On the Heegaard Floer Homology of Dehn Surgery and Unknotting Number

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by

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I declare that this thesis and all the research it contains are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Pretzel Knots with Unknotting Number One.
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To my parents, with love and gratitude.

ABSTRACT

In this thesis we generalise three theorems from the literature on Heegaard Floer homology and Dehn surgery: one by Ozsváth and Szabó on deficiency symmetries in half-integral L-space surgeries, and two by Greene which use Donaldson's diagonalisation theorem as an obstruction to integral and half-integral L-space surgeries. Our generalisation is two-fold: first, we eliminate the L-space conditions, opening these techniques up for use with much more general 3-manifolds, and second, we unify the integral and half-integral surgery results into a broader theorem applicable to non-zero rational surgeries in S^3 which bound sharp, simply connected, negative-definite smooth 4-manifolds. Such 3-manifolds are quite common and include, for example, a huge number of Seifert fibred spaces.

Over the course of the first three chapters, we begin by introducing background material on knots in 3-manifolds, the intersection form of a simply connected 4-manifold, Spin- and Spin^c-structures on 3- and 4-manifolds, and Heegaard Floer homology (including knot Floer homology). While none of the results in these chapters are original, all of them are necessary to make sense of what follows. In Chapter 4, we introduce and prove our main theorems, using arguments that are predominantly algebraic or combinatorial in nature. We then apply these new theorems to the study of unknotting number in Chapter 5, making considerable headway into the extremely difficult problem of classifying the 3-strand pretzel knots with unknotting number one. Finally, in Chapter 6, we present further applications of the main theorems, ranging from a plan of attack on the famous Seifert fibred space realisation problem to more biologically motivated problems concerning rational tangle replacement. An appendix on the implications of our theorems for DNA topology is provided at the end.

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INTRODUCTION

Since its inception in the early 20th century, Dehn surgery has consistently been one of the most fundamental techniques in the study of 3-manifolds. Its significance is perhaps best demonstrated by the famous theorem of Lickorish [33] and Wallace [71] which states that every closed, connected, orientable 3-manifold can be obtained by ± 1-surgeries on the components of a link in S^3. Since this theorem first appeared, mathematicians have expended a huge amount of effort trying to understand the connection between the isotopy class of a link $L \subset Y$ and the 3-manifolds resulting from surgery on L. It is perhaps a reflection of the difficulty this goal poses that, even many decades on, we are still far from achieving it.

Approximately 40 years ago, Moser posed the following innocuous looking question [41]. She asked for a list of the lens spaces Y' which could be obtained by surgery on non-trivial knots $C \subset S^3$, and for a description of the surgeries (i.e. the knots C and surgery coefficients p/q) that would yield those lens spaces. As harmless as this question sounds, its solution has proved quite the opposite. The first half, now known on its own terms as the *lens space realisation problem*, had to wait several decades until a seminal piece of work by Greene in 2010 finally pinned down the correct lens spaces, while the second half, on the surgeries that produce these manifolds, remains unclear even today. It has been posited (and is almost universally believed) that the Berge conjecture answers the problem fully, but a proof of this conjecture continues to elude us.

The reason Moser's question is so significant is that, in a very real sense, lens spaces are the simplest of all closed, connected, orientable 3-manifolds. Given that Dehn surgery is also one of the fundamental techniques in 3-manifold construction, Moser's question is about as simple a version of the following basic and natural question as one could possibly ask.

Question 1. *Given a 3-manifold Y, which other 3-manifolds Y' can be obtained from Y via surgery? Which knots C in Y (and coefficients) yield Y'?*

Anyone who has ever worked on Dehn surgery has almost certainly considered this question, at least in passing. The main aim of this thesis is to provide a powerful new tool for the case when $Y = S^3$, inspired by Greene's solution to the lens space realisation problem in [26]. In that paper, Greene was able to use the fact that if a lens space is obtained by surgery in S^3 on a non-toroidal knot, then the surgery coefficient must be integral (see the famous cyclic surgery theorem of Culler, Gordon, Luecke, and Shalen [12]). He then used the correction terms of Ozsváth and Szabó [46] and an approach similar to that of Lisca in [36] to finish off the problem. His core theorem and its proof make critical use of Donaldson's diagonalisation theorem [16].

Written explicitly, Greene's theorem (Theorem 1.6 from [26]) says that if $L(p', q')$ is a lens space obtained by k-surgery in S^3, and if $p'/q' = [b_1, \ldots, b_n]^-$ in Hirzebruch-Jung continued fraction notation, then there must exist some matrix A such that

$$-AA^t = \begin{pmatrix} -b_1 & 1 & & & & \\ 1 & -b_2 & 1 & & & \\ & 1 & -b_3 & & & \\ & & & \ddots & 1 & \\ & & & 1 & -b_n & \\ & & & & & -k \end{pmatrix},$$

and whose last row has the form

$$\begin{pmatrix} \sigma_n & \sigma_{n-1} & \ldots & \sigma_1 & \sigma_0 \end{pmatrix},$$

where the σ_j are non-negative, non-decreasing integers satisfying $\sigma_0 \leq 1$ and $\sigma_j \leq \sigma_0 + \sigma_1 + \cdots + \sigma_{j-1} + 1$ for all j. This last condition is called the *changemaker condition*, since it guarantees that if one has coins of value σ_j, one can use those coins to make up any value from zero to their sum. This, it turns out, was more or less sufficient to solve the lens space problem by some ingenious and difficult combinatorics.

The main theorem of this thesis is a direct generalisation of Greene's result. Though the proof must wait until Chapter 4 (see Theorem 4.1.3), we can still quote it here for comparison. We require $p/q = [a_1, \ldots, a_\ell]^-$ in canonical Hirzebruch-Jung continued fraction notation, and set $n = a_1 = \lceil p/q \rceil$ and $p = nq - r$.

Theorem 1. *Suppose that $Y' = S^3_{-p/q}(C)$ for some knot $C \subset S^3$ and coprime $p, q > 0$, and that $-Y'$ bounds a sharp, simply connected, negative-definite smooth 4-manifold X with intersection form Q_X. Suppose also that W is the trace of the surgery on C, and that Q_W is its intersection form. Then if*

$$d(Y', i) - d(S^3_{-p/q}(U), i) = 0$$

for either (a) one value of i if n is odd; or (b) $q - r + 1$ values of i if n is even, there exists an integral matrix A such that

$$-AA^t = Q_X \oplus Q_W.$$

In addition, if $q \neq 1$, one can choose A so that its last ℓ rows are:

$$\begin{pmatrix} \sigma_r & \ldots & \sigma_1 & 1 & 0 & & & & & & & & & \\ & & & -1 & 1 & \ldots & 1 & & & & & & & \\ & & & & & & & \ddots & & & & & & \\ & & & & & & & & -1 & 1 & \ldots & 1 & 0 & & & \\ & & & & & & & & & & & -1 & 1 & \ldots & 1 \end{pmatrix},$$

where there are exactly a_i non-zero entries in rows $i = 2, \ldots, \ell$, all ± 1 as above, and the first row consists of non-negative integers satisfying $\sigma_1 \leq 1$, $\sigma_1 \leq \cdots \leq \sigma_r$, and the changemaker condition

$$\sigma_j \leq \sigma_1 + \sigma_2 + \cdots + \sigma_{j-1} + 1$$

for all $j = 2, \ldots, r$. If instead $q = 1$, then one can choose A so that its last row is:

$$\begin{pmatrix} \sigma'_r & \ldots & \sigma'_1 & \sigma'_0 \end{pmatrix},$$

and the elements of $\{\sigma'_j\}_{j=0}^r$ satisfy the same relations as those of $\{\sigma_j\}_{j=1}^r$ above.

Although the similarities between Theorem 1 and Greene's theorem should be obvious, it is worth emphasising the two main advantages which make Theorem 1 more applicable.

1. Our new theorem stipulates that Y' is obtained by $-p/q$ surgery for non-zero coefficients. That is, we are no longer restricted to integral surgeries.

2. The assumption that Y' be a lens space is no longer necessary. In Greene's proof, the main requirement was that Y' be an L-space in the sense of Heegaard Floer homology (of which lens spaces are examples), but in the stronger form given above, even this hypothesis is unnecessary. Instead, we have two different Heegaard Floer-related requirements, neither of which are particularly taxing. They are satisfied, for instance, by a huge number of Seifert fibred spaces.

It is our belief that Theorem 1 provides a significant new tool for tackling problems such as Question 1, and we are particularly confident that it will yield good results if applied to the case when $Y = S^3$ and Y' is a small Seifert fibred space. Specifically, we believe that it is capable of identifying a large proportion of the small Seifert fibred spaces that result from surgeries on non-toroidal knots in S^3. This broader problem, which subsumes the lens space question posed by Moser, has unsurprisingly also received considerable attention from the topological community; any resolution, even if only partial, would potentially prove ground-breaking. We discuss this subject more towards the beginning of Chapter 6.

Stepping down from these lofty airs to the more grounded level of this thesis, however, there are many other applications of this theorem. One in particular, presented in Chapter 5, concerns the unknotting number $u(K)$, an invariant which has attained a considerable degree of knot theoretic infamy: though easy to define, it often resists calculation. Indeed, even knots with $u(K) = 1$ are difficult to detect. Courtesy of the Montesinos theorem [40], which states that the double branched cover $\Sigma(K)$ of a knot K with unknotting number one can be obtained by half-integral surgery on a knot $C \subset S^3$ [40], one common method of proving that $u(K) \neq 1$ has historically been to show that the equation

$$\Sigma(K) = S^3_{\pm D/2}(C) \tag{1}$$

cannot be satisfied for any $C \subset S^3$ and $D = \det K$. This idea lies, for instance, at the heart of Lickorish's linking form criterion [34], but also underpins more sophisticated and recent techniques by Ozsváth and Szabó [51] (and others, [24, 45]), techniques best known for their success in classifying all the alternating knots with ten or fewer crossings that satisfy $u(K) = 1$. In fact, it is a generalisation of the correction term symmetries implied by (1) and described in [51] which underscores the proof of Theorem 1. In this sense, our work generalises a very powerful existing obstruction to unknotting number one.

Given the difficulty involved in a straight-out computation of $u(K)$, much of the work in the area, as hinted above, has followed the general trend of choosing a family F of knots and classifying those $K \in F$ which satisfy $u(K) = 1$. (There are of course some notable exceptions, such as the case when F is the family of torus knots, see Kronheimer and Mrowka [31] and Rasmussen [56].) Examples of this trend include the 2-bridge knots [28], large algebraic knots [23], knots of genus one [10], and the alternating 3-braids [24]. The work presented in Chapter 5 is similar, tackling the notoriously difficult case of the 3-strand pretzel knots $P(p, q, r)$. In this way, our applications of Theorem 1 and its underlying correction term symmetries represent significant progress in and of themselves: the pretzels $P(p, q, r)$ have not only defied almost all the classical obstructions to unknotting number one, but also represent, to the best of the author's knowledge, the first time the correction term symmetries have been directly applied to an infinite family of knots.

It should be stated at this point that Greene, in his paper [24], had already established a version of Theorem 1 in the case $q = 2$, though still with the very restrictive requirement that Y' be an L-space. While he was able to use it to classify the alternating 3-braids with $u(K) = 1$, it was also the L-space requirement which forced him to consider *alternating*, rather than general, 3-braids. Our theorem, on the other hand, is true sans this requirement. In short, Theorem 1 represents a generalisation of the main theorems of three papers: two by Greene [26, 24], and one by Ozsváth and Szabó [51].

There are other applications of Theorem 1 beyond the $q = 2$ case discussed above. In fact, as we shall see towards the end, it is useful in a large number of rational tangle replacements, and provides a new and fundamentally different obstruction to unknotting number one, quite distinct from those revolving around (1). We leave a detailed discussion of these subjects to the sixth and final chapter.

Organisation

This thesis is divided into six chapters according to a linear structure. The first three provide what is essentially background material: preliminaries about knots, Dehn surgery, and 3- and 4-manifolds, but also an introduction to Heegaard Floer homology and its relative, knot Floer homology. After that, the rest of the material is new work.

Chapter 4 introduces the main theorem discussed above, proving our matrix-based obstruction to Dehn surgeries in S^3. Its proof occupies the majority of the chapter and draws extensively on the material set up beforehand. If the reader is pressed for time or wishes to skip this proof, little harm will come to him or her in later chapters provided they have digested the theorem's statement.

Chapter 5 then presents the most significant application of Theorem 1 we have achieved to date: a near-complete classification of the pretzel knots $P(p, q, r)$ with unknotting number one. This question, significant both mathematically and biologically, is discussed in some detail, before the final chapter, Chapter 6, provides further applications of the theorems of Chapter 4. An Appendix is provided on their biological implications.

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CHAPTER 1: KNOTS AND 3-MANIFOLDS

The fundamental ideas and definitions required to make sense of our main theorem are collected together in this first chapter, which covers the rudiments of knot theory and Dehn surgery. Since the majority of the material in this chapter should be familiar to most low-dimensional topologists, the treatment we provide here is sparser on proofs than the treatment in later chapters dedicated to our new results. It should be regarded only as a summary of the relevant material; we will not, for example, explain the names of various common knots. The chapter is biased towards the tools relevant for unknotting number since this invariant will be of central focus in Chapter 5.

Good references on similar material are Lickorish [35] and Rolfsen [58]. The latter is particularly helpful for the sections on Dehn surgery.

1.1 Preliminaries

When we speak about a knot K in a 3-manifold Y, we will technically mean a smooth, oriented embedding $K : S^1 \to Y$, but we will also use the word to refer to the image of this embedding or its equivalence class modulo ambient isotopy. These isotopy classes may also be called *knot types.* Throughout the thesis, Y should be taken to be a closed, connected, oriented 3-manifold; any deviation from this convention, in particular to the case when Y has boundary, will be noted as appropriate. The smoothness assumption on K ensures that we avoid pathological examples (e.g. wild knots) which might possess an infinite number of crossings or exhibit fractal-like self-replication. Our knots, without exception, will always be tame (see [58]).

Without a doubt, the most common way of identifying knot types is with algebraic or topological invariants. These range from the very coarse (such as the homology class of a knot in $H_1(Y)$; consider, for example, the case $Y = S^3$), to the so-called perfect knot invariants which give a 1-1 classification of knots by isotopy class. The following theorem by Gordon and Luecke provides us with an example of the latter. As one might expect, it is very difficult to manipulate. Since it will be relevant for our purposes, we spell it out explicitly.

Theorem 1.1.1. *Let $K \subset S^3$ be a knot, and let $N(K)$ be a tubular neighbourhood of K. Then K is classified by the homeomorphism class of its* knot complement $M(K)$, *defined by*

$$M(K) := S^3 \setminus N(K).$$

Seen in this way, the unknot U, which is characterised by the property that it bounds a disc in S^3, is the unique knot $K \subset S^3$ with solid torus complement; it is therefore (after some additional reasoning) also characterised by the property $\pi_1(M(U)) = \mathbb{Z}$. On the basis of this evidence, one might hazard even further that it is classified by the property $H_1(M(U)) = \mathbb{Z}$, but a quick application of the Mayer-Vietoris sequence finds that this equality holds for every $K \subset S^3$.

Naturally, it is worth asking whether or not Theorem 1.1.1 remains true for ℓ-component *links* L (where $\ell \geq 2$). As it turns out, however, these latter objects (defined as disjoint unions of ℓ knots) do not posses unique complements, and explicit examples of homeomorphic complements belonging to non-isotopic links are known to exist. Hence, even if $M(K)$ is a perfect knot invariant, it is not a perfect link invariant. To the best of the author's knowledge, the search for a perfect link invariant besides the trivial one (i.e. the links themselves) is currently well beyond our reach.

Given the computational difficulties involved in defining an invariant using the embedding of K or L itself, the more usual way of studying knots (or links), at least in S^3, is via *knot diagrams*: projections of K into $\mathbb{R}^2$ which are at most 2-1 at isolated points and which remember, at the double-points, which arc passed over which. The double-points are referred to as *crossings*, and we attribute signs to each crossing according to the conventions in Figures 1.1(a) and 1.1(b). Figures 1.1(c) and 1.1(d) also show the different *resolutions* possible at that crossing (only one of which is compatible with the knot's orientation). A knot is said to be *alternating*

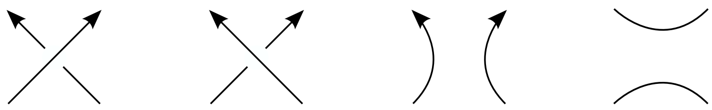


Figure 1.1: (a) A positive crossing; (b) a negative crossing; (c) a resolution compatible with the orientation; and (d) a resolution incompatible with the orientation.

if it possesses a diagram D such that by following the strands in D, one alternates between over and under crossings.

Using these diagrams, we obtain two easy invariants. The first, called the *crossing number* $c(D)$ of a diagram D for K, is defined to be the number of crossings in D; the second, called the *unknotting number* $u(D)$, is defined to be the minimal number of crossings in D that must be changed in order to obtain a diagram of the unknot. From these diagrammatic calculations, we define the *bona fide* knot invariants

$$c(K) := \min_D \{c(D) | D \text{ is a diagram for } K\}$$
$$u(K) := \min_D \{u(D) | D \text{ is a diagram for } K\},$$

called respectively the *crossing* and *unknotting numbers* of K. Diagrams D for which $c(D) = c(K)$ are called *minimal diagrams*. Although one might expect that if D is minimal it realises the unknotting number (i.e. that $u(D) = u(K)$), this is often not true, and in general $c(K)$ and $u(K)$ are difficult to compute. A more detailed discussion of $u(K)$ is provided in Chapter 5.

Since S^3 has an orientation reversing homeomorphism (reflection), one of the most natural operations we can perform on a knot K is the taking of its mirror image, denoted $\overline{K}$. The diagram for $\overline{K}$ is identical to K except that all crossings are reversed (sign included); clearly $c(\overline{K}) = c(K)$ and $u(\overline{K}) = u(K)$. One must be careful, however, of falling into the trap of thinking that $\iota(\overline{K}) = \iota(K)$ for any invariant ι and any knot K, unless of course $K = \overline{K}$. In this very special circumstance, K is said to be *achiral*; otherwise, it is *chiral*. The operation of *connect sum* is defined by taking two knots K_1 and K_2 separated by a 2-sphere and performing the usual manifold connect sum across the 2-sphere. This produces a well-defined knot $K_1 \# K_2$.

A knot K is said to be *prime* if the equation $K = K_1 \# K_2$ implies that one of K_1 or K_2 is the unknot.

1.2 Seifert Invariants

A remarkable property of knots $K \subset S^3$ (or, more generally, of knots K in integral homology spheres) is that there always exists an orientable surface F, called a *Seifert surface* for K, such that $\partial F = K$. Indeed, this can easily be proved in S^3 by the following algorithm due to Seifert:

1. Orient K and fix a diagram D for K;

2. Resolve all the crossings in D in the orientation-compatible way so that what remains is a collection of oriented circles, each of which bounds a disc with compatible orientation;

3. Glue the discs from the previous step together with half-twisted bands to recover the crossings, again matching orientation.

The result should be an oriented surface F with K as boundary. One must be careful, however, in restricting one's attention to such surfaces, since it is not true that this algorithm exhausts all the possible Seifert surfaces for K.

Once we have chosen a Seifert surface F for K, a number of invariants of K follow naturally. We begin with the most classical.

Definition 1.2.1 (Signed intersection number)**.** *Let A, B be two complementary dimensional oriented submanifolds of a compact, oriented manifold M. Moreover, suppose that A and B intersect transversally (by isotoping if necessary). Then for all $x \in A \cap B$, define*

$$\epsilon : A \cap B \longrightarrow \{\pm 1\}$$

by the equation

$$T_x A \oplus T_x B = \epsilon(x) T_x M.$$

The value $\epsilon(x)$ *is called the* sign *of the intersection point* x*. The* (signed) intersection number *between* A *and* B *is the value*

$$A \cdot B := \sum_{x \in A \cap B} \epsilon(x),$$

which is finite by compactness of M*.*

Definition 1.2.2 (Linking number)**.** *Let* (K, K') *be a 2-component link in* S^3 *and let* F *be a Seifert surface for* K'*. Then the* linking number *of* K *and* K' *is defined as*

$$\mathrm{lk}(K, K') := K \cdot F.$$

It can be verified that this definition is independent of the chosen Seifert surface F*, and that* $\mathrm{lk}(\cdot, \cdot)$ *is symmetric.*

The verification that $\mathrm{lk}(K, K')$ is independent of F considers the homology class $[K] \in H_1(M(K')) = \mathbb{Z}\mu'$ (where μ' is a generator). It is not too difficult to see that $[K] = \pm(K \cdot F)\mu'$ (the sign ambiguity arises from the ambiguity in our choice of generator μ'). Since $[K]$ is clearly independent of F, so too must $K \cdot F = \mathrm{lk}(K, K')$ be.

Having defined the linking number, it now becomes possible to define a symmetric bilinear form, called the *Seifert form* of F, using generators α, β of $H_1(F)$. Explicitly, since α, β are generators, they are represented by embedded closed, oriented curves on F which we shall also refer to as α, β. Pushing β slightly away from F according to the normal specified by the orientation on F, we obtain a second curve β^+, disjoint from α, and may therefore define the following bilinear form:

$$\begin{aligned} V_F : H_1(F) \times H_1(F) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longrightarrow \mathrm{lk}(\alpha, \beta^+). \end{aligned}$$

Note that V_F is not a symmetric form, but can by symmetrised by taking $V_F + V_F^t$; this symmetrised form is the one we will refer to as the *Seifert form* of F.

It should be pointed out that the Seifert form is not an invariant of K. For example, it is possible to change the rank of $H_1(F)$ by taking a connect sum with another, closed surface F'. However, one can check that this move is equivalent to taking a direct sum with g copies of $\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, where g is the genus of F', and that it leaves both the signature and determinant of the Seifert form invariant (the latter up to sign). This is one example of the fact that even though the Seifert form may not be a knot invariant, many of its associated algebraic invariants are.

Definition 1.2.3 (Knot signature and determinant). *Let $K \subset S^3$ be a knot with Seifert surface F. Then the* signature, *$\sigma(K)$, and the* determinant, *$\det K$, of K are defined by*

$$\sigma(K) := \operatorname{sig}(V_F + V_F^t) \qquad\qquad \det K := \left|\det(V_F + V_F^t)\right|.$$

These are independent of the choice of surface F, as well as the basis chosen for $H_1(F)$.

Of these two, the signature is the more useful for unknotting number computations since it can be used to give a lower bound for $u(K)$. Explicitly, $\frac{1}{2}|\sigma(K)| \leq u(K)$, which can be proved by comparing Seifert surfaces before and after a crossing change.

Theorem 1.2.4. *Suppose that K' can be obtained from K by changing a crossing of sign ϵ. Then*

$$\sigma(K') \in \{\sigma(K), \sigma(K) + 2\epsilon\}.$$

In particular, if K has unknotting number one, then we can choose K' so that it is the unknot U. Since the unknot bounds a disc F with trivial $H_1(F)$, it follows that $\sigma(U) = 0$. The following corollary is immediate.

Corollary 1.2.5. *Suppose that $K \subset S^3$ has unknotting number one, realised by changing a crossing of sign ϵ. Then $\sigma(K) \in \{0, -2\epsilon\}$.*

Because of this result, the signature is often the first port of call when investigating unknotting number. We will follow this trend when we attempt to classify the 3-strand pretzel knots with unknotting number one in Chapter 5.

1.3 Covering Spaces

As useful as the signature and determinant are for a variety of purposes, they are far from the only invariants afforded us by Seifert surfaces. Indeed, suppose that $K \subset S^3$ is a knot with Seifert surface F, and suppose that S_i is a copy of $M(K)$ cut along F for all $i \in \mathbb{Z}_m$ (where $m \in \mathbb{N}$). Then S_i has two boundary components, each homeomorphic to F, which we label $S_i^{\pm}$ according to the orientation of F.

Definition 1.3.1 (Cyclic covers of the knot exterior)**.** *Using the notation above, we define the 3-manifold $X_m(K)$ by taking the union $\bigcup_{i\in\mathbb{Z}_m} S_i$ and gluing S_i^+ to S_{i+1}^- for all i. The resulting manifold is called the m-fold* cyclic cover *of $M(K)$, and is indeed a covering space of $M(K)$ with deck transformation group $\mathbb{Z}_m$.*

If we repeat the above construction with $\mathbb{Z}$ instead of $\mathbb{Z}_m$, then we obtain the infinite cyclic cover *$X_\infty(K)$ of $M(K)$ with deck transformation group $\mathbb{Z}$.*

The procedure described above is illustrated diagrammatically for $m = 7$ in Figure 1.2. Significantly, we observe that $\partial X_m(K) = S^1 \times S^1$, and that the *meridian* μ of $\partial M(K)$ (the curve which generates the kernel of the homology map induced by $\partial N(K) \hookrightarrow N(K)$) lifts to a portion of the meridian $\widetilde{\mu}$ in $\partial X_m(K)$ (the curve illustrated in red in Figure 1.2). If we then attach a solid torus $T = S^1 \times D^2$ to $X_m(K)$ along their boundaries and extend the covering to

$$\rho : X_m(K) \cup T \longrightarrow M(K) \cup N(K) = S^3$$

via the map $T \to N(K) \simeq T$ given by $(w, z) \mapsto (w, z^m/\left|z\right|^{m-1})$ (in complex number notation, taking the limit as $z \to 0$), then ρ becomes an m-fold cover branched over $K \subset S^3$.

Definition 1.3.2 (Double branched cover)**.** *Set $m = 2$ in the above discussion. Then the closed double cover of S^3 described above, branched over K, is called the* double (branched) cover *of K, denoted $\Sigma(K)$.*

From this description, it is fairly clear that $\Sigma(U) = S^3$. Other double branched covers will be discussed later, since they are of particular importance to the unknotting

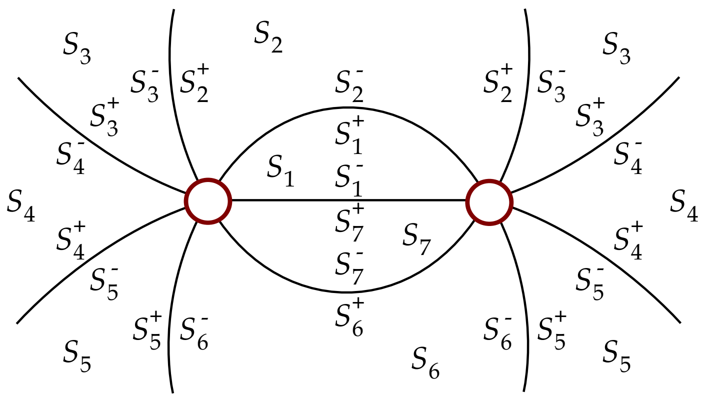


Figure 1.2: A diagrammatic representation of $X_7(K)$. The knot K can be imagined as sitting at the centre of the toroidal hole outlined in red.

number of K. In the meantime, we observe that double branched covers satisfy the following properties:

1. $\Sigma(K)$ comes equipped with an obvious involution $h : \Sigma(K) \to \Sigma(K)$ such that $\Sigma(K)/h = S^3$ and $\text{Fix}(h) = K$. This map characterises $\Sigma(K)$ and its existence is sometimes used to define $\Sigma(K)$;

2. Any 3-manifold $\Sigma(K)$ satisfying the alternative definition just given is unique (that is, there is only one double branched cover $\Sigma(K)$ for any knot $K \subset S^3$). This is established in Section 10.F of Rolfsen [58];

3. $H_1(\Sigma(K))$ is of order $\det K$, which is odd. This can be proved by applying the Mayer-Vietoris sequence to the decomposition $\Sigma(K) = S_1 \cup S_2$ (see Theorem 8.D.1 of Rolfsen [58]); and

4. All of this can be repeated for links instead of knots, giving us a unique $\Sigma(L)$ for each link L. Once we introduce tangles T later in Section 1.5.2, the same can be said of $\Sigma(T)$.

Although $\Sigma(K)$ will be our main concern through most of the thesis, we conclude this section with some remarks on the case $m = \infty$ from above.

Theorem 1.3.3 (Alexander module and ideals)**.** *Let $X = X_\infty(K)$ for some knot $K \subset S^3$. Then $H_1(X)$ is a knot invariant called the* Alexander module *of K, and is* *presented by any matrix representing* $tV_F - V_F^t$*. As such, the elementary ideals* A_i *of* $A := \mathbb{Z}[t, t^{-1}]$ *generated by the* $(n+1-i) \times (n+1-i)$*-minors of* $tV_F - V_F^t$ *are also knot invariants, called the* Alexander ideals.

The first Alexander ideal in particular is very familiar to knot theorists. If we symmetrise the generating polynomial of A_1, the result is the *Alexander polynomial* of K, denoted $\Delta_K(t) := \det(t^{\frac{1}{2}}V_F - t^{-\frac{1}{2}}V_F^t)$, and has the general form

$$\Delta_K(t) = a_0 + \sum_{i=1}^{\infty} a_i(t + t^{-1}),$$

where a_0 is odd (see Corollary 6.11 of Lickorish [35]). Phrased this way, we see that $\det K = |\Delta_K(-1)|$ is odd, and hence that the size of $H_1(\Sigma(K))$ is also odd by the third property above.

This connection aside, our main interest in the Alexander ideals will later rest on their relevance to unknotting number via the theorem below. Since the proof requires more technology than we have already set up, we refer the reader to Theorem 7.10 in Lickorish [35]. The main idea behind the proof goes back to Nakanishi [43].

Theorem 1.3.4. *Let* $K \subset S^3$ *be a knot. Then if* $A/A_i \neq 0$*, it follows that* $u(K) \geq i$*.*

1.4 Dehn Surgery

So far, we have described at least one method for producing 3-manifolds out of knots in S^3, namely the m-fold branched covers, but this is far from the only possible 3-manifold recipe involving knots. As foreshadowed in the Introduction, it is a second construction, Dehn surgery, that is most important for this thesis.

Before giving a definition, some preliminaries are in order. To motivate the reader through these, we give the gentle spoiler that the general idea behind Dehn surgery on $K \subset Y$ is the excavation of a tubular neighbourhood of K followed by the gluing-in of replacement solid torus.

1.4.1 Anatomy of a Torus

Consider a tubular neighbourhood $N(K) \subset Y$ of a knot K, and observe that $N(K)$ is homeomorphic to a standard solid torus. Thus $\partial N(K) \simeq S^1 \times S^1$, and $H_1(\partial N(K)) = \mathbb{Z} \oplus \mathbb{Z}$. In order to identify curves on $\partial N(K)$, which are classified by their homology classes, we would ideally like to specify our choice of generators in as canonical a way as possible. One generator is always given to us: the inclusion $\partial N(K) \hookrightarrow N(K)$ induces a map $H_1(\partial N(K)) \to H_1(N(K)) = \mathbb{Z}$ whose kernel $\mathbb{Z}$ is generated by an element $\mu \in H_1(\partial N(K))$. This choice, unique up to sign (and therefore as an unoriented curve), is called the *meridian* of K. To complete our basis, we define a *longitude* for K as any other homology class λ such that $\{\mu, \lambda\}$ forms a basis for $H_1(\partial N(K))$, and $\mu \cdot \lambda = 1$.

It is clear that the longitudes of K are in bijection with $\mathbb{Z}$, for if λ is a generator, then so too is $p\mu + \lambda$ for all $p \in \mathbb{Z}$. A more subtle point, however, is the fact that none of these λ are canonically preferred over the others; a specification must be made. We say that a knot K is *framed* once a particular longitude λ has been chosen, and we call this choice of λ the *framing* of K.

The one exception to the previous paragraph is the following situation. Suppose Y is an integral homology sphere. Then there is a portion of the Mayer-Vietoris sequence which reads

$$0 \longrightarrow H_1(\partial N(K)) \longrightarrow H_1(N(K)) \oplus H_1(Y \setminus \operatorname{Int} N(K)) \longrightarrow 0,$$

and from this we can conclude that $H_1(Y \setminus \operatorname{Int} N(K)) = \mathbb{Z}$, generated by the image of μ, and that the kernel of the map $H_1(\partial N(K)) \to H_1(Y \setminus \operatorname{Int} N(K))$ is also $\mathbb{Z}$. This kernel is generated by a distinguished longitude λ^0 called the *canonical longitude* of K. Since μ was only determined up to sign, so too is λ^0, though its orientation relative to μ is determined by the equation $\mu \cdot \lambda^0 = 1$.

An alternative way of realising this canonical longitude uses the following trick: let F be a Seifert surface for K, and let λ^0 be the curve $\partial N(K) \cap F$. Then λ^0 represents the canonical longitude, at least up to orientation, since it is null-homologous in $H_1(Y \setminus \operatorname{Int} N(K))$ and wraps once around K.

1.4.2 Definition and Properties

By now, we are finally ready to define Dehn surgery.

Definition 1.4.1 (Dehn surgery on a knot)**.** *Let $K \subset Y$ be a knot with meridian μ and framing λ (if Y is an integral homology sphere, take λ to be the canonical longitude). Then the operation gluing a generic solid torus T to $Y \setminus \operatorname{Int} N(K)$ via the boundary homeomorphism which sends μ_T to the knot specified by $\pm(p\mu + q\lambda)$ on $\partial N(K) = -\partial(Y \setminus \operatorname{Int} N(K))$ is called p/q-*(Dehn) surgery *on K, and the resulting manifold is denoted $Y_{p/q}(K)$. The extended rational number $p/q \in \mathbb{Q}^* := \mathbb{Q} \cup \{\infty\}$ is called the* (surgery) coefficient *of K (we set $1/0 = \infty$).*

Definition 1.4.2 (Kirby diagram)**.** *If $K \subset S^3$ is a canonically framed knot with surgery coefficient p/q, we call the pair $(K, p/q)$ a* Kirby diagram *for $S^3_{p/q}(K)$. We consider Kirby diagrams equivalent if their associated 3-manifolds are homeomorphic as oriented manifolds.*

Before discussing any of the basic properties this construction enjoys, some remarks on the definition are in order. First off, notice that while the homeomorphisms $h : \partial T \to \partial N(K)$ between 2-tori are classified up to isotopy by elements of $SL(2; \mathbb{Z})$, the 3-manifold determined by such h depends only on the image of μ_T in $\partial N(K)$, for what remains thereafter is an essentially unique gluing of a 2-disc. Stated differently, the 3-manifolds obtained by considering general homeomorphisms h are classified by unoriented knots in $\partial N(K)$, or by the coprime integers $\pm(p, q)$ described above. Second, although μ was defined only up to orientation, since the orientation of λ is determined by μ, all ambiguities are removed by forming the extended rational number $p/q \in \mathbb{Q}^*$. Thus, the definition of Dehn surgery given above genuinely accounts for all possible h. And third, observe that as oriented curves on $\partial N(K)$, μ and λ satisfy $\mu \cdot \lambda = 1$; as oriented curves on $\partial(Y \setminus \operatorname{Int} N(K)) = -\partial N(K)$, they instead satisfy $\mu \cdot \lambda = -1$.

Proposition 1.4.3. *Dehn surgery on $K \subset Y$ satisfies the following properties.*

1. *$Y_\infty(K) = Y$ for all $K \subset Y$;*

2. *If* $Y = S^3$*, then* $-S^3_r(K) = S^3_{-r}(\overline{K})$*;*

3. *If* Y *is an integral homology sphere, then* $H_1(Y_{p/q}(K)) = \mathbb{Z}_p$*.*

Proof. In order:

1. To construct $Y_\infty(K)$, we map μ_T to μ, and thus are merely refilling the excavated tubular neighbourhood of K. This clearly recovers Y.

2. Reflect S^3, mapping K to $\overline{K}$ and reversing the orientation on μ. However, since the orientation on S^3 has also been reversed, our insistence that $\mu \cdot \lambda = 1$ preserves the sign of λ, whence r becomes $-r$. The result follows.

3. We consider the Mayer-Vietoris sequence:

$$H_1(\partial T) \xrightarrow{\alpha} H_1(T) \oplus H_1(Y \setminus \operatorname{Int} N(K)) \longrightarrow H_1(Y_{p/q}(K)) \longrightarrow 0,$$

   in which $\alpha(\mu_T) = (0, \pm p\mu)$ and $\alpha(\lambda_T) = (\lambda_T, 0)$. From this, we can see that if Y is an integral homology sphere, $H_1(Y \setminus \operatorname{Int} N(K)) = \mathbb{Z}\mu$, whence

$$H_1(Y_{p/q}(K)) = \frac{H_1(T) \oplus H_1(Y \setminus \operatorname{Int} N(K))}{\operatorname{im} \alpha} = \frac{\mathbb{Z}\lambda_T \oplus \mathbb{Z}\mu}{\langle \lambda_T, p\mu \rangle} = \mathbb{Z}_p,$$

   as required. □

1.4.3 Kirby Calculus

Now that we have defined Dehn surgery, several very important questions immediately spring to mind. To what extent can a closed, connected, orientable 3-manifold Y be written as a surgery in S^3? Can every such 3-manifold be obtained by surgery on a knot? To what extent is that surgery unique? The third part of Proposition 1.4.3 tells us that any manifold with non-cyclic $H_1(Y)$ cannot be obtained by a single surgery, but what happens if we allow surgery on multiple-component links? Does one obtain a different answer? Before investigating these questions, one must first make more precise what one means by surgery on a link.

Definition 1.4.4 (Dehn surgery on a link). *Suppose that L is an ℓ-component link in Y, that its components $K_1, \ldots, K_\ell$ are labelled in an arbitrary but fixed way, and that we have given each K_i a framing λ_i (canonically, if possible) and surgery coefficient r_i. Then we make the following inductive definition, performing all surgeries in tubular neighbourhoods $N(K_i)$ of sufficient thinness that they are pairwise disjoint.*

Let $M_i = S^3_{r_1,\ldots,r_i}(K_1, \ldots, K_i)$ be the result of surgery on $K_1, \ldots, K_i$, and define M_{i+1} as the result of r_{i+1}-surgery on $K_{i+1} \subset M_i$ (thinking of $\lambda_{i+1} \subset M_i$). Then

$$S^3_{p_1/q_1,\ldots,p_\ell/q_\ell}(K_1, \ldots, K_n) := M_n.$$

Definition 1.4.5 (Kirby diagram of a link). *A* Kirby diagram *for $S^3_{r_1,\ldots,r_n}(K_1, \ldots, K_n)$ is the collection $\{(K_1, r_1), \ldots, (K_n, r_n)\}$ of canonically framed knots K_i together with coefficients r_i. We consider Kirby diagrams equivalent if their associated 3-manifolds are homeomorphic as oriented manifolds.*

Because the K_i are disjoint, this definition is independent of the labelling of the K_i.

Definition 1.4.6 (Linking matrix). *Suppose that we have a (perhaps non-canonically) framed link $L = (K_1, \ldots, K_\ell) \subset S^3$. Suppose also that the framing on K_i is $\lambda_i = n_i\mu_i + \lambda_i^0$, where λ_i^0 is the canonical framing. Then the* linking matrix *of L is the matrix whose (i,j)-th entry is $\mathrm{lk}(K_i, K_j)$ if $i \neq j$ or n_i otherwise.*

As a remark, observe that since λ_i^0 sits on a Seifert surface for K_i, we must have $\mathrm{lk}(K_i, \lambda_i^0) = 0$. Moreover, as $H_1(M(K_i))$ is generated by the meridian μ_i (see our remarks after Definition 1.2.2), it follows that $\mathrm{lk}(K_i, \lambda_i) = n_i$. Thus, since $\lambda_i \sim K_i$, we can view the diagonal entries of the linking matrix as the "self-linking numbers" determined by the framing of the corresponding knots. If the framing is canonical on all components of L, then the diagonal is zero.

Allowing for multiple surgeries in this way, the answer to our question about which 3-manifolds can be obtained by surgery on knots and links is given by the following famous theorem of Lickorish [33] and Wallace [71].

Theorem 1.4.7 (Lickorish-Wallace)**.** *Let Y be a closed, connected, orientable 3-manifold. Then there exists some link $L \subset S^3$ such that Y is the result of surgery on L. Moreover, we can ensure that all the components of L are unknots, and that the surgery coefficients are ± 1.*

The implications of this theorem are huge: most strikingly, Dehn surgery in S^3 provides us with a means of studying all closed, connected, orientable 3-manifolds, provided we can identify the associated Kirby diagrams. With this caveat, our second question on the uniqueness of surgery presentations becomes even more pertinent. To answer it, we will need the notion of Dehn twisting.

Definition 1.4.8. *Let $U \subset S^3$ be an oriented unknot spanning a disc D and $L \subset S^3$ an oriented link disjoint from U. Then we say that we have performed a* $+1$-Dehn twist *around U if we cut L at the points of $L \cap D$, apply a full rotation to L within a $[-1, 1]$-neighbourhood of D (in the direction specified by the orientation on U), and reglue the severed ends.*

A k-Dehn twist *around U is an application of k Dehn twists around U, where a negative number k indicates that we have twisted L in the opposite direction to U a total of $-k$ times.*

This procedure is illustrated in Figures 1.3(a), 1.3(b), and 1.3(c). It is just one example of the many possible manipulations we can make to a Kirby diagram, together referred to as the *Kirby calculus*, which leave the resulting manifold unchanged. The following theorem, due originally to Kirby [29] and later strengthened by Rolfsen [59], illustrates a complete set of such moves.

Theorem 1.4.9. *Two Kirby diagrams are equivalent if and only if they are related by a sequence of the following operations:*

1. *Isotopy within S^3;*

2. *Addition or deletion of components with coefficient ∞;*

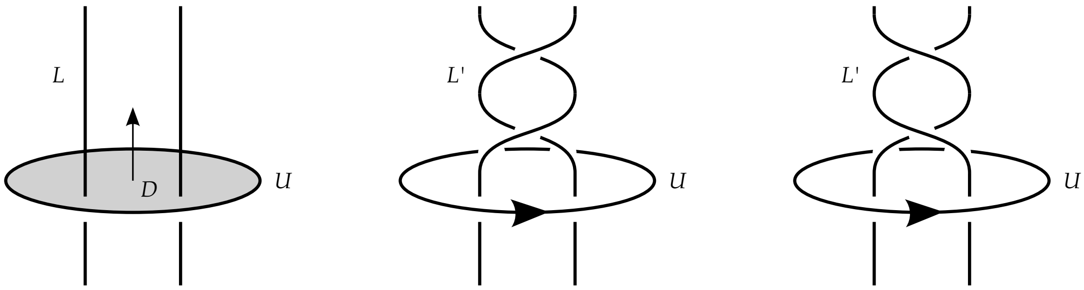


Figure 1.3: (a) The set-up for a Dehn twist; (b) our knots after a +1-Dehn twist; and (c) our knots after a −1-Dehn twist.

3. *Dehn twists about unknots U, provided that we change the coefficient r on each knot K in the Kirby diagram to $r' \in \mathbb{Q}^*$ defined as follows:*

$$r' := \begin{cases} r + k \cdot \mathrm{lk}(K,U)^2 & \text{if } K \neq U \\ \frac{1}{k+r^{-1}} & \text{if } K = U \end{cases}.$$

Although every other move can be reduced to some combination of these three, there are nevertheless some particularly useful combinations worth keeping in mind. The one below is especially important (its colloquial name notwithstanding).

Definition 1.4.10 (Slam dunk)**.** *Suppose $\{(K,n),(U,r)\}$ is a Kirby diagram such that U is a meridian of K and $n \in \mathbb{Z}$, as shown in Figure 1.4(a). Then an equivalent Kirby diagram is provided by $(K, n-1/r)$, shown in Figure 1.4(b). The move from the first diagram to the second is known as* slam dunking*, and the inverse move as* reverse slam dunking.

The two Kirby diagrams in Figures 1.4(a) and 1.4(b) are equivalent since U, being a meridian of K, is equivalent to a longitude of K after we have performed the integral surgery on K. Hence, the second surgery, on U, is equivalent to a second surgery on K. One must simply work out the modified coefficient.

The main reason that we will be interested in slam-dunking is the following corollary. We recall that the *Hirzebruch-Jung continued fraction* $[a_1,\ldots,a_n]^-$ is defined

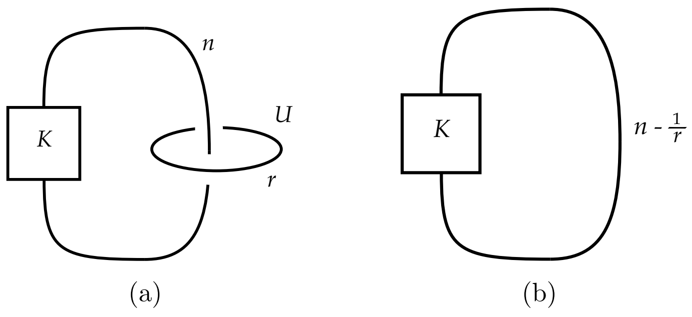


Figure 1.4: (a) A Kirby diagram before slam dunking; and (b) the same Kirby diagram after slam dunking.

inductively by

$$[a_1, a_2]^- := a_1 - \frac{1}{a_2} \qquad\qquad [a_1, \ldots, a_n]^- = a_1 - \frac{1}{[a_2, \ldots, a_n]^-},$$

and that if

$$\begin{aligned} p/q &= a_1 - r_1 \qquad \text{where} \qquad 0 \leq r_1 < 1; \text{ and} \\ r_i^{-1} &= a_{i+1} - r_{i+1} \qquad \text{where} \qquad 0 \leq r_{i+1} < 1. \end{aligned}$$

for all $i \geq 0$ (i.e. the a_i are obtained from a modified Euclidean algorithm), then $[a_1, \ldots, a_n]^-$ is called the *canonical* Hirzebruch-Jung continued fraction for p/q.

Corollary 1.4.11. *Suppose that $Y = S^3_{p/q}(K)$. Then Y has a Kirby diagram*

$$\{(K, a_1), (U_2, a_2), \ldots, (U_n, a_n)\},$$

where $U_2, \ldots, U_n$ are unknots, U_2 is a meridian for K, and U_{i+1} a meridian for U_i. The integers $a_1, \ldots, a_n$ are taken from any Hirzebruch-Jung continued fraction expansion $p/q = [a_1, \ldots, a_n]^-$.

Proof. We reverse slam dunk the diagram $(K, p/q)$, peeling off unknots one by one until all the coefficients are integers. The claim about the coefficients of $K, U_2, \ldots, U_n$ follows by construction. □

1.5 Examples of Dehn Surgery

To conclude this chapter, it is worth pausing to discuss some examples of Dehn surgery: so far, we have only given abstract definitions, precious few of which have been made concrete. These examples will be the central objects of interest in Chapters 4 and beyond.

1.5.1 Lens Spaces

The family of 3-manifolds below, which includes S^3, consists of the most fundamental closed, connected, orientable 3-manifolds one can find.

Definition 1.5.1 (Lens spaces). *Let $U \subset S^3$ be the unknot. The* lens space $L(p,q)$ *is defined by*

$$L(p,q) := S^3_{-p/q}(U).$$

A quick calculation using the van Kampen theorem tells us that $\pi_1(L(p,q)) = \mathbb{Z}_p$, so these 3-manifolds are exceptional in that they have finite cyclic fundamental group. Since they are obtained by Dehn surgery on the unknot in S^3, they are in some sense the simplest 3-manifolds; it is therefore astonishing that Moser's question (see the Introduction) from some four decades ago still remains unanswered. It is a problem worth keeping in mind for Chapter 4 when we prove our main theorem.

1.5.2 Rational Tangle Calculus

Aside from the description in Definition 1.5.1, lens spaces also arise as the double-branched covers of the so-called *2-bridge*, or *rational knots*. Though there are many equivalent definitions of these knots, the easiest one for our purposes uses the theory of tangles.

Definition 1.5.2 (Tangles). *A* tangle *is a pair (B,T) (or merely T, as convenient) consisting of a closed 3-ball B with four marked points on the boundary, NE, SE, SW, and NW, and a pair of embedded arcs T such*

$$T \cap \partial B = \partial T = \{NE, SE, SW, NW\}.$$

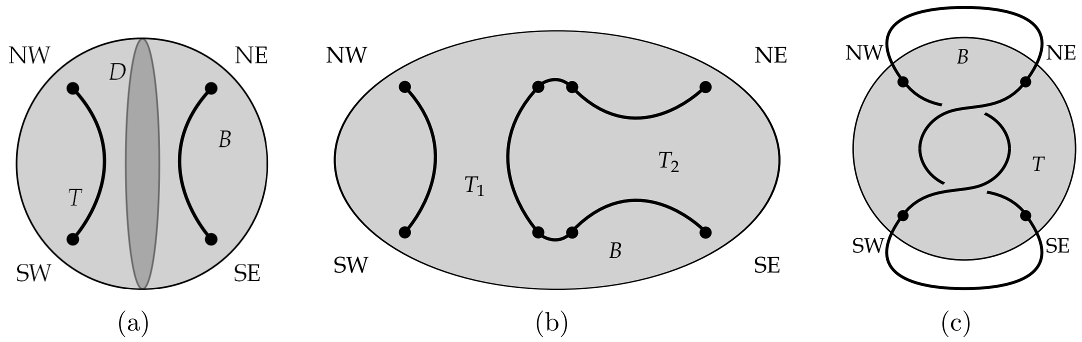


Figure 1.5: (a) A (rational) tangle T (notice the disc D which separates the two arcs); (b) a tangle sum $T_1 + T_2$; and (c) the numerator closure $N(T)$ of a tangle T.

If, moreover, $(B, T) \simeq (D^2 \times I, \{*, *\} \times I)$*, then we say that* T *is a* rational tangle *(see Figure 1.5(a)). We consider tangles equivalent up to isotopy relative to* ∂T*, and, just as with knots, we define a double branched cover* $\Sigma(T)$ *over* B*.*

Definition 1.5.3 (Tangle operations)**.** *The* tangle sum $T_1 + T_2$ *of two tangles* T_1, T_2 *is the tangle obtained by joining the NE and SE points of* T_1 *to the NW and SW points of* T_2 *respectively and including* $B_1 \cup B_2$ *inside a larger ball* B *such that* $\partial(T_1 + T_2)$ *consists of the NW and SW points of* T_1 *and the NE and SE points of* T_2 *(see Figure 1.5(b)).*

We form the numerator closure $N(T)$ *of a tangle* T *by adding arcs between the NE and NW points and the SE and SW points. If* T *is a rational tangle, then the resulting knot or link is called a* rational *or* 2-bridge knot *or* link *(see Figure 1.5(c)).*

Since the name "2-bridge knot" is more common in the literature, in later chapters we will primarily refer to rational knots as 2-bridge knots. However, for the moment we prefer the name "rational knot" (and similarly for links).

One of the remarkable properties of rational tangles (and, indeed, the justification for their name) is the fact that they are in 1-1 correspondence with the extended rational numbers [9]. To describe this bijection, we begin with a sequence of integers $(a_1, \ldots, a_n)$, where $n \geq 1$, and construct on the one hand an associated element of

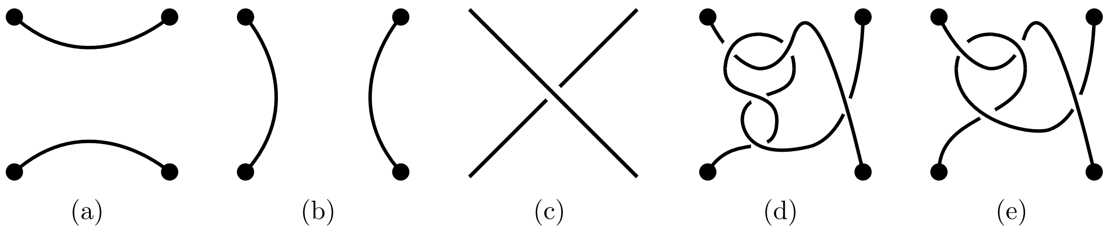


Figure 1.6: (a) The 0-tangle; (b) the ∞-tangle; (c) a positive twist (horizontal and vertical); (d) $T(-2,2,1)$; and (e) $T(2,1,1)$.

$\mathbb{Q}^*$ by taking the continued fraction $[a_n, \ldots, a_1]^+$ defined inductively by

$$[a_2, a_1]^+ = a_2 + \frac{1}{a_1} \qquad\qquad [a_n, \ldots, a_1]^+ := a_n + \frac{1}{[a_{n-1}, \ldots, a_1]^+}.$$

On the other hand, the same sequence also determines a rational tangle according to the parity of n. Starting with one of the two trivial tangles, either the 0-tangle shown in Figure 1.6(a) or the ∞-tangle shown in Figure 1.6(b), we perform the following sequence of operations.

1. If n is odd, begin with the 0-tangle and perform a_1 horizontal twists (each time rotating the NE point over the SE point as shown in Figure 1.6(c), or in the reverse direction if $a_1 < 0$). Then perform a_2 vertical twists in a similar manner (rotating the SW point over the SE point, also shown in the same figure) and repeat as necessary, alternating between horizontal and vertical twists, until the final a_n horizontal twists are finished.

2. If n is even, apply the same procedure, only this time starting with the ∞-tangle and vertical twists.

The tangle one obtains after following this procedure is denoted $T(a_1, \ldots, a_n)$. Examples are provided in Figures 1.6(d) and 1.6(e).

Theorem 1.5.4 (Conway)**.** *Every rational tangle T can be written in the form $T = T(a_1, \ldots, a_n)$. Moreover, there is a bijection*

$$T(a_1, \ldots, a_n) \longleftrightarrow [a_n, \ldots, a_1]^+$$

under which $T(a_1, \ldots, a_n)$ *is isotopic to* $T(b_1, \ldots, b_m)$ *if and only if* $[a_n, \ldots, a_1]^+ = [b_m, \ldots, b_1]^+$. *Thus,* $T(a_1, \ldots, a_n)$ *is also referred to as the* p/q-tangle, *where* $p/q = [a_n, \ldots, a_1]^+$.

By way of example, consider the tangles shown in Figures 1.6(d) and 1.6(e). As $[1, 2, -2]^+ = [1, 1, 2]^+ = \frac{5}{3}$, these two tangles are the same (this can be seen geometrically by twisting the clasp on the left hand side). Observe also that the 0-tangle corresponds to the fraction 0, while ∞ corresponds to $[0, 0]^+ = \infty$, justifying their names. In general, rotation by 90 degrees around the axis into the page transforms the n-tangle into the $-n^{-1}$-tangle (for $n \in \mathbb{Z}$).

One note of caution: despite the conclusions of Theorem 1.5.4, when considering rational knots or links $N(T)$, as opposed to rational tangles T, one must be careful not to assume that the bijection with rational numbers carries through. It is quite possible that $N(T_1) = N(T_2)$ even if $T_1 \neq T_2$. For example, the tangles $T_n := T(n, 0)$ for $n \in \mathbb{Z}$ are all distinct, but $N(T_n) = U$ for all n.

1.5.3 The Montesinos Trick

As was mentioned in the previous section, if K is a rational knot, then $\Sigma(K)$ is a lens space. In order to see why, it is our aim in this section to illustrate what happens to $\Sigma(K)$ if we replace a tangle $T \subset K$ with another tangle T'. An understanding of this kind of tangle replacement, when lifted to the level of double branched covers, will also be helpful for our work on unknotting number (see Chapter 5).

Theorem 1.5.5 (Montesinos trick). *If* $T \subset B$ *is a rational tangle, then* $\Sigma(T)$ *is a solid torus whose core projects to an arc between the two components of* T. *Moreover, the homeomorphism on* ∂B *which effects a horizontal twist in* T *lifts to the homeomorphism of* $\partial\Sigma(T)$ *with matrix* $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, *written in the* $H_1(\partial\Sigma(T))$*-basis* $\{\mu, \lambda\}$ *for some choice of longitude* λ. *The exact sign of the off-diagonal entry is given by the choice of orientation on* μ. *Similarly, a vertical twist on* ∂B *corresponds to* $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$.

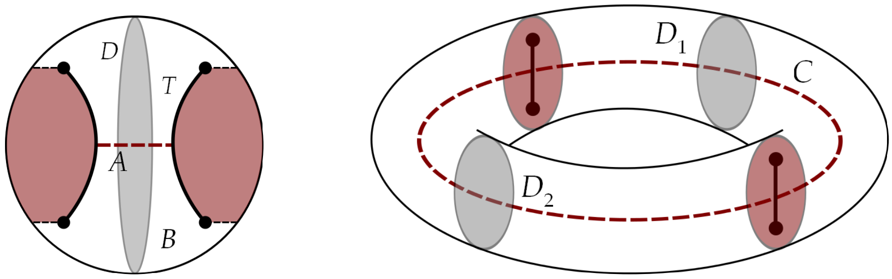


Figure 1.7: (a) Isotoping T to the boundary of the ball B, cutting as we go (notice the arc A joining the two components of T); and (b) $\Sigma(T)$ composed of two copies of the cut ball, the disc D having lifted to two meridinal discs D_1, D_2 (notice also that A has lifted to the core C of $\Sigma(T)$).

Proof. This is a simple exercise in the uniqueness of $\Sigma(T)$. Take two copies of (B, T) and observe that because T is rational we can isotope the arcs of T to the boundaries of their 3-balls, cutting as we do so. After gluing these two nicked balls together, we obtain a solid torus with an obvious involution, and the uniqueness of $\Sigma(T)$ tells us that this solid torus must be the double cover. The procedure is illustrated in Figures 1.7(a) and 1.7(b). The claims about horizontal and vertical twists should be clear from the picture. □

It is extremely important to observe that we cannot tell, in general, which longitude the λ described above represents (i.e. we cannot compute $\text{lk}(\lambda, C)$, where C is the core of T). If $T \subset K$, and $\Sigma(T) \subset \Sigma(K) = S^3$ is standardly embedded, then we can conclude that C is an unknot and λ is the canonical longitude, but beyond that it is difficult to say much more. Consequently, one must be very careful when applying Theorem 1.5.5.

Theorem 1.5.6. *If T is the p/q-tangle, then $\Sigma(N(T)) = -L(p,q)$. What is more, if p is odd, then $N(T)$ is the unique knot with this double branched cover. If p is even, then $N(T)$ is a link.*

Sketch of Proof. We shall only prove the first part; the statement that $N(T)$ is the unique knot with the double branched cover can be found in [63], and the statement about links follows from the third part of Proposition 1.4.3 and the third and fourth properties of $\Sigma(K)$ listed after Definition 1.3.2.

Suppose that we transform $U = N(T(0,0))$ into $K = N(T)$ by replacing the central tangle. Then by the Montesinos trick, Theorem 1.5.5, it follows that $\Sigma(K)$ is obtained by surgery on a knot C in $\Sigma(U) = S^3$. It also follows, from the construction of $\Sigma(U)$, that C is an unknot. We claim for the moment that the surgery coefficient p'/q' is p/q. If this is true, then on comparing this with the second part of Proposition 1.4.3 and Definition 1.5.1, we find that we have just described $-L(p,q)$.

We must therefore prove our claim about the coefficients. The crucial observation here is that the λ specified in Theorem 1.5.5 is the canonical longitude of C (see the remarks after the theorem). Since we began constructing the p/q-tangle with the ∞-tangle, if $p/q = [a_n, \ldots, a_1]^+$ then n must be even. On noting that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ (together with the transposed formula), we find that the image of μ is given by the $p'\mu + q'\lambda$ curve on $\partial\Sigma(T)$, where

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} 1 & \pm a_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm a_{n-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \pm a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The signs $\pm$ are consistent throughout and depend only on the orientation of μ. A straightforward induction on n then shows that p' and q' are respectively the numerator and denominator of the continued fraction expansion, at least up to sign, and hence that $p'/q' = p/q$. □

Having now identified the lens spaces as the double covers of rational knots, the following theorem, originally due to Schubert [63], tells us which rational knots are equivalent. It can also be proved by appealing to Theorem 1.5.6 and applying work by Reidemeister on the classification of lens spaces [57].

Theorem 1.5.7. *The rational knots $N(T_1)$ and $N(T_2)$, where T_i is the p_i/q_i-tangle, are isotopic if and only if $p_1 = p_2$ and $q_1 \equiv q_2^{\pm 1} \mod p_1$.*

1.5.4 Montesinos Knots and Seifert Fibred Spaces

As a last example, we extend the results of the previous section to the so-called Montesinos knots and their double branched covers, the Seifert fibred spaces. These examples are the last ones we will require to make sense of Chapter 5. Much of this work, as suggested by the presence of his name, goes back to Seifert [64], though a similar treatment of the material to the one here can be found in Bleiler [3].

Definition 1.5.8 (Montesinos knots and Seifert fibred spaces)**.** *Let T_i be the rational tangle associated with $r_i \in \mathbb{Q}^*$, for $i = 1, \dots, n$. Then knots of the form*

$$M(r_1, \dots, r_n) := N(T_1 + \cdots + T_n)$$

are referred to as Montesinos knots *of length n. Their double branched covers are called the* Seifert fibred spaces *with n exceptional fibres. If $n = 3$, we say that the double covers are* small *Seifert fibred spaces.*

Proposition 1.5.9. *Let $K = M(r_1, \dots, r_n)$, where $r_i \in \mathbb{Q}^*$. Then $\Sigma(K)$ has a Kirby diagram given by Figure 1.8.*

Proof. By Theorem 1.5.6, we know that $\Sigma(M(r)) = S^3_r(U)$. Hence, by reverse slam dunking the Kirby diagram (U, r), we find that $\Sigma(M(r))$ can also be achieved by surgery on the Hopf link, one component of which has coefficient 0, the other of which $-r^{-1}$. In particular, this means that $\Sigma(M(r))$ can be obtained from $\Sigma(M(0)) = S^2 \times S^1$ by $-r^{-1}$-surgery on an S^1 fibre.

With this first step done, the claim for general $M(r_1, \dots, r_n)$ follows by induction: because $M(r_1, \dots, r_{n-1}) = M(r_1, \dots, r_{n-1}, 0)$, the move from $\Sigma(M(r_1, \dots, r_{n-1}, 0))$ to $\Sigma(M(r_1, \dots, r_n))$ is achieved by another S^1 fibre surgery with coefficient $-r_n^{-1}$. All that remains is to identify these fibres in a Kirby diagram of $S^2 \times S^1$. They are the meridians of the central unknot displayed in Figure 1.8. □

Notice that the set of Seifert fibred spaces includes the set of lens spaces. Indeed, if we suspect a given Seifert fibred space of being a lens space, it is generally not difficult to work out which one it is thanks to Propositions 1.5.9 and 1.5.6.

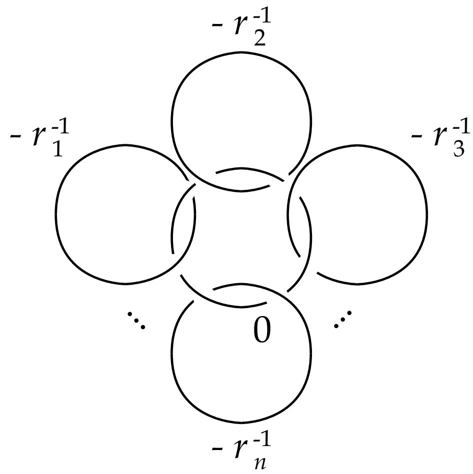


Figure 1.8: $\Sigma(K)$ for $K = M(r_1, \ldots, r_n)$.

CHAPTER 2: 4-MANIFOLDS AND SPIN^C-STRUCTURES

In recent decades, there has been an explosion of productivity in the 4-manifold world on topics ranging from questions about the existence and uniqueness of smooth structures to attempted classifications of the simply connected 4-manifolds. In many ways, it is one of the richest and most exciting fields of topology active today, as evidenced by any list of the eminent mathematicians who have contributed to its diversity.

Sadly, however, it is this same richness which also makes the subject impossible to treat properly within the confines of this thesis. Instead, this chapter endeavours to sketch only the bare bones of what we will need. The first part concentrates primarily on the intersection form, building to the statement of Donaldson's theorem, and paves the way for a discussion of Spin^c-structures on 3- and 4-manifolds. This in turn leads us nicely into the Heegaard Floer homology of the following chapter. Prerequisite material for this discussion, mainly on characteristic classes, can be found in [39].

Unlike our survey in the last chapter, this time we include a substantial number of proofs since we feel that they are enlightening for the chapters that will follow. References to the omitted proofs are given as appropriate.

2.1 The Intersection Form

One of the most important oriented smooth 4-manifold invariants, especially in the simply connected category, is the intersection form. Indeed, given an X in that category, all the interesting cohomology of X is confined to $H^2(X)$ and $H^2(X, \partial X)$; it is therefore on these groups that the intersection form is defined. From this point on, all 4-manifolds will implicitly be oriented.

Definition 2.1.1 (Intersection form)**.** *Let X be a smooth 4-manifold, possibly with boundary, and let $[X] \in H_4(X, \partial X)$ be the fundamental class of X. Then the* intersection form Q_X *of X is the symmetric bilinear form*

$$\begin{aligned} Q_X : H^2(X, \partial X) \times H^2(X, \partial X) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto \langle \alpha \cup \beta, [X] \rangle \,. \end{aligned}$$

It can easily be shown that the intersection form vanishes if either α or β are torsion elements, as a result of which Q_X is often defined modulo torsion. We have chosen not to do so here since we will be exclusively concerned with the case when X is simply connected and hence are guaranteed not to encounter torsion in the second cohomology. We have also chosen to define Q_X on $H^2(X, \partial X)$ instead of $H_2(X)$ as in some books in order to streamline notation in subsequent chapters. The two definitions are equivalent by virtue of Poincaré duality.

In a similar vein to Definition 2.1.1 (i.e. evaluation of the cup product on the fundamental class), it is also possible to define other intersection forms:

$$\begin{aligned} Q_X : H^2(X, \partial X) \times H^2(X) &\longrightarrow \mathbb{Z} \\ Q_X : H^2(X) \times H^2(X, \partial X) &\longrightarrow \mathbb{Z} \\ Q_X : H^2(X) \times H^2(X) &\longrightarrow \mathbb{Q}. \end{aligned}$$

Of these, only the last requires careful explanation (and the extra hypothesis that ∂X be a rational homology 3-sphere). The problem in this case is that $\alpha \cup \beta$ is not an element of $H^2(X, \partial X)$ and therefore cannot be evaluated directly on the fundamental class. To get around this, one must instead notice that the map $\jmath : H^2(X, \partial X; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ from the long exact sequence in relative cohomology is an isomorphism and apply $\jmath^{-1}$ to elements of $H^2(X)$ (included in $H^2(X; \mathbb{Q})$) before evaluating their cup product on the image of $[X]$ under the map $H_4(X, \partial X) \hookrightarrow H_4(X, \partial X; \mathbb{Q})$. Since $\jmath^{-1}$ is defined only with $\mathbb{Q}$-coefficients, it is important to realise that the output may not be integral, explaining the enlarged codomain (that said, if ∂X is an integral homology sphere, the form will still be $\mathbb{Z}$-valued). In a similar way, we may also take the $\mathbb{Z}_2$-valued intersection form of any of the $\mathbb{Z}$-valued ones above. We will denote all these intersection forms the same way, as Q_X, hoping that our meaning will be clear from whatever context we are working in.

The following is one of the fundamental properties of Q_X from which the form derives its name.

Proposition 2.1.2. *Suppose that $\alpha, \beta \in H^2(X, \partial X)$ (or $H^2(X)$ as appropriate) and that A and B are surfaces representing $\mathrm{PD}(\alpha)$ and $\mathrm{PD}(\beta)$ respectively. Then*

$$Q_X(\alpha, \beta) = A \cdot B.$$

Thus, we may also write $\alpha \cdot \beta$ for $Q_X(\alpha, \beta)$.

Suppose now that X is simply connected (so that the universal coefficients theorem tells us $H^2(X) = \mathrm{Hom}(H_2(X), \mathbb{Z})$) and that X has a rational homology sphere Y as boundary. Then we have the following long exact sequence in relative cohomology:

$$0 \longrightarrow H^2(X, Y) \xrightarrow{\jmath} H^2(X) \longrightarrow H^2(Y) \longrightarrow 0.$$

Supposing that we take $\{\alpha_i\}$ as a basis for $H^2(X, Y)$, and letting $\beta_j := (\mathrm{PD}(\alpha_j))^* \in \mathrm{Hom}(H_2(X), \mathbb{Z}) = H^2(X)$ (where the $*$ denotes dualisation), it is straightforward to see that

$$\alpha_i \cdot \beta_j = \langle \alpha_i \cup \beta_j, [X] \rangle = \beta_j([X] \cap \alpha_i) = (\mathrm{PD}(\alpha_j))^*(\mathrm{PD}(\alpha_i)) = \delta_{ij}.$$

Thus, if $\jmath$ has matrix $(Q_{ij})_{i,j}$, we find:

$$\alpha_i \cdot \alpha_j = \alpha_j \cdot \alpha_i = \alpha_j \cdot \jmath(\alpha_i) = \alpha_j \cdot \sum_{i,k} Q_{ik} \beta_k = Q_{ij}.$$

Hence, with our choice of bases above, $\jmath$ has the same matrix as Q_X, a fact which will be extremely useful in Chapters 4 and 5. It is summarised below.

Proposition 2.1.3. *Given a choice of basis for $H^2(X, Y)$, where X is simply connected and $Y = \partial X$ is a rational homology sphere, any matrix representing Q_X is a* *presentation matrix for* $H^2(Y) = H_1(Y)$. *Consequently,* Q_X *is non-degenerate.*

Notice that if $H_1(X) = 0$, so that $H^2(X) = \mathrm{Hom}(H_2(X), \mathbb{Z})$, then two elements $K, K' \in H^2(X)$ written in the dual basis of $H_2(X)$ must satisfy

$$K \cdot K' = K Q_X^{-1} (K')^t,$$

where Q_X is the matrix for the intersection form on X.

The main reason why Q_X is so important for simply connected 4-manifolds is explained by the following very deep theorem [18]. It tells us, in essence, that the intersection form is almost enough to identify closed, simply connected, topological 4-manifolds. The ambiguity in the word "almost" can be made more precise via the Kirby-Siebenmann invariant, but we refrain from doing so in the interests of brevity.

Definition 2.1.4. *If* Q *is an integral symmetric, bilinear form on some* $\mathbb{Z}$*-module* V*, and* $Q(v, v) \equiv 0 \mod 2$ *for all* $v \in V$*, then we say that* Q *is* even*; else, it is* odd.

Definition 2.1.5. *We say that* X *is* positive- *(respectively* negative-*)* definite *if* Q_X *is positive- or negative-definite as a symmetric bilinear form.*

Theorem 2.1.6 (Freedman)**.** *Given any unimodular, symmetric bilinear form* Q *over* $\mathbb{Z}$*, there exists a closed, simply connected, topological 4-manifold* X *with* Q *as intersection form. What is more, if* Q *is even, then there is precisely one such* X*; if* Q *is odd, then there are two such* X*, at most one of which admits a smooth structure.*

Corollary 2.1.7 (Freedman)**.** *Any two closed, simply connected, smooth 4-manifolds with the same intersection form are homeomorphic.*

These theorems, while astonishing in their own right, are mainly included here to illustrate the huge importance of Q_X. They also lead nicely into an equally astonishing result due to Donaldson [16] which will be of critical importance in Chapters 4, 5, and 6.

Theorem 2.1.8 (Donaldson)**.** *Let* X *be a closed, simply connected, positive- (respectively negative-)definite smooth 4-manifold. Then there is a basis for* $H^2(X)$ *such that* $Q_X = \mathrm{id}$ *(respectively* $-\mathrm{id}$*).*

This theorem puts a very strict constraint on the number of simply connected, definite smooth 4-manifolds in existence; it becomes even more remarkable when one realises that the simply connected hypothesis has been removed in the years since Donaldson's original proof. We have chosen to retain it here for historical accuracy since it suffices for our purposes.

2.2 Examples of Intersection Forms

A natural question at this point is, "How computable is Q_X?" As luck would have it, the answer is "surprisingly tractable" in a large number of cases. The aim of this section is to sketch some of these in detail.

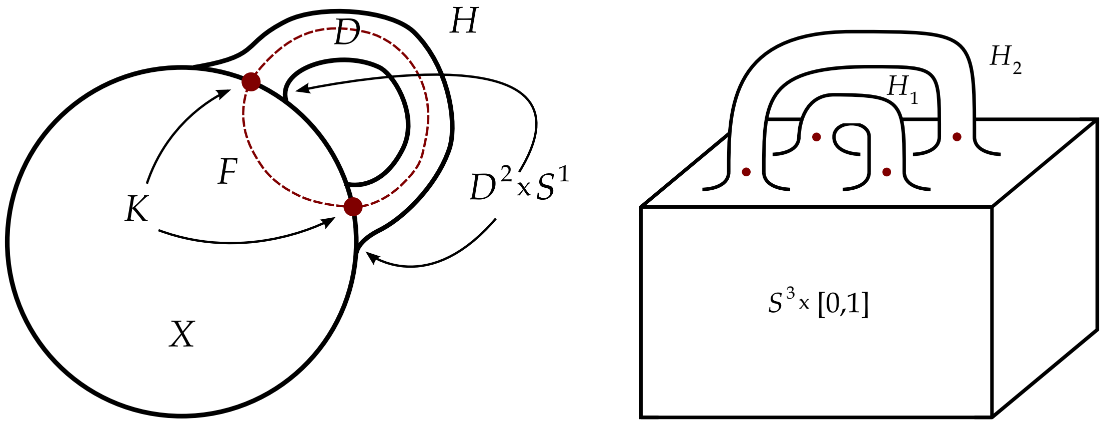
2.2.1 From Surgery

Our first examples come from Dehn surgery, which provides us a plethora of computable intersection forms. In order to explain how, though, we must first specify what we mean by handle addition (see [20] for a definition of a general n-handle).

Definition 2.2.1 (Handle addition)**.** *Suppose that X is a smooth 4-manifold and suppose that $H = D^2 \times D^2$ is a 2-handle. Then $\partial H = (\underbrace{S^1 \times D^2}_{\partial_1 H}) \cup (\underbrace{D^2 \times S^1}_{\partial_2 H})$, and we can attach H to ∂X via $\partial_1 H$. This requires us to specify a framed knot K (i.e. the image of $S^1 \times \{0\}$) in ∂X; the framing specifies the number of twists in the diffeomorphism from $\partial_1 H$ to its image around K. The disc $D^2 \times \{*\} \subset \partial_2 H$ is referred to as the* core *of H.*

This process is illustrated diagrammatically in Figure 2.1(a) and can be smoothed as discussed in [20]. In light of it, we have the following proposition, which tells us, loosely speaking, that 2-handle addition to D^4 is equivalent to integral Dehn surgery in S^3.

Proposition 2.2.2. *Suppose we attach ℓ smooth 2-handles H_i to the upper boundary component of $S^3 \times [0,1]$ along a (possibly non-canonically) framed link L. Then the*



(a) (b)

Figure 2.1: (a) Adding a handle H to a 4-manifold X: a cross-section of the knot K, which forms the core of $\partial_1 H$, is depicted as two red dots, and bounds both a Seifert surface F and the core D of H; and (b) the addition of two handles H_1 and H_2 to the upper boundary of $S^3 \times [0,1]$ yields a cobordism from S^3 to $S^3_{n_1,n_2}(K_1, K_2)$. The knots K_1 and K_2 are also depicted as red dots.

result is a smooth, simply connected cobordism

$$W : S^3 \longrightarrow S^3_{n_1,\ldots,n_\ell}(K_1, \ldots, K_\ell),$$

where $\lambda_i = n_i\mu_i + \lambda_i^0$ *is the framing of* K_i *given relative to its meridian* μ_i *and canonical longitude* λ_i^0*. Moreover, there is a basis for the free group* $H^2(W, \partial W)$ *such that the matrix representing* Q_W *is the linking matrix of* L*.*

Sketch of Proof. For the first part, consider the trivial cobordism $S^3 \times [0,1]$ which has two boundary components, each S^3; we must show that after handle addition, the upper boundary gives the surgered manifold. Clearly, this upper boundary is given by

$$S^3 \setminus \bigsqcup_{i=1}^{\ell} N(K_i) \cup \bigsqcup_{i=1}^{\ell} \partial_2 H_i,$$

where the the union takes place at the $S^1 \times S^1$ boundaries of the $\partial_2 H_i$. To obtain our result, we need only remark that this description is nothing more than Dehn surgery on the link $(K_1, \ldots, K_\ell)$, and that the surgery coefficient on K_i is n_i (with respect to λ_i^0). The second claim, about the cobordism being simply connected, follows as a simple corollary of the fact that this construction can be made without resorting to any 1-handles (see [20]).

For the second part, consider Seifert surfaces F_i for each knot K_i, and let D_i be the core of H_i. Then, by inducting on the number of handles and applying the Mayer-Vietoris sequence, $S_i := F_i \cup D_i$ represents a generator $[S_i] \in H_2(W)$. We can compute their intersections as follows. First, observe that all the intersections occur in $S^3 \times \{1\}$, since the handles are disjoint (i.e. that only the F_i contribute any intersections). Second, observe that to compute the intersections of F_i and F_j (or F_i and a slightly displaced copy of F_i if we want to compute its self-intersection), there is enough space to isotope $\operatorname{Int} F_i$ into $S^3 \times [0,1)$, leaving only $\partial F_i = K_i$ in $S^3 \times \{1\}$. Consequently, using $\{\mathrm{PD}[S_1], \ldots, \mathrm{PD}[S_\ell]\}$ as a basis for $H^2(W, \partial W)$, we find that

$$Q_W(\mathrm{PD}[S_i], \mathrm{PD}[S_j]) = S_i \cdot S_j = \begin{cases} K_i \cdot F_j & \text{if } i \neq j \\ n_i & \text{if } i = j \end{cases}.$$

Since this right hand side recovers the linking matrix of the non-canonically framed link L, we apply Poincaré duality and are done. □

Let us consider the implications of this proposition. From Theorem 1.4.7, we know that every closed, connected, oriented 3-manifold Y can be obtained via integral surgery in S^3. Consequently, after capping the S^3 component of the cobordism in Proposition 2.2.2, we now know that Y bounds a smooth 4-manifold (i.e. that any such Y is smoothly cobordant to the empty manifold). Even better, this 4-manifold comes with an easily computable intersection form.

It is worth noting, at this point, that the 4-manifolds W predicted by Proposition 2.2.2 all have non-empty boundary and are thus not covered by Corollary 2.1.7 or Theorem 2.1.8. In particular, two links with the same linking matrix may yield non-homeomorphic 4-manifolds, and the intersection forms so obtained need not diagonalise to $\pm\,\mathrm{id}$ if they are definite. This point is an essential subtlety in the proofs in Chapter 4.

Proposition 2.2.3. *Suppose that $Y = S^3_{-p/q}(C)$. Then Y bounds a compact, con-*

nected, oriented, simply connected smooth 4-manifold W with intersection form

$$Q_W = \begin{pmatrix} -a_1 & 1 & & & \\ 1 & -a_2 & 1 & & \\ & 1 & -a_3 & & \\ & & & \ddots & 1 \\ & & & 1 & -a_\ell \end{pmatrix},$$

where $[a_1, \ldots, a_\ell]^-$ *is any Hirzebruch-Jung continued fraction for* p/q.

Proof. This is an exercise in reverse slam dunking. A quick application of Corollary 1.4.11 and Proposition 2.2.2, again capping the S^3 component of ∂W, and we are done. □

Definition 2.2.4 (Trace). *In the special case when* $[a_1, \ldots, a_\ell]^-$ *is the canonical Hirzebruch-Jung continued fraction of* p/q, *the 4-manifold described in Proposition 2.2.3 is called the* trace *of the* ($-p/q$-)*surgery on* C.

2.2.2 From Plumbing

Another fertile source of 4-manifolds with predictable intersection form is the operation known as plumbing. Here, the principal ingredient is a vertex-weighted, simple graph. Since we will never mean anything else by the word "graph," these two conditions should be taken as part of the word's definition.

Definition 2.2.5 (Plumbing of disc bundles). *Let* B *and* B' *be* D^2*-bundles over* S^2, *and let* p *and* p' *be points on their base spaces possessing* D^2*-neighbourhoods* D *and* D' *with local trivialisations* $D \times D^2$ *and* $D' \times D^2$. *Then the operation which glues* D *to the* D^2*-fibre of* B' *and the* D^2*-fibre of* B *to* D' *within these local trivialisations is called* plumbing B *and* B'.

We can also plumb *a disc bundle* B *onto another smooth 4-manifold* X *which contains* B' *as a submanifold with properly embedded fibres by plumbing* B *and* B'.

Now for the 4-manifold recipe. Suppose G is a graph with vertex set $V(G)$ and weights $w(v)$ on each $v \in V(G)$. Then we can construct a 4-manifold $\mathcal{X}(G)$ from G

by taking the D^2-bundle $B(v)$ over S^2 with Euler number $w(v)$ for each $v \in V(G)$ and plumbing $B(v)$ and $B(v')$ if and only if v and v' are adjacent in G.

Proposition 2.2.6. *$X := \mathcal{X}(G)$ described above has free $H_2(X)$ generated by the vertices of G and is simply connected if G is a tree. Moreover, there exists a basis of $H^2(X, \partial X)$ such that Q_X is represented by the weighted adjacency matrix of G.*

Proof. We proceed by induction on $|V(G)|$, beginning with the supposition that $V(G) = \{v\}$. In this case, the claim is trivial since $B(v)$ is contractible to S^2, meaning that $H_2(X)$ is free and generated by the homology class represented by the zero-section S of $B(v)$, denoted $[S]$. This also implies that $\pi_1(B(v)) = \pi_1(S^2) = 0$. Taking two copies of S, one of them displaced slightly into the fibres, we see that

$$Q_X(\mathrm{PD}[S], \mathrm{PD}[S]) = S \cdot S = w(v),$$

and we are done.

We now assume that the proposition holds for all graphs $H \leq G$ such that $|V(H)| = |V(G)| - 1$. Let v be the extra vertex, so that $\mathcal{X}(G) = \mathcal{X}(H) \cup B(v)$, and let $\mathcal{B} = \mathcal{X}(H) \cap B(v)$. Then the Mayer-Vietoris sequence tells us that

$$H_2(\mathcal{B}) \longrightarrow H_2(\mathcal{X}(H)) \oplus H_2(B(v)) \longrightarrow H_2(\mathcal{X}(G)) \longrightarrow H_1(\mathcal{B}).$$

By construction, $\mathcal{B}$ is a disjoint union of copies of $D^2 \times D^2$, whence the above portion of the sequence becomes an isomorphism

$$H_2(\mathcal{X}(H)) \oplus H_2(B(v)) \simeq H_2(\mathcal{X}(G)),$$

proving the first claim. The second claim follows a similar argument using the van Kampen theorem and the fundamental group. Lastly, just as we saw with $B(v)$, if S_v denotes the zero-section of $B(v)$, then we claim $\{\mathrm{PD}[S_v]\}_{v \in V(G)}$ provides the required basis for $H^2(X, \partial X)$. Just as in the previous paragraph, self-intersections are given by the Euler numbers $w(v)$, and non-adjacent vertices v and v' determine disjoint spheres S_v and $S_{v'}$. If, on the other hand, v and v' are adjacent, then their intersection is, by construction, the intersection number of $\{*\} \times D^2$ and $D^2 \times \{*\}$ inside $D^2 \times D^2$, which is to say 1. □

Aside from having a computable intersection form, just like the integral Kirby diagrams of the previous example, one of the very pleasant properties of a plumbing diagram is the fact that the boundary can be read off G if G is a tree. Explicitly, the following theorem is true.

Definition 2.2.7. *If G is a graph, define $\mathcal{Y}(G)$ to be the 3-manifold whose Kirby diagram consists of unknots U_v for $v \in V(G)$ such that:*

1. *U_v has coefficient $w(v)$; and*
2. *U_v and $U_{v'}$ are linked if and only if v and v' are adjacent in $V(G)$, in which case U_v is a meridian of $U_{v'}$.*

Proposition 2.2.8. *If G is a tree, $\partial\mathcal{X}(G) = \mathcal{Y}(G)$.*

Proof. Again, the proof is by induction. We show that if G is a tree, then $\mathcal{X}(G)$ is the 4-manifold obtained by attaching 2-handles $H(v)$ to $S^3 = \partial D^4$ along each unknot U_v, framed according to the integer $w(v)$. Proposition 2.2.2 (on capping the S^3 boundary component) then yields the result.

We start the induction with a single vertex v with weight $w(v)$ and consider the trace X of $w(v)$-surgery on U_v. By construction, $\partial X = \mathcal{Y}(G)$, so we must check that $X = \mathcal{X}(G)$ by showing that X is a D^2-bundle over S^2 with Euler number $w(v)$. To this end, notice that $X \setminus \operatorname{Int} H(v) \simeq D^4 \simeq F \times D^2$, where $F \subset S^3$ is a spanning disc for U_v. Consequently, if D is the core of $H(v)$, then it is clear that X contracts through D^2-fibres to $S := F \cup D$; since we already know from Proposition 2.2.2 that $S \cdot S = w(v)$, this must also be the Euler number of X.

Now let us suppose that $H = G \setminus \{v\}$, where v is a leaf of H connected only to v'. Then by induction we can suppose that $\partial\mathcal{X}(H) = \mathcal{Y}(H)$ and $B(v') \subset \mathcal{X}(H)$ (with properly embedded fibres). In order to plumb $B(v)$ to $B(v')$, we let F be a spanning disc for $U_{v'}$ and glue the local trivialisation $D \times D^2 \subset B(v)$ described in Definition 2.2.5 to some neighbourhood $D^2 \times F \subset B(v')$ of F via a map h as in Figure 2.2. Notice two things about this procedure:

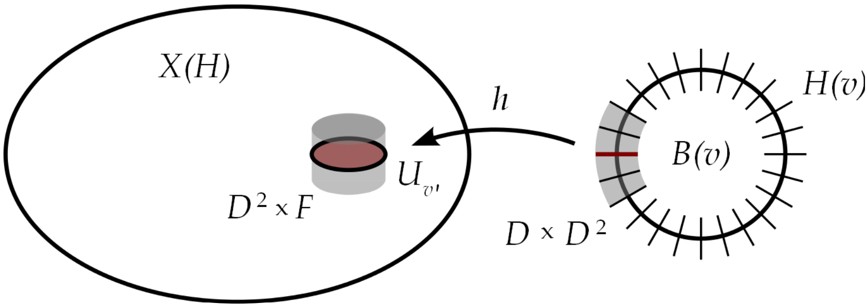


Figure 2.2: Plumbing $B(v)$ onto $\mathcal{X}(H)$. The locally trivialising disc $D \subset B(v)$ is shown as the thick curved segment of the shaded region on the right, and its fibre $D^2 \times \{0\}$ shown in red is glued via h to the spanning disc F of $U_{v'}$ also shown in red. Notice the handle $H(v) := B(v) \setminus \mathrm{Int}(D \times D^2)$.

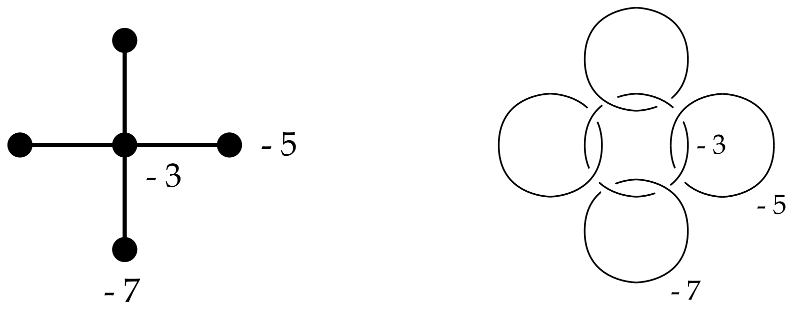
1. The reversal in the position of the D^2-fibre, as required for plumbing; and
2. The disc F is guaranteed to exist since all the coefficients in the Kirby diagram of $\mathcal{Y}(H)$ are integers.

Phrased like this, it is not difficult to see that an equivalent procedure would instead be to excise $\mathrm{Int}(D \times D^2)$ from $B(v)$ and regard the result as a 2-handle $H(v)$ to be attached via the map

$$h|_{(\partial D)\times D^2} : (\partial D) \times D^2 \longrightarrow (\partial D^2) \times F$$

This, in turn, is equivalent to an integral Dehn surgery on the knot $(\partial D^2) \times \{0\}$, which is a meridian of $U_{v'}$ (i.e. the unknot U_v in our theorem). To recover the Euler number of $B(v)$, the coefficient of this surgery must clearly be $w(v)$. □

Because we will usually be interested in negative-definite 4-manifolds, we will by and large require $w(v) < 0$ for all $v \in V(G)$. Bearing this in mind, we will employ the convention that unlabelled vertices have weight -2. As an example, the graph shown in Figure 2.3(a), whose vertices are labelled in an inward, clockwise spiral starting on



(a) (b)

Figure 2.3: (a) An example of a plumbing graph G; and (b) the Kirby diagram for $\mathcal{Y}(G) = \partial\mathcal{X}(G)$.

the left, has intersection form

$$\begin{pmatrix} -2 & & & & 1 \\ & -2 & & & 1 \\ & & -5 & & 1 \\ & & & -7 & 1 \\ 1 & 1 & 1 & 1 & -3 \end{pmatrix},$$

and a boundary 3-manifold given by the surgery presentation in Figure 2.3(b).

In a large number of cases when $\mathcal{X}(G)$ is negative-definite and G is a tree, the Heegaard Floer homology of $\mathcal{Y}(G) = \partial\mathcal{X}(G)$ can be computed algorithmically. Since we have not yet discussed Heegaard Floer homology properly, we defer a discussion of this algorithm until the next chapter. The specific reference is Section 3.5.

2.3 Spin^c-Structures on 3- and 4-manifolds

In addition to the Dehn surgery and intersection form material presented so far, the Heegaard Floer homology of the next chapter also requires an understanding of Spin^c-structures on both 3- and 4-manifolds. In fact, most of the content in Chapters 4 and 5 is essentially a combinatorial argument involving these structures. In view of this fact, we now present the classical take on Spin^c-structures, though an alternative due to Turaev [68] will also be given in Chapter 3.

2.3.1 Spin(n) and Spin$^c(n)$

Recall that n-dimensional fibre bundles ξ over a connected base space M are well-defined provided we have specified a fibre F of dimension n, a structure group G, and a locally trivialising open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ for M equipped with transition functions

$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow G$$

that satisfy the cocycle condition

$$\varphi_{\alpha,\beta} \cdot \varphi_{\beta,\gamma} \cdot \varphi_{\gamma,\alpha} = 1.$$

Here, the $\cdot$ indicates pointwise multiplication in G. The bundle ξ can be reconstituted by patching together local trivialisations $U_\alpha \times F$ and $U_\beta \times F$ via $\varphi_{\alpha,\beta}$. Two different sets of transition functions $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta}$ and $\{\varphi'_{\alpha,\beta}\}_{\alpha,\beta}$ determine the same bundle if and only if there are maps $f_\alpha : U_\alpha \to G$ such that

$$\varphi'_{\alpha,\beta} = f_\alpha \cdot \varphi_{\alpha,\beta} \cdot (f_\beta)^{-1},$$

and if this occurs, we say that $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta}$ and $\{\varphi'_{\alpha,\beta}\}_{\alpha,\beta}$ are equivalent. Note that $(f_\beta)^{-1}(u) = f_\beta(u)^{-1}$ for $u \in U_\beta$ (i.e. that $(f_\beta)^{-1}$ is not the inverse map of f_β).

Suppose that $G = GL(n)$. Then by reducing the structure group to $H \leq G$, we may impose extra structure on our bundle. For example, reduction to $O(n)$ allows the introduction of a metric and a further reduction to $SO(n)$ allows the introduction of an orientation. Analogously, the construction of Spin- and Spin^c-structures requires us to lift the structure group of an oriented n-plane bundle to the Spin(n) and Spin$^c(n)$ groups. Since these are less familiar objects than $SO(n)$, we describe them below.

Definition 2.3.1. *The group* Spin(n) *is the unique double cover of* $SO(n)$.

It is a well-known fact that isomorphism classes of coverings of a connected space M are classified by conjugacy classes of subgroups of $\pi_1(M)$. Since $\pi_1(SO(n)) = \mathbb{Z}_2$ (when $n > 2$), it follows that there are only two coverings of $SO(n)$: the trivial and the universal. Spin(n) is defined as the latter. We let $h :$ Spin$(n) \to$ Spin(n) be the covering automorphism.

Calculating for $n = 3$ and $n = 4$, we have:

$$\text{Spin}(3) = SU(2) \qquad\qquad \text{Spin}(4) = SU(2) \times SU(2).$$

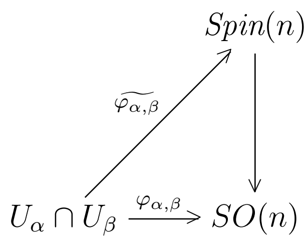
Definition 2.3.2. *We define*

$$\mathit{Spin}^c(n) := \frac{\mathit{Spin}(n) \times U(1)}{(h, -\,\text{id})} = \mathit{Spin}(n) \times_{\mathbb{Z}_2} U(1),$$

which admits a map $\mathit{Spin}^c(n) \to SO(n)$ *which factors through* $\mathit{Spin}(n)$ *by projection onto the first factor.*

The groups Spin(n) and Spin^c(n) now dealt with, we are ready to give the definition of Spin- and Spin^c-structures.

Definition 2.3.3 (Spin- and Spin^c-structures)**.** *Let M be a manifold, and suppose that ξ is an n-plane bundle over M with transition functions* $\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \to SO(n)$*. Suppose moreover that we can construct a lift* $\widetilde{\varphi_{\alpha,\beta}}$ *such that the diagram*



commutes and such that the cocycle condition

$$\widetilde{\varphi_{\alpha,\beta}} \cdot \widetilde{\varphi_{\beta,\gamma}} \cdot \widetilde{\varphi_{\gamma,\alpha}} = 1$$

holds. Then the bundle determined by the equivalence class of $\{\widetilde{\varphi_{\alpha,\beta}}\}_{\alpha,\beta}$ *is called a* Spin-structure *on ξ. If $\xi = TM$, then the bundle determined by such an equivalence class of lifts is called a* Spin-structure *on M. The set of Spin-structures on M is denoted Spin(M).*

If instead of Spin(n) we use Spin^c(n), then we have also just defined Spin^c-structures *on ξ and M. The set of Spin^c-structures is denoted Spin^c(M).*

Because the cocycle condition has already been imposed on $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta}$, it follows that $\widetilde{\varphi_{\alpha,\beta}} \cdot \widetilde{\varphi_{\beta,\gamma}} \cdot \widetilde{\varphi_{\gamma,\alpha}} = \pm 1$, but we can be no more precise using only the diagram. Thus, there is no guarantee that any $\{\widetilde{\varphi_{\alpha,\beta}}\}_{\alpha,\beta}$ satisfying only the diagram above will define a legitimate bundle.

It is worth streamlining some notation at this point. We will usually be interested in Spin- and Spin^c-structures on 3-manifolds Y and 4-manifolds X (often related by $Y = \partial X$). To avoid confusion, we will try to use the letter $\mathfrak{t}$ exclusively for elements of Spin$^c(Y)$ and the letter $\mathfrak{s}$ for elements of Spin$^c(X)$. If $Y = \partial X$, then we will denote the restriction of $\mathfrak{s}$ to Y by $\mathfrak{s}|_Y$. Note the subtlety involved here: although $\mathfrak{s}$ consists of Spin$^c(4)$-valued cocycles, $\mathfrak{t}$ consists of Spin$^c(3)$-valued cocycles. Hence, the restriction is not a simple fibre bundle restriction. The actual construction requires us to use the vector field of unit normals to ∂Y to include $T\partial Y$ as a sub-bundle of TY (and thus to restrict double covers of TY to double covers of $T\partial Y$). We refer the reader to Proposition 2.15 in Chapter II of [32].

2.3.2 Čech Cohomology

In order to make any deductions about the existence or classification of Spin- and Spin^c-structures, it is convenient to use some Čech cohomology. We remark that much of what follows in this section can be done with continuous functions in lieu of smooth ones.

Recall our cover $\mathcal{U}$ from before and let φ be a collection of smooth maps

$$\varphi_{\alpha_0,\dots,\alpha_n} : U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \longrightarrow G,$$

for some $n \geq 0$. Then we would ideally like to build a chain complex $\check{C}^*(\mathcal{U}, G)$ by taking the $\mathbb{Z}$-span of such collections and equipping it with a graded differential. If G is abelian, this is certainly viable: we let $\check{C}^n(\mathcal{U}, G)$ be the $\mathbb{Z}$-span of the φ defined on sets $U_{\alpha_0} \cap \dots \cap U_{\alpha_n}$ and define the differential

$$\partial : \check{C}^n(\mathcal{U}, G) \longrightarrow \check{C}^{n+1}(\mathcal{U}, G)$$

such that

$$(\partial\varphi)_{\alpha_0,\ldots,\alpha_{n+1}} = \sum_{i=0}^{n+1}(-1)^i \varphi_{\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_{n+1}}.$$

The hat here indicates omission. Arguing as for singular cohomology, the terms in $\partial^2\varphi$ cancel in pairs, allowing us to define

$$\check{H}^n(\mathcal{U}, G) := \ker\partial / \operatorname{im}\partial.$$

Since this definition depends on the covering, to obtain a proper definition of the Čech cohomology for M we must finally take a direct limit.

Definition 2.3.4 (Čech cohomology, abelian case)**.** *If G is abelian, we define the* Čech cohomology *of a topological space M in degree n by*

$$\check{H}^n(M, G) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, G),$$

where the direct limit is taken using refinement of coverings.

Now consider what changes if G is non-abelian. Most severely, we are no longer able to take the $\mathbb{Z}$-span, which renders it difficult to define the order in which the "sum" in ∂ should be taken. Consequently, we will only define the Čech cohomology in degrees 0 and 1. Let the 0-cochains be collections of maps $f = \{f_\alpha : U_\alpha \to G\}$, and define the differential ∂ by

$$(\partial f)_{\alpha,\beta} := f_\alpha \cdot (f_\beta)^{-1}.$$

We then define $\check{H}^0(M, G)$ as before. Notice that it is clearly the set of G-valued sections on M.

In degree 1, however, more complexities arise. Although the 1-cochains are still collections of maps $\varphi = \{\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G\}$, the differential must take a different form. It is defined instead by

$$(\partial\varphi)_{\alpha,\beta,\gamma} := \varphi_{\alpha,\beta} \cdot \varphi_{\beta,\gamma} \cdot \varphi_{\gamma,\alpha},$$

which allows us to construct $\check{H}^1(\mathcal{U}, G)$ in the usual fashion. It is worth remarking, however, that the cohomology so produced is just a set, not a group, though it still possesses a distinguished element analogous to a unit (the trivial cocycle).

Definition 2.3.5 (Čech cohomology, non-abelian case)**.** *We define the first* Čech cohomology *of a topological space M by*

$$\check{H}^1(M, G) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G),$$

where the direct limit is taken using refinement of coverings.

Proposition 2.3.6. *Let M be a manifold. Then $\check{H}^1(M, G)$ is the set of isomorphism classes of n-plane bundles over M with structure group G.*

Proof. This is nothing more than the observation that the base, fibre, and transition functions satisfying the cocycle condition (i.e. $\partial\varphi = 1$) determine the bundle. (The term cocycle in a bundle context is motivated by the fact that they are *bona fide* 1-cocycles in the Čech cohomological sense.) The quotient by the image of ∂ ensures that we are looking at isomorphism classes of bundles. □

Proposition 2.3.7. *If M is a smooth manifold and G a discrete group, then*

$$\check{H}^n(M, G) = H^n(M; G).$$

Sketch of Proof. Since M is smooth, it is triangulable. Triangulate M with vertices v_α, and let U_α be the open star of v_α. Then, after refining the triangulation if necessary, $\mathcal{U} = \{U_\alpha\}_\alpha$ gives us a locally trivialising open covering. Observing that the intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ is non-empty if and only if $v_{\alpha_0}, \ldots, v_{\alpha_n}$ span an n-simplex, and observing that $\mathcal{C}^\infty(G)$ is the set of locally constant functions since G is discrete, our Čech cochains now correspond with simplicial cochains. Since their boundary maps also match, the proposition is proved. □

2.3.3 Existence of Spin-Structures

Just as the long exact sequences in singular cohomology provide useful information, so too do those in Čech cohomology. We will use them in this section to provide a criterion for the existence of Spin-structures on a smooth manifold M in terms of the homology of M.

Theorem 2.3.8. *An oriented n-plane bundle ξ on a smooth manifold M admits a Spin-structure if and only if $w_2(\xi) = 0$. That is, the second Stiefel-Whitney class is an obstruction to "spinnability" of ξ.*

Proof. By definition of $\mathrm{Spin}(n)$, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\pi} SO(n) \longrightarrow 0$$

which gives rise to a long exact sequence in Čech cohomology:

$$\ldots \longrightarrow \check{H}^1(M, \mathrm{Spin}(n)) \xrightarrow{\pi^*} \check{H}^1(M, SO(n)) \xrightarrow{w} \check{H}^2(M, \mathbb{Z}_2)\cdot$$

In this context, exactness is defined using the distinguished element (trivial cocycle) in each set: the preimage of this distinguished element under the relevant map must coincide with the image of the preceding map. Applying Proposition 2.3.7, this sequence becomes:

$$\ldots \longrightarrow \check{H}^1(M, \mathrm{Spin}(n)) \xrightarrow{\pi^*} \check{H}^1(M, SO(n)) \xrightarrow{w} H^2(M; \mathbb{Z}_2)\cdot$$

By verifying that w satisfies the axioms of w_2, the second Stiefel-Whitney class (see [39]), it follows that $w = w_2$. Thus, an element $\xi \in \check{H}^1(M, SO(n))$ (i.e. an oriented n-plane bundle over M) is in the image of π^* (i.e. admits a Spin-structure) if and only if $w_2(\xi) = 0$, exactly as claimed in the theorem. □

Proposition 2.3.9. *Suppose that M is a smooth n-manifold. Then M admits a Spin-structure if and only if $w_2(M) = 0$. Moreover, should a Spin-structure exist and $H^1(M; \mathbb{Z}_2) = 0$, then that Spin-structure is unique.*

Proof. This is a scholium of the previous theorem. Truncating the long exact sequence, we find that

$$H^1(M;\mathbb{Z}_2) \longrightarrow \check{H}^1(M,\mathrm{Spin}(n)) \xrightarrow{\pi^*} \check{H}^1(M,SO(n)) \xrightarrow{w_2} H^2(M;\mathbb{Z}_2),$$

whence $w_2(\xi)=0$ if and only if $\xi \in \ker w_2 = \operatorname{im}\pi^*$, or if and only if ξ lifts to $\mathrm{Spin}(n)$. Under the hypothesis $H^1(M;\mathbb{Z}_2)=0$, the map π^* is an injection, meaning that there is at most one lift for ξ. Applying these statements in the case $\xi = TM$, we recover the proposition. □

This proposition is particularly relevant to us for 3-manifolds ($M=Y$) and 4-manifolds ($M=X$). Since the situation is quite different depending on the dimension, we outline the implications separately.

1. If Y is an oriented 3-manifold, then $w_2(Y)=0$ since TY is trivial. Hence an oriented 3-manifold Y always admits a Spin-structure, and this Spin-structure is unique if Y is a rational homology 3-sphere with no 2-torsion (e.g. $\Sigma(K)$ for some knot $K\subset S^3$). It can be shown in general (Proposition 1.4.25 in [20]) that the number of Spin-structures for 3-manifolds Y are in bijection with $H^1(Y;\mathbb{Z}_2)$. Therefore, if we drop the 2-torsion hypothesis, rational homology 3-spheres admit 2^k Spin-structures, where $H_1(Y;\mathbb{Z}_2)=\mathbb{Z}_2^k$.

2. If X is a simply connected smooth 4-manifold and ∂X is a rational homology 3-sphere, then Corollary 2.3.14 below tells us that $w_2(X)=0$ if and only if Q_X is even. Thus, such X have precisely one Spin-structure.

We will soon see how to identify the related Spin^c-structures on X which correspond with these Spin-structures.

2.3.4 Spin^c-Structures on Simply Connected Smooth 4-manifolds

If Spin-structures are to be thought of as lifts of $SO(n)$ bundles to $\mathrm{Spin}(n)$ bundles, then Spin^c-structures can be thought of as "complexifications" of such lifts. Our main goals in this section are to provide a classification for the Spin^c-structures on a 4-manifold, and to point out which of these Spin^c-structures are in fact Spin-structures.

Recall that $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\mathbb{Z}_2} U(1)$. Hence, $p : \mathrm{Spin}^c(4) \longrightarrow SO(4) \times U(1)$ is a double cover built up of the double covers $\pi : \mathrm{Spin}(4) \to SO(4)$ (already discussed) and $U(1) \to U(1)$. The latter cover is given by $z \mapsto z^2$ (in complex number notation).

Let $\det : \mathrm{Spin}^c(4) \to U(1)$ be the composition of p and projection onto $U(1)$ (so that $\det(A, \lambda) = \lambda^2$). Then the cocycles $\varphi_{\alpha,\beta}$ determining a given Spin^c-structure $\mathfrak{s}$ on X project via det to $U(1)$, yielding a new set of cocycles $\det \varphi_{\alpha,\beta}$ which act on $\mathbb{C}$. Consequently, the new cocycles $\det \varphi_{\alpha,\beta}$ determine a complex line bundle $\mathcal{L}_{\mathfrak{s}}$ on X called the *canonical line bundle* of $\mathfrak{s}$.

Definition 2.3.10. *We define the* first Chern class *of* $\mathfrak{s} \in \mathit{Spin}^c(X)$ *by*

$$c_1(\mathfrak{s}) := c_1(\mathcal{L}_{\mathfrak{s}}).$$

Suppose we have an $\mathfrak{s} \in \mathrm{Spin}^c(X)$ whose cocycles are the maps

$$u \longmapsto (A(u), \lambda(u)) \in \mathrm{Spin}^c(4),$$

for $u \in U_{\alpha,\beta}$. Then we can also define the *conjugate* Spin^c-structure $\bar{\mathfrak{s}}$ as the Spin^c-structure with cocycles

$$u \longmapsto (A(u), \lambda(u)^{-1}) \in \mathrm{Spin}^c(4),$$

after we check that the cocycle condition is still satisfied. It follows that the determinant line bundle of $\bar{\mathfrak{s}}$ is conjugate to that of $\mathfrak{s}$, and so clearly $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$.

Importantly, as c_1 gives us a bijection between (complex) line bundles on X and $H^2(X)$, there is, given $x \in H^2(X)$, always a line bundle $\mathcal{L}_x$ with $c_1(\mathcal{L}_x) = x$. It is not necessarily true, however, that for any line bundle $\mathcal{L}$ there is a corresponding Spin^c-structure on X. In fact, as we will soon see, a great many line bundles do *not* determine Spin^c-structures.

Since X is simply connected, we know that $H^1(X; \mathbb{Z}_2) = 0$. Thus, the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(4) \xrightarrow{\ p\ } SO(4) \times U(1) \longrightarrow 0$$

yields a long exact sequence in Čech cohomology:

$$0 \longrightarrow \check{H}^1(X, \mathrm{Spin}^c(4)) \xrightarrow{\ p^*\ } \check{H}^1(X, SO(4)) \oplus \check{H}^1(X, U(1)) \xrightarrow{\ w_2\ } H^2(X; \mathbb{Z}_2)\cdot$$

As before, injectivity of p^* implies that for any pair $(\xi, \mathcal{L}) \in \ker w_2$ there is exactly one Spin^c-structure. Therefore, let us fix $\xi \in \check{H}^1(X, SO(4))$ and consider those $(\xi, \mathcal{L}) \in \ker w_2$. Such pairs must satisfy $w_2(\xi) + w_2(\mathcal{L}) = 0$, since both ξ and $\mathcal{L}$ are orientable. Thus the Spin^c-structures for ξ are in bijection with those $\mathcal{L}$ such that $w_2(\mathcal{L}) = w_2(\xi)$, or equivalently $c_1(\mathcal{L}) \equiv w_2(\xi) \mod 2$ (since $c_1(\mathcal{L})$ has image $w_2(\mathcal{L})$ under the cohomological map induced by $\mathbb{Z} \to \mathbb{Z}_2$). Since we have already discussed the correspondence between line bundles on X and $H^2(X)$, the question about the existence of a Spin^c-structure for $(\xi, \mathcal{L})$ then becomes a question about whether or not $w_2(\xi)$ has an integral lift in $H^2(X)$ (that is, a preimage under the cohomological map induced by $\mathbb{Z} \to \mathbb{Z}_2$).

To determine the answer, we need consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \times 2\ } \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0\cdot$$

which gives us a long exact sequence

$$\ldots \longrightarrow H^2(X) \xrightarrow{\ q\ } H^2(X; \mathbb{Z}_2) \xrightarrow{\ \beta\ } H^3(X) \xrightarrow{\ \times 2\ } H^3(X) \longrightarrow \ldots\cdot$$

To ensure that $w_2(\xi)$ has an integral lift (i.e. a preimage under q), we require that q be surjective. Stated differently, that the Bockstein map β vanish. This is guaranteed if and only if the map $\times 2$ on $H^3(X)$ is injective.

Corollary 2.3.11. *Every orientable, simply connected smooth 4-manifold X admits a Spin^c-structure.*

Proof. Poincaré duality and the relative homology sequence tell us that $H^3(X) =$

$H_1(X, \partial X) = 0$. The map $\times 2$ on $H^3(X)$ is therefore trivially injective. Applying the above discussion to $\xi = TX$, we are done. $\square$

This result is very useful and a stark contrast to the existence of Spin-structures on the same 4-manifolds. We have essentially seen that the extra freedom given to us by the determinant line bundle enables us to "fix" $SO(4)$ bundles which do not lift to Spin(4) so that they lift to Spin^c(4) instead. It would therefore be ideal to give a classification of the Spin^c-structures on X. The following propositions lead us in this direction.

Proposition 2.3.12. *If $\mathfrak{s} \in Spin^c(X)$ and $\mathfrak{s} = \overline{\mathfrak{s}}$, then $\mathfrak{s} \in Spin(X)$.*

Proof. Observe that if $\mathfrak{s} = \overline{\mathfrak{s}}$, then $c_1(\mathfrak{s}) = 0$. Consequently, $w_2(TX) = 0$, whence the pair $(TX, \mathcal{L}_{\mathfrak{s}})$ determines a cocycle lift from $SO(4) \times U(1)$ to Spin^c(4) which is trivial on the $U(1)$ factor. In other words, a lift to Spin(4), or a Spin-structure. $\square$

Proposition 2.3.13. *Suppose that X is an oriented, simply connected smooth 4-manifold. Then*

$$Q_X(w_2(X), \alpha) \equiv Q_X(\alpha, \alpha) \mod 2,$$

for all $\alpha \in H^2(X, \partial X)$, where on the left we have used the $\mathbb{Z}_2$-valued intersection form Q_X and the image of α under the map $H^2(X, \partial X) \hookrightarrow H^2(X, \partial X; \mathbb{Z}_2)$.

Proof. Let A be the corresponding embedded surface for PD(α). Then the left hand side becomes

$$\begin{aligned} \langle w_2(X), [A] \rangle &= \langle w_2(TX), [A] \rangle \\ &= \langle w_2(TA \oplus \nu(A)), [A] \rangle \\ &= \langle w_2(TA), [A] \rangle + \langle w_2(\nu(A)), [A] \rangle , \end{aligned}$$

where $\nu(A)$ is the normal bundle to A. Since $w_2(TA)$ is the image in $H^2(X; \mathbb{Z}_2)$ of the Euler class of TA (i.e. the Euler characteristic of A), which is even, this term vanishes; similarly, the second term evaluates to the image of the Euler class of $\nu(A)$, which is the self-intersection $A \cdot A$. The claim follows. $\square$

The following corollary of this result was used in the previous section to show that not all simply connected smooth 4-manifolds admit Spin-structures.

Corollary 2.3.14. *If X is a simply connected smooth 4-manifold and ∂X is a rational homology 3-sphere, then Q_X is even if and only if $w_2(X) = 0$.*

Proof. By Proposition 2.3.13,

$$Q_X(w_2(X), \alpha) \equiv Q_X(\alpha, \alpha) \mod 2,$$

for all $\alpha \in H^2(X, \partial X)$. Consequently, the right hand side vanishes if and only if the left hand side vanishes for all α, which occurs if and only if $w_2(X) = 0$, since Q_X is non-degenerate (see Proposition 2.1.3). □

Proposition 2.3.15. *Let X be simply connected smooth 4-manifold, and identify $H^2(X)$ with $\mathrm{Hom}(H_2(X), \mathbb{Z})$ via the universal coefficients theorem. Then the Spin^c-structures on X are in bijection, via c_1, with the set*

$$\left\{K \in H^2(X) \,|\, \langle K, v\rangle \equiv v \cdot v \textit{ for all } v \in H_2(X)\right\}.$$

Such K are referred to as the characteristic covectors *of X. If Q_X is even, then the Spin-structure on X corresponds with $K = 0$.*

Proof. We observe that $K \in H^2(X)$ determines a Spin^c-structure on X if and only if its associated line bundle $\mathcal{L}_K$ satisfies $w_2(TX) \equiv c_1(\mathcal{L}_K) = K$. Applying Proposition 2.3.13,

$$\langle K, v\rangle = Q_X(K, \mathrm{PD}(v)) \equiv Q_X(\mathrm{PD}(v), \mathrm{PD}(v)) = v \cdot v \mod 2,$$

and we are done with our first claims. The statement about the Spin-structure, is an immediate consequence of Corollary 2.3.14 (which establishes that it exists), Proposition 2.3.9 (which establishes that it is unique), and Proposition 2.3.12. □

Knowing what the Spin^c-structures on X look like, it remains for us to remark that $\mathrm{Spin}^c(X)$ is an affine space over $H^2(X)$. Indeed, recall that for any $x \in H^2(X)$, there is a line bundle $\mathcal{L}_x$ with $c_1(\mathcal{L}_x) = x$. Let its cocycles be $\lambda_{\alpha,\beta} : U_\alpha \cap U_\beta \to U(1)$, and lift $\lambda_{\alpha,\beta}$ to Spin$^c(4)$-valued cocycles $\lambda'_{\alpha,\beta}$ in the obvious way (the identity on $SO(4)$).

Definition 2.3.16. *If $\mathfrak{s} \in Spin^c(X)$ has cocycle $\varphi_{\alpha,\beta}$, we define $\mathfrak{s}+x$ to be the Spin^c-structure with cocycle $\varphi_{\alpha,\beta} \cdot \lambda'_{\alpha,\beta}$.*

In this way, we have an explicit affine $H^2(X)$ structure on Spin$^c(X)$. Notice that the Chern class of $\mathfrak{s}+x$ is easy to determine since

$$\mathcal{L}_{\mathfrak{s}+x} = \mathcal{L}_{\mathfrak{s}} \otimes \mathcal{L}_x^{\otimes 2},$$

whence

$$c_1(\mathfrak{s}+x) = c_1(\mathfrak{s}) + 2x.$$

It therefore makes sense to write $\mathfrak{s}-\mathfrak{s}'$ for $\mathfrak{s}, \mathfrak{s}' \in$ Spin$^c(X)$, since this difference defines an element of $H^2(X)$. As an immediate corollary, we have the following.

Corollary 2.3.17. *If $\mathfrak{s} \in Spin^c(X)$, then $\mathfrak{s} - \overline{\mathfrak{s}} = c_1(\mathfrak{s})$.*

2.3.5 Spin^c-Structures on Closed Rational Homology 3-spheres

If instead of simply connected 4-manifolds we are interested in closed, oriented rational homology 3-spheres, the results of the previous section are somewhat different. The crucial point of divergence concerns the map $c_1 : \text{Spin}^c(Y) \to H^2(Y)$, which is no longer bijective onto its image unless $H^2(Y)$ is of odd order.

Having said this, much of our previous work still applies. We can still define the determinant line bundle the same way, we still have an affine $H^2(Y)$ structure on Spin$^c(Y)$, and, as before, a pair $(TY, \mathcal{L})$ still determines a Spin^c-structure if and only if $w_2(TY) = w_2(\mathcal{L})$. The difference is that $w_2(TY) = 0$ for all Y, as mentioned at the end of Section 2.3.3. Hence $\mathcal{L}$ determines a $\mathfrak{t} \in$ Spin$^c(Y)$ if and only if $w_2(\mathcal{L}) = 0$; that is, if and only if the modulo 2 reduction of $c_1(\mathcal{L})$ vanishes. Hence $c_1(\mathcal{L}) = 2y$ for some $y \in H^2(Y)$. If $H^2(Y)$ is of odd order, we have the following.

Proposition 2.3.18. *Suppose that* Y *is a closed rational homology 3-sphere and that* $H^2(Y)$ *is of odd order. Then*

$$\tfrac{1}{2}c_1 : Spin^c(Y) \longrightarrow H^2(Y)$$

is a canonical isomorphism.

Proof. Analogously to the latter part of Proposition 2.3.15, Y has a unique Spin-structure $\mathfrak{t}_0$ corresponding to the line bundle with zero Chern class. Fixing this as the zero element of $\mathrm{Spin}^c(Y)$, the affine structure on $\mathrm{Spin}^c(Y)$ gives us the isomorphism required, for if y is a generator of $H^2(Y)$ and $k \in \mathbb{Z}$, then

$$\tfrac{1}{2}c_1(\mathfrak{t}_0 + ky) = \tfrac{1}{2}c_1(\mathfrak{t}_0) + ky = ky,$$

which enables us to define $\mathfrak{t}_1 + \mathfrak{t}_2$, for $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathrm{Spin}^c(Y)$, to be the unique $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ such that $\frac{1}{2}c_1(\mathfrak{t}) = \frac{1}{2}c_1(\mathfrak{t}_1) + \frac{1}{2}c_1(\mathfrak{t}_2)$. □

In the case that $H^2(Y)$ has 2-torsion, we no longer have a unique Spin-structure with which to construct the isomorphism, but one can still show (using techniques described in [20]) that there is a bijection

$$\mathrm{Spin}^c(Y) \longleftrightarrow 2H^2(Y) \oplus H^2(Y; \mathbb{Z}_2).$$

Since $|2H^2(Y)| = \frac{1}{2^k}\left|H^2(Y)\right|$, where $H_1(Y;\mathbb{Z}_2) = \mathbb{Z}_2^k$, there must therefore also exist a bijection

$$\mathrm{Spin}^c(Y) \longleftrightarrow H^2(Y),$$

but this bijection is not canonical. To summarise, we now know the following.

Proposition 2.3.19. *Suppose that* Y *is a closed rational homology 3-sphere and that* $H^2(Y)$ *is of even order. Then there is a non-canonical bijection*

$$Spin^c(Y) \longleftrightarrow H^2(Y),$$

and the map $c_1 : Spin^c(Y) \to H^2(Y)$ *is* 2^k*-to-one onto* $2H^2(Y)$*, the kernel consisting* *of the* 2^k *Spin-structures on* Y.

It is worth noting that whenever $H_1(X)$ is torsion free and $Y = \partial X$ is a rational homology sphere, the restriction map $\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y)$ is a surjection. That is, for any $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, there exists an $\mathfrak{s} \in \mathrm{Spin}^c(X)$ such that $\mathfrak{s}|_Y = \mathfrak{t}$. This can be seen from the long exact sequence in relative cohomology:

$$\ldots \longrightarrow H^2(X) \longrightarrow H^2(Y) \longrightarrow H^3(X, Y) \longrightarrow \ldots.$$

Since $H^2(Y) = H_1(Y)$ is torsion and $H^3(X, Y) = H_1(X)$ is free, it follows that the map $H^2(Y) \to H^3(X, Y)$ vanishes. Hence $H^2(X) \to H^2(Y)$ surjects. Fixing $\mathfrak{s}_0 \in \mathrm{Spin}^c(X)$ and setting $\mathfrak{t}_0 = \mathfrak{s}_0|_Y$, we let $y = \mathfrak{t} - \mathfrak{t}_0 \in H^2(Y)$ and $x \in H^2(X)$ map to y. Then the $\mathfrak{s}$ we desire is given by $\mathfrak{s} = \mathfrak{s}_0 + x$. In particular, this applies if X is a simply connected smooth 4-manifold and Y is a rational homology 3-sphere.

As a final remark, we warn the reader that many authors use different labellings of Spin^c-structures, even in the case when $H^2(Y)$ is of odd order. Indeed, one of the challenges in this area (as we will see in Chapter 5) is the task of relating one labelling to another. We will endeavour to be as explicit as possible about all our labellings from here on in.

CHAPTER 3: HEEGAARD FLOER HOMOLOGY

Around the turn of the millennium, Ozsváth and Szabó introduced a package of new 3-manifold invariants which revolutionised low-dimensional topology. These invariants, known collectively as Heegaard Floer homology, have since been extended to associated 4-manifold and knot invariants, the latter of which has had spectacular success both as a categorification of the Alexander polynomial and in detecting the genus, fibredness, and other topological properties of knots. The 3-manifold invariants, when applied to $\Sigma(K)$, have also enjoyed considerable success detecting knots K with unknotting number one.

At the heart of the new theory is a familiar object that goes back to Lagrangian Floer homology: a chain complex built out of intersection points between complementary dimensional submanifolds of a larger manifold, and a differential that counts holomorphic discs between these points. Where the Heegaard Floer theory differs, however, is that one can modify this construction to produce a knot invariant $HFK(Y, K)$ by letting the knot $K \subset Y$ induce a filtration on the chain complex. It is the hope of this chapter to illustrate sufficiently many details of both invariants that the reader unfamiliar with Heegaard Floer homology will at least be able to follow the exposition in later chapters. As such, almost nothing is proved here; indeed, many of the proofs are the domain of a series of lengthy papers by Ozsváth and Szabó, to which we refer the interested reader [50, 49, 53, 46, 48].

Although Heegaard Floer homology $HF^\circ(Y)$ is defined for general closed, connected, oriented 3-manifolds Y, the definitions are simpler when restricted to rational homology spheres. In these cases, the homology also takes on a more refined structure. Since we will never need to compute $HF^\circ(Y)$ for any other type of Y, we restrict our attention to this type of 3-manifold. The reader is referred to [50] for the more general case.

3.1 Constructing Heegaard Floer Homology

3.1.1 Heegaard Diagrams

The main ingredient in the definition of $HF^\circ(Y)$ is a Heegaard diagram for Y, an object which encodes all the raw data used to assemble Y.

Definition 3.1.1 (Heegaard splitting). *Suppose that Y is a closed, connected, oriented 3-manifold. Then a* Heegaard splitting *for Y is a triple (U_1, U_2, Σ), where U_1 and U_2 are handlebodies of the same genus (i.e. boundary connect sums of copies of $S^1 \times D^2$), $\Sigma = \partial U_1$, and $Y = U_1 \cup_\Sigma U_2$.*

Critically, every closed, connected, oriented 3-manifold Y possesses a Heegaard splitting. This is most easily seen by noting that all such 3-manifolds are triangulable. By thickening up the 1-skeleton of any such triangulation, we obtain the first handlebody U_1; since its complement U_2 is itself a thickening of the dual 1-skeleton, it too is a handlebody of the same genus. An alternative perspective takes a self-indexing Morse function $f : Y \to [0, 3]$ for Y and sets $U_1 = f^{-1}([0, \frac{3}{2}])$, $U_2 = f^{-1}([\frac{3}{2}, 3])$, and $\Sigma = f^{-1}(\frac{3}{2})$. Some effort is required to establish that all Heegaard splittings for Y arise in this manner.

Whichever way one views the matter, the main piece of information required to construct a 3-manifold from two handlebodies U_1 and U_2 is the map $h : \partial U_1 \to \partial U_2$ performing the gluing. Since maps can be tricky things to play with directly, an easier approach is to relate ∂U_1, ∂U_2, and h to a standard surface Σ of the same genus in what is called a system of attaching circles.

Definition 3.1.2 (System of attaching circles). *Let Σ be a genus g surface. Then a* system of attaching circles *on Σ, denoted $\boldsymbol{\alpha}$, is a collection of pairwise disjoint simple closed curves $\{\alpha_1, \ldots, \alpha_g\}$ such that the α_i determine independent homology classes in $H_1(\Sigma)$.*

By attaching discs to the α_i and capping the resulting 2-skeleton with a 3-ball, we obtain a genus g handlebody $U_{\boldsymbol{\alpha}}$. It is clear that all handlebodies with boundary Σ arise in this fashion. With this in mind, we can now define a Heegaard diagram.

Definition 3.1.3 (Heegaard diagram)**.** *A* (pointed) Heegaard diagram *for a closed, connected, oriented 3-manifold Y is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ where*

1. *Σ is a genus g surface with two systems of attaching circles $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^g$ and $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^g$ determining handlebodies $U_{\boldsymbol{\alpha}}$ and $U_{\boldsymbol{\beta}}$ respectively which form a Heegaard splitting $(U_{\boldsymbol{\alpha}}, U_{\boldsymbol{\beta}}, \Sigma)$ for Y; and*

2. *z is a point (called the* basepoint*) on Σ disjoint from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.*

While it is possible to specify Heegaard diagrams without the basepoint, or indeed with two or perhaps more basepoints, until we reach Section 3.6 we will only be dealing with single pointed Heegaard diagrams.

Figures 3.1(a), 3.1(b), 3.1(c), and 3.1(d) provide some examples of genus one Heegaard diagrams. Viewing Σ in each diagram as $\partial N(C)$ for the knot C obtained by closing-up the cores of the handles in our diagrams with straight lines, we can write $H_1(\Sigma) = \mathbb{Z}\mu \oplus \mathbb{Z}\lambda$, and every curve K on Σ is specified by its homology class. Equivalently, by integers p, q such that $[K] = \pm(p\mu + q\lambda)$. Our examples can then be summarised as follows.

1. Setting $\alpha = \lambda$ and $\beta = \mu$, one obtains a Heegaard diagram of S^3;

2. Setting $\alpha = \beta = \lambda$, one obtains $S^2 \times S^1$;

3. Setting $\alpha = p\mu + q\lambda$ and $\beta = \lambda$, one obtains $-L(p, q)$.

The intersection points $\mathbf{x}_i$ labelled in the figures will be relevant in the next section. They are the principal ingredients in the Heegaard Floer chain complex.

3.1.2 Symmetric Products and Whitney Discs

While the previous section should have provided ample justification for the appearance of Heegaard's name in the theory, this section should justify Floer's. As mentioned in the preamble, Floer homology theories usually count intersection points between k-dimensional submanifolds of a $2k$-dimensional supermanifold and relate

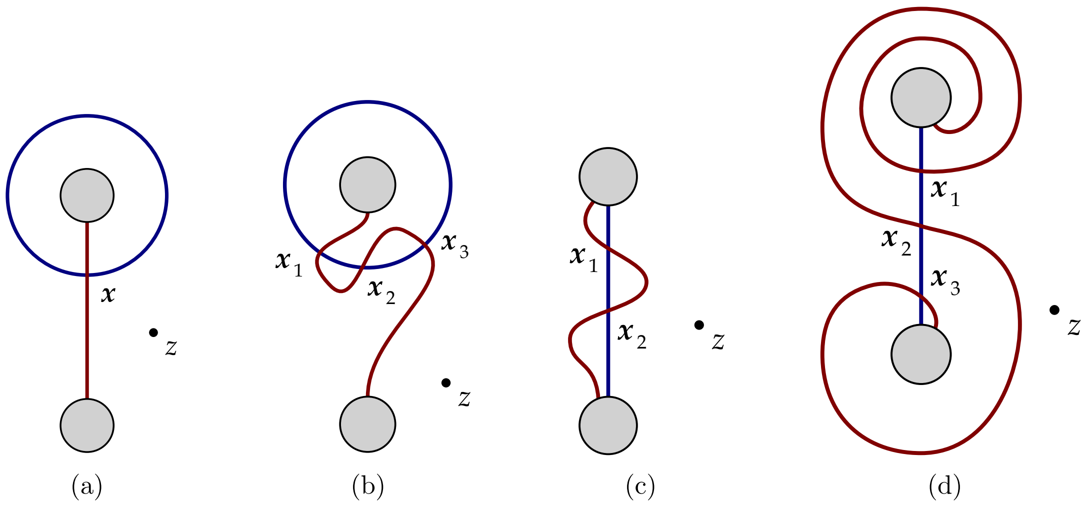


Figure 3.1: (a) The standard genus one Heegaard diagram for S^3: the shaded circles represent the handle, while the red and blue circles represent $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively; (b) an alternative Heegaard diagram for S^3; (c) a Heegaard diagram for $S^2 \times S^1$; and (d) a Heegaard diagram for $L(3,1)$.

these points via connecting holomorphic discs. Heegaard Floer homology is no different. In this case, the arena in which the action takes place is an appropriately constructed symmetric product.

Definition 3.1.4 (Symmetric product)**.** *Let* Σ *be a genus* g *surface. Then the* symmetric product $\mathrm{Sym}^g(\Sigma)$ *of* Σ *is defined as the quotient*

$$\mathrm{Sym}^g(\Sigma) := \overbrace{\Sigma \times \cdots \times \Sigma}^{g} / Sym(g),$$

where the action of the symmetric group $Sym(g)$ *on* g *objects is given by permutation.*

Since Σ is a closed surface, and thus also a complex curve (once given a complex structure), the symmetric product $\mathrm{Sym}^g(\Sigma)$ can be viewed (at least locally) as an unordered collection of g complex numbers. By the fundamental theorem of algebra, it therefore resembles a collection of monic polynomials of degree g (whose roots are the unordered complex numbers); the space of such polynomials being homeomorphic to $\mathbb{C}^g$, we see that $\mathrm{Sym}^g(\Sigma)$ is a g-dimensional complex manifold (i.e. a $2g$-dimensional real manifold). The two g-dimensional real submanifolds required for a Floer-based approach are the following.

Definition 3.1.5 (Heegaard torus)**.** *Let* $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ *be a system of attaching circles on a genus* g *surface* Σ*. Then we define the* Heegaard torus $\mathbb{T}_{\boldsymbol{\alpha}}$ *to be the image of* $\alpha_1 \times \cdots \times \alpha_g$ *inside* $\text{Sym}^g(\Sigma)$*.*

Although it is certainly true that $\alpha_1 \times \cdots \times \alpha_g$ is a g-dimensional torus (each α_i is homeomorphic to S^1), it is not immediately obvious that $\mathbb{T}_{\boldsymbol{\alpha}}$ shares this property. However, since the α_i are pairwise disjoint, if $x, y \in \alpha_1 \times \cdots \times \alpha_g$, then there does not exist $\sigma \in S_g$ such that $y = \sigma \cdot x$ unless $x = y$. Consequently, distinct points of $\alpha_1 \times \cdots \times \alpha_g$ are in distinct orbits of S_g, and $\mathbb{T}_{\boldsymbol{\alpha}}$ is homeomorphic to the product of g circles, proving our claim. Note, however, that $\mathbb{T}_{\boldsymbol{\alpha}}$ is only a real manifold; it is entirely possible that g might be odd.

The points of $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ are called the *intersection points* of the Heegaard diagram. In general, these points are ordered g-tuples $(x_1, \ldots, x_g)$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ and $\sigma \in S_g$, and will form the generators of our eventual Heegaard Floer chain complex. The differential will count holomorphic discs between them.

Definition 3.1.6. *Let* $\mathbb{D}$ *be the unit disc in* $\mathbb{C}$ *and* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$*. A* Whitney disc *from* $\mathbf{x}$ *to* $\mathbf{y}$ *is a map* $\phi : \mathbb{D} \to \text{Sym}^g(\Sigma)$ *such that*

1. $\phi(-i) = \mathbf{x}$ *and* $\phi(i) = \mathbf{y}$*;*
2. $\phi(z) \in \mathbb{T}_{\boldsymbol{\alpha}}$ *for* $|z| = 1$ *and* $\Re(z) \geq 0$*; and*
3. $\phi(z) \in \mathbb{T}_{\boldsymbol{\beta}}$ *for* $|z| = 1$ *and* $\Re(z) \leq 0$*.*

We let $\pi_2(\mathbf{x}, \mathbf{y})$ *be the set of homotopy classes of Whitney discs from* $\mathbf{x}$ *to* $\mathbf{y}$*. In an abuse of notation, we will continue to write* ϕ *for both the Whitney disc and its equivalence class in* $\pi_2(\mathbf{x}, \mathbf{y})$*.*

A Whitney disc is illustrated in Figure 3.2. Given any complex structure J on Σ (and hence an induced almost complex structure on $\text{Sym}^g(\Sigma)$), we can also speak about *pseudoholomorphic* representatives of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. For suitable choices of J and suitable perturbations of the induced almost complex structure on $\text{Sym}^g(\Sigma)$, it

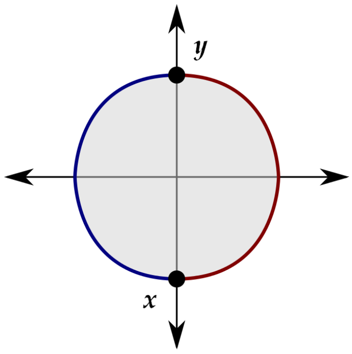


Figure 3.2: The image of $\mathbb{D}$ under a Whitney disc.

turns out that the *moduli space* $M(\phi)$ of pseudoholomorphic representatives for ϕ is a manifold, and that its associated *Maslov index* $\mu(\phi)$ computes $\dim_{\mathbb{R}} M(\phi)$. What is more, we can define an $\mathbb{R}$-action on $M(\phi)$ as follows. Given a pseudoholomorphic representative ϕ' for ϕ, the conformal equivalence

$$\mathbb{D} \setminus \{\pm i\} \longrightarrow \mathbb{S} := \{z \in \mathbb{C} \,|\, 0 \leq \Re(z) \leq 1\}$$

allows us to think of ϕ' as a map $\mathbb{S} \cup \{\pm\infty\} \to \mathrm{Sym}^g(\Sigma)$. Since the domain $\mathbb{S} \cup \{\pm\infty\}$ now has a clear $\mathbb{R}$-action by vertical translation, and since this translation is homotopic to the identity, any translate of ϕ' yields a different pseudoholomorphic representative of ϕ. Quotienting $M(\phi)$ by this action, we obtain

$$\mathcal{M}(\phi) := M(\phi)/\mathbb{R},$$

which, for suitable J, has (real) dimension $\mu(\phi) - 1$. It is a theorem that if $\mu(\phi) = 1$, then $\mathcal{M}(\phi)$ is compact.

Armed with this reduced moduli space, there remains only one prerequisite for Heegaard Floer homology left to discuss. It concerns the basepoint $z \in \Sigma$ which has so far gone unnoticed. Lifting any point $z \in \Sigma$ to the symmetric product, we define $V_z \subset \mathrm{Sym}^g(\Sigma)$ to be the image of the projection

$$\{z\} \times \overbrace{(\Sigma \times \cdots \times \Sigma)}^{g-1} \longrightarrow \mathrm{Sym}^g(\Sigma).$$

If $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we also define the intersection number of ϕ and z by

$$n_z(\phi) := \operatorname{im} \phi' \cdot V_z,$$

where ϕ' is any pseudoholomorphic representative of ϕ. Note that $\operatorname{im} \phi'$ and V_z have complementary dimensions in $\mathrm{Sym}^g(\Sigma)$ and can therefore be isotoped to intersect transversally in points. These points are finite in number due to the compactness of $\operatorname{im} \phi'$ and V_z.

3.1.3 Spin^c-Structures Revisited

Recall from Chapter 2 that every rational homology 3-sphere Y comes equipped with Spin^c-structures in bijection with $H^2(Y)$, and that $\mathrm{Spin}^c(Y)$ is an affine space over $H^2(Y)$. Moreover, if X is a 4-manifold with Y as boundary, we can restrict Spin^c-structures from X to Y, and this restriction surjects if X is simply connected.

In most of the papers on Heegaard Floer homology, however, the Spin^c-structures on Y are not defined in the manner outlined in Chapter 2: while the formalism of that chapter is certainly necessary to make sense of $\mathrm{Spin}^c(X)$ and restrictions to $\mathrm{Spin}^c(Y)$, it is often more convenient to use an equivalent 3-manifold formulation due to Turaev [68] when dealing with Heegaard Floer-type questions.

Definition 3.1.7 (Spin^c-structures, Turaev)**.** *Let v and v' be nowhere vanishing vector fields on Y. We say that v is* homologous *to v' (written $v \sim v'$) if they are homotopic through nowhere vanishing vector fields on $Y \setminus B$ for some 3-ball $B \subset Y$. It can be shown that the equivalence classes defined by $\sim$ are in bijection with $\mathrm{Spin}^c(Y)$; consequently, they themselves are referred to as* Spin^c-structures*. It can be shown that if $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ corresponds to the nowhere vanishing vector field v, then $\bar{\mathfrak{t}}$ corresponds to $-v$.*

The equivalence between this definition and our previous one is proved in the early pages of [68]. As a quick sketch, observe that a nowhere vanishing unit-vector field (which must exist on Y as $\chi(Y) = 0$) determines a splitting of the tangent bundle TY into $v^\perp \oplus \mathbb{R}v$. Consequently, the structure group of TY has been reduced to

$U(1)$ (that of $v^{\perp}$), which embeds diagonally into $U(2) = \mathrm{Spin}^c(3)$, thus giving us a Spin^c-structure. For the details in the other direction, and the relevance of the ball B, we refer the reader to [68].

The main reason that we will be interested in Spin^c-structures from a Heegaard Floer point of view is because they induce a splitting of the Heegaard Floer homology

$$HF^{\circ}(Y) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} HF^{\circ}(Y, \mathfrak{t}). \tag{3.1}$$

This splitting, as we will shortly see, arises because the differential will involve a count of appropriate elements of $\pi_2(\mathbf{x}, \mathbf{y})$; $\mathbf{x}$ and $\mathbf{y}$ will be associated (in some sense) with Spin^c-structures, and when $\mathbf{x}$ and $\mathbf{y}$ are related to different Spin^c-structures, we will find that $\pi_2(\mathbf{x}, \mathbf{y}) = \emptyset$.

Now to make this more precise. As it turns out (see Section 2.4 of [50]), the emptiness of $\pi_2(\mathbf{x}, \mathbf{y})$ depends on an element $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y)$ defined by taking two paths $a : [0,1] \to \mathbb{T}_{\boldsymbol{\alpha}}$ and $b : [0,1] \to \mathbb{T}_{\boldsymbol{\beta}}$ from $\mathbf{x}$ to $\mathbf{y}$ and letting $\epsilon(\mathbf{x}, \mathbf{y})$ be the image of $a * b^{-1}$ under the isomorphism

$$\frac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_{\boldsymbol{\alpha}}) \oplus H_1(\mathbb{T}_{\boldsymbol{\beta}})} \simeq \frac{H_1(\Sigma)}{[\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g]} \simeq H_1(Y).$$

Since this isomorphism factors through the middle group, another way to realise $\epsilon(\mathbf{x}, \mathbf{y})$ is as follows. Let $\mathbf{x} = (x_1, \ldots, x_g)$ and $\mathbf{y} = (y_1, \ldots, y_g)$. Starting at x_1, follow some curve in $\boldsymbol{\alpha}$ until we reach a point of $\mathbf{y}$, then follow some curve in $\boldsymbol{\beta}$ until we reach a new point of $\mathbf{x}$ and repeat, alternating between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ until we return to x_1. If all points x_i and y_j have been visited, then we are finished and let $c_{\mathbf{x},\mathbf{y}}$ be the loop so generated. If, however, there are unvisited points, we choose any one of them and repeat the process; $c_{\mathbf{x},\mathbf{y}}$ is then taken to be union of all these closed paths. Its image in $H_1(Y)$ is $\epsilon(\mathbf{x}, \mathbf{y})$.

Generally speaking, although both the loop $a * b^{-1}$ and $c_{\mathbf{x},\mathbf{y}}$ depend on a lot of choices, their image $\epsilon(\mathbf{x}, \mathbf{y})$ does not (again, see [50]).

Proposition 3.1.8. $\pi_2(\mathbf{x}, \mathbf{y}) = \emptyset$ *if* $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$.

Given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, it is trivial to verify that $c_{\mathbf{x},\mathbf{z}} = c_{\mathbf{x},\mathbf{y}} + c_{\mathbf{y},\mathbf{z}}$, and hence that

$$\epsilon(\mathbf{x}, \mathbf{z}) = \epsilon(\mathbf{x}, \mathbf{y}) + \epsilon(\mathbf{y}, \mathbf{z}).$$

Consequently, ϵ defines a relative $H_1(Y)$-grading on $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and partitions the intersection points into equivalence classes by the relation $\mathbf{x} \sim \mathbf{y}$ if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$.

To apply this relative grading and realise the splitting in (3.1), we first need to establish a map $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(Y)$ which lifts $\epsilon(\mathbf{x}, \mathbf{y})$. As in [50], the assignation we will use depends on our choice of basepoint z, and is therefore written $\mathfrak{t}_z(\mathbf{x})$. It is constructed as follows. Take any self-indexing Morse function $f : Y \to [0, 3]$ which defines the Heegaard splitting of Y as discussed earlier. If $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ is the ordered g-tuple $(x_1, \ldots, x_g)$, then we can always arrange for x_i to lie on the gradient flow of f from an index 1 critical point to an index 2 critical point [50]. Taking pairwise disjoint neighbourhoods of the flowlines for each x_i, as well as a non-overlapping neighbourhood of the flowline containing z from the index 0 critical point to the index 3 critical point, we now have $g + 1$ balls contained in Y whose union we denote $\mathcal{B}$. Since there are no critical points in $Y_0 := Y \setminus \mathcal{B}$, it follows that ∇f is nowhere vanishing on Y_0, and since each ball of $\mathcal{B}$ contains a pair of complementary-index critical points, we can extend $\nabla f|_{Y_0}$ to a nowhere vanishing vector field on Y. We call the associated Spin^c-structure $\mathfrak{t}_z(\mathbf{x})$.

It is proved in Section 2.6 of [50] that this assignation $\mathfrak{t}_z : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(Y)$ lifts $\epsilon(\mathbf{x}, \mathbf{y})$ in the manner we desire.

Proposition 3.1.9. *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$,

$$\mathfrak{t}_z(\mathbf{x}) - \mathfrak{t}_z(\mathbf{y}) = \mathrm{PD}(\epsilon(\mathbf{x}, \mathbf{y})).$$

This means that $\mathfrak{t}_z(\mathbf{x}) = \mathfrak{t}_z(\mathbf{y})$ if and only if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$, a fact will be crucial in the section ahead.

3.1.4 The Definition

We are finally ready to define Heegaard Floer homology for a rational homology sphere Y. Though much of what we are about to state can be extended to the case

$b_1(Y) > 0$, the zero-Betti number hypothesis is particularly amenable because of the following proposition, proved in [50]. Its relevance will become clear once the definition is given.

Proposition 3.1.10. *Let Y be a rational homology sphere with Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Then between any two points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, there is at most one $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ satisfying $\mu(\phi) = 1$.*

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a Heegaard diagram for an oriented rational homology 3-sphere Y with implied (almost) complex structures on Σ and $\mathrm{Sym}^g(\Sigma)$. Then we define $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to be the $\mathbb{Z}$-span of pairs $[\mathbf{x}, i]$, where $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $i \in \mathbb{Z}$, and define a differential ∂ on $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ by

$$\partial[\mathbf{x}, i] := \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#\mathcal{M}(\phi)[\mathbf{y}, i - n_z(\phi)].$$

Notice that the sum is finite by Proposition 3.1.10.

Constructing $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ in this fashion, there is a natural ∂-equivariant automorphism

$$\begin{aligned} U : CF^\infty(Y) &\longrightarrow CF^\infty(Y) \\ [\mathbf{x}, i] &\longmapsto [\mathbf{x}, i - 1] \end{aligned}$$

which turns $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ into a $\mathbb{Z}[U, U^{-1}]$-module with generators $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Since it can be proved (Theorem 4.3 of [50]) that $\partial^2 = 0$, it follows that there is a well defined homology

$$HF^\infty(Y) := \ker \partial / \operatorname{im} \partial,$$

called the ∞-flavoured *Heegaard Floer homology* of Y. This homology is itself a $\mathbb{Z}[U, U^{-1}]$-module, and, as proved over the second half of [50], is independent of both the choice of Heegaard diagram and the complex structure on Σ. Therefore, where the meaning is clear, we may write $CF^\infty(Y)$ for the chain complex, safe in the knowledge that the chain complex is unique up to chain homotopy.

Now, recall from Proposition 3.1.8 that $\pi_2(\mathbf{x}, \mathbf{y}) = \emptyset$ if $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$, which occurs, by Proposition 3.1.9, if and only if $\mathfrak{t}_z(\mathbf{x}) \neq \mathfrak{t}_z(\mathbf{x})$. In the light of the definition above, this tells us that $[\mathbf{y}, j]$ does not appear with non-zero coefficient in the expression for $\partial[\mathbf{x}, i]$ whenever the Spin^c-structures associated with $\mathbf{x}$ and $\mathbf{y}$ are different. Consequently, if we define

$$CF^\infty(Y, \mathfrak{t}) := \langle [\mathbf{x}, i] \,|\mathfrak{t}_z(\mathbf{x}) = \mathfrak{t} \rangle_{\mathbb{Z}} \,,$$

then ∂ is in fact an endomorphism on $CF^\infty(Y, \mathfrak{t})$, and

$$HF^\infty(Y) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} HF^\infty(Y, \mathfrak{t}).$$

This is the first flavour of Heegaard Floer homology. From it, we are also able to define three other flavours by applying various algebraic constructions to the complex $CF^\infty(Y, \mathfrak{t})$.

1. $\widehat{CF}(Y, \mathfrak{t})$ is defined as the subcomplex of $CF^\infty(Y, \mathfrak{t})$ spanned by those generators $[\mathbf{x}, i]$ with $i = 0$. It inherits a $\mathbb{Z}$-module structure;

2. $CF^-(Y, \mathfrak{t})$ is defined as the subcomplex of $CF^\infty(Y, \mathfrak{t})$ spanned by those generators $[\mathbf{x}, i]$ with $i \leq 0$. It inherits a $\mathbb{Z}[U]$-module structure; and

3. $CF^+(Y, \mathfrak{t})$ is defined as the quotient complex $CF^\infty(Y, \mathfrak{t})/CF^-(Y, \mathfrak{t})$. It inherits both a $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$-module structure and a $\mathbb{Z}[U]$-module structure.

Associated with each of these chain complexes is a homology, $\widehat{HF}(Y, \mathfrak{t})$, $HF^-(Y, \mathfrak{t})$, or $HF^+(Y, \mathfrak{t})$. In fact, the U-equivariant short exact sequence

$$0 \longrightarrow CF^-(Y, \mathfrak{t}) \xrightarrow{\ i\ } CF^\infty(Y, \mathfrak{t}) \xrightarrow{\ \pi\ } CF^+(Y, \mathfrak{t}) \longrightarrow 0$$

induces a U-equivariant long exact sequence in homology

$$\ldots \longrightarrow HF^-(Y, \mathfrak{t}) \xrightarrow{\ i_*\ } HF^\infty(Y, \mathfrak{t}) \xrightarrow{\ \pi_*\ } HF^+(Y, \mathfrak{t}) \longrightarrow \ldots \tag{3.2}$$

This long exact sequence will become relevant later on.

We conclude this section with the remark that if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is a Heegaard diagram for Y, so too is $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$; both diagrams should compute the same homology. However, if f is a self-indexing Morse function for the first diagram, then $3 - f$ is a similar function for the second, and hence the Spin^c-structure $\mathfrak{t}_z(\mathbf{x})$ determined by the first diagram is in fact the Spin^c-structure $\overline{\mathfrak{t}_z(\mathbf{x})}$ in the second. Since all $\mathfrak{t} \in \text{Spin}^c(Y)$ with non-zero $HF^\circ(Y, \mathfrak{t})$ arise in this fashion, we have the following result (see Theorem 2.4 in [49]).

Proposition 3.1.11. *If Y is a closed, connected, oriented 3-manifold, and $\mathfrak{t} \in$ Spin$^c(Y)$, then*

$$HF^\circ(Y, \mathfrak{t}) = HF^\circ(Y, \bar{\mathfrak{t}}).$$

3.2 Example Calculations

To illustrate these constructions, we return momentarily to our examples of Figures 3.1(a), 3.1(b), and 3.1(d). We will not treat Figure 3.1(c), which represents $S^2 \times S^1$, since $b_1(S^2 \times S^1) \neq 0$ and we have have not covered the definition of $HF^\circ(Y)$ in this case. Our calculations are simplified by the fact that $g = 1$, implying that $\text{Sym}^g(\Sigma) = \Sigma$.

Starting with Figure 3.1(a), observe that there is precisely one intersection point $\mathbf{x}$, and as the diagram reconstructs S^3, there is only one Spin^c-structure. We claim that $\partial[\mathbf{x}, i] = 0$ for all i. By Proposition 3.1.10, there is at most one ϕ relevant to the differential. If there are none, the claim is trivial; otherwise, there is a unique ϕ and we can define $m := \#\mathcal{M}(\phi)$. It follows that

$$\partial^2[\mathbf{x}, i] = m^2[\mathbf{x}, i - 2n_z(\phi)].$$

On the other hand, $\partial^2 = 0$, forcing $m^2 = 0$, and thus $m = 0$. Hence $\partial[\mathbf{x}, i] = 0$. This immediately tells us that

$$HF^\infty(S^3) = \mathbb{Z}[U, U^{-1}] \qquad \widehat{HF}(S^3) = \mathbb{Z}$$
$$HF^-(S^3) = \mathbb{Z}[U] \qquad HF^+(S^3) = \frac{\mathbb{Z}[U, U^{-1}]}{U \cdot \mathbb{Z}[U]}.$$

We should obtain the same results if we repeat this calculation using Figure 3.1(b) instead. This time, there are unique discs ϕ from $\mathbf{x}_1$ to $\mathbf{x}_2$ and from $\mathbf{x}_3$ to $\mathbf{x}_2$ satisfying $\mu(\phi) = 1$ (uniqueness follows from Proposition 3.1.10, and the orientation is pinned down once we remember that $\partial\mathbb{D}$ must map to $\mathbb{T}_{\boldsymbol{\alpha}}$ or $\mathbb{T}_{\boldsymbol{\beta}}$ appropriately). Observe that $n_z(\phi) = 0$ for both discs. One can also compute that $\partial[\mathbf{x}_2, i] = 0$ by a slight modification of the previous argument (one observes that $\partial^2[\mathbf{x}_1, i] = 0$, so $\partial[\mathbf{x}_2, i] = 0$). Putting this together, we find that $\ker\partial$ is generated by $[\mathbf{x}_1 - \mathbf{x}_3, i]$ and $[\mathbf{x}_2, i]$, while $\operatorname{im}\partial$ is generated by $[\mathbf{x}_2, i]$. Thus, we recover the same results as in the previous case.

Turning to the lens space example, Figure 3.1(d), observe that any choice of $c_{\mathbf{x}_i,\mathbf{x}_j}$ for $i \neq j$ yields a non-zero homology class $\epsilon(\mathbf{x}_i, \mathbf{x}_j)$ (in general, the result is a generator of $H_1(Y)$). Consequently, the Heegaard Floer homology of $L(3,1)$ in each Spin^c-structure is isomorphic to that of S^3, and

$$HF^\circ(L(3,1), \mathfrak{t}) = HF^\circ(S^3, \mathfrak{t}_0),$$

where $\mathfrak{t} \in \text{Spin}^c(L(3,1))$ and $\mathfrak{t}_0$ is the unique element of $\text{Spin}^c(S^3)$. A little extra effort establishes that this is a general fact for lens spaces. Phrased otherwise, the Heegaard Floer homology of $L(p,q)$ is as simple as possible.

Definition 3.2.1. *A rational homology 3-sphere Y is called an L-*space *if*

$$HF^+(Y, \mathfrak{t}) \simeq \frac{\mathbb{Z}[U, U^{-1}]}{U \cdot \mathbb{Z}[U]}$$

for all $\mathfrak{t} \in Spin^c(Y)$.

Notice that our calculations here were greatly simplified by the fact that $g = 1$. If instead $g > 1$, it can be very difficult to visualise the relevant Whitney discs, though computational tricks do exist. We refer the interested reader to Section 2.4 of [50] for a discussion on domains. Since domains will never be mentioned again in this work, we omit them for brevity.

3.3 Cobordisms and Gradings

The reader making comparison between Heegaard Floer homology and singular homology will notice one crucial difference: whereas singular homology comes with an absolute $\mathbb{Z}$-grading, our definition above so far does not. We will soon see that it is still possible to produce such a grading on $HF^{\circ}(Y)$, but only because we have chosen to work within the restricted class of rational homology spheres. We will also see that the grading must generally be $\mathbb{Q}$-valued, unless Y is an integral homology sphere (in which case it is $\mathbb{Z}$-valued).

Given two intersection points $\mathbf{x}$ and $\mathbf{y}$, and a Whitney disc $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we define their *relative grading* by

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) := \mu(\phi) - 2n_z(\phi).$$

Proposition 2.15 and Lemma 3.3 in [50] then show that $\mathrm{gr}(\mathbf{x}, \mathbf{y})$ is independent of our choice of ϕ. We define the grading between $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ by the formula

$$\mathrm{gr}([\mathbf{x}, i], [\mathbf{y}, j]) := \mathrm{gr}(\mathbf{x}, \mathbf{y}) + 2i - 2j.$$

Thus, ∂ reduces the grading of $[\mathbf{x}, i]$ by 1, as one might expect, and the automorphism U reduces the grading by 2. We therefore stipulate that $U \in \mathbb{Z}[U, U^{-1}]$ has degree -2.

Definition 3.3.1. *We say* $\xi \in CF^{\circ}(Y, \mathfrak{t})$ *is* homogeneous *if* $\mathrm{gr}([\mathbf{x}, i], [\mathbf{y}, j]) = 0$ *for all* $[\mathbf{x}, i]$ *and* $[\mathbf{y}, j]$ *appearing with non-zero coefficient in* ξ*. If* $\mathrm{gr}([\mathbf{x}, i], [\mathbf{y}, j]) < 0$ *for all* $[\mathbf{y}, j] \in CF^{\circ}(Y, \mathfrak{t})$*, then we say* $[\mathbf{x}, i]$ *has* least grading.

Before lifting the relative $\mathbb{Z}$-grading defined above to an absolute grading as promised, it is important to mention one of the main results from [53]. In that paper, Ozsváth and Szabó prove that if $W : Y_1 \to Y_2$ is a certain type of cobordism and $\mathfrak{s} \in \mathrm{Spin}^c(W)$, then there are maps $F^{\circ}_{W,\mathfrak{s}} : HF^{\circ}(Y_1, \mathfrak{s}|_{Y_1}) \to HF^{\circ}(Y_2, \mathfrak{s}|_{Y_2})$ which are invariants of W. To avoid going into details, we merely mention that all the handle cobordisms discussed in the previous chapter satisfy the requirements. The maps $F^\circ_{W,\mathfrak{s}}$ are also natural, in that we have the following commutative diagram (where $\mathfrak{t}_i := \mathfrak{s}|_{Y_i}$):

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\delta} & HF^-(Y_1,\mathfrak{t}_1) & \xrightarrow{i_*} & HF^\infty(Y_1,\mathfrak{t}_1) & \xrightarrow{\pi_*} & HF^+(Y_1,\mathfrak{t}_1) & \xrightarrow{\delta} & \ldots \\ & & \Big\downarrow{\scriptstyle F^-_{W,\mathfrak{s}}} & & \Big\downarrow{\scriptstyle F^\infty_{W,\mathfrak{s}}} & & \Big\downarrow{\scriptstyle F^+_{W,\mathfrak{s}}} & & \\ \ldots & \xrightarrow{\delta} & HF^-(Y_2,\mathfrak{t}_2) & \xrightarrow{i_*} & HF^\infty(Y_2,\mathfrak{t}_2) & \xrightarrow{\pi_*} & HF^+(Y_2,\mathfrak{t}_2) & \xrightarrow{\delta} & \ldots \end{array} \tag{3.3}$$

With these maps in mind, we have Theorem 7.1 from [53].

Theorem 3.3.2. *Suppose that Y is a rational homology sphere. Then the relative grading* gr *on $HF^\circ(Y,\mathfrak{t})$ lifts to a $\mathbb{Q}$-valued absolute grading $\widetilde{\mathrm{gr}}$ on homogeneous elements satisfying the following properties:*

1. *If $\xi \in HF^+(S^3)$ is homogeneous with least grading, then $\widetilde{\mathrm{gr}}(\xi) = 0$;*

2. *The maps i_* and π_* preserve $\widetilde{\mathrm{gr}}$, while δ and U drop $\widetilde{\mathrm{gr}}$ by 1 and 2 respectively;*

3. *If $W : Y_1 \to Y_2$ is a cobordism as above and $\mathfrak{s} \in Spin^c(W)$, then*

$$\widetilde{\mathrm{gr}}(F^+_{W,\mathfrak{s}}(\xi)) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\,\mathrm{sig}(W)}{4}.$$

Definition 3.3.3 (Correction term)**.** *Let Y be a rational homology sphere. Then the* correction term *$d(Y,\mathfrak{t})$ of Y in $Spin^c$-structure $\mathfrak{t}$ is defined as the lowest grading of any homogeneous element in* $\operatorname{im}\pi_* \subset HF^+(Y,\mathfrak{t})$.

Proofs of the following two properties are found in Section 4 of [46].

Proposition 3.3.4. *Correction terms satisfy the following two properties.*

1. *Conjugation symmetry: $d(Y,\mathfrak{t}) = d(Y,\bar{\mathfrak{t}})$; and*

2. *Orientation sensitivity: $d(-Y,\mathfrak{t}) = -d(Y,\mathfrak{t})$.*

Correction terms are particularly important when it comes to cobordisms. For example, suppose that W is a cobordism $W : Y_1 \to Y_2$ between rational homology 3-spheres with Spin^c-structure $\mathfrak{s}$ restricting to $\mathfrak{t}_i$ on Y_i. Then the third part of Theorem 3.3.2 tells us that if $\xi \in HF^+(Y_1, \mathfrak{t}_1)$ is homogeneous, then

$$\widetilde{\mathrm{gr}}(F^+_{W,\mathfrak{s}}(\xi)) - \widetilde{\mathrm{gr}}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\,\mathrm{sig}(W)}{4}.$$

Supposing also that $b_2^+(W) = 0$, it can be proved that $F^\infty_{W,\mathfrak{s}}$ is an isomorphism (see the proof of Theorem 9.1 of [46]). Hence, on looking at the second square in (3.3), it follows that we can always choose some $\xi \in \pi_*(HF^\infty(Y_1, \mathfrak{t}_1))$ so that $\widetilde{\mathrm{gr}}(F^+_{W,\mathfrak{s}}(\xi)) = d(Y_2, \mathfrak{t}_2)$. Since $\widetilde{\mathrm{gr}}(\xi) \geq d(Y_1, \mathfrak{t}_1)$ by definition, we have essentially proved the following.

Proposition 3.3.5. *If $W : Y_1 \to Y_2$ is a cobordism, $b_2^+(W) = 0$, and $\mathfrak{s} \in Spin^c(W)$ restricts to $\mathfrak{t}_i$ on Y_i, then*

$$\frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\,\mathrm{sig}(W)}{4} \leq d(Y_2, \mathfrak{t}_2) - d(Y_1, \mathfrak{t}_1).$$

Corollary 3.3.6. *If Y bounds a negative-definite 4-manifold X, and $\mathfrak{t} \in Spin^c(Y)$ lifts to $\mathfrak{s} \in Spin^c(X)$, then*

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y, \mathfrak{t}). \tag{3.4}$$

Proof. Apply Proposition 3.3.5 to the cobordism $W : S^3 \to Y$ obtained by removing a small 4-ball from X. In particular, notice that $\chi(W) = b_2(W) = b_2(X)$ (an easy computation), $\mathrm{sig}(W) = -b_2(W)$ (by the negative-definite hypothesis), and $Q_W = Q_X$. Since S^3 only has one Spin^c-structure $\mathfrak{t}_0$, it follows that $\mathfrak{s}|_{S^3} = \mathfrak{t}_0$, and by the first part of Theorem 3.3.2, $d(S^3, \mathfrak{t}_0) = 0$. Putting all this information together, we are done. □

Definition 3.3.7 (Sharp 4-manifold)**.** *A negative-definite 4-manifold X with boundary Y is* sharp *if every $\mathfrak{t} \in Spin^c(Y)$ extends to some $\mathfrak{s} \in Spin^c(X)$ which gives equality in* (3.4)*.*

3.4 Structure of $HF^+(Y)$ for Rational Homology Spheres

Having defined the absolutely graded Heegaard Floer homology $HF^+(Y)$ of a closed, connected, oriented rational homology 3-sphere Y, we are now in a position to make some more detailed observations about the form taken by $HF^+(Y)$. Indeed, the structure of $HF^+(Y)$ can often be reduced to a set of numerical data.

First off, we have the following theorem, which is established in [50].

Theorem 3.4.1. *If Y is a closed, connected, oriented rational homology 3-sphere, then $HF^\infty(Y, \mathfrak{t}) \simeq \mathbb{Z}[U, U^{-1}]$ for all $\mathfrak{t} \in Spin^c(Y)$.*

As a result of this theorem, it is clear that the only interesting data to be extracted from $HF^\infty(Y, \mathfrak{t})$ is the grading in $[0, 2)$ of any homogeneous element whose grading falls in that interval (all homogeneous elements ξ and $\eta \in HF^\infty(Y, \mathfrak{t})$ are related by the equation $\xi = U^n \cdot \eta$ for some $n \in \mathbb{Z}$).

On the other hand, $HF^+(Y, \mathfrak{t})$ possesses a more refined structure. Indeed, if the reader were wondering why we defined multiple flavours of Heegaard Floer homology, these next few comments should prove enlightening.

Definition 3.4.2. *Let $\mathcal{T}^+ := HF^+(S^3, \mathfrak{t}_0)$, where $\mathfrak{t}_0$ is the unique $Spin^c$-structure on S^3. Then we set $\mathcal{T}^+_d := \mathcal{T}^+[d]$, where $[d]$ denotes a $\mathbb{Q}$-grading shift of d, and define the* reduced *Heegaard Floer homology $HF^+_{\text{red}}(Y, \mathfrak{t})$ by*

$$HF^+_{\text{red}}(Y, \mathfrak{t}) := HF^+(Y, \mathfrak{t}) / \operatorname{im} \pi_*.$$

It can be shown without too much difficulty that $HF^+_{\text{red}}(Y, \mathfrak{t})$ is finitely generated as a $\mathbb{Z}$-module. In fact, since $HF^+(Y, \mathfrak{t})$ is finitely generated as a $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$-module, Lemma 4.6 of [50] tells us that $\operatorname{im} U^k = \operatorname{im} \pi_*$ for sufficiently large k, and our claim follows. Putting this together with Theorem 3.4.1, we obtain the following structural theorem (again, see [50]).

Theorem 3.4.3. *Let Y be a rational homology 3-sphere. Then*

$$HF^+(Y, \mathfrak{t}) = \mathcal{T}^+_{d(Y, \mathfrak{t})} \oplus HF^+_{\text{red}}(Y, \mathfrak{t}).$$

Notice that the $\mathcal{T}^+_{d(Y,\mathfrak{t})}$ component is simply $\operatorname{im}\pi_*$. Phrased like this, we can see that an L-space Y is characterised by the property that $HF^+_{\text{red}}(Y,\mathfrak{t}) = 0$ for all $\mathfrak{t} \in \text{Spin}^c(Y)$. Its Heegaard Floer homology is therefore entirely determined by the correction terms $d(Y,\mathfrak{t})$, which in turn can be computed by calculating the $\mathbb{Q}$-gradings of the elements of $\ker U$.

3.5 Computations for Plumbed 3-manifolds

We say that a 3-manifold is *plumbed* if it is the boundary of a 4-manifold X obtained by plumbing (see Section 2.2.2). Since these sorts of manifolds occur often (for example, all Seifert fibred spaces), it is worth having some understanding of their Heegaard Floer homology. Luckily for us, this exact problem has been studied extensively, and algorithms exist for computing $HF^+(-Y)$ in various different circumstances. The one we will use goes back to Ozsváth and Szabó in [51].

Let G be a tree, and let X be its associated 4-manifold (denoted $\mathcal{X}(G)$ in Section 2.2.2). We let $Y = \partial X$, and recall from Proposition 2.2.6 that $H_2(X)$ is freely generated by $[S_v]$, where v is a vertex of G, and $H^2(X,Y)$ by $\text{PD}[S_v]$. For ease of notation, we will write $[v]$ and $\text{PD}[v]$ for $[S_v]$ and $\text{PD}[S_v]$ respectively. Pushing this through the short exact sequence underlying Proposition 2.1.3,

$$\begin{array}{ccccccccc} & & & & \text{Spin}^c(X) & \longrightarrow & \text{Spin}^c(Y) & & \\ & & & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} & & \\ 0 & \longrightarrow & H^2(X,Y) & \xrightarrow{Q} & H^2(X) & \xrightarrow{\alpha} & H^2(Y) & \longrightarrow & 0, \end{array} \tag{3.5}$$

we see that $\ker\alpha$ is generated by the images of the $\text{PD}[v]$, which in turn are visible (in $H^2(X)$) as the rows of Q. Remember also from that proposition that with our chosen bases for $H^2(X,Y)$ and $H^2(X)$, the map Q has the same matrix as Q_X. We will think of $H^2(X)$ as $\text{Hom}(H_2(X),\mathbb{Z})$, since X is simply connected, and hence write elements of $H^2(X)$ in the dual basis $[v]^*$.

To see how the Spin^c-structures fit into this picture, we consider all the characteristic covectors $\mathrm{Char}(G) \subset H^2(X)$. That is, those K which satisfy

$$\langle K, [v] \rangle \equiv v \cdot v = w(v) \mod 2 \text{ for all } v \in V(G).$$

By Proposition 2.3.15, the map $c_1 : \mathrm{Spin}^c(X) \to \mathrm{Char}(G)$ is a bijection, so we will think of these covectors as being the Spin^c-structures of X themselves. We must be more careful, however, in identifying Spin^c-structures on Y with their Chern classes, for according to Section 2.3.5, the map $c_1 : \mathrm{Spin}^c(Y) \to H^2(Y)$ has image $2H^2(Y)$ and may not be injective.

1. If $H^2(Y)$ contains no 2-torsion (i.e. is of odd order), then c_1 is bijective and so any two characteristic K_1, K_2 determine the same Spin^c-structure on ∂X if and only if $K_1 - K_2 \in \ker \alpha$. Equivalently, if and only if $(K_1 - K_2)Q^{-1} \in \mathbb{Z}^{|G|}$.

2. If $H^2(Y)$ does contain 2-torsion (i.e. is of even order), then c_1 is 2^k-to-one, where $H_1(Y; \mathbb{Z}_2) = \mathbb{Z}_2^k$. Consequently, if $(K_1 - K_2)Q^{-1} \in \mathbb{Z}^{|G|}$, then the best we can conclude is that K_1 and K_2 determine Spin^c-structures with the same first Chern class.

In light of these remarks, instead of thinking of $\alpha(K)$, where $K \in \mathrm{Char}(G)$, as being a Spin^c-structure on ∂X, we partition $\mathrm{Char}(G)$ according to the following rule. Take $K \in \mathrm{Char}(G)$ and let $\mathfrak{s} \in \mathrm{Spin}^c(X)$ be such that $c_1(\mathfrak{s}) = K$. We define $[K] := \mathfrak{s}|_Y$, and say that $K_1 \sim K_2$ if $[K_1] = [K_2]$. It is now safe to think of the equivalence classes of $\sim$ as the elements of $\mathrm{Spin}^c(Y)$ since $\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y)$ surjects when G is a tree.

Having set this framework up, the algorithm in Section 3 of Ozsváth and Szabó's paper [47] gives us an efficient method for identifying representatives of the equivalence classes of $\sim$ when G is a disjoint union of trees, is negative-definite, and has at most one overweight vertex (meaning that $d(v) > -w(v)$, where $d(v)$ is the degree of vertex v). Their result says that $\ker U \subset HF^+(Y)$ is given by some subset of those $K \in \mathrm{Char}(G)$ satisfying

$$w(v) + 2 \leq \langle K, [v] \rangle \leq -w(v) \text{ for all } v \in V(G). \tag{3.6}$$

To identify which subset is relevant, we begin by taking some K which satisfies (3.6) and letting $K_0 := K$. Then if $\langle K_i, [v] \rangle = -w(v)$ for some v, we let $K_{i+1} = K_i + 2PD[v]$ (which determines the same Spin^c-structure on ∂X, as noted above). This operation is called *pushing down* (the co-ordinate of) K_i at v. Continuing like this, we conclude with either some $L := K_m$ such that

$$w(v) \leq \langle L, [v] \rangle \leq -w(v) - 2 \text{ for all } v \in V(G),$$

or else with an L such that there exists a v satisfying $\langle L, [v] \rangle > -w(v)$. If we conclude in the first way, we say that K initiates a *maximising* path, while if we conclude in the second we say that K initiates a *non-maximising* path. Theorem 3.2 of [47] tells us that $\ker U$ is given by those K satisfying (3.6) which initiate maximising paths, and Corollary 1.5 of the same tells us that the correction terms of Y are computed as

$$d(Y, \mathfrak{t}) = \max_{K:[K]=\mathfrak{t}} \frac{KQ^{-1}K^t + |G|}{4}. \tag{3.7}$$

This equation proves immediately that X is sharp (provided there is at most one overweight vertex in G).

In the special instance when Y is an L-space, we already know from our remarks after Theorem 3.4.3 that $\ker U$ is in bijection with Spin$^c(Y)$. Hence, if Y is an L-space, the algorithm above outputs a complete set of representatives for the Spin^c-structures on Y without repetition.

3.6 Knot Floer Homology

The last concept we need to introduce before beginning our new work is the filtration a knot $K \subset Y$ induces on the complex $CF^\infty(Y)$ (again taking Y to be a rational homology sphere). This extra filtration will allow us to define the knot Floer homology of K, which has considerably more structure than the Heegaard Floer homology of Y. Indeed, as mentioned at the start of this chapter, the resulting homology is a categorification of the Alexander polynomial.

To introduce this filtration, we must first modify the type of Heegaard diagrams we are interested in. We say that a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a *Heegaard diagram for* $K \subset Y$ if

1. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0)$ is a Heegaard diagram for $Y \setminus \operatorname{Int} N(K)$, where $\boldsymbol{\beta}_0 := \boldsymbol{\beta} \setminus \{\beta_g\}$; and

2. β_g is a meridian for K, and intersects precisely one curve in $\boldsymbol{\alpha}$ (conventionally, this is taken to be α_g).

Such diagrams exist for all knots $K \subset Y$ (see [48]), and in the case $Y = S^3$ they can be constructed explicitly using bridge presentations (see Section 3.2 of [55]). Due to space constraints, we must unfortunately invite the interested reader to consult these sources at his own leisure.

Once we are given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $K \subset Y$, together with a choice of almost-complex structure J on Σ, we can associate with Y a double pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ defined by taking any longitude λ of K and letting w and z be points on λ disjoint from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that w and z lie at a small distance either side of β_g, and the interval joining them on λ and intersecting β_g runs from w to z. Recall that β_g is a meridian for K, so we are assured that $\beta_g \cdot \lambda = 1$ and that such a construction is possible. These two basepoints now allow us to make the following definition.

Definition 3.6.1 (Knot Floer homology)**.** *Suppose that* $K \subset Y$ *is a knot with an associated double pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$*. Then we define the* knot Floer complex $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ *to be the* $\mathbb{Z}$*-span of* $[\mathbf{x}, i, j]$ *where* $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ *and* $i, j \in \mathbb{Z}$*, and define its differential* ∂ *by the formula*

$$\partial[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#\mathcal{M}(\phi)[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)].$$

Since $\partial^2 = 0$*, it follows that there is a well-defined homology*

$$HFK(Y, K) := \frac{\ker \partial}{\operatorname{im} \partial},$$

which is an invariant of the knot K *independent of our choices of Heegaard diagram* *and complex structure on* Σ.

We shall refer to $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ as $CFK^\infty(Y, K)$ for concision. One can show, as per [50], that $n_w(\phi)$ and $n_z(\phi)$ are non-negative if ϕ admits holomorphic representatives, so the indices i and j do indeed give us legitimate filtrations on $CFK^\infty(Y, K)$. The automorphism U is defined this time as the map

$$\begin{aligned} U : CFK^\infty(Y, K) &\longrightarrow CFK^\infty(Y, K) \\ [\mathbf{x}, i, j] &\longmapsto [\mathbf{x}, i-1, j-1]. \end{aligned}$$

Although much more can be said on the structure of $HFK(Y, K)$, in particular that it can be split according to Spin^c-structures on the 0-surgery of K (see Section 3.1 of [48]), this extra material is not necessary for our purposes. We will therefore content ourselves with one final observation: by restricting to $i = 0$ and $j = 0$ in the case $Y = S^3$, one can define subcomplexes $C_{a,b}(K)$ of the restricted chain complex $CFK^{\{i=0\},\{j=0\}}(S^3, K)$ consisting of those elements with so called *Alexander grading* a (defined by an appropriate lift of the relative grading $A(\mathbf{x}, \mathbf{y}) := n_z(\phi) - n_w(\phi)$ using the unique $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ if it exists), and absolute S^3-grading b (the $\mathbb{Q}$-grading previously discussed). Each $C_{a,b}(K)$ then has an associated homology $H_{a,b}(K)$, and

$$\Delta_K(t) = \sum_a \sum_b (-1)^b \operatorname{rank} H_{a,b}(K) t^a.$$

That is, $HFK^{\{i=0\},\{j=0\}}(S^3, K) = \bigoplus_{a,b} H_{a,b}(K)$ categorifies the Alexander polynomial. Viewed this way, the well-known theorem that the degree of $\Delta_K(t)$ is a lower bound on the knot genus $g(K)$ is strengthened into a theorem which tells us that $g(K)$ is the maximal grading $a \in \mathbb{Z}$ such that $H_{a,b}(K) \neq 0$ for some $b \in \mathbb{Z}$. We invite the reader to pursue these interesting avenues on his or her own in [48].

CHAPTER 4: DEFICIENCY SYMMETRIES OF SURGERIES IN S^3

We are now finally ready to discuss and prove the main theorem of this thesis (Theorem 1 from the Introduction). At its core, this theorem relies on certain symmetries in the correction terms of 3-manifolds Y obtained by Dehn surgery on knots in S^3. These symmetries, which first appeared in the half-integral case in [51], are generalised here into similar symmetries on a much broader class of rational surgeries (Theorem 4.1.1).

While it is technically possible, as in [51], to use the symmetries described by Theorem 4.1.1 to prove that a given Y does not arise by Dehn surgery on a knot, it is typically extremely difficult to do so. This point is best illustrated in Chapter 5, which furnishes us with an entire family of examples whose deficiencies are difficult to calculate, even though the manifolds themselves are L-spaces. This difficulty is the motivation for our second theorem, Theorem 4.1.3, which recasts Theorem 4.1.1 in a more accessible form provided certain other assumptions on the topology of the 4-manifolds bounded by Y are also satisfied.

The content of this chapter is taken from the author's paper [19], currently awaiting publication.

4.1 Statement of the Theorems

The key objects in this chapter are the *deficiencies* of a $-p/q$-surgery on a knot C in S^3 in the case when $p, q > 0$. These objects are constructed using the bijection

$$\mathrm{Spin}^c(S^3_{-p/q}(C)) \longleftrightarrow \mathrm{Spin}^c(S^3_{-p/q}(U))$$

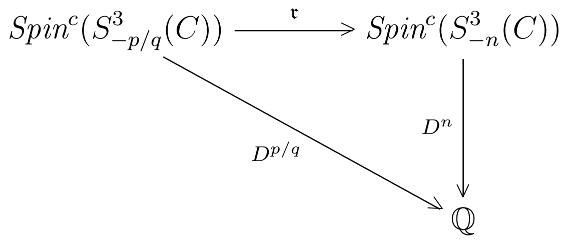
outlined in the next section, and are defined as the differences

$$D_C^{p/q}(\mathfrak{t}) := d(S^3_{-p/q}(C), \mathfrak{t}) - d(S^3_{-p/q}(U), \mathfrak{t}),$$

where U is the unknot, $\mathfrak{t}$ is a Spin^c-structure, and $d(\cdot, \cdot)$ denotes the appropriate correction term. When it is clear what we mean, we may sometimes drop the C from the notation $D_C^{p/q}$.

As suggested in the preamble, the first of our main theorems is a study of the symmetries exhibited by $D_C^{p/q}(\mathfrak{t})$. We let $n = \lceil p/q \rceil$ and define r by $p = nq - r$.

Theorem 4.1.1. *Let C be a knot in S^3, and let p, q be coprime, non-negative integers. Then there is a function $\mathfrak{r} : Spin^c(S^3_{-p/q}(C)) \to Spin^c(S^3_{-n}(C))$ such that the following diagram commutes:*



and the fibres of $\mathfrak{r}$ are of size q, with one exception over an element of $Spin^c(S^3_{-n}(C))$ which minimises the value of D^n. In particular, conjugation on $Spin^c(S^3_{-n}(C))$, under which D^n is invariant, lifts to a function on $Spin^c(S^3_{-p/q}(C))$ under which $D^{p/q}$ is invariant.

Stated thus, the theorem is in fact not difficult to prove; what requires more effort is the exhibition of such an $\mathfrak{r}$. We provide one example towards the middle of the chapter, and use it explicitly to convert Theorem 4.1.1 into a computable obstruction to negative rational surgeries. This obstruction is motivated by and generalises Greene's work in [26], [25], and [24]. As in the Introduction, we must use the canonical Hirzebruch-Jung continued fraction $p/q = [a_1, a_2, \ldots, a_\ell]^-$, in which $n = a_1$; we let W be the trace described in Proposition 2.2.3.

Definition 4.1.2 (Changemaker set)**.** *A set of non-negative integers $\{\sigma_j\}_{j=1}^r$ is called a* changemaker set *if the σ_j are non-decreasing and satisfy both $\sigma_1 \leq 1$ and the* changemaking condition

$$\sigma_j \leq \sigma_1 + \sigma_2 + ... + \sigma_{j-1} + 1$$

for all $j = 2, \dots, r$.

Theorem 4.1.3. *Suppose that* $Y = S^3_{-p/q}(C)$ *for some knot* $C \subset S^3$ *and coprime* $p, q > 0$, *that* W *is the manifold above (with intersection form* Q_W*), and that* $-Y$ *bounds a sharp, simply connected, negative-definite smooth 4-manifold* X *with intersection form* Q_X. *Then if*

$$d(Y, \mathfrak{t}) - d(S^3_{-p/q}(U), \mathfrak{t}) = 0$$

for either (a) one choice of $\mathfrak{t}$ *if* n *is odd; or (b)* $q - r + 1$ *choices of* $\mathfrak{t}$ *if* n *is even, there exists an integral matrix* A *such that*

$$-AA^t = Q_X \oplus Q_W.$$

Moreover, if $q \neq 1$, *one can choose* A *so that the last* ℓ *rows are:*

$$\begin{pmatrix} \sigma_r & \dots & \sigma_1 & 1 & 0 & & & & & & & & \\ & & & -1 & 1 & \dots & 1 & & & & & & \\ & & & & & & & \ddots & & & & & \\ & & & & & & & & -1 & 1 & \dots & 1 & 0 & & \\ & & & & & & & & & & & -1 & 1 & \dots & 1 \end{pmatrix},$$

where there are exactly a_i *non-zero entries in rows* $i = 2, \dots, \ell$, *all* ± 1 *as above, and* $\{\sigma_j\}_{j=1}^r$ *forms a changemaker set. If instead* $q = 1$, *one can choose* A *so that the last row reads:*

$$\begin{pmatrix} \sigma'_r & \dots & \sigma'_1 & \sigma'_0 \end{pmatrix},$$

and $\{\sigma'_j\}_{j=0}^r$ *forms a changemaker set.*

As a brief remark, the expansion $p/q = [a_1, \dots, a_\ell]^-$ used above does not need to be the canonical one. It must, however, have an associated 4-manifold W from Proposition 2.2.3 which is negative-definite and an associated plumbing graph with at most one overweight vertex. Since this is quite a mouthful to state, we have opted for the canonical choice (which satisfies both requirements).

4.2 Relative Spin^c-Structures

Implicit in the very definition of a deficiency is a bijection

$$\mathrm{Spin}^c(S^3_{-p/q}(C)) \longleftrightarrow \mathrm{Spin}^c(S^3_{-p/q}(U)).$$

Before proceeding any further, therefore, we should describe how this bijection works.

Let C be any oriented knot in S^3 and consider its exterior $M(C)$. Then the relative Spin^c-structures on $M(C)$, denoted $\underline{\mathrm{Spin}}^c(S^3, C)$ are enumerated as follows. Following similar lines to the reasoning in Section 2.3.5, a $\mathfrak{t}^* \in \underline{\mathrm{Spin}}^c(S^3, C)$ determines a relative Chern class $c_1(\mathfrak{t}^*) \in H^2(M(C), \partial M(C))$ which evaluates to an even integer $2i$ on a Seifert surface for C (the homology class of such a surface generates the relative homology). We label $\mathfrak{t}^*$ by $i = \frac{1}{2}c_1(\mathfrak{t}^*)$. As this can be done independent of the isotopy class of C, we now have a bijection

$$\underline{\mathrm{Spin}}^c(S^3, C) \longleftrightarrow \underline{\mathrm{Spin}}^c(S^3, U).$$

According to the interpretation given by Turaev [68] (see Section 3.1.3), $\mathfrak{t}^* \in \underline{\mathrm{Spin}}^c(S^3, C)$ corresponds to an equivalence class of nowhere-vanishing vector fields represented by a vector field $V_{\mathfrak{t}^*}$ on $M(C)$ such that $V_{\mathfrak{t}^*}|_{\partial M(C)} = U$, where U is the vector field on $\partial M(C)$ obtained by parallel translating any given vector in $T_x\partial M(C)$ (for any $x \in \partial M(C)$). This choice of U is unique up to homotopy, and in this formulation the conjugation map $\mathfrak{t}^* \mapsto \overline{\mathfrak{t}^*}$ gives us $V_{\overline{\mathfrak{t}^*}} = -V_{\mathfrak{t}^*}$.

Now suppose that we have performed a p/q-Dehn surgery on C, stitching a solid torus $S^1 \times D^2$ to $M(C)$ along its boundary to obtain a closed 3-manifold Y; we want to extend relative Spin^c-structures on $M(C)$ to Y. There is an essentially unique vector field V on $S^1 \times D^2$ with $V|_{\partial(S^1 \times D^2)} = U$, obtained by isotoping U to the inward-pointing normal at a shallow depth inside $S^1 \times D^2$ and setting V in the rest of the interior transverse to the D^2 factor until it runs parallel to C along the core. By gluing V to $V_{\mathfrak{t}^*}$, we obtain a Spin^c-structure on Y, and on the cohomological level this realises

$$\mathrm{Spin}^c(Y) = \frac{\underline{\mathrm{Spin}}^c(S^3, C)}{\langle p \cdot \mathrm{PD}[\mu] \rangle} = \mathbb{Z}_p,$$

where μ is the meridian of C. Thus, the integer $\frac{1}{2}c_1(\mathfrak{t}^*) = i$ determines an element of $H^2(Y)$ by its image in the quotient map $\mathbb{Z} \to \mathbb{Z}_p$, and this gives us a labelling of Spin$^c(Y)$. In this labelling, we can see that the conjugate $\overline{\mathfrak{t}^*}$ determines $-i \in \mathbb{Z}_p$, and that the Spin-structures (as the self-conjugate Spin^c-structures, see Proposition 2.3.12) are $0, \frac{p}{2} \in \mathbb{Z}_p$ if p is even, or just $0 \in \mathbb{Z}_p$ if p is odd. Again noting that this was independent of C, we have our bijection

$$\mathrm{Spin}^c(S^3_{p/q}(C)) \longleftrightarrow \mathrm{Spin}^c(S^3_{p/q}(U)).$$

In light of this, if we wish to enumerate the Spin^c-structures on $S^3_{-p/q}(C)$, we need only enumerate them on $S^3_{-p/q}(U)$.

4.3 An Application of Knot Floer Homology

In [54], Ozsváth and Szabó outline a variety of tools for computing the Heegaard Floer homology of $S^3_{-p/q}(C)$ in terms of the knot Floer homology of C. This work has since been extended by Ni and Wu in [44] to calculate $D^{p/q}_C(\mathfrak{t})$. One of their results is important for us now.

If $C \subset S^3$ is a knot, recall from Section 3.6 that there is an associated knot Floer chain complex $CFK := CFK^\infty(S^3, C)$. Let S be a subset of $\mathbb{Z} \oplus \mathbb{Z}$ such that if $(i, j) \in S$ then $(i+1, j), (i, j+1) \in S$, and define $CFK\{S\}$ to be the quotient of the knot Floer complex by the subcomplex generated by those $[\mathbf{x}, i, j]$ satisfying $(i, j) \in S$. With this notation, we let $k \in \mathbb{Z}$, and set

$$A^+_k := CFK\{i \geq 0 \text{ or } j \geq k\} \qquad \text{and} \qquad B^+ := CFK\{i \geq 0\}.$$

As per [54], there exist canonical U-equivariant chain maps

$$v_k^+, h_k^+ : A_k^+ \longrightarrow B^+;$$

v_k^+ is projection onto $CFK\{i \geq 0\}$, while h_k^+ is a composition of projection onto $CFK\{j \geq k\}$, identification with $CFK\{j \geq 0\}$, and chain homotopy equivalence with $CFK\{i \geq 0\}$. At sufficiently high gradings, these two maps are isomorphisms and hence behave as multiplication by U^{V_k} and U^{H_k} respectively, where $V_k, H_k \geq 0$ are integers.

Using the labelling of Spin^c-structures given in Section 7 of [54], we have the following result due to Ni and Wu. It can be found in [44] under Proposition 2.11, though as stated here we have applied it to $\overline{C}$.

Proposition 4.3.1 (Ni-Wu). *Let C be any knot in S^3, and let p, q be coprime, positive integers. Then*

$$D_C^{p/q}(\mathfrak{t}_i) = 2\max\left\{V_{\left\lfloor \frac{i}{q} \right\rfloor}, H_{\left\lfloor \frac{i-p}{q} \right\rfloor}\right\}.$$

As it stands, the labelling $\mathfrak{t}_i$ used above is a difficult one to play with on general rational surgeries. However, in the case of an integral n-surgery, it simplifies considerably. In this instance, the Spin^c-structure $\mathfrak{t}_i$ is the one that admits an extension $\mathfrak{s}$ on the cobordism $S^3 \to S_n^3(C)$ which satisfies

$$\langle c_1(\mathfrak{s}), [S] \rangle \equiv n + 2i \mod 2n,$$

where S is the closed surface obtained by gluing a Seifert surface for C to the core of the attached handle. This fact will be useful for us later in our proofs.

One extra comment: before applying Proposition 4.3.1, we will need certain extra properties of the V_i and H_i. These are also proved in [44] and summarised here.

Lemma 4.3.2. *The V_i and H_i satisfy the following properties:*

1. *$V_0 = H_0$; and*

2. *The V_i are a non-increasing sequence, while the H_i are a non-decreasing sequence.*

We are now in a position to prove the following result.

Lemma 4.3.3. *Let C be a knot in S^3. Then*

$$\sum_{\mathfrak{t}\in Spin^c(S^3_{-p/q}(C))} D_C^{p/q}(\mathfrak{t}) = q\cdot \sum_{\mathfrak{t}\in Spin^c(S^3_{-n}(C))} D_C^n(\mathfrak{t}) - r\cdot \min_{\mathfrak{t}\in Spin^c(S^3_{-n}(C))}\left\{D_C^n(\mathfrak{t})\right\}.$$

Proof. We consider the integral surgery first. By a direct application of Proposition 4.3.1 we obtain

$$\sum_{\mathfrak{t}\in \mathrm{Spin}^c(S^3_{-n}(C))} D_C^n(\mathfrak{t}) = 2\sum_{i=0}^{n-1} \max\left\{V_i, H_{i-n}\right\}. \tag{4.1}$$

Our goal is to compare this with the rational surgery.

Labelling the Spin^c-structures on the rational surgery as $\mathfrak{t}_{iq+j}$, we have the following bounds:

1. j ranges from 0 to $q-1$;
2. i ranges from 0 to $n-1$ if $j < q-r$, or from 0 to $n-2$ if $j \geq q-r$.

Rephrasing the second of these, i ranges from 0 to $n-1-\delta(j)$, where $\delta(j) := \left\lfloor \frac{j+r}{q} \right\rfloor$. Consequently, using Proposition 4.3.1,

$$\sum_{j=0}^{q-1}\sum_{i=0}^{n-1-\delta(j)} D_C^{p/q}(\mathfrak{t}_{iq+j}) = 2\sum_{j=0}^{q-1}\sum_{i=0}^{n-1-\delta(j)} \max\{V_i, H_{i-n+\delta(j)}\}. \tag{4.2}$$

We fix j and observe that

$$\sum_{i=0}^{n-1-\delta(j)} \max\{V_i, H_{i-n+\delta(j)}\} = \begin{cases} \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} & \text{if } \delta(j) = 0 \\ \sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} & \text{if } \delta(j) = 1 \end{cases}. \tag{4.3}$$

Clearly, if $\delta(j) = 0$, then the RHS is the same as the RHS of (4.1). This happens for the first $q-r$ values of j, meaning that the situations of interest are the r larger cases when $\delta(j) = 1$. In effect, to obtain our result we need to establish that

$$\sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} = \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} - m,$$

where $2m$ is the value of the minimum deficiency.

Suppose that i' is chosen to be the largest integer such that the integral deficiency in Spin^c-structure $\mathfrak{t}_{i'}$ is minimal. That is, that $\max\{V_{i'}, H_{i'-n}\}$ is minimal. Then there are two possibilities.

1. Suppose that $V_{i'} \geq H_{i'-n}$. As V_* is non-increasing and H_* is non-decreasing, it follows that $V_i \geq H_{i-n}$ for all $i \leq i'$, and hence that $V_i \geq V_{i+1} \geq H_{i-n+1}$ for all $i < i'$.

   Going in the other direction, suppose that $V_{i'+1} > H_{i'-n+1}$. Then our choice of i' implies that $V_{i'+1} = \max\{V_{i'+1}, H_{i'-n+1}\} > \max\{V_{i'}, H_{i'-n}\} = V_{i'}$, a contradiction to the non-increasing behaviour of V_*. Thus $H_{i'-n+1} \geq V_{i'+1}$, and $H_{i-n+1} \geq H_{i-n} \geq V_i$ for all $i > i'$.

   In the case $i = i'$, observe that

   $$V_{i'} = \max\{V_{i'}, H_{i'-n}\} < \max\{V_{i'+1}, H_{i'-n+1}\} = H_{i'-n+1}.$$

   Putting this together with the conclusions of the previous two paragraphs, we deduce that

   $$\begin{aligned}\sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} &= \sum_{i=0}^{i'-1} V_i + \sum_{i=i'}^{n-2} H_{i-n+1} \\ &= \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} - V_{i'}.\end{aligned}$$

   Observe that $V_{i'} = m$.

2. Suppose instead that $H_{i'-n} \geq V_{i'}$. This case is similar, though a little more complicated; we define j' to be the smallest integer such that $\max\{V_k, H_{k-n}\}$ is

minimal for $j' \leq k \leq i'$, and end with the conclusion

$$\begin{aligned}\sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} &= \sum_{i=0}^{j'-1} V_i + \sum_{i=j'}^{i'-1} H_{i'-n} + \sum_{i=i'}^{n-2} H_{i-n+1} \\ &= \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} - H_{i'-n}\end{aligned}$$

and the observation that $H_{i'-n} = m$, by definition of i'.

To complete the proof, one puts the above information into (4.2) via (4.3) and compares with (4.1). □

If one reads this argument carefully, one will find that it can be modified slightly to give a proof of Theorem 4.1.1. However, since this modified argument provides no insight as to the nature of $\mathfrak{r}$ without a deeper knowledge of the labelling $\mathfrak{t}_i$, it is of limited use to us.

In light of the above lemma, a natural question at this point is: Which Spin^c-structures on $S^3_{-n}(C)$ minimise the deficiency? We answer it below.

Lemma 4.3.4. *According to the parity of n,*

1. *If n is even, then $\mathfrak{t}_{\frac{n}{2}}$ realises the minimal deficiency; and*
2. *If instead n is odd, then $\mathfrak{t}_{\frac{n\pm1}{2}}$ do the same.*

Proof. Recall that $\mathfrak{t}_i$ evaluates, modulo $2n$ to $n + 2i$. Hence $\mathfrak{t}_i$ and $\mathfrak{t}_{n-i}$ are conjugates, and so by the conjugation symmetry of correction terms (see the first part of Proposition 3.3.4), if $i \neq 0$ and $\mathfrak{t}_i$ realises the minimum, so does $\mathfrak{t}_{n-i}$. Assuming that $i \leq n - i$, we claim that the same is true for all $\mathfrak{t}_j$ with $i \leq j \leq n - i$. Indeed, let the minimum deficiency be $2m$, so that $\max\{V_i, H_{i-n}\} = \max\{V_{n-i}, H_{-i}\} = m$. We know that $m \geq V_i \geq V_j$ and $m \geq H_{-i} \geq H_{j-n}$, so $\max\{V_j, H_{j-n}\} \leq m$, and as m is minimal it follows that we have equality. Consequently, $\mathfrak{t}_j$ also realises the minimum.

Thus, if $\mathfrak{t}_i$ realises the minimum for $i \neq 0$, so do the $\mathfrak{t}_j$ for the centralmost values of j, namely $\frac{n}{2}$ or $\frac{n\pm1}{2}$, depending on parity. The other possibility, of course, is that $\mathfrak{t}_0$ realises the minimum. In this case, observe that $D^n_C(\mathfrak{t}_0) = 2\max\{V_0, H_{-n}\}$. By

Lemma 4.3.2, $V_0 = H_0 \geq H_{-n}$, and we see that the deficiency is in fact $2V_0$. Since $V_i \leq V_0$ and $H_{i-n} \leq H_0 = V_0$,

$$D^n_C(\mathfrak{t}_i) = 2\max\{V_i, H_{i-n}\} \leq 2V_0 = D^n_C(\mathfrak{t}_0),$$

and $\mathfrak{t}_0$ is in fact the Spin^c-structure with the maximal deficiency. □

4.4 Preliminaries to the Proofs

Our goal now is to exhibit a function $\mathfrak{r} : \mathrm{Spin}^c(S^3_{-p/q}(C)) \to \mathrm{Spin}^c(S^3_{-n}(C))$ that satisfies Theorem 4.1.1. This will we require some enumeration of the Spin^c-structures on $S^3_{-p/q}(C)$, but as mentioned at the end of Section 4.2, this enumeration need only consider the case $C = U$.

4.4.1 A Plumbing Diagram for $S^3_{-p/q}(U)$

Recall from Section 3.5 that, given a negative-definite graph G which is a disjoint union of trees with at most one overweight vertex, there is an algorithm for determining $\ker U \subset HF^+(Y)$, where $Y = \partial\mathcal{X}(G)$ (see Section 2.2.2). As per the remarks following Theorem 3.4.3, if Y is an L-space, then this is enough to determine $HF^+(Y)$ completely, and the elements of $\ker U$ enumerate the Spin^c-structures on Y. Their corresponding correction terms are obtained by computing their $\mathbb{Q}$-gradings.

Consider the linear graph G in Figure 4.1, where $p/q = [a_1, \ldots, a_\ell]^- > 0$ is written in the canonical Hirzebruch-Jung continued fraction notation. We would like to apply the aforementioned algorithm to the boundary of the associated 4-manifold $W' := \mathcal{X}(G)$. This requires us to check the various conditions of the previous paragraph. To see that $\partial W' = S^3_{-p/q}(U)$, we repeatedly slam-dunk the Kirby diagram from Proposition 2.2.8 until all that remains is a single unknot. The following lemma covers the remaining hypotheses. (Observe also that W' is simply connected since G is a tree, see Proposition 2.2.6.)

Lemma 4.4.1. *Let G be the graph in Figure 4.1. Then $W' = \mathcal{X}(G)$ has a negative-definite intersection form.*

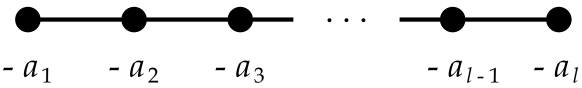


Figure 4.1: A graph G determining the 4-manifold $W' := \mathcal{X}(G)$ by plumbing. The integers a_i are taken from the canonical Hirzebruch-Jung continued fraction expansion $p/q = [a_1, \ldots, a_\ell]^-$. Notice that $\partial W' = S^3_{-p/q}(U)$.

Proof. As $[a_1, \ldots, a_\ell]^-$ is the canonical Hirzebruch-Jung continued fraction for p/q, the integers $a_i \geq 0$ must satisfy

$$\begin{aligned} p/q &= a_1 - r_1 \qquad \text{where} \qquad 0 \leq r_1 < 1; \text{ and} \\ r_i^{-1} &= a_{i+1} - r_{i+1} \qquad \text{where} \qquad 0 \leq r_{i+1} < 1. \end{aligned}$$

As $p/q \neq 0$, it follows that $a_1 \geq 1$; if $p/q > 1$, we can improve this to $a_1 \geq 2$. Moreover, as $r_i^{-1} > 1$, it follows that $a_i \geq 2$ for $i \geq 2$.

Now, consider the intersection form of W', which, under some choice of basis, is represented by the matrix

$$Q = \begin{pmatrix} -a_1 & 1 & & \\ 1 & -a_2 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & -a_\ell \end{pmatrix}.$$

According to Sylvester's criterion this matrix is negative-definite if $\operatorname{sgn} A_i = (-1)^i$, where A_i is the determinant of the upper left $i \times i$ submatrix. We claim two things by induction:

1. $\operatorname{sgn} A_i = (-1)^i$ (i.e. that $A_i = (-1)^i |A_i|$); and
2. $|A_i| \geq |A_{i-1}|$.

To start the induction, we check these two claims for $i = 1, 2$. Notice that $A_1 = -a_1 < 0$ and $A_2 = a_1 a_2 - 1 > 0$, verifying the first, and that

$$|A_2| = (a_2 - 1)a_1 + (a_1 - 1) \geq a_1 = |A_1|,$$

verifying the second.

We now proceed to the inductive step. By expanding along the i-th row of the i-th submatrix, $A_i = -a_i A_{i-1} - A_{i-2}$. Using this recurrence relation and our inductive hypotheses,

$$\begin{aligned}(-1)^i A_i &= (-1)^{2i} a_i \left|A_{i-1}\right| - (-1)^{2i-2} \left|A_{i-2}\right| \\ &= (a_i - 1) \left|A_{i-1}\right| + \left|A_{i-1}\right| - \left|A_{i-2}\right| \\ &\geq (a_i - 1) \left|A_{i-1}\right| .\end{aligned}$$

In particular, since $a_i \geq 2$ for $i \geq 2$, it follows that $(-1)^i A_i > 0$, so $\operatorname{sgn}(A_i) = (-1)^i$, and $|A_i| \geq |A_{i-1}|$. This concludes our proof. □

4.4.2 Combinatorics of the Plumbing Diagram

Because we are interested in obtaining a convenient enumeration of the elements of $\mathrm{Spin}^c(S^3_{-p/q}(C)) \leftrightarrow \mathrm{Spin}^c(S^3_{-p/q}(U))$, the natural question at this point is: What do the Spin^c-structures of $S^3_{-p/q}(U)$ look like after working through the algorithm in Section 3.5? Our answer will require a few more definitions.

Definition 4.4.2. *Suppose that Y is a closed 3-manifold contained in X, a smooth 4-manifold. Then given some $\mathfrak{t} \in Spin^c(Y)$, we say that $c \in H^2(X)$ is a* maximiser *for $\mathfrak{t}$ if $c = c_1(\mathfrak{s})$ for some $\mathfrak{s} \in Spin^c(X)$ which satisfies $\mathfrak{s}|_Y = \mathfrak{t}$ and which maximises $c_1(\mathfrak{s})^2$.*

Definition 4.4.3. *Let K be a characteristic covector for the linear graph G in Figure 4.1 which satisfies*

$$w(v) \leq \langle K, [v] \rangle \leq -w(v) \text{ for all } v \in V(G).$$

Then we say that a vertex v_i is a peak *for K if $\langle K, [v_i] \rangle = a_i$, and call $K + 2\,\mathrm{PD}[v_i]$ the* push-down *of K at v_i (we also call any covector obtained by a sequence of such moves a* push-down *of K). We say that K contains no* full tanks *if there do not exist $i < j$ such that v_i and v_j are peaks and $\langle K, [v_k] \rangle = a_k - 2$ for all $i < k < j$. We say* *that K is* left-full *if there exists a peak v_i such that $\langle K, [v_k]\rangle = a_k - 2$ for all $k < i$. To make notation simpler we will write $b_k := 2 - a_k$.*

Lemma 4.4.4. *The Spin^c-structures on $S^3_{-p/q}(U)$ are represented by those characteristic covectors K satisfying* (3.6) *that contain no full tanks. We call the collection of such characteristic covectors $\mathcal{K}$.*

Proof. We first prove that if K has a full tank then it initiates a non-maximising path. Indeed, observe the following path (presenting only the relevant section of K):

$$\begin{aligned}(a_i, -b_{i+1}, -b_{i+2}, \dots, -b_{j-1}, a_j) &\longrightarrow (-a_i, a_{i+1}, -b_{i+2}, \dots, -b_{j-1}, a_j)\\ &\longrightarrow (b_i, -a_{i+1}, a_{i+2}, \dots, -b_{j-1}, a_j)\\ &\longrightarrow (b_i, b_{i+1}, -a_{i+2}, \dots, -b_{j-1}, a_j)\\ &\longrightarrow \dots\\ &\longrightarrow (b_i, b_{i+1}, b_{i+2}, \dots, a_{j-1}, a_j)\\ &\longrightarrow (b_i, b_{i+1}, b_{i+2}, \dots, -a_{j-1}, a_j + 2).\end{aligned}$$

Here $\langle L, [v_j]\rangle > -w(v_j)$, so the initiated path is non-maximising.

What remains to be shown is that if K does not have a full tank, then it initiates a maximising path. We do this by inducting on the number of peaks in K and its push-downs. If there are none, we have a (trivial) maximising path. Thus, we presume there is at least one peak at v_i. Now, push down at v_i. Depending on whether $\langle K, [v_{i\pm1}]\rangle = a_{i\pm1} - 2$, there are three possibilities for the new $K' = K + 2\,\mathrm{PD}[v_i]$:

1. v_{i-1} and v_{i+1} are not peaks of K'. Then K' has one peak fewer than K and also contains no full tanks as $\langle K', [v_i]\rangle = -a_i \neq a_i - 2$ since $a_i \geq 2$ for $i \geq 2$. Hence, we apply the inductive hypothesis.

2. v_{i-1} is not a peak, v_{i+1} is (or the reverse situation). In this case, push down at v_{i+1}, and continue pushing down at any further peaks this generates, necessarily heading to the right. As K had no full tanks, this process must stop without initiating a non-maximising path. As in the previous case, the resulting covector has one peak fewer than K and no full tanks, so apply the induction hypothesis.

3. v_{i-1} and v_{i+1} are peaks. This situation is the same as the one above, pushing down in both directions unilaterally until the process halts. If $a_i = 2$, we will have to repeat this whole procedure multiple times, but eventually it will halt.

In all situations we have a maximising path. This completes our proof. $\square$

Since $S^3_{-p/q}(U)$ is an L-space, we have now isolated a collection $\mathcal{K} \subset \mathrm{Char}(G)$ in bijection with $\mathrm{Spin}^c(S^3_{-p/q}(U))$. Hence, we have the following proposition (the last part of which is an application of (3.7)).

Proposition 4.4.5. *Given $\mathfrak{t} \in Spin^c(S^3_{-p/q}(U))$, there is a unique $K \in \mathcal{K}$ such that $[K] = \mathfrak{t}$ and K is a maximiser for $[K]$. Moreover,*

$$d(S^3_{-p/q}(U), \mathfrak{t}) = \frac{KQ^{-1}K^t + b_2(W')}{4}.$$

4.4.3 Comparing the Rational and Integral Surgeries

We are now ready to make comparisons between the $-p/q$ and $-n$ surgeries on C and U. In doing so, it is very important to keep track of which coefficients and which knots we are considering. Thus, we observe the following:

1. Let $W(C) : S^3 \to S^3_{-p/q}(C)$ be the cobordism described by Proposition 2.2.2 using the reverse slam-dunked Kirby diagram with canonical Hirzebruch-Jung continued fraction, and let $W := W(C) \cup_{S^3} D^4$ and $W' := W(U) \cup_{S^3} D^4$ (i.e. these manifolds are the traces of the surgeries on C and U). As a scholium of Proposition 2.2.8, this W' and the W' of the previous section are identical;

2. The intersection form of the cobordism $W(C)$ is independent of C. Ergo, Q_W and $Q_{W'}$ have the same intersection form, represented in some bases by the adjacency matrix Q of the graph G in Figure 4.1;

3. Courtesy of this fact, a $K \in \mathrm{Char}(G)$ is a maximiser for $[K] \in \mathrm{Spin}^c(S^3_{-p/q}(U))$ if and only if K is also a maximiser for the corresponding Spin^c-structure on $S^3_{-p/q}(C)$, which we shall also denote by $[K]$. The crucial difference is that K might not compute the correction term for $S^3_{-p/q}(C)$. Henceforth, we shall

think of $\mathrm{Char}(G)$ as the Spin^c-structures on either 4-manifold W or W', and any complete collection $\mathcal{F}$ of representatives of equivalence classes of $\sim$ as the Spin^c-structures on either 3-manifold $S^3_{-p/q}(C)$ or $S^3_{-p/q}(U)$;

4. $W(C)$ splits naturally into two cobordisms,

$$S^3 \xrightarrow{W_1(C)} S^3_{-n}(C) \xrightarrow{W_2(C)} S^3_{-p/q}(C).$$

All three cobordisms are negative definite and have intersection forms independent of C.

Now consider $W_2(C)$. Given a maximiser K which determines $\mathfrak{t} \in \mathrm{Spin}^c(S^3_{-p/q}(C))$, this K also determines a $\mathfrak{t}' \in \mathrm{Spin}^c(S^3_{-n}(C))$ by considering the value of $k := \langle K, [v_1] \rangle$ modulo $2n$. Comparing this with the labelling used in Proposition 4.3.1, we see that $\mathfrak{t}'$ corresponds with $\mathfrak{t}_{\frac{k+n}{2}} \in \mathrm{Spin}^c(S^3_{-n}(C))$, unless $k = n$, in which case $\mathfrak{t}'$ corresponds with $\mathfrak{t}_0 \in \mathrm{Spin}^c(S^3_{-n}(C))$.

Lemma 4.4.6. *The map $\mathcal{K} \to \mathrm{Char}(v_1)$ given by restriction to the first co-ordinate satisfies the following two properties:*

1. *The fibre over $(j) \in \mathrm{Char}(v_1)$ has size q if $-n < j < n$; and*

2. *The fibre over $(n) \in \mathrm{Char}(v_1)$ has size $q - r$.*

Proof. Let $K \in \mathcal{K}$ and consider the case when $\langle K, [v_1] \rangle \neq n$. Then the number of covectors K that restrict to $\langle K, [v_1] \rangle$ is equal to the number of characteristic covectors on the linear graph $G - v_1$ which satisfy (3.6), initiate maximising paths, and have no full tanks. This is just the number of Spin^c-structures on the lens space given by $[a_2, \dots, a_\ell]^-$-surgery on the unknot. As $[a_2, \dots, a_\ell]^- = q/r$, we are done.

For the second claim, observe that there are $n - 1$ values of $\langle K, [v_1] \rangle$ covered by the above case, together accounting for $(n-1)q$ of the elements of $\mathcal{K}$. Therefore, the remaining $q - r$ must restrict to $(n) \in \mathrm{Char}(v_1)$. □

4.4.4 Adjusting the Maximisers

As a special case of Proposition 3.3.5 (c.f. the proof of Proposition 3.3.6), we observe that if $\overline{W}$ is a negative-definite cobordism from Y_1 to Y_2, both rational homology spheres, then for any $\mathfrak{s} \in \mathrm{Spin}^c(\overline{W})$,

$$\left\{\frac{c_1(\mathfrak{s})^2 + b_2(\overline{W})}{4}\right\} \leq d(Y_2, \mathfrak{s}|_{Y_2}) - d(Y_1, \mathfrak{s}|_{Y_1}). \tag{4.4}$$

To apply this, let $K \in \mathcal{K}$ satisfy $K = c_1(\mathfrak{s})$ for some $\mathfrak{s} \in \mathrm{Spin}^c(W')$. Then by construction K is a maximiser for $\mathfrak{t} = \mathfrak{s}|_{S^3_{-p/q}(U)}$, and $K|_{v_1}$ is a maximiser for $\mathfrak{t}' = \mathfrak{s}|_{S^3_{-n}(U)}$. Thus, if we define

$$\mathfrak{t}_0 := \mathfrak{s}|_{S^3} \qquad \mathfrak{s}_0 := \mathfrak{s}|_{W(U)} \qquad \mathfrak{s}_i := \mathfrak{s}|_{W_i(U)},$$

and if $\xi \in HF^+(S^3, \mathfrak{t}_0)$ has minimal grading (i.e. grading 0), then by the third part of Theorem 3.3.2 and Proposition 4.4.5,

$$\widetilde{\mathrm{gr}}(F^+_{W(U),\mathfrak{s}_0}(\xi)) = \frac{c_1(\mathfrak{s}_0)^2 + b_2(W(U))}{4} = \frac{c_1(\mathfrak{s})^2 + b_2(W')}{4} = d(S^3_{-p/q}(U), \mathfrak{t}).$$

Similarly, noting that $c_1(\mathfrak{s}_1) = K|_{v_1}$,

$$\widetilde{\mathrm{gr}}(F^+_{W_1(U),\mathfrak{s}_1}(\xi)) = \frac{c_1(\mathfrak{s}_1)^2 + b_2(W_1(U))}{4} = d(S^3_{-n}(U), \mathfrak{t}').$$

Putting these two facts together, we find that

$$\begin{aligned}\frac{c_1(\mathfrak{s}_2)^2 + b_2(W_2(U))}{4} &= \widetilde{\mathrm{gr}}(F^+_{W_2(U),\mathfrak{s}_2} \circ F^+_{W_1(U),\mathfrak{s}_1}(\xi)) - \widetilde{\mathrm{gr}}(F^+_{W_1(U),\mathfrak{s}_1}(\xi)) \\ &= \widetilde{\mathrm{gr}}(F^+_{W(U),\mathfrak{s}_0}(\xi)) - \widetilde{\mathrm{gr}}(F^+_{W_1(U),\mathfrak{s}_1}(\xi)) \\ &= d(S^3_{-p/q}(U), \mathfrak{t}) - d(S^3_{-n}(U), \mathfrak{t}').\end{aligned}$$

Substituting this into (4.4) with $\overline{W} = W_2(C)$ and noting that $Q_{W_2(C)}$ is independent of C, we conclude that if $\mathfrak{t}'$ and $\mathfrak{t}$ are cobordant by an $\mathfrak{s}$ such that $c_1(\mathfrak{s}) \in \mathcal{K}$, then

$$D_C^n(\mathfrak{t}') \leq D_C^{p/q}(\mathfrak{t}). \tag{4.5}$$

A very similar argument, applying (4.4) to $W_1(C)$, tells us that

$$0 \leq D_C^n(\mathfrak{t}'). \tag{4.6}$$

This argument on $\mathcal{K}$ is just a special case of the following lemma.

Lemma 4.4.7. *Suppose that $\mathcal{F} \subset \mathrm{Char}(G)$ is a complete set of representatives of equivalence classes of $\sim$, and that every $K_i \in \mathcal{F}$ is a maximiser for $[K_i]$. Then on defining $\mathfrak{s}_i$ by $c_1(\mathfrak{s}_i) = K_i$ and setting $w_i = \mathfrak{s}_i|_{S^3_{-p/q}(C)}$ and $v_i = \mathfrak{s}_i|_{S^3_{-n}(C)}$, it follows that*

$$0 \leq D_C^n(v_i) \leq D_C^{p/q}(w_i).$$

Proof. This is a combination of (4.6) on the left and (4.5) on the right. □

As it turns out, $\mathcal{K}$ is not the optimal choice of representatives for our purposes, since it does not yield a function $\mathfrak{r}$ satisfying Theorem 4.1.1. We therefore ask: If K is a maximiser, are there any other $K' \sim K$ that are maximisers? The answer is yes.

Lemma 4.4.8. *Let $\langle K, [v_i] \rangle = a_i$, where $K \in \mathrm{Char}(G)$ (not necessarily in $\mathcal{K}$). Then $K' := K + 2\,\mathrm{PD}[v_i]$ satisfies $(K')^2 = K^2$.*

Proof. Recall that $\mathrm{PD}[v_i]$, viewed as an element of $H^2(W)$, is the i-th row of Q. Hence, $\mathrm{PD}[v_i]Q^{-1} = e_i$, the i-th standard basis vector. Thus,

$$\begin{aligned}(K + 2\,\mathrm{PD}[v_i])^2 &= (K + 2\,\mathrm{PD}[v_i])Q^{-1}(K + 2\,\mathrm{PD}[v_i])^t \\ &= KQ^{-1}K^t + 4\,\mathrm{PD}[v_i]Q^{-1}K^t + 4\,\mathrm{PD}[v_i]Q^{-1}\,\mathrm{PD}[v_i]^t \\ &= KQ^{-1}K^t + 4e_iK^t + 4e_i\,\mathrm{PD}[v_i]^t \\ &= KQ^{-1}K^t + 4\,\langle K, [v_i] \rangle - 4a_i,\end{aligned}$$

and as $\langle K, [v_i] \rangle = a_i$, we are done. □

Corollary 4.4.9. *If K' is a push-down of $K \in \mathcal{K}$, then K' is a maximiser of $[K]$.*

Corollary 4.4.10. *Let $\mathcal{M}$ be the set of all maximisers in* Char(G)*. Then if $K \in \mathcal{M}$, so are all its push-downs.*

4.5 Proofs of the Theorems

Now that all the machinery is in place, we can finally prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Construct a family $\mathcal{K}'$ of characteristic covectors for use in Lemma 4.4.7 as follows. If $K \in \mathcal{K}$ satisfies

1. $\langle K, [v_1] \rangle = j$ for some $-1 \leq j < n$; and
2. $K|_{G-v_1}$ is left full,

then let $K' := K + 2\sum_{i=2}^{k} \mathrm{PD}[v_i]$ be a member of $\mathcal{K}'$. Here $k \geq 2$ is the smallest integer such that v_k is a peak for K (guaranteed to exist by the second condition above). This clearly determines the same Spin^c-structure on the boundary manifolds, and is a maximiser by Corollary 4.4.9.

For all other $K \in \mathcal{K}$, let K be a member of $\mathcal{K}'$. The family $\mathcal{K}'$ is now clearly a complete set of representatives for the equivalence classes of $\sim$, each element of which is a maximiser. We claim that the desired result is obtained by adding up all inequalities in Lemma 4.4.7, using $\mathcal{F} = \mathcal{K}'$.

To prove this claim, let us consider what the pushing down does. Our first piece of information is that $\langle K', [v_1] \rangle = j + 2$, so we are "nudging K up" the values in the first co-ordinate. We claim that, for a given j, we have nudged up precisely r different K. Indeed, recall from Lemma 4.4.6 that there are $q - r$ elements of $\mathcal{K}$ with $\langle K, [v_1] \rangle = n$. Another way of computing this number is:

$$\#\mathcal{K}|_{G-v_1} - \#\left\{ \begin{array}{c} \text{left-full elements of} \\ \mathcal{K}|_{G-v_1} \end{array} \right\}.$$

Since the first term here is q (as a scholium of Lemma 4.4.6), the second term must be r, as required.

This calculation completed, we now observe by Lemma 4.4.6 that $\mathcal{K}'$ has q elements which restrict to $(j) \in \mathrm{Char}(v_1)$ for any $-n+2 \leq j \leq n$ except $j=-1$ if n is odd or $j=0$ if n is even. In these cases, there are only $q-r$ such elements. Adding up all the right hand inequalities in Lemma 4.4.7, and noticing that the exceptional element of $\mathrm{Spin}^c(S^3_{-n}(C))$ is the one with minimal deficiency (see Lemma 4.3.4), we obtain

$$\sum_{\mathfrak{t} \in \mathrm{Spin}^c(S^3_{-p/q}(C))} D_C^{p/q}(\mathfrak{t}) \geq q \cdot \sum_{\mathfrak{t} \in \mathrm{Spin}^c(S^3_{-n}(C))} D_C^n(\mathfrak{t}) - r \cdot \min_{\mathfrak{t} \in \mathrm{Spin}^c(S^3_{-n}(C))} \{D_C^n(\mathfrak{t})\},$$

which we already know to be an equality by Lemma 4.3.3. Hence the right hand inequalities in Lemma 4.4.7 were in fact equalities induced by the members of $\mathcal{K}'$, and the result follows. $\square$

As remarked after the proof of Lemma 4.3.3, we had actually already proved Theorem 4.1.1 some time ago. However, this more recent proof has the advantage that it gives us insights the previous one did not: it allows us to see how $\mathfrak{r}$ behaves. Indeed, take any $\mathfrak{t} \in \mathrm{Spin}^c(S^3_{-p/q}(C))$ and some maximiser K for $\mathfrak{t}$. Then $\mathfrak{r}(\mathfrak{t})$ is determined by finding the $K' \in \mathcal{K}'$ such that $K \sim K'$; it is the Spin^c-structure on $S^3_{-n}(C)$ determined by the maximiser $(\langle K', [v_1]\rangle)$.

Corollary 4.5.1. *With notation as above, $D^{p/q}(\mathfrak{t})$ is minimal if $\langle K', [v_1]\rangle = 0, \pm 1$. If, additionally, n is even and there are $q-r+1$ choices of $\mathfrak{t}$ such that $D^{p/q}(\mathfrak{t})$ is minimal, then this extends to $\langle K', [v_1]\rangle = \pm 2$.*

In either case, if $D^{p/q}(\mathfrak{t}) = 0$ for some $\mathfrak{t}$, then the minimal deficiency is zero.

Proof. The first statement is an immediate consequence of Theorem 4.1.1 and Lemma 4.3.4. The second arises because there are only $q-r$ such $\mathfrak{t}$ with first co-ordinate 0, and because those K' with $\langle K', [v_1]\rangle = \pm 2$ have the next smallest deficiencies (by an argument very similar to the one in Lemma 4.3.4). The final comment is a trivial by-product of Lemma 4.4.7 as the deficiencies are non-negative. $\square$

As mentioned in the preamble, this knowledge of $\mathfrak{r}$ allows us to turn Theorem 4.1.1 into an obstruction, given Y, to $Y = S^3_{-p/q}(C)$ (under certain extra circumstances). We have already stated this obstruction as Theorem 4.1.3, but to prove it we must establish some algebraic preliminaries. For greater detail on these preliminaries, we refer the reader to Lemma 2.3 of [25] and Section 3.2 of [24]. We have summarised the key results below.

Proposition 4.5.2. *Let Y be a rational homology 3-sphere resulting from integral surgery on an ℓ-component link L with negative-definite linking matrix Q and trace W, and let Y' be the result of the same surgery on a (possibly different) link L', also with linking matrix Q, but whose trace W' is sharp. Moreover, suppose that $\partial X = -Y$, where X is a sharp, simply connected, negative-definite smooth 4-manifold. Then since c_1 commutes with the restriction maps on $Spin^c(\cdot)$ and $H^2(\cdot)$ induced by inclusion of a 3- or 4-manifold into a 4-manifold, there is a bijection*

$$\{\mathfrak{s} \in Spin^c(X \cup_Y W)\,|\,\mathfrak{s}|_Y = \mathfrak{t}\} \longrightarrow \left\{ (\mathfrak{s}_X, \mathfrak{s}_W) \in Spin^c(X) \times Spin^c(W) \;\middle|\; \begin{array}{c} \mathfrak{s}_X|_Y = \mathfrak{t} \\ \mathfrak{s}_W|_Y = \mathfrak{t} \end{array} \right\}$$
$$\mathfrak{s} \longmapsto (\mathfrak{s}|_X, \mathfrak{s}|_W) \tag{4.7}$$

such that

$$c_1(\mathfrak{s})^2 = c_1(\mathfrak{s}|_X)^2 + c_1(\mathfrak{s}|_W)^2.$$

Moreover, given $\mathfrak{t} \in Spin^c(Y)$,

$$\max_{\substack{\mathfrak{s} \in Spin^c(X \cup_Y W) \\ \mathfrak{s}|_Y = \mathfrak{t}}} c_1(\mathfrak{s})^2 + b_2(X \cup_Y W) = 4d(Y', \mathfrak{t}) - 4d(Y, \mathfrak{t}). \tag{4.8}$$

Note that in [24], the additional assumption was made that $\det(Q)$ is odd. This, however, is not necessary: its only function was to ensure that $\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y)$ and $\mathrm{Spin}^c(W) \to \mathrm{Spin}^c(Y)$ surject. This is assured by the fact that $H_1(X)$ and $H_1(W)$ are torsion-free (c.f. the discussion after Proposition 2.3.19, or [25]).

As a consequence of this proposition, a maximiser $c_1(\mathfrak{s})$ decomposes into a pair of maximisers $(c_1(\mathfrak{s}|_X), c_1(\mathfrak{s}|_W))$. To see what this decomposition looks like, at least on W, we use the diagram

$$H_2(X) \oplus H_2(W) \longrightarrow H_2(X \cup_Y W) \simeq H^2(X \cup_Y W) \longrightarrow H^2(X) \oplus H^2(W).$$

If we employ a basis $\{u_1, \dots, u_\ell\}$ for $H_2(W)$ with images $\{\overline{u_1}, \dots, \overline{u_\ell}\}$ in $H_2(X \cup_Y W)$, a class $\alpha \in H^2(X \cup_Y W)$ restricts to the class $(\langle \alpha, \overline{u_1}\rangle, \dots, \langle \alpha, \overline{u_\ell}\rangle) \in H^2(W)$ when written in the dual basis $\{u_1^*, \dots, u_\ell^*\}$. In particular this applies when $\alpha = c_1(\mathfrak{s})$: the restriction $c_1(\mathfrak{s}|_W)$ has the form $(\langle c_1(\mathfrak{s}), \overline{u_1}\rangle, \dots, \langle c_1(\mathfrak{s}), \overline{u_\ell}\rangle)$.

Now suppose that $K \in H^2(W)$ is a maximiser for $[K]$ and that $K = c_1(\mathfrak{s}_W)$ for some $\mathfrak{s}_W \in \mathrm{Spin}^c(W)$. Suppose moreover that if we put $\mathfrak{t} = [K]$ in (4.8), the RHS vanishes. Then there is some $\mathfrak{s}' \in \mathrm{Spin}^c(X \cup_Y W)$ satisfying $\mathfrak{s}'|_Y = \mathfrak{t}$ with $\alpha' = c_1(\mathfrak{s}')$ such that

$$-b_2(X \cup_Y W) = (\alpha')^2 = c_1(\mathfrak{s}'|_X)^2 + c_1(\mathfrak{s}'|_W)^2.$$

Since we know from Proposition 4.5.2 that $c_1(\mathfrak{s}'|_W)$ is also a maximiser for $\mathfrak{t}$, it follows that its square is K^2. Thus, letting $\mathfrak{s} \in \mathrm{Spin}^c(X \cup_Y W)$ correspond to $(\mathfrak{s}'|_X, \mathfrak{s}_W)$ under (4.7), and putting $\alpha = c_1(\mathfrak{s})$, we have

$$(\alpha')^2 = c_1(\mathfrak{s}'|_X)^2 + c_1(\mathfrak{s}'|_W)^2 = c_1(\mathfrak{s}'|_X)^2 + K^2 = c_1(\mathfrak{s}'|_X)^2 + c_1(\mathfrak{s}_W)^2 = \alpha^2.$$

Hence $\alpha^2 + b_2(X \cup_Y W) = 0$.

Since $X \cup_Y W$ is a closed, negative-definite smooth 4-manifold, it follows from Donaldson's theorem (Theorem 2.1.8) that

$$(H^2(X \cup_Y W), Q_{X \cup_Y W}) \simeq (\mathbb{Z}^{b_2(X)+\ell}, -\mathrm{id})$$

as lattices. Hence, we fix the bases on $H_2(X \cup_Y W)$ and $H^2(X \cup_Y W)$ as the diagonalising bases. Since α is a characteristic covector of $Q_{X \cup_Y W}$, it follows that $\alpha \equiv (1, 1, \dots, 1) \mod 2$, whence all entries of α are ± 1. Summarised, we have the following lemma.

Lemma 4.5.3. *Let K be a maximiser for $\mathfrak{t} = [K]$ such that*

$$d(Y', \mathfrak{t}) - d(Y, \mathfrak{t}) = 0.$$

Then there is some $\alpha \in \{\pm 1\}^{b_2(X)+\ell}$ such that $K = (\langle \alpha, \overline{u_1}\rangle, \dots, \langle \alpha, \overline{u_\ell}\rangle)$, written in the dual basis $\{u_1^*, \dots, u_\ell^*\}$.

To apply this lemma, we let $Y' = S^3_{-p/q}(U)$ (whose trace W' is sharp), and $Y = S^3_{-p/q}(C)$ (with trace W). Theorem 4.1.3 is now finally within reach.

4.5.1 Proof of Theorem 4.1.3 when $p > q > 1$

Let our basis $\{u_1, \ldots, u_\ell\}$ for $H_2(W)$ above be given by the vertices $[v_i]$ of our graph G, and consider the diagram below:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H^1(Y) & \longrightarrow & H^2(X, \partial X) \oplus H^2(W, \partial W) & \longrightarrow & H^2(X \cup_Y W) & \longrightarrow \ldots \\
& \Big\downarrow{\scriptstyle \mathrm{PD}} & & \Big\downarrow{\scriptstyle \mathrm{PD}} & & \Big\downarrow{\scriptstyle \mathrm{PD}} & \\
\ldots \longrightarrow & H_2(Y) & \longrightarrow & H_2(X) \oplus H_2(W) & \xrightarrow{A} & H_2(X \cup_Y W) & \longrightarrow \ldots
\end{array}$$

Here, the two groups on the left must vanish because the vertical maps are isomorphisms and $H^1(Y) = \mathbb{Z}^{b_1(Y)} \oplus \mathrm{Tors}(H_0(Y)) = 0$. Hence, the lattice underlying $Q_X \oplus Q_W$ embeds in the one underlying $Q_{X \cup_Y W}$, and on passing to the lower row and fixing bases for $H_2(X)$ and $H^2(X)$, there must exist some matrix A with integral entries such that $-AA^t = Q_X \oplus Q_W$. When expressed like this, the last ℓ rows of A are the images of the $[v_i]$ in $H_2(X \cup_Y W)$, and we shall use Lemma 4.5.3 to prove our claims about their structure. It is helpful to keep in mind that the (i, j)-th entry of $Q_X \oplus Q_W$ is in fact the standard negative-definite inner product of the i-th and j-th rows of A. We label the last ℓ rows of A by $x, y_2, \ldots, y_\ell$.

Our first task is to establish the structure of y_i for $i = 2, \ldots, \ell$. This has three parts: first, we show that all the non-zero entries are unital; second, that non-adjacent rows have no non-zero entries in the same spots; and third, that adjacent rows share only one spot with non-zero entries and that these overlapping entries are opposite in sign.

To achieve the first of these objectives, consider $y_i = (y_{i,j})_j$ for $2 \leq i \leq \ell$. Recall that $b_j := 2 - a_j$ and define $K \in \mathrm{Char}(G)$ by

$$K = \begin{cases} (0, b_2, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_\ell) & \text{if } n \text{ is even} \\ (-1, b_2, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_\ell) & \text{if } n \text{ is odd} \end{cases}.$$

Since there are no full tanks in K it is clear that $K \in \mathcal{K}$ and hence it is a maximiser. To determine the value of $D^{p/q}([K])$, we need to find the first co-ordinate of the corresponding $K' \in \mathcal{K}'$ such that $K' \sim K$ (by Theorem 4.1.1). If there is some $a_j \neq 2$ for $2 \leq j < i$ then $K \in \mathcal{K}'$. Otherwise, consider $K' = K + 2\sum_{j=2}^{i} \mathrm{PD}[v_j] \in \mathcal{K}'$, which has first co-ordinate 2 or 1 depending on parity. By our hypothesis on the deficiencies, it follows that $D^{p/q}([K]) = 0$ (via Corollary 4.5.1). Thus by Lemma 4.5.3, there is an $\alpha \in \{\pm 1\}^{b_2(X)+\ell}$ such that $\langle \alpha, y_i \rangle = a_i$. Rephrased,

$$-\sum_j \alpha_j y_{i,j} = a_i = \sum_j y_{i,j}^2,$$

where the right hand side comes from the fact that $y_i^2 = a_i$. Consequently,

$$\sum_j (y_{i,j}^2 + \alpha_j y_{i,j}) = 0,$$

and since $\alpha_j = \pm 1$, each summand is non-negative and therefore must vanish. This in turn requires $y_{i,j} \in \{-1, 0, 1\}$. It is clear that for exactly a_i values of j, $y_{i,j} \neq 0$.

With this step done, we now need to establish how the rows line up with each other. Thus, consider $2 \leq i < j \leq \ell$ such that $j - i \geq 2$, set $m := a_i$, and permute the basis of $H_2(X \cup_Y W)$, changing signs as necessary, so that $y_i = (1, \ldots, 1, 0, \ldots, 0)$. As before, there must be an $\alpha = c_1(\mathfrak{s})$ such that $c_1(\mathfrak{s}|_W) = K$ where

$$K = \begin{cases} (0, b_2, \ldots, b_{i-1}, a_i, -a_{i+1}, b_{i+2}, \ldots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \ldots, b_\ell) & \text{if } n \text{ is even} \\ (-1, b_2, \ldots, b_{i-1}, a_i, -a_{i+1}, b_{i+2}, \ldots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \ldots, b_\ell) & \text{if } n \text{ is odd} \end{cases},$$

which is obtained by pushing down at v_{j-1} in

$$(*, b_2, \ldots, b_{i-1}, a_i - 2, -b_{i+1}, -b_{i+2}, \ldots, -b_{j-2}, a_{j-1}, a_j - 2, b_{j+1}, \ldots, b_\ell) \in \mathcal{K}$$

and repeating to the left. In either case, exactly as before we find that $D^{p/q}([K]) = 0$ by showing the first co-ordinate of the corresponding K' is one of $0, \pm 1, 2$. Then there is an α such that $\langle \alpha, y_i \rangle = a_i$ and $\langle \alpha, y_j \rangle = a_j$. The first of these statements tells us that $\alpha_k = -1$ for all $k = 1, \dots, m$.

Now, let $I = \{k \leq m | y_{j,k} \neq 0\}$. We claim that $I = \emptyset$. Indeed, as

$$-a_j = -\langle \alpha, y_j \rangle = \sum_{k \in I} \alpha_k y_{j,k} + \sum_{k > m} \alpha_k y_{j,k},$$

each summand on the RHS must be -1 or 0. We know that $\alpha_k = -1$ for $k \in I$, so $y_{j,k} = 1$ for $k \in I$. Yet $y_i \cdot y_j = 0$, so $\sum_{k \in I} 1 = 0$, and $I = \emptyset$.

We repeat a similar argument for $j = i + 1$, though our goal is to show that there is a unique element $k \in I$ and that $y_{i+1,k} = -1$. For $2 \leq i \leq \ell$ we take

$$K = \begin{cases} (0, b_2, \dots, b_{i-1}, a_i, a_{i+1} - 2, b_{i+2}, \dots, b_\ell) & \text{if } n \text{ is even} \\ (-1, b_2, \dots, b_{i-1}, a_i, a_{i+1} - 2, b_{i+2}, \dots, b_\ell) & \text{if } n \text{ is odd} \end{cases},$$

and note again that $D^{p/q}([K]) = 0$. Permuting and changing signs as necessary, we may assume that $y_i = (1, \dots, 1, 0, \dots, 0)$ and define I as before. This time, however,

$$-a_{i+1} + 2 = -\langle \alpha, y_{i+1} \rangle = \sum_{k \in I} \alpha_k y_{i+1,k} + \sum_{k > m} \alpha_k y_{i+1,k},$$

and exactly one summand on the RHS is 1. If that summand is in the second sum then all summands in the first are negative, so $y_{i+1,k} = 1$ for all $k \in I$. But then $-1 = y_i \cdot y_{i+1} = \sum_{k \in I} 1$, a contradiction. Therefore $y_{i+1,k} = -1$ for precisely one $k \in I$, and by an argument similar to the one just made, this k is the unique element of I.

At this point, up to permuting the basis of $H_2(X \cup_Y W)$ and changing signs as necessary, we have established the form of the last $\ell - 1$ rows of A. What remains is to establish x. With this in mind, our first goal is to prove that it has the shape $(*, \dots, *, 1, 0, \dots, 0)$ as outlined in the statement of the theorem.

Fix $i \in \{3, \dots, \ell\}$, let $x_1, \dots, x_m$ be the entries of x in the same spots as the non-zero entries of row y_i, and let $x_{m+1}, \dots, x_{b_2(X)+\ell}$ be the rest. Note that $m = a_i$. Again, change signs as necessary so that $y_{i,k} \geq 0$ for all k. Then as $x \cdot y_i = 0$, it follows that

$$x_1 + \cdots + x_{m-1} + x_m = 0. \tag{4.9}$$

Our goal is to show that $x_k = 0$ for all $k \leq m$.

Define a set

$$\mathcal{S} = \left\{ K \in \mathcal{M} \,\middle|\, \langle K, [v_i] \rangle = m, D^{p/q}([K]) = 0 \right\} .$$

Then for any $K \in \mathcal{S}$, we find an α according to Lemma 4.5.3 such that $\langle \alpha, y_i \rangle = a_i = m$. Hence, $\alpha_1 + \cdots + \alpha_m = -m$, whence $\alpha_k = -1$ for $k \leq m$. Indeed, all $\alpha \in \{\pm 1\}^{b_2(X)+\ell}$ satisfying these equations determine some $K \in \mathcal{S}$ (to check that $D^{p/q}([K]) = 0$, observe that $\alpha^2 = -b_2(X) - \ell$ and that this is the maximal square possible). Let $j = \langle K, [v_1] \rangle = \langle \alpha, x \rangle$. Then

$$j = x_1 + \cdots + x_m - \sum_{k>m} \alpha_k x_k = - \sum_{k>m} \alpha_k x_k,$$

where we used (4.9) to obtain the last equality. Thus the maximum value for j as we vary $K \in \mathcal{S}$ is

$$\sum_{k>m} |x_k| . \tag{4.10}$$

We now suppose without loss of generality that $x_m \leq x_k$ for all $k < m$. Define

$$\mathcal{S}' = \left\{ K \in \mathcal{M} \,\middle|\, \langle K, [v_i] \rangle = m-2, D^{p/q}([K]) = 0 \right\} .$$

Then similarly there is some $\beta \in \{\pm 1\}^{b_2(X)+\ell}$ such that $\langle \beta, y_i \rangle = m - 2$, whence $\beta_k = -1$ for all values of $k \leq m$ except one. As before, all such β determine a $K \in \mathcal{S}'$. Hence

$$j = - \sum_{k=1}^{m} \beta_k x_k - \sum_{k>m} \beta_k x_k \tag{4.11}$$

attains its maximal value when $\beta_k = -1$ for all $k < m$ and $\beta_m = 1$ (by choice of x_m). This maximal value is

$$\sum_{k<m} x_k - x_m + \sum_{k>m} |x_k| . \tag{4.12}$$

We claim that the two maxima given by (4.10) and (4.12) are in fact identical. Indeed, let $j_{\max} \leq n$ be the maximal integer j such that $D^n([j]) = 0$ (note that $j_{\max} \geq 1$ by assumption on the number of vanishing deficiencies). Then if $j_{\max}$ can be attained by elements of $\mathcal{S}$ and $\mathcal{S}'$, the claim must be true (since larger values are ruled out by the deficiency condition). Observe that

$$K = (j_{\max}, -a_2, b_3, \ldots, b_{i-2}, b_{i-1}, m, b_{i+1}, \ldots, b_\ell)$$

satisfies $D^{p/q}([K]) = 0$, since $K \in \mathcal{K}'$: it is obtained by pushing down

$$(j_{\max} - 2, -b_2, -b_3, \ldots, -b_{i-2}, a_{i-1}, m - 2, b_{i+1}, \ldots, b_\ell) \in \mathcal{K}$$

at v_{i-1} and to the left. Similarly,

$$K' = (j_{\max}, b_2, \ldots, b_{i-1}, m - 2, b_{i+1}, \ldots, b_\ell) \in \mathcal{S}' \cap \mathcal{K}'.$$

Thus the maximal values of j are the same in both families. Hence,

$$\sum_{k<m} x_k - x_m = 0.$$

However, using (4.9) to rewrite the first term, we find that $x_m = 0$. Therefore $x_k \geq 0$ for all $k \leq m$, by choice of x_m, and from (4.9) again we find that $x_k = 0$.

Now consider row $i = 2$ and set $m = a_2$. We wish to show that $x_k = 0$ for all $k \leq m$ except one, for which $x_k = -1$. In this case, (4.9) becomes

$$x_1 + \cdots + x_{m-1} + x_m = -1. \tag{4.13}$$

Although we will keep the set $\mathcal{S}'$ as defined before, this time we use

$$\mathcal{S} = \left\{ K \in \mathcal{M} \,\middle|\, \langle K, [v_2] \rangle = -m, D^{p/q}([K]) = 0 \right\},$$

so that the maximum (4.10) becomes

$$1 + \sum_{k>m} |x_k| \, .$$

while the second maximum (4.12) remains unchanged after we have defined x_m to be the smallest of the x_k for $k \leq m$. We claim that these two maxima are equal. Indeed, $(j_{\max}, -m, b_3 + 2, b_4, \ldots, b_\ell) \in \mathcal{S} \cap \mathcal{K}'$, and $(j_{\max}, m-2, b_3, \ldots, b_\ell) \in \mathcal{S}' \cap \mathcal{K}'$. Hence, comparing the maxima, we obtain:

$$\sum_{k<m} x_k - x_m = 1.$$

Rearranging (4.13) as before, we find that $x_m = -1$, and if $m = 2$, then (4.13) yields the result. If, on the other hand, $m > 2$, then repeat this process with

$$\mathcal{S}'' = \left\{ K \in \mathcal{M} \,\middle|\, \langle K, [v_2] \rangle = m - 4, D^{p/q}([K]) \right\}$$

and x_{m-1} defined to be the next smallest after x_m. We find that $x_{m-1} + x_m = -1$, from which $x_{m-1} = 0$. Hence $x_k \geq 0$ for all $k \leq m - 1$, and it follows from (4.13) that $x_k = 0$ for $k \leq m - 1$, as required.

By this point we are finally almost there. What remains to establish is the change-maker condition on x. Using the labels σ_i established, and defining σ_0 to be the other unital entry, change signs as usual so that $\sigma_i \geq 0$. Let

$$J := \left\{ \langle K, [v_1] \rangle \,\middle|\, K \in \mathcal{M}, \langle K, [v_2] \rangle = -a_2, D^{p/q}([K]) = 0 \right\} ,$$

and observe that J consists of all values $j \equiv n$ from $2 - j_{\max}$ to $j_{\max}$. The asymmetry is a result of the fact that if $K = (j, -a_2, *, \ldots, *)$ is a relevant maximiser with appropriate values *, then $K \in \mathcal{K}'$ if $j \geq 1$, whereas $K \sim K' := (j - 2, a_2, *, \ldots, *) \in \mathcal{K}'$ if $j < 1$. Thus, in light of the evaluation on $[v_2]$,

$$j = \sigma_0 - \sum_{i \geq 1} \alpha_i \sigma_i = 1 - \sum_{i \geq 1} \alpha_i \sigma_i$$

attains these values too. By writing $\alpha_i = -1 + 2\chi_i$ (where $\chi_i \in \{0,1\}$), we obtain

$$j = 1 + \sum_{i\geq 1} \sigma_i - 2\sum_{i\geq 1} \chi_i\sigma_i = j_{\max} - 2\sum_{i\geq 1} \chi_i\sigma_i,$$

and thus $\left\{ \sum_{i\geq 1} \chi_i\sigma_i \middle| \chi_i \in \{0,1\} \right\}$ consists of all integers from 0 to $\sum_{i\geq 1} \sigma_i$. As in [24], this is readily equivalent to the changemaker condition specified in the statement of our theorem.

4.5.2 Proof of Theorem 4.1.3 when $0 < p < q$

This proof is extremely similar to the previous one, so we only outline the differences. Crucially, $n = 1$, so via Theorem 4.1.1 it follows that all the deficiencies $D^{p/q}([K])$ vanish for any maximiser K. This fact makes the proof much easier, as does the fact that, by Lemma 4.4.1, we still have $a_i \geq 2$ for $i \geq 2$.

To ensure that all non-zero entries in y_i are ± 1 for all i, we use the maximiser

$$K = (1, b_2, \dots, b_{i-2}, -a_{i-1}, a_i, b_{i+1}, \dots, b_\ell).$$

We know that this choice of K is in fact a maximiser since it is a push-down of the maximiser $(1, -b_2, \dots, -b_{i-2}, -b_{i-1}, -b_i, b_{i+1}, \dots, b_\ell) \in \mathcal{K}$. The same argument as above then yields our results.

To show that rows y_i and y_j where $j - i \geq 2$ do not overlap (i.e. share non-zero entries in the same spots), one must be a little more careful. Supposing that $a_i \neq 2$ (i.e. that $a_i > 2$), one uses the maximiser

$$K = (1, b_2, \dots, b_{i-2}, -a_{i-1}, a_i, -a_{i+1}, b_{i+2}, \dots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \dots, b_\ell),$$

which, by pushing-down at v_1 to the right and at v_{j-1} to the left, can be obtained from $(1, -b_2, \dots, -b_{i-2}, -b_{i-1}, a_i - 4, -b_{i+1}, -b_{i-2}, \dots, -b_{j-2}, a_{j-1}, a_j - 2, b_{j+1}, \dots, b_\ell) \in \mathcal{K}$. If instead $a_i = 2$ and there is some $k \in \{2, \dots, i-1\}$ such that $a_k \neq 2$, then we use

$$K = (1, b_2, \dots, b_{i-1}, a_i, -a_{i+1}, b_{i+2}, \dots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \dots, b_\ell),$$

a push-down of

$$(1, b_2, \ldots, b_{i-1}, a_i - 2, -b_{i+1}, -b_{i+2}, \ldots, -b_{j-2}, a_{j-1}, a_j - 2, b_{j+1}, \ldots, b_\ell) \in \mathcal{K}.$$

Finally, if $a_k = 2$ for all $k = 2, \ldots, i$, the fact that $y_i \cdot y_j = 0$ implies that if y_j and y_i overlap, then they overlap in two places. Consequently, since $y_{i-1} \cdot y_i = -1$, it follows that y_j also overlaps with y_{i-1}, and similarly as $y_{i-1} \cdot y_j = 0$, that y_j overlaps in two places with y_{i-1}. Iterating this, we find eventually that y_j and x overlap, violating the condition $x \cdot y_j = 0$, since x contains precisely one non-zero entry (as $x^2 = 1$). Hence, y_i and y_j cannot overlap.

To show that y_i and y_{i+1} have only one overlap (in which they are opposite in sign), one uses

$$K = (1, b_2, \ldots, b_{i-2}, -a_{i-1}, a_i, a_{i+1} - 2, b_{i+2}, \ldots, b_\ell),$$

which is a push-down of $(1, -b_2, \ldots, -b_{i-2}, -b_{i-1}, a_i - 2, a_{i+1} - 2, b_{i+2}, \ldots, b_\ell) \in \mathcal{K}$.

Because $x^2 = n = 1$, the rest of the computation is trivial, and the theorem is proved.

4.5.3 Proof of Theorem 4.1.3 when $q = 1$

This last proof is even easier than in the previous section. Since none of the rows y_i exist, we need only prove the statement about x; in the absence of the other rows, the only adjustments we need make to the proof of the changemaker statement are to define instead

$$J := \left\{ \langle K, [v_1] \rangle \,\middle|\, K \in \mathcal{M}, D^{p/q}([K]) = 0 \right\},$$

and remove the assumption that $\sigma_0 = 1$. Once this is done, the modified statement follows easily.

4.5.4 A Remark on Vanishing Deficiencies

In its current form, the reader will hopefully have noticed the asymmetry in Theorem 4.1.3 concerning the number of deficiencies which vanish. If n is odd, we only require one to vanish, but if n is even, then we require $q-r+1$. It is possible that by choosing a different function $\mathfrak{r}$ we can remove this asymmetry, but as of the current writing we have been unable to achieve this.

What we can say, however, is that in the special case when $q = 2$, some simplifications are possible. Since this case will be the one of most interest to us in the next chapter, we have the following proposition. In practice, we will often apply it to the case $\mathfrak{t} = \mathfrak{t}_0$, the unique Spin-structure.

Proposition 4.5.4. *In the case $q = 2$, Theorem 4.1.3 applies if we use a weaker assumption on the number of vanishing deficiencies. Namely, even if n is even, we require only that*

$$d(Y, \mathfrak{t}) - d(S^3_{-p/q}(U), \mathfrak{t}) = 0,$$

for some $\mathfrak{t} \in Spin^c(Y)$.

Proof. When $q = 2$, notice that $p/q = [n, 2]^-$. We relabel row y_2 as y for convenience. Since $y^2 = 2$, we can set $y = (1, 1, 0, \ldots, 0)$ without loss of generality. Then $x \cdot y = -1$ tells us that

$$x_1 + x_2 = -1, \tag{4.14}$$

and we let $x_2 \leq x_1$, also without loss of generality. Observe that $x_1 \geq 0$, else $x_1 + x_2 \leq -2$.

Now define a set

$$\mathcal{S} = \left\{ K \in \mathcal{M} \,\middle|\, \langle K, [v_2] \rangle = 0, D^{p/q}([K]) = 0 \right\},$$

and observe that the maximal value $j_{\max}$ of $\langle K, [v_1] \rangle$ obtained by letting K range over $\mathcal{S}$ satisfies $j_{\max} \geq 0$, since we know that at least one deficiency vanishes. If $j_{\max} > 0$, however, then this means that at least $q-r+1$ deficiencies vanish, and Theorem 4.1.3 applies. Thus, suppose $j_{\max} = 0$. By arguments similar to those in Section 4.5.1, we find that

$$j_{\max} = x_1 - x_2 + \sum_{i \geq 3} |x_i| = 0,$$

and on substituting from (4.14),

$$2x_1 + 1 + \sum_{i \geq 3} |x_i| = 0.$$

Since none of the terms on the LHS are negative, we have a contradiction. Hence $j_{\max} \neq 0$, and Theorem 4.1.3 applies. $\square$

CHAPTER 5: PRETZEL KNOTS WITH UNKNOTTING NUMBER ONE

The author's original motivation for studying the symmetries of Theorem 4.1.1 came from the half-integral case discussed in Theorem 4.1 of [51]. In that paper, Ozsváth and Szabó were able to use their result to obstruct unknotting number one in alternating knots with low crossing numbers. The author, at the time, was interested in seeing to what extent those symmetries could be applied to an infinite family of non-alternating knots; the results of these investigations are presented in this chapter.

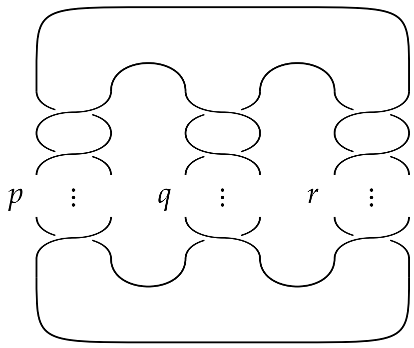
Aside from the intrinsic value in our results, this chapter is also designed to illustrate two things. First, that Theorem 4.1.3 provides an efficient, computable obstruction to rational surgeries when compared with Theorem 4.1.1; as we shall see, the latter can be extremely difficult to apply, even to L-spaces. And second, that there are instances when Theorem 4.1.3 is not applicable. We shall thus come to understand both the benefits of our theorem and something of its limitations.

The majority of the results in this chapter can be found in [5], which was joint work with Buck and Staron. However, since that paper was written before Theorems 4.1.1 and 4.1.3, our new technology allows us not only to simplify some of the proofs from that paper, but also to extend its results into previously unresolved cases.

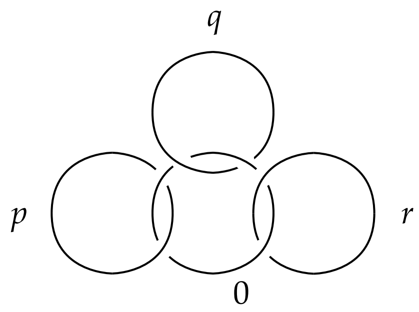
5.1 Introduction to Pretzel Knots

Our principal aim for this chapter is a partial classification of the 3-strand pretzels $K = P(p, q, r)$ with unknotting number one (where $p, q, r \in \mathbb{Z}$). These knots, depicted in Figure 5.1(a), are defined by

$$P(p, q, r) := M\left(-\tfrac{1}{p}, -\tfrac{1}{q}, -\tfrac{1}{r}\right).$$



(a)



(b)

Figure 5.1: (a) A pretzel knot $P(p, q, r)$ with $p, r > 0$, and $q < 0$; and (b) the Seifert fibred space $\Sigma(P(p, q, r))$.

As a consequence of Proposition 1.5.9, we know that $\Sigma(K)$ is presented by the Kirby diagram in Figure 5.1(b).

Since general n-strand pretzels (defined similarly) can be drawn with n-gon symmetry, it follows that the 3-strand pretzels are unchanged by permutations of their parameters (Sym(3) is isomorphic to the dihedral group of order 6). Their reflections are given by

$$\overline{P(p, q, r)} = P(-p, -q, -r).$$

It is worth remarking that for some values of p, q, r, the pretzel $P(p, q, r)$ is in fact a link. If we want K to be a *bona fide* knot, then we must require either that all three parameters be odd, or that exactly one of them be even (say $r = 2m$). The first of these cases (all odd) has been studied independently by Kobayashi [30] and Scharlemann and Thompson [61], who give the criterion that

$$u(K) = 1 \iff \pm\{1, 1\} \text{ or } \pm\{3, -1\} \subset \{p, q, r\},$$

so our work concentrates on the case $P(p, q, 2m)$. As $u(\overline{K}) = u(K)$, we assume that $2m$ is non-negative, and, as we can deal with the case $m = 0$ early on, we restrict our attention primarily to $m > 0$.

Before beginning our serious analysis, it is worth observing that Kanenobu and Murakami [28] and Torisu [67] have given a complete description of the 2-bridge knots with unknotting number one. If any of p, q, or r is ±1 in Figure 5.1(b), then we can perform a ∓1-Dehn twist about the corresponding unknot, resulting in a linear chain of unknots with integral coefficients. This, we already know, describes a lens space. By Theorem 1.5.6 it follows that $P(p, q, r)$ is a 2-bridge knot and therefore covered by the work just cited. In recognition of this fact, we will focus on the case when $p, q, r \neq \pm1$. Since r is even, this reduces to $p, q \neq \pm1$.

5.1.1 Motivation

As described in the Introduction, a large body of the work on unknotting number has followed the trend of taking a particular family of knots and working out which of its members satisfy $u(K) = 1$. Examples include the 2-bridge knots already discussed [28, 67], the large algebraic knots [23], knots with genus one [10], and the alternating 3-braids [24]. Such decisions are usually motivated in part by acceptance of the fact that $u(K)$ is difficult to compute precisely, but also by a desire to provide examples supporting the conjecture that if K satisfies $u(K) = 1$, then there is a minimal diagram D of K such that $u(D) = 1$. Our results are no different in this regard: all the subfamilies in which we are able to complete our classification have minimal diagrams with $u(K) = 1$, and we conjecture the same for those subfamilies in which we are not.

It bears mentioning that the pretzels $P(p, q, r)$ are a particularly tough family of knots. As is so often the case in 3-manifold topology, the fact that their double covers are small Seifert fibred spaces causes all manner of complications, and the knots themselves resist almost all the classical obstructions to unknotting number one. What is more, though calculations with the Rasmussen s-invariant (which bounds the slice genus [56], in turn a bound on the unknotting number [42]) have shown that $P(p, q, r)$ usually has quite a high unknotting number [66], in the four candidate families we identify these trends do not apply. In fact, within these families, the Rasmussen s-invariant often tells us only that $u(K) \geq 0$ or $u(K) \geq 1$, nothing more. The pretzels are therefore extremely interesting for the pure challenge that they pose.

There is an additional biological motivation for this problem, but to avoid dis-

Family	Conditions for $u(K) = 1$	Conjecture
$p + q = -2$	$P(1, -3, 2m), P(-1, -1, 2m)$	–
$p + q = 0$	$P(3, -3, 2)$	–
$p + q = 2$	undone by changing a positive crossing, or $\det K = 1, 5$	$P(3, -1, 2m), P(1, 1, 2m)$
$p + q = 4$	$\det K = 3, 11$	$P(5, -1, 4), P(5, -1, 2), P(3, 1, 2)$

Table 5.1: Summary of pretzels $P(p, q, 2m)$, $m > 0$, with unknotting number one (up to reflection).

tracting ourselves from the mathematical narrative, we refer the interested reader to the Appendix.

5.1.2 Results and Conjectures

Since this chapter is a long one, it is well worth summarising its main results. Our main theorem is the following.

Theorem 5.1.1. *Suppose that $K = P(p, q, 2m)$, $m \neq 0$, is a pretzel knot with unknotting number one. Then, up to reflection, $p + q = 0, \pm 2, 4$ and $m > 0$. Moreover:*

1. *If $p + q = -2$, then $K = P(1, -3, 2m), P(-1, -1, 2m)$ (all 2-bridge);*
2. *If $p + q = 0$, then $K = P(3, -3, 2)$ (which is not 2-bridge);*
3. *If $p + q = 2$, then either K is undone by changing a positive crossing (in an arbitrary diagram), or $\det K = 1$ or $\det K = 5$; and*
4. *If $p + q = 4$, then either $\det K = 3$ or $\det K = 11$.*

Table 5.1 summarises the list of pretzels in each family that have unknotting number one, together with our conjectures. On noting that most of the entries are 2-bridge knots (at least one parameter is ± 1), we conjecture the following.

Conjecture 5.1.2. *The only 3-strand pretzel knots $P(p, q, r)$ with unknotting number one that are not 2-bridge knots are $P(3, -3, 2)$ and its reflection.*

5.2 Calculating the Signature

In this section, we identify the four families above by calculating the signature of $K = P(p, q, 2m)$. There are several ways to go about this. The computation we present here differs from the one in [5] based on work by Gordon and Litherland [22] in favour of an explicit calculation by Jabuka [27].

Suppose that we define a symmetric product $D(p, q, r)$ by

$$D(p, q, r) := pq + qr + pr.$$

Then it is shown in [27], by explicit construction of a Seifert surface for K, that $\det K = |D(p, q, 2m)|$. The following proposition is a special case of Theorem 1.18 in [27].

Proposition 5.2.1. *The signature of $K = P(p, q, 2m)$, where p and q are odd, is given by*

$$\sigma(K) = \operatorname{sgn}(p)+\operatorname{sgn}(q)-(p+q)-\operatorname{sgn}(p)\operatorname{sgn}(q)\operatorname{sgn}(p+q)+\operatorname{sgn}(p+q)\operatorname{sgn}(D(p, q, 2m)).$$

Let $K = P(p, q, 2m)$. Then since $u(K) = u(\overline{K})$, it follows that we only need to classify the $P(p, q, 2m)$ with unknotting number one up to reflection. We can thus assume that $m > 0$. Since p and q are interchangeable, it follows that we have three cases to deal with: both negative, both positive, and the two opposite in sign.

If $p > 0$ and $q > 0$, then the above proposition tells us that

$$\sigma(K) = 2 - (p + q) - 1 + \operatorname{sgn}(D(p, q, 2m)).$$

By Proposition 1.2.5, if $u(K) = 1$, we require $\sigma(K) \in \{0, \pm 2\}$. So, supposing that $p + q \geq 6$, we find that $\sigma(K) \leq -4$, which is too low. But if $p + q = 2$ or 4, then one of $p = 1$ or $q = 1$, and K is a 2-bridge knot. Since this case is already dealt with in [28] and [67], we find solutions $P(1, 1, 2m)$ for $p + q = 2$ and $P(3, 1, 2)$ for $p + q = 4$. It is easy to see that by changing a crossing in the central columns of these examples, we obtain the unknot. Similarly, if $p < 0$ and $q < 0$, we find solutions $P(-1, -1, 2m)$

(again undone by changing the central crossing).

This leaves the case when $p > 0$ and $q < 0$. In this instance,

$$\sigma(K) = -(p+q) + \operatorname{sgn}(p+q) + \operatorname{sgn}(p+q)\operatorname{sgn}(D(p,q,2m)),$$

from which we can make a variety of deductions.

1. If $p + q = 0$, then $\sigma(K) = 0$;
2. If $p + q = 2$, we find that $\sigma(K) = 0$ when $D(p, q, 2m) > 0$, or $\sigma(K) = -2$ otherwise;
3. If $p + q = -2$, then $\sigma(K) = 2$, since $D(p, q, 2m) = -p^2 - 2p - 4m < 0$;
4. If $p + q = 4$, then $\sigma(K) = -2$ if $D(p, q, 2m) > 0$, and $\sigma(K) = -4$ otherwise, which is too low;
5. If $p + q < -2$, then $D(p, q, 2m) < 0$, whence $\sigma(K) > 2$, which is too high;
6. If $p + q > 4$, then $\sigma(K) < -2$, which is too small.

This exhausts all cases, giving us the following lemma.

Lemma 5.2.2. *If $K = P(p, q, 2m)$ for p and q odd satisfies $u(K) = 1$, then, up to reflection, $p + q = 0, \pm 2, 4$ and $m > 0$.*

Once we have identified our particular families according to the value of $N := p+q$, each of p and q are determined by the other. To reflect this, we make a notational change. From now on, we set $k := p$, and the knot $P(p, q, 2m)$ will be referred to as the knot $P(k, N - k, 2m)$, given our choice of $N \in \{0, \pm 2, 4\}$. Notice that

$$D(k, N-k, 2m) = -k^2 + Nk + 2Nm.$$

Despite this change of notation, the four families of interest will continue to be referred to $p + q = 0, \pm 2, 4$. They are summarised in Table 5.2.

As a final remark in this section, although we mentioned that we will only be considering $m > 0$, for completeness we can also resolve the case $m = 0$. This

Case	$D(p,q,2m)$	$\sigma(K)$
$p+q=-2$		2
$p+q=0$		0
$p+q=2$	<0	-2
	>0	0
$p+q=4$	>0	-2

Table 5.2: The four candidate families for $P(p,q,2m)$, $m>0$, identified according to the signatures.

argument appears in [5]. In this instance, $P(p,q,0) = T(p,2)\#T(q,2)$, and since unknotting number one knots are prime (see Scharlemann [62] or Zhang [72]), it follows that either $p=\pm1$ or $q=\pm1$. Since the signature of torus knots is a tight bound on $u(K)$ (see [31] and [56]), and $\sigma(T(k,2)) = \frac{1}{2}(k-1)$ for $k \geq 1$, we obtain the result below.

Lemma 5.2.3. *If $K = P(p,q,0)$ and $u(K)=1$, for p,q odd, then $pq = \pm3$.*

5.3 The Cases $p+q=\pm2,4$

In this section, we consider the knots $K = P(k, N-k, 2m)$ for $k>1$, $m>0$, and $N=\pm2,4$. Our main ingredient is the following theorem due to Montesinos [40].

Theorem 5.3.1 (Signed Montesinos theorem)**.** *Suppose that $K \subset S^3$ is a knot which can be undone by changing a negative crossing (so $\sigma(K)=0,2$). Then $\Sigma(K) = S^3_{-\epsilon D/2}(C)$ for some other knot $C \subset S^3$, where $D = \det K$ and $\epsilon = (-1)^{\frac{1}{2}\sigma(K)}$. In particular, $-\Sigma(K) = S^3_{\epsilon D/2}(\overline{C})$ bounds a simply connected smooth 4-manifold W with ϵ-definite intersection form $-\epsilon R_n$, where*

$$R_n := \begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}$$

and $D = 2n-1$.

This theorem can be proved by careful application of Theorem 1.5.5. Using that theorem, we know that $\Sigma(K)$ is obtained by surgery on some knot C; care must be taken, however, since we do not know how to relate the λ of Theorem 1.5.5 to the canonical longitude of C. The statements about the definite 4-manifold W follow from Proposition 2.2.3.

Occasionally, it will be useful to know which $\Sigma(K)$ are L-spaces. The following answer is a direct application of the notes in Section 3.1 of [7] (alternatively, some of this proposition can be deduced from their Theorem 3.2).

Proposition 5.3.2. *If $p+q=0,-2$, then $\Sigma(K)$ is an L-space. If $N=2,4$, then there are particular choices of p, q, and m such that $\Sigma(K)$ is not an L-space.*

5.3.1 Resolution of $p+q=-2$

We now specialise to the case $p+q=-2$. Our ultimate goal is to prove the following theorem via Theorem 4.1.3. This argument is the one available in [5].

Theorem 5.3.3. *Suppose that $k,m>0$ and that k is odd. Then $P(k,-2-k,2m)$ has unknotting number one if and only if $k=1$.*

We already know that $\sigma(K)=2$, so if $u(K)=1$, then K must be undone by changing a negative crossing (see Proposition 1.2.5). Applying Theorem 5.3.1, $-\Sigma(K)=S^3_{-D/2}(C)$ for some knot $C\subset S^3$, and therefore bounds a simply connected, negative-definite smooth 4-manifold W with intersection form R_n. We will apply Theorem 4.1.3 to $Y=-\Sigma(K)$.

As per the conditions of Theorem 4.1.3, we must now check that $-Y=\Sigma(K)$ bounds a sharp, simply connected, negative-definite smooth 4-manifold X, and that the correction term condition given in Proposition 4.5.4 is satisfied. The first of these is easy: Figure 5.2(a) provides us with a Kirby diagram for $\Sigma(K)$ which can be converted into Figure 5.2(b) by -1-Dehn twists around the unknots with positive coefficients. We also observe that

$$\frac{k}{k-1}=\overbrace{[2,2,\ldots,2]}^{k-1}{}^{-} \qquad\qquad \frac{2m}{2m-1}=\overbrace{[2,2,\ldots,2]}^{2m-1}{}^{-}.$$

Hence, by Proposition 2.2.8, $\Sigma(K)$ is the boundary of the plumbing $X=\mathcal{X}(G)$ given in Figure 5.3. By (3.7), and since there is only one overweight vertex, this plumbing

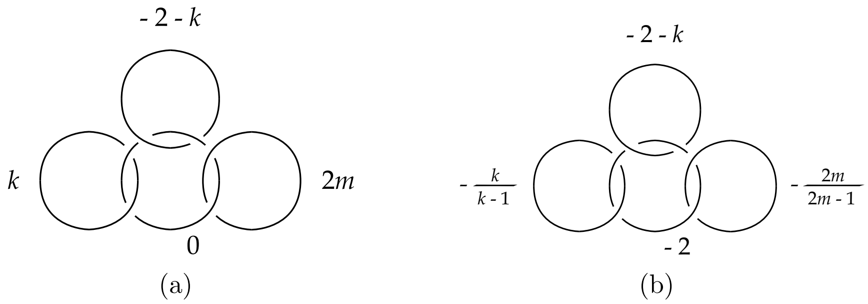


Figure 5.2: (a) A Kirby diagram for $\Sigma(P(k,-2-k,2m))$; and (b) an alternative Kirby diagram for the same manifold.

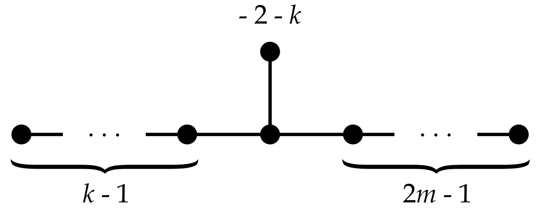


Figure 5.3: A plumbing graph G for a simply connected, negative-definite smooth 4-manifold X with $\Sigma(P(k,-2-k,2m))$ as boundary. The vertices are labelled $v_1,\ldots,v_{k+2m-1}$ along the horizontal segment, left to right; the top vertex is labelled v_{k+2m}.

is also sharp. Simple connectedness of X follows from Proposition 2.2.6 (since G is a tree).

To check that Q_X is negative-definite, the easiest way is via Sylvester's criterion, as we did in Section 4.4.1. The determinants of the upper left $(i\times i)$-submatrices of Q_X, with the exception of Q_X itself, are easily computed by induction. They are $(-1)^i(i+1)$, and therefore manifestly alternate in sign. Once we observe that $\det Q_X = D(k,-2-k,2m) < 0$, and that the rank of Q_X is odd, we are finished.

The remaining condition, concerning correction terms, is more difficult to check. Recall from Section 3.5 that we have an exact sequence

$$0 \longrightarrow H^2(X,Y) \xrightarrow{Q_X} H^2(X) \xrightarrow{\alpha} H^2(Y) \longrightarrow 0,$$

and that since $H^2(Y)$ is of odd order, a $K\in \mathrm{Char}(G)$ determines the unique Spin-structure on Y if and only if $K\in\ker\alpha = \operatorname{im} Q_X$. Hence, if and only if K is a sum of rows of Q_X. We set

$$\begin{aligned} K &= -\sum_{i=1}^{m} 2i\,\mathrm{PD}[v_i] - 2m \sum_{i=m+1}^{k} \mathrm{PD}[v_i] - \sum_{i=k+1}^{k+2m-1} (k+2m-i)\,\mathrm{PD}[v_i] - \mathrm{PD}[v_{k+2m}] \\ &= (0,\ldots,0,2_{(m)},0,\ldots,0,k-2m+2) \end{aligned}$$

if $k \geq m$, or

$$\begin{aligned} K &= -\sum_{i=1}^{k} 2i\,\mathrm{PD}[v_i] - \sum_{i=k+1}^{m} (k+i)\,\mathrm{PD}[v_i] - \sum_{i=m+1}^{k+2m-1} (k+2m-i)\,\mathrm{PD}[v_i] - \mathrm{PD}[v_{k+2m}] \\ &= (0,\ldots,0,2_{(m)},0,\ldots,0,-k+2) \end{aligned}$$

if $k \leq m$, and check that these choices initiate maximising paths. Since $\Sigma(K)$ is an L-space by Proposition 5.3.2, these K must be the unique representatives of the Spin-structure with this property, and we know from (3.7) that

$$d(\Sigma(K),\mathfrak{t}_0) = \frac{KQ_X^{-1}K^t + (k+2m)}{4}.$$

To calculate $KQ_X^{-1}K^t$ without computing Q_X^{-1}, observe that $K = \sum_{i=1}^{k+2m} k_i\,\mathrm{PD}[v_i]$, where $\mathrm{PD}[v_i]$ is the i-th row of Q_X, from which it follows that $KQ_X^{-1} = \sum_{i=1}^{k+2m} k_i e_i$, where e_i is the i-th standard basis vector for $H^2(X)$. Thus $K^2 = \sum_{i=1}^{k+2m} k_i \langle K, [v_i]\rangle$, and hence $d(\Sigma(K),\mathfrak{t}_0) = -\frac{1}{2}$. By the second part of Proposition 3.3.4, it follows that $d(Y,\mathfrak{t}_0) = \frac{1}{2}$. Similarly, we can repeat this sort of calculation for the plumbing of $S^3_{-D/2}(U)$ shown in Figure 4.1, and as $\det K \equiv 3 \mod 4$, we deduce that $d(S^3_{-D/2}(U),0) = \frac{1}{2}$, verifying the deficiency condition of Proposition 4.5.4.

Proof of Theorem 5.3.3. We apply Theorem 4.1.3 by showing that if $k \geq 3$ then there is no matrix A of the prescribed form satisfying

$$-AA^t = \left(\begin{array}{ccccccccc|cc} -2 & 1 & & & & & & & & & \\ 1 & -2 & \ddots & & & & & & & & \\ & \ddots & \ddots & 1 & & & & & & & \\ & & 1 & -2 & 1 & & & & 1 & & \\ & & & 1 & \ddots & \ddots & & & & & \\ & & & & \ddots & -2 & 1 & & & & \\ & & & & & 1 & -2 & & & & \\ & & & 1 & & & & & -k-2 & & \\ \hline & & & & & & & & & -n & 1 \\ & & & & & & & & & 1 & -2 \end{array}\right).$$

In the matrix above, the $(k+2m,k)$ and $(k,k+2m)$ entries are both 1. Disproving the existence of such an A will then tell us that $u(K) \geq 2$ if $k \geq 3$.

Observe, as we did during the proof of Theorem 4.1.3, that the (i,j)-th entry on the RHS is the dot product $-v'_i \cdot v'_j$, where v'_i is the i-th row of A. Since the diagonal entries are all of magnitude 2, except the $(k+2m-2)$-th and $(k+2m-1)$-th, it follows that the corresponding rows of A have precisely two entries, each of magnitude 1. Without loss of generality, we take $v'_1 = (1,-1,0,\ldots,0)$, so that, up to a basis permutation of $H_2(X \cup W)$, we can arrange for A to have the form:

$$A = \left(\begin{array}{cccccc|cc} 1 & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & & & & & \\ & & & & 1 & -1 & & \\ * & * & \ldots & \ldots & * & * & * & * \\ \hline * & * & \ldots & \ldots & * & * & 1 & \\ & & & & & & -1 & 1 \end{array}\right).$$

Implicitly, we are using the fact that $k+2m-1 \geq 4$ by virtue of our assumptions on

k and m. This inequality implies that the top $k+2m-1$ rows overlap as shown.

Next let $A_{k+2m+1,1} = \alpha$. Since $v'_i \cdot v'_{k+2m+1} = 0$ for $i = 1, 2, \ldots, k+2m-1$, the leading $k+2m$ entries of v'_{k+2m+1} are also α.

$$A = \left(\begin{array}{cccccc|cc} 1 & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & & & & & \\ & & & & 1 & -1 & & \\ * & * & \ldots & \ldots & * & * & * & * \\ \hline \alpha & \alpha & \ldots & \ldots & \alpha & \alpha & 1 & \\ & & & & & & -1 & 1 \end{array}\right).$$

According to Theorem 4.1.3, then, either $|\alpha| = 0, 1$.

1. If $\alpha = 0$, then $1 = (v'_{k+2m+1})^2 = n = \frac{1}{2}(\det K + 1)$, whence $\det K = 1$. However, we already know that $\det K = k^2 + 2k + 4m$, so clearly $\det K > 1$.

2. If $\alpha = \pm 1$, then $k + 2m + 1 = (v'_{k+2m+1})^2 = n = \frac{1}{2}(\det K + 1)$, but this can only occur if $k^2 = 1$, which contradicts our assumption that $k \geq 3$.

Since we obtain contradictions in either situation, the assumption that $k \geq 3$ is wrong. If $k = 1$, then it is clear that $P(1, -3, 2m)$ has unknotting number one; by changing any crossing in the central column, we obtain the unknot $P(1, -1, 2m)$. Our proof is thus complete. □

5.3.2 Partial Resolution of $p + q = 4$

We now consider the case $p + q = 4$. For reasons we will explain shortly, our result is slightly weaker.

Theorem 5.3.4. *Suppose that $k, m > 0$ and that k is odd. Then $P(k, 4-k, 2m)$ has unknotting number one only if $\det K = 3$ or 11.*

Notice that, unlike the previous case, this result is an "only if" rather than an "if and only if". We are unfortunately unable to make further progress as, for both choices of $\det K$ implied in the above theorem, there is always an A satisfying Theorem 4.1.3.

To prove Theorem 5.3.4, we proceed in much the same manner as before: this time, $\sigma(K) = -2$, so $\sigma(\overline{K}) = 2$, and thus by Theorem 5.3.1, $\Sigma(\overline{K}) = S^3_{D/2}(C)$ for some knot $C \subset S^3$. Additionally, $\Sigma(K) = -\Sigma(\overline{K})$ bounds a negative-definite 4-manifold with intersection form R_n. We will apply Theorem 4.1.3 to $Y = \Sigma(K)$.

As before, we need a sharp, simply connected, negative-definite smooth 4-manifold X with $-\Sigma(K) = \Sigma(\overline{K})$ as boundary. Using the same methods as in the previous case, we can verify that the 4-manifold $X = \mathcal{X}(G)$ corresponding to the graph G in Figure 5.4 fulfils these conditions. The only substantial difference is the verification of Proposition 4.5.4. This time, we use

$$\begin{aligned} K &= -\sum_{i=1}^{k-3} 2i\,\mathrm{PD}[v_i] - \mathrm{PD}[v_{k-2}] \\ &= (0, \ldots, 0, 1, 7-k, 8-2k), \end{aligned}$$

which both initiates a maximising path and represents the Spin-structure $\mathfrak{t}_0$. Using (3.7), we see that $d(\Sigma(\overline{K}), \mathfrak{t}_0) \geq -\frac{1}{2}$, and by the second part of Proposition 3.3.4, it follows that $d(Y, \mathfrak{t}_0) \leq \frac{1}{2}$. Since $\det K = -k^2 + 4k + 8m \equiv 3 \mod 4$, we already know that $d(S^3_{-D/2}(U), \mathfrak{t}_0) = \frac{1}{2}$, so

$$0 \leq d(Y, \mathfrak{t}_0) - d(S^3_{-D/2}(U), \mathfrak{t}_0) \leq 0,$$

where the left hand inequality comes from Lemma 4.4.7. Hence, Proposition 4.5.4 is satisfied.

Proof of Theorem 5.3.4. We must show that if an A exists satisfying Theorem 4.1.3,

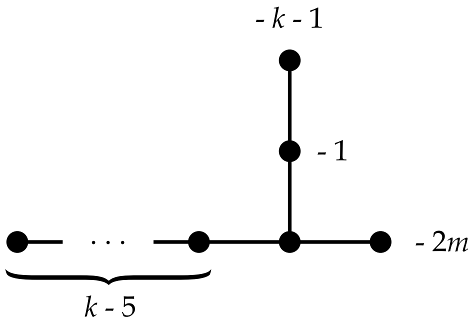


Figure 5.4: A plumbing graph G for a simply connected, negative-definite smooth 4-manifold X with $-\Sigma(P(k, 4-k, 2m))$ as boundary. The vertices are labelled $v_1, \ldots, v_{k-4}$ along the horizontal segment starting at the left and continuing until the central node; the vertex with weight -1 is labelled v_{k-3}, the top vertex v_{k-2}, and the rightmost vertex v_{k-1}.

then $\det K$ has the required form. Using the 4-manifold from Figure 5.4, we want

$$-AA^t = \left(\begin{array}{cccccccc|cc}
-2 & 1 & & & & & & & & \\
1 & -2 & 1 & & & & & & & \\
 & 1 & -2 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & -2 & 1 & & 1 & & \\
 & & & & 1 & -1 & 1 & & & \\
 & & & & & 1 & -k-1 & & & \\
 & & & & 1 & & & -2m & & \\
\hline
 & & & & & & & & -n & 1 \\
 & & & & & & & & 1 & -2
\end{array}\right).$$

This implies that

$$A = \left(\begin{array}{ccccccc|cc}
1 & -1 & & & & & & & \\
 & 1 & -1 & & & & & & \\
 & & \ddots & \ddots & & & & & \\
 & & & 1 & -1 & & & & \\
 & & & & 1 & & & & \\
* & * & \dots & * & * & * & * & * & \\
* & * & \dots & * & * & * & * & * & \\
\hline
\gamma & \gamma & \dots & \gamma & \gamma & \alpha & \beta & 1 & \\
 & & & & & & & -1 & 1
\end{array}\right).$$

The relation $v'_{k-3} \cdot v'_k = 0$ then tells us that $\gamma = 0$. Consequently, the changemaker condition tells us that either $\alpha = 0, 1$ and $\beta \leq \alpha + 1$, or vice versa. Either way, once we have enumerated the various possibilities for α and β, the relation $(v'_k)^2 = n = \frac{1}{2}(\det K + 1)$ informs us that $\det K \in \{1, 3, 5, 11\}$. Since $\det K = -k^2 + 4k + 8m \equiv 3 \mod 4$, the result follows. □

Before concluding this section, we should discuss the extent to which Theorem 5.3.4 could be improved. If one reads the proof of Theorem 4.1.3 carefully, one will notice that the full symmetries of Theorem 4.1.1 were not required, only the symmetries that exist among the deficiency-minimising Spin^c-structures. Thus, one might rightly conclude that Theorem 4.1.1 has more to say than Theorem 4.1.3, and indeed we shall see this to be the case over the next couple of sections as we tackle $p + q = 0$.

On the other hand, since $\Sigma(K)$ is not always an L-space (see Proposition 5.3.2), Theorem 4.1.1 throws up several difficulties of its own. The most grievous of these is the fact that the covectors K produced by the algorithm in Section 3.5 are no longer the unique representatives of the corresponding Spin^c-structures $[K]$, rendering a calculus of the correction terms a formidable undertaking indeed: an undertaking which, as of the time of this writing, we have been unable to complete. For the reader with doubts about this claim, the case $p + q = 0$ should hopefully provide a convincing demonstration of the subtleties and complexities involved in Theorem 4.1.1, even if Y is an L-space.

That being said, we are confident that when $\det K = 11$, a sufficiently clever or stubborn application of Theorem 4.1.1 will eventually show that $u(K) = 1$ if and only if $k = 3$ or 5 (i.e. $K = P(5, -1, 4)$, $P(5, -1, 2)$, or $P(3, 1, 2)$). Sadly, though, we believe it unlikely that the case $\det K = 3$ can be resolved in this manner, for if $\det K \leq 3$, Theorem 4.1.1 yields no more information than the conjugation symmetry which is true of every 3-manifold Y.

5.3.3 Difficulties with $p + q = 2$

As with the previous case, there is room for some progress when $p + q = 2$, though not much. The method of proof is almost exactly the same as the one used in the $p + q = 4$ case, and suffers from the same sort of limitations. As such, we will not give the full proofs, only sketches.

Lemma 5.3.5. *Suppose that $k, m > 0$ and that k is odd. Then if $K = P(k, 2-k, 2m)$ has unknotting number one and $\sigma(K) = 0$, either*

1. *K is undone by changing a positive crossing; or*
2. $\det K = 1$ *or* 5.

Proof. Under the hypotheses stated, notice that $D(k, 2 - k, 2m) > 0$. If K is undone by changing a negative crossing, then $\Sigma(K) = S^3_{-D/2}(C)$ by Theorem 5.3.1. Using the manifold $X = \mathcal{X}(G)$ where G is a similar graph to Figure 5.4 (except that the left hand arm has $k - 3$ vertices), we apply Theorem 4.1.3 and derive similar results as in the $p + q = 4$ case. Notice this time that $\det K = -k^2 + 2k + 4m \equiv 1 \mod 4$. □

In the other situation, when $\sigma(K) = -2$, it follows from Proposition 1.2.5 that K is undone by changing a positive crossing. The problem this time is in finding an appropriate negative-definite 4-manifold X whose orientation is compatible with W, something we have not yet been able to do.

We believe that just as with $p + q = 4$, a sufficiently sustained application of Theorem 4.1.1 to all the leftover cases here will prove our conjectured result that $u(K) = 1$ if and only if $K = P(1, 1, 2m)$ or $P(3, -1, 2m)$, regardless of whether or not $\sigma(K) = 0$ or -2. The only exception is when $\det K = 1$; in this case, as noted at the end of the previous section, there are not enough Spin^c-structures for Theorem 4.1.1 to be meaningful.

5.4 The Case $p + q = 0$: First Results

We now tackle the knots $K = P(k, -k, 2m)$ for $k > 1$ and $m > 0$, a family which illustrates very well both the usefulness and limitations of Theorem 4.1.3. The process involves two steps. First, we pin down the value of m using the Alexander module, concluding that $m = 1$. Second, we employ Theorem 4.1.1 to deduce the values of k which give $u(K) = 1$. One naturally wonders if the methods of the previous section will help us in this endeavour, but unfortunately Theorem 4.1.3 only allows us to identify the sign of the crossing change involved.

Our ultimate goal over the next two sections is the following theorem.

Theorem 5.4.1. *Suppose that $k, m > 0$ and that k is odd. Then $P(k, -k, 2m)$ has unknotting number one if and only if $k = 3$ and $m = 1$.*

The arguments that follow on the $p + q = 0$ case are taken from [5]. Observe that $\det P(k, -k, 2m) = k^2$. Hence, in Theorem 5.3.1, we set $D = k^2$, so that $n = \frac{k^2+1}{2}$.

5.4.1 The Alexander Module

Recall from Chapter 1 that we can construct the infinite cyclic cover $X_\infty(K)$ of a knot K. This has a deck transformation group $\mathbb{Z}$, generated by some element t, and $H_1(X_\infty(K))$ is a $\mathbb{Z}[t, t^{-1}]$-module A, called the *Alexander module*, from which a considerable amount of topological information can be mined. This is done via the r-th *elementary ideal*, denoted A_r, which we defined in Section 1.3 as the ideal in $\mathbb{Z}[t, t^{-1}]$ spanned by the $(n - r + 1) \times (n - r + 1)$-minors of any $n \times n$ presentation matrix for A.

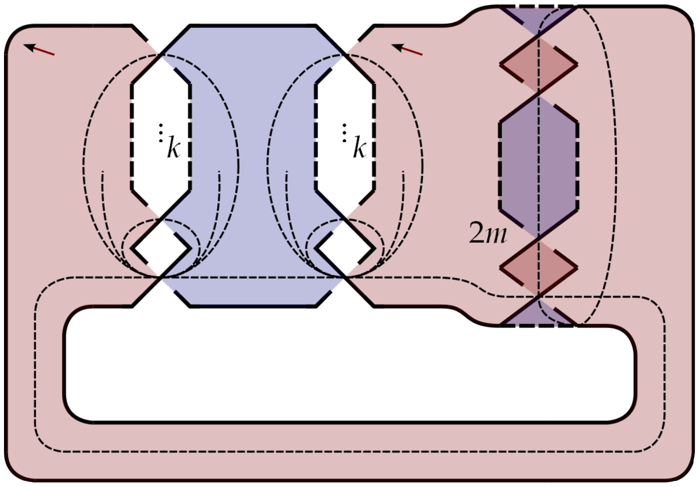


Figure 5.5: A Seifert surface for $P(k, -k, 2m)$. All curves are oriented clockwise except for the big curve around the hole. The normal to the surface is also indicated (out of the page in red sections, into the page in blue ones).

From Nakanishi [43], in the form cited in Lickorish [35], we also know that the Alexander module can be used to bound the unknotting number (see Theorem 1.3.4). Specifically, if $A/A_r \neq 0$, then $u(K) \geq r$. Using this theorem, we can now prove the following lemma.

Lemma 5.4.2. *Suppose that $k \geq 3$, $m > 0$, and k is odd. Then if $P(k, -k, 2m)$ has unknotting number one, $m = 1$.*

Proof. We take the Seifert surface F for $P(k, -k, 2m)$ shown in Figure 5.5. The curves are indexed starting with the central column of loops, largest to smallest, followed by the same labelling on the first column, then the big loop around the hole and finally the loop crossing the "bridge".

Using this surface F, we construct the bilinear form V_F described in Section 1.2. With our chosen basis, it has matrix

$$V_F = \begin{pmatrix} X_k & 0 & \mathbf{1}^t & 0 \\ 0 & -X_k & -\mathbf{1}^t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & m \end{pmatrix},$$

where X_k is the $(k-1)\times(k-1)$ lower triangular matrix of 1's, and $\mathbf{1}$ is a suitably sized row of 1's.

Consequently, the Alexander module is presented by

$$A = \begin{pmatrix} M_k & 0 & \mathbf{t}^t & 0 \\ 0 & -M_k & -\mathbf{t}^t & 0 \\ -\mathbf{1} & \mathbf{1} & 0 & -1 \\ 0 & 0 & t & m(t-1) \end{pmatrix},$$

from which we can compute the relevant minors. Here, $M_k = tX_k - X_k^t$, and $\mathbf{t}$ is a row with all entries t.

We claim that when $k \geq 3$, the second elementary ideal A_2 generated by the principal minors of A, is precisely given by

$$A_2 = \langle \mathcal{P}_k(t), m(t-1) \rangle,$$

where $\mathcal{P}_k(t) = \sum_{i=0}^{k-1}(-1)^i t^{k-1-i}$. For the moment, we take this as given and show that $A_2 = \mathbb{Z}[t,t^{-1}]$ if and only if $m = 1$. Indeed, the quotient $\mathbb{Z}[t,t^{-1}]/\langle \mathcal{P}_k(t) \rangle$ is the $\mathbb{Z}$-module consisting of all Laurent polynomials

$$a_{k-2}t^{k-2} + a_{k-3}t^{k-3} + \cdots + a_1 t + a_0,$$

together with their unit multiples (recall that the units are elements of the form t^n for n an integer). These Laurent polynomials vanish in A/A_2 if and only if they fall in the ideal $\langle m(t-1) \rangle$. In particular, if and only if a_i is divisible by m for all i. This statement then implies $m = 1$.

When $m = 1$, observe that

$$\mathcal{P}_k(t) = t^{k-1} - t^{k-3}(t-1) - t^{k-5}(t-1) - \cdots - (t-1),$$

which means that in the quotient $\mathbb{Z}[t,t^{-1}]/\langle m(t-1) \rangle$, the polynomial $\mathcal{P}_k(t)$ is a unit. Hence, $\mathbb{Z}[t,t^{-1}]/A_2 = 0$, and there is no obstruction to unknotting number one:

Theorem 1.3.4 guarantees only that $u(K) \geq 1$.

We must therefore check our claim about A_2. As a first step, we compute $\det M_k$ as follows, using the notation $\mathbf{v}^*$ to indicate a square matrix whose every row is $\mathbf{v}$.

$$\begin{aligned}
\det M_k &= \det \begin{pmatrix} t-1 & -1 & -1 & \dots & -1 \\ t & t-1 & -1 & \dots & -1 \\ t & t & t-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \dots & t-1 \end{pmatrix} \\
&= t^{-1} \det \begin{pmatrix} t^2 & 0 & 0 & \dots & -1 \\ t & t-1 & -1 & \dots & -1 \\ t & t & t-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \dots & t-1 \end{pmatrix} \\
&= t \det M_{k-1} + (-1)^{k-1} t^{-1} \det \begin{pmatrix} \mathbf{t}^t & M_{k-2} \\ t & \mathbf{t} \end{pmatrix} \\
&= t \det M_{k-1} + (-1)^{k-1} \det \begin{pmatrix} 0 & M_{k-2} - \mathbf{t}^* \\ 1 & \mathbf{1} \end{pmatrix} \\
&= t \det M_{k-1} + \det(M_{k-2} - \mathbf{t}^*).
\end{aligned}$$

Using the fact that $M_{k-1} - \mathbf{t}^*$ is an upper-triangular $(k-2) \times (k-2)$ matrix, each diagonal entry of which is -1, we obtain the recurrence relation

$$\det M_k = t \det M_{k-1} + (-1)^{k-1}.$$

This recurrence, when solved, shows that $\det M_k = \mathcal{P}_k(t)$.

Now, let us suppose we wish to compute $\det A^{i,j}$ when $i, j < 2k$. Expanding down the final column or row, we obtain two terms, one of which is a multiple of $m(t-1)$, and the other of which is the determinant of a block diagonal matrix featuring either M_k or $-M_k$ (both of whose determinants are $\mathcal{P}_k(t)$ up to sign). Hence $\det A^{i,j} \in \langle \mathcal{P}_k(t), m(t-1) \rangle$. The other determinants $\det A^{i,j}$ when either $i = 2k$ or

$j = 2k$ are calculations very much like the one presented below, and therefore are multiples of $\mathcal{P}_k(t)$. Hence, in order to prove that A_2 is spanned by these two key polynomials, all we must do now is ensure that they are both actually in the ideal. This is proved by the following two example minors.

First, we delete the first row and final column:

$$\begin{aligned}
\det A^{1,2k} &= \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & \mathbf{t}^t \\ 0 & 0 & -M_k & -\mathbf{t}^t \\ -1 & -\mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \\
&= t \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 \\ 0 & 0 & -M_k \\ -1 & -\mathbf{1} & \mathbf{1} \end{pmatrix} \\
&= t \det \begin{pmatrix} M_{k-1} - \mathbf{t}^* & \mathbf{t}^* \\ 0 & -M_k \end{pmatrix} \\
&= (-1)^{k-1} t \det(M_{k-1} - \mathbf{t}^*) \det M_k \\
&= -t\mathcal{P}_k(t).
\end{aligned}$$

The last equality uses our previous calculation. Since $-t$ is a unit, we know that $\mathcal{P}_k(t) \in A_2$.

Now delete the first row and k-th column:

$$\det A^{1,k} = \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & \mathbf{t}^t & 0 \\ 0 & 0 & \mathbf{1} & -t & 0 \\ 0 & 0 & -M_{k-1} & -\mathbf{t}^t & 0 \\ -1 & -\mathbf{1} & \mathbf{1} & 0 & -1 \\ 0 & 0 & 0 & t & m(t-1) \end{pmatrix}$$

$$= m(t-1) \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & \mathbf{t}^t \\ 0 & 0 & \mathbf{1} & -t \\ 0 & 0 & -M_{k-1} & -\mathbf{t}^t \\ -1 & -\mathbf{1} & \mathbf{1} & 0 \end{pmatrix}$$

$$= m(t-1) \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & -\mathbf{1}^t \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & -M_{k-1} & \mathbf{1}^t \\ t & \mathbf{t} & -\mathbf{t} & 0 \end{pmatrix}$$

$$= m(t-1) \det \begin{pmatrix} 0 & M_{k-1} - \mathbf{t}^* & \mathbf{t}^* & -\mathbf{1}^t \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & -M_{k-1} & \mathbf{1}^t \\ t & \mathbf{t} & -\mathbf{t} & 0 \end{pmatrix}.$$

This last determinant evaluates as

$$-t \det(M_{k-1} - \mathbf{t}^*) \det \begin{pmatrix} \mathbf{1} & 1 \\ -M_{k-1} & \mathbf{1}^t \end{pmatrix} = (-1)^{k-1} \det \begin{pmatrix} M_{k-1} & -\mathbf{1}^t \\ \mathbf{t} & t \end{pmatrix},$$

which in turn is almost M_k. Up to sign, it is

$$\det M_k - (t-1) \det M_{k-1} + t \det M_{k-1} = \det M_k + \det M_{k-1} = t^{k-1}.$$

It follows that $m(t-1) \in A_2$, and our proof is at last complete. □

5.4.2 Theorem 4.1.3 and $\Sigma(P(k,-k,2))$

The previous section is unusual in that it is the only time that the Alexander module (or any other classical invariant besides the signature, for that matter), gives any useful information. Having exhausted the classical avenues, then, it is natural to wonder if Theorem 4.1.3 yields anything. We will see, however, that since the signature of K vanishes in this case, the only progress we can make with the theorem is to pin down the sign of the unknotting crossing (a piece of information which, while far from ideal, still cuts down our workload in the next section by half). This is due to orientation incompatibilities when it comes to gluing W and X together.

Lemma 5.4.3. *Suppose $k \geq 3$ is odd. Then if $K = P(k,-k,2)$ has unknotting number one, it is undone by changing a negative crossing.*

Suppose that K is undone with a positive crossing. Then, $\overline{K}$ is undone by changing a negative crossing. Hence, by Theorem 5.3.1, $\Sigma(\overline{K}) = S^3_{-D/2}(C)$, where C is a knot in S^3 and $D = \det \overline{K} = \det K$. So, $-\Sigma(K)$ bounds a negative-definite 4-manifold W with intersection form R_n. We apply Theorem 4.1.3 to the case $Y = -\Sigma(K)$.

As in Section 5.3, we construct $X = \mathcal{X}(G)$ using the graph G in Figure 5.6; this X is negative-definite and sharp. Proposition 4.5.4 is checked using the covector

$$\begin{aligned} K &= -2\sum_{i=1}^{k} \mathrm{PD}[v_i] - \mathrm{PD}[v_{k+1}] - \mathrm{PD}[v_{k+2}] \\ &= (2, 0, \dots, 0, k-2), \end{aligned}$$

which initiates a maximising path and is the unique representative of $\mathfrak{t}_0$ with this property (since $\Sigma(K)$ is an L-space, see Proposition 5.3.2). This tells us that $d(Y, \mathfrak{t}_0) = 0$, and since $\det K = k^2 \equiv 1 \mod 4$, we also find $d(S^3_{-D/2}(U), \mathfrak{t}_0) = 0$.

Proof of Lemma 5.4.3. Knowing that Theorem 4.1.3 is applicable, the matrix A has

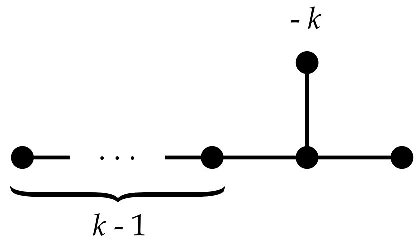


Figure 5.6: A plumbing graph G for a simply connected, negative-definite smooth 4-manifold X with $\Sigma(P(k,-k,2))$ as boundary. The vertices are labelled $v_1, \ldots, v_{k+1}$ along the horizontal segment, left to right; the top vertex is labelled v_{k+2}.

the form

$$\left(\begin{array}{ccccccc|cc}
1 & -1 & & & & & & & \\
 & 1 & -1 & & & & & & \\
 & & 1 & -1 & & & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & & 1 & -1 & & & \\
 & & & & & 1 & -1 & & \\
a & a & a & \ldots & a & b & b & c & c \\
\hline
d & d & d & \ldots & d & d & d & 1 & \\
 & & & & & & & -1 & 1
\end{array}\right).$$

Denote the $k+4$ different rows v_i'. Then $v_k' \cdot v_{k+2}' = -1$, so $b = a+1$, and $(v_{k+2}')^2 = k$ implies

$$ka^2 + 2(a+1)^2 + 2c^2 = k, \tag{5.1}$$

whence we must have $a = 0, -1$ (else the LHS is too big). We split the cases:

1. If $a = 0$, then from (5.1), $2 + 2c^2 = k$. This is nonsense for parity reasons.

2. If $a = -1$, then from (5.1), $c = 0$. Hence $v_{k+2}' \cdot v_{k+3}' = 0$ tells us that $kd = 0$, and so $d = 0$. The fact that $(v_{k+3}')^2 = n$ yields up $n = 1$, so $k^2 = 1$, contradicting $k \geq 3$.

This completes the proof. □

5.5 Heegaard Floer Homology of $\Sigma(P(k,-k,2))$

To complete the work started in the previous section we now employ Theorem 4.1.1. As mentioned in the Introduction, this result is a direct generalisation of Theorem 4.1 in [51]. If we compare the function $\mathfrak{r} : \mathrm{Spin}^c(S^3_{-D/2}(C)) \to \mathrm{Spin}^c(S^3_{-n}(C))$ from Theorem 4.1.1 with the corresponding restriction in Section 4 of [51], we find that they coincide (though our formulation has noticeably fewer conditions that need to be checked). For convenience, we summarise the relevant details below, having set $\Sigma := \Sigma(P(k,-k,2))$ and $L := S^3_{-D/2}(U)$. Since $D \equiv 1 \mod 4$ and $D = 2n-1$, it follows that n is odd, and we write $n = 2s+1$.

Theorem 5.5.1. *If C is a knot in S^3 and $D \equiv 1 \mod 4$, then*

$$d(S^3_{-D/2}(C), i) - d(L, i) = d(S^3_{-D/2}(C), 2s-i) - d(L, 2s-i), \tag{5.2}$$

for $i = 0, 1, \ldots, s$, where the labelling on the Spin^c-structures is by $\frac{1}{2}c_1$.

To apply this, observe that we know from Lemma 5.4.3 that $K = P(k,-k,2)$ must be undone by changing a negative crossing; Theorem 5.3.1 then tells us that $\Sigma(K) = S^3_{-D/2}(C)$ for some knot $C \subset S^3$. Our goal is to show that at least one of the equations (5.2) must fail if $k \geq 5$. Once that is done, we can conclude that $u(K) \geq 2$ for $k \geq 5$.

If the reader is making a comparison with Theorem 1.1 of [51], the reason we do not bother with the conditions on positive and even matchings is that they are not strong enough to obstruct our pretzels. The symmetry condition, however, is, and this is essentially just Theorem 5.5.1.

As mentioned earlier, all work in this section is taken from [5].

5.5.1 Correction Terms of $\Sigma(P(k,-k,2))$

Recall the algorithm for computing Heegaard Floer homology of plumbed 3-manifolds from Section 3.5. We shall apply this to the graph G in Figure 5.6; the associated manifold $X = \mathcal{X}(G)$, as has already been discussed, has Σ as boundary. Because Σ is an L-space (Proposition 5.3.2), it necessarily follows that the algorithm will output a complete set of representatives for the Spin^c-structures on Σ without repetition, and because $H^2(\Sigma)$ is of odd order, given $K_1, K_2 \in \mathrm{Char}(G)$, we know that $K_1 \sim K_2$ if and only if $(K_1 - K_2)Q^{-1} \in \mathbb{Z}^{k+2}$, where $Q := Q_X$.

Proposition 5.5.2. *The following are the only elements of* $\mathrm{Char}(G)$ *which initiate maximising paths:*

1. $(0, 0, \ldots, 0, 2_{(i)}, 0, \ldots, 0, j)$*, where* j *is an odd integer satisfying* $2-k \leq j \leq k-4$*;*
2. $(2, 0, \ldots, 0, 0, k-2)$ *and* $(0, \ldots, 0, 2, k-2)$*; and*
3. $(0, \ldots, 0, j)$ *where* j *is an odd integer satisfying* $2 - k \leq j \leq k$*.*

Proof. Let $K \in \mathrm{Char}(G)$ satisfy (3.6). We show that if K satisfies either of two conditions below then it must initiate a non-maximising path.

First suppose that there are two $v \in V(G)$ such that $\langle K, [v] \rangle = -w(v)$. Then, pushing down at v_{k+2} if necessary, there must be a substring of K that resembles $(2, 0, \ldots, 0, 2)$. On pushing down the 2's at either end of this substring and iterating with the 2's so created, we eventually obtain a value 4 in the substring. Thus, K initiates a non-maximising path.

We now consider $K = (0, \ldots, 0, 2_{(i)}, 0, \ldots, 0, k-2)$ for $i = 2, \ldots, k$. Pushing down the 2 to create two more 2's on either side, and continuing to push down these newly created 2's down in either direction, we end up with $(-2, 0, \ldots, 0, 2_{(i)}, 0, \ldots, -2, k)$. On repeating this procedure, we end up with $k + 2$ in the final co-ordinate. As this is too large, K initiates a non-maximising path.

After eliminating these two possibilities, the remaining K are precisely those listed in the proposition. Since there are $(k+1)(k-2) = k^2 - k - 2$ covectors of the first kind, 2 of the second, and k of the last, we have k^2 in total. That being the order of $H^2(\Sigma)$, these must initiate maximising paths and enumerate the different Spin^c-structures on Σ. □

We call the set of these specific maximisers $\mathrm{Char}^*(G)$, and observe that we have an isomorphism $\mathrm{Spin}^c(\Sigma) \simeq \mathrm{Char}^*(G)$ (see Proposition 2.3.18 for the group structure on

$\mathrm{Spin}^c(\Sigma)$). Indeed, given $K_1, K_2 \in \mathrm{Char}^*(G)$, we know that $[K_1] + [K_2] = [K]$ if and only if $\frac{1}{2}c_1(K_1) + \frac{1}{2}c_1(K_2) = \frac{1}{2}c_1(K)$. Phrased in the notation of (3.5), this condition is satisfied if and only if $\alpha(K_1 + K_2 - K) = 0$, or, equivalently, $(K_1 + K_2 - K)Q^{-1} \in \mathbb{Z}^{k+2}$. We emphasise that this is only possible as $H^2(\Sigma)$ is of odd order.

We now label the elements of $\mathrm{Char}^*(G)$ as follows:

$$K^1_{i,j} := (0, \ldots, 2_{(i)}, \ldots, 0, j) \text{ for } j \text{ odd, } 2 - k \leq j \leq k - 4$$

$$K^2_1 := (2, 0, \ldots, 0, 0, k - 2) \qquad\qquad K^2_2 := (0, \ldots, 0, 2, k - 2)$$

$$K^3_j := (0, \ldots, 0, j) \text{ for } j \text{ odd, } 2 - k \leq j \leq k$$

It will sometimes be convenient to write $K^1_{i,j}$ for the same type of covector as above, but with a value for j outside the range and parity specified. In this case we emphasise that it does not represent a maximiser useful for calculating the corresponding correction term. In a similar vein, we set

$$K^1_{0,j} := K^3_j.$$

To compute the correction terms, then, we need Q^{-1}. This calculation is surprisingly tractable so we present the result directly: $Q^{-1} = \frac{1}{k^2}(c_{ij})$, where

$$c_{ij} = \begin{cases} -i(k^2 - jk + 2j) & \text{if } i \leq j \leq k-1 \\ -2jk & \text{if } i = k, j \leq k \\ -jk & \text{if } i = k+1, j \leq k \\ -k^2 & \text{if } i = j = k+1 \\ -2j & \text{if } i = k+2, j \leq k \\ -k & \text{if } i = k+2, j = k+1 \\ -(k+2) & \text{if } i = j = k+2 \\ c_{ji} & \text{for all } i, j \end{cases}.$$

This in turn permits an explicit calculus of $K^2 = KQ^{-1}K^t$:

$$(K^1_{i,j})^2 = \begin{cases} -\frac{1}{k^2}(4i(k^2 - ik + 2i) + (k+2)j^2 + 8ij) & \text{for } i = 0, \ldots, k \\ -\frac{1}{k^2}(4k^2 + (k+2)j^2 + 4kj) & \text{for } i = k+1 \end{cases}$$

$$(K^2_i)^2 = \begin{cases} -(k+2) & \text{for } i = 1 \\ -\frac{1}{k^2}(k^3 + 6k^2 - 12k + 8) & \text{for } i = 2 \end{cases}.$$

The computation of $d(\Sigma, [K])$ is then trivial. From now on, we will write $d(\cdot, K)$ in place of $d(\cdot, [K])$ as the meaning is clear.

5.5.2 Correction Terms of $S^3_{-D/2}(U)$

In order to compute the deficiencies we wish to investigate, we need to repeat this procedure for the corresponding lens space L, which has an associated plumbing given by the linear graph H on two vertices, weighted $-n$ and -2 (recall $k^2 = 2n - 1 = 4s + 1$). The plumbing therefore has intersection form with matrix R_n, whose inverse is trivially

$$R_n^{-1} = -\frac{1}{k^2}\begin{pmatrix} 2 & 1 \\ 1 & n \end{pmatrix}.$$

We have already identified the relevant maximisers; their correspondence with $H^2(L)$ is given in Section 4 of [51].

Lemma 5.5.3. *The lens space L has characteristic covectors given by the map $\psi : \mathbb{Z}_D \longrightarrow \mathrm{Char}^*(H)$, defined below.*

$$\psi(i) = \begin{cases} (2i-1, 2) & 0 \le i \le s \\ (2i - 4s - 1, 0) & s+1 \le i \le 3s+1 \\ (2i - 8s - 3, 2) & 3s+2 \le i \le 4s \end{cases}. \tag{5.3}$$

Applying this lemma, the correction terms are

$$d(L,\psi(i)) = \begin{cases} -\frac{1}{k^2}(2i^2) & 0 \leq i \leq s \\ -\frac{1}{2k^2}((2i-k^2)^2 - k^2) & s+1 \leq i \leq 3s+1 \\ -\frac{1}{k^2}(2(k^2-i)^2) & 3s+2 \leq i \leq 4s \end{cases}.$$

5.5.3 First Application of Theorem 5.5.1

To compare our correction terms for Σ and L we will need the isomorphism $\varphi : \mathbb{Z}_D \to \mathrm{Char}^*(G)$ implicit in Theorem 5.5.1. Since φ was only implicit in our statement of the theorem, let us be clear what we are doing. Assuming that we have identified the labelling φ of $\mathrm{Spin}^c(\Sigma)$ required for Theorem 5.5.1, we will then have two labellings of Spin^c-structures: one, ψ, for $\mathrm{Spin}^c(L)$, the other, φ, for $\mathrm{Spin}^c(\Sigma)$. Theorem 5.5.1 tells us that

$$d(\Sigma,\varphi(i)) - d(\Sigma,\varphi(2s-i)) = d(L,\psi(i)) - d(L,\psi(2s-i)) \tag{5.4}$$

for $i = 0, 1, \ldots, s$. We want to show, by way of contradiction, that such a φ cannot exist when $k \geq 5$: if φ *did* exist, then we could pre-compose any other isomorphism $\phi : \mathbb{Z}_D \to \mathrm{Char}^*(G)$ with some automorphism of $\mathbb{Z}_D$ such that the equations (5.4) are satisfied. This automorphism must be multiplication by some ℓ coprime to k^2. That is, there must exist some ℓ such that

$$d(\Sigma,\phi(i\ell)) - d(\Sigma,\phi(2s\ell - i\ell)) = d(L,\psi(i)) - d(L,\psi(2s-i)). \tag{5.5}$$

To prove that $u(K) \geq 2$ when $k \geq 5$, we claim that no such ℓ exists, and it is in this direction that we proceed over the course of the following pages. As a first step, we must specify some ϕ by choosing a unit $K \in \mathrm{Char}^*(G)$ and setting $\phi(1) = K$. Since ϕ is an isomorphism, this choice will determine ϕ completely.

As we saw in (3.5), the kernel of α is generated by PD$[v]$. Thus, on observing that

$$K_1^2 = -2\sum_{i=1}^{k} \mathrm{PD}[v_i] - \mathrm{PD}[v_{k+1}] - \mathrm{PD}[v_{k+2}], \tag{5.6}$$

we see that K_1^2 is the zero element of $\mathrm{Char}^*(G)$. In order to find a unit in $\mathrm{Char}^*(G)$, we must find a K such that $m[K] = [K_1^2]$ if and only $k^2|m$. Equivalently, a K such that $(mK - K_1^2)Q^{-1} \in \mathbb{Z}^{k+2}$ if and only $k^2|m$. Setting $K = K_{1,-1}^1$, we have

$$mK - K_1^2 = (2(m-1), 0, \ldots, 0, -m - (k-2)),$$

and on computing the $(k+2)$-th co-ordinate of $(mK - K_1^2)Q^{-1}$ we find

$$((mK - K_1^2)Q^{-1})_{k+2} = \tfrac{1}{k^2}(k^2 - m(k-2)) \in \mathbb{Z}.$$

It follows that $k^2|m$ since k^2 and $k-2$ are coprime. Hence $K_{1,-1}^1$ is a unit.

Our choice of ϕ, which we now fix, is thus determined by

$$\phi(0) = K_1^2 \qquad\qquad \phi(1) = K_{1,-1}^1.$$

We are now ready to make our first applications of Theorem 5.5.1.

Proposition 5.5.4. *Suppose $P(k,-k,2)$ has unknotting number one. Then there exists an ℓ coprime to k^2 such that $\ell^2(3k-2) \equiv -8 \mod k^2$. Equivalently, the same ℓ satisfies*

$$\ell^2 \equiv 6k + 4 \mod k^2. \tag{5.7}$$

Proof. We observe that if $\phi(i) = [K]$, then $\phi(i\ell) = \ell\phi(i) = \ell[K] = [\ell K]$. Thus $\phi(i\ell)Q^{-1} \equiv \ell\phi(i)Q^{-1} \mod \mathbb{Z}$, and so $\phi(i\ell)^2 \equiv \ell^2\phi(i)^2 \mod \mathbb{Z}$. Consequently,

$$d(\Sigma, \phi(i\ell)) - d(\Sigma, \phi(2s\ell - i\ell)) \equiv \ell^2(d(\Sigma, \phi(i)) - d(\Sigma, \phi(2s - i))) \mod \mathbb{Z}. \tag{5.8}$$

It is now routine, using our previous calculation of L's correction terms, to compute the difference

$$d(L, \psi(0)) - d(L, \psi(2s)) = -\tfrac{1}{2k^2}(k^2 - 1).$$

and equally routine to find

$$\phi(2s) = \begin{cases} K^1_{k+1, -\frac{1}{2}(k+1)} & k \equiv 1 \mod 4 \\ K^3_{\frac{1}{2}(k-1)} & k \equiv 3 \mod 4 \end{cases},$$

from which we deduce that

$$d(\Sigma, \phi(0)) - d(\Sigma, \phi(2s)) = \begin{cases} \tfrac{1}{16k^2}(5k^3 - 3k + 2) & k \equiv 1 \mod 4 \\ \tfrac{1}{16k^2}(-3k^3 + 4k^2 - 3k + 2) & k \equiv 3 \mod 4 \end{cases}.$$

Applying (5.8) to (5.5) in the case when $i = 0$ and substituting in the above calculations, we find that we must have

$$-8(k^2 - 1) \equiv \begin{cases} \ell^2(5k^3 - 3k + 2) \mod k^2 & \text{if } k \equiv 1 \mod 4 \\ \ell^2(-3k^3 + 4k^2 - 3k + 2) \mod k^2 & \text{if } k \equiv 3 \mod 4 \end{cases},$$

which transforms into the equivalent statement (5.7) after a simple rearrangement (to make ℓ^2 the subject). □

One might think that by considering (5.5) modulo $\mathbb{Z}$ for any other value of i we could obtain a different congruence. Sadly, however, this is not true, and no further information is to be gained along these lines. However, even on its own, (5.7) cuts down the number of possible ℓ considerably. In the case that k is a prime power, for instance, it determines k up to sign (see next section).

5.5.4 Precise Applications of Theorem 5.5.1

The rest of the proof that $u(K) \geq 2$ for $k \geq 5$ follows the following line of reasoning. We show that no ℓ satisfying (5.7) can simultaneously satisfy (5.5) for $i = 0$ and r, where r is the residue of ℓ modulo k. This requires us to do the following:

1. Pinpoint the values of $\phi(2s\ell)$, $\phi(r\ell)$, and $\phi(2s\ell - r\ell)$, and compute their squares;
2. Compute the differences

$$Z(i) := d(\Sigma, \phi(i\ell)) - d(\Sigma, \phi(2s\ell - i\ell)) - d(L, \psi(i)) + d(L, \psi(2s - i))$$

for $i = 0, r$;

3. Obtain a good reason why $Z(0)$ and $Z(r)$ cannot simultaneously be zero for $k \geq 5$.

For the reader who does not like results plucked out of thin air, the following formulae are the tools used to compute the values of ϕ called for in the first step above.

Lemma 5.5.5. *We have the following equivalences:*

$$\textbf{\textit{(A):}}\ K^3_{J+kB} \sim K^1_{-B,J+2B} \qquad \textbf{\textit{(B):}}\ K^1_{I,J} \sim K^1_{I+1,J+k-2} \qquad \textbf{\textit{(C):}}\ K^1_{I,J} \sim K^1_{I,J+k^2}.$$

where $B \leq 0$ and J are arbitrary integers.

Proof. This is an easy calculation: simply verify that $(K - K')Q^{-1} \in \mathbb{Z}^{k+2}$, for the above K, K'. □

As mentioned before, if k is a prime power then there is an essentially unique choice of ℓ, but the situation becomes much more complicated if k has several different prime factors. To deal with this complexity, we introduce some auxiliary notation.

Proposition 5.5.6. *Let $\ell = ak + r$, where $0 \leq a < k$ and $0 < r < k$. Then we can choose r even and set $r^2 = Ak + 4$, where*

$$A + 2ar \equiv 6 \mod k, \tag{5.9}$$

and $0 \leq r - A < \frac{k}{4} + 1$.

Proof. Since $\pm\ell$ have the same effect on the correction terms, and k is odd, one of $\pm\ell$ will have even r and we make this choice. Notice that as ℓ is coprime to k, we cannot have $r = 0$.

From (5.7), $\ell^2 \equiv 6k + 4 \mod k^2$. However, it is also clear that $\ell^2 \equiv 2ark + r^2 \mod k^2$. Comparing these expressions, we obtain the desired congruence (5.9).

For the inequality, we have $r - A = r - \frac{r^2-4}{k}$. By considering this quadratic in the range from 0 to k, we find it is always positive, maximises when $r = \frac{k}{2}$, and has maximum $\frac{k}{4} + \frac{4}{k}$. Since $r - A$ is an integer, and as $k \geq 5$, the upper bound follows. □

Proposition 5.5.7. *In the case that k is a prime power, then $r = 2$, $A = 0$, and $a = \frac{k+3}{2}$.*

Proof. This is a direct calculation using (5.7) and the observation that when k is prime power, square roots modulo k are unique up to sign. □

In order to carry out our prescribed programme, we now have to branch into several different cases. Because the condition that r be even implies nothing about a, and because the parity of a becomes important in what follows, we divide the rest of our proof into two sections: a even and a odd. Before we do so, however, we remark that one value of ϕ called for in Step 1 is independent of a.

Proposition 5.5.8. *For $k \geq 5$,*

$$\phi(r\ell) = -K^1_{\frac{A}{2},k-4-A}.$$

Proof. By direct verification. Check that $(-r\ell K^1_{1,-1} - K^1_{\frac{A}{2},k-4-A})Q^{-1} \in \mathbb{Z}^{k+2}$, which is easy. □

Strictly speaking, we want to compute $d(\Sigma, \phi(i))$, but because of the conjugation symmetry $d(Y, \phi(i)) = d(Y, \phi(-i))$ (see the first part of Proposition 3.3.4), we will sometimes in fact compute $\phi(-i)$ instead of $\phi(i)$. In order to streamline our notation, we will write $\phi(i) = -K$ to mean $\phi(-i) = K$, just as we have done above.

The Case a Even

According to Step 1, we must now compute the values of $\phi(2s\ell)$ and $\phi(2s\ell - r\ell)$. This is done in the following two propositions. For the interested reader, these calculations were performed originally by assuming that K had the form K^3_j, and then applying Lemma 5.5.5 until the subscripts fitted the required conditions.

Proposition 5.5.9 ($r \equiv 2 \mod 4$)**.** *If $r \equiv 2 \mod 4$ and a is even, then we define parameters $B := 1 + \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \in (-\frac{k}{2}, \frac{k}{2})$ and $J := -\frac{r}{2} - 4 < 0$. These give*

$$\phi(2s\ell - r\ell) = \begin{cases} K^1_{-B,J+2B} & \text{if } B \leq 0, J + 2B > -k \\ K^1_{2-B,J+2B+2k-4} & \text{if } B \leq 0, J + 2B \leq -k \\ -K^1_{B,-J-2B} & \text{if } B \geq 0 \end{cases},$$

and also

$$\phi(2s\ell) = \begin{cases} K^1_{\frac{a-r}{2},\frac{r}{2}-a} & \text{if } a \geq r \\ -K^1_{\frac{r-a}{2},a-\frac{r}{2}} & \text{if } a \leq r \end{cases}.$$

Proposition 5.5.10 ($r \equiv 0 \mod 4$)**.** *If, on the other hand, $r \equiv 0 \mod 4$, then we instead define $B := \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \in (-\frac{k}{2}, \frac{k}{2})$ and $J := k - \frac{r}{2} - 4 > 0$, giving*

$$\phi(2s\ell - r\ell) = \begin{cases} K^1_{-B,J+2B} & \text{if } B \leq 0 \\ -K^1_{B,-J-2B} & \text{if } B \geq 0, J + 2B < k \\ -K^1_{B+2,-J-2B+2k-4} & \text{if } B \geq 0, J + 2B \geq k \end{cases},$$

and also

$$\phi(2s\ell) = \begin{cases} K^1_{\frac{a-r+2}{2},\frac{r}{2}-a+k-2} & \text{if } a \geq r - 2 \\ -K^1_{\frac{r-a-2}{2},a-\frac{r}{2}-k+2} & \text{if } \frac{r}{2} \leq a \leq r - 2 \\ -K^1_{\frac{r-a+2}{2},a-\frac{r}{2}+k-2} & \text{if } a < \frac{r}{2} \end{cases}.$$

Proof (of both propositions). This is a straightforward verification. To perform it, one need only check that $(mK^1_{1,-1} - K)Q^{-1} \in \mathbb{Z}^{k+2}$ for the right choices of m and

Case	r (mod 4)	Conditions	$16k^2Z(r)$
A	2	$B \leq 0$ $J + 2B > -k$	$(4kr + 8k^2)A + (2 - 3k)r^2 + ((4a - 24)k - 8k^2)r$ $- 4k^3 + (8a + 16)k^2 + 32ak - 8$
B	2	$B \leq 0$ $J + 2B \leq -k$	$(4kr - 8k^2)A + (2 - 3k)r^2 + (4a - 24)kr$ $+ 12k^3 + (-8a - 16)k^2 + 32ak - 8$
C	2	$B \geq 0$	$(4kr - 8k^2)A + (2 - 3k)r^2 + ((4a - 24)k + 8k^2)r$ $- 4k^3 + (-8a + 48)k^2 + 32ak - 8$
D	0	$B \leq 0$	$4Akr + (2 - 3k)r^2 + ((4a - 24)k - 4k^2)r$ $+ 8k^2 + 32ak - 8$
E	0	$B \geq 0$ $J + 2B < k$	$(4kr - 16k^2)A + (2 - 3k)r^2 + ((4a - 24)k + 12k^2)r$ $+ (-16a + 8)k^2 + 32ak - 8$
F	0	$B \geq 0$ $J + 2B \geq k$	$4Akr + (2 - 3k)r^2 + ((4a - 24)k + 4k^2)r$ $+ 72k^2 + 32ak - 8$

Table 5.3: Computation of $Z(r)$ when a is even.

Case	r (mod 4)	Conditions	$16k^2Z(0)$
1	2	$a \geq r$	$(2 - 3k)r^2 + (4ak - 8k^2)r - 4k^3 + 8ak^2 - 8$
2	2	$a \leq r$	$(2 - 3k)r^2 + (4ak + 8k^2)r - 4k^3 - 8ak^2 - 8$
3	0	$a \geq r - 2$	$(2 - 3k)r^2 + (4ak - 4k^2)r + 8k^2 - 8$
4	0	$\frac{r}{2} \leq a \leq r - 2$	$(2 - 3k)r^2 + (4ak + 12k^2)r - (16a + 24)k^2 - 8$
5	0	$a < \frac{r}{2}$	$(2 - 3k)r^2 + (4ak + 4k^2)r + 8k^2 - 8$

Table 5.4: Computation of $Z(0)$ when a is even.

K listed above, using (5.9). The numerous cases occur to fit the various constraints imposed on i, j in $K^1_{i,j}$; the parity of $\frac{r}{2}$ is relevant because j must be odd. □

This completes Step 1. The next step is to compute $Z(i)$ for $i = 0, r$. As this is straightforward, if tedious, we present the results in Tables 5.3 and 5.4.

Proposition 5.5.11. *If a is even, then no ℓ exists which ensures that $Z(r) = Z(0) = 0$.*

Proof. The idea is to show that the $Z(r) = 0$ equation in case α is incompatible with the $Z(0) = 0$ equation in case β (for appropriate choices of α and β). If both the α and β equations are satisfied, then we should have

$$Z(r) \pm Z(0) = 0.$$

We compare cases $\alpha = A, B, C$ with cases $\beta = 1, 2$ (six combinations), and cases

$\alpha = D, E, F$ with cases $\beta = 3, 4, 5$ (nine more combinations). In each of these combinations, both of the new equations generally involve A, a, and r, so obtaining contradictions can be difficult. The following method, however, is generally useful:

1. Cancel sufficient common factors from all the terms;
2. Substitute $A = \frac{r^2-4}{k}$;
3. Solve the $Z(r) + Z(0) = 0$ equation for a and substitute the solution into the $Z(r) - Z(0) = 0$ equation. Since the "+" equation is generally linear in a, and the coefficient of a is generally non-vanishing, this task is not difficult, and yields a new equation $f_{\alpha,\beta}(r) = 0$ which we must satisfy;
4. Find an argument establishing that $f_{\alpha,\beta}(r)$ is positive or negative over the range $2, 4 \leq r < k$. The choice of 2 or 4 depends on the minimum value of r allowed by α and β;
5. Hence, conclude that α, β are not compatible.

We illustrate the procedure once, then summarise the relevant $f_{\alpha,\beta}$ in different cases. Take $\alpha = A$ and $\beta = 1$. Cancelling terms, we obtain:

$$\begin{aligned} Z(r) + Z(0) = 0 &= (2r + k + 2)(r^2 - 4) - (8k + 12)kr - 4k^3 \\ &\quad + 8k^2 - 12k + 4ak(r + 2k + 4) \\ Z(r) - Z(0) = 0 &= (r + 2k)(r^2 - 4) - 6rk + 4k^2 + 8ak. \end{aligned}$$

Since the coefficient of a in the "+" equation is non-zero, we solve it for a and substitute into the "−" equation, giving

$$f_{A,1}(r) = r^4 + 4kr^3 + (4k^2 - 8)r^2 + (8k^2 - 16k)r + 16k^3 - 16k^2 + 16 = 0.$$

Since $k \geq 5$, the coefficients of r are all positive, whence $f_{A,1}(r) > 0$ on $0 < r < k$. This is the contradiction we require.

In a similar vein, Table 5.5 summarises the data for the other cases. We attack them one by one.

α, β	$f_{\alpha,\beta}(r)$
$A, 1$	$r^4 + 4kr^3 + (4k^2 - 8)r^2 + (8k^2 - 16k)r + 16k^3 - 16k^2 + 16$
$B, 1$	$r^4 - (5k^2 - 2k + 8)r^2 + 4k^4 + 16k^2 - 8k + 16$
$C, 1$	$r^4 - (3k^2 - 2k + 8)r^2 + 16k^2 r - 4k^4 + 32k^3 + 16k^2 - 8k + 16$
$A, 2$	$r^4 - (5k^2 + 2k + 8)r^2 + 4k^4 + 16k^2 + 8k + 16$
$B, 2$	$r^4 - 4kr^3 + (2k^2 - 8)r^2 + (8k^3 - 8k^2 + 16k)r - 8k^4 + 16k^3 - 16k^2 + 16$
$C, 2$	$r^4 - 4kr^3 + (4k^2 - 8)r^2 + (8k^2 + 16k)r - 16k^3 - 16k^2 + 16$
$D, 3$	$r^4 - 8r^2 + 8k^2 r - 16k^2 + 16$
$E, 3$	$r^4 - 4kr^3 + (k^2 + 2k - 8)r^2 - (4k^3 - 8k^2 - 16k)r + 8k^3 - 16k^2 - 8k + 16$
$F, 3$	$r^4 + (2k^2 - 8)r^2 + 24k^2 r - 16k^2 + 16$
$D, 4$	$r^4 - 4kr^3 - (k^2 + 2k + 8)r^2 + (4k^3 + 8k^2 + 16k)r - 8k^3 + 48k^2 + 8k + 16$
$E, 4$	$r^4 - 8kr^3 + (16k^2 - 8)r^2 + (8k^2 + 32k)r - 32k^3 - 16k^2 + 16$
$F, 4$	$r^4 - 4kr^3 + (k^2 - 2k - 8)r^2 - (4k^3 - 24k^2 - 16k)r - 72k^3 + 48k^2 + 8k + 16$
$D, 5$	$r^4 - (2k^2 + 8)r^2 - 8k^2 r - 16k^2 + 16$
$E, 5$	$r^4 - 4kr^3 - (k^2 - 2k + 8)r^2 + (4k^3 - 8k^2 + 16k)r + 8k^3 - 16k^2 - 8k + 16$
$F, 5$	$r^4 - 8r^2 + 8k^2 r - 16k^2 + 16$

Table 5.5: The functions $f_{\alpha,\beta}$ when a is even.

A1 Already done.

B1, A2 In both situations, $f_{\alpha,\beta}(r) = r^4 - Nr^2 + M$. The turning points of this quartic occur when $r = 0$ or $r^2 = \frac{N}{2}$, so provided that $\frac{N}{2} \geq k^2$, we know that $f_{\alpha,\beta}(r)$ is decreasing on $0 < r < k$. As it happens, $\frac{N}{2}$ is indeed greater than k^2; since also

$$f_{\alpha,\beta}(k-1) = \begin{cases} 8k^3 + 5k^2 + 6k + 9 & \text{if } \alpha = B, \beta = 1 \\ 4k^3 + 13k^2 + 18k + 9 & \text{if } \alpha = A, \beta = 2 \end{cases},$$

we see that $f_{\alpha,\beta}(r) > 0$ on $0 < r < k$, as required.

C1 $f_{C,1}$ is not obviously useful, but since $1 + \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \geq 0$ (by case C) and $a \geq r$ (by case 1), we are forced to conclude that $A = 0$. However, then $r = 2$ and $a = \frac{k+3}{2}$ by direct computation, and the condition from case C fails. Contradiction.

B2 We aim to show that $f(r) := f_{B,2}(r) < 0$ on $0 < r < k$ when $k \geq 7$. We begin with its derivatives:

$$\begin{aligned}\frac{df}{dr}(r) &= 4r^3 - 12kr^2 + (4k^2 - 16)r + (8k^3 - 8k^2 + 16k)\\ \frac{d^2f}{dr^2}(r) &= 12r^2 - 24kr + (4k^2 - 16)\\ \frac{d^3f}{dr^3}(r) &= 24r - 24k.\end{aligned}$$

As we can see, $\frac{d^3f}{dr^3}(r) < 0$, whence $\frac{d^2f}{dr^2}$ is decreasing. Noticing that $\frac{d^2f}{dr^2}(0) = 4k^2 - 16 > 0$ while $\frac{d^2f}{dr^2}(k) = -8k^2 - 16 < 0$, we see that $\frac{d^2f}{dr^2}$ has precisely one zero in the range $0 < r < k$. Hence, $\frac{df}{dr}$ has precisely one turning point in that range, a maximum by the negativity of $\frac{d^3f}{dr^3}$. Checking at both extremes of the range, we find that $\frac{df}{dr}(r) > 0$, and so f is increasing. However,

$$f(k) = -k^4 + 8k^3 - 8k^2 + 16$$

is negative for $k \geq 7$, so $f_{B,2}(r) = f(r) < 0$ on the range prescribed. If $k = 5$, then $r = 2$ by Proposition 5.5.7, and direct computation finds $f_{B,2}(2) < 0$.

C2 We play around with the "$-$" equation:

$$2k = r + \tfrac{8a}{A-6}.$$

Since $\frac{r}{2}$ is odd, we know that $\frac{A}{4}k + 1 = \frac{r^2}{4} \equiv 1 \mod 4$, and so $A \equiv 0 \mod 16$. Thus, if $A \geq 16$, then $2k \leq r + \frac{4}{5}a < 2k$, which is nonsense. If instead $A = 0$, then we find that $2k = r - \frac{4}{3}a < 2k$, which is also nonsense.

D3, F5 Write

$$f_{D,3}(r) = (r^4 - 8r^2) + (8k^2r - 16k^2 + 16).$$

The two bracketed expressions are both positive once $r \geq 4$, and since $r \equiv 0 \mod 4$, it is clear that $r = 4$ is the smallest value for r allowed. Hence we have our contradiction.

E3, F3 From condition 3 we know that $B = \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \leq 1 - \frac{A}{2} < 0$ unless $A = 0$.

This contradicts conditions E and F. However, if $A = 0$, then $r = 2$, and we violate the condition that $r \equiv 0 \mod 4$.

D4 Write

$$f_{D,4}(r) = \underbrace{(r^4 - 4kr^3 - k^2r^2 + 4k^3r - 8k^3)}_{g(r)} + (8k^2r - 2kr^2) + (48k^2 - 8r^2) + 16kr + 8k + 16,$$

and observe that except possibly $g(r)$, all the terms are positive. We aim to show that $g(r) > 0$ on the range $2 < r < k$ provided $k \geq 7$. Indeed, consider its derivatives:

$$\frac{dg}{dr}(r) = 4r^3 - 12kr^2 - 2k^2r + 4k^3$$
$$\frac{d^2g}{dr^2}(r) = 12r^2 - 24kr - 2k^2.$$

The second derivative is clearly negative on $0 < r < k$, and so $\frac{dg}{dr}$ is decreasing on our range of interest. From $\frac{dg}{dr}(0) = 4k^3 > 0$ and $\frac{dg}{dr}(k) = -6k^3 < 0$, we deduce that there is precisely one zero to $\frac{dg}{dr}(r)$ on $0 < r < k$. That is, that g has precisely one turning point; since $\frac{d^2g}{dr^2}(r) < 0$, it is a local maximum. We compute:

$$g(4) = 8k^3 - 16k^2 - 256k + 256 \qquad g(k-2) = 4k^3 - 28k^2 + 16.$$

When $k \geq 7$, both values are positive, so $g(r)$ is positive over the range $4 \leq r \leq k - 2$. Thus, because the requirements that $r \equiv 0 \mod 4$ and $r^2 \equiv 4 \mod k$ imply that we need not consider $r = 2, k - 1$, this range is sufficient to obtain our contradiction. If, on the other hand, $k = 5$, then Proposition 5.5.7 tells us that $r = 2$, $A = 0$, and $a = 4$, violating condition 4.

E4 Rearrange the "−" equation to obtain

$$4k = r - \tfrac{4}{A-2}(r - 2a).$$

We already know from condition 4 that $a \leq r-2$, whence $4k \leq r + \frac{4k}{A-2} < 3k$ if $A \neq 0$, since $A \equiv 0 \mod 4$. If $A = 0$, then $a = \frac{k+3}{2} > 0 = r-2$, a contradiction.

F4 Write

$$\begin{aligned} f_{F,4}(r) = (r^4 + k^2r^2 - 2k^3r) - 4kr^3 - (2k+8)r^2 \\ - (2k^3 - 24k^2 - 16k)r - (72k^3 - 48k^2 - 8k - 16). \end{aligned}$$

Once $k \geq 13$, all the bracketed terms are negative. For $k < 13$, we obtain contradictions to conditions F and 4 by way of Proposition 5.5.7 (since k must be prime power).

D5 Write

$$f_{D,5}(r) = (r^4 - 2k^2r^2 - 8r^2) - 8k^2r - (16k^2 - 16),$$

and note that all bracketed terms are negative.

E5 Write

$$f_{E,5}(r) = g(r) + (2k-8)r^2 + (8k^3 - 8k^2r + 16kr) + (8k^3 - 16k^2 - 8k + 16),$$

where $g(r)$ is as in case D4, and all bracketed terms are positive if $k \geq 7$. If $k = 5$, we are not in this case by Proposition 5.5.7.

Now that all possible combinations have been exhausted, we have finished our proof. □

The Case a Odd

We now repeat all the work of the previous section when a is an odd integer. As this is naturally a very similar process, we omit those proofs which are virtually identical, beginning with the proofs of the following two results.

Proposition 5.5.12 ($r \equiv 2 \mod 4$)**.** *If* $r \equiv 2 \mod 4$ *and* a *is odd, then we define*

Case	r (mod 4)	Conditions	$16k^2 Z(r)$
A	2	$J+2B<k$	$(4kr-8k^2)A+(2-3k)r^2+((4a-24)k+4k^2)r + 4k^3+(-8a+16)k^2+32ak-8$
B	2	$J+2B\geq k$	$(4kr+8k^2)A+(2-3k)r^2+((4a-24)k-4k^2)r + 4k^3+(8a+48)k^2+32ak-8$
C	0	$J+2B<k$	$(4kr-16k^2)A+(2-3k)r^2+((4a-24)k+8k^2)r + 16k^3+(-16a-24)k^2+32ak-8$
D	0	$J+2B\geq k$	$4Akr+(2-3k)r^2+(4a-24)kr+40k^2+32ak-8$

Table 5.6: Computation of $Z(r)$ when a is odd.

parameters $B := 1+\frac{r}{2}-\frac{a-k}{2}-\frac{A}{2}\in[0,k-1)$ *and* $J := -\frac{r}{2}-4<0$, *giving*

$$\phi(2s\ell - r\ell) = \begin{cases} -K^1_{B,-J-2B} & \text{if } J+2B<k \\ -K^1_{B+2,-J-2B+2k-4} & \text{if } J+2B\geq k \end{cases},$$

and also

$$\phi(2s\ell) = \begin{cases} K^1_{\frac{a-r+k}{2},\frac{r}{2}-a-k} & \text{if } r>2a \\ K^1_{\frac{a-r+k}{2}+2,\frac{r}{2}-a+k-4} & \text{if } r\leq 2a \end{cases}.$$

Proposition 5.5.13 ($r\equiv 0 \mod 4$)**.** *If, on the other hand,* $r\equiv 0 \mod 4$, *then we instead define* $B := \frac{r}{2}-\frac{a-k}{2}-\frac{A}{2}\in[0,k-1)$ *and* $J := k-\frac{r}{2}-4>0$, *giving*

$$\phi(2s\ell - r\ell) = \begin{cases} -K^1_{B,-J-2B} & \text{if } J+2B<k \\ -K^1_{B+2,-J-2B+2k-4} & \text{if } J+2B\geq k \end{cases},$$

and lastly

$$\phi(2s\ell) = K^1_{\frac{a-r+k}{2}+1,\frac{r}{2}-a-2}.$$

With these computed, we next construct the same tables as before. Table 5.6 shows the values of $Z(r)$, while Table 5.7 shows the values of $Z(0)$. We use this data to obtain the following proposition.

Proposition 5.5.14. *If a is odd, then no ℓ exists which ensures that* $Z(r)=Z(0)=0$.

Proof. Exactly as before, we construct functions $f_{\alpha,\beta}$; they are summarised in Table 5.8. The consequent case-by-case analysis is outlined below.

Case	r (mod 4)	Conditions	$16k^2Z(0)$
1	2	$r > 2a$	$(2-3k)r^2 + (4ak - 4k^2)r + 4k^3 + 8ak^2 - 8$
2	2	$r \le 2a$	$(2-3k)r^2 + (4ak + 4k^2)r + 4k^3 - 8ak^2 - 8$
3	0	$-$	$(2-3k)r^2 + 4akr + 8k^2 - 8$

Table 5.7: Computation of $Z(0)$ when a is odd.

α, β	$f_{\alpha,\beta}(r)$
$A, 1$	$r^4 - (5k^2 - 2k + 8)r^2 + 4k^4 + 16k^2 - 8k + 16$
$B, 1$	$r^4 + 4kr^3 + (4k^2 - 8)r^2 + (8k^2 - 16k)r + 16k^3 - 16k^2 + 16$
$A, 2$	$r^4 - 4kr^3 + (4k^2 - 8)r^2 + (8k^2 + 16k)r - 16k^3 - 16k^2 + 16$
$B, 2$	$r^4 - (3k^2 + 2k + 8)r^2 + 16k^2r - 4k^4 - 32k^3 + 16k^2 + 8k + 16$
$C, 3$	$r^4 - 4kr^3 - (k^2 - 2k + 8)r^2 + (4k^3 - 8k^2 + 16k)r + 8k^3 - 16k^2 - 8k + 16$
$D, 3$	$r^4 - 8r^2 + 8k^2r - 16k^2 + 16$

Table 5.8: The functions $f_{\alpha,\beta}$ when a is odd.

A1 Since $f_{A,1}$ has the same structure as cases A2 and B1 from the previous section, and since $f_{A,1}(k-1) = 8k^3 + 5k^2 + 6k + 9 > 0$, we are done.

B1 All the coefficients of r in $f_{B,1}(r)$ are clearly positive.

A2 The "$-$" equation tells us that

$$k = \tfrac{4a-2r}{A-2} + \tfrac{r}{2},$$

and since $A \equiv 0 \mod 16$ (c.f. case C2 in the previous section), we discover, with the exception of the case when $A = 0$, that $k < \frac{1}{7}(2a - r) + \frac{r}{2} < \frac{11}{14}k$, a contradiction. If instead $A = 0$, we see $k = r - 2a + \frac{r}{2}$, which is also a contradiction since $r \le 2a$ by condition 2.

B2 Condition B implies that $J + 2B \ge k$, so $\frac{r}{2} - a - A - 2 \ge 0$. However, from condition 2, we know that $r \le 2a$. Contradiction.

C3 Write

$$f_{C,3}(r) = g(r) + (2k-8)r^3 + (8k^3 - 8k^2r + 16kr) + (8k^3 - 16k^2 - 8k + 16),$$

where $g(r)$ is the same function as in case D4 from the previous section. It is then clear that all the terms are positive for $k \geq 7$, and if $k = 5$ we use Proposition 5.5.7 to obtain the usual contradiction (namely, that one of the conditions is violated).

D3 Identical to cases D3 and F5 from the previous section.

With all cases resolved, the proof is complete. □

The Proof of Theorem 5.4.1

For any m, it is clear that $k = 1$ yields the unknot. Otherwise, Lemma 5.4.2 tells us that $m = 1$, and from the previous two sections we know that $k \geq 5$ implies that $u(P(k, -k, 2)) \neq 1$. Since $u(P(3, -3, 2)) = 1$, realised by changing any crossing in the central column of twists, the theorem is proved.

5.5.5 Examples

To illustrate the above working, we focus on the case when k is prime power. Recall from Proposition 5.5.7 that there is an essentially unique ℓ, and that a is even or odd according to the congruence of k modulo 4 (cases A1 and A2 respectively). Explicitly, we have:

$$\phi(2\ell) = K^3_{k-4} \qquad \phi((2s-2)\ell) = \begin{cases} -K^3_k & k = 5 \\ K^1_{\frac{1}{4}(k-5), -\frac{1}{2}(k+5)} & k > 5 \text{ and } k \equiv 1 \mod 4 \\ -K^1_{\frac{1}{4}(k+5), -\frac{1}{2}(k-5)} & k \equiv 3 \mod 4 \end{cases},$$

which implies that

$$d(\Sigma, \phi(2\ell)) = -\tfrac{1}{k^2}(-2k^2 + 8) \qquad d(\Sigma, \phi((2s-2)\ell)) = -\tfrac{1}{2k^2}(-k^2 + 25).$$

Grinding all this into (5.5), we should find $Z(2) = 0$, but instead obtain

$$Z(2) = \tfrac{1}{2k^2}(3k^2 + 9) + \tfrac{1}{2k^2}(k^2 - 9) = 2.$$

This tells us that $P(k,-k,2)$ does not have unknotting number one when $k \geq 5$.

We can see this even more concretely if we set $k = 5$. The correction terms for L in this case are:

$$d(L,\psi(i)) = (0, -\tfrac{2}{25}, -\tfrac{8}{25}, -\tfrac{18}{25}, -\tfrac{32}{25}, -2, -\tfrac{72}{25}, -\tfrac{48}{25}, -\tfrac{28}{25}, -\tfrac{12}{25}, 0, \tfrac{8}{25}, \tfrac{12}{25}, \dots).$$

We have only presented the first half since $d(\cdot,\psi(i)) = d(\cdot,\psi(-i))$. On the other hand, labelling Spin^c-structures according to our isomorphism ϕ, the correction terms of Σ are:

$$d(\Sigma,\phi(i')) = (0, \tfrac{22}{25}, -\tfrac{12}{25}, -\tfrac{2}{25}, \tfrac{2}{25}, 0, \tfrac{42}{25}, \tfrac{28}{25}, \tfrac{8}{25}, -\tfrac{18}{25}, 0, \tfrac{12}{25}, \tfrac{18}{25}, \dots).$$

On solving (5.7), we find that $\ell = \pm 3$. As per the approach above, we let $\ell = 22$, and check that we indeed have $r = 2$, $A = 0$, and $a = \frac{k+3}{2} = 4$.

$$d(\Sigma,\phi(22i')) = (0, -\tfrac{2}{25}, \tfrac{42}{25}, -\tfrac{18}{25}, \tfrac{18}{25}, 0, \tfrac{28}{25}, \tfrac{2}{25}, \tfrac{22}{25}, -\tfrac{12}{25}, 0, \tfrac{8}{25}, \tfrac{12}{25}, \dots).$$

Multiplying these numbers by 25, we can tabulate the corresponding sides of (5.5):

i	$\Sigma(k,-k,2)$	$-L(k^2,2)$
0	-12	-12
1	-10	-10
2	42	-8
3	-6	-6
4	-4	-4
5	-2	-2
6	0	0

Notice that the two columns are congruent modulo 25, but not equal, implying that $u(P(5,-5,2)) \neq 1$. Notice also the explicit failure of $Z(2) = 0$, and that the correct value is indeed $Z(2) = 2$.

For those who wish to compare this with Theorem 1.1 of [51], our choice of ℓ produces a positive, even matching. This matching, however, fails to be symmetric.

5.6 Concluding Remarks

Over the course of this lengthy chapter, we should hopefully have illustrated the point that Theorem 4.1.3 provides a clean and efficient obstruction to rational surgeries in S^3. Even a superficial comparison based on page numbers should show that it is faster than its competitors, the Alexander module and Theorem 4.1.1. However, given the fact that it involves gluing two 4-manifolds together, we have also seen that incompatibilities on their boundaries may render it inapplicable. In these instances, we have seen that the natural fallback, Theorem 4.1.1, though at times difficult to apply, may yield additional results, especially when dealing with L-spaces.

In addition to these points, we have also seen that the classification of pretzel knots with unknotting number one remains incomplete. Specifically, some entirely new method will be required to make any further progress on the cases when $p+q=4$ and $\det K=3$, and $p+q=2$ and $\det K=1$; in these circumstances, neither Theorem 4.1.3 nor Theorem 4.1.1 provide us with any useful information. Other tools, including the classical invariants (such as Lickorish's linking form and the Alexander module), bounds on the 4-ball genus (such as the Rasmussen s- and Ozsváth and Szabó τ-invariants), and mutation (which leaves the property $u(K)=1$ invariant, see [23]) have proved equally unhelpful in the author's personal investigations. It remains yet to be seen if the ideas outlined in the next chapter, specifically Section 6.3, will prove fruitful.

CHAPTER 6: FURTHER APPLICATIONS

In this final chapter, we present a summary of some future applications for Theorem 4.1.3. Since this is largely speculative, we will omit most of the proofs in favour of describing the general ideas.

6.1 The Seifert Fibred Space Realisation Problem

As mentioned in the Introduction, Theorem 4.1.3 is a generalisation of Theorem 1.6 from [26], used by Greene in the same paper to solve the lens space realisation problem. That is, the problem of determining which lens spaces can be obtained by p/q-Dehn surgery on non-trivial knots $C \subset S^3$. In his approach, he benefited from the cyclic surgery theorem, which allowed him to assume that $q = 1$.

Now that we have generalised this technology away from the assumption $q = 1$, and to spaces much more general than lens spaces or even L-spaces, we are well poised to take on a similar problem concerning small Seifert fibred spaces. The following are reasons for encouragement:

1. We no longer have to worry about whether or not our small Seifert fibred space is an L-space. Indeed, the vast majority are not;
2. A large number of small Seifert fibred spaces are negative-definite. Of these, most have associated plumbing diagrams with at most one overweight vertex (the central node). This gives us not only an appropriate 4-manifold X with our small Seifert fibred space as boundary, but also a good algorithm for determining if any Spin-deficiencies vanish; and

3. We have already investigated a class (albeit a small class) of small Seifert fibred spaces arising as the double covers of 3-strand pretzel knots and found Theorem 4.1.3 quite successful at proving that they cannot be obtained by half-integral surgeries.

In light of the above, we propose Theorem 4.1.3 (and Theorem 4.1.1 if absolutely necessary) as a tool for tackling the following problem.

Question 2. *Which of the small Seifert fibred spaces that bound negative-definite smooth 4-manifolds can be obtained by p/q-surgery on a knot $C \subset S^3$, where $p/q > 0$? Is it necessarily true that if C is a hyperbolic knot, then $q = 1$?*

The latter question here is a special case of a general conjecture (see Conjecture 4.8 in [21]) that any non-integral surgery that yields a small Seifert fibred space must be performed on a non-hyperbolic knot.

It is also possible that Theorem 4.1.3 could be modified to provide information on $HFK(S^3, C)$, where C is a knot with small Seifert fibred surgeries. This is another of the remarkable achievements in Greene's paper [26]: he was able to prove that the knot Floer homologies of lens space knots match precisely with the knot Floer homologies of the Berge knots. However, as we have not yet generalised the theorems underlying this part of his paper, any analogous undertaking in the small Seifert fibred case would likely prove considerably more difficult than a straightforward resolution to Question 2.

6.2 Generalised Unknotting Operations

The notion of an unknotting operation is somewhat arbitrary. Why, for instance, are we only allowed to change crossings? Why can we not perform more complicated manoeuvres? One answer, which until recently has been amply borne out in a lot of examples, is that the standard unknotting operation is difficult enough. However, now that we have a computable obstruction to more general rational surgeries, there is no reason not to consider applying it to more general operations.

Definition 6.2.1 (Rational tangle replacement)**.** *Let T_1 and T_2 be rational tangles with associated extended rational numbers r_1 and r_2, and let K be a knot. Then we define a* (r_1, r_2)-(rational tangle) replacement *on K as the operation which locates a copy of T_1 in K and replaces it with T_2.*

Comparing this with the standard unknotting operations, it is not too hard to convince oneself that the standard unknotting operations are not only $(+1, -1)$- and $(-1, +1)$-replacements, but also $(0, \pm 2)$- and $(\pm 2, 0)$-replacements. Indeed, it has been proved in [13] that any (r_1, r_2)-replacement is equivalent to some $(0, r)$-replacement, which we shall from now on refer to as an r-replacement. We define $u_r(K)$ to be the minimal number of r-replacements required to turn K into the unknot.

Now, suppose that we define the distance $\Delta(\cdot, \cdot)$ between elements of $\mathbb{Q}^*$ by

$$\Delta\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) := |p_1 q_2 - p_2 q_1| \,.$$

By analogy with the Montesinos theorem, it can be shown that if K' is obtained from K by an (r_1, r_2)-replacement, then $\Sigma(K) = \Sigma(K')_{\pm D/\Delta(r_1, r_2)}(C)$ for some knot $C \subset \Sigma(K)$ and some $D \in \mathbb{Z}$ coprime to $\Delta(r_1, r_2)$. Applying this in the case when K can be unknotted by a p'/q'-replacement, we find $\Sigma(K) = S^3_{\pm D/p'}(C)$, where $D = \det K$.

The significance of this is the following. When K is alternating, there is a well-established method for constructing a sharp, simply connected, negative-definite smooth 4-manifold X with $\Sigma(K)$ as boundary (see Section 3 of [51], which references proofs in [52], as well as the latter half of Section 2 in [24]); moreover, the mirror image $\overline{K}$ is also alternating, and thus shares the same property. It is shown in Section 4.2 of [24] (via Theorem 1.2 of [38]) that Proposition 4.5.4 is always satisfied. Hence, if we wish to prove that $u_r(K) > 1$, where $r = p'/q'$, then we can apply Theorem 4.1.3 twice, the first time to $Y = \Sigma(K)$ and the second time to $Y = \Sigma(\overline{K})$; if the matrix A fails to exist in either situation, then we will have obstructed $\Sigma(K) = S^3_{\pm D/p'}(C)$, and thus $u_r(K) = 1$.

This obstruction, while certainly interesting and useful mathematically, is of particular interest in biology. We refer the interested reader to the Appendix.

6.3 A New Obstruction to Unknotting Number One

As has already been remarked, the Montesinos theorem underlies a huge number of the existing obstructions to unknotting number one, so it is worth considering whether it has any generalisations to higher unknotting numbers. Indeed, applying Theorem 1.5.5 multiple times, it seems credible that if $u(K) = r$, then $\Sigma(K)$ can be obtained by half-integral surgeries on an r-component link $L \subset S^3$. The subtlety and difficulty in this argument arises in determining the linking numbers of L's components.

As it turns out, this idea has already been pursued quite extensively by Owens in [45]. In that paper, he manages to compute the unknotting numbers of a large number of nine and ten-crossing knots. His theorem, however, comes with the requirement that if K is undone by changing p positive and n negative crossings, where $p + n = 2$, then $n = \frac{1}{2}\sigma(K)$. That is, he requires that sufficiently many negative crossings be changed in order to unknot K.

A general scan of the literature shows that this sort of requirement is not uncommon, and the main result of this section is no different. We provide an obstruction to $u(K) = p$; that is, an obstruction to the property that K can be undone by changing positive crossings only. It is based on the following theorem of Bao [1], who defines the set $\Omega \subset \mathbb{Q}$ as being the set of rational numbers p/q possessing a Hirzebruch-Jung continued fraction expansion $p/q = [a_1, \ldots, a_\ell]^-$ in which $a_i \geq 2$ for all $1 \leq i \leq \ell$ and equality occurs at most twice.

Theorem 6.3.1. *Suppose that the unknotting number $u(K)$ of $K \subset S^3$ can be realised by changing only positive crossings. Then $D_K^{p/q}(\mathfrak{t}) = 0$ for all $\mathfrak{t} \in Spin^c(S^3_{-p/q}(K))$ and all $p/q \in \Omega$.*

In view of this result, we let W be the 4-manifold described in Proposition 2.2.3 when applied to $-p/q$-surgery on K for $p/q \in \Omega$ (choosing the a_i so that satisfy the conditions described above), and recast Theorem 4.1.3 as follows.

Theorem 6.3.2. *Suppose that K is a knot which can be undone by changing only positive crossings, that W is the 4-manifold above (so that $p/q \in \Omega$), and that $S^3_{p/q}(\overline{K})$*

bounds a sharp, simply connected, negative-definite smooth 4-manifold X with intersection form Q_X. Then there exists a matrix A satisfying

$$-AA^t = Q_X \oplus Q_W,$$

whose last ℓ rows have the form

$$\begin{pmatrix} 1 & \dots & 1 & 1 & 0 & & & & & & & & & \\ & & & -1 & 1 & \dots & 1 & & & & & & & \\ & & & & & & & \ddots & & & & & & \\ & & & & & & & & -1 & 1 & \dots & 1 & 0 & & & \\ & & & & & & & & & & & -1 & 1 & \dots & 1 \end{pmatrix},$$

where there are exactly a_i non-zero entries in each row, all ± 1 as above.

Proof. This is a straightforward application of Theorem 4.1.3, on noting that $p/q \in \Omega$ implies $n \geq 2$. The changemaker condition has vanished because Theorem 6.3.1 guarantees that all the deficiencies vanish; hence $j_{\max} = n$, and

$$1 + \sum_i \sigma_i = n = 1 + \sum_i \sigma_i^2.$$

This in turn, by comparison with similar statements about y_i, means that $\sigma_i = 1$. □

Despite the apparent similarities with the approach used in Chapter 5, this theorem is fundamentally different: it has nothing to do with the Montesinos theorem, nothing to do with half-integral surgeries (indeed, our choice of p/q is not even prescribed), and at no point requires us to compute any deficiencies. Instead, its main requirement is a decent understanding of the surgeries on the knot K whose unknotting number we wish to test.

Given the difficulties encountered in the various "loose end" cases of Chapter 5 (in particular when $p + q = 4$ and $\det K = 3$ or $p + q = 2$ and $\det K = 1$), we believe that Theorem 6.3.2 will allow us to finish off the pretzel question once and for all. At the time of this writing, we are still investigating its potential, but have as yet not been able to make much progress in isolating the correct choices of p/q.

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APPENDIX: BIOLOGICAL IMPLICATIONS

In this appendix, we provide a very brief introduction to the sort of biological questions for which Theorem 4.1.3 is likely to be helpful. We stress that this introduction is biased heavily towards mathematical terminology, and is not intended to be a survey of DNA topology. The reader interested in a more comprehensive survey is advised to consult [2] or [4].

It is now well known that DNA possesses a beautiful double-helical structure first discovered by Crick, Watson, and Franklin [11]. As illustrated diagrammatically in Figure 1, the double-helix has a ladder-like structure, each rung of which consists of a pair of nucleobases together referred to as *basepairs*. These nucleobases, known individually as *adenine* A, *cytosine* C, *guanine* G, and *thymine* T, are paired canonically so that an A on one backbone corresponds with a T on the other, and similarly with C and G. Replication of this code is usually achieved by unzipping the DNA down the central axis; the canonical correspondence allows the cell to construct two identical copies of the original DNA.

While the central axis of the DNA is often linear, in certain cells, such as bacterial cells, the central axis forms a closed loop, thus allowing for the possibility of knots in the central axis. If these knots are non-trivial, they can cause considerable problems for a variety of cellular processes. One such example is replication. Phrased mathematically, if K represents the DNA axis, then after replication, the copied DNA will

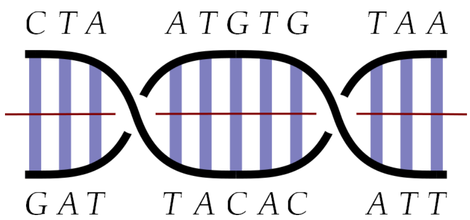

Figure 1: The double-helical structure of DNA. The basepairs, shown in blue, consist of canonically paired nucleobases, one on each of the thick, black backbones. The central axis, shown in red, may be open-ended and linear, or closed and knotted.

have an axis represented by a longitude K_n of K which satisfies $\mathrm{lk}(K, K_n) = n$ for some $n \in \mathbb{Z}$ (i.e. the integer n is determined by the number of twists involved in the double helix). While it is certainly conceivable that certain enzymes exist (indeed, they are called *Type I topoisomerases*) to adjust the linking number of the DNA molecules, the fundamental problem still remains that even K and K_0 are linked unless K is the unknot. This physical constraint prevents the replicated DNA from being passed on to daughter cells, and eventually kills the cell [14].

Given this information, it is natural to ask how DNA knots form in the first place: presumably, if DNA is successfully passed on from a parent cell to its daughter cells, this new DNA is untangled. However, it is now well understood that a lot of the observed DNA knotting occurs as an unwanted by-product of enzymatic action. In fact, many enzymes, whose primary purpose is to rearrange the $A - C - G - T$ sequence, either by reordering the basepairs, splicing in new sections, or excising unwanted ones, have been found to alter the topology of the DNA.

Type IIα Topoisomerase

As opposed to the infinitely stretchable, infinitely flexible knots K we have been considering until now, DNA is more rigid: it comes with limits on its flexibility, and has a prescribed length in basepairs. These geometric constraints therefore impose certain limitations on the complexity of the knots one finds. As a rough guide, Table 1 provides a decent sized sample of the knots that have been observed (see [4]). In particular, we draw attention to the pretzel family, which occurs with unusual frequency.

Because of the potentially lethal impact these knots might have on the cell containing them, it should come as no surprise that nature has evolved a specific type of enzyme, known as *Type IIα topoisomerase* (TopoIIα), whose purpose is to perform crossing changes on the DNA before it is replicated (or various other cellular processes take place) [60]. On the mathematical side, this means that the most relevant invariant from the point of view of TopoIIα is the unknotting number, and when viewed in this way, Chapter 5 can be seen as the first part of a bigger biological

Family	Notation	Specialisation
Torus knots	$T(p+q,2)$	$s=0$
Clasp knots	$C(r,s)$	$p=0$, $q=\pm 1$
Pretzel knots	$P(p,q,r\pm 1)$	$s=\pm 1$
Connected sum 1	$T(p,2)\#T(r\pm 1,2)$	$q=0$, $s=\pm 1$
Connected sum 2	$T(p,2)\#C(r,s)$	$q=0$

Table 1: Common DNA knots. All these knots are special cases of the Montesinos knot $M\left(\frac{1}{p},\frac{1}{q},\frac{s}{rs+1}\right)$, once the choices listed in the column headed "specialisation" have been made.

project which asks for a classification of those pretzels which are adjacent under the action of TopoIIα. What is more, since all the knots in Table 1 can be written as Montesinos knots of the form $M\left(\frac{1}{p},\frac{1}{q},\frac{s}{rs+1}\right)$, the project described in Section 6.1, were it to succeed, would mean that Theorem 4.1.3 would have genuinely enhanced our understanding of the unknotting numbers of all the knots in Table 1 (indeed, of even more knots than are presented in the table).

To delve a little deeper into the biology behind all this, the following question is perhaps the greatest unsolved mystery about TopoIIα: How, given that TopoIIα must latch on to the DNA in order to perform a local crossing change, does the enzyme know which crossing to change? Although experiments have shown that TopoIIα almost invariably favours crossing changes which simplify the DNA in the most efficient way possible [60] (we are able to confirm that they are the most efficient thanks to results such as those in Chapter 5), the details of how TopoIIα identifies these crossings remain hotly debated (see [6], [70], and [65] for different theories, and [69] for a review). It is ultimately hoped that a better understanding of the effects of TopoIIα, on both a biological and a mathematical level, will lead us to greater insight as to how TopoIIα solves this problem. Indeed, should we finally manage to understand the mechanism properly, then the current set of antibiotic, anti-tumor, and anti-cancer drugs which inhibit the action of TopoIIα in harmful cells to prevent them from replicating may just be improved another step further [8, 37, 15].

Site-Specific Recombination

One of the commonly accepted models for the type of enzyme activity known as *site-specific recombination* is the *tangle model* given by Ernst and Sumners [17]. In this model, the original DNA sample K, called the *substrate*, is transformed into a *product* sample K' via rational tangle replacement (the term "recombination" is used because the DNA is cleaved at the endpoints of T, rearranged, and then reglued). That is, if we write

$$K = N(O + T),$$

where T is a rational tangle and O is some general tangle, then

$$K' = N(O + R),$$

for some other rational tangle R. The enzymes responsible for this replacement are usually classified by two properties:

1. They require a certain "constrained" structure in the unchanged outer tangle O, typically by specifying that $O = O_f + O_c$ for some prerequisite tangle O_c and some other "free" tangle O_f; and

2. They latch onto a particular tangle T and replace it with another tangle R.

Thus, for example, the enzyme Tn3 resolvase (see [17]) requires that O_c be the $-\frac{1}{3}$-tangle. In fact, the requirement that O_c be a $-\frac{1}{k}$-tangle for $k \in \mathbb{N}^+$ is not uncommon.

Putting this first requirement to the side, Theorem 4.1.3 is particularly relevant to these systems. Given any biological enzyme E, and ignoring any prerequisite structure in O, we can associate with E an abstract (t, r)-"enzyme" which performs a (t, r)-tangle replacement (where $t, r \in \mathbb{Q}^*$ correspond with T and R). At least mathematically, then, E is equivalent to a p/q-"enzyme" which performs the equivalent p/q-tangle replacement. If we wish to know, conversely to our desires with TopoIIα, whether or not an unknotted substrate can be converted into a non-trivial knot K by our p/q-enzyme (and which knots may result), this statement then translates into the equation

$$\Sigma(K) = S^3_{\pm \det K/p}(C),$$

where $C \subset S^3$. Thus, if K is alternating (as are many low-crossing knots, and therefore many biologically observed knots), we can use Theorem 4.1.3 in the manner discussed in Section 6.2 to obstruct the above equation and hence answer questions about site-specific recombination. Indeed, the theorem is even more powerful than this specific example might suggest: by letting p vary, we could even prove that certain products can never be obtained in one go by the action of any enzyme on an unknotted substrate (or, conversely, that certain knotted substrates can never be unknotted in one go).

In the case that E acts more than once, there are two alternative models. The first, called *distributive recombination*, asserts that E, having identified its prerequisite structure in O, then latches onto the substrate K, recombines it, and releases it. In order to act a second time, E must then isolate a second copy of T inside the product. Mathematically stated, if our enzyme converts $K := K_0$ into K_1, and thereafter converts K_i into K_{i+1}, then we must satisfy the following equations for tangles O_i:

$$\begin{aligned} K_i &= N(O_i + T) \\ K_{i+1} &= N(O_i + R) = N(O_{i+1} + T) \\ K_{i+2} &= N(O_{i+1} + R). \end{aligned}$$

In the second model, called *processive recombination*, E instead latches on to T once, and adds an extra copy of R each time it acts. Thus, mathematically,

$$\begin{aligned} K_0 &= N(O + T) \\ K_i &= N(O + \underbrace{R + \cdots + R}_{i \text{ times}}). \end{aligned}$$

This latter model, as noted in [17], is easier to analyse mathematically.

We believe that either of these models, as per the instructions outlined in Section 6.2, is amenable to analysis with Theorem 4.1.3, provided that the sequence $(K_i)_{i\geq 0}$

features unknots along the way. We are hopeful that such analysis, when applied to real biological systems, will provide us with valuable new insights into what is and is not possible in the enzymatic world.