A Nash Game Approach to Mixed H_2/H_∞ Control for Input-Affine Nonlinear Systems

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Abstract: With the aim of designing controllers to simultaneously ensure robustness *and* optimality properties, the mixed H_2/H_∞ control problem is considered. A class of input-affine nonlinear systems is considered and the problem is formulated as a nonzero-sum differential game, similar to what has been done earlier by Limebeer et al. for linear systems. A heuristic algorithm for obtaining solutions for the coupled algebraic Riccati equations which are characteristic of the linear quadratic problem is provided together with a systematic method for constructing approximate solutions for the general, nonlinear problem. A few numerical examples are provided.

Keywords: Differential games, robust control, optimal control, nonlinear control, game theory.

1. INTRODUCTION

In this paper we consider the problem of controlling an input-affine system with two conflicting aims, namely to achieve a certain objective in an *optimal* way using minimal control efforts, while simultaneously ensuring certain *robustness* properties with respect to an external disturbance. The former objective is similar to the objective of optimal control, where a control input is designed to optimise a given objective function subject to the dynamics of the system. This is closely related to the problem of H_2 control, where the goal is to design a control input which minimises the energy of a certain control variable, see Vinter [2000]. The latter is a problem of H_∞ control and is equivalent to a worst-case, or *robust*, problem (see, for example, Isidori and Astolfi [1992], Astolfi and Colaneri [2001]). In certain situations it is of interest to seek both optimality *and* robustness and in these cases both performance criteria are considered simultaneously. This is the topic of mixed H_2/H_∞ control.

It is well-known that the H_∞ control problem can be formulated as a two-player, zero-sum differential game, see Isidori [1992], Isidori and Astolfi [1992], Orlov and Aguilar [2014], Basar and Bernhard [1995]. However, the more complex nature of the mixed H_2/H_∞ control problem calls for a somewhat different approach. In Limebeer et al. [1992, 1994] linear systems are considered and a novel approach for solving the problem is proposed by recognising that it can be described as a two-player, nonzero-sum differential game. Thus, solutions are found using a game theoretic approach in which Nash equilibrium solutions, which rely on solving a system of coupled algebraic Riccati equations (AREs), are sought. Another game theoretic approach is taken in Jungers et al. [2008] where a solution, again for linear systems, is sought in terms of Stackelberg solutions, instead of the more typical Nash solutions. In Khargonekar and Rotea [1991] a suboptimal solution (for linear systems) is found by reducing the problem to a convex optimisation one. For nonlinear systems, different approaches exist, see for example Astolfi and Colaneri [2001] and references therein. In Lin [1995] nonlinear systems and the mixed H_2/H_∞ problem are considered and as in Limebeer et al. [1994] the problem is formulated as a differential game, for which the solution relies on solving a system of coupled partial differential equations (PDEs) which are typical of nonlinear differential games. However, solving the PDEs characterising the differential game is not generally a trivial task, an issue which is not addressed in Lin [1995].

In general there is a trade-off between robustness and optimality, in the sense that an increase in robustness often comes at the cost of optimality and vice versa, as seen in Astolfi and Colaneri [2001], Megretski [1994]. A consequence of this trade-off is that obtaining solutions for mixed H_2/H_∞ control problems usually is a challenging task. Despite the challenges associated with mixed H_2/H_∞ control problems the topic is of significant practical interest and consequently it is of interest to continue to seek ways of obtaining solutions to such problems. In Chen and Chang [1997] robust tracking of robotic systems is posed as a nonlinear mixed H_2/H_∞ control problems which is solved using game theoretic ideas and exploiting the particular structure of the system. A similar problem

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is considered in Tseng and Chen [2003] for non-holonomic systems.

In this paper the problem of mixed H_2/H_∞ control for nonlinear systems is solved using a differential game approach, similar to what has been done in Limebeer et al. [1992, 1994] for linear systems and in Lin [1995] for nonlinear systems. The nonlinear problem is formulated as a two-player nonzero-sum differential game for which a systematic method of constructing approximate solutions is developed using ideas similar to those introduced in Mylvaganam et al. [2015].

The remainder of the paper is structured as follows. In Section 2 the mixed H_2/H_∞ control problem is defined for a class of nonlinear systems and formulated as a Nash game. For linear systems (or linear approximations of a system about an operating point) the problem can be formulated as a linear-quadratic Nash game as in Limebeer et al. [1992, 1994]. A brief summary of this linear setting is given in Section 3 in which a novel algorithm for determining solutions for the resulting coupled AREs is also provided. The notion of *algerbraic* $\bar{P}$ *matrix solutions* for mixed H_2/H_∞ control is given in Section 4 and used to constructed approximate solutions for the nonlinear differential game in Section 5. The developed methods are then illustrated by a numerical example in Section 6 before some concluding remarks and directions for future work are given in Section 7.

2. THE MIXED H_2/H_∞ CONTROL PROBLEM: NONLINEAR SETTING

Consider an input-affine, nonlinear, system with dynamics given by

$$\dot{x} = f(x) + g_1(x)w + g_2(x)u \,, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input, $u(t) \in \mathbb{R}^{m_2}$ is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field and $g_i : \mathbb{R}^{m_i} \to \mathbb{R}^n$ are continues mappings, with [1] $m_i \in \mathbb{N}^+$, for $i = 1, 2$. Without loss of generality, the following assumption is made.

Assumption 1. The origin is an equilibrium of the system, *i.e.* $f(0) = 0$.

Assumption 1 implies that there exists a matrix-valued mapping $F(x)$ (non-unique) such that $f(x) = F(x)x$.

In what follows let u denote feedback strategies, *i.e.* $u = u(x(t))$ and consider the performance variables

$$z_1 = h_1(x) + k_1(x)u \,,\ z_2 = h_2(x) + k_2(x)u \,, \tag{2}$$

where $z_1 \in \mathbb{R}^{p_1}$ and $z_2 \in \mathbb{R}^{p_2}$, $p_i \in \mathbb{N}^+$, for $i = 1, 2$, h_1, k_1, h_1 and k_1 are continuous mappings, and $z_1 \in \mathbb{R}^{p_1}$ and $z_2 \in \mathbb{R}^{p_2}$, with $p_i \in \mathbb{N}^+$, $i = 1, 2$, relate to the H_∞ and the H_2 criteria, respectively.

In the remainder of the paper we make the following standing assumptions, similar to those made in Astolfi and Colaneri [2001].

Assumption 2. The mappings in (2) satisfy the following conditions.

(i) The mappings $h_1(x)$ and $h_2(x)$ are such that $h_1(0) = 0$ and $h_2(0) = 0$.
(ii) The conditions $h_1(x)^\top k_1(x) = 0$, $k_1(x)^\top k_1(x) = I$, $h_2(x)^\top k_2(x) = 0$ and $k_2(x)^\top k_2(x) = I$ are satisfied for all x.

[1] The set of positive natural numbers is denoted by $\mathbb{N}$.

Remark 1. The second conditions in Assumption 2 are such that the cross terms in $z_1^\top z_1$ and $z_2^\top z_2$ are zero. In Limebeer et al. [1994] this is ensured by defining the performance variables as

$$z_1 = z_2 = z = \begin{bmatrix} Cx \\ Du \end{bmatrix} ,$$

where C and D are matrices of appropriate dimensions. ▲

We now introduce the following two cost functionals:

$$\begin{aligned} J_1(u, w) &\triangleq \frac{1}{2}\int_0^\infty \left(-z_1^\top z_1 + \gamma^2 w^\top w\right) \mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty \left(-h_1(x)^\top h_1(x) - u_2^\top u_2 + \gamma^2 w^\top w\right) \mathrm{d}t \,, \end{aligned} \tag{3}$$

and

$$\begin{aligned} J_2(u, w) &\triangleq \frac{1}{2}\int_0^\infty z_2^\top z_2 \mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty \left(h_2(x)^\top h_2(x) + u^\top u\right) \mathrm{d}t \,, \end{aligned} \tag{4}$$

where $\gamma > 0$ is a “disturbance attenuation” level. The cost functions J_1 and J_2 are to be minimised by the disturbance w and the control input u, respectively.

Remark 2. Define $q_1(x) = h_1(x)^\top h_1(x)$ and $q_2(x) = h_2(x)^\top h_2(x)$. It follows from Assumptions 2 (i) that these can be written in the form $q_i(x) = x^\top Q_i(x)x$, $i = 1, 2$, where $Q_i(x)$, $i = 1, 2$ are matrix-valued functions. ▲

The infinite-horizon, mixed H_2/H_∞ control problem is defined in the following statement, similar to what has been done in Limebeer et al. [1994].

Problem 1. Given the system (1) determine a feedback control law u^* such that

(i) The inequality $z_1^\top z_1 \le \gamma^2 w^\top w$ is satisfied for all $w(t) \neq 0$ when $x(0) = 0$.
(ii) The state x is regulated while minimising the control effort when the worst-case disturbance w^* is applied to the system, *i.e.* the cost functional J_2 is minimised when $w = w^*$.

Similarly to what has been done for linear systems in Limebeer et al. [1992, 1994] and for nonlinear systems in Lin [1995], Problem 1 can be formulated as a nonzero-sum differential game characterised by the dynamics (1) and the cost functionals (3) and (4) as follows.

Problem 2. Consider the system (1) and the cost functionals (3) and (4). Solving the resulting two-player differential game constitutes to determining a set of admissible [2] feedback strategies (u^*, w^*) satisfying the *Nash equilibrium inequalites*

$$\begin{aligned} J_1(u^*, w^*) &\le J_1(u^*, w) \,, \\ J_2(u^*, w^*) &\le J_2(u, w^*) \,, \end{aligned} \tag{5}$$

[2] A set of strategies (u, w) is said to be admissible if it renders the zero equilibrium of the system in closed-loop with (u, w) (locally) asymptotically stable.

for all admissible sets of strategies (u, w^*) and (u^*, w), where $u \neq u^*$ and $w \neq w^*$.

The *Nash equilibrium strategies* u^* and w^* are said to be the optimal control and the worst-case disturbance, respectively.

Remark 3. From (3) it is clear that provided $J_1(u^*, w) \geq 0$ when $x(0) = 0$, for all w (such that the set of strategies (u^*, w) is admissible), the first criterion in Problem 1 is satisfied. The second criterion in Problem 1 follows directly from the second Nash equilibrium inequality in (5) when $w = w^*$. Thus, as remarked in Limebeer et al. [1994], Lin [1995], Problem 1 can be described by the nonzero sum differential game in Problem 2 provided $J_1(u^*, w) \geq 0$, for all admissible (u^*, w). ▲

Suppose $V_1(x) \leq 0$ and $V_2(x) \geq 0$ such that $V_i(0) = 0$, $i = 1, 2$, $V_2(x) - V_1(x) > 0$, for all $x \neq 0$ and satisfy the so-called Hamilton-Jacobi-Isaacs PDEs

$$\begin{aligned} & -\frac{1}{2}h_1(x)^\top h_1(x) - \frac{1}{2}\frac{\partial V_2}{\partial x}g_2(x)^\top g_2(x)\frac{\partial V_2}{\partial x}^\top \\ & - \frac{1}{2\gamma^2}\frac{\partial V_1}{\partial x}g_1(x)^\top g_1(x)\frac{\partial V_1}{\partial x}^\top + \frac{\partial V_1}{\partial x}f(x) \\ & - \frac{\partial V_1}{\partial x}g_2(x)^\top g_2(x)\frac{\partial V_2}{\partial x}^\top = 0\,, \\ & \frac{1}{2}h_2(x)^\top h_2(x) - \frac{1}{2}\frac{\partial V_2}{\partial x}g_2(x)^\top g_2(x)\frac{\partial V_2}{\partial x}^\top \\ & + \frac{\partial V_2}{\partial x}f(x) - \frac{1}{2\gamma^2}\frac{\partial V_2}{\partial x}g_1(x)^\top g_1(x)\frac{\partial V_1}{\partial x}^\top = 0\,. \end{aligned} \tag{6}$$

Moreover, suppose the solution is such that the following assumptions hold.

Assumption 3. Consider the system (1) with output $y(x) = q_1(x) + q_2(x)$. Then the following conditions are satisfied

(i) The pair $\{f, y\}$ is zero-state detectable.

(ii) The pair $\{f - \frac{1}{\gamma^2}g_1^\top \frac{\partial V_1}{\partial x}^\top, y\}$ is zero-state detectable.

The Nash equilibrium strategies for the differential game in Problem 2 are then given by

$$u^* = -g_2(x)^\top \frac{\partial V_2}{\partial x}^\top\,, \quad w^* = -\frac{1}{\gamma^2}g_1(x)^\top \frac{\partial V_1}{\partial x}^\top\,. \tag{7}$$

Remark 4. The Nash equilibrium strategies derive from the dynamic programming principle. The admissibility of the pair of strategies $\{u^*, w^*\}$ can be proved by taking $W = V_2 - V_1 > 0$ as a candidate Lyapunov function. It follows from (6) that $\dot{W} = -\frac{1}{2}(q_1(x) + q_2(x)) - \|u^*\|^2 + \frac{1}{2}\gamma^2\|w^*\|^2$ and, consequently, (local) asymptotic stability follows from Assumption 3 and LaSalle's invariance principle. ▲

Remark 5. The value functions $V_i(x)$ are such that

$$V_i(x(0), u^*, w^*) = J_i(u^*, w^*)\,,$$

for $i = 1, 2$ (see Basar and Olsder [1982] for details). Since $V_1(0, u^*, w^*) = 0$ this implies that $J_1(u^*, w^*) = 0$ when $x(0) = 0$. It follows from the first of the Nash inequalities (5) that $J_i(u^*, w) \geq 0$ for this initial condition. Thus, as stated in Remark 3, the solution to the differential game in Problem 2 solves Problem 1. ▲

In general, closed-form solutions to the HJI PDEs associated with a differential game are not easily obtained. As a consequence it is often necessary to seek approximate solutions for differential games, for instance as done in Mylvaganam et al. [2015].

3. THE MIXED H_2/H_∞ CONTROL PROBLEM: LINEAR SETTING

A brief summary of the theory concerning mixed H_2/H_∞ control problems for linear systems is provided in this section. In this case the differential game in Problem 2 reduces to a linear quadratic differential game. For more details regarding this class of problems see Limebeer et al. [1992, 1994] and for details regarding general linear quadratic differential games see, for example, Basar and Olsder [1982], Starr and Ho [1969].

Consider the linear system

$$\dot{x} = Ax + B_1 w + B_2 u\,, \tag{8}$$

with $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$ and $B_2 \in \mathbb{R}^{n\times m_2}$, and the performance variables

$$z_1 = C_1 x + D_1 u\,, \quad z_2 = C_2 x + D_2 u\,, \tag{9}$$

with $C_1 \in \mathbb{R}^{p_1\times n}$, $D_1 \in \mathbb{R}^{p_1\times m_1}$, $C_2 \in \mathbb{R}^{p_2\times n}$ and $D_2 \in \mathbb{R}^{p_2\times m_2}$. The corresponding cost functionals are quadratic in the state and Problem 2 simplifies to a *linear quadratic* differential game. Considering linear feedback strategies only [3], the solution to Problem 2 relies on coupled AREs. Suppose matrices $P_1 = P_1^\top \leq 0$, and $P_2 = P_2^\top \geq 0$, such that $P_2 - P_1 > 0$, satisfying

$$\begin{aligned} & -C_1^\top C_1 - P_2 B_2 B_2^\top P_2 - \frac{1}{\gamma^2}P_1 B_1 B_1^\top P_1 + P_1 A \\ & + A^\top P_1 - \left(P_1 B_2 B_2^\top P_2 + P_2 B_2 B_2^\top P_1\right) = 0\,, \\ & C_2^\top C_2 - P_2 B_2 B_2^\top P_2 + P_2 A + A^\top P_2 \\ & - \frac{1}{\gamma^2}\left(P_2 B_1 B_1^\top P_1 + P_1 B_1 B_1^\top P_2\right) = 0\,, \end{aligned} \tag{10}$$

can be found. Furthermore suppose (A, C_2) is detectable and $((A - \frac{1}{\gamma^2}B_1 B_1^\top P_1), C_1)$ is detectable. Then the Nash equilibrium strategies, *i.e.* the optimal control and worst-case disturbance, are given by

$$u^* = -B_2^\top P_2 x\,, \quad w^* = -\frac{1}{\gamma^2}B_1^\top P_1 x\,. \tag{11}$$

Stability of the zero equilibrium can then be shown using $W = \frac{1}{2}x^\top(P_2 - P_1)x$ as a candidate Lyapunov function as in the nonlinear case.

Remark 6. The matrix-valued functions $Q_i(x)$ in Remark 2 are such that $Q_i(0) = C_i^\top C_i$, for $i = 1, 2$. ▲

Although in Limebeer et al. [1994] a scalar problem is solved using a standard integration procedure, it is not generally straightforward to obtain a solution for the coupled AREs (10) arising in linear quadratic, nonzero-sum differential games. See, for example, Engwerda [2005], Papavassilopoulos and Olsder [1979] for details regarding

[3] The Nash equilibrium strategies for a linear quadratic differential game may, in general, be nonlinear. However, it is common to restrict the attention to linear feedback strategies Engwerda [2005], Limebeer et al. [1994].

solutions for the coupled AREs. In the following statement a heuristic method for obtaining solutions for (10) is provided. First, let $\tilde{P}_i(t)$, $L_i(t)$ denote time-varying matrices with the same dimensions as P_i, $i = 1, 2$.

Proposition 1. Let $\tilde{P}_i(0) = \tilde{P}_i(0)^\top$ and $L_i(0) = L_i(0)^\top$, for $i = 1, 2$ and consider the matrix differential equations

$$\begin{aligned}\dot{\tilde{P}}_1 = \Big(& - C_1^\top C_1 - \frac{1}{2}(\tilde{P}_2 B_2 B_2^\top L_2 + L_2 B_2 B_2^\top \tilde{P}_2) \\ & - \frac{1}{2\gamma^2}(\tilde{P}_1 B_1 B_1^\top L_1 + L_1 B_1 B_1^\top \tilde{P}_1) + \tilde{P}_1 A \\ & + A^\top \tilde{P}_1 - \frac{1}{2}(L_1 B_2 B_2^\top \tilde{P}_2 + \tilde{P}_1 B_2 B_2^\top L_2) \\ & - \frac{1}{2}(L_2 B_2 B_2^\top \tilde{P}_1 + L_1 B_2 B_2^\top \tilde{P}_2)\Big) , \\ \dot{\tilde{P}}_2 = \Big(& C_2^\top C_2 - \frac{1}{2}(\tilde{P}_2 B_2 B_2^\top L_2 + L_2 B_2 B_2^\top \tilde{P}_2) \\ & - \frac{1}{2\gamma^2}(L_1 B_1 B_1^\top \tilde{P}_2 + \tilde{P}_1 B_1 B_1^\top L_2) + \tilde{P}_2 A \\ & + A^\top \tilde{P}_2 - \frac{1}{2\gamma^2}(L_2 B_1 B_1^\top \tilde{P}_1 + L_1 B_1 B_1^\top \tilde{P}_2)\Big) ,\end{aligned} \tag{12}$$

and

$$\begin{aligned}\dot{L}_1 &= \dot{P}_1 + \kappa(\tilde{P}_1 - L_1) , \\ \dot{L}_2 &= \dot{P}_2 + \kappa(\tilde{P}_2 - L_2) ,\end{aligned} \tag{13}$$

where $\kappa > 0$. Any converging trajectories of the system is such that the convergent values of $\tilde{P}_i$ (and L_i) correspond to a solution of the coupled AREs (10). ◇

Proof: Defining the error matrices $E_i = \tilde{P}_i - L_i$, $i = 1, 2$, it follows that $\dot{E}_i = -\kappa(E_i)$. It follows that any $\kappa > 0$ is such that $\lim_{t\to\infty} E_i(t) = 0$. Moreover, when $E_i = 0$ the right-hand sides of (12) are identical to the left-hand sides of the coupled AREs in (10). By assumption the trajectories are convergent, *i.e.* $\lim_{t\to\infty} \dot{\tilde{P}}_i = 0$, $i = 1, 2$. It follows that the dynamic matrices $\tilde{P}_i$ (and L_i), $i = 1, 2$ converge to a solution of (10). □

Remark 7. If $L_i = \tilde{P}_i$, the matrix dynamic equations (12) are similar to the AREs characterising the finite-horizon problem with the same dynamics and cost functionals as the differential game in Problem 2. For such problems, solved over an interval $[0, T]$, $P_i(T)$ is known and the resulting ordinary differential equations (ODEs) (12) are solved backwards in time. Thus, the equations (12) can easily be solved backwards in time to seek solutions of finite-horizon problems [4]. Provided T is sufficiently large, $\tilde{P}_i$ (and L_i), $i = 1, 2$, will converge to a solution (if it exists) of the finite-horizon problem.

It follows that Proposition 1 can be interpreted as a way of identifying solutions to an infinite-horizon problem by considering the convergent behaviour of the ODEs characterising the corresponding finite-horizon problem. Note, however, that this limit may not give all solutions for the infinite-horizon problem, which can in general have no solutions, a unique solution or several solutions (see Weeren et al. [1999]). ▲

[4] Whereas the finite-horizon problem is solved in Limebeer et al. [1994] we are interested in obtaining solutions for the infinite-horizon linear quadratic differential game. Therefore, we solve the problem forwards in time and consider the asymptotic behaviour of the dynamic variables $\tilde{P}_i$ and L_i, $i = 1, 2$.

4. ALGEBRAIC $\bar{P}$ MATRIX SOLUTION

In this section we give the notion of *algebraic $\bar{P}$ solutions* for the mixed H_2/H_∞ control problem. This is a *mathematical tool* which is instrumental in constructing approximate solutions for the differential game defined in Problem 2.

Definition 1. Consider the system (1) and the cost functionals (4) and (3). Let $\Sigma_i : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be matrix-valued functions such that $\Sigma_i(x) = \Sigma_i(x)^\top \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, and $\Sigma_i(0) = \bar{\Sigma}_i > 0$, for $i = 1, 2$. The $\mathcal{C}^1$ matrix-valued functions $P_i : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $P_i(x) = P_i(x)^\top$, $i = 1, 2$, are said to be *$\mathcal{X}$-algebraic $\bar{P}$ matrix solutions* [5] of (6) provided the following conditions hold.

(i) For all $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $i = 1, 2$, $j \neq i$,

$$\begin{aligned}& - Q_1(x) - P_2(x)g_2(x)g_2(x)^\top P_2(x) \\ & - \frac{1}{\gamma^2}P_1(x)g_1(x)g_1(x)^\top P_1(x) + P_1(x)F(x) \\ & + F(x)^\top P_1(x) - P_1(x)g_2(x)g_2(x)^\top P_2(x) \\ & - P_2(x)g_1(x)g_1(x)^\top P_1(x) - \Sigma_1(x) = 0 , \\ & Q_2(x) - P_2(x)g_2(x)g_2(x)^\top P_2(x) \\ & + P_2(x)F(x) + F(x)^\top P_2(x) \\ & - \frac{1}{\gamma^2}P_2(x)g_1(x)g_1^\top(x)P_1(x) \\ & - \frac{1}{\gamma^2}P_1(x)g_1(x)g_1^\top(x)P_2(x) + \Sigma_2(x) = 0 ,\end{aligned} \tag{14}$$

(ii) $P_i(0) = \bar{P}_i$, such that $\bar{P}_2 - \bar{P}_1 > 0$, with $\bar{P}_i$, $i = 1, 2$, symmetric solutions of the *coupled Riccati equations*

$$\begin{aligned}& - C_1^\top C_1 - \bar{P}_2 B_2 B_2^\top \bar{P}_2 - \frac{1}{\gamma^2}\bar{P}_1 B_1 B_1^\top \bar{P}_1 + \bar{P}_1 A \\ & + A^\top \bar{P}_1 - \left(\bar{P}_1 B_2 B_2^\top \bar{P}_2 + \bar{P}_2 B_1 B_1^\top \bar{P}_1\right) - \bar{\Sigma}_1 = 0 , \\ & C_2^\top C_2 - \bar{P}_2 B_2 B_2^\top \bar{P}_2 + \bar{P}_2 A + A^\top \bar{P}_2 \\ & - \frac{1}{\gamma^2}\left(\bar{P}_2 B_1 B_1^\top \bar{P}_1 + \bar{P}_1 B_1 B_1^\top \bar{P}_2\right) + \bar{\Sigma}_2 = 0 ,\end{aligned} \tag{15}$$

If $x \in \mathbb{R}^n$, *i.e.* $\mathcal{X} = \mathbb{R}^n$, then P_1 and P_2 are said to be *algebraic $\bar{P}$ matrix solutions.*

Remark 8. An algorithm similar to that of Proposition 1 can be used to determine the matrix-valued functions $P_1(x)$ and $P_2(x)$ satisfying (14) for each given value of x. ▲

In what follows the existence of algebraic $\bar{P}$ matrix solutions is assumed and used to construct approximate solutions for the differential game, similarly to what has been done for a class of N-player differential games in Mylvaganam et al. [2015].

[5] Provided the set $\mathcal{X}$ contains the origin.

5. CONSTRUCTIVE APPROXIMATE SOLUTIONS

Let $P_1(x)$ and $P_2(x)$ denote an algebraic $\bar{P}$ matrix solution, as introduced in the previous section. Introduce a dynamic variable $\xi(t) \in \mathbb{R}^n$ and consider the extended state-space $(x^\top, \xi^\top)^\top$. Furthermore, let $R_i = R_i^\top > 0$ and define the *extended value functions*

$$
\begin{aligned}
V_1(x,\xi) &= \frac{1}{2}x^\top P_1(\xi)x - \frac{1}{2}\|x-\xi\|_{R_1}, \\
V_2(x,\xi) &= \frac{1}{2}x^\top P_2(\xi)x + \frac{1}{2}\|x-\xi\|_{R_2},
\end{aligned}
\tag{16}
$$

defined in the extended state-space. Let $\Phi_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be continuous matrix-valued functions such that $x^\top(P_i(x) - P_i(\xi)) = (x-\xi)^\top \Phi_i(x,\xi)$ and let $\Phi(x,\xi)$ denote the Jacobian matrices of $\frac{1}{2}P_i(\xi)x$ with respect to ξ, for $i = 1, 2$. Using this notation it follows that

$$
\begin{aligned}
\frac{\partial V_1}{\partial x} &= x^\top P_1(x) - (x-\xi)(R_1 + \Phi(x,\xi)), \\
\frac{\partial V_1}{\partial \xi} &= x^\top \Psi_1(x,\xi) + (x-\xi)^\top R_1, \\
\frac{\partial V_2}{\partial x} &= x^\top P_2(x) + (x-\xi)(R_2 + \Phi(x,\xi)), \\
\frac{\partial V_2}{\partial \xi} &= x^\top \Psi_2(x,\xi) - (x-\xi)^\top R_2.
\end{aligned}
$$

Finally, let $A_{cl}(x)$ describe the closed-loop system (1) when the optimal control and the worst-case disturbance (7) are applied, *i.e.* $A_{cl}(x) = F(x) - \frac{1}{\gamma^2}g_1(x)g_1(x)^\top P_1(x) - g_2(x)g_2(x)^\top P_2(x)$.

Using the properties of the algebraic $\bar{P}$ matrix solution dynamic feedback strategies which satisfy partial differential inequalities in place of the PDEs (6) can be constructed as demonstrated in the following statement.

Theorem 1. Consider the system (1), the cost functionals (3) and (4) and the resulting nonzero-sum differential game described in Problem 2. Let R_1 and R_2 be such that $R_2 > R_1$ and

$$
\begin{aligned}
R_1(R_1+R_2) + (R_1+R_2)R_1 &> 0, \\
R_2(R_1+R_2) + (R_1+R_2)R_2 &> 0.
\end{aligned}
\tag{17}
$$

There exists a neighbourhood Ω, containing the origin, and $\bar{k} > 0$ such that for all $k \geq \bar{k}$ the inequalities

$$
\begin{aligned}
\mathcal{HJI}_1 = &-\frac{1}{2}h_1(x)^\top h_1(x) - \frac{1}{2}\frac{\partial V_2}{\partial x}g_2(x)^\top g_2(x)\frac{\partial V_2}{\partial x}^\top \\
&- \frac{1}{2\gamma^2}\frac{\partial V_1}{\partial x}g_1(x)^\top g_1(x)\frac{\partial V_1}{\partial x}^\top + \frac{\partial V_1}{\partial x}f(x) \\
&- \frac{\partial V_1}{\partial x}g_2(x)^\top g_2(x)\frac{\partial V_2}{\partial x}^\top + \frac{\partial V_1}{\partial \xi}\dot{\xi} \geq 0, \\
\mathcal{HJI}_2 = &\frac{1}{2}h_2(x)^\top h_2(x) - \frac{1}{2}\frac{\partial V_2}{\partial x}g_2(x)^\top g_2(x)\frac{\partial V_2}{\partial x}^\top \\
&+ \frac{\partial V_2}{\partial x}f(x) - \frac{1}{2\gamma^2}\frac{\partial V_2}{\partial x}g_1(x)^\top g_1(x)\frac{\partial V_1}{\partial x}^\top + \frac{\partial V_2}{\partial \xi}\dot{\xi} \leq 0,
\end{aligned}
\tag{18}
$$

with

$$
\dot{\xi} = -k\left(\frac{\partial V_2}{\partial \xi} - \frac{\partial V_1}{\partial \xi}\right),
$$

are satisfied for all $(x,\xi) \in \Omega$. Suppose Assumption 3 is satisfied with V_i, $i = 1, 2$, given by (16). Then, the dynamic feedback strategy

$$
\begin{aligned}
u^* &= -g_2(x)^\top \frac{\partial V_2}{\partial x}^\top, \quad w^* = -\frac{1}{\gamma^2}g_1(x)^\top \frac{\partial V_1}{\partial x}^\top, \\
\dot{\xi} &= -k\left(\frac{\partial V_2}{\partial \xi} - \frac{\partial V_1}{\partial \xi}\right).
\end{aligned}
\tag{19}
$$

is such that the closed-loop system (1)-(19) is (locally) asymptotically stable. ◇

Proof: The left-hand-sides of the inequalities (18) can be written in *quadratic form*

$$
\begin{aligned}
\mathcal{HJI}_1 &= \frac{1}{2}\left[\, x^\top \ (x-\xi)^\top \,\right](M_1 + kD_1)\begin{bmatrix} x \\ (x-\xi)\end{bmatrix}, \\
\mathcal{HJI}_2 &= -\frac{1}{2}\left[\, x^\top \ (x-\xi)^\top \,\right](M_2 + kD_2)\begin{bmatrix} x \\ (x-\xi)\end{bmatrix},
\end{aligned}
$$

where M_i are given by

$$
M_i(x,\xi) = \begin{bmatrix} \Sigma_i & \Gamma_{i,12} \\ \Gamma_{i,12}^\top & \Gamma_{i,22}\end{bmatrix},
$$

with $\Gamma_{1,12} = -A_{cl}(x)^\top(R_1 - \Phi_1) - (P_1 + P_2)g_2g_2^\top(R_2 - \Phi_2)$,

$$
\begin{aligned}
\Gamma_{1,22} = &(R_1 - \Phi_1)^\top\left(g_2g_2^\top(R_2 - \Phi_2) - \frac{1}{\gamma^2}g_1g_1^\top(R_1 - \Phi_1)\right) \\
&(R_2 - \Phi_2)^\top g_2g_2^\top((R_1 - \Phi_1) - (R_2 - \Phi_2)),
\end{aligned}
$$

$\Gamma_{2,12} = A_{cl}(x)^\top(R_2 - \Phi_2) + \frac{1}{\gamma^2}P_2g_1g_1^\top(R_1 - \Phi_1)$, and

$$
\begin{aligned}
\Gamma_{2,22} = &(R_2 - \Phi_2)^\top\left(\frac{1}{\gamma^2}g_1g_1^\top(R_1 - \Phi_1) - g_2g_2^\top(R_2 - \Phi_2)\right) \\
&+ \frac{1}{\gamma^2}(R_1 - \Phi_1)^\top g_1g_1^\top(R_2 - \Phi_2),
\end{aligned}
$$

and D_i are given by

$$
D_i(x,\xi) = \begin{bmatrix} \Delta_{i,11} & \Delta_{i,12} \\ \Delta_{i,12}^\top & \Delta_{i,22}\end{bmatrix},
$$

with $\Delta_{i,11} = \Psi_i(\Psi_2 - \Psi_1)^\top + (\Psi_2 - \Psi_1)^\top\Psi_i$, $\Delta_{1,12} = -\Psi_1(R_2 + R_1)^\top + (\Psi_2 - \Psi_1)^\top R_1$, $\Delta_{2,12} = -\Psi_2(R_2 + R_1)^\top - (\Psi_2 - \Psi_1)^\top R_2$, $\Delta_{1,12} = -\Psi_2(R_2 + R_1)^\top - (\Psi_2 - \Psi_1)^\top R_2$, $\Delta_{i,22} = R_i(R_1 + R_2) - (R_1 + R_2)R_i$, $i = 1, 2$. Noting that $\Psi(0,\xi) = 0$, it follows that D_i, $i = 1, 2$, are positive semidefinite in a neighbourhood $\mathcal{W}_i$ of the origin of the extended state space $\mathbb{R}^n \times \mathbb{R}^n$. The columns of the matrix $Z = [I, 0]^\top$ spans the kernels of $D_i(x, 0)$, $i = 1, 2$, and $Z^\top M_i\big|_{0,0}Z = \Sigma_i > 0$. It follows from Anstreicer and Wright [2000] that there exists a non-empty set Ω containing the origin and $\bar{k} \geq 0$ such that the inequalities (18) are satisfied for all $k \geq \bar{k}$ and $(x,\xi) \in \Omega$.

Noting that, by the definition of algebraic $\bar{P}$ matrix solution, $W = V_2 - V_1 > 0$ in a neighbourhood $\mathcal{W}$ of the origin, let W be a candidate Lyapunov function. Its time derivative is given by $\dot{W} \leq -\frac{1}{2}(q_1(x) + q_2(x)) - \|u^*\|^2 + \frac{1}{2}\|w^*\|^2$. By Assumption 3 and LaSalle's invariance principle it follows that $\lim_{t\to\infty} x(t) = 0$. Furthermore, the zero equilibrium of the system $\dot{\xi} = -k(\frac{\partial V_2}{\partial \xi} - \frac{\partial V_2}{\partial \xi})(0,\xi)^\top$ is asymptotically stable. Thus, asymptotic stability of $(x,\xi) = (0,0)$ follows from standard arguments on interconnected systems. ☐

Remark 9. The dynamic feedback strategies (19) are the Nash equilibrium strategies of a *modified* two-player,

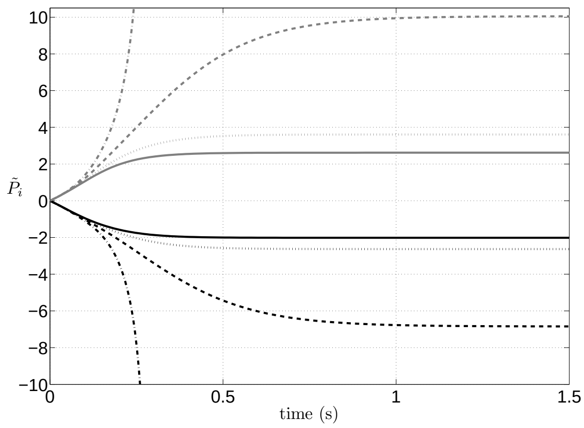

Fig. 1. Time histories of $\tilde{P}_1$ (black lines) and $\tilde{P}_2$ (grey lines) for different values of γ.

nonzero-sum differential game characterised by the dynamics (1) and the *modified cost functionals*

$$\tilde{J}_1(u,w) \triangleq \frac{1}{2}\int_0^\infty \left(-z_1^\top z_1 + \gamma^2 w^\top w - c_1(x,\xi)\right) \mathrm{d}t$$

$$\tilde{J}_2(u,w) \triangleq \frac{1}{2}\int_0^\infty \left(z_2^\top z_2 + c_2(x,\xi)\right) \mathrm{d}t\,,$$

where $c_i(x,\xi) : R^n \times R^n \to R$ are continuous mappings given by

$$c_1(x,\xi) = 2\mathcal{HJI}_1 \geq 0\,,$$

and

$$c_2(x,\xi) = -2\mathcal{HJI}_2 \geq 0\,.$$

▲

6. NUMERICAL EXAMPLES

Numerical examples illustrating the theoretical results of Proposition 1 and Theorem 1 are provided in this section.

6.1 A scalar linear system

In this section we revisit the scalar example considered in Limebeer et al. [1994]. Using the same four values for γ considered therein we show that the algorithm proposed in Proposition 1 yields solutions to the coupled AREs which are consistent with those found in Limebeer et al. [1994].

Consider the case in which the state $x \in \mathbb{R}$ and the performance variables are given by

$$\dot{x} = 2x + w + 3u\,, \quad z_1 = z_2 = \left[\, 3x \;\; u \,\right]^\top . \qquad (20)$$

The initial conditions for the matrices exploited in Proposition 1 are selected as $\tilde{P}_i(0) = L_i(0) = 0$, $i = 1,2$ and $\kappa = 1$ is selected. The resulting time histories of $\tilde{P}_i$ (top) and M_i (bottom) are shown for $\gamma = 0.35$ (dash-dotted lines), $\gamma = 0.4$ (dashed lines), $\gamma = 0.45$ (dotted lines) and $\gamma = 0.5$ (solid lines) in Figure 1, for $i = 1$ (black lines) and $i = 2$ (grey lines). It can be seen that they converge to a solution of (10) for all but the smalles value of γ.

6.2 A two-dimensional linear system

Consider now the case in which the state is described by a vector $x = [x_1, x_2]^\top$, with the following dynamics

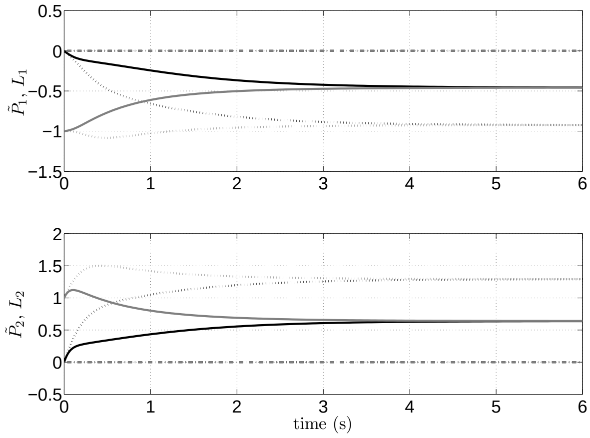

Fig. 2. Time histories of the components of $\tilde{P}_i$ (black lines) and L_i (grey lines) for $i = 1$ (top) and $i = 2$ (bottom).

$$\dot{x} = \mathrm{diag}\{2,1\}x + \left[\, 0 \;\; 0.1 \,\right]^\top w + \mathrm{diag}\{4,2\}u\,,$$

$$z_1 = \begin{bmatrix} x \\ u \end{bmatrix}, \quad z_2 = \begin{bmatrix} 2x \\ u \end{bmatrix},$$

and $\gamma = 0.5$. A solution P_1 and P_2 to the coupled AREs (10) is found using the algorithm proposed in Proposition 1 with $\kappa = 1$. The notation $L_i = \begin{bmatrix} l_{11}^i & l_{12}^i \\ l_{12}^i & l_{22}^i \end{bmatrix}$ and $\tilde{P}_i = \begin{bmatrix} \tilde{p}_{11}^i & \tilde{p}_{12}^i \\ \tilde{p}_{12}^i & \tilde{p}_{22}^i \end{bmatrix}$ is used in what follows. The initial conditions for the matrices are selected as $\tilde{P}_i(0) = 0$, $i = 1,2$, $L_1(0) = -I$ and $L_2(0) = I$. The time histories of p_{11}^i (solid, black lines), p_{12}^i (dash-dotted lines, black lines), p_{22}^i (dotted, black lines), l_{11}^i (solid, grey lines), l_{12}^i (dash-dotted lines, grey lines) and l_{22}^i (dotted, grey lines) are shown in Figure 2, for $i = 1$ (top) and $i = 2$ (bottom). The matrices converge to $\tilde{P}_1 = L_1 = \mathrm{diag}\{-0.4585, -0.9252\}$ and $\tilde{P}_2 = L_2 = \mathrm{diag}\{0.6404, 1.292\}$, and it is easily verified that these matrices are a solution for (10).

6.3 A nonlinear system

Consider now the nonlinear scalar system with dynamics

$$\dot{x} = ax(1 + \sin x) + b_1 w + b_2 u\,,$$

with $a \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, the performance variables $z_1 = z_2 = z$,

$$z = \left[\, cx \;\; u \,\right]^\top ,$$

and the resulting Problem 2. Suppose $a = -2$, $b_1 = 0.4$, $b_2 = 1$ and $c = 2$. The functions $p_1(x) = l_1(1 + \sin x)$ and $p_2(x) = l_2(1 + \sin x)$ with $l_1 = -1.5$ and $l_2 = 1.8$ constitute an algebraic $\bar{P}$ matrix solution with $\Sigma_1(x) > 0$ and $\Sigma_2(x) > 0$ in a neighbourhood of the origin. The dynamic feedback strategies (19) are constructed using this algebraic $\bar{P}$ matrix solution with the selection of parameters $R_1 = 1$, $R_2 = 1.5$, $k = 0.1$ and $\xi(0) = 0$.

Consider first the scenario in which $x(0) = 0$ and the system is subject to the disturbance w shown in the top graph of Figure 3. Note that this disturbance is such that the trajectory of the system remains in the neighbourhood Ω in which the inequalities (18) are satisfied. The time

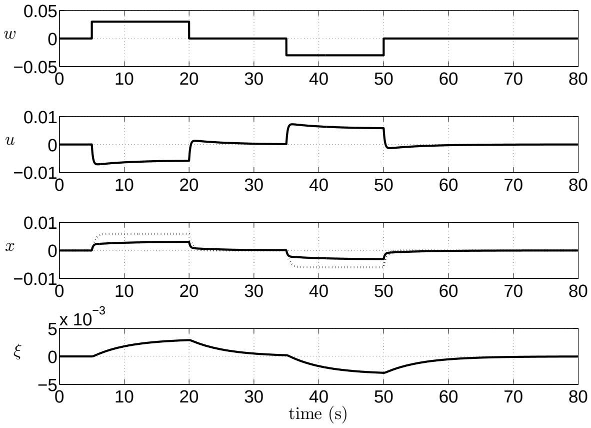

Fig. 3. Time histories of the the disturbance w, dynamic feedback control u^*, x and ξ (top to bottom).

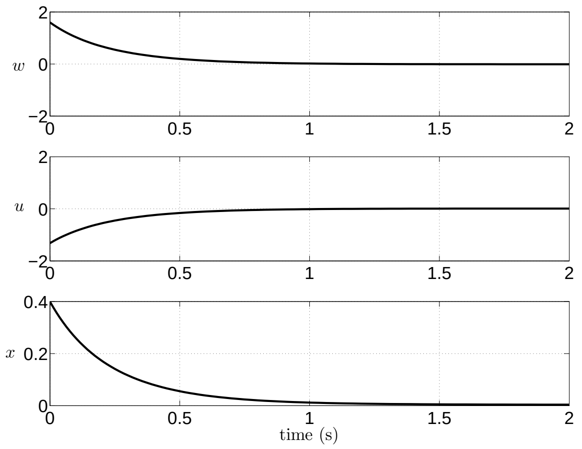

Fig. 4. Time histories of the the disturbance w^* (top), dynamic feedback control u^* (middle), x (bottom).

history of the dynamic feedback u^* given in (19), corresponding to this selection of algebraic $\bar{P}$ matrix solution and disturbance, is shown in the second graph of Figure 3, whereas the time histories of the state x when $u = u^*$ (solid line) and $u = 0$ (dotted line) is shown in the third graph of the same figure. The time history of ξ is shown in the bottom graph of Figure 3.

Consider now the second scenario in which the system has been perturbed so that the initial state is nonzero. In particular consider the case in which $x(0) = 0.4$. The resulting time histories of w^* (top), u^* (middle) and x (bottom) when the dynamic feedback strategy u^* and dynamic worst-case feedback w^* given in (19) are applied are shown in Figure 4. For completeness the time history of ξ is shown in Figure 5.

7. CONCLUSIONS AND FUTURE WORK

The problem of mixed H_2/H_∞ control for a class of nonlinear systems is considered. Drawing inspiration from Limebeer et al. [1992, 1994], Lin [1995] the problem is formulated as a two-player, nonzero-sum differential game. A

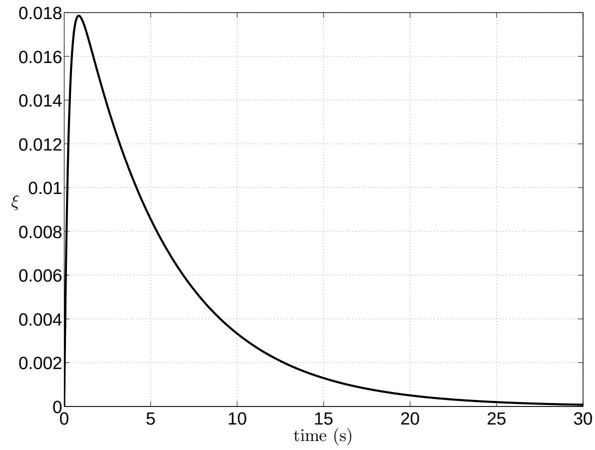

Fig. 5. Time history of the the disturbance ξ.

heuristic algorithm for obtaining solutions for the coupled AREs which characterise the linear problem considered in Limebeer et al. [1994] is provided before constructive approximate solutions for the general nonlinear case are given. A series of simple numerical examples for linear and nonlinear systems is then provided.

Directions for future research include determining a strict relationship between solutions to the *modified* differential game in Remark 9 and the original differential game and the mixed H_2/H_∞ control problem in Problems 1 and 2. This may rely on notions similar to the so-called ϵ_α-Nash equilibrium solution introduced in Mylvaganam et al. [2015]. It is also of interest to consider multi-player mixed H_2/H_∞ control as well as comprehensive simulation studies. It is also of interest to study the convergence properties of the algorithm proposed in Proposition 1.

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