

L^∞ Estimates on Trajectories Confined to a Closed Subset

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Abstract

This paper concerns the validity of estimates on the distance of an arbitrary state trajectory from the set of state trajectories which lie in a given state constraint set. These so called distance estimates have wide-spread application in state constrained optimal control, including justifying the use of the Maximum Principle in normal form and establishing regularity properties of value functions. We focus on linear, L^∞ distance estimates which, of all the available estimates have, so far, been the most widely used. Such estimates are known to be valid for general, closed state constraint sets, provided the functions defining the dynamic constraint are Lipschitz continuous, with respect to the time and state variables. We ask whether linear, L^∞ distance estimates remain valid when the Lipschitz continuity hypothesis governing t -dependence of the data is relaxed. We show by counter-example that these distance estimates are not valid in general if the hypothesis of Lipschitz continuity is replaced by continuity. We also provide a new hypothesis, ‘absolute continuous from the left’, for the validity of linear, L^∞ estimates. The new hypothesis is less restrictive than Lipschitz continuity and even allows discontinuous time dependence in certain cases. It is satisfied, in particular, by differential inclusions exhibiting non-Lipschitz t -dependence at isolated points, governed, for example, by a fractional-power modulus of continuity. The relevance of distance estimates for state constrained differential inclusions permitting fractional-power time dependence is illustrated by an example in engineering design, where we encounter an isolated, square-root type singularity, concerning the t -dependence of the data.

Keywords: Optimal Control, Differential Inclusions, State Constraints, Sensitivity.

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1 Introduction

Consider the state-constrained differential inclusion, described as follows:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{for a.e. } t \in [S, T] \\ x(t) \in A & \text{for all } t \in [S, T], \end{cases}$$

in which $[S, T]$ is a given interval ($T > S$), $F(., .) : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multifunction with closed, non-empty values and $A \subset \mathbb{R}^n$ is a given closed set.

Given a subinterval (possibly closed or left open) $I \subset [S, T]$, we shall refer to an absolutely continuous function $x(.) : I \rightarrow \mathbb{R}^n$ which satisfies $\dot{x}(t) \in F(t, x(t))$ a.e. as an F -trajectory (on I). An F -trajectory $x(.)$ on I is said to be ‘feasible’ (on I) if $x(t) \in A$ for all $t \in I$, and ‘strictly feasible’ (on I) if $x(t) \in \text{int } A$ for all $t \in I$.

In this paper, attention focuses on hypotheses for the validity of estimates of the type: given a ball $r_0\mathbb{B}$ in \mathbb{R}^n there exists a constant $K > 0$ such that, for any F -trajectory $\hat{x}(.)$ on a closed subinterval $I \subset [S, T]$, emanating from $r_0\mathbb{B} \cap A$, we have

$$\|\hat{x}(.) - x(.)\|_{L^\infty} \leq K \max_{t \in I} d_A(\hat{x}(t)),$$

for some *feasible* F -trajectory $x(.)$ with the same initial value. Such estimates are referred to as linear L^∞ estimates (on the distance of a general F -trajectory $\hat{x}(.)$, from the set of feasible F -trajectories with shared left endpoint, expressed in terms of $\max_{t \in I} d_A(\hat{x}(t))$, which is interpreted as a measure of the state constraint violation by $\hat{x}(.)$). The significance of such estimates, in studying regularity of the value function, establishing validity of ‘normal’ forms of the state constrained Maximum Principle, characterizing the value function in terms of solutions to the Hamilton Jacobi equation, in other areas, is well documented. (See for instance [4], [5], [7], [9], [10], [13], [14], [15], [16], [20]; and for related results cf. also [2], [3], [6].) While other, related, estimates, involving stronger norms on the left side and different measures of state constraint violation, are of interest, linear L^∞ distance estimates have, so far, found most widespread application and are therefore currently of greatest interest.

In the case when A has a C^{1+} boundary (i.e. A , locally, has the representation $\{x \mid h(x) \leq 0\}$, for some C^1 function with non-vanishing on the boundary of A , locally Lipschitz gradient), linear L^∞ distance estimates of the type described above are known to be valid under the following hypotheses:

- $F(., x)$ is measurable, and $F(t, .)$ has linear growth and is Lipschitz continuous,
- $F(t, x)$ satisfies a ‘strictly inward pointing’ condition near the boundary of A .

See, e.g., [4]. We refer to these hypotheses as the ‘basic’ hypotheses.

In this paper we examine the validity of linear, L^∞ estimates when no assumptions are made about the nature of the state constraint sets A considered, except that they are closed and non-empty (‘general’ state constraint sets). We provide answers to two questions. First:

Are linear, L^∞ distance estimates valid merely under the basic hypotheses, for general state constraint sets A ?

We show, by exhibiting two counter-examples, that the answer is in general ‘no’. The counter-examples, besides answering the above question, reveal more subtle limitations on the validity of

distance estimates for state constrained differential inclusions, for general state constraint sets. The first counter-example demonstrates that not even a weaker, super-linear, Hölder distance estimate is in general valid, under the basic hypotheses. A second counter-example shows that, even if we additionally assume that $F(., .)$ is a continuous multi-function, the linear estimate still fails to be valid in general; nor is a super-linear ‘ $\rho |\ln(\rho)|$ ’ distance estimate valid, where ρ is the state constraint violation. The question then arises:

What supplementary hypothesis regarding t -regularity of F is required for validity of linear, L^∞ estimates, for general state constraint sets A ?

It is already known (see [9]) that such estimates are valid when F is Lipschitz w.r.t. x and does not depend on t , and techniques are provided in [13] for establishing linear distance estimates in some cases involving time-dependent F 's, including the case when $F(., .)$ is Lipschitz continuous in both variables. In the present paper we propose a new supplementary hypothesis for the validity of linear, L^∞ distance estimates, namely the requirement that $F(., x)$ is ‘absolutely continuous from the left’.

The new supplementary hypothesis on the t -dependence of F is significantly weaker than Lipschitz continuity. It allows F to depend on t according to fractional powers of t , but it also covers some situations where F fails to be Hölder continuous w.r.t. t for *any* Hölder index $\alpha \in (0, 1)$. Since the new supplementary hypothesis requires merely absolute continuity *from the left*, it is satisfied in some situations in which F is discontinuous.

The relevance of the new supplementary condition in engineering design is illustrated by reference to an optimal design problem in civil engineering, where the object is to determine the distribution of constituent materials in a beam to maximize rigidity. The design problem takes the form of a state constrained optimal control problem, in which the functions defining the control system dynamics are not Lipschitz continuous with respect to the time-like variable, but have a square-root type dependency covered by the new supplementary hypothesis. This paper provides linear L^∞ distance estimates for such control systems, which in turn can be used to derive the Maximum Principle in the normal form for solution of the problem.

The analytical techniques employed to construct ‘neighboring’ feasible F -trajectories, and thereby to prove the desired distance estimates, are based on directing the velocity into the interior of A over an initial period of time which is proportional to the state constraint violation, and then introducing a time-delay. They are akin to the techniques earlier used by [20], [10], [13] and [9]. But adapting these techniques, to give stronger conclusions (‘strict’ feasibility of the constructed F -trajectory) and to take account of the weaker hypotheses imposed (‘absolute continuity’ from the left), is far from straightforward.

Notation. For a given interval $[t_0, t_1] \subset \mathbb{R}$ the space $L^p([t_0, t_1]; \mathbb{R}^n)$, $p = 1$ or $p = \infty$, is written briefly $L^p(t_0, t_1)$ or L^p . \mathbb{B} denotes the closed unit ball in Euclidean space. The Euclidean distance is written $|\cdot|$, $\text{int } C$ denotes the interior of a set C in Euclidean space. Take a closed set $D \subset \mathbb{R}^n$ and $x \in D$. We write $\text{co } D$ for the convex hull of D . The Clarke tangent cone to D at x is written $T_D(x)$ (cf. [8]). We denote by $\chi_D(\cdot)$ the indicator function of D , that is the function taking value 1 on D and 0 elsewhere. $d_D(x)$ denotes the Euclidean distance of the point x from the set D , namely $\min_{y \in D} |x - y|$. We denote by $\Pi_D(x)$ the possibly set-valued projection of the point $x \in \mathbb{R}^n$ into D . For arbitrary non-empty closed sets in \mathbb{R}^n , D' and D , we denote by

$d_D(D')$ the excess from set D to a set D' :

$$d_D(D') = \inf\{\beta > 0 \mid D' \subset D + \beta\mathbb{B}\},$$

(alternatively referred to as the ‘asymmetric Hausdorff distance’ of the set D' from the set D .)

Given a multifunction $G : D \rightsquigarrow \mathbb{R}^n$ and $x \in D$ (where D is closed), we define the limit inferior (in the Kuratowski sense) of G at x to be (cf. [1] or [21])

$$\liminf_{x' \xrightarrow{D} x} G(x') := \{v \in \mathbb{R}^n : \limsup_{x' \xrightarrow{D} x} d_{G(x')}(v) = 0\}.$$

The notation $x' \xrightarrow{D} x$ indicates consideration of convergent sequences $x' \rightarrow x$, all elements of which belong to D . An alternative, and often useful, characterization of the lim inf operator on a set-valued function $G(\cdot)$ is as follows: $v \in \liminf_{x' \xrightarrow{D} x} G(x')$ if and only if for every $\varepsilon > 0$ there exists $\eta > 0$ such that $(v + \varepsilon\mathbb{B}) \cap G(x') \neq \emptyset$ for every $x' \in (x + \eta\mathbb{B}) \cap D$. We denote the Lebesgue subsets of $[S, T]$ by \mathcal{L} .

2 Distance Estimates

We state conditions for the validity of linear, L^∞ estimates relating to state constrained differential inclusions of the Introduction,

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{for a.e. } t \in [S, T] \\ x(t) \in A & \text{for all } t \in [S, T], \end{cases}$$

for a general closed, non-empty state constraint set A , under the ‘basic’ hypotheses of the introduction (now precisely described) and a supplementary hypothesis ‘absolute continuity from the left’, regarding the t -dependence of F .

Definition 2.1 *Given a set $X_0 \subset \mathbb{R}^n$ and a multifunction $F(\cdot, \cdot) : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, we say that $F(\cdot, x)$ is absolutely continuous from the left, uniformly over $x \in X_0$ if and only if the following condition is satisfied: given any $\varepsilon > 0$ we may find $\delta > 0$ such that, for any finite partition of $[S, T]$*

$$S \leq s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq s_m < t_m \leq T$$

satisfying $\sum_{i=1}^m (t_i - s_i) < \delta$, we have

$$\sum_{i=1}^m d_{F(t_i, x)}(F(s_i, x)) < \varepsilon.$$

A convenient characterization of absolute continuity from the left is provided by the following lemma, stated without proof.

Lemma 2.2 *Given a subset $X_0 \subset \mathbb{R}^n$ and a multifunction $F(\cdot, \cdot) : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, then $F(\cdot, x)$ is absolutely continuous from the left uniformly over X_0 if and only if there exists $\gamma(\cdot) \in L^1(S, T)$ such that*

$$F(s, x) \subset F(t, x) + \int_s^t \gamma(s') ds' \mathbb{B} \tag{1}$$

for all subintervals $[s, t] \subset [S, T]$ and $x \in X_0$.

For an interval $I \subset [S, T]$ and an arc $x(\cdot) : I \rightarrow \mathbb{R}^n$ we define

$$\rho_I(x(\cdot)) := \sup_{t \in I} d_A(x(t)).$$

Theorem 2.3 Fix $r_0 > 0$. Assume that, for some constant $c > 0$ and some $k_F(\cdot) \in L^1$ and for $R := e^{c(T-S)}(r_0 + 1)$, the following hypotheses (H1), (H2), (CQ) and (ACL) are satisfied:

(H1): $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, non-empty values, $F(\cdot, x)$ is \mathcal{L} -measurable for all $x \in \mathbb{R}^n$, and

$$F(t, x) \subset c(1 + |x|) \mathbb{B} \quad \text{for all } (t, x) \in [S, T] \times \mathbb{R}^n .$$

(H2):

$$F(t, x') \subset F(t, x) + k_F(t)|x - x'| \mathbb{B} \quad \text{for all } x, x' \in R\mathbb{B} \text{ and a.e. } t \in [S, T] .$$

(CQ): For each $(t, x) \in [S, T] \times (R\mathbb{B} \cap \partial A)$,

$$\left(\liminf_{(t', x') \xrightarrow{D} (t, x)} \text{co } F(t', x') \right) \cap \text{int } T_A(x) \neq \emptyset ,$$

where $D = [S, T] \times A$.

(ACL): There exists $\eta > 0$ such that $F(\cdot, x)$ is absolutely continuous from the left, uniformly over $x \in (\partial A + \eta\mathbb{B}) \cap R\mathbb{B}$.

Then, there exists a constant $K > 0$ with the following property:

Given any interval $[t_0, t_1] \subset [S, T]$, any F -trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap (e^{c(t_0-S)}(r_0 + 1) - 1)\mathbb{B}$, and any $\rho > 0$ such that

$$\rho \geq \rho_{[t_0, t_1]}(\hat{x}(\cdot)) ,$$

we can find an F -trajectory $x(\cdot)$ on $[t_0, t_1]$ such that $x(t_0) = \hat{x}(t_0)$,

$$x(t) \in \text{int } A \quad \text{for all } t \in (t_0, t_1]$$

and

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} \leq K \rho .$$

The assertions of the theorem cover two cases, each of independent interest:

Case A: $\rho_{[t_0, t_1]}(\hat{x}(\cdot)) > 0$ ($\hat{x}(\cdot)$ is not feasible).

In this case, an F -trajectory $x(\cdot)$, with initial value $\hat{x}(t_0)$ and strictly feasible on $(t_0, t_1]$ exists, which satisfies the linear distance estimate

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} \leq K \rho_{[t_0, t_1]}(\hat{x}(\cdot)) .$$

(This follows from the theorem statement, after setting $\rho := \rho_{[t_0, t_1]}(\hat{x}(\cdot))$.)

Case B: $\rho_{[t_0, t_1]}(\hat{x}(\cdot)) = 0$ ($\hat{x}(\cdot)$ is feasible).

In this case, for arbitrary $\epsilon > 0$, there exists an F -trajectory $x(\cdot)$, with initial value $\hat{x}(t_0)$ and strictly feasible on $(t_0, t_1]$ such that

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} \leq \epsilon$$

(This follows from the theorem statement, after setting $\rho := \epsilon/K$.)

3 Discussion of the Supplementary Hypothesis

The concept of ‘absolute continuity from the left’ of a set valued function has been considered earlier in the control theory literature as a hypothesis on a time varying constraint set in theorems asserting the existence of viable trajectories for differential inclusions with measurable time dependence. (See [11], [12] and [17].) It is used here for the first time, as a hypothesis regarding the t -dependence of $F(t, x)$ for validity of distance estimates.

It is clear from Lemma 2.2 that hypothesis (ACL), ‘absolute continuity from the left uniformly over $(\partial A + \eta\mathbb{B}) \cap R\mathbb{B}$ ’, is a weaker hypothesis than ‘ $F(\cdot, \cdot)$ is locally Lipschitz continuous’, and therefore improves on earlier invoked hypotheses for the validity of linear, L^∞ distance estimates. Indeed, if $F(\cdot, \cdot)$ is locally Lipschitz continuous then the conditions of Lemma 2.2 are satisfied with $\gamma(\cdot) \equiv K$, where K is a Lipschitz constant for $F(\cdot, \cdot)$ on $[S, T] \times R\mathbb{B}$.

The fact that (ACL) imposes the requirement that $F(\cdot, x)$ is absolutely continuous merely *from the left*, i.e. it is defined in terms of the asymmetric distance function rather than the Hausdorff distance function, means that (ACL) permits examples of $F(\cdot, \cdot)$ which are discontinuous.

Example 1: Consider the control system

$$\begin{cases} \dot{x}(t) = f(x) + b(t)u(t) & a.e. \ t \in [0, 1] \\ u(t) \in [-1, 1] \end{cases}$$

where f is a locally Lipschitz function on \mathbb{R} and $b(\cdot)$ is the discontinuous function

$$b(t) = \begin{cases} 0.5 & \text{if } t \in [0, 0.5] \\ 1 & \text{if } t \in (0.5, 1] \end{cases}.$$

This control system generates the same trajectories as the differential inclusion with discontinuous velocity set

$$F(t, x) = f(x) + b(t)[-1, 1].$$

This multifunction is absolutely continuous from the left uniformly over $x \in R\mathbb{B}$ for any R . Yet it is discontinuous.

The relation between (ACL) and the hypothesis ‘ $F(\cdot, x)$ is Hölder continuous from the left with index $\alpha \in (0, 1)$ uniformly over $x \in R\mathbb{B}$ ’, in the sense that:

$$\left\{ \begin{array}{l} \text{there exists } K > 0 \text{ such that for all intervals } [s, t] \subset [S, T] \text{ and } x \in R\mathbb{B} \\ F(s, x) \subset F(t, x) + K(t - s)^\alpha \mathbb{B}, \end{array} \right.$$

is not a simple one. Classical constructions of functions that are Hölder continuous for some index α , yet are nowhere differentiable (consider for example ‘space-filling’ Peano curves, [19]), permit us to conclude that $F(\cdot, x)$ may be Hölder continuous from the left, yet fail to satisfy (ACL). On the other hand, the function $f : [0, 1] \rightarrow \mathbb{R}$, vanishing at 0 and with derivative expressible as the absolutely convergent sum

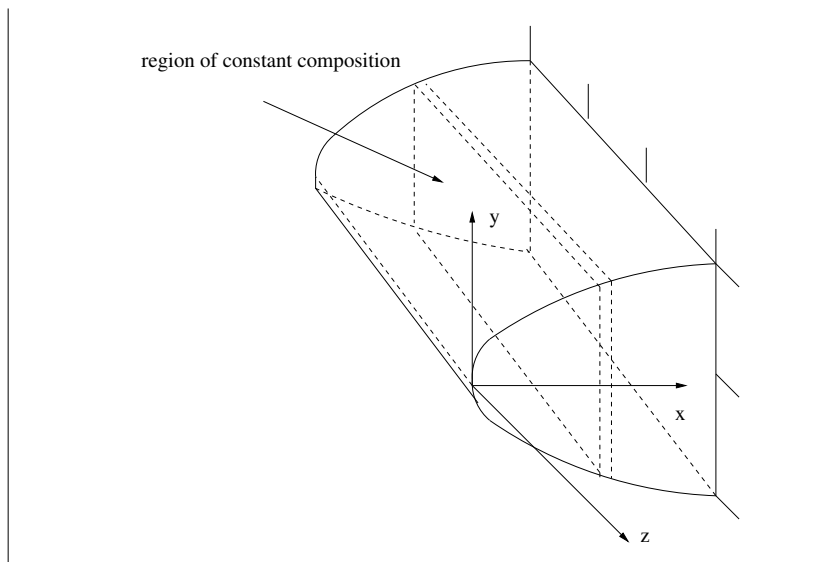
$$df(t)/dt = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k t^{\frac{1}{k}},$$

of $L^1(0, 1)$ functions, yields a multifunction $F(t, x) = \{f(t)\}$ satisfying (ACL) but which is not Hölder continuous for any index α . More generally, (ACL) permits multifunctions for which the

modulus of absolute continuity $\gamma(\cdot)$ in (1) is a weighted sum of fractional powers of t on the interval $[0, 1]$, thus

$$\gamma(t) = \sum_{k=1}^{\infty} c_k t^{\beta_k}. \quad (2)$$

Here, all the exponents β_1, β_2, \dots are assumed to lie in $(0, 1)$ and the c_k 's are non-negative numbers satisfying $\sum_{k=1}^{\infty} c_k < \infty$.



Example 2: *The purpose of this example is to illustrate the potential relevance of the weakened supplementary hypothesis (ACL) in applications. In this example, concerning civil engineering design, the object is to design a beam in 3D space, of infinite length, with a smooth surface and having a constant cross-section in the direction of the z -axis, to maximize bending rigidity (which we may interpret as minimizing the displacement of the free edge, for a fixed uniform load per unit length along this edge). The beam is to be constructed from a composition of two materials A and B ; the composition varies along the x axis, but is constant on any plane normal to the x axis. We can think of A as a material which adds stiffness to the structure, but which must be blended with the less expensive material B to reduce cost.*

Suppose that the cross-section of the beam orthogonal to the z axis is a parabola, and the free edge is located at $(x, y) = (0, 0)$. Thus points (x, y) on the surface of the beam satisfy

$$y = x^{1/2}, \quad 0 \leq x \leq 1.$$

Let $w(x) \in [0, 1]$ denote variation of the proportion of material A w.r.t. x . We assume there is a bound V of the volume per unit length of material A in the beam. This gives rise to the isoperimetric constraint

$$\int_0^1 2w(x)|x|^{1/2} dx \leq V.$$

A restriction is placed on the rate of variation of the composition along the x axis, giving rise to the constraint

$$|dw(x)/dx| \leq k \quad \text{for all } x \in [0, 1]. \quad (3)$$

Finally, the proportion of $w(x)$ material A , for any x , must satisfy the constraint

$$w(x) \in [0, 1] \quad \text{for all } x \in [0, 1]. \quad (4)$$

The cost function will be a complicated function of additional variables, whose values are obtained by solving differential equations which depend on $w(\cdot)$.

This problem can be set up as an optimal control problem in which x is a time-like variable and $u(x) = dw(x)/dx$ is the control, involving the control constraint (3) and the state constraint (4). To investigate the solution with the help of the Maximum Principle involves replacing the isoperimetric constraint with a differential equation for an augmented state variable e satisfying the differential equation

$$de(x)/dx = 2w(x)|x|^{1/2}.$$

The key point here is that the augmented dynamics above involve data exhibiting non-Lipschitz behavior w.r.t. the time-like variable. But data of this nature is permitted by hypothesis (ACL), because the x -dependence is governed by a fractional power modulus of absolute continuity, as in (2). Notice that, whatever way the smooth profile of the beam is modeled (here, by a parabola), the x -dependence of the position of the upper surface of the beam will have an infinite derivative at the free edge, and will fail to conform to hypotheses requiring Lipschitz continuous dependence.

4 Limitations on the Validity of Linear Distance Estimates for State Constrained Differential Inclusions

We recall that for state constraint sets with smooth boundaries, linear, L^∞ distance estimates are valid for differential inclusions $\dot{x} \in F(t, x)$ and A satisfying hypotheses (H1), (H2), (CQ) alone. For general closed state constraint sets A however, currently available proofs of such distance estimates require the imposition of a supplementary hypothesis on the regularity of $F(\cdot, x)$. Concerning the need for a supplementary hypothesis, we note:

Proposition 4.1 *Data $F(\cdot, \cdot)$, and A , satisfying hypotheses (H1), (H2), (CQ) (for some r_0) can be chosen with the following property: given any $K > 0$, $\alpha \in (0, 1)$ and $\delta > 0$, there exists an interval $[t_0, t_1] \subset I$ of length not greater than δ and an F -trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap (e^{c(t_0-S)}(r_0 + 1) - 1)\mathbb{B}$ such that $\rho_{[t_0, t_1]}(\hat{x}(\cdot)) > 0$ and*

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} > K |\rho_{[t_0, t_1]}(\hat{x}(\cdot))|^\alpha,$$

for all feasible F -trajectories $x(\cdot)$ on $[t_0, t_1]$ with initial state $\hat{x}(t_0)$.

This proposition confirms that linear, L^∞ distance estimates are not valid in general under the basic hypotheses, not even ‘in the small’, i.e. over a sufficiently small time interval. It tells furthermore that not even weaker, Hölder-type distance estimates, with arbitrary Hölder index $\alpha \in (0, 1)$, are valid.

It might be thought that lack of continuity is the obstacle to obtaining linear distance estimates. The following proposition confirms however that this is not the case.

Proposition 4.2 *Data $F(\cdot, \cdot)$, $[S, T]$ and A , satisfying hypotheses (H1), (H2), (CQ) (for some constant r_0) and also the supplementary hypothesis*

(C): $F(\cdot, \cdot)$ is continuous,

can be chosen with the following property: given any $K > 0$ and $\delta > 0$, there exists an interval $[t_0, t_1] \subset [S, T]$ and an F -trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap (e^{c(t_0-S)}(r_0 + 1) - 1)\mathbb{B}$ such that $t_1 - t_0 \leq \delta$, $\rho_{[t_0, t_1]}(\hat{x}(\cdot)) > 0$ and

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} \geq K \theta(\rho_{[t_0, t_1]}(\hat{x}(\cdot))) ,$$

for all feasible F -trajectories $x(\cdot)$ on $[t_0, t_1]$ with initial state $\hat{x}(t_0)$. Here, $\theta(\cdot)$ is the modulus

$$\theta(\rho) = (1 + |\ln(\rho)|) \rho \quad \text{for } \rho > 0 .$$

This proposition tells us that adding the supplementary hypothesis ‘ F is continuous’ to the basic hypotheses is not enough to furnish linear L^∞ distance estimates; not even $(1 + |\ln(\rho)|)\rho$ -type distance estimates are in general valid under such hypotheses. (Note that $(1 + |\ln(\rho)|)\rho$ estimates are intermediate between linear estimates and Hölder estimates: i.e. they are weaker than linear estimates, yet stronger than Hölder estimates, of any index.)

The proofs of the above propositions are based on the construction of two counter-examples to the conjecture ‘linear, L^∞ distance estimates are valid under the basic hypotheses’, details of which are given in the Appendix.

5 Preliminary Analysis

In this section we take some preliminary steps towards the proof of Thm. 2.3. We show that some additional, simplifying assumptions on the data can be made, and examine some useful implications of hypothesis (CQ). Throughout $r_0 > 0$ is fixed. c is the constant of hypotheses (H1) and R is the constant of the theorem statement.

We begin by recalling an important existence theorem with accompanying estimates, known as Filippov’s Existence Theorem (see [1] or [21]), which is frequently invoked in our analysis.

Theorem 5.1 (Filippov’s Existence Theorem) *Consider a multi-function $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ taking closed non-empty values such that $F(\cdot, x)$ is \mathcal{L} -measurable for all $x \in \mathbb{R}^n$ and satisfies:*

(H2)’: *There exists $k_F(\cdot) \in L^1(S, T)$ such that*

$$F(t, x') \in F(t, x) + k_F(t)|x - x'| \mathbb{B} \quad \text{for all } x, x' \in \mathbb{R}^n \text{ and a.e. } t \in [S, T] .$$

Take any absolutely continuous arc $y : [S, T] \rightarrow \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. If $d_{F(\cdot, y(\cdot))}(\dot{y}(\cdot)) \in L^1(S, T)$, then, there exists an F -trajectory $x(\cdot)$ satisfying $x(S) = \xi$ such that for all $t \in (S, T]$

$$\begin{aligned} \|y(\cdot) - x(\cdot)\|_{L^\infty(S, t)} &\leq |y(S) - x(S)| + \int_S^t |\dot{y}(s) - \dot{x}(s)| ds \\ &\leq e^{\int_S^t k_F(s) ds} \left(|\xi - y(S)| + \int_S^t d_{F(s, y(s))}(\dot{y}(s)) ds \right) . \end{aligned}$$

Lemma 5.2 (Hypothesis Reduction) *Fix $\delta > 0$ and $\bar{\rho} > 0$. Assume that the assertions of Thm. 2.3 (for the given r_0) are valid under hypotheses (H1), (H2)’ (the strengthened version of (H2) defined in the statement of Thm. 5.1), (CQ) and (ACL) and under the additional hypothesis on the data:*

(H3): $F(t, x)$ is convex for all $(t, x) \in [S, T] \times \mathbb{R}^n$,

and when the following conditions are imposed on the reference F -trajectory $\hat{x}(\cdot) : [s, t] \rightarrow \mathbb{R}^n$, with $\hat{x}(s) \in A \cap (e^{c(s-S)}(r_0 + 1) - 1)\mathbb{B}$, and the positive number $\rho \geq \rho(\hat{x}(\cdot))$:

(i): $\rho \leq \bar{\rho}$, and

(ii): $t - s \leq \delta$.

Then the assertions are valid under (H1), (H2), (CQ) and (ACL) alone.

Proof. In what follows, $r_0 > 0$ and $R = e^{c(T-S)}(r_0 + 1)$.

Step 1: Assume that the assertions are valid (with constant K) under hypotheses (H1), (H2)', (H3), (CQ) and (ACL), and when it is assumed that $\hat{x}(\cdot)$ on $[s, t]$ satisfies conditions (i) and (ii). We show that they remain valid (with a modified K) even if $\hat{x}(\cdot)$ violates condition (i).

By assumption, the assertions are valid (with constant K) if $\rho \leq \bar{\rho}$. Suppose that $\rho > \bar{\rho}$. By standard viability theorems (see for instance [11] or [12]), there exists some feasible F -trajectory $x(\cdot)$ on $[s, t]$, with $x(s) = \hat{x}(s)$. Now apply the special case of the theorem we assume to be valid, treating $x(\cdot)$ as the reference trajectory, to justify replacing $x(\cdot)$ by an F -trajectory (we do not re-label) that is *strictly* feasible on $(s, t]$. Then by (H1)

$$\|x(\cdot) - \hat{x}(\cdot)\|_{L^\infty} \leq 2c(1 + R)(T - S) \leq 2\bar{\rho}^{-1}c(1 + R)(T - S) \times \rho.$$

So the assertions of the theorem are valid, in absence of the condition (i), with the larger constant K

$$\max\{K, 2\bar{\rho}^{-1}c(1 + R)(T - S)\}.$$

Step 2: Assume that the assertions are valid (with constant K) under hypotheses (H1), (H2)', (H3), (CQ) and (ACL), and when it is assumed that the reference trajectory $\hat{x}(\cdot)$ on $[s, t]$ satisfies condition (ii). We show that they remain valid (with a modified K) even if condition (i) is violated.

Choose N to be the smallest integer such that $N^{-1}(T - S) \leq \delta$. Write $x_0(\cdot) = \hat{x}(\cdot)$. Partition $[s, t]$ as a family of N contiguous intervals $\{[t_0^i, t_1^i]\}_{i=1}^N$ with $t_0^1 = s$ and $t_1^N = t$, each of length at most δ . Now apply the special case of the theorem (in which condition (ii) is assumed to hold) with $\hat{x}(\cdot)|_{[t_0^1, t_1^1]}$ as reference trajectory, to yield an F -trajectory $x_1(\cdot)$ on $[t_0^1, t_1^1]$ such that $x_1(\cdot)$ is strictly feasible on $(t_0^1, t_1^1]$, $x_1(t_0^1) = \hat{x}(t_0^1)$ and

$$\|x_1(\cdot) - x_0(\cdot)\|_{L^\infty(t_0^1, t_1^1)} \leq K\rho.$$

Invoking the Filippov Existence Theorem (Thm. 5.1), we can extend $x_1(\cdot)$ as an F -trajectory to $[t_0^1, t_1^N]$ (we do not re-label) such that

$$\|x_1(\cdot) - x_0(\cdot)\|_{L^\infty(t_0^1, t_1^N)} \leq K_1K\rho = K_1K(\rho(x_0(\cdot)) \vee \rho),$$

for some constant K_1 that does not depend on the choice of $\hat{x}(\cdot)$. Now apply the special case of the theorem (in which condition (ii) is satisfied), taking as reference trajectory $x_1(\cdot)$ restricted to $[t_0^2, t_1^2]$, to yield an F -trajectory $x_2(\cdot)$ on $[t_0^1, t_1^2]$ that is strictly feasible on $(t_0^1, t_1^2]$, which we

extend to $[t_0^1, t_1^N]$, and so on. We thereby generate a sequence of F -trajectories $x_i(\cdot)$ on $[t_0^1, t_1^N]$, $i = 1, \dots, N$, such that for each $i = 1, \dots, N$, $x_i(\cdot)$ is strictly feasible on (t_0^1, t_1^i) and

$$\|x_i(\cdot) - x_{i-1}(\cdot)\|_{L^\infty(t_0^1, t_1^N)} \leq K_1 K (\rho(x_{i-1}(\cdot)) \vee \rho) .$$

We also have

$$\rho(x_i(\cdot)) \vee \rho \leq (\rho(x_{i-1}(\cdot)) \vee \rho) + \|x_i(\cdot) - x_{i-1}(\cdot)\|_{L^\infty(t_0^1, t_1^N)} .$$

Now write $x(\cdot) = x_N(\cdot)$. Then $x(\cdot)$ is strictly feasible on $(t_0^1, t_0^N) = (s, t)$ and, from the preceding relations,

$$\|x(\cdot) - \hat{x}(\cdot)\|_{L^\infty(s, t)} \leq \bar{K} \rho ,$$

in which $\bar{K} = (1 + K_1 K)^N - 1$. The assertions are therefore valid even if (ii) is not satisfied, when we replace K by \bar{K} .

Step 3: Assume that the assertions are valid (with constant K) under hypotheses (H1), (H2)', (H3), (CQ) and (ACL). We show that they remain valid even if (H3) is violated, i.e. F is not convex valued.

Assume that the above hypotheses are satisfied, with the exception of (H3). Replace F by $\text{co } F$. Then the above hypotheses, including (H3), are satisfied. The special case of the theorem yields a constant K (independent of the choice of reference trajectory $\hat{x}(\cdot)$ on $[s, t]$) and a $\text{co } F$ trajectory $x'(\cdot) : [s, t] \rightarrow \mathbb{R}^n$, which is strictly feasible on $(s, t]$, such that

$$\|x'(\cdot) - \hat{x}(\cdot)\|_{L^\infty(s, t)} \leq K \rho .$$

Choose a decreasing sequence $\{s_i\}$ in $(s, t]$, with $s_1 = t$, such that $s_i \downarrow s$. Since $x'(\cdot)$ is strictly feasible on $(s, t]$ we can find a sequence of positive numbers $\epsilon_i \in (0, \rho)$ such that $\epsilon_i \downarrow 0$ and, for $i = 1, 2, \dots$

$$x'(\sigma) + \epsilon_i \mathbb{B} \subset A \quad \text{for all } \sigma \in [s_i, t] . \quad (5)$$

Take a sequence of positive numbers $\{\alpha_i\}$. (We shall place restrictions on the α_i 's presently.) By the Relaxation Theorem (which asserts the density, with respect to the L^∞ norm, of the set of F -trajectories with a fixed initial state in the set of $\text{co } F$ -trajectories, with the same initial state; cf. [1] or [21]), there exists a sequence of F -trajectories $x_i(\cdot) : [s_i, t] \rightarrow \mathbb{R}^n$ such that, for all integer $i \geq 2$, we have $x_i(s_i) = x'(s_i)$ and

$$\|x_i(\cdot) - x'(\cdot)\|_{L^\infty(s_i, t)} \leq \alpha_i . \quad (6)$$

For each integer $j \geq 2$, we construct an F -trajectory $y_j(\cdot) : [s_j, t] \rightarrow \mathbb{R}^n$ as follows:

$y_j(\cdot)$ restricted to $(s_j, s_{j-1}]$ coincides with $x_j(\cdot)$. $y_j(\cdot)$ restricted to $(s_{j-1}, s_{j-2}]$ is an F -trajectory with initial state $y_j(s_{j-1})$, obtained by applying Theorem 5.1 with reference trajectory $x_{j-1}(\cdot)$, and so on, until $y_j(\cdot)$ has been constructed on the whole interval $[s_j, s_1 = t]$.

Now fix an integer $j > 2$. We deduce from Theorem 5.1 that, for each $2 \leq i < j$,

$$\begin{aligned} \|y_j(\cdot) - x_i(\cdot)\|_{L^\infty(s_i, s_{i-1})} &\leq M |y_j(s_i) - x'(s_i)| \\ \|\dot{y}_j(\cdot) - \dot{x}_i(\cdot)\|_{L^1(s_i, s_{i-1})} &\leq M |y_j(s_i) - x'(s_i)| , \end{aligned}$$

where $M := \exp \int_S^T k_F(\sigma) d\sigma$. (We have also used the fact that $x_i(s_i) = x'(s_i)$). From these relations and (6) it follows that for each $2 \leq i < j$ and any integer m , we have

$$\|y_j(\cdot) - x'(\cdot)\|_{L^\infty(s_i, s_{i-1})} \leq \sum_{k=i}^j M^{k-i} \alpha_k \quad (7)$$

$$\|\dot{y}_{j+m}(\cdot) - \dot{y}_j(\cdot)\|_{L^1(s_i, s_{i-1})} \leq 2 \sum_{k=i+1}^{j+m} M^{k-i} \alpha_k. \quad (8)$$

Notice that for each $j \geq 2$, $y_j(s_j) = x'(s_j)$. So we can extend each F -trajectory $y_j(\cdot)$ as an $\text{co}F$ trajectory to all of $[s, t]$, setting $y_j(\sigma) = x'(\sigma)$ for $\sigma \in [s, s_j]$. (We do not re-label.)

Now choose the sequence $\{\alpha_k\}$ to satisfy

$$\sum_{k=i}^{\infty} M^k \alpha_k < \epsilon_i/2, \quad \text{for all } i \geq 2. \quad (9)$$

This condition is satisfied, in particular, if we assume that $\epsilon_i < 1/3$, for all $i \geq 2$, and we chose $\alpha_k = (\epsilon_k/M)^k$.

Since the $y_i(\cdot)$'s have initial value $\hat{x}(s)$ and in view of hypothesis (H1), we can extract a subsequence (we do not re-label) converging uniformly to a $\text{co}F$ -trajectory $x(\cdot)$ on $[s, t]$, with initial value $\hat{x}(s)$. We conclude from (5), (7) and (9) that $x(\cdot)$ is strictly feasible on $(s, t]$. To see this, take any $\sigma \in (s, t]$ and note that $\sigma \in (s_i, s_{i-1}]$ for some $i \geq 2$. But then from (7) and (9) we have

$$y_j(\sigma) \in x'(\sigma) + \frac{\epsilon_i}{2} \mathbb{B} \subset \text{int } A, \quad \text{for all } j \geq i.$$

Since the $y_j(\cdot)$'s converge uniformly to $x(\cdot)$,

$$x(\sigma) \in x'(\sigma) + \frac{\epsilon_i}{2} \mathbb{B} \subset \text{int } A.$$

On the other hand, for each $k \geq 2$, the y_i 's, restricted to $[s_k, t]$, are F -trajectories, which, owing to (8), define a Cauchy sequence in $W^{1,1}(s_k, s_{k-1})$. It may be deduced that the limiting $\text{co}F$ -trajectory $x(\cdot)$ is actually an F -trajectory. Finally we note that, since each $\epsilon_i \leq \rho$,

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(s,t)} \leq \|\hat{x}(\cdot) - x'(\cdot)\|_{L^\infty(s,t)} + \|x(\cdot) - x'(\cdot)\|_{L^\infty(s,t)} \leq \bar{K} \rho,$$

where $\bar{K} = K + 1$. This is the desired distance estimate, with the modified constant \bar{K} .

Step 4: Assume that the assertions are valid (with constant K) under hypotheses (H1), (H2)', (CQ) and (ACL). We show that they remain valid when (H2)' is replaced by (H2).

Define the multivalued function $\tilde{F}(t, x) := F(t, \Pi_{R\mathbb{B}}(x))$ (we recall that $\Pi_{R\mathbb{B}}(x)$ is the unique projection of the point $x \in \mathbb{R}^n$ into the closed ball $R\mathbb{B}$). Since the projection on the (convex) set $R\mathbb{B}$ is Lipschitz with Lipschitz constant 1, it follows that \tilde{F} is globally integrably Lipschitz with respect to x and satisfies (H2)' w.r.t. the same function $k_F(\cdot)$.

Consider an arbitrary reference \tilde{F} -trajectory $\hat{x}(\cdot)$ on $[s, t]$ with $\hat{x}(s) \in A \cap (e^{c(s-S)}(r_0 + 1) - 1)\mathbb{B}$. The special case of Thm. 2.3 yields an \tilde{F} -trajectory $x(\cdot)$ with the desired properties. But, since $\hat{x}(\cdot)$ and $x(\cdot)$ stay in $R\mathbb{B}$, $\hat{x}(\cdot)$ and $x(\cdot)$ are in fact F -trajectories. We are therefore justified in assuming that (H2) has been replaced by the stronger hypothesis (H2)'. The proof of Lemma 5.2 is complete.

Hypothesis (CQ) is a local condition on the existence of velocities in F pointing into the interior of A , at each point $(t, x) \in [S, T] \times (R\mathbb{B} \cap \partial A)$, in some uniform sense conveyed by a parameter $\epsilon > 0$. The proof of the following lemma exploits the properties of the lim inf operation on a sequence of sets, to show that (CQ) implies a related global property, in which the same uniformity parameter ϵ serves for *all* points $(t, x) \in [S, T] \times ((\partial A + \eta\mathbb{B}) \cap R\mathbb{B} \cap A)$ for some $\eta > 0$:

Lemma 5.3 *Suppose the multifunction $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and the closed set A satisfy hypothesis (CQ) (for some $R \geq 0$). Then there exist $M > 0$, $\epsilon > 0$ and $\eta > 0$ with the following property: for any $(t, x) \in [S, T] \times ((\partial A + \eta\mathbb{B}) \cap R\mathbb{B} \cap A)$, there exists $v \in \text{co} F(t, x)$ such that $|v| \leq M$ and*

$$y + [0, \epsilon](v + \epsilon\mathbb{B}) \subset A$$

for all $y \in (x + \epsilon\mathbb{B}) \cap A$.

Proof.

Step 1: We claim that for each $(t, x) \in [S, T] \times (R\mathbb{B} \cap \partial A)$ there exist $M_{t,x} > 0$, $\epsilon_{t,x} \in (0, 1)$ and $\delta_{t,x} \in (0, \epsilon_{t,x}]$ such that, given any $(t', x') \in ((t, x) + \delta_{t,x}\mathbb{B}) \cap ([S, T] \times A)$, a vector $v' \in \text{co} F(t', x')$ can be found such that $|v'| \leq M_{t,x}$ and

$$y' + [0, \epsilon_{t,x}](v' + \epsilon_{t,x}\mathbb{B}) \subset A, \quad \text{for all } y' \in (x' + \epsilon_{t,x}\mathbb{B}) \cap A.$$

Indeed, take any $(t, x) \in [S, T] \times (R\mathbb{B} \cap \partial A)$ and chose any vector

$$v \in \left(\liminf_{(t', x') \xrightarrow{[S, T] \times A} (t, x)} \text{co} F(t', x') \right) \cap \text{int } T_A(x).$$

By the characterization of the interior of the Clarke tangent cone (see for instance [18]), there exists $\epsilon \in (0, 1)$ such that

$$y + [0, \epsilon](v + 2\epsilon\mathbb{B}) \subset A \quad \text{for all } y \in (x + 2\epsilon\mathbb{B}) \cap A. \quad (10)$$

On the other hand, by definition of the limit inferior operation, there exists $\delta \in (0, \epsilon]$ such that, given any $(t', x') \in ((t, x) + \delta\mathbb{B}) \cap ([S, T] \times A)$, there exists $v' \in \text{co} F(t', x')$ satisfying $|v - v'| \leq \epsilon$. Then, $|v'| \leq |v| + 1 (=: M_{t,x})$.

Now take any $y' \in (x' + \epsilon\mathbb{B}) \cap A$. Then, since $x' + \epsilon\mathbb{B} \subset x + 2\epsilon\mathbb{B}$ and $v' \in v + \epsilon\mathbb{B}$, we may conclude from (10)

$$y' + [0, \epsilon](v' + \epsilon\mathbb{B}) \subset A \quad \text{for all } y' \in (x' + \epsilon\mathbb{B}) \cap A.$$

Step 2: By a standard compactness argument, we can find a finite number of points $(t_i, x_i) \in [S, T] \times (R\mathbb{B} \cap \partial A)$ and numbers $M_i > 0$, $\epsilon_i > \delta_i > 0$, for $i = 1, \dots, m$, such that

$$\bigcup_{i=1, \dots, m} \left((t_i, x_i) + \delta_i \overset{\circ}{\mathbb{B}} \right) \supset [S, T] \times (R\mathbb{B} \cap \partial A) \quad (11)$$

(here $\overset{\circ}{\mathbb{B}}$ denotes the open unit ball), and for each $(t', x') \in ((t_i, x_i) + \delta_i\mathbb{B}) \cap ([S, T] \times A)$, there exists a vector $v' \in \text{co} F(t', x')$ such that $|v'| \leq M_i$ and

$$y' + [0, \epsilon_i](v' + \epsilon_i\mathbb{B}) \subset A, \quad \text{for all } y' \in (x' + \epsilon_i\mathbb{B}) \cap A.$$

Notice also that there exists $\eta \in (0, \min_{i=1, \dots, m} \delta_i)$ such that

$$\bigcup_{i=1, \dots, m} \left((t_i, x_i) + \delta_i \overset{\circ}{\mathbb{B}} \right) \supset [S, T] \times ((\partial A + \eta \mathbb{B}) \cap R\mathbb{B}),$$

otherwise we could find a sequence of points $(s_j, y_j) \in R\mathbb{B} \setminus \bigcup_{i=1, \dots, m} \left((t_i, x_i) + \delta_i \overset{\circ}{\mathbb{B}} \right)$ such that $(s_j, y_j) \rightarrow (s, y) \in [S, T] \times (R\mathbb{B} \cap \partial A)$, which would contradict (11).

To conclude we just take $\epsilon = \min_{i=1, \dots, m} \epsilon_i$, $M = \max_{i=1, \dots, m} M_i$ and the assertions of the lemma immediately follow.

6 Proof of Theorem 2.3

Fix $r_0 > 0$. Set $R := e^{c(T-S)}(1 + r_0)$ and $\bar{R} := c(1 + R)$. (The constants R and \bar{R} bound, respectively, magnitudes and velocities of arcs $x(\cdot)$ on subintervals of $[S, T]$ originating in $r_0\mathbb{B}$ and satisfying $|\dot{x}| \leq c(1 + |x|)$.)

We know (see Lemma 5.3) that there exist $M > 0$, $\epsilon > 0$ and $\eta > 0$ with the property: given any $(t, x) \in [S, T] \times ((\partial A + \eta \mathbb{B}) \cap R\mathbb{B} \cap A)$, $v \in \text{co} F(t, x)$ can be found such that $|v| \leq M$ and

$$x' + [0, \epsilon](v + \epsilon \mathbb{B}) \subset A \quad (12)$$

for all $x' \in (x + \epsilon \mathbb{B}) \cap A$. Notice that assumption (H1) yields $|v| \leq \bar{R}$ and so, in fact, we can take $M = \bar{R}$. From assumption (ACL) we also know that $\eta > 0$ can be chosen such that

$$F(t, x) \subset F(s, x) + \int_s^t \gamma(s') ds' \mathbb{B} \quad (13)$$

for all points $x \in (\partial A + \eta \mathbb{B}) \cap R\mathbb{B}$ and subintervals $[s, t] \subset [S, T]$. Here $\gamma(\cdot)$ is the summable function of Lemma 2.2.

Let $k_F(\cdot)$ be the integrable function of hypothesis (H2). Define the non-negative functions $\theta(\cdot)$ and $\omega(\cdot)$ on $[0, T - S]$

$$\theta(\sigma) := \sup \left\{ \int_I \gamma(s) ds \right\} \quad \text{and} \quad \omega(\sigma) := \sup \left\{ \int_I k_F(s) ds \right\}$$

where (in both definitions) the supremum is taken over sub-intervals $I \subset [S, T]$ of length not greater than σ . By properties of integrable functions, $\theta(\sigma) \rightarrow 0$ and $\omega(\sigma) \rightarrow 0$, as $\sigma \downarrow 0$.

Take $k > 0$ such that $k > \epsilon^{-1}$ and choose $\Delta > 0$ and $\bar{\rho} > 0$ such that

$$\Delta \leq \epsilon, \quad \bar{\rho} + \bar{R}\Delta < \epsilon, \quad k\bar{\rho} < \epsilon, \quad \bar{\rho} \leq \eta, \quad 4\Delta\bar{R} \leq \eta, \quad (14)$$

and

$$e^{\omega(\Delta)}(\theta(\Delta) + \omega(\Delta)\bar{R}(T - S)) < \epsilon, \quad 2e^{\omega(\Delta)}(\theta(\Delta) + \omega(\Delta)\bar{R})k < (k\epsilon - 1). \quad (15)$$

To prove the theorem we must find $K > 0$ such that, given any sub-interval $[t_0, t_1] \subset [S, T]$, F -trajectory $\hat{x}(\cdot)$ on $[t_0, t_1]$ with $\hat{x}(t_0) \in A \cap (e^{c(t_0-S)}(r_0 + 1) - 1)\mathbb{B}$ and $\rho > 0$ satisfying $\rho \geq \rho_{[t_0, t_1]}(\hat{x}(\cdot))$, there exists a feasible F -trajectory $x(\cdot)$ on $[t_0, t_1]$ with the same initial state satisfying

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} \leq K \rho.$$

In view of Lemma 5.2, we can assume, without loss of generality, that $F(.,.)$ is convex valued, $\rho \leq \bar{\rho}$ and $t_1 - t_0 \leq \Delta$.

Notice that we can restrict attention to the case $\hat{x}(t_0) \in (\partial A + \frac{\eta}{2}\mathbb{B}) \cap A \cap (e^{c(t_0-S)}(r_0+1) - 1)\mathbb{B}$. Indeed, if $\hat{x}(t_0) \in (A \cap (e^{c(t_0-S)}(r_0+1) - 1)\mathbb{B}) \setminus (\partial A + \frac{\eta}{2}\mathbb{B})$, then it follows from condition (14) on Δ that $x(\cdot) = \hat{x}(\cdot)$ has the required properties.

Since $F(.,.)$ is now convex valued and

$$\hat{x}(t_0) \in (\partial A + \frac{\eta}{2}\mathbb{B}) \cap A \cap (e^{c(t_0-S)}(r_0+1) - 1)\mathbb{B} \subset (\partial A + \frac{\eta}{2}\mathbb{B}) \cap R\mathbb{B} \cap A,$$

we can chose a vector $v \in F(t_0, \hat{x}(t_0))$ satisfying condition (12) when $(t, x) = (t_0, \hat{x}(t_0))$. Define $y(\cdot) : [t_0, t_1] \rightarrow \mathbb{R}^n$ to be the arc satisfying $y(t_0) = \hat{x}(t_0)$ and

$$\dot{y}(t) = \begin{cases} v & \text{if } t \in [t_0, (t_0 + k\rho) \wedge t_1] \\ \hat{x}(t - k\rho) & \text{if } t \in (t_0 + k\rho, t_1] \text{ and if } \hat{x}(t - k\rho) \text{ exists.} \end{cases}$$

Note that, for $t \geq t_0 + k\rho$,

$$y(t) = \hat{x}(t - k\rho) + k\rho v. \quad (16)$$

Observing that both v and $\|\dot{\hat{x}}(\cdot)\|_{L^\infty}$ are bounded by \bar{R} , we conclude that

$$\|\hat{x}(\cdot) - y(\cdot)\|_{L^\infty(t_0, t_1)} \leq 2\bar{R}k\rho. \quad (17)$$

Take any $s \in [t_0, (t_0 + k\rho) \wedge t_1]$. In view of (13), and since $\|\dot{y}\|_{L^\infty} \leq \bar{R}$ and $v \in F(t_0, x_0)$, we have

$$\begin{aligned} d_{F(s, y(s))}(\dot{y}(s)) &\leq \theta(\Delta) + d_{F(t_0, y(s))}(v) \\ &\leq \theta(\Delta) + k_F(s)\bar{R}(s - t_0). \end{aligned} \quad (18)$$

By Theorem 5.1, and using the estimate (18), there exists an F -trajectory $x(\cdot)$ on $[t_0, (t_0 + k\rho) \wedge t_1]$ such that $x(t_0) = y(t_0)$ and, for any $t \in [t_0, (t_0 + k\rho) \wedge t_1]$

$$\|x(\cdot) - y(\cdot)\|_{L^\infty(t_0, t)} \leq e^{\omega(\Delta)}(\theta(\Delta) + \omega(\Delta)\bar{R})(t - t_0). \quad (19)$$

If $t_0 + k\rho < t_1$, then from (13), (14) and (16) it follows that, for a.e. $s \in [t_0 + k\rho, t_1]$,

$$\begin{aligned} d_{F(s, y(s))}(\dot{y}(s)) &= d_{F(s, k\rho v + \hat{x}(s - k\rho))}(\dot{\hat{x}}(s - k\rho)) \\ &\leq k_F(s)\bar{R}k\rho + d_{F(s, \hat{x}(s - k\rho))}(\dot{\hat{x}}(s - k\rho)) \\ &\leq k_F(s)\bar{R}k\rho + \int_{s - k\rho}^s \gamma(s') ds' + d_{F(s - k\rho, \hat{x}(s - k\rho))}(\dot{\hat{x}}(s - k\rho)) \\ &= k_F(s)\bar{R}k\rho + \int_{s - k\rho}^s \gamma(s') ds' + 0. \end{aligned}$$

But by Fubini's Theorem

$$\int_{t_0 + k\rho}^t \int_{s - k\rho}^s \gamma(s') ds' ds = \int_{t_0}^t \int_{t_0}^{t_1} \chi_{[t_0 + k\rho, t]}(s) \chi_{[s - k\rho, s]}(s') ds \gamma(s') ds' \leq \theta(\Delta)k\rho.$$

We deduce that, for any $t \in [t_0 + k\rho, t_1]$,

$$\int_{t_0 + k\rho}^t d_{F(s, y(s))}(\dot{y}(s)) ds \leq (\omega(\Delta)\bar{R} + \theta(\Delta))k\rho.$$

Appealing once again to Theorem 5.1, we can extend the F -trajectory $x(\cdot)$ from $[t_0, (t_0 + k\rho) \wedge t_1]$ to $[t_0, t_1]$ (we do not re-label) such that

$$\|x(\cdot) - y(\cdot)\|_{L^\infty(t_0, t_1)} \leq 2e^{\omega(\Delta)}(\theta(\Delta) + \omega(\Delta)\bar{R})k\rho. \quad (20)$$

It follows from (17) that

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(t_0, t_1)} \leq K\rho$$

where

$$K = 2 \left(\bar{R} + e^{\omega(\Delta)}(\theta(\Delta) + \omega(\Delta)\bar{R}) \right) k.$$

It remains then to show that

$$x(t) \in \text{int } A \quad \text{for } t \in (t_0, t_1].$$

We need to consider two cases:

Case (A): $t \in (t_0, t_0 + k\rho]$. Since $y(t) = \hat{x}(t_0) + (t - t_0)v$ and $t - t_0 \leq \epsilon$, it follows from (12) that

$$y(t) + (t - t_0)\epsilon\mathbb{B} = \hat{x}(t_0) + (t - t_0)(v + \epsilon B) \subset A.$$

But then, by (19) and (15), $x(t) \in \text{int } A$.

Case (B): $t \in (t_0 + k\rho, t_1]$. Let $\pi(\cdot)$ be a projection on A of the arc $t \rightarrow \hat{x}(t - k\rho)$:

$$\pi(t) \in \Pi_A(\hat{x}(t - k\rho)).$$

We have $\pi(t) \in A$ and

$$|\hat{x}(t - k\rho) - \pi(t)| = d_A(\hat{x}(t - k\rho)) \leq \rho.$$

From (16)

$$y(t) \in \pi(t) + k\rho v + \rho\mathbb{B}. \quad (21)$$

Since $|\hat{x}(t - k\rho) - \hat{x}(t_0)| \leq \bar{R}(t_1 - t_0)$,

$$|\pi(t) - \hat{x}(t_0)| \leq \bar{\rho} + \bar{R}\Delta.$$

Taking note of (12) and (14), we see that

$$\pi(t) + k\rho v + k\rho\epsilon\mathbb{B} \subset A.$$

So, by (21),

$$y(t) + (k\epsilon - 1)\rho\mathbb{B} \subset A.$$

But then, in view of (15) and (20), $x(t) \in \text{int } A$ in this case also. The proof is complete.

7 Appendix

In this appendix we provide proofs of the two propositions in Section 4, concerning the validity, or otherwise, of distance estimates, in the absence of a supplementary hypothesis on the regularity of $F(\cdot, x)$.

Proof of Proposition 4.1. To prove the proposition it suffices to construct an interval $[S, T]$, multifunction $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and a closed set $A \subset \mathbb{R}^n$, satisfying (for some $r_0 > 0$)

hypotheses (H1), (H2) and (CQ) (but not (ACL)) with the following properties: for any $K > 0$, $\delta > 0$ and $\alpha \in (0, 1)$, there exist an interval $I \subset [S, T]$ of length not greater than δ , and an F -trajectory $\hat{x}(\cdot)$ on I with initial state in $r_0\mathbb{B}$ such that

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(I)} > K |\rho_I(\hat{x}(\cdot))|^\alpha,$$

for all feasible F -trajectories $x(\cdot)$ on I with the same initial state. We proceed to construct such an F and A . Notice that F , defined below, depends only on t , and the A is the intersection of two closed half-spaces (which might be thought of as the simplest kind of state constraint set with non-smooth boundary).

Fix an integer $N \geq 3$ and a real number $\nu \in (0, \frac{1}{4}]$. Let $y(\cdot) : [0, 1] \rightarrow \mathbb{R}$ be the function defined by the properties: $y(0) = 0$,

$$y(t_k) := (-1)^k \frac{t_k}{2},$$

and

$$\dot{y}(t) = (-1)^k \frac{N+1}{2(N-1)} \quad \text{for } (t_{k+1}, t_k],$$

where t_k is the decreasing sequence of times

$$t_k := \frac{1}{N^k}, \quad k = 0, 1, 2, 3, \dots$$

For each k , we write s_k for the time when the piecewise affine function $y(\cdot)$ takes value zero in the interval $[t_{k+1}, t_k]$:

$$s_k = \frac{2t_k}{N+1} \quad \text{for } k = 0, 1, 2, 3, \dots$$

Let v_0, v_1 and v_3 be the vectors in \mathbb{R}^3 :

$$v_0 = (\nu, 0, 0), \quad v_1 = \left(1/2, \frac{N+1}{2(N-1)}, 0\right) \quad \text{and} \quad v_2 = \left(1/2, -\frac{N+1}{2(N-1)}, 0\right).$$

Set $[S, T] = [0, 1]$, $r_0 = 1$ and take the time-dependent multi-function $F(\cdot) : [0, 1] \rightsquigarrow \mathbb{R}^3$

$$F(t) := \begin{cases} \{v_0\} \cup \{v_1\} & \text{if } t \in (t_{k+1}, t_k] \\ \{v_0\} \cup \{v_2\} & \text{if } t \in (t_{k+2}, t_{k+1}], \end{cases}$$

where $k = 0, 2, 4, \dots$. Consider the state-constrained differential inclusion

$$\begin{aligned} \dot{x}(t) &\in F(t) \quad \text{a.e. } t \in [0, 1] \\ x(t) &\in A \quad \text{for all } t \in [0, 1], \end{aligned} \tag{22}$$

in which the state constraint A is the closed set

$$A = \{x \in \mathbb{R}^3 \mid x_2 - x_1 + x_3 \leq 0, -x_2 - x_1 + x_3 \leq 0\}.$$

For these choices of F and A , hypotheses (H1)-(H2) and (CQ) are satisfied (for the given r_0). But condition (ACL) is not satisfied.

Consider now the family of F -trajectories $\{\hat{x}_i(\cdot) : [s_i, 1] \rightarrow \mathbb{R}^3 \mid i = 1, 2, \dots\}$:

$$\hat{x}_i(t) = \left(\hat{x}_{1,i}(t), \hat{x}_{2,i}(t), \hat{x}_{3,i}(t)\right) = \left(\frac{t + s_i}{2}, y(t), s_i\right), \quad \text{for } t \in [s_i, 1], \quad i = 1, 2, \dots \tag{23}$$

For each i , $\hat{x}_i(s_i) = \xi_i$, where

$$\xi_i = (s_i, 0, s_i).$$

Observe that, for each i , $\xi \in A \cap r_0\mathbb{B}$ and $\hat{x}_i(\cdot)$ is an F -trajectory which is not feasible, with violation

$$\rho_{[s_i, 1]}(\hat{x}_i(\cdot)) = \frac{\sqrt{3}}{2} s_i \quad (= \max_{k \geq i} d_A(\hat{x}_i(t_k))) . \quad (24)$$

Notice also that for any i such that $t_i \leq \delta$ the state constraint violation of the arc $\hat{x}_i(\cdot)$ restricted to $[s_i, \delta]$ has the same value: $\rho_{[s_i, \delta]}(\hat{x}_i(\cdot)) = \frac{\sqrt{3}}{2} s_i$.

Now take $\nu = \nu(N) := \frac{1}{N+1}$ and define

$$\beta(N) := \left(\frac{1}{2} - \nu(N) \right) \left(\frac{N-1}{1 + [N-1]\nu(N)} \right) .$$

Note that $\frac{\beta(N)}{N} \rightarrow 1/4$ as $N \rightarrow \infty$.

Next take $x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot))$ to be any feasible F -trajectory on $[s_i, 1]$ with initial data $x(s_i) = \xi_i$. It has to satisfy the inequality $|x_2(t)| \leq x_1(t) - x_3(t)$, for all $t \in [s_i, 1]$, that is

$$|x_2(t)| \leq x_1(t) - s_i \quad \text{for all } t \in [s_i, 1] . \quad (25)$$

Moreover, for all $1 \leq j \leq i$ we have

$$\begin{aligned} \hat{x}_{1,i}(t_{j-1}) - x_1(t_{j-1}) &= \hat{x}_{1,i}(t_j) - x_1(t_j) + \int_{t_j}^{t_{j-1}} \left(\dot{\hat{x}}_{1,i}(t) - \dot{x}_1(t) \right) dt \\ &= \hat{x}_{1,i}(t_j) - x_1(t_j) + \left(\frac{1}{2} - \nu(N) \right) d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) , \end{aligned} \quad (26)$$

where, given an interval $I \subset [S, T]$, we write

$$d_I(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) := \text{meas} \{t \in I \mid \dot{\hat{x}}_i(\cdot) \neq \dot{x}(\cdot)\} .$$

From (25) and (26), we can deduce

$$|x_2(t_{j-1})| \leq |\hat{x}_{2,i}(t_{j-1})| - \frac{s_i}{2} - (\hat{x}_{1,i}(t_j) - x_1(t_j)) - \left(\frac{1}{2} - \nu(N) \right) d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot))$$

and therefore

$$|\hat{x}_{2,i}(t_{j-1}) - x_2(t_{j-1})| \geq \frac{s_i}{2} + (\hat{x}_{1,i}(t_j) - x_1(t_j)) + \left(\frac{1}{2} - \nu(N) \right) d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) . \quad (27)$$

We also know that

$$|(\hat{x}_{2,i}(t_{j-1}) - x_2(t_{j-1})) - (\hat{x}_{2,i}(t_j) - x_2(t_j))| = \frac{N+1}{2(N-1)} d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) . \quad (28)$$

Thus (27) and (28) imply that

$$\frac{N+1}{2(N-1)} d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) \geq s_i + (\hat{x}_{1,i}(t_j) - x_1(t_j)) + \left(\frac{1}{2} - \nu(N) \right) d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot))$$

and therefore

$$d_{[t_j, t_{j-1}]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) \geq \frac{N-1}{1 + [N-1]\nu(N)} \left(s_i + (\hat{x}_{1,i}(t_j) - x_1(t_j)) \right) . \quad (29)$$

Using (24), (26) and (29), for all $j = 1, \dots, i$, we have

$$|\hat{x}_{1,i}(t_{j-1}) - x_1(t_{j-1})| \geq \frac{2}{\sqrt{3}}\beta(N) \rho(\hat{x}_i(\cdot)) + (1 + \beta(N)) |\hat{x}_{1,i}(t_j) - x_1(t_j)|. \quad (30)$$

On the other hand, from (24) and (25), similarly as in (30), we obtain

$$|\hat{x}_{1,i}(t_i) - x_1(t_i)| = \left(\frac{1}{2} - \nu(N)\right) d_{[s_i, t_i]}(\dot{\hat{x}}_i(\cdot), \dot{x}(\cdot)) \geq \frac{1}{\sqrt{3}} \left(\frac{1}{2} - \nu(N)\right) \rho(\hat{x}_i(\cdot)). \quad (31)$$

Fix any $\delta \in (0, 1)$. Then there exists $k \in \mathbb{N}$ such that $t_{k+1} \leq \delta \leq t_k$. Consider now any $i \geq k+3$. The following estimates can be deduced from (30) and (31)

$$\begin{aligned} |\hat{x}_{1,i}(t_{k+1}) - x_1(t_{k+1})| &\geq \frac{1}{4\sqrt{3}} \sum_{j=0}^{i-k-1} (1 + \beta(N))^j \rho(\hat{x}_i(\cdot)) \\ &> \frac{1}{4\sqrt{3}} (1 + \beta(N))^{\log_N(\delta)-1} \left(\frac{\sqrt{3}}{N+1}\right)^{\log_N(1+\beta(N))} [\rho(\hat{x}_i(\cdot))]^{1-\log_N(1+\beta(N))}. \end{aligned} \quad (32)$$

From the properties of log functions, we deduce that

$$\log_N(1 + \beta(N)) = \frac{\ln(1 + \beta(N))}{\ln(N)} = 1 + \frac{\ln\left(\frac{1}{N} + \frac{\beta(N)}{N}\right)}{\ln(N)} \rightarrow 1,$$

as $N \rightarrow \infty$, since $\frac{1}{N} + \frac{\beta(N)}{N} \rightarrow 1/4$. (Taking the logarithm to base e here was an arbitrary choice.)

Now take any $\alpha \in (0, 1)$, $\alpha' \in (0, \alpha)$ and $K > 0$. In view of the preceding relation, we can choose N such that

$$1 - \log_N(1 + \beta(N)) < \alpha'.$$

But then, for $i \geq k+3$,

$$\|\hat{x}_i(\cdot) - x(\cdot)\|_{L^\infty(s_i, t_{k+1})} \geq c |\rho_{[s_i, t_{k+1}]}(\hat{x}_i(\cdot))|^{\alpha'},$$

in which $c > 0$ is some number that does not depend on i . Since $\rho_{[s_i, t_{k+1}]}(\hat{x}_i(\cdot)) \rightarrow 0$, as $i \rightarrow \infty$, it follows that

$$\|\hat{x}_i(\cdot) - x(\cdot)\|_{L^\infty(I)} \geq K |\rho_I(\hat{x}_i(\cdot))|^\alpha$$

in which $I = [s_i, t_{k+1}]$ (an interval of length not greater than δ), if we further arrange that

$$c > K |\rho_{[s_i, t_{k+1}]}(\hat{x}_i(\cdot))|^{(\alpha-\alpha')}.$$

Proof of Proposition 4.2. To prove the proposition it suffices find an interval $[S, T]$, multi-function $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and a closed set $A \subset \mathbb{R}^n$ satisfying (for some $r_0 > 0$) hypotheses (H1), (H2), (CQ) and (C) (but not (ACL)), with the following properties: for any $K > 0$ and $\delta > 0$, there exist an interval $I \subset [S, T]$ and an F -trajectory $\hat{x}(\cdot)$ on I with initial state in $r_0\mathbb{B}$ such that

$$\|\hat{x}(\cdot) - x(\cdot)\|_{L^\infty(I)} > K (1 + |\ln(\rho_I \hat{x}(\cdot))|) \rho_I(\hat{x}(\cdot)),$$

for all feasible F -trajectories $x(\cdot)$ on I with the same initial state.

As before, we set $[S, T] = [0, 1]$ and take A to be the set

$$A = \{x \in \mathbb{R}^3 \mid x_2 - x_1 + x_3 \leq 0, -x_2 - x_1 + x_3 \leq 0\}.$$

But we replace the earlier multifunction $F(\cdot)$ by a new multifunction $F_c(\cdot) : [0, 1] \rightsquigarrow \mathbb{R}^3$ (still a function only of t), which is continuous. $F_c(\cdot)$ is constructed using the vectors

$$v_0 = (0, 1, 0), \quad v_1 = (1/2, 1, 0) \quad \text{and} \quad v_2 = (1/2, -1, 0),$$

the decreasing sequences of times $\{t_k\}$ and $\{s_k\}$ of the earlier proof, the sequence of functions $\{\mu_k : [t_k, t_{k-1}] \rightarrow \mathbb{R}\}$

$$\mu_k(t) := \begin{cases} \frac{t-t_k}{\tau_k} & \text{if } t \in [t_k, t_k + \tau_k] \\ 1 & \text{if } t \in [t_k + \tau_k, t_{k-1} - \tau_k] \\ \frac{t_{k-1}-t}{\tau_k} & \text{if } t \in [t_{k-1} - \tau_k, t_{k-1}], \end{cases} \quad (33)$$

in which $\tau_k = \frac{1}{16 \times 3^{2k}}$, and some sequence of positive numbers $\epsilon_k \downarrow 0$. Writing

$$\eta_k(\cdot) = \epsilon_k \mu_k(\cdot),$$

we define the multifunction $F_c(\cdot)$ to be

$$F_c(t) := \begin{cases} \{v_1\} \cup \{v_1 + (v_0 - v_1)\eta_k(t)\} & \text{if } t \in (t_{k+1}, t_k] \\ \{v_2\} \cup \{v_2 + (-v_0 - v_2)\eta_k(t)\} & \text{if } t \in (t_{k+2}, t_{k+1}], \end{cases}$$

where $k = 0, 2, 4, \dots$

(Note that the new multifunction $F_c(\cdot)$ coincides with $F(\cdot)$ at ‘mesh’ points $t = t_1, t_2, \dots$; intermediate values are generated by means of a continuous interpolation scheme.)

The data $F_c(\cdot)$ and A verify (H1), (H2), (CQ), and also (C) (for $r_0 = 1$, say). Notice that, if the series $\sum_{k=1}^{\infty} \epsilon_k$ diverges, then $F_c(\cdot)$ does not satisfy assumption (ACL).

For $i = 1, 2, \dots$ take the F_c -trajectory $\hat{x}_i(\cdot)$ on $[s_i, 1]$ with initial value $\xi_i = (s_i, 0, s_i) \in A$ to be the same as before with $N = 3$. Recall that the state constraint violation of $\hat{x}_i(\cdot)$ is $\frac{\sqrt{3}}{2}s_i$.

Using a similar analysis to that of the proof of Prop. 4.1 (cf. in particular formulas (26)-(31)), we can show that for any positive integer k such that $t_{k+1} \leq \delta \leq t_k$ and any even integer $i \geq 4k + 16$,

$$|\hat{x}_{1,i}(t_{k+1}) - x_1(t_{k+1})| > \frac{1}{\sqrt{3}} \left[(i/2 + 1 - k) \frac{\epsilon_{i+1}}{8} + (i/2 - k) \frac{(i/2 - k + 1)}{2} \left(\frac{\epsilon_{i+1}}{8} \right)^2 \right] \rho_{I_i}(\hat{x}_i(\cdot)),$$

for any feasible F_c -trajectory $x(\cdot)$ on $I_i := [s_i, t_k]$ with initial state $x(s_i) = \hat{x}_i(s_i)$. Then each I_i has length not greater than δ . It can be deduced from the above inequality that there exists a number c , which do not depend on i , such that

$$\begin{aligned} \|\hat{x}_i(\cdot) - x(\cdot)\|_{L^\infty(I_i)} &> c(|\ln(\rho_{I_i}(\hat{x}_i(\cdot)))|)^2 \epsilon_{i+1}^2 \rho_{I_i}(\hat{x}_i(\cdot)), \\ &= K_i \times (1 + |\ln(\rho_{I_i}(\hat{x}_i(\cdot)))|) \rho_{I_i}(\hat{x}_i(\cdot)), \end{aligned}$$

for all feasible F_c -trajectories on I_i with initial state $\hat{x}_i(s_i)$ and even i sufficiently large, where

$$K_i := \frac{c(|\ln(\rho_{I_i}(\hat{x}_i(\cdot)))|)^2}{(1 + |\ln(\rho_{I_i}(\hat{x}_i(\cdot)))|)^{3/2}},$$

when we choose

$$\epsilon_{i+1}^2 := (1 + |\ln(\rho_{I_i}(\hat{x}_i(\cdot)))|)^{-1/2}.$$

Since $\epsilon_i \geq \frac{1}{(2\ln(N) \times i)^{\frac{1}{4}}}$ for each i , clearly the series $\sum_{i=1}^{\infty} \epsilon_i$ diverges and, as a consequence, F_c does not satisfy (ACL). Noting that $K_i \uparrow \infty$, we see that the sequence of F_c -trajectories $\{\hat{x}_i : I_i \rightarrow \mathbb{R}^3\}$ has the required properties for completion of the proof.

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