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Jump-Diffusion Asset-Liability Management Via Risk-Sensitive Control

Mark H. A. Davis · Sébastien Lleo

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Abstract In this paper, we use risk-sensitive control methods to solve a jump-diffusion Asset-Liability Management (ALM) problem. We show that the ALM problem admits a unique classical ($C^{1,2}$) solution under two different sets of assumptions.

Keywords Asset and Liability Management, Risk-Sensitive Asset Management, Risk-Sensitive Control, Classical Solutions, Viscosity Solutions, Jump Diffusion Processes, Fund Separataion Theorems

1 Introduction

Effective Asset and Liability Management (ALM) models is crucial for funded investors such as endowment funds and pension funds, but also for investors who have the ability to grow their asset base by borrowing such as banks and hedge funds. In this paper, we solve an ALM problem in a jump-diffusion setting under two sets of assumptions. Under both sets of assumptions, the asset

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M.H.A.Davis
Department of Mathematics, Imperial College London
E-mail: mark.davis@imperial.ac.uk

S. Lleo
Finance Department and Behavioral Sciences Research Centre, NEOMA Business School, 59 rue Pierre Taittinger, 51100 Reims, France
E-mail: sebastien.lleo@NEOMA-bs.fr

prices and liability value depend on a random factor process X_t, the components of which can be interpreted either as macroeconomic factors or simply as a statistical representation of the uncertainty of asset returns. Under the first set of assumptions, the growth rate of assets and liabilities are affine functions of the factors, while the diffusions are constant. The factors are modelled as Gaussian diffusion processes. Under the second set of assumptions, both the growth rate and diffusion depend on the factors, making it possible to incorporate stochastic volatility. The factors are modelled as jump-diffusion processes. We also consider investment constraints in our model.

We formulate the ALM problem as a risk-sensitive control problem: the investor's objective is to jointly select an optimal amount of leverage $\rho(t)$ and an optimal asset allocation $h(t)$ to maximize the criterion

$$J(h, \rho; \theta) := -\frac{1}{\theta} \ln \mathbf{E} \left[e^{-\theta R_T(h,\rho)} \right] \tag{1.1}$$

where $R_T(h, \rho)$ is a reward function at a fixed final time T corresponding to a pair of control processes (h, ρ), and the exogenous parameter $\theta > 0$ represents the decision maker's degree of risk aversion. In optimal investment problems (as in Bielecki and Pliska (1999) or Kuroda and Nagai (2002)) it is customary to take $R_T = \ln V_T$, where V_T is the value of the investment portfolio corresponding to an asset allocation strategy h. For the ALM problem discussed here, we use $R_T = \ln E_T = \ln(V_T - L_T)$ to measure the investor's return on equity or on surplus.

The solution technique presented in this paper extends the approaches proposed by Davis and Lleo (2011, 2013a) in an investment management context and by Davis and Lleo (2013b) for benchmarked investment management. The first step is to use a change-of-measure, an idea introduced by Kuroda and Nagai (2002) that reduces the risk-sensitive optimization problem to a stochastic control problem in the factor process. Under the first set of assumptions, affine asset growth rates with constant volatility, the factor process X_t has no jumps and the associated Bellman equation is a partial differential equation (PDE) which can be shown to admit a unique classical ($C^{1,2}$) solution.

Under the second set of assumptions, we have a fully nonlinear controlled jump-diffusion, and the Bellman equation is a partial integro-differential (PIDE) for which no analytical solution exists. In such situation, viscosity solutions are generally used to show that the value function is the unique continuous solution of the Hamilton Jacobi Bellman (HJB) PIDE; see in particular Crandall et al (1992), Barles and Imbert (2008) for an overview of viscosity solutions and Fleming and Soner (2006), Øksendal and Sulem (2005) or Touzi (2002) for a discussion of their application to stochastic control, as well as Davis and Lleo (2010) for a viscosity approach to risk-sensitive asset management. In the context of optimal control, a limitation of viscosity solutions is that they are weak solutions. Proving that the HJB PIDE admits a unique strong ($C^{1,2}$) solution requires the development of a more sophisticated argument combining viscosity solutions and classical solutions. The key references are due to Pham (1998), Davis et al (2009) and Davis and Lleo (2013a).

Our model is in three main respects a generalisation of the works by Rudolf and Ziemba (2004) and Benk (2012). First, the jump-diffusion setting we propose in this paper permits the implementation of a wide range of jump specifications as well as factor-dependent stochastic drift and volatility. Second, we take the degree of leverage into account explicitly by modelling it as a control variable. Third, we consider investment constraints.

The paper is organised as follows. We introduce the analytical setting in Section 2 before formulating the control problem in Section 3 and formulating the HJB equation in Section 4. The main result, Theorem 4.7, which addresses the questions of the existence of classical solution to the HJB P(I)DE under both sets of assumptions, is stated in Section 4.3 and proved in the next two sections: in Sections 5 under affine assumptions and in Section 6 under standard control assumptions.

2 Analytical Setting

In our model, the investor selects an asset allocation and a degree of leverage with the objective of maximising a given measure of their equity, that is, the difference between the value of the investor's asset and the value of the liability. The three key components of the model are an asset market comprising m risky securities $S_i,\ i = 1, \dots m$ and a money market account process S_0, an exogenous liability L and n factors $X_1(t), \dots, X_n(t)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and define an $\mathbb{R}^M$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_k(t)$, $k = 1, \dots, M$, and $M := m + n + 1$. Moreover, let N be a $(\mathcal{F}_t)$-Poisson point process on $(0, \infty) \times \mathbf{Z}$, independent of $W(t)$, where $(\mathbf{Z}, \mathcal{B}_{\mathbf{Z}})$ is a given Borel space. Define

$$\mathfrak{Z} := \{U \in \mathcal{B}_{\mathbf{Z}}, \mathbf{E}\left[N(t, U)\right] < \infty \ \forall t\} \tag{2.1}$$

We fix throughout the paper a set $\mathbf{Z}_0 \in \mathcal{B}_{\mathbf{Z}}$ such that $\nu(\mathbf{Z} \backslash \mathbf{Z}_0) < \infty$ and define, as in Øksendal and Sulem (2005)

$$\begin{aligned} &\bar{N}(dt, dz) \qquad (2.2) \\ &= \begin{cases} N(dt, dz) - \tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt =: \tilde{N}(dt, dz) & \text{if } z \in \mathbf{Z}_0 \\ N(dt, dz) & \text{if } z \in \mathbf{Z} \backslash \mathbf{Z}_0 \end{cases} \end{aligned}$$

For $t \in [0, T]$ let $\mathcal{F}_t$ be the σ-field generated by the Brownian motions $W_k(s)$ and Poisson processes $N(A, s)$ for $k = 1, \dots, M$, $A \in \mathcal{B}_{\mathbf{Z}}$ and $0 \leq s \leq t$, completed with all null sets of $\mathcal{F}_T$. It is well known that the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ satisfies the 'usual conditions'.

2.1 Asset Market Dynamics

Let S_0 denote the wealth invested in the money market account with dynamics given by the equation:

$$\frac{dS_0(t)}{S_0(t)} = a_0\left(t, X(t)\right)dt, \qquad S_0(0) = s_0 \tag{2.3}$$

Denote by $S_i(t)$ the price at time t of the ith risky security, with $i = 1, \dots, m$. The dynamics of $S(t)$ is:

$$\frac{dS_i(t)}{S_i(t^-)} = \left[a\left(t, X(t^-)\right)\right]_i dt + \sum_{k=1}^{N} \Sigma_{ik}(t, X(t))dW_k(t) + \int_{\mathbf{Z}} \gamma_i(t,z)\bar{N}(dt,dz),$$
$$S_i(0) = s_i, \quad i = 1, \dots, m \tag{2.4}$$

Throughout this paper, we consider two different sets of standing assumptions: 1. affine drift and constant diffusion with no jumps in the factor, and 2. standard stochastic control assumptions of bounded, Lipschitz continuous drift and diffusion, with jumps in both assets and factors. These assumptions are standard (see for example Davis and Lleo (2011, 2013a)).

Assumption 2.1 (Affine Drift and Constant Diffusion)

(i) $a_0(t,x) = a_0 + A_0 x$ *where* $a_0 \in \mathbb{R}^m$, $A_0 \in \mathbb{R}^{m \times n}$
(ii) $a(t,x) = a + Ax$ *where* $a \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$
(iii) $\Sigma(t,x) = \Sigma$ *with* $\Sigma \in \mathbb{R}^{m \times M}$
(iv) $\Sigma\Sigma' > 0$
(v) $\gamma(t,z) = \gamma(z) \in \mathbb{R}^m$ *satisfies:*

$$\int_{\mathbf{Z}_0} |\gamma(z)|^2 \nu(dz) < \infty \tag{2.5}$$

Assumption 2.2 (Standard Control Assumptions)

(i) the function a_0 *defined as* $a_0 : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is bounded, of class* $C^{1,1}\left([0,T] \times \mathbb{R}^n\right)$ *and is Lipshitz continuous*

$$|a_0(t,y) - a_0(s,x)| \leq K_0\left(|t-s| + |y-x|\right) \tag{2.6}$$

for some constant $K_0 > 0$.

(ii) There exists $K_0' > 0$ *such that*

$$\left|\frac{\partial a_0}{\partial t}\right| + |Da_0| \leq K_0' \tag{2.7}$$

(iii) the function $a : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ *is bounded and Lipshitz continuous:*

$$|a(t,y) - a(s,x)| \leq K_a\left(|t-s| + |y-x|\right) \tag{2.8}$$

for some constant $K_a > 0$.

(iv) *the function* $\Sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{m\times M}$ *is bounded and Lipshitz continuous, i.e.*

$$|\Sigma(t,y) - \Sigma(s,x)| \le K_\Sigma\left(|t-s| + |y-x|\right) \tag{2.9}$$

for some constant $K_\Sigma > 0$.

(v) *There exists* $\psi_\Sigma > 0$ *such that*

$$\zeta'\Sigma\Sigma'(t,x)\zeta \ge \psi_\Sigma|\zeta|^2 \tag{2.10}$$

for all $\zeta \in \mathbb{R}^m$

(vi) *There exists* $K_a' > 0$ *and* $K_\Sigma' > 0$ *such that*

$$|a_t| + |a_x| \le K_a' \tag{2.11}$$

$$|\Sigma_t| + |\Sigma_x| \le K_\Sigma' \tag{2.12}$$

(vii) *the function* $\gamma : [0,T]\times\mathbf{Z} \to \mathbb{R}^m$ *is bounded, continuous and satisfies the growth condition*

$$|\gamma(t,z) - \gamma(s,z)| \le K_\gamma\left(|t-s|\right) \tag{2.13}$$

for some constant $K_\gamma > 0$.

(viii) *The vector valued function* $\gamma(t,z)$ *satisfy:*

$$\int_{\mathbf{Z}_0} |\gamma(t,z)|^2\nu(dz) < \infty, \qquad \forall (t,x) \in [0,T]\times\mathbb{R}^n \tag{2.14}$$

(ix) $|\Lambda\Sigma'(t,y) - \Lambda\Sigma'(s,x)| \le K_{\Lambda\Sigma}\left(|t-s| + |y-x|\right)$ *for some constant* $K_{\Lambda\Sigma} > 0$

Conditions (2.11) and (2.12) follow respectively from (2.8) and (2.9) when a and Σ are differentiable.

2.2 Liability Modelling

We model the dynamics of the liability $L(t)$ using the same type of geometric jump-diffusion process as for the asset prices. Specifically,

$$\frac{dL(t)}{L(t)} = c(t,X(t^-))dt + \varsigma'(t,X(t))dW(t) + \int_{\mathbf{Z}} \eta(t,z)\bar{N}(dt,dz), \qquad L(0) = l$$

where c is a scalar, ς is a n-element column vector, and η is a N-element column vector. The standing assumptions for the affine model and for the standard stochastic control model are presented below.

Assumption 2.3 (Affine Drift and Constant Diffusion)

(i) $c(t,x) = c + Cx$ *where* $c \in \mathbb{R}$, $C \in \mathbb{R}^n$

(ii) $\varsigma(t,x) = \varsigma$ *with* $\varsigma \in \mathbb{R}^M$

(iii) $\eta(t,z) = \eta(z) \in \mathbb{R}$ *satisfies:*

$$\int_{\mathbf{Z}_0} |\eta(z)|^2 \nu(dz) < \infty \tag{2.15}$$

Assumption 2.4 (Standard Control Assumptions)

(i) the function $c : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ *is bounded and Lipshitz continuous:*

$$|c(t,y) - c(s,x)| \leq K_c \left(|t-s| + |y-x| \right) \tag{2.16}$$

for some constant $K_c > 0$.

(ii) the function $\varsigma : [0,T] \times \mathbb{R} \to \mathbb{R}^M$ *is bounded and Lipshitz continuous, i.e.*

$$|\varsigma(t,y) - \varsigma(s,x)| \leq K_\varsigma \left(|t-s| + |y-x| \right) \tag{2.17}$$

for some constant $K_\varsigma > 0$.

(iii) $\varsigma(t,x) > 0 \forall (t,x) \in [0,T] \times \mathbb{R}^n$

(iv) There exists $K_c' > 0$ *and* $K_\varsigma' > 0$ *such that*

$$|c_t| + |c_x| \leq K_c' \tag{2.18}$$

$$|\varsigma_t| + |\varsigma_x| \leq K_\varsigma' \tag{2.19}$$

(v) the function $\eta : [0,T] \times \mathbf{Z} \to \mathbb{R}$ *is bounded, continuous and satisfies the growth condition*

$$|\eta(t,z) - \eta(s,z)| \leq K_\eta \left(|t-s| \right) \tag{2.20}$$

for some constant $K_\eta > 0$.

(vi) The vector valued function $\eta(t,z)$ *satisfy:*

$$\int_{\mathbf{Z}_0} |\eta(t,z)|^2 \nu(dz) < \infty, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n \tag{2.21}$$

(vii) $|\Lambda\varsigma'(t,y) - \Lambda\varsigma'(s,x)| \leq K_{\Lambda\varsigma} \left(|t-s| + |y-x| \right)$ *for some constant* $K_{\Lambda\varsigma} > 0$

Assumption 2.5 $\gamma(t,z)\xi'(t,x,z) = \eta(t,z)\xi'(t,x,z) = 0 \quad \forall (t,x,z) \in [0,T] \times \mathbb{R}^n \times \mathfrak{S}$.

This implies that there are no simultaneous jumps in the factor process and the asset price processor in the factor process and the liability process. This imposes some restriction, but appears essential to identify an optimal control.

2.3 Factor Dynamics

The factor process $X(t) \in \mathbb{R}^n$ is allowed to have a full jump-diffusion dynamics, satisfying the stochastic differential equation:

$$dX(t) = b\left(t, X(t^-)\right) dt + \Lambda(t, X(t))dW(t) + \int_{\mathbf{Z}} \xi\left(t, X(t^-), z\right) \bar{N}(dt, dz),$$
$$X(0) = x_0 \in \mathbb{R}^n. \tag{2.22}$$

Assumption 2.6 (Affine Drift and Constant Diffusion)

(i) $b(t,x) = b + Bx$ *where* $b \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$
(ii) $\Lambda \in \mathbb{R}^{n\times n}$,
(iii) $\Lambda\Lambda' > 0$
(iv) $\xi(t,x,z) = 0$

Assumption 2.7 (Standard Control Assumptions)

(i) The function $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *is bounded and Lipshitz continuous*

$$|b(t,y) - b(s,x)| \leq K_b \left(|t-s| + |y-x|\right) \tag{2.23}$$

for some constant $K_b > 0$.
(ii) the function $\Lambda : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times M}$ *is bounded and Lipschitz continuous,*

$$|\Lambda(t,y) - \Lambda(s,x)| \leq K_\Lambda \left(|t-s| + |y-x|\right) \tag{2.24}$$

for some constant $K_\Lambda > 0$.
(iii) There exists $\eta_\Lambda > 0$ *such that*

$$\zeta'\Lambda\Lambda'(t,x)\zeta \geq \eta_\Lambda|\zeta|^2 \tag{2.25}$$

for all $\zeta \in \mathbb{R}^n$
(iv) There exists $K_b' > 0$ *and* $K_\Lambda' > 0$ *such that*

$$|b_t| + |b_x| \leq K_b' \tag{2.26}$$

$$|\Lambda_t| + |\Lambda_x| \leq K_\Lambda' \tag{2.27}$$

(v) The function $\xi : [0,T]\times\mathbb{R}^n\times\mathbf{Z} \to \mathbb{R}$ *is bounded and Lipshitz continuous:*

$$|\xi(t,y,z) - \xi(s,x,z)| \leq K_\xi \left(|t-s| + |y-x|\right) \tag{2.28}$$

for some constant $K_\xi > 0$.
(vi) The vector valued function $\xi(t,x,z)$ *satisfies:*

$$\int_{\mathbf{Z}} |\xi(t,x,z)|\nu(dz) < \infty, \qquad \forall (t,x) \in [0,T]\times\mathbb{R}^n \tag{2.29}$$

and for some constant c

$$\int_{\mathbf{Z}} |\xi(t,x,z)|^2\nu(dz) < c(1+|x|)^2. \tag{2.30}$$

The minimal condition on ξ under which the factor equation (2.22) is well posed is

$$\int_{\mathbf{Z}_0} |\xi(t,x,z)|^2 \nu(dz) < \infty,$$

see Definition II.4.1 in Ikeda and Watanabe (1981). As discussed by Davis et al (2009) and Davis and Lleo (2013a,b), we need the stronger condition (2.29) to connect the viscosity solution of the HJB PIDE to the viscosity solution of a related parabolic PDE.

Note that (2.26) and (2.27) follow respectively from (2.23) and (2.24) when b and Λ are differentiable.

3 Formulation of the Asset and Liability Mangement Problem

3.1 Asset Portfolio Dynamics

Let $\mathcal{G}_t := \sigma((S(s), X(s)), L(s), 0 \leq s \leq t)$ be the sigma-field generated by the security, factor and liability processes up to time t. The investment strategy $h(t) \in \mathbb{R}^m$ is represented by a vector containing the percentage of assets allocation to each of the m risks securities. The exact definition of the investment strategy $h(t)$ is closely related to the following assumptions on the functions γ and η used to parametrize the jumps in asset price and liability value.

Assumption 3.1 *Define*

$$\mathfrak{S} := \mathrm{supp}(\nu) \in \mathcal{B}_{\boldsymbol{Z}}, \qquad \tilde{\mathfrak{S}} = \mathrm{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m)$$

where supp$(\cdot)$ *denotes the support of the measure, and let* $\prod_{i=1}^m [\gamma_i^{min}, \gamma_i^{max}]$ *be the smallest closed hypercube containing* $\tilde{\mathfrak{S}}$. *We assume that* $\gamma(t,z) \in \mathbb{R}^m$ *satisfies*

$$-1 \leq \gamma_i^{min} \leq \gamma_i(t,z) \leq \gamma_i^{max} < +\infty, \qquad \gamma_i^{min} < 0 < \gamma_i^{max}, \qquad i = 1, \dots, m$$

We also assume that $\eta(t,z) \in \mathbb{R}$ *satisfies*

$$-1 < \eta^{min} \leq \eta_i(t,z) \leq \eta^{max} < +\infty, \qquad \eta^{min} < 0 < \eta^{max}$$

Furthermore, define the set $\mathcal{J}_0$ as

$$\mathcal{J}_0 := \left\{ h \in \mathbb{R}^m : 1 + h'\psi > 0 \quad \forall \psi \in \tilde{\mathfrak{S}} \right\} \tag{3.1}$$

For a given $z \in \mathfrak{S}$, the equation $h'\gamma(t,z) = -1$ describes a hyperplane in $\mathbb{R}^m$. Under Assumption 3.1, $\mathcal{J}_0$ is a convex subset of $\mathbb{R}^m$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

The investment strategy $h(t)$ is defined as follows:

Definition 3.2 (Class $\mathcal{H}_0$) An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{H}_0$ if the following conditions are satisfied:

(i) $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t\geq 0}$ and is càdlàg;
(ii) $h(t) \in \mathcal{J}_0 \quad \forall t$ a.s.

Under Assumption 3.1, a control process $h(t)$ satisfying (ii) is bounded.

By the budget equation, the proportion invested in the money market account equals $h_0(t) = 1 - \sum_{i=1}^m h_i(t)$. Thus, the wealth $V(t)$ of the investor in response to an investment strategy $h(t) \in \mathcal{H}_0$, follows the dynamics

$$\begin{aligned}\frac{dV(t)}{V(t^-)} &= (a_0\,(t, X(t)))\,dt + h'(t)\tilde{a}\,(t, X(t))\,dt + h'(t)\Sigma(t, X(t))dW_t \\ &\quad + \int_{\mathbf{Z}} h'(t)\gamma(t,z)\bar{N}(dt,dz) \end{aligned} \tag{3.2}$$

where $V(0) = v_0$ is the initial endowment and $\tilde{a} := a - a_0\mathbf{1}$, $\mathbf{1} \in \mathbf{R}^m$ denotes the m-element unit column vector.

3.2 Equity Dynamics and Leverage Ratio

Next, we define the equity at time t, $E(t)$, defined as the difference between the investor's asset and liabilities and representing the wealth belonging directly to the investor. In the ALM literature, the equity is also referred to as surplus.

$$E(t) = V(t) - L(t), \qquad E(0) = e_0 = v - l > 0$$

In the rest of the paper we assume that $e_0 = 1$ WLOG. The dynamics of the equity is given in differential form by

$$\begin{aligned} dE(t) &= dV(t) - dL(t) \\ &= V(t^-)\left[(a_0\,(t, X(t)))\,dt + h'(t)\tilde{a}\,(t, X(t))\,dt + h'(t)\Sigma(t, X(t))dW_t \right.\\ &\quad \left. + \int_{\mathbf{Z}} h'(t)\gamma(t,z)\bar{N}(dt,dz)\right] \\ &\quad - L(t^-)\left[c(t, X(t^-))dt + \varsigma'(t, X(t))dW(t) + \int_{\mathbf{Z}} \eta(t,z)\bar{N}(dt,dz)\right] \end{aligned}$$

The time t degree of leverage, or leverage ratio, $\rho(t)$, is defined as the ratio of asset value to equity value, i.e.

$$\rho(t) = \frac{V(t)}{E(t)}$$

As a result $V(t) = \rho(t)E(t)$ and $L(t) = (\rho(t) - 1)E(t)$ and we can simplify the SDE for the equity $E(t)$ as:

$$\begin{aligned}\frac{dE(t)}{E(t^-)} &= \alpha(t, X(t), h(t), \rho(t))dt + \beta(t, X(t), h(t), \rho(t))dW(t), \\ &\quad + \int_{\mathbf{Z}} \zeta((t, z, h(t), \rho(t)))\bar{N}(dt,dz) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\alpha(t,x,h,\rho) &:= c(t,x) + \rho\left[h'\tilde{a}(t,x) - \tilde{c}(t,x)\right]\\ \beta(t,x,h,\rho) &:= \varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x))\\ \zeta(t,z,h,\rho) &:= \eta(t,z) + \rho(t)\left[h'(t)\gamma(t,z) - \eta(t,z)\right]\\ \tilde{c} &:= c - a_0\end{aligned}$$

In this model, the leverage ratio is a control variable: the investor's objective is to choose both an optimal level of leverage and an optimal investment strategy. The leverage ratio $\rho(t)$ is defined as:

Definition 3.3 A leverage process $\rho(t)$ is in class $\mathcal{R}_0$ if the following conditions are satisfied:

1. $\rho(t) \in \mathbb{R}$;
2. $\rho(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t\geq 0}$;
3. $P\left(\int_0^T |\rho(s)|^2 \, ds < +\infty\right) = 1$;

3.3 Constraints on Investment and Leverage

To limit risk taking, the Asset and Liability Committee usually sets investment constraints defining the minimum and maximum proportion of assets and limits the amount of leverage. To model the investment constraints, we consider $r \in \mathbb{N}$ fixed constraints expressed in the form

$$\Upsilon' h(t) \leq \upsilon \tag{3.4}$$

where $\Upsilon \in \mathbb{R}^m \times \mathbb{R}^r$ is a matrix and $\upsilon \in \mathbb{R}^r$ is a column vector. For the constrained control problem to be sensible, we need Υ and υ to satisfy the following condition:

Assumption 3.4 *The system*

$$\Upsilon' y \leq \upsilon$$

for the variable $y \in \mathbb{R}^m$ admits at least two solutions.

We define the feasible region $\mathcal{J}$ as

$$\mathcal{J} := \{h \in \mathcal{J}_0 : \Upsilon' h \leq \upsilon\} \tag{3.5}$$

The feasible region $\mathcal{J}$ is a a convex subset of $\mathbb{R}^m$ and as a result of Assumption 3.4, $\mathcal{J}$ has at least one interior point. The leverage limits are modelled through the following constraint:

$$\mathcal{K} := \left\{\rho \in (0,\infty) : 0 < \rho_- \leq \rho(t) \leq \rho^+ < \infty\right\} \tag{3.6}$$

where ρ_-, ρ^+ are two real constants.

3.4 The ALM Stochastic Control Problem

The investor's objective is to maximise the risk-sensitive criterion $J(h, \rho; \theta)$

$$J(h, \rho; \theta) := -\frac{1}{\theta} \ln \mathbf{E}\left[e^{-\theta \ln E_T(h,\rho)}\right] \tag{3.7}$$

where $\ln E_T(h)$ can be interpreted as the log return on equity. From (3.2) and the general Itô formula we find that the term $e^{-\theta \ln E(T)}$ can be expressed as

$$e^{-\theta \ln E(T)} = \exp\left\{\theta \int_0^T g(t, X_t, h(t))dt\right\} \chi^\pi(T) \tag{3.8}$$

where

$$\begin{aligned} g(t,x,h) = &\frac{1}{2}(\theta+1)\,\beta\beta'(t,x,h,\rho) - \alpha(t,x,h,\rho) \\ &+ \int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[(\zeta(t,z,h,\rho))^{-\theta} - 1\right] + \zeta(t,z,h,\rho)\mathit{1}_{\mathbf{Z}_0}(z)\right\}\nu(dz) \end{aligned} \tag{3.9}$$

and the Doléans exponential $\chi^\pi(T)$ is defined for $t \in [0,T]$ by

$$\begin{aligned} \chi^\pi(t) := \exp\Bigg\{ &-\theta \int_0^t \beta(s, X(s), h(s), \rho(s))dW_s \\ &-\frac{1}{2}\theta^2 \int_0^t \beta(s, X(s), h(s), \rho(s))\beta(s, X(s), h(s), \rho(s))'ds \\ &+ \int_0^t \int_{\mathbf{Z}} \ln\left(1 - G(s,z,h(s),\rho(s))\right)\tilde{N}(ds,dz) \\ &+ \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - G(s,z,h(s),\rho(s))\right) + G(s,z,h(s),\rho(s))\right\}\nu(dz)ds \Bigg\}, \end{aligned} \tag{3.10}$$

with

$$G(t,z,h,\rho) = 1 - (1+\zeta(t,z,h,\rho))^{-\theta} = 1 - (1 + \eta(t,z) + \rho(t)\left[h'(t)\gamma(t,z) - \eta(t,z)\right])^{-\theta} \tag{3.11}$$

The maximisation of the risk-sensitive criterion $J(h, \rho; \theta)$ is performed over the class of admissible control $\mathcal{A}$, which is defined as follows:

Definition 3.5 (Admissible Controls) A control process $\pi(t) = (h(t), \rho(t))$ is in class $\mathcal{A}$ if the following conditions are satisfied:
(i) $h \in \mathcal{H}$, where

$$\mathcal{H} := \{h(t) \in \mathcal{H}_0 : h(t) \in \mathcal{J} \ \forall t \in [0,T], \text{a.s.}\} \tag{3.12}$$

(ii) $\rho \in \mathcal{R}$, where

$$\mathcal{R} := \left\{\rho(t) \in \mathcal{R}_0 : \rho_- \leq \rho(t) \leq \rho^+ \ \forall t \in [0,T]\right\} \tag{3.13}$$

(iii) $\mathbf{E}\chi^\pi(T) = 1$ where $\chi^\pi(t)$ is the Doléans exponential defined in (3.10)

Remark 3.6 For a given, fixed h, the functional g is bounded and Lipschitz continuous in the state variable x. This follows easily by boundedness and Lipschitz continuity of the coefficients a_0, a, Σ and γ.

For $h \in \mathcal{A}$, $\rho \in \mathcal{R}$ and $\theta > 0$ let $\mathbb{P}^\pi$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative

$$\frac{d\mathbb{P}^\pi}{d\mathbb{P}} = \chi^\pi(T), \tag{3.14}$$

and let $\mathbf{E}^\pi$ denote the corresponding expectation. Then from (3.8) we see that the criterion J is given by

$$J(h, \rho; \theta) = -\frac{1}{\theta} \ln \mathbf{E}^\pi \left[\exp\left(\theta \int_0^T g(t, X_t, h(t)) dt \right) \right]. \tag{3.15}$$

Moreover, under $\mathbb{P}^\pi$,

$$W_t^\pi = W_t + \theta \int_0^t \beta'(s, X(s), h(s), \rho(s)) ds$$

is a standard Brownian motion and the $\mathbb{P}^\pi$-compensated Poisson random measure is given by

$$\int_0^t \int_{\mathbf{Z}_0} \tilde{N}^\pi(ds, dz) = \int_0^t \int_{\mathbf{Z}_0} N(ds, dz) - \int_0^t \int_{\mathbf{Z}_0} \left\{ (\zeta(s, z, h(s), \rho(s)))^{-\theta} \right\} \nu(dz) ds$$

Under $\mathbb{P}^\pi$ the factor process $X(s)$, $0 \le s \le t$ satisfies the SDE:

$$dX(s) = f(s, X(s), h(s)) ds + \Lambda(s, X(s)) dW_s^\theta + \int_{\mathbf{Z}} \xi\left(s, X(s^-), z\right) \tilde{N}^\pi(ds, dz), \quad X(0) = x_0 \tag{3.16}$$

where

$$f(t, x, h, \rho) := b(t, x) - \theta \Lambda \beta'(t, x, h, \rho) + \int_{\mathbf{Z}} \xi(t, x, z) \left[(\zeta(t, z, h, \rho))^{-\theta} \right] \nu(dz) \tag{3.17}$$

and b is the $\mathbb{P}$-measure drift of the factor process (see (2.22)).

Remark 3.7 The drift function f is Lipschitz continuous with coefficient $K_f = K_b + \theta(K_{\Lambda\Sigma} + K_{\Lambda\varsigma}) + K_\xi K_0$ where $K_0 > 0$ is a constant.

The generator of the state process $X(t)$ for a constant control h is

$$\begin{aligned} \mathcal{L}u(t, x) := & f(t, x, h, \rho)' Du + \frac{1}{2} \mathrm{tr}\left(\Lambda \Lambda'(t, x) D^2 u \right) \\ & + \int_{\mathbf{Z}} \left\{ u\left(x + \xi(t, x, z)\right) - u(x) - \xi(t, x, z)' Du \right\} \nu(dz) ds \end{aligned} \tag{3.18}$$

So far, we have shown that the risk-sensitive asset allocation problem is equivalent to the stochastic control problem of minimizing the cost criterion

$$\tilde{J}(h) = \mathbf{E}^{\pi}\left[\exp\left(\theta \int_0^T g(t, X_t, h(t), \rho(t))dt\right)\right] \tag{3.19}$$

over the control set $\mathcal{A}$ for a controlled process X_t satisfying (in 'weak solution' form) the jump-diffusion SDE (3.16). The next three sections of the paper, sections 4, 5 and 6, are devoted to solving this stochastic control problem.

4 Dynamic programming and the value function

We solve the control problem by studying the Hamilton-Jacobi-Bellman (HJB) dynamic programming equation, which involves embedding the original problem in a family of problems indexed by time-space points (s, x), the starting time and position of the controlled process X_t. The following description is in the same spirit as Bouchard and Touzi (2011).

For fixed $s \in [0, T]$ we define the filtration $\{\mathcal{F}_t^s, t \in [s, T]\}$ by

$$\mathcal{F}_t^s = \sigma\{W_k(r) - W_k(t), N(A, r) - N(A, t), k = 1, \ldots, M, A \in \mathcal{B}_{\mathbf{Z}}, s \le r \le t\}$$

and note that $\mathcal{F}_t^s$ is independent of $\mathcal{F}_t$. $X(t)$ denotes the solution of (2.22) on $[s, t]$ with initial condition $X(s) = x$ and $\mathbb{P}_{s,x}$ the measure on $\mathcal{F}_T^s$ such that $\mathbb{P}_{s,x}[X_s = x] = 1$. The class of admissible controls $\mathcal{A}^s$ is defined analogously to $\mathcal{A}$ above with h adapted to $\mathcal{F}_t^s$, leading to a change of measure on $\mathcal{F}_T^s$ defined by the Radon-Nikodým derivative

$$\frac{d\mathbb{P}_{s,x}^{\pi}}{d\mathbb{P}_{s,x}} = \chi_s^{\pi}(T).$$

The next step is to introduce two auxiliary criterion functions under the measure $\mathbb{P}_{s,x}^{\pi}$:

$$\tilde{I}(s, x, h, \rho) = \mathbf{E}_{s,x}^{\pi}\left[\exp\left\{\theta \int_s^T g(t, X_t, h(t), \rho(t))dt\right\}\right] \tag{4.1}$$

$$I(s, x, h, \rho) = -\frac{1}{\theta} \ln \tilde{I}(s, x, h, \rho). \tag{4.2}$$

The value functions associated with these auxiliary criterion are

$$\tilde{\Phi}(s, x) = \inf_{(\rho,h)\in\mathcal{A}^s} \tilde{I}(s, x, h, \rho); \qquad \Phi(s, x) = \sup_{(\rho,h)\in\mathcal{A}^s} I(s, x, h, \rho) \tag{4.3}$$

Lemma 4.1 $\tilde{\Phi}(s, x) = \inf_{(\rho,h)\in\mathcal{A}} \tilde{I}(s, x, h)$. *That is, the infimum is unchanged if the class $\mathcal{A}^s$ is replaced by the larger class $\mathcal{A}$.*

Proof This uses exactly the argument of Remark 2, page 958 of Bouchard and Touzi (2011). We condition on the initial filtration and use the independence of $\mathcal{F}_s$ and $\mathcal{F}_t^s$.

4.1 The Risk-Sensitive Control Problems under $\mathbb{P}_h$

We will show that the value function Φ defined in (4.3) satisfies the HJB PIDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h\in\mathcal{J},\rho\in\mathcal{K}} L^{\pi}\left(t,x,\Phi,D\Phi,D^2\Phi\right) = 0 \tag{4.4}$$

where $\mathcal{J}$ is defined in (3.5),

$$\begin{aligned} L^{\pi}\left(t,x,u,p,M\right) &= f(t,x,h)'p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)M\right) - \frac{\theta}{2}p'\Lambda\Lambda'(t,x)p \\ &\quad -g(t,x,h) + \mathcal{I}_{NL}\left[t,x,u,p\right] \qquad (4.5)\\ \mathcal{I}_{NL}\left[t,x,u,p\right] &= \int_{\mathbf{Z}}\left\{-\frac{1}{\theta}\left(e^{-\theta[u(t,x+\xi(t,x,z))-u(t,x)]}-1\right)-\xi(t,x,z)'p\right\}\nu(dz) \end{aligned}$$

subject to the terminal condition $\Phi(T,x)=0$ (recall our normalization $v_0=1$). Condition (2.29) ensures that $\mathcal{I}_{NL}$ is well defined, at least for bounded u. For $\tilde{\Phi}$, the corresponding HJB PIDE is

$$\begin{aligned} &\frac{\partial\tilde{\Phi}}{\partial t}(t,x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)D^2\tilde{\Phi}(t,x)\right) + H(t,x,\tilde{\Phi},D\tilde{\Phi}) \\ &+\int_{\mathbf{Z}}\left\{\tilde{\Phi}\left(t,x+\xi(t,x,z)\right)-\tilde{\Phi}(t,x)-\xi(t,x,z)'D\tilde{\Phi}(t,x)\right\}\nu(dz) = 0 \qquad (4.6)\end{aligned}$$

subject to terminal condition $\tilde{\Phi}(T,x)=1$ where for $r\in\mathbb{R}$, $p\in\mathbb{R}^n$

$$H(s,x,r,p) = \inf_{h\in\mathcal{J},\rho\in\mathcal{K}}\left\{f(s,x,h)'p+\theta g(s,x,h)r\right\} \tag{4.7}$$

Remark 4.2 The function H satisfies a Lipschitz condition as well as the linear growth condition

$$|H(s,x,r,p)| \le C\left(1+|p|\right), \quad \forall (s,x)\in Q_0$$

The value functions Φ and $\tilde{\Phi}$ are related through the strictly monotone continuous transformation $\tilde{\Phi}(t,x)=\exp\left\{-\theta\Phi(t,x)\right\}$. Thus an admissible (optimal) strategy for the exponentially transformed problem is also admissible (optimal) for the risk-sensitive problem. In the remainder of the article, we refer to the control problem and HJB PIDE related to the value function Φ as the *risk sensitive* control problem and the *risk sensitive* HJB PIDE, and to the control problem and HJB PIDE related to the value function $\tilde{\Phi}$ as the *exponentially transformed* control problem and the *exponentially transformed* HJB PIDE.

4.2 Properties of the Value Function $\tilde{\Phi}$

In this section, we establish two *a priori* properties of the value function.

Proposition 4.3 *The exponentially transformed value function $\tilde{\Phi}$ is positive and bounded, i.e. there exists $M > 0$ such that*

$$0 \leq \tilde{\Phi}(t,x) \leq M \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

Proof By definition,

$$\tilde{\Phi}(t,x) = \inf_{h \in \mathcal{A}} \mathbf{E}^{\pi}_{t,x} \left[\exp \left\{ \theta \int_t^T g(s, X_s, h(s), \rho(s)) ds - \theta \ln v \right\} \right] \geq 0$$

Moreover, the strategy of investing only in the money-market account, i.e. taking $\rho \equiv 1$ and $h \equiv 0$ is sub-optimal, and hence

$$\tilde{\Phi}(t,x) \leq \mathbf{E}^{0}_{t,x} e^{\theta \int_t^T g(X(s),0) ds} = \mathbf{E}^{0}_{t,x} e^{\theta \int_t^T a_0(s,X(s) ds} \leq e^{\theta \check{a}_0 (T-t)}$$

where $\check{a}_0$ is a bound for $|a_0(t,x)|$ (see Assumption 2.2(i)). This concludes the proof. □

Proposition 4.4 *The value function $\tilde{\Phi}$ is Lipschitz continuous in the state variable x.*

Proof The proof follows closely Proposition 3.5 in Davis and Lleo (2013a). □

Proposition 4.5 *Under either Assumption 2.1 (v) or both Assumption 2.2 (v) and Assumption 2.5, the supremum in* (4.4), (4.5) *admits a unique Borel measurable maximizer $\tilde{h}(t,x,p)$ for $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.*

Proof We present the proof under Assumption 2.2 (v) and Assumption 2.5. The proof under Assumption 2.1 (v) follows as a special case.

The supremum in (4.4) can be expressed as

$$\begin{aligned}
&\sup_{h \in \mathcal{J}, \rho \in \mathcal{K}} L^{\pi}(t,x,u,p,M) \\
&= b'(t,x)p + \frac{1}{2} \mathrm{tr}\left(\Lambda \Lambda'(t,x) M \right) - \frac{\theta}{2} p' \Lambda \Lambda'(t,x) p + c(t,x) + \mathcal{I}_{NL}[t,x,u,p] \\
&\sup_{h \in \mathcal{J}, \rho \in \mathcal{K}} \left\{ -\frac{1}{2} (\theta+1) \left[\varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x)) \right] \left[\varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x)) \right]' \right. \\
&\quad -\theta \left[\varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x)) \right] \Lambda' p + \rho \left[h'\tilde{a}(t,x) - \tilde{c}(t,x) \right] \\
&\quad -\frac{1}{\theta} \int_{\mathbf{Z}} \left\{ (1 - \theta \xi(t,x,z)'p) \left[(1 + \eta(t,z) + \rho(t) \left[h'(t)\gamma(t,z) - \eta(t,z) \right])^{-\theta} - 1 \right] \right. \\
&\quad \left. \left. +\theta \left(\rho(t) \left[h'(t)\gamma(t,z) - \eta(t,z) \right] \right) \mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz) \right\}
\end{aligned} \tag{4.8}$$

Define the auxiliary functional

$$\begin{aligned}
\ell(h,\rho;x,p) &= \frac{1}{2} (\theta+1) \left[\varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x)) \right] \left[\varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x)) \right]' \\
&\quad +\theta \left[\varsigma'(t,x) + \rho(h'\Sigma(t,x) - \varsigma'(t,x)) \right] \Lambda' p - \rho \left[h'\tilde{a}(t,x) - \tilde{c}(t,x) \right] \\
&\quad +\frac{1}{\theta} \int_{\mathbf{Z}} \left\{ (1 - \theta \xi(t,x,z)'p) \left[(1 + \eta(t,z) + \rho(t) \left[h'(t)\gamma(t,z) - \eta(t,z) \right])^{-\theta} - 1 \right] \right. \\
&\quad \left. +\theta \left(\eta(t,z) + \rho(t) \left[h'(t)\gamma(t,z) - \eta(t,z) \right] \right) \mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz)
\end{aligned} \tag{4.9}$$

for $h \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ and $\theta \in (0, \infty)$. Under Assumption (2.10), for any $p \in \mathbb{R}^n$ the term

$$\frac{1}{2}(\theta+1)\left[\varsigma'(t,x)+\rho(h'\Sigma(t,x)-\varsigma'(t,x))\right]\left[\varsigma'(t,x)+\rho(h'\Sigma(t,x)-\varsigma'(t,x))\right]' \\ +\theta\left[\varsigma'(t,x)+\rho(h'\Sigma(t,x)-\varsigma'(t,x))\right]\Lambda'p-\rho\left[h'\tilde{a}(t,x)-\tilde{c}(t,x)\right]$$

is strictly convex in h and ρ $\forall (t,x,z) \in [0,T] \times \mathbb{R}^n \times \mathbf{Z}$ a.s. $d\nu$. Under Assumption 2.5, the nonlinear jump-related term

$$\frac{1}{\theta}\int_{\mathbf{Z}}\Big\{(1-\theta\xi(t,x,z)'p)\left[(1+\eta(t,z)+\rho(t)\left[h'(t)\gamma(t,z)-\eta(t,z)\right])^{-\theta}-1\right] \\ +\theta\left(\eta(t,z)+\rho(t)\left[h'(t)\gamma(t,z)-\eta(t,z)\right]\right)\mathit{1}_{\mathbf{Z}_0}(z)\Big\}\nu(dz)$$

simplifies to

$$\frac{1}{\theta}\int_{\mathbf{Z}}\Big\{\left[(1+\eta(t,z)+\rho(t)\left[h'(t)\gamma(t,z)-\eta(t,z)\right])^{-\theta}-1\right] \\ +\theta\left(\eta(t,z)+\rho(t)\left[h'(t)\gamma(t,z)-\eta(t,z)\right]\right)\mathit{1}_{\mathbf{Z}_0}(z)\Big\}\nu(dz)$$

which is also convex in h and ρ $\forall (t,x,z) \in [0,T] \times \mathbb{R}^n \times \mathbf{Z}$ a.s. $d\nu$.

As a function of the variables h and ρ, $\ell(h,\rho;x,p)$ can be defined more precisely as a map from the vector space $\mathbb{R}^{m+1}$ into $\mathbb{R}$. Moreover, ℓ is continuous in h and ρ $\forall (h,\rho) \in \mathbb{R}^{m+1}$, twice differentiable and with continuous derivatives in both variables. Holding h (ρ) constant, ℓ attains a finite infimum in ρ (h). As a result, ℓ attains its infimum in (h,ρ), and the infimum is finite.

Looking at the constraints, the matrix Υ defines a map from the vector space $\mathbb{R}^m$ into the normed space generated by associating to the constraint vector space $\mathcal{U}$ the Euclidian norm. Under Assumption 3.4, there exists an h_1 such that $\Upsilon' h_1 < \upsilon$. Similarly, $\rho_- \leq \rho \leq \rho^+$

As a result, the auxiliary constrained optimization problem

$$\min_{h\in\mathcal{U},\rho\in[\rho_-,\rho^+]}\ell(h,\rho;x,p)$$

is a convex programming problem satisfying the assumptions of Lagrange Duality (see for example Theorem 1 in Section 8.6 in Luenberger (1969)). We conclude that the supremum is reached for a unique pair of maximizers $(\tilde{h}(t,x,p),\tilde{\rho}(t,x,p))$, which is an interior point of the set $\mathcal{J}\cap\mathcal{K}$ defined in equations (3.5) and (3.6), and the supremum, evaluated at $(\tilde{h}(t,x,p),\tilde{\rho}(t,x,p)) \in \mathbb{R}^{n+1}$is finite. By measurable selection, $\tilde{h}$ and $\tilde{\rho}$ can be taken as Borel measurable functions on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $[0,T] \times \mathbb{R}^n \times \mathbb{R}$ respectively.

□

Corollary 4.6 *Under either Assumption 2.1 (v) or both Assumption 2.2 (v) and Assumption 2.5, , the infimum in* (4.7) *admits a unique Borel measurable minimizing pair* $\left(\check{h}(t,x,r,p),\check{\rho}(t,x,r,p)\right)$ *for* $(t,x,r,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

4.3 Main result

The main result of this paper is the following:

Theorem 4.7 *Under either of*

1. *Affine drift assumptions 2.6, 2.1, 2.5 and 3.4; or*
2. *Standard control assumptions 2.7, 2.2, 2.5, 3.1 and 3.4;*

the following hold:

1. *The exponentially transformed value function* $\tilde{\Phi}$ *defined at* (4.3) *is the unique* $C^{1,2}([0,T]\times\mathbb{R}^n)$ *solution of the RS HJB PIDE* (4.6) *with terminal condition* $\tilde{\Phi}(T,x)=1$.
2. *The value function* Φ, *also defined at* (4.3), *is the unique* $C^{1,2}([0,T]\times\mathbb{R}^n)$ *solution of the RS HJB PIDE* (4.4)-(**??**).
3. *The asset allocation* $h^*(t)=\tilde{h}(t,X_t,D\Phi(t,X_t))$ *and the leverage ratio* $\rho^*(t)=\tilde{\rho}(t,X_t,D\Phi(t,X_t))$, *where* $\tilde{h}$ *and* $\tilde{\rho}$ *are the functions introduced in Proposition 4.5, are optimal in the class* $\mathcal{A}$ *of admissible controls.*

Proof (Proof of Theorem 4.7)
The proof is different under the two sets of assumptions.

1. <u>*Proof under affine drift assumptions 2.6, 2.1, 2.5 and 3.4*</u>

Existence of a classical ($C^{1,2}$) solution - the key points of the proof are discussed in Section 5.

Existence of an optimal control - by Proposition 4.5, the supremum in (4.4) admits a unique Borel measurable maximizer. By Proposition 6.2, this maximizer is admissible. Moreover, by Theorem 6.3 and Corollary 6.4 the maximiser is the pair of optimal controls (h^*,ρ^*).

Verification and uniqueness of the classical solution - $\tilde{\Phi}$ is bounded by Proposition 4.3. Part (i). of Verification Theorem 5.6 therefore applies. Choosing as optimal control the unique maximizer of the supremum (4.8), part (ii). of Theorem 5.6 also applies: $\tilde{\Phi}$ is the unique solution to the HJB PIDE. It then follows that Φ is the unique classical solution to the HJB PIDE (4.4) with terminal condition $\Phi(T,x)=0$.

2. <u>*Proof under standard control assumptions 2.7, 2.2, 2.5, 3.1 and 3.4*</u>

Existence of a classical ($C^{1,2}$) solution - Section 6 presents an outline of the approach developed to prove that $\tilde{\Phi}$ is a $C^{1,2}([0,T]\times\mathbb{R}^n)$ solution of the RS HJB PDE (4.6) with terminal condition $\tilde{\Phi}(T,x)=1$.

Existence of an optimal control - by Proposition 4.5, the supremum in (4.4) and infimum in (4.6) admit the same unique Borel measurable maximizing/minimizing pair $(h^*(t,X_t),\rho^*(t,X_t))$. By Proposition 6.2, the pair of controls (h^*,ρ^*) defined by $(h^*(t,X(t)),\rho^*(t,X_t))$ is admissible, i.e. belongs to the class $\mathcal{A}$. Theorem 6.3 shows by a martingale argument that this control is optimal.

□

5 Solving the ALM Problem Under Affine Drift Assumptions

5.1 Existence of a Classical ($C^{1,2}$) Solution

Under Assumptions 2.6, 2.1, 2.5 and 3.4, the problem reduces to solving a stochastic control problem in the factor process, which has no jumps. As a result, the HJB equation is a PDE rather than a PIDE:

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathcal{J}, \rho \in \mathcal{K}} L^{\pi}\left(t, x, \Phi, D\Phi, D^2\Phi\right) = 0 \tag{5.1}$$

where $\mathcal{J}$ is defined in (3.5),

$$\begin{aligned} L^{\pi}(t,x,u,p,M) = {} & f(t,x,h)'p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)M\right) - \frac{\theta}{2}p'\Lambda\Lambda'(t,x)p \\ & - g(t,x,h) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} g(t,x,h) = {} & \frac{1}{2}(\theta+1)\left[\varsigma' + \rho(h'\Sigma - \varsigma')\right]\left[\varsigma' + \rho(h'\Sigma - \varsigma')\right]' - (c + Cx) - \rho\left[h'(\tilde{a} + \tilde{A}x) - (\tilde{c} + \tilde{C}x)\right] \\ & + \int_{\mathbf{Z}} \left\{ \frac{1}{\theta}\left[\left(\eta(z) + \rho(t)\left[h'(t)\gamma(z) - \eta(z)\right]\right)^{-\theta} - 1\right] + \left(\eta(z) + \rho(t)\left[h'(t)\gamma(z) - \eta(z)\right]\right) \mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz) \end{aligned} \tag{5.3}$$

$$f(t,x,h) := b + Bx - \theta\Lambda\left[\varsigma' + \rho(h'\Sigma - \varsigma')\right] \tag{5.4}$$

and subject to the terminal condition $\Phi(T,x) = 0, x \in \mathbb{R}^n$.

For $\tilde{\Phi}$, the corresponding HJB PDE is

$$\frac{\partial \tilde{\Phi}}{\partial t}(t,x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)D^2\tilde{\Phi}\right) + H(t,x,\tilde{\Phi},D\tilde{\Phi}) = 0 \tag{5.5}$$

subject to terminal condition $\tilde{\Phi}(T,x) = 1$, where for $r \in \mathbb{R}$, $p \in \mathbb{R}^n$

$$H(s,x,r,p) = \inf_{h \in \mathcal{J}} \left\{ f(s,x,h)'p + \theta g(s,x,h)r \right\} \tag{5.6}$$

The proof of existence of a classical solution follows a similar argument to those developed by Fleming and Rishel (1975) (Theorem 6.2 and Appendix E) based on PDE results from Ladyzenskaja et al (1968). Namely, we can use an approximation in policy space alongside results on linear parabolic partial differential equations to prove that the exponentially transformed value functions $\tilde{\Phi}$ is of class $C^{1,2}((0,T)\times\mathbb{R}^n)$. Then it follows that the value functions Φ is also of class $C^{1,2}((0,T)\times\mathbb{R}^n)$. The approximation in policy space algorithm was originally proposed by Bellman in the 1950s (see Bellman (1957) for details) as a numerical method to compute the value function.

The approach proposed by Davis and Lleo (2011) in an asset management context was extended to take into account investment benchmarks (see Davis and Lleo (2013b)). We can also use it, with minor modifications, to solve an ALM problem. The first step is to use the approximation in policy space algorithm to show existence of a classical solution in a bounded region. Then, we need to extend this argument to unbounded state space.

A first difference between the asset management problem considered by Davis and Lleo (2011) and the ALM problem presented in this article lies in the definition of 'zero beta' policies (see Black (1972)), which now needs to take into account the pair of controls $\pi = (h, \rho)$.

Definition 5.1 (0β-policy) By reference to the definition of the function g in equation (3.9), a *'zero beta' (0β) control policy* pair $(\check{h}(t), \check{\rho})$ is an admissible pair of control policies for which the function g is independent from the state variable x.

The obvious choice for the 0β portfolio is an asset portfolio fully funded by equity (leverage $\check{\rho} = 1$). This choice brings us back to the pure asset management problem considered in Davis and Lleo (2011). From (3.2) we see that the set $\mathcal{Z}$ of 0β-policies is the set of admissible policies $\check{h}$ which satisfy the equation

$$\check{h}'\tilde{A} = -A_0$$

Note that since $m > n$, there is potentially an infinite number of 0β-policies even for a fixed $\check{\rho} = 1$ as long as the following assumption is satisfied

Assumption 5.2 *The matrix $\tilde{A}$ has rank n.*

Assumption 5.3 $\mathcal{Z} \cap \mathcal{J} \neq \{\emptyset\}$

Assumption 5.3 ensures that at least one zero beta policy $\check{h}$ is in $\mathcal{J}$. This assumption forces some consistency between the drift coefficients $A_0, \tilde{A}$ and the jump coefficient γ, but is consistent with the 'spirit' of zero beta policies: zero beta policies are proxies for the risk-free asset and should not result in a highly risky portfolio allocation. Without loss of generality, we can fix a 0β pair of controls $(\check{h}, \check{\rho})$ as a constant function of time so that $g(x, \check{h}, \check{\rho}) = \check{g}$, where $\check{g}$ is a constant. The main result for this section is:

Theorem 5.4 (Existence of a Classical Solution for the Exponentially Transformed Control Problem) *The RS HJB PDE* (5.5) *with terminal condition $\tilde{\Phi}(T, x) = 1$ has a solution $\tilde{\Phi} \in C^{1,2}\left((0, T) \times \mathbb{R}^n\right)$ with $\tilde{\Phi}$ continuous in $[0, T] \times \mathbb{R}^n$.*

Proof The poof follows along the same line as the proof of Theorem 7.2 in Davis and Lleo (2011). □

Corollary 5.5 (Existence of a Classical Solution for the Risk-Sensitive Control Problem) *The RS HJB PDE* (5.1) *with terminal condition $\Phi(T, x) = 0$ has a solution $\Phi \in C^{1,2}\left([0, T] \times \mathbb{R}^n\right)$ with Φ continuous in $\overline{[0, T] \times \mathbb{R}^n}$.*

5.2 Verification Theorem

First, we prove a verification theorem for the exponentially transformed problem with HJB PDE (5.5) and value function $\tilde{\Phi}$. As a corollary, we obtain a verification theorem for the control problem with HJB PDE (5.1) and value function Φ.

Theorem 5.6 (Verification Theorem for the Exponentially Transformed Control Problem)

(i) Assume that there exists a $C^{1,2}\left([0,T]\times\mathbb{R}^n\right)$ solution $\tilde{\Phi}$ to the HJB PIDE (4.6):

$$\begin{aligned}&\frac{\partial\tilde{\Phi}}{\partial t}(t,x)+\frac{1}{2}tr\left(\Lambda\Lambda'(t,x)D^2\tilde{\Phi}(t,x)\right)+H(t,x,\tilde{\Phi},D\tilde{\Phi})\\&+\int_{\mathbf{Z}}\left\{\tilde{\Phi}\left(t,x+\xi(t,x,z)\right)-\tilde{\Phi}(t,x)-\xi(t,x,z)'D\tilde{\Phi}(t,x)\right\}\nu(dz)\\&=0\end{aligned}$$

then $\tilde{\Phi}(t,x)\leq\tilde{I}(v,x,\tilde{h};t;T;\theta)$ for any $\tilde{h}\in\mathcal{A}(T)$.

(ii) Further assume that there exists a Borel-measurable minimizing pair $\tilde{\pi}=(h^*(t,x),\rho^*(t,x))$ of $(\tilde{h},\tilde{\rho})\mapsto\tilde{L}^{\tilde{\pi}}\tilde{\phi}$ defined as

$$\tilde{L}_t^{\pi}\varphi(t,x):=f(t,x,h,\rho)'D\varphi(t,x)+\theta g(x,h,\rho)\varphi(t,x)$$

then $\tilde{\Phi}(t,x)=\tilde{I}(t,x,h^*,\rho^*;T;\theta)$ and $(h^*(t,x),\rho^*(t,x))$ is optimal.

Proof The proof is standard (see for example Davis and Lleo (2013a)). □

Corollary 5.7 (Verification Theorem for the Risk-Sensitive Control Problem)

(i) Assume that there exists a $C^{1,2}\left([0,T]\times\mathbb{R}^n\right)$ solution Φ to the HJB PIDE (4.4):

$$\frac{\partial\Phi}{\partial t}+\sup_{h\in\mathcal{J}}L_t^{\pi}\Phi(t,X(t))=0$$

where L is defined in (4.5) then $\Phi(t,x)\geq I(t,x,h,\rho;T;\theta)$ for any $\tilde{\pi}\in\mathcal{A}(T)$.

(ii) Further assume that there exists a Borel-measurable minimizing pair $(h^*(t,x),\rho^*(t,x))$ of $\tilde{\pi}=(\tilde{h},\tilde{\rho})\mapsto L^{\pi}\Phi$ defined in (4.5) then $\Phi(t,x)=I(t,x,h^*,\rho^*;\theta;T)$ and $(h^*(t,x),\rho^*(t,x))$ is optimal.

Proof This corollary follows from the relation between Φ and $\tilde{\Phi}$ and from the fact that an admissible (optimal) strategy for the exponentially transformed problem is also admissible (optimal) for the risk-sensitive problem. □

Proposition 5.8 *The process $(h^*(t),\rho^*(t))$ is admissible: $(h^*(t),\rho^*(t))\in\mathcal{A}(T)$.*

Proof The proof is a slight extension of Proposition 4.3 in Davis and Lleo Davis and Lleo (2011). The class of admissible controls is presented in Definition 3.5. □

Applying Proposition 5.8 we deduce that the pair of controls $(h^*(t), \rho^*(t))$ is optimal for the auxiliary problems (4.3) resulting from the change of measure. However, this proposition is not sufficient to conclude that $(h^*(t), \rho^*(t))$ is optimal for the original problem (3.7) set under the $\mathbb{P}$-measure. The next result shows that this is indeed the case.

Proposition 5.9 *The optimal control* $(h^*(t), \rho^*(t))$ *for the auxiliary problem*

$$\sup_{h \in \mathcal{A}(T)} I(t, x; h, \rho; T; \theta)$$

where I *is defined in* (4.1) *is also optimal for the initial problem* $\sup_{h \in \mathcal{A}(T)} J(x, t, h, \rho)$ *where* J *is defined in* (3.7).

Proof Consider the control problem $\inf_{(h,\rho) \in \mathcal{A}(T)} \tilde{J}(x, t, h)$ where

$$\tilde{J}(x, t, h, \rho) := \ln \mathbf{E}\left[e^{-\theta \ln V(t,x,h,\rho)}\right] \tag{5.7}$$

The term $e^{-\theta \ln V(t,x,h)}$ is bounded from below by 0 and therefore $\inf_{h \in \mathcal{A}(T)} \tilde{J}(x, t, h)$ is well defined. This implies the existence of at least one minimizer $\tilde{h}$.

$$\mathbf{E}\left[e^{-\theta \ln V(t,x,h,\rho)}\right] = \mathbf{E}^{\pi}_{t,x}\left[\exp\left\{\theta \int_t^T g(s, X_s, h(s), \rho(s))ds - \theta \ln v\right\}\right]$$

(see for example Lemma 8.6.2. in Øksendal (2003)) and hence

$$\begin{aligned}\inf_{h \in \mathcal{A}(T)} \mathbf{E}\left[e^{-\theta \ln V(t,x,h,\rho)}\right] &= \inf_{h \in \mathcal{A}(T)} \mathbf{E}^{h,\rho}_{t,x}\left[\exp\left\{\theta \int_t^T g(s, X_s, h(s), \rho(s))ds - \theta \ln v\right\}\right] \\ &= I(v, x; h^*(t), \rho^*(t); t, T)\end{aligned}$$

This proves that the optimal control pair $(h^*(t), \rho^*(t))$ for the auxiliary problem $\sup_{(h,\rho) \in \mathcal{A}(T)} I(v, x; h, \rho; t, T)$ is optimal for the problem $\sup_{(h,\rho) \in \mathcal{A}(T)} J(x, t, h, \rho)$. □

6 Solving the ALM Problem Under Standard Control Assumptions

6.1 Existence of a Classical ($C^{1,2}$) Solution

In this section we prove that the value functions Φ and $\tilde{\Phi}$ are smooth under standard control assumptions. The process involves the six steps proposed by Davis and Lleo (2013a) for an asset management problem.

Step 1: $\tilde{\Phi}$ *is a Lipschitz Continuous Viscosity Solution (VS-PIDE) of* (4.6). Following Theorem 4.4 in Davis and Lleo (2013a), we can show that $\tilde{\Phi}$ is a (possibly discontinuous) viscosity solution of the PIDE (4.6). However, we know *a priori* from Proposition 4.4 that $\tilde{\Phi}$ is Lipschitz.

Step 2: From PIDE to PDE. Assumption 1(vi) (2.29) implies that we can write the non-local term in (4.6) as

$$\int_{\mathbf{Z}} \{\tilde{\Phi}(t,x+\xi(t,x,z)) - \tilde{\Phi}(t,x)\}\nu(dz) + \int_{\mathbf{Z}} \xi'(t,x,z)\nu(dz)D\tilde{\Phi}(t,x).$$

Change notation to rewrite the HJB PIDE as a PDE *à la* Pham (1998):

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t,x)D^2u\right) + H_a(t,x,u,Du) + d_a^{\tilde{\Phi}}(t,x) = 0 \tag{6.1}$$

subject to terminal condition $u(T,x) = 1$ and with

$$H_a(s,x,r,p) = \inf_{h\in\mathcal{U}} \{f_a(s,x,h,\rho)'p + \theta g(s,x,h,\rho)r\} \tag{6.2}$$

for $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and where

$$f_a(s,x,h,\rho) := f(s,x,h,\rho) - \int_{\mathbf{Z}} \xi(s,x,z)\nu(dz) \tag{6.3}$$

$$d_a^{\tilde{\Phi}}(t,x) = \int_{\mathbf{Z}} \left\{\tilde{\Phi}\left(t,x+\xi(t,x,z)\right) - \tilde{\Phi}(t,x)\right\}\nu(dz). \tag{6.4}$$

Step 3: Viscosity Solution to PDE (6.1). We consider viscosity solutions u of the semi-linear PDE (6.1) (interpreted as an equation for 'unknown' u with the last term prespecified, with $\tilde{\Phi}$ defined as in Step 1.) The key point is that $\tilde{\Phi}$ *is a viscosity solution of the PDE* (6.1). Indeed, due to definition (6.4), PIDE (4.6) and PDE (6.1) are in essence the same equation. Hence, if $\tilde{\Phi}$ satisfies the PIDE in the viscosity sense, which from Step 1 we know that it does, then $u = \tilde{\Phi}$ is a viscosity solution of the PDE (6.1). This last statement depends crucially on the Definition of 'viscosity solution' for the PIDE suggested by Pham (1998) and formalised by Davis et al (2009).

Step 4: Uniqueness of the Viscosity Solution to the PDE (6.1). If a function u solves the PDE (6.1) it does not mean that u also solves the PIDE (4.6) because the term d_a in the PDE (4.6) depends on $\tilde{\Phi}$ regardless of the choice of u. Thus, if we were to show the existence of a classical solution u to PDE (6.1), we would not be sure that this solution is the value function $\tilde{\Phi}$ unless we can show that PDE (6.1) admits a unique solution. This only requires applying a "classical" comparison result for viscosity solutions (see Theorem 8.2 in Crandall et al (1992)) provided appropriate conditions on f_a and $d_a^{\tilde{\Phi}}$ are satisfied.

Step 5: Existence of a Classical Solution to the HJB PDE (4.6). The proof in Davis and Lleo (2013a) uses an argument similar to that Fleming and Rishel (1975) (Appendix E) together with a result from Davis et al (2009) to show the existence of a classical solution to the PDE (6.1) with $d_a^{\tilde{\Phi}}(t,x)$ regarded as an autonomous term.

Step 6: Existence of a Classical Solution to the HJB PIDE (4.6). Combining Steps 4 and 5, we conclude that $\tilde{\Phi}$ and Φ are respectively a classical ($C^{1,2}$) solution of (4.6) and a classical ($C^{1,2}$) solution of (4.4).

6.2 Identifying the Optimal Strategy

All that remains now is to show that the controls derived from the Hamiltonian-minimizing function $(\check{h}, \check{\rho})$ of Corollary 4.6 and from the maximiser $\tilde{h}$ of Proposition 4.5 correspond to the optimal policy.

Lemma 6.1 *The Hamiltonian-minimizing function $(\check{h}, \check{\rho})$ of Corollary 4.6 and the maximiser $(\tilde{h}, \tilde{\rho})$ of Proposition 4.5 correspond to the same control $(h^*(t, X_t), \rho^*(t, X_t))$, that is:*

$$\tilde{h}(t, X_t, D\Phi(t, X_t)) = \check{h}(t, X_t, \tilde{\Phi}(t, X_t), D\tilde{\Phi}(t, X_t)) =: h^*(t, X_t)$$
$$\tilde{\rho}(t, X_t, D\Phi(t, X_t)) = \check{\rho}(t, X_t, \tilde{\Phi}(t, X_t), D\tilde{\Phi}(t, X_t)) =: \rho^*(t, X_t)$$

Proof The Hamiltonian-minimizing function $(\check{h}, \check{\rho})$ of Corollary 4.6 and the maximizing pair $(\tilde{h}, \tilde{\rho})$ of Proposition 4.5 are both unique. Moreover Φ and $\tilde{\Phi}$ are related through a monotone transformation.This proves that $\tilde{h}(t, X_t, D\Phi(t, X_t)) = \check{h}(t, X_t, \tilde{\Phi}(t, X_t), D\tilde{\Phi}(t, X_t))$ and $\tilde{\rho}(t, X_t, D\Phi(t, X_t)) = \check{\rho}(t, X_t, \tilde{\Phi}(t, X_t), D\tilde{\Phi}(t, X_t))$. □

Proposition 6.2 *The pair of controls $(h^*(t, X_t), \rho^*(t, X_t))$ is admissible: $(h^*(t, X_t), \rho^*(t, X_t)) \in \mathcal{A}$.*

Proof The proof follows closely Proposition 4.3 in Davis and Lleo (2011). The class of admissible controls is presented in Definition 3.5. □

Theorem 6.3 *The pair of controls $(h^*(t, X_t), \rho^*(t, X_t))$ is optimal. In particular $\tilde{\Phi}(t, x) = \tilde{I}(t, x, h^*, \rho^*; T; \theta)$.*

Proof Consider the Borel-measurable minimizing pair of controls $(h^*(t, X_t), \rho^*(t, X_t))$ with associated measure $\mathbb{P}^*$ and let $X(s), s \geq t$ be the state process with initial data $X(t) = x$.

Define the process $Z(s) := \theta \int_t^s g(u, X_u, h_u^*, \rho_u^*) du$ and use the general Itô formula to calculate $Z(s)\tilde{\Phi}(s, X(s))$. We find that

$$\tilde{\Phi}(s, X_s)e^{Z_s} = \tilde{\Phi}(t, x) + \int_t^s D\tilde{\Phi}' \Lambda(u, X(u)) dW_u^\theta + \int_t^s \int_{\mathbf{Z}} \left\{ \tilde{\Phi}\left(u, X(u^-) + \xi\left(u, X(u^-), z\right)\right) - \tilde{\Phi}(u, X(u^-)) \right\} \tilde{N}(du, dz) \tag{6.5}$$

(The drift term is equal to zero since (h^*, ρ^*) achieves the minimum in the HJB PIDE (4.6).) We claim that both stochastic integrals in (6.5) are martingales.

Indeed, under Assumption 2.7, Theorem 1.19 of Øksendal and Sulem (2005) implies that

$$\mathbf{E}^* \left[\int_t^T |X(s)|^2 ds \right] < \infty. \tag{6.6}$$

This is enough to show that the Brownian integral is a martingale, since $D\tilde{\Phi}$ is bounded and Λ is Lipschitz in x. For the Poisson random measure integral we have, since $\tilde{\Phi}$ is Lipschitz (with constant K),

$$\alpha(u, X(u^-), z) \equiv |\tilde{\Phi}(u, X(u^-) + \xi(u, X(u^-), z) - \tilde{\Phi}(u, X(u^-))| \leq K|\xi(u, X(u^-), z)|,$$

so, by Assumption (2.30),

$$\int_{\mathbf{Z}} \alpha^2(u, X(u^-), z)\nu(dz) < 3c^2(1 + |X(u^-)|^2).$$

Hence by (6.6)

$$\mathbf{E}^* \int_0^T \int_{\mathbf{Z}} \alpha^2(u, X(u^-), z)\nu(dz)du < \infty,$$

and this is a sufficient condition for the stochastic integral to be a martingale (see Øksendal and Sulem (2005), (1.1.13)). Thus, from (6.5),

$$\begin{aligned}
\tilde{\Phi}(t, x) &= \mathbf{E}^*[\tilde{\Phi}(T, X(T)) \exp(Z(T))] \\
&= \mathbf{E}^* \left[e^{\theta \int_t^T g(s, X_s, h_s^*, \rho_s^*) ds} \right] \\
&= \tilde{I}(t, x, h^*, \rho^*; T; \theta).
\end{aligned}$$

□

Corollary 6.4 *The pair (h^*, ρ^*) defined above is optimal for the logarithmically-transformed problem, and*

$$\Phi(t, x) = I(t, x, h^*, \rho^*; T; \theta).$$

Proof This corollary follows from the relation between Φ and $\tilde{\Phi}$ and from the fact that an admissible (optimal) strategy for the exponentially transformed problem is also admissible (optimal) for the risk-sensitive problem. □

7 Conclusion

In this paper, we have used risk-sensitive control methods to solve an ALM problem in a general jump-diffusion setting where asset prices and the value of the liability are influenced by a number of valuation factors. The investor's objective is to jointly select an optimal amount of leverage $\rho(t)$ and an optimal asset allocation $h(t)$ to maximise a function of the investor's equity or surplus, defined as the difference between the investor's assets and liabilities.

To solve the problem, we extended the solution techniques proposed by Davis and Lleo (2011, 2013a) to show that the Bellman equation for the control problem admits a unique classical ($C^{1,2}$) solution.

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