DOA and Range Estimation of Multiple Sources Under the Wideband Assumption

Zexi Fang and Athanassios Manikas
Department of Electrical and Electronic Engineering
Imperial College London
{zexi.fang12, a.manikas}@imperial.ac.uk

***Abstract*—In this paper, two novel channel parameter estimation algorithms are proposed under the "wideband assumption" (WBA), where a wavefront varies significantly when traversing through the sensors of an array. The first is a covariance-based algorithm that utilizes the cross-covariance matrix between two subvectors of the received signal vector and its singular value decomposition to reconstruct the parameter-dependent signal subspace. The second employs the rotation of the array reference point to transform the WBA problem to its "narrowband assumption" (NBA) counterpart so that estimation techniques under the NBA are readily applicable. Through computer simulation studies, the two proposed approaches are shown to successfully estimate the channel parameters under the WBA with outstanding accuracy in terms of the root mean squared error.**

***Index Terms*—Array processing, estimation problem, wideband assumption, covariance matrix, array reference point.**

NOMENCLATURE

a, A	Scalar
$\underline{a}, \underline{A}$	Column vector
$\mathbb{A}$	Matrix
$(\cdot)^T$	Transpose
$(\cdot)^H$	Hermitian
$\|\cdot\|$	Euclidean norm
$\lfloor\cdot\rfloor$	Floor function
$\odot$	Hadamard product
$\oslash$	Hadamard quotient
$\otimes$	Kronecker product
$\boxtimes$	Khatri–Rao product
$\underline{A}^b$	Element-wise b-th power of $\underline{A}$
$\text{diag}(\underline{A})$	Diagonal matrix formed from $\underline{A}$
$\exp(\underline{A})$	Element-wise exponential of $\underline{A}$
$\text{rank}(\mathbb{A})$	Rank of $\mathbb{A}$
$\mathcal{E}\{\cdot\}$	Expectation operator
$\underline{1}_N$	Column vector of N ones
$\underline{0}_N$	Column vector of N zeros
$\mathbb{I}_N$	$N \times N$ identity matrix
$\mathbb{O}_{M\times N}$	$M \times N$ zero matrix
$\mathcal{Z}$	Set of integers
$\mathcal{R}$	Set of real numbers
$\mathcal{C}$	Set of complex numbers

I. INTRODUCTION

In the field of wireless communications using array processing, the estimation problem (i.e., the estimation of the channel parameters of interest) has been an active and important area of research. Directions of arrival (DOAs) of wireless signals exemplify the common channel parameters to be estimated in this problem. A widely used family of techniques in this domain is the signal subspace technique, where the signals of multiple sources are mapped to a signal subspace, which is embedded in a high-dimensional complex observation space. The signal subspace is spanned by the array manifold vectors of the sources. Therefore, with the estimate of the signal subspace, projection operators can be exploited to determine the parameters (e.g., DOAs) of the array manifold vectors that reside in the signal subspace. Moreover, the subspace techniques enjoy a substantial superresolution performance advantage compared to other parameter estimation methods like maximum likelihood, maximum entropy, and conventional beamforming [1].

However, subspace techniques like multiple signal classification (MUSIC) [1] or root-MUSIC [2] are only applicable under the "narrowband assumption" (NBA), which is defined as the case where a wavefront remains unchanged when it traverses through the sensors of an array [3]. This is in general valid in compact (small aperture and collocated) arrays. Conversely, if the array elements are distributed in space with large intersensor spacing, the NBA does not generally hold because the transmitted wavefront may change significantly when it traverses through the sensors of the array. This is defined as the "wideband assumption" (WBA)[1]. In this case, algorithms such as MUSIC fail to operate and estimate channel parameters correctly.

In this paper, two novel channel parameter estimators under the WBA are proposed. The first is a covariance-based approach that makes use of the cross-covariance matrix between two subvectors of the received signal vector and its singular value decomposition to estimate the parameter-dependent signal subspace. On the other hand, the second is a reference-based approach that employs the concept of the rotation of the array reference point to transform the WBA problem to its NBA counterpart before the application of the methods under the NBA. In this paper, the channel parameters of interest under the WBA are the DOAs and ranges. Also, the performance of the proposed approaches is examined in terms of the root mean squared error (RMSE) using computer

[1]Note that the concept of the WBA should not be confused with that of the wideband signal.

simulation studies.

The organization of the rest of this paper is as follows. In Section II, the signal model under the WBA is presented. In Section III, two novel channel parameter estimators under this assumption are proposed and explained. In Section IV, their performance is assessed via computer simulation studies. Finally, in Section V, the paper is concluded.

II. Signal Model

Consider an array of N widely distributed sensors (i.e., a large aperture array) with a known array geometry. The array geometry is described as

$$[\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N] = \left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right]^T \in \mathcal{R}^{3\times N} \tag{1}$$

where $\underline{r}_k \in \mathcal{R}^{3\times 1}$ denotes the Cartesian coordinates of the k-th sensor and $\underline{r}_x$, $\underline{r}_y$, and $\underline{r}_z \in \mathcal{R}^{N\times 1}$ represent the coordinates of the x-, y-, and z-axis. Without loss of generality, the first sensor is selected as the array reference point and is located at the origin of the coordinate system. The array receives the signals from M uncorrelated sources that follow the WBA via the line-of-sight paths solely with $M < N$. In addition, it is supposed that these sources are temporally uncorrelated. With reference to Fig. 1, spherical wave propagation is assumed. Furthermore, the position (i.e., Cartesian coordinates) of the i-th source can be expressed as a vector

$$\underline{\bar{r}}_i \triangleq \underline{\bar{r}}(\theta_i, \phi_i, \rho_i) = \rho_i \underline{u}_i \in \mathcal{R}^{3\times 1} \tag{2}$$

with

$$\begin{aligned}\underline{u}_i &\triangleq \underline{u}(\theta_i, \phi_i)\\ &= [\cos\theta_i \cos\phi_i, \sin\theta_i \cos\phi_i, \sin\phi_i]^T \in \mathcal{R}^{3\times 1}\end{aligned} \tag{3}$$

where θ_i is its azimuth angle measured counterclockwise from the positive x-axis, ϕ_i is its elevation angle, and ρ_i is its range between the array reference point and the source itself. Without loss of generality, in Fig. 1 and overall in this paper, the elevation angle ϕ_i is assumed equal to zero.

The baseband signal $\underline{x}(t) \in \mathcal{C}^{N\times 1}$ received at the array under the WBA can be modeled as

$$\begin{aligned}\underline{x}(t) &= [x_1(t), x_2(t), \ldots, x_N(t)]^T\\ &= \sum_{i=1}^{M} \underline{S}_i \odot \underline{m}_i(t) + \underline{n}(t)\end{aligned} \tag{4}$$

where, for the i-th source, $\underline{S}_i \in \mathcal{C}^{N\times 1}$ is the spherical wave array manifold vector and

$$\underline{m}_i(t) = [m_i(t-\tau_{i1}), m_i(t-\tau_{i2}), \ldots, m_i(t-\tau_{iN})]^T \in \mathcal{C}^{N\times 1} \tag{5}$$

contains the delayed versions of the message received at all the sensors with τ_{ik} denoting the relative delay between the array reference point and the k-th sensor. Note that $\tau_{i1} = 0$ for all the sources as the first sensor is the array reference point. Also, if the i-th source follows the NBA, then $\underline{m}_i(t) = m_i(t)\,\underline{1}_N$. Besides, $\underline{n}(t) \in \mathcal{C}^{N\times 1}$ represents the complex additive white Gaussian noise with zero mean and noise power σ_n^2. Further, the spherical wave array manifold vector is given as [4], [5]

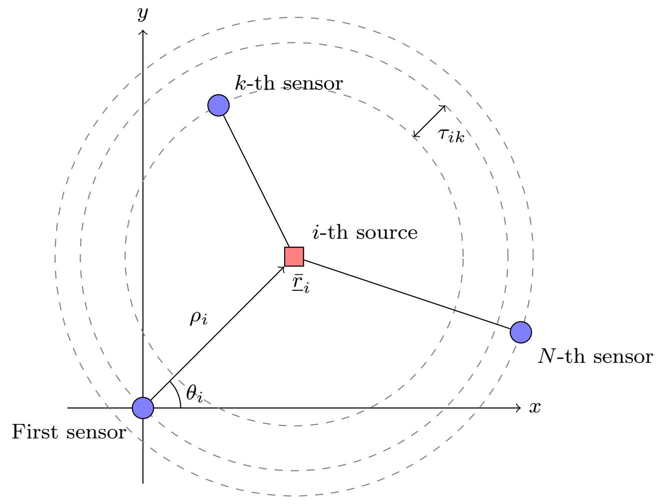

Fig. 1. Array and source geometry. A two-dimensional representative geometry of the sensor array (represented by the blue circles) and the i-th source (represented by the red square) of the azimuth angle θ_i and range ρ_i. The term τ_{ik} is the relative delay between the array reference point and the k-th sensor for the i-th source.

$$\underline{S}_i = \rho_i^{\alpha} \underline{\rho}^{-\alpha}(\theta_i, \rho_i) \odot \exp\left(-j2\pi \frac{F_c}{c}\left(\rho_i \underline{1}_N - \underline{\rho}(\theta_i, \rho_i)\right)\right) \tag{6}$$

with $\underline{\rho}(\theta_i, \rho_i) \in \mathcal{R}^{N\times 1}$ defined as follows

$$\underline{\rho}(\theta_i, \rho_i) \triangleq \sqrt{\rho_i^2 \underline{1}_N + \underline{r}_x^2 + \underline{r}_y^2 + \underline{r}_z^2 - 2\rho_i \left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right] \underline{u}_i} \tag{7}$$

where α represents the known path loss exponent, F_c expresses the carrier frequency, and c denotes the speed of light. Equation (4) can be rewritten in a more compact matrix format as

$$\underline{x}(t) = (\mathbb{S} \odot \mathbb{M}(t))\, \underline{1}_M + \underline{n}(t) \tag{8}$$

where

$$\mathbb{S} = [\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_M] \in \mathcal{C}^{N\times M}; \tag{9}$$

$$\mathbb{M}(t) = [\underline{m}_1(t), \underline{m}_2(t), \ldots, \underline{m}_M(t)] \in \mathcal{C}^{N\times M}. \tag{10}$$

It can be proven (given in Appendix A) that the covariance matrix of the received signal vector $\underline{x}(t)$ is given, under the WBA, as

$$\begin{aligned}\mathbb{R}_{xx} &= \mathcal{E}\left\{\underline{x}(t)\, \underline{x}^H(t)\right\}\\ &= \sum_{i=1}^{M} \underline{S}_i \underline{S}_i^H \odot \mathbb{R}_{m_i m_i} + \mathbb{R}_{nn} \in \mathcal{C}^{N\times N}\end{aligned} \tag{11}$$

where $\mathbb{R}_{m_i m_i}$ and $\mathbb{R}_{nn} = \sigma_n^2 \mathbb{I}_N$ are the respective covariance matrices of the i-th message and the noise. With reference to Equ. (11), if Q denotes the number of the most significant eigenvalues of $\mathbb{R}_{xx}$, then Q is determined by the rank of $\mathbb{R}_{m_i m_i}$ but also bounded from above by N, which is the dimension of $\mathbb{R}_{xx}$; that is,

$$Q = \min\left(\sum_{i=1}^{M} \text{rank}(\mathbb{R}_{m_i m_i}), N\right). \tag{12}$$

Based on Equ. (12), there are the following two cases.

- If $Q < N$, then the sources span solely a subspace of the observation space. In this case, a source under the NBA spans the same one-dimensional subspace as its corresponding manifold vector. By contrast, a source under the WBA spans a multidimensional subspace that is the transformed[2] version of the most significant eigenspace of its covariance matrix. This means that only the parameters of the sources under the NBA can be estimated using subspace techniques like MUSIC.
- If $Q = N$, then the sources span the entire observation space and the dimension of the noise subspace is zero. This means that no parameters can be estimated using subspace techniques like MUSIC.

In either case, the parameters of the sources under the WBA cannot be estimated using the subspace techniques directly. Hence, two approaches are proposed and explained in the next section for solving the estimation problem under the WBA.

III. Proposed Approaches Under the Wideband Assumption

A. Covariance-Based Approach

From the phase of the spherical wave manifold vector given in Equ. (6), the relative delay vector $\underline{\tau}_i \in \mathcal{R}^{N\times 1}$ associated with the i-th source can be derived as

$$\begin{aligned}\underline{\tau}_i &\triangleq \underline{\tau}(\theta_i, \rho_i) = [\tau_{i1}, \tau_{i2}, \ldots, \tau_{iN}]^T \\ &= \frac{1}{c}\left(\rho \underline{1}_N - \underline{\rho}(\theta_i, \rho_i)\right). \end{aligned} \quad (13)$$

If $\underline{\tau}_i$ was known, then the received signal vector $\underline{x}(t)$ could be reversely delayed using $\underline{\tau}_i$, forming the vector $\underline{\bar{x}}_i(t) \in \mathcal{C}^{N\times 1}$ defined as follows

$$\underline{\bar{x}}_i(t) = [x_1(t+\tau_{i1}), x_2(t+\tau_{i2}), \ldots, x_N(t+\tau_{iN})]^T. \quad (14)$$

Consequently, the corresponding reversely delayed message vector of the i-th source is

$$\underline{\bar{m}}_i(t) = \begin{bmatrix} m_i(t-\tau_{i1}+\tau_{i1}) \\ m_i(t-\tau_{i2}+\tau_{i2}) \\ \vdots \\ m_i(t-\tau_{iN}+\tau_{iN}) \end{bmatrix} = m_i(t)\,\underline{1}_N \in \mathcal{C}^{N\times 1}. \quad (15)$$

This means that for the i-th source having the parameters θ_i and ρ_i, the reversely delayed i-th message is aligned at time zero and follows the NBA. Meanwhile, $\underline{\tau}_i$ is not equal to the delay vectors of the other sources (provided that the delay vectors of all the sources are different, which is valid in general). Hence, the other sources still remain under the WBA.

The signal vector given in Equ. (14) can be partitioned into two nonoverlapping (but possibly interlacing) subvectors. Without loss of generality, assume that the two subvectors contain the first N_A and last N_B elements respectively with $N = N_A + N_B$; that is,

$$\underline{\bar{x}}_i(t) = \left[\underline{\bar{x}}_{iA}^T(t), \underline{\bar{x}}_{iB}^T(t)\right]^T \quad (16)$$

[2]The transformation is governed by the array manifold vector of this source.

where $\underline{\bar{x}}_{iA}(t) \in \mathcal{C}^{N_A\times 1}$ and $\underline{\bar{x}}_{iB}(t) \in \mathcal{C}^{N_B\times 1}$. The cross-covariance matrix between these two subvectors can be calculated as

$$\begin{aligned}\mathbb{R}_i &= \mathcal{E}\left\{\underline{\bar{x}}_{iA}(t)\,\underline{\bar{x}}_{iB}^H(t)\right\} \\ &= \underline{S}_{iA}\underline{S}_{iB}^H \odot \mathbb{R}_{\bar{m}_{iA}\bar{m}_{iB}} \in \mathcal{C}^{N_A\times N_B} \end{aligned} \quad (17)$$

where $\underline{S}_{iA} \in \mathcal{C}^{N_A\times 1}$ and $\underline{S}_{iB} \in \mathcal{C}^{N_B\times 1}$ are the subvectors of the manifold vector $\underline{S}_i$ and $\mathbb{R}_{\bar{m}_{iA}\bar{m}_{iB}} \in \mathcal{R}^{N_A\times N_B}$ denotes the cross-covariance matrix between the subvectors of the reversely delayed i-th message. Note that the cross-covariance matrices of the other sources and noise become zero due to the property of their temporal statistics. Since the reversely delayed i-th message follows the NBA, its cross-covariance matrix is $\mathbb{R}_{\bar{m}_{iA}\bar{m}_{iB}} = P_i \underline{1}_{N_A}\underline{1}_{N_B}^T$ with rank one where P_i is its signal power. Therefore, $\mathbb{R}_i$ is simplified to

$$\mathbb{R}_i = P_i \underline{S}_{iA}\underline{S}_{iB}^H. \quad (18)$$

This is a rank one matrix and is completely determined by the subvectors $\underline{S}_{iA}$ and $\underline{S}_{iB}$ of the manifold vector $\underline{S}_i$. Moreover, it is not Hermitian and may not be square. Thus, its singular value decomposition can be written as

$$\mathbb{R}_i = \mathbb{U}_i \mathbb{D}_i \mathbb{V}_i^H \quad (19)$$

where $\mathbb{U}_i \in \mathcal{C}^{N_A\times N_A}$ is a unitary matrix containing the left singular vectors, $\mathbb{D}_i \in \mathcal{R}^{N_A\times N_B}$ is a rectangular diagonal matrix containing the singular values on its diagonal, and $\mathbb{V}_i \in \mathcal{C}^{N_B\times N_B}$ is a unitary matrix containing the right singular vectors. The structure of the matrix containing the singular values is

$$\mathbb{D}_i = \begin{bmatrix} P_i \,\|\underline{S}_{iA}\|\,\|\underline{S}_{iB}\|, & \underline{0}_{N_B-1}^T \\ \underline{0}_{N_A-1}, & \mathbb{O}_{(N_A-1)\times(N_B-1)} \end{bmatrix}. \quad (20)$$

In other words, among the $\min(N_A, N_B)$ singular values, there is only one nonzero singular value; i.e., the minimum singular value is zero and its multiplicity is $\min(N_A, N_B)-1$. Further, the left singular vector corresponding to the most significant singular value spans the same subspace as $\underline{S}_{iA}$ and, similarly, this applies to the right singular vector and $\underline{S}_{iB}$. This feature can be employed to estimate the parameters of the i-th source, regardless it following the NBA or WBA before the received signal is reversely delayed.

According to the theory explained above, the estimation algorithm is designed as follows. Suppose that L snapshots are collected at the array at the sampling frequency F_s and sampling period $T_s = 1/F_s$. The snapshots can be written in a matrix format as

$$\begin{aligned}\mathbb{X} &= [\underline{x}(t_1), \underline{x}(t_2), \ldots, \underline{x}(t_L)] \\ &= [\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N]^T \in \mathcal{C}^{N\times L} \end{aligned} \quad (21)$$

where $t_\ell = \ell T_s$ is the ℓ-th time instance and $\underline{x}_k \in \mathcal{C}^{L\times 1}$ is the vector of L snapshots collected at the k-th sensor. In order to estimate the parameters θ and ρ of all the sources, a cost function of these two parameters is to be maximized through a two-dimensional search of them in a parameter space. In contrast to the MUSIC algorithm in [1], the null subspace in the proposed covariance-based approach is parameter-dependent. Based on the received signal matrix $\mathbb{X}$ of Equ. (21), the steps of the proposed approach for estimating the channel parameters are outlined below.

Step 1: For a particular (θ, ρ), calculate the discrete relative delay vector based on Equ. (13) as

$$\begin{aligned}\underline{\ell}(\theta,\rho) &= [\ell_1(\theta,\rho), \ell_2(\theta,\rho), \dots, \ell_N(\theta,\rho)]^T \\ &= \lfloor F_s \underline{\tau}(\theta,\rho) \rfloor \in \mathcal{Z}^{N\times 1} \end{aligned} \quad (22)$$

where $\ell_k(\theta,\rho)$ denotes its k-th element.

Step 2: Reversely delay the signal received at the k-th sensor (i.e., the k-th row of $\mathbb{X}$, in the format of a column vector) using $\ell_k(\theta,\rho)$ as

$$\underline{\bar{x}}_k(\theta,\rho) = \mathbb{C}_L^{L-\ell_k(\theta,\rho)} \underline{x}_k \in \mathcal{C}^{L\times 1} \quad (23)$$

where $\mathbb{C}_L$ is the L-dimensional circular shift matrix given as

$$\mathbb{C}_L = \begin{bmatrix} \underline{0}_{L-1}^T, & 1 \\ \mathbb{I}_{L-1}, & \underline{0}_{L-1} \end{bmatrix} \in \mathcal{Z}^{L\times L}. \quad (24)$$

By repeating this for all k (i.e., all the rows of $\mathbb{X}$), form the reversely delayed signal matrix associated with this (θ,ρ) as

$$\begin{aligned}\bar{\mathbb{X}}(\theta,\rho) &= [\underline{\bar{x}}_1(\theta,\rho), \underline{\bar{x}}_2(\theta,\rho), \dots, \underline{\bar{x}}_N(\theta,\rho)]^T \\ &= \left[\bar{\mathbb{X}}_A^T(\theta,\rho), \bar{\mathbb{X}}_B^T(\theta,\rho)\right]^T \in \mathcal{C}^{N\times L} \end{aligned} \quad (25)$$

where $\bar{\mathbb{X}}_A \in \mathcal{C}^{N_A\times L}$ and $\bar{\mathbb{X}}_B \in \mathcal{C}^{N_B\times L}$ are its two submatrices containing its first N_A and last N_B rows respectively.

Step 3: Calculate the cross-covariance matrix between the two submatrices $\bar{\mathbb{X}}_A$ and $\bar{\mathbb{X}}_B$; that is,

$$\mathbb{R}_{\bar{x}_A\bar{x}_B}(\theta,\rho) = \frac{1}{L}\bar{\mathbb{X}}_A(\theta,\rho)\,\bar{\mathbb{X}}_B^H(\theta,\rho) \in \mathcal{C}^{N_A\times N_B}. \quad (26)$$

Step 4: Obtain the null subspace of $\mathbb{R}_{\bar{x}_A\bar{x}_B}(\theta,\rho)$ using its singular value decomposition. Taking its left singular vectors as an example, its null subspace, denoted as $\mathbb{E}_n(\theta,\rho) \in \mathcal{C}^{N_A\times(N_A-1)}$, consists of its left singular vectors corresponding to its $N_A - 1$ least significant singular values.

Step 5: Evaluate the following cost function

$$\xi(\theta,\rho) = \frac{\underline{S}_A^H(\theta,\rho)\,\underline{S}_A(\theta,\rho)}{\underline{S}_A^H(\theta,\rho)\,\mathbb{E}_n(\theta,\rho)\,\mathbb{E}_n^H(\theta,\rho)\,\underline{S}_A(\theta,\rho)}. \quad (27)$$

Step 6: Repeat Steps 1 to 5 $\forall\theta$ and $\forall\rho$ in the parameter space.

B. Reference-Based Approach

In addition to the covariance-based method proposed above, the channel parameters of the sources under the WBA can also be estimated by exploiting the concept of the rotation of the array reference point.

As described in Section II, the first sensor is selected as the array reference point without loss of generality. This is defined as the primary reference point, utilizing which the manifold vector associated with the i-th source is $\underline{S}_i$. Now consider that the array reference point is changed to the k-th sensor. In this case, the new reference point is $\underline{r}_k$, and the array geometry and azimuth angles and ranges of all the sources are measured with respect to $\underline{r}_k$. Moreover, the manifold vector of the i-th source under the new reference point is $S_{ik}^{-1}\underline{S}_i$ where S_{ik} is the k-th element of $\underline{S}_i$ [4], [5].

In the presence of M uncorrelated sources, the signal received at the array system when the k-th sensor is the reference point can be expressed as [4], [5]

$$\underline{x}_k(t) = \sum_{i=1}^{M} S_{ik}^{-1}\underline{S}_i \odot \underline{m}_i(t) + \underline{n}(t) \in \mathcal{C}^{N\times 1}. \quad (28)$$

Poll the reference point from the first sensor to the last and preprocess (concatenate and average) all the received signals as (the derivation is given in Appendix B)

$$\begin{aligned}\underline{\bar{x}}(t) &= \frac{1}{\sqrt{N}}\left(\mathbb{I}_N \otimes \underline{1}_N\right)^T \left[\underline{x}_1^T(t), \underline{x}_2^T(t), \dots, \underline{x}_N^T(t)\right]^T \\ &= \mathbb{A}\underline{\bar{m}}(t) + \underline{\bar{n}}(t) \in \mathcal{C}^{N\times 1} \end{aligned} \quad (29)$$

where $\mathbb{A} = \underline{1}_N\underline{1}_M^T \oslash \mathbb{S} \in \mathcal{C}^{N\times M}$ and

$$\underline{\bar{m}}(t) = \frac{1}{\sqrt{N}}\left(\mathbb{S}\odot\mathbb{M}(t)\right)^T \underline{1}_N \in \mathcal{C}^{M\times 1}; \quad (30)$$

$$\underline{\bar{n}}(t) = \frac{1}{\sqrt{N}}\underline{1}_N\underline{1}_N^T\underline{n}(t) \in \mathcal{C}^{N\times 1}. \quad (31)$$

Furthermore, the covariance matrix of the preprocessed signal $\underline{\bar{x}}(t)$ is given as

$$\mathbb{R}_{\bar{x}\bar{x}} = \mathcal{E}\left\{\underline{\bar{x}}(t)\,\underline{\bar{x}}^H(t)\right\} = \mathbb{A}\mathbb{R}_{\bar{m}\bar{m}}\mathbb{A}^H + \mathbb{R}_{\bar{n}\bar{n}} \in \mathcal{C}^{N\times N} \quad (32)$$

where

$$\mathbb{R}_{\bar{m}\bar{m}} = \mathcal{E}\left\{\underline{\bar{m}}(t)\,\underline{\bar{m}}^H(t)\right\} \in \mathcal{R}^{M\times M} \quad (33)$$

is a diagonal matrix with M nonzero elements on its diagonal and

$$\mathbb{R}_{\bar{n}\bar{n}} = \mathcal{E}\left\{\underline{\bar{n}}(t)\,\underline{\bar{n}}^H(t)\right\} = \sigma_n^2\underline{1}_N\underline{1}_N^T \in \mathcal{R}^{N\times N} \quad (34)$$

has the rank of one. According to the structure of $\mathbb{R}_{\bar{x}\bar{x}}$, its eigenspace comprises the $(M+1)$-dimensional signal subspace corresponding to the preprocessed messages and noise as well as the complementary $(N-M-1)$-dimensional null subspace. Thus, the null subspace can be exploited to estimate the channel parameters with the employment of the subspace estimation techniques under the NBA.

In practice, let $\mathbb{X}_k \in \mathcal{C}^{N\times L}$ denote the received signal matrix when the k-th sensor is the reference point and

$$\bar{\mathbb{X}} = \frac{1}{\sqrt{N}}\left(\mathbb{I}_N\otimes\underline{1}_N\right)^T\left[\mathbb{X}_1^T, \mathbb{X}_2^T, \dots, \mathbb{X}_N^T\right]^T \in \mathcal{C}^{N\times L} \quad (35)$$

denote the preprocessed signal matrix. Its covariance matrix can be constructed as

$$\mathbb{R}_{\bar{x}\bar{x}} = \frac{1}{L}\bar{\mathbb{X}}\bar{\mathbb{X}}^H \in \mathcal{C}^{N\times N}. \quad (36)$$

Its eigenspace corresponding to the zero eigenvalues is the null subspace and is denoted as $\mathbb{E}_n \in \mathcal{C}^{N\times(N-M-1)}$. The channel parameters can be estimated by maximizing the following cost function

$$\xi(\theta, \rho) = \frac{\underline{A}^H(\theta, \rho)\, \underline{A}(\theta, \rho)}{\underline{A}^H(\theta, \rho)\, \mathbb{E}_n \mathbb{E}_n^H\, \underline{A}(\theta, \rho)} \tag{37}$$

where $\underline{A}(\theta, \rho) = \underline{1}_N \oslash \underline{S}(\theta, \rho) \in \mathcal{C}^{N \times 1}$.

Briefly, the two proposed channel estimators can be summarized as the following steps.

- Covariance-based approach
  1) For a particular (θ, ρ) calculate the discrete relative delay vector using Equ. (22).
  2) Reversely delay the signals received at all the sensors using Equ. (23). Form the reversely delayed signal matrix and partition it into two submatrices using Equ. (25).
  3) Calculate the cross-covariance matrix between the two submatrices using Equ. (26).
  4) Find the null subspace of the cross-covariance matrix as its left or right singular vectors corresponding to its least significant singular values.
  5) Evaluate the cost function Equ. (27).
  6) Repeat Steps 1 to 5 $\forall\theta$ and $\forall\rho$ in the parameter space to estimate the channel parameters.
- Reference-based approach
  1) Poll the array reference point from the first sensor to the last and construct the respective matrices of the received signals using Equ. (35).
  2) Preprocess (concatenate and average) the received signals and calculate the covariance matrix of the preprocessed signal using Equ. (36).
  3) Find the null subspace of the covariance matrix as its eigenvectors corresponding to its least significant eigenvalues.
  4) Estimate the channel parameters by evaluating the cost function Equ. (37).

IV. Computer Simulation Studies

The performance of the two proposed channel estimators presented in Section III is assessed through computer simulation studies. In the simulations, a 20-element uniform circular array with $100\,\mathrm{m}$ intersensor spacing is utilized. The two subvectors in the covariance-based approach are chosen as two 10-element vectors that comprise alternating elements of the original received signal vector. Other simulation parameters are summarized in Table I.

First, consider a scenario in which the array receives the signals from four equipowered and uncorrelated sources under the WBA with the signal-to-noise ratio (SNR) of $20\,\mathrm{dB}$. The azimuth angles and ranges of the sources are listed in Table I while their elevation angles are assumed to be equal to $0°$. The joint azimuth and range estimation results of the covariance- and reference-based approaches are shown in Figs. 2 and 3 respectively. Four peaks can be clearly observed in both cases, indicating a successful estimation of the azimuth angles and ranges. The performance of the reference-based approach is better than its covariance-based counterpart. This is because

TABLE I
Simulation Parameters

Parameter	Value	Source	Azimuth	Range
Carrier frequency	3 GHz	1	18°	578 m
Sampling frequency	30 MHz	2	59°	551 m
Number of snapshots	200	3	137°	563 m
SNR	20 dB	4	156°	521 m

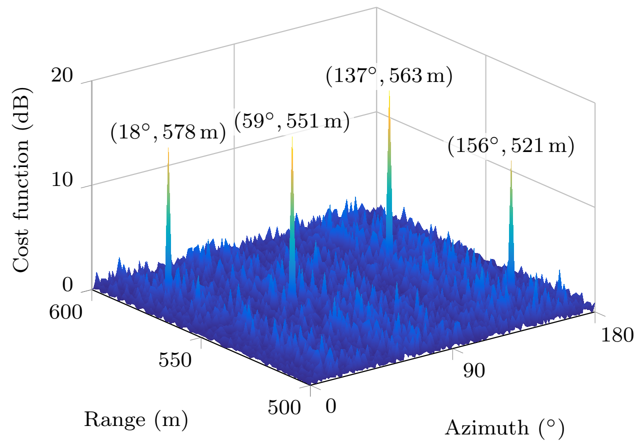

Fig. 2. Joint azimuth and range estimation of the covariance-based approach.

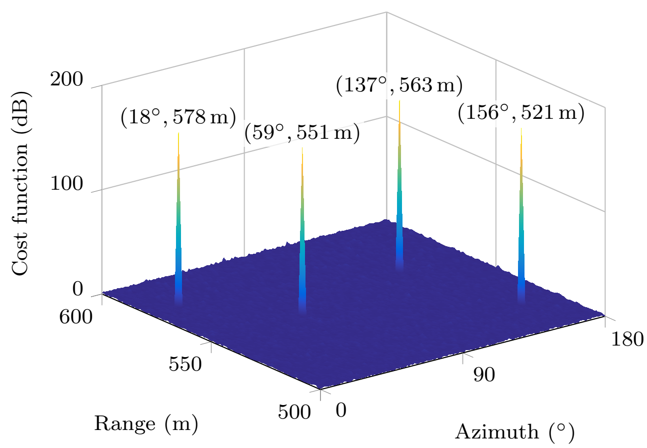

Fig. 3. Joint azimuth and range estimation of the reference-based approach.

in the covariance-based approach, there exists the residual of the undesired sources and noise in the cross-covariance matrix. Meanwhile, in the reference-based approach, the noise is transformed into an independent source that follows the NBA and does not contribute to the null subspace.

Second, consider a scenario in which only a single source under the WBA is present. Its azimuth angle, elevation angle, and range are $18°$, $0°$, and $578\,\mathrm{m}$ respectively. The RMSE of the azimuth and range estimates of the proposed approaches versus the product of the SNR and number of snapshots is shown in Fig. 4. The RMSE curves of the covariance-based approach decline as the product of the SNR and number of snapshots increases. This is an expected result as the

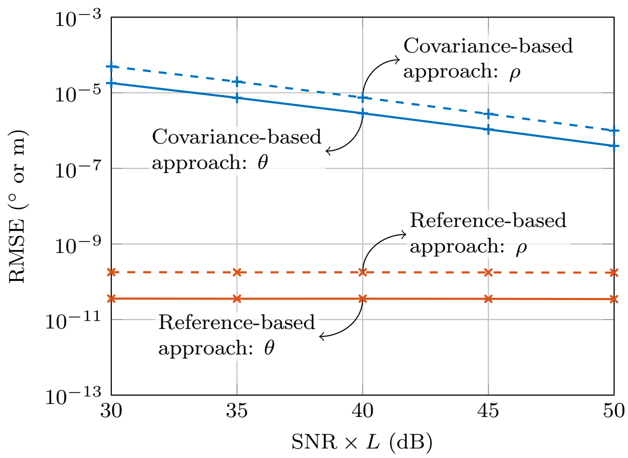

Fig. 4. Estimation RMSE. The true azimuth angle and range are 18° and 578 m respectively. The results are averaged over 10 000 realizations.

covariance-based approach belongs to the family of subspace techniques and the estimation error tends to zero as L tends to infinity. Furthermore, it is clear from Fig. 4 that the reference-based approach enjoys very small RMSE curves. This is due to the fact that the observation space is big (bigger than the covariance-based approach) and the preprocessor averages even further the noise effect.

In summary, both proposed approaches can estimate the channel parameters (azimuth angles and ranges) of multiple sources under the WBA very accurately.

V. Conclusions

In this paper, two channel parameter estimators under the WBA are proposed. The covariance-based approach utilizes the cross-covariance matrix between two nonoverlapping sub-vectors of the received signal vector in conjunction with its singular value decomposition to recover the parameter-dependent signal subspace. Meanwhile, the reference-based approach employs the concept of the rotation of the array reference point so that the algorithms under the NBA are readily applicable. Both proposed approaches are shown to successfully estimate the DOAs and ranges under the WBA with substantial accuracy in terms of the estimation RMSE.

Appendix A
Derivation of the Covariance Matrix

The covariance matrix of the received signal in the presence of M uncorrelated sources under the WBA is

$$\begin{aligned}
\mathbb{R}_{xx} &= \mathcal{E}\left\{\underline{x}(t)\,\underline{x}^H(t)\right\} \\
&= \mathcal{E}\left\{(\mathbb{S}\odot\mathbb{M}(t))\,\underline{1}_M\underline{1}_M^T\,(\mathbb{S}\odot\mathbb{M}(t))^H\right\} \\
&\quad + \underbrace{\mathcal{E}\left\{\underline{n}(t)\,\underline{n}^H(t)\right\}}_{\triangleq\mathbb{R}_{nn}} \\
&= \mathcal{E}\left\{\left(\sum_{i=1}^{M}\underline{S}_i\odot\underline{m}_i(t)\right)\left(\sum_{j=1}^{M}\underline{S}_j\odot\underline{m}_j(t)\right)^H\right\}+\mathbb{R}_{nn} \\
&= \mathcal{E}\left\{\sum_{i=1}^{M}\sum_{j=1}^{M}\underline{S}_i\underline{S}_j^H\odot\underline{m}_i(t)\,\underline{m}_j^H(t)\right\}+\mathbb{R}_{nn} \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\underline{S}_i\underline{S}_j^H\odot\underbrace{\mathcal{E}\left\{\underline{m}_i(t)\,\underline{m}_j^H(t)\right\}}_{\triangleq\mathbb{R}_{m_im_j}}+\mathbb{R}_{nn} \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}\underline{S}_i\underline{S}_j^H\odot\mathbb{R}_{m_im_j}+\mathbb{R}_{nn} \qquad (38)
\end{aligned}$$

where $\mathbb{R}_{m_im_j}$ denotes the covariance matrix between the i-th and j-th messages and $\mathbb{R}_{nn}$ denotes covariance matrix of the noise. Since all the sources are uncorrelated, the covariance matrix $\mathbb{R}_{m_im_j} = \mathbb{O}_{N\times N}$ if $i \neq j$. Therefore, Equ. (38) is simplified to

$$\begin{aligned}
\mathbb{R}_{xx} &= \sum_{\substack{i=1\\ j=i}}^{M}\underline{S}_i\underline{S}_j^H\odot\mathbb{R}_{m_im_j}+\mathbb{R}_{nn} \\
&= \sum_{i=1}^{M}\underline{S}_i\underline{S}_i^H\odot\mathbb{R}_{m_im_i}+\mathbb{R}_{nn}. \qquad (39)
\end{aligned}$$

Appendix B
Derivation of the Preprocessed Signal

The preprocessed (concatenated and averaged) signal in the reference-based approach is

$$\begin{aligned}
\bar{\underline{x}}(t) &= \frac{1}{\sqrt{N}}\left(\mathbb{I}_N\otimes\underline{1}_N\right)^T\left[\underline{x}_1^T(t),\ldots,\underline{x}_N^T(t)\right]^T \\
&= \frac{1}{\sqrt{N}}\left(\mathbb{I}_N\otimes\underline{1}_N\right)^T\left(\mathbb{A}\boxtimes(\mathbb{S}\odot\mathbb{M}(t))\right)\underline{1}_N \\
&\quad + \frac{1}{\sqrt{N}}\left(\mathbb{I}_N\otimes\underline{1}_N\right)^T\left(\underline{1}_N\otimes\underline{n}(t)\right) \\
&= \frac{1}{\sqrt{N}}\left(\mathbb{A}\boxtimes\underline{1}_N^T(\mathbb{S}\odot\mathbb{M}(t))\right)\underline{1}_N \\
&\quad + \frac{1}{\sqrt{N}}\left(\underline{1}_N\otimes\underline{1}_N^T\underline{n}(t)\right) \\
&= \mathbb{A}\frac{1}{\sqrt{N}}\,\mathrm{diag}\Big((\mathbb{S}\odot\mathbb{M}(t))^T\,\underline{1}_N\Big)\,\underline{1}_N+\frac{1}{\sqrt{N}}\underline{1}_N\underline{1}_N^T\underline{n}(t) \\
&= \mathbb{A}\frac{1}{\sqrt{N}}\left(\mathbb{S}\odot\mathbb{M}(t)\right)^T\underline{1}_N+\frac{1}{\sqrt{N}}\underline{1}_N\underline{1}_N^T\underline{n}(t) \\
&\triangleq \mathbb{A}\bar{\underline{m}}(t)+\bar{\underline{n}}(t) \qquad (40)
\end{aligned}$$

where $\mathbb{A} = \underline{1}_N\underline{1}_M^T \oslash \mathbb{S}$. Hence, this model follows the NBA with the manifold vectors being the columns of $\mathbb{A}$.

References

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Antennas and Propagation*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[2] J. Zhuang, W. Li, and A. Manikas, "Fast root-MUSIC for arbitrary arrays," *Electronics Letters*, vol. 46, no. 2, pp. 174–176, Jan. 2010.

[3] A. Manikas, *Differential Geometry in Array Processing*. London, UK: Imperial College Press, 2004.

[4] A. Manikas, Y. I. Kamil, and P. Karaminas, "Positioning in wireless sensor networks using array processing," in *2008 Global Communications Conference*, New Orleans, LA, 2008, pp. 1–5.

[5] A. Manikas, Y. I. Kamil, and M. Willerton, "Source localization using sparse large aperture arrays," *IEEE Transactions on Signal Processing*, vol. 60, no. 12, pp. 6617–6629, Dec. 2012.