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Taming Animal Spirits:

Risk Management with Behavioural Factors

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Abstract In several countries a major factor contributing to the current economic crisis was massive borrowing to fund investment projects on the basis of, in retrospect, grossly optimistic valuations. The purpose of this paper is to initiate an approach to project valuation and risk management in which 'behavioural' factors—Keynes' 'animal spirits' or Greenspan's 'irrational exuberance'—can be explicitly included. An appropriate framework is risk-neutral valuation based on the use of the numéraire portfolio—the 'benchmark' approach advocated by Platen and Heath (2006). In the paper, we start by discussing the ingredients of the problem: 'animal spirits', financial instability, market-consistent valuation, the numéraire portfolio and structural models of credit risk. We then study a project finance problem in which a bank lends money to an entrepreneur, collateralized by the value of the latter's investment project. This contains all the components of our approach in a simple setting and illustrates what steps are required. In a final section, we briefly discuss the econometric problems that need to be solved next.

Keywords Collateralized loans · animal spirits · Confidence indices · market-consistent valuation · numéraire portfolio · structural credit risk models

JEL Classification F47 – G02 – G11 – G32

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1 Introduction

The goal of this paper is to initiate a quantitative theory of credit risk relevant to scenarios that contributed to the credit crunch of 2008. We have in mind specifically the experience of Ireland and Spain, in which banks funded massive investments in property developments on the basis of heroically optimistic valuations of the return on these investments. At the World Economic Forum, Davos (2012), Enda Kenny, Taoiseach of Ireland, noted that 'Ireland's problems stem from a kind of madness that led to the country borrowing \$60 billion at unrealistically high rates'. It is obvious that any explanation must include behavioural factors.

In this paper we study a project finance problem involving two parties,

- Bank (B), which borrows from other commercial banks or a central bank, at funding rate r^f and lends to entrepreneurs;
- Entrepreneur (E) who borrows funds from B at a contract rate r_c in order to finance a project that will deliver a product of value G at time T.

E's loan will be paid off (with interest) in a bullet repayment at T. The investment project is the collateral for the loan, but of course its value is uncertain until T. B will insist that an over-collateralization ratio $\kappa > 1$ be maintained at all times $t \in [0, T]$, based on B's current assessment of the value G, and will insist on margin payments should this ratio be breached. This is what makes the loan so risky: E has to use other capital (assumed to be invested in the financial markets) to make margin payments, and if this capital is insufficient the loan will be foreclosed and the project sold off at a 'fire sale' price—some fraction of its pre-default assessed value at time t.

Clearly, the key question here is how B assesses the value of the project. E is, as Keynes says in the quote below, an optimist, but B should take a rational view. We assume B abides by the principles of market-consistent valuation, i.e. uses a model such that no arbitrage would be introduced if the project were traded at the model price in addition to existing traded asset in the market. This principle allows (see §2.2 below) a wide range of estimates, and we assume that B's valuation is affected by 'confidence' as represented by published business or consumer confidence indices.

In our analysis we find ourselves at the intersection of six lines of thought, namely (i) 'animal spirits', (ii) confidence indices, (iii) the financial instability hypothesis (iv) market-consistent valuation, (v) the numéraire portfolio, and (vi) structural models of credit risk. In Section 2 we give some background information on these topics that informs the models we construct and analyse in subsequent sections. Our project finance model is introduced in Section 3, and results for a simply computable example are described in Section 4. This section also discusses the computational requirements for larger-scale problems. The final section 5 gives a brief discussion of the problems concerning quantification of 'animal spirits'.

2 Background

2.1 Animal spirits.

John Maynard Keynes is not always recognized as a founding father of behavioural finance but, as in so many areas, the great man got there first. Indeed, in his Nobel Prize lecture, George Akelof (2003) states that 'Keynes' General Theory was the greatest contribution to behavioral economics before the present era'. The key to Keynes' thinking can be found in the General Theory (Keynes 2007, page 161):

> [A] large proportion of our positive activities depend on spontaneous optimism rather than on a mathematical expectation, whether moral or hedonistic or economic. Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as a result of animal spirits[1]— of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.

While nobody, surely, could disagree with the basic point, there is a mixed message here in that Keynes appears to be warning us off probabilistic and statistical analysis, and indeed he was quite sceptical about it, as reported by Akerlof and Shiller (2009, page 16). This point of view was in fact prevalent at the time. Frank Knight, whose seminal work substantiated the distinction between risk and uncertainty, noted in Knight (1921):

> It is a world of change in which we live, and a world of uncertainty. We live only by knowing something about the future; while the problems of life, or of conduct at least, arise from the fact that we know so little. This is as true of business as of other spheres of activity. The essence of the situation is action according to opinion, of greater or less foundation and value, neither entire ignorance nor complete and perfect information, but partial knowledge. If we are to understand the workings of the economic system we must examine the meaning and significance of uncertainty; and to this end some inquiry into the nature and function of knowledge itself is necessary.

Knight categorizes 'probabilities' into

1. *A priori probability*, a probability that can be computed exactly and objectively because the exact nature and structure of the underlying experiment is known;
2. *Statistical probability*, an empirical probability;
3. *Estimates*

Concerning estimates, Knight writes:

[1] The phrase is an allusion to the classical term *spiritus animalis* conveying the idea of animation, not atavism!

> It is this third type of probability or uncertainty which has been neglected in economic theory, and which we propose to put in its rightful place. As we have repeatedly pointed out, an uncertainty which can by any method be reduced to an objective, quantitatively determinate probability, can be reduced to complete certainty by grouping cases. [...] The present and more important task is to follow out the consequences of that higher form of uncertainty not susceptible to measurement and hence to elimination. It is this true uncertainty which by preventing the theoretically perfect outworking of the tendencies of competition gives the characteristic form of 'enterprise' to economic organization as a whole and accounts for the peculiar income of the entrepreneur.

This message is markedly different from the ideas promoted by standard financial economics starting in the 1950s: uncertainty appears as irrelevant because it can be diversified away or hedged against. The implication is that rational decision makers should rely on *a priori* probabilities when known or on statistical probabilities to form their opinions. Decision makers who are not rational, such as noise traders, will be arbitraged out of the economy.

The flourishing field of behavioural finance has demonstrated convincingly that the view held by standard finance theory is not tenable *stricto sensu*. Noise traders are alive and well and arbitrage is fraught with difficulties (see for example Shleifer (2000) for a discussion). Although most of the ideas and tests performed by behavioural finance theorist focus on pure decision theory and on psychology, behavioural finance does not exclude quantitative models and methods. In fact the scope for quantitative methods is much greater now than it was in the 1920s and 1930s, or even in the 1950s and 1960s thanks to the creation of confidence indices designed to gauge 'animal spirits,' the development of a theory of financial economics and advances in computational technology.

2.2 Confidence indices.

Data on confidence indices is now widely available, see for example Markit (2011). There are two varieties, the *consumer confidence index* and the *purchasing managers' index* (PMI). Both are based on surveys, and represent respectively the propensity of consumers to go out and spend, and the propensity of businesses to invest. In the United States, consumer confidence is measured by the Conference Board and by the University of Michigan. The main difference between the two surveys is in the time horizon: while the Conference Board polls households on their expectations over the next six months, the University of Michigan looks at expectations over the coming year. On the other hand, purchasing manager expectations are assessed regionally: the Chicago PMI is widely regarded as the most representative of nationwide sentiment.

There are a number of empirical studies—see Akerlof and Shiller (2009), footnote 9, page 179— aimed at testing whether confidence actually 'causes' economic growth (interpreted in the sense of 'Granger causality' Granger (1969)). These include Matsusaka and Sbordone (1995) who produce quite convincing evidence that this link exists[2].

Following in the footsteps of Matsusaka and Sbordone, Howrey (2001) investigates the predictive power of the University of Michigan consumer confidence index over the period 1961 to 1999. He finds that the consumer confidence index is a statistically significant predictor of the future rate of growth of real GDP and of recessions. He also finds that consumer confidence provides a good point estimate of future consumer spending, albeit with a large standard error. In the case of Japan, Utaka (2003) finds that consumer confidence has a short term impact on GDP growth, but no short term effect.

If confidence is a good predictor of macroeconomic trends and cycles, could it also have an impact on asset prices? In an event study, Rigobon and Sack (2008) test the impact of unexpected changes in 13 macroeconomic data series including the Chicago PMI and consumer confidence on eurodollar futures contracts, treasury yields and the S&P 500. They find that surprises in the Chicago PMI and in consumer confidence have a statistically significant impact on the rate of six-month and 12-months eurodollar futures contracts and on the yields of 2-year and 10-year Treasuries, but not on the S&P 500.

Still, sentiment by and large plays a significant role in the behaviour of stock markets, as evidenced for example by Baker and Wurgler (2006). It is therefore natural to investigate specifically the relation between consumer confidence and stock market returns. Jansen and Nahuis (2003) study the relationship between stock market developments and consumer confidence in eleven European countries over the period 1986-2001. Although consumer confidence is positively correlated with stock market returns in nine countries, they did not find statistical evidence that consumer confidence Granger-causes stock market returns. To the contrary, stock market returns appears to Granger-cause consumer confidence over a short horizon of two weeks to one month. This result is intriguing, especially when we consider studies on leading economic indicators. Hertzberg and Beckman (1989) find that consumer confidence has a lead time of 14 months with respect to economic peaks while the S&P has a shorter lead time of 8.5 months. The gap is narrower for economic troughs: 4.5 months for consumer confidence versus 4 months for the S&P 500.

Fisher and Statman (2000, 2003) find that statistically significant increases in the bullishness of individual investors follow increases in consumer confidence. Over the period 1989 to 2002, large improvements in consumer confidence appear to have be followed by high returns on the S&P 500 index, NASDAQ index and among small caps. Lemmon and Portniaguina (2006) find that consumer confidence is useful in forecasting the returns on small stocks. Their view is that consumer confidence reflects not only current and expected fundamentals but also excessive sentiment such as overoptimism and

[2] Akerlof and Shiller (2009) somehow understate the reach of Matsusaka and Sbordone's argument.

pessimism. As a result of excessive optimism (pessimism), investors will overvalue (undervalue) small stocks relative to large stocks.

The relation between investor or entrepreneur sentiment and asset market is both important and complex, and more research is needed to understand their connection. This is particularly true for real estate, for which the literature linking confidence and real estate prices is scarcer.

2.3 The Financial Instability Hypothesis

Recent crises in the U.S. mortgage market, Ireland or Spain, have demonstrated that animal spirits and sentiment have a considerable influence on the long-term evolution of financial markets and on the build-up of speculative bubbles. Keynesian economist Hyman Minsky (1992, 2008) proposed a Financial Instability Hypothesis to describes the journey of a financial market from safe (hedge units) to speculative and then to untenably speculative (Ponzi units[3]) investments. Minsky argues that investors tend to build long-term predictions based on current conditions[4]: if current conditions are good, investors will tend to extrapolate that financial conditions will remain favorable over the long run. As their risk tolerance declines, investors seek riskier assets. This in turns pushes asset prices up and makes current financial conditions look even more favorable. This feedback loop pushes investments into ever more speculative assets up until the Ponzi stage is reached, triggering a collapse of financial markets. To add to this instability, Minsky postulate that longer periods of stability will ultimately generate higher risk taking and ultimately more profound instability.

McCulley (2009) illustrates Minsky's financial instability hypothesis in the case of the subprime crisis in the United States. In first, or hedged unit, stage of development, both lenders and investors are cautious. Loans are only extended to home buyers who have enough equity and earning power to pay back both the principal and the interest. The liability inherent in the mortgage is therefore properly hedged. As the market grows and house prices increase, we enter the second, or speculative unit, stage. Lenders and investors have gained in confidence. Loans are now extended to home buyers who have enough equity and earning power to pay back the interest on the mortgage, but necessarily the principal. Implicitly, lenders and investors speculate that mortgage rates will not rise, mortgage terms will not deteriorate and house prices will not decline. As optimism reaches unrealistic levels, the market transitions form the second stage to the third, or Ponzi unit, stage. Lenders and investors take an unsustainable amount of risk. Loans are now offered to home buyers who

[3] Ponzi units are named after 1920s con artist Charles Ponzi.

[4] Minsky's view that investors predominently form their opinion based on recent data is related to the availability heuristics first analyzed by Tversky and Kahneman (1973): to make their decisions, individuals use short cuts (heuristics) such as extrapolating from immediately available information, and in particular from the recent past.

can neither afford to repay the principal nor the interest. Implicitly, lenders and investors speculate that mortgage rates will not rise, mortgage terms will not deteriorate and most importantly that house prices will keep on increasing.

2.4 Market-consistent valuation

Suppose we have a universe of liquidly-tradable assets whose current prices are $\{S_0^j,\ j = 1, \dots, n\}$. It is assumed that no arbitrage opportunities are presented by trading at these prices. Now consider a further contract, denoted Y. This could be another traded asset about to be launched, such as a call option on one of the existing assets S^j, or more generally a non-traded or illiquid asset such as a basket of insurance policies or mortgages. We say that a 'price' Y_0 is *market consistent* if offering Y for sale or purchase for Y_0 does not introduce arbitrage into the existing market. As an elementary example, consider the 1-period trinomial tree shown in Figure 1 modelling an asset S and suppose there is a riskless account paying zero interest. By considering possible 'risk-neutral' or 'martingale' measures (q_1, q_2, q_3) readers can convince themselves that 90 is an arbitrage-free price for S and that the no-arbitrage range of prices for an at-the-money call option (strike $K = 90$) is the open interval $(5, 7.5)$. For any $p \in (5, 7.5)$ there is a risk-neutral measure (q_1^p, q_2^p, q_3^p) such that $p = 30q_1^p + 10q_2^p$, the risk-neutral call option price. Thus any option price within this range is 'market consistent'. It is in fact true in some generality, see Föllmer and Schied (2011), that for general multi-period models the range of market-consistent prices coincides with the set of discounted expectations corresponding to all possible 'martingale' measures.

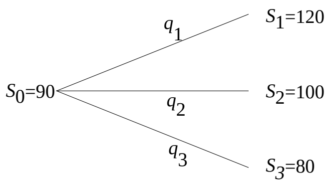

Fig. 1 1-period trinomial tree.

In general, *market-consistent valuation* refers to the process of building a model for (some section of) the market, calibrating it—i.e., choosing model parameters so that the model reproduces prices of market-traded assets—and then using the model to price another contract by risk-neutral valuation. By definition, the enlarged market is then arbitrage-free. This process has been routinely used for many years in investment banks under the name of *marking to market* and traders are well aware of 'model risk', the fact that different models for the underlying assets S^j, all perfectly calibrated to the same market data, may give widely differing prices for the new asset Y.

Recent developments in financial regulation and accounting practice have forced the principle of market consistency onto a wider segments of the financial services industry, notably insurance and pensions. An excellent account of the current state of play is given by Kemp (2009). How useful, or indeed credible, this process is depends on 'how complete' the market is. If there is no S^j that is sufficiently closely related to Y as to be useful for hedging then the range of market-consistent prices will be too wide to be of any practical significance without further modelling input or constraints. In addition, it is important to realize that market consistent pricing is not synonymous with rational pricing. Market consistent pricing provides consistency across securities prices at a time t by preventing instantaneous arbitrage. As a result, market consistent pricing is procylical and as such it is susceptible to bubbles and crashes over periods of time. The optimistic prices of loans and credit products booked by banks before the crisis and subsequent writedowns starting 2008 illustrate this points.

2.5 The numéraire portfolio.

In elementary treatments of mathematical finance, call option prices are generally expressed, in conventional notation, as $C(K) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - K)^+]$, where $\mathbb{E}_{\mathbb{Q}}$ denotes expectation with respect to 'the' risk-neutral measure. This is equivalent to $C(K) = \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+/B_T]$ where $B_t = e^{rt}$, showing that $C(K)$ is the expected value of the option payoff expressed in units of the 'savings account' B_t. It turns out that using the savings account as numéraire is an arbitrary choice. The modern view, stated explicitly in Geman et al (1995) and clearly expounded by Hunt and Kennedy (2004) for example, is to think in terms of *numéraire pairs* $(N, \mathbb{Q})$, where N_t is a tradable asset with strictly positive price, conventionally normalized to $N_0 = 1$, and $\mathbb{Q}$ is a measure such that for any traded asset S the price ratio S_t/N_t is a martingale. A key point is that if one fixes the measure $\mathbb{Q}$ and searches for an asset price process $N^{\mathbb{Q}}$ such that $(N^{\mathbb{Q}}, \mathbb{Q})$ is a numéraire pair, then there is a unique solution, namely that $N^{\mathbb{Q}}$ is the *growth-optimal* portfolio[5] when the asset prices are governed by the probability law $\mathbb{Q}$. The growth-optimal portfolio maximizes, over investment strategies π, the expected log-utility $\mathbb{E}_{\mathbb{Q}}[\log W_T^\pi]$ at some fixed time T, where W_T^π is the value of the investment portfolio at time T using strategy π, starting conventionally at $W_0^\pi = 1$. If $(N^{\mathbb{Q}}, \mathbb{Q})$ is a numéraire pair then using the inequality $\log x \leq x - 1$ and the numéraire property of $N^{\mathbb{Q}}$ we have

$$\mathbb{E}_Q \log W_T^\pi - \mathbb{E}_Q \log N_T^{\mathbb{Q}} = \mathbb{E}_Q[\log(W_T^\pi/N_T^{\mathbb{Q}})] \leq \mathbb{E}_Q[W_T^\pi/N_T^{\mathbb{Q}}] - 1 = 0.$$

Thus $N_T^{\mathbb{Q}}$ maximises logarithmic utility under $\mathbb{Q}$.

J.B. Long (1990) first realized the significance of this fact, namely that there is nothing stopping us choosing $\mathbb{Q} = \mathbb{P}$, the real-world 'statistical' measure governing asset prices, and then $N^{\mathbb{P}}$ is the optimal investment portfolio

[5] See MacLean et al (2011) for a comprehensive account of investment based on the growth-optimal or 'Kelly' criterion

for an investor with logarithmic utility, which is easily computed in many cases. This approach has the decisive advantage that *all modelling is carried out under the statistical measure*. It is the basis for Platen's 'benchmark approach' to financial valuation Platen and Heath (2006). In our case we want to include econometric factors such as confidence indices, GDP growth etc. in our modelling framework. If we use the benchmark approach then econometric models for these quantities, estimated using historical data, can be plugged right into our model without worrying about the distinction between real-world and risk-neutral measures since these two things are now one and the same.

2.6 Structural models of credit risk.

Below we shall be considering investment funded by collateralized loans, where the investor may default if he is unable to post sufficient additional margin in case of a fall in the value of the collateral. In our model, the time at which this happens will be a stopping time of some filtration. This is true of all models in the modern theory of credit risk, although in this theory, as is seen in textbooks such as Lando (2004), there are two distinct classes of model, 'reduced form' and 'structural form'. The latter, which contains our model, is ultimately derived from early work by Robert Merton (1974) in which the default risk on corporate debt is represented as a put option on the value of the firm. Modelling firm value accurately is not an easy thing to do (it is not the same thing as market capitalization), and later modellers such as Hull and White (2001) or Longstaff and Schwartz (1995) have concentrated on stylized models in which default occurs at the first hitting time of a possibly time-varying boundary by some stochastic process, where parameters specifying the process and/or the boundary are calibrated from market credit default swap quotes. Our model is in the same vein mathematically, but because we model explicitly the collateral value and the evolution of the margin account we return to a Merton-like picture where the credit model has economic as well as mathematical content.

3 The Project Finance Model

We now proceed to a formal specification of our model.

3.1 Financial Market

We consider a simple model in which tradable asset prices $S_i(t), i = 1, \ldots, m$ satisfy SDEs of the form

$$dS_i(t) = S_i(t)\mu_i(X(t))dt + S_i(t)\sigma_i(X(t))dW_t, \quad i = 1, \ldots, m, \qquad (3.1)$$

$$dX(t) = \alpha(X(t))dt + \Lambda(X(t))dW_t, \quad X(0) = x \qquad (3.2)$$

for $t \in [0,T]$, where W_t is $\mathbb{P}$-Brownian motion in $\mathbb{R}^{n+m}$ and $X(t)$ is an n-dimensional factor process on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. We assume that α, Λ are Lipschitz continuous so that a unique strong solution of (3.2) exists. $S_i(t)$ is then given explicitly by

$$S_i(t) = S_i(0)\exp\left(\int_0^t (\mu_i - \frac{1}{2}|\sigma_i|^2)ds + \int_0^t \sigma_i dW_s\right). \tag{3.3}$$

The short rate of interest available to E is $r(X(t))$ for some given function $r(\cdot)$. We will describe the components of the factor process $X(t)$ below. The integrability conditions on μ_i, σ_i are such that the integrals in (3.3) are well defined and $\mathbb{E}[S_i(t)] < \infty$ for all i, t. The main point is that $X(t)$ includes confidence indices.

The log-optimal portfolio for this model is (omitting the X-dependence)

$$dY(t) = Y(t)\,(r + \pi_* \Sigma\Sigma' \pi_*')\,dt + Y(t)\pi_* \Sigma\, dW, \tag{3.4}$$

where $\pi_*(t) = (\mu - \mathbf{1}\,r)'(\Sigma\Sigma')^{-1}$ is the optimal asset allocation (the allocation to the money market account being $\pi_*^0 = 1 - \pi_*\mathbf{1}$). Here μ [Σ] is the vector [matrix] with rows μ_i [σ_i] and $\mathbf{1}$ is the vector with all entries equal to 1. We assume that, for some $\epsilon > 0$, $s'\Sigma\Sigma'(x)s \geq \epsilon|s|^2$ for all $(s,x) \in \mathbb{R}^{m+n}$, which is equivalent to saying that there are no redundant assets. Equation (3.4) can be expressed as

$$dY(t) = Y(t)(r + \beta^2)dt + Y(t)\beta\, dB_t, \tag{3.5}$$

where B_t is the scalar Brownian motion

$$B_t = \int_0^t \frac{\pi_*\Sigma(s)}{|\pi_*\Sigma(s)|} dW_s$$

and $\beta(t) = |\pi_*(t)\Sigma(t)|$. With initial endowment $Y(0) = 1$, equation (3.5) has explicit solution

$$Y(t) = \exp\left(\int_0^t (r + \frac{1}{2}\beta^2)ds + \int_0^t \beta\, dB_s\right). \tag{3.6}$$

We are going to use the log-optimal portfolio $Y(t)$ given by (3.5) for two different purposes:

(a) It is assumed that entrepreneur E is a log-optimal ("Kelly") investor, so that his surplus wealth (capital not invested in the project) is just $x_0 Y(t)$ if his initial surplus wealth is x_0. (This will hold up to the time of B's first margin call; see below.)

(b) $Y(t)$ is the numéraire asset, so the risk-neutral value at t of an $\mathcal{F}_T$-measurable payment H paid at T is

$$H_t = Y(t)\mathbb{E}\left[\frac{H}{Y(T)}\middle|\mathcal{F}_t\right]. \tag{3.7}$$

This is the valuation formula used by B.

A discrete-time formulation. Most of the econometric data we consider, such as confidence indices, is posted monthly. Let us suppose that T is an integer number n of months, and denote by $0 = t_0, t_1, \ldots, t_{n-1}$ the first day of each monthly period. If $X_i(\cdot)$, the ith component of the factor process $X(\cdot)$, is an econometric variable based on monthly data then we simply define $X_i(t) = X_i(t_k)$ for $t \in [t_k, t_{k+1})$. If every component of $X(\cdot)$ is obtained from discrete data in this way then equation (3.6) has piecewise-constant coefficients, and the solution $Y(t_k)$ can be expressed as

$$Y(t_k) = \prod_{j=1}^{k} U_j, \qquad U_j = \exp\left((r_j + \frac{1}{2}\beta_j^2)\delta_j + \beta_j\sqrt{\delta_j}Z_j\right) \tag{3.8}$$

where $\delta_j = t_j - t_{j-1}$, $r_j = r(X(t_{j-1}))$, $\beta_j = \beta(X(t_{j-1}))$, and the Z_j are independent $N(0,1)$ random variables.

Let $\{\mathcal{G}_k, k = 0, \ldots, n\}$ be the discrete filtration where $\mathcal{G}_0$ is the trivial σ-field and $\mathcal{G}_k = \sigma\{B(t_j) - B(t_{j-1}), j = 1, \ldots, k\}$ for $k = 1, \ldots, n$. If H is a $\mathcal{G}_n$-measurable random variable then we see from (3.7) and (3.8) that the value at time t_k is

$$H_{t_k} = \mathbb{E}\left[\left(\prod_{j=k+1}^{n} U_j^{-1}\right) H \,\middle|\, \mathcal{G}_k\right]. \tag{3.9}$$

3.2 Project Finance

The project finance valuation problem was informally described in Section 1. The entrepreneur E has initial capital x and can, for a payment \$$A$, invest in a venture which, at time $T = n$ months, will yield a reward $G(X(T))$ as above. G is a function that will be specified below, but it is a function of $X(T)$ only and hence is $\mathcal{G}_T$-measurable. He invests \$$a$ of his own money and borrows \$$b = A - a$ from a bank at a term rate of interest r_c (expressed for convenience in continuously-compounding terms), repayable by a bullet payment at T. Thus effectively his initial capital is reduced to $x_0 = x - a$ while the eventual reward is $G(X(T)) - e^{r_c T}b$. The capital x_0 is invested in the Kelly portfolio described above.

The value of the project at an intermediate time t_k is deemed by the bank B to be the market-consistent value $G_k(X(t_k))$ given by (3.9) as

$$G_k = \mathbb{E}\left[\left(\prod_{j=k+1}^{n} U_j^{-1}\right) G(X(T)) \,\middle|\, \mathcal{G}_k\right]. \tag{3.10}$$

The loan is collateralized by the value of the project, and B stipulates over-collateralization with factors $\kappa > \kappa' > 1$, checked at monthly intervals. Thus $G_0 \geq \kappa b$ and in any subsequent verification time k we have

$$G_k(X(t)) \geq \kappa' b e^{r_c t_k}.$$

Defining $H_k(x) = e^{-r_c t_k} G_k(x)$ this is equivalent to $H_k \geq \kappa' b$. Let

$$\theta_1 = \min\{k : H_k \leq \kappa' b\}, \quad \tau_1 = t_{\theta_1},$$
$$b_1 = \frac{H_{\theta_1}}{\kappa}.$$

At τ_1, E is contractually obliged to provide additional collateral to restore the collateral level to κ by paying off the amount $d_1 = e^{r_c \tau_1}(b - b_1)$ of the loan. Since the project is illiquid, he can only do this from his investment portfolio, and hence this experiences a jump of $-d_1$. In general, we define for $j = 2, 3, \ldots$

$$\theta_j = \min\{k : \theta_{j-1} < k < n, \, H_k \leq \kappa' b_{j-1}\}, \quad \tau_j = t_{\theta_j},$$
$$b_j = \frac{H_{\theta_j}}{\kappa}$$

giving a jump in V of $-d_j = -e^{r_c \tau_j}(b_{j-1} - b_j)$. The entrepreneur's market investment portfolio evolves as follows:

$$V(t) = V(0) + \int_0^t (r + \beta^2) V(s) ds + \int_0^t V(s) \beta dB_s - \sum_{\tau_j \leq t} d_j.$$

Let $\tau^* = \min\{\tau_j : V(\tau_j) < 0\}$, with $\tau^* = +\infty$ if there is no such t_j, and let $\theta^* = j$ when $\tau^* = t_j$. If $\tau^* < T$ the entrepreneur is insolvent at τ^* and the project must be liquidated at 'fire sale' value $F(\tau^*) = \phi(\tau^*) G_{\theta^*}(X(\tau^*))$, where ϕ is an increasing function of time with values in $[0, 1[$. The bank receives $F(\tau^*) + V(\tau^* -)$.

The market-consistent value of the loan to the bank is therefore

$$\text{MCV} = \mathbb{E}\left[\sum_j \frac{d_j}{Y(\tau_j)} \mathbf{1}_{(\tau_j < \tau^* \wedge T)} + \frac{b_n \wedge (V(T) + G(X(T)))}{Y(T)} \mathbf{1}_{(\tau^* > T)} \right.$$
$$\left. + \frac{F(\tau^*) + V(\tau^* -)}{Y(\tau^*)} \mathbf{1}_{(\tau^* < T)} \right]. \tag{3.11}$$

We thus have a credit risk model. The bank loses value because of early partial repayment of the loan together with the risk of actual default. Define $p_k = (1 + r_k^f/12)^{-1}$ where r_k^f is the Bank's (annualized) funding cost for the kth month, and $p_{0,k} = \prod_{l=1}^k p_l$. Then, recalling that the initial loan amount is b, the Bank's P&L along one sample path, discounted to time 0, is

$$\Xi(\omega) = \sum_j p_{0,\theta_j} d_j \mathbf{1}_{(\tau_j < \tau^* \wedge T)} + p_{0,n} [b_n \wedge (V(T) + G(T))] \mathbf{1}_{(\tau^* > T)}$$
$$+ \, p_{0,\theta^*} [F(\tau^*) + V(\tau^* -)] \mathbf{1}_{(\tau^* < T)} - b.$$

The Bank will be interested in the *expected profit* $\mathbf{e} = \mathbb{E}[\Xi]$, the *value at risk* $\text{VaR} = \mathbf{e} - \mathbf{q}$, where $\mathbf{q}$ is the (say) 5% quantile of the P&L distribution, and the *expected shortfall*

$$\text{CVaR} = \mathbf{e} - \mathbb{E}[\Xi | \Xi < \mathbf{q}] = \mathbf{e} - \frac{1}{\mathbf{q}} \mathbb{E}\left[\Xi \mathbf{1}_{(\Xi < \mathbf{q})}\right].$$

4 A simply computable example

In this section we demonstrate the computations required in a simple example where the factor process is a scalar Ornstein-Uhlenbeck process. This is a stylized model intended mainly to illustrate the computational process. We do not attempt to connect the factor variable to econometric data, an entirely separate matter.

4.1 Model specification

Recall that the numéraire asset is Y_t satisfying

$$dY_t = (r(X_t) + \beta^2(X_t))Y_t dt + \beta(X_t)Y_t dB_t, \qquad Y_0 = 1. \tag{4.1}$$

In this example we suppose that X_t is scalar, $\beta(x) = b_0 + b_1 x$ and $r(x) = r_0 + r_1 x$. X_t is the mean-reverting Gaussian process

$$dX_t = -\alpha X_t dt + \gamma dW_t, \qquad X_0 = x_0, \tag{4.2}$$

where $\alpha, \gamma > 0$ are constant and W_t is a Brownian motion with $\mathbb{E}[dW\, dB] = \rho\, dt$. The project value at completion is defined by $G(X_T) = e^{\eta+\xi X_T}$. This is analogous to conventional modelling of commodity prices as exponentials of mean-reverting processes. Note that ξ represents, up to a constant, the volatility of project value.

The Bank's valuation of the project at $t < T$ is

$$G(t, X_t) = \mathbb{E}\left[\frac{Y_t}{Y_T} e^{\eta+\xi X_T} \middle| \mathcal{F}_t \right]. \tag{4.3}$$

Proposition 4.1 *Let $c_0 = \gamma\rho b_0$, $c_1 = \alpha + \gamma\rho b_1$, $d = r_1/c_1$. Then*

$$G(t, x) = \exp(v_0(t) + v_1(t)x) \tag{4.4}$$

where

$$v_1(t) = (\xi + d)e^{-c_1(T-t)} - d, \tag{4.5}$$

$$\begin{aligned} v_0(t) = \eta - \left(r_0 - c_0 d - \frac{1}{2}\gamma^2 d^2\right)(T-t) - (\xi+d)(c_0+\gamma^2 d)\left[\frac{1}{c_1}(1-e^{-c_1(T-t)})\right] \\ +\frac{1}{2}\gamma^2(\xi+d)^2\left[\frac{1}{2c_1}(1-e^{-2c_1(T-t)})\right]. \end{aligned} \tag{4.6}$$

Proof. The result follows from the fact that (4.1),(4.2) is an affine factor model Duffie et al (2000). We outline the steps, which can be completed by routine—if tedious—computations.
(i) The risk-neutral measure $\mathbb{Q}$ with money-market account as numéraire is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \beta\, dB - \frac{1}{2}\int_0^T \beta^2 dt\right).$$

If we express W_t in (4.2) as $W_t = \rho B_t + \sqrt{1-\rho^2}\, W_t^0$, where B, W^0 are $\mathbb{P}$-independent Brownian motions, then $d\tilde{B} = dB + \beta\, dt$, W_t^0 and $dW_t^1 = \rho\, d\tilde{B}_t + \sqrt{1-\rho^2}\, dW^0$ are $\mathbb{Q}$-independent Brownian motions and X_t satisfies

$$\begin{aligned} dX_t &= -(\gamma\rho b_0 + (\alpha + \gamma\rho b_1)X_t)dt + \gamma\, dW_t^1 \\ &= -\left(c_0 + c_1 X_t\right) dt + \gamma\, dW_t^1. \end{aligned}$$

(ii) The project value is expressed under the measure $\mathbb{Q}$ as $G(t, X_t)$ where

$$G(t,x) = \mathbb{E}_{t,x}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} e^{\eta + \xi X_T} \right].$$

(iii) By the Feynman-Kac formula, $v(t,x)$ satisfies the backward equation

$$\frac{\partial G}{\partial t} - (c_0 + c_1 x)\frac{\partial G}{\partial x} + \frac{1}{2}\gamma^2 \frac{\partial^2 G}{\partial x^2} - (r_0 + r_1 x)G = 0, \quad G(T,x) = e^{\eta + \xi x}. \tag{4.7}$$

(iv) The PDE (4.7) has solution (4.4) where v_1, v_0 are given respectively by (4.5), (4.6). Indeed, one can check that a solution of the form (4.4) satisfies (4.7) if v_1 satisfies the ODE

$$\frac{d}{dt} v_1(t) = c_1 v_1(t) + r_1, \quad v_1(T) = \xi,$$

whose solution is (4.5). v_0 is then given by direct integration of the following expression involving v_1:

$$\frac{d}{dt} v_0(t) = c_0 v_1(t) - \frac{1}{2}\gamma^2 \left(v_1(t)\right)^2 + r_0, \quad v_0(T) = \eta,$$

Working this out gives (4.6). □

With Proposition 4.1 in hand, we can estimate the project value MCV of (3.11), and the VaR and CVaR, by Monte Carlo simulation. Note that simulation is exact, because of the discrete-time formulation, in that the result ultimately depends only on a finite vector of $N(0,1)$ random variables.

4.2 Results

We consider a project with initial cost $A = \$12$ (or $\$12,000,000$) and an entrepreneur with initial cash of $x_0 = \$10$. We assume that the project price is "fair", in the sense that it coincides with the risk-neutral value at time 0. The parameters of the model are presented in Table 1. Note that η can be determined given the initial price of the project and the other parameters.

The entrepreneur can be considered rather respectable—he has almost enough cash to finance the entire project without resorting to loans. Given the overcollateralization requirement imposed by the bank, the entrepreneur may choose to borrow between \$2 and \$10, and he invests the surplus of between \$8 and \$0 respectively in the market. Figure 2 shows CVaR of the loan

r_0	0.03	α	0.90
r_1	0.01	γ	0.80
b_0	0.01	ρ	0.70
b_1	0.01	r_c	0.05
T	5	dt	1/12
κ	1.2	κ'	1.1
ξ	1	η	1.948

Table 1 Parameter Values

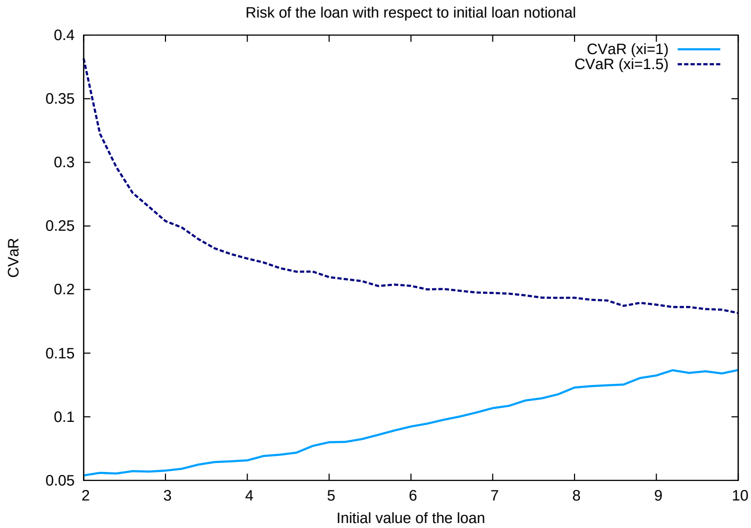

Fig. 2 CVaR for different initial loan values, expressed as a fraction of the initial value of the loan.

from the bank's perspective, plotted against the notional of the loan. All the values are presented as the fraction of the initial value of the loan.

The immediate thing we notice in Figure 2 is that the risk behaves in a completely different way for different levels of project volatility ξ. Under normal market conditions ($\xi = 1$, which gives the project a similar volatility as the stock market) the bank prefers, from the risk management perspective, lower loans. This is perfectly intuitive, because the initial value of the collateral is the same for loans with different notionals and equal to $A = \$12$. The lower-value loans hardly ever default—around 1.1% of them compared to 6.1% for loans with value \$10.

In the case of high volatility ($\xi = 1.5$, when the market for the project is possibly in crisis or distress), the situation becomes very different. Despite the lower-value loans having a lower probability of default—9% for \$3 loans comparing to 14.5% for \$10 loans, the higher-value loans have smaller risk. The key to understanding this seeming paradox is to notice that loans with higher values leave the entrepreneur with more liquid assets. These assets are used to cover the margin payments, so the bank can recover large proportion of the loan before default, whereas in the case of minimal loan first margin call immediately causes insolvency. Moreover, the entrepreneur invests his liquid assets in the market. The extra leverage causes more volatility and defaults, but also increases his average return, and hence the amount of money the bank can recover.

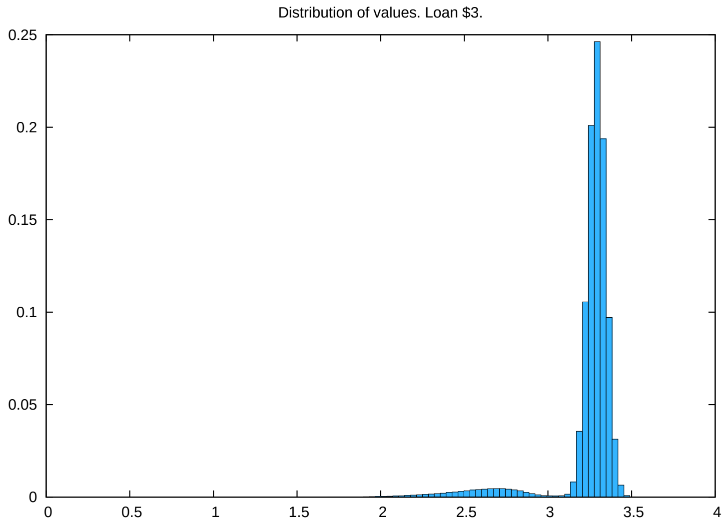

Fig. 3 Distribution of values for loan with initial value \$3 and $\xi = 1.5$. The average is 3.26 and CVaR = 0.77.

Figures 3 and 4 show the distribution of the value of the loan with notional of \$3 and \$10 respectively. In both cases the graph has two distinct peaks: the one with values above the notional corresponds to no-default scenarios, whereas the other one contains values after default. The values are dispersed because of the effect of stochastic discount rates, random early repayments and—for the relevant cases—different default times. We immediately notice that in the \$3 notional case more mass is in the no-default peak comparing to the \$10 case, but the returns if default occurs are proportionately much lower. Figures 5 and 6 stress this point even more. They depict the distribution of the outstanding loan value for the cases that ended up in default (taken just before the default time and discounted suitably). In these figures we can clearly see that in most cases almost half of the \$10 loan is repaid early, whereas in a considerable percentage of cases the first margin call made the \$3 loan default. Note that these defaults are much more costly for the bank (in terms of percentage of the initial loan value).

For all considered ξ and loan values the defaults happen mostly just before the maturity of the project, see for example Figure 7.

The entrepreneur always prefers to borrow more, so that he has more leverage and more potential to earn money. His losses are only limited to

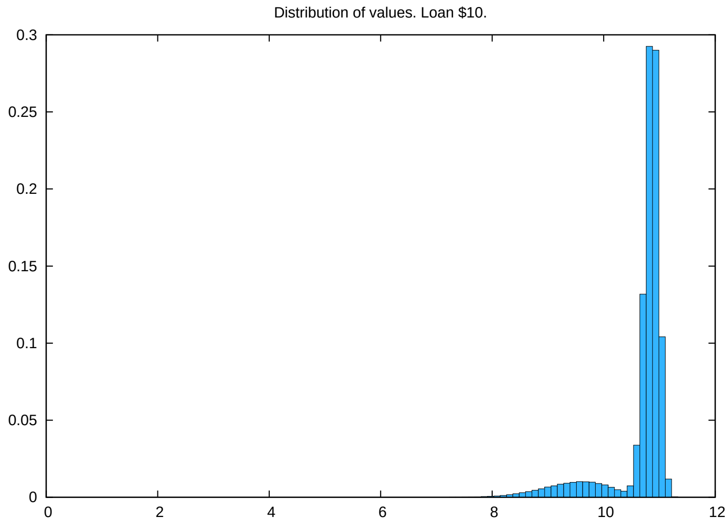

Fig. 4 Distribution of values for loan with initial value \$10 and $\xi = 1.5$. The average is 10.74 and CVaR 1.82.

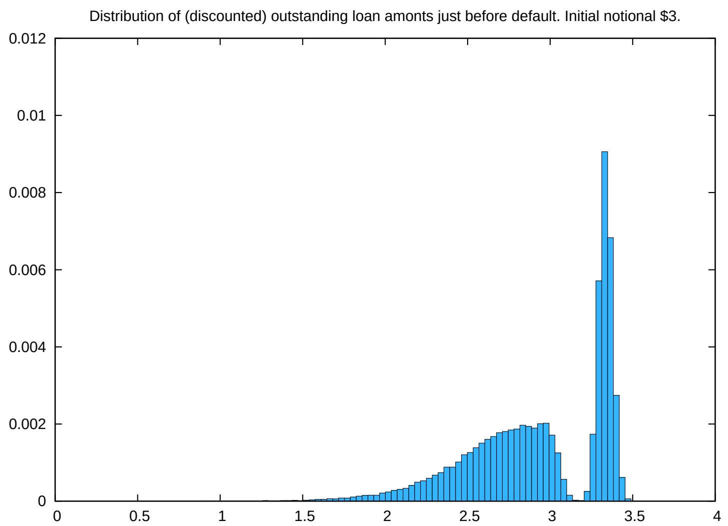

Fig. 5 Distribution of (discounted) outstanding loan just before default for the initial loan value \$3 and $\xi = 1.5$. This graph contains only cases that ended up in default. Total default probability is 0.08.

his initial capital $x_0 = \$10$, and—as is apparent in Figure 8—the potential gains are very high. For the initial loan value of \$6, even in the normal risk circumstances ($\xi = 1$) the entrepreneur has a positive probability to earn more than ten times his initial investment. In high risk case this goes up to almost

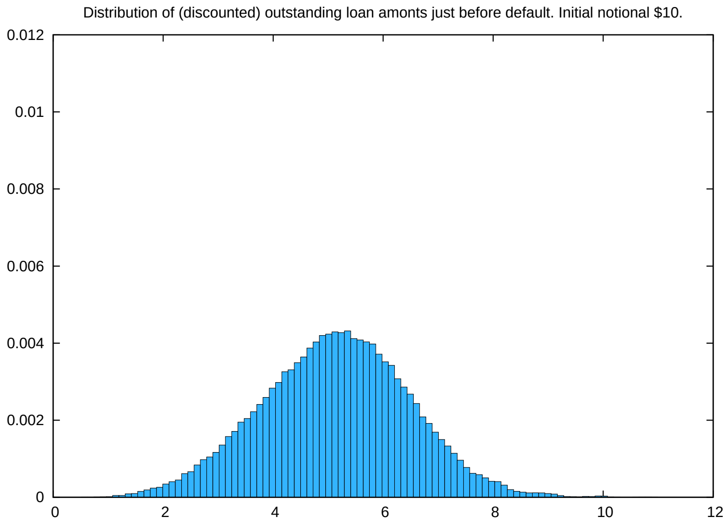

Fig. 6 Distribution of (discounted) outstanding loan just before default for the initial loan value $10 and $\xi = 1.5$. This graph contains only cases that ended up in default. Total default probability is 0.13.

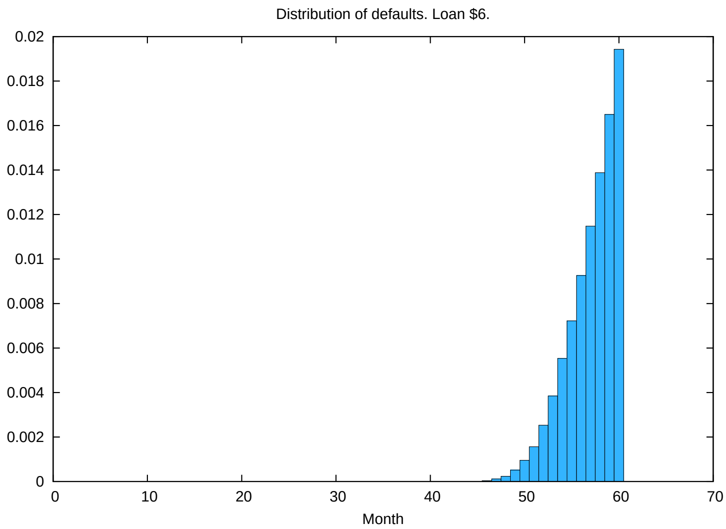

Fig. 7 Distribution of default times for loan with initial value $6 and $\xi = 1.5$. Total default probability is 0.10.

thirty times his initial investment. Despite the single most probable outcome in both cases is a loss (a default in high risk case), the entrepreneur has a positive profit expectation of $3.57 in the low risk case and $4.53 in the high risk case. This kind of highly asymmetric payoff characteristics foster the entrepreneurs

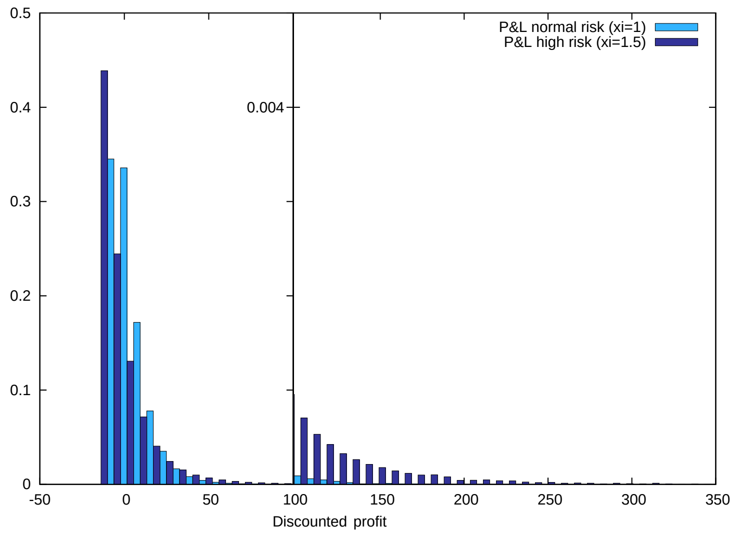

Fig. 8 Distribution of P&L of the entrepreneur with the initial loan value \$6. In case of default P&L is set to −\$10, otherwise it is the difference between the discounted value of the project and the stock account at maturity minus the outstanding loan, and the initial capital of the entrepreneur $x_0 = \$10$. The average profit in the case $\xi = 1$ is \$3.57 and in the case $\xi = 1.5$ it is \$4.53. The right-hand panel has a 100 times smaller scale.

drive to invest and provide examples of others who succeeded in a spectacular way—even though most of them failed. Being an "optimist" in Keynes' sense is perfectly rational in this model.

When agreeing to the loan amount in normal market circumstances the entrepreneur will prefer to borrow as much as possible, but the risk-optimizing bank will prefer to lend much less—because the risk increases with the notional of the loan. Figure 9 shows that this statement is only true in the case of ξ being around unity. Based on this parameter we can distinguish three market regimes: very low risk regime with $\xi < 0.5$, normal market circumstances $(0.5 < \xi < 1.2)$ and high risk market $(\xi > 1.5)$. In the first case the project is bound to succeeded and the bank only faces interest rate risk. Hence the risk doesn't depend on the value of the loan. As discussed, in the normal case the bank prefers to have more collateral compared to the amount of the loan. In the high risk case, however, things change dramatically. It becomes less risky for the bank to offer maximum loans to the entrepreneurs. For the economy it may have severe consequences: on the one hand these loans have much higher probability of default—which can be further aggravated by contagion effects, and on the other the bank starts having big items on its balance sheet. Even one default of these big loans could deplete the bank's Tier 1 capital and cause its collapse.

Although not explicit in here, the ξ parameter can be assumed to be linked to the agents' perception of the state of the market, with higher ξ meaning

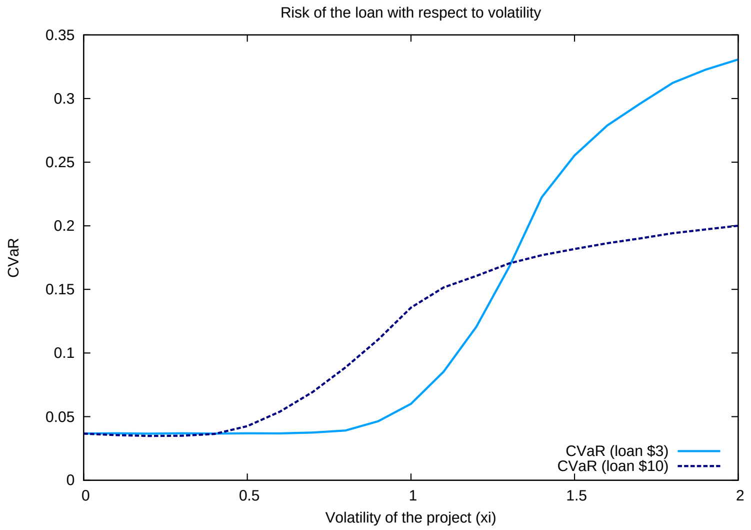

Fig. 9 CVaR for different values of project volatility (ξ), expressed as a fraction of the initial loan amount.

more volatility and uncertainty. Then a crisis would mean a shift from the lower risk regime to a higher one, with all the economic implications.

The mechanism of margin payments is effective way to minimize the risk for the bank only for large loans in the volatile case, but somewhat surprisingly not in other cases. In all cases introduction of forced early repayments increases the number of defaults between 2.5 and 6 times and in normal market conditions it increases the bank's risk as well. In particular, if a bank decides to give a "safe" loan of $3 under normal volatility conditions then the margin payments are inefficient from the very start. But if a crisis begins and volatility rises, it gets much worse. Looking at Figure 2 the risk will increase more than threefold and the mechanism of early repayments becomes even greater burden for the bank and the economy.

4.3 Computations for the general case

The above computation is in two stages: solve the backward equation (4.7) to determine the project valuation function G of (4.3), then simulate (forwards) to determine the value of the project and the risk parameters. In our example the first stage is easy because the backward equation has the closed-form solution (4.4). However, when we have a general factor model with, say, 5 or 6 factors, a numerical method will be required, and the dimensionality stretches standard finite-difference methods up to, or beyond, their normal limits. The best candidates seem to be stochastic mesh methods, see Glasserman (2003), and specifically the basis function approach originally devised by Longstaff and Schwartz (2001) for American options. While computationally intensive these methods have the advantage that the same set of sample paths used for the forward simulation is also used to solve the backward equation. They are becoming the methods of choice for large-scale credit risk calculations, see Cesari et al (2010).

5 Quantifying Animal Spirits

With our computational framework in place, the next step is to return to the discussion of confidence indices in Section 2.2 above in an effort to determine what are the leading indicators of confidence, or over-confidence or irrational exuberance. The papers cited in that section show that there is some debate about the relationship of confidence indices to stock market returns. While we have not analysed this in detail, anecdotal evidence such as the data presented in Figure 10 indicates that there is some appreciable connection between confidence indices and non-financial indicators of economic activity, in this case the growth rate of the composite Case-Shiller house price index. It would be meaningless to impute any causal relationship on the basis of this data alone, but clearly there is *some* relationship. Possibly a better approach would be to regard animal spirits as a *hidden variable* (Bhar and Hamori 2004),(Mamon and Elliott 2007), to be tracked on the basis of observed series. This would encapsulate the idea that optimism/pessimism are relatively long-lived 'states'; in a model, switches between them could be picked up in minimal time using 'quickest detection' algorithms (Poor and Hadjiliadis 2009).

All of these suggestions are topics for future research.

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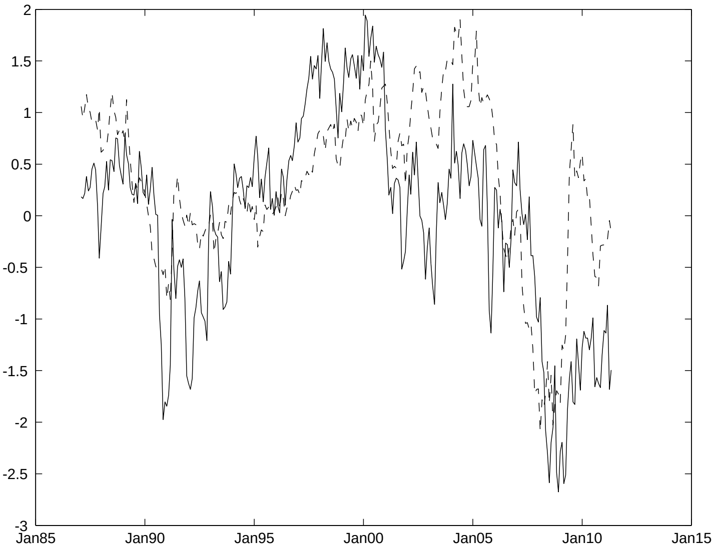

Fig. 10 Consumer Sentiment Index (solid line) and "returns" series for Case-Shiller House Price Index (dashed line). Monthly data 1987-2011. Both series rescaled to zero mean and unit variance.

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