Coupled consolidation in unsaturated soils: An alternative approach to deriving the Governing Equations

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Abstract

The equations governing coupled consolidation in unsaturated soils are known to contain additional parameters when compared to the equations for saturated soils. Nonetheless, the variation of these parameters with suction or degree of saturation is not generally agreed upon. The paper introduces a novel approach to deriving general equations for each of these parameters and their variation, and explains that, for consistency with the constitutive and soil-water retention curve models adopted, these general equations need to be transformed into case-specific expressions. Finally, a conceptual model is presented highlighting how the behaviour of unsaturated soil reflects aspects of its water content.

Keywords: coupled consolidation; conceptual model; unsaturated

Introduction

Several attempts have been made to model hydro-mechanical coupling in unsaturated soil states, usually based on extending Biot's theory (Biot, 1941). Different simplifications can be made in extending Biot's approach, such as assuming that the air phase is drained and air pressure is equal to atmospheric pressure (e.g. Meroi et al., 1995; Wong et al., 1998; Sheng et al., 2003; Sheng et al., 2008; Wang et al., 2009). This assumption applies to the whole of the unsaturated region, independently of the water and air content of the soil, and the continuity equation for the air phase is ignored. The assumption of a static air phase may impede the applicability of the Governing Equations to a series of Geotechnical Engineering problems involving air pressure, such as Reservoir Engineering, where both water and gas exist under pressure. In certain cases the flow of air is explicitly modelled and air pressure is an additional primary variable (e.g. Darkshanamurthy et al., 1984; Schrefler & Xiaoyong, 1993; Schrefler et al., 1995; Gatmiri et al., 1998; Schrefler & Scotta, 2001; Laloui et al., 2003), yet transition between fluid phases may be ignored. The effect of temperature can be included in thermo-hydro-mechanical coupling in unsaturated soils (e.g. Olivella et al., 1994; Thomas & He, 1995).

The work presented here is an extension of the work done by Darkshanamurthy et al. (1984) and Wong et al. (1998). Although in Darkshanamurthy et al. (1984) the flow of the gas phase is modelled, "the effect of air diffusing through water, air dissolving in the water phase and the movement of water vapour are ignored". Wong et al. (1998) extended the work of Darkshanamurthy et al. (1984) to multidimensional cases and presented the Finite Element (FE) formulation for coupled consolidation problems, assuming that "(i) the pore-air pressure is atmospheric and remains unchanged during an analysis, and (ii) water flows through the soil skeleton in accordance with Darcy's law". These two assumptions were also made in the proposed approach.

The original formulations of the governing equations in Biot (1941), Darkshanamurthy et al. (1984) and Wong et al. (1998) are explained in Appendix A and the main points are discussed in the subsequent section, in order to highlight the differences of the proposed approach. All the relevant equations reported from the literature have been reproduced in Appendix A employing the same symbols as in the original publication, with the exception of Poisson's ratio which is always μ. These equations are summarised in Table 1.

The ground-breaking work of Biot (1941) set the basis for coupled consolidation analysis in saturated soils and in soils containing air in the form of occluded bubbles (i.e. not in a continuous form). The two equations proposed, i.e. the constitutive relationship for the soil structure and the constitutive relationship for the water phase, contain four moduli: E, which is Young's modulus (the shear modulus G is used for shear strains); H, which is similar to Young's modulus and shows the effect of changing pore water pressure on the direct strains in the soil; H_1, which is a physical constant describing the effect of changes in applied total stress on the water content; and R, which is a physical constant describing the effect of incremental pore water pressure on the water content.

Biot (1941) demonstrated that $H = H_1$ for the particular stress condition where $\sigma_x = \sigma_y = \sigma_z$

and $\tau_{xy} = \tau_{yz} = \tau_{xz} = 0$, i.e. no distinction is necessary between the modulus governing the effect of changing pore water pressure on the direct strains and the modulus governing the effect of incremental pore water pressure on the water content.

In extending Biot's work to unsaturated soil conditions, Darkshanamurthy et al. (1984) and Wong et al. (1998) also assumed, either explicitly or implicitly that the two moduli remain equal and, as in Biot (1941), are physical constants. However, there is no particular reason why the modulus governing the effect of matric suction on the direct strains should be equal to the modulus governing the effect of net stress on the volumetric water content under all circumstances and especially for truly unsaturated soils containing continuous air, where the effect on soil behaviour of changing suction is distinctively and fundamentally different from the effect of changing applied stress (Burland & Jennings, 1962, Fredlund & Morgerstern, 1964). As discussed in the subsequent section, not making a clear distinction between the moduli controlling the effect of suction on direct strains and the effect of net stresses on volumetric water content oversimplifies the complex behaviour of unsaturated soils.

This drawback is addressed in the present paper, where a distinction between the two moduli is explicitly made. In addition to moduli E and R, which are similar to the moduli in Biot (1941), modulus E_w, which governs the effect of net stress on the volumetric water content, and modulus H, which governs the effect of matric suction on direct strains, are required for the formulation of the Governing Equations in unsaturated states. Following an approach similar to Biot (1941) and Wong et al. (1998), the constitutive relationship for the water phase is rewritten in a form containing three parameters, Ω, ω and H, which are related to the four moduli and which are required to extend coupled consolidation to unsaturated soil states.

The main differences of the current approach, which constitute the innovative aspects of this work, are:

(a) a clear distinction is made between the two moduli controlling the effect of matric suction on direct strains, H, and the effect of net stress on the volumetric water content, E_w, as explained above. As a result, the three additional parameters, Ω, ω and H, which are required to extend coupled consolidation to unsaturated soil states, relate to four moduli, E, E_w, R and H rather than to three as in Wong et al. (1998) (E, R and H, as no distinction is made between E_w and H);

(b) Parameters Ω, ω and H are not soil constants, and their variation with suction and degree of saturation is obtained in a consistent manner. Other Authors (e.g Gatmiri et al., 1998; Sheng et al., 2003; Khalili et al., 2008) have obtained similar expressions following different approaches to deriving the Governing Equations to the one presented here. However, what is shown for the first time is that the Governing Equations need to be adjusted to the soil-water retention (SWR) and compressibility relationships (i.e. SWR and constitutive model) used to reproduce soil behaviour, as these will affect individual terms in the equations obtained for parameters Ω, ω and H. This is more fundamental than updating the value of parameters Ω, ω and H, which vary in a highly non-linear manner with suction; it is shown that essentially

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different Equations correspond to different SWR and constitutive models. More specifically, if the SWR model accounts for the effect of specific volume v on the degree of saturation, S_r, in addition to matric suction, s, i.e. S_r is a function f of both v and s, $S_r = f(s, v)$, parameter Ω is no longer the same as in the case where S_r is a function f of s only, $S_r = f(s)$. This is an aspect of the Governing Equations commonly ignored in the literature.

To illustrate the effect of the constitutive and SWR model on parameters Ω, ω and H, four cases are considered, where the compressibility and SWR relationships differ slightly but result in measurable differences in the values of parameters Ω, ω and H and how these vary with suction and therefore degree of saturation. Of these cases one is further considered and used to present a conceptual model, highlighting how the behaviour of unsaturated soil reflects aspects of its water content.

The FE element formulation of the governing equations proposed here and their use in non-linear boundary value problems are discussed in Tsiampousi et al. (2016).

Table 1: Summary of constitutive relationships proposed for the soil structure and for the water phase

	Constitutive relat. for the soil structure	Constitutive relat. for the water phase
Biot (1941)	$\varepsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\mu}{E}(\sigma_y + \sigma_z) + \dfrac{\sigma}{3H}$ (A.1) and: $\gamma_{xy} = \dfrac{\tau_{xy}}{G}$ (A.2)	$\theta = \dfrac{1}{3H_1}(\sigma_x + \sigma_y + \sigma_z) + \dfrac{\sigma}{R}$ (A.3) or: $\theta = \alpha\varepsilon + \dfrac{\sigma}{Q}$ (A.4) where: $\alpha = \dfrac{2(1+\mu)}{3(1-2\mu)} \cdot \dfrac{G}{H}$ (A.5) $Q = \dfrac{1}{R} - \dfrac{\alpha}{H}$ (A.6)
	Asumption: $H = H_1$	
	Approach based on the use of moduli G, R and H	
Darkshanamurthy et al. (1984)	$\varepsilon_x = \dfrac{(\sigma_x - u_a)}{E_1} - \dfrac{\mu}{E_1}(\sigma_y + \sigma_z - 2u_a)$ $\quad + \left(\dfrac{u_a - u_w}{H_1}\right)$ (A.10) and: $\gamma_{xy} = \dfrac{\tau_{xy}}{G}$ (A.2)	$\theta_w = \dfrac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{3H_1'}$ $\quad + \dfrac{u_a - u_w}{R_1}$ (A.11) or: $\theta_w = \dfrac{\beta}{3}\varepsilon + \gamma(u_a - u_w)$ (A.12) where: $\beta = \dfrac{E_1}{H_1}\dfrac{1}{1-2\mu}$ (A.13) $\gamma = \dfrac{1}{R_1} - \dfrac{\beta}{H_1'}$ (A.14)
	Asumption: $H_1 = H_1'$	
	Approach based on the use of moduli E_1, R_1 and H_1	
Wong et al. (1998)	$\varepsilon_{ij} = \dfrac{1+\mu}{E}\sigma_{ij}^n - \dfrac{\mu}{E}\sigma^n\delta_{ij}$ $\quad + \dfrac{u_a - u_w}{H}\delta_{ij}$ (A.17)	$\theta_w = \beta\varepsilon_v + \omega(u_a - u_w)$ (A.19) where: $\beta = \dfrac{E}{H} \cdot \dfrac{1}{1-2\mu}$ (A.20) $\omega = \dfrac{1}{R} - \dfrac{3\beta}{H}$ (A.21)
	Asumption: No distinction between modulus H in Eq. A.17 and A.20	
	Approach based on the use of moduli E, R and H	
Pproposed approach	$\varepsilon_x = \dfrac{(\sigma_x - u_a)}{E} - \dfrac{\mu}{E}(\sigma_y + \sigma_z - 2u_a)$ $\quad + \left(\dfrac{u_a - u_w}{H}\right)$ (1) and: $\gamma_{xy} = \dfrac{\tau_{xy}}{G}$ (1)	$\theta_w = \dfrac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{E_w}$ $\quad + \dfrac{(u_a - u_w)}{R}$ (2) or: $\theta_w = \Omega\varepsilon_{vol} + \omega(u_a - u_w)$ (10) where: $\Omega = \dfrac{E}{E_w}\dfrac{1}{(1-2\mu)}$ (9) $\omega = \left(\dfrac{1}{R} - \dfrac{3\Omega}{H}\right)$ (11)
	Assumption: moduli H and E_w are independent	
	Approach based on the use of moduli E, E_w, R and H	

Original development of the constitutive relationships for saturated and unsaturated soils

As noted above, the original formulation of the governing equations in Biot (1941), Darkshanamurthy et al. (1984) and Wong et al. (1998) is explained in Appendix A and the relevant equations are summarised in Table 1 for ease of reference. All symbols are defined in Appendix A. To aid the discussion and highlight the differences of the proposed approach, the main points from the original development of the equations are discussed here.

Biot (1941) proposed two constitutive relationships to describe coupled consolidation in fully saturated soils and in soils containing air in the form of occluded bubbles; the constitutive relationship for the soil structure (Equations A.1 and A.2 in Table 1) and the constitutive relationship for the water phase (Equation A.3 in Table 1). These contain the moduli E (or G), H, H_1 and R, as explained in the Introduction and detailed in Appendix A. Biot (1941) demonstrated that modulus H governing the effect of changing pore water pressure on the direct strains and modulus H_1 governing the effect of incremental pore water pressure on the water content ($H = H_1$) are equal for the particular stress condition where $\sigma_x = \sigma_y = \sigma_z$ and $\tau_{xy} = \tau_{yz} = \tau_{xz} = 0$, thus reducing the number of required moduli to three.

Equation A.3 can be rewritten in the form given by Equation A.4 , from which Biot (1941) concluded that for the case of a fully saturated soil $\theta = \varepsilon$, i.e. the increment of volumetric water content, θ, is equal to the volumetric strain, ε. It is shown in Appendix A that if compression is considered positive, as is the convention in Soil Mechanics, this relationship should actually be written as $\theta = -\varepsilon$.

Darkshanamurthy et al. (1984) and Fredlund & Rahardjo (1993) among others extended Biot's coupled theory to unsaturated soils containing air in a continuous form (rather than in occluded bubbles as in Biot, 1941). In particular, Darkshanamurthy et al. (1984) altered Biot's equations in order to make use of the two stress variables net stress and matric suction (Equations A.10 and A.11 in Table 1). As demonstrated in Appendix A, Darkshanamurthy et al. (1984) made the implicit assumption that the modulus governing the effect of matric suction on direct strains (H_1 in Equation A.10) is equal to the modulus governing the effect of net stress on the volumetric water content (H_1' in Equation A.11).

Later, Wong et al. (1998) described the numerical implementation of the coupled formulation presented by Darkshanamurthy et al. (1984) (Equations A.17 and A.19 in Table 1). It is clear that no distinction was made between modulus H in Equation A.17 and modulus H in Equation A.20.

However, there is no particular reason why the two moduli should be equal under all circumstances and especially for truly unsaturated soils containing continuous air, where the effect on soil behaviour of changing suction is distinctively and fundamentally different from the effect of changing applied stress (Burland & Jennings, 1962, Fredlund & Morgerstern, 1977). Indeed, in fully saturated conditions, $\Delta\varepsilon_{vol} = \Delta\theta_w = \Delta n$ (where ε_{vol} is the volumetric

strain, θ_w is the volumetric water content and n is porosity), as $\theta_w = n \cdot S_r$, S_r being the degree of saturation and equal to 1. Changes in porosity, Δn, can result from changes in pore water pressures and/or changes in total stress, with the two having an equal effect on porosity. In unsaturated conditions, however, $\Delta\varepsilon_{vol} \neq \Delta\theta_w$, as some deforming voids may be empty of water. As $\theta_w = n \cdot S_r$, changes in θ_w may result from changes in porosity n and/or changes in degree of saturation S_r. In turn, each of these may result from changes in suction and/or changes in net stress, with net stress and suction affecting both n and S_r by different amounts. For example, it may be expected that suction changes have a greater effect on S_r than net stress changes. Similarly, drying a soil sample (suction changes) may be expected to affect its volume to a lesser extent than loading (net stress changes). Not making a clear distinction between the modulus controlling the effect of suction on direct strains and the modulus controlling the effect of net stresses on volumetric water content oversimplifies the complex behaviour of unsaturated soils.

The above drawback is addressed in the present paper, where a distinction between the two moduli is explicitly made and the constitutive relationship for the water phase is re-derived to reflect the inclusion of the additional modulus. To avoid further confusion with the notation which has been used in previous publications, the modulus in the constitutive relation for the soil structure controlling the effect of suction on direct strains is denoted by the letter H, whereas the modulus in the constitutive equation for the water phase controlling the effect of net stresses on volumetric water content is denoted by E_w.

Re-deriving of the constitutive relationship for the water phase

The constitutive relationships for the soil structure and the water phase can be written as:

$$\varepsilon_x = \frac{(\sigma_x - u_a)}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z - 2u_a) + \left(\frac{u_a - u_w}{H}\right) \quad \& \quad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad (1)$$

$$\theta_w = \frac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{E_w} + \frac{(u_a - u_w)}{R} \qquad (2)$$

and similar for ε_y, ε_z, γ_{yz} and γ_{zx}, where ε_x, ε_y, ε_z are direct strains and σ_x, σ_y, σ_z are total direct stresses, γ_{xy}, γ_{yz}, γ_{zx} are shear strains and τ_{xy}, τ_{yz}, τ_{zx} are shear stresses, $u_a - u_w$ is matric suction, u_a being the pore air pressure and u_w the pore water pressure, E is Young's modulus and G is the shear modulus. H is a modulus accounting for the effect of changing matric suction on the direct strains in the soil, and E_w and R are additional moduli controlling the effect of applied net stress and matric suction on the volumetric water content, θ_w, of the soil, respectively. $1/R$ can be interpreted as the slope of the SWR curve in terms of volumetric water content, i.e. $1/R = \partial\theta_w/\partial(u_a - u_w)$. Equations 1 and 2 differ from Equations A.10 and A.11 of Darkshanamurthy et al. (1984) (see Appendix A and Table 1) in that the modulus E_w is distinct from modulus H (note that in Darkshanamurthy et al. (1984) $H_1 = H_1'$ and are both equivalent to E_w in Equation 2).

Equation 1 can be written as

$$(\sigma_x - u_a) = 2G(\varepsilon_x + \alpha\varepsilon_{vol}) - \beta(u_a - u_w) \tag{3}$$

where ε_{vol} is the volumetric strain.

:

$$\alpha = \frac{\mu}{1 - 2\mu} \tag{4}$$

and:

$$\beta = \frac{E}{H} \cdot \frac{1}{1 - 2\mu} \tag{5}$$

Therefore:

$$\begin{aligned}
\sigma_x + \sigma_y + \sigma_z - 3u_a &= 2G(\varepsilon_x + \varepsilon_y + \varepsilon_z + 3\alpha\varepsilon_{vol}) - 3\beta(u_a - u_w) = \\
&= \frac{E}{(1 + \mu)}\varepsilon_{vol}(1 + 3\alpha) - 3\beta(u_a - u_w) = \\
&= \frac{E}{(1 + \mu)}\varepsilon_{vol}\left(\frac{1 - 2\mu}{1 - 2\mu} + \frac{3\mu}{1 - 2\mu}\right) - 3\beta(u_a - u_w) = \\
&= \frac{E\varepsilon_{vol}}{(1 - 2\mu)} - 3\beta(u_a - u_w)
\end{aligned} \tag{6}$$

Equation 2 thus becomes:

$$\begin{aligned}
\theta_w &= \frac{\frac{E\varepsilon_{vol}}{(1-2\mu)} - 3\beta(u_a - u_w)}{E_w} + \frac{(u_a - u_w)}{R} = \\
&= \frac{E\varepsilon_{vol}}{(1 - 2\mu)E_w} - \frac{3\beta(u_a - u_w)}{E_w} + \frac{(u_a - u_w)}{R} = \\
&= \frac{HE\varepsilon_{vol}}{(1 - 2\mu)HE_w} + (u_a - u_w)(\frac{1}{R} - \frac{3\beta}{E_w})
\end{aligned} \tag{7}$$

Substituting Equation 5 into the one above:

$$\theta_w = \frac{H\beta\varepsilon_{vol}}{E_w} + (u_a - u_w)\left(\frac{1}{R} - \frac{3\beta}{E_w}\right) \tag{8}$$

Introducing the new parameter Ω, such that:

$$\Omega = \frac{\beta H}{E_w} = \frac{E}{E_w}\frac{1}{(1 - 2\mu)} \tag{9}$$

the water phase constitutive equation becomes:

$$\theta_w = \Omega\varepsilon_{vol} + \omega(u_a - u_w) \tag{10}$$

where:

$$\omega = \left(\frac{1}{R} - \frac{3\Omega}{H}\right) \tag{11}$$

Equation 10 represents the newly proposed constitutive relationship for the water phase for the case of unsaturated soils. The constitutive relationship for the soil structure is given by Equation 1. The finite element formulation of these two equations is explained in Tsiampousi et al. (2016).

Equations 1 and 10 need to be considered simultaneously as in Biot's approach and they replace Equations 1 and 2 in the modelling of the fully-coupled water flow in unsaturated soils. Therefore, instead of defining parameters E_w, R and H as implied by Equations 1 and 2, the proposed approach requires the alternative parameters ω, Ω and H to be defined. Parameters Ω and ω in Equation 10 show the effect of volumetric strains and matrix suction, respectively, on the volumetric water content, θ_w. This effect is expected to become progressively smaller as the soil becomes more unsaturated, continuity of bulk water ceases and residual conditions are approached. Additionally, the soil response to suction changes, defined by modulus H in Equation 1, is expected to become progressively stiffer as the soil becomes more unsaturated. The variation of parameters Ω, ω and H with suction should reflect this behaviour. This is discussed later in the paper in the form of a conceptual model.

Parameters Ω, ω and H

As explained above, the proposed approach requires that parameters ω, Ω and H are defined. The three parameters are related through Equation 11. Parameters ω and Ω can be defined as follows.

The volumetric water content is given by the following phase relationship:

$$\theta_w = n \cdot S_r \tag{12}$$

where n is porosity and S_r is the degree of saturation. Clearly:

$$\frac{\partial\theta_w}{\partial t} = n \cdot \frac{\partial S_r}{\partial t} + S_r \cdot \frac{\partial n}{\partial t} \tag{13}$$

t being time. The same differential can be obtained from Equation 10:

$$\frac{\partial\theta_w}{\partial t} = \Omega\frac{d\varepsilon_{vol}}{dt} + \omega\frac{ds}{dt} \tag{14}$$

where s is the matric suction equal to $u_a - u_w$. Equating Equations 13 and 14 parameters Ω and ω can be defined, as shown in Appendix B. Clearly, parameters Ω and ω should be

adjusted to the SWR model used, as this will affect the differentials $\frac{\partial S_r}{\partial t}$ and $\frac{\partial n}{\partial t}$ in Equation 13. For example, for the simple case where the degree of saturation is a function of suction, s, but not of specific volume v (e.g. Van Genuchten, 1980), it can be shown that (see Appendix B):

$$\Omega = -S_r \tag{15}$$

and:

$$\omega = n \cdot \frac{\partial S_r}{\partial s} \tag{16}$$

In recent literature (e.g. Gallipoli et al., 2003; Tsiampousi et al., 2013) the SWR curve is expressed in the three-dimensional space $s - S_r - v$, so that S_r is a function f of both s and v, i.e. $S_r = f(s, v)$. Therefore, the differential $\frac{\partial S_r}{\partial t}$ in Equation 13 will also depend on specific volume according to the function f that a particular SWR model adopts. In that case, Ω becomes (see Appendix B):

$$\Omega = -S_r - e \frac{\partial S_r}{\partial v} \tag{17}$$

whereas the expression for ω remains unchanged, as explained in detail in Appendix B. The above equations are summarised in Table 2.

The fact that parameter Ω is given by a different equation when $S_r = f(s, v)$ in comparison to the case where $S_r = f(s)$, is an aspect of the Governing Equations not explicitly discussed in the literature. Clearly, using $\Omega = -S_r$ when $S_r = f(s, v)$ leads to inconsistencies, As Equations 13 and 14 are no longer equivalent.

Parameter H can then be defined from Equation 11. It is shown in Appendix B that:

$$H = -\frac{1}{\frac{1}{v_0} \cdot \frac{\partial v}{\partial s}} \tag{18}$$

v_0 being the specific volume, v, at the beginning of an increment of suction change. The negative sign in the above equation cancels out with the negative sign of the partial differential $\frac{\partial v}{\partial s}$ in the same equation, as positive changes in suction (i.e. increase) cause a reduction in the specific volume, v. For example, in the case of the Barcelona Basic Model (BBM, Alonso et al., 1990), for elastic changes of suction:

$$\frac{\partial v}{\partial s} = -\frac{\kappa_s}{(s + p_{atm})} \tag{19}$$

where κ_s is the coefficient of compressibility with respect to suction changes and p_{atm} is the atmospheric pressure. Clearly, the partial differential $\frac{\partial v}{\partial s}$ depends on the particular constitutive model adopted and should be adjusted accordingly.

In Table 3 the equations obtained for parameters Ω, ω and H are compared to equivalent expressions in the literature to explore differences and similarities. The expressions for parameters Ω and H are similar to the equivalent expressions in Gatmiri et al. (1998), the main difference in the second term for Ω being associated with the fact that the degree of saturation is a function of suction and net stress in Gatmiri et al. (1998). Nonetheless, the second term in the expression for ω in Gatmiri et al. (1998) does not seem to be justified based on the current approach. It is not clearly explained by Gatmiri et al. (1998) why the term is needed.

The expressions for Ω and ω are similar to the equivalent ones in Khalili et al. (2008). Nonetheless, the term $n \frac{\partial S_r}{\partial \varepsilon_{vol}}$ in Khalili et al. (2008) is not explicitly related to the degree of saturation being a function of suction and volume, whereas the term $e \frac{\partial S_r}{\partial v}$ in the expression derived here for Ω is shown to arise as a result of $S_r = f(s, v)$. In fact, Khalili et al. (2008) adopt a SWR model similar to Brooks and Corey (1964) extended to include hysteresis and accounting for the initial pore size distribution, but ignoring the effect of subsequent volume changes on the degree of saturation. Parameter H in the current approach is very different to the equivalent parameter in Khalili et al. (2008), where in effect $1/H = \Omega$. In the proposed approach, this would be equivalent to assuming that the modulus governing the effect of suction on the direct strains is the same as the modulus governing the effect of volumetric strains on the volumetric water content.

Finally, similar differences can be observed between parameter H in the current approach and in Sheng et al. (2003). Moreover, it should be noted that the equivalent parameter to H in Sheng et al. (2003) includes a term relating to plasticity. Parameter ω is the same in the two approaches. On the contrary, in the equivalent to parameter Ω in Sheng et al. (2003) the term $e \frac{\partial S_r}{\partial v}$ has been ignored and Sheng et al. (2003) clearly state that the degree of saturation if a function of suction only.

Although comparison with the literature cannot be exhaustive, the main differences are that in the approach described in this paper: (a) a clear distinction is made between the moduli controlling the effect of matric suction on direct strains and the effect of net stress on the volumetric water content, and (b) it is explicitly shown that the Governing Equations need to be consistent with the SWR model, as this will affect individual terms in the equation obtained for parameter Ω. This highlights the fact that when extending the capabilities of a numerical tool, e.g. of a Finite Element code, to model coupled consolidation in unsaturated soils, it is not merely sufficient to implement Governing Equations found in the literature, but the Equations need to be consistent with the particular SWR and constitutive models used, including the stress variables adopted.

At full saturation it is expected that:

$$\theta_w = -\varepsilon_{vol} \tag{20}$$

for a compression-positive sign convention. This implies that $\Omega = -1$ and $\omega = 0$. However, for the general case in Table 2 (Case 2; $S_r = f(s, v)$), when $s = 0$ kPa and $S_r = 1$:

$$\Omega = -1 - e\frac{\partial S_r}{\partial v} \tag{21}$$

and:

$$\omega = n \cdot \frac{\partial S_r}{\partial s} \tag{22}$$

and therefore it is required that both $\frac{\partial S_r}{\partial s}$ and $\frac{\partial S_r}{\partial v}$ are equal to zero at full saturation. Both differentials are usually assumed to be equal to zero for compressive pore water pressures or, depending on the SWR model, for suctions s lower than the air-entry value, s_{air}. Nonetheless, the slope of the SWR curve may not be equal to zero at $s = 0$ or $s = s_{air}$, depending not only on the SWR model employed but also on the model parameters used. For example, Figure 1 (a) illustrates four SWR curves generated by Tinjum et al. (1997) using the Van Genuchten (1980) expression with the model parameters a and n indicated in the legend of the figure. Parameter m was taken equal to $1 - 1/n$. The variation of the respective $\frac{\partial S_r}{\partial s}$ with suction is shown in Figure 1 (b). It can be seen that, depending on the model parameters employed, the partial differential $\frac{\partial S_r}{\partial s}$ is not always sufficiently small at s=0 to be considered equal to zero. Consequently, although parameters Ω and ω should ideally tend to -1 and 0, respectively, as full saturation is approached, this may not always be the caser and an abrupt change may occur at the interface between fully saturated and unsaturated conditions. This is further discussed in the conceptual model presented later. This abrupt transition from one condition to the other is a consequence of the SWR model used and is common to all Governing Equations where $\omega = n\frac{\partial S_r}{\partial s}$. It should not be confused with the transition from net stresses in unsaturated conditions to effective stresses in fully saturated conditions. The choice of net stress here was made so that the Governing Equations are consistent with the constitutive models available in the computer code ICFEP (Potts & Zdravkovic, 1999) into which the equations were implemented, as explained in Tsiampousi et al. (2016).

Table 2: Expressions for parameters Ω, ω and H

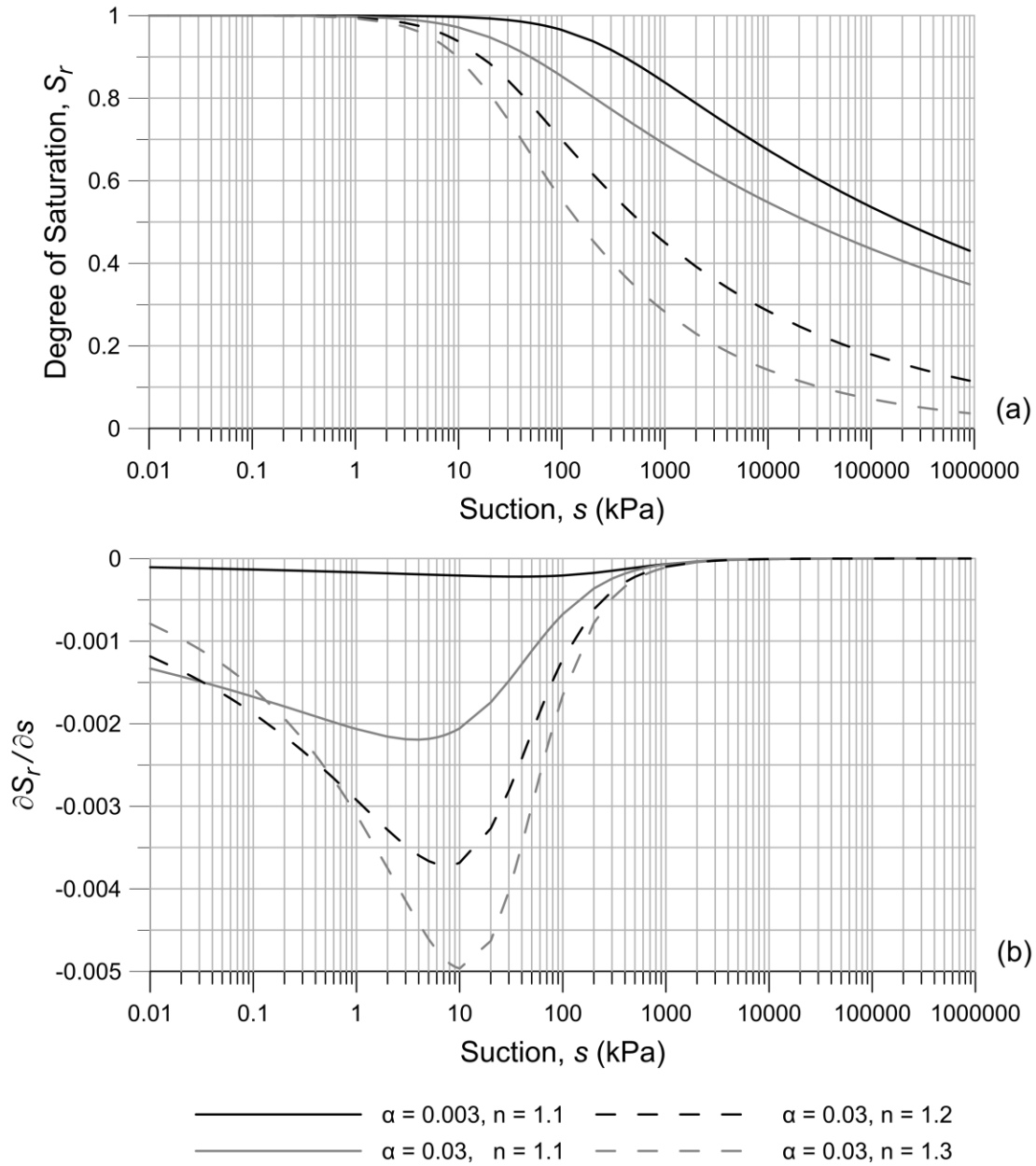| Case	Parameter Ω	Parameter ω	Parameter H
Case 1 $S_r = f(s)$	$\Omega = -S_r$	$\omega = n \cdot \frac{\partial S_r}{\partial s}$	$H = -\dfrac{3}{\frac{1}{v_0} \cdot \frac{\partial v}{\partial s}}$
Case 2 $S_r = f(s, v)$	$\Omega = -S_r - e\frac{\partial S_r}{\partial v}$	$\omega = n \cdot \frac{\partial S_r}{\partial s}$	$H = -\dfrac{3}{\frac{1}{v_0} \cdot \frac{\partial v}{\partial s}}$

Figure 1: (a) Soil water retention curves (SWR curves) generated by Tinjum et al. (1997) using the van Genuchten (1980) expression; (b) $\frac{\partial s_r}{\partial s}$ versus s for the same SWR curves. Parameters a and n in the legend refer to the van Genuchten (1980) parameters, while parameter m was taken equal to 1-1/n.

Table 3: Comparison of expressions for parameters Ω, ω and H obtained here to literature

|			
Current approach	$\Omega = -S_r - e\dfrac{\partial S_r}{\partial v}$	$\omega = n \cdot \dfrac{\partial S_r}{\partial s}$	$H = -\dfrac{3}{\dfrac{1}{v_0} \cdot \dfrac{\partial v}{\partial s}}$
Gatmiri et al. (1998) Note: formulated in terms of net stress $(\sigma - u_a)$; the equation for the air phase has been ignored here	$S_r - n\dfrac{\partial S_r}{\partial(\sigma - u_a)}C$ C being the elasticity matrix	$n \cdot \dfrac{\partial S_r}{\partial s} - n \cdot \dfrac{\partial S_r}{\partial(\sigma - u_a)}m_1$ m_1 being a constant	$-\dfrac{1 + e_0}{\dfrac{\partial e}{\partial s}}$
Khalili et al. (2008) Note: formulated in terms of Bishop's effective stress $\sigma' = \sigma_{net} - \chi s$; the equation for the air phase has been ignored here	$\psi = S_r - n\dfrac{\partial S_r}{\partial \varepsilon_{vol}}$ ψ being the effective stress parameter in Bishop's effective stress Although this expression is similar to Ω in the current approach, it is not explicitly related to the case where $S_r = f(s, v)$. In fact, it appears that $S_r = f(s)$.	$n \cdot \dfrac{\partial S_r}{\partial s}$	$\dfrac{1}{\psi}$ The effect of suction on direct strains is the same as the effect of volumetric strains on the volumetric water content.
Sheng et al. (2003) Note: formulated in terms of Bishop's effective stress $\sigma' = \sigma_{net} - \varphi(S_r)s$, i.e. $\psi = \varphi(S_r)$; the equation for the air phase has been ignored here	$-S_r$ $S_r = f(s)$ in Sheng et al. (2003)	$n \cdot \dfrac{\partial S_r}{\partial s}$	Related to $\dfrac{1}{\varphi(S_r)}$ and includes a term for plasticity If plasticity is ignored, the effect of suction on direct strains appears to be the same as the effect of volumetric strains on the volumetric water content.
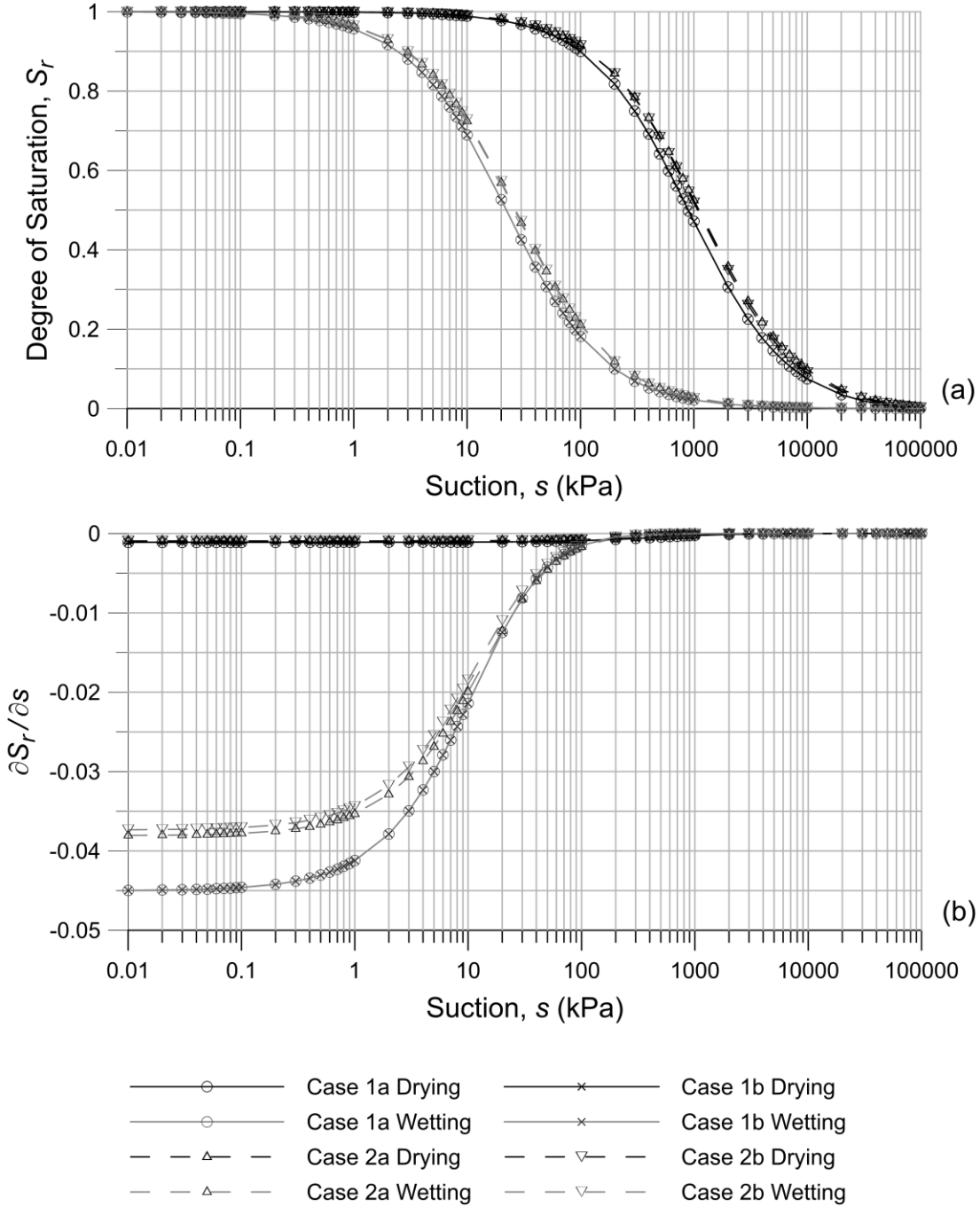
Figure 2: (a) Soil water retention curves (SWR curves); (b) $\frac{\partial s_r}{\partial s}$ versus s for Case 1a, 1b, 2a and 2b

Variation of parameters Ω, ω and H with suction

The variation of parameters Ω, ω and H with suction is subsequently presented. Each of the two cases in Table 2 has been coupled with two scenarios regarding the variation of specific volume with suction (see also Table 4): (a) in the first scenario the variation is similar to the BBM, i.e. κ_s in Equation 19 is constant, and (b) in the second scenario κ_s varies as a function of the degree of saturation, S_r (i.e. $\kappa_s = g(S_r)$) according to the expression proposed by Tsiampousi et al. (2013):

$$\kappa_s = \chi \cdot (S_r)^{\omega_s} \tag{23}$$

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where χ and ω are fitting parameters (see also Table 4). To facilitate the discussion, the hysteretic SWR model by Tsiampousi et al. (2013) was used to determine the functions $S_r = f(s)$ and $S_r = f(s, v)$. The model equations and respective parameters are summarised in Table 4. Similar model parameters were used in Tsiampousi et al. (2013) to reproduce the behaviour of an artificial soil consisting of 70% HPF4 silt, 20% speswhite kaolin and 10% London clay, which was tested by Jotisankasa (2005). Note that although the SWR model accounts for the effect of specific volume on the retention behaviour of the soil, by setting parameter ψ in Table 4 to zero this effect is removed (i.e. $S_r = f(s)$). The SWR curves obtained for each case and scenario examined are illustrated in Figure 2 (a). Figure 2 (b) illustrates the slope of the SWR curves (i.e. $\frac{\partial S_r}{\partial s}$) plotted versus suction. Evidently, Cases 1a and 1b produce the exact same hysteretic SWR curve. Cases 2a and 2b produce hysteretic SWR curves which are similar in between themselves but distinct when compared to Cases 1a and 1b. Figure 3 presents the associated drying/wetting compression curves. In this latter figure, Cases 1a and 2a, for which κ_s is constant, coincide and, additionally, exhibit no hysteresis. On the contrary, Cases 1b and 2b, for which κ_s is a function of the degree of saturation, S_r, produce hysteretic compression curves on drying and wetting, which are also distinct in between themselves, as the corresponding SWR curves are distinct (see fig. 2a for Cases 1b and 2b), one employing a simpler function (i.e. $S_r = f(s)$) than the other (i.e. $S_r = f(s, v)$). The variation of parameters Ω, ω and $1/H$ with suction are shown in Figure 4 to Figure 6.
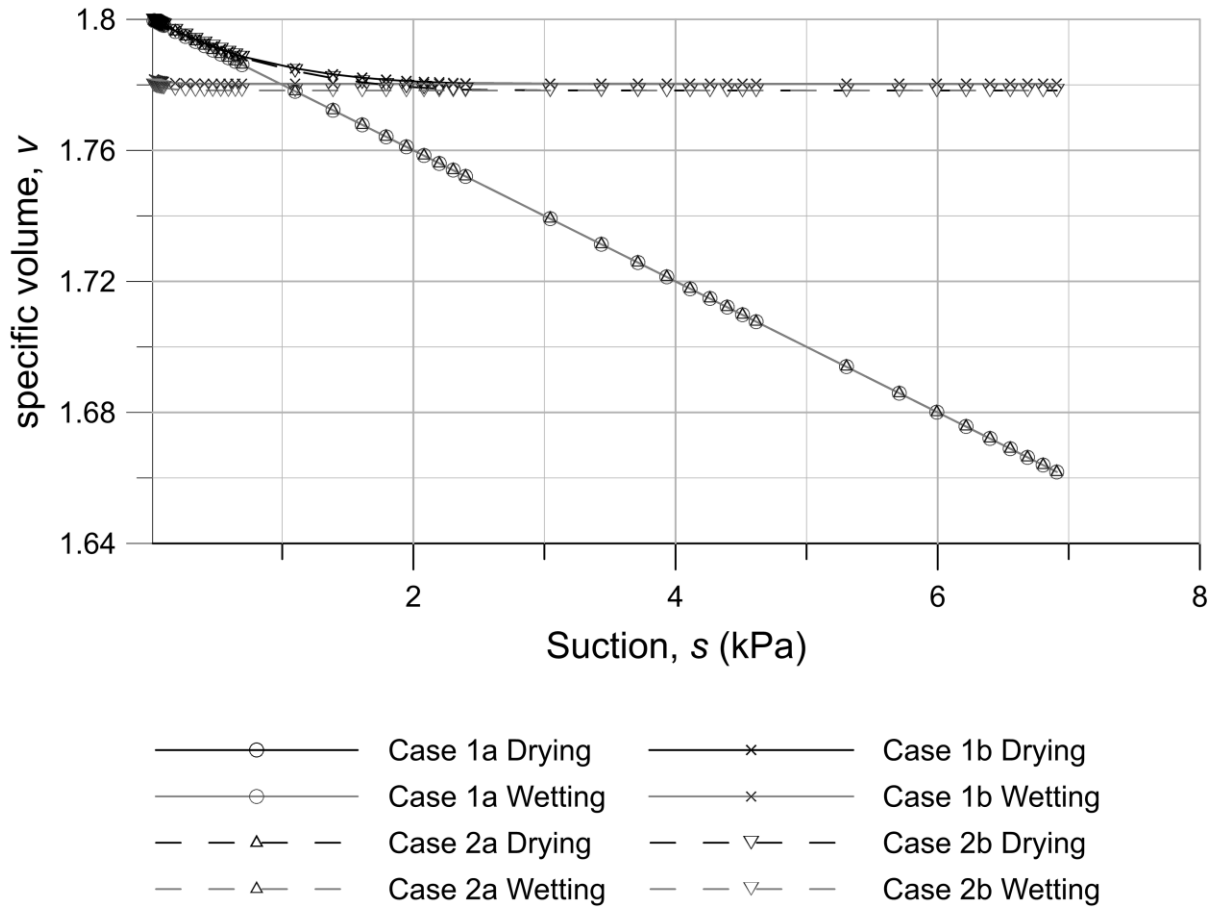

Figure 3: Drying/wetting compression curves for Case 1a, 1b, 2a and 2b

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Equations (Tsiampousi et al. (2013)	
$$\kappa_s = \chi \cdot (S_r)^{\omega_s}$$	(23)
where χ and ω are fitting parameters.	
$$S_{r,pr}^{dr,wet} = \frac{1 - \frac{1}{s_0^*} \cdot s^*}{1 + \alpha_{d,w} \cdot s^*}$$	(24)
$$\frac{\partial S_{r,pr}^{dr,wet}}{\partial s^*} = \frac{\frac{1}{s_0^*} + \alpha_{d,w}}{\left(1 + \alpha_{d,w} \cdot s^*\right)^2}$$	(25)
where $S_{r,pr}$ is the degree of saturation on the primary SWR curve, bearing the superscript dr for drying and wet for wetting;	
α is a fitting parameter bearing the subscript d for drying and w for wetting;	
$s^* = (v-1)^\psi \cdot (s - s_{air})$ is the combined suction, s being the current matric suction, v being specific volume and ψ being a fitting parameter;	
$s_0^* = (v-1)^\psi \cdot (s_0 - s_{air})$ and;	
s_{air}, s_0 are the air-entry value of suction and the suction at which $S_r = 0$, respectively.	

Parameters				
	Case 1a $S_r = f(s)$ $\kappa_s = constant$	**Case 1b** $S_r = f(s)$ $\kappa_s = g(S_r)$	**Case 2a** $S_r = f(s,v)$ $\kappa_s = constant$	**Case 2b** $S_r = f(s,v)$ $\kappa_s = g(S_r)$
s_{air}	0 kPa	0 kPa	0 kPa	0 kPa
s_0	100,000 kPa	100,000 kPa	100,000 kPa	100,000 kPa
ψ	**0**	**0**	**0.75**	**0.75**
α_d	0.0011	0.0011	0.0011	0.0011
α_w	0.0045	0.0045	0.0045	0.0045
χ	0.02	0.02	0.02	0.02
ω_s	**0**	**4**	**0**	**4**
$v_{initial}$	1.8	1.8	1.8	1.8

As the SWR curves in Figure 2 (a) are all hysteretic, so are the Ω and ω curves obtained. Parameter Ω appears in Equation 10 and provides a link between changes of volumetric strain, ε_{vol}, and volumetric water content, θ_w. The variation of parameter Ω with suction for Case 1a is shown in Figure 4 (a). The parameter exhibits a maximum absolute value at low suction levels, where air is present within the soil pores in the form of occluded bubbles and can flow with bulk water. Its value continuously decreases in absolute terms with increasing suction (drying curve shown in black in Figure 4 (a)) and becomes zero when continuity of bulk water ceases. On the wetting curve (shown in grey in Figure 4 (a)), parameter Ω steadily increases in absolute terms with decreasing suction and acquires its maximum absolute value of 1 as full saturation is approached. It should be noted that this is particular to the SWR model adopted, which does not allow for values of S_r smaller than 1 to be maintained when wetting to full saturation, as is often the case in laboratory experiments.

The hysteresis observed implies that the application of the same change in ε_{vol} will affect the volumetric water content θ_w by a different amount for a soil element which has been undergoing wetting in its recent history than for an element undergoing drying, even if both elements experience the same suction level. This is to be expected, as the amount of bulk water present and potentially available to flow within a soil element depends on its degree of saturation and void ratio.

The above observations are common for all cases and scenarios examined, as demonstrated in Figure 4 (b). Whereas Cases 1a and 1b coincide in Figure 4 (b) both on drying and on wetting, Cases 2a and 2b produce curves which are distinct between themselves and when compared to Cases 1a and 1b. The small difference between Cases 2a and 2b is due to the slight difference in the corresponding SWR curves (see Figure 2), but also to the term $e \frac{\partial S_r}{\partial v}$ in Equation 21 (see also Table 2). This term is defined from the SWR curve and the constitutive model used to simulate soil behaviour. This emphasises the need to adjust parameter Ω in the governing equation both to the SWR and the constitutive models adopted.

Similar observations can be made for the variation of parameter ω with suction, s. This parameter also appears in Equation 10 and provides a link between suction changes and volumetric water content, θ_w. As shown in Figure 5 (a) for Case 1a, this is clearly hysteretic. Both on drying and on wetting the maximum absolute value corresponds to low suction levels (and high degrees of saturation), implying, that suction changes produce larger variations in the volumetric water content when the degree of saturation and, therefore, the amount of bulk water available to flow within the soil pores, is higher. Nonetheless, as discussed in the previous section, ω should ideally tend to zero as full saturation is approached for a smooth transition from full to partial saturation to occur. This is clearly not the case in the SWR model by Tsiampousi et al. (2013); the partial differential $\frac{\partial S_r}{\partial s}$ (see eq. 25 in Table 4) is not zero at $s^* = 0$, i.e. when $s = s_{air}$. This is a particularity of the equation used to reproduce the primary drying and wetting paths (eq. 24 in Table 4) and implies that an abrupt change in the governing equations for fully saturated and unsaturated conditions occurs.
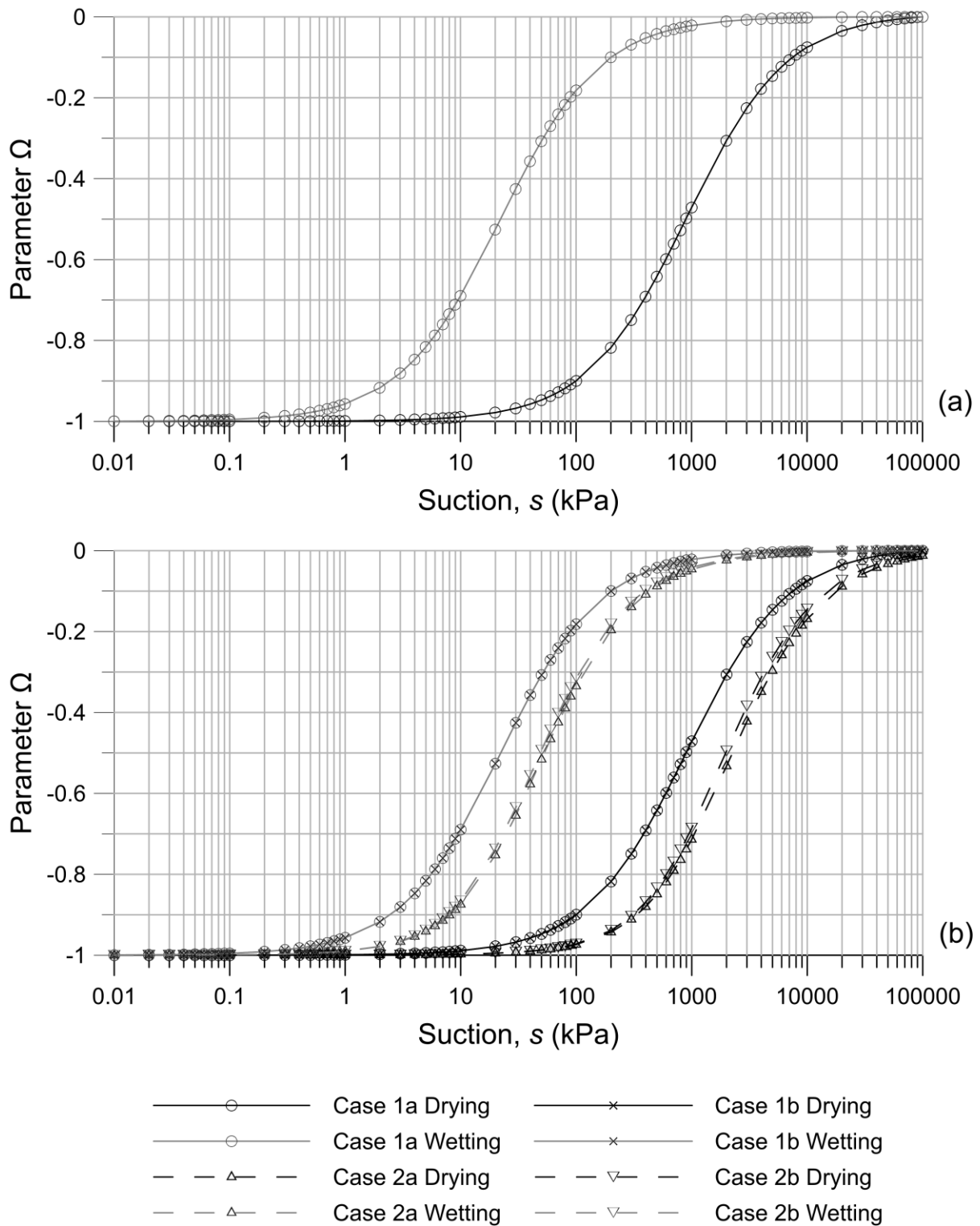
Figure 4: Variation of parameter Ω (a) for Case 1a; (b) for all cases

With reference to Figure 5 (a), on drying, parameter ω reduces in absolute terms with increasing suction and eventually becomes zero, reflecting the diminishing effect of suction changes on the variation of volumetric water content, as bulk water retreats in smaller voids and eventually loses its continuity. On subsequent wetting, ω gradually increases from zero, following a distinct path back to full saturation, where it obtains its maximum absolute value.

This is significantly larger than the maximum absolute value observed on the drying ω curve, as the primary wetting curve is significantly steeper than the primary drying curve at low suctions (parameter ω depends heavily on the slope of the SWR curve defined as $\partial S_r/\partial s$ – see also Table 2 and fig. 2b). This reflects the fact that in the low suction range (lower than 100 kPa in the particular example examined), the degree of saturation is changing significantly faster for the primary wetting curve although its actual value is smaller than for the primary drying curve, as evident from Figure 2 (a) and (b).
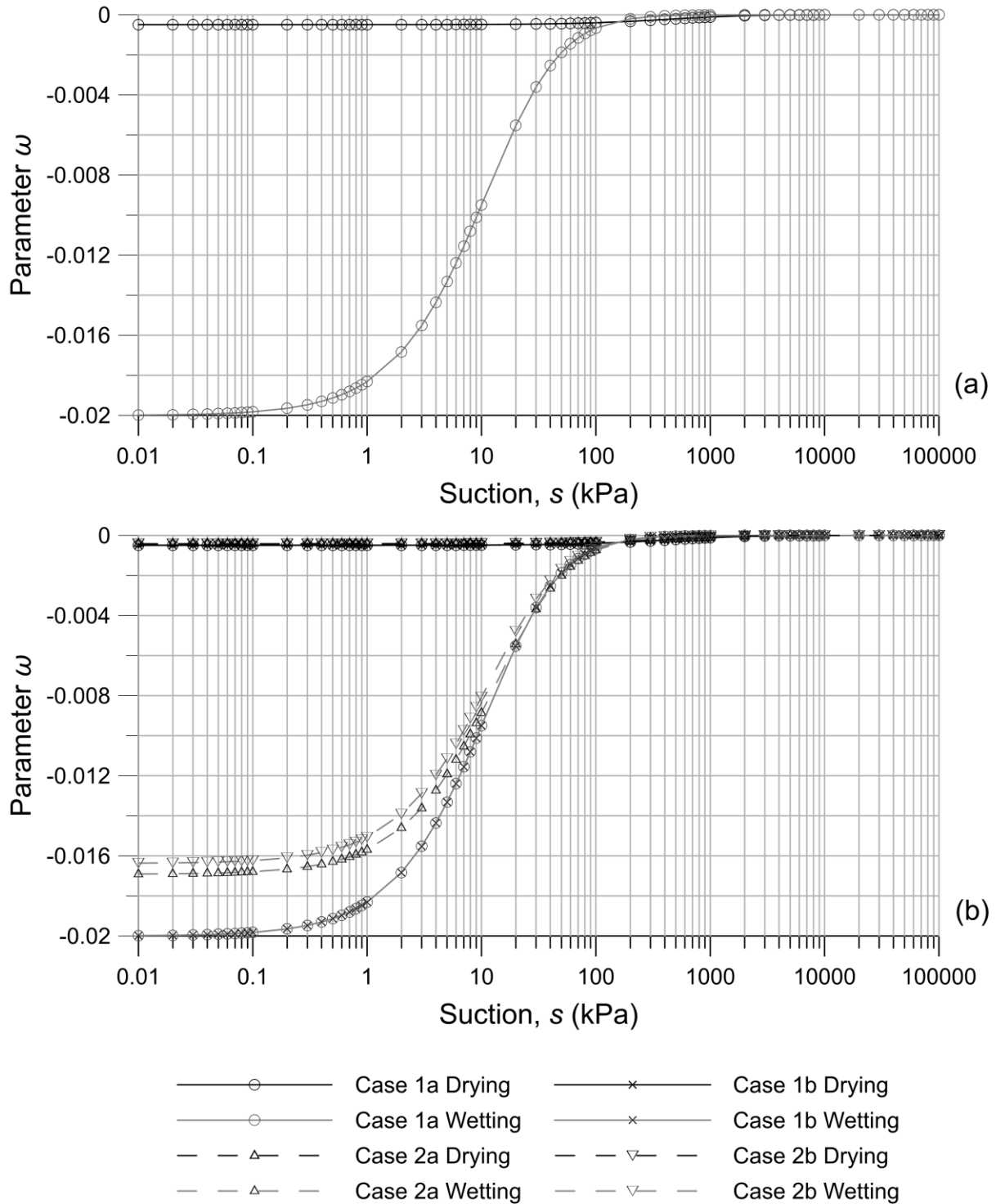
Figure 5: Variation of parameter ω (a) for Case 1a; (b) for all cases

The above observations are common to all cases and scenarios examined, as demonstrated in Figure 5 (b). Although the four ω curves corresponding to drying in this figure do not exhibit any noticeable difference, the difference between the four ω curves is significant on wetting. With reference to wetting, Cases 1a and 1b coincide similar to the Ω variation, whereas Cases 2a and 2b produce distinct curves, reflecting the effect of porosity n (see Table 2 and Figure 3) on the simulated behaviour. Once again, the need to adjust the parameters in the governing equations, in this case parameter ω, to the constitutive model adopted in addition to the SWR model is highlighted.

Figure 6: Variation of parameter 1/H (a) for Case 1a; (b) for all cases

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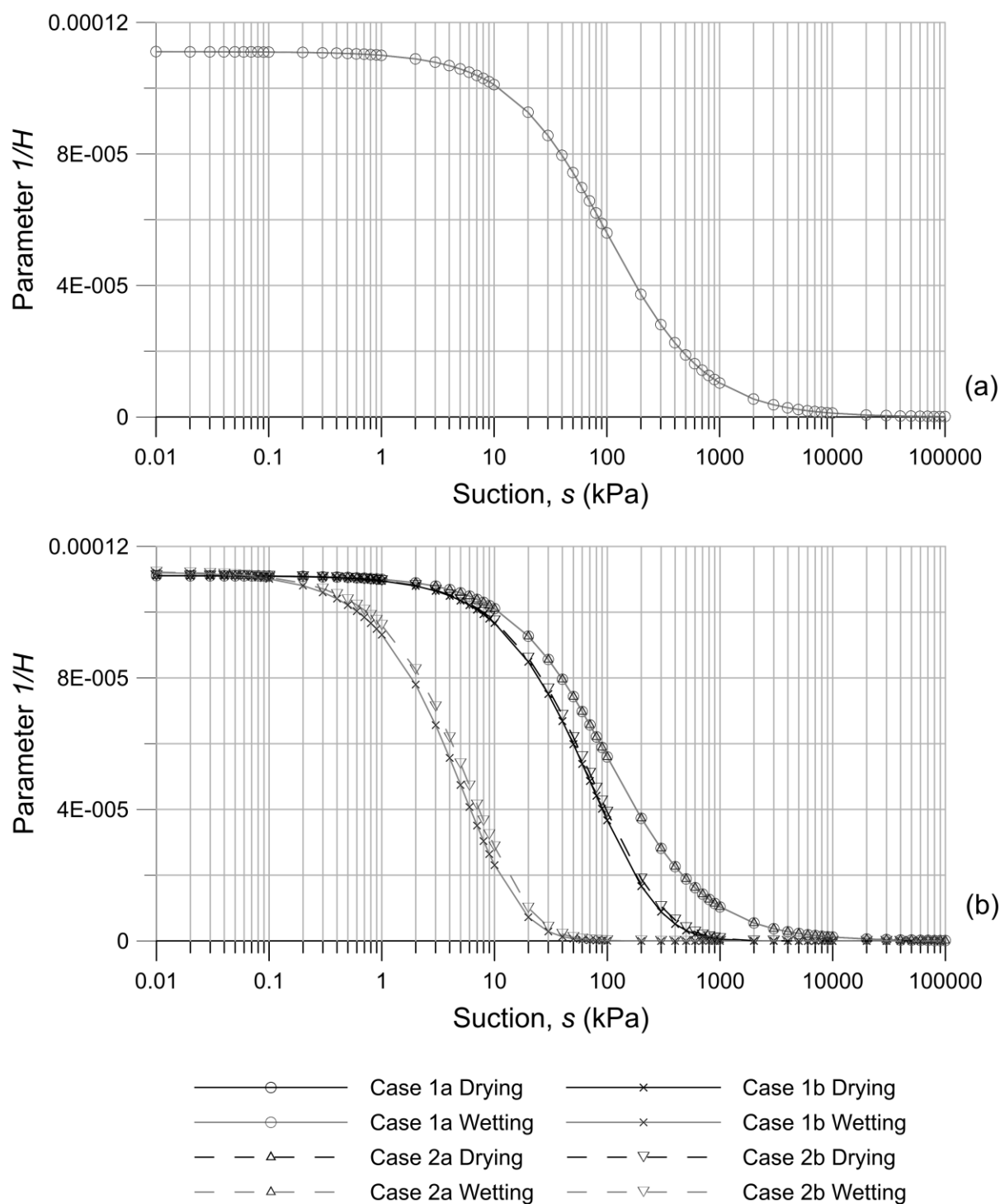
The variation of the term $1/H$ with suction is shown in Figure 6 (a) for Case 1a. The term appears in Equation 1 and defines the effect of changing suction on the direct strains ε_x, ε_y and ε_z and therefore on the volume of soil. From Figure 6 (a) it can be deducted that suction increase (i.e. drying) contributes to soil stiffness increase, as the term $1/H$ approaches zero. Changes in suction will have a gradually smaller effect on the strains ε_x, ε_y and ε_z, reflecting the stabilising effect of menisci forming at interparticle contacts. On subsequent wetting, stiffness is gradually reduced, as pores are flooded with water and menisci disappear. For Case 1a where κ_s is constant, the curve obtained on wetting coincides with the curved followed on drying: the partial differential $\frac{\partial v}{\partial s}$ in Equation 18 (see Table 2) is common for drying and wetting as the drying/wetting curve for Case 1a exhibits no hysteresis (see Figure 3). Furthermore, the drying/wetting curve for Case 1a coincides with the drying/wetting curve for Case 2a (Figure 3), as discussed above and, therefore, the two produce the same $1/H$ variation with suction, as evident in Figure 6 (b). On the contrary, Cases 1b and 2b produce distinct curves (Figure 6 (b)) when compared to Cases 1a and 2a and also when compared in between themselves, reflecting the differences exhibited by the corresponding drying/wetting curves in Figure 3. Additionally, the curves for Cases 1b and 2b are hysteretic, reflecting the hysteresis of the drying/wetting curves (Figure 3). Again, parameter H needs to be adjusted to the particular constitutive model adopted to reproduce soil behaviour.

Conceptual model

The above discussion emphasises how the behaviour of an unsaturated soil reflects aspects of its water content. It is possible to extend the above to a conceptual model for the coupled hydro-mechanical behaviour of unsaturated soils. The conceptual model presented below draws heavily on the work of White et al. (1970) and divides the soil into the four principle zones illustrated in Figure 7. Zone 3 is further subdivided into two zones of behaviour.

To facilitate the description of the conceptual model, Case 1b is further considered and the corresponding SWR curve is repeated in Figure 8 (a). As the SWR curve can be obtained experimentally, it is convenient to try and identify the principle zones from its shape. These have been marked on Figure 8 (a). The zones have been additionally marked on Figure 8 (b) illustrating the variation of the slope of the SWR curve with suction, and on Figure 9 to Figure 11, which illustrate, respectively, the variation of parameters Ω, ω and H with suction for Case 1b.

On drying, the soil remains fully saturated within zone 1 and no air is present as shown in Figure 7 (a). The behaviour of the soil within this zone is governed by conventional soil mechanics.

Within zone 2 air is present in the soil pores in the form of occluded bubbles, having come out of solution. Air may have also started to penetrate into the soil forming air-boundaries but will have not yet penetrated past the outer-most soil particles, as shown in Figure 7 (b). The

degree of saturation is smaller than but very close to unity (Figure 8 (a)).

In zone 3, air will have penetrated significantly into the soil. It is reasonable to identify the boundary between zones 2 and 3 as the air-entry value of suction, s_{air}. Nonetheless, in numerous SWR models (e.g. Van Genuchten, 1980; Fredlund & Xing, 1994; Gallipoli et al., 2003; Li, 2005; Tarantino, 2009; Pedroso & Williams, 2010), s_{air} is not explicitly a model parameter, while in others (e.g. Tsiampousi et al., 2013) desaturation occurs at suction s_{air} for modelling purposes (i.e. $S_r = 1$ at $s = s_{air}$). In the example used here, it was assumed that $s_{air} = 0$ kPa and both primary curves correspond to $S_r = 1$ at s = 0 kPa. Setting the boundary between zones 2 and 3 at the air-entry value of suction would not be appropriate in this case, as it would eliminate zone 2. Instead, the boundary between the two zones is set here at the point where the degree of saturation becomes visibly smaller than 1.

Zone 3 can be further divided in two zones, A and B. Zone 3A distinguishes the situation where, although the air phase is continuous from any point at which it is present within the soil back to an air boundary, it is not continuous all the way across the element (Figure 7 (c)). On the contrary, the water phase is continuous across the element and free to flow in all directions. The switch to zone 3B occurs when the air phase becomes continuous across the element (see Figure 7 (d)), while the water phase is also still continuous. Note that Figure 7 (d) illustrates a two-dimensional slice through a three-dimensional element. Although in the figure it appears that the water phase has become discontinuous this is not actually the case. The switch from zone 3A to 3B is assumed to occur at the point of inflection of the SWR curve when plotted on a semi-logarithmic plane (Figure 8 (a)).

Once the soil element is desaturated to the point that there can be no further flow of water, the model enters zone 4 (Figure 7 (e)). The degree of saturation reaches its residual value. In the SWR model by Tsiampousi et al. (2013) used as an example here (Figure 8 (a)), this is assumed to be zero (Figure 8 (a)) but it is appreciated that other SWR models may allow for a certain residual value of S_r larger than 0 to be maintained. Despite exhibiting a practically zero slope (Figure 8 (b)) from earlier on, changes in S_r are visible up until $s = 100,000$ kPa when the SWR curve is plotted in a semi-logarithmic plane (Figure 8 (a)). Therefore, the onset of zone 4 is assumed at this particular value of suction.

On subsequent wetting, zone 4 extends to smaller suctions than those marking the onset of this zone on drying; S_r is practically zero up until $s = 4,000$ kPa (Figure 8 (a)), from where it increases steadily as wetting continues. The above value of suction marks the boundary between zones 4 and 3 on wetting. The point of inflection of the wetting curve on the semi-logarithmic plane marks the transition between sub-zones 3A and 3B (Figure 8 (a)) and again is obtained at a smaller suction than the corresponding one for drying. Finally, zone 2 is reached at a low suction where the degree of saturation becomes practically 1. Again, this is a simplification of the SWR model used as an example here; there is no particular reason why S_r should return to 1 on wetting to full saturation, whether this corresponds to zero suction or to the air-entry value of suction defined on drying. Indeed, there is sufficient experimental evidence to support the opposite.
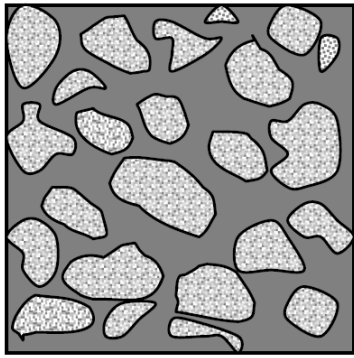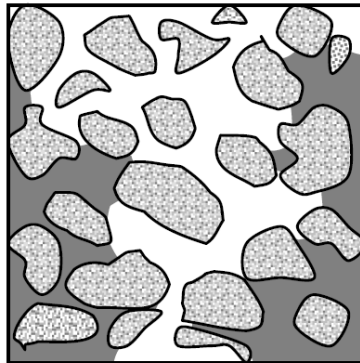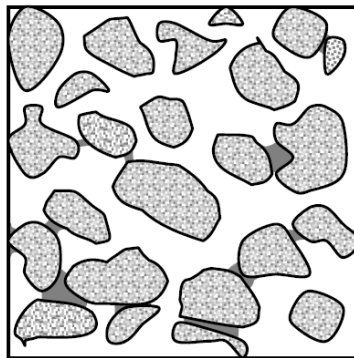
Zone 1   (a)
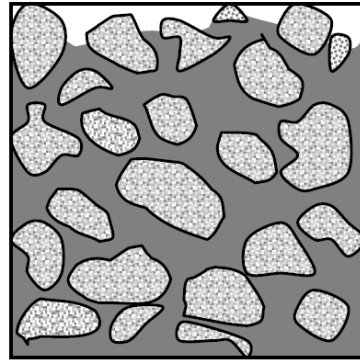
Zone 2   (b)
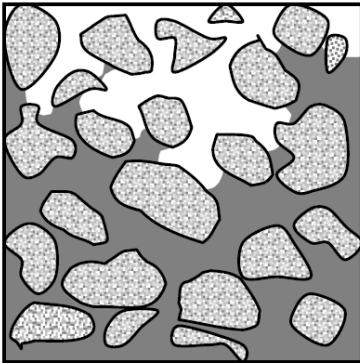
Zone 3A   (c)

Zone 3B   (d)

Zone 4   (e)

Figure 7: Conceptual zones of behaviour (a) zone 1; (b) zone 2; (c) zone 3A; (d) zone 3B; (e) zone 4 (after Smith, 2003)
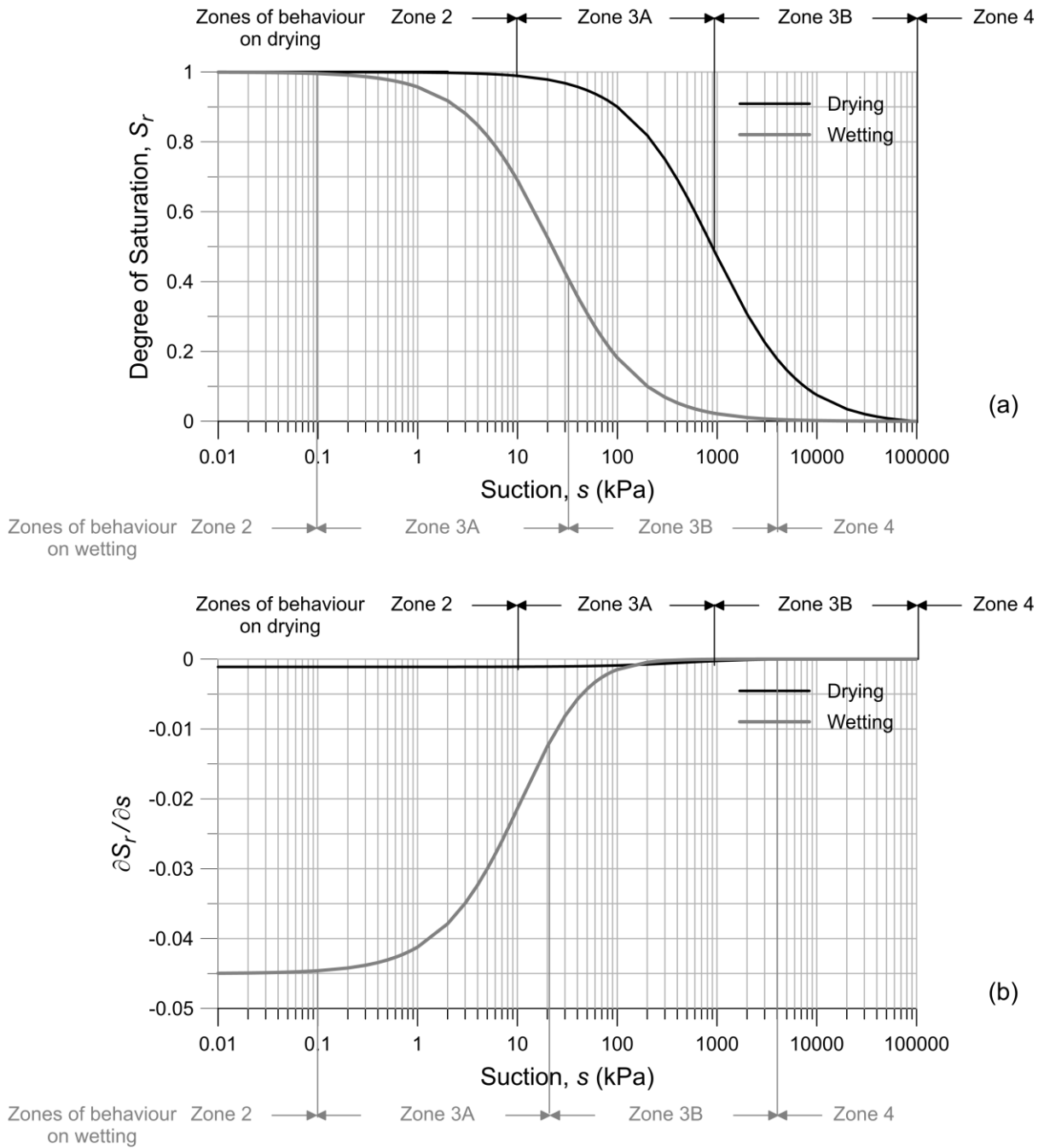
Figure 8: Conceptual zones of behaviour on drying and on wetting for Case 1b: (a) SWR curve; (b)
$$\frac{\partial S_r}{\partial s} \text{ versus } s$$

The zones of behaviour as identified on the SWR curve in Figure 8 (a) are also marked on Figure 9 to Figure 11, illustrating the variation of the corresponding parameters Ω, ω and H with suction. It can be observed that within zone 2, parameter Ω (Figure 9) exhibits a maximum absolute value of 1, both for drying and wetting. This implies that changes in volumetric strain are equal to changes in the volumetric water content and any air bubbles flow readily with bulk water. Within zone 3A, the absolute value of parameter Ω is smaller than 1, indicating that changes in volumetric water content are smaller than the associated changes in volumetric strain, despite the fact that the water phase is continuous and able to flow freely if the soil element in Figure 7 is subjected to external compression. As air has penetrated the larger pores, these are compressible and deform under the applied

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compression. Additionally, water flow is restricted through the smaller pores to which the water phase has retreated. Within zone 3B, both water and air are continuous and both will flow if the element is subjected to a change in volume. However, it is to be expected that air will flow much more readily than water, so the value of parameter Ω, while still greater than zero in absolute value, will increasingly tend towards zero. Within zone 4, where water is present only in the form of menisci, flow of water is impossible and parameter Ω is zero.
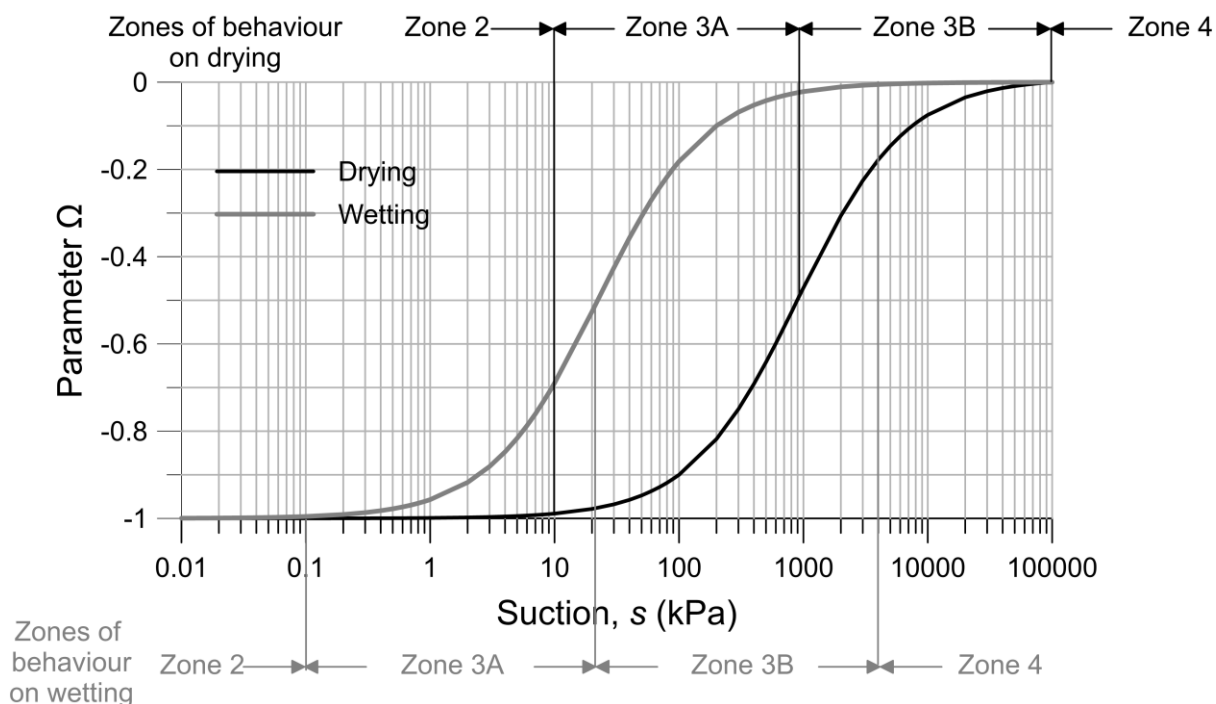

Figure 9: Variation of parameter Ω with suction for Case 1b and zones of behaviour on drying (in black) and wetting (in grey)

With reference to Figure 10, within zone 2 parameter ω exhibits its maximum absolute value both for drying and wetting, implying that suction changes produce the largest changes in volumetric water content in this zone. Indeed, this is the reason why the SWR curve is usually plotted in a semi-logarithmic plane. The reduced absolute values of parameter ω within zone 3 indicate that as air penetrates the larger pores and water retreats to smaller voids, suction changes have a progressively smaller effect on the volumetric water content changes produced, until zone 4 is encountered, where water is only present in the form of menisci and the volumetric water content no longer changes with changes in suction. As pointed out in the previous section, the absolute value of parameter ω is larger on wetting than on drying for suction values smaller than about 100 kPa, as the slope of the primary wetting curve in Figure 8 (b) becomes larger than that of the primary drying curve.
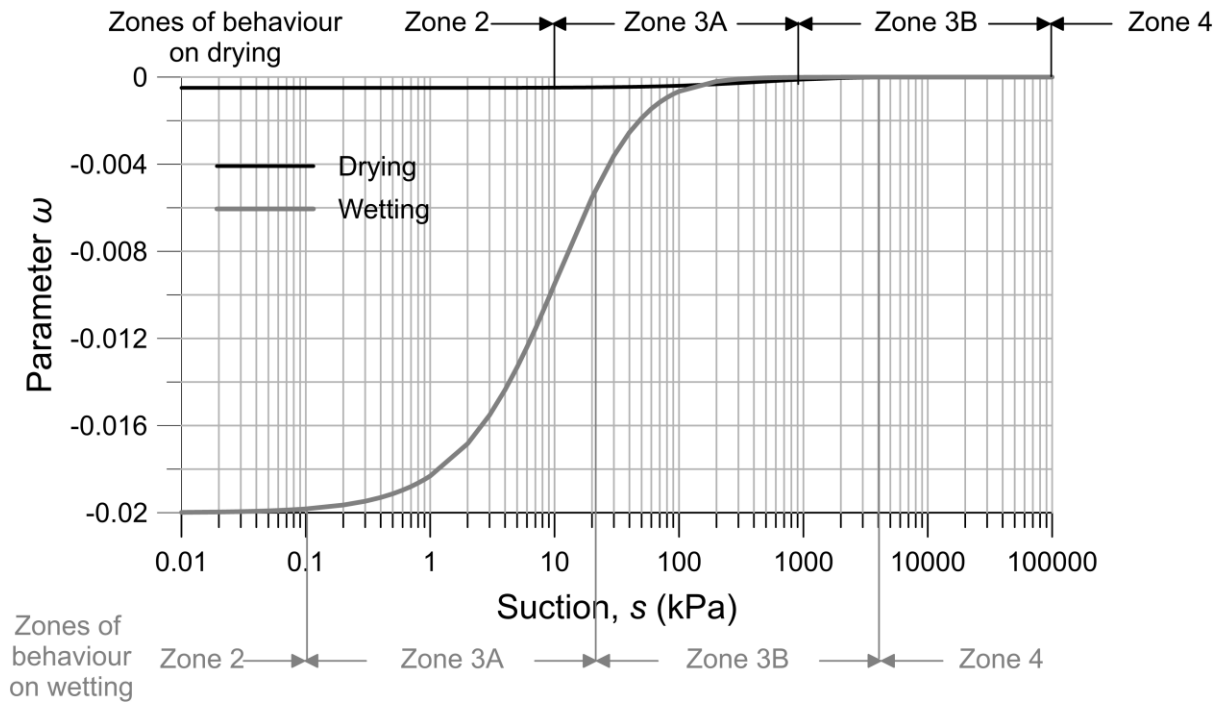
Figure 10: Variation of parameter ω with suction for Case 1b and zones of behaviour on drying (in black) and wetting (in grey)

Finally, with reference to Figure 11, the soil response to suction changes becomes progressively stiffer from zone 2 to zone 4, and it is visibly stiffer on wetting than on drying. This justifies the irreversibility observed in the volume of soil samples subjected to cycles of drying and wetting. Within Zone 2, where air is present in the pores in the form of occluded bubbles, there is little effect of suction on the value of stiffness. This is so both for drying and wetting. The soil response to changes in suction is significantly stiffer within zones 3A and 3B, as menisci form at interparticle contacts. In the particular example examined, where $\kappa_s = g(S_r)$, the partial differential $\frac{\partial v}{\partial s}$ tends to zero as S_r tends to zero, thus H in Equation 18 tends to infinity in zone 4. Although this prediction is clearly model dependent, it reflects the situation often observed in the laboratory where changes in suction no longer produce changes in the volume of a soil sample.

The boundaries between the different zones of the conceptual model are shifted to the left (i.e. lower suctions) for wetting in comparison to drying. This is a result of the hydraulic hysteresis exhibited by most unsaturated soils. To avoid this additional complication, it is reasonable to explain and model soil behaviour with reference to the degree of saturation rather than the value of suction experienced by soils. Nonetheless, customarily, methods of analysis such as the Finite Element Method employ nodal pore water pressures as the primary variables (degrees of freedom). In this case, expressing parameters Ω, ω and H as functions of S_r helps overcome the complexity arising from the hydraulic hysteresis.
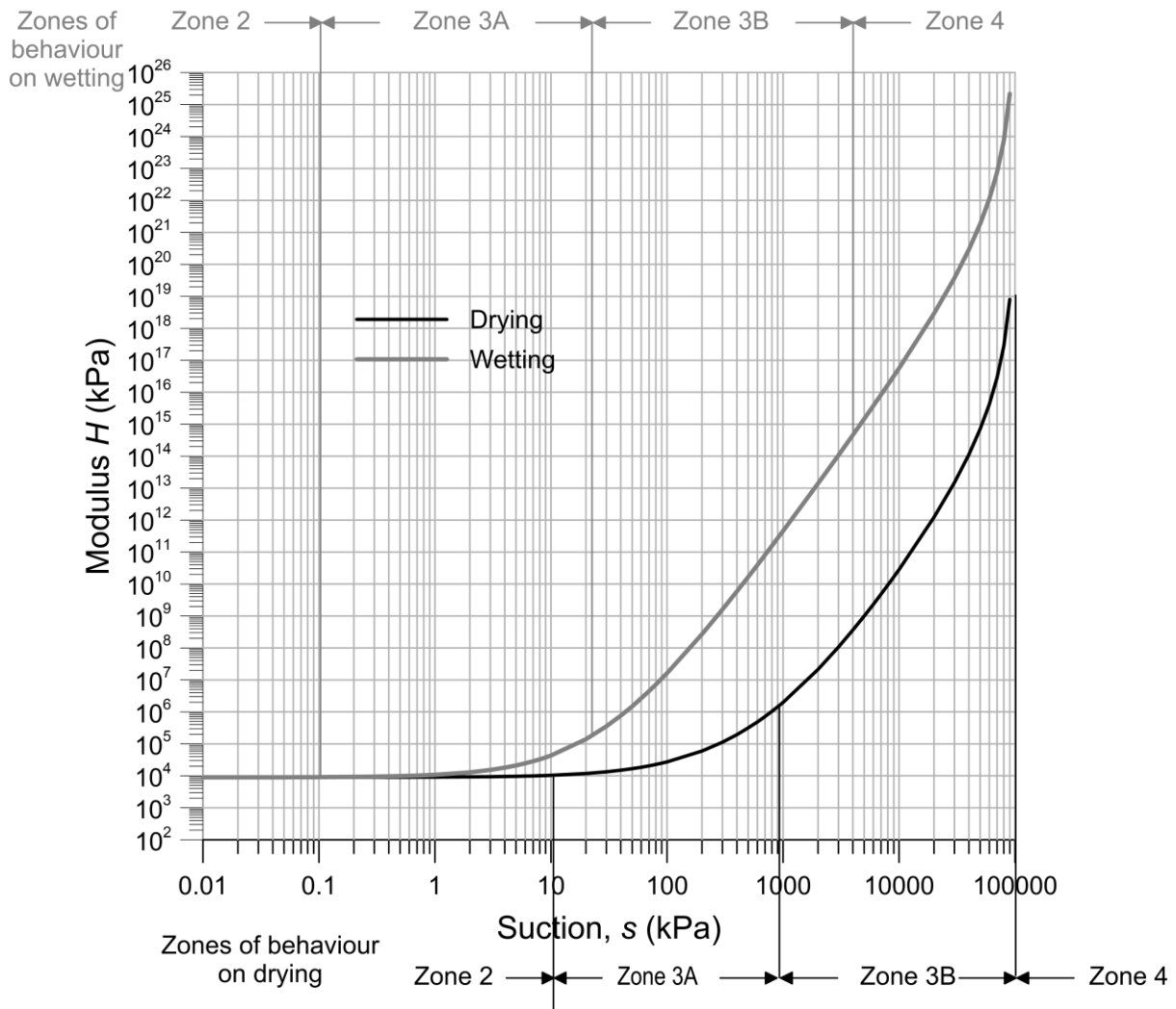
Figure 11: *Variation of parameter H with suction for Case 1b and zones of behaviour on drying (in black) and wetting (in grey)*

Conclusions

The governing equations presented here are an extension of those proposed by Darkshanamurthy et al. (1984) and Wong et al. (1998). The air is assumed to be continuous and at atmospheric pressure and its flow is not explicitly modelled (i.e. air is not an additional primary variable in the governing equations).

The main difference with the earlier work of Wong et al. (1998) is that a clear distinction between the modulus controlling the effect of matric suction on direct strains and the modulus governing the effect of net stress on the volumetric water content is made, consistent with unsaturated soil behaviour. Therefore, four moduli are required in order to formulate the Governing Equations, rather than three as in Wong et al. (1998).The Governing Equations were rewritten in a form containing three additional parameters, Ω, ω and H, which are required to extend coupled consolidation to unsaturated soil states and are related to the four moduli. Parameter Ω governs the effect of volume changes on the volumetric water content and parameter ω governs the effect of suction changes on the volumetric water content. Parameter H represents the stiffness exhibited on changes of suction.

These additional parameters are not soil constants; however, it was shown that, for modelling purposes, their variation with suction and degree of saturation can be obtained in a consistent manner. It was actually shown that parameters Ω, ω and H, and therefore, the Governing Equations need to be adjusted to the SWR and constitutive models used to reproduce soil behaviour. In particular, it was shown parameter Ω is given by a different equation when the SWR curve is expressed in the three-dimensional space $s - S_r - v$, so that S_r is a function f of both suction s and specific volume v, i.e. $S_r = f(s, v)$, in comparison to the case where $S_r = f(s)$. The effect of the constitutive and SWR model on parameters Ω, ω and H was demonstrated considering a total of four cases, where the compressibility and SWR relationships differ slightly but result in measurable differences in the variation of parameters Ω, ω and H with suction and therefore degree of saturation.

A conceptual model was presented to illustrate how the behaviour of unsaturated soils reflects aspects of its water and air content and how this is captured by the variation of parameters Ω, ω and H. Four zones of behaviour were identified: in zone 1 the soil is fully saturated, in zone 2 air is present in the form of occluded bubbles, in zone 3A and B the water and the water and air, respectively, exist in a continuous form and in zone 4 no further flow of water is possible. The boundaries between the different zones were identified on the SWR curve and more specifically on a hysteretic but idealised model. The various assumptions and simplifications of the model were therefore transferred to the conceptual model. Nonetheless, the same principles can be applied to any SWR model and experimental data to explain aspects of the complex unsaturated soil behaviour in a simplified manner.

Most importantly, it was shown that the variation of parameters Ω, ω and H in the governing equations also reflects aspects of the water and air content in the soil, even though the flow of air is not explicitly modelled. Also, parameters Ω, ω and H may vary in a hysteretic manner with suction. For this reason it is particularly convenient to express these parameters as functions of the degree of saturation S_r, as in the proposed approach.

As the two governing equations need to be satisfied simultaneously in a coupled analysis, it is convenient to solve them employing a numerical method. The theoretical requirements of equilibrium, compatibility, constitutive behaviour and boundary conditions can then be satisfied. The FE element formulation of the governing equations proposed here is presented in Tsiampousi et al. (2016).

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Appendix A

The work of Biot (1941)

Biot (1941) based his approach on the following assumptions: (1) the soil is isotropic (anisotropy is considered in Biot, 1941 but has been neglected here for simplicity); (2) the stress-strain relationships are linear and (3) reversible under final equilibrium conditions (i.e. assumption of linear elasticity); (4) strains remain small (Biot does not explain the range of strains he considers "small"); (5) the water contained within the pores is incompressible but (6) may contain air bubbles (clearly the assumption is made that the mixture of water and occluded bubbles is incompressible); (7) Darcy's law is applicable. Extending Hooke's law for an isotropic elastic body to include the effect of the water pressure, for the case of a soil containing air bubbles, Biot (1941) proposes that:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z) + \frac{\sigma}{3H} \tag{A.1}$$

and similar for ε_y and ε_z, and:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{A.2}$$

and similar for γ_{yz} and γ_{xz}, where:

- ε_x, ε_y and ε_z are direct strains in x, y and z directions, respectively;

- γ_{xy} is the shear strain acting on the x-plane in the y-direction (similar for γ_{yz} and γ_{xz});

- σ_x, σ_y and σ_z are the direct total stresses in the x, y and z directions, respectively;

- τ_{xy} is the shear stress acting on the x-plane in the y-direction (similar for τ_{yz} and τ_{xz});

- σ is the increment of pore water pressure;

- E is Young's modulus of the soil structure and μ is Poisson's ratio;

- G is the shear modulus, $G = {E}/{2(1 + \mu)}$ and;

- H is an additional physical constant.

H is similar to Young's modulus and shows the effect of changing pore water pressure on the direct strains in the soil (as opposed to the shear strains on which it has no effect).

Equations A.1 and A.2 give the constitutive relationship for the soil structure, linking the six strain components to the six total stress components in the soil and the pore water pressure. Biot (1941) highlights that, additionally to the above constitutive relationship, the dependence of the increment of water content on the stress variables has to be considered. He proposes that:

$$\theta = \frac{1}{3H_1}\left(\sigma_x + \sigma_y + \sigma_z\right) + \frac{\sigma}{R} \tag{A.3}$$

where:

- θ is the increment of volumetric water content;

- H_1 and R are two physical constants describing the effect of changes in applied total stress (H_1) and incremental pore water pressure (R) on the water content.

Equation A.3 describes the constitutive relation for the water phase and together with Equations A.1 and A.2 set the basis for coupled consolidation analysis. It can be deduced that changes in the applied total stress and in the pore water pressure will affect both the strains within the soil and the water content.

Based on the assumption of the existence of a potential energy for the soil, Biot (1941) argues that the physical constant H_1 in the constitutive relationship for the water phase (Equation A.3) equals H, the additional physical constant in the constitutive relationship for the soil structure (Equation A.1). He demonstrates that indeed $H = H_1$ for the particular stress condition where $\sigma_x = \sigma_y = \sigma_z$ and $\tau_{xy} = \tau_{yz} = \tau_{xz} = 0$.

Biot (1941) rearranged Equation A.3 to:

$$\theta = \alpha\varepsilon + \frac{\sigma}{Q} \tag{A.4}$$

where:

$$\alpha = \frac{2(1+\mu)}{3(1-2\mu)} \cdot \frac{G}{H} \tag{A.5}$$

$$Q = \frac{1}{R} - \frac{\alpha}{H} \tag{A.6}$$

and explained that the coefficient α relates the changes in the volume of water to the volume changes of a soil element which is compressed under $\sigma = 0$ (i.e. full drainage). The coefficient Q is a measure of the volume of water that can flow into or out of the soil element under a given pore water pressure change when its volume is kept constant.

For the case of a fully saturated soil, standard tests show that the volume change of a soil sample is equal to the volume of water extracted from the sample. Biot (1941) interpreted this observation as equivalent to adopting $Q = \infty$ and $\alpha = 1$, in which case Equation A.4 is reduced to:

$$\theta = \varepsilon \tag{A.7}$$

indicating that the increment of volumetric water content, θ, is equal to the volumetric strain, ε. It is expected that for elastic deformations, compression of a unit volume of soil is

associated with a decrease in volumetric water content, as the volume of voids and therefore the volume of water reduces, whereas dilation of a unit volume of soil is followed by an increase in volumetric water content. Consequently, dilative strains in Equation A.7 should be positive. Indeed, Biot (1941) states that "ε represents the volume increase of the soil per unit initial volume". Nonetheless, conventionally compression is considered positive in Soil Mechanics, in which case the above equation can be written as:

$$\theta = -\varepsilon \tag{A.8}$$

It can therefore be concluded that for fully saturated conditions:

$$\alpha = -1 \tag{A.9}$$

The work of Darkshanamurthy et al. (1984)

Darkshanamurthy et al. (1984) and Fredlund & Rahardjo (1993) among others extended Biot's coupled theory to unsaturated soils containing air in a continuous form (rather than in occluded bubbles as in Biot, 1956). In particular, Darkshanamurthy et al. (1984) altered Biot's equation for the soil structure (Equation A.1) in order to make use of the two stress variables net stress and matric suction:

$$\varepsilon_x = \frac{(\sigma_x - u_a)}{E_1} - \frac{\mu}{E_1}(\sigma_y + \sigma_z - 2u_a) + \left(\frac{u_a - u_w}{H_1}\right) \tag{A.10}$$

and similar for ε_y and ε_z, where:

- u_a is the pore air pressure (gauge pressure: above atmospheric);

- $(u_a - u_w)$ is the matric suction, u_w being the pore water pressure;

- E_1 is Young's modulus of the soil structure with respect to the net stresses $(\sigma_x - u_a)$, $(\sigma_y - u_a)$ and $(\sigma_z - u_a)$ and;

- H_1 is the elastic modulus of the soil structure with respect to $(u_a - u_w)$, similar to $3H$ in the equivalent equation by Biot (1941) (Equation A.1).

For the water phase Darkshanamurthy et al. (1984) employed the following equation:

$$\theta_w = \frac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{3H_1'} + \frac{u_a - u_w}{R_1} \tag{A.11}$$

where H_1' and R_1 are similar to H_1 and R in the equivalent equation by Biot (1941) (Equation A.3), and θ_w is the volumetric water content (referred to as net inflow and outflow of water in the original paper).

Darkshanamurthy et al. (1984) rewrote θ_w as:

$$\theta_w = \frac{\beta}{3}\varepsilon + \gamma(u_a - u_w) \tag{A.12}$$

where:

$$\beta = \frac{E_1}{H_1}\frac{1}{1 - 2\mu} = \frac{2G}{H_1}\frac{1 + \mu}{1 - 2\mu} \tag{A.13}$$

$$\gamma = \frac{1}{R_1} - \frac{\beta}{H_1'} \tag{A.14}$$

However, this implies that the authors have interchanged H_1 and H_1': rearranging Equation A.10 to obtain $(\sigma_x - u_a)$ (and similar for $(\sigma_y - u_a)$ and $(\sigma_z - u_a)$), and substituting into Equation A.11:

$$\theta_w = \frac{E_1}{3H_1'}\varepsilon\left(\frac{1}{1 - 2\mu}\right) + \left(\frac{1}{R_1} - \frac{\beta}{H_1'}\right)\cdot(u_a - u_w) \tag{A.15}$$

If $H_1 = H_1'$ then:

$$\beta = \frac{E_1}{H_1}\cdot\frac{1}{1 - 2\mu} = \frac{E_1}{H_1'}\cdot\frac{1}{1 - 2\mu} \tag{A.16}$$

Then and only then can Equation A.12 be obtained. It is not clear whether the assumption that $H_1 = H_1'$ was made intentionally in the original paper, similar to Biot (1941) employing $H_1 = H$.

Darkshanamurthy et al. (1984) carry on to explain the constitutive relation for the air phase (three phase flow), but as air is assumed herein to be continuous and at atmospheric pressure, the details of this relation are omitted.

The work of Wong et al. (1998)

Wong et al. (1998) described the numerical implementation of the coupled formulation presented by Biot (1941) and extended by Darkshanamurthy et al. (1984). They rewrote the constitutive relationships for the soil structure and the water phase in a different form and with slightly different notation. Equation A. 10 was rewritten in the general form:

$$\varepsilon_{ij} = \frac{1 + \mu}{E}\sigma_{ij}^n - \frac{\mu}{E}\sigma^n\delta_{ij} + \frac{u_a - u_w}{H}\delta_{ij} \tag{A.17}$$

where:

- ε_{ij} are strain components;

- σ_{ij}^n are net stress components;

- σ^n is the mean net stress equal to $\sum \sigma^n_{ii}/3$;

- u_w is the pore water pressure;

- u_a is the pore air pressure;

- E is Young's modulus and μ is Poisson's ratio for the soil skeleton;

- δ_{ij} is the Kronecker δ and

- H is the elastic modulus of the soil structure with respect to $(u_a - u_w)$. The same modulus was denoted as H_1 in Darkshanamurthy et al. (1984) and is equivalent to $3H$ in Biot (1941).

As explained above, Wong et al. (1998) state that σ^n is the mean net stress and is equal to $\sum \sigma^n_{ii}/3$. However, substituting $\sigma^n = \sum \sigma^n_{ii}/3$ into Equation A.17:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1+\mu}{E}\sigma^n_{xx} - \frac{\mu}{E}\cdot\frac{\sigma^n_{xx} + \sigma^n_{yy} + \sigma^n_{zz}}{3} + \frac{u_a - u_w}{H}\\
&= \left(\frac{1+\mu}{E} - \frac{\mu}{3E}\right)\sigma^n_{xx} - \frac{\mu}{E}\cdot\frac{\sigma^n_{yy} + \sigma^n_{zz}}{3} + \frac{u_a - u_w}{H}\\
&= \frac{3+2\mu}{E}\sigma^n_{xx} - \frac{\mu}{E}\cdot\frac{\sigma^n_{yy} + \sigma^n_{zz}}{3} + \frac{u_a - u_w}{H}
\end{aligned}
\tag{A.18}
$$

which is not equivalent to Equation A.10, as intended. Instead, if $\sigma^n = \sum \sigma^n_{ii}$, Equation A.17 is equivalent to Equation A.10, provided that $E = E_1$ and $H = H_1$.

Similar to the constitutive relation for the soil structure, Wong et al. (1998) use the constitutive relation given by Darkshanamurthy et al. (1984) (i.e. Equation A.12) for the water phase, but rewritten in the following form:

$$
\theta_w = \beta\varepsilon_v + \omega(u_a - u_w)
\tag{A.19}
$$

ε_v being the volumetric strain (i.e. equal to ε in Darkshanamurthy et al., 1984) and where:

$$
\beta = \frac{E}{H}\cdot\frac{1}{1-2\mu}
\tag{A.20}
$$

and:

$$
\omega = \frac{1}{R} - \frac{3\beta}{H}
\tag{A.26}
$$

It is clear that no distinction was made between modulus H in Equation A.17 and modulus H in Equation A.20.

For Equation A.19 to be equivalent to Equation A.12 of Darkshanamurthy et al. (1984), it is required that $3E = E_1$, $R = R_1$ and $H = H_1 = H'_1$, where E, R and H are moduli in Wong et

al. (1998) and E_1, R_1, H_1 and H_1' are moduli in Darkshanamurthy et al. (1984). Indeed, substituting β and γ from Equations A.13 and A.14 into Equation A.12:

$$
\begin{aligned}
\theta_w &= \frac{1}{3}\frac{E_1}{H_1}\frac{1}{1-2\mu}\varepsilon + \left(\frac{1}{R_1} - \frac{\beta}{H_1'}\right)(u_a - u_w) \\
&= \frac{1}{3}\frac{E_1}{H_1}\frac{1}{1-2\mu}\varepsilon + \left(\frac{1}{R_1} - \frac{\frac{E_1}{H_1}\frac{1}{1-2\mu}}{H_1'}\right)(u_a - u_w) \\
&= \frac{1}{3}\frac{E_1}{H_1}\frac{1}{1-2\mu}\varepsilon + \left(\frac{1}{R_1} - \frac{E_1}{H_1 H_1'(1-2\mu)}\right)(u_a - u_w)
\end{aligned}
\tag{A.22}
$$

Nonetheless, substituting β and γ from Equations A.20 and A.21 into Equation A.19:

$$
\theta_w = \frac{E}{H}\frac{1}{1-2\mu}\varepsilon + \left(\frac{1}{R} - \frac{3E}{H^2(1-2\mu)}\right)(u_a - u_w)
\tag{A.23}
$$

For Equation A.23 to be equivalent to A.22 it is required that:

$$3E = E_1$$

$$R = R_1 \tag{A.24}$$

$$H = H_1 = H_1'$$

However, as discussed above for Equation A.17 to be equivalent to Equation A.10, it is required that $E = E_1$ and $\sigma^n = \sum \sigma_{ii}^n$. Indeed, if $3E = E_1$ and $\sigma^n = \sum \sigma_{ii}^n/3$ as proposed by Wong et al. (1998):

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1+\mu}{\frac{E_1}{3}}\sigma_{xx}^n - \frac{\mu}{\frac{E_1}{3}}\cdot\left(\frac{\sigma_{xx}^n + \sigma_{yy}^n + \sigma_{zz}^n}{3}\right) + \frac{u_a - u_w}{H} \\
&= \frac{3(1+\mu)}{E_1}\sigma_{xx}^n - \frac{\mu}{E_1}\cdot\left(\sigma_{xx}^n + \sigma_{yy}^n + \sigma_{zz}^n\right) + \frac{u_a - u_w}{H} \\
&= \frac{3+2\mu}{E_1}\sigma_{xx}^n - \frac{3\mu}{E_1}\cdot\left(\sigma_{yy}^n + \sigma_{zz}^n\right) + \frac{u_a - u_w}{H}
\end{aligned}
\tag{A.25}
$$

which is not equivalent to Equation A.10. It is not clear whether Wong et al. (1998) intended to alter the equations by Dakshanamurthy et al. (1984).

Appendix B

As discussed in the text:

$$\theta_w = \Omega \varepsilon_{vol} + \omega(u_a - u_w) \qquad\text{(B.1)}$$

Therefore:

$$\frac{d\theta_w}{dt} = \Omega \frac{d\varepsilon_{vol}}{dt} + \omega \frac{d(u_a - u_w)}{dt} \qquad\text{(B.2)}$$

where:

$$\frac{d\varepsilon_{vol}}{dt} = -\frac{1}{v_0} \cdot \frac{dv}{dt} \qquad\text{(B.3)}$$

and:

$$d(u_a - u_w) = ds \qquad\text{(B.4)}$$

where s is the matric suction and v_0 is the specific volume, v, at the beginning of the increment.

It follows that:

$$\frac{d\theta_w}{dt} = \Omega \left(-\frac{1}{v_0} \cdot \frac{dv}{dt} \right) + \omega \frac{ds}{dt} \qquad\text{(B.5)}$$

The volumetric water content is also given by the following relationship:

$$\theta_w = n \cdot S_r \qquad\text{(B.6)}$$

and therefore, for the general case where $S_r = f(s, v)$:

$$\frac{d\theta_w}{dt} = n \cdot \frac{\partial S_r}{\partial t} + S_r \cdot \frac{\partial n}{\partial t} = n \cdot \left(\frac{\partial S_r}{\partial s} \cdot \frac{ds}{dt} + \frac{\partial S_r}{\partial v} \frac{dv}{dt} \right) + S_r \cdot \frac{\partial n}{\partial v} \cdot \frac{dv}{dt} \qquad\text{(B.7)}$$

Noting that:

$$n = \frac{e}{1 + e_0} \qquad\text{(B.8)}$$

e_0 being the void ratio at the beginning of the increment, Equation B.7 can be written as:

$$\frac{d\theta_w}{dt} = n \cdot \frac{\partial S_r}{\partial s} \cdot \frac{ds}{dt} + \frac{e}{1 + e_0} \cdot \frac{\partial S_r}{\partial v} \frac{dv}{dt} + S_r \cdot \frac{\partial n}{\partial v} \cdot \frac{dv}{dt} \qquad\text{(B.9)}$$

where:

$$\frac{\partial n}{\partial v} = \frac{1}{v_0} \tag{B.10}$$

Finally:

$$\frac{d\theta_w}{dt} = n \cdot \frac{\partial S_r}{\partial s} \cdot \frac{ds}{dt} - \left(e\frac{\partial S_r}{\partial v} + S_r\right) \cdot \left(-\frac{1}{1+e_0} \cdot \frac{dv}{dt}\right) \tag{B.11}$$

Equating Equations B.5 and B. 11, it is possible to calculate parameters Ω and ω:

$$\omega = n \cdot \frac{\partial S_r}{\partial s} \tag{B.12}$$

and:

$$\Omega = -S_r - e\frac{\partial S_r}{\partial v} \tag{B.13}$$

Clearly, both parameters are dependent on the soil-water retention curve (SWRC) employed and in the case where $S_r = f(s)$, and not $S_r = f(s, v)$ as above:

$$\Omega = -S_r \tag{B.14}$$

The negative sign in Equations B.13 and B.1 signifies that compression of a unit volume of soil causes a decrease in volumetric water content (see Eq. B.2).

Parameter H can now be calculated from Equation 11, as follows.

$$\omega = \frac{1}{R} - \frac{3\Omega}{H} \Rightarrow H = \frac{3\Omega}{\frac{1}{R} - \omega} \tag{B.15}$$

For the general case where $S_r = f(s, v)$:

$$\begin{aligned}
\frac{1}{R} &= \frac{\partial \theta_w}{\partial s} = \frac{\partial \theta_w}{\partial S_r}\left(\frac{\partial S_r}{\partial s} + \frac{\partial S_r}{\partial v}\frac{\partial v}{\partial s}\right) + \frac{\partial \theta_w}{\partial n}\frac{\partial n}{\partial v}\frac{\partial v}{\partial s} \\
&= n\left(\frac{\partial S_r}{\partial s} + \frac{\partial S_r}{\partial v}\frac{\partial v}{\partial s}\right) + S_r\frac{1}{v_0}\frac{\partial v}{\partial s}
\end{aligned} \tag{B.16}$$

It has already been shown that $\Omega = -S_r - e\frac{\partial S_r}{\partial v}$ and $\omega = n \cdot \frac{\partial S_r}{\partial s}$, therefore:

$$H = \frac{3\left(-S_r - e\frac{\partial S_r}{\partial v}\right)}{n\left(\frac{\partial S_r}{\partial s} + \frac{\partial S_r}{\partial v}\frac{\partial v}{\partial s}\right) + S_r\frac{1}{v_0}\frac{\partial v}{\partial s} - n \cdot \frac{\partial S_r}{\partial s}} = \frac{3\left(-S_r - e\frac{\partial S_r}{\partial v}\right)}{\frac{e}{v_0}\frac{\partial S_r}{\partial v}\frac{\partial v}{\partial s} + S_r\frac{1}{v_0}\frac{\partial v}{\partial s}} \tag{B.17}$$

and finally:

$$H = -\frac{3}{\frac{1}{v_0}\frac{\partial v}{\partial s}}$$ (B.18)

It can easily be shown that the above equation is also true for the case where $S_r = f(s)$.

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