

Integrability of the hyperbolic reduced Maxwell-Bloch equations for strongly correlated Bose-Einstein condensates

Alexis Arnaudon* and John D. Gibbon

Department of Mathematics, Imperial College, London SW7 2AZ, UK.

(Dated: September 5, 2016)

We derive and study the hyperbolic reduced Maxwell-Bloch equations (HRMB) which acts as a simplified model for the dynamics of strongly correlated Bose-Einstein condensates. A proof of their integrability is found by the derivation of a Lax pair which is valid for both the hyperbolic and standard cases of the reduced Maxwell-Bloch equations. The origin of the latter lies in quantum optics. We derive explicit solutions of the HRMB equations that correspond to kinks propagating on the Bose-Einstein condensate (BEC). These solutions are different from Gross-Pitaevskii solitons because the nonlinearity of the HRMB equations arises from the interaction of the BEC and excited atoms.

I. INTRODUCTION

The excitation and propagation of solitons in Bose-Einstein condensates (BECs) has been an active area of study for a number of years. Two reviews cover the more general area of BECs [1, 2] while two more put greater emphasis on soliton excitation [3, 4]. Experimental studies of strongly correlated BECs have very recently become possible [5–8] and new phenomena have emerged [9, 10]. The fundamental parameter in these experiments is the correlation of the BEC represented as a scattering length a_{scatt} . An abrupt change of the value of a_{scatt} from small to large creates a stable intermediate state that would usually have evaporated due to the strong interactions [9].

A mathematical description of this highly correlated BEC was recently initiated by Kira [11, 12] who derived the so-called hyperbolic Bloch equations (HBE) to represent the excited atoms of the BEC. The method is the same as that for the semiconductor Bloch equations and is based on a cluster expansion approach of the normal component of the strongly correlated BEC. The application of this method to the full BEC dynamics will not work, because all orders in the cluster expansion are necessary. For the excitations, this difficulty is circumvented by the application of a non-unitary transformation that uses the normal component alone by representing the BEC as the vacuum state of the normal component. Then the expansion can be carried out to arbitrary orders. The first order describes singlet dynamics which is already nonlinear and can capture interesting features of the dynamics. We refer the reader to [10–12] for a complete description of this method for the BECs, and to [13] for its initial use for semiconductors in quantum optics.

Although the physics is different in these two cases, the HBE have the same structure as the semiconductor Bloch equations (SBE). The only difference is a minus sign that transforms the Bloch sphere into a hyperboloid on which

the solutions evolve. This important observation is central to our investigation of the complete integrability of the HBE equations following the method used for the SBE equations [12].

In the case of semi-conductor optics, the SBE is coupled electromagnetically through a wave equation for the electric field which contains a term dependent on the polarisation of the semi-conductor. Recall that the HBE does not describe the BEC dynamics which thus requires a coupling to the Gross-Pitaevskii equation (GP) in order to describe the complete dynamics. In fact, the BEC wavefunction replaces the wave equation of the electric field and the coupling is performed via a source term that describes the local loss or gain of atoms in the BEC.

The aim of this paper is to derive a particular approximation of the coupled HBE and GP equations that will be shown to be completely integrable. This approximation roughly corresponds to considering solutions of those systems that have small amplitude with respect to the average amplitude of the BEC. In order to emphasize the parallel with optics, a similar approximation made more than 40 years ago yielded the reduced Maxwell-Bloch equations (RMB) in quantum optics, a completely integrable equation: see [14–17] and references therein. The resulting equations in the present context of BECs, will be called the hyperbolic reduced Maxwell-Bloch equations (HRMB). We will also study the simplest soliton solution (kink) that can be observed, provided the present approximation remains valid.

II. REVIEW OF THE RMB EQUATIONS

The Maxwell-Bloch equations appear in two different contexts. It first appeared in quantum optics in the context of the phenomenon called self-induced transparency: see for example [18, 19] for reviews on this topic. More recently, the quantum semi-conductor optics uses a more general form for these equations that can be reduced to the Maxwell-Bloch equations after neglecting the extra higher order terms. We refer to [13] for a recent monograph on this topic. For the purpose of this work we will prefer the semiconductor description of the Maxwell-

* alexis.arnaudon@ic.ac.uk

Bloch equations as their derivation uses the same method that for the derivation of the HBE equations in [12].

Let us first recall the SBE in its simplest form

$$\begin{aligned} i\dot{P} &= \omega_0 P + (2f - 1)\Omega \\ \dot{f} &= -2 \operatorname{Im}(\Omega P^*). \end{aligned} \quad (1)$$

P is a complex field that represents the transition amplitude between the state of an electron and a hole. The scalar f is the occupation number of the electrons that varies between -1 and 1 . The complex number Ω is the Rabi energy, which is proportional to the electric field applied to the system. Notice that these equations conserve the quantity $\eta = (f - \frac{1}{2})^2 + |P|^2$, which corresponds to the Bloch sphere of radius $\sqrt{\eta}$. These equations are already a simplified version because they incorporate the sharp line approximation. This amounts to writing the equations with only a resonance frequency ω_0 and no frequency averaging with a response function. These equations can be derived with the cluster-expansion approach, a method similar to the BBGKY hierarchy that allows the computation of the many-body interactions between electrons up to some order. Equations (1) only contain singlet terms where more physically realistic doublet or triplet dynamics have been neglected.

Similar interesting phenomena occur in semiconductor quantum optics where the Bloch equations (1) are coupled to the standard Maxwell wave equation. The result is the semiconductor Maxwell-Bloch equations, where the wave equation for the electric field E is coupled to the polarization P via a small material parameter α_0 . The smallness of α_0 together with the use of short intense pulses allows one to neglect backscattering of waves in the Maxwell equation. The resulting wave equation is

$$E_t + cE_x = \alpha_0 P, \quad (2)$$

where c the speed of light. We refer the reader to [14–18] and references therein for more details on the derivation of these equations.

III. THE HYPERBOLIC BLOCH EQUATIONS

The hyperbolic counterpart of the Bloch equations can be derived in the context of a strongly interacting BEC. Here we will sketch the main steps of the derivation that can be found in full in [12]. The successful method of cluster-expansions can be used for BEC only in the strongly interacting regime for the obvious reason that the expansion for the BEC itself will contain all orders in particle interactions. In the strongly interacting regime, a non-negligible number of atoms are not condensed and form the normal component of the BEC. The dynamics of these atoms can be described by a cluster-expansion with only a few orders of interactions for relatively short times after the strong correlation appears. Indeed, higher order correlations are created from the original lower order ones. In order to take into account only the normal

component, a non-unitary transformation is applied to the BEC wave function in order to replace the BEC by a ground state, and to concentrate only on the dynamics of the atoms in the normal component. This technique developed in [10] together with the cluster-expansion produces dynamical equations called the HBE that can contain various order of approximations. This equation derived in [12] has been shown to be more precise than the classical Hartree-Fock approximation for strongly correlated BECs.

In this work we will use the simplest HBE equations that neglect all correlations higher than the singlets. Furthermore we will approximate the potential describing the inter-atomic interactions by a contact potential; that is, a Dirac delta function, which is equivalent to expressing the Fourier transform of the contact potential as $V_k = 4\pi\hbar^2 m^{-1} a_{\text{scatt}}$, where a_{scatt} is now only an effective scattering length. The strongly interacting regime, is characterised by the limit $a_{\text{scatt}} \rightarrow \infty$. In practice means that the scattering length is saturated. This scattering length is experimentally controlled by the application of an external uniform magnetic field that triggers the so-called Feshbach resonance: see [12] or [1, 2] for more details on this interaction potential.

In summary, the HBE equations that will be used in the rest of this work is given by

$$\begin{aligned} i\dot{P} &= \omega_0 P + (2f + 1)\Omega \\ \dot{f} &= 2 \operatorname{Im}(\Omega P^*), \end{aligned} \quad (3)$$

where f is the occupation number of non-condensed atoms and P is the transition amplitude between the BEC and atoms in the normal component. ω_0 is the transition energy, and $\Omega = 4\pi\hbar^2 m^{-1} a_{\text{scatt}} N_c$ is the quantum-depletion source, proportional to the number of condensed atoms

$$N_c = N_{\text{tot}} - f. \quad (4)$$

Here, N_{tot} is the total number of atoms in the system, taken to be constant. Due to a sign flip in the f equation, the hyperboloid $\eta = (f + \frac{1}{2})^2 - |s|^2$ is preserved by the solution instead of the sphere for the Bloch equations.

IV. COUPLING WITH THE GROSS-PITAEVSKII EQUATION

The next step is to couple the HBE equations (3) with the Gross-Pitaevskii (GP) equation in order to include the internal BEC dynamics. Recall that in the case of the SBE (1), the coupling with the Maxwell equation is achieved using the Rabi frequency and the electric field. In the case of BEC, the condensed atom number plays the role of the electric field and the GP equation of the wave equation.

The first approximation is the standard local-density approximation (LDA) which consists at studying the

BEC dynamics locally, thus neglecting the exterior trapping potential and use the approximation of locally homogeneous BEC. The second approximation used here is to consider a one-dimensional condensate, that could still be valid in appropriate experiments: see for example [5]. The coupling between the BEC dynamics and the HBE is implemented as a source term in the GP equation, which then reads

$$i\hbar\psi_t + \alpha\psi_{xx} + \beta\psi|\psi|^2 = i\beta\text{Im}(P^*)\psi, \quad (5)$$

where ψ is a complex-valued wavefunction, $\alpha = \hbar^2/2m$, $\beta = 8\pi a_{\text{scatt}}\alpha$ and m the mass of a boson. First recall that the interaction length $a_{\text{scatt}} \rightarrow \infty$ for strongly interacting BEC. Let us now write the GP equation in amplitude phase variables using the Madelung transformation $\psi(x, t) = \sqrt{n(x, t)} \exp\{i\phi(x, t)\}$ for the amplitude $n(x, t)$ and the phase $\phi(x, t)$

$$n_t + (2n\phi_x)_x = \beta\text{Im}(P^*)n \quad (6a)$$

$$\phi_t = \alpha \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}} - \phi_x^2 \right) + 2\beta n. \quad (6b)$$

Using the LDA, we can decompose the amplitude as $n(x, t) = n_0 + n_1(x, t)$, where n_0 is a constant background with $n_1 \ll n_0$. The steady solution is given by $n_1 = 0$ and a time independent phase $\phi_0(x)$ found by solving $\alpha\phi_{0,x}^2 = 2\beta n_0$. The full phase solution can thus be written as

$$\phi(x, t) = x\sqrt{2\frac{\beta}{\alpha}n_0} + \phi_1(x, t) := \kappa x + \phi_1(x, t), \quad (7)$$

where κ is a large number as $\beta/\alpha \rightarrow \infty$. The phase $\phi_0(x)$ is thus highly oscillating and ϕ_1 can be considered as a slowly varying phase. The equation (6a) for the amplitude n is then approximated at first order in κ , and together with the LDA, one obtains the wave equation

$$n_{1,t} + 8\sqrt{\pi a_{\text{scatt}} n_0} n_{1,x} = \hbar m^{-1} 4\pi a_{\text{scatt}} n_0 \text{Im}(P^*). \quad (8)$$

For large values of n_0 and a_{scatt} , the right hand side will contribute more to the dynamics than the wave equation.

V. THE HRMB AND RMB EQUATIONS

The system of equations derived previously is equivalent to the HRMB equations (10) below by using the following change of variables

$$\begin{aligned} Q &= \text{Re}(P), & P &= -\text{Im}(P), \\ N &= 2\pi \frac{\hbar}{m} (2f + 1) & \text{and} & \quad E = n_0 + n_1. \end{aligned} \quad (9)$$

The RMB and the hyperbolic HRMB equations can be written together as

$$\begin{aligned} E_x &= P \\ P_t &= EN + \sigma_2 \omega_0 Q \\ N_t &= -\sigma_1 EP \\ Q_t &= -\omega_0 P, \end{aligned} \quad (10)$$

where we have changed frames in (8) to absorb all constants. We introduced $\sigma_{1,2} = \pm 1$ that selects the RMB for $\sigma_1 = \sigma_2 = 1$ and the two possible HRMB equations for $\sigma_1 = -1$ and $\sigma_2 = \pm 1$. They are both integrable, as shown below, but only $\sigma_2 = 1$ has been derived before. The physical interpretation for the RMB and the hRMB with $\sigma_2 = -1$ is an open problem. Notice that we have performed changes of independent and dependent variables to simplify the form of the equations in the study of integrability while only keeping track of the parameter ω_0 .

From (10) we see that the generalised Bloch sphere is given by

$$P^2 + \sigma_2 Q^2 + \sigma_1 N^2 = \eta, \quad (11)$$

which is a hyperboloid when either or both of the σ_i are negative.

Notice that setting $\omega_0 = 0$ recovers the Sine-Gordon equation from the RMB equations and the Sinh-Gordon equation from both HRMB equations. Indeed, for the HRMB equations, the change of variables $E = \phi_x$, $P = \sinh(\phi)$ and $Q = \cosh(\phi)$ gives

$$\phi_{xt} = 2 \sinh(\phi), \quad (12)$$

After the KdV and NLS equations [20], the Sine-Gordon equation was the next to be shown to be completely integrable [21–23].

VI. COMPLETE INTEGRABILITY

We will now show that all of the RMB and HRMB equations are integrable by mean of the inverse scattering transform (IST). Remarkably, the spectral problem associated to these equations is the Zakharov-Shabat spectral problem [20, 23]; that is

$$\begin{aligned} \Psi_x &= L_{\sigma_1} \Psi \\ \Psi_t &= M_{\sigma_1, \sigma_2} \Psi, \end{aligned} \quad (13)$$

where $\Psi = (\psi_1, \psi_2)^T$ is the scattering wavefunction and L the spectral operator

$$L_{\sigma_1} = \lambda \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} 0 & E \\ -\sigma_1 E & 0 \end{bmatrix}, \quad (14)$$

for the spectral parameter λ . Well-known equations such as the KdV or NLS equations can be written in such a spectral problem with the operator M having only positive powers of λ . These would be the so-called positive AKNS hierarchy [23]. Here, we will use the negative part of the hierarchy, where the M operator has negative powers of λ . It is given for the RMB ($\sigma_1 = 1$) and HRMB ($\sigma_1 = -1, \sigma_2 = 1$) by

$$M_{\sigma_1, +} = \frac{1}{2(\lambda^2 - \omega_0^2)} \left(i\lambda \begin{bmatrix} -iN & P \\ \sigma_1 P & iN \end{bmatrix} - \omega_0 \begin{bmatrix} 0 & Q \\ -\sigma_1 Q & 0 \end{bmatrix} \right), \quad (15)$$

whereas the HRMB or RMB case with $\sigma_2 = -1$ has a different M operator

$$M_{\sigma_1,-} = \frac{1}{2(\lambda^2 - \omega_0^2)(\lambda - \omega_0)} \left(i\lambda^2 \begin{bmatrix} 0 & P \\ -\sigma_1 P & 0 \end{bmatrix} - \lambda\omega_0 \begin{bmatrix} 0 & Q \\ \sigma_1 Q & 0 \end{bmatrix} + i\omega_0^2 \begin{bmatrix} N & 0 \\ 0 & -N \end{bmatrix} \right). \quad (16)$$

The RMB and HRMB equations appear from computing the compatibility condition between the two equations in (13), that is

$$\partial_t L_{\sigma_1} - \partial_x M_{\sigma_1,\sigma_1} + [L_{\sigma_1}, M_{\sigma_1,\sigma_1}] = 0. \quad (17)$$

This allows for the use of the IST by first solving the scattering problem, i.e., compute the eigenvalues of the scattering problem with the operator L , thus evolving them with the M operator to finally reconstruct the solution by inverting the scattering problem. We will not do this here, but leave this computation for future work but just comment on the spectral problems. In the case $\sigma_1 = 1$, the L operator is anti-Hermitian, which means that the spectrum can have isolated eigenvalues in the case of vanishing boundary conditions, i.e. $E(\pm\infty) = 0$. In the hyperbolic case this operator is Hermitian, and so no discrete eigenvalues exist unless the boundary conditions are non-vanishing. This feature is also found in the nonlinear Schrödinger equation, where $\sigma = 1$ corresponds to the focusing NLS, and $\sigma_1 = -1$ to the defocussing case. The solitons in the latter equation are of a different type than the first and could be either dark or grey solitons, or even kinks, as in the HRMB equations – see below.

An interesting feature of this spectral problem is that although the RMB and HRMB equations, with either $\sigma_2 = \pm 1$, share the same L operator, the M operator differs. Because only the operator L describes the shape of the solitons, the shapes are uniform on the sign of σ_2 . The main difference between the two M operators is in the position of the poles in the λ -plane. If $\sigma_2 = 1$, there are two simple poles $\lambda = \pm\omega$, and if $\sigma_2 = -1$ there is a double pole at $\lambda = \omega_0$. Notice here that this is an arbitrary choice, and that $\lambda = -\omega_0$ could have also been to be the double pole. This is an unusual feature not present in the NLS equation which only contains positive powers of λ in the M operator. Shifting of the zeros by

some parameter ω_0 will only produce a gauge equivalent equation.

VII. KINK SOLITONS

As already mentioned, soliton solutions can be derived with the IST method, but here we will find them by simply using the travelling wave ansatz $E(x,t) = E(t-c^{-1}x)$ for a constant parameter c . We find the following ODEs, when using the boundary condition $N(\pm\infty) = N_\infty$,

$$E_{xx} = -E \left(\frac{1}{2}\sigma_1 E^2 + cN_\infty + \sigma_2 \omega^2 \right). \quad (18)$$

The sign in front of the E^3 terms changes the type of solution, from sech-profile to a tanh-profile, as expected. For the solution of the RMB equations, we obtain with $N_\infty = -1$,

$$E(x,t) = E_0 \operatorname{sech} \left(\frac{1}{2} E_0 \left(t - \frac{4}{E_0^2 + 4\sigma_2 \omega^2} x \right) \right). \quad (19)$$

For $\sigma_1 = -1$, thus the hyperbolic case, the solution is given by

$$E(x,t) = \pm E_\infty \tanh \left(2E_\infty \left(t - \frac{N_\infty}{E_\infty^2/2 - \sigma_2 \omega^2} x \right) \right) \quad (20)$$

where E_∞ is the value of E at $x \rightarrow \pm\infty$. This solution is often called a kink, and propagates on top of n_0 , the constant background of the BEC. In the physical case, with $\sigma_1 = 1$, the speed can change sign, with 0 speed if $\omega^2 E_\infty^2 = \frac{1}{2}$. In the HRMB case, with $\omega \neq 0$, non-zero constants solutions can exist. Notice also the possible 0 speed soliton for the RMB with $\sigma_2 = -1$.

ACKNOWLEDGMENTS

We acknowledge, with thanks, discussions with R. Barnett, M. Kira, D. Holm and A. Hone. AA acknowledges partial support from an Imperial College London Roth Award and from the European Research Council Advanced Grant 267382 FCCA.

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- [1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
 [2] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).
 [3] Y. S. Kivshar and B. Luther-Davies, Phys. Reports **298**, 81 (1998).
 [4] D. J. Frantzeskakis, J. Phys. A: Mathematical and Theoretical **43**, 213001 (2010).
 [5] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. **83**, 5198 (1999).
 [6] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature **417**, 150 (2002).
 [7] S. Wüster, B. J. Dkabrowska-Wüster, A. S. Bradley, M. J. Davis, P. B. Blakie, J. J. Hope, and C. M. Savage, Phys. Rev. A **75**, 043611 (2007).
 [8] K. Xu, Y. Liu, D. E. Miller, J. K. Chin, W. Setiawan, and S. J. Kim, Phys. Rev. Lett. **96**, 180405 (2006).
 [9] P. Makotyn, C. E. Klauss, D. L. Goldberger, E. Cornell, and D. S. Jin, Nature Physics **10**, 116 (2014).
 [10] M. Kira, Nat. comm. **6** (2015).

- [11] M. Kira, *Ann. Phys.* **351**, 200 (2014).
- [12] M. Kira, *Ann. Phys.* **356**, 185 (2015).
- [13] M. Kira and S. W. Koch, *Semiconductor quantum optics* (Cambridge University Press, 2011).
- [14] J. C. Eilbeck, J. D. Gibbon, P. J. Caudrey, and R. K. Bullough, *J. Phys. A: Mathematical, Nuclear and General* **6**, 1337 (1973).
- [15] J. D. Gibbon, P. J. Caudrey, R. K. Bullough, and J. C. Eilbeck, *Lett. Nuovo Cim.* (1971–1985) **8**, 775 (1973).
- [16] P. J. Caudrey, J. C. Eilbeck, and J. D. Gibbon, *IMA J. Appl. Math.* **14**, 375 (1974).
- [17] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and nonlinear wave equations* (Academic Press, Inc., London-New York, 1982).
- [18] R. Bullough, P. Caudrey, J. Eilbeck, and J. Gibbon, *J. opto-electron.* **6**, 121 (1974).
- [19] A. Maimistov, A. Basharov, S. Elyutin, and Y. M. Sklyarov, *Phys. Repts.* **191**, 1 (1990).
- [20] V. Zakharov and A. Shabat, *Sov. Phys.-JETP (Engl. Transl.)* **34** (1972).
- [21] P. J. Caudrey, J. C. Eilbeck, J. D. Gibbon, and R. K. Bullough, *Phys. Rev. Lett.* **30**, 273 (1973).
- [22] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Phys. Rev. Lett.* **30**, 1262 (1973).
- [23] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Phys. Rev. Lett.* **31**, 125 (1973).