A}$, the map a^τ is called a *Majorana involution* corresponding to a. Note that, by (M1) and (M4), W decomposes into an orthogonal sum of $ad(a)$-eigenspaces, hence (M6) actually implies that every Majorana involution is an element of $Aut(W)$. Let

$$\tau : \mathcal{A} \to Aut(W)$$

be the map $a \mapsto a^\tau$. Note that $\mathcal{A}$ is invariant under $Aut(W)$ and, for $a \in \mathcal{A}$ and $\delta \in Aut(W)$, we have

$$(a^\delta)^\tau = \delta^{-1} a^\tau \delta,$$

so that the set $\mathcal{A}^\tau$ of Majorana involutions is invariant under conjugation by elements of $Aut(W)$.

The fundamental examples of Majorana involutions are given by the $2A_M$-involutions (i.e. those centralised by the double cover of the Baby Monster) of the Monster group M acting on the 196884-dimensional Conway-Norton-Griess algebra W_M. A key result, in this context, is the Norton-Sakuma Theorem, that classifies and describes the *Norton-Sakuma algebras*, i.e. the algebras that are generated by a pair of Majorana axes [19] (see also [9, Section 2.6]). By S. Sakuma's classification, every Norton-Sakuma algebra is isomorphic to a subalgebra of W_M generated by a pair of Majorana axes a_0, a_1 corresponding via τ to $2A_M$-involutions in M. In [17] S. Norton proved that the latter algebras (hence all Norton-Sakuma algebras) fall into nine isomorphism types, labelled $1A$, $2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$, and $6A$, accordingly to the conjugacy class in the Monster of the element $a_0^\tau a_1^\tau$. Further, Norton produced, for each type, a basis (the *Norton basis*), the relative structure constants and the Gram matrix. Table 1 (which is an extract from Table 3 in [9]) summarises the results from the Norton-Sakuma Theorem we need for this paper: more precisely, for each pair of distinct Majorana axes a_0, a_1, we give the Norton basis of the algebra generated by a_0 and a_1, and the relevant (for this paper) scalar products (with the same scaling as in [9]):

Type	Norton basis	Scalar Products
$2A$	a_0, a_1, a_ρ	$(a_0, a_1)_W = \frac{1}{2^3}$
$2B$	a_0, a_1	$(a_0, a_1)_W = 0$
$3A$	a_0, a_1, a_{-1}, u_ρ	$(a_0, a_1)_W = \frac{13}{2^8}$
$3C$	a_0, a_1, a_{-1}	$(a_0, a_1)_W = \frac{1}{2^6}$
$4A$	$a_0, a_1, a_{-1}, a_2, v_\rho$	$(a_0, a_1)_W = \frac{1}{2^5}$
$4B$	$a_0, a_1, a_{-1}, a_2, a_{\rho^2}$	$(a_0, a_1)_W = \frac{1}{2^6}$
$5A$	$a_0, a_1, a_{-1}, a_2, a_{-2}, w_\rho$	$(a_0, a_1)_W = \frac{3}{2^7}$
$6A$	$a_0, a_1, a_{-1}, a_2,$ $a_{-2}, a_3, a_{\rho^3}, u_{\rho^2}$	$(a_0, a_1)_W = \frac{5}{2^8}$

TABLE 1

Here, for $\rho := a_0^\tau a_1^\tau$ in each Norton-Sakuma algebra,

- $a_{-1} := a_1^{\rho^{-1}}$, $a_{-2} := a_0^{\rho^{-1}}$, $a_2 := a_0^\rho$, $a_3 := a_1^\rho$, in particular they are Majorana axes.
- The vectors u_ρ, v_ρ, $\pm w_\rho$, resp. u_{ρ^2}, appearing in the algebras of type $3A$, $4A$, $5A$, resp. $6A$, are called $3A$-, $4A$-, $5A$-, resp. $3A$-, *axes* and, in each Norton-Sakuma algebra, they are defined as follows,

$$u_\rho := \frac{2^6}{3^3 5}(2a_0 + 2a_1 + a_{-1}) - \frac{2^{11}}{3^3 5} a_0 \cdot a_1,$$
$$v_\rho := a_0 + a_1 + \frac{1}{3}(a_{-1} + a_{-1}) - \frac{2^6}{3} a_0 \cdot a_1,$$
$$w_\rho := -\frac{1}{2^7}(3a_0 + 3a_1 - a_{-1} - a_{-1} - a_{-2}) + a_0 \cdot a_1,$$

$u_{\rho^2} := \frac{2^6}{3^3 5}(2a_0 + 2a_{-1} + a_{-2}) - \frac{2^{11}}{3^3 5} a_0 \cdot a_{-1}.$

The indexing with powers of ρ is justified by the fact that, in the action of M on W_M, for $3 \leq N \leq 5$, the NA-axes are essentially determined (up to the sign in the $5A$-case) by the cyclic groups $\langle \rho \rangle$ in M of order N (see [9, p. 2450]). It is not clear if that property follows from Axioms (M1)-(M7), therefore axiom (M8)(b) was added in [2] in the definition of Majorana representations.

- The vectors a_ρ, a_{ρ^2}, resp. a_{ρ^3} appearing in the algebras of type $2A$-, $4B$-, resp. $6A$ are further Majorana axes. As above, the indexing is suggested by the action of M on W_M since, in that case, whenever a_0 and a_1 generate a subalgebra of type $2A$, the product $\rho = a_0^\tau a_1^\tau$ is the Majorana involution corresponding to a_ρ. As in the previous paragraph, that property will be axiomatised in (M8)(a). Finally, by the Norton-Sakuma Theorem (see [9, Lemma 2.20 (iv) and (v)]), a_0 and a_2 (resp. a_0 and a_3) generate a subalgebra of type $2A$ in the algebra of type $4B$ (resp. $6A$) and, for $i \in \{2,3\}$, by the definition of a_i, the product $a_0^\tau a_i^\tau$ is equal to ρ^i.

The Norton-Sakuma Theorem inspired the definition of Majorana representations, introduced by A. A. Ivanov in [10] in order to provide an axiomtic framework for studying the actions of $2A_M$-generated subgroups of M on W_M.

Let G be a finite group, $\mathcal{T}$ a G-invariant set of involutions generating G,

$$\phi\colon G \to Aut(W)$$

a faithful representation of G on W, and

$$\psi\colon \mathcal{T} \to \mathcal{A}$$

be an injective map such that for every $g \in G$ and $t \in \mathcal{T}$,

$$(t^\psi)^\tau := t^\phi \tag{1}$$

and

$$(t^\psi)^{g^\phi} = (g^{-1}tg)^\psi. \tag{2}$$

The quintet

$$\mathcal{R} := (G, \mathcal{T}, W, \phi, \psi)$$

is called a *Majorana representation* (or, to put evidence on the set $\mathcal{T}$, a $\mathcal{T}$-*Majorana representation*) of G, if $\mathcal{R}$ satisfies the following condition (see [2, Axiom **M8**]):

(M8) (a) For t_1 and t_2 in $\mathcal{T}$, the Norton-Sakuma algebra generated by t_1^ψ and t_2^ψ has type $2A$ if and only if $t_1 t_2 \in \mathcal{T}$.
(b) Suppose that t_1, t_2, t_3, and t_4 are elements of $\mathcal{T}$ such that $t_1t_2 = t_3t_4$ and the subalgebras generated by t_1^ψ, t_2^ψ and t_3^ψ, t_4^ψ have both type $3A$, $4A$, or $5A$. Then $u_{(t_1t_2)^\phi} = u_{(t_3t_4)^\phi}$, $v_{(t_1t_2)^\phi} = v_{(t_3t_4)^\phi}$, or $w_{(t_1t_2)^\phi} = w_{(t_3t_4)^\phi}$, respectively.

Axiom (M8)(a) and Norton-Sakuma Theorem (see [9, Lemma 2.20]) imply that,

if t_1^ψ and t_2^ψ generate a Norton-Sakuma subalgebra of W of type $2A$, $4B$, or $6A$, then t_1t_2, $(t_1t_2)^2$, or $(t_1t_3)^3$ belongs to $\mathcal{T}$, and $(t_1t_2)^\psi$, $((t_1t_2)^2)^\psi$, or $((t_1t_3)^3)^\psi$ coincides with $a_{(t_1t_2)^\phi}$, $a_{((t_1t_2)^2)^\phi}$, or $a_{((t_1t_2)^3)^\phi}$, respectively.

An immediate consequence of that definition is that, given a Majorana representation

$$\mathcal{R} := (G, \mathcal{T}, W, \phi, \psi)$$

of a group G and a nonempty subset $\mathcal{T}_0$ of $\mathcal{T}$, such that $\mathcal{T}_0$ is $\langle \mathcal{T}_0 \rangle$-invariant, the quintet

$$\mathcal{R}_{\langle \mathcal{T}_0 \rangle} := (\langle \mathcal{T}_0 \rangle, \mathcal{T}_0, W, \phi|_{\langle \mathcal{T}_0 \rangle}, \psi|_{\mathcal{T}_0}) \tag{3}$$

is a $\mathcal{T}_0$-Majorana representation of $\langle \mathcal{T}_0 \rangle$. Further, if we replace W with the subalgebra $W_{\mathcal{T}_0}$ generated by the set of Majorana axes $\mathcal{T}_0^{\psi}$ in the quintet (3), we still have a Majorana representation of $\langle \mathcal{T}_0 \rangle$ provided $\langle \mathcal{T}_0 \rangle$ acts nontrivially on $W_{\mathcal{T}_0}$ (which is the case, e.g., when $\langle \mathcal{T}_0 \rangle$ has trivial centre). In particular, if ϵ is an embedding of a group H in M and H^{ϵ} is generated by a subset $\mathcal{T}$ of $2A_M$, then H inherits a $(\mathcal{T} \cap H^{\epsilon})^{\epsilon^{-1}}$-Majorana representation $\mathcal{R}_{\epsilon}$ obtained by composing ϵ with the restriction of $\mathcal{R}_M$ to H^{ϵ}. In that case, the Majorana representation $\mathcal{R}_{\epsilon}$ of H is said to be *based on the embedding* ϵ. In this paper, whenever a Majorana representation of a group G is based on an embedding ϵ in the Monster, we shall always identify G with G^{ϵ}.

For a pair (a, b) of elements in W, denote the subalgebra they generate with $\langle\langle a, b \rangle\rangle$. Let $\mathcal{R}$ be as above, the *shape* of $\mathcal{R}$ is a function $sh_{\mathcal{R}}$ from the set of the nondiagonal orbitals of G on $\mathcal{T}$ to the set of types of the Norton-Sakuma algebras so that

(1) $sh_{\mathcal{R}}((t, s)^G) = NX$ if and only if ts has order N and the algebra $\langle\langle t^{\psi}, s^{\psi} \rangle\rangle$ is a Norton-Sakuma algebra of type NX.

(2) $sh_{\mathcal{R}}$ must respect the embeddings of the algebras:

$$2A \hookrightarrow 4B, 2A \hookrightarrow 6A, 2B \hookrightarrow 4A, 3A \hookrightarrow 6A$$

in the sense that, for $t, r_1, r_2 \in \mathcal{T}$, if $\langle\langle t^{\psi} \rangle\rangle < \langle\langle t^{\psi}, r_1^{\psi} \rangle\rangle < \langle\langle t^{\psi}, r_2^{\psi} \rangle\rangle$, then

$$(sh_{\mathcal{R}}((t, r_1)^G),\ sh_{\mathcal{R}}((t, r_2)^G)) \in \{(2A, 4B), (2A, 6A), (2B, 4A), (3A, 6A)\}.$$

Remark: Clearly, if $\mathcal{T}_0$ is a $\langle \mathcal{T}_0 \rangle$-invariant nonempty subset of $\mathcal{T}$, the shape of $\mathcal{R}_{\langle \mathcal{T}_0 \rangle}$ is the restriction of $sh_{\mathcal{R}}$ to $\mathcal{T}_0 \times \mathcal{T}_0$.

Majorana representations of several groups have already been investigated (see [9], [11], [12, 13], [14], [5], [2] and [6]).

In this paper we study the $2A$-Majorana representations of the Harada-Norton group F_5, where $2A$ is the set of the involutions of F_5 whose centraliser is $(2HS) \cdot 2$, the double cover of the Higman-Sims group extended by its outer automorphism group of order 2. We shall show that every $2A$-Majorana representation of F_5 has the same shape as the Majorana representations of F_5 based on its embedding into M as the subgroup generated by the set of involutions in $2A_M$ that centralise an element of type $5A$ (here $2A = 2A_M \cap F_5$, see [4]). By [18, Theorem 21], that one is the unique embedding of F_5 into M (up to conjugation in M), hence, since F_5 is transitive on $2A$, there is (up to conjugation in M) only one Majorana representation of F_5 based on an embedding in M. We prove the following result.

Theorem 1. *Let W be as above and $\mathcal{R} := (F_5, 2A, W, \phi, \psi)$ be a $2A$-Majorana representation of F_5 on W. Then*

(i) $\mathcal{R}$ has the shape given in Table 3*;*

(ii) *The* $\mathbb{R}$*-linear span* $\langle 2A^\psi \rangle$ *of* $2A^\psi$ *has dimension* 18 316*;*
(iii) $\langle 2A^\psi \rangle$ *is the direct sum of three irreducible* $\mathbb{R}[F_5]$*-submodules of dimensions* 1*,* 8910 *and* 9405*, respectively;*
(iv) *if* $\mathcal{R}$ *is based on the embedding of* F_5 *in* M*, then* $W_{2A} = \langle 2A^\psi \rangle$*.*

Unless explicitly stated, for the remainder of this paper we shall stick to the notations introduced in this section. We shall also set $\mathcal{T} := 2A$.

2. The First Eigenmatrix

By [4, p. 166], we have $|\mathcal{T}| = 1539000$, and it seems hard, at present, to perform a direct computation of the dimension of the linear span of $\mathcal{T}^\psi$. We therefore apply the theory of association schemes as in [14] and [6] to reduce ourselves to a more manageable situation. The first step is to compute the first eigenmatrix of the association scheme relative to the permutation action of F_5 on $\mathcal{T}$ (see [1, pp. 59-60]). For that purpose, we need to recover some information about the action F_5 induces by conjugation on $\mathcal{T}$.

Let $n := |\mathcal{T}|$ and let $t_1, \ldots, t_n$ be the distinct elements of $\mathcal{T}$, so that

$$\mathcal{B} := (t_1, \ldots, t_n)$$

is an ordered basis for the complex permutation module V of F_5 on $\mathcal{T}$. With respect to $\mathcal{B}$, we identify $End_{\mathbb{C}}(V)$ with the set of $n \times n$ matrices with complex entries. Let $T_0, \ldots, T_8$ be the orbitals of F_5 on $\mathcal{T}$ and, for every $k \in \{0, \ldots, 8\}$, let A_k be the *adjacency matrix* associated to the orbital T_k, that is

$$(A_k)_{ij} = \begin{cases} 1 & \text{if the pair } (t_i, t_j) \text{ is in } T_k \\ 0 & \text{otherwise.} \end{cases}$$

By [1, Theorem 1.3], the 9-tuple $(A_0, \ldots, A_8)$ is a basis for the *centralizer algebra*

$$\mathcal{C} := End_{\mathbb{C}[F_5]}(V).$$

For $i, j, k \in \{0, \ldots, 8\}$, let p_{ij}^k be the number of elements z in $\mathcal{T}$ such that for a fixed pair (x, y) in T_k we have $(x, z) \in T_i$ and $(z, y) \in T_j$. By definition, the p_{ij}^k's are all non negative integers and, by [1, §2.2], they are the structure constants of $\mathcal{C}$ relative to the basis $(A_0, \ldots, A_8)$, that is

$$A_i A_j = \sum_{k=0}^{8} p_{ij}^k A_k. \tag{4}$$

The matrix B_i of size 9 whose j, k entry is p_{ij}^k is called *ith intersection matrix.* Clearly, B_i^t is the matrix associated to the endomorphism induced by A_i on $\mathcal{C}$ via left multiplication with respect to the basis $(A_0, \ldots, A_8)$, in particular B_i has the same eigenvalues as A_i. By [8, Lemma 2.18.1*(ii)*] we may choose the indexes of the orbitals $T_0, \ldots, T_8$ in such a way that T_0 is the diagonal orbital (hence B_0 is the identity matrix), T_1 is the non-diagonal orbital of smallest size, and the first intersection matrix B_1 is as follows:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 53 & 32 & 18 & 4 & 2 & 0 & 0 & 0 \\ 0 & 50 & 0 & 2 & 12 & 0 & 2 & 0 & 0 \\ 0 & 450 & 32 & 100 & 32 & 50 & 32 & 0 & 0 \\ 0 & 350 & 672 & 112 & 160 & 100 & 92 & 160 & 0 \\ 0 & 504 & 0 & 504 & 288 & 356 & 312 & 320 & 0 \\ 0 & 0 & 672 & 672 & 552 & 650 & 720 & 640 & 1280 \\ 0 & 0 & 0 & 0 & 360 & 250 & 240 & 288 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 128 \end{pmatrix}.$$

By [1, Theorem 3.1], we have that V decomposes into the direct sum

$$V = V_0 \oplus \ldots \oplus V_8 \tag{5}$$

of nine irreducible $\mathbb{C}[F_5]$-submodules. Since F_5 is transitive on $\mathcal{T}$, the subspace linearly spanned by the sum of all elements of $\mathcal{T}$ is the unique trivial submodule of V. As usual, we shall denote it by V_0. Since the action of F_5 on $\mathcal{T}$ is multiplicity free (see [8, Lemma 2.18.1.(ii)]), the V_j's are minimal common eigenspaces for the adjacency matrices A_i. It follows that there is a complex invertible matrix D that simultaneously diagonalises the matrices A_i's. By the definition of the adjacency matrices, we have that, for each i, the sums (say k_i) of the entries in each row of the matrices A_i are constant, whence V_0 is a k_i-eigenspace for A_i, for each i.

For $i, j \in \{0, \ldots, 8\}$, let p_{ij} be the eigenvalue of A_j on V_i. The 9×9 matrix $P := (p_{ij})$ is called the *first eigenmatrix* of the association scheme $(\mathcal{T}, \{T_0, \ldots, T_8\})$.

Lemma 2. *With the above notations,*

$$P = \begin{pmatrix} 1 & 1408 & 2200 & 35200 & 123200 & 354816 & 739200 & 277200 & 5775 \\ 1 & 128 & 200 & 0 & 1600 & -2304 & 0 & 0 & 375 \\ 1 & 28 & -50 & -50 & -100 & 396 & -750 & 450 & 75 \\ 1 & 16 & 4 & -56 & -136 & -288 & 504 & 0 & -45 \\ 1 & -32 & 40 & -80 & 80 & 576 & -240 & -360 & 15 \\ 1 & -47 & -50 & 250 & 350 & -504 & 0 & 0 & 0 \\ 1 & -112 & 300 & 1000 & -2200 & -864 & -1800 & 3600 & 75 \\ 1 & 208 & -50 & 2200 & -2800 & 2016 & 4200 & -6300 & 525 \\ 1 & 208 & 100 & 1000 & 1400 & 2016 & -4200 & 0 & -525 \end{pmatrix}$$

Proof. Note that, since A_0 is the identity matrix, $p_{i0} = 1$ for all i's. Straightforward computation shows that the eigenvalues of B_1 are 1408, 128, 28, 16, -32, -47, -112, 208, and 208, giving the first two columns of P. Set

$$(\lambda_0, \ldots, \lambda_8) = (1408, 128, 28, 16, -32, -47, -112, 208, 208).$$

For each $h \in \{0, \ldots, 8\}$, let $\mathcal{S}_h$ be the linear system

$$(B_1 - \lambda_h Id)\,{}^t(1, \lambda_h, x_2, \ldots, x_8) = 0 \tag{6}$$

in the indeterminates $x_2, \ldots, x_8$. Taking $i = 1$ in Equation (4) and multipling each term by D on the right and by D^{-1} on the left, we get

$$(D^{-1}A_1D)(D^{-1}A_jD) = \sum_{h=0}^{8} p_{1j}^h (D^{-1}A_hD). \tag{7}$$

Since the matrices $D^{-1}A_hD$ are diagonal with eigenvalues p_{kh} on the common eigenspaces V_k, for each $k \in \{0, \ldots, 8\}$, from Equation (7) we obtain that the relation

$$\lambda_k p_{kj} = \sum_{h=0}^{8} p_{1j}^h p_{kh} \tag{8}$$

holds for every $k \in \{0, \dots, 8\}$. Note that the second member is the jth entry of the vector $B_1{}^t(1, \lambda_k, p_{k2}, \dots, p_{k8})$, therefore Equation (8) implies that the 9-tuple $(1, \lambda_k, p_{k2}, \dots, p_{k8})$ is an eigenvector for B_1 relative to the eigenvalue λ_k, for every $k \in \{0, \dots, 8\}$. Since, for $k \neq 7, 8$, the eigenvalue λ_k has multiplicity 1, it follows that the first seven rows of the matrix P can be obtained computing the unique solution $(p_{k2}, \dots, p_{k8})$ of the system $\mathcal{S}_k$ for each $k \in \{1, \dots, 6\}$.

We are now left with the last two rows of the matrix P, corresponding to the eigenvalue 208 of B_1. The set of solutions of the system $\mathcal{S}_7$,

$$(B_1 - 208Id)\,{}^t(1, 208, x_2, \dots, x_8) = 0,$$

is

$$\{(25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x) \mid \text{ where } x \in \mathbb{R}\}.$$

Therefore, for suitable $x, y \in \mathbb{R}$, we can write the last two rows of the matrix P as follows

$$1, 208, 25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x$$

$$1, 208, 25 - \frac{y}{7}, 1600 + \frac{8y}{7}, -700 - 4y, 2016, 8y, -3150 - 6y, y.$$

Set $m_i = \dim_{\mathbb{R}}(V_i)$. Then $m_0 = 1$ and, for $1 \leq i \leq 6$, m_i can be computed from the rows of P using the following formula (see [1, Theorem 4.1]):

$$m_i = \frac{n}{\sum_{j=0}^{8} k_j^{-1} p_{ij}^2}$$

from which we get $m_1 = 16929$, $m_2 = 267520$, $m_3 = 653125$, $m_4 = 365750$, $m_5 = 214016$, $m_6 = 8910$, whence

$$m_7 + m_8 = n - \sum_{i=0}^{6} m_i = 12749.$$

Comparing that value with the decomposition of the permutation module of F_5 on $\mathcal{T}$ into irreducible submodules given in [8, Lemma 2.18.1.(*ii*)], we obtain that, modulo interchanging the indices 7 and 8,

$$m_7 = 3344 \text{ and } m_8 = 9405.$$

By the Column Orthogonality Relation of the first eigenmatrix,

$$\sum_{k=0}^{8} m_k p_{ki} p_{kj} = n k_i \delta_{ij}$$

(see [1, Theorem 3.5]), applied with $(i, j) = (0, 8)$ and $(i, j) = (8, 8)$, we get the quadratic system

$$\begin{cases} 3344x + 9405y = -3182025 \\ 3344x^2 + 9405y^2 = 3513943125 \end{cases}$$

whose solutions are

$$(x, y) = (525, -525) \text{ or } (x, y) = (1575/61, 62475/61).$$

By [3, Theorem 3.5(b)], the matrices A_i's are symmetric, since, by [4], the Frobenius-Schur indices of the irreducible constituents of the permutation character of F_5 on $\mathcal{T}$ is +1 (and the action is multiplicity free). Thus, recalling that the p_{ij}^k's are all non negative integers, in order to determine which of the two solutions is the right one, we may use the formula

$$p_{ij}^h = \frac{1}{nk_h} tr(A_i A_j A_h) \tag{9}$$

(see [1, Theorem 3.6(ii)]). Since the trace is invariant by matrix conjugation, $tr(A_i A_j A_h)$ can be obtained by multiplying, entry-wise, the ith, jth, and hth columns of the matrix P and adding the entries of the resulting column. In that way, we get that the entries p_{2j}^k are integers only in the case when $(x, y) = (525, -525)$. □

3. The shape

We continue with the notations of the last section. The next lemma recalls some known facts about conjugacy classes in M and F_5 (see [16, 15]). For the remainder of this paper let H be the centraliser in M of an A_5-subgroup of type $(2A, 3A, 5A)$. By [15], we have that $H \cong A_{12}$ and we may w.l.o.g. assume that F_5 centralizes a $5A$-element in that A_5-subgroup, in particular $H \leq F_5$.

Lemma 3. *Denoting the conjugacy classes of M and F_5 as in [4], the correspondences between the conjugacy classes of the elements of order less or equal to 6 in M, F_5 and H are as in Table 2:*

Conj. class in M	$2A$	$2B$	$3A$	$4A$	$4B$	$5A$	$6A$
Conj. class in F_5	$2A$	$2B$	$3A$	$4A$	$4B$	$5A$	$6A$
Cycle type in H	$2^2, 2^6$	2^4	$3, 3^2, 3^4$	$4^2, 4^2 \cdot 2^2$	$4 \cdot 2, 4 \cdot 2^2$	$5, 5^2$	$3 \cdot 2^2, 6 \cdot 2^3, 6^2, 3^2 \cdot 2^2$

Table 2

Let $(t_1, \ldots, t_n)$ be as in the previous section. For $i, j \in \{1, \ldots n\}$, set

$$\gamma_{ij} := (t_i^\psi, t_j^\psi)_W.$$

Lemma 4. *If (t_i, t_j) and (t_h, t_k) belong to the same orbital of F_5 on $\mathcal{T}$, then $\gamma_{ij} = \gamma_{hk}$*

Proof. That follows immediately from Equation (2) and the definition of γ_{ij}. □

Thus, we can set, for $k \in \{0, \ldots, 8\}$ and $(t, s) \in T_k$,

$$\gamma_k := (t^\psi, s^\psi)_W. \tag{10}$$

Lemma 5. *For every $x \in \{2^2, 3, 4 \cdot 2, 2^4, 5\}$ there are pairs of involutions of type 2^2 in A_{12} such that their product has cycle type x. Every element of cycle type $4^2 \cdot 2^2$ in A_{12} is the product of two elements of cycle type 2^6.*

Proof. That is an elementary computation (note that two elements of cycle type 2^6 whose product has cycle type $4^2 \cdot 2^2$ are explicitely given in the proof of Lemma 6). □

Lemma 6. *With the above notations, for every $k \in \{0, \dots, 8\}$ and $(t, s) \in T_k$, the scalar products γ_k's are given in the following table*

k	$\lvert t^{C_{F_5}(s)} \rvert$	$(st)^{F_5}$	$sh_{\mathcal{R}}(T_k)$	γ_k
0	1	1	–	1
1	1408	$5A$	$5A$	$3/2^7$
2	2200	$2A$	$2A$	$1/2^3$
3	35200	$3A$	$3A$	$13/2^8$
4	123200	$4B$	$4B$	$1/2^6$
5	354816	$5E$	$5A$	$3/2^7$
6	739200	$6A$	$6A$	$5/2^8$
7	277200	$4A$	$4A$	$1/2^5$
8	5775	$2B$	$2B$	0

TABLE 3

Proof. The first two columns of Table 3 follow from Lemma 2. The correspondence that associates to each orbital T_k of F_5 on $\mathcal{T}$ the F_5-conjugacy class x_k of the products ts, where $(t, s) \in T_k$, has been determined by Segev in [20], giving the third column.

Assume $sh_{\mathcal{R}}(T_k) = NX$, where $N \in \{1, \dots, 6\}$ and $X \in \{A, B, C\}$. By the definition of shape, for $(t, s) \in T_k$, we have that $|st| = N$. In particular, for k equal to 1, 5 and 6, we have that $sh_{\mathcal{R}}(T_k)$ is equal to $5A$, $5A$, and $6A$, respectively.

Let $k \in \{2, 3, 4, 8\}$. By the second and third rows of Table 2 and Lemma 5 there are involutions s and t of cycle type 2^2 in $\mathcal{T} \cap H$ such that $st \in x_k$, whence, by the first and third columns of Table 3,

$$(s, t) \in T_k \cap (H \times H).$$

By the remark in the introduction, we have that

$$sh_{\mathcal{R}}(T_k) = sh_{\mathcal{R}_H}((s, t)^H),$$

whence Lemma 8 and Table 10 in [6] give the entry in the fourth column corresponding to k.

Assume now $k = 7$. Choose the elements

$$s = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \text{ and } t = (1, 3)(2, 4)(5, 7)(6, 9)(8, 11)(10, 12)$$

in H. Then st has cycle type $4^2 \cdot 2^2$. By Table 2, s and t are contained in $\mathcal{T}$ and $(st)^{F_5} = 4A$, hence, by the third column of Table 3, $(s, t) \in T_7$ and, by the Norton-Sakuma Theorem, $sh_{\mathcal{R}}(T_7) \in \{4A, 4B\}$. By Equation (2),

$$(t^{\psi})^{(ts)^{\phi}} = (t^{ts})^{\psi} = (t^s)^{\psi},$$

so we have that t^{ψ} and $(t^s)^{\psi}$ are contained in the subalgebra generated by t^{ψ} and s^{ψ}, which is $\langle s, t \rangle$-invariant. Since tt^s has cycle type 2^4, by Table 2 it belongs to the class $2B$ of F_5, whence, by the third column of Table 3, $(t, t^s) \in T_8$ and the subalgebra generated by $t^{\psi}, (t^s)^{\psi}$ is of type $2B$, by the previous paragraph. By the second condition of the definition of the shape, $sh_{\mathcal{R}}(T_7) = 4A$.

Finally, the last column follows from Table 1. □

4. CLOSURE

Lemma 7. *Suppose that $\mathcal{R}$ is based on the embedding of F_5 in M. Then*

$$\langle \mathcal{T}^{\psi} \rangle = W_{\mathcal{T}}.$$

Proof. Let H be the subgroup of F_5 isomorphic to A_{12} defined as in the previous section. Let t, s be distinct elements of $\mathcal{T}$, set $\rho = (ts)^{\phi}$ and let N be the order of ρ. Let U be the Norton-Sakuma algebra generated by t^{ψ} and s^{ψ}, and let NX be its type. By Table 1, if NX is contained in $\{2A, 2B, 4B\}$, then U is linearly spanned by elements in $\mathcal{T}^{\psi}$, otherwise, by Lemma 6, $NX \in \{3A, 4A, 5A, 6A\}$ and U has a basis all of whose elements but the NX-axis are Majorana axes. Therefore, with the notations of Table 1, we may assume that $NX \in \{3A, 4A, 5A, 6A\}$ and show that, in all those cases, the NX-axes u_ρ, v_ρ, w_ρ, u_{ρ^2} are contained in $\langle \mathcal{T}^{\psi} \rangle$.

If ts has order $3, 4$, or 5, then by Lemma 3, there is $g \in F_5$, depending on ts, such that ts is an element of cycle type respectively 3, $4^2 \cdot 2^2$, and 5 in H^g. By Lemma 5, there are elements t' and s' of cycle type 2^2 or 2^6 in H^g such that $ts = t's'$. By Lemma 3, $(t')^{\psi}$ and $(s')^{\psi}$ generate a Norton-Sakuma algebra of the same type as U, thus, by Axiom (M8)(b), we have that $u_\rho = u_{(t's')^\phi}$, $v_\rho = v_{(t's')^\phi}$, and $w_\rho = w_{(t's')^\phi}$, respectively.

Assume $NX = 3A$. By [2, Corollary 3.2], $u_{(t's')^\phi}$ is a linear combination of elements of $(\mathcal{T} \cap H^g)^{\psi}$ and we are done.

Similarly, assume $NX = 4A$ (resp. $NX = 5A$). By [2], second formula in the abstract, or Section 6 (resp. Lemma 5.1), we have that $v_{(t's')^\phi}$ (resp. $w_{(t's')^\phi}$) is a linear combination of elements in $(\mathcal{T} \cap H^g)^{\psi}$ and $3A$-axes, and we are done by the previous case.

Finally assume $NX = 6A$. Then, by the remarks after Table 1, u_{ρ^2} is a $3A$-axis and again we are done by the $3A$ case. □

Note that in the previous proof we require that $\mathcal{R}$ is based on the embedding of F_5 in M only to deal with the case $4A$, all the other cases following from results of [2] that depend only on the shape of that representation of A_{12}.

5. Proof of Theorem 1

The first claim of Theorem 1 follows from Lemma 6 and the last is the content of Lemma 7. To prove the second and the third claims, let

$$\Gamma = (\gamma_{ij})$$

be the Gram matrix of $(\,,\,)_W$ associated to the n-tuple $(t_1^{\psi}, \ldots, t_n^{\psi})$. By an elementary result on Euclidean spaces, we have that

$$\operatorname{rank}(\Gamma) = \dim_{\mathbb{R}}(\langle \mathrm{t}^{\psi} \mid \mathrm{t} \in \mathcal{T} \rangle). \tag{11}$$

Since $T_0, \ldots, T_8$ is a partition of $\mathcal{T} \times \mathcal{T}$ and, by Equation (10), $\gamma_k = \gamma_{ij}$, for $(t_i, t_j) \in T_k$, we have that

$$\Gamma = \sum_{k=0}^{8} \gamma_k A_k. \tag{12}$$

.

Let D be as in Section 2. From Equation (12) we get:

$$\overline{\Gamma} := D^{-1} \Gamma D = \sum_{k=0}^{8} \gamma_k D^{-1} A_k D, \tag{13}$$

where all the matrices $\overline{\Gamma}$, and $\overline{A}_k := D^{-1} A_k D$ for $k \in \{0, \ldots, 8\}$, are diagonal. Now, clearly, the rank of Γ is equal to the rank of $\overline{\Gamma}$, hence (being $\overline{\Gamma}$ diagonal) to the number of nonzero entries of $\overline{\Gamma}$. By Lemma 6 (Table 3), Equation (13) becomes

$$\overline{\Gamma} = \overline{A}_0 + \frac{3}{2^7}\overline{A}_1 + \frac{1}{8}\overline{A}_2 + \frac{13}{2^8}\overline{A}_3 + \frac{1}{2^6}\overline{A}_4 + \frac{3}{2^7}\overline{A}_5 + \frac{5}{2^8}\overline{A}_6 + \frac{1}{2^5}\overline{A}_7 + 0\overline{A}_8,$$

which, by Lemma 2, gives the eigenvalues

$$70875/2, 0, 0, 0, 0, 0, 875/8, 0, 225/4$$

of $\overline{\Gamma}$ on the subspaces $V_0, \ldots, V_8$, respectively. Hence

$$\dim_{\mathbb{R}}(\langle \mathcal{T}^{\psi} \rangle) = m_0 + m_6 + m_8 = 1 + 9405 + 8910 = 18\,316.$$

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