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L^∞ Estimates on Trajectories Confined to a Closed Subset, for Control Systems With Bounded Time Variation

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Dedicated to R. Tyrrell Rockafellar

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Abstract The term 'distance estimate' for state constrained control systems refers to an estimate on the distance of an arbitrary state trajectory from the subset of state trajectories that satisfy a given state constraint. Distance estimates have found widespread application in state constrained optimal control. They have been used to establish regularity properties of the value function, to establish the non-degeneracy of first order conditions of optimality, and to validate the characterization of the value function as a unique solution of the HJB equation. The most extensively applied estimates of this nature are so-called linear L^∞ distance estimates. The earliest estimates of this nature were derived under hypotheses that required the multifunctions, or controlled differential equations, describing the dynamic constraint, to be locally Lipschitz continuous w.r.t. the time variable. Recently, it has been shown that the Lipschitz continuity hypothesis can be weakened to a one-sided absolute continuity hypothesis. This paper provides new, less restrictive, hypotheses on the time-dependence of the dynamic constraint, under which linear L^∞ estimates are valid. Here, one-sided absolute continuity is replaced by the requirement of one-sided bounded variation. This refinement of hypotheses is significant because it makes possible the application of analytical techniques based on distance estimates to important, new classes of discontinuous systems including some hybrid control systems. A number of examples are investigated showing that, for control systems that do not have bounded variation the w.r.t. time, the desired estimates are not in general valid, and thereby illustrating the important role of the bounded variation hypothesis in distance estimate analysis.

Keywords Differential Inclusions · State Constraints · Optimal Control

1 Introduction

Linear L^∞ estimates for control systems, which govern the distance of an arbitrary state trajectory from the set of state trajectories confined to a closed set A, have widespread applications in the theory of state constrained optimal control. They have been used to establish regularity properties of the value function (cf. [21], [2], [3], [4], [11], [14], [12]). They also have an important role in the formulation of hypotheses under which first order necessary conditions of optimality are non-degenerate (see e.g. [17], [18], [7]), under which the value function may be identified as the unique generalized solution of the Hamilton-Jacobi equation ([21], [15]), and to investigate sensitivity relations ([8], [13], [6]).

The question of what hypotheses need to be imposed, concerning the time-dependence of the underlying differential inclusion, in order for such estimates to be valid, is a delicate one. Suppose that A is a general closed set and an appropriate inward pointing condition is satisfied. Then a linear L^∞ estimate is valid if the time dependence is Lipschitz continuous. On the other hand, examples are known, showing that if the

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time dependence is assumed merely to be continuous, then linear estimates may fail to be valid.

Recently, it has been shown that the Lipschitz continuity hypothesis can be replaced by a weaker, one-sided absolute continuity hypothesis. In this paper we show that a further weakening of the hypotheses is possible; linear L^∞ estimates are derived for differential inclusions, whose time dependence has bounded variation from the left. This is significant, because it allows control systems whose time dependence involves two-sided discontinuities, fractional singularities, etc. Linear L^∞ estimates have been previously established by Frankowska and Rampazzo [14], under similar, but non-equivalent, hypotheses concerning the time-dependence of the differential inclusion, which also cover some of these cases. Properties of control systems with bounded variation time dependence are also explored in [16] and [23].

Consider the state-constrained differential inclusion, described as follows:

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) & \text{for a.e. } \; t \in [S,T] \\ x(t) \in A & \text{for all } \; t \in [S,T]\,, \end{cases}$$

in which $[S,T]$ is a given interval ($T > S$), $F(.,.) : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multifunction with closed, non-empty values and $A \subset \mathbb{R}^n$ is a given closed set.

Given a subinterval (possibly closed or left open) $I \subset [S,T]$, we shall refer to an absolutely continuous function $x(.) : I \to \mathbb{R}^n$ which satisfies $\dot{x}(t) \in F(t,x(t))$ a.e. as an F-trajectory (on I). An F-trajectory $x(.)$ on I is said to be 'feasible' (on I) if $x(t) \in A$ for all $t \in I$, and 'strictly feasible' (on I) if $x(t) \in \operatorname{int} A$ for all $t \in I$.

This paper concerns hypotheses under which estimates, and their refinements, of the following type are valid: given a ball $r_0\mathbb{B}$ in $\mathbb{R}^n$ there exists a constant $K > 0$ such that, for any F-trajectory $\hat{x}(.)$ on a closed subinterval $I \subset [S,T]$, emanating from $r_0\mathbb{B} \cap A$, we have

$$||\hat{x}(.) - x(.)||_{L^\infty(I)} \;\leq\; K \max_{t \in I} d_A(\hat{x}(t))\;,$$

for some *feasible* F-trajectory $x(.)$ with the same initial value. Such estimates are referred to as linear L^∞ distance estimates for F-trajectories confined to a closed set. They provide an upper bound on the distance of a given F-trajectory $\hat{x}(.)$ from the set of F-trajectories satisfying a state constraint, in terms of the expression $\max_{t\in I} d_A(\hat{x}(t))$, which can be interpreted as a measure of the state constraint violation by $\hat{x}(.)$.

Notation. For a given interval $[t_0,t_1] \subset \mathbb{R}$ the space $L^p([t_0,t_1];\mathbb{R}^n)$, $p = 1$ or $p = \infty$, is written briefly $L^p(t_0,t_1)$ or L^p. We denote the Lebesgue subsets of $[S,T]$ by $\mathcal{L}$. $\mathbb{B}$ denotes the closed unit ball in Euclidean space. The Euclidean distance is written $|\;.\;|$, $\operatorname{int} C$ denotes the interior of a set C in Euclidean space. We write $\operatorname{co} C$ for the convex hull of C. Take a closed set $D \subset \mathbb{R}^n$ and $x \in D$. The Clarke tangent cone to D at x is written $T_D(x)$ (cf. [10]). We denote by $\chi_D(.)$ the indicator function of D, that is the function taking value 1 on D and 0 elsewhere. $d_D(x)$ denotes the Euclidean distance of the point x from the set D, namely $\min_{y\in D} |x-y|$. For arbitrary non-empty closed sets in $\mathbb{R}^n$, D' and D, we denote by $d_D(D')$ the excess from set D to a set D':

$$d_D(D') \;:=\; \inf\{\beta > 0 \,|\, D' \subset D + \beta\mathbb{B}\}\;,$$

(alternatively referred to as the 'asymmetric Hausdorff distance' of the set D' from the set D.) The Hausdorff distance between the sets D and D' is written $d_H(D,D') := \max\{d_D(D'), d_{D'}(D)\}$.

Given a multifunction $G : D \rightsquigarrow \mathbb{R}^n$ and $x \in D$ (where D is closed), we define respectively the limit inferior and the limit superior (in the Kuratowski sense) of G at x to be (cf. [1], [22])

$$\liminf_{x' \xrightarrow{D} x} G(x') := \{v \in \mathbb{R}^n \;:\; \limsup_{x' \xrightarrow{D} x} \; d_{G(x')}(v) = 0\}\;,$$

$$\limsup_{x' \xrightarrow{D} x} G(x') \;:=\; \{v \in \mathbb{R}^n \;:\; \liminf_{x' \xrightarrow{D} x} \; d_{G(x')}(v) = 0\}\;.$$

The notation $x' \xrightarrow{D} x$ indicates consideration of convergent sequences $x' \to x$, all elements of which belong to D. An alternative, and often useful, characterization of the $\liminf$ operator on a set-valued function $G(.)$ is as follows: $v \in \liminf_{x' \xrightarrow{D} x} G(x')$ if and only if for every $\varepsilon > 0$ there exists $\eta > 0$ such that $(v + \varepsilon\mathbb{B}) \cap G(x') \neq \emptyset$ for every $x' \in (x + \eta\mathbb{B}) \cap D$.

2 Distance Estimates

This Section provides conditions for the validity of linear L^∞ distance estimates for the state constrained differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) & \text{for a.e. } t \in [S,T] \\ x(t) \in A & \text{for all } t \in [S,T]\,, \end{cases}$$

in which A is a given closed, non-empty subset of $\mathbb{R}^n$.

The distinctive feature of the first theorem is the unrestricitve nature of hypothesis (BVL) under which, together with hypotheses of a more standard nature, linear L^∞ estimates are asserted. (BVL) merely requires that the multifunction $F(t,x)$ has bounded variation from the left, w.r.t. the t variable, in a sense made precise in the hypothesis statement. (BVL) contrasts with the comparable hypothesis invoked in [5], requiring that $F(t,x)$ is absolutely continuous from the left w.r.t. the t variable.

Theorem 1 *Fix $r_0 > 0$. Assume that, for some constant $c > 0$ and some $k_F(.) \in L^1(S,T)$ and for $R := e^{c(T-S)}(r_0+1)$, the following hypotheses (H1), (H2), (CQ) and (BVL) are satisfied:*

(H1): $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, non-empty values, $F(.,x)$ is $\mathcal{L}$-measurable for all $x \in \mathbb{R}^n$, and

$$F(t,x) \subset c(1+|x|)\,\mathbb{B} \quad \textit{for all } (t,x) \in [S,T] \times \mathbb{R}^n\,.$$

(H2):

$$F(t,x') \subset F(t,x) + k_F(t)|x-x'|\,\mathbb{B} \quad \textit{for all } x,x' \in R\mathbb{B} \textit{ and a.e. } t \in [S,T]\,.$$

(CQ): For each $(t,x) \in [S,T] \times (R\mathbb{B} \cap \partial A)$,

$$\Big(\liminf_{(t',x') \xrightarrow{D} (t,x)} \operatorname{co} F(t',x') \Big) \cap \operatorname{int} T_A(x) \neq \emptyset\,,$$

where $D = [S,T] \times A$.

(BVL): $F(.,x)$ has bounded variation from the left uniformly over $x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}$ for some $\bar{\eta} > 0$, in the following sense: there exists a non-decreasing bounded variation function $\eta(.) : [S,T] \to \mathbb{R}$ (called a 'modulus of variation of $F(.,x)$') such that, for every $[s,t] \subset [S,T]$ and $x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}$,

$$F(s,x) \subset F(t,x) + (\eta(t) - \eta(s)) \times \mathbb{B}\,.$$

Then there exists a constant $K > 0$ with the following property:

Given any interval $[t_0,t_1] \subset [S,T]$, any F-trajectory $\hat{x}(.)$ on $[t_0,t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{c(t_0-S)}(r_0+1)-1\big)\mathbb{B}$, and any $\rho > 0$ such that

$$\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}\,,$$

we can find an F-trajectory $x(.)$ on $[t_0,t_1]$ such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \in \operatorname{int} A \quad \textit{for all } t \in (t_0,t_1]$$

and

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \leq K\,\rho\,. \tag{1}$$

The assertions of the theorem cover two cases, each of independent interest:

Case A: $\max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\} > 0$ ($\hat{x}(.)$ is *not* feasible).

In this case, an F-trajectory $x(.)$ with initial value $\hat{x}(t_0)$ and strictly feasible on $(t_0,t_1]$ exists, which satisfies the linear distance estimate

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \leq K\,\max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}\,.$$

(This follows from the theorem statement, after setting $\rho := \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}$.)

Case B: $\max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\} = 0$ ($\hat{x}(.)$ is feasible).

In this case, for arbitrary $\epsilon > 0$, there exists an F-trajectory $x(.)$, with initial value $\hat{x}(t_0)$ and strictly feasible on $(t_0, t_1]$ such that

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \;\leq\; \epsilon$$

(This follows from the theorem statement, after setting $\rho := \epsilon/K$.)

The next theorem is a variant on the preceding distance estimate, in which it is asserted that, if $F(t, x)$ has a two-sided bounded variation property w.r.t. the time variable (in place of the one-sided property required in Thm. 1), a linear L^∞ is still valid, under a less restrictive version of the inward pointing condition.

Theorem 2 *The assertions of Thm. 1 remain valid when assumptions (CQ) and (BVL) are replaced by the following hypotheses (CQ)′ and (BV):*

(CQ)′: For each $t \in [S,T)$, $s \in (S,T]$ and $x \in R\mathbb{B} \cap \partial A$,

$$\Big(\liminf_{x' \xrightarrow{A} x} \; \limsup_{t' \downarrow t} \; \text{co}\, F(t', x')\Big) \cap \text{int}\, T_A(x) \;\neq\; \emptyset\,,$$

$$\Big(\liminf_{x' \xrightarrow{A} x} \; \limsup_{s' \uparrow s} \; \text{co}\, F(s', x')\Big) \cap \text{int}\, T_A(x) \;\neq\; \emptyset\,.$$

(BV): $F(., x)$ has bounded variation uniformly over $x \in (\partial A + \bar{\eta}\mathbb{B}) \cap \mathbb{B}$ for some $\bar{\eta} > 0$, in the following sense: there exists a non-decreasing bounded variation function $\eta(.) : [S,T] \to \mathbb{R}$ such that, for every $[s,t] \subset [S,T]$ and $x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}$,

$$d_H(F(s, x), F(t, x)) \leq \eta(t) - \eta(s)\;.$$

3 Examples

In this section we take $[S,T] = [0,1]$.

Example 1. Choose an integer $N \geq 4$ and define the times

$$\tau_k = \frac{1}{N^k} \quad \text{for } k = 1, 2, \ldots \tag{2}$$

Consider the 3-vectors

$$v_0 = (\frac{1}{N+1}, 0, 0), \;\; v_1 = (1, \frac{N+1}{N-1}, 0), \;\; v_2 = (1, -\frac{N+1}{N-1}, 0)\,. \tag{3}$$

Now define the multifunction $F(.) : [0,1] \rightsquigarrow \mathbb{R}^3$

$$F(t) \;:=\; \begin{cases} \{v_0\} \cup \{v_1\} \cup \{v_2\} & \text{for } t = 0 \\ \{v_0\} \cup \{v_1\} & \text{for } t \in (\tau_i, \tau_{i-1}] \\ \{v_0\} \cup \{v_2\} & \text{for } t \in (\tau_{i+1}, \tau_i] \end{cases} \tag{4}$$

$i = 1, 3, \ldots$ Take A to be the set

$$A = \{x \in \mathbb{R}^3 \mid \; x_2 - x_1 + x_3 \leq 0,\; -x_2 - x_1 + x_3 \leq 0\}\,. \tag{5}$$

The following information about $(F(.), A)$ is proved in [5].

Take any $\alpha \in (0,1)$ and $r_0 > 0$. Then the integer parameter N in (2) and (3) can be chosen such that $(F(.), A)$, defined by (4) and (5), have the following property: given any $K > 0$ and $\delta > 0$, there exists an interval $[t_0, t_1] \subset [0.1]$ of length not greater than δ and an F-trajectory $\hat{x}(.)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap (r_0\mathbb{B})$ such that

$$\max_{t \in [t_0,t_1]} d_A(\hat{x}(t)) > 0$$

and

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \;>\; K \,\Big|\max_{t \in [t_0,t_1]} d_A(\hat{x}(t))\Big|^\alpha\,,$$

for all feasible F-trajectories $x(.)$ on $[t_0, t_1]$ with initial state $\hat{x}(t_0)$.

According to the preceding statement, a linear L^∞ estimate is not valid for the choice (4) and (5) of $(F(.), A)$. Nor even a weaker, Hölder-type distance estimate is valid, in which $|\max_{t\in[t_0,t_1]} d_A(\hat{x}(t))|^\alpha$ replaces $|\max_{t\in[t_0,t_1]} d_A(\hat{x}(t))|$, for α an arbitrary constant in the interval $(0, 1)$.

The statement is consistent with Thm. 1. Indeed, for $(F(.), A)$ given by (4) and (5), hypotheses (H1), (H2) and (CQ) are satisfied. Hypothesis (BVL) is not satisfied however. This is because, for this choice of $(F(.), A)$ in which $F(.)$ is a function of t alone, condition (BVL) can be equivalently expressed:

$$\sup\{\sum_{j=0}^{M-1} \max_{v\in F(s_j)} d_{F(s_{j+1})}(v)\} < \infty\,, \tag{6}$$

in which the supremum is taken over all partitions $\{s_0 = 0, s_1, \ldots, s_M = 1\}$ of $[0, 1]$. But, for any integer M and partition

$$\{s_0 = 0,\ s_1 = \tau_{M-1}, s_2 = \tau_{M-2}, \ldots, s_M = \tau_0\}\,,$$

in which the τ_j's are given by (2), it can be deduced from the analysis in [5] that

$$\sum_{j=0}^{M-1} \max_{v\in F(s_j)} d_{F(s_{j+1})}(v) > M\,.$$

Since M is an arbitrary integer, (6) cannot be valid.

The example of $(F(.), A)$ given by (4) and (5) is revealing because it tells that the assertions of Thm. 1 are not valid under hypotheses (H1), (H2), (CQ) alone; they must be supplemented by some hypothesis on the t-dependence of $F(t, x)$. In this paper we show that it suffices to assume $F(., x)$ has bounded variation (from the left).

The essential role of a 'bounded variation' hypothesis on the t dependence of $F(t, x)$, for purposes of establishing linear L^∞ distance estimates, is further illustrated by the next example. This shows that if all the hypotheses Thm. 2 are satisfied, but the bounded variation hypothesis is discarded, then the consequences are even more severe than those revealed in the previous example; now it is no longer possible, in general to prove a distance estimate of the form

$$||\hat{x}(.) - x(.)||_{L^\infty(I)} \le d(\max_{t\in I} d_A(\hat{x}(t)))\,,$$

where $d(.) : [0, \infty) \to [0, \infty)$ is *any* continuous function such that $d(0) = 0$. This example is the first example, to the authors' knowledge, in which such discontinuous behaviour is revealed.

Example 2. Let N be any integer such that $N \ge 4$ and take the decreasing sequence of times $\tau_k := \frac{1}{N^k}$, for $k = 0, 1, 2, 3, \ldots$, as in Example 1. Consider the function $z(.) : [0, 1] \to \mathbb{R}$, which is defined by the following properties: $z(0) = 0$,

$$z(\tau_k) := (-1)^k \tau_k\,,$$

and

$$\dot{z}(t) = (-1)^k \frac{N+1}{N-1} \quad \text{for} \quad (\tau_{k+1}, \tau_k]\,.$$

For each k, we write s_k for the time when the piecewise affine function $z(.)$ takes value zero in the interval $[\tau_{k+1}, \tau_k]$:

$$s_k = \frac{2\tau_k}{N+1} \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots\,.$$

We shall consider a differential inclusion employing vectors $w_1 = w_1(t)$, $w_2 = w_2(x)$, $w_3 = w_3(x)$ and w_4 in $\mathbb{R}^4$:

$$w_1 := (\nu(N), 0, \nu(N) - \varepsilon_k, 1), \qquad \text{for}\ \ t \in (\tau_{k+1}, \tau_k]\,,$$

$$w_2(x) := \left(1, \frac{N+1}{N-1}(1 + x_4), 0, 0\right) \ \text{ and } \ w_3(x) := \left(1, -\frac{N+1}{N-1}(1 + x_4), 0, 0\right)\,,$$

$$w_4 := (\nu(N), 0, 0, 0) \ ,$$

in which

$$\nu(N) := \frac{1}{N+1} \qquad \text{and} \qquad \varepsilon_k := \frac{1}{N^{k+2}} \ .$$

Define the (t, x)-dependent multi-function $F : [0, 1] \times \mathbb{R}^4 \rightsquigarrow \mathbb{R}^4$

$$F(t, x) := \begin{cases} \{w_1(t)\} \cup \{w_2(x)\} & \text{if } \ t \in (\tau_i, \tau_{i-1}) \\ \{w_1(t)\} \cup \{w_3(x)\} & \text{if } \ t \in (\tau_{i+1}, \tau_i) \\ \{w_1(t)\} \cup \{w_2(x)\} \cup \{w_3(x)\} \cup \{w_4\} & \text{if } \ t = 0, \tau_i, \tau_{i-1} \ , \end{cases}$$

for $i = 1, 3, 5, \ldots$. Observe that the multivalued function F is Lipschitz w.r.t. x, and it is measurable but not of bounded time variation w.r.t. t (for the same argument employed in Example 1 applies in this case).

Consider the state constrained differential inclusion

$$(CS)_{t_0,x_0} \begin{cases} \dot{x}(t) \ \in \ F(t, x(t)) \quad \text{a.e. } t \in [t_0, 1] \, , \\ x(t) \ \in \ A \quad \text{for all } t \in [t_0, 1] \, , \\ x(t_0) \ = \ x_0 \ , \end{cases}$$

where the state constraint set A is defined by

$$A = \{x \in \mathbb{R}^4 \mid \ x_2 - x_1 + x_3 \leq 0, \ -x_2 - x_1 + x_3 \leq 0\} \, .$$

Notice that the constrained differential inclusion (F, A) satisfies hypotheses (H1), (H2) and (CQ)$'$ of Thm. 2, but not (BV).

Fix any integer $i \geq 10$. Consider the state constrained differential inclusion $(CS)_{s_i,x_i}$ defined on the time interval $[s_i, 1]$, where

$$x_i \ := \ (2s_i, 0, 2s_i, 0) \in A \ .$$

Take the F-trajectory $\hat{x}(.) : [s_i, 1] \to \mathbb{R}^4$:

$$\hat{x}(t) = \Big(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t), \hat{x}_4(t)\Big) = (t + s_i, z(t), 2s_i, 0) \, , \ \ \text{for } t \in [s_i, 1], \qquad i = 1, 2, \ldots \ . \tag{7}$$

Observe that $\hat{x}(s_i) = x_i \in A$, and that $\hat{x}(.)$ is an F-trajectory which is not feasible, with violation rate

$$\max_{t \in [s_i, 1]} d_A(\hat{x}(t)) = \frac{s_i}{\sqrt{3}} \quad \big(= \max_{k \leq i} d_A(\hat{x}(\tau_k)) \big) \ . \tag{8}$$

For any feasible F-trajectory $x(.)$ on $[s_i, 1]$, starting from $x_i \ = \ (2s_i, 0, 2s_i, 0)$, we obtain

$$||x(.) - \hat{x}(.)||_{L^\infty(s_i,1)} \ \geq \ \frac{\sqrt{2}(N-1)}{4(N+1)} \ .$$

Details of the derivation appear in the Appendix (Section 7).

We draw the following conclusions from the above estimates: *data for the state constrained control system* $(F(.,.), A)$ *can be chosen, which satisfy hypotheses (H1), (H2), (CQ)$'$ of Thm. 2, but not (BV), for which the following assertions are true.*

There exists $c > 0$ *such that for any* $\rho > 0$ *it is possible to find a non-feasible* F*-trajectory* $\hat{x}(.)$ *on an interval* $[t_0, t_1] \subset [0, 1]$ *satisfying:*

$$\max_{t \in [t_0, t_1]} d_A(\hat{x}(t)) \leq \rho$$

and

$$||x(.) - \hat{x}(.)||_{L^\infty(t_0,t_1)} > c \ ,$$

for all feasible trajectories $x(.)$ *with the same initial state. The example reveals a fundamental discontinuity phenomenon regarding the distance of an arbitrary state trajectory from the set of feasible state trajectories (emanating from the same initial state), when* $F(t, x)$ *fails to have certain regularity properties w.r.t.* t*.*

Consider next an optimal control problem in which we seek to minimize a 'terminal cost' $g(x(T))$ over feasible state trajectories emanating from a fixed initial state, under the hypotheses of either Thm. 1 or Thm. 2. Then is easy to deduce from either of these theorems that the value function (the infimum cost regarded as a function of the initial state and time) is Lipschitz continuous.

The purpose of the last example is to show that, if all the hypotheses of Thm. 2 are satisfied, with the exception of the bounded variation hypothesis (BV), it is possible that the value function is not even continuous.

Example 3. Consider the following optimal control problem

$$(P)_{t_0,x_0} \begin{cases} \text{Minimize } g(x(1)) = x_4(1) \\ \text{over arcs } x(.) = (x_1(.), x_2(.), x_3(.), x_4(.)) \in W^{1,1}([t_0,1];\mathbb{R}^4) \text{ s.t.} \\ \dot{x}(t) \;\in\; F(t,x(t)) \quad \text{a.e. } t \in [t_0,1]\,, \\ x(t) \;\in\; A \quad \text{for all } t \in [t_0,1]\,, \\ x(t_0) \;=\; x_0\;, \end{cases}$$

where the state constrained control system $(F(.,.),A)$ is as described in Example 2.
The *value function* $V : [0,1]\times\mathbb{R}^4 \to \mathbb{R}\cup\{+\infty\}$ is

$$V(t,x) \;=\; \inf\,(P)_{t,x}\;, \tag{9}$$

where the right hand side is interpreted as the infimum cost (under the hypotheses we shall impose it cannot be $-\infty$) in the case that admissible F-trajectories for $(P)_{t,x}$ exist and as $+\infty$ otherwise. Clearly

$$V(0,(0,0,0,0)) \;=\; g(\bar{x}(1)) = 0$$

where $\bar{x}(.)$ is the F-trajectory

$$\bar{x}(t) \;:=\; (t,z(t),0,0), \quad \text{for all } t\in[0,1]\;.$$

On the other hand, from the analysis of Example 2, it follows that

$$V(s_i,x_i) \;>\; \inf_{x(.)\ \text{feasible}} x_4(1) \;>\; \frac{N-1}{2(N+1)}\;.$$

These inequalities confirm the discontinuity of $V(.,.)$ at $(0,0)$.

4 Preliminary Analysis

We shall make extensive use of the following well-known theorem (see, e.g., [1] or [22]), providing conditions for existence F-trajectories with accompanying estimates.

Theorem 3 (Filippov's Existence Theorem) *Consider a multi-function $F : [S,T]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ taking closed, non-empty values such that $F(.,x)$ is $\mathcal{L}$-measurable for all $x\in\mathbb{R}^n$ and satisfies:*

(H2)′: There exists $k_F(.)\in L^1(S,T)$ such that

$$F(t,x') \;\in\; F(t,x) + k_F(t)|x-x'|\,\mathbb{B} \quad \textit{for all } x,x'\in\mathbb{R}^n \textit{ and a.e. } t\in[S,T]\,.$$

Take any absolutely continuous arc $y : [S,T]\to\mathbb{R}^n$ and $\xi\in\mathbb{R}^n$. If $d_{F(.,y(.))}(\dot{y}(.))\in L^1(S,T)$, then, there exists an F-trajectory $x(.)$ satisfying $x(S)=\xi$ such that for all $t\in(S,T]$

$$\begin{aligned} ||y(.)-x(.)||_{L^\infty(S,t)} &\le |y(S)-x(S)| + \int_S^t |\dot{y}(s)-\dot{x}(s)|\; ds \\ &\le e^{\int_S^t k_F(s)\;ds}\left(|\xi-y(S)| + \int_S^t d_{F(s,y(s))}(\dot{y}(s))\;ds\right)\;. \end{aligned}$$

Henceforth we take $F(.,.)$ to be the multifunction in the statement of Thm. 1, satisfying (H1), (H2), (CQ) and (BVL) for some positive numbers r_0, c, $\bar{\eta}$ (and $R := e^{c(T-S)}(r_0+1)$), some function $k_F(.)$, and some non-decreasing function $\eta(.)$. In the subsequent analysis we assume that hypothesis (H2) has been replaced by the stronger hypothesis (H2)$'$. There is no loss of generality involved since, if (H2)$'$ is violated we can redefine $F(t,x)$ for $|x| > R$ so that (H2)$'$ is satisfied, and (H1) continues to hold, and the assertions of the theorem are the same for the original constant c and function $k_F(.)$.

The following lemma summarises some consequences of the 'bounded variation' hypothesis (BVL):

Lemma 1 *Take a multifunction $F(.,.) : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ that satisfies hypotheses (H1), (H2)$'$ and (BVL) of the theorem statement. Take also any interval $[a,b] \subset [S,T]$. Define $\tilde{F}(.,.) : [a,b] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ to be*

$$\tilde{F}(t,x) \;=\; \begin{cases} \textit{lim sup}_{t\downarrow a}\, F(t,x) & \textit{if } t = a \\ F(t,x) & \textit{if } t \in (a,b) \\ \textit{lim sup}_{t\uparrow b}\, F(t,x) & \textit{if } t = b\,, \end{cases} \tag{10}$$

for any $x \in \mathbb{R}^n$. Then $F(.,.)$ takes values, closed non-empty sets, and

$$\tilde{F}(t,x) \subset \tilde{F}(t,x') + k_F(t)|x - x'|\mathbb{B} \quad \textit{and} \quad \tilde{F}(t,x) \in c(1+|x|)\mathbb{B}$$

for all $x, x' \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}$ and $t \in [a,b]$. Furthermore

$$\sup_{x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}} d_{\tilde{F}(t,x)}(\tilde{F}(s,x)) \;\leq\; \tilde{\eta}(t) - \tilde{\eta}(s)\,, \tag{11}$$

for any $[s,t] \subset [a,b]$, where

$$\tilde{\eta}(t) \;=\; \begin{cases} \eta(a^+) & \textit{for } t = a \\ \eta(t) & \textit{for } t \in (a,b) \\ \eta(b^-) & \textit{for } t = b\,, \end{cases}$$

where $\eta(.)$ is the 'modulus of variation' in hypothesis (BVL).
Assume, in addition, that also (BV) is satisfied. Then, we can supplement the assertions above with the following extra information

$$\sup_{x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}} d_H(\tilde{F}(s,x), \tilde{F}(t,x)) \;\leq\; \tilde{\eta}(t) - \tilde{\eta}(s)\,. \tag{12}$$

Proof.
Suppose first that assumptions (H1), (H2)$'$ and (BVL) are satisfied. We omit proof of the assertions that the constructed $\tilde{F}(.,.)$ takes values non-empty closed sets, and inherits the Lipschitz continuity and boundedness properties of the original multifunction $F(.,.)$ (with the same constants), since this is a straightforward exercise. Consider the final assertion. Take any $[s,t] \subset [a,b]$. We must show (11). This relation is obviously true for $a < s < t < b$. Consider the case $[s,t] = [a,b]$. (The remaining cases $[s,t] = [a,t']$ or $[t',b]$, for some $t' \in (a,b)$ are similar, but simpler, to deal with.) Fix $x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}$. Take any $v \in \tilde{F}(a,x)$. We must find $w \in \tilde{F}(b,x)$ such that

$$|v - w| \;\leq \eta(b^-) - \eta(a^+)\,. \tag{13}$$

By definition of the 'lim sup' operation, there exist $s_i \downarrow a$ and $v_i \to v$ such that $v_i \in F(s_i,x)$ for each i. Now take any sequence $t_i \uparrow b$. Then by the properties of $F(t,x)$, which coincides with $\tilde{F}(t,x)$ for $t \in (a,b)$ we have: for each x, there exists $w_i \in F(t_i,x)$ such that

$$|v_i - w_i| \leq \eta(t_i) - \eta(s_i) \tag{14}$$

The sequence $\{w_i\}$ is bounded, in view of hypothesis (H1). So, by restricting attention to a subsequence, we can arrange that $w_i \to w$ for some $w \in \mathbb{R}^n$. But then, again by definition of the 'lim sup' operation, $w \in \tilde{F}(b,x)$. Passing to the limit in (14), and noting that $\eta(.)$, as a monotone function, has left and right limits everywhere, we arrive at (13).

If the stronger assumption (BV) is also satisfied, then (12) follows from the argument above in view of the symmetric property of the Hausdorff distance.
The proof is complete. □

The next lemma, a proof of which appears in [5], states implications of the inward pointing condition (CQ):

Lemma 2 *Suppose the multifunction $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and the closed set A satisfy assumption (CQ) (for some $R \geq 0$). Then there exist $M > 0$, $\epsilon > 0$ and $\bar{\eta} > 0$ with the following property: for any $(t,x) \in [S,T] \times \big((\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B} \cap A\big)$, there exists $v \in \text{co}\, F(t,x) \cap M\mathbb{B}$ such that*

$$y + [0,\epsilon](v + \epsilon\mathbb{B}) \ \subset \ A \tag{15}$$

for all $y \in (x + \epsilon\mathbb{B}) \cap A$.

Comment: We take note, for future use, of an implication of this lemma. Take any $[t_0, t_1] \subset [S,T]$ and define $\tilde{F}(.,.) : [t_0, t_1,] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ according to (10). Take an arbitrary point $(t,x) \in [S,T] \times \big((\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B} \cap A\big)$. Then there exists a point $v \in \text{co}\, \tilde{F}(t,x)$ such that (15) is true for all $y \in (x + \epsilon\mathbb{B}) \cap A$. Here ϵ is the positive constant as in the lemma statement. This follows from the way (CQ) is formulated and the fact that, for any $(t,x) \in [t_0, t_1] \times \mathbb{R}^n$:

$$\liminf_{(t',x') \xrightarrow{D} (t,x)} \ \text{co}\, F(t',x') \ \subset \ \liminf_{(t',x') \xrightarrow{D} (t,x)} \ \text{co}\, \tilde{F}(t',x')\,,$$

where $D = [S,T] \times A$.

Henceforth, we take $\epsilon > 0$ and $\bar{\eta} > 0$ to be the constants referred to in Lemma 2. We can arrange (by reducing its size if necessary) that $\bar{\eta}$ is also the constant appearing in hypothesis (BVL). (This is true since, if (BVL)) is valid for some $\bar{\eta}$, then it remains valid for any lower positive value of this parameter.)

The lemmas below are aimed at simplifying the proof of Thm. 1, by showing that attention can be restricted to a special case.

Lemma 3 *Assume that hypotheses (H1), (H2)$'$, (CQ) and (BVL) are satisfied (for the given r_0, c etc.). Assume also*

(H3): $F(t,x)$ is convex for all $(t,x) \in [S,T] \times \mathbb{R}^n$,

Suppose that, for $\delta > 0$ and $\bar{\rho} > 0$ sufficiently small, there exists a constant $K > 0$ (whose magnitude depends on the magnitude of r_0, c, $k_F(.)$, $1/\epsilon$ and $1/\bar{\eta}$), with the following property:

(A): 'Given any interval $[t_0, t_1] \subset [S,T]$, any F-trajectory $\hat{x}(.)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{c(t_0 - S)}(r_0 + 1) - 1\big)\mathbb{B}$, and a positive number $\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0, t_1]\}$, which satisfy the conditions

(i) $\quad |t_0 - t_1| \leq \delta$

(ii) $\quad \rho \leq \bar{\rho}$

we can find an F-trajectory $x(.)$ on $[t_0, t_1]$ such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \ \in \ \text{int}\, A \quad \textit{for all } t \in (t_0, t_1] \quad \textit{and}$$

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \ \leq \ K\ \rho\,.\text{'}$$

Then, for δ and $\bar{\rho}$ sufficiently small, we can find an 'adjusted' constant K (whose magnitude depends on the magnitude of r_0, c, $k_F(.)$, $1/\epsilon$ and $1/\bar{\eta}$) that possesses a modification of property (A), in which the conditions (i) and (ii) are no longer imposed, and when the extra hypothesis (H3) is no longer required to be satisfied.

This lemma appears in [5, Lemma 5.2], with two minor differences. First (BVL) is replaced by a stronger hypothesis (ACL) 'absolute continuity from the left'. We can ignore the change from (ACL) to (BVL), because it does not affect the proof in [5, Lemma 5.2]. The other is the information in the above lemma that the magnitude of the constant K depends on the parameters r_0 , c, etc., which is proved in [5], though not explicitly stated in this reference.

Lemma 4 *Assume that, for $\delta > 0$, $\bar{\rho} > 0$ and $\gamma > 0$ sufficiently small, the assertions of Thm. 1 are valid under hypotheses (H1), (H2)$'$, (CQ) and (BVL) and under the additional hypothesis:*

(H3): $F(t,x)$ is convex for all $(t,x) \in [S,T] \times \mathbb{R}^n$,

and when the following conditions are imposed on the reference F-trajectory $\hat{x}(.) : [s,t] \to \mathbb{R}^n$, with $\hat{x}(s) \in A \cap \big(e^{c(s-S)}(r_0+1)-1\big)\mathbb{B}$, and the positive number $\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}$:

(i): $\rho \;\leq\; \bar{\rho}$

(ii): $t-s \leq \delta$.

(iii): $\eta(t_0)-\eta(t_1) \;\leq\; \gamma$.

Then the assertions are valid under (H1), (H2)$'$, (CQ) and (BVL) alone, and without conditions (i) - (iii) on $\hat{x}(.)$ and ρ.

Proof. Assume (H1), (H2)$'$, (BVL) and (H3). Let us assume that, for $\delta>0$, $\bar{\rho}>0$ and $\gamma>0$ sufficiently small, there exists a number K (whose magnitude depends on the magnitude of the constants r_0, c, k_F, δ, $\bar{\rho}$, γ and $1/\epsilon$) with the property:

(B): *'Given any interval $[t_0,t_1] \subset [S,T]$, any F-trajectory $\hat{x}(.)$ on $[t_0,t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{c(t_0-S)}(r_0+1)-1\big)\mathbb{B}$ satisfying conditions (i)-(iii) above, and any $\rho>0$ such that*

$$\rho \;\geq\; \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}\ , \tag{16}$$

we can find an F-trajectory $x(.)$ on $[t_0,t_1]$ such that $x(t_0)=\hat{x}(t_0)$,

$$x(t) \;\in\; \operatorname{int} A \quad \textit{for all } t \in (t_0,t_1] \tag{17}$$

and

$$||\hat{x}(.)-x(.)||_{L^\infty(t_0,t_1)} \;\leq\; K\ \rho\ .\text{'}$$

In view of the preceding lemma, our task is to show:

'For sufficiently small values of $\delta>0$ and $\bar{\rho}>0$, there exists an adjusted constant K (whose magnitude depends on the same parameters) possessing a modification of property (B), in which (iii) is no longer required to be satisfied.'

Choose any $\gamma>0$, $\delta>0$, $\bar{\rho}>0$ and $K>0$, such that K which possesses property (B) above.

The function $\eta(.)$ is increasing, and can therefore be decomposed into the sum of a continuous functions $\eta^c(.)$ and a countable family of functions $\{s_i(.)\}$ satisfying, for each i,

$$s_i(t) \;= a_i \times \begin{cases} 1 \text{ if } t > \sigma_i \\ 0 \text{ if } t < \sigma_i\ , \end{cases}$$

in which $\{\sigma_i\}$ is a sequence of distinct points in $[S,T]$ (the 'jump times') and $\{a_i\}$ is a sequence of non-negative numbers (the 'jumps'). (In the analysis to follow, we do not have to take account of the value of $s_i(.)$ at its jump time σ_i.) The collection of jumps $\{a_i\}$ satisfies

$$\sum_i a_i \;<\; \infty\ .$$

In view of this last relation, there exists a finite index set $J \subset \{1,2,\ldots\}$ such that

$$\sum_{i \notin J} a_i \;<\; \gamma/2\,.$$

Since the jump times $\{\sigma_i\}$ are distinct, there exists $\bar{\alpha}>0$ such that

$$|\sigma_i-\sigma_j| > \bar{\alpha} \text{ for } i,j \in J,\ i \neq j\ .$$

By reducing the size of $\delta>0$, if necessary, we can also ensure that

$$\eta^c(t')-\eta^c(s') < \gamma/2$$

for all subinterval $[s',t'] \subset [S,T]$ such that $t'-s' \leq \delta$. Now further reduce the size of δ, if necessary, to ensure also that $\delta < \bar{\alpha}$.

Take any interval $[t_0, t_1] \subset [S, T]$ and F-trajectory $\hat{x}(.)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap \left(e^{c(t-S)}(r_0+1)-1\right)\mathbb{B}$ such that conditions (i) and (ii) are satisfied, i.e.

$$t_1 - t_0 \le \delta \text{ and } \max\{d_A(\hat{x}(t)) \,|\, t \in [s,t]\} \le \bar{\rho}\,.$$

Since we have arranged that $\delta < \bar{\alpha}$, there is at most one jump time $\bar{t} = \sigma_j$, with $j \in J$, located in the interval $[t_0, t_1]$. It follows that

$$\eta(\bar{t}^-) - \eta(t_0^+) \le \gamma \text{ and } \eta(t_1^-) - \eta(\bar{t}^+) \le \gamma\,. \tag{18}$$

Case 1: $\bar{t} \in (t_0, t_1)$.

Define $\tilde{F}_1(.,.) : [t_0, \bar{t}] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ according to (10) when $[a,b] = [t_0, \bar{t}]$, and $\tilde{F}_2(.,.) : [\bar{t}, t_1] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ according to (10) when $[a,b] = [\bar{t}, t_1]$.

$\tilde{F}_1(.,.)$ satisfies the hypotheses (H1), (H2)$'$, (BVL), (H3) of $F(.,.)$, restricted to $[t_0, \bar{t}] \times \mathbb{R}^n$, with the same constants c, R and $\bar{\eta}$, and function $k_F(.)$. Also, the assertions of Lemma 1 continue to be true, with the same constants $\epsilon > 0$ and $\bar{\eta}$, when $\tilde{F}_1(.,.)$ replaces $F(.,.)$. By Lemma 1 and (18) however, the variation modulus $\tilde{\eta}_1(.)$ of $\tilde{F}_1(.,.)$ is such that

$$\tilde{\eta}_1(\bar{t}) - \tilde{\eta}_1(t_0) = \eta(\bar{t}^-) - \eta(t_0^+) < \gamma\,.$$

We have shown that $\hat{x}(.)$ restricted to $[t_0, \bar{t}]$ satisfies not just condition (i) and (ii), but also condition (iii), in the lemma statement, when $\hat{x}(.)$ is interpreted as a $\tilde{F}_1$-trajectory. We have confirmed that K possesses property (B). This means that, if we take any $\rho > 0$ such that

$$\rho \;\ge\; \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0, t_1]\}\,.$$

and $\rho \le \bar{\rho}$, there exists an $\tilde{F}_1(.,.)$ trajectory $x_1(.)$ (which is also an F-trajectory) satisfying $x_1(t_0) = \hat{x}(t_0)$ and

$$||\hat{x}(.) - x_1(.)||_{L^\infty(t_0,\bar{t})} \;\le\; K\;\rho\,.$$

Also, $x_1(t) \in \operatorname{int} A$ for $t \in (t_0, \bar{t}]$. By Filippov's Existence Theorem 3, there exists an F-trajectory $y(.)$ on $[\bar{t}, t_1]$ such that $y(\bar{t}) = x_1(\bar{t})$ and

$$||y(.) - \hat{x}(.)||_{L^\infty(\bar{t},t_1)} \le k_1 K \rho\,,$$

where $k_1 := e^{\int_S^T k_F(t)}\,dt$. We see that

$$\max\{d_A(y(t)) \,|\, t \in [t_0,\ \bar{t}]\} \le (1 + k_1 K)\rho\,.$$

Define $\rho_1 > 0$ to be

$$\rho_1 := (1 + k_1 K)\rho\,.$$

Making further use of property (B), now in relation to $\tilde{F}_2(.,.)$, we conclude that there exists an $\tilde{F}_2$-trajectory $x_2(.)$ on $[\bar{t}, t_1]$ (which is also an F-trajectory) such that

$$||y(.) - x_2(.)||_{L^\infty(\bar{t},t_1)} \le K\rho_1 = (k_1 K + 1)K\rho\;,$$

such that $x_2(t) \in \operatorname{int} A$ for all $t \in [\bar{t}, t_1]$. (We have used the fact that $y(\bar{t}) = x_1(\bar{t}) \in \operatorname{int} A$.) But then, by the triangle inequality,

$$||\hat{x}(.) - x_2(.)||_{L^\infty(\bar{t},t_1)} \le (1 + k_1 + k_1 K)K\rho\,.$$

Now define $x(.)$ to be the F-trajectory on $[t_0, t_1]$ obtained by concatenating $x_1(.)$ (on $[t_0, \bar{t}]$) and $x_2(.)$ (on $[\bar{t}, t_1]$). Then $x(t_0) = \hat{x}(0)$ and $x(t) \in \operatorname{int} A$ for $t \in (t_0, \bar{t}]$. Furthermore

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \le \max\{k_1 K\rho, (1 + k_1 + k_1 K)K\rho\} = K'\rho$$

where

$$K' := (1 + k_1 + k_1 K)K\;.$$

Case 2: $\bar{t} \notin (t_0, t_1)$.

In this case, $\bar{t}$ coincides with either t_0 or t_1, so either $[t_0, \bar{t}]$ or $[\bar{t}, t_1]$ degenerate to a single point. Invoking the hypothesis of the lemma just once yields an F-trajectory $x(.)$ on $[t_0, t_1]$ such that $x(0) = \hat{x}(t_0)$, $x(t) \in \operatorname{int} A$ for $t \in (t_0, t_1]$ and satisfying

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \le K\rho \le \max\{k_1 K\rho, (1 + k_1 + k_1 K)K\rho\} = K'\rho\,.$$

In either case, we have exhibited a new K, written K', with the desired properties. □

5 Proof of Theorem 1

Fix $r_0 > 0$. Assume that the multifunction $F(.,.)$ and set A in the theorem statement satisfy (H1), (H2)$'$, (H3), (CQ) and (BVL) with constant $c > 0$ and function $k_F(.) \in L^1(S,T)$, for $R := e^{c(T-S)}(1+r_0)$. Write also $\bar{R} := c(1+R)$. (The constants R and $\bar{R}$ bound, respectively, magnitudes and velocities of arcs $x(.)$ on subintervals of $[S,T]$ originating in $r_0\mathbb{B}$ and satisfying $|\dot{x}| \le c(1+|x|)$.) Let $\bar{\eta} > 0$ and $\eta(.)$ be the constant and modulus of variation appearing in (BVL) (and also in Lemma 2), and let $\epsilon > 0$ and $M > 0$ be the constants from Lemma 2. Increase the size of $\bar{R}$, if necessary, we can also ensure that $\bar{R} \ge M$.

We know (see Lemma 2) that, given any $(t,x) \in [S,T] \times \big((\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B} \cap A\big)$, $v \in \mathrm{co}\, F(t,x) \cap \bar{R}\mathbb{B}$ can be found such that

$$x' + [0,\epsilon](v + \epsilon\mathbb{B}) \ \subset\ A \tag{19}$$

for all $x' \in (x + \epsilon\mathbb{B}) \cap A$. By hypothesis (BVL)

$$F(s,x) \ \subset\ F(t,x) + (\eta(t) - \eta(s))\, \mathbb{B} \tag{20}$$

for all points $x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}$ and subintervals $[s,t] \subset [S,T]$.

Let $\omega(.) : [0, T-S] \to [0,\infty)$ be the function

$$\omega(\alpha) := \sup\{\int_I k_F(s)ds\}$$

where the supremum is taken over sub-intervals $I \subset [S,T]$ of length not greater than α. Since $k_F(.) \in L^1(S,T)$, by properties of integrable functions, $\omega(.)$ is well-defined on $[0, T-S]$, and $\omega(\alpha) \to 0$, as $\alpha \downarrow 0$.

Fix $k > 0$ such that $k > \epsilon^{-1}$ and take constants $\delta > 0$, $\bar{\rho} > 0$ and $\gamma > 0$ in such a manner that

$$\delta \le \epsilon, \quad \bar{\rho} + \bar{R}\delta < \epsilon, \quad k\bar{\rho} < \delta, \quad \bar{\rho} \le \bar{\eta}, \quad 4\delta\bar{R} \le \bar{\eta}, \tag{21}$$

and

$$e^{\omega(\delta)}(\gamma + \omega(\delta)\bar{R}(T-S)) < \epsilon, \quad 2e^{\omega(\delta)}(\gamma + \omega(\delta)\bar{R})\ k < (k\epsilon - 1)\ . \tag{22}$$

The assertions of the theorem will be confirmed provided that we establish the existence a constant $K > 0$ such that, for any interval $[t_0,t_1] \subset [S,T]$, given any F-trajectory $\hat{x}(.)$ on $[t_0,t_1]$ with $\hat{x}(t_0) \in A \cap \big(e^{c(t_0-S)}(r_0+1) - 1\big)\mathbb{B}$, and any $\rho > 0$ satisfying $\rho \ge \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}$, we can find a feasible F-trajectory $x(.)$ on $[t_0,t_1]$ with $x(t_0) = \hat{x}(t_0)$ and such that

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \ \le\ K\ \rho$$

and

$$x(t) \in \mathrm{int}\, A \quad \text{for } t \in (t_0, t_1]\,.$$

Owing to the reduction Lemmas 3 and 4, we can restrict attention, without loss of generality, to the case in which $F(.,.)$ is convex valued,

(i): $\rho \le \bar{\rho}$,

(ii): $t_1 - t_0 \le \delta$ and

(iii): $\eta(t_1) - \eta(t_0) \le \gamma$.

Observe that, if $\hat{x}(t_0) \in \big(A \cap \big(e^{c(t_0-S)}(r_0+1) - 1\big)\mathbb{B}\big) \setminus (\partial A + \frac{\bar{\eta}}{2}\mathbb{B})$, then the third and the fourth conditions in (21), together with (ii) above, imply that $x(.) = \hat{x}(.)$ is a feasible F-trajectory having the required properties. Therefore, it is not restrictive to impose also that $\hat{x}(t_0) \in (\partial A + \frac{\bar{\eta}}{2}\mathbb{B}) \cap A \cap \big(e^{c(t_0-S)}(r_0+1) - 1\big)\mathbb{B}$. An immediate consequence (from the definition of R) is that

$$\hat{x}(t_0) \in (\partial A + \frac{\bar{\eta}}{2}\mathbb{B}) \cap R\mathbb{B} \cap A.$$

Then, recalling the fact that the multifunction $F(.,.)$ can be considered convex valued, there exists a vector $v \in F(t_0, \hat{x}(t_0)) \cap \bar{R}\mathbb{B}$ satisfying property (19) for $(t, x) = (t_0, \hat{x}(t_0))$. Now, consider the arc $y(.) : [t_0, t_1] \to \mathbb{R}^n$ such that $y(t_0) = \hat{x}(t_0)$ and

$$\dot{y}(t) \;=\; \begin{cases} v & \text{if } t \in [t_0, (t_0 + k\rho) \wedge t_1] \\ \dot{\hat{x}}(t - k\rho) & \text{if } t \in (t_0 + k\rho, t_1] \quad \text{and if} \quad \dot{\hat{x}}(t - k\rho) \quad \text{exists} . \end{cases}$$

If $t_0 + k\rho < t_1$, then it immediately follows that, for all $t \geq t_0 + k\rho$,

$$y(t) \;=\; \hat{x}(t - k\rho) + k\rho v \,. \tag{23}$$

Recalling that $\bar{R}$ constitutes an upper bound for the magnitude for both v and $||\dot{\hat{x}}(.)||_{L^\infty}$, we deduce that

$$||\hat{x}(.) - y(.)||_{L^\infty(t_0,t_1)} \;\leq\; 2\bar{R}k\rho \,. \tag{24}$$

In addition, from (20) we also obtain that, for all $s \in [t_0, (t_0 + k\rho) \wedge t_1]$,

$$\begin{aligned} d_{F(s,y(s))}(\dot{y}(s)) &\leq (\eta(s) - \eta(t_0)) + d_{F(t_0,y(s))}(v) \\ &\leq \gamma + k_F(s)\bar{R}\,(s - t_0) \,. \end{aligned} \tag{25}$$

Invoking Filippov's existence theorem and taking into account condition (25), we can find an F-trajectory $x(.)$ on $[t_0, (t_0 + k\rho) \wedge t_1]$ with $x(t_0) = y(t_0)$ and such that, for any $t \in [t_0, (t_0 + k\rho) \wedge t_1]$

$$||x(.) - y(.)||_{L^\infty(t_0,t)} \;\leq\; e^{\omega(\delta)}(\gamma + \omega(\delta)\bar{R})\,(t - t_0) \,. \tag{26}$$

Consider now the case in which $t_0 + k\rho < t_1$. Conditions (20), (21) and (23) imply that, for a.e. $s \in [t_0 + k\rho, t_1]$,

$$\begin{aligned} d_{F(s,y(s))}(\dot{y}(s)) &= d_{F(s,k\rho v + \hat{x}(s-k\rho))}(\dot{\hat{x}}(s - k\rho)) \\ &\leq k_F(s)\bar{R}k\rho + d_{F(s,\hat{x}(s-k\rho))}(\dot{\hat{x}}(s - k\rho)) \\ &\leq k_F(s)\bar{R}k\rho + (\eta(s) - \eta(s - k\rho)) + d_{F(s-k\rho,\hat{x}(s-k\rho))}(\dot{\hat{x}}(s - k\rho)) \\ &= k_F(s)\bar{R}k\rho + \gamma + 0 \,. \end{aligned}$$

Then it follows that, for any $t \in [t_0 + k\rho, t_1]$,

$$\int_{t_0+k\rho}^{t} d_{F(s,y(s))}(\dot{y}(s))\; ds \leq (\omega(\delta)\bar{R} + \gamma)k\rho \,.$$

Thus, the F-trajectory $x(.)$ can be extended from $[t_0, (t_0 + k\rho) \wedge t_1]$ to $[t_0, t_1]$ by applying again Filippov's Theorem, in such a manner that

$$||x(.) - y(.)||_{L^\infty(t_0,t_1)} \leq 2e^{\omega(\delta)}(\gamma + \omega(\delta)\bar{R})k\rho \,. \tag{27}$$

From (24) and (27) we deduce the required estimate:

$$||\hat{x}(.) - x(.)||_{L^\infty(t_0,t_1)} \;\leq\; K\rho$$

in which

$$K \;=\; 2\left(\bar{R} + e^{\omega(\delta)}(\gamma + \omega(\delta)\bar{R})\right)k \,.$$

To complete the proof we must show that

$$x(t) \;\in\; \text{int}\, A \qquad \text{for } t \in (t_0, t_1] \,.$$

There are two cases to consider (we are assuming that $t_0 + k\rho < t_1$, otherwise only the first case occurs):

Case 1.: $t \in (t_0, t_0 + k\rho]$. Since $y(t) = \hat{x}(t_0) + (t - t_0)v$ and $t - t_0 \leq \epsilon$, it follows from (19) that

$$y(t) + (t - t_0)\epsilon\mathbb{B} \;=\; \hat{x}(t_0) + (t - t_0)(v + \epsilon\mathbb{B}) \;\subset\; A \,.$$

Then, conditions (26) and (22) immediately yield $x(t) \in \text{int}\, A$, for all $t \in (t_0, t_0 + k\rho]$.

Case 2.: $t \in (t_0 + k\rho, t_1]$. Write $z(t)$ a projection on A of the arc $t \to \hat{x}(t - k\rho)$. It means that, for each $t \in (t_0 + k\rho, t_1]$, we select $z(t) \in A$ such that

$$|\hat{x}(t - k\rho) - z(t)| \ = \ d_A(\hat{x}(t - k\rho)) \ \ (\leq \ \rho) \ .$$

As a consequence, invoking (23), we obtain

$$y(t) \ \in \ z(t) + k\rho v + \rho\mathbb{B} \ , \tag{28}$$

and, since $|\hat{x}(t - k\rho) - \hat{x}(t_0)| \ \leq \ \bar{R}(t_1 - t_0)$ for all $t \in (t_0 + k\rho, t_1]$, appealing once again (21), we also have

$$|z(t) - \hat{x}(t_0)| \ \leq \ \bar{\rho} + \bar{R}\delta \ < \ \epsilon \ .$$

Thus bearing in mind (19) and (21), we see that

$$z(t) + k\rho v + k\rho\epsilon\mathbb{B} \ \subset \ A \ ,$$

and, owing to (28),

$$y(t) + (k\epsilon - 1)\rho\mathbb{B} \ \subset \ A \ .$$

Taking into account (22) and (27), we deduce that $x(t) \in \operatorname{int} A$ in this case as well, confirming all the assertions of the theorem. □

6 Proof of Theorem 2

If the state constrained control system $(F(.,.), A)$ satisfy assumptions (H1), (H2) and (CQ), then, employing 'stability' properties of the interior of Clarke tangent cone, Lemma 2 guarantees the existence of vectors in co $F(t, x)$ pointing uniformly inward A, whenever x is close to the boundary of A. The crucial step in this proof is to show that, if we impose just condition (CQ)$'$, which is weaker than condition (CQ), still we can obtain implications similar to those ones stated in Lemma 2. The next lemma ensures that the required properties are valid if, in addition, we are allowed to make use of condition (BV).

Lemma 5 *Suppose the multifunction $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and the closed set A satisfy hypotheses (H1), (H2), (BV) and (CQ)$'$. Then there exist $M > 0$, $\epsilon > 0$, $\bar{\eta} > 0$ and a finite time set $\{\tau_j\}_{j\in J} \subset [S, T]$ with the following property: for any $(t, x) \in [S, T] \times \big((\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B} \cap A\big)$, there exists*

$$v \in \begin{cases} \operatorname{co} F(t, x) \cap M\mathbb{B} & \text{if } t \notin \{\tau_j\}_{j\in J} \\ \big(\limsup_{\tau\downarrow t} \operatorname{co} F(\tau, x)\big) \cap M\mathbb{B} & \text{if } t \in \{\tau_j\}_{j\in J} \ , \end{cases}$$

such that

$$y + [0, \epsilon](v + \epsilon\mathbb{B}) \ \subset \ A$$

for all $y \in (x + \epsilon\mathbb{B}) \cap A$.

Proof.

Fix any $(t, x) \in [S, T] \times (R\mathbb{B} \cap \partial A)$. From (CQ)$'$ we can find vectors

$$v_1 \in \Big(\liminf_{x' \xrightarrow{A} x} \ \limsup_{t'\downarrow t} \ \operatorname{co} F(t', x')\Big) \cap \operatorname{int} T_A(x) \,,$$

and

$$v_2 \in \Big(\liminf_{x' \xrightarrow{A} x} \ \limsup_{t'\uparrow t} \ \operatorname{co} F(t', x')\Big) \cap \operatorname{int} T_A(x) \,.$$

Write $M_{t,x} := \max\{|v_1|, |v_2|\} + 1$.
In view of Rockafellar's characterization of the interior of the Clarke tangent cone (cf. [19], [20]), there exist $\gamma \in (0, 1/2)$ and $r > 0$ such that

$$y + [0, \gamma](v_1 + 2\gamma\mathbb{B}) \ \subset \ A \ , \qquad \text{for all } y \in (x + 2r\mathbb{B}) \cap A \tag{29}$$

and

$$y + [0,\gamma](v_2 + 2\gamma\mathbb{B}) \ \subset \ A \ , \qquad \text{for all } y \in (x + 2r\mathbb{B}) \cap A \ . \tag{30}$$

From the definition of the lim inf operator (in the sense of Kuratowski), we can deduce the existence of $\bar{r} \in (0, r)$ such that

$$\sup_{x' \in (x+\bar{r}\mathbb{B})\cap A} \ \limsup_{t' \downarrow t} \ d_{\text{co}\, F(t',x')}(v_1) < \frac{\gamma}{2} \tag{31}$$

and

$$\sup_{x' \in (x+\bar{r}\mathbb{B})\cap A} \ \limsup_{t' \uparrow t} \ d_{\text{co}\, F(t',x')}(v_2) < \frac{\gamma}{2} \ . \tag{32}$$

Taking into account assumption (BV) and the same argument employed in the proof of Lemma 4, we denote $\eta(.)$ the modulus of variation of $F(.,.)$, which can be decomposed into a sum of a continuous function $\eta^c(.)$ and a singular (discontinuous) component. Write $\{\sigma_i\}$ and $\{a_i\}$ respectively the sequence of the countable distinct jump times and the sequence of the (countable) non-negative jumps of $\eta(.)$ in $[S,T]$. We can find numbers $\delta > 0$ and $\bar{\alpha} > 0$ such that

$$\eta^c(t') - \eta^c(s') < \gamma/2 \tag{33}$$

for each subinterval $[s',t'] \subset [S,T]$ with $t' - s' \le \delta$, and

$$\sum_{i \notin J} a_i \ < \ \gamma/2 \, , \tag{34}$$

where $J \subset \{1, 2, \ldots\}$ is a finite index set for which

$$|\sigma_i - \sigma_j| > \bar{\alpha} \text{ for } i,j \in J, \ i \neq j \ .$$

Take $\delta_t \in (0,\delta)$ such that $[t-\delta_t, t+\delta_t] \cap \{\sigma_i \ : \ i \in J\} \neq \emptyset$. Define $\tilde{F}_1(.,.) : [t-\delta_t, t] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ according to (10) when $[a,b] = [t-\delta_t, t]$, and $\tilde{F}_2(.,.) : [t, t+\delta_t] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ according to (10) when $[a,b] = [t, t+\delta_t]$. $\tilde{F}_1(.,.)$ and $\tilde{F}_2(.,.)$ satisfy the hypotheses (H1), (H2)$'$, (H3) of $F(.,.)$, restricted respectively to $[t-\delta_t, t] \times \mathbb{R}^n$ and $[t, t+\delta_t] \times \mathbb{R}^n$, with the same constants $c > 0$ and $\bar{\eta} > 0$, and function $k_F(.)$. Moreover, the assertions of Lemma 1 hold true when $\tilde{F}_1(.,.)$ and $\tilde{F}_2(.,.)$ replace $\tilde{F}(.,.)$. Then, from (12), for all $s' \in [t-\delta_t, t]$, we have

$$\sup_{x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}} d_H(\tilde{F}_1(s',x), \tilde{F}_1(t,x)) \le \tilde{\eta}_1(t) - \tilde{\eta}_1(s') \tag{35}$$

and for all $t' \in [t, t+\delta_t]$, we have

$$\sup_{x \in (\partial A + \bar{\eta}\mathbb{B}) \cap R\mathbb{B}} d_H(\tilde{F}_2(t,x), \tilde{F}_2(t',x)) \le \tilde{\eta}_2(t') - \tilde{\eta}_2(t) \ . \tag{36}$$

Also conditions (31) and (32) can be rephrased in terms of $\tilde{F}_1(.,.)$ and $\tilde{F}_2(.,.)$: for all $x' \in (x + \bar{r}\mathbb{B}) \cap A$

$$(v_1 + \frac{\gamma}{2}\mathbb{B}) \cap \text{co}\, \tilde{F}_1(t,x') \neq \emptyset \quad \text{and} \quad (v_2 + \frac{\gamma}{2}\mathbb{B}) \cap \text{co}\, \tilde{F}_2(t,x') \neq \emptyset \ .$$

Thus, for all $x' \in (x + \bar{r}\mathbb{B}) \cap A$ and for all $s' \in [t-\delta_t, t)$, there exists $v' \in \text{co}\, \tilde{F}_1(s',x') \cap M_{t,x}\mathbb{B}$ such that

$$|v_1 - v'| \le \frac{\gamma}{2} + \tilde{\eta}_1(t) - \tilde{\eta}_1(s') \le \frac{3}{2}\gamma \ ,$$

where, taking account of (33) and (34), we have used the following fact:

$$\tilde{\eta}_1(t) - \tilde{\eta}_1(s') \le \eta(t^-) - \eta(s'^+) < \gamma \ .$$

Similarly, for all $x' \in (x + \bar{r}\mathbb{B}) \cap A$ and for all $t' \in (t, t+\delta_t]$, there exists $w' \in \text{co}\, \tilde{F}_2(t',x') \cap M_{t,x}\mathbb{B}$ such that

$$|v_2 - w'| \le \frac{\gamma}{2} + \tilde{\eta}_2(t) - \tilde{\eta}_2(t') \le \frac{3}{2}\gamma \ .$$

(Observe that in the relations above, $\tilde{F}_1(s',x') = F(s',x')$ for all $s' \in (t-\delta_t, t)$, and $\tilde{F}_2(t',x') = F(t',x')$ for all $t' \in (t, t+\delta_t)$.)

Therefore, for all $t' \in (t-\delta_t, t+\delta_t)$, $t' \neq t$, and for all $x' \in (x+\bar{r}\mathbb{B})\cap A$, we can find $v' \in \mathrm{co}\, F(t',x')\cap M_{t,x}\mathbb{B}$ such that

$$y + [0,\gamma](v' + \frac{\gamma}{2}\mathbb{B}) \ \subset\ A\ , \qquad \text{for all } y \in (x' + \bar{r}\mathbb{B}) \cap A\ .$$

And, if $t' = t$, for all $x' \in (x + \bar{r}\mathbb{B}) \cap A$, there exists $\tilde{v} \in \tilde{F}_2(t, x') \cap M_{t,x}\mathbb{B}$ satisfying

$$y + [0,\gamma](\tilde{v} + \frac{\gamma}{2}\mathbb{B}) \ \subset\ A\ , \qquad \text{for all } y \in (x' + \bar{r}\mathbb{B}) \cap A\ .$$

Set $\varepsilon_{t,x} := \min\{\delta_t, \bar{r}, \gamma/2\}$. Then we have proved the following property: for any $(t,x) \in [S,T] \times (R\mathbb{B} \cap \partial A)$ there exist $\varepsilon_{t,x} > 0$ and $M_{t,x}$ such that for all $(t',x') \in [S,T] \times A$ with $|x - x'| < \varepsilon_{t,x}$ and $|t - t'| < \varepsilon_{t,x}$, we can find

$$v' \in \begin{cases} \mathrm{co}\, F(t',x') \cap M_{t,x}\mathbb{B} & \text{if } t' \neq t \\ \\ \mathrm{co}\, \tilde{F}_2(t',x') \cap M_{t,x}\mathbb{B} & \text{if } t' = t\ , \end{cases}$$

satisfying

$$y + [0, \varepsilon_{t,x}](v' + \varepsilon_{t,x}\mathbb{B}) \ \subset\ A\ , \quad \text{for all } \ y \in (x' + \varepsilon_{t,x}\mathbb{B}) \cap A\ .$$

To conclude we apply a standard compactness argument. □

Proof of Theorem 2. In view of Lemma 5, we can define the following multifunction

$$\tilde{F}(t,x) := \begin{cases} F(t,x) & \text{if } t \notin \{\tau_j\}_{j\in J} \\ \\ \limsup_{\tau \downarrow t} F(\tau, x) & \text{if } t \in \{\tau_j\}_{j \in J}\ . \end{cases}$$

Then, the approach used in the proof of Thm. 1 is applicable to $(\tilde{F}, A)$. Since an $\tilde{F}$-trajectory is also an F-trajectory and vice versa, we obtain the validity of the required properties. □

7 Appendix

In this appendix we give the details of Example 2 of Section 3. For a reference F-trajectory $x(.) = (x_1(.), x_2(.), x_3(.), x_4(.))$ and time interval $[\sigma_1, \sigma_2] \subset [0,1]$, we shall define the number

$$d_{[\sigma_1,\sigma_2]}(x(.)) := \ \mathrm{meas}\ \{s \in [\sigma_1, \sigma_2]\ :\ \dot{x}(s) \neq \dot{\hat{x}}(s)\} \ = \ \mathrm{meas}\ \{s \in [\sigma_1,\sigma_2]\ :\ \dot{x}(s) = w_0(s)\}\ ,$$

which provides the measure of the subset of $[\sigma_1, \sigma_2] \subset [0,1]$ in which the choice for the velocity of $x(.)$ differs from that one of $\hat{x}(.)$.
Take any $i \geq 10$. We notice that for any feasible F-trajectory $x(.)$ on $[s_i, 1]$, starting from $x_i \ = \ (2s_i, 0, 2s_i, 0)$, we cannot use just vectors w_1 and w_2 for its dynamics (as for $\hat{x}(.)$). Indeed, if we chose only vectors w_1 and w_2 for the F-trajectory $x(.)$, clearly it would violate the state constraint A. More precisely, on the time interval $[s_i, \tau_i]$, we have:

$$\begin{aligned} \hat{x}_1(\tau_i) - x_1(\tau_i) &= \int_{\tau_k}^{\tau_{k-1}} \left(\dot{\hat{x}}_1(s) - \dot{x}_1(s) \right)\ ds \\ &= (1 - \nu(N))\ \ d_{[s_i,\tau_i]}(x(.))\ , \end{aligned} \tag{37}$$

$$|(\hat{x}_2(\tau_i)| - |x_2(\tau_i))| = \frac{N+1}{N-1} \times \left(d_{[s_i,\tau_i]}(x(.)) - \in \tau_{\{s \in [s_i,\tau_i]\ :\ \dot{x}_2(s) \neq 0\}}\ x_4(s)\ ds \right)\ , \tag{38}$$

$$x_3(\tau_i) \ = \ x_3(s_i) + (\nu(N) - \varepsilon_i)\ d_{[s_i,\tau_i]}(x(.)) \ = \ 2s_i + (\nu(N) - \varepsilon_i)\ d_{[s_i,\tau_i]}(x(.))\ , \tag{39}$$

and

$$x_4(\tau_i) \ = \ x_4(s_i) + d_{[s_i,\tau_i]}(x(.))\ . \tag{40}$$

Therefore, a lower-bound of the measure of the set in which we are 'forced' to employ v_0 on $[s_i, \tau_i]$ is given by:

$$d_{[s_i,\tau_i]}(x(.)) \ > \ \frac{N-1}{2 + \varepsilon_i(N-1)} \ > \ \frac{N^2-1}{3N+1} s_i\ . \tag{41}$$

Whereas, for all $k = 1, \dots, i$, we obtain

$$\begin{aligned} \hat{x}_1(\tau_{k-1}) - x_1(\tau_{k-1}) &= \hat{x}_1(\tau_k) - x_1(\tau_k) + \int_{\tau_k}^{\tau_{k-1}} \left(\dot{\hat{x}}_1(s) - \dot{x}_1(s)\right)\, ds \\ &= \hat{x}_1(\tau_k) - x_1(\tau_k) + (1-\nu(N))\ d_{[\tau_k,\tau_{k-1}]}(x(.)) \ , \end{aligned} \tag{42}$$

$$\begin{aligned} &\frac{N+1}{N-1} \times \left(d_{[\tau_k,\tau_{k-1}]}(x(.)) - \int_{\{s\in[\tau_k,\tau_{k-1}]\,:\,\dot{x}_2(s)\neq 0\}} x_4(s)\ ds \right) = \\ &= |(\hat{x}_2(\tau_{k-1}) - x_2(\tau_{k-1})) - (\hat{x}_2(\tau_k) - x_2(\tau_k))| \\ &= |\hat{x}_2(\tau_{k-1})| - |x_2(\tau_{k-1})| + |\hat{x}_2(\tau_k)| - |x_2(\tau_k)| \ . \end{aligned}$$

In addition

$$x_3(\tau_{k-1}) \ = \ x_3(\tau_k) + (\nu(N) - \varepsilon_{k-1})\ d_{[\tau_k,\tau_{k-1}]}(x(.)) \ , \tag{43}$$

and

$$x_4(\tau_{k-1}) \ = \ x_4(\tau_k) + d_{[\tau_k,\tau_{k-1}]}(x(.)) \ , \tag{44}$$

Bearing in mind the state constraint relation $|x_2(t)| \le x_1(t) - x_3(t)$, for all $t \in [s_i, 1]$, from the above inequalities, (42), (43), (43) and (44), we derive

$$\begin{aligned} &\frac{N+1}{N-1} \times d_{[\tau_k,\tau_{k-1}]}(x(.)) - \frac{N+1}{N-1} \times \int_{\{s\in[\tau_k,\tau_{k-1}]\,:\,\dot{x}_2(s)\neq 0\}} x_4(s)\ ds > \\ &\quad > \ 2[\hat{x}_1(\tau_k) - x_1(\tau_k) + x_3(\tau_k) - 2s_i] + 2s_i \ , \end{aligned}$$

and so

$$\begin{aligned} \left[\frac{2}{N-1} + \varepsilon_{k-1}\right] \times d_{[\tau_k,\tau_{k-1}]}(x(.)) &> 2[\hat{x}_1(\tau_k) - x_1(\tau_k) + x_3(\tau_k) - 2s_i] + 2s_i + \\ &+ \frac{N+1}{N-1} \times \left[(N-1)\tau_k - d_{[\tau_k,\tau_{k-1}]}(x(.))\right]\ x_4(\tau_k) \ . \end{aligned} \tag{45}$$

Observe that from (42) and (43) we also have:

$$\hat{x}_1(\tau_{k-1}) - x_1(\tau_{k-1}) + x_3(\tau_{k-1}) - 2s_i = \hat{x}_1(\tau_k) - x_1(\tau_k) + x_3(\tau_k) - 2s_i + (1-\varepsilon_{k-1})\ d_{[\tau_k,\tau_{k-1}]}(x(.)) \ . \tag{46}$$

Write, for all $k = 0, \dots, i$

$$y_k := \hat{x}_1(\tau_k) - x_1(\tau_k) + x_3(\tau_k) - 2s_i \quad (>0) \ .$$

Then applying recursively (44) and (46) we respectively obtain:

$$x_4(\tau_k) \ = \ x_4(\tau_{k+1}) + d_{[\tau_{k+1},\tau_k]}(x(.)) = \sum_{j=k}^{i-1} d_{[\tau_{j+1},\tau_j]}(x(.)) + d_{[s_i,\tau_i]}(x(.))$$

and

$$y_k = y_{k+1} + (1-\varepsilon_k)\ d_{[\tau_{k+1},\tau_k]}(x(.)) = \left(\sum_{j=k}^{i-1} (1-\varepsilon_j)\ d_{[\tau_{j+1},\tau_j]}(x(.)) + (1-\varepsilon_i)\ d_{[s_i,\tau_i]}(x(.))\right) \ . \tag{47}$$

Thus

$$x_4(\tau_k) \ > \ \frac{1}{(1-\varepsilon_i)}\ y_k \ . \tag{48}$$

Now only one of the following two cases may occur:
Case 1: for all $k = 1, \dots, i$

$$d_{[\tau_k,\tau_{k-1}]}(x(.)) < \frac{N-1}{2}\tau_k \ . \tag{49}$$

Case 2: for some $k_0 \in \{0, \dots, i-1\}$ we have

$$d_{[\tau_{k_0},\tau_{k_0-1}]}(x(.)) \ge \frac{N-1}{2}\tau_{k_0} \ . \tag{50}$$

We start considering the first case. Condition (49) implies that, for all $k = 1, \dots, i$

$$(N-1)\tau_k - d_{[\tau_k,\tau_{k-1}]}(x(.)) \;>\; \frac{N-1}{2}\tau_k \;. \tag{51}$$

From the choice of ε_k it follows that, for all $k = 1, \dots, i$

$$(1-\varepsilon_k)\frac{N-1}{2+(N-1)\varepsilon_k} \times \left(2 + \frac{(N+1)}{2(1-\varepsilon_i)}\;\tau_k\right) > N-1 \;. \tag{52}$$

Therefore, using (51) and (48) in inequality (45), and bearing in mind (52), we deduce the validity of the following estimate for all $k = 1, \dots, i$:

$$\begin{aligned} y_{k-1} &> y_k + (1-\varepsilon_{k-1})\frac{N-1}{2+(N-1)\varepsilon_{k-1}} \times \left[\left(2 + \frac{(N+1)}{2(1-\varepsilon_i)}\;\tau_{k-1}\right) y_k + 2s_i\right] \\ &> \left[1 + (1-\varepsilon_{k-1})\frac{N-1}{2+(N-1)\varepsilon_{k-1}} \times \left(2 + \frac{(N+1)}{2(1-\varepsilon_i)}\;\tau_{k-1}\right)\right] y_k \\ &\quad +2(1-\varepsilon_{k-1})\frac{N-1}{2+(N-1)\varepsilon_{k-1}} s_i \\ &> N \times y_k + 2(1-\varepsilon_{k-1})\frac{N-1}{2+(N-1)\varepsilon_{k-1}} s_i \;. \end{aligned} \tag{53}$$

Recalling the definition of y_i and estimate (41) we also have

$$\begin{aligned} y_i = (1-\varepsilon_i)\; d_{[s_i,\tau_i]}(x(.)) \;&>\; (1-\varepsilon_i)\frac{N-1}{2+(N-1)\varepsilon_i}\; s_i \\ &> \frac{N(N^2-1)}{(N+1)(3N+1)}\; s_i \;=\; \frac{2N(N-1)}{(N+1)(3N+1)}\;\frac{1}{N^i} \;. \end{aligned} \tag{54}$$

Combining (53) and (54) yields

$$y_0 \;>\; N^i \times y_i \;>\; \frac{2N(N-1)}{(N+1)(3N+1)} \;.$$

Suppose now that Case 2 occurs, and write k_0 the smaller integer in $\{0, \dots, i-1\}$ such that we have

$$d_{[\tau_{k_0},\tau_{k_0-1}]}(x(.)) \geq \frac{N-1}{2}\tau_{k_0} \;.$$

It means that for all $k = 1, \dots, k_0 - 1$

$$d_{[\tau_k,\tau_{k-1}]}(x(.)) < \frac{N-1}{2}\tau_k \;,$$

that is

$$(N-1)\tau_k - d_{[\tau_k,\tau_{k-1}]}(x(.)) \;>\; \frac{N-1}{2}\tau_k \;.$$

Then we can use for $k = 1, \dots, k_0 - 1$ estimate (53), obtaining:

$$y_0 \;\geq\; N^{k_0-1}\; y_{k_0-1} + \sum_{j-0}^{k_0-2} N^j s_i \;. \tag{55}$$

But, from (47), we deduce in particular that

$$y_{k_0-1} \;>\; (1-\varepsilon)\; d_{[\tau_{k_0},\tau_{k_0-1}]}(x(.)) \;\geq\; (1-\varepsilon)\;\frac{N-1}{2}\tau_{k_0} \;.$$

As a consequence

$$y_0 \;>\; N^{k_0-1}\;\frac{N(N-1)}{2(N+1)}\frac{1}{N^{k_0}} \;\geq\; \frac{N-1}{2(N+1)} \;.$$

In conclusion, observing that $||x(.) - \hat{x}(.)||_{L^\infty} \geq y_0$, we have proved that, for all $i \geq 10$:

$$||x(.) - \hat{x}(.)||_{L^\infty([s_i,1])} \;\geq\; \frac{\sqrt{2}}{2} \times y_0 \;\geq\; \frac{\sqrt{2}}{2}\min\left\{\frac{N-1}{2(N+1)}; \frac{2N(N-1)}{(N+1)(3N+1)}\right\} = \frac{\sqrt{2}(N-1)}{4(N+1)} \;.$$

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