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Spatial logic of tangled closure operators and modal mu-calculus

Robert Goldblatt [a], Ian Hodkinson [b,*]

[a] School of Mathematics and Statistics, Victoria University, Wellington, New Zealand
[b] Department of Computing, Imperial College London, UK

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ABSTRACT

There has been renewed interest in recent years in McKinsey and Tarski's interpretation of modal logic in topological spaces and their proof that S4 is the logic of any separable dense-in-itself metric space. Here we extend this work to the modal mu-calculus and to a logic of tangled closure operators that was developed by Fernández-Duque after these two languages had been shown by Dawar and Otto to have the same expressive power over finite transitive Kripke models. We prove that this equivalence remains true over topological spaces.

We extend the McKinsey–Tarski topological 'dissection lemma'. We also take advantage of the fact (proved by us elsewhere) that various tangled closure logics with and without the universal modality $\forall$ have the finite model property in Kripke semantics. These results are used to construct a representation map (also called a d-p-morphism) from any dense-in-itself metric space X onto any finite connected locally connected serial transitive Kripke frame.

This yields completeness theorems over X for a number of languages: (i) the modal mu-calculus with the closure operator $\Diamond$; (ii) $\Diamond$ and the tangled closure operators $\langle t\rangle$ (in fact $\langle t\rangle$ can express $\Diamond$); (iii) $\Diamond, \forall$; (iv) $\Diamond, \forall, \langle t\rangle$; (v) the derivative operator $\langle d\rangle$; (vi) $\langle d\rangle$ and the associated tangled closure operators $\langle dt\rangle$; (vii) $\langle d\rangle, \forall$; (viii) $\langle d\rangle, \forall, \langle dt\rangle$. Soundness also holds, if: (a) for languages with $\forall$, X is connected; (b) for languages with $\langle d\rangle$, X validates the well-known axiom G_1. For countable languages without $\forall$, we prove strong completeness. We also show that in the presence of $\forall$, strong completeness fails if X is compact and locally connected.

1. Introduction

Modal logic can be given semantics over topological spaces. In this setting, the modality $\Diamond$ can be interpreted in more than one way. The first and most obvious way is as *closure*. Writing $[\![\varphi]\!]$ for the set of points (in a topological model) at which a formula φ is true, $[\![\Diamond\varphi]\!]$ is defined to be the *closure* of $[\![\varphi]\!]$, so that $\Diamond\varphi$ holds at a point x if and only if every open neighbourhood of x contains a point y satisfying φ.

* Corresponding author.
E-mail addresses: Rob.Goldblatt@msor.vuw.ac.nz (R. Goldblatt), i.hodkinson@imperial.ac.uk (I. Hodkinson).
URLs: http://sms.vuw.ac.nz/~rob (R. Goldblatt), http://www.doc.ic.ac.uk/~imh/ (I. Hodkinson).

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Then, $\Box$ becomes the interior operator: $[\![\Box\varphi]\!]$ is the interior of $[\![\varphi]\!]$. Early studies of this semantics include [40,41,26–29].

In a seminal result, McKinsey and Tarski [27] proved that the logic of any given separable[1] dense-in-itself metric space in this semantics is S4: it can be axiomatised by the basic modal Hilbert system K augmented by the two axioms $\Box\varphi \to \varphi$ (T) and $\Box\varphi \to \Box\Box\varphi$ (4).

Motivated perhaps by the current wide interest in spatial logic, a wish to present simpler proofs in 'modern language', growing awareness of the work of particular groups such as Esakia's and Shehtman's, or involvement in new settings such as dynamic topology, interest in McKinsey and Tarski's result has revived in recent years. A number of new proofs of it have appeared, some for specific spaces or embodying other variants [30,4,1,31,39,24,17]. Very recently, strong completeness (every countably infinite S4-consistent set of modal formulas is satisfiable in every dense-in-itself metric space) was established by Kremer [20].

In this paper, we seek to extend McKinsey and Tarski's theorem to more powerful languages. We will extend the modal syntax in two separate ways: first, to the mu-calculus, which adds least and greatest fixed points to the basic modal language, and second, by adding an infinite sequence of new modalities $\Diamond_n$ of arity n ($n \geq 1$) introduced in the context of Kripke semantics by Dawar and Otto [7]. The semantics of $\Diamond_n$ is given by the mu-calculus formula

$$\Diamond_n(\varphi_1, \ldots, \varphi_n) \equiv \nu q \bigwedge_{1 \leq i \leq n} \Diamond(\varphi_i \wedge q),$$

for a new atom q not occurring in $\varphi_1, \ldots, \varphi_n$. The order and multiplicity of arguments of $\Diamond_n$ is immaterial, so we will abbreviate $\Diamond_n(\gamma_1, \ldots, \gamma_n)$ to $\langle t\rangle\{\gamma_1, \ldots, \gamma_n\}$. Fernández-Duque used this to give the modalities topological semantics, dubbed them *tangled closure modalities* (this is why we use the notation $\langle t\rangle$), and studied them in [9–12].

Dawar and Otto [7] showed that, somewhat surprisingly, the mu-calculus and the tangled modalities have exactly the same expressive power over finite Kripke models with transitive frames. We will prove that this remains true over topological spaces. So the tangled closure modalities offer a viable alternative to the mu-calculus in both these settings.

We go on to determine the logic of an arbitrary dense-in-itself metric space X in these languages. We will show that in the mu-calculus, the logic of X is axiomatised by a system called S4μ comprising Kozen's basic system for the mu-calculus augmented by the S4 axioms, and the tangled logic of X is axiomatised by a system called S4t similar to one in [10]. We will establish strong completeness for countable sets of formulas.

We will also consider the extension of the tangled language with the *universal modality,* '$\forall$'. (Earlier work on the universal modality in topological spaces includes [36,25].) This language can express connectedness: there is a formula C valid in precisely the connected spaces. Adding this and some standard machinery for $\forall$ to the system S4t gives a system called 'S4t.UC'. We will show that every S4t.UC-consistent formula is satisfiable in every dense-in-itself metric space. Thus, the logic of an arbitrary connected dense-in-itself metric space is S4t.UC. We also show that strong completeness fails in general, even for the modal language plus the universal modality.

A second and more powerful spatial interpretation of $\Diamond$ is as the *derivative operator.* Following tradition, when considering this interpretation we will generally write the modal box and diamond as $[d]$ and $\langle d\rangle$. In this interpretation, $[\![\langle d\rangle\varphi]\!]$ is defined to be the set of *strict limit points* of $[\![\varphi]\!]$: so $\langle d\rangle\varphi$ holds at a point x precisely when every open neighbourhood of x contains a point $y \neq x$ satisfying φ. The original closure diamond is expressible by the derivative operator: $\Diamond\varphi$ is equivalent in any topological model to $\varphi \vee \langle d\rangle\varphi$, and $\Box\varphi$ to $\varphi \wedge [d]\varphi$. So in passing to $\langle d\rangle$, we have not reduced the power of the language.

[1] The separability assumption was removed in [32].

Already in [27, Appendix I], McKinsey and Tarski discussed the derivative operator and asked a number of questions about it. It has since been studied by, among others, Esakia and his Tbilisi group ([8,3], plus many other publications), Shehtman [34,38], Lucero-Bryan [25], and Kudinov–Shehtman [23], section 3 of which contains a survey of results.

In the derivative semantics, determining the logic of a given dense-in-itself metric space is not a simple matter, for the logic can vary with the space. As McKinsey and Tarski observed, $\langle d\rangle\big((x \wedge \langle d\rangle\neg x) \vee (\neg x \wedge \langle d\rangle x)\big) \leftrightarrow \langle d\rangle x \wedge \langle d\rangle\neg x$ is valid in $\mathbb{R}^2$ but not in $\mathbb{R}$ ([34] attributes this observation to Kuratowski (1922)). This formula is valid in the same topological spaces as the formula G_1, where for each integer $n \geq 1$,

$$\mathrm{G}_n = \Big([d] \bigvee_{0\leq i\leq n} \Box Q_i\Big) \to \bigvee_{0\leq i\leq n} [d]\neg Q_i.$$

Here, $p_0, \dots, p_n$ are pairwise distinct atoms, and for $i = 0, \dots, n$,

$$Q_i = p_i \wedge \bigwedge_{i\neq j\leq n} \neg p_j.$$

The formulas G_n were introduced by Shehtman [34, p. 43]. A sufficient (but not necessary) condition for G_n to be valid in a space is that every open neighbourhood of an arbitrary point x contains an open neighbourhood N of x such that $N \setminus \{x\}$ can be partitioned into at most n non-empty open sets (cf. [34, lemma 2, p. 3]). So, for example, G_1 is valid in $\mathbb{R}^2$, and G_2 in $\mathbb{R}$. See Remark 8.6 below for further discussion.

We now recall some relevant results on $[d]$-logics.

R1. In [34], Shehtman proved that the logic of every separable zero-dimensional dense-in-itself metric space (such as $\mathbb{Q}$ and the Cantor space) is just KD4, axiomatised by the basic system K together with the axioms $\langle d\rangle\top$ (D) and $[d]p \to [d][d]p$ (4). This is the smallest possible logic of a dense-in-itself metric space in the derivative semantics.

R2. [34] also proved that the logic of $\mathbb{R}^n$ for finite $n \geq 2$ is KD4G$_1$, axiomatised by KD4 plus G_1. In fact, rather more is shown: see Remark 8.6 below.

R3. The logic of $\mathbb{R}$ was shown by Shehtman [38] and Lucero-Bryan [25] to be KD4G$_2$, and KD4G$_2$.UC if $\forall$ is added.

R4. In [16], R1 and R3 were extended to tangled closure modalities and the separability assumption in R1 was eliminated.

R5. [5] proved that there are continuum-many logics of subspaces of the rationals in the language with $[d]$.

R6. It is plain that $\mathrm{G}_1 \vdash \mathrm{G}_2 \vdash \mathrm{G}_3 \vdash \cdots$, so the logics KD4G$_1 \supseteq$ KD4G$_2 \supseteq \cdots$ form a decreasing chain, and by [25, corollary 3.11], its intersection is KD4.

Shehtman [34, problem 1] asked if KD4G$_1$ is the largest possible logic of a dense-in-itself metric space in the derivative semantics. In this paper, we answer Shehtman's question affirmatively: every KD4G$_1$-consistent formula of the language with $\langle d\rangle$ is satisfiable in every dense-in-itself metric space. Thus, the logic of every dense-in-itself metric space that validates G_1 is exactly KD4G$_1$. This strengthens R2 above. We also establish strong completeness for such spaces.

Adding the tangled closure operators, we prove similarly that the logic of every dense-in-itself metric space that validates G_1 is axiomatised by KD4G$_1t$ (including the tangle axioms). We also prove strong completeness.

Further adding the universal modality, we show similarly that KD4G$_1t$.UC (and KD4G$_1$.UC if the tangle closure operators are dropped) axiomatises the logic of every connected dense-in-itself metric space that validates G_1. Strong completeness fails in general, as a consequence of the proof that it already fails for the weaker language with $\Box$ and $\forall$.

The reader can find a summary of our results in Table 2 in section 10.

Our proof works in a fairly familiar way, similar in spirit to McKinsey and Tarski's original argument in [27]. There are two main steps.

1. We establish Tarski's 'dissection lemma' [41, satz 3.10], [27, theorem 3.5] and a variant of it.
2. These topological results are used to construct a map from an arbitrary dense-in-itself metric space onto any finite connected $\mathrm{KD4G_1}$ Kripke frame, preserving the required formulas.

Step 2, together with results from [14] and the mu-calculus canon establishing the *finite model property* for the various logics in Kripke semantics, proves completeness for all the languages, which is then lifted by a separate argument to strong completeness for languages without $\forall$.

It can be seen that our results concern the logic of each individual space within a large class of spaces (the dense-in-themselves metric spaces), rather than the logic of a large class of spaces, or of particular spaces such as $\mathbb{R}$. This is as in [27]. We do not assume separability, we consider languages that have not previously been much studied in the topological setting, and we obtain some results on strong completeness, a matter that has only recently been investigated in this arena.

2. Basic definitions

In this section, we lay out the main definitions, notation, and some basic results.

2.1. Notation for sets and binary relations

Let X, Y, Z be sets. We let $\wp(X)$ denote the power set (set of all subsets) of X. We write $X \setminus Y$ for $\{x \in X : x \notin Y\}$. Note that $(X \cap Y) \setminus Z = X \cap (Y \setminus Z)$, so we may omit the parentheses in such expressions. For a partial function $f : X \to Y$, we let $\operatorname{dom} f$ denote the domain of f, and $\operatorname{rng} f$ its range.

A *binary relation* on a set W is a subset of $W \times W$. Let R be a binary relation on W. We write any of $R(w_1, w_2)$, Rw_1w_2, w_1Rw_2 to denote that $(w_1, w_2) \in R$. We say that R is *reflexive* if $R(w, w)$ for all $w \in W$, and *transitive* if $R(w_1, w_2)$ and $R(w_2, w_3)$ imply $R(w_1, w_3)$. We write R^* for the *reflexive transitive closure* of R: the smallest reflexive transitive binary relation that contains R. We also write

$$\begin{aligned} R^{-1} &= \{(w_2, w_1) \in W \times W : R(w_1, w_2)\}, \\ R^{\circ} &= \{(w_1, w_2) \in W \times W : R(w_1, w_2) \wedge R(w_2, w_1)\} &= R \cap R^{-1}, \\ R^{\bullet} &= \{(w_1, w_2) \in W \times W : R(w_1, w_2) \wedge \neg R(w_2, w_1)\} &= R \setminus R^{-1}. \end{aligned}$$

The notation is loosely motivated by the traditional use of $\circ$ for a reflexive world and $\bullet$ for an irreflexive world in diagrams of frames in modal logic. For $w \in W$, we say that w is *reflexive* if Rww, and *irreflexive* otherwise. We let $R(w)$ denote the set $\{w' \in W : R(w, w')\}$, sometimes called the set of *R-successors* or *R-alternatives* of w. For $W' \subseteq W$, we write $R \restriction W'$ for the binary relation $R \cap (W' \times W')$ on W'.

We write $\mathbb{Z}$ for the set of integers, $\mathbb{Q}$ for the set of rational numbers, $\mathbb{R}$ for the set of real numbers, and ω for the first infinite ordinal. A set will be said to be *countable* if its cardinality is at most ω.

2.2. Kripke frames

A *(Kripke) frame* is a pair $\mathcal{F} = (W, R)$, where W is a non-empty set of 'worlds' (sometimes referred to as the *domain* of $\mathcal{F}$), and R is a binary relation on W. We attribute properties to a frame by the usual extrapolation from the frame's components. So, we say that $\mathcal{F}$ is *finite* if W is finite, *reflexive* if R is reflexive, *serial* if $R(w) \neq \emptyset$ for every $w \in W$, and *transitive* if R is transitive. Two frames are said to be *disjoint* if their respective sets of worlds are disjoint. And so on.

A *root of* $\mathcal{F}$ is an element $w \in W$ such that $W = R^*(w)$. Roots of a frame may not exist, nor be unique when they do. We say that $\mathcal{F}$ is *rooted* if it has a root. At the other end, an element $w \in W$ is said to be *R-maximal* if $R^\bullet(w) = \emptyset$. Such an element has no 'proper' R-successors, of which it is not itself an R-successor.

If $\mathcal{F}$ is transitive, a *cluster in* $\mathcal{F}$ is an equivalence class of the equivalence relation $R^\circ \cup \{(w, w) : w \in W\}$ on W. A cluster consists either of a single irreflexive world, in which case we say it is *degenerate,* or a non-empty set of reflexive worlds, in which case we say it is *nondegenerate.* For example, if w is R-maximal then $R^*(w)$ is a cluster.

A *subframe* of $\mathcal{F}$ is a frame of the form $\mathcal{F}' = (W', R \restriction W')$, for non-empty $W' \subseteq W$. It is simply a substructure of $\mathcal{F}$ in the usual model-theoretic sense. We call $\mathcal{F}'$ the *subframe of* $\mathcal{F}$ *based on* W'*.* We say that $\mathcal{F}'$ is a *proper subframe* of $\mathcal{F}$ if $W' \neq W$. We say that $\mathcal{F}'$ is a *generated* or *inner subframe* of $\mathcal{F}$ if $R(w) \subseteq W'$ for every $w \in W'$ — equivalently, $R \restriction W' = R \cap (W' \times W)$. For $w \in W$, we write:

- $\mathcal{F}(w)$ for the subframe $(R(w), R \restriction R(w))$ of $\mathcal{F}$ based on $R(w)$,
- $\mathcal{F}^*(w)$ for the subframe $(R^*(w), R \restriction R^*(w))$ of $\mathcal{F}$ generated by w.

For an integer $n \geq 1$, we say that $\mathcal{F}$ is *connected* if it is not the union of two pairwise disjoint generated subframes (recall that subframes are non-empty), and *locally connected* if for each $w \in W$, the subframe $\mathcal{F}(w)$ is connected. Note that $\mathcal{F}$ is connected iff the equivalence relation $(R \cup R^{-1})^*$ on W has a single equivalence class (i.e., it is the global relation $W \times W$). Every rooted frame is connected.

2.3. Topological spaces

We will assume some familiarity with topology, but we take a little time to reprise the main concepts and notation. A *topological space* is a pair (X, τ), where X is a set and $\tau \subseteq \wp(X)$ satisfies:

1. if $\mathcal{S} \subseteq \tau$ then $\bigcup \mathcal{S} \in \tau$,
2. if $\mathcal{S} \subseteq \tau$ is finite then $\bigcap \mathcal{S} \in \tau$, on the understanding that $\bigcap \emptyset = X$.

So τ is a set of subsets of X closed under unions and finite intersections. By taking $\mathcal{S} = \emptyset$, it follows that $\emptyset, X \in \tau$. The elements of τ are called *open subsets* of X, or just *open sets.* An *open neighbourhood* of a point $x \in X$ is an open set containing x. A subset $C \subseteq X$ is called *closed* if $X \setminus C$ is open, and *clopen* if it is both closed and open. The set of closed subsets of X is closed under intersections and finite unions. If O is open and C closed then $O \setminus C$ is open and $C \setminus O$ is closed.

We use the signs int, cl, $\langle d \rangle$ to denote the *interior, closure,* and *derivative* operators, respectively. So for $S \subseteq X$,

- $\operatorname{int} S = \bigcup\{O \in \tau : O \subseteq S\}$ — the largest open set contained in S,
- $\operatorname{cl} S = \bigcap\{C \subseteq X : C \text{ closed}, S \subseteq C\}$ — the smallest closed set containing S; we have $\operatorname{cl} S = \{x \in X : S \cap O \neq \emptyset$ for every open neighbourhood O of $x\}$,
- $\langle d \rangle S = \{x \in X : S \cap O \setminus \{x\} \neq \emptyset$ for every open neighbourhood O of $x\}$.

Then $\operatorname{int} S \subseteq S \subseteq \operatorname{cl} S \supseteq \langle d \rangle S$. For all subsets A, B of X, we have

$$\begin{aligned} \operatorname{cl}(A \cup B) &= \operatorname{cl} A \cup \operatorname{cl} B, \\ \langle d \rangle (A \cup B) &= \langle d \rangle A \cup \langle d \rangle B, \\ \operatorname{int}(A \cap B) &= \operatorname{int} A \cap \operatorname{int} B. \end{aligned}$$

That is, *closure and* $\langle d\rangle$ *are additive and interior is multiplicative.* It follows that each of these three operators is *monotonic:* if $A \subseteq B$ then $\mathrm{cl}\, A \subseteq \mathrm{cl}\, B$, $\langle d\rangle A \subseteq \langle d\rangle B$, and $\mathrm{int}\, A \subseteq \mathrm{int}\, B$. It is also standard that $\mathrm{int}(X \setminus A) = X \setminus \mathrm{cl}\, A$ and $\mathrm{cl}(X \setminus A) = X \setminus \mathrm{int}\, A$.

We follow standard practice and identify (notationally) the space (X, τ) with X. The reader should note that we do allow empty topological spaces, where $X = \emptyset$. This is particularly useful when dealing with subspaces.

A *subspace* of X is a topological space of the form $(Y, \{O \cap Y : O \in \tau\})$, for (possibly empty) $Y \subseteq X$. It is a subset of X, made into a topological space by endowing it with what is called the *subspace topology.* It is said to be an *open subspace* if Y is an open subset of X. As with X, we identify (notationally) the subspace with its underlying set, Y. We write $\mathrm{int}_Y, \mathrm{cl}_Y$ for the operations of interior and closure in the subspace Y. It can be checked that for every $S \subseteq Y$ we have $\mathrm{cl}_Y\, S = Y \cap \mathrm{cl}\, S$, and if Y is an open subspace then $\mathrm{int}_Y\, S = \mathrm{int}\, S$.

We will be considering various properties that a topological space X may have. We leave most of them for later, but we mention now that X is said to be *dense in itself* if no singleton subset is open, *connected* if it is not the union of two disjoint non-empty open sets, and *separable* if it has a countable subset D with $X = \mathrm{cl}\, D$. We say that X is *T1* if every singleton subset $\{x\}$ is closed, and T_D if the derivative $\langle d\rangle\{x\}$ of every singleton is closed, which is equivalent to requiring $\langle d\rangle\langle d\rangle\{x\} \subseteq \langle d\rangle\{x\}$. The T_D property, introduced in [2], is strictly weaker than T1.

2.4. Metric spaces

A *metric space* is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a 'distance function' (having nothing to do with the operator $\langle d\rangle$ above) satisfying, for all $x, y, z \in X$,

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ iff $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, y) + d(y, z)$ (the 'triangle inequality').

We assume some experience of working with this definition, in particular with the triangle inequality. Examples of metric spaces abound and include the real numbers $\mathbb{R}$ with the standard distance function $d(x, y) = |x - y|$, $\mathbb{R}^n$ with Pythagorean distance, etc. As usual, we often identify (notationally) (X, d) with X.

Let (X, d) be a metric space, and $x \in X$. For non-empty $S \subseteq X$, define

$$d(x, S) = \inf\{d(x, y) : y \in S\}.$$

We leave $d(x, \emptyset)$ undefined. For a real number $\varepsilon > 0$, we let $N_\varepsilon(x)$ denote the so-called 'open ball' $\{y \in X : d(x, y) < \varepsilon\}$. A metric space (X, d) gives rise to a topological space (X, τ_d) in which a subset $O \subseteq X$ is declared to be open (i.e., in τ_d) iff for every $x \in O$, there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq O$. In other words, the open sets are the unions of open balls. We frequently regard a metric space (X, d) equally as a topological space (X, τ_d). So, we will say that a metric space has a given topological property (such as being dense in itself) if the associated topological space has the property. As an example, it can be checked that every metric space is T_D.

A *subspace* of a metric space (X, d) is a pair of the form $(Y, d \restriction Y \times Y)$, where $Y \subseteq X$. It is plainly a metric space, and the topological space $(Y, \tau_{d \restriction Y \times Y})$ is a subspace of (X, τ_d).

2.5. Fixed points

Let X be a set and $f : \wp(X) \to \wp(X)$ be a map. We say that f is *monotonic* if $f(S) \subseteq f(S')$ whenever $S \subseteq S' \subseteq X$. By a well known theorem of Knaster and Tarski [42], actually formulated for complete lattices, every monotonic $f : \wp(X) \to \wp(X)$ has a *least and a greatest fixed point* — there is a unique $\subseteq$-minimal subset $L \subseteq X$ such that $f(L) = L$, and a unique $\subseteq$-maximal $G \subseteq X$ such that $f(G) = G$. We write $L = \mathrm{LFP}(f)$ and $G = \mathrm{GFP}(f)$.

There is a useful way to 'compute' these fixed points. A subset $S \subseteq X$ is said to be a *pre-fixed point* of f if $f(S) \subseteq S$, and a *post-fixed point* if $f(S) \supseteq S$. Now, the Knaster–Tarski theorem [42] states that $\mathrm{LFP}(f)$ is the intersection of all pre-fixed points of f, and dually for $\mathrm{GFP}(f)$:

$$\begin{aligned}\mathrm{LFP}(f) &= \bigcap\{S \subseteq X : f(S) \subseteq S\},\\ \mathrm{GFP}(f) &= \bigcup\{S \subseteq X : f(S) \supseteq S\}.\end{aligned}$$

For $f : \wp(X) \to \wp(X)$, define $f' : \wp(X) \to \wp(X)$ by $f'(S) = X \setminus f(X \setminus S)$. It is an exercise to check that f is monotonic iff f' is, and in that case, $\mathrm{GFP}(f) = X \setminus \mathrm{LFP}(f')$.

Least fixed points are used in the semantics of the mu-calculus, coming up next.

2.6. Languages

We assume some familiarity with modal languages and the mu-calculus. We fix an infinite set Var of *propositional variables,* or *atoms.* We will be considering various logical languages. The biggest of them is denoted by $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, which is a set of formulas defined as follows:

1. each $p \in \mathsf{Var}$ is a formula (of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$),
2. $\top$ is a formula,
3. if φ, ψ are formulas then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $\Box\varphi$, $[d]\varphi$, and $\forall\varphi$,
4. if Δ is a non-empty finite set of formulas then $\langle t\rangle\Delta$ and $\langle dt\rangle\Delta$ are formulas,
5. if $q \in \mathsf{Var}$ and φ is a formula that is *positive in* q (that is, every free occurrence of q as an atomic subformula of φ is in the scope of an even number of negations in φ; *free* means 'not in the scope of any μq in φ'), then $\mu q\varphi$ is a formula, in which all occurrences of q are *bound.* Bound atoms arise only in this way.

For formulas φ, ψ, and $q \in \mathsf{Var}$, the expression $\varphi(\psi/q)$ denotes the result of replacing every free occurrence of q in φ by ψ, where the result is well formed — that is, all of its subformulas of the form $\mu p\theta$ are such that θ is positive in p. We leave $\varphi(\psi/q)$ undefined if the result is not well formed. For example, if $\varphi = \mu p\, q$ then $\varphi(\neg p/q)$ is undefined, since $\mu p\,\neg p$ is not well formed.

We use standard abbreviations: $\bot$ denotes $\neg\top$, $(\varphi \vee \psi)$ denotes $\neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \to \psi)$ denotes $\neg(\varphi \wedge \neg\psi)$, $(\varphi \leftrightarrow \psi)$ denotes $(\varphi \to \psi) \wedge (\psi \to \varphi)$, $\Diamond\varphi$ denotes $\neg\Box\neg\varphi$, $\langle d\rangle\varphi$ denotes $\neg[d]\neg\varphi$, $\exists\varphi$ denotes $\neg\forall\neg\varphi$, and if φ is positive in q then $\nu q\varphi$ denotes $\neg\mu q\neg\varphi(\neg q/q)$ (this is well formed). We let $\Box^*\varphi$ abbreviate $\varphi \wedge \Box\varphi$, and $[d]^*\varphi$ abbreviate $\varphi \wedge [d]\varphi$. For a non-empty finite set $\Delta = \{\delta_1, \ldots, \delta_n\}$ of formulas, we let $\bigwedge\Delta$ denote $\delta_1 \wedge \ldots \wedge \delta_n$ and $\bigvee\Delta$ denote $\delta_1 \vee \ldots \vee \delta_n$ (the order and bracketing of the conjuncts and disjuncts will always be immaterial). We set $\bigwedge\emptyset = \top$ and $\bigvee\emptyset = \bot$. Parentheses will be omitted where possible, by the usual methods.

The connectives $\langle t\rangle, \langle dt\rangle$ are called *tangle connectives,* or (more fully) *tangled closure operators.*

We will be using various *sublanguages* of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, and they will be denoted in the obvious way by omitting prohibited operators from the notation. So for example, $\mathcal{L}^{\mu}_{\Box\forall}$ denotes the language consisting of all $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formulas that do not involve $[d]$, $\langle t\rangle$, or $\langle dt\rangle$.

2.7. Kripke semantics

An *assignment* or *valuation* into a frame $\mathcal{F} = (W, R)$ is a map $h : \mathsf{Var} \to \wp(W)$. A *Kripke model* is a triple $\mathcal{M} = (W, R, h)$, where (W, R) is a frame and h an assignment into it. The *frame of* $\mathcal{M}$ is (W, R), and we say that $\mathcal{M}$ is finite, reflexive, transitive, etc., if its frame is.

For every Kripke model $\mathcal{M} = (W, R, h)$ and every world $w \in W$, we define the notion $\mathcal{M}, w \models \varphi$ of a formula φ of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ being *true at w in $\mathcal{M}$*. The definition is by induction on φ, as follows:

1. $\mathcal{M}, w \models p$ iff $w \in h(p)$, for $p \in \mathsf{Var}$.
2. $\mathcal{M}, w \models \top$.
3. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$.
4. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.
5. $\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, v \models \varphi$ for every $v \in R(w)$.
6. The truth condition for $[d]\varphi$ is exactly the same as for $\Box\varphi$.[2]
7. $\mathcal{M}, w \models \forall\varphi$ iff $\mathcal{M}, v \models \varphi$ for every $v \in W$.
8. $\mathcal{M}, w \models \langle t\rangle\Delta$ iff there are worlds $w = w_0, w_1, \ldots \in W$ with $R(w_n, w_{n+1})$ for each $n < \omega$ and such that for each $\delta \in \Delta$ there are infinitely many $n < \omega$ with $\mathcal{M}, w_n \models \delta$.
9. The truth condition for $\langle dt\rangle\Delta$ is exactly the same as for $\langle t\rangle\Delta$.
10. The truth condition for $\mu q\varphi$ takes longer to explain. For an assignment $h : \mathsf{Var} \to \wp(W)$ and $S \subseteq W$, define a new assignment $h[S/q] : \mathsf{Var} \to \wp(W)$ by

$$h[S/q](p) = \begin{cases} S, & \text{if } p = q, \\ h(p), & \text{otherwise,} \end{cases}$$

for $p \in \mathsf{Var}$. Suppose that φ is positive in q and (inductively) that the set $[\![\varphi]\!]_h = \{w \in W : (W, R, h), w \models \varphi\}$ is well defined, for every assignment h into (W, R). Define a map $f : \wp(W) \to \wp(W)$ by

$$f(S) = [\![\varphi]\!]_{h[S/q]} \quad \text{for } S \subseteq W.$$

Since φ is positive in q, it can be shown that f is monotonic, so it has a least fixed point, $\mathrm{LFP}(f)$ (see section 2.5). We define $\mathcal{M}, w \models \mu q\varphi$ iff $w \in \mathrm{LFP}(f)$.

In the notation of the last clause, it can be checked that $\mathcal{M}, w \models \nu q\varphi$ iff $w \in \mathrm{GFP}(f)$.

For a set Γ of formulas, we write $\mathcal{M}, w \models \Gamma$ if $\mathcal{M}, w \models \gamma$ for every $\gamma \in \Gamma$.

A word on the semantics of $\langle t\rangle$ and $\langle dt\rangle$. Let us temporarily write $\varphi \equiv \psi$ to mean that $\mathcal{M}, w \models \varphi \leftrightarrow \psi$ for every *transitive* Kripke model $\mathcal{M} = (W, R, h)$ and every $w \in W$. Then it can be checked that for every non-empty finite set Δ of formulas,

$$\begin{aligned} \langle t\rangle\Delta &\equiv \nu q \bigwedge_{\delta\in\Delta} \Diamond(\delta \wedge q), \\ \langle dt\rangle\Delta &\equiv \nu q \bigwedge_{\delta\in\Delta} \langle d\rangle(\delta \wedge q), \end{aligned} \tag{2.1}$$

if $q \in \mathsf{Var}$ is a 'new' atom that does not occur in any formula in Δ. For more details, see Lemma 4.2. In a sense, (2.1) is the 'official' definition of the semantics of the tangle connectives, which boils down to clause 8 above in the case of transitive Kripke models.

[2] There is no distinction between $\Box$ and $[d]$ or between $\langle t\rangle$ and $\langle dt\rangle$ in Kripke semantics. This is not so in topological semantics, our next topic.

Let $\mathcal{M} = (W, R, h)$ be a Kripke model. A *generated submodel* of $\mathcal{M}$ is a model of the form $\mathcal{M}' = (W', R', h')$, where (W', R') is a generated subframe of (W, R) and $h' : \mathsf{Var} \to \wp(W')$ is given by $h'(p) = h(p) \cap W'$ for $p \in \mathsf{Var}$. The following is an easy extension to $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]}$ of a well known result in modal logic:

Lemma 2.1. *Let $\mathcal{M}' = (W', R', h')$ be a generated submodel of $\mathcal{M} = (W, R, h)$. Then for each $\varphi \in \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]}$ and $w \in W'$, we have $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w \models \varphi$.*

2.8. Topological semantics

Given a topological space X, an *assignment* into X is simply a map $h : \mathsf{Var} \to \wp(X)$. A *topological model* is a pair (X, h), where X is a topological space and h an assignment into X. We will also be considering topological models where Var is replaced by some other set of atoms. Details will be given later. As with Kripke models, we attribute a topological property to a topological model if the underlying topological space has the property.

For every topological model (X, h) and every point $x \in X$, we define $(X, h), x \models \varphi$, for a $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ, by induction on φ:

1. $(X, h), x \models p$ iff $x \in h(p)$, for $p \in \mathsf{Var}$.
2. $(X, h), x \models \top$.
3. $(X, h), x \models \neg\varphi$ iff $(X, h), x \not\models \varphi$.
4. $(X, h), x \models \varphi \wedge \psi$ iff $(X, h), x \models \varphi$ and $(X, h), x \models \psi$.
5. $(X, h), x \models \Box\varphi$ iff there is an open neighbourhood O of x with $(X, h), y \models \varphi$ for every $y \in O$.
6. $(X, h), x \models [d]\varphi$ iff there is an open neighbourhood O of x with $(X, h), y \models \varphi$ for every $y \in O \setminus \{x\}$. We do not require φ to hold at x itself.
7. $(X, h), x \models \forall\varphi$ iff $(X, h), y \models \varphi$ for every $y \in X$.
8. For a non-empty finite set Δ of formulas for which we have inductively defined semantics, write $[\![\delta]\!] = \{x \in X : (X, h), x \models \delta\}$, for each $\delta \in \Delta$. Then define:
   - $(X, h), x \models \langle t\rangle\Delta$ iff there is some $S \subseteq X$ such that $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$,
   - $(X, h), x \models \langle dt\rangle\Delta$ iff there is some $S \subseteq X$ such that $x \in S \subseteq \bigcap_{\delta\in\Delta} \langle d\rangle([\![\delta]\!] \cap S)$.
9. Suppose that φ is positive in q and (inductively) that $[\![\varphi]\!]_h = \{x \in X : (X, h), x \models \varphi\}$ is well defined, for every assignment h into X. Define a map $f : \wp(X) \to \wp(X)$ by
$$f(S) = [\![\varphi]\!]_{h[S/q]} \quad \text{for } S \subseteq X,$$
where $h[S/q]$ is defined as in Kripke semantics. Again, f is monotonic, and we define $(X, h), x \models \mu q\varphi$ iff $x \in \mathrm{LFP}(f)$.

The definition makes sense but has no content if X is empty: there are no points $x \in X$ to evaluate at. Writing $[\![\varphi]\!] = \{x \in X : (X, h), x \models \varphi\}$, we have $[\![\Box\varphi]\!] = \mathrm{int}([\![\varphi]\!])$, $[\![\Diamond\varphi]\!] = \mathrm{cl}([\![\varphi]\!])$, and $[\![\langle d\rangle\varphi]\!] = \langle d\rangle([\![\varphi]\!])$ for each φ, h. Again, $[\![\nu q\varphi]\!] = \mathrm{GFP}(f)$, where φ, f are as in the last clause.

As with Kripke semantics, for a set Γ of formulas we write $(X, h), x \models \Gamma$ if $(X, h), x \models \gamma$ for every $\gamma \in \Gamma$.

Remark 2.2. Again we briefly discuss the semantics of $\langle t\rangle$ and $\langle dt\rangle$ (see clause 8 above). With $\varphi \equiv \psi$ redefined to mean that $(X, h), x \models \varphi \leftrightarrow \psi$ for every topological model (X, h) and $x \in X$, the equivalences in (2.1) above continue to hold, and indeed they motivate clause 8. However, there is a perhaps more intuitive meaning for $\langle t\rangle$ and $\langle dt\rangle$ in terms of *games,* which are used extensively in the mu-calculus. Let players $\forall$, $\exists$ play a game of length ω on X. Initially, the position is x. In each round, if the current position is $y \in X$, player $\forall$ chooses an open neighbourhood O of y and a formula $\delta \in \Delta$. Player $\exists$ must select a point $z \in O$

at which δ is true (and with $z \neq y$ in the case of $\langle dt\rangle$). If she cannot, player $\forall$ wins. That is the end of the round, and the next round commences from position z. Player $\exists$ wins if she survives every round. It can be checked that $(X, h), x \models \langle t\rangle\Delta$ (respectively, $(X, h), x \models \langle dt\rangle\Delta$) iff $\exists$ has a winning strategy in this game (respectively, the game where she must additionally choose $z \neq y$).

As an aside, a transitive Kripke frame (W, R) can be made into a topological space $X = (W, \tau)$, where $\tau = \{S \subseteq W : R(x) \subseteq S \text{ for every } x \in S\}$. Then for each assignment $h : \mathsf{Var} \to \wp(W)$ and $w \in W$, if R is reflexive then $(W, R, h), w \models \varphi$ iff $(X, h), w \models \varphi$ for every $\varphi \in \mathcal{L}^{\mu\langle t\rangle}_{\Box\forall}$, and if every $v \in W$ is irreflexive then $(W, R, h), w \models \varphi$ iff $(X, h), w \models \varphi$ for every $\varphi \in \mathcal{L}^{\mu\langle dt\rangle}_{[d]\forall}$.

2.9. Topological semantics in open subspaces

Let X be a topological space and Y a subspace of X. Each assignment $h : \mathsf{Var} \to \wp(X)$ into X induces an assignment h_Y into Y, via $h_Y(p) = Y \cap h(p)$, for each $p \in \mathsf{Var}$. Thus, we can evaluate formulas at points in Y in both (X, h) and (Y, h_Y). Because the semantics of the connectives $\Box, [d], \langle t\rangle, \langle dt\rangle$ depend on only arbitrarily small open neighbourhoods of the evaluation point, it is easily seen that if Y is an *open* subspace of X, we get the same result for every formula not involving $\forall$. That is, the following analogue of Lemma 2.1 holds:

Lemma 2.3. *Whenever Y is an open subspace of X, we have $(X, h), y \models \varphi$ iff $(Y, h_Y), y \models \varphi$, for every $y \in Y$ and $\varphi \in \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]}$.*

(This holds vacuously if Y is empty.)

2.10. Hilbert systems

These are familiar, and we will be informal. A *Hilbert system* H in a given language $\mathcal{L} \subseteq \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ is a set of *axioms,* which are $\mathcal{L}$-formulas, and *inference rules,* which have the form

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}, \tag{2.2}$$

for $\mathcal{L}$-formulas $\varphi_1, \ldots, \varphi_n, \psi$. A *derivation in H (of length l)* is a sequence $\varphi_1, \ldots, \varphi_l$ of $\mathcal{L}$-formulas such that each φ_i $(1 \leq i \leq l)$ is either an H-axiom or is derived from earlier φ_j by an H-rule — that is, there are $1 \leq j_1, \ldots, j_n < i$ such that

$$\frac{\varphi_{j_1}, \ldots, \varphi_{j_n}}{\varphi_i}$$

is an instance of a rule of H.

A *theorem of* H is a formula that occurs in some derivation in H. An *H-logic* is a set of $\mathcal{L}$-formulas that contains all H-axioms and is closed under all H-rules. The set of theorems of H is the smallest H-logic. Sometimes we identify (notationally) H with this set, or present H implicitly by defining an H-logic.

A formula φ is *consistent* with H if $\neg\varphi$ is not a theorem of H. A set Γ of formulas is *consistent* with H if $\bigwedge \Gamma_0$ is consistent with H, for every finite $\Gamma_0 \subseteq \Gamma$.

2.11. Satisfiability, validity, equivalence

Let $\mathcal{F} = (W, R)$ be a Kripke frame and X a topological space. A set Γ of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formulas is said to be *satisfiable in* $\mathcal{F}$ if there exist an assignment h into $\mathcal{F}$ and a world $w \in W$ such that $(W, R, h), w \models \Gamma$. Similarly, Γ is said to be *satisfiable in* X if there exist an assignment h into X and a point $x \in X$ such that $(X, h), x \models \Gamma$.

Let φ be an $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula. We say that φ is *satisfiable in* $\mathcal{F}$*, or in* X*,* if the set $\{\varphi\}$ is so satisfiable. We say that φ is *valid in* $\mathcal{F}$ *(respectively, in* X*)* if $\neg\varphi$ is not satisfiable in $\mathcal{F}$ (respectively, in X). We may also say in this case that $\mathcal{F}$ or X *validates* φ.

We also say that φ is *equivalent* to a formula ψ in $\mathcal{F}$ (respectively, X) if $\varphi \leftrightarrow \psi$ is valid in $\mathcal{F}$ (respectively, X).

2.12. Logics

Let $\mathcal{K}$ be a class of Kripke frames or topological spaces. In the context of a given language $\mathcal{L} \subseteq \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, the *(*$\mathcal{L}$*)-logic of* $\mathcal{K}$ is the set of all $\mathcal{L}$-formulas that are valid in every member of $\mathcal{K}$. A Hilbert system H for $\mathcal{L}$ whose set of theorems is T, say, is said to be

- *sound over* $\mathcal{K}$ if T is a subset of the logic of $\mathcal{K}$ (all H-theorems are valid in $\mathcal{K}$),
- *weakly complete*, or simply *complete, over* $\mathcal{K}$ if T contains the logic of $\mathcal{K}$ (all $\mathcal{K}$-valid formulas are H-theorems),
- *strongly complete over* $\mathcal{K}$ if every countable H-consistent set Γ of $\mathcal{L}$-formulas is satisfiable in some structure in $\mathcal{K}$. Recall that in this paper, 'countable' means 'of cardinality at most ω'. The restriction to countable sets will be discussed at the beginning of section 10.2.

The logic of a single frame $\mathcal{F}$ is defined to be the logic of the class $\{\mathcal{F}\}$; similar definitions are used for the other terms here.

We say that a Kripke frame $\mathcal{F}$ is an H *frame,* or that $\mathcal{F}$ *validates* H*,* if H is sound over $\mathcal{F}$. To establish this, it is enough to check that each axiom of H is valid in $\mathcal{F}$, and that each rule of H preserves $\mathcal{F}$-validity (in the notation in (2.2) above, this means that if $\varphi_1, \ldots, \varphi_n$ are valid in $\mathcal{F}$ then so is ψ).

It can be checked that H is weakly complete over $\mathcal{K}$ iff every *singleton* H-consistent set is satisfiable in some structure in $\mathcal{K}$. Hence, every strongly complete Hilbert system is also weakly complete. The main aim of this paper is to provide Hilbert systems that are (where possible) sound and strongly complete over various topological spaces, with respect to various sublanguages of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$.

3. Hilbert systems for mu-calculus

We now present a very brief diversion on a Hilbert system 'S4μ' for the mu-calculus that is (sound and) complete over the class of finite reflexive transitive Kripke frames. This will be used in two places: in Theorem 8.3, to show that in the language $\mathcal{L}^{\mu}_{\Box}$, the system S4μ is complete over every dense-in-itself metric space; and in Corollary 4.7, together with deep results of Dawar–Otto [7] and the evident soundness of S4μ over topological spaces, to show that $\mathcal{L}^{\mu}_{\Box}$ is no more expressive than $\mathcal{L}^{\langle t\rangle}_{\Box}$ over topological spaces, a fact used in Theorem 9.3 to establish *strong* completeness of S4μ over every dense-in-itself metric space in the language $\mathcal{L}^{\mu}_{\Box}$.

In this section, all formulas are $\mathcal{L}^{\mu}_{\Box}$-formulas, and all Hilbert systems are for this language.

Definition 3.1. Consider the Hilbert systems:

K: the axioms comprise (i) all instances of propositional tautologies (e.g., $\varphi \to (\psi \to \varphi)$, etc.) and (ii) all formulas of the form $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (the so-called 'normality' scheme). The inference rules are:

$$\text{modus ponens: } \frac{\varphi,\ \varphi \to \psi}{\psi} \qquad \Box\text{-generalisation: } \frac{\varphi}{\Box\varphi}$$

The well known substitution rule $\dfrac{\varphi}{\varphi(\psi/q)}$ is not always sound in the mu-calculus and is not needed in other systems, so we omit it.

Kμ: this is K augmented with the following for each formula φ positive in q:

- *fixed point axiom:* $\varphi(\mu q\varphi/q) \to \mu q\varphi$, provided that no free occurrence of an atom in $\mu q\varphi$ gets bound in $\varphi(\mu q\varphi/q)$ — consequently, $\varphi(\mu q\varphi/q)$ is well formed. The idea is roughly that $\mu q\varphi$ is a pre-fixed point of φ.
- *fixed point rule:* $\dfrac{\varphi(\psi/q) \to \psi}{\mu q\varphi \to \psi}$, provided that no free occurrence of an atom in ψ gets bound in $\varphi(\psi/q)$ — hence, $\varphi(\psi/q)$ is well formed. The idea this time is roughly that $\mu q\varphi$ is the least pre-fixed point of φ.

We write $\mathrm{K}\mu \vdash \varphi$ if φ is a theorem of this system. It is well known (see, e.g., [6, §6]) that the system is equivalent to the original equational system of Kozen [19].

K4μ: this is Kμ plus the '4' scheme $\Box\varphi \to \Box\Box\varphi$. We write $\mathrm{K4}\mu \vdash \varphi$ if φ is a theorem of this system. K4μ is not needed in our spatial completeness results, but it is used in proving equivalence of $\mathcal{L}^{\mu}_{[d]}$ and $\mathcal{L}^{\langle dt\rangle}_{[d]}$ over T_D spaces (Remark 4.8).

S4μ: this is Kμ plus the S4 schemes $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$. We write $\mathrm{S4}\mu \vdash \varphi$ if φ is a theorem of this system.

The following combines some famous and difficult work in the mu-calculus.

Fact 3.2 *([19,44,18]).* Kμ is sound and complete over the class of all finite Kripke frames.

We are going to extend it to show that S4μ is sound and complete over the class of finite reflexive transitive frames (and, much later, over every dense-in-itself metric space). First, a form of the substitution rule can be established.

Lemma 3.3. *Suppose φ, ψ are formulas such that for each atom s occurring free in ψ, there is no subformula of φ of the form $\mu s\theta$. If $\mathrm{S4}\mu \vdash \varphi$, then $\mathrm{S4}\mu \vdash \varphi(\psi/p)$ for any atom p.*

Proof (sketch). Let φ, ψ, p be as stipulated. For a formula α, write $\alpha^\dagger = \alpha(\psi/p)$. We show that $\mathrm{S4}\mu \vdash \varphi \Rightarrow \mathrm{S4}\mu \vdash \varphi^\dagger$ (when the stipulation holds) by induction on the length of a derivation of φ in S4μ.

If φ is an instance of one of the S4 axiom schemes, then $\varphi^\dagger$ is an instance of the same scheme, so $\mathrm{S4}\mu \vdash \varphi^\dagger$. Suppose that φ is an instance $\alpha(\mu q\alpha/q) \to \mu q\alpha$ of the fixed point axiom. If $p = q$, then $\varphi^\dagger = \varphi$, so certainly $\mathrm{S4}\mu \vdash \varphi^\dagger$. If $p \neq q$, then $\varphi^\dagger = \alpha^\dagger(\mu q\alpha^\dagger/q) \to \mu q\alpha^\dagger$, another instance of the fixed point axiom; hence again, $\mathrm{S4}\mu \vdash \varphi^\dagger$.

If φ is obtained by MP from formulas ψ, $\psi \to \varphi$ occurring earlier in the derivation, then inductively, $\mathrm{S4}\mu \vdash \psi^\dagger$ and $\mathrm{S4}\mu \vdash (\psi \to \varphi)^\dagger$ — that is, $\mathrm{S4}\mu \vdash \psi^\dagger \to \varphi^\dagger$. That $\mathrm{S4}\mu \vdash \varphi^\dagger$ now follows by MP. Similarly, if $\varphi = \Box\psi$, where ψ occurs earlier in the derivation, then inductively, $S4\mu \vdash \psi^\dagger$, so by generalisation, $S4\mu \vdash \Box\psi^\dagger$ — that is, $S4\mu \vdash \varphi^\dagger$, as required.

Suppose that φ is derived by the fixed point rule, so that $\varphi = \mu q\alpha \to \beta$ for some α, β, q meeting the condition of the rule, and $\alpha(\beta/q) \to \beta$ occurs earlier in the derivation. If s occurs free in ψ then there is no μs in $\mu q\alpha \to \beta$, so none in $\alpha(\beta/q) \to \beta$ either. So the inductive hypothesis applies, to give $\mathrm{S4}\mu \vdash (\alpha(\beta/q) \to \beta)^\dagger$. Let us evaluate this. If $p = q$, it is $\mathrm{S4}\mu \vdash \alpha(\beta^\dagger/q) \to \beta^\dagger$. By our stipulation, the fixed point rule applies, giving $\mathrm{S4}\mu \vdash \mu q\alpha \to \beta^\dagger$. But $(\mu q\alpha)^\dagger = \mu q\alpha$. So $\mathrm{S4}\mu \vdash \varphi^\dagger$ as required. If instead $p \neq q$, then it is $\mathrm{S4}\mu \vdash \alpha^\dagger(\beta^\dagger/q) \to \beta^\dagger$. Again, the rule applies, to give $\mathrm{S4}\mu \vdash \mu q\alpha^\dagger \to \beta^\dagger$. But this is exactly $\mathrm{S4}\mu \vdash \varphi^\dagger$. $\square$

Plainly, the analogous result for $\mathrm{K4}\mu$ and indeed $\mathrm{K}\mu$ can be proved in the same way.

Definition 3.4. For a formula φ, define a new formula φ^* by induction:

- $p^* = p$ for $p \in \mathsf{Var}$;
- $-^*$ *commutes* with the boolean connectives and μ. That is, $\top^* = \top$, $(\neg\varphi)^* = \neg\varphi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, and $(\mu q\varphi)^* = \mu q\varphi^*$.
- $(\Box\varphi)^* = \nu q(\varphi^* \wedge \Box q)$, where $q \in \mathsf{Var}$ is a 'new' atom not occurring in φ^*.

The formula φ^* is plainly well formed, for all $\varphi \in \mathcal{L}^\mu_\Box$.

Lemma 3.5. *Let φ be any formula. Then for every Kripke model (W, R, h) and $w \in W$, we have $(W, R, h), w \models \varphi^*$ iff $(W, R^*, h), w \models \varphi$, where (recall) R^* is the reflexive transitive closure of R.*

Proof. The proof is by induction on φ. The atomic and boolean cases are easy. Assuming the result for φ, it is a well-known exercise in the mu-calculus to check that $(W, R, h), w \models (\Box\varphi)^*$ iff $(W, R, h), u \models \varphi^*$ for every $u \in R^*(w)$. Inductively, this is iff $(W, R^*, h), u \models \varphi$ for every $u \in R^*(w)$, iff $(W, R^*, h), w \models \Box\varphi$ as required.

Finally assume that the result holds for φ, positive in q, for every Kripke model. For a formula ψ and Kripke model (W, R, h), write $[\![\psi]\!]_{(W,R,h)} = \{w \in W : (W, R, h), w \models \psi\}$. Then $(W, R, h), w \models (\mu q\varphi)^*$ iff $(W, R, h), w \models \mu q\varphi^*$, iff w is in the least fixed point of the map $f : \wp(W) \to \wp(W)$ given by $f(S) = [\![\varphi^*]\!]_{(W,R,h[S/q])}$. But inductively, $f(S) = [\![\varphi]\!]_{(W,R^*,h[S/q])}$. So this is iff $(W, R^*, h), w \models \mu q\varphi$ as required. $\square$

Remark 3.6. For each formula φ, define a formula φ^+ in the same way as for φ^* but using the clause $(\Box\varphi)^+ = \nu q\Box(\varphi^+ \wedge q)$. It can then be shown that $(W, R, h), w \models \varphi^+$ iff $(W, R^+, h), w \models \varphi$, for every formula φ, Kripke model (W, R, h), and $w \in W$, where R^+ is the transitive closure of R.

Definition 3.7. For a formula $\mu q\varphi$, define formulas φ^n $(n < \omega)$ by induction: $\varphi^0 = \bot$, and $\varphi^{n+1} = \varphi(\varphi^n/q)$. (Below, the relevant q will be determined by context.)

Lemma 3.8. *For every formula $\mu q\varphi$, if φ^2 is well formed then $\mathrm{K}\mu \vdash \varphi^n \to \mu q\varphi$ for every $n < \omega$.*

Proof. By induction on n. (It can be checked that every φ^n is well formed.) The case $n = 0$ is trivial. Assuming inductively that $\mathrm{K}\mu \vdash \varphi^n \to \mu q\varphi$, we can use [19, proposition 5.7(iii)] ('monotonicity') to obtain $\mathrm{K}\mu \vdash \varphi(\varphi^n/q) \to \varphi(\mu q\varphi/q)$ — that is, $\mathrm{K}\mu \vdash \varphi^{n+1} \to \varphi(\mu q\varphi/q)$. By the fixed point axiom, $\mathrm{K}\mu \vdash \varphi(\mu q\varphi/q) \to \mu q\varphi$, and $\mathrm{K}\mu \vdash \varphi^{n+1} \to \mu q\varphi$ now follows by propositional reasoning. $\square$

Lemma 3.9. $\mathrm{S4}\mu \vdash \varphi \leftrightarrow \varphi^*$ *for every* φ.

Proof. Again, the proof is by induction on φ. We write just '$\vdash$' for '$\mathrm{S4}\mu \vdash$' in the proof. We also write $\alpha \equiv \beta$ for $\vdash \alpha \leftrightarrow \beta$. First, replace all bound atoms in φ by fresh ones, to give a formula $\overline{\varphi}$. More formally, $\overline{\psi}$ is defined for each subformula ψ of φ by induction: $\overline{\mu q \psi} = \mu s(\overline{\psi}(s/q))$, where s is a new atom associated with ψ and not occurring in φ, and $\overline{\cdot}$ commutes with all other operators. By Fact 3.2 or [19, proposition 5.7(i)], $\overline{\varphi} \equiv \varphi$ and $(\overline{\varphi})^* \equiv \varphi^*$. So, replacing φ by $\overline{\varphi}$, we can suppose without loss of generality that for each atom q that occurs free in φ, there is no subformula of φ of the form $\mu q\theta$. The $-^*$ operator preserves this condition, so it holds for φ^* as well.

For atomic φ, the result is trivial since $\varphi^* = \varphi$, and booleans are fine.

Assume inductively that $\varphi \equiv \varphi^*$ and consider $\Box\varphi$. We need to show that $\Box\varphi \equiv \nu q(\varphi^* \wedge \Box q)$, for 'new' q — that is, $\Box\varphi \equiv \neg\mu q\neg(\varphi^* \wedge \Box\neg q)$. By a tautology, it is enough to show $\neg\Box\varphi \equiv \mu q\neg(\varphi^* \wedge \Box\neg q)$. By Fact 3.2, $\neg\Box\varphi \equiv \Diamond\neg\varphi$ and $\mu q\neg(\varphi^* \wedge \Box\neg q) \equiv \mu q(\neg\varphi^* \vee \Diamond q)$. So, letting $\psi = \neg\varphi$, it is enough to prove

$$\Diamond\psi \equiv \mu q\chi, \text{ where } \chi = \psi^* \vee \Diamond q. \tag{3.1}$$

Note that the inductive hypothesis gives $\psi \equiv \psi^*$. Towards (3.1), we first show that $\vdash \Diamond\psi \to \mu q\chi$. Observe that inductively, $\chi^1 = \psi^* \vee \Diamond\bot \equiv \psi$ and $\chi^2 = \psi^* \vee \Diamond\chi^1 \equiv \psi \vee \Diamond\psi$. By Lemma 3.8, $\mathrm{K}\mu \vdash \chi^2 \to \mu q\chi$. As $\mathrm{S4}\mu$ extends $\mathrm{K}\mu$, we get $\vdash \chi^2 \to \mu q\chi$. So by propositional logic, $\vdash \Diamond\psi \to \mu q\chi$.

Now we show $\vdash \mu q\chi \to \Diamond\psi$. By the fixed point rule, it is enough to show $\vdash \chi(\Diamond\psi/q) \to \Diamond\psi$. That is, $\vdash \psi^* \vee \Diamond\Diamond\psi \to \Diamond\psi$. But given the inductive hypothesis, this is just what the S4 axioms say. This proves (3.1) and completes the case of $\Box\varphi$.

Finally assume the result for φ positive in q, and consider the case $\mu q\varphi$. All formulas below meet all necessary conditions because of our initial assumption on φ. By the inductive hypothesis and Lemma 3.3 we get $\vdash \varphi(\mu q\varphi^*/q) \to \varphi^*(\mu q\varphi^*/q)$. The fixed point axiom gives $\vdash \varphi^*(\mu q\varphi^*/q) \to \mu q\varphi^*$. Putting the two together gives $\vdash \varphi(\mu q\varphi^*/q) \to \mu q\varphi^*$. This says that $\mu q\varphi^*$ is a pre-fixed point of φ, so the fixed point rule gives $\vdash \mu q\varphi \to \mu q\varphi^*$. The converse, $\vdash \mu q\varphi^* \to \mu q\varphi$, is similar. $\Box$

Theorem 3.10. *The system* $\mathrm{S4}\mu$ *is sound and complete over the class of finite reflexive transitive Kripke frames.*

Proof. Soundness is easily checked. Conversely, assume that φ is consistent with $\mathrm{S4}\mu$. By Lemma 3.9, φ^* is consistent with $\mathrm{S4}\mu$ and hence with $\mathrm{K}\mu$ as well. By Fact 3.2, there is a finite Kripke model $\mathcal{M} = (W, R, h)$ and a world $w \in W$, with $\mathcal{M}, w \models \varphi^*$. We do not know that (W, R) is reflexive or transitive. However, by Lemma 3.5 we have $(W, R^*, h), w \models \varphi$ as well, and R^* is reflexive and transitive. $\Box$

Remark 3.11. Continuing Remark 3.6, it can be shown in a similar way to Lemma 3.9 that $\mathrm{K4}\mu \vdash \varphi \leftrightarrow \varphi^+$ for each formula φ. We leave this as an exercise, since we will use it only in Remark 4.8. It then follows as in Theorem 3.10 that $\mathrm{K4}\mu$ is sound and complete over the class of finite transitive frames.

4. Translations

The language $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ has some redundancy. We can express $\Box$ with $[d]$, and $\langle t\rangle$ with $\langle dt\rangle$ (but not vice versa). We can also express $\langle t\rangle, \langle dt\rangle$ with μ — and often vice versa, using results of Dawar and Otto [7].

Later, we will need translations that work in both topological spaces and (possibly restricted) Kripke models. In this section, we will explore translations — but only to the extent needed later.

4.1. Translating $\langle d\rangle$ and $\langle dt\rangle$ to μ

This is the simplest case. We have already seen the idea, in the equivalence of $\langle t\rangle$- and $\langle dt\rangle$-formulas to ν-formulas given in (2.1).

Definition 4.1. For each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ, we define a $\mathcal{L}^{\mu}_{\Box[d]\forall}$-formula φ^μ as follows:

1. $p^\mu = p$ for $p \in \mathsf{Var}$.
2. $-^\mu$ commutes with the boolean connectives, $\Box$, $[d]$, $\forall$, and μ (cf. Definition 3.4).
3. $(\langle t\rangle\Delta)^\mu = \nu q \bigwedge_{\delta\in\Delta} \Diamond(\delta^\mu \wedge q)$, where $q \in \mathsf{Var}$ does not occur in any δ^μ $(\delta \in \Delta)$.
4. $(\langle dt\rangle\Delta)^\mu = \nu q \bigwedge_{\delta\in\Delta} \langle d\rangle(\delta^\mu \wedge q)$, where $q \in \mathsf{Var}$ does not occur in any δ^μ $(\delta \in \Delta)$.

These formulas can be checked to be well formed. The translation simply replaces $\langle t\rangle$ by an expression using μ and $\Box$, and similarly for $\langle dt\rangle$. So if $\varphi \in \mathcal{L}^{\langle t\rangle}_{\Box}$ then $\varphi^\mu \in \mathcal{L}^{\mu}_{\Box}$, if $\varphi \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ then $\varphi^\mu \in \mathcal{L}^{\mu}_{[d]}$, etc.

This translation is faithful in all relevant semantics:

Lemma 4.2. *Let φ be any $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula. Then φ is equivalent to φ^μ in every transitive Kripke frame and in every topological space. (See section 2.11 for the definition of equivalence.)*

Proof. An easy induction on φ. We consider only the case $\langle t\rangle\Delta$ (for finite $\Delta \neq \emptyset$), in Kripke semantics (the case $\langle dt\rangle\Delta$ is of course identical). Assume the lemma for each $\delta \in \Delta$. Take any transitive Kripke model $\mathcal{M} = (W, R, h)$ and any $w \in W$. Inductively, $\mathcal{M}, w \models (\langle t\rangle\Delta)^\mu$ iff $\mathcal{M}, w \models \nu q \bigwedge_{\delta\in\Delta} \Diamond(\delta \wedge q)$. By the post-fixed point characterisation of greatest fixed points given in section 2.5, this holds iff $(*)$ there is $S \subseteq W$ with $w \in S$ and such that for every $s \in S$ and $\delta \in \Delta$, there is $t \in S$ with sRt and $\mathcal{M}, t \models \delta$.

Assuming $(*)$, it is easy to choose a sequence $w = s_0 R s_1 R s_2 \dots$ in S by induction so that $\{n < \omega : \mathcal{M}, s_n \models \delta\}$ is infinite for every $\delta \in \Delta$. It follows that $\mathcal{M}, w \models \langle t\rangle\Delta$. Conversely, if $\mathcal{M}, w \models \langle t\rangle\Delta$ then there are worlds $w = w_0 R w_1 R w_2 \dots$ in W with $\{n < \omega : \mathcal{M}, w_n \models \delta\}$ infinite for every $\delta \in \Delta$. Let $S = \{w_n : n < \omega\}$. Then $w \in S$, and for each $w_n \in S$ and $\delta \in \Delta$, there is $m > n$ with $\mathcal{M}, w_m \models \delta$. Then $w_m \in S$, and by transitivity of R we have $w_n R w_m$. So $(*)$ holds. $\Box$

4.2. Translating $\Box$ to $[d]$ and $\langle t\rangle$ to $\langle dt\rangle$

Just replacing $\Box$ by $[d]$ and $\langle t\rangle$ by $\langle dt\rangle$ in a formula $\varphi \in \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ yields an $\mathcal{L}^{\mu\langle dt\rangle}_{[d]\forall}$-formula equivalent to φ in all Kripke frames. But the two are not equivalent in topological spaces, so we seek a better translation that works in both semantics.

Definition 4.3. For each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ, we define a $\mathcal{L}^{\mu\langle dt\rangle}_{[d]\forall}$-formula φ^d as follows:

1. $p^d = p$ for $p \in \mathsf{Var}$.
2. $-^d$ commutes with the boolean connectives, $[d]$, $\langle dt\rangle$, $\forall$, and μ.
3. $(\Box\varphi)^d = \varphi^d \wedge [d]\varphi^d$.
4. $(\langle t\rangle\Delta)^d = (\bigwedge \Delta^d) \vee \langle d\rangle(\bigwedge \Delta^d) \vee \langle dt\rangle\Delta^d$, where $\Delta^d = \{\delta^d : \delta \in \Delta\}$.

Again, φ^d is always well formed. It turns out that the translation $-^d$ is faithful in reflexive frames and T_D spaces (recall from section 2.3 that a space is T_D if the derivative of every singleton is closed).

Lemma 4.4. *Each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ is equivalent to φ^d in every reflexive Kripke frame.*

Proof. An easy induction on φ. To show, e.g., that $\Box\varphi$ implies $(\Box\varphi)^d$, we need reflexivity. We also note that $\bigwedge\Delta$ and $\langle d\rangle\bigwedge\Delta$ both imply $\langle t\rangle\Delta$ in reflexive Kripke models. $\Box$

Lemma 4.5. *Each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ is equivalent to φ^d in a topological space X if, and only if, X is T_D.*

Proof. Let X be a T_D topological space. We prove by induction on φ that each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ is equivalent to φ^d in X. We consider only two cases: $\Box\varphi$ and $\langle t\rangle\Delta$. Inductively assume the result for φ and each formula in the finite set Δ of formulas, let h be an assignment into X, and let $x \in X$. In the proof, we write '$x \models$ ' as short for '$(X,h), x \models$', and for a formula φ, we write $[\![\varphi]\!] = \{y \in X : y \models \varphi\}$.

We prove that $x \models \Box\varphi \leftrightarrow (\Box\varphi)^d$. We have $x \models \Box\varphi$ iff for some open neighbourhood O of x, we have $(X,h), y \models \varphi$ for every $y \in O$. This is plainly iff $x \models \varphi \wedge [d]\varphi$. Inductively, this is iff $x \models \varphi^d \wedge [d]\varphi^d$ — i.e., iff $x \models (\Box\varphi)^d$.

Now we prove that $x \models \langle t\rangle\Delta \leftrightarrow (\langle t\rangle\Delta)^d$. Recall that

$$(\langle t\rangle\Delta)^d = (\bigwedge \Delta^d) \vee \langle d\rangle(\bigwedge \Delta^d) \vee \langle dt\rangle\Delta^d.$$

First we prove that $x \models (\langle t\rangle\Delta)^d \to \langle t\rangle\Delta$. Suppose that $x \models (\langle t\rangle\Delta)^d$. To show that $x \models \langle t\rangle\Delta$, we need to find $S \subseteq X$ with $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$. If $x \models \bigwedge \Delta^d$, take $S = \{x\}$. If $x \models \langle d\rangle \bigwedge \Delta^d$, take $S = \{x\} \cup [\![\bigwedge \Delta^d]\!]$. And if $x \models \langle dt\rangle\Delta^d$, there is $S \subseteq X$ with $x \in S \subseteq \bigcap_{\delta\in\Delta} \langle d\rangle([\![\delta]\!] \cap S)$; then $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$ as required.

It remains to prove that $x \models \langle t\rangle\Delta \to (\langle t\rangle\Delta)^d$. So suppose that $x \models \langle t\rangle\Delta$. If $x \models (\bigwedge \Delta^d) \vee \langle d\rangle(\bigwedge \Delta^d)$, we are done.

So suppose not. Thus, there is an open neighbourhood U of x with $y \models \neg \bigwedge \Delta^d$ for every $y \in U$. So for every $y \in U$, there is $\delta_y \in \Delta$ with $y \models \neg\delta_y^d$.

We prove that $x \models \langle dt\rangle\Delta^d$.

Since $x \models \langle t\rangle\Delta$, there is $S \subseteq X$ with $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$.

Claim 4.5.1. *Put $S' = U \cap S$. Then $x \in S' \subseteq \bigcap_{\delta\in\Delta} \langle d\rangle([\![\delta^d]\!] \cap S')$.*

Proof of claim. Plainly, $x \in S'$. For the other half, let $y \in S'$ and $\delta \in \Delta$ be arbitrary; we show that $y \in \langle d\rangle([\![\delta^d]\!] \cap S')$. So let O be any open neighbourhood of y. As X is T_D, $\langle d\rangle\{y\}$ is closed, so since it does not contain y, $O \cap U \setminus \langle d\rangle\{y\}$ is an open neighbourhood of y too. As $y \in S' \subseteq S \subseteq \mathrm{cl}([\![\delta_y]\!] \cap S)$, there is some $z \in O \cap U \cap S \setminus \langle d\rangle\{y\}$ with $z \models \delta_y$. But $y \models \neg\delta_y^d$, so inductively, $y \models \neg\delta_y$. It follows that $z \neq y$.

Now we have $z \notin \{y\} \cup \langle d\rangle\{y\} = \mathrm{cl}\{y\}$, so $O \cap U \setminus \mathrm{cl}\{y\}$ is an open neighbourhood of z. Since $z \in S \subseteq \mathrm{cl}([\![\delta]\!] \cap S)$, there is some $t \in O \cap U \cap S \setminus \mathrm{cl}\{y\} = O \cap S' \setminus \mathrm{cl}\{y\}$ with $t \models \delta$. Then $t \neq y$. Since O was arbitrary, this shows that $y \in \langle d\rangle([\![\delta]\!] \cap S')$. Since inductively, $[\![\delta]\!] = [\![\delta^d]\!]$, this proves the claim. □

By definition of the semantics, Claim 4.5.1 immediately yields $x \models \langle dt\rangle\Delta^d$ as required. This completes the induction and the proof that each φ is equivalent to φ^d. (The reader may like to construct an alternative proof using the games described in Remark 2.2.)

Conversely, to show that the T_D hypothesis is necessary, we first prove

Claim 4.5.2. *In any space X, for any $x \in X$, $\mathrm{cl}\,\langle d\rangle\{x\} \setminus \langle d\rangle\{x\} \subseteq \{x\}$. Hence $\langle d\rangle\{x\}$ is closed iff $x \notin \mathrm{cl}\,\langle d\rangle\{x\}$.*

Proof of claim. For the first part, since $\langle d\rangle\{x\} \subseteq \mathrm{cl}\{x\}$ and the latter is closed, $\mathrm{cl}\,\langle d\rangle\{x\} \subseteq \mathrm{cl}\{x\} = \langle d\rangle\{x\} \cup \{x\}$. This implies $\mathrm{cl}\,\langle d\rangle\{x\} \setminus \langle d\rangle\{x\} \subseteq \{x\}$.

For the second part, $\langle d\rangle\{x\}$ is closed iff $\mathrm{cl}\,\langle d\rangle\{x\} \setminus \langle d\rangle\{x\} = \emptyset$. By the first part, this holds iff $x \notin \mathrm{cl}\,\langle d\rangle\{x\} \setminus \langle d\rangle\{x\}$. But $x \notin \langle d\rangle\{x\}$, so $x \notin \mathrm{cl}\,\langle d\rangle\{x\} \setminus \langle d\rangle\{x\}$ iff $x \notin \mathrm{cl}\,\langle d\rangle\{x\}$. This proves the claim. □

Now suppose the space X is not T_D. Then there is some point x of X with $\langle d\rangle\{x\}$ not closed. By Claim 4.5.2, $x \in \mathrm{cl}\,\langle d\rangle\{x\}$. Hence $\mathrm{cl}\{x\} \subseteq \mathrm{cl}\,\langle d\rangle\{x\}$. Let $p \in \mathsf{Var}$ and $h : \mathsf{Var} \to \wp X$ satisfy $h(p) = \{x\}$. Then $(X,h), x \models \langle t\rangle\{p, \langle d\rangle p\}$, but $(X,h), x \not\models (\langle t\rangle\{p, \langle d\rangle p\})^d$, i.e. $(X,h), x \not\models (p\wedge\langle d\rangle p)\vee\langle d\rangle(p\wedge\langle d\rangle p)\vee\langle dt\rangle\{p, \langle d\rangle p\}$, giving a case of φ not being equivalent to φ^d. That $x \not\models (p \wedge \langle d\rangle p) \vee \langle d\rangle(p \wedge \langle d\rangle p)$ follows because $[\![p \wedge \langle d\rangle p]\!] =$

$\{x\} \cap \langle d\rangle\{x\} = \emptyset$. That $x \not\models \langle dt\rangle\{p, \langle d\rangle p\}$ follows as no 'punctured neighbourhood' $O \setminus \{x\}$ contains a point of $[\![p]\!] = \{x\}$. To see that $x \models \langle t\rangle\{p, \langle d\rangle p\}$, let $S = \mathrm{cl}\{x\}$. Then S is included in both $\mathrm{cl}([\![p]\!] \cap S) = \mathrm{cl}\{x\} = S$ and $\mathrm{cl}([\![\langle d\rangle p]\!] \cap S) = \mathrm{cl}(\langle d\rangle\{x\})$ (because $\mathrm{cl}\{x\} \subseteq \mathrm{cl}\,\langle d\rangle\{x\}$ as noted above). Since $x \in S$, it follows that $x \models \langle t\rangle\{p, \langle d\rangle p\}$. □

4.3. Translating μ to $\langle t\rangle$

We use this translation only to prove strong completeness for $\mathcal{L}^{\mu}_{\Box}$ in Theorem 9.3(2). Fortunately, most of the hard work involved has already been done by others. We will need only the fact below, but its proof was a major enterprise.

Fact 4.6 *(Dawar–Otto, [7, theorem 4.57(5)]).* For each formula φ of $\mathcal{L}^{\mu}_{\Box}$, there is a formula φ^t of $\mathcal{L}^{\langle t\rangle}_{\Box}$ that is equivalent to φ in every finite transitive Kripke frame.

To lift this to topological spaces, we will use the proof theory from section 3.

Corollary 4.7. *Each $\mathcal{L}^{\mu}_{\Box}$-formula φ is equivalent to φ^t in every topological space.*

Proof. By Fact 4.6 and Lemma 4.2, $\varphi \leftrightarrow (\varphi^t)^{\mu}$ is an $\mathcal{L}^{\mu}_{\Box}$-formula valid in every finite transitive Kripke frame. By Theorem 3.10, $\mathrm{S4}\mu \vdash \varphi \leftrightarrow (\varphi^t)^{\mu}$.

Now it is easy to check that S4μ is sound over every topological space. (The S4 axioms are sound by definition of the topological semantics of $\Box$, and the fixed point axiom and rule are sound by the semantics of μ.) Hence, $\varphi \leftrightarrow (\varphi^t)^{\mu}$ is valid in every topological space. But by Lemma 4.2, $(\varphi^t)^{\mu}$ is equivalent to φ^t in every topological space. We conclude that φ is equivalent to φ^t in every topological space, as required. □

By the corollary and Lemma 4.2, $\mathcal{L}^{\mu}_{\Box}$ and $\mathcal{L}^{\langle t\rangle}_{\Box}$ have the same expressive power over the class of all topological spaces.

Remark 4.8. We can translate $\mathcal{L}^{\mu}_{[d]}$ into $\mathcal{L}^{\langle dt\rangle}_{[d]}$ in a similar way. Since $\Box, [d]$ and $\langle t\rangle, \langle dt\rangle$ are indistinguishable in Kripke semantics, each formula φ of $\mathcal{L}^{\mu}_{[d]}$ is equivalent to $(\varphi^t)^{\mu}$ on the class of finite transitive frames, where $\varphi^t \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ is obtained exactly as in Fact 4.6. By Remark 3.11, K4μ is complete over this class, so $\mathrm{K4}\mu \vdash \varphi \leftrightarrow (\varphi^t)^{\mu}$. Now as Esakia showed (e.g., [8, proposition 2]), the '4' scheme $[d]\varphi \to [d][d]\varphi$ is valid in a topological space precisely when the space is T_D. So K4μ is sound over every T_D space, and hence its theorem $\varphi \leftrightarrow (\varphi^t)^{\mu}$ is valid in every such space. Lemma 4.2 now yields that $\varphi \in \mathcal{L}^{\mu}_{[d]}$ is equivalent to $\varphi^t \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ in every T_D space.

5. More topology

Not surprisingly, for our completeness theorems we will need some simple and standard topological definitions and results. They are collected here. We will see some more substantial ones in the next section. We begin with the following very simple fact.

Lemma 5.1. *Let X be a topological space, and suppose that $N \subseteq X$ has empty interior.*

1. *If $C \subseteq X$ is closed, then* $\mathrm{int}(C \cup N) = \mathrm{int}\, C$.
2. *If $O \subseteq X$ is open, then* $\mathrm{cl}(O \setminus N) = \mathrm{cl}\, O$.

Proof. For the first part, $\mathrm{int}\, C \subseteq \mathrm{int}(C \cup N)$ by monotonicity of int. For the converse, if $\mathrm{int}(C \cup N) \not\subseteq \mathrm{int}\, C$, then $\mathrm{int}(C \cup N) \not\subseteq C$. So $\mathrm{int}(C \cup N) \setminus C$ is a non-empty open subset of $(C \cup N) \setminus C$ and so of N, a contradiction.

The second part follows from the first, since $X\setminus \mathrm{cl}(O\setminus N) = \mathrm{int}(X\setminus(O\setminus N)) = \mathrm{int}((X\setminus O)\cup N) = \mathrm{int}(X\setminus O) = X\setminus \mathrm{cl}\, O$. □

5.1. The $\langle d\rangle$ operator on sets

Let X be a topological space. For a set $S \subseteq X$, recall that $\langle d\rangle S = \{x \in X : S \cap O \setminus \{x\} \neq \emptyset$ for every open neighbourhood O of $x\}$, the set of strict limit points of S. The $\langle d\rangle$ operator has the following basic properties.

Lemma 5.2. *Let $S, T \subseteq X$.*

1. $\mathrm{cl}\, S = S \cup \langle d\rangle S$.
2. *$\langle d\rangle$ is additive: $\langle d\rangle(S \cup T) = \langle d\rangle S \cup \langle d\rangle T$.*
3. *$\langle d\rangle S = \emptyset$ iff every subset of S is closed.*
4. *If X is T_D then $\langle d\rangle\langle d\rangle S \subseteq \mathrm{cl}\,\langle d\rangle S = \langle d\rangle S = \langle d\rangle\, \mathrm{cl}\, S$.*
5. *If X is dense in itself, then (i) $\mathrm{int}\, S \subseteq \langle d\rangle S$ and (ii) if S is open then $\langle d\rangle S = \mathrm{cl}\, S$.*

Proof. We prove only part 4, leaving the other parts to the reader. Recall from section 2.3 that X is T_D if $\langle d\rangle\{x\}$ is closed for every $x \in X$. Aull and Thron showed in [2, theorem 5.1] that in fact, X is T_D iff $\langle d\rangle S$ is closed for every $S \subseteq X$. (They say that this theorem is due to C.T. Yang and that it motivated their definition of T_D.) With part 1, this yields $\langle d\rangle\langle d\rangle S \subseteq \mathrm{cl}\,\langle d\rangle S = \langle d\rangle S$. (Esakia showed that T_D is necessary here: see Remark 4.8.) With parts 1 and 2, this gives $\langle d\rangle\, \mathrm{cl}\, S = \langle d\rangle(S \cup \langle d\rangle S) = \langle d\rangle S \cup \langle d\rangle\langle d\rangle S = \langle d\rangle S$. □

This leads to the following.

Lemma 5.3. *Suppose that X is dense-in-itself and T_D. Then every non-empty open subset of X is infinite.*

Proof. Suppose not. Let $O \subseteq X$ be a non-empty finite open set of least possible cardinality. Take any $x \in O$. By Lemma 5.2(2, 5), $O = \mathrm{int}\, O \subseteq \langle d\rangle O = \bigcup_{y\in O} \langle d\rangle\{y\}$. Choose $y \in O$ with $x \in \langle d\rangle\{y\}$, and let $Q = O \setminus \langle d\rangle\{y\}$. As X is T_D, $\langle d\rangle\{y\}$ is closed, so Q is open. Moreover, $Q \subsetneq O$ since $x \notin Q$; and $Q \neq \emptyset$ (as plainly $y \notin \langle d\rangle\{y\}$, so $y \in Q$). This contradicts the minimality of O. □

5.2. Regular open sets

Let X be a topological space. A regular open subset of X is one equal to the interior of its closure. We will mainly be interested in regular open subsets of open subspaces of X, so we give definitions directly for such situations.

Definition 5.4. Let U be an open subset of X. A subset S of X is said to be a *regular open subset of U* if $S = \mathrm{int}(U \cap \mathrm{cl}\, S)$.

As 'int' is multiplicative and U is open, it is equivalent to say that $S = U \cap \mathrm{int}\,\mathrm{cl}\, S$, and we sometimes prefer this formulation. In such a case, $S \subseteq U$ and S is open. So $S = \mathrm{int}_U\, \mathrm{cl}_U\, S$: that is, S is a regular open subset of the subspace U of X. It is worth noting that if $S \subseteq U$ is arbitrary then $\mathrm{int}_U\, \mathrm{cl}_U\, S$ is a regular open subset of U.

It is known (see, e.g., [13, chapter 10]) that for every open subset U of X, the set $RO(U)$ of regular open subsets of U is closed under the operations $+, \cdot, -, 0, 1$ defined by

- $S + S' = U \cap \mathrm{int}\,\mathrm{cl}(S \cup S')$

- $S \cdot S' = S \cap S'$
- $-S = U \setminus \operatorname{cl} S$
- $0 = \emptyset$ and $1 = U$,

and $(RO(U), +, \cdot, -, 0, 1)$ is a (complete) boolean algebra. We will also use the notation $RO(U)$ to denote this boolean algebra. The standard boolean ordering $\leq$ on $RO(U)$ coincides with set inclusion, because for $S, T \in RO(U)$ we have $S \leq T$ iff $S \cdot T = S$, iff $S \cap T = S$, iff $S \subseteq T$. We will need the following general lemma.

Lemma 5.5. *Let $V \subseteq U$ be open subsets of X, and S, S' be regular open subsets of U.*

1. *If $T = U \setminus \operatorname{cl} S$, then T is also a regular open subset of U, with $S = U \setminus \operatorname{cl} T$ and $U \setminus S \subseteq \operatorname{cl} T$.*
2. *If $U \cap \operatorname{cl} S \cap \operatorname{cl} S' = \emptyset$, then $S + S' = S \cup S'$.*
3. *If $S \subseteq V$, then S is a regular open subset of V.*
4. *Every regular open subset of S is a regular open subset of U.*

Proof.

1. The first two points follow from boolean algebra considerations, and can easily be shown directly; of course, the first point is equivalent to $RO(U)$ being closed under $-$. The third point, $U \setminus S \subseteq \operatorname{cl} T$, follows from $U \setminus \operatorname{cl} T = S$.
2. Since $S, S' \leq S + S'$ and $\leq$ coincides with $\subseteq$, we obtain $S, S' \subseteq S + S'$ and so $S \cup S' \subseteq S + S'$. Conversely, it is easy to check[3] that

$$\operatorname{int}\operatorname{cl}(S \cup S') \subseteq \operatorname{int}\operatorname{cl} S \cup \operatorname{int}\operatorname{cl} S' \cup (\operatorname{cl} S \cap \operatorname{cl} S').$$

   Since $U \cap \operatorname{cl} S \cap \operatorname{cl} S' = \emptyset$,

$$S + S' = U \cap \operatorname{int}\operatorname{cl}(S \cup S') \subseteq (U \cap \operatorname{int}\operatorname{cl} S) \cup (U \cap \operatorname{int}\operatorname{cl} S') = S \cup S',$$

   as required.
3. $V \cap \operatorname{int}\operatorname{cl} S = (V \cap U) \cap \operatorname{int}\operatorname{cl} S = V \cap (U \cap \operatorname{int}\operatorname{cl} S) = V \cap S = S$.
4. Let T be a regular open subset of S. Clearly, $\operatorname{int}\operatorname{cl} T \subseteq \operatorname{int}\operatorname{cl} S$. So $U \cap \operatorname{int}\operatorname{cl} T = U \cap (\operatorname{int}\operatorname{cl} S \cap \operatorname{int}\operatorname{cl} T) = (U \cap \operatorname{int}\operatorname{cl} S) \cap \operatorname{int}\operatorname{cl} T = S \cap \operatorname{int}\operatorname{cl} T = T$. $\square$

5.3. Normal spaces

Definition 5.6. A topological space X is said to be *Hausdorff* (or T2) if for every two distinct points $x_0, x_1 \in X$, there are disjoint open sets O_0, O_1 with $x_0 \in O_0$ and $x_1 \in O_1$, and *normal* (or T4) if it is Hausdorff and for every two disjoint closed subsets C_0, C_1 of X, there are disjoint open sets O_0, O_1 with $C_0 \subseteq O_0$ and $C_1 \subseteq O_1$.

Equivalently, X is normal iff it is Hausdorff and if $C \subseteq O \subseteq X$, C is closed, and O is open, then there is open Q with $C \subseteq Q \subseteq \operatorname{cl} Q \subseteq O$. It is standard that every T2 (and hence every normal) space is T1, and hence also T_D.

[3] Indeed, $\Box\Diamond(p \vee q) \to \Box\Diamond p \vee \Box\Diamond q \vee (\Diamond p \wedge \Diamond q)$ is valid in reflexive transitive frames, so by classical modal logic, it is provable in S4. Since S4 is sound over X, the formula is valid in X.

Lemma 5.7. *Let C_0, C_1 be disjoint closed subsets of an open subset Q of a normal topological space X. Then there are regular open subsets O_0, O_1 of X with disjoint closures, such that $C_0 \subseteq O_0 \subseteq Q$ and $C_1 \subseteq O_1 \subseteq Q$.*

Proof. Since $Q \setminus C_1$ is open and contains C_0, by normality there is open O_0^- with $C_0 \subseteq O_0^- \subseteq \operatorname{cl} O_0^- \subseteq Q \setminus C_1$. Let $O_0 = \operatorname{int} \operatorname{cl} O_0^-$. Then O_0 is regular open in X and $C_0 \subseteq O_0^- \subseteq O_0 \subseteq \operatorname{cl} O_0 = \operatorname{cl} O_0^- \subseteq Q \setminus C_1$. Then $C_1 \subseteq Q \setminus \operatorname{cl} O_0$, an open set, so repeating the argument gives a regular open subset O_1 of X with $C_1 \subseteq O_1 \subseteq \operatorname{cl} O_1 \subseteq Q \setminus \operatorname{cl} O_0$. Now O_0, O_1 are as required. □

The following is well known (see, e.g., [32, III, 6.1]), but is so important for us that we include a quick proof.

Lemma 5.8. *Every metric space is normal.*

Proof. Let X be a metric space. It is easy to check that X is Hausdorff, and we leave this to the reader. Let C, D be disjoint closed subsets of X. It is enough to show that $(*)$ there is open $O \supseteq C$ with $\operatorname{cl}(O) \cap D = \emptyset$. For then, applying $(*)$ to the disjoint closed sets D and $\operatorname{cl}(O)$, we find open $P \supseteq D$ with $\operatorname{cl}(P) \cap \operatorname{cl}(O) = \emptyset$, as required.

We proceed to prove $(*)$. If $C = \emptyset$, take $O = \emptyset$. If $D = \emptyset$ take $O = X$. So we can suppose $C, D \neq \emptyset$, and thus define

$$O = \{x \in X : d(x, C) < d(x, D)/2\}$$

(recall from section 2.4 that $d(x, S) = \inf\{d(x, s) : s \in S\}$ for non-empty $S \subseteq X$). Then $C \subseteq O$, because if $x \in C$ then $d(x, C) = 0$, while $x \notin D$, so $d(x, D) > 0$ as D is closed. It is easily seen that O is open. Let $K = \{x \in X : d(x, C) \leq d(x, D)/2\}$. Then K is closed, so $\operatorname{cl}(O) \subseteq K$. So it is enough to show that $K \cap D = \emptyset$. But if $x \in D \cap K$ then $d(x, C) \leq d(x, D)/2 = 0$, so $x \in C$ as C is closed. This contradicts the assumption that $C \cap D = \emptyset$. □

6. Tarski's 'dissection theorem' and relatives

A 'dissection' of a space (or a non-empty open subset of it) is a partition of it into subsets that have topological relationships allowing them to represent the structure of certain Kripke frames. The original dissection results of Tarski, developed further by others, involved finitely many partition sets. Here we strengthen the analysis by allowing (countably) infinitely many partition sets; permitting each partition set to contain any given starting set, so long as any union of the starting sets is closed and nowhere dense; and making each partition set be within some prescribed distance of any point. We also develop a closely related result in which the subsets need not partition the space but each of them has the same predetermined set of limit points.

We will use these results in Proposition 7.10, to represent finite Kripke frames. We will state them and discuss them in section 6.1, and prove them in section 6.2. Section 6.3 contains a corollary also needed in Proposition 7.10.

6.1. *The dissection theorems*

The first 'dissection theorem' is as follows. We will use it in Corollary 6.5, and in our main Proposition 7.10 to handle frames with irreflexive roots. It is also used in [16].

Theorem 6.1. *Let X be a dense-in-itself metric space. Let there be given open subsets $\mathbb{T}, \mathbb{U}$ of X with $\emptyset \neq \mathbb{T} \subseteq \mathbb{U}$; a non-empty countable*[4] *index set $\mathcal{E}$; pairwise disjoint subsets $E_i \subseteq \mathbb{T}$ ($i \in \mathcal{E}$) with $\langle d \rangle \bigcup_{i\in\mathcal{E}} E_i = \emptyset$; and a real number $\varepsilon > 0$. Then there are pairwise disjoint non-empty sets $\mathbb{I}_i$ with $E_i \subseteq \mathbb{I}_i \subseteq \mathbb{T}$ (for each $i \in \mathcal{E}$), such that for every $i \in \mathcal{E}$ we have $d(x, \mathbb{I}_i) < \varepsilon$ for every $x \in \operatorname{cl}\mathbb{T}$, and*

$$\langle d \rangle \mathbb{I}_i = \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}.$$

Example 6.2. We give an instance of Theorem 6.1 for $X = \mathbb{R}$, ignoring the E_i and ε. Suppose that $\mathbb{T} = (0, 1)$ and $\mathbb{U} = (0, 2)$. Choose pairwise disjoint infinite sets K_i ($i \in \mathcal{E}$) of positive integers, and let $\mathbb{I}_i = \{1/n : n \in K_i\}$ for each $i \in \mathcal{E}$. Then the $\mathbb{I}_i$ are non-empty and pairwise disjoint subsets of $\mathbb{T}$ with $\langle d \rangle \mathbb{I}_i = \{0\} = \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$ for each $i \in \mathcal{E}$.

Next is our second dissection result, which will be used in Proposition 7.10 to handle frames with reflexive roots. Recall that a subset $N \subseteq X$ is *nowhere dense* if $\operatorname{int}\operatorname{cl} N = \emptyset$. Any such set plainly has empty interior.

Theorem 6.3. *Let X be a dense-in-itself metric space. Let $\mathbb{G}$ be a non-empty open subset of X. Let $\mathcal{G}$, $\mathcal{B}$ be disjoint countable index sets with $\mathcal{B} \neq \emptyset$. Let E_i ($i \in \mathcal{G} \cup \mathcal{B}$) be pairwise disjoint subsets of $\mathbb{G}$ such that $\bigcup_{i\in S} E_i$ is closed and nowhere dense for every $S \subseteq \mathcal{G} \cup \mathcal{B}$. Let a real number $\varepsilon > 0$ be given. Then there are non-empty subsets $\mathbb{G}_i, \mathbb{B}_j \subseteq \mathbb{G}$ ($i \in \mathcal{G}$, $j \in \mathcal{B}$) with the following properties.*

1. *$E_i \subseteq \mathbb{G}_i$ for each $i \in \mathcal{G}$, and $E_j \subseteq \mathbb{B}_j$ for each $j \in \mathcal{B}$.*
2. *$(\mathbb{G}_i : i \in \mathcal{G}) \cup (\mathbb{B}_j : j \in \mathcal{B})$ is a partition of $\mathbb{G}$.*
3. *Each $\mathbb{G}_i$ ($i \in \mathcal{G}$) is open. (The $\mathbb{B}_j$ need not be open.)*
4. *Letting*

$$\mathbb{D} = \operatorname{cl}(\mathbb{G}) \setminus \bigcup_{i\in\mathcal{G}} \mathbb{G}_i,$$

   we have $\operatorname{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \mathbb{D}$ for each $i \in \mathcal{G}$, and $\langle d \rangle \mathbb{B}_j = \mathbb{D}$ for each $j \in \mathcal{B}$.
5. *$d(x, \mathbb{G}_i) < \varepsilon$ and $d(x, \mathbb{B}_j) < \varepsilon$ for every $x \in \operatorname{cl}\mathbb{G}$, $i \in \mathcal{G}$, and $j \in \mathcal{B}$.*

This theorem is largely known, and has a long history. Paraphrasing slightly, Tarski [41, satz 3.10] proved the following, which was perhaps the original 'dissection theorem'. (He credited the proof to Samuel Eilenberg, noting that he had originally proven the result himself for $\mathbb{R}$ and its dense-in-themselves subspaces.)

> Let X be a dense-in-itself normal topological space with a countable basis of open sets (see below). Then for every $r < \omega$, every non-empty open subset $\mathbb{G}$ of X can be partitioned into non-empty open sets $\mathbb{G}_1, \ldots, \mathbb{G}_r$ and a non-empty set $\mathbb{B}_0$ such that $\operatorname{cl}(\mathbb{G}) \setminus \mathbb{G} \subseteq \operatorname{cl}\mathbb{B}_0 \subseteq \operatorname{cl}\mathbb{G}_1 \cap \ldots \cap \operatorname{cl}\mathbb{G}_r$.

Here and below, the empty intersection (when $r = 0$) is taken to be X. This statement is equivalent to the statement of Theorem 6.3 (parts 2–4) above, when $r = |\mathcal{G}| < \omega$, $|\mathcal{B}| = 1$, and with $\langle d \rangle \mathbb{B}_j$ replaced by $\operatorname{cl}\mathbb{B}_j$. We observe that if $r \geq 2$ then actually $\operatorname{cl}\mathbb{B}_0 = \operatorname{cl}\mathbb{G}_1 \cap \ldots \cap \operatorname{cl}\mathbb{G}_r$.

A topological space (X, τ) has a countable basis of open sets iff there is countable $\tau_0 \subseteq \tau$ such that τ is the smallest topology on X containing τ_0. Given this and normality, Urysohn's theorem [43] yields that $\tau = \tau_d$ for some metric d on X. Any metric space is normal, and has a countable basis of open sets iff it is separable (see section 2.3). So Tarski's stipulation on X boils down to stipulating that X is a separable dense-in-itself metric space.

[4] Recall from section 2.12 that we use 'countable' to mean 'of cardinality at most ω'.

Example 6.4. We give an instance of Tarski's result for $X = \mathbb{R}$ and $r = 1$. Take a copy of $\mathbb{R}$. Replace each rational in it by a copy of the real interval $[0, 1]$. Let $\mathbb{G}_1$ be the union of the interiors of these intervals, and $\mathbb{B}_0$ be the set of all other points. (Formally, let $I_x = [0, 1]$ for $x \in \mathbb{Q}$, $I_x = \{0\}$ for $x \in \mathbb{R} \setminus \mathbb{Q}$, and consider $I = \bigcup_{x\in\mathbb{R}}(\{x\} \times I_x)$, ordered lexicographically by $\langle x, p\rangle < \langle y, q\rangle$ iff $x < y$ or ($x = y$ and $p < q$). Let $\mathbb{G}_1 = \mathbb{Q} \times (0, 1)$ and $\mathbb{B}_0 = I \setminus \mathbb{G}_1$.) The resulting linear order (formally, $(I, <)$) is Dedekind complete, separable, and without endpoints, and hence order-isomorphic to the open interval $\mathbb{G} = (0, 1)$ of $\mathbb{R}$ (see, e.g., [33, theorem 2.30]). We identify it with this interval. It can be checked that $\mathrm{cl}\,\mathbb{G} \setminus \mathbb{G} \subseteq \mathrm{cl}\,\mathbb{B}_0 \subseteq \mathrm{cl}\,\mathbb{G}_1$, and (cf. Theorem 6.3) that $\mathrm{cl}\,\mathbb{G} \setminus \mathbb{G}_1 = \mathrm{cl}\,\mathbb{G}_1 \setminus \mathbb{G}_1 = \langle d\rangle\mathbb{B}_0$.

Removing the restriction to $|\mathcal{B}| = 1$ but with the same hypotheses on X, McKinsey and Tarski [27, theorem 3.5] proved that

> for every $r, s < \omega$, every non-empty open set $\mathbb{G}$ can be partitioned into non-empty open sets $\mathbb{G}_1, \dots, \mathbb{G}_r$ and non-empty sets $\mathbb{B}_0, \dots, \mathbb{B}_s$ with $\mathrm{cl}(\mathbb{G}) \setminus \mathbb{G} \subseteq \mathrm{cl}\,\mathbb{B}_0 = \cdots = \mathrm{cl}\,\mathbb{B}_s \subseteq \mathrm{cl}\,\mathbb{G}_1 \cap \ldots \cap \mathrm{cl}\,\mathbb{G}_r$.

This statement is equivalent to parts 2–4 of the statement of Theorem 6.3 above, with $r = |\mathcal{G}| < \omega$, $s = |\mathcal{B}| < \omega$, and with $\langle d\rangle\mathbb{B}_j$ replaced by $\mathrm{cl}\,\mathbb{B}_j$. It was used in [27] to prove (in modal terminology) that the $\mathcal{L}_\Box$-logic of X is S4; a form of the result 'readily available to those whose main interest lies in sentential calculus rather than in topology or algebra' was given in [29, theorem 1.3].

Removing the assumption of separability, Rasiowa and Sikorski [32, III, 7.1] proved parts 2–4 of Theorem 6.3 essentially as formulated above, but for finite $\mathcal{G}, \mathcal{B}$ and with $\langle d\rangle\mathbb{B}_j$ replaced by $\mathrm{cl}\,\mathbb{B}_j$. Our use of $\langle d\rangle\mathbb{B}_j$ strengthens this, since part 4 implies that $\mathbb{B}_j \subseteq \mathbb{D} = \langle d\rangle\mathbb{B}_j$, so by Lemma 5.2(1), $\mathrm{cl}\,\mathbb{B}_j = \langle d\rangle\mathbb{B}_j$. But it is only a formal strengthening, since the same effect can be achieved by first obtaining disjoint sets $\mathbb{B}^k_j$ with $\mathrm{cl}\,\mathbb{B}^k_j = \mathbb{D}$ for $j \in \mathcal{B}$ and $k = 0, 1$, and then defining $\mathbb{B}_j = \mathbb{B}^0_j \cup \mathbb{B}^1_j$ for each j. As $\mathbb{B}^0_j \cap \mathbb{B}^1_j = \emptyset$, using Lemma 5.2(1–2) we have

$$\mathbb{D} \subseteq (\mathbb{D} \setminus \mathbb{B}^0_j) \cup (\mathbb{D} \setminus \mathbb{B}^1_j) = (\mathrm{cl}\,\mathbb{B}^0_j \setminus \mathbb{B}^0_j) \cup (\mathrm{cl}\,\mathbb{B}^1_j \setminus \mathbb{B}^1_j) \subseteq \underbrace{\langle d\rangle\mathbb{B}^0_j \cup \langle d\rangle\mathbb{B}^1_j}_{\langle d\rangle\mathbb{B}_j} \subseteq \mathrm{cl}\,\mathbb{B}^0_j \cup \mathrm{cl}\,\mathbb{B}^1_j = \mathbb{D}.$$

So $\langle d\rangle\mathbb{B}_j = \mathbb{D}$ as required.

In this paper, we will not need the ε-conditions (Theorem 6.3(5)), nor infinite index sets. However, they may be useful in future work. Indeed, analogous ε-conditions (for finite $\mathcal{G}, \mathcal{B}$) have already been proved by Kremer [20, lemma 6.1], and infinite index sets can also be helpful in arguments like Kremer's.

Other recent related results include [23, proposition 6.7], which (roughly speaking and among other things) replaces part 5 in Theorem 6.3 by the statement that each $\mathbb{G}_i$ is the union of pairwise disjoint open balls $\mathbb{O}_{ik}$ ($k \in K_i$) such that every open neighbourhood of every point in $\bigcup_{j\in\mathcal{B}} \mathbb{B}_j$ contains some $\mathbb{O}_{ik}$.

6.2. Proof of Theorems 6.1 and 6.3

We will sometimes use without mention Lemma 5.2(1–2) and the consequent additivity and monotonicity of $\langle d\rangle$ and cl. We will get to the theorems shortly, but first, fix a non-empty countable index set $\mathcal{E}$ and pairwise disjoint subsets $E_i \subseteq X$ ($i \in \mathcal{E}$) such that $\bigcup_{i\in S} E_i$ is closed for every $S \subseteq \mathcal{E}$. (Hence each E_i is closed.) Two players, $\forall$ (male) and $\exists$ (female), play a game, $\mathfrak{G}(E_i : i < \omega)$, to build pairwise disjoint subsets $\mathbb{I}_i$ ($i \in \mathcal{E}$) of X with $E_i \subseteq \mathbb{I}_i$ for each i.

The game is co-operative, with no winners or losers. It has ω rounds, numbered $0, 1, 2, \dots$. At the start of round n (for each $n < \omega$), subsets $I^n_i \subseteq X$ ($i \in \mathcal{E}$), whose closures are pairwise disjoint, are in play. Initially — at the start of round 0 — we put $I^0_i = E_i$ for each $i \in \mathcal{E}$. Then $\mathrm{cl}\,I^0_i$ ($i \in \mathcal{E}$) are pairwise disjoint by assumption on the E_i. Round n is played as follows.

1. $\forall$ plays a pair (O_n, i_n), where $i_n \in \mathcal{E}$ and O_n is an open subset of X of his choice, satisfying

$$O_n \cap \langle d \rangle \bigcup_{i\in\mathcal{E}} I_i^n = \emptyset. \tag{6.1}$$

2. $\exists$ responds by defining $I_j^{n+1} = I_j^n$ for all $j \in \mathcal{E} \setminus \{i_n\}$, and extending $I_{i_n}^n$ to a set $I_{i_n}^{n+1}$ of her choice, such that

$$\mathrm{cl}\, I_{i_n}^n \subseteq I_{i_n}^{n+1} \subseteq O_n \cup \mathrm{cl}\, I_{i_n}^n. \tag{6.2}$$

   So $\exists$ must include $\mathrm{cl}\, I_{i_n}^n$ in $I_{i_n}^{n+1}$, but all other points she includes must lie in O_n. These inclusions are needed in Claim 6.2.3. As the game requires that $\mathrm{cl}\, I_j^{n+1}$ $(j \in \mathcal{E})$ are pairwise disjoint, she must also ensure that $\mathrm{cl}\, I_{i_n}^{n+1}$ is disjoint from $\mathrm{cl}\, I_j^n$ for each $j \in \mathcal{E} \setminus \{i_n\}$. Since she can satisfy these requirements by simply playing $I_{i_n}^{n+1} = \mathrm{cl}\, I_{i_n}^n$, she never gets stuck.

That completes the round, and the sets I_i^{n+1} $(i \in \mathcal{E})$ are passed to the start of round $n + 1$.
Plainly, $I_i^0 \subseteq I_i^1 \subseteq \cdots$ for every $i \in \mathcal{E}$. The following claim will be useful later.

Claim 6.2.1. $\bigcup_{i\in S} \mathrm{cl}\, I_i^n$ *is closed — equivalently,* $\mathrm{cl}(\bigcup_{i\in S} I_i^n) = \bigcup_{i\in S} \mathrm{cl}\, I_i^n$ *— for each* $S \subseteq \mathcal{E}$ *and* $n < \omega$.

Proof of claim. Fix n, S as stated. Let $N = \{i_m : m < n\}$. By the game rules, only $I_{i_m}^m$ changes in round m, so $I_i^n = E_i$ for each $i \in S \setminus N$. So $\bigcup_{i\in S} \mathrm{cl}\, I_i^n = \bigcup_{i\in S\cap N} \mathrm{cl}\, I_i^n \cup \bigcup_{i\in S\setminus N} E_i$. But $\bigcup_{i\in S\setminus N} E_i$ is closed by assumption on the E_i, so $\bigcup_{i\in S} \mathrm{cl}\, I_i^n$ itself is a finite union of closed sets, and so closed. The equivalence to $\mathrm{cl}(\bigcup_{i\in S} I_i^n) = \bigcup_{i\in S} \mathrm{cl}\, I_i^n$ is easy. This proves the claim. □

After ω rounds, the game ends. Its outcome is the sequence $(\mathbb{I}_i : i \in \mathcal{E})$ of subsets of X, where $\mathbb{I}_i = \bigcup_{n<\omega} I_i^n$ for each $i \in \mathcal{E}$. Since the I_i^n $(i \in \mathcal{E})$ are pairwise disjoint for each n, the $\mathbb{I}_i$ $(i \in \mathcal{E})$ are also pairwise disjoint, and $E_i = I_i^0 \subseteq \mathbb{I}_i$ for each i.

We say that $\forall$ *plays well* if:

A1. $\{n < \omega : i_n = i\}$ is infinite for each $i \in \mathcal{E}$,
A2. $O_n \neq \emptyset$ whenever $n < \omega$ and $i_n \neq i_m$ for every $m < n$,
A3. $O_m \subseteq O_n$ whenever $n < m < \omega$ and $i_n = i_m$,
A4. $\bigcap_{n<\omega} O_n = \bigcap\{O_n : n < \omega,\ i_n = i\}$ for each $i \in \mathcal{E}$.

In short, $\forall$ chooses each index in $\mathcal{E}$ infinitely often, and his choices of O_n whenever he picks a particular index form a decreasing chain whose largest member is non-empty and whose intersection is independent of the index.

Now fix $\varepsilon > 0$. Let $\varepsilon_n = \varepsilon/(n + 2)$ for each $n < \omega$. Then $\varepsilon > \varepsilon_0 > \varepsilon_1 > \cdots > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. We say that $\exists$ *plays well* (this notion is dependent on ε but we do not make ε explicit in the notation) if in each round n, she defines

$$P_n = O_n \setminus \mathrm{cl}\Big(\bigcup_{i\in\mathcal{E}} I_i^n\Big), \tag{6.3}$$

chooses (using Zorn's lemma) a maximal subset $Z_n \subseteq P_n$ such that $d(x, y) \geq \varepsilon_n$ for each distinct $x, y \in Z_n$, and ensures that $Z_n \subseteq I_{i_n}^{n+1}$.

Let us make some observations about P_n and Z_n.

Z1. $\langle d \rangle Z_n = \emptyset$ (because for each $x \in X$, the set $N_{\varepsilon_n/2}(x) \cap Z_n$ has at most one element). So by Lemma 5.2(3, 5), Z_n is closed and $\operatorname{int} Z_n = \emptyset$.

Claim 6.2.2. $\operatorname{cl} P_n = \operatorname{cl} O_n$.

Proof of claim. Let $S = \bigcup_{i \in \mathcal{E}} I_i^n$ and $N = \operatorname{cl}(S) \setminus \langle d \rangle S$. By (6.3) and (6.1), $P_n = O_n \setminus \operatorname{cl} S$ and $O_n \cap \langle d \rangle S = \emptyset$. So $P_n = O_n \setminus (\operatorname{cl} S \setminus \langle d \rangle S) = O_n \setminus N$.

By Lemma 5.2(1), $N \subseteq S$, so by Lemma 5.2(5), $\operatorname{int} N \subseteq N \cap \operatorname{int} S \subseteq N \cap \langle d \rangle S = \emptyset$. So $\operatorname{int} N = \emptyset$, and by Lemma 5.1(2) we obtain $\operatorname{cl} P_n = \operatorname{cl}(O_n \setminus N) = \operatorname{cl} O_n$. This proves the claim. □

If $O_n = \emptyset$ then plainly $P_n = Z_n = \emptyset$. Suppose then that $O_n \neq \emptyset$. By the claim, $P_n \neq \emptyset$, and:

Z2. Z_n is non-empty (because P_n is non-empty and any singleton subset of P_n satisfies the ε_n condition),

Z3. $d(x, Z_n) < \varepsilon_n$ for every $x \in P_n$ (else x can be added to Z_n, contradicting its maximality). Recall that $d(x, Z_n) = \inf\{d(x, z) : z \in Z_n\}$, which is defined because Z_n is non-empty. By Claim 6.2.2 we get $d(x, Z_n) \leq \varepsilon_n < \varepsilon$ for every $x \in \operatorname{cl} P_n = \operatorname{cl} O_n$.

Clearly, if $\exists$ plays well, the Z_n $(n < \omega)$ are pairwise disjoint and $Z_m \subseteq I_{i_m}^n$ for all $m < n < \omega$.

Claim 6.2.3. *Suppose that both players play well. Then for each $i \in \mathcal{E}$:*

1. $\mathbb{I}_i \neq \emptyset$.
2. *If $n < \omega$ and $i_n = i$, then $d(x, \mathbb{I}_i) < \varepsilon$ for every $x \in \operatorname{cl} O_n$.*
3. $\displaystyle\bigcup_{n<\omega} \operatorname{cl} I_i^n \subseteq \mathbb{I}_i$.
4. $\displaystyle\langle d \rangle \mathbb{I}_i = \bigcup_{n<\omega} \langle d \rangle I_i^n \cup \bigcap_{n<\omega} \operatorname{cl} O_n$.

Proof of claim. Fix $i \in \mathcal{E}$. For part 1, using A1, let $n < \omega$ be least such that $i_n = i$. By A2, $O_n \neq \emptyset$. So by Z2, $Z_n \neq \emptyset$, and $Z_n \subseteq I_{i_n}^{n+1} \subseteq \mathbb{I}_i$ since $\exists$ plays well.

For part 2, let $x \in \operatorname{cl} O_n$. So $O_n \neq \emptyset$, and hence $Z_n \neq \emptyset$. By Z3, $d(x, Z_n) < \varepsilon$. But again, $Z_n \subseteq I_{i_n}^{n+1} \subseteq \mathbb{I}_i$ as $\exists$ plays well, so $d(x, \mathbb{I}_i) \leq d(x, Z_n) < \varepsilon$ as required.

For part 3, let $n < \omega$. By A1, there is $m \geq n$ with $i_m = i$. By (6.2), $\operatorname{cl} I_i^n \subseteq \operatorname{cl} I_i^m \subseteq I_i^{m+1} \subseteq \mathbb{I}_i$.

We move to the last part. First we prove that $\bigcup_{n<\omega} \langle d \rangle I_i^n \cup \bigcap_{n<\omega} \operatorname{cl} O_n \subseteq \langle d \rangle \mathbb{I}_i$. For each $n < \omega$ we have $I_i^n \subseteq \mathbb{I}_i$ and so $\langle d \rangle I_i^n \subseteq \langle d \rangle \mathbb{I}_i$. It follows that $\bigcup_{n<\omega} \langle d \rangle I_i^n \subseteq \langle d \rangle \mathbb{I}_i$. To show that $\bigcap_{n<\omega} \operatorname{cl} O_n \subseteq \langle d \rangle \mathbb{I}_i$ as well, take $x \in \bigcap_{n<\omega} \operatorname{cl} O_n$. Let $\delta > 0$. By A1 and since the Z_n are pairwise disjoint, we may choose $n < \omega$ such that $\varepsilon_n < \delta$, $i_n = i$, and $x \notin Z_n$. By assumption, $x \in \operatorname{cl} O_n$, so by Z3, $d(x, Z_n) \leq \varepsilon_n < \delta$. So there is $z \in Z_n \cap N_\delta(x)$ with $x \neq z$ (since $x \notin Z_n$). Since $Z_n \subseteq \mathbb{I}_i$, the point z witnesses $N_\delta(x) \cap \mathbb{I}_i \setminus \{x\} \neq \emptyset$. This holds for all $\delta > 0$, and it follows that $x \in \langle d \rangle \mathbb{I}_i$. So $\bigcap_{n<\omega} \operatorname{cl} O_n \subseteq \langle d \rangle \mathbb{I}_i$ as required.

To prove the converse inclusion $\langle d \rangle \mathbb{I}_i \subseteq \bigcup_{n<\omega} \langle d \rangle I_i^n \cup \bigcap_{n<\omega} \operatorname{cl} O_n$, fix $n < \omega$ with $i_n = i$. We first show by induction on m that $I_i^m \subseteq \operatorname{cl} I_i^n \cup \operatorname{cl} O_n$ for each $m \geq n$. For $m = n$ it is clear. Assume inductively that it holds for m. If $i_m \neq i$ then $I_i^{m+1} = I_i^m \subseteq \operatorname{cl} I_i^n \cup \operatorname{cl} O_n$ by the induction hypothesis. If instead $i_m = i$, then

$$\begin{aligned}
I_i^{m+1} &\subseteq O_m \cup \operatorname{cl} I_i^m && \text{by (6.2)} \\
&\subseteq O_n \cup \operatorname{cl} I_i^m && \text{since } i_m = i_n \text{ and so (by A3) } O_m \subseteq O_n \\
&\subseteq O_n \cup \operatorname{cl}(\operatorname{cl} I_i^n \cup \operatorname{cl} O_n) && \text{by induction hypothesis and monotonicity of cl} \\
&= O_n \cup \operatorname{cl} I_i^n \cup \operatorname{cl} O_n && \text{as } \operatorname{cl} I_i^n \cup \operatorname{cl} O_n \text{ is closed} \\
&= \operatorname{cl} I_i^n \cup \operatorname{cl} O_n && \text{since } O_n \subseteq \operatorname{cl} O_n.
\end{aligned}$$

This completes the induction. Hence, $\mathbb{I}_i = \bigcup_{m \geq n} I_i^m \subseteq \operatorname{cl} I_i^n \cup \operatorname{cl} O_n$. Now we obtain

$$\begin{aligned}
\langle d\rangle \mathbb{I}_i &\subseteq \langle d\rangle(\operatorname{cl} I_i^n \cup \operatorname{cl} O_n) && \text{by the above and monotonicity of } \langle d\rangle \\
&= \langle d\rangle \operatorname{cl} I_i^n \cup \langle d\rangle \operatorname{cl} O_n && \text{by additivity of } \langle d\rangle \\
&= \langle d\rangle I_i^n \cup \langle d\rangle O_n && \text{by Lemma 5.2(4), since } X \text{ is } \mathrm{T}_D \\
&\subseteq \Big(\bigcup_{m<\omega} \langle d\rangle I_i^m\Big) \cup \operatorname{cl} O_n && \text{by definition of union, and Lemma 5.2(1)}
\end{aligned}$$

This holds for all n with $i_n = i$, so

$$\langle d\rangle \mathbb{I}_i \subseteq \bigcap_{\substack{n<\omega \\ i_n=i}} \Big(\Big(\bigcup_{m<\omega} \langle d\rangle I_i^m\Big) \cup \operatorname{cl} O_n\Big) = \Big(\bigcup_{m<\omega} \langle d\rangle I_i^m\Big) \cup \bigcap_{\substack{n<\omega \\ i_n=i}} \operatorname{cl} O_n = \Big(\bigcup_{m<\omega} \langle d\rangle I_i^m\Big) \cup \bigcap_{n<\omega} \operatorname{cl} O_n,$$

the last step using A4. This proves the claim. □

With these results in hand, we can prove our two theorems. First, Theorem 6.1.

Proof of Theorem 6.1. As in the theorem's statement, let there be given open sets $\emptyset \neq \mathbb{T} \subseteq \mathbb{U}$, pairwise disjoint subsets $E_i \subseteq \mathbb{T}$ ($i \in \mathcal{E}$) with $\langle d\rangle \bigcup_{i\in\mathcal{E}} E_i = \emptyset$, and a real number $\varepsilon > 0$. By Lemma 5.2(3), every subset of $\bigcup_{i\in\mathcal{E}} E_i$ is closed, so certainly $\bigcup_{i\in S} E_i$ is closed for every $S \subseteq \mathcal{E}$. We are also given that $\mathcal{E}$ is non-empty and countable. So $\forall$ and $\exists$ can play $\mathfrak{G}(E_i : i \in \mathcal{E})$. $\exists$ will play so that

$$\langle d\rangle \bigcup_{i\in\mathcal{E}} I_i^n = \emptyset \quad \text{for each } n < \omega. \tag{6.4}$$

We have $I_i^0 = E_i$ for each i, so (6.4) is true for $n = 0$ by assumption on the E_i. Recall that $\varepsilon_n = \varepsilon/(n+2)$ for each $n < \omega$. In round n, $\forall$ picks (O_n, i_n), where

$$O_n = \begin{cases} \mathbb{T}, & \text{if } i_n \neq i_m \text{ for all } m < n, \\ \mathbb{T} \cap \bigcup_{x\in\operatorname{cl}(\mathbb{T})\setminus\mathbb{U}} N_{\varepsilon_n}(x), & \text{otherwise.} \end{cases} \tag{6.5}$$

By (6.4), this trivially satisfies (6.1). He also arranges to let $i_n = i$ for infinitely many n, for each $i \in \mathcal{E}$, so condition A1 will hold. A2 holds because $\mathbb{T} \neq \emptyset$. Given A1, it follows from (6.5) that A3 and A4 also hold, and thus, he plays well.

$\exists$ responds to $\forall$'s move in round n by choosing Z_n as above and letting $I_{i_n}^{n+1} = I_{i_n}^n \cup Z_n$ (so she plays well), and $I_i^{n+1} = I_i^n$ for every $i \in \mathcal{E} \setminus \{i_n\}$. We check (6.2). By (6.4) for n and Lemma 5.2(3), $I_{i_n}^n$ is closed; by Z1, Z_n is closed too; and by (6.3), $Z_n \subseteq P_n \subseteq O_n$. Hence, $\operatorname{cl} I_{i_n}^n = I_{i_n}^n \subseteq I_{i_n}^n \cup Z_n = I_{i_n}^{n+1} \subseteq \operatorname{cl} I_{i_n}^n \cup O_n$, as required for (6.2). Also, $Z_n \subseteq P_n = O_n \setminus \bigcup_{i\in\mathcal{E}} \operatorname{cl} I_i^n$, and it follows that $\operatorname{cl} I_i^{n+1}$ ($i \in \mathcal{E}$) are pairwise disjoint. Moreover, since $\langle d\rangle Z_n = \emptyset$ by Z1, her move preserves (6.4).

Let the outcome of play be the sets $\mathbb{I}_i$ ($i \in \mathcal{E}$). We check that these sets meet the conditions of the theorem. Being the game's outcome, they are pairwise disjoint, and $E_i \subseteq \mathbb{I}_i$ for each i. Let $i \in \mathcal{E}$. First, we check that $\emptyset \neq \mathbb{I}_i \subseteq \mathbb{T}$ and $d(x, \mathbb{I}_i) < \varepsilon$ for each $x \in \operatorname{cl}\mathbb{T}$. By Claim 6.2.3, $\mathbb{I}_i \neq \emptyset$. Let $n < \omega$ be least such that $i_n = i$. By the first clause of (6.5), $O_n = \mathbb{T}$, and Claim 6.2.3 yields $d(x, \mathbb{I}_i) < \varepsilon$ for every $x \in \operatorname{cl}\mathbb{T}$. Also, $\mathbb{I}_i = E_i \cup \bigcup\{Z_n : n < \omega,\ i_n = i\} \subseteq \mathbb{T} \cup \bigcup_{n<\omega} O_n = \mathbb{T}$.

Second, we check that $\langle d\rangle \mathbb{I}_i = \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$. By (6.4), $\langle d\rangle I_i^n = \emptyset$ for each n. So Claim 6.2.3 yields $\langle d\rangle \mathbb{I}_i = \bigcap_{n<\omega} \operatorname{cl} O_n$. It is therefore sufficient to prove the next claim.

Claim 6.2.4. $\bigcap_{n<\omega} \operatorname{cl} O_n = \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$.

Proof of claim. Certainly, each $x \in \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$ lies in $\operatorname{cl} O_n$ for each n, because for every $\delta > 0$,

$$O_n \cap N_\delta(x) \supseteq \Big(\mathbb{T} \cap \bigcup_{y \in \mathrm{cl}(\mathbb{T})\setminus\mathbb{U}} N_{\varepsilon_n}(y)\Big) \cap N_{\min(\delta,\varepsilon_n)}(x) = \mathbb{T} \cap N_{\min(\delta,\varepsilon_n)}(x) \neq \emptyset.$$

So $\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} \subseteq \bigcap_{n<\omega} \mathrm{cl}\, O_n$.

Now we prove the converse, $\bigcap_{n<\omega} \mathrm{cl}\, O_n \subseteq \mathrm{cl}(\mathbb{T}) \setminus \mathbb{U}$. First note that if $\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} = \emptyset$ then by A1 and (6.5), infinitely many O_n are empty as well, so $\bigcap_{n<\omega} \mathrm{cl}\, O_n = \emptyset$ and we are done.

Suppose then that $\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} \neq \emptyset$. Certainly, $\bigcap_{n<\omega} \mathrm{cl}\, O_n \subseteq \mathrm{cl}\, O_0 = \mathrm{cl}\, \mathbb{T}$. It remains to show that $\mathbb{U} \cap \bigcap_{n<\omega} \mathrm{cl}\, O_n = \emptyset$. Suppose for contradiction that there is some $x \in \mathbb{U} \cap \bigcap_{n<\omega} \mathrm{cl}\, O_n$. As $\mathbb{U}$ is open, we can choose $\delta > 0$ with $N_\delta(x) \subseteq \mathbb{U}$. As $\forall$ played well, we can pick $m < n < \omega$ such that $i_m = i_n$ and $\varepsilon_n < \delta$. Then $O_n = \mathbb{T} \cap \bigcup_{y \in \mathrm{cl}(\mathbb{T})\setminus\mathbb{U}} N_{\varepsilon_n}(y)$ by (6.5). So $d(y, \mathrm{cl}(\mathbb{T}) \setminus \mathbb{U}) < \varepsilon_n$ for each $y \in O_n$ — note that this is defined since $\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} \neq \emptyset$. Since $x \in \mathrm{cl}\, O_n$, it follows that $d(x, \mathrm{cl}(\mathbb{T}) \setminus \mathbb{U}) \leq \varepsilon_n < \delta$. As $N_\delta(x) \subseteq \mathbb{U}$, this is a contradiction, and proves the claim, and the theorem. □

We can also prove Theorem 6.3.

Proof of Theorem 6.3. Let $\mathbb{G}, \mathcal{G}, \mathcal{B}, E_i, \varepsilon$ be as in the theorem's statement: so $\mathbb{G}$ is non-empty and open, $|\mathcal{G}|, |\mathcal{B}| \leq \omega$, $\mathcal{G} \cap \mathcal{B} = \emptyset$, and $\mathcal{B} \neq \emptyset$. Fix arbitrary $b \in \mathcal{B}$, and let $\mathcal{E} = \mathcal{G} \cup \mathcal{B}$; so $\mathcal{E}$ is non-empty and countable. It is given that $E_i \subseteq \mathbb{G}$ $(i \in \mathcal{E})$ are pairwise disjoint and $\bigcup_{i\in S} E_i$ is closed and nowhere dense for each $S \subseteq \mathcal{E}$. So $\forall$ and $\exists$ can play $\mathfrak{G}(E_i : i \in \mathcal{E})$.

They play as follows. In round n, $\forall$ plays (O_n, i_n), where

$$O_n = \mathbb{G} \setminus \langle d \rangle \Big(\bigcup_{i \in \mathcal{E}} I_i^n \Big). \tag{6.6}$$

Condition (6.1) is trivially met. He chooses $i_0 = b$, and also arranges to choose each index in $\mathcal{E}$ infinitely often. As we will see below, he plays well.

$\exists$ will play so that the following properties hold for each $n < \omega$:

P1. $\mathrm{cl}\, I_i^n$ $(i \in \mathcal{E})$ are pairwise disjoint subsets of $\mathbb{G}$.
P2. $\bigcup_{i\in\mathcal{B}} I_i^n$ is nowhere dense.
P3. I_i^n is open for each $i \in \mathcal{G} \cap \{i_m : m < n\}$ (that is, for each $i \in \mathcal{G}$ that $\forall$ already picked in some round earlier than n).

When $n = 0$, we have $I_i^0 = E_i$ for each $i \in \mathcal{E}$. P1 and P2 are then given, and P3 holds vacuously.

Assume that P1–P3 hold for n. $\exists$ responds to $\forall$'s move (O_n, i_n) in round n as follows. Of course she sets $I_i^{n+1} = I_i^n$ for $i \in \mathcal{E} \setminus \{i_n\}$, and defines Z_n as described above.

Case 1: $i_n \in \mathcal{B}$**.** Then $\exists$ sets $I_{i_n}^{n+1} = \mathrm{cl}(I_{i_n}^n) \cup Z_n$. This clearly satisfies (6.2). The set Z_n is a closed (by Z1) subset of $\mathbb{G}$ (since $Z_n \subseteq O_n \subseteq \mathbb{G}$ by (6.3) and (6.6)), and it is disjoint from $\mathrm{cl} \bigcup_{i\in\mathcal{E}} I_i^n$ (by (6.3) and $Z_n \subseteq P_n$), so P1 for $n+1$ follows. P2 is kept, since

$$\begin{aligned}
\mathrm{int}\,\mathrm{cl} \textstyle\bigcup_{i\in\mathcal{B}} I_i^{n+1} &= \mathrm{int}\,\mathrm{cl}\left(\textstyle\bigcup_{i\in\mathcal{B}\setminus\{i_n\}} I_i^n \cup \mathrm{cl}\, I_{i_n}^n \cup Z_n\right) && \text{by } \exists\text{'s move} \\
&= \mathrm{int}\left(\mathrm{cl}(\textstyle\bigcup_{i\in\mathcal{B}} I_i^n) \cup Z_n\right) && \text{by additivity of cl, and Z1} \\
&= \mathrm{int}\,\mathrm{cl} \textstyle\bigcup_{i\in\mathcal{B}} I_i^n && \text{by Z1 and Lemma 5.1(1)} \\
&= \emptyset && \text{by P2 for } n.
\end{aligned}$$

P3 is unchanged because $i_n \notin \mathcal{G}$.

Case 2: $i_n \in \mathcal{G}$**.** Then $\exists$ chooses an open set $I_{i_n}^{n+1}$ satisfying

$$L \stackrel{\text{def}}{=} \mathrm{cl}(I_{i_n}^n) \cup Z_n \subseteq I_{i_n}^{n+1} \subseteq \mathrm{cl}\, I_{i_n}^{n+1} \subseteq \mathbb{G} \setminus \bigcup_{j \in \mathcal{E}\setminus\{i_n\}} \mathrm{cl}\, I_j^n \stackrel{\text{def}}{=} R. \tag{6.7}$$

We need to check some things here. First, L is closed, since (by Z1) Z_n is closed. Second, R is open, since $\mathbb{G}$ is open and (by Claim 6.2.1) $\bigcup_{j\in\mathcal{E}\setminus\{i_n\}} \operatorname{cl} I_j^n$ is closed. Third, $L \subseteq R$, since $\operatorname{cl} I_{i_n}^n \subseteq R$ by P1, and $Z_n \subseteq P_n = O_n \setminus \operatorname{cl} \bigcup_{i\in\mathcal{E}} I_i^n \subseteq \mathbb{G} \setminus \bigcup_{i\in\mathcal{E}} \operatorname{cl} I_i^n \subseteq R$ by definition of Z_n, (6.3), (6.6), and (6.7). So as X is normal (Lemma 5.8), an open set Q with $L \subseteq Q \subseteq \operatorname{cl} Q \subseteq R$ can be found. $\exists$ lets $I_{i_n}^{n+1}$ be any such Q, so satisfying (6.7).

Next we check that $\exists$'s move satisfies (6.2). We have $\operatorname{cl} I_{i_n}^n \subseteq I_{i_n}^{n+1}$ by (6.7). Also,

$$\begin{aligned}
I_{i_n}^{n+1} \subseteq R &= \mathbb{G} \setminus \textstyle\bigcup_{j\in\mathcal{E}\setminus\{i_n\}} \operatorname{cl} I_j^n && \text{by (6.7)}\\
&\subseteq \left(\mathbb{G} \setminus \textstyle\bigcup_{j\in\mathcal{E}} \operatorname{cl} I_j^n\right) \cup \operatorname{cl} I_{i_n}^n && \text{as } A \subseteq (A\setminus B)\cup B\\
&= \left(\mathbb{G} \setminus \operatorname{cl} \textstyle\bigcup_{j\in\mathcal{E}} I_j^n\right) \cup \operatorname{cl} I_{i_n}^n && \text{by Claim 6.2.1}\\
&\subseteq \left(\mathbb{G} \setminus \langle d\rangle \textstyle\bigcup_{j\in\mathcal{E}} I_j^n\right) \cup \operatorname{cl} I_{i_n}^n && \text{by Lemma 5.2(1)}\\
&= O_n \cup \operatorname{cl} I_{i_n}^n && \text{by (6.6).}
\end{aligned}$$

This confirms (6.2). Finally we check that P1–P3 still hold. P1 follows from (6.7) and since it holds for n. P2 is unchanged, and P3 holds since $\exists$ chose $I_{i_n}^{n+1}$ to be open.

Since $Z_n \subseteq I_{i_n}^{n+1}$ in both cases, $\exists$ plays well. We will soon see that $\forall$ plays well too, but first, a handy claim.

Claim 6.2.5. $\mathbb{G} \subseteq \operatorname{cl}(O_n) \cup \bigcup_{i\in\mathcal{G}} \operatorname{cl} I_i^n$ *for each* $n < \omega$.

Proof of claim. Let $n < \omega$ be given. By (6.6) and Lemma 5.2(1), $\mathbb{G} \setminus \operatorname{cl} O_n \subseteq \mathbb{G} \setminus O_n \subseteq \langle d\rangle \bigcup_{i\in\mathcal{E}} I_i^n \subseteq \operatorname{cl} \bigcup_{i\in\mathcal{E}} I_i^n$. So

$$\begin{aligned}
\mathbb{G} \setminus \operatorname{cl} O_n &\subseteq \operatorname{int} \operatorname{cl} \textstyle\bigcup_{i\in\mathcal{E}} I_i^n && \text{as } \mathbb{G} \setminus \operatorname{cl} O_n \text{ is open}\\
&= \operatorname{int}\left(\operatorname{cl} \textstyle\bigcup_{i\in\mathcal{B}} I_i^n \cup \operatorname{cl} \textstyle\bigcup_{i\in\mathcal{G}} I_i^n\right) && \text{by additivity of cl and } \mathcal{E} = \mathcal{B} \cup \mathcal{G}\\
&= \operatorname{int} \operatorname{cl} \textstyle\bigcup_{i\in\mathcal{G}} I_i^n && \text{by P2 and Lemma 5.1(1)}\\
&\subseteq \textstyle\bigcup_{i\in\mathcal{G}} \operatorname{cl} I_i^n && \text{by Claim 6.2.1}
\end{aligned}$$

The claim now follows. $\square$

We can now see that $\forall$ plays well. He chooses each $i \in \mathcal{E}$ infinitely often, so A1 holds. A2 holds because *every* O_n is non-empty, as we now show. Write $E = \bigcup_{i\in\mathcal{E}} E_i$. By (6.6), $O_0 = \mathbb{G} \setminus \langle d\rangle \bigcup_{i\in\mathcal{E}} I_i^0 = \mathbb{G} \setminus \langle d\rangle E$. Since E is nowhere dense, we have $\operatorname{int} \langle d\rangle E \subseteq \operatorname{int} \operatorname{cl} E = \emptyset$. So by Lemma 5.1(2),

$$\operatorname{cl} O_0 = \operatorname{cl} \mathbb{G}. \tag{6.8}$$

As $\mathbb{G} \neq \emptyset$, this yields $O_0 \neq \emptyset$. So as $\exists$ plays well, by Z2 we have $\emptyset \neq Z_0 \subseteq I_b^1$. Now let $n > 0$. Then I_b^n is non-empty because it contains I_b^1. Also, by P1 and Claim 6.2.5 we have $I_b^n \subseteq \mathbb{G} \setminus \bigcup_{i\in\mathcal{G}} \operatorname{cl} I_i^n \subseteq \operatorname{cl} O_n$. It follows that $\operatorname{cl} O_n$, and hence O_n, are non-empty as required. So indeed A2 holds. Since $I_i^n \subseteq I_i^{n+1}$ for each n, i, it follows from (6.6) that $O_0 \supseteq O_1 \supseteq \cdots$. So, using A1, we see that A3–A4 hold, and therefore $\forall$ indeed plays well.

At the end of the game we define $\mathbb{I}_i = \bigcup_{n<\omega} I_i^n$ for $i \in \mathcal{E}$, and let

$$\left.\begin{aligned}
\mathbb{G}_i &= \mathbb{I}_i && \text{for } i \in \mathcal{G}\\
\mathbb{B}_j &= \mathbb{I}_j && \text{for } j \in \mathcal{B} \setminus \{b\}\\
\mathbb{B}_b &= \mathbb{G} \setminus \bigcup_{i\in\mathcal{E}\setminus\{b\}} \mathbb{I}_i &&\\
\mathbb{D} &= \operatorname{cl}(\mathbb{G}) \setminus \bigcup_{i\in\mathcal{G}} \mathbb{G}_i. &&
\end{aligned}\right\} \tag{6.9}$$

We check that the requirements of the theorem are met. Being the game's outcome, the $\mathbb{I}_i$ are pairwise disjoint and $E_i \subseteq \mathbb{I}_i$ for each $i \in \mathcal{E}$. As both players played well, by Claim 6.2.3 each $\mathbb{I}_i$ is non-empty. By P1, $\mathbb{I}_i = \bigcup_{n<\omega} I_i^n \subseteq \mathbb{G}$ for each i. It follows by (6.9) that $\mathbb{I}_b \subseteq \mathbb{B}_b$ — so $\mathbb{I}_j \subseteq \mathbb{B}_j$ for every $j \in \mathcal{B}$. This is enough to show that the $\mathbb{G}_i, \mathbb{B}_j$ are non-empty and pairwise disjoint subsets of $\mathbb{G}$. So $(\mathbb{G}_i, \mathbb{B}_j : i \in \mathcal{G},\ j \in \mathcal{B})$ is a partition of $\mathbb{G}$. Also, $E_i \subseteq \mathbb{I}_i = \mathbb{G}_i$ for each $i \in \mathcal{G}$, and $E_j \subseteq \mathbb{I}_j \subseteq \mathbb{B}_j$ for each $j \in \mathcal{B}$. For each $i \in \mathcal{G}$, by A1 there is $m < \omega$ with $i_m = i$; then $\mathbb{G}_i = \bigcup\{I_i^n : m < n < \omega\}$; by P3, this is a union of open sets, and so is open.

For the remaining requirements (points 4–5) of the theorem, we need a claim.

Claim 6.2.6. $\mathbb{D} \subseteq \langle d\rangle \mathbb{I}_i$ *for each* $i \in \mathcal{E}$.

Proof of claim. Let $n < \omega$ be arbitrary. By Claim 6.2.5, $\mathbb{G} \subseteq \mathrm{cl}(O_n) \cup \bigcup_{i\in\mathcal{G}} \mathrm{cl}\, I_i^n$. By Claim 6.2.1, $\bigcup_{i\in\mathcal{G}} \mathrm{cl}\, I_i^n$ is closed, so we actually have $\mathrm{cl}\,\mathbb{G} \subseteq \mathrm{cl}(O_n) \cup \bigcup_{i\in\mathcal{G}} \mathrm{cl}\, I_i^n$. So by (6.9),

$$\mathbb{D} = \mathrm{cl}\,\mathbb{G} \setminus \bigcup_{i\in\mathcal{G}} \mathbb{G}_i \subseteq \Big(\mathrm{cl}(O_n) \cup \bigcup_{i\in\mathcal{G}} \mathrm{cl}\, I_i^n\Big) \setminus \bigcup_{i\in\mathcal{G}} \mathbb{G}_i. \tag{6.10}$$

But by Claim 6.2.3(3) and (6.9), $\bigcup_{i\in\mathcal{G}} \mathrm{cl}\, I_i^n \subseteq \bigcup_{i\in\mathcal{G}} \mathbb{I}_i = \bigcup_{i\in\mathcal{G}} \mathbb{G}_i$, so by (6.10) we obtain $\mathbb{D} \subseteq \mathrm{cl}\, O_n$. This holds for all n, so by Claim 6.2.3, $\mathbb{D} \subseteq \bigcap_{n<\omega} \mathrm{cl}\, O_n \subseteq \langle d\rangle \mathbb{I}_i$ for each $i \in \mathcal{E}$. This proves the claim. □

Now we can finish easily. For point 4 of the theorem, we first show that $\mathbb{D} = \mathrm{cl}\,\mathbb{G}_i \setminus \mathbb{G}_i$ for $i \in \mathcal{G}$, and $\mathbb{D} = \langle d\rangle \mathbb{B}_j$ for $j \in \mathcal{B}$. For each $j \in \mathcal{B}$, by (6.9) we have $\mathbb{B}_j \subseteq \mathbb{G} \setminus \bigcup_{i\in\mathcal{G}} \mathbb{G}_i \subseteq \mathbb{D}$. Since $\mathbb{D}$ is clearly closed, $\langle d\rangle \mathbb{B}_j \subseteq \mathrm{cl}\,\mathbb{B}_j \subseteq \mathbb{D}$. Conversely, since $\mathbb{I}_j \subseteq \mathbb{B}_j$, by Claim 6.2.6 we get $\mathbb{D} \subseteq \langle d\rangle \mathbb{I}_j \subseteq \langle d\rangle \mathbb{B}_j$.

Similarly, take $i \in \mathcal{G}$. Since the $\mathbb{G}_l$ $(l \in \mathcal{G})$ are pairwise disjoint open subsets of $\mathbb{G}$, we have $\mathrm{cl}\,\mathbb{G}_i \subseteq \mathrm{cl}(\mathbb{G}) \setminus \bigcup_{l\in\mathcal{G}\setminus\{i\}} \mathbb{G}_l$ and hence $\mathrm{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i \subseteq \mathrm{cl}(\mathbb{G}) \setminus \bigcup_{l\in\mathcal{G}} \mathbb{G}_l = \mathbb{D}$. Conversely, by Claim 6.2.6 and Lemma 5.2(5) we have $\mathbb{D} \subseteq \langle d\rangle \mathbb{I}_i = \langle d\rangle \mathbb{G}_i = \mathrm{cl}\,\mathbb{G}_i$. By (6.9), $\mathbb{D} \cap \mathbb{G}_i = \emptyset$. So $\mathbb{D} \subseteq \mathrm{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i$, as required.

Finally, for point 5, let $x \in \mathrm{cl}\,\mathbb{G}$. By (6.8), $x \in \mathrm{cl}\, O_0$. Since $i_0 = b$, Claim 6.2.3 yields $d(x, \mathbb{I}_b) < \varepsilon$. So there is $y \in \mathbb{I}_b$ with $d(x, y) < \varepsilon$. Now let $i \in \mathcal{G}$. We showed that $\mathbb{I}_b \subseteq \mathbb{B}_b \subseteq \mathbb{D} \subseteq \mathrm{cl}\,\mathbb{G}_i$. So $y \in \mathrm{cl}\,\mathbb{G}_i$, and hence we can take $z \in \mathbb{G}_i$ with $d(y, z) < \varepsilon - d(x, y)$. Then $d(x, \mathbb{G}_i) \le d(x, z) \le d(x, y) + d(y, z) < \varepsilon$ as required. The proof that $d(x, \mathbb{B}_j) < \varepsilon$ for $j \in \mathcal{B}$ is similar, using that $\mathbb{D} \subseteq \mathrm{cl}\,\mathbb{B}_j$. □

6.3. A corollary

We will use the following corollary in Proposition 7.10 to handle non-rooted frames. In the simple case where $\mathbb{S}_0 = \mathbb{S}_1 = \emptyset$ and (so) $\mathbb{T} = \mathbb{U}$, it says that any non-empty open set $\mathbb{U}$ has regular open subsets $\mathbb{U}_0, \mathbb{U}_1$ whose closures (1) are disjoint within $\mathbb{U}$ and (2) contain all 'boundary points' of $\mathbb{U}$ (points in $\mathrm{cl}\,\mathbb{U} \setminus \mathbb{U}$). It is proved using Lemma 5.7 (essentially normality) to 'fatten' two sets (obtained from Theorem 6.1) whose derivatives are exactly the set of boundary points.

Corollary 6.5. *Let* $\mathbb{U}$ *be an open subspace of a dense-in-itself metric space* X*, and suppose that* $\mathbb{S}_0$, $\mathbb{S}_1$ *are open subsets of* $\mathbb{U}$ *such that* $\mathbb{U} \cap \mathrm{cl}\,\mathbb{S}_0 \cap \mathrm{cl}\,\mathbb{S}_1 = \emptyset$ *and* $\mathbb{T} = \mathbb{U} \setminus \mathrm{cl}(\mathbb{S}_0 \cup \mathbb{S}_1) \neq \emptyset$. *Then there are regular open subsets* $\mathbb{U}_0, \mathbb{U}_1$ *of* $\mathbb{U}$ *such that* $\mathbb{U} \cap \mathrm{cl}\,\mathbb{U}_0 \cap \mathrm{cl}\,\mathbb{U}_1 = \emptyset$*, and for each* $i = 0, 1$*:*

1. $\mathbb{U} \cap \mathrm{cl}\,\mathbb{S}_i \subseteq \mathbb{U}_i$,
2. *writing* $\mathbb{T}_i = \mathbb{U}_i \setminus \mathrm{cl}\,\mathbb{S}_i$*, we have* $\mathbb{T}_i \neq \emptyset$ *and* $\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} \subseteq \mathrm{cl}\,\mathbb{T}_i$.

Proof. Since $\mathbb{T}$ is a non-empty open subset of $\mathbb{U}$, we can use Theorem 6.1 to choose disjoint non-empty subsets $\mathbb{I}_0, \mathbb{I}_1 \subseteq \mathbb{T}$ such that $\langle d\rangle \mathbb{I}_0 = \langle d\rangle \mathbb{I}_1 = \mathrm{cl}(\mathbb{T}) \setminus \mathbb{U}$.

We now work in the subspace $\mathbb{U}$. Recall that $\mathrm{cl}_{\mathbb{U}}$ denotes the closure operator in the subspace topology on $\mathbb{U}$, so $\mathrm{cl}_{\mathbb{U}} K = \mathbb{U} \cap \mathrm{cl}\, K$ for subsets $K \subseteq \mathbb{U}$. The sets

$$\mathrm{cl}_{\mathbb{U}}\, \mathbb{S}_0,\ \mathrm{cl}_{\mathbb{U}}\, \mathbb{S}_1,\ \mathbb{I}_0,\ \mathbb{I}_1$$

are pairwise disjoint (by assumptions) and closed in $\mathbb{U}$. (Each $\mathbb{I}_i$ is closed in $\mathbb{U}$ because by Lemma 5.2(1), $\mathrm{cl}_{\mathbb{U}}\, \mathbb{I}_i = \mathbb{U} \cap \mathrm{cl}\, \mathbb{I}_i = \mathbb{U} \cap (\mathbb{I}_i \cup \langle d\rangle \mathbb{I}_i) = \mathbb{U} \cap (\mathbb{I}_i \cup (\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U})) = \mathbb{U} \cap \mathbb{I}_i = \mathbb{I}_i$.) Hence, $\mathbb{I}_0 \cup \mathrm{cl}_{\mathbb{U}}\, \mathbb{S}_0$ and $\mathbb{I}_1 \cup \mathrm{cl}_{\mathbb{U}}\, \mathbb{S}_1$ are disjoint closed subsets of $\mathbb{U}$. The subspace $\mathbb{U}$ is a metric space in its own right, and so, by Lemma 5.8, normal. Using Lemma 5.7 in $\mathbb{U}$ (taking $Q = \mathbb{U}$), we can find regular open subsets $\mathbb{U}_0, \mathbb{U}_1$ of $\mathbb{U}$ with

$$\mathbb{I}_i \cup \mathrm{cl}_{\mathbb{U}}\, \mathbb{S}_i \subseteq \mathbb{U}_i \subseteq \mathbb{U} \quad \text{for } i = 0, 1, \tag{6.11}$$

and $\mathrm{cl}_{\mathbb{U}}\, \mathbb{U}_0 \cap \mathrm{cl}_{\mathbb{U}}\, \mathbb{U}_1 = \emptyset$. Working back in X again, this says that

$$\mathbb{U} \cap \mathrm{cl}\, \mathbb{U}_0 \cap \mathrm{cl}\, \mathbb{U}_1 = \emptyset. \tag{6.12}$$

For each $i = 0, 1$, write $\mathbb{T}_i = \mathbb{U}_i \setminus \mathrm{cl}\, \mathbb{S}_i$. By (6.11), $\mathbb{I}_i \subseteq \mathbb{U}_i$, and by definition of $\mathbb{I}_i$ we have $\mathbb{I}_i \subseteq \mathbb{T} = \mathbb{U} \setminus \mathrm{cl}(\mathbb{S}_0 \cup \mathbb{S}_1) \subseteq \mathbb{U} \setminus \mathrm{cl}\, \mathbb{S}_i$. Hence, $\mathbb{I}_i \subseteq \mathbb{U}_i \cap \mathbb{U} \setminus \mathrm{cl}\, \mathbb{S}_i = \mathbb{U}_i \setminus \mathrm{cl}\, \mathbb{S}_i = \mathbb{T}_i$. This gives $\mathbb{T}_i \neq \emptyset$ (since $\mathbb{I}_i \neq \emptyset$), and also

$$\mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} = \langle d\rangle \mathbb{I}_i \subseteq \mathrm{cl}\, \mathbb{I}_i \subseteq \mathrm{cl}\, \mathbb{T}_i. \tag{6.13}$$

By (6.11), $\mathbb{U} \cap \mathrm{cl}\, \mathbb{S}_i = \mathrm{cl}_{\mathbb{U}}\, \mathbb{S}_i \subseteq \mathbb{U}_i$. With (6.12), (6.13), and $\mathbb{T}_i \neq \emptyset$, this proves the corollary. □

7. Representations of frames over topological spaces

Our next aim is to use the results of the preceding section to construct a 'representation' from an arbitrary dense-in-itself metric space to any given finite connected locally connected transitive serial Kripke frame. The notion of representation is chosen so as to preserve $\mathcal{L}^{\mu}_{[d]\forall}$-formulas, and this will allow us to prove completeness theorems in the next two sections.

Until the end of section 7.5, we fix a topological space X and a finite Kripke frame $\mathcal{F} = (W, R)$. We will frequently regard the elements of W as propositional atoms.

7.1. Representations

The following definition seems to originate with Shehtman: see equation (71) in [34, §5, p. 25].

Definition 7.1. A map $\rho : X \to W$ is said to be a *representation of $\mathcal{F}$ over X* if for every $w \in W$ we have

$$\langle d\rangle \rho^{-1}(w) = \rho^{-1}(R^{-1}(w)).$$

Here, recall from section 2.1 that $R^{-1}(w) = \{u \in W : Ruw\}$. There are numerous equivalent formulations of this definition. One is $\langle d\rangle \rho^{-1}(S) = \rho^{-1}(R^{-1}(S))$ for every $S \subseteq W$, where $R^{-1}(S) = \bigcup_{w \in S} R^{-1}(w)$ (see [34, lemma 20, p. 25]; the proof of equivalence uses finiteness of $\mathcal{F}$ and distributivity properties). Another is therefore $(X, \rho^{-1} \circ h), x \models \langle d\rangle p$ iff $(\mathcal{F}, h), \rho(x) \models \langle d\rangle p$ for each $x \in X$, $p \in \mathsf{Var}$, and assignment $h : \mathsf{Var} \to \wp W$, using that $\rho : X \to W$ induces a dual map $\rho^{-1} : \wp W \to \wp X$. In the light of this formulation, Proposition 7.5 below is not surprising.

One more equivalent formulation, which we will use frequently, is

$$(X, \rho^{-1}), x \models \langle d\rangle w \iff R(\rho(x), w) \quad \text{for every } x \in X \text{ and } w \in W. \tag{7.1}$$

Here, ρ^{-1} assigns an atom $w \in W$ to the possibly empty subset $\{x \in X : \rho(x) = w\}$ of X. The condition says that for every $x \in X$, the set of points of W with preimages under ρ in every open neighbourhood of x but distinct from x itself is precisely $R(\rho(x))$.

Note that ρ need not be surjective. Indeed, the empty map is vacuously a representation of $\mathcal{F}$ over the empty space — and we definitely do allow empty representations.

It can be checked that if $\rho : X \to W$ is a representation then $\operatorname{rng}\rho$ is the domain of a transitive generated subframe of $\mathcal{F}$. Endow W with the topology generated by $\{R(w) : w \in W\}$ (so the open sets are those $A \subseteq W$ such that $a \in A$ implies $R(a) \subseteq A$). Then every representation of $\mathcal{F}$ over X is an *interior map* from X to W: that is, a map that is both continuous and open. The converse fails in general. For example, let $\mathcal{F}$ be the two-world reflexive frame $(\{0,1\}, \leq)$, and let $X = \mathbb{R}$ with its usual topology. Let $\rho : \mathbb{R} \to \{0,1\}$ be given by $\rho(x) = 0$ if $x \in \mathbb{Z}$, and $\rho(x) = 1$ otherwise. Then ρ is an interior map, but not a representation, since $\langle d\rangle\rho^{-1}(0) = \langle d\rangle\mathbb{Z} = \emptyset$, but $\rho^{-1}(\leq^{-1}(0)) = \rho^{-1}(0) = \mathbb{Z}$. Alternatively, using (7.1), for $x = 0 \in \mathbb{R}$ we have $\rho(x) \leq 0$ but $(X, \rho^{-1}), x \not\models \langle d\rangle 0$.

We remark that if $\mathcal{F}$ is reflexive, then $\rho : X \to W$ is an interior map iff $\rho^{-1}(R^{-1}(w)) = \operatorname{cl}\rho^{-1}(w)$ for each $w \in W$. Indeed, interior maps are suitable notions of representation for $\mathcal{L}_\Box$, and many topological completeness proofs use them. See [3,25] for more information.

Although Shehtman uses the term 'd-p-morphism' (when ρ is surjective), here we call ρ a 'representation' because it is closely related to the representations of algebras of relations seen in algebraic logic. Indeed, if ρ is a surjective representation of (W, R) over X then ρ^{-1} induces an embedding from $\wp(W)$ into $\wp(X)$ that preserves the algebraic structure with which these power sets can be naturally endowed.

7.2. Representations over subspaces

Our main interest is in representations over X itself, but representations over subspaces are also useful in proofs. Given a subspace U of X, a map $\rho : U \to W$ induces a well defined assignment $\rho^{-1} : W \to \wp(X)$ by $\rho^{-1}(w) = \{x \in X : x \in U \text{ and } \rho(x) = w\}$, for $w \in W$. Put simply, preimages under ρ of elements of W are obviously subsets of U, but they are also subsets of X, and so ρ^{-1} can be regarded equally as an assignment into U or X, as appropriate. The following easy lemma gives some connections between the two views. It is a specialisation of a more general result in which ρ^{-1} is replaced by any assignment and w by any atom (see also Lemma 2.3).

Lemma 7.2. *Let U be a subspace of X and let $\rho : U \to W$ be a map. Let $x \in U$ and $w \in W$ be arbitrary.*

1. *If $(U, \rho^{-1}), x \models \langle d\rangle w$ then $(X, \rho^{-1}), x \models \langle d\rangle w$.*
2. *If U is open in X, then $(U, \rho^{-1}), x \models \langle d\rangle w$ iff $(X, \rho^{-1}), x \models \langle d\rangle w$.*

Proof. For the first part, assume that $(U, \rho^{-1}), x \models \langle d\rangle w$ and let O be any open neighbourhood of x in X. Then $O \cap U$ is an open neighbourhood of x in U, so by assumption, there is $y \in O \cap U \setminus \{x\}$ with $(U, \rho^{-1}), y \models w$. Then $y \in O \setminus \{x\}$ and $(X, \rho^{-1}), y \models w$. Hence, $(X, \rho^{-1}), x \models \langle d\rangle w$.

The second part is a special case of Lemma 2.3. For a proof, assume that $(X, \rho^{-1}), x \models \langle d\rangle w$. Let N be an arbitrary open neighbourhood of x in U, so that $N = O \cap U$ for some open neighbourhood O of x in X. As U is assumed open in X, we see that N is also open in X, so by assumption, there is $y \in N \setminus \{x\}$ with $(X, \rho^{-1}), y \models w$. Plainly, $(U, \rho^{-1}), y \models w$. This shows that $(U, \rho^{-1}), x \models \langle d\rangle w$, and the converse follows from the first part. $\square$

By part 2 of the lemma, if ρ is a representation of $\mathcal{F}$ over an open subspace U of X, then $(X, \rho^{-1}), x \models \langle d\rangle w$ iff $R(\rho(x), w)$ for every $x \in U$ and $w \in W$. So we can work in (X, ρ^{-1}) instead of (U, ρ^{-1}). To avoid too much jumping around between subspaces, we will do this below, often without mention. Part 3 of the next lemma makes it a little more explicit. The lemma gives some general information on the relationships between representations of different generated subframes of $\mathcal{F}$ over different subspaces of X.

Lemma 7.3. *Let $\mathcal{G} = (W', R')$ be a generated subframe of $\mathcal{F}$. Let T, U, and U_i $(i \in I)$ be open subspaces of X, with $T \subseteq U = \bigcup_{i\in I} U_i$. Finally, let $\rho : U \to W'$ be a map. Then:*

1. *ρ is a representation of $\mathcal{F}$ over U iff it is a representation of $\mathcal{G}$ over U.*
2. *ρ is a representation of $\mathcal{F}$ over U iff for each $i \in I$, the restriction $\rho \restriction U_i$ is a representation of $\mathcal{F}$ over U_i.*
3. *If $\rho \restriction T$ is a representation of $\mathcal{F}$ over T, then $(X, \rho^{-1}), x \models \langle d\rangle w$ iff $R(\rho(x), w)$, for each $x \in T$ and $w \in W$.*

Proof. Simple. $\square$

7.3. Representations preserve formulas

Here, we will show that surjective representations preserve all formulas of $\mathcal{L}^{\mu}_{[d]\forall}$. Since representations are like p-morphisms, albeit between different kinds of structure, this is entirely expected and the proof is essentially quite standard — see [34, lemma 20] and [3, corollary 2.9], for example. We do need, however, that $\mathcal{F}$ is finite. We will be able to handle larger sublanguages of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ by using the translations of section 4.

Let us explain the setting. Suppose we are given a representation $\rho : X \to W$ of $\mathcal{F}$ over X. Recall that Var is our fixed base set of propositional variables, or atoms. For each assignment $h : \mathsf{Var} \to \wp(W)$ of atoms in Var into W, the map $\rho^{-1} \circ h : \mathsf{Var} \to \wp(X)$ is an assignment of atoms into X, given of course by

$$(\rho^{-1} \circ h)(p) = \{x \in X : \rho(x) \in h(p)\}, \qquad \text{for each } p \in \mathsf{Var}.$$

So ρ, or rather ρ^{-1}, gives us a way to transform an assignment into $\mathcal{F}$ to one into X, and then to evaluate a formula in the resulting model on X. Clearly, we would like to get the same result as in the original model on $\mathcal{F}$, and this leads to the following definition.

Definition 7.4. Let $\rho : X \to W$ be a map, and let φ be a formula of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$. We say that ρ *preserves* φ if for every assignment $h : \mathsf{Var} \to \wp(W)$ and every $x \in X$,

$$(X, \rho^{-1} \circ h), x \models \varphi \quad \text{iff} \quad (W, R, h), \rho(x) \models \varphi. \tag{7.2}$$

We are now ready for our main preservation result.

Proposition 7.5. *Let $\rho : X \to W$ be a surjective representation of $\mathcal{F}$ over X. Then ρ preserves every formula of $\mathcal{L}^{\mu}_{[d]\forall}$.*

Proof. The proof is by induction on φ. The atomic and boolean cases are easy and left to the reader. Let φ be a formula, and inductively assume (7.2) for every assignment $h : \mathsf{Var} \to \wp(W)$ and every $x \in X$. It is sufficient to consider the cases $\langle d\rangle\varphi$, $\forall\varphi$, and $\mu q\varphi$.

First, consider $\langle d\rangle\varphi$. Fix h, x. Suppose that $(W, R, h), \rho(x) \models \langle d\rangle\varphi$. Choose $w \in R(\rho(x))$ with $(W, R, h), w \models \varphi$. As ρ is a representation, $(X, \rho^{-1}), x \models \langle d\rangle w$. So for every open neighbourhood O of x, there is $y \in O \setminus \{x\}$ with $\rho(y) = w$. Since $(W, R, h), w \models \varphi$, for any such y we inductively have $(X, \rho^{-1} \circ h), y \models \varphi$. It follows that $(X, \rho^{-1} \circ h), x \models \langle d\rangle\varphi$.

Conversely, suppose that $(X, \rho^{-1} \circ h), x \models \langle d \rangle \varphi$. Let $[\![\varphi]\!] = \{y \in X : (X, \rho^{-1} \circ h), y \models \varphi\}$. As $\mathcal{F}$ is finite and $\langle d \rangle$ is additive (Lemma 5.2(2)), we have

$$\begin{aligned} x \in \langle d \rangle [\![\varphi]\!] = \langle d \rangle ([\![\varphi]\!] \cap X) &= \langle d \rangle \Big([\![\varphi]\!] \cap \bigcup_{w \in W} \rho^{-1}(w) \Big) \\ &= \langle d \rangle \Big(\bigcup_{w \in W} ([\![\varphi]\!] \cap \rho^{-1}(w)) \Big) = \bigcup_{w \in W} \langle d \rangle ([\![\varphi]\!] \cap \rho^{-1}(w)). \end{aligned}$$

So we can take $w \in W$ with $x \in \langle d \rangle ([\![\varphi]\!] \cap \rho^{-1}(w))$. Then $(X, \rho^{-1}), x \models \langle d \rangle w$, so as ρ is a representation, $R(\rho(x), w)$. Moreover, $[\![\varphi]\!] \cap \rho^{-1}(w) \neq \emptyset$. Take any $y \in [\![\varphi]\!] \cap \rho^{-1}(w)$. Then $(X, \rho^{-1} \circ h), y \models \varphi$ and $\rho(y) = w$. Inductively, $(W, R, h), w \models \varphi$. By Kripke semantics, $(W, R, h), \rho(x) \models \langle d \rangle \varphi$, as required.

Next, consider $\forall \varphi$. Then $(X, \rho^{-1} \circ h), x \models \forall \varphi$ iff $(X, \rho^{-1} \circ h), y \models \varphi$ for all $y \in X$, iff $(W, R, h), \rho(y) \models \varphi$ for all $y \in X$ (by the inductive hypothesis (7.2)), iff $(W, R, h), w \models \varphi$ for all $w \in W$ (since ρ is surjective), iff $(W, R, h), \rho(x) \models \forall \varphi$.

Finally consider the case $\mu q \varphi$, assumed well formed. For a formula ψ and assignments $h : \mathsf{Var} \to \wp(W)$ and $h' : \mathsf{Var} \to \wp(X)$, write

$$\begin{aligned} [\![\psi]\!]_h &= \{w \in W : (W, R, h), w \models \psi\}, \\ [\![\psi]\!]_{h'} &= \{x \in X : (X, h'), x \models \psi\}. \end{aligned}$$

Fix arbitrary $h : \mathsf{Var} \to \wp(W)$. Define $f : \wp(W) \to \wp(W)$ by $f(S) = [\![\varphi]\!]_{h[S/q]}$, for $S \subseteq W$. Similarly define $g : \wp(X) \to \wp(X)$ by $g(U) = [\![\varphi]\!]_{(\rho^{-1} \circ h)[U/q]}$, for $U \subseteq X$. These functions are monotonic. By semantics, $[\![\mu q \varphi]\!]_h = \mathrm{LFP}(f)$ and $[\![\mu q \varphi]\!]_{\rho^{-1} \circ h} = \mathrm{LFP}(g)$, so we need to show that $\rho^{-1}(\mathrm{LFP}(f)) = \mathrm{LFP}(g)$.

Inductively, for every $S \subseteq W$ we have

$$g(\rho^{-1}(S)) = [\![\varphi]\!]_{(\rho^{-1} \circ h)[\rho^{-1}(S)/q]} = [\![\varphi]\!]_{\rho^{-1} \circ (h[S/q])} = \rho^{-1}([\![\varphi]\!]_{h[S/q]}) = \rho^{-1}(f(S)). \tag{7.3}$$

By the Knaster–Tarski pre-fixed point characterisation of LFP (see section 2.5),

$$\begin{aligned} \mathrm{LFP}(f) &= \bigcap \{S \subseteq W : f(S) \subseteq S\}, \\ \mathrm{LFP}(g) &= \bigcap \{U \subseteq X : g(U) \subseteq U\}. \end{aligned} \tag{7.4}$$

By (7.3), $g(\rho^{-1}(\mathrm{LFP}(f))) = \rho^{-1}(f(\mathrm{LFP}(f))) = \rho^{-1}(\mathrm{LFP}(f))$. By (7.4), $\mathrm{LFP}(g) \subseteq \rho^{-1}(\mathrm{LFP}(f))$.

Conversely, let $S = \{w \in W : \rho^{-1}(w) \subseteq \mathrm{LFP}(g)\}$. Plainly, $\rho^{-1}(S) \subseteq \mathrm{LFP}(g)$. So by (7.3) and because g is monotonic, $\rho^{-1}(f(S)) = g(\rho^{-1}(S)) \subseteq g(\mathrm{LFP}(g)) = \mathrm{LFP}(g)$. This says that every $w \in f(S)$ satisfies $\rho^{-1}(w) \subseteq \mathrm{LFP}(g)$, and hence $w \in S$ by definition of S. That is, $f(S) \subseteq S$. By (7.4), $\mathrm{LFP}(f) \subseteq S$. So $\rho^{-1}(\mathrm{LFP}(f)) \subseteq \rho^{-1}(S) \subseteq \mathrm{LFP}(g)$.

So indeed, $\rho^{-1}(\mathrm{LFP}(f)) = \mathrm{LFP}(g)$. This completes the induction and the proof. □

We proved the proposition only for $\mathcal{L}^{\mu}_{[d]\forall}$, but later we will apply it to larger languages, using the translations of section 4.

7.4. Basic representations

Certain very primitive representations called *basic representations* will play an important role later, because they can easily be extended to more interesting representations.

Definition 7.6. Let S, U be open subspaces of X, with $S \subseteq U$, and let $\sigma : S \to W$ be a representation of $\mathcal{F}$ over S. We say that σ is U-*basic* if for every $x \in U$ and $w, v \in W$, if $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$, then Rwv.

Note that we use $\Diamond$ and not $\langle d \rangle$ here.

Remark 7.7. In the setting of this definition:

1. Vacuously, if σ is empty then it is U-basic.
2. More generally, but equally trivially, if $\operatorname{rng}\sigma$ is contained in a nondegenerate cluster C in $\mathcal{F}$, then σ is U-basic. For, $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$ implies that $w, v \in \operatorname{rng}\sigma \subseteq C$, and so Rwv as C is a nondegenerate cluster.

We remark (but will not formally use) that σ is U-basic iff $\operatorname{rng}\sigma$ is a (possibly empty) union of R-maximal clusters in $\mathcal{F}$ whose preimages under σ have pairwise disjoint closures within U. Moreover, each such preimage is a clopen subset of S.

7.5. Full representations and full representability

In induction proofs, we often need a stronger inductive hypothesis than formally required for the final result. This will be the case in Proposition 7.10 below, where we build a representation by combining several 'smaller' representations obtained inductively. For this to work, we will need these smaller representations to be well behaved on the boundaries of their domains. The following definition will help to do this.

Definition 7.8. Let $T \subseteq U$ be open subspaces of X. A representation $\rho : U \to W$ of $\mathcal{F}$ over U is said to be *T-full* if

1. ρ is surjective,
2. for every $x \in \operatorname{cl}(T) \setminus U$ and $w \in W$, we have $(X, \rho^{-1}), x \models \langle d \rangle w$.

Every surjective representation is vacuously $\emptyset$-full.

Definition 7.9. We say that $\mathcal{F}$ is *fully representable (over X)* if whenever

1. $U \subseteq X$ is open,
2. S is a regular open subset of U,
3. $\sigma : S \to W$ is a U-basic representation of $\mathcal{F}$ over S,
4. $T \stackrel{\text{def}}{=} U \setminus \operatorname{cl} S \neq \emptyset$,

then σ extends to a T-full representation $\rho : U \to W$ of $\mathcal{F}$ over U. Moreover, if $\mathcal{F}$ is rooted then for any given root w_0 of $\mathcal{F}$ and $x_0 \in T$, we can choose ρ so that $\rho(x_0) = w_0$.

Notice that in the boolean algebra $RO(U)$ of regular open subsets of U, we have $T = -S$, so $\{S, T\}$ is a partition of 1. That is, $S, T \in RO(U)$, $S \cdot T = 0$, and $S + T = 1$.

In Proposition 7.10 below, we will fulfil our main aim, to prove (surjective) representability of every finite connected locally connected serial transitive frame. We are going to do it by induction on the size of the frame. We appear to need a stronger inductive hypothesis, namely full representability, than is needed for the conclusion. T-fullness and extending σ are mainly to do with this, but the extending of σ is also helpful in the proof of strong completeness in Theorem 9.1 later. Note that if $\mathcal{F}$ is fully representable over X, and $X \neq \emptyset$, then by taking $U = X$ and $S = \sigma = \emptyset$, we see that there exists a surjective representation of $\mathcal{F}$ over X. So we do obtain our desired conclusion from the stronger hypothesis of full representability.

7.6. Main proposition

The following proposition has relatives in the literature: see, e.g., [27, theorem 3.7], [34, proposition 22], [25, lemma 4.4], and [23, lemma 6.9]. It actually holds for any dense-in-itself topological space X for which Theorems 6.3 and 6.1 and Corollary 6.5 can be proved.

Proposition 7.10. *Suppose that X is a dense-in-itself metric space. Then every finite connected locally connected serial transitive frame $\mathcal{F} = (W, R)$ is fully representable over X.*

Proof. The proof is by induction on the number of worlds in $\mathcal{F}$. Let $\mathcal{F} = (W, R)$ be a finite connected locally connected serial transitive frame, and assume the result inductively for all smaller frames. Recall from sections 2.1–2.2 that we write

- $R^\circ = \{(w, v) \in W^2 : Rwv \wedge Rvw\}$,
- $R^\bullet = \{(w, v) \in W^2 : Rwv \wedge \neg Rvw\}$.

and, for $w \in W$,

- $\mathcal{F}(w)$ for the subframe $(R(w), R \restriction R(w))$ of $\mathcal{F}$ with domain $R(w)$,
- $\mathcal{F}^*(w)$ for the subframe $(R^*(w), R \restriction R^*(w)) = (R(w) \cup \{w\}, R \restriction R(w) \cup \{w\})$ of $\mathcal{F}$ generated by w.

Let $U \subseteq X$ be open, let S be a regular open subset of U, and let $\sigma : S \to W$ be a U-basic representation of $\mathcal{F}$ over S. Write

$$T = U \setminus \operatorname{cl} S,$$

and suppose that $T \neq \emptyset$. We need to extend σ to a T-full representation $\rho : U \to W$ of $\mathcal{F}$ over U. Further, if $\mathcal{F}$ is rooted and we are given a root w_0 of $\mathcal{F}$ and $x_0 \in T$, we wish to choose ρ so that $\rho(x_0) = w_0$. There are three cases.

Case 1: $\mathcal{F} = \mathcal{F}^*(w_0)$ for some reflexive $w_0 \in W$.

Choose any such w_0 (it may not be unique). Then w_0 is a root of $\mathcal{F}$, and since w_0 is reflexive, $R(w_0) = W$ and $w_0 \in R^\circ(w_0)$. So $R^\circ(w_0) \neq \emptyset$. Since T is clearly a non-empty open set, we can use Theorem 6.3 to partition T into non-empty open sets $G_{v^\bullet}$ ($v^\bullet \in R^\bullet(w_0)$) and other non-empty sets B_{v° ($v^\circ \in R^\circ(w_0)$) such that for each $v^\bullet \in R^\bullet(w_0)$ and $v^\circ \in R^\circ(w_0)$ we have

$$\operatorname{cl}(G_{v^\bullet}) \setminus G_{v^\bullet} = \langle d \rangle B_{v^\circ} = \operatorname{cl}(T) \setminus \bigcup_{v \in R^\bullet(w_0)} G_v \stackrel{\text{def}}{=} D. \tag{7.5}$$

By taking $E_{w_0} = \{x_0\}$ and $E_v = \emptyset$ for $v \in W \setminus \{w_0\}$ in Theorem 6.3, we can suppose that $x_0 \in B_{w_0}$.

For each $v^\bullet \in R^\bullet(w_0)$, the frame $\mathcal{F}^*(v^\bullet)$ is connected (as it is rooted) and locally connected, serial, and transitive (as it is a generated subframe of $\mathcal{F}$). Since w_0 is a world of $\mathcal{F}$ but not of $\mathcal{F}^*(v^\bullet)$, the frame $\mathcal{F}^*(v^\bullet)$ is smaller than $\mathcal{F}$. By the inductive hypothesis, $\mathcal{F}^*(v^\bullet)$ is fully representable over X. So, taking the regular open subset 'S' of $G_{v^\bullet}$ to be $\emptyset$ and 'T' to be $G_{v^\bullet} \setminus \operatorname{cl} \emptyset = G_{v^\bullet}$, which is non-empty, we can find a $G_{v^\bullet}$-full representation $\rho_{v^\bullet}$ of $\mathcal{F}^*(v^\bullet)$ over $G_{v^\bullet}$.

Define $\rho : U \to W$ by:

$$\rho(x) = \begin{cases} \rho_{v^\bullet}(x), & \text{if } x \in G_{v^\bullet} \text{ for some (unique) } v^\bullet \in R^\bullet(w_0), \\ v^\circ, & \text{if } x \in B_{v^\circ} \text{ for some (unique) } v^\circ \in R^\circ(w_0), \\ \sigma(x), & \text{if } x \in S, \\ w_0, & \text{otherwise,} \end{cases}$$

for each $x \in U$. The map ρ is well defined because the $G_{v^\bullet}$, the B_{v°, and S are pairwise disjoint, and plainly it is total, extends σ, and satisfies $\rho(x_0) = w_0$.

We aim to show that ρ is a T-full representation of $\mathcal{F}$ over U. The following claim will help.

Claim 7.10.1. *Let $x \in D$ (see (7.5)). Then $(X, \rho^{-1}), x \models \langle d\rangle w$ for every $w \in W$.*

Proof of claim. Let $x \in D$ and $w \in W$ be given. There are two cases. The first is when $w \in R^\bullet(w_0)$. Now (7.5) gives $x \in \operatorname{cl} G_w \setminus G_w$. As ρ_w is a G_w-full representation of $\mathcal{F}^*(w)$, a frame of which w is a world, we have $(X, \rho_w^{-1}), x \models \langle d\rangle w$, and hence $(X, \rho^{-1}), x \models \langle d\rangle w$ (since $\rho_w \subseteq \rho$).

The second case is when $w \notin R^\bullet(w_0)$. Since $w \in W = R(w_0) = R^\bullet(w_0) \cup R^\circ(w_0)$, we have $w \in R^\circ(w_0)$. By (7.5), $x \in \langle d\rangle B_w$ (since $x \in D$). Since $\rho \restriction B_w$ has constant value w, we obtain again that $(X, \rho^{-1}), x \models \langle d\rangle w$. This proves the claim. □

We now check that ρ is a representation of $\mathcal{F}$ over U. Let $x \in U$ and $w \in W$. We require $(X, \rho^{-1}), x \models \langle d\rangle w$ iff $R(\rho(x), w)$. There are four cases.

1. Suppose that $x \in G_{v^\bullet}$ for some $v^\bullet \in R^\bullet(w_0)$. Since $G_{v^\bullet}$ is open and $\rho \restriction G_{v^\bullet} = \rho_{v^\bullet}$, a representation over $G_{v^\bullet}$ of the generated subframe $\mathcal{F}^*(v^\bullet)$ of $\mathcal{F}$, Lemma 7.3 yields $(X, \rho^{-1}), x \models \langle d\rangle w$ iff $R(\rho(x), w)$.
2. Suppose that $x \in B_{v^\circ}$ for some $v^\circ \in R^\circ(w_0)$. Then $\rho(x) = v^\circ$. As $v^\circ \in R^\circ(w_0)$, we have $Rv^\circ w_0$. As w_0 is a root of $\mathcal{F}$, by transitivity of R we have $R(\rho(x), w)$ for every $w \in W$. So we need to prove that $(X, \rho^{-1}), x \models \langle d\rangle w$ for every $w \in W$. But $x \in B_{v^\circ} \subseteq D$ by definition of D (7.5), so this follows from Claim 7.10.1.
3. If $x \in S$, then since S is open and $\rho \restriction S = \sigma$, a representation of $\mathcal{F}$ over S, the result follows from Lemma 7.3 again.
4. Suppose finally that $x \in U \setminus (S \cup T)$. Then $\rho(x) = w_0$. Since $R(w_0, w)$ for all $w \in W$, we require that $(X, \rho^{-1}), x \models \langle d\rangle w$ for all $w \in W$ as well.
   Now as S is a regular open subset of U, by Lemma 5.5 we obtain $U \setminus S \subseteq \operatorname{cl} T$. Hence, $x \in \operatorname{cl} T \setminus T \subseteq D$ by (7.5). As in case 2, Claim 7.10.1 now gives $(X, \rho^{-1}), x \models \langle d\rangle w$ for all $w \in W$.

So ρ is indeed a representation of $\mathcal{F}$ over U. We check that it is T-full. First let $x \in \operatorname{cl} T \setminus U$. Then $x \in D$ by (7.5). By Claim 7.10.1, $(X, \rho^{-1}), x \models \langle d\rangle w$ for every $w \in W$, as required. We also need that ρ is surjective. Take any $x \in B_{w_0}$. Then $x \in D$ by definition of D in (7.5). By Claim 7.10.1, $(X, \rho^{-1}), x \models \langle d\rangle w$, and so $\rho^{-1}(w) \neq \emptyset$, for every $w \in W$. Hence, ρ is surjective.

Case 2: $\mathcal{F} = \mathcal{F}^*(w_0)$ for some irreflexive $w_0 \in W$.

Choose such a w_0 (it is unique this time). Then w_0 is the root of $\mathcal{F}$, and W is the disjoint union of $\{w_0\}$ and $R(w_0)$. Using Theorem 6.1, select non-empty $I \subseteq T$ with $E = \{x_0\} \subseteq I$ and

$$\langle d\rangle I = \operatorname{cl} T \setminus U. \tag{7.6}$$

Write

$$U' = U \setminus I,$$
$$T' = T \setminus I.$$

We aim to use the inductive hypothesis on these sets and $\sigma : S \to \mathcal{F}(w_0)$, so we check the necessary conditions.

Claim 7.10.2. *U' is open, S is a regular open subset of U', and $T' = U' \setminus \mathrm{cl}\, S \neq \emptyset$.*

Proof of claim. First, U' is open. For, by Lemma 5.2(1) and (7.6),

$$U \setminus \mathrm{cl}\, I = U \setminus (I \cup \langle d\rangle I) = U \setminus (I \cup (\mathrm{cl}(T) \setminus U)) = U \setminus I = U',$$

and the left-hand side is open.

We are given that S is a regular open subset of U. Since $S \subseteq U$ and $I \subseteq T = U \setminus \mathrm{cl}\, S$, we have $S \subseteq U \setminus I = U'$. By Lemma 5.5(3), S is a regular open subset of U'.

Next, $U' \setminus \mathrm{cl}\, S = (U \setminus I) \setminus \mathrm{cl}\, S = (U \setminus \mathrm{cl}\, S) \setminus I = T \setminus I = T'$.

Finally, we check that $T' \neq \emptyset$. Well, $\langle d\rangle I = \mathrm{cl}\, T \setminus U$ by (7.6), while $\langle d\rangle T = \mathrm{cl}\, T$ by Lemma 5.2(5), since T is open by definition. But these are distinct sets, because $\langle d\rangle I \cap T = (\mathrm{cl}\, T \setminus U) \cap T = \emptyset$, while $\langle d\rangle T \cap T = \mathrm{cl}\, T \cap T = T$, which is non-empty by assumption. So $I \neq T$, whence $I \subsetneq T$ and $T' = T \setminus I \neq \emptyset$. This proves the claim. □

Claim 7.10.3. *σ is a U'-basic representation of $\mathcal{F}(w_0)$ over S.*

Proof of claim. First we show that $\sigma : S \to R(w_0)$. We know that $\sigma : S \to W = \{w_0\} \cup R(w_0)$. Assume for contradiction that there is some $x \in S$ with $\sigma(x) = w_0$. Then plainly, $x \in U$ and $(X, \sigma^{-1}), x \models \Diamond w_0$. As σ is a U-basic representation of $\mathcal{F}$ over S, we obtain Rw_0w_0, contradicting the choice of w_0 as irreflexive. So indeed, $\mathrm{rng}\, \sigma \subseteq W \setminus \{w_0\} = R(w_0)$. Since σ is a representation of $\mathcal{F}$ over S, by Lemma 7.3 it is also a representation (over S) of the generated subframe $\mathcal{F}(w_0)$ of $\mathcal{F}$. It is trivially U'-basic, since if $x \in U'$, $w, v \in R(w_0)$, and $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$, then $x \in U$ and $w, v \in W$ as well, so Rwv since σ is U-basic. This proves the claim. □

In summary, U' is open, S is a regular open subset of U', σ is a U'-basic representation of $\mathcal{F}(w_0)$ over S, and $T' = U' \setminus \mathrm{cl}\, S \neq \emptyset$.

Now $\mathcal{F}(w_0)$ is smaller than $\mathcal{F}$ (since $w_0 \notin R(w_0)$), connected (since $\mathcal{F}$ is locally connected), and locally connected, serial, and transitive (since it is a generated subframe of $\mathcal{F}$). By the inductive hypothesis, $\mathcal{F}(w_0)$ is fully representable over X. So σ extends to a T'-full representation $\rho' : U' \to R(w_0)$ of $\mathcal{F}(w_0)$ over U'. By T'-fullness,

$$(X, \rho'^{-1}), x \models \langle d\rangle v \text{ for every } v \in R(w_0) \text{ and } x \in \mathrm{cl}\, T' \setminus U'. \tag{7.7}$$

We extend ρ' to a map $\rho : U \to W$ by defining

$$\rho(x) = \begin{cases} \rho'(x), & \text{if } x \in U', \\ w_0, & \text{if } x \in I, \end{cases}$$

for $x \in U$. This is plainly well defined and total, with $\rho(x_0) = w_0$. Since ρ extends ρ', it also extends σ. We will show that ρ is a T-full representation of $\mathcal{F}$ over U. To do it, we need another claim.

Claim 7.10.4. $\mathrm{cl}\, T \setminus U \subseteq \mathrm{cl}\, I \subseteq \mathrm{cl}\, T' \setminus U'$.

Proof of claim. By (7.6) and Lemma 5.2(1), we have $\mathrm{cl}\, T \setminus U = \langle d\rangle I \subseteq \mathrm{cl}\, I$.

Using openness of $T = T' \cup I$, the assumption that X is dense in itself, and Lemma 5.2(5, 2), we have $I \subseteq T \subseteq \operatorname{cl} T = \langle d \rangle T = \langle d \rangle T' \cup \langle d \rangle I$. But by (7.6), $I \cap \langle d \rangle I \subseteq U \cap \operatorname{cl} T \setminus U = \emptyset$. So in fact, $I \subseteq \langle d \rangle T' \subseteq \operatorname{cl} T'$. Hence, $\operatorname{cl} I \subseteq \operatorname{cl} T'$. Since $I \cap U' = \emptyset$ and U' is open (Claim 7.10.2), we have $\operatorname{cl} I \cap U' = \emptyset$. So $\operatorname{cl} I \subseteq \operatorname{cl} T' \setminus U'$, proving the claim. □

Claim 7.10.5. *ρ is a representation of $\mathcal{F}$ over U.*

Proof of claim. Let $x \in U$. We require $(X, \rho^{-1}), x \models \langle d \rangle w$ iff $R(\rho(x), w)$, for each $w \in W$.

There are two cases here. The first is when $x \in I$. Then $\rho(x) = w_0$, so we require first that $(X, \rho^{-1}), x \models \langle d \rangle w$ for each $w \in R(w_0)$. So pick any $w \in R(w_0)$. By Claim 7.10.4, $x \in I \subseteq \operatorname{cl} I \subseteq \operatorname{cl} T' \setminus U'$, so by (7.7), $(X, {\rho'}^{-1}), x \models \langle d \rangle w$. As $\rho' \subseteq \rho$, the result follows.

We also require that $(X, \rho^{-1}), x \not\models \langle d \rangle w$ for each $w \in W \setminus R(w_0)$ — that is, $(X, \rho^{-1}), x \not\models \langle d \rangle w_0$. But as $x \in U$, we have $x \notin \operatorname{cl} T \setminus U = \langle d \rangle I$ by (7.6). Since $\rho^{-1}(w_0) = I$, we do indeed have $(X, \rho^{-1}), x \not\models \langle d \rangle w_0$.

The second case is when $x \notin I$. In this case, $x \in U'$, an open set, and $\rho \restriction U' = \rho'$, a representation over U' of the generated subframe $\mathcal{F}(w_0)$ of $\mathcal{F}$. By Lemma 7.3, $(X, \rho^{-1}), x \models \langle d \rangle w$ iff $R(\rho(x), w)$ for every $w \in W$, as required. The claim is proved. □

Claim 7.10.6. *ρ is T-full.*

Proof of claim. Let $x \in \operatorname{cl} T \setminus U$ and $w \in W$. We require $(X, \rho^{-1}), x \models \langle d \rangle w$.

Suppose first that $w = w_0$. By (7.6), $x \in \langle d \rangle I$. Since $I = \rho^{-1}(w_0)$, we obtain $(X, \rho^{-1}), x \models \langle d \rangle w_0$. Suppose instead that $w \in R(w_0)$. By Claim 7.10.4, $x \in \operatorname{cl} T' \setminus U'$. So by (7.7), $(X, {\rho'}^{-1}), x \models \langle d \rangle w$. As $\rho' \subseteq \rho$, we obtain $(X, \rho^{-1}), x \models \langle d \rangle w$ as required.

We must also show that $\rho(U) = W$. Well, $I \neq \emptyset$. Take $x \in I$. Then $\rho(x) = w_0$, and by the proof of Claim 7.10.5, $(X, \rho^{-1}), x \models \langle d \rangle w$ for every $w \in R(w_0)$. This can only be if ρ is surjective.

This proves the claim and completes case 2 of Proposition 7.10. Only case 3 remains, but this is the hardest case. □

Case 3: otherwise — that is, $\mathcal{F}$ is not rooted.

By the case assumption, $\mathcal{F}$ has proper connected generated subframes — for example, $\mathcal{F}^*(w)$ for any $w \in W$. So let $\mathcal{F}_0 = (W_0, R \restriction W_0)$ be a *maximal* proper connected generated subframe of $\mathcal{F}$. Then $W \setminus W_0 \neq \emptyset$. Since $\mathcal{F}$ is connected, $(W \setminus W_0, R \restriction W \setminus W_0)$ is not a generated subframe of $\mathcal{F}$. So there are $a \in W \setminus W_0$ and $b \in W_0$ with Rab. Let $\mathcal{F}_1 = \mathcal{F}^*(a) = (W_1, R \restriction W_1)$, where $W_1 = R^*(a)$.

Claim 7.10.7. *$\mathcal{F}_0$ and $\mathcal{F}_1$ are proper connected generated subframes of $\mathcal{F}$. Also, $W_0 \cap W_1 \neq \emptyset$ and $W = W_0 \cup W_1$.*

Proof of claim. By definition, $\mathcal{F}_0$ is a proper connected generated subframe of $\mathcal{F}$. Since $\mathcal{F}_1$ is rooted, it is connected, and a proper subframe of $\mathcal{F}$ (which by the case assumption is not rooted). It is a generated subframe of $\mathcal{F}$ by definition of $\mathcal{F}^*(a)$. Clearly $b \in W_0 \cap W_1$, so $W_0 \cap W_1 \neq \emptyset$. The subframe $(W_0 \cup W_1, R \restriction W_0 \cup W_1)$ of $\mathcal{F}$ is plainly a generated subframe of $\mathcal{F}$. It is connected, since $\mathcal{F}_0, \mathcal{F}_1$ are connected and $W_0 \cap W_1 \neq \emptyset$. It properly extends $\mathcal{F}_0$ since $a \in W_1 \setminus W_0$. By maximality of $\mathcal{F}_0$, we have $W_0 \cup W_1 = W$. This proves the claim. □

Being generated subframes, $\mathcal{F}_0$ and $\mathcal{F}_1$ are locally connected serial transitive frames. Since they are proper connected subframes of $\mathcal{F}$, by the inductive hypothesis they are fully representable over X. Our plan is to combine suitable representations of them to give a representation of $\mathcal{F}$ over U.

Recall that S is a regular open subset of U and $\sigma : S \to W$ is a U-basic representation of $\mathcal{F}$. We use W_0, W_1 to split S (and, later, σ) in two. Let

$$S_0 = \sigma^{-1}(W_0) = \{x \in S : \sigma(x) \in W_0\},$$
$$S_1 = S \setminus S_0.$$

So $\sigma(S_0) \subseteq W_0$ and $\sigma(S_1) \subseteq W \setminus W_0 \subseteq W_1$. Also, $S_0 = S \setminus S_1$.

Claim 7.10.8. *S_0 and S_1 are regular open subsets of U, and $U \cap \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1) = \emptyset$.*

Proof of claim. We prove the last point first. Suppose for contradiction that there is some $x \in U \cap \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1)$. As $x \in \mathrm{cl}\, S_0$, we have $(X, \sigma^{-1}), x \models \Diamond \bigvee_{w \in W_0} w$. As $\Diamond$ is additive and W_0 finite, it follows that there is some $w_0 \in W_0$ such that $(X, \sigma^{-1}), x \models \Diamond w_0$. Similarly, as $x \in \mathrm{cl}\, S_1$ and $\sigma(S_1) \subseteq W \setminus W_0$, there is some $w_1 \in W \setminus W_0$ with $(X, \sigma^{-1}), x \models \Diamond w_1$. As σ is a U-basic representation, we obtain Rw_0w_1. Since $\mathcal{F}_0$ is a generated subframe of $\mathcal{F}$, this implies that $w_1 \in W_0$, a contradiction. So $U \cap \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1) = \emptyset$ as required.

Now let $i < 2$. We show that S_i is regular open in U. First note that S_i is open. To see this, observe that $S_i = S \setminus \mathrm{cl}\, S_{1-i}$, an open set. For,

$$\begin{aligned} S_i &\subseteq S \cap U \cap \mathrm{cl}\, S_i && \text{as } S_i \subseteq S \subseteq U \text{ by definition and assumption} \\ &\subseteq S \cap U \setminus \mathrm{cl}\, S_{1-i} && \text{by the above} \\ &= S \setminus \mathrm{cl}\, S_{1-i} && \text{as } S \subseteq U \text{ by assumption; and} \\ S \setminus \mathrm{cl}\, S_{1-i} &\subseteq S \setminus S_{1-i} && \text{as } S_{1-i} \subseteq \mathrm{cl}\, S_{1-i} \\ &= S_i && \text{by definition of } S_i. \end{aligned}$$

Similarly, S_{1-i} is open. It follows that $\mathrm{cl}(S_i) \cap S_{1-i} = \emptyset$, so $S_i \subseteq S \cap \mathrm{cl}\, S_i \subseteq S \setminus S_{1-i} = S_i$. Thus, $S \cap \mathrm{cl}\, S_i = S_i$, and so $\mathrm{int}(S \cap \mathrm{cl}\, S_i) = \mathrm{int}\, S_i = S_i$ as S_i is open. So S_i is regular open in S, and as S is regular open in U, Lemma 5.5(4) yields that S_i is regular open in U. The claim is proved. □

The claim and the assumption at the outset that $T \neq \emptyset$ are more than enough to apply Corollary 6.5, to obtain open subsets U_i, T_i of U, for $i = 0, 1$, satisfying the following conditions:

C1. $U \cap \mathrm{cl}\, U_0 \cap \mathrm{cl}\, U_1 = \emptyset$,
C2. $U \cap \mathrm{cl}\, S_i \subseteq U_i$,
C3. $T_i = U_i \setminus \mathrm{cl}\, S_i \neq \emptyset$,
C4. $\mathrm{cl}(T) \setminus U \subseteq \mathrm{cl}(T_i)$,
C5. U_i is a regular open subset of U.

We now work in the boolean algebra $RO(U)$ of regular open subsets of U. By C5, we have $U_0, U_1 \in RO(U)$. We define further elements of $RO(U)$:

C6. $M = -(U_0 + U_1)$,
C7. $V_i = M + U_i$ for $i = 0, 1$.

The main property of these sets is as follows.

Claim 7.10.9. *$\{M, S_0, S_1, T_0, T_1\}$ is a partition of 1 in the boolean algebra $RO(U)$. That is, the five elements are pairwise disjoint regular open subsets of U, with*

$$U = \underbrace{\overbrace{S_0 + T_0}^{U_0} + M}_{V_0} + \overbrace{M + \underbrace{S_1 + T_1}_{U_1}}^{V_1} . \tag{7.8}$$

Proof of claim. Let $i < 2$. By Claim 7.10.8 and condition C5 above, $S_i, U_i \in RO(U)$. By this and condition C3,

$$T_i = U_i \setminus \operatorname{cl} S_i = U_i \cap U \setminus \operatorname{cl} S_i = U_i \cdot -S_i \in RO(U). \tag{7.9}$$

So $S_i \cdot T_i = \emptyset$ and, since $S_i \subseteq U_i$ by condition C2, also $U_i = U_i \cdot S_i + U_i \cdot -S_i = S_i + T_i$. Condition C1 above gives $U_0 \cdot U_1 = \emptyset$. By definition, $M = -(U_0 + U_1)$, so $M \in RO(U)$ and M is disjoint from T_i, S_i. Also, $U = U_0 + U_1 + M = S_0 + T_0 + S_1 + T_1 + M$. It is now plain that $M + S_i + T_i = M + U_i = V_i$. This proves the claim. $\square$

We aim to apply the inductive hypothesis to $V_i, M + S_i, T_i, \mathcal{F}_i$, for each $i = 0, 1$. We will construct a V_i-basic representation of $\mathcal{F}_i$ over $M + S_i$, and extend it inductively to a representation over V_i. We will arrange that these two representations over V_0 and V_1 agree on M, so their union will be our desired representation over U.

Our first step, then, is to find a V_i-basic representation of $\mathcal{F}_i$ over $M + S_i$, and the next claim helps us get one.

Claim 7.10.10. *For each $i < 2$ we have $U \cap \operatorname{cl} M \cap \operatorname{cl} S_i = \emptyset$, and $M + S_i = M \cup S_i$ in $RO(U)$.*

Proof of claim. By definition, $M = -(U_0 + U_1) = U \setminus \operatorname{cl}(U_0 + U_1) \subseteq U \setminus U_i$. Since U_i is open, $\operatorname{cl} M \cap U_i = \emptyset$. But $U \cap \operatorname{cl} S_i \subseteq U_i$ by condition C2 above, so $U \cap \operatorname{cl} M \cap \operatorname{cl} S_i = \emptyset$. By Lemma 5.5, $M + S_i = M \cup S_i$. This proves the claim. $\square$

By the claim, in order to find a V_i-basic representation of $\mathcal{F}_i$ over $M + S_i$, all we need is to find suitable representations over M and S_i and take their union.

By Claim 7.10.7, $W_0 \cap W_1 \neq \emptyset$. Fix some R-maximal $b_0 \in W_0 \cap W_1$. So $R^\bullet(b_0) = \emptyset$. Clearly, $\mathcal{F}^*(b_0)$ is a proper subframe of $\mathcal{F}$. It is obviously connected (since rooted), and a generated subframe of $\mathcal{F}$, so a locally connected serial transitive frame. By the inductive hypothesis, it is fully representable over X. So we can find an (M-full) representation

$$\beta : M \to R(b_0)$$

of $\mathcal{F}^*(b_0)$ over M, by using the definition of 'fully representable' if M is non-empty, and trivially by taking $\beta = \emptyset$ if M is empty. Also, for each $i < 2$ let

$$\sigma_i = (\sigma \restriction S_i) \ : \ S_i \to W_i.$$

Claim 7.10.11. *For each $i < 2$, $\beta \cup \sigma_i : M \cup S_i \to W_i$ is a well defined V_i-basic representation of $\mathcal{F}_i$ over $M \cup S_i$.*

Proof of claim. Since $\mathcal{F}^*(b_0)$ is a generated subframe of $\mathcal{F}_i$, it follows from Lemma 7.3(1) that β is a representation of $\mathcal{F}_i$ over M. Similarly, σ_i is a representation of $\mathcal{F}_i$ over S_i. Since M and S_i are disjoint open sets, $\beta \cup \sigma_i : M \cup S_i \to W_i$ is well defined and, by Lemma 7.3(2), a representation of $\mathcal{F}_i$ over $M \cup S_i$.

To prove that it is V_i-basic, let $x \in V_i$ and $v, w \in W_i$ be given, and suppose that $(X, (\beta \cup \sigma_i)^{-1}), x \models \Diamond w \wedge \Diamond v$. We require Rwv.

Plainly, $x \in \operatorname{cl}(M \cup S_i) = \operatorname{cl} M \cup \operatorname{cl} S_i$, and $x \in V_i \subseteq U$. But $U \cap \operatorname{cl} M \cap \operatorname{cl} S_i = \emptyset$ by Claim 7.10.10. So there are two possibilities.

The first one is that $x \notin \operatorname{cl} M$. In this case, we must have $(X, \sigma_i^{-1}), x \models \Diamond w \wedge \Diamond v$. As $\sigma_i \subseteq \sigma$, we also have $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$. As σ is U-basic, we obtain Rwv.

The other possibility is that $x \notin \mathrm{cl}\, S_i$. So $(X, \beta^{-1}), x \models \Diamond w \wedge \Diamond v$. Since β is a representation of $\mathcal{F}^*(b_0)$, we have $w, v \in R(b_0)$. But b_0 is R-maximal, so $R^\bullet(b_0) = \emptyset$. Hence, $w \in R^\circ(b_0)$, so Rwb_0, and since Rb_0v, we deduce Rwv by transitivity. (Essentially we are using that $\mathcal{F}^*(b_0)$ is a nondegenerate cluster.) This proves the claim. □

In summary, for each $i < 2$:

- V_i is open (by Claim 7.10.9).
- $M + S_i, V_i \in RO(U)$ and $M + S_i \subseteq V_i$, so by Lemma 5.5, $M + S_i$ is a regular open subset of V_i.
- working in $RO(U)$, we have $V_i = (M+S_i)+T_i$ and $(M_i+S_i)\cdot T_i = \emptyset$ by Claim 7.10.9. In a boolean algebra, if $v = s+t$ and $s \cdot t = 0$ then $t = v \cdot -s$. So $T_i = V_i \cdot -(M + S_i) = V_i \cap U \setminus \mathrm{cl}(M + S_i) = V_i \setminus \mathrm{cl}(M + S_i)$. Also, $T_i \neq \emptyset$ by condition C3.
- $M + S_i = M \cup S_i$ (by Claim 7.10.10), and $\beta \cup \sigma_i : M \cup S_i \to W_i$ is a V_i-basic representation of $\mathcal{F}_i$ over $M + S_i$ (by Claim 7.10.11).

So for each $i < 2$, recalling that $\mathcal{F}_i$ is fully representable, we see that $\beta \cup \sigma_i : M \cup S_i \to W_i$ extends to a T_i-full representation $\rho_i : V_i \to W_i$ of $\mathcal{F}_i$ over V_i. We have

$$(X, \rho_i^{-1}), x \models \langle d \rangle w \quad \text{for every } w \in W_i \text{ and } x \in \mathrm{cl}\, T_i \setminus V_i. \tag{7.10}$$

Finally define

$$\rho = \rho_0 \cup \rho_1 : U \to W. \tag{7.11}$$

We check first that ρ is well defined and total. Working in $RO(U)$ again, we have $\mathrm{dom}\, \rho_0 \cap \mathrm{dom}\, \rho_1 = V_0 \cap V_1 = V_0 \cdot V_1 = (M + U_0) \cdot (M + U_1) = M$ by Claim 7.10.9. But $\rho_0 \upharpoonright M = \beta = \rho_1 \upharpoonright M$. So ρ is well defined. Also, $V_i = -U_{1-i} = U \setminus \mathrm{cl}\, U_{1-i}$ (for $i = 0, 1$) by (7.8), and $U \cap \mathrm{cl}\, U_0 \cap \mathrm{cl}\, U_1 = \emptyset$ by condition C1 above, so

$$\mathrm{dom}\, \rho = V_0 \cup V_1 = (U \setminus \mathrm{cl}\, U_1) \cup (U \setminus \mathrm{cl}\, U_0) = U \setminus (\mathrm{cl}\, U_1 \cap \mathrm{cl}\, U_0) = U. \tag{7.12}$$

Hence, ρ is total. Plainly, ρ extends σ, since $\rho = \rho_0 \cup \rho_1 \supseteq (\beta \cup \sigma_0) \cup (\beta \cup \sigma_1) = \beta \cup \sigma$.

Claim 7.10.12. *ρ is a representation of $\mathcal{F}$ over U.*

Proof of claim. Let $i < 2$. Then $\rho \upharpoonright V_i = \rho_i$, a representation of $\mathcal{F}_i$ over V_i. By Lemma 7.3(1), this is also a representation of $\mathcal{F}$ over V_i, which is an open set by Claim 7.10.9. By (7.12), $U = V_0 \cup V_1$, so by Lemma 7.3(2), ρ is a representation of $\mathcal{F}$ over U, proving the claim. □

Claim 7.10.13. *ρ is T-full.*

Proof of claim. Let $x \in \mathrm{cl}\, T \setminus U$. We require $(X, \rho^{-1}), x \models \langle d \rangle w$ for every $w \in W$.

For each $i < 2$, as $\mathrm{cl}\, T \setminus U \subseteq \mathrm{cl}\, T_i$ by condition C4 above, and $x \notin U \supseteq V_i$, we have $x \in \mathrm{cl}\, T_i \setminus V_i$. Since $\rho_i \subseteq \rho$, it follows from (7.10) that $(X, \rho^{-1}), x \models \langle d \rangle w$ for every $w \in W_i$. This holds for each $i = 0, 1$. Since $W_0 \cup W_1 = W$, we have $(X, \rho^{-1}), x \models \langle d \rangle w$ for every $w \in W$.

Finally, we show that $\rho(U) = W$. Since each ρ_i is a T_i-full representation of $\mathcal{F}_i$ over V_i, it is surjective, and by (7.12) we obtain $\rho(U) = \rho(V_0) \cup \rho(V_1) = \rho_0(V_0) \cup \rho_1(V_1) = W_0 \cup W_1 = W$. This proves the claim, and with it, Proposition 7.10 (as $\mathcal{F}$ is not rooted, the value of $\rho(x_0)$ is immaterial in this case). □

Remark 7.11. We end with some technical remarks on the definition of 'fully representable' (Definition 7.9) and its relation to the proof just completed. They are not needed later, and the reader can of course skip them if desired.

It is very helpful throughout the proof that U is open — see, e.g., Lemma 7.3. However, we cannot assume in Definition 7.9 that U is regular open in X. For if we did, then in case 2 of the proof, we have $\mathrm{cl}\, I \subseteq \mathrm{cl}\, T' \subseteq \mathrm{cl}\, U'$ by Claim 7.10.4 and $T' \subseteq U'$, so $U' \neq U = \mathrm{int}\,\mathrm{cl}\, U = \mathrm{int}(\mathrm{cl}\, U' \cup \mathrm{cl}\, I) = \mathrm{int}\,\mathrm{cl}\, U'$. Therefore, U' is not regular open in X, and we can not apply the inductive hypothesis to it. We use that X is dense in itself to show that $I \subseteq \mathrm{cl}\, T'$, as well as to use the results of section 6.

At least according to the construction we gave, S should be open. In case 1, if S is not open then there is $x \in S \setminus \mathrm{int}\, S \subseteq \mathrm{cl}(U \setminus S)$, and a little thought shows that $(X, \rho^{-1}), x \models \langle d \rangle w_0$ for any such x. For ρ to be a representation, we would need $R(\rho(x), w_0)$. Since $\rho \supseteq \sigma$ and $x \in S$, this says that $R(\sigma(x), w_0)$, which we have no reason to suppose is true.

The problem if S is not regular open in U is that, again in case 1, we used that $U \setminus S = \mathrm{cl}\, T$. If this were to fail, there may be points $x \in U \setminus (S \cup \mathrm{cl}\, T)$ (so $x \in U \cap \mathrm{int}\,\mathrm{cl}\, S$). We have to define ρ on these x, and defining $\rho(x) = w_0$ as in the proof may not give a representation. However, as σ is U-basic, it is possible to define $\rho(x)$ using σ instead. This effectively extends σ to $U \cap \mathrm{int}\,\mathrm{cl}\, S$. So we can assume without loss of generality that S is regular open in U. It is therefore easier to do so and avoid the problem completely.

We could just suppose in Definition 7.9 that S is regular open in X, but we cannot suppose this of U, and we have to work in $RO(U)$, so there is little gain in doing so.

We need that σ is U-basic in order that in case 3, the subsets S_0, S_1 have disjoint closures in U (Claim 7.10.8). This in turn is needed to apply normality in the proof of Corollary 6.5.

We cannot assume instead in Definition 7.9 that σ is X-basic, because in case 3, we cannot guarantee that $\beta \cup \sigma_i$ is X-basic. This is because we do not know that $M \cap \mathrm{cl}\, S_i = \emptyset$, but only that $U \cap M \cap \mathrm{cl}\, S_i = \emptyset$. We could solve this problem by assuming further that $\mathrm{cl}\, S \subseteq U$ (which implies that S is regular open in X), but this weakens the proposition sufficiently to cause trouble in Theorem 9.1 later, where we would need to ensure that $\mathrm{cl}\, S_n \cup \mathrm{cl}\, S_{n+1} \subseteq U_n$ for each n.

We require that $T \neq \emptyset$ in Definition 7.9 because Proposition 7.10 trivially fails without this condition, unless σ is already surjective. We include surjectivity in the definition of 'full representation' (Definition 7.8) because surjective representations preserve $\forall$ (see Proposition 7.5). We might try to drop surjectivity from Definition 7.8 and simply prove it from the second part of the definition, as in cases 1 and 2 of the proof, but it is not clear how to do this in case 3.

Finally, we mention that actually $\rho(T) = W$ — not only ρ but also $\rho \restriction T$ is surjective.

8. Weak completeness

We are now ready to prove our first tranche of main results, showing that Hilbert systems for various sublanguages of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ are sometimes sound and always complete over any non-empty dense-in-itself metric space. Several of the proofs use the translations $-^d$ and $-^\mu$ of section 4. We establish only weak completeness here. We will discuss strong completeness later, in section 9.4.

8.1. The Hilbert systems

We will use the Hilbert systems for the mu-calculus in Definition 3.1, and also the following ones. The two basic systems are

K: as in Definition 3.1. The axioms comprise all instances of propositional tautologies and all formulas of the form $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. The inference rules are modus ponens and $\Box$-generalisation.

S4: this is K plus all instances of the S4 schemes: $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$.

As usual, we denote particular Hilbert systems extending K or S4 by sequences of letters and numbers indicating the axioms mustered. For example, S4.UC denotes the extension of S4 by the axioms generated by the two schemes given in U and C below. The relevant schemes are as follows. Recall that $\Box^*\varphi$ abbreviates $\varphi \wedge \Box\varphi$.

4: all instances of the '4' scheme $\Box\varphi \to \Box\Box\varphi$

D: $\Diamond\top$

***t*:** all instances of the following schemes, sometimes referred to as the *tangle axioms.*

Fix: $\langle t\rangle\Gamma \to \Diamond(\gamma \wedge \langle t\rangle\Gamma)$, for each $\gamma \in \Gamma$,

Ind: $\Box^*(\varphi \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \varphi)) \to (\varphi \to \langle t\rangle\Gamma)$.

U: all instances of the S5 schemes and rules for $\forall$ ($\forall\varphi \to \varphi$, $\varphi \to \forall\exists\varphi$, $\forall\varphi \to \forall\forall\varphi$, and the $\forall$-generalisation rule $\dfrac{\varphi}{\forall\varphi}$), plus the scheme $\forall\varphi \to \Box\varphi$.

C: all instances of the scheme $\forall(\Box^*\varphi \vee \Box^*\neg\varphi) \to \forall\varphi \vee \forall\neg\varphi$.

$\mathbf{G_1}$: all instances of the scheme $\Box(\Box^*\varphi \vee \Box^*\neg\varphi) \to \Box\varphi \vee \Box\neg\varphi$.

We have formulated these Hilbert systems using $\Box$ and $\langle t\rangle$, and we will refer to them below as the $\Box$*-form* of the systems. Analogous systems written with $[d]$ and $\langle dt\rangle$ can be obtained by replacing each $\Box$ above by $[d]$ and each $\langle t\rangle$ by $\langle dt\rangle$ (recall that $\Diamond$ abbreviates $\neg\Box\neg$). Below, we will refer to these as the $[d]$*-form* of the systems. Which form is meant in a particular context will sometimes be determined by the ambient language. For example, Theorem 8.5 concerns $\mathcal{L}_{[d]}$ and $\mathcal{L}_{[d]}^{\langle dt\rangle}$ and so the $[d]$-form of systems is intended.

8.2. Soundness

Soundness of (the $\Box$-form of) $S4\mu$ over the class of all topological spaces was already observed in Corollary 4.7. We proceed to examine soundness of the Hilbert systems just introduced. First, we consider validity of the axioms. (We postpone discussion of G_1 to Remark 8.6.) For short, we say that a scheme is valid in a topological space X (or class $\mathcal{K}$ of spaces) if all instances of the scheme are valid in X (resp., $\mathcal{K}$).

Lemma 8.1.

1. *In $\Box$-form, the axioms of* S4*t*D.U *are valid in every topological space.*
2. *In $[d]$-form,*
   (a) *the axioms of* K*t*.U *are valid in every topological space.*
   (b) *the* 4*-scheme is valid in every* T_D *topological space.*
   (c) *the* D *axiom ($\langle d\rangle\top$) is valid in every dense-in-itself topological space.*
3. *In both $\Box$- and $[d]$-forms, the* C *scheme is valid in every connected topological space.*

Proof. It is easy to check that in both $\Box$- and $[d]$-forms, the axioms of K.U are valid in every topological space. Every instance of $\Box\varphi \to \varphi$ is trivially valid in every topological space. (This is not true for $[d]\varphi \to \varphi$, of course.) The 4-scheme in its $\Box$-form is easily seen to be valid in every topological space, while its $[d]$-form was shown to be valid in precisely the T_D spaces by Esakia (see, e.g., [8, proposition 2]). Turning to the D axiom, plainly $\Diamond\top$ is valid in every space, and $\langle d\rangle\top$ is valid in precisely the dense-in-themselves spaces.

Next, working in any model (X, h) on an arbitrary topological space X, we show that the tangle axioms in both $\Box$- and $[d]$-forms are true at all points. We write $[\![\varphi]\!]$ for the set $\{x \in X : (X, h), x \models \varphi\}$, as usual. The result for Fix is immediate from the fixed-point semantics, which tells us that

Table 1
Finite model property.

Logic(s)	Has the finite model property over the class of	Proved in
S4t	Reflexive transitive frames	[14, §9]
S4.UC, S4t.UC	Reflexive transitive connected frames	[14, §11]
KD4G$_1$, KD4G$_1t$	Serial transitive locally connected frames	[14, §14]
KD4G$_1$.UC, KD4G$_1t$.UC	Serial, transitive, connected and locally connected frames	[14, §14]

$$\begin{aligned}[\![\langle t\rangle\Gamma]\!] &= \textstyle\bigcap_{\gamma\in\Gamma} \Diamond([\![\gamma]\!] \cap [\![\langle t\rangle\Gamma]\!]),\\ [\![\langle dt\rangle\Gamma]\!] &= \textstyle\bigcap_{\gamma\in\Gamma} \langle d\rangle([\![\gamma]\!] \cap [\![\langle dt\rangle\Gamma]\!]).\end{aligned}$$

For the $[d]$-form of Ind, suppose $(X, h), x \models [d]^*(\varphi \to \bigwedge_{\gamma\in\Gamma} \langle d\rangle(\gamma \wedge \varphi))$ and $(X, h), x \models \varphi$. Then there is an open neighbourhood O of x such that for every $\gamma \in \Gamma$ we have $O \subseteq [\![\varphi \to \langle d\rangle(\gamma \wedge \varphi)]\!]$. Let $Q = O \cap [\![\varphi]\!]$. Then $Q \subseteq \langle d\rangle([\![\gamma]\!] \cap [\![\varphi]\!])$ for each γ.

But then $Q \subseteq \langle d\rangle([\![\gamma]\!] \cap Q)$ for each γ, because if $y \in Q$, then for any open neighbourhood O' of y, $O \cap O' \setminus \{y\}$ intersects $[\![\gamma]\!] \cap [\![\varphi]\!]$, hence $O' \setminus \{y\}$ intersects $[\![\gamma]\!] \cap O \cap [\![\varphi]\!] = [\![\gamma]\!] \cap Q$.

Since by definition, $[\![\langle dt\rangle\Gamma]\!] = \bigcup\{S \subseteq X : S \subseteq \bigcap_{\gamma\in\Gamma} \langle d\rangle([\![\gamma]\!] \cap S)\}$, we see that $Q \subseteq [\![\langle dt\rangle\Gamma]\!]$. But $x \in O \cap [\![\varphi]\!] = Q$, so then $(X, h), x \models \langle dt\rangle\Gamma$, confirming that Ind is true at x.

The argument for the $\Box$-form of Ind is similar.

Finally we show that the C scheme is valid in every *connected* topological space. The meaning of C is the same in $\Box$- and $[d]$-form, since $\Box^*\varphi$ is equivalent to $[d]^*\varphi$ (and to $\Box\varphi$) in any space. Suppose that X is a connected topological space. Let h be an assignment into X. Let $O = \{x \in X : (X, h), x \models \Box^*\varphi\}$ and $O' = \{x \in X : (X, h), x \models \Box^*\neg\varphi\}$. By semantics of $\Box$, these sets are open and disjoint. Suppose that $(X, h), x \models \Box^*\varphi \vee \Box^*\neg\varphi$ for every $x \in X$. This says that $O \cup O' = X$, so X is partitioned by O, O'. As X is connected, one of them must be X. If $O = X$, then $(X, h), x \models \varphi$ for all $x \in X$, while if $O' = X$ then $(X, h), x \models \neg\varphi$ for all $x \in X$. Either way, we have $(X, h), x \models \forall\varphi \vee \forall\neg\varphi$ for all $x \in X$. This establishes that C is valid in X. $\Box$

This lemma is sufficient to prove soundness theorems, for the following reason. Let $\mathcal{K}$ be a class of topological spaces, and H a Hilbert system whose rules are at most those listed above (modus ponens and the two generalisation rules). These rules plainly preserve validity over $\mathcal{K}$. So as mentioned in section 2.12, if the axioms of H are valid in $\mathcal{K}$ then H is sound over $\mathcal{K}$. For example, if G$_1$ in $[d]$-form is valid in $\mathcal{K}$ then we can conclude from the lemma that Kt.UG$_1$ in $[d]$-form is sound over $\mathcal{K}$.

8.3. Finite model property

Given a class $\mathcal{K}$ of frames, a Hilbert system is said to have the *finite model property over* $\mathcal{K}$ if it is sound and complete over the class of *finite* frames in $\mathcal{K}$.

Our completeness theorems rely critically on several results on the finite model property. Two of them come from Fact 3.2 and Theorem 3.10, but the majority were proved in [14,15], using special kinds of filtration: we recall the relevant ones in Fact 8.2 below. Related earlier results on the finite model property include [10,12], [34, theorem 15], [36, theorem 10], and [45].

Fact 8.2. The finite model property results shown in Table 1 hold.

So armed, we can proceed to prove soundness and completeness theorems.

8.4. Weak completeness for $\mathcal{L}_\Box^\mu$ and $\mathcal{L}_\Box^{\langle t\rangle}$

The pioneering result in this field was the theorem of [27] that the $\mathcal{L}_\Box$-logic of every separable dense-in-itself metric space is S4. The assumption of separability was removed in [32]. We begin by generalising this theorem, establishing (weak) completeness results for $\mathcal{L}_\Box^\mu$ and $\mathcal{L}_\Box^{\langle t\rangle}$ over any dense-in-itself metric space. We will go on to prove strong completeness in Theorem 9.3.

Theorem 8.3. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* S4μ *is sound and complete over X for $\mathcal{L}_\Box^\mu$-formulas.*
2. *The Hilbert system* S4t *is sound and complete over X for $\mathcal{L}_\Box^{\langle t\rangle}$-formulas.*

Proof. For part 1, soundness is easy to check and indeed we have already mentioned it in Corollary 4.7. For completeness, let φ be an $\mathcal{L}_\Box^\mu$-formula that is not a theorem of S4μ. By Theorem 3.10, we can find a finite reflexive transitive frame $\mathcal{F} = (W, R)$, an assignment h into $\mathcal{F}$, and a world $w \in W$ with $(W, R, h), w \models \neg\varphi$. By replacing $\mathcal{F}$ by $\mathcal{F}(w)$, we can suppose that w is a root of $\mathcal{F}$ — this can be justified in a standard way using Lemma 2.1. Since $\mathcal{F}$ is rooted, it is clearly connected. Since it is reflexive and transitive, it is locally connected and serial. So by Proposition 7.10, it is fully representable over X. So, taking $U = X$ and $S = \sigma = \emptyset$ in the definition of 'fully representable' (Definition 7.9), for any $x \in X$ we may choose an X-full, hence surjective, representation ρ of $\mathcal{F}$ over X with $\rho(x) = w$. Then

$$\begin{array}{llll}(W,R,h),w\models\varphi & \text{iff} & (W,R,h),w\models\varphi^d & \text{by Lemma 4.4, since } \mathcal{F} \text{ is reflexive,}\\ & \text{iff} & (X,\rho^{-1}\circ h),x\models\varphi^d & \text{by Proposition 7.5, since } \varphi^d\in\mathcal{L}^\mu_{[d]\forall},\\ & \text{iff} & (X,\rho^{-1}\circ h),x\models\varphi & \text{by Lemma 4.5, since } X \text{ is } \mathrm{T}_D.\end{array}$$

We obtain $(X, \rho^{-1} \circ h), x \models \neg\varphi$. Thus, φ is not valid in X, proving completeness.

The proof of part 2 is similar. The differences are: for soundness, use Lemma 8.1; for completeness, φ is assumed to be an $\mathcal{L}_\Box^{\langle t\rangle}$-formula that is not a theorem of S4t; we use Fact 8.2 in place of Theorem 3.10 to obtain a finite reflexive transitive Kripke model satisfying $\neg\varphi$ at a root; and having obtained, for any $x \in X$, a surjective representation ρ of $\mathcal{F}$ over X with $\rho(x) = w$, we use the additional translation $-^\mu$ from section 4, as follows. Note that $\varphi \in \mathcal{L}_\Box^{\langle t\rangle}$, $\varphi^d \in \mathcal{L}_{[d]}^{\langle dt\rangle}$, and $(\varphi^d)^\mu \in \mathcal{L}_{[d]}^\mu \subseteq \mathcal{L}_{[d]\forall}^\mu$.

$$\begin{array}{llll}(W,R,h),w\models\varphi & \text{iff} & (W,R,h),w\models\varphi^d & \text{by Lemma 4.4, since } \mathcal{F} \text{ is reflexive,}\\ & \text{iff} & (W,R,h),w\models(\varphi^d)^\mu & \text{by Lemma 4.2, since } \mathcal{F} \text{ is transitive,}\\ & \text{iff} & (X,\rho^{-1}\circ h),x\models(\varphi^d)^\mu & \text{by Proposition 7.5, since } (\varphi^d)^\mu\in\mathcal{L}^\mu_{[d]\forall},\\ & \text{iff} & (X,\rho^{-1}\circ h),x\models\varphi^d & \text{by Lemma 4.2 again,}\\ & \text{iff} & (X,\rho^{-1}\circ h),x\models\varphi & \text{by Lemma 4.5, since } X \text{ is } \mathrm{T}_D.\end{array}$$

Note that we have shown that in each case, any consistent formula is satisfiable in X *at any chosen point.* □

8.5. Weak completeness for $\mathcal{L}_{\Box\forall}$ and $\mathcal{L}_{\Box\forall}^{\langle t\rangle}$

Completeness for languages with $\forall$ follows the same lines, although soundness requires that the space be connected.

Theorem 8.4. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* S4.UC *is complete over X for $\mathcal{L}_{\Box\forall}$-formulas, and sound if X is connected.*[5]
2. *The Hilbert system* S4t.UC *is complete over X for $\mathcal{L}_{\Box\forall}^{\langle t\rangle}$-formulas, and sound if X is connected.*

Proof. For part 1, soundness was shown in Lemma 8.1. For completeness, even when X is not connected, suppose that $\varphi \in \mathcal{L}_{\Box\forall}$ is not a theorem of S4.UC. By Fact 8.2, or by [36, theorem 10], S4.UC has the finite model property, so we can find a finite reflexive (hence serial and locally connected) transitive connected frame $\mathcal{F} = (W, R)$, an assignment h into $\mathcal{F}$, and a world $w \in W$ such that $(W, R, h), w \models \neg\varphi$. ($\mathcal{F}$ may not be rooted.) As in Theorem 8.3, we may take a surjective representation ρ of $\mathcal{F}$ over X. Using surjectivity, take $x \in X$ with $\rho(x) = w$. Then as before, $(X, \rho^{-1} \circ h), x \models \neg\varphi$, so φ is not valid in X.

Part 2 is proved similarly. □

We have no results for $\mathcal{L}_{\Box\forall}^{\mu}$ because we are not aware of any completeness theorem for this language with respect to finite reflexive transitive connected frames. If one is proved in future, we could take advantage of it.

8.6. Weak completeness for $\mathcal{L}_{[d]}$ and $\mathcal{L}_{[d]}^{\langle dt\rangle}$

In one way this is even easier, as we do not need the translation φ^d. But again, soundness requires a condition on the space.

Theorem 8.5. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* KD4G$_1$ *is complete over X for $\mathcal{L}_{[d]}$-formulas, and sound if* G$_1$ *is valid in X.*
2. *The Hilbert system* KD4G$_1t$ *is complete over X for $\mathcal{L}_{[d]}^{\langle dt\rangle}$-formulas, and sound if* G$_1$ *is valid in X.*

Proof. For part 1, soundness follows from Lemma 8.1 and the assumed validity of G$_1$. For completeness, even when X does not validate G$_1$, suppose that $\varphi \in \mathcal{L}_{[d]}$ is not a theorem of KD4G$_1$. Now KD4G$_1$ has the finite model property (see [34, theorem 15] or Fact 8.2), so we can find a finite serial transitive locally connected frame $\mathcal{F} = (W, R)$, an assignment h into $\mathcal{F}$, and a world $w \in W$ such that $(W, R, h), w \models \neg\varphi$. As usual, by replacing $\mathcal{F}$ by $\mathcal{F}^*(w)$, we can suppose that $\mathcal{F}$ is connected and w is a root of it. Let $x \in X$ be arbitrary. By Proposition 7.10, $\mathcal{F}$ is fully representable over X, so there is a surjective representation ρ of $\mathcal{F}$ over X with $\rho(x) = w$. Then $(X, \rho^{-1} \circ h), x \models \neg\varphi$ by Proposition 7.5. So φ is not valid in X.

The proof of part 2 is similar, but in order to apply Proposition 7.5, we first use the translation $-^{\mu}$ to turn $\varphi \in \mathcal{L}_{[d]}^{\langle dt\rangle}$ into an $\mathcal{L}_{[d]}^{\mu}$-formula φ^{μ} equivalent to φ in transitive frames and in X.

Again, we have shown that any consistent formula is satisfiable at any given point of X. □

Remark 8.6. Theorem 8.5(1) is related to earlier work of Shehtman [34]. In [34, theorem 23, p. 39], the following is proved for the language $\mathcal{L}_{[d]}$:

(i) Let X be a topological space having an open set homeomorphic to some $\mathbb{R}^n$, $n > 0$. Then $L(D(X)) \subseteq D4\mathrm{G}_1$ [the $\mathcal{L}_{[d]}$-logic of X is contained in KD4G$_1$].

(ii) If additionally X satisfies conditions of lemma 2 then $L(D(X)) = D4\mathrm{G}_1$.

[5] In [36, theorem 18], Shehtman states this result when X is additionally assumed separable. However, [23, footnote 7] states that [36] "contains a stronger claim: [the $\mathcal{L}_{\Box\forall}$-logic of X is S4.UC] for any connected dense-in-itself separable metric X. However, recently we found a gap in the proof of Lemma 17 from that paper. Now we state the main result only for the case $X = \mathbb{R}^n$; a proof can be obtained by applying the methods of the present Chapter, but we are planning to publish it separately."

Lemma 2 [34, p. 3] states the following.

> Let X be a topological space satisfying the following condition: for any open U and any $x \in U$ there is open $V \subseteq U$ such that $x \in V$ and $(V \setminus \{x\})$ is connected [as a subspace of X]. Then $X \models \mathrm{G}_1$.

Shehtman's results (i), (ii) above follow from Theorem 8.5(1). We remark that the converse of his lemma 2 fails in general — the reader may check that the subspace $\mathbb{R}^2 \setminus \{(1/n, y) : n$ a positive integer, $y \in \mathbb{R}\}$ of $\mathbb{R}^2$ validates G_1, but for no open neighbourhood V of $(0, 0)$ is $V \setminus \{(0, 0)\}$ connected. [25, theorems 3.12, 3.14] give a characterisation of when a topological space validates G_n, for $n \geq 1$.

Shehtman [34, p. 43] also states two open problems:

1. To describe all $[\mathcal{L}_{[d]}$-]logics [of] dense-in-itself metric spaces X. In particular, is $[K]D4\mathrm{G}_1$ the greatest of them?
2. Is theorem 23(ii) extended to the infinite dimensional case? In particular, does it hold for Hilbert space ℓ_2 (with the weak or with the strong topology)?

Theorem 8.5(1) appears to resolve problem 2 and the second part of problem 1, both positively.

Shehtman also proved in [34, theorem 29] that the $\mathcal{L}_{[d]}$-logic of any separable zero-dimensional dense-in-itself metric space is KD4. This does not follow from Theorem 8.5. The separability assumption was removed, and the result extended to tangled closure operators, in [16].

8.7. Weak completeness for $\mathcal{L}_{[d]\forall}$ and $\mathcal{L}^{\langle dt\rangle}_{[d]\forall}$

The following is now purely routine.

Theorem 8.7. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* $\mathrm{KD4G_1.UC}$ *is complete over X for $\mathcal{L}_{[d]\forall}$-formulas, and sound if X is connected and validates* G_1.
2. *The Hilbert system* $\mathrm{KD4G}_1t\mathrm{.UC}$ *is complete over X for $\mathcal{L}^{\langle dt\rangle}_{[d]\forall}$-formulas, and sound if X is connected and validates* G_1.

Proof. There are no new elements in the proof, so we leave it to the reader. □

9. Strong completeness

Here, we will prove that $\mathrm{KD4G}_1t$ is strongly complete over any non-empty dense-in-itself metric space X: any countable $\mathrm{KD4G}_1t$-consistent set of $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas is satisfiable over X. The analogous results for $\mathcal{L}^{\mu}_{\square}$ and the weaker languages $\mathcal{L}_{[d]}$ and $\mathcal{L}^{\langle t\rangle}_{\square}$ will follow. The analogous result for $\mathcal{L}_{\square}$ also follows, but this is a known result, proved recently by Kremer [20].[6] We will then show that strong completeness frequently fails for languages with $\forall$.

[6] Kremer's argument does not appear to work in our situation. One difficulty is that strong completeness even for $\mathcal{L}^{\langle t\rangle}_{\square}$ fails in Kripke semantics (an example in [14, §5] can be used to show this). Even without the tangled closure operators, satisfying an infinite set of formulas over a connected locally connected frame presents further difficulties.

9.1. The problem

Let us outline a naïve approach to the problem. It does not work, but it will illustrate the difficulty we face and motivate the formal proof later.

Let Γ be a countable KD4G$_1t$-consistent set of $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas. For simplicity, assume that Γ is maximal consistent. Write Γ as the union of an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ of finite sets. Fix $x \in X$. As we saw at the end of the proof of Theorem 8.5, each Γ_n $(n < \omega)$ is satisfiable *at the point* x. So we can find an assignment g_n on X with $(X, g_n), x \models \Gamma_n$. Suppose we could build a new assignment g that *behaves like* g_n *for larger and larger* n, as we approach x. Then we might hope that $(X, g), x \models \Gamma_n$ for all n, and so $(X, g), x \models \Gamma$.

To define such a g, we choose a countable sequence $X = S_0 \supseteq S_1 \supseteq \cdots$ of open neighbourhoods of x, such that

S1. every open neighbourhood of x contains some S_n (that is, the S_n form a 'base of open neighbourhoods' of x).

X is a metric space, so we can do this. Since we can make the S_n as small as we like, and the Γ_n are finite sets, we can suppose that for each $n < \omega$:

S2. for each $[d]\varphi \in \Gamma_n$, we have $(X, g_n), y \models \varphi$ for every $y \in S_n \setminus \{x\}$,
S3. for each $\langle d\rangle\varphi \in \Gamma_n$, there is $y \in S_n \setminus \operatorname{cl} S_{n+1}$ with $(X, g_n), y \models \varphi$.

We can now define a new assignment g by 'using g_n within S_n', for each $n < \omega$. More precisely, we let

$$g(p) \cap (S_n \setminus S_{n+1}) = g_n(p) \cap (S_n \setminus S_{n+1})$$

for each atom p and each $n < \omega$. We also need to define g at x itself, but we can use Γ to determine truth values of atoms there.

Now we try to prove that $\varphi \in \Gamma$ iff $(X, g), x \models \varphi$ for all formulas φ, by induction on φ. The atomic and boolean cases are easy. Consider the case $\langle d\rangle\varphi$.

If $\langle d\rangle\varphi \in \Gamma$, then $\langle d\rangle\varphi \in \Gamma_n$ for all large enough n, so by S3, there is $y \in S_n \setminus \operatorname{cl} S_{n+1}$ with $(X, g_n), y \models \varphi$. As $S_n \setminus \operatorname{cl} S_{n+1}$ is open and g_n agrees with g on it, it follows that $(X, g), y \models \varphi$. This holds for cofinitely many n, so $(X, g), x \models \langle d\rangle\varphi$.

Conversely, if $(X, g), x \models \langle d\rangle\varphi$, then for infinitely many n, there is $y \in S_n \setminus S_{n+1}$ with $(X, g), y \models \varphi$. *If we could find such a* $y \in S_n \setminus \operatorname{cl} S_{n+1}$, then as above, $(X, g_n), y \models \varphi$, and it would follow by S2 and maximality of Γ that $\langle d\rangle\varphi \in \Gamma$.

But it may be that we can only find such $y \in \operatorname{cl} S_{n+1}$. The truth of φ at such y may not be preserved when we change from g to g_n, because it may depend on points in S_{n+1}, and at such points, g agrees with g_{n+1}, not g_n. (We cannot just make S_{n+1} smaller to take the witnesses y out of $\operatorname{cl} S_{n+1}$, because g will then change, and we may no longer have $(X, g), y \models \varphi$.)

So we would like to arrange a *smooth transition* between g_n and g_{n+1}, avoiding unpleasant discontinuities. It would be sufficient if there is some closed $T_{n+1} \subseteq S_{n+1}$ such that g_n and g_{n+1} agree on the 'buffer zone' $S_{n+1} \setminus T_{n+1}$. Much of the formal proof below is aimed at achieving something like this for atoms occurring in Γ_n — see Claim 9.1.3 especially.

However, the argument clearly would work if we could arrange that the S_n are *clopen*. This can easily be done for 0-dimensional spaces [16].

9.2. Strong completeness for $\mathcal{L}^{\langle dt\rangle}_{[d]}$

Theorem 9.1 *(Strong completeness). Let X be a non-empty dense-in-itself metric space. Then the Hilbert system* $\mathrm{KD4G_1}t$ *is strongly complete over X for $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas, and sound if* G_1 *is valid in X.*

Proof. For soundness, see Lemma 8.1. For strong completeness, let Γ be a countable $\mathrm{KD4G_1}t$-consistent set of $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas. We show that Γ is satisfiable in X. We can suppose without loss of generality that Γ is maximal consistent. Since Γ is countable, we can write it as $\Gamma = \bigcup_{n<\omega}\Gamma_n$, where $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ is a chain of finite sets. Let Var_n be the finite set of atoms occurring in formulas in Γ_n, for each $n < \omega$. So $\mathsf{Var}_0 \subseteq \mathsf{Var}_1 \subseteq \cdots$. For each $n < \omega$, as Γ_n is $\mathrm{KD4G_1}t$-consistent, by Fact 8.2 there is a finite serial transitive locally connected Kripke model $\mathcal{M}_n = (W_n, R_n, h_n)$ and a world $w_n \in W_n$ with

$$\mathcal{M}_n, w_n \models \Gamma_n.$$

We can assume without loss of generality that the W_n $(n < \omega)$ are pairwise disjoint. For each n, fix an arbitrary $e_n \in W_n$ with $R_n w_n e_n$ and such that e_n is R_n-maximal — that is, $R^\bullet_n(e_n) = \emptyset$.

For $i \le j < \omega$ and $w \in W_j$ write

$$\begin{aligned} \mathrm{tp}_i(w) &= \{p \in \mathsf{Var}_i : \mathcal{M}_j, w \models p\} \in \wp\mathsf{Var}_i, \\ \tau^j_i &= \{\mathrm{tp}_i(w) : w \in R_j(e_j)\} \quad \in \wp\wp\mathsf{Var}_i. \end{aligned}$$

So $\mathrm{tp}_i(w)$ is the 'atomic type' of w in $\mathcal{M}_j$ with respect to the finite set Var_i of atoms. We do not need to write $\mathrm{tp}^j_i(w)$ since the W_n are pairwise disjoint so j is determined by w. And τ^j_i is the set of such types that occur as types of points in the cluster $R_j(e_j)$.

The following claim shows that we can actually assume without loss of generality that $\tau^j_i = \tau^i_i$ whenever $i \le j < \omega$, so that τ^j_i is independent of j.

Claim 9.1.1. *There are $s_0 < s_1 < \cdots < \omega$ such that $s_n \ge n$ and $\tau^{s_n}_n = \tau^{s_m}_n$ whenever $n \le m < \omega$.*

Proof of claim. Essentially König's tree lemma. We will define by induction infinite sets $\omega = S_{-1} \supseteq S_0 \supseteq S_1 \supseteq \cdots$. We let $s_n = \min S_n$, and we will arrange that $0 = s_{-1} < s_0 < s_1 < \cdots$ and $s_n \ge n$ for all n. Let $n < \omega$ and suppose that we are given S_{n-1} and $s_{n-1} = \min S_{n-1} \ge n-1$ inductively. Using that $\wp\wp\mathsf{Var}_n$ is finite and S_{n-1} infinite, choose infinite $S_n \subseteq S_{n-1} \setminus \{s_{n-1}\}$ such that $\tau^s_n \in \wp\wp\mathsf{Var}_n$ is constant for all $s \in S_n$. The term τ^s_n is defined for all $s \in S_n$, because $s \ge \min S_n > s_{n-1} \ge n-1$ and so $s \ge n$. Of course define $s_n = \min S_n$. Then $s_n > s_{n-1}$ and $s_n \ge n$ as required. This completes the definition. Then for any $n \le m < \omega$ we have $s_n \in S_n$ and $s_m \in S_m \subseteq S_n$, so $\tau^{s_n}_n = \tau^{s_m}_n$, as required. This proves the claim. □

Now replace $\mathcal{M}_n, w_n, e_n$ by $\mathcal{M}_{s_n}, w_{s_n}, e_{s_n}$ for each $n < \omega$. Do not change Γ_n or Var_n. Since $n \le s_n$, we have $\Gamma_n \subseteq \Gamma_{s_n}$, and consequently we still have $\mathcal{M}_n, w_n \models \Gamma_n$ for each n. Moreover, if $i \le j < \omega$ we have $\tau^{s_i}_i = \tau^{s_j}_i$, and consequently after replacement, $\tau^i_i = \tau^j_i$.

For each $n < \omega$, define the frames

$$\begin{aligned} \mathcal{F}_n &= (R_n(w_n), R_n \restriction R_n(w_n)), \\ \mathcal{C}_n &= (R_n(e_n), R_n \restriction R_n(e_n)). \end{aligned}$$

$\mathcal{F}_n$ is a generated subframe of (W_n, R_n), so inherits its serial, transitive, and locally connected properties. Also, $\mathcal{F}_n$ is connected since (W_n, R_n) validates G_1. The reason for considering $\mathcal{F}_n$ instead of just (W_n, R_n) or $(R^*_n(w_n), R_n \restriction R^*_n(w_n))$ will be seen in Claim 9.1.5. As e_n is R_n-maximal, $\mathcal{C}_n$ is a nondegenerate cluster, so trivially a connected serial transitive locally connected frame, and (as R_n is transitive) a generated subframe of $\mathcal{F}_n$. We conclude from Proposition 7.10 that $\mathcal{F}_n$ and $\mathcal{C}_n$ are fully representable over X, for all $n < \omega$.

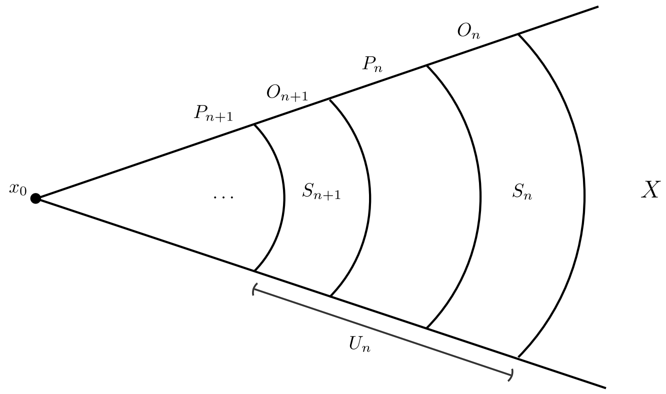

Fig. 1. Rough guide to the sets O_n, P_n, U_n, S_n.

Now fix arbitrary $x_0 \in X$. Let O be an open neighbourhood of x_0. Since X is a metric space, all singletons are closed, and since it is dense in itself, we can pick $y \in O \setminus \{x_0\}$. By Lemma 5.7, there is a regular open subset P of X with $x_0 \in P \subseteq \operatorname{cl} P \subseteq O \setminus \{y\} \subsetneq O$. So *every open neighbourhood of x_0 properly contains the closure of some regular open neighbourhood of x_0*. Using this repeatedly, we may choose regular open subsets O_n, P_n of X (for $n < \omega$) containing x_0, with $O_0 = X$, and with the following properties:

1. $\operatorname{cl} O_{n+1} \subsetneq P_n$ and $\operatorname{cl} P_n \subsetneq O_n$ for each $n < \omega$.
2. $O_n \subseteq N_{1/n}(x_0)$ for each $n > 0$.

It follows that for every open neighbourhood O of x_0, there is $n < \omega$ with $O_n \subseteq O$. That is, the O_n form a base of open neighbourhoods of x_0. For each $n < \omega$ define open sets

$$\begin{aligned} U_n &= O_n \setminus \operatorname{cl} P_{n+1}, \\ S_n &= O_n \setminus \operatorname{cl} P_n. \end{aligned}$$

See Fig. 1. It is easily seen that

$$\bigcup_{n<\omega} (O_n \setminus O_{n+1}) = X \setminus \{x_0\}, \tag{9.1}$$

$$\bigcup_{n\le m<\omega} U_m = O_n \setminus \{x_0\} \quad \text{for each } n < \omega. \tag{9.2}$$

The following claim lists some other basic facts about our situation.

Claim 9.1.2. *For each $n < \omega$:*

1. $U_n \cap U_{n+1} = S_{n+1} \neq \emptyset$.
2. $S_n \cup S_{n+1} \subseteq U_n$.
3. $\operatorname{cl} S_n \cap \operatorname{cl} S_{n+1} = \emptyset$.
4. S_n, S_{n+1}, *and* $S_n \cup S_{n+1}$ *are regular open subsets of* U_n.
5. $U_n \setminus \operatorname{cl}(S_n \cup S_{n+1}) \neq \emptyset$.

Proof of claim.

1. Easy.
2. From the definitions we have $S_n = O_n \setminus \operatorname{cl} P_n \subseteq O_n \setminus \operatorname{cl} P_{n+1} = U_n$ and $S_{n+1} = O_{n+1} \setminus \operatorname{cl} P_{n+1} \subseteq O_n \setminus \operatorname{cl} P_{n+1} = U_n$.

3. It is clear that

$$\operatorname{cl} S_n \subseteq \operatorname{cl} O_n \setminus P_n. \tag{9.3}$$

Applying this for $n+1$ and n gives $\operatorname{cl} S_{n+1} \cap \operatorname{cl} S_n \subseteq \operatorname{cl} O_{n+1} \setminus P_n \subseteq P_n \setminus P_n = \emptyset$.

4. O_n and P_n are regular open subsets of X, so by Lemma 5.5, $S_n = O_n \setminus \operatorname{cl} P_n$ is a regular open subset of X too. Since $\operatorname{cl} S_n \cap \operatorname{cl} S_{n+1} = \emptyset$ by part 2, Lemma 5.5(2) yields that $S_n \cup S_{n+1}$ is also a regular open subset of X. Since each of these three sets is a subset of U_n by part 2, by Lemma 5.5(3) it is also regular open in U_n.
5. By (9.3) (for n and $n+1$), $\operatorname{cl} S_n$ and $\operatorname{cl} S_{n+1}$ are disjoint from $P_n \setminus \operatorname{cl} O_{n+1}$, so by additivity of closure, $U_n \setminus \operatorname{cl}(S_n \cup S_{n+1}) = U_n \setminus (\operatorname{cl} S_n \cup \operatorname{cl} S_{n+1}) \supseteq P_n \setminus \operatorname{cl} O_{n+1} \neq \emptyset$. □

Claim 9.1.3. *There are surjective representations ρ_n of $\mathcal{F}_n$ over U_n $(n < \omega)$ such that*

$$\operatorname{tp}_n(\rho_n(x)) = \operatorname{tp}_n(\rho_{n+1}(x)) \quad \textit{for every } x \in S_{n+1}. \tag{9.4}$$

Proof of claim. We define the ρ_n by induction on n to satisfy (9.4) and additionally

$(*)$ $\rho_n \restriction S_{n+1}$ is a representation of $\mathcal{C}_n$ over S_{n+1}.

First let $n = 0$. Since $\mathcal{C}_0$ is fully representable over X, we can choose a representation $\sigma : S_1 \to \mathcal{C}_0$. Because $\mathcal{C}_0$ is a nondegenerate cluster, σ is actually a U_0-basic representation (see Remark 7.7). By Claim 9.1.2, S_1 is a regular open subset of U_0, and $U_0 \setminus \operatorname{cl} S_1 \neq \emptyset$. Now $\mathcal{F}_0$ is also fully representable over X, so σ extends to a surjective representation ρ_0 of $\mathcal{F}_0$ over U_0. Clearly, condition $(*)$ above is met.

Let $n < \omega$ and assume inductively that for each $m \leq n$, a surjective representation ρ_m of $\mathcal{F}_m$ over U_m has been constructed, such that $\rho_m \restriction S_{m+1}$ is a representation of $\mathcal{C}_m$ over S_{m+1}, and $tp_m(\rho_m(x)) = tp_m(\rho_{m+1}(x))$ for all $x \in S_{m+1}$ when $m < n$. We will define ρ_{n+1} to continue the sequence.

Note first that since $\mathcal{C}_n$ is a nondegenerate cluster, $\rho_n \restriction S_{n+1}$ is U_n-basic — see Remark 7.7. It is also surjective. For, let $w \in R_n(e_n)$ be given. Take $x \in S_{n+1}$ (note that S_{n+1} is non-empty by Claim 9.1.2). As $\mathcal{C}_n$ is a nondegenerate cluster, $R_n(\rho_n(x), w)$, so as $\rho_n \restriction S_{n+1}$ is a representation, $(S_{n+1}, (\rho_n \restriction S_{n+1})^{-1}), x \models \langle d \rangle w$. This certainly implies that $\rho_n(y) = w$ for some $y \in S_{n+1}$.

For each $w \in R_n(e_n)$, define

$$\begin{aligned} D_w &= \{x \in S_{n+1} : \rho_n(x) = w\} && \subseteq S_{n+1}, \\ H_w &= \{v \in R_{n+1}(e_{n+1}) : \operatorname{tp}_n(v) = \operatorname{tp}_n(w)\} && \subseteq W_{n+1}, \\ \mathcal{H}_w &= (H_w, R_{n+1} \restriction H_w). \end{aligned}$$

See Fig. 2. Because $\rho_n \restriction S_{n+1}$ is surjective onto $\mathcal{C}_n$, each set D_w is non-empty, and plainly, S_{n+1} is partitioned by the D_w $(w \in R_n(e_n))$. Because $\tau_n^{n+1} = \tau_n^n$, each H_w is non-empty and $\bigcup_{w \in R_n(e_n)} H_w = R_{n+1}(e_{n+1})$. (The sets H_w may not be pairwise disjoint, but any two of them are equal or disjoint.) Observe that

$$S_{n+1} \subseteq \langle d \rangle D_w \qquad \text{for each } w \in R_n(e_n). \tag{9.5}$$

To see this, let $x \in S_{n+1}$ and $w \in R_n(e_n)$. Because $\mathcal{C}_n$ is a nondegenerate cluster, $R_n(\rho_n(x), w)$. As $\rho_n \restriction S_{n+1}$ is a representation of $\mathcal{C}_n$ over S_{n+1}, we have $(S_{n+1}, (\rho_n \restriction S_{n+1})^{-1}), x \models \langle d \rangle w$. Since $(\rho_n \restriction S_{n+1})^{-1}(w) = D_w$, this says exactly that $x \in \langle d \rangle D_w$.

Let $w \in R_n(e_n)$ and consider D_w as a subspace of X. We show that it is dense in itself. Let $x \in D_w$ and suppose for contradiction that $\{x\}$ is open in D_w. So there is open $O \subseteq X$ with $D_w \cap O = \{x\}$, and as S_{n+1}

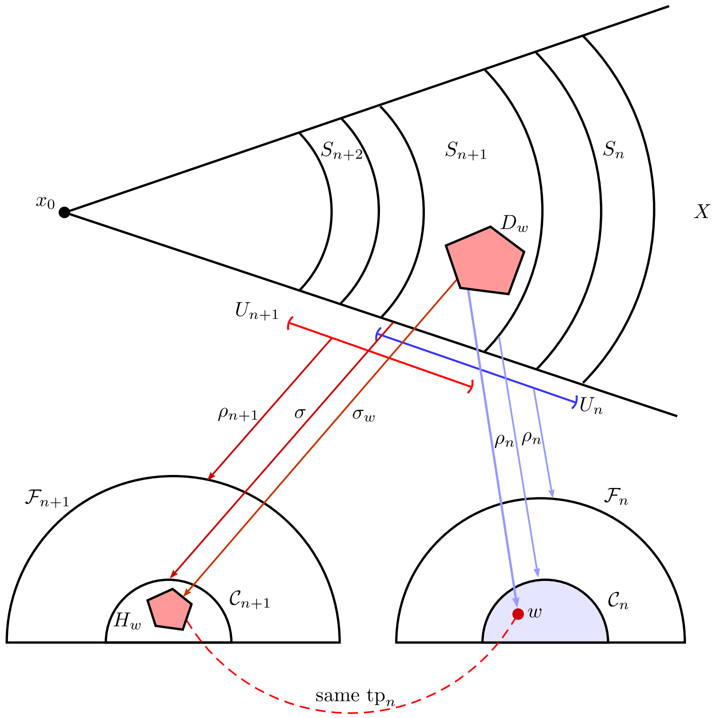

Fig. 2. Illustration for Claim 9.1.3.

is open, we can suppose that $O \subseteq S_{n+1}$. But by (9.5), $x \in D_w \subseteq S_{n+1} \subseteq \langle d\rangle D_w$, so $D_w \cap O \setminus \{x\} \neq \emptyset$. This contradicts $D_w \cap O = \{x\}$.

So D_w is a dense-in-itself metric space in its own right. Since $\mathcal{C}_{n+1}$ is a nondegenerate cluster, so is its subframe $\mathcal{H}_w$. Hence, $\mathcal{H}_w$ is trivially a finite connected locally connected serial transitive frame. So by Proposition 7.10, there is a surjective representation

$$\sigma_w : D_w \to H_w$$

of $\mathcal{H}_w$ over D_w. We have $(D_w, \sigma_w^{-1}), x \models \langle d\rangle v$ for every $x \in D_w$ and $v \in H_w$. By Lemma 7.2,

$$(X, \sigma_w^{-1}), x \models \langle d\rangle v \quad \text{ for every } x \in D_w \text{ and } v \in H_w. \tag{9.6}$$

Now let

$$\sigma = \Big(\bigcup_{w \in R_n(e_n)} \sigma_w \Big) \; : \; S_{n+1} \to R_{n+1}(e_{n+1}).$$

The sets D_w partition S_{n+1}, so σ is a well defined and total map. It has the following property. Let $x \in S_{n+1}$. Writing $\rho_n(x) = w$, say, we have $x \in D_w$ and $\sigma(x) = \sigma_w(x) \in H_w$, so $\mathrm{tp}_n(\sigma(x)) = \mathrm{tp}_n(w)$ by definition of H_w. That is,

$$\mathrm{tp}_n(\sigma(x)) = \mathrm{tp}_n(\rho_n(x)) \quad \text{ for each } x \in S_{n+1}. \tag{9.7}$$

We show that σ is a representation of $\mathcal{C}_{n+1}$ over S_{n+1}. Since $\mathcal{C}_{n+1}$ is a nondegenerate cluster, we need show only that $(X, \sigma^{-1}), x \models \langle d\rangle v$ for every $x \in S_{n+1}$ and $v \in R_{n+1}(e_{n+1})$.

So take such x, v. Suppose that $\rho_n(x) = w$, say, so $x \in D_w$. Choose $w' \in R_n(e_n)$ such that $v \in H_{w'}$ (it may not be unique). By (9.5), $x \in \langle d\rangle D_{w'}$. But by (9.6), $(X, \sigma^{-1}), y \models \langle d\rangle v$ for every $y \in D_{w'}$. It follows that $(X, \sigma^{-1}), x \models \langle d\rangle\langle d\rangle v$, and hence $(X, \sigma^{-1}), x \models \langle d\rangle v$ as required.

So σ is indeed a representation of $\mathcal{C}_{n+1}$ over S_{n+1}. As $\mathcal{C}_{n+1}$ is fully representable over X, we may choose a representation σ' of $\mathcal{C}_{n+1}$ over S_{n+2}. By Claim 9.1.2, $S_{n+1} \cap S_{n+2} = \emptyset$, so by Lemma 7.3, $\sigma \cup \sigma'$ is a well defined representation of $\mathcal{C}_{n+1}$ over the regular open subset $S_{n+1} \cup S_{n+2}$ of U_{n+1}. And since $\mathcal{C}_{n+1}$ is a nondegenerate cluster, $\sigma \cup \sigma'$ is U_{n+1}-basic (see Remark 7.7 again). Also, $U_{n+1} \setminus \mathrm{cl}(S_{n+1} \cup S_{n+2}) \neq \emptyset$ by Claim 9.1.2 again. We can now use the fact that $\mathcal{F}_{n+1}$ is fully representable over X to extend $\sigma \cup \sigma'$ to a surjective representation ρ_{n+1} of $\mathcal{F}_{n+1}$ over U_{n+1}. Then $\rho_{n+1} \restriction S_{n+2} = \sigma'$ is a representation of $\mathcal{C}_{n+1}$ over S_{n+2}, and by (9.7), $tp_n(\rho_n(x)) = \mathrm{tp}_n(\sigma(x)) = tp_n(\rho_{n+1}(x))$ for each $x \in S_{n+1}$, proving (9.4). This proves Claim 9.1.3. $\square$

Let $n < \omega$. Define an assignment $g_n : \mathsf{Var} \to \wp U_n$ by

$$g_n(p) = \rho_n^{-1}(h_n(p)) \text{ for each atom } p \in \mathsf{Var}. \tag{9.8}$$

By Claim 9.1.3, if $p \in \mathsf{Var}_n$ and $x \in S_{n+1}$, we have $x \in g_n(p)$ iff $\rho_n(x) \in h_n(p)$, iff $p \in \mathrm{tp}_n(\rho_n(x)) = \mathrm{tp}_n(\rho_{n+1}(x))$, iff $\rho_{n+1}(x) \in h_{n+1}(p)$, iff $x \in g_{n+1}(p)$. So g_n and g_{n+1} agree on S_{n+1} with respect to atoms in Var_n:

$$S_{n+1} \cap g_n(p) = S_{n+1} \cap g_{n+1}(p) \quad \text{ for each } p \in \mathsf{Var}_n. \tag{9.9}$$

Finally, define an assignment g on X as follows. Let p be an atom.

- For $x \in X \setminus \{x_0\}$, define $x \in g(p)$ iff $x \in g_n(p)$, where $x \in O_n \setminus O_{n+1}$.
  Since the $O_n \setminus O_{n+1}$ are pairwise disjoint, and $\bigcup_{n<\omega}(O_n \setminus O_{n+1}) = X \setminus \{x_0\}$ by (9.1), this is well defined.
- Define $x_0 \in g(p)$ iff $p \in \Gamma$.

Claim 9.1.4. *Let $n < \omega$, let $x \in U_n$, and let $\varphi \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ be a formula whose atoms lie in Var_n. Then $(X, g), x \models \varphi$ iff $\mathcal{M}_n, \rho_n(x) \models \varphi$.*

Proof of claim. Let $p \in \mathsf{Var}_n$ be arbitrary. Recall that $U_n = O_n \setminus \mathrm{cl}\, P_{n+1}$. By definition of g, if $x \in O_n \setminus O_{n+1}$ then $x \in g(p)$ iff $x \in g_n(p)$. If instead $x \in O_{n+1}$, then $x \in O_{n+1} \setminus \mathrm{cl}\, P_{n+1} = S_{n+1} \subseteq O_{n+1} \setminus O_{n+2}$, and the definition of g gives $x \in g(p)$ iff $x \in g_{n+1}(p)$. But by (9.9), this is iff $x \in g_n(p)$ again. So g and g_n agree on U_n as far as atoms in Var_n are concerned, and it follows by a trivial induction on formulas that $(U_n, g_n), x \models \varphi$ iff $(U_n, g_{U_n}), x \models \varphi$, where φ is as given, and $g_{U_n} : \mathsf{Var} \to \wp(U_n)$ is given by $g_{U_n}(p) = U_n \cap g(p)$ for each $p \in \mathsf{Var}$. Since ρ_n is a representation over U_n of the generated subframe $\mathcal{F}_n$ of (W_n, R_n), by Lemma 7.3 it is also a representation of (W_n, R_n) over U_n. The claim now follows by observing that

$$\begin{array}{llll}
(X, g), x \models \varphi & \text{iff} & (U_n, g_{U_n}), x \models \varphi & \text{by Lemma 2.3, as } U_n \text{ is open} \\
 & \text{iff} & (U_n, g_n), x \models \varphi & \text{by the above} \\
 & \text{iff} & (U_n, g_n), x \models \varphi^\mu & \text{by Lemma 4.2} \\
 & \text{iff} & \mathcal{M}_n, \rho_n(x) \models \varphi^\mu & \text{by Proposition 7.5 and (9.8), since } \varphi^\mu \in \mathcal{L}^\mu_{[d]\forall} \\
 & \text{iff} & \mathcal{M}_n, \rho_n(x) \models \varphi & \text{by Lemma 4.2 again, since } \mathcal{M}_n \text{ is transitive.} \quad \square
\end{array}$$

We are now ready to prove a ‘truth lemma’. We begin with formulas of the form $\langle d\rangle\varphi$.

Claim 9.1.5. *For every $\varphi \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ we have $(X, g), x_0 \models \langle d\rangle\varphi$ iff $\langle d\rangle\varphi \in \Gamma$.*

Proof of claim. Suppose first that $\langle d\rangle\varphi \in \Gamma$. Choose $n < \omega$ such that $\langle d\rangle\varphi \in \Gamma_n$. Let $i \geq n$ be arbitrary. Then $\langle d\rangle\varphi \in \Gamma_i$, so $\mathcal{M}_i, w_i \models \langle d\rangle\varphi$, and hence there is $v \in R_i(w_i)$ with $\mathcal{M}_i, v \models \varphi$. As $\rho_i : U_i \to R_i(w_i)$ is surjective (see Claim 9.1.3), there is $x \in U_i$ with $\rho_i(x) = v$. Since $\langle d\rangle\varphi \in \Gamma_i$, the atoms of φ lie in Var_i, so

Claim 9.1.4 applies: $(X, g), x \models \varphi$. We conclude that for every $i \geq n$ there is $x \in U_i$ with $(X, g), x \models \varphi$. As $U_i \subseteq O_i \setminus \{x_0\}$ and the O_i form a base of neighbourhoods of x_0, it follows that $(X, g), x_0 \models \langle d\rangle\varphi$.

Conversely, suppose that $(X, g), x_0 \models \langle d\rangle\varphi$. Since Γ is maximal consistent, either $\langle d\rangle\varphi \in \Gamma$ or $\neg\langle d\rangle\varphi \in \Gamma$. Choose $n < \omega$ such that either $\langle d\rangle\varphi \in \Gamma_n$ or $\neg\langle d\rangle\varphi \in \Gamma_n$. As O_n is an open neighbourhood of x_0, there is $x \in O_n \setminus \{x_0\}$ with $(X, g), x \models \varphi$. Since $O_n \setminus \{x_0\} = \bigcup_{n \leq i < \omega} U_i$ by (9.2), we have $x \in U_i$ for some $i \geq n$. The atoms of φ lie in Var_i, and $\rho_i : U_i \to R_i(w_i)$, so $\mathcal{M}_i, v \models \varphi$ for some $v \in R_i(w_i)$ (by Claim 9.1.4). So by Kripke semantics, $\mathcal{M}_i, w_i \models \langle d\rangle\varphi$ (we defined $\mathcal{F}_i$ as based on $R_i(w_i)$ rather than on W_i or $R_i^*(w_i)$ so that we can take this step). Since $\mathcal{M}_i, w_i \models \Gamma_i$, we have $\neg\langle d\rangle\varphi \notin \Gamma_i \supseteq \Gamma_n$. So $\langle d\rangle\varphi \in \Gamma_n \subseteq \Gamma$, proving the claim. □

The general case now follows:

Claim 9.1.6. *For every $\varphi \in \mathcal{L}_{[d]}^{\langle dt\rangle}$ we have $(X, g), x_0 \models \varphi$ iff $\varphi \in \Gamma$.*

Proof of claim. By induction on φ. For atoms, the result follows from the definition of g. The boolean operators are handled in the usual way by induction, using the maximal consistency of Γ; they are the only cases in which the inductive hypothesis is used. The case $[d]\varphi$ follows from Claim 9.1.5. Finally, consider the case $\langle dt\rangle\Delta$, where Δ is any non-empty finite set of formulas. It was shown in [14, section 4] that $\mathrm{K4}t \vdash \langle dt\rangle\Delta \leftrightarrow \langle d\rangle\langle dt\rangle\Delta$. It follows by soundness (Lemma 8.1) that $\langle dt\rangle\Delta \leftrightarrow \langle d\rangle\langle dt\rangle\Delta$ is valid in X, so $(X, g), x_0 \models \langle dt\rangle\Delta$ iff $(X, g), x_0 \models \langle d\rangle\langle dt\rangle\Delta$. By Claim 9.1.5, this is iff $\langle d\rangle\langle dt\rangle\Delta \in \Gamma$. Since Γ is maximal $\mathrm{KD4G_1}t$-consistent, this is iff $\langle dt\rangle\Delta \in \Gamma$, as required. The claim is proved. Hence, $(X, g), x_0 \models \Gamma$, so the theorem is proved as well. □

9.3. Strong completeness for $\mathcal{L}_{[d]}$

We can now easily derive the analogous result for 'modal' $\mathcal{L}_{[d]}$-formulas, essentially by showing that $\mathrm{KD4G_1}t$ is a conservative extension of $\mathrm{KD4G_1}$.

Theorem 9.2. *Let X be a non-empty dense-in-itself metric space. Then the Hilbert system* $\mathrm{KD4G_1}$ *is strongly complete over X for $\mathcal{L}_{[d]}$-formulas, and sound if* $\mathrm{G_1}$ *is valid in X.*

Proof. For soundness, see Theorem 8.5. For strong completeness, let Γ be a countable $\mathrm{KD4G_1}$-consistent set of $\mathcal{L}_{[d]}$-formulas. Let $\Gamma_0 \subseteq \Gamma$ be finite and put $\gamma = \bigwedge \Gamma_0$. Then γ is $\mathrm{KD4G_1}$-consistent, so by Fact 8.2 it is satisfiable in some finite serial transitive locally connected frame $\mathcal{F}$. It is easily seen that $\mathcal{F}$ is a $\mathrm{KD4G_1}t$ frame, and it follows that γ is $\mathrm{KD4G_1}t$-consistent. Since Γ_0 was arbitrary, Γ is $\mathrm{KD4G_1}t$-consistent. By Theorem 9.1, Γ is satisfiable over X. □

9.4. Strong completeness for $\mathcal{L}_{\Box}^{\langle t\rangle}$ and $\mathcal{L}_{\Box}^{\mu}$

This also follows, using the translations $-^d$ and $-^t$ of section 4.

Theorem 9.3. *Let X be any dense-in-itself metric space.*

1. *The Hilbert system* $\mathrm{S4}t$ *is sound and strongly complete over X for $\mathcal{L}_{\Box}^{\langle t\rangle}$-formulas.*
2. *The Hilbert system* $\mathrm{S4}\mu$ *is sound and strongly complete over X for $\mathcal{L}_{\Box}^{\mu}$-formulas.*
3. *(Kremer, [20]) The Hilbert system* $\mathrm{S4}$ *is sound and strongly complete over X for $\mathcal{L}_{\Box}$-formulas.*

Proof. Soundness is clear in all cases: cf. Theorem 8.3. We prove strong completeness. For part 1, let φ be an $\mathrm{S4}t$-consistent $\mathcal{L}_{\Box}^{\langle t\rangle}$-formula. By Fact 8.2, φ is satisfiable in some finite reflexive transitive Kripke frame $\mathcal{F}$.

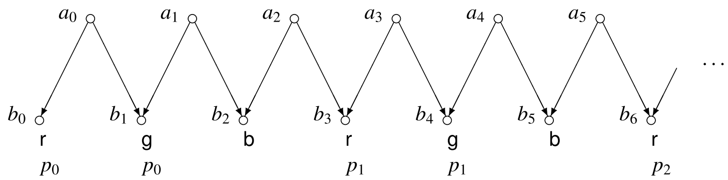

Fig. 3. $\mathcal{M}$.

Recall from section 4 the translation $-^d$ of $\mathcal{L}_\Box^{\langle t\rangle}$-formulas to $\mathcal{L}_{[d]}^{\langle dt\rangle}$-formulas. Since $\mathcal{F}$ is reflexive, it follows from Lemma 4.4 that φ^d is equivalent to φ in $\mathcal{F}$. So φ^d is satisfiable in $\mathcal{F}$. Plainly, $\mathcal{F}$ is a KD4G$_1t$ frame, so φ^d is KD4G$_1t$-consistent.

Since $-^d$ commutes with $\wedge$, it is now easily seen that if $\Gamma \subseteq \mathcal{L}_\Box^{\langle t\rangle}$ is a countable S4t-consistent set then $\Gamma^d = \{\gamma^d : \gamma \in \Gamma\} \subseteq \mathcal{L}_{[d]}^{\langle dt\rangle}$ is a countable KD4G$_1t$-consistent set. By Theorem 9.1, Γ^d is satisfiable over X. Since X is T$_D$, by Lemma 4.5 each $\gamma \in \Gamma$ is equivalent to γ^d in X, so Γ is also satisfiable over X.

For part 2, for a set $\Gamma \subseteq \mathcal{L}_\Box^\mu$ we write $\Gamma^t = \{\gamma^t : \gamma \in \Gamma\} \subseteq \mathcal{L}_\Box^{\langle t\rangle}$, where the translation $-^t : \mathcal{L}_\Box^\mu \to \mathcal{L}_\Box^{\langle t\rangle}$ is as in Fact 4.6. Let $\Gamma \subseteq \mathcal{L}_\Box^\mu$ be a countable S4μ-consistent set. Let $\Gamma_0 \subseteq \Gamma$ be any finite subset. By assumption, the formula $\bigwedge \Gamma_0$ is S4μ-consistent. So by Theorem 3.10, there is a finite reflexive transitive frame $\mathcal{F}$ in which $\bigwedge \Gamma_0$ is satisfiable. By Fact 4.6, φ^t is equivalent to φ in $\mathcal{F}$, for each $\varphi \in \mathcal{L}_\Box^\mu$. So $\bigwedge(\Gamma_0^t)$ is also satisfiable in $\mathcal{F}$. Since $\mathcal{F}$ is clearly an S4t frame, it follows that $\bigwedge(\Gamma_0^t)$ is S4t-consistent. As Γ_0 was arbitrary, Γ^t is S4t-consistent.

By part 1, Γ^t is satisfiable in X. But by Corollary 4.7, each $\gamma \in \Gamma$ is equivalent to γ^t in X. So Γ is also satisfiable in X.

Part 3 can be proved similarly, by showing in the same way that for $\mathcal{L}_\Box$-formulas, S4-consistency implies S4t-consistency, and then appealing to part 1. $\Box$

9.5. Universal modality

We do not include the universal modality in our strong completeness results, for good reason.

Theorem 9.4. *There is a countable set Σ of $\mathcal{L}_{\Box\forall}$-formulas such that for every non-empty compact locally connected dense-in-itself metric space X, each finite subset of Σ is satisfiable in X, but Σ as a whole is not.*

Compact means that if $\mathcal{S}$ is a set of open sets with $\bigcup \mathcal{S} = X$, then $X = \bigcup \mathcal{S}_0$ for some finite $\mathcal{S}_0 \subseteq \mathcal{S}$. Every compact space X is *sequentially compact* — for every sequence x_i $(i < \omega)$ of points of X, there is $z \in X$ such that for every open neighbourhood O of z, the set $\{i < \omega : x_i \in O\}$ is infinite. *Locally connected* means that every open neighbourhood of a point x contains a connected (in the subspace topology) open neighbourhood of x. An example of a compact locally connected dense-in-itself metric space is the subspace $[0, 1]$ of $\mathbb{R}$.

Proof. The proof is based on the following model $\mathcal{M} = (W, R, h)$, where we suppose that $\mathsf{Var} = \{\mathsf{r}, \mathsf{g}, \mathsf{b}\} \cup \{p_i : i < \omega\}$.

1. $W = \{a_n, b_n : n < \omega\}$, where the a_n and b_n are pairwise distinct.
2. R is the reflexive closure of $\{(a_n, b_n), (a_n, b_{n+1}) : n < \omega\}$.
3. $h(\mathsf{r}) = \{b_{3n} : n < \omega\}$, $h(\mathsf{g}) = \{b_{3n+1} : n < \omega\}$, $h(\mathsf{b}) = \{b_{3n+2} : n < \omega\}$, and $h(p_n) = \{b_{3n}, b_{3n+1}\}$ for each $n < \omega$.

The model is shown in Fig. 3 — it goes off to the right forever, roughly repeating after every three steps. Of course R is reflexive. Note that the underlying frame (W, R) is connected.

We let Σ be the set comprising the following formulas:

$\Sigma 1$. $\exists(\Diamond p_i \wedge \Diamond \mathsf{r} \wedge \Diamond \mathsf{g})$ for each $i < \omega$
$\Sigma 2$. $\forall\neg(\Diamond p_i \wedge \Diamond p_j)$ for $i < j < \omega$
$\Sigma 3$. $\forall\neg(\Diamond \mathsf{r} \wedge \Diamond \mathsf{g} \wedge \Diamond \mathsf{b})$
$\Sigma 4$. $\forall(\Diamond p_i \wedge \Box\neg \mathsf{b} \to \Box\Diamond p_i)$ for $i < \omega$.

They are plainly true at every world in $\mathcal{M}$. So for every finite subset $\Sigma_0 \subseteq \Sigma$, we have $\mathcal{M}, a_0 \models \Sigma_0$. As can be checked, the frame of $\mathcal{M}$ validates S4.UC, and it follows that Σ_0 is S4.UC-consistent. Hence, by Theorem 8.4, Σ_0 is satisfiable in X.

Assume for contradiction that Σ is true at some point of some model (X, h) on X. Below, we will write $x \models \varphi$ instead of $(X, h), x \models \varphi$. By $\Sigma 1$, for each $i < \omega$ there is $x_i \in X$ with $x_i \models \Diamond p_i \wedge \Diamond \mathsf{r} \wedge \Diamond \mathsf{g}$. As X is compact, it is sequentially compact and contains a point z such that for every open neighbourhood O of z, the set $\{i < \omega : x_i \in O\}$ is infinite. Then $z \models \Diamond \mathsf{r} \wedge \Diamond \mathsf{g}$ as well. By $\Sigma 3$, $z \models \Box\neg \mathsf{b}$. As X is locally connected, there is a connected open neighbourhood N of z with $y \models \neg \mathsf{b}$ for all $y \in N$.

Take $i < j < \omega$ with $x_i, x_j \in N$. Let $U = \{x \in N : x \models \Diamond p_i\}$. Then U is an open subset of N, because for every $u \in U$ we have $u \models \Diamond p_i \wedge \Box\neg \mathsf{b}$, and $\Sigma 4$ gives $u \models \Box\Diamond p_i$. And $N \setminus U$ is also open, because $U' = \{x \in X : x \models \Diamond p_i\}$ is closed and $N \setminus U = N \setminus U'$. We have $x_i \in U$, but by $\Sigma 2$, $x_j \in N \setminus U$. So N is the union of two disjoint non-empty open sets (U and $N \setminus U$), contradicting its connectedness. □

Corollary 9.5. *Let X be a non-empty compact locally connected dense-in-itself metric space, and $\mathcal{L} \subseteq \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ a language containing $\mathcal{L}_{\Box\forall}$ or $\mathcal{L}_{[d]\forall}$. Then no Hilbert system for $\mathcal{L}$ is sound and strongly complete over X.*

Proof. Assume for contradiction that the Hilbert system H is sound and strongly complete over X. Let Σ be as in Theorem 9.4 (use the translation $-^d$ if necessary to ensure it is a set of $\mathcal{L}$-formulas). Since every finite subset of Σ is satisfiable in X, and H is sound over X, it follows that Σ is H-consistent. But H is strongly complete over X, so Σ is satisfiable over X, contradicting the theorem. □

This does not rule out the possibility of strong completeness of a system having inference rules (2.2) with infinitely many premises.

10. Conclusion

This paper has presented some completeness theorems for various spatial logics over dense-in-themselves metric spaces. Table 2 summarises them. The numbers in parentheses refer to our earlier results. The first line of the table is of course known, included here to give a more complete picture. For handy reference, Table 3 summarises the ingredients of each logic.

There are of course many problems left open by our work, and we present some of them here. For simplicity, in this section we take metric spaces to be non-empty.

10.1. Extensions

Problem 10.1. *Can the results be extended to more general topological spaces?*

For example, consider the topological space T defined as follows. For ordinals α, β write ${}^{\alpha}\beta$ for the set of all maps $f : \alpha \to \beta$. The set of points of T is $\bigcup_{n \leq \omega} {}^{n}2$, and the open sets are unions of sets of the form $\{f \in T : f \supseteq g\}$ for some $g \in \bigcup_{n < \omega} {}^{n}2$. This space is dense in itself, and T0 — that is, no two distinct points have the same open neighbourhoods. It is not even T_D, but still it may be that the methods in this paper can be applied to it. So we ask:

Table 2
Soundness and completeness for a non-empty dense-in-itself metric space X.

Language	Logic	Sound	Complete	Strongly complete
$\mathcal{L}_\Box$	S4	yes	yes [27]	yes [20]
$\mathcal{L}_\Box^\mu$	S4μ	yes	yes (8.3)	yes (9.3)
$\mathcal{L}_\Box^{\langle t\rangle}$	S4t	yes	yes (8.3)	yes (9.3)
$\mathcal{L}_{\Box\forall}$	S4.UC	if X connected	yes (8.4)	not in general (9.5)
$\mathcal{L}_{\Box\forall}^{\langle t\rangle}$	S4t.UC	if X connected	yes (8.4)	not in general (9.5)
$\mathcal{L}_{[d]}$	KD4G$_1$	if G$_1$ valid in X	yes (8.5)	yes (9.2)
$\mathcal{L}_{[d]}^{\langle dt\rangle}$	KD4G$_1t$	if G$_1$ valid in X	yes (8.5)	yes (9.1)
$\mathcal{L}_{[d]\forall}$	KD4G$_1$.UC	if X connected & validates G$_1$	yes (8.7)	not in general (9.5)
$\mathcal{L}_{[d]\forall}^{\langle dt\rangle}$	KD4G$_1t$.UC	if X connected & validates G$_1$	yes (8.7)	not in general (9.5)

Table 3
Parts of the logics.

S4	$\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$
S4μ	fixed point axiom and rule: see Definition 3.1
KD4	$\langle d\rangle\top$, $[d]\varphi \to [d][d]\varphi$
t	tangle axioms from section 8.1
U	$\forall\varphi \to \Box\varphi$ or $\forall\varphi \to [d]\varphi$, S5 axioms for $\forall$, $\forall$-generalisation rule
C	$\forall(\Box^*\varphi \vee \Box^*\neg\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$, where $\Box^*\varphi = \varphi \wedge \Box\varphi$ or $\forall([d]^*\varphi \vee [d]^*\neg\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$, where $[d]^*\varphi = \varphi \wedge [d]\varphi$
G$_1$	$[d]([d]^*\varphi \vee [d]^*\neg\varphi) \to [d]\varphi \vee [d]\neg\varphi$

Problem 10.2. *What is the logic of T in the various languages discussed above?*

Problem 10.3. *Can the results be extended to stronger languages, for example, the mu-calculus with $[d]$ and/or $\forall$, languages with the difference modality or graded modalities, hybrid languages, and so on?*

Results of Kudinov [21,22] are relevant. Recently, Kudinov and Shehtman [23] proved numerous results about logics of topology with $\Box$, $[d]$, $\forall$, and the 'difference modality' $[\neq]$. In particular, they determine the logic of $\mathbb{R}^n$ for $n \geq 2$ in the language with $[d]$ and $[\neq]$. However, results for general dense-in-themselves metric spaces appear to be lacking.

10.2. Strong completeness

Our strong completeness results for languages with $[d]$ are limited to logics with G$_1$. We could ask for more:

Problem 10.4. *Let X be a dense-in-itself metric space and let $\mathcal{L}$ be $\mathcal{L}_{[d]}$ or $\mathcal{L}_{[d]}^{\langle dt\rangle}$. Is the $\mathcal{L}$-logic of X strongly complete over X?*

By Theorems 9.1 and 9.2, the answer is 'yes' if X validates G$_1$.

We saw in Corollary 9.5 that in the language $\mathcal{L}_{\Box\forall}$, there are many dense-in-themselves metric spaces over which S4.UC is not strongly complete. So we ask:

Problem 10.5. *Can strong completeness for languages with $\forall$ be proved for each dense-in-itself metric space in some reasonably large class, and for $\mathbb{R}^n$ for $n \geq 1$?*

Problem 10.6. *In the language $\mathcal{L}_{\Box\forall}$, is* S4.UC *strongly complete over the class of connected reflexive transitive Kripke frames?*

Even without $\forall$, an example in [14, §5] can be used to show that strong completeness fails in Kripke semantics for all our systems for languages containing $\mathcal{L}_{\Box}^{\langle t\rangle}$. But we saw that strong completeness does hold for some of these systems over dense-in-themselves metric spaces. Taking the example of S4t for $\mathcal{L}_{\Box}^{\langle t\rangle}$, it is striking that this logic is sound and complete for two different semantics (the class of S4 frames, and any non-empty dense-in-itself metric space), but strongly complete for only the latter. For more information about different notions of modal strong completeness, see, e.g., [35,37].

Our definition of strong completeness is limited to countable sets of formulas. We have not investigated the extent to which the strong completeness results in section 9 generalise to uncountable sets, but strong completeness will fail over any given dense-in-itself topological space X for any Hilbert system H that is sound over X, for large enough sets of formulas.

To see this, let $\kappa > |\wp(X)|$, and let $\Gamma = \{\Diamond p_i : i < \kappa\} \cup \{\neg\Diamond(p_i \wedge p_j) : i < j < \kappa\}$. Then Γ is H-consistent, because every finite subset of Γ is satisfiable in X. But given any assignment $h : \{p_i : i < \kappa\} \to \wp(X)$, by the pigeonhole principle there are $i < j < \kappa$ with $h(p_i) = h(p_j)$, so that $\Diamond p_i \wedge \neg\Diamond(p_i \wedge p_j)$ is everywhere false under h. Hence, Γ is not satisfiable in X (we thank the referee for this simple proof).

Definition 10.7. For each language $\mathcal{L}$ with $\mathcal{L}_{\Box} \subseteq \mathcal{L} \subseteq \mathcal{L}_{\Box[d]\forall}^{\mu\langle t\rangle\langle dt\rangle}$, and each dense-in-itself metric space X, let $\sigma(\mathcal{L}, X)$ be the least cardinal κ such that some set Γ of formulas with $|\Gamma| = \kappa$ is unsatisfiable over X but every finite subset of Γ is satisfiable over X.

So σ measures the degree of strong completeness of a language over a space. The larger $\sigma(\mathcal{L}, X)$ is, the more strong completeness we have. Here are some facts about σ.

1. By the proof just given, $\omega \le \sigma(\mathcal{L}, X) \le (2^{|X|})^+$ for any $\mathcal{L}, X$, so σ is well defined.
2. If $\mathcal{L}_{\Box} \subseteq \mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}_{\Box[d]\forall}^{\mu\langle t\rangle\langle dt\rangle}$ then $\sigma(\mathcal{L}, X) \ge \sigma(\mathcal{L}', X)$.
3. In terms of section 2.12, a Hilbert system in a language $\mathcal{L}$ that is sound and complete over X is strongly complete over X iff $\sigma(\mathcal{L}, X) > \omega$.
4. By Theorem 9.3, $\sigma(\mathcal{L}_{\Box}^{\langle t\rangle}, X) > \omega$ and $\sigma(\mathcal{L}_{\Box}^{\mu}, X) > \omega$.
5. By Theorems 9.1 and 9.2, if X validates G_1 then $\sigma(\mathcal{L}_{[d]}^{\langle dt\rangle}, X) > \omega$ and $\sigma(\mathcal{L}_{[d]}, X) > \omega$.
6. By Theorem 9.4, $\sigma(\mathcal{L}_{\Box\forall}, X) = \omega$ whenever X is compact and locally connected.

Problem 10.8. *Determine $\sigma(\mathcal{L}_{\Box}, \mathbb{Q})$ and $\sigma(\mathcal{L}_{\Box}, \mathbb{R})$. Do the same for $\mathcal{L}_{[d]}$. More generally, determine the function σ.*

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