SUM RULES AND NON-PERTURBATIVE PARAMETERS IN QCD

by

Martin Joseph Lavelle

A thesis presented for the Degree of Doctor of Philosophy of the University of London and the Diploma of Membership of Imperial College

Department of Physics
The Blackett Laboratory
Imperial College
London SW7 2BZ

September 1985

ABSTRACT

The technique of sum rules for light-cone vertex functions is applied to a vertex involving a proton state. The resulting sum rules give information about the form of the non-perturbative, low energy, distribution amplitude or wavefunction of the proton. The non-asymptotic antisymmetric component of the wavefunction is clearly indicated to be significant and of the opposite sign to the symmetric component. One such antisymmetric moment is predicted, in agreement with the result from two-point functions. A mixture of moments, symmetric and antisymmetric, is also predicted and the result is in agreement again with that of the two-point functions.

In the course of the above work a discrepancy in an earlier, similar, analysis of the pion wavefunction was discovered. A more sophisticated investigation of this wavefunction was therefore performed and the radiative corrections and power corrections to the $\pi\rho\omega$ vertex incorporated into the sum rules. One of the higher-twist wavefunctions that appear in the power corrections was identified as the major source of uncertainty in the vertex sum rules and its non-asymptotic form was then calculated from the two-point function sum rules. This was substituted into the vertex sum rules. The final result was again in agreement with the results from two-point sum rules and

unambiguously indicated the non-convex nature of the lowest-twist pion wavefunction.

PREFACE

The work presented in this thesis was carried out in the Theoretical Physics group of the Department of Physics, Imperial College, London between October 1982 and September 1985, under the supervision of Dr. H.F. Jones. Unless otherwise stated, the work is original and has not been submitted before for a degree of this or any other university.

I would especially like to thank my supervisor Hugh Jones, for introducing me to the subject and for his guidance and help throughout the course of this work. I should also like to thank Dr. A. Andrikopoulou, Dr. E.A. Bartnik, Dr. I.A. Fox, Dr. S. Generalis, Dr. G. Launer, Dr. J. Namysłowski and A. Vladikas for many useful discussions.

I would also like to thank Miss S. Nicolaou for typing the manuscript.

The financial support of the SERC is also gratefully acknowledged.

In a manner of speaking objects are colourless

Wittgenstein

To my parents

CONTENTS

CHAPTER ONE

INTRODUCTION

It is now generally believed that the observed strong interactions and their participant hadrons are governed by Quantum Chromodynamics (QCD). QCD is a renormalisable Lagrangian quantum field theory, based upon an unbroken non-Abelian gauge symmetry; SU(3).

Historically QCD came as a development of the quark model. In the early 1960´s a succesful description of the low energy spectroscopy of particles was obtained from the Eightfold Way (SU(3) flavour) classification of hadrons as composite objects made up of quarks and anti-quarks.

The parton model (1), using the naive assumption that the quarks behaved freely, was an initial attempt to explain the results from deep inelastic scattering experiments. These experiments use a high energy lepton to probe a nucleon, and the surprising result was found that the nucleon structure functions were approximately scale invariant. Under the assumptions of the parton model that the nucleon is made up of free, point like constituents, each of which carries a fraction x_i of the nucleon momentum and which take part in the interaction, all of the deep inelastic structure functions could be expressed in terms of the parton distributions (Fig.

1). In the Bjorken limit (where $x = Q^2/2\nu$, $\nu = 2p.q$ and Q^2 $(=-q^2)$, $\nu \longrightarrow \infty$ with x fixed) the moments of the structure functions

$$M_i^n = \int_0^1 dx \ x^{n-2} F_i(x) \qquad n \geqslant 2 \qquad 1.1$$

were found to be roughly independent of Q^2. The parton model agreed well with experiment for certain values of $x(0.15 \leqslant x \leqslant 0.25)$ for $2 \leqslant Q^2 \leqslant 100 \text{ GeV}^2$ (2), but outside this range of x a breakdown of scaling was observed. It should be noted that the model assumed that the confinement mechanism is independent of the particular process under consideration and that all the confining physics is contained in the parton distribution functions.

The absence of a theoretical grounding for the parton model means that these parton distribution functions are not calculable in the model. A picture in which we hope both to have an explanation of the short distance parton-like physics and also of the long distance confining interactions is QCD.

The need for a new quantum number so as to explain baryon statistics, the $\pi^0 \longrightarrow \gamma\gamma$ decay rate and the ratio R

$$R = \frac{\sigma(e^+e^- \longrightarrow \text{hadrons})}{\sigma(e^+e^- \longrightarrow \mu^+\mu^-)} \qquad 1.2$$

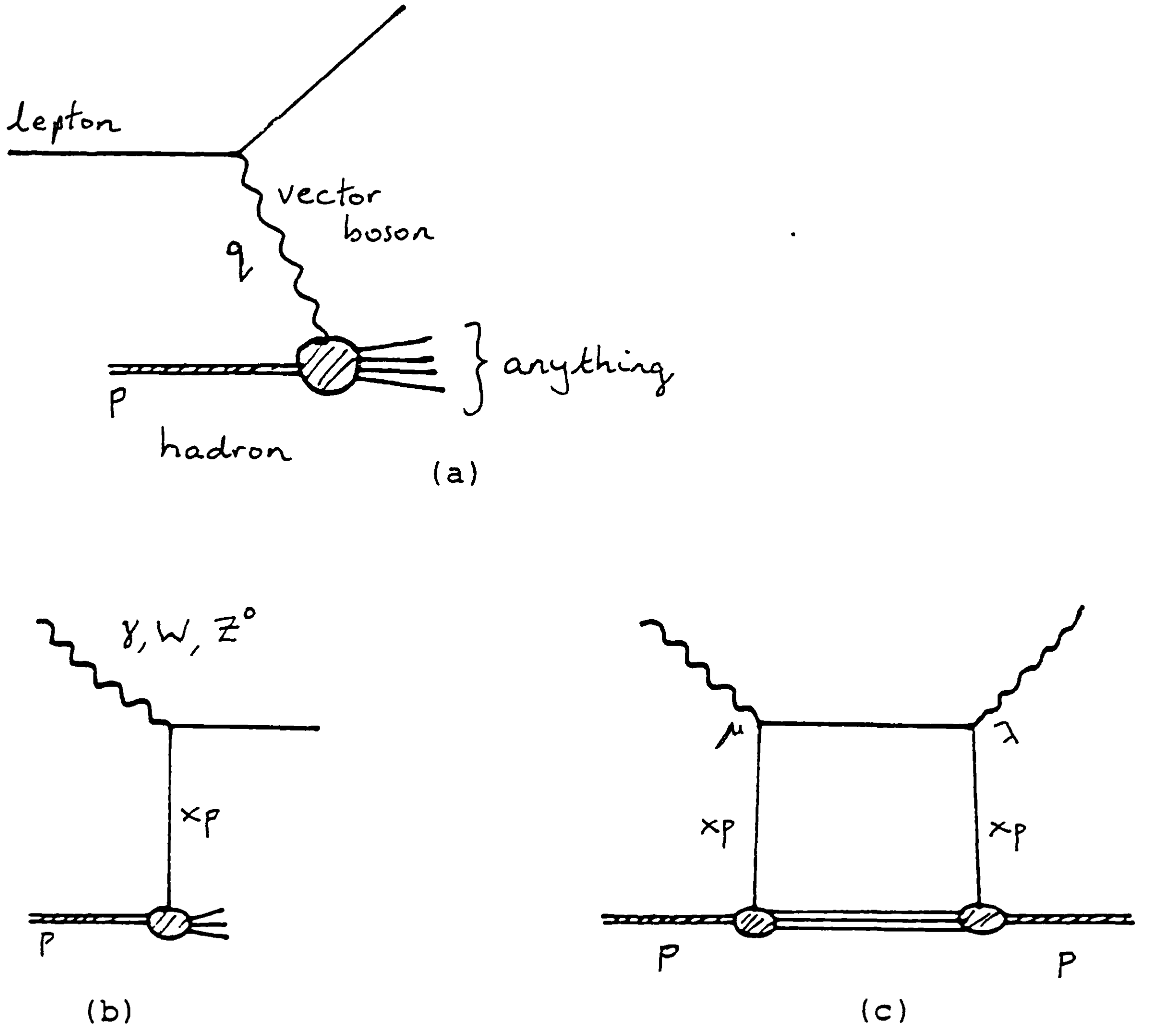


Fig. 1

(a) Deep inelastic lepton-hadron scattering

(b) Vector boson-parton scattering in the parton model

(c) The corresponding virtual Compton amplitude had led to the postulation of colour. Thus far physicists were considering that the role of the new quantum number was to produce the interaction that held the quarks together (via gluons) in some, not understood, qualitative fashion. However, with the discovery of asymptotic freedom (3) the situation changed dramatically and quantitative calculations became possible. This remarkable property of QCD says that the effective coupling constant vanishes at short distances. It was also shown that the coupling constant appeared to increase at greater distances.

Asymptotic freedom, through its implication that the strong interaction becomes weak at high energies, means that perturbation theory becomes applicable to hadronic physics in this regime. A large number of perturbative calculations have been performed over the intervening years, attempting to test QCD. The predictions so far obtained often agree with QCD, but much of the experimental data remains unexplained. This is particularly true at lower energies, where perturbation theory is expected to break down.

QCD was applied to deep inelastic scattering using the operator product expansion (OPE) and the renormalisation group (4). In this approach to inclusive processes the OPE permits us to factorise out the terms depending on large momenta. We obtain a sum of products of coefficient functions, which we calculate in perturbation theory, and matrix elements of operators between hadronic states, which last are non-perturbative. The Q^2 dependence of the coefficient functions is then calculated via the renormalisation group equations and the non-perturbative parts can be extracted from experiment. The results of this analysis showed that the parton model is, essentially, preserved in QCD, with only a slow Q^2 dependence now appearing in the hadronic structure functions. Indeed the results of this approach have been recast in parton model language by Altarelli and Parisi (5) and this equivalence has been further supported by calculations in the leading log approximation of ladder diagrams in an axial gauge (6). The treatment can be extended to deal with semi-inclusive processes, where we detect one hadron in the final state (7).

The extension of factorisation to exclusive processes by Brodsky and Lepage (8) gave QCD a new set of reactions to calculate and predict. The assumptions behind the factorisation of exclusive processes are the same as in earlier work; for any process involving sufficiently high momentum transfers, one can isolate all of the non-perturbative, confining, physics in the hadronic wave functions, which are then convoluted with a hard subprocess, T_H, calculable inside perturbation theory. (Fig. 2). The overall amplitude is therefore given by

$$\mathcal{M} = \int [dx][dY]\, \varphi^*(Y_i, P_\perp)\, T_H(x_i, Y_i, P_\perp)\, \varphi(x_i, P_\perp) \qquad (1.3)$$

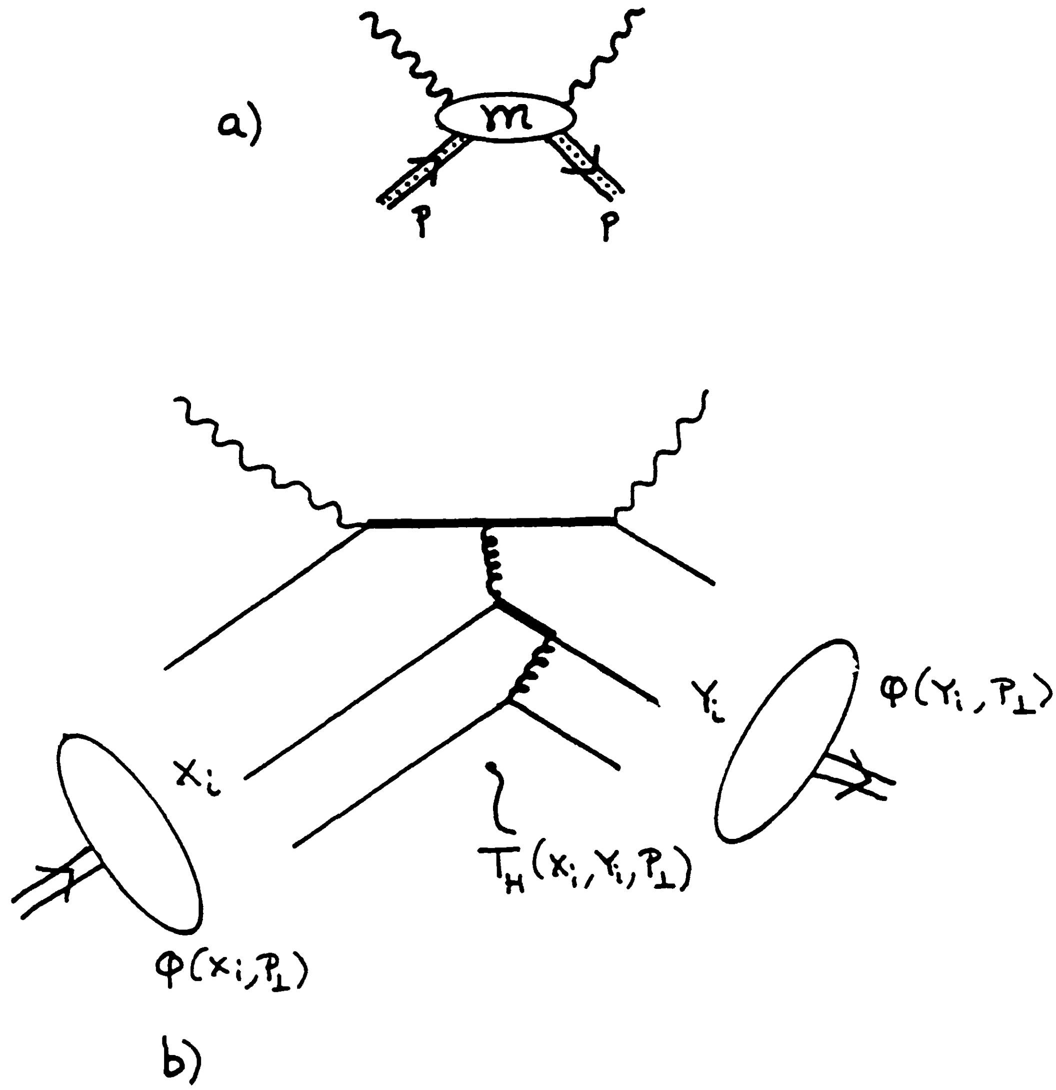


Figure 2: a) Proton Compton scattering amplitude
b) Short distance behaviour in infinite momentum frame. (Thick lines denote quark propagators at large Q , x and y are the momentum fractions carried by the quarks and p is the average transverse momentum.)

The Q^2 dependence of M comes both from T_H (which has the standard logarithmic deviations from scaling) and from the wavefunctions. In the hard subprocess we use the effective coupling constant g_{eff}, which depends upon the characteristic momentum transfer of the process in question. To leading order in perturbation theory g is given by

$$\alpha_s(Q^2) = \frac{g_{eff}^2}{4\pi} = \frac{4\pi}{\beta \ln(Q^2/\Lambda^2)} \tag{1.4}$$

where $\beta = 11 - 2/3n_f$, n_f being the number of flavours and Λ is the scale of QCD beyond which perturbation theory becomes reliable. The variation of $\varphi(x_i, Q)$ with Q is less drastic than that of T, but is rather more complicated. From a consideration of gluon exchange between the quarks we obtain an evolution equation for the wavefunction of the form

$$Q^2 \frac{d}{dQ^2} \varphi(x_i, Q) = \frac{\alpha_s(Q^2)}{4\pi} \int_0^1 [dY] V(x_i, Y_i) \varphi(Y_i, Q) \tag{1.5}$$

where $V(x_i, y_i)$ is the symmetric one-gluon exchange kernel. The general solution for a baryon is

$$\varphi(x_i, Q) = x_1 x_2 x_3 \sum_{n=0}^{\infty} a_n \tilde{\varphi}_n(x_i) \left[\ln\left(\frac{Q^2}{\Lambda^2}\right)\right]^{-\gamma_n} \tag{1.6}$$

where the $\tilde{\Phi}_n$ are polynomials in X_i, the γ_n are the anomalous dimensions of the lowest twist three quark operators for the baryon and the a_n are not calculable perturbatively. Asymptotically we are only left with n = 0 and, for the proton, we have

$$\Phi(x_i, Q) = C x_1 x_2 x_3 \left[\ln\left(\frac{Q^2}{\Lambda^2}\right)\right]^{-2/3\beta} \tag{1.7}$$

C is again a constant incalculable in perturbation theory, but using the universal nature of the wavefunction C can, in principle, be obtained from a comparison of two exclusive processes (9). It should be noted that perturbation theory only gives the asymptotic form of a wavefunction (1.7).

The most persuasive evidence that perturbative QCD can indeed be applied to large momentum transfer exclusive processes is the very good agreement for $Q^2 > 5\ GeV^2$ of the experimental data with the predicted power-law energy dependence (10); from dimensional counting the prediction is

$$\mathcal{M} \sim \frac{1}{(Q^2)^{\frac{n-4}{2}}} f(\theta_{cm}) \tag{1.8}$$

where n is the minimum number of quanta interacting (8 for proton compton scattering). Agreement with (1.8) is obtained for the pion and proton form factors, Compton scattering, photoproduction, meson-nucleon and nucleon-nucleon scattering.

On top of the above however, QCD can be used to obtain: numerical values for branching ratios, hadronic form factors, fixed-angle elastic scattering and exclusive processes from photon-photon collisions. This implies an ability to test the spin effects of QCD and to verify working hypotheses such as the massless spinor formalism.

An example where the above procedure is very succesful is $\pi^+ \pi^-$ and $K^+ K^-$ production from photon-photon collisions. The normalization and angular dependence of $\gamma\gamma \rightarrow \pi^+ \pi^-$ is predicted to be insensitive to the form of the pion wavefunction and the results can in fact be written in terms of the pion form factor, which may then be taken directly from experiment. Recent data (11) are in excellent agreement with both the normalization and energy dependence predictions of QCD. However, the experimental data for $\gamma\gamma \rightarrow \rho^0 \rho^0$ are much larger than the QCD predictions in similar energy ranges (and there is a suggestion of possible resonance enhancement of the cross-section). For $\gamma\gamma \rightarrow p\bar{p}$ there are two calculations (12) which, unfortunately, disagree. Neither, however, is in agreement with experiment (a factor of 60 disagreement with the result of Farrar et al., for example).

One of the most hotly debated areas in this field is the proton electromagnetic form factor. As stated earlier this observes the Q^{-4} power law dependence, indicating that it is probably governed by perturbation theory for $Q^2 \geqslant 5$ GeV . However, Isgur and Llewellyn-Smith (13) have claimed that this is not the case; they base their arguments upon a calculation of an upper bound to the perturbative contribution, which they find to be one to two orders of magnitude below the experimental data. The validity of their argument has, in turn, been questioned by Farrar (14) who points out that their result is sensitive to arbitrary assumptions; they use a totally symmetric wavefunction and calculate in the Born approximation, which produces an exact cancellation between two terms, which cancellation disappears should an element of antisymmetry be introduced into the wavefunction, if the coupling constant is allowed to run (with different Q^2's for different graphs) or if we keep one-loop corrections to the Born approximation.

It is in fact perfectly consistent that the wavefunction of the proton, and inded those of other hadrons, are quite different from their asymptotic forms, but that perturbative QCD is legitimate. (Possibly even in the Born approximation, though running coupling constants could be used there to partially incorporate higher order effects). However, the low energy form of the wavefunction is a non-perturbative object, so the question arises of how it can be calculated. Four approaches have so far been suggested. A study of inclusive meson production in e^+e^- annihilation (15) has shown the process to be linear under certain restrictions in the mesonic wavefunction; thus the wavefunction can, in principle, be extracted from experiment. An extension of this to the baryonic case has been made by Fontannaz and Jones (16). Chernyak, Zhitnitsky and Zhitnitsky have, via a study of QCD sum rules for two-point functions involving derivative currents, predicted moments of various wavefunctions and put forward model wavefunctions for a variety of mesons and baryons (17). Consideration of the vertex function (18)

$$i\int d^4z\, e^{iq\cdot z}\langle \pi(p)|T(J_\mu(\tfrac{z}{2})\, J_\nu(-\tfrac{z}{2}))|0\rangle \qquad (1.9)$$

where J_μ is an electromagnetic current shows that the amplitude can be written as a functional of the pion wavefunction. It was noted by Fontannaz and Jones (16) that this is also linear in the wavefunction and so provides another opportunity to determine the wavefunction from experiment. It is possible, in the absence of such data, to adopt a phenomenological approach, as was done by Craigie and Stern (19), who considered the resonance - saturated vertex function (1.9) thus obtaining a set of QCD sum rules which lead to predictions for the moments of the pion wavefunction of lowest twist.

In this thesis the problem of determining low energy, non-perturbative, hadronic wavefunctions is further investigated. Chapter 2 is an extension of the work of Craigie and Stern to the case of the proton. A set of sum rules is obtained which predicts one moment of the antisymmetric part of the proton wavefunction unambiguously (it is of course zero aymptotically) and further predicts a mixture of moments of both the symmetric and antisymmetric parts of the protons wavefunction. The results are shown to be in accord with those obtained from two-point sum rules.

In the third chapter, prompted by the discovery of an ambiquity in the work of Ref. 19, we reconsider the pion wavefunction in ther vertex sum rule approach. Radiative corrections, power corrections (corresponding to higher twist parts of the pion) and an improved ansatz for continuum effects are included. The two-point sum rule technique is used to calculate the low energy form of a higher twist wavefunction and this is substituted back into the vertex sum rules. The result for the lowest twist pion wave function is, once more, in agreement with the results of Chernyak and Zhitnitsky.

Finally in the concluding chapter the general outlook is briefly discussed and possible extensions of this work are mentioned.

CHAPTER TWO

THE PROTON WAVE FUNCTION IN QCD

2.1 QCD Sum Rules

The QCD sum rule approach (20) to hadronic physics has over the last few years been the most productive of predictions for various non-perturbative parameters. The basis of the approach is to incorporate power corrections, due to non-perturbative effects, and thus approach resonance physics from the "short distance side". A phenomenological indication that power corrections are more important than higher order perturbative effects comes from the observed spectra in the vector and axial vector channels with isotopic spin, I = 1. In the vector channel we have the ρ meson and in the axial vector case we have both the π (much lighter than the ρ) and the A_1 (much heavier). However, in the chiral limit, the perturbative graphs (Fig. 3) do not differentiate between the currents. It is argued that chiral symmetry breaking is responsible for this mass splitting and that the spontaneous production of a non-zero condensate $\langle 0 | \bar{q}q | 0 \rangle$, with a pion as the Goldstone boson of the symmetry, is at work. On dimensional grounds it is obvious that only power corrections can incorporate this effect.

Shifman, Vainshtein and Zakharov (20) proceed to

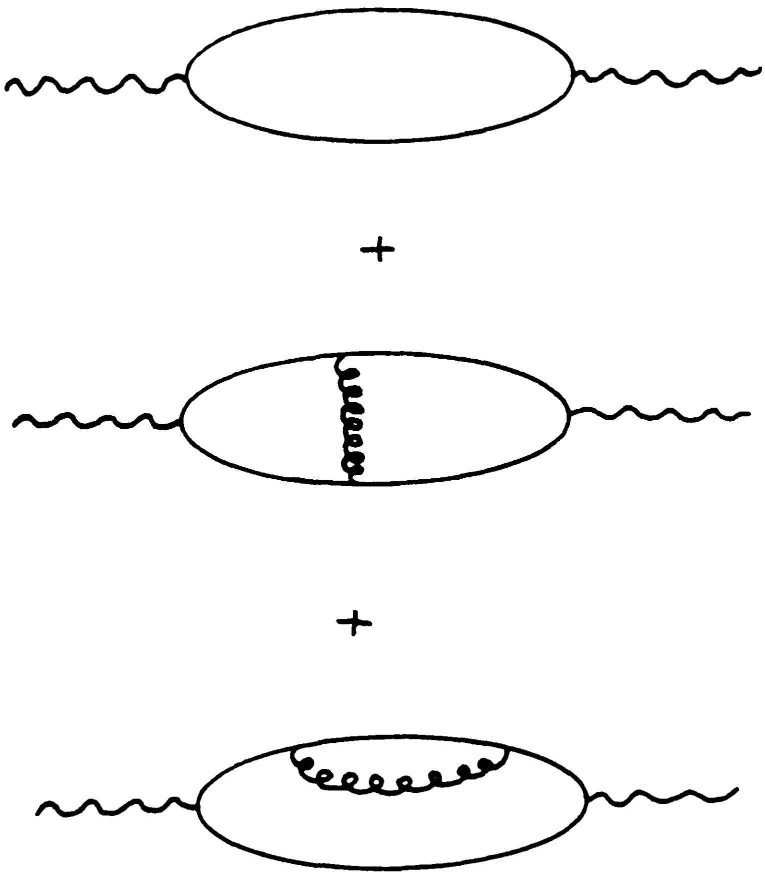


Figure 3: Purely perturbative graphs for two-
point functions with mesonic currents.

include power corrections due to the non-perturbative effects. These corrections signal the breakdown of asymptotic freedom and are introduced via non-vanishing vacuum expectation values of higher dimensional operators. The hope is to find an intermediate region, or "window", where there is an overlap between the asymptotic freedom regime(where it is possible to perform calculations) and the long distance part (where resonance saturation is reliable).

The procedure is based upon the assumption that the OPE is valid; defining two currents J_A and J_B which can be made from light or heavy quarks. Then for large external momentum q or for a large quark mass the OPE gives

$$i\int dx\, e^{iq\cdot x}\, T\{ j_A(x)\, j_B(0)\} = \sum_n C_n^{AB}\, O_n \qquad (2.1.1)$$

where C_n^{AB} are coefficients and the O_n are local operators made from light quarks or gluon fields. Taking the vacuum expectation value ensures that only spin zero operators survive. The higher dimensional operators are suppressed by extra powers of Q^2 (or quark mass squared), and so we give below the operators with zero Lorentz spin and dimension $d \leqslant 6$.

$$I \quad (\text{unit operator}), \qquad (d=0)$$

$$O_M = \overline{\Psi} M \psi, \qquad (d=4)$$

$$O_G = G^a_{\mu\nu} G^a_{\mu\nu}, \qquad (d=4)$$

$$O_\sigma = \overline{\psi} \sigma_{\mu\nu} t^a \widetilde{M} \psi G^a_{\mu\nu}, (d=6)$$

$$O_\Gamma = \overline{\psi} \Gamma_1 \psi \overline{\psi} \Gamma_2 \psi, \qquad (d=6)$$

$$O_f = f^{abc} G^a_{\mu\nu} G^b_{\nu\lambda} G^c_{\lambda\mu}, (d=6) \qquad (2.1.2)$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, M and $\widetilde{M}$ are quark mass matrices, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, the t^a are the Gell-Man SU(3) matrices (normalized by the requirement $tr(t^a t^b) = 2\delta^{ab}$) and $\Gamma_{1,2}$ are some matrices acting on the colour, flavour and spinor indices of the quark fields.

The condensate values are, of course, universal and once known can be used in any sum rule. The correct values have been a source of controversy throughout, however (20, 21), though agreement appears to be approaching. The calculation of the coefficients of the condensates requires the evaluation of certain diagrams (Fig. 4). The standard Feynman diagram technique is, in principle, adequate to this task but that language is not the most economical for the case. A more suitable calculational scheme is the Schwinger approach (22), and the Fock-Schwinger gauge

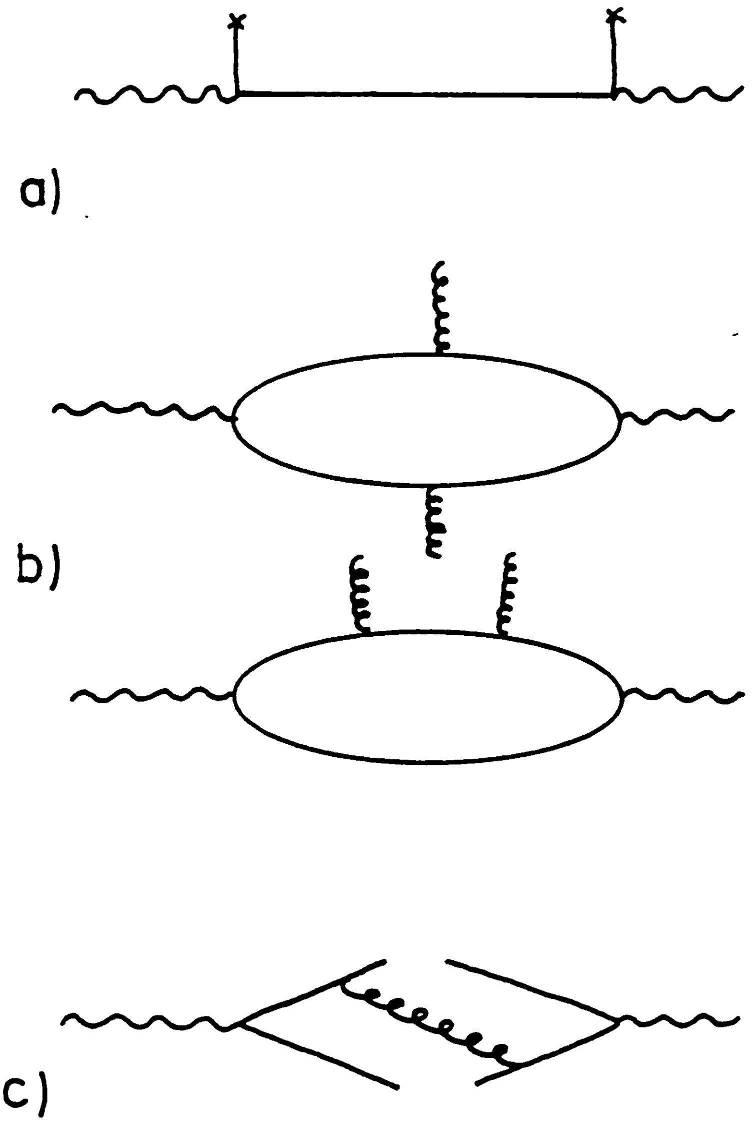


Figure 4: a)Diagrams required for the calculation of the coefficient of $\langle 0|\bar{\psi}\psi|0\rangle$
b)Diagrams contributing to $\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle$ coefficient.
c)Diagrams contributing to $\langle 0|\bar{\psi}\Gamma_1\psi\,\bar{\psi}\Gamma_2\psi|0\rangle$ coefficient.

(Appendix A; this is used for such calculations in this thesis).

Once the calculation of the QCD side of the correlation function (2.1) is complete we may consider the resonance saturation of the currents. In general, the vacuum polarization induced by a current is given by the correlation function

$$T_{\mu\nu\ldots}\Pi^{j}(Q^2) = i\int d^4x\, e^{iq\cdot x}\langle 0|T(j(x)\,j(0))|0\rangle \qquad (2.1.3)$$

where $T_{\mu\nu\ldots}$ is a tensor depending on the current and $\Pi_j(Q^2)$ is a scalar function. $\Pi_j(Q^2)$ obeys a dispersion relation

$$\Pi^{j}(Q^2) = \frac{(q^2)^n}{\pi}\int \frac{\mathrm{Im}\,\Pi^{j}(s)\,ds}{s^n\,(s-q^2)} + \sum_{k=0}^{n-1} a_k\,(q^2)^k \qquad (2.1.4)$$

where the a_k are, unknown, subtraction constants. (They can be removed by taking the appropriate number of derivatives with respect to Q^2). The analysis for mesonic currents (20) of the form

$$j_{\Gamma}(x) = \bar{q}_i(x)\,\Gamma\, q_i(x)\ ,\quad \Gamma = 1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu \ldots \qquad (2.1.5)$$

was later extended to baryonic currents by Ioffe (23)

$$j_{\Gamma_{1,2}}(x) = \varepsilon^{abc}\left(q_i^{a}(x)\,\Gamma_1\, q_j^{b}(x)\right)\Gamma_2\, q_k^{c}(x) \qquad (2.1.6)$$

In both cases i, j, k denotes flavour indices, a, b, c are colour indices and the Γ_i denote the tensor structure. Choosing the Γ_i and picking suitable combinations of quark flavours, the currents can be given definite quantum numbers (J, P, I). For example

$$j_\mu(x) = \frac{1}{2}\left(\bar{u}(x)\gamma_\mu u(x) - \bar{d}(x)\gamma_\mu d(x)\right) \qquad (2.1.7)$$

has quantum numbers $I = 1$, $J^{PC} = 1^{--}$, ie. it has the quantum numbers of the ρ, ρ', ...channel.

The imaginary part of $\Pi^j(Q^2)$ is related to a cross-section, but the standard parametrization is to feed hadronic states (plus a continuum term) into it. Usually a narrow resonance approximation is applied, with the imaginary part written as a sum over δ functions. For example, the vector current of flavour q with change e_q gives the expression

$$\mathrm{Im}\,\Pi_{phys}(s) = \mathrm{Im}\,\Pi^V(s) =$$

$$= \frac{\pi}{e_q^2}\sum_{RES}\frac{m_R^2}{g_R^2}\,\delta(s-m_R^2) + \frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right)\theta(s-s_0) \qquad (2.1.8)$$

where the coefficient of the θ term is chosen so as to provide an asymptotic matching of the continuum, which it represents, with the perturbative QCD predictions of the unit operator coefficient.

The substitution of (2.1.8) into (2.1.4) and (2.1.3) and the equating of this to the operator product expansion yields a QCD sum rule. However, there exists a freedom in the summation procedure for the power terms which enables us to fix the weight function entering the integration over the spectral density. (The series is of course truncated in practice; both because of finite time to perform calculations in and, more importantly, because the OPE breaks down at a critical operator dimension (20); but the results used are sufficiently general to remain valid). The most generally used procedure is to take the Borel, or Laplace, transform of the sum rule (Appendix B). This transform gives a weight function which suppresses higher-energy resonance contributions exponentially, giving integrals such as

$$\int e^{-s/M^2} \, \mathrm{Im}\, \Pi_{phys}(s)\, ds \tag{2.1.9}$$

where M^2 is the Borel variable, and also factorially suppresses higher power corrections on the QCD side. There do exist a variety of other transforms (24, 25) but they will not be further considered here.

The resulting, Borel transformed, sum rule is then equated over a range of M^2 where the power corrections are (typically) kept between 5-10% and 30-35% and the resonance contribution is at least the equal of the continuum contribution (20, 26). It is hoped that thus unaccounted power corrections are not too large and that the conclusions drawn have not been swamped by continuum effects. The resonance parameters and the duality interval, S_0 , (2.8) are then fitted so as to get the best agreement with the QCD predictions.

This procedure has by now been applied to many channels and many resonance parameters have been predicted. (For a review see (27)). Here we shall just mention the applications to the light quark channels (with L = 0 (20) and L = 1 (28), charmonium (20), bottomium (29), light quark baryons (23), heavy quark baryons (30), hybrid mesons (31) and glueball states (32). The natural extension to three point functions (33) of the form

$$i\int d^4x\, d^4Y\, e^{ip'\cdot x - iq\cdot Y} \langle 0|T(j_1(x)j_2(Y)j_3(0))|0\rangle \quad (2.1.10)$$

has also been performed, giving constraints on coupling constants amongst hadrons.

2.2 The Proton Wave Function and Experiment

Experiment is linked with the proton wave function in two ways. Firstly any model wave function can be used in the calculation of an exclusive process and the result compared with experiment. The results obtained using the asymptotic form of the proton wave function are, with few exceptions (37), in disagreement with the data, the proton form factor and $\gamma\gamma \longrightarrow p\bar{p}$ being the two most striking examples (12, 14, 38). However, these disagreements indicate a low energy wavefunction quite different to the asymptotic form, with a strong antisymmetric component (38).

Secondly the wavefunction can in principle be extracted from experiment. Baier and Grozin (15) have shown in the case of inclusive meson production in e^+e^- annihilation where the meson is isolated by a cone of half-angle θ that when $\theta \ll 1$ the cross section becomes proportional to the square of $f(z, \varphi]$, where

$$f(z,\varphi] \equiv \int dx\, K(z,x)\, \varphi(x, Q^2\theta^2) \tag{2.2.1}$$

and x is the quark scaling variable, φ is the mesonic wave function and K is a non-trivial kernel. Because of the linearity of this amplitude with respect to φ one can, in principle, invert (2.2.1) and from a detailed knowledge of the cross-section obtain the wave function. Fontannaz and Jones have extended this idea to the proton and from their consideration of the electroproduction of an isolated proton produced a similar result. However, it is necessary that the antisymmetric part of the proton wave function be negligible to directly determine the form (16). These authors have also pointed out the potential to extract the wavefunction from experimental data concerning the vertex function (1.9).

2.3 The Proton Wave Function - General Definition

Because the experimental data differ so greatly from predictions using the asymptotic proton wavefunction it is important to know the non-asymptotic form of the wavefunction. The experiments required to obtain direct information on the proton wavefunction at lower energies are not imminent; we must therefore search for other ways of ascertaining it.

The application of QCD sum rules to the problem was first carried out by Zhitnitsky (39). The procedure starts by defining the matrix element of the tri-local operator (as $P_z \longrightarrow \infty$, Ref. 40)

$$\langle 0 | u^i_\alpha(z_1) u^j_\beta(z_2) d^k_\gamma(z_3) | P\rangle_{\mu^2} \varepsilon^{ijk} = -\frac{1}{4}\Big\{ (\not{p}C)_{\alpha\beta}(\gamma_5 N)_\gamma \cdot V_{\mu^2}(z_i p) - i(\not{p}\gamma_5 C)_{\alpha\beta} N_\gamma A_{\mu^2}(z_i p) + (\sigma_{\mu\nu} p_\nu C)_{\alpha\beta}(i\gamma_\mu \gamma_5 N)_\gamma T_{\mu^2}(z_i p) \tag{2.3.1}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, $|p\rangle$ is the proton state with momentum p, N_γ is the proton spinor, C is the charge conjugation matrix, u and d are quark fields, i, j and k are colour indices and V, A and T are the leading twist nucleon wave functions. The wave functions, $V_{\mu^2}(x_i)$ etc. are introduced through the Fourier decomposition:

$$V_{\mu^2}(z_i p) = \int_0^1 [dx] \exp\Big\{-i \sum_j x_j (z_j p)\Big\} V_{\mu^2}(x_i) \tag{2.3.2}$$

where $[dx] \equiv dx_1 dx_2 dx_3 \delta(1-\sum_i x_i)$ and the x_i are longitudinal momentum fractions: $0 < x_i < 1$, $\sum_i x_i = 1$. The moments of these wave functions are defined as

$$V^{(n_1 n_2 n_3)} = \int_0^1 [dx] V(x_i)\, x_1^{n_1} x_2^{n_2} x_3^{n_3} \tag{2.3.3}$$

The generality of (2.3.1) is reasonably simple to prove. On dimensional grounds alone one can see that a factor p is required on the right hand side. This implies the structures must come from the set of first-rank covariants (with three indices) of Ref. 41, which have either zero or two γ_5's from chirality considerations

$$\begin{aligned}
&1)\ (\gamma_\mu)_{\alpha\beta} N_\gamma, \quad 2)\ (1)_{\alpha\beta} (\gamma_\mu N)_\gamma \\
&3)\ (\gamma_5 \gamma_\mu)_{\alpha\beta} (\gamma_5 N)_\gamma, \quad 4)\ (\gamma_5)_{\alpha\beta} (\gamma_5 \gamma_\mu N)_\gamma \\
&5)\ (\sigma_{\mu\nu})_{\alpha\beta} (\gamma_\nu N)_\gamma, \quad 6)\ (\gamma_\nu)_{\alpha\beta} (\sigma_{\mu\nu} N)_\gamma \\
&7)\ (\gamma_5 \sigma_{\mu\nu})_{\alpha\beta} (\gamma_5 \gamma_\nu N)_\gamma, \quad 8)\ (i\gamma_5 \gamma)_{\alpha\beta} (\gamma_5 \sigma_{\mu\nu} N)_\gamma
\end{aligned} \tag{2.3.4}$$

However, contracting structures (2) and (4) with P_μ gives zero, because in the $P \longrightarrow \infty$ frame the proton appears massless and the Dirac equation implies $\not{P} N$ is negligible. Similarly contracting with P_μ leads to structures (6) being equivalent with (1), (8) is identical to (3) and (7) becomes (5). (The last requires use of the Dirac equation and the identity $\sigma_{\kappa\lambda}\gamma_5 = \frac{1}{2}\varepsilon_{\kappa\lambda\mu\nu}\sigma_{\mu\nu}$). Thus we see that our three structures V, A and T are general.

In fact, V, A and T are overcomplete. Two things make this clear. Firstly we have the identity of the two u-quarks and secondly there is the requirement that the total isospin is 1/2. In the first case we interchange the u-quarks, producing terms like $(\not{P}C)_{\beta\alpha}(\gamma_5 N)_\gamma V_\mu^2(z_i p)$. Now using the properties of C (42)

$$\left(1, \gamma_\mu, \sigma_{\mu\nu}, i\gamma_\mu\gamma_5, \gamma_5\right) C = C\left(1, -\gamma_\mu, -\sigma_{\mu\nu}, i\gamma_\mu\gamma_5, \gamma_5\right)^T \tag{2.3.5}$$

we see that

$$\left(C\Gamma\right)_{\alpha\beta} = -\lambda\left(C\Gamma\right)_{\beta\alpha} \tag{2.3.6}$$

where λ is ± 1 as in eq. 2.3.5. Thus we see

$$V(1,2,3) = V(2,1,3), \quad A(1,2,3) = -A(2,1,3)$$
$$T(1,2,3) = T(2,1,3) \tag{2.3.7}$$

The total isospin constraint can be expressed in terms of the orthogonality of the I = 3/2 state to the proton, $\langle 0 | \Delta^+ | p \rangle = 0$. Expressing the Δ^+ in terms of a two quark state and a one quark state, using the relevant Clebsch-Gordan coefficients, we get

$$\Delta^+ = \frac{1}{\sqrt{3}}\left(|1,1\rangle |\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{2}\, |1,0\rangle |\tfrac{1}{2},\tfrac{1}{2}\rangle \right) \tag{2.3.8}$$

Orthogonality therefore requires

$$\langle 0 | udu + duu + uud | p \rangle = 0 \tag{2.3.9}$$

in more detail

$$\langle 0 | u_\alpha(1) d_\beta(2) u_\gamma(3) + d_\alpha(1)\, u_\beta(2) u_\gamma(3) + u_\alpha(1)\, u_\beta(2) d_\gamma(3) | p \rangle = 0 \tag{2.3.10}$$

defining

$$-4 M_{\alpha\beta\gamma} = (\not{p} C)_{\alpha\beta} (\gamma_5 N)_\gamma\, V(123) - i (\not{p} \gamma_5 C)_{\alpha\beta} N_\gamma\, A(123)$$

$$+ (\sigma_{\mu\nu} p^\nu C)_{\alpha\beta} (i \gamma^\mu \gamma^5 N)_\gamma\, T(123)$$

(2.3.10) can be written as

$$M_{\alpha\beta\gamma}(123) = -\left(M_{\alpha\gamma\beta}(132) + M_{\beta\gamma\alpha}(231) \right) \tag{2.3.11}$$

Now, $M_{\alpha\beta\gamma}$ can be expressed as

$$M_{\alpha\beta\gamma} = \sum_R B_R (\Gamma_R C)_{\alpha\beta} (\Gamma_R' N)_\gamma \tag{2.3.12}$$

and we can extract the B_R coefficients by a Fierz reshuffle, ie. using the relation

$$\frac{1}{4}\left(C^{-1}\Gamma_S\right)^{\beta\alpha}\left(\bar{N}\gamma_0\Gamma_S^{\prime}\right)^{\gamma} M_{\alpha\beta\gamma} = B_S \bar{N}\gamma_0 N \tag{2.3.13}$$

where the factor of γ_0 is introduced to avoid producing $\bar{N}N$, which is zero in this frame and leads to ambiguities. Some algebra then yields the relationship

$$2T(1,2,3) = V(1,3,2) - A(1,3,2) + V(3,2,1) + A(3,2,1) \tag{2.3.14}$$

Thus we see that we can define the nucleon wave function in terms of just one function, say $\varphi_N(x_i) = V(x_i) - A(x_i)$ The logarithmic corrections, due to perturbation theory, give, via the renormalization group, a weak μ^2 dependence to the wavefunction (40, 42)

$$\begin{aligned} V(x_i,\mu^2) &= \varphi_{as}(x_i)\sum_n f_n P_n(x_i)\, e^{-\varepsilon_n/\tau}, \\ \varphi_{as}(x_i) &= 120 x_1 x_2 x_3, \\ f_n &= \int_0^1 [dx]\, P_n(x_i) V(x_i,\mu_0^2), \quad \tau = \frac{1}{\beta}\ln\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right) \end{aligned} \tag{2.3.15}$$

where $\{P_n(x_i)\}$ is the orthogonal system of Appel polynomials and ε_n are the corresponding anomalous dimensions. (For an explanation of why they are Appel polynomials and a method used to obtain the forms of various asymptotic wavefunctions see Appendix C.)

2.4 QCD Sum Rules and the Proton Wavefunction

From the definition of the moments and of the wavefunctions (2.3.1, 2.3.2, 2.3.3) we see that a suitable choice of currents can separate out the various parts of the wavefunction. The matrix elements of the local operators below can be used to select out both V and A:

$$\langle 0|(i z_\mu \overrightarrow{D}_\mu)^{n_1} u_i(0)\, C\not{z}\, (i z_\nu \overrightarrow{D}_\nu)^{n_2} u_j(0) (i z_\lambda \overrightarrow{D}_\lambda)^{n_3} \cdot (\gamma_5 d^k(0))_\tau |p\rangle\, \varepsilon^{ijk} = -(z.p)^{n_1+n_2+n_3} N_\tau V^{(n_1 n_2 n_3)}$$

$$\langle 0|(i z_\mu \overrightarrow{D}_\mu)^{n_1} u_i(0)\, C\not{z}\gamma_5 (i z_\nu \overrightarrow{D}_\nu)^{n_2} u_j(0) (i z_\lambda \overrightarrow{D}_\lambda)^{n_3} \cdot (d^k(0))_\tau |p\rangle\, \varepsilon^{ijk} = -(z.p)^{n_1+n_2+n_3} N_\tau A^{(n_1 n_2 n_3)} \quad (2.4.1)$$

$$, z^2 = 0 .$$

A correlation function of one of the currents above and a current chosen to represent the proton can now be used to make a study of the proton wavefunction. However, there is no firm agreement as to which current best represents the proton, and indeed there exists a considerable body of literature on this point (23, 47). Zhitnitsky considered the current:

$$\langle 0| J_\lambda^{(n_1)} |p\rangle = \langle 0|(i \overrightarrow{D}_\mu z_\mu)^{n_1} u_i(0)\, C\not{z}\, u^j(0) (\gamma_5 d(0))^k_\lambda - (i D_\nu \overrightarrow{z}_\nu)^{n_1} u_i(0) C\not{z}\, d^j(0) (\gamma_5 u(0))^k_\lambda |p\rangle\, \varepsilon^{ijk}$$

(2.4.2)

$$= -(z \cdot p)^{n_1+1} N_\lambda \left[\frac{1}{2} (V-A)^{(n_1 00)} + T^{(n_1 00)} \right]$$

The arguments for this current being strongly correlated with the proton are that it has isospin 1/2 and that it gives a spectral density proportional to S (rather than S^2 as in (23, 47)) at large S, which reduces the dependence of the sum rules on the form of the continuum. The choice of n_1 is such as to maximise the sensitivity of the sum rules to the proton. The correlator considered was of the form:

$$\underline{I}^{(n_1 n_2 n_3)} = i \int d^4x \, e^{iq \cdot x} \langle 0 | T \left(V_\gamma^{(n_1 n_2 n_3)} \eta_{\gamma'}^{(1)} \right) | 0 \rangle \not{z}_{\gamma\gamma'} \quad (2.4.3)$$

and an analogous correlator with $A_\gamma^{(n_1 n_2 n_3)}$ was also used. The factor $\not{z}_{\gamma\gamma'}$ was introduced for the same reason as the γ_0 factor in equation (2.3.13).

The QCD side of the sum rules is given by the diagrams of figure 5. The sum rules have the following form:

$$4 |f_N|^2 \left[\frac{1}{2} \varphi_N^{(1,0,0)} + T^{(1,0,0)} \right] \varphi_N^{(n_1 n_2 n_3)} \exp\left(-\frac{m_N^2}{M^2}\right)$$

$$= \frac{\beta_1^{(n_1 n_2 n_3)}}{160\pi^4} M^4 \left\{ 1 - (1+H) e^{-H} \right\} + \quad (2.4.4)$$

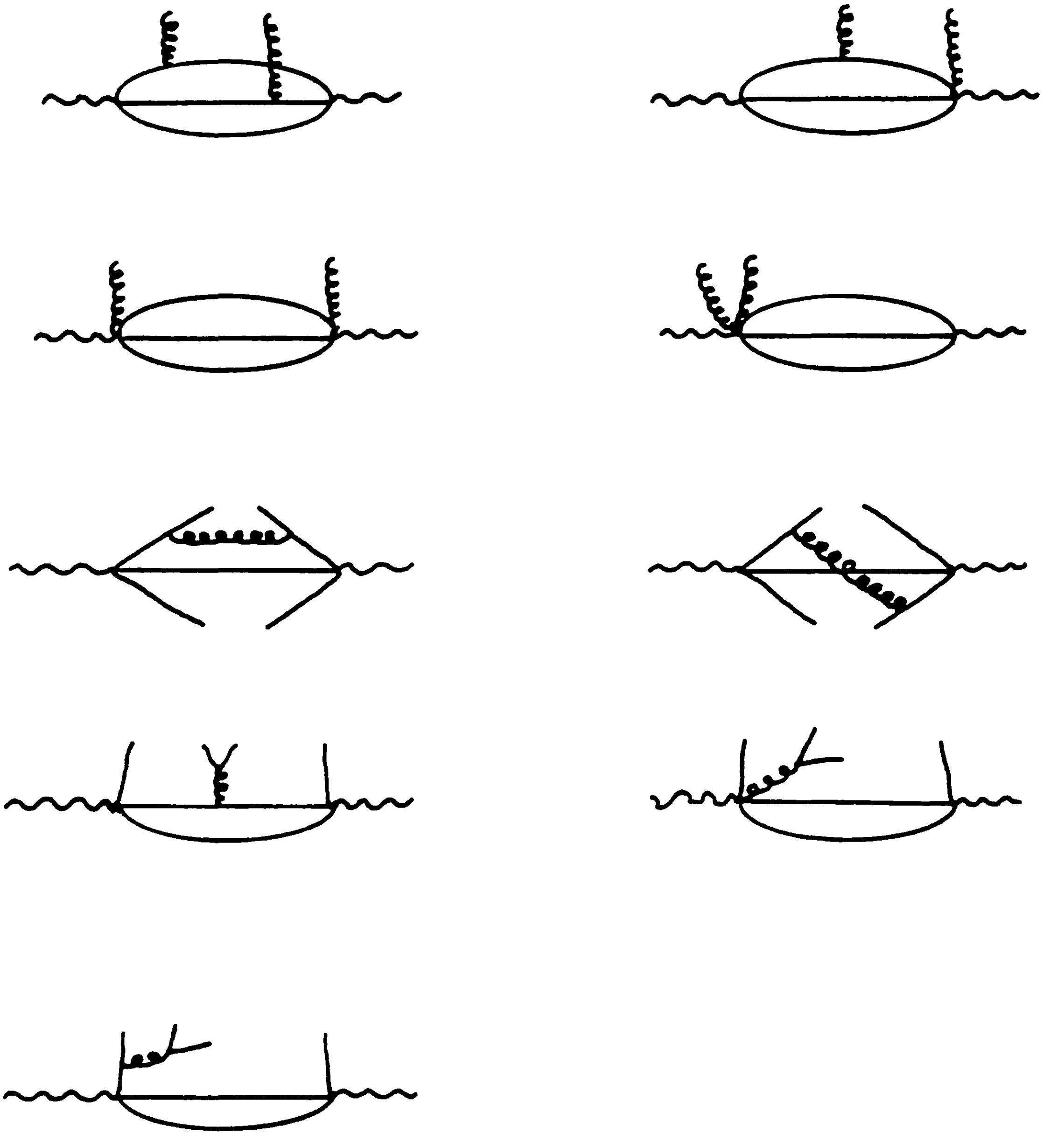


Figure 5: Diagrams contributing to the sum rules (2.4.4)

$$+ \frac{\beta_2^{(n_1 n_2 n_3)}}{48\pi^2} \langle 0 | G^a_{\mu\nu} G^a_{\mu\nu} | 0 \rangle$$

$$+ \frac{\beta_3^{(n_1 n_2 n_3)}}{35\pi M^2} \langle 0 | \sqrt{\alpha_s}\, \bar{u} u | 0 \rangle^2$$

where $H = S^{(n_1 n_2 n_3)}/M^2$, M_N is the nucleon mass, M is the Borel variable and the $S^{(n_1 n_2 n_3)}$ are the duality intervals of the sum rules. These duality intervals follow from the use of the standard ansatz for the resonance saturation of the sum rules, i.e.:

$$\frac{1}{\pi} \operatorname{Im} I^{(n_1 n_2 n_3)}(s) = \Gamma^{(n)} \delta(s - m_N^2) + \theta(s - s_0) \frac{\beta_1^{(n_1 n_2 n_3)} s}{640\pi^4} \quad (2.4.5)$$

where

$$\Gamma^{(n_1 n_2 n_3)} := |f_N|^2 \varphi_N^{(n_1 n_2 n_3)} \left[\tfrac{1}{2} \varphi^{(100)} + T^{(100)} \right]$$

The values of the β_i coefficients are given in table 1.

The sum rules (2.4.4) were then fitted over a range of M^2 where the non-perturbative contributions were of the order of 10 - 40% of the perturbation theory contribution. An effective resonance contribution term was added to (2.4.5) with the effective resonance mass at

Table 1

	β_1		β_2		β_3	
$n_1\ n_2\ n_3$	$\varphi=V$	$\varphi=\varphi_N$	$\varphi=V$	$\varphi=\varphi_N$	$\varphi=V$	$\varphi=\varphi_N$
0 0 0	1/3	1/3	1/12	1/12	36	36
1 0 0	5/42	1/7	11/240	7/60	313/20	148/5
0 1 0	5/42	2/21	11/240	−1/40	313/20	17/10
0 0 1	2/21	2/21	−1/120	−1/120	47/10	47/10
2 0 0	3/56	1/14	1/20	1/10	247/20	24
0 2 0	3/56	1/28	1/20	0	247/20	7/10
0 0 2	1/28	1/28	1/90	1/90	5/2	5/2
1 1 0	1/28	1/28	1/180	1/180	11/5	11/5
1 0 1	5/168	1/28	−7/720	1/90	11/10	17/5
0 1 1	5/168	1/42	−7/720	−11/360	11/10	−6/5

Table 2

$n_1 n_2 n_3$	SUM RULES	MODEL	SUM RULES	MODEL	$\varphi_{AS} = 120 x_1 x_2 x_3$
0 0 0	1	1	1	1	1
1 0 0	.38–.42	.39	.60–.75	.63	$1/3 \simeq .33$
0 1 0	.38–.42	.39	.09–.16	.15	1/3
0 0 1	.18–.24	.22	.18–.24	.22	1/3
2 0 0	.18–.25	.21	.25–.40	.40	$1/7 \simeq .14$
0 2 0	.18–.25	.21	.03–.08	.03	1/7
0 0 2	.03–.12	.08	.08–.12	.08	1/7
1 1 0	.07–.12	.11	.07–.12	.11	$2/21 \simeq .10$
1 0 1	.04–.08	.07	.09–.14	.12	2/21
0 1 1	.04–.08	.07	−.03–.03	.03	2/21

1.5 GeV in accord with the experimental spectrum. The best fit was made over the range of M^2 with the various moments and duality intervals as free parameters. The $S^{(n_1 n_2 n_3)}$ parameters were then varied within about 15% of their "best" values which gave the uncertainty in the moment values due to the unprecise nature of the continuum ansatz. The errors are sligtly smaller than might at first sight appear to be the case because of the identities $\sum_i x_i = 1$, $\langle x_i \rangle = \sum_j \langle x_i x_j \rangle$; an analysis of the results using these constraints implied the moment values given in Table 2.

As can be seen from the most cursory inspection of table 2 the asymptotic wavefunction (Appendix C and (39)) moments are totally different from the sum rules results. Therefore a new model wavefunction is proposed in (39) which has moments close to those of table 2. From the discussion of Appendix C and the work of (40) the form of the wavefunction must be:

$$\varphi(x_i, \mu^2) = \varphi_{as}(x_i) \sum_{n=0}^{\infty} C_n P_n(x_i) \quad . \tag{2.4.6}$$

ignoring logarithmic corrections, where the P_n are the orthogonalised system of Appel's polynomials. The P_n systems vary with the wavefunctions ($V-A$, $V+A$, T) used, but are connected by the relations (2.3.7) and (2.3.14). Assuming that only the first three polynomials contribute (i.e. $n \leq 2$) a fit is made for the forms of the wavefunctions. This gives for V, A and T the following result:

$$V(x_i, \mu^2 \simeq 1\,GeV^2) = \varphi_{as}(x_i)\left[11.35(x_1^2 + x_2^2) + 8.82x_3^2 - 1.68x_3 - 2.94\right]f_0$$

$$A(x_i, \mu^2 \simeq 1\,GeV^2) = \varphi_{as}(x_i)\left[6.72(x_2^2 - x_1^2)f_0\right]$$

(2.4.7)

$$f_0 = 5.3 \times 10^{-3}\,GeV^2$$

The moments of the model wavefunctions are given in table 2 for comparison with the sum rule results.

A check of these results was made in Ref. (39) by consideration of sum rules for the correlator:

$$I^{(n_1 n_2 n_3)} = i\int d^4x\, e^{iq\cdot x}\langle 0|T(T_\lambda^{(n_1 n_2 n_3)}\bar{\eta}_{\lambda'}^{(000)})|0\rangle \not{z}_{\lambda\lambda'}$$

(2.4.8)

where

$$T_\alpha^{(n_1 n_2 n_3)} = \varepsilon^{ijk}\left\{(iz_\mu \vec{D}_\mu)^{n_1} u^i C\sigma_{\mu\nu} z_\nu (iz_\nu \vec{D}_\nu)^{n_2} u^j (iz_\lambda \vec{D}_\lambda)^{n_3} (\gamma_\mu \gamma_5 d)_\alpha\right\}$$

and

$$\langle 0|T_\lambda^{(n_1 n_2 n_3)}(0)|p\rangle = (z.p)^{n_1+n_2+n_3+2} N_\lambda\, 2T^{(n_1 n_2 n_3)}$$

Evidently the results obtained must agree with those of table 2 by virtue of (2.3.14); this was indeed the case, for details see (39). A check of these calculations is under way.*

2.5 The Form of the Sum Rules Wavefunction

The results for the moments clearly show the difference between the asymptotic and non-perturbative wavefunctions. Considering a few specific discrepancies will display the physical changes between them. The values for $V^{(100)}$ and $V^{(200)}$ are larger in the non-asymptotic case and those of $V^{(001)}$, $V^{(002)}$ and $V^{(101)}$ are smaller compared with those of the asymptotic wavefunction. Also the ratios:

$$\frac{\varphi_N^{(100)}}{\varphi_N^{(001)}} \simeq \frac{\varphi_N^{(100)}}{\varphi_N^{(010)}} \simeq 4 \sim 5 \qquad (2.5.1)$$

are much larger than the ratios found with the perturbative wavefunction (where they are unity).

To make the interpretation of this clear we now consider the wavefunction rewritten as (40):

---

* D. King - private communication

$$|p^{\uparrow}\rangle = \frac{1}{\sqrt{48}}\int_0^1 d^3x \Big\{ \frac{1}{2}\big[V(x)-A(x)\big]\,|u^{\uparrow}(x_1)u^{\downarrow}(x_2)d^{\uparrow}(x_3)\rangle$$
$$+\frac{1}{2}\big[V(x)+A(x)\big]\,|u^{\downarrow}(x_1)u^{\uparrow}(x_2)d^{\uparrow}(x_3)\rangle$$
$$-T(x)\,|u^{\uparrow}(x_1)u^{\uparrow}(x_2)d^{\downarrow}(x_3)\rangle \Big\} \qquad (2.5.2)$$

Study of this wavefunction form and the results noted above makes it clear that the largest part of the proton longitudinal momentum is carried by one u-quark with its spin directed parallel to that of the resonance. To illustrate the form of the wavefunction consider figure 6, where the Mandelstam plane for the X_i is depicted. A maximum is denoted by $\oplus$ and a minimum by $\ominus$; figure 6a shows the totally symmetric asymptotic wavefunction, 6b depicts the, symmetric in $X_1 \leftrightarrow X_2$, V wavefunction and 6c is of φ_N , which is an overall wavefunction. The highly antisymmetric nature of φ_N and the dramatic change in character from φ_{as} is apparent. The general character of the wavefunction can be seen by the statement that about 60-70% of the momentum is carried by one u-quark with spin parallel to the proton and the remainder is shared amongst the remaining quarks. (This distribution is the same for the neutron with $u \leftrightarrow d$).

2.6 Vertex Sum Rules and Hadronic Wavefunctions

The great variation between the asymptotic

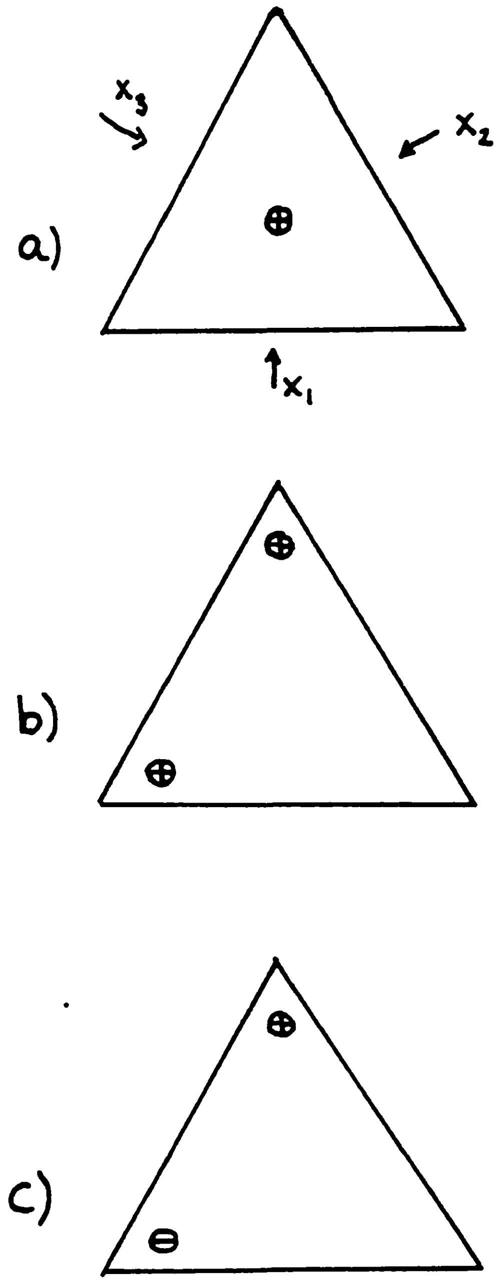


Figure 6: The nucleon wavefunctions and the Mandelstam plane for the x .

wavefunction and the non-asymptotic proposed wavefunction of Chernyak and Zhitnitsky is in one sense a source of relief; there is hope that this provides an explanation of the problems with exclusive processes and a vindication of the evidence supplied by the power-law behaviour of the data that perturbative QCD is indeed applicable. However, the accuracy in making predictions of exclusive processes with non-perturbative wavefunctions is not high and an optimistic estimate is roughly a factor of two (17). Whilst much better than the situation with asymptotic wavefunctions there is still much to be desired here; the need for a direct check of the wavefunctions suggested in (39) is evident, and this is especially true in the case of the proton where the experimental data are so much at odds with simple predictions.

One possible non-perturbative test would be to consider three-point functions (33) with derivative currents. However, an approach ideally suited to the study of wavefunctions is the technique of vertex function sum rules developed by Craigie and Stern (19). Here one considers sum rules for vertex functions of the form:

$$i\int d^4x\, e^{iq\cdot x} \langle \mathrm{hadron} | T\left(J_1\left(\tfrac{x}{2}\right) J_2\left(-\tfrac{x}{2}\right)\right)|0\rangle \qquad (2.6.1)$$

the advantages of this approach over the usual three-point functions technique are that we have only one q variable to Borel transform and that the hadronic wavefunction is directly probed via the use of a state; the currents no longer need to incorporate derivatives. This last point together with the three-point coupling (which uses the experimentally known coupling constants as a source of non-perturbative information rather than the, less accessible, vacuum expectation values of the two-point functions) also sharply differentiate this method from the sum rules of Chernyak and Zhitnitsky. The wavefunction now enters directly from the QCD side of the sum rules rather than from the resonance saturation model.

For these reasons it may be hoped that a consideration of a suitable vertex on the lines of (2.6.1) will provide an independent check of the "CZ wavefunctions".

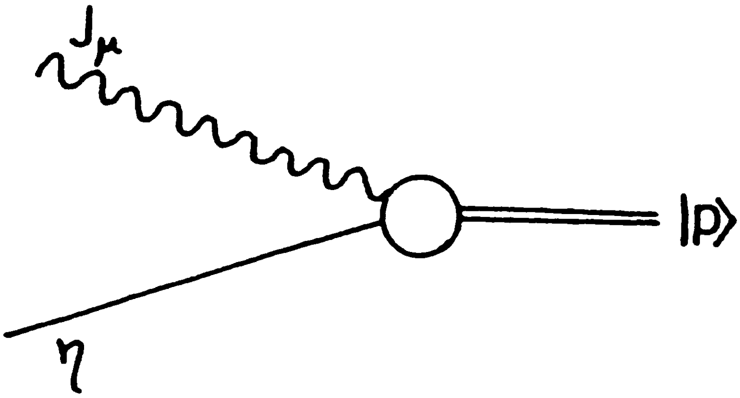
2.7 Vertex Sum Rules For the Proton

In this section we derive sum rules for the vertex:

$$V_{\alpha\mu} = i\int d^4z\, e^{iq\cdot z} \langle p | T\left(\eta_\alpha\left(\tfrac{z}{2}\right) J_\mu\left(-\tfrac{z}{2}\right)\right)|0\rangle \tag{2.7.1}$$

where η_α is a current representing the proton and J_μ is the electromagnetic current. This vertex has a good chance of providing information about the proton wavefunction, figure 7.

The derivation of the sum rules we present follows the lines of Craigie and Stern; the differences will mainly arise from the facts that a) the final hadron is massive in



· Figure 7: The baryonic vertex function (q
are the momenta of the currents.)

our case (they consider the pion in the chiral limit), b) the resonances used to saturate the currents are no longer degenerate in mass (as are to a good approximation the ρ and ω mesons used in (19)), and c) the QCD diagrams (figure 8) are now rather more intricate functions because of the far more complicated wavefunction structure of a baryon.

Before commencing the calculation there is one more decision to make; which current should be used to represent the proton? We use the current suggested by Ioffe (23, 47) which we hope will interpolate well with the proton; this is:

$$\eta_\alpha = (u^{aT} C \gamma_\mu u^b)(\gamma_5 \gamma_\mu d^c)_\alpha \, \varepsilon^{abc} \tag{2.7.2}$$

The electromagnetic current is of course:

$$J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d \tag{2.7.3}$$

The diagrams of figure 8 are now calculated in QCD. Using the shorthand notation that the current (2.7.2) is represented as:

$$\eta_\alpha = (u^{aT} C \Gamma_\eta u^b)(\Gamma'_\eta d^c)_\alpha \, \varepsilon^{abc} \quad . \tag{2.7.4}$$

and that the general form of the wavefunction is (rewriting (2.3.1)):

$$\langle 0|u_\alpha^a(z_1)\,u_\beta^b(z_2)\,d_\gamma^c(z_3)|p\rangle\varepsilon^{abc} = -\frac{1}{4}\sum_R (\Gamma_R C)_{\alpha\beta}\cdot \cdot(\Gamma_R' N)_\gamma R \tag{2.7.5}$$

where R runs over V, A and T; we get for the three diagrams of figure 8:

$$V_{\alpha\mu}^{(1)} = C_d \sum_R \frac{1}{l_3^2}(\bar{N}\Gamma_R'\gamma_\mu \not{l}_3 \Gamma_\eta')_\alpha \, tr(\overline{C\Gamma_R}\, C\Gamma_\eta) R$$

$$V_{\alpha\mu}^{(2)} = C_u \sum_R \frac{1}{l_1^2}(\bar{N}\Gamma_R'\Gamma_\eta')_\alpha \, tr(\overline{C\Gamma_R}\,\gamma_\mu^T \not{l}_1^T C\Gamma_\eta) R \tag{2.7.6}$$

$$V_{\alpha\mu}^{(3)} = C_u \sum_R \frac{1}{l_2^2}(\bar{N}\Gamma_R'\Gamma_\eta')_\alpha \, tr(C\Gamma_\eta(-\not{l}_2)\gamma_\mu \overline{C\Gamma_R}) R$$

where the C_u and C_d are the charges of the u- and d-quarks. Evidently the traces will only permit certain parts of the proton wavefunction to contribute. Using the variables:

$$l_i = -q_1 + x_i p \tag{2.7.7}$$

where p is the proton state momentum, q_1 is the photon current momentum and the x_i are the momentum fractions of the quarks (see figure 8), we get for the overall

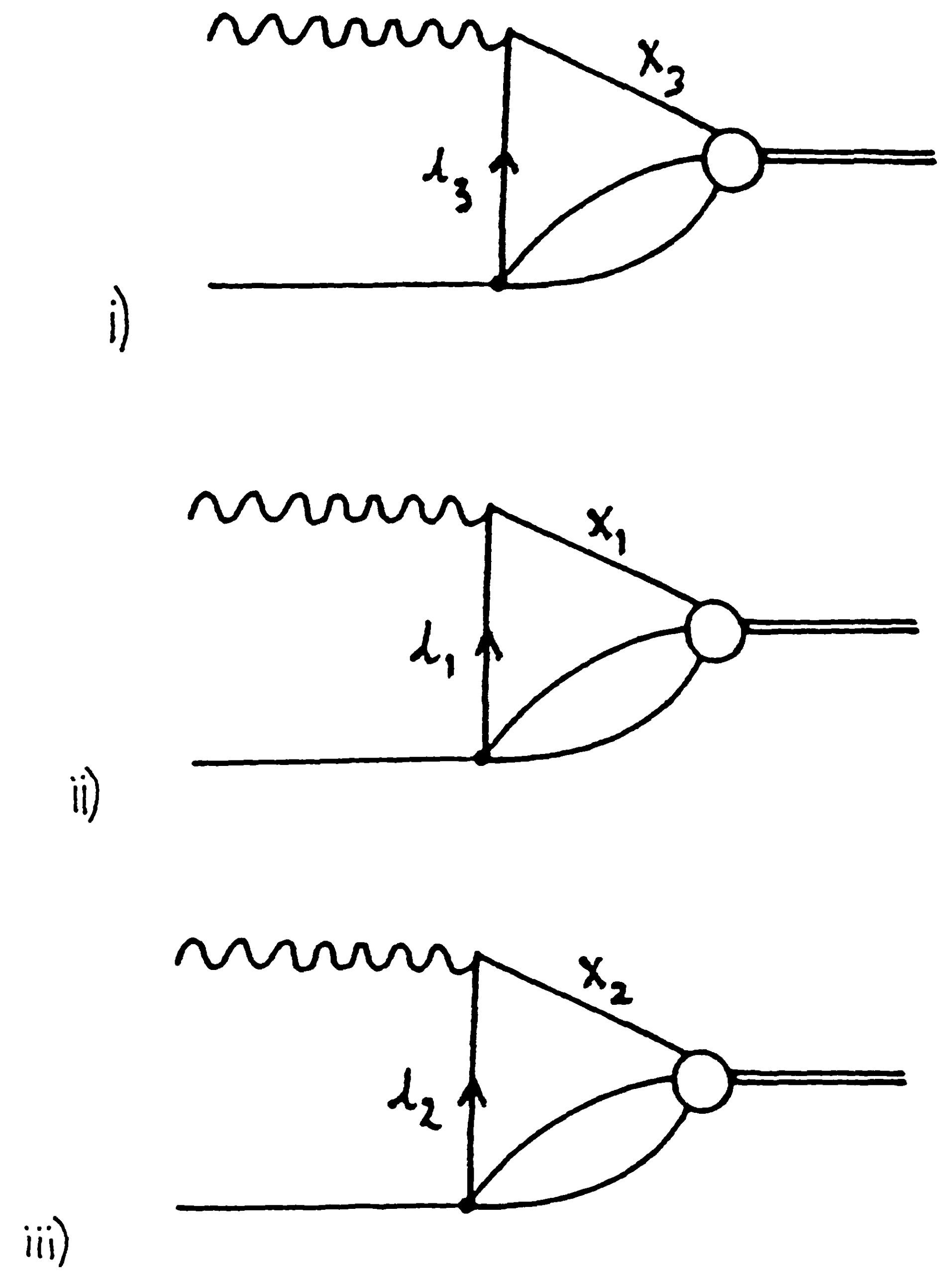


Figure 8: The QCD diagrams for the baryonic vertex function.

contribution of the diagrams:

$$V^{(1)}_{\alpha\mu} + V^{(2)}_{\alpha\mu} + V^{(3)}_{\alpha\mu} = \int d\mu_3 \left[(\bar{N}\gamma_1)_\alpha P_\mu - \right.$$

$$\left. - (\bar{N}\gamma_\mu)_\alpha P\cdot q_1 \right] \frac{1}{3} \left\{ \frac{2V(x_i)}{3\lambda_3^2} + \right. \tag{2.7.8}$$

$$\left. + \frac{2}{3}\left(\frac{V(x_i) - A(x_i)}{\lambda_1^2} + \frac{V(x_i)+A(x_i)}{\lambda_2^2} \right) \right\}$$

where $d\mu_3 = dx_1\, dx_2\, dx_3\, \delta(1 - \sum_i x_i)$

The amplitude depends on two scalar variables, which we may choose to be:

$$q_1^2 = (q + \tfrac{1}{2}P)^2 \quad , \quad q_2^2 = (q - \tfrac{1}{2}P)^2 \tag{2.7.9}$$

The Bjorken limit ($Q^2 \to \infty$, $P\cdot q \to \infty$) allows us to asymptotically expand the vertex function. We now do this and consider the amplitude dispersed over a straight line in the $q_1^2 - q_2^2$ plane. Taking the (resonance-saturated) amplitude we will then equate it to the expansion of the QCD amplitude.

The dispersion relation required is of the light-cone variety. A proof of the validity of this is to be found in Appendix D; the final result in our case is that we can write the amplitude as:

$$V_{\alpha\mu} = \frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds}{s-q^2-i\varepsilon}W_{\alpha\mu} \tag{2.7.10}$$

where

$$W_{\alpha\mu} = \frac{i}{2}\int d^4z\, e^{iq.z}\langle p|\left\{\eta_\alpha\left(\tfrac{z}{2}\right), J_\mu\left(-\tfrac{z}{2}\right)\right\}_+|0\rangle$$

The above result is dependent on the assumption that the parameter $|\omega| < 1$, where:

$$\omega = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \tag{2.7.11}$$

and the result may than be used generally, provided we work in the zero-width approximation for our resonances (see below).

We now wish to resonance saturate $W_{\alpha\mu}$ and must therefore insert physical states. This implies:

$$W_{\alpha\mu} = \frac{i}{2}\int d^4z\, e^{iq.z}\Big\{\langle p|J_\mu\left(-\tfrac{z}{2}\right)|N^\lambda\rangle\langle N^\lambda|\eta_\alpha\left(+\tfrac{z}{2}\right)|0\rangle$$

$$+ \langle p|\eta_\alpha\left(\tfrac{z}{2}\right)|\lambda,\nu\rangle\langle\lambda,\nu|J_\mu\left(-\tfrac{z}{2}\right)|0\rangle\Big\} \tag{2.7.12}$$

where $|N^{\lambda}\rangle$ denotes states with the quantum number of the nucleon and $|\lambda,V\rangle$ denotes vector mesons; in both cases λ is summed over. The evaluation of (2.7.12) requires the use of the following matix elements (39, 48):

$$\langle N^{\lambda}|\eta_{\alpha}(q_2)|0\rangle = f_N \bar{N}^{\lambda\alpha}(q_2)\,\delta(q_2^2 - m_N^2)$$

$$\langle \lambda,V|J_{\mu}(q_1)|0\rangle = \frac{\varepsilon_{\mu}^{(\lambda)}(q_1)\,m_V^2}{f_V}\,\delta(q_1^2 - m_V^2) \qquad (2.7.13)$$

ε_{μ} is the polarization vector of the meson and obeys:

$$\sum_{\lambda} \varepsilon_{\mu}^{(\lambda)}\varepsilon_{\nu}^{*(\lambda)} = -g_{\mu\nu} + \frac{q_{1\mu}q_{1\nu}}{m^2} \qquad (2.7.14)$$

we also use:

$$N^{\lambda}(q_2)_{\beta}\,\bar{N}^{\lambda\alpha}(q_2) = (\not{q}_2 + m_N)_{\beta}^{\ \alpha} \qquad (2.7.15)$$

These are combined with the use of vector meson dominance for the systems' coupling. We thus write the vertex as:

$$\left[G^{V}_{\rho,\omega}\gamma_{\mu} + G^{T}_{\rho,\omega}\frac{\sigma_{\mu\nu}}{2im_N}q_{1\nu}\right]\varepsilon_{\mu}^{*(\lambda)} \qquad (2.7.16)$$

Combining (2.7.13) - (2.7.16) leads to (figure 9):

$$W_{\alpha\mu}^{RES} = \frac{2\pi f_N m_V^2}{m_N f_V} \sum_{\rho,\omega}^{*} G_{\rho,\omega}^{T} \left[\frac{\delta(q_2^2 - m_N^2)}{q_1^2 - m_V^2} + \frac{\delta(q_1^2 - m_V^2)}{q_2^2 - m_N^2} \right] \cdot \left[\overline{N} \left(\not{q}_1 P_\mu - \gamma_\mu P \cdot q_1 \right) \right]_\alpha \tag{2.7.17}$$

as the result for the lowest order diagram structure to leading order in q^2 (28, 49). $\sum^{*}$ is a weighted sum, due to the different contributions of ρ and ω mesons to the electromagnetic current J_μ :

$$J_\mu = \rho_\mu + \frac{1}{3} \omega_\mu \tag{2.7.18}$$

We now substitute (2.7.17) into (2.7.10) and equate the result with (2.7.8) to give us a sum rule. However, there exist many ways of summing such a result (as argued in section 2.1 and in reference (20)). In our computations we use the Borel transform (Appendix B) of the sum rule; the resulting sum rule is:

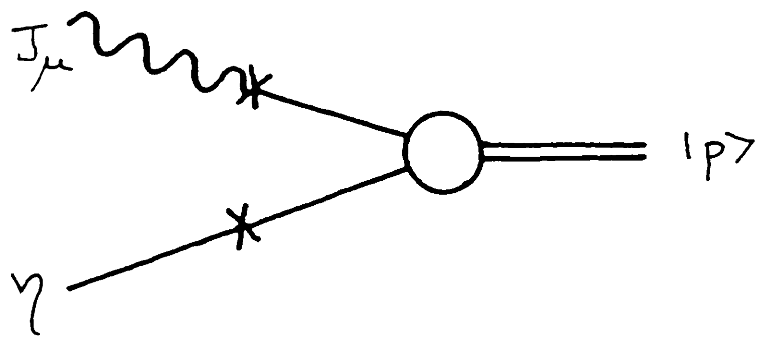


Figure 9: Resonance saturation of the baryonic vertex function; an x denotes a resonance.

$$\frac{1}{9}\int d\mu_3 \left\{ \frac{V}{l_3^2} + \frac{V-A}{l_1^2} + \frac{V+A}{l_2^2} \right\} \frac{q^2}{M^2}$$

$$= \frac{f_N m_V^2}{m f_V} \sum_{\rho,\omega}^{*} G^T_{\rho,\omega} \int_{S_0}^{\infty} \frac{ds\, e^{-s/M^2}}{M^2} \cdot$$

$$\cdot \left[\frac{\delta(q_2^2 - m_N^2)}{q_1^2 - m_V^2} + \frac{\delta(q_1^2 - m_V^2)}{q_2^2 - m_N^2} \right] \qquad (2.7.19)$$

where M^2 is the Borel variable (the q^2 factor is an artefact introduced to cancel the q^{-2} dependence of the l_i^{-2}). We hope that such a sum rule will have a reduced dependence both on higher lying resonances and on power corrections.

We now wish to change our variables so as to yield a set of sum rules. Our two scalar variables q_1^2 and q_2^2 (2.7.9) are re-expressed by (Appendix D):

$$\begin{aligned} q_1^2 &= (1+\omega)s + \tfrac{1}{4} m_N^2 \\ q_2^2 &= (1-\omega)s + \tfrac{1}{4} m_N^2 \end{aligned} \qquad (2.7.20)$$

where ω is as in (2.7.11). As argued in Appendix D we are now free to expand our sum rule (2.7.19) in powers of ω and equate coefficients. On the QCD side this yields for the first three coefficients:

$$\omega^0 \sim \frac{2V^{(000)}}{3M^2}, \quad \omega^1 \sim \frac{-4}{9M^2}\left[V^{(000)} + A^{(100)}\right]$$

(2.7.21)

$$\omega^2 \sim \frac{2}{3M^2}\left[-V^{(000)} + 8V^{(200)} + 4V^{(002)} + 8A^{(100)} - 8A^{(200)}\right]$$

The resonance saturation of the sum rules is now carried out; if we use $m_\omega = m_\rho = m_N$ as a symmetry then we get:

$$\text{const.} \int_{S_0}^{\infty} ds\, e^{-s/M^2} \sum_{\rho,\omega}^{*} \left[\frac{1}{1+\omega}\, \delta\left(s + \frac{m_N^2/4 - m_V^2}{1+\omega}\right) \cdot \frac{1}{(1-\omega)s - \frac{3}{4}m_N^2} + (\omega \to -\omega)\right]. \tag{2.7.22}$$

This evidently implies that, insofar we have the symmetry $m_{\rho,\omega}(\approx .78\,\text{GeV}) = m_N(\simeq .939\,\text{GeV})$, the coefficient of ω^1 must be zero. Since $V^{(000)}$ is defined to be 1 this implies that $A^{(100)} = -.25$. Although very naive, this result is extremely encouraging. Firstly it signals a very strong symmetry breaking effect (remember that is zero in the asymptotic limit) and secondly the symmetry breaking is of exactly the same sign as in the results of Chernyack and Zhitnitsky. Moreover the extent of the breaking is of the order predicted in (39), where the sum rules yielded $A^{(100)} = -.17 \sim -.25$!

For a more quantitative interpretation of our sum rules we now use $m_N^2 - m_V^2 = \alpha$ (keeping only the lowest lying resonances). This gives the following sum rules for the first three coefficients:

$$\frac{4}{3} V^{(000)} = \frac{-m_\rho^2 G_\rho^T f_N}{m_N f_\rho}\left[1 - e^{\alpha Y/m^2}\right] \frac{e^{-3/4 y}}{\alpha} \qquad (2.7.23)$$

where $y = m_N^2 / M^2$

$$\frac{8}{9}\left(V^{(000)} + 4A^{(100)}\right) = \frac{m_\rho^2 G_\rho^T f_N}{m_N f_\rho}\left(\frac{1}{\alpha}\left(\frac{3}{2} m_N^2 - \alpha\right)\left(\exp\left(\frac{\alpha Y}{m^2}\right) - 1\right)\right.$$

$$\left. - \exp\left(\frac{\alpha Y}{m^2}\right) y \left(\frac{3}{4} - \frac{\alpha}{m^2}\right) - \frac{3}{4} y\right) \qquad (2.7.24)$$

which goes to zero as α approaches zero, and

$$\frac{4}{3}\left[-V^{(000)} + 8V^{(200)} + 4V^{(002)} + 8A^{(100)} - 8A^{(200)}\right] =$$

$$= \frac{f_N m_\rho^2 G_\rho^T}{m_N f_\rho}\left[\frac{e^{-3/4 y}}{\alpha}\right]\left[\frac{2\left(\alpha - \frac{3}{2} m^2\right)^2}{\alpha^2}\left(e^{\frac{\alpha Y}{m^2}} - 1\right) + \right. \qquad (2.7.25)$$

$$+ \frac{2\left(\alpha - \frac{3}{2} m^2\right)}{\alpha}\left(\left(\frac{3}{4} Y - \frac{\alpha Y}{m^2}\right) e^{\frac{\alpha Y}{m^2}} + \frac{3}{4} Y\right) +$$

$$\left. + \left(\frac{3Y}{4} - \frac{\alpha Y}{m^2}\right)^2 - 2\left(\frac{3}{4} y - \frac{\alpha y}{m^2}\right) e^{\frac{\alpha Y}{m^2}} + \left(\frac{3y}{2} - \left(\frac{3y}{4}\right)^2\right)\right]$$

In the above we have neglected the ω meson contribution altogether: this can be done because the experimental data (see below) for G^T_ω and f_ω indicate that its contribution is of the order of one percent of that of the ρ meson.

The experimental data we use (48) are given below:

$$\frac{G^2_{\rho T}}{4\pi} = 20.5 \pm 2.1 \quad , \quad \frac{f^2_\rho}{4\pi} = 2.26$$

$$m_N = 0.939 \text{ GeV} \quad , \quad \alpha = 0.280 \text{ GeV} \qquad (2.7.26)$$

Following Craigie and Stern, our procedure is to search for a stability point for equation (2.7.23) and take our sum rule at that value of M^2. The stability point gives:

$$y = \frac{m_N^2}{\alpha} \ln\left[\frac{3}{4}\left(\frac{3}{4} - \frac{\alpha}{m^2}\right)^{-1}\right] = 1.73 \qquad (2.7.27)$$

(it may be noted that in the $\alpha = 0$ limit this would imply $y = 4/3$, a quite reasonable value). The value of y we use (2.7.27) implies that $M^2 \simeq 0.5 \text{ GeV}^2$ and $Q^2 = \frac{1}{2}(q_1^2 + q_2^2) \simeq .75 \text{ GeV}^2$ These seem sensible values for our approximations to be valid. As for direct comparisons with the results of Chernyak and Zhitnitsky, they worked at 1 GeV^2 and it can readily be seen that the logarithmic corrections are extremely small. (For $V^{(000)}$ they are of the order of one percent and for $A^{(100)}$ roughly five percent; in neither case is our result changed to two significant figures). We therefore feel justified in neglecting these corrections for simplicity.

Now, $V^{(000)}$ is, of course, unity and the scale of this wavefunction is set by f_N , which is experimentally known. Thus our first sum rule is indeed a consistency condition. We substitute (2.7.27), together with the experimental data into (2.7.23) to find that the QCD and resonance sides are respectively 1.333 and 1.296, a good agreement. We now use (2.7.23) to specify a value for G_ρ^T which gives exact agreement between the two sides. This value is:

$$\frac{{G_\rho^T}^2}{4\pi} = 21.7 \qquad (2.7.28)$$

and we use it henceforth. (Note that this policy further justifies the neglect of the ω-meson; we are not investigating coupling constants, but the experimentally unknown matrix elements of the proton wavefunction).

The next task is to take a more precise look at the sum rule for the ω^1 coefficients. Substituting the data into this equation, we get:

$$\frac{A^{(100)}}{V^{(000)}} = -0.18 \tag{2.7.29}$$

Once more we should stress that this quantity is zero in the asymptotic limit. The result of (39) is consistent with (2.7.29); they obtained -0.25 < $A^{(100)}/V^{(000)}$, -0.17, and the model wavefunction they proposed had the value -0.24 The result we obtain is quite stable with respect to both M^2 (as seen above for y = 4/3) and G_ρ^T (G_ω^T) , which can be varied inside the errors without damaging our conclusions (if G_ρ^T is minimized and G_ω^T still neglected the value of $A^{(100)}$ is unchanged to two significant figures).

The next sum rule (ω^2 , equation (2.7.25)) involves a complicated mixture of moments. Using the experimental information and the value of $A^{(100)}$ obtained above, we get:

$$0.38 = -2.44 + \frac{4\left(2V^{(200)} + V^{(002)} - 2A^{(200)}\right)}{V^{(000)}} \tag{2.7.30}$$

Substituting the central values of Chernyak and Zhitnitsky into this gives the right hand side a value of 0.88. Should we minimize the right hand side (inside Chernyak and Zhitnitsky´s range of allowed values) this would become 0.2. So although we cannot exactly predict a moment this constraint is satisfied by their results. (It is amusing to note that the moments of their model wavefunction yield a value of 0.349!). By contrast using the moments of the asymptotic wavefunction in (2.7.30), but still taking the $A^{(100)}$ value we have obtained, gives -0.73; using only the asymptotic values means killing the first term on the right hand side, which gives us a result of +1.7. Evidently our result is extremely sensitive to the antisymmetric components of the wavefunction and it is no easy matter to obtain agreement between the two sides of the sum rule.

2.8 Discussion

The results obtained from the above equations provide strong evidence for the low-energy, non-perturbative, proton wavefunction being greatly different from the asymptotic form. It has an antisymmetric component which must be taken into account to get agreement between the two sides of our sum rules. This alone is encouraging, but furthermore the magnitude and sign of the antisymmetric component is in accord with the results obtained independently from sum rules for two-point functions (39).

It is evidently interesting to try to improve upon this result. For example, one might try to better the simple pole model, used here, by adding polynomial corrections to the resonance terms, as was done by Craigie and Stern for the $\rho\omega\pi$ vertex (19). This, however, appears impossible in our case, as they had two sum rules (one extra by the use of the ABJ anomaly to specify a subtraction coefficient), which enabled them to use extra variables. We do not have enough equations to specify the coefficients of the correction terms, and also using the second sum rule one is, in fact, combining two inconsistent equations, thus making us sceptical of the results so obtained.

The second idea one might consider is to study the vertex function with a different mesonic current. An obvious candidate here is the pseudoscalar current. However, there is no stability point in this case because of the great difference between the masses of the proton and the pion (an equation like (2.7.27) involves the logarithm of a negative number). Should one merely take M^2 such as to satisfy the consistency condition (ω^0 sum rule), we would have to use $M^2 \simeq 4\ GeV^2$, which is so large as to make background (continuum) effects very important.

CHAPTER THREE

THE PION WAVE FUNCTION

3.1 Results from Perturbation Theory and from Two-Point Functions

As stated earlier, the predictions of perturbation theory for the process $\gamma\gamma \rightarrow \pi^+\pi^-$ are in excellent agreement with the experiment data (11), (with regard to both normalization and the energy dependence). Paradoxically this does not conflict with the belief that it is our lack of knowledge of the wavefunctions that most hinders us from predicting exclusive processes. This is due to the, fortuitous, insensitivity of this particular process to the details of the pion distribution amplitude; the results can be written in terms of the (experimentally known) pion form factor.

However, the decays of heavy mesons into pionic states do indicate that once again the asymptotic wavefunction is inadequate. (The branching ratios using the perturbation theory wavefunction are too small (17).)

The results of a study of this wavefunction, via the two-point sum rules, are to be found in references(17, 26). The calculation is essentially similar to that of the proton by Chernyak and Zhitnitsky (see section 2.4). Starting from the definition of the wavefunction for the lowest twist pion component:

$$\langle 0|\bar{d}(z)\gamma_\nu\gamma_5 \exp\left\{ig\int_{-z}^{z} d\sigma_\mu B_\mu(\sigma)\right\}u(-z)|\pi^+(q)\rangle_{\mu^2} =$$

$$= \sum_n \frac{(-1)^n}{n!}\langle 0|\bar{d}(0)\gamma_\nu\gamma_5(z_\mu \overleftrightarrow{D}_\mu)^n u(0)|\pi^+(q)\rangle_{\mu^2} \tag{3.1.1}$$

$$= i q_\nu \tilde{\varphi}(z.q, \mu^2) + \ldots \qquad z^2 = 0, \; n = 0, 2, 4 \ldots$$

where $\tilde{\varphi}(z.q, \mu^2) = \int_{-1}^{1} d\xi \, e^{i\xi(z.q)} \tilde{\varphi}(\xi, \mu^2)$

and $\tilde{\varphi}(\xi, \mu^2) = \tilde{\varphi}(-\xi, \mu^2)$

sum rules for two-point functions with derivative currents are considered. The diagrams of figures 3 and 4 were calculated and sum rules derived. (Note that the radiative correction diagrams of figure 3 were not considered originally, but they have been calculated in (50) and shown to leave the results unchanged).

Defining the normalized wavefunction:

$$\tilde{\varphi}(\xi, \mu^2) = f_\pi \, \varphi(\xi, \mu^2) \tag{3.1.2}$$

we then define moments of the normalized wavefunction, such that:

$$\langle \xi^n \rangle = \int_{-1}^{1} d\xi \, \varphi(\xi, \mu^2)\xi^n, \quad \langle \xi^0 \rangle = 1 \tag{3.1.3}$$

From Appendix C we know that the asymptotic wavefunction for the lowest twist pion wavefunction is:

$$\varphi_{as}(\xi) = \frac{3}{4}(1-\xi^2) \tag{3.1.4}$$

(see figure 10). The moments of this wavefunction are:

$$\langle \xi^2 \rangle = 0{\cdot}20, \ \langle \xi^4 \rangle = 0{\cdot}086, \langle \xi^6 \rangle = 0{\cdot}048 \tag{3.1.5}$$

and differ greatly from the non-perturbative wavefunction proposed by Chernyak and Zhitnitsky. As a result of their sum rules they find moments:

$$\begin{aligned} \langle \xi^2 \rangle_{\mu^2 = 1.5\,\text{GeV}^2} &= 0{\cdot}40 \\ \langle \xi^4 \rangle_{\mu^2 = 2{\cdot}2\,\text{GeV}^2} &= 0{\cdot}24 \\ \langle \xi^6 \rangle_{\mu^2 = 3\,\text{GeV}^2} &= 0{\cdot}17 \end{aligned} \tag{3.1.6}$$

(An estimate of the reliability of their results comes from the sum rule without any derivatives. This predicts $f_\pi = 129\,\text{MeV}$, which is to be compared with the experimental value of 133 MeV.) This leads Chernyak and Zhitnitsky to propose a model wavefunction:

$$\varphi(\xi, \mu^2 \approx (500\,\text{MeV})^2) = \frac{15}{4}\xi^2(1-\xi^2) \tag{3.1.7}$$

with moments:

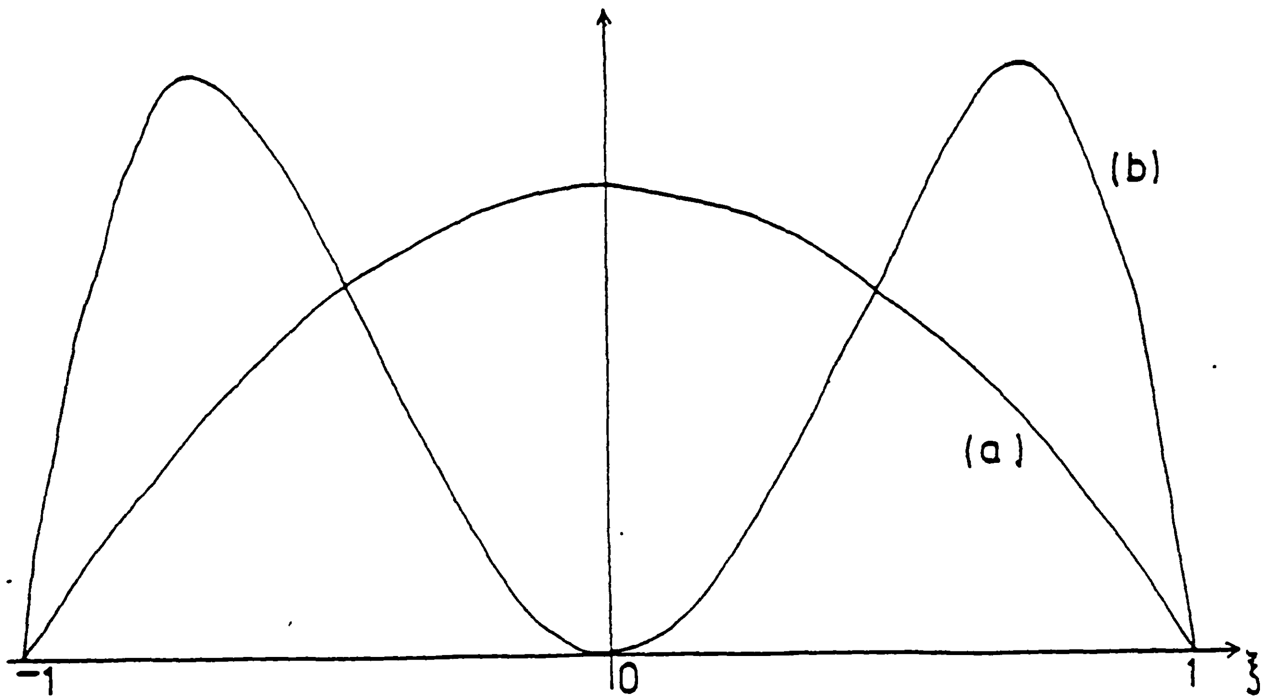


Figure 10: a) Asymptotic pion wavefunction.
b) Wavefunction from QCD sum rules.

$$\langle \xi^2 \rangle = 0.43 \, , \, \langle \xi^4 \rangle = 0.24 \, , \, \langle \xi^6 \rangle = 0.15 \tag{3.1.8}$$

The non-convex nature of this wavefunction (figure 10) goes against one's initial prejudices, as voiced, for example by Ioffe (27), who claims that the true non-asymptotic form of the wavefunction should be convex (although perhaps broader than the asymptotic form).

3.2 The Pion Wavefunction and Experiment

The sum-rule wavefunction (3.1.7) can be applied to a variety of exclusive processes in an attempt to improve upon the agreement between theory and experiment. The process $\gamma\gamma \to \pi^+\pi^-$ is, of course, useless as such a testing ground, because of the previously noted insensitivity to the wavefunction form. The heavy mesonic decays which earlier gave us motivation for our doubts as to the validity of the asymptotic form at lower energies provide an interesting probe of the wavefunction. The results (17, 26) of such calculations give reasonable agreement for charmonium decay widths, whilst the use of wavefunctions with a convex shape is, seemingly, ruled out.

As noted in section (2.2), electroproduction of an isolated meson (15, 16) can provide a determination of the mesonic wavefunction, because the amplitude for this process is linear in the particle wavefunction. Another process which shares this desireable quality of linearity is the vertex $\gamma^* \gamma^* \to M$, where M denotes the meson in question. This process has been considered in QCD in reference (18) and its suitability as a source of the determination of the pion wavefunction pointed out in (16). (It does not admit of a baryon analogue, because of baryon number conservation preventing a linear amplitude). The amplitude for the process is of the form:

$$F(z, \varphi] = \int_0^1 dx\, K(z, x)\varphi(x) \tag{3.2.1}$$

where $z = (q_1^2 - q_2^2)/(q_1^2 + q_2^2)$ and K is a non-trivial kernel which can, in principle, be inverted (14, 16).

There has been no sign of the experiments above being carried out, so there is still a need to independently check the pion wavefunction results of Chernyak and Zhitnitsky.

3.3 Lowest Order Vertex Sum Rules for the Pion

The Chase process (14, 16) mentioned above can be approached by means of resonance saturation (as an alternative to experimental data). This was indeed done by Craigie and Stern (19), who considered the vertex function of figure 11.

Their calculation followed the lines of the proton calculation given earlier. The diagrams of figure 11.b were

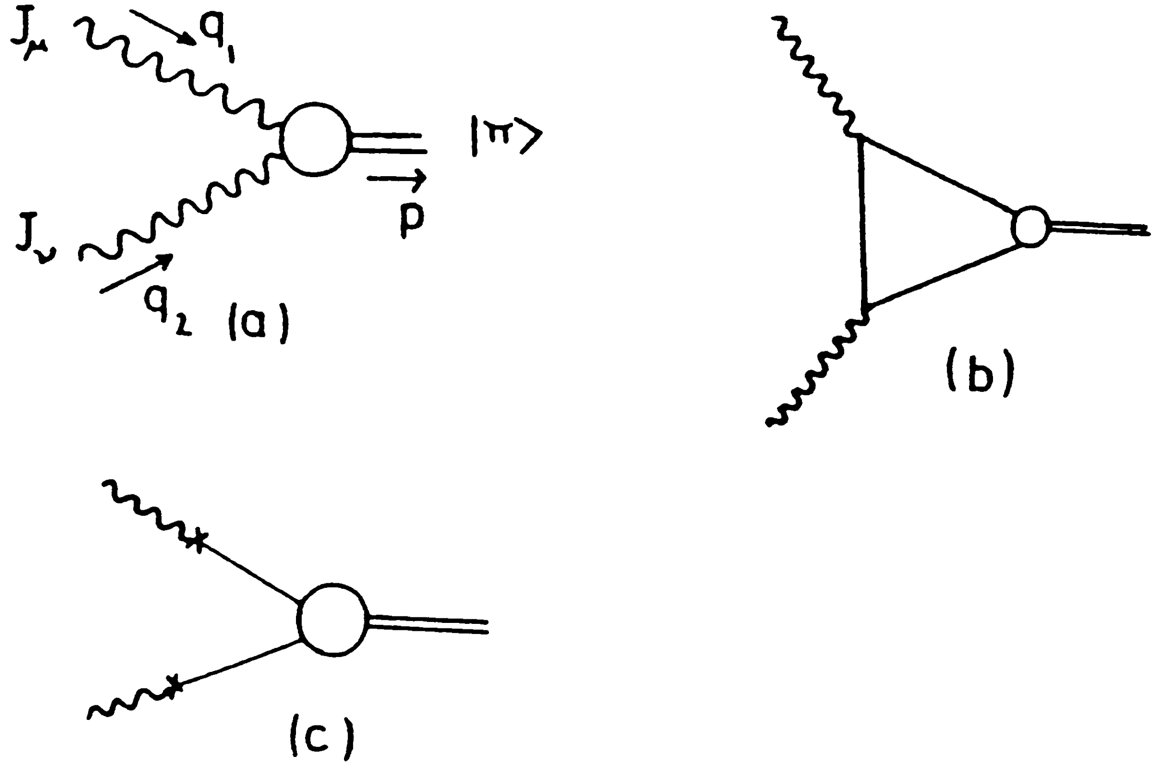


Figure 11: a)The pion vertex function
b)Lowest order QCD for the vertex.
c)Resonance saturation; $\times$ denotes a resonance.

calculated as:

$$\frac{1}{4}\int_{-1}^{1} dx \, \frac{tr(\gamma_\mu \not{\lambda} \gamma_\nu \not{p} \gamma_5)}{\lambda^2} \varphi(\xi, Q^2) \tag{3.3.1}$$

where $Q^2 = \frac{1}{2}(q_1^2 + q_2^2)$

Now we see that $\lambda = q_1 - \frac{1}{2}(1+x)\,p$, so the diagram evidently yields:

$$\int_{-1}^{1} dx \, \frac{\varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha p^\beta \varphi(\xi, Q^2)}{Q^2(1-\omega\xi)} \tag{3.3.2}$$

where:

$$\omega = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}$$

Using the fact that $|\omega| \leq 1$, we expand the denominator to give, for our vertex:

$$\frac{1}{Q^2}\varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha p^\beta \sum_{n=0,2,4..} \omega^n \langle \xi^n \rangle \tag{3.3.3}$$

where

$$\langle \xi^n \rangle = \int_{-1}^{1} \varphi(\xi)\, \xi^n\, d\xi$$

(Our definition of $\langle \xi^n \rangle$ is related to the notation of Craigie and Stern by a factor of 2^n ; $\langle \xi^n \rangle = a_n / 2^n$.)

On the other side of the sum rules, a light-cone dispersion relation (Appendix D) gave:

$$\frac{1}{Q^2} \sum_{n=0,2,4..} \omega^n \langle \xi^n \rangle = \frac{1}{\pi} \int_{s_0}^{\infty} \frac{ds}{s+Q^2}\, W(q_1^2, q_2^2) \tag{3.3.4}$$

$$\varepsilon_{\mu\nu\alpha\beta}\, q^\alpha p^\beta\, W(q_1^2, q_2^2) = \frac{1}{2} \int d^4 z\, e^{iq\cdot z} \langle 0 | \{ J_\mu(\tfrac{z}{2}), J_\nu(-\tfrac{z}{2}) \}_+ | \pi^+(p) \rangle$$

Equation (3.3.4) is a sum rule, which after resonance saturation and substitution of the data will yield results for the $\langle \xi^n \rangle$. However, in this vertex we can take advantage of the anomaly, to supply a subtraction constant (in the $\lambda = 0$ limit, Appendix D); this gives a subtracted sum rule (51):

$$\frac{1}{\pi} \int_{s_0}^{\infty} \frac{ds}{s(s+Q^2)}\, W(q_1^2, q_2^2) = \frac{1}{Q^2} \frac{N_c}{8\pi^2 f_\pi} - \frac{1}{Q^4} \sum_{n=0,2,4..} \omega^n \langle \xi^n \rangle$$

With the simple model for resonance saturation (19), that both currents are saturated with ρ and ω -mesons, we can write $W(q_1^2, q_2^2)$ as (m is the vector meson mass):

$$W(q_1^2, q_2^2) = \frac{\pi}{2} g m^2 f_\rho \left\{ \frac{1}{\omega m^2} \delta\left(s - m^2 - \frac{\omega m^2}{1+\omega}\right) + (\omega \to -\omega) \right\}$$

After the Borel transform has been applied to these equations we get two sum rules:

$$\frac{1}{1+\omega} m f_\rho G\left[m^2\left(\frac{1-\omega}{1+\omega}\right)\right] e^{-y/1+\omega} + \omega \to -\omega =$$

$$\underset{\omega \to 0}{=} m f_\rho g e^{-y}\left[y + \omega^2\left(y - y^2 + \frac{y^3}{6}\right) + O(\omega^4)\right] \qquad (3.3.5)$$

$$= f_\pi\left[1 + \omega^2 \langle \xi^2 \rangle + O(\omega^4)\right] .$$

and

$$m f_\rho g e^{-y}\left[1 + y + \omega^2\left(-\frac{1}{2} y^2 + \frac{1}{6} y^3\right) + O(\omega^4)\right]$$

$$(3.3.6)$$

$$= \frac{N_c m^2}{8\pi^2 f_\pi} - y f_\pi\left[1 + \omega^2 \langle \xi^2 \rangle + O(\omega^4)\right]$$

where we have defined $y = m^2 / M^2$.

If we look for a stability point for y and then take this as a function of ω we see that:

$$y_0 = 1 - \frac{2}{3}\omega^2 + \ldots \tag{3.3.7}$$

Now, the first sum rule is the y derivative of the second, so if the first sum rule is satisfied at some y_0 this is also the stability point of the second. Thus we evaluate our sum rules at $y_0 = 1 - \frac{2}{3}\omega^2$. Expanding (3.3.5) to order ω^2 and considering the ω^0 sum rule for (3.3.6) Craigie and Stern get three sum rules:

$$e^{-1} m f_\rho g = f_\pi \tag{3.3.8}$$

$$\frac{1}{6} e^{-1} m f_\rho g = \langle \xi^2 \rangle \tag{3.3.9}$$

$$2 e^{-1} m f_\rho g = \frac{N_c m^2}{8\pi^2 f_\pi} - f_\pi \tag{3.3.10}$$

They find satisfaction from (3.3.8), which predicts the coupling constant $g = 2.26\,GeV^{-1}$ (in good agreement with the experimental data we shall use below), and use (3.3.10) combined with (3.3.8) to predict:

$$f_\pi = \sqrt{\frac{N_c m^2}{24\pi^2}} = 88\,\mathrm{MeV} \qquad (3.3.11)$$

which is in very good agreement with the experimental value of 93 MeV. (Note – the factor of $\sqrt{2}$ difference from the definition on p.66.)

More important for us this leads to the prediction $\langle \xi^2 \rangle = \frac{1}{6} \simeq 0.17$! This value disagrees entirely with Chernyak and Zhitnitsky, predicting a wavefunction even narrower than the asymptotic form. However, this is not their final answer; they proceed to improve upon their simple pole model by adding polynomial corrections of the form:

$$\frac{g m^2}{m^2 - t} \longrightarrow \frac{g m^2}{m^2 - t} + g \sum_{r=0}^{\infty} \delta_r \left(\frac{m^2 - t}{m^2} \right)^r \qquad (3.3.12)$$

The δ_r are initially unknown constants. By truncating the series after the first three δ_r, Craigie and Stern hope to increase the accuracy of their predictions. The values of the δ_r can be found from a generalized stability equation, together with combinations of the two sum rules. Values for the δ_r are now used, together with a non-zero λ parameter (appendix D) which also requires some algebraic combination of the two sum rules, to give a fresh prediction for $\langle \xi^2 \rangle$; this is:

$$\langle \xi^2 \rangle = 0.40 \tag{3.3.13}$$

The agreement between this and Chernyak and Zhitnitsky is excellent.

Unfortunately, we do not have a great deal of faith in this result. This is because if one expands (3.3.6) to order ω^2 one gains another sum rule. Craigie and Stern say that we should evaluate this at $Y = 1 + C_2\omega^2$, where C_2 is such as to produce consistency between the two sum rules. However, expanding (3.3.6) to ω^2 gives an equation where the C_2 dependence has cancelled. The result of this equation is that:

$$\langle \xi^2 \rangle = \frac{1}{3}\,\frac{e^{-1} m f_\rho g}{f_\pi} = \frac{1}{3} \simeq 0.33 \tag{3.3.14}$$

evidently inconsistent with their earlier result. Indeed many of their intermediary equations, en route to obtaining their final result, can be used to produce any value of $\langle \xi^2 \rangle$ between 0.17 and 0.4.

It can be argued therefore that they have two final conclusions; $\langle \xi^2 \rangle = 0.17$, $\langle \xi^2 \rangle = 0.33$. One could indeed say that the result $\langle \xi^2 \rangle = 0.33$ is better because the subtraction should help control the behaviour of the series, rendering it more convergent.

We would tend to argue that the sum rules, by differing so markedly in their predictions for $\langle \xi^2 \rangle$, betray the importance of corrections in this particular case. As one studies higher powers of ω this is bound to occur and we shall now consider such effects.

3.4 Improved Sum Rules for the Pion Vertex

In this section we shall add three new terms to the sum rules of Craigie and Stern; radiative corrections (figure 12), power corrections which are connected with higher twist parts of the pion wavefunction (figure 13), and an ansatz for continuum effects. The hope is that these results will give a better result for the lowest non-trivial pion moment.

The first effect to be considered is that of the radiative corrections to the vertex. These have been calculated by Chase (18) and Voloshin (52) for particular values of ω (corresponding to the photons carrying equal momenta, $\omega = 0$, and to one being real $\omega = 1$). These two calculations disagree, however, at $\omega = 0$ (by a factor of 5/6). The calculation for general ω has been performed by Braaten (53); his result agreed with that of Chase in this limit. We repeated this calculation of the diagrams of figure 12. The calculation was performed in $n=4-2\varepsilon$ dimensions and, though tedious, is essentially straightforward; one subtlety, however, must be taken account of,

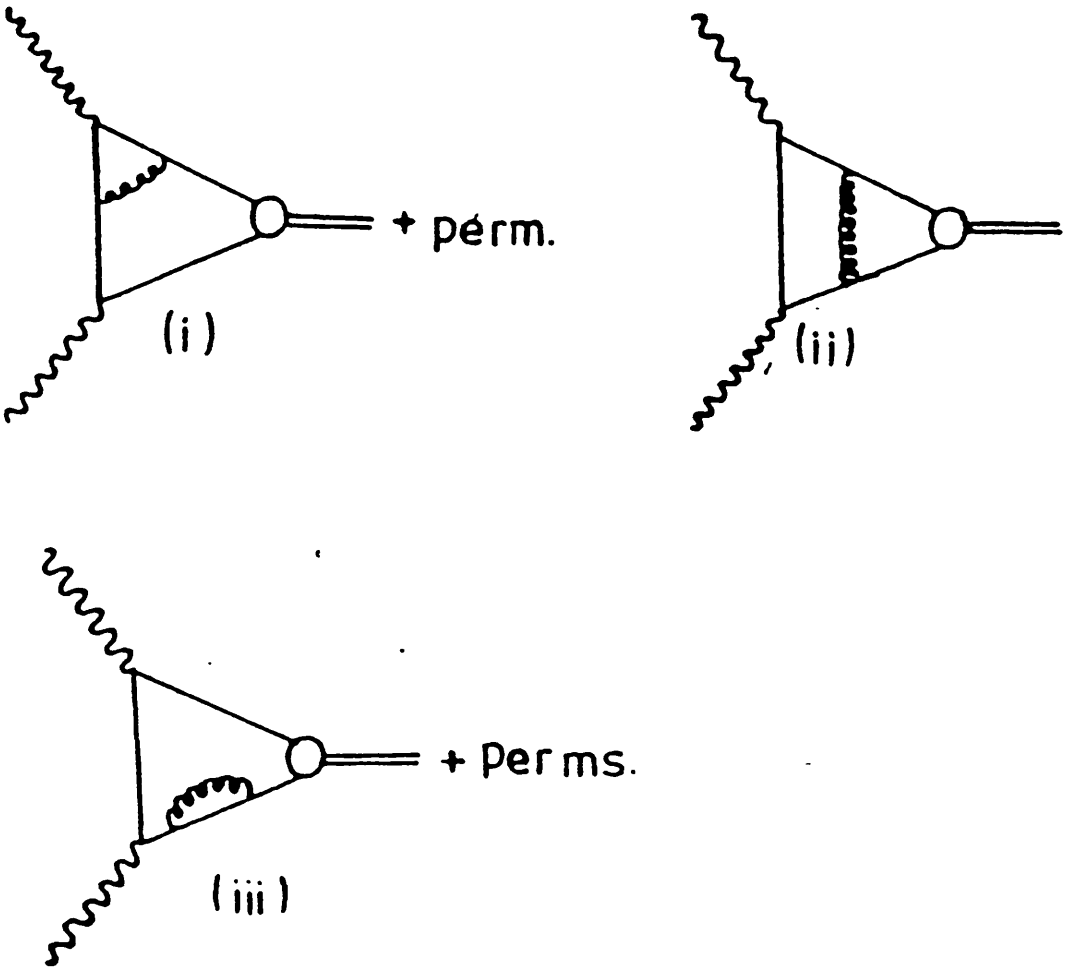


Figure 12: Radiative corrections to the pion vertex.

and that is the use of γ_5 in n dimensions. The trace below appears straightforward:

$$\mathrm{tr}\left(\gamma_5 \not{a}\, \gamma_\mu \not{b} \not{c} \not{d}\, \gamma_\mu\right) = (n-6)\, \mathrm{tr}\left(\gamma_5 \not{a} \not{b} \not{c} \not{d}\right) \tag{3.4.1}$$

but, if first one uses the anticommutation of γ_μ with γ_5 one gets:

$$\mathrm{tr}\left(\gamma_5 \not{a}\, \gamma_\mu \not{b} \not{c} \not{d}\, \gamma_\mu\right) = (2-n)\, \mathrm{tr}\left(\gamma_5 \not{a} \not{b} \not{c} \not{d}\right) \tag{3.4.2}$$

and (3.4.1) and (3.4.2) differ for ε non-zero. The resolution of this difficulty was provided by Braaten using mass regularization; for our purposes the solution is that the "box diagram" (figure 12 (ii)) must be calculated with the γ_μ's contracted through the γ_5 and that the other diagrams should be calculated directly as though there were no γ_5 factor.

The result of the calculation is to be found in appendix E. For simplicity we only give the coefficients of an expansion in ω . Modulo the factor of the lowest order diagram, we get:

$$\begin{aligned} \omega^0 &: \quad -\alpha_s/\pi \\ \omega^2 &: \quad \frac{5\alpha_s}{12\pi}\left(\xi^2-1\right) \end{aligned} \tag{3.4.3}$$

the ω^0 coefficient agrees with the result of Chase. (Overall we agree with Braaten). The results given above are, of course, UV finite, and we have subtracted the IR divergences to obtain the result we employ (such a subtraction is only necessary for ω non-zero).

The next corrections we consider are the power corrections. In two-point sum rules these correspond to the vacuum expectation values of the higher dimensional operators being non-zero. Here we also have contributions of higher twist components of the pion to the vertex, which give the leading corrections to the sum rules.

The power corrections to this vertex have been considered by Gorsky (54). There are two types of higher twist contribution; a wavefunction dependent on the sampling of the transverse momentum of the hadron, and a three-particle wavefunction, which involves a gluonic insertion. (The other higher twist wavefunction $\overline{\psi}\,\gamma_5\,\psi$ does not contribute, as can be seen from the trace). So, the leading corrections are determined by the wavefunctions φ_2 and φ_3, which are defined in a particular frame (54) as:

$$\langle \pi^-(p)|\,\overline{u}(y)\,\overleftarrow{\partial}_\perp^{\,2}\,\gamma_\kappa\gamma_5\, d(0)|0\rangle_{\mu^2} = i\sqrt{2}\,C\,P_\kappa f_\pi\,\varphi_2(yp,\mu^2) \tag{3.4.4}$$

and

$$\langle \pi^0(p)|\,\overline{u}(y)\gamma_\rho\,\widetilde{G}_{\alpha\beta}(z)\,d(0)\,|0\rangle_{\mu^2} =$$
$$= -if_\pi\,C'(P_\beta\, g_{\alpha\rho} - P_\alpha\, g_{\beta\rho})\int_0^1 [dx]\,\varphi_3(x,\mu^2)\, e^{ip(x_1y+x_2z)} \tag{3.4.5}$$

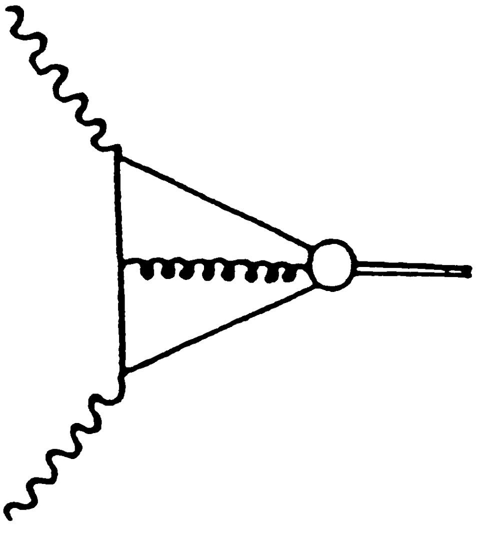


Figure 13: Power corrections to the pion vertex.

where $[dx] = dx_1 dx_2 dx_3 \delta(1-\Sigma x_i)$, $\tilde{G}_{\alpha\beta} = \frac{g}{4} \varepsilon_{\alpha\beta\mu\nu} \cdot \lambda^a G^a_{\mu\nu}$ and the X_i are the momentum fractions carried by the constituents and X_3 is that of the gluon. C and C' are non-perturbative constants related to each other via the equations of motion:

$$C = -\frac{5}{18} C' \tag{3.4.6}$$

Essentially, we must calculate the average transverse momentum in the pion (55), $\langle K_\perp \rangle^A_\pi$ where

$$\langle 0 | \bar{d} \gamma_\mu \gamma_5 (i \vec{D}_\nu)^2 u | \pi \rangle = i f_\pi q_\mu \langle K_\perp^2 \rangle^A_\pi \tag{3.4.7}$$

This can be related to the matrix element (55):

$$\langle 0 | \bar{d} \gamma_\mu \gamma_5 (i \vec{D}_\nu)^2 u | \pi \rangle = -\frac{1}{2} \langle 0 | \bar{d} \gamma_\mu \gamma_5 \, i g \sigma_{\alpha\beta} G^a_{\alpha\beta} \frac{\lambda^a}{2} u | \pi \rangle \tag{3.4.8}$$

The determination of the right hand side of (3.4.8) was carried out in (55), by retaining only the pion contribution to the resonance saturation of the correlator:

$$q_\mu I(q^2) = i \int d^4x \, e^{iq\cdot x} \langle 0 | T \left(\bar{d}(x) \gamma_\mu \gamma_5 \, g \sigma_{\alpha\beta} G^a_{\alpha\beta} \frac{\lambda^a}{2} u(x), \; \bar{u}(0) \, i \gamma_5 \, d(0) \right) | 0 \rangle \tag{3.4.9}$$

In the $|q^2| \longrightarrow \infty$ limit this leads to the relation (in the chiral limit) that:

$$\langle K_\perp^2 \rangle_\pi^A = \frac{5}{36} \frac{\langle 0 | \bar{u} \sigma_{\mu\nu} i g G^a_{\mu\nu} \frac{\lambda^a}{2} u | 0 \rangle}{\langle 0 | \bar{u} u | 0 \rangle}$$

$$\simeq (320 \text{ MeV})^2 \qquad (3.4.10)$$

Using (3.4.6) and (3.4.10) we have the extra non-perturbative information required to exploit our sum rules. (The use of ordinary and covariant derivatives in (3.4.4) and (3.4.7) is consistent thanks to the virtues of the fixed-point or Fock-Schwinger gauge, appendix A).

The contributions made to the vertex function by the higher twist wavefunctions are (54):

$$\frac{2 c' F_0(q_1^2, q_2^2)}{q^2} \left\{ -\frac{1}{2} \int_{-1}^{1} \frac{\frac{5}{18} \varphi_2(\xi)\, d\xi}{(1+\omega\xi)^2} \right.$$

$$\left. -\frac{1}{3} \int_0^1 \frac{[dx]\, \varphi_3(x_i)}{1+\omega(1-2x_2)} \int_0^1 \frac{\alpha\, d\alpha}{(1+\omega(1-2(x_2-\alpha x_3))} \right. \qquad (3.4.11)$$

where $F_0(q_1^2, q_2^2) = -2\sqrt{2}(e_u^2 - e_d^2) f_\pi$ and e_i is the i^{th}

quark charge.

We now expand this result in ω to get the result for the power corrections; modulo the lowest order factor, we get:

$$\omega^0 : \quad -\frac{8}{9}\,\frac{C'}{Q^2} \tag{3.4.12}$$

$$\omega^2 : \quad \Delta^{\omega^2} = -\frac{C'}{Q^2}\Big\{ 1 + \frac{5}{9}\langle \xi^2 \rangle_{\varphi_2} + 4\langle 020 \rangle - 4\langle 010 \rangle$$

$$+ \frac{8}{9}\langle 001 \rangle + \frac{1}{6}\langle 002 \rangle - \frac{16}{9}\langle 011 \rangle \Big\}$$

where we have defined:

$$\langle n_1 n_2 n_3 \rangle = \int_0^1 [dx]\, \varphi_3(x_i)\, x_1^{n_1} x_2^{n_2} x_3^{n_3} \tag{3.4.13}$$

The power corrections and radiative corrections complete the leading QCD corrections to the vertex. However, we now wish to improve upon the resonance side of our sum rules. The use of polynomial corrections (3.3.12) to the simple poles used above is unfortunately now impossible, as we cannot specify the δ_r coefficients. So, we must take a simpler ansatz for the background (continuum) effects.

We use the asymptotic properties of QCD in a manner inspired by the work on two-point sum rules. We add a term which in the $|q^2| \longrightarrow \infty$ limit agrees with the QCD prediction and introduce one new parameter, s_0, the duality interval which we fit later. The sum rules thus take the form (after the Borel transform):

$$\frac{g f_\rho m}{f_\pi} \left\{ \frac{m^2}{M^2} \exp\left(-\frac{m^2}{M^2}\right)\right\} + \left(1 - \frac{\alpha_s}{\pi}\right) \exp\left(-\frac{s_0}{M^2}\right) = \tag{3.4.14}$$

$$= \left(1 - \frac{\alpha_s}{\pi}\right) - \frac{8}{9}\frac{c'}{M^2}$$

and

$$\frac{m f_\rho g}{6 f_\pi} \left\{ \frac{m^2}{M^2} \exp\left(-\frac{m^2}{M^2}\right)\right\} + \left\{ \langle \xi^2 \rangle + \frac{5}{4}\left(\langle \xi^2 \rangle - 1\right)\frac{\alpha_s}{3\pi} \right\} \times$$

$$\times \exp\left(-\frac{s_0}{M^2}\right) = \langle \xi^2 \rangle + \frac{5}{4}\left(\langle \xi^2 \rangle - 1\right)\frac{\alpha_s}{3\pi} + \Delta^{\omega^2} \tag{3.4.15}$$

where Δ^{ω^2} was defined in (3.4.12). These equations complete our vertex function sum rules. The major difference between such sum rules and the lowest order sum rules is that we now hope to find an equivalence between the two sides over a range of M^2. This means that instead of looking at the sum rules at a single stability point we can vary our parameters so as to obtain a best fit over a whole range of M^2, as in customarily done in sum rules for two-point functions. Before attempting such a procedure it is best, however, to look at the uncertainties in (3.4.14) and (3.4.15).

3.5 The Higher Twist Wave Functions

The major sources of possible error in our sum rules are the experimental data for g (which has quite a wide range of allowed values) and the forms of the higher twist wavefunctions. We are not in a position to improve the accuracy of experiment and so we shall here study the forms of the higher twist wavefunctions. (The ω^0 sum rule will give us a prediction for g anyway, and we shall later find that our conclusions will not be changed by any value of g within the experimentally allowed range).

The asymptotic form of the higher twist wavefunctions can be found from the Born diagrams of the relevant two-point correlators. The results are (Appendix C):

$$\varphi_2^{as}(\xi) = \frac{35}{32}\,(1-\xi^2)^3 \;, \quad \varphi_3^{as}(x_i) = 120\,x_1 x_2 x_3 \qquad (3.5.1)$$

It should be noted that the asymptotic form of φ_2 used in (54) differs from the above; it is a simple matter to calculate the Born diagram and check the validity of (3.5.1).

However, from our point of view it would not be utterly satisfactory to use the asymptotic forms of the wavefunctions. The low-energy, realistic, forms may well be expected to differ from these asymptotic wavefunctions, and so the use of (3.5.1) may introduce an error into the calculation. The ω^0 sum rule does not depend on the wavefunction shape, of course ($\langle\xi^0\rangle \equiv \langle\xi^0\rangle_{\varphi_2} \equiv \langle 000\rangle \equiv 1$) and this problem only affects the ω^2 sum rule through the power corrections Δ^{ω^2} , (3.4.12). It is extremely fortuitous that Δ^{ω^2} is insensitive to the form of φ_3 . This is because the combination of the $\langle n_1 n_2 n_3\rangle$ found in (3.4.12) is such that the constraints $x_1+x_2+x_3 \equiv 1$ and $\langle n_1 n_2 n_3\rangle = \langle n_2 n_1 n_3\rangle$ tend to minimize the effects of changes in Δ^{ω^2} . For example: using the asymptotic values of φ_2 and φ_3 we get $\Delta^{\omega^2} = -0.094$, with $C' = 0.2$ (55); if we use for φ_3 the non-asymptotic wavefunction proposed in (17):

$$\varphi = 360\, x_1 x_2 x_3^2 \cdot 14 \cdot \left[1.5\left(x_1^2+x_2^2\right) + 4.5\, x_3^2 - 3.72\, x_3 + 0.38\right] \qquad (3.5.2)$$

the value of Δ^{ω^2} is only changed to $-0.092/M$. This leads us to feel safe in our use of φ_3^{as}.

However, φ_2 enters the sum rules via $\langle\xi^2\rangle_{\varphi_2}$ only and so this is the greatest source of uncertainty for the value of Δ^{ω^2} . Therefore we now proceed to consider this wavefunction inside the framework of sum rules for two-point functions. Using derivative currents we can calculate the non-asymptotic moments of φ_2 by considering the correlator:

$$T_{no}(q^2) = i\int d^4x\, e^{iq.x}\langle 0|T\left(J_\perp^{(n)}(x)\, J_\perp^{(0)}(0)\right)|0\rangle$$
$$= (z.q)^{n+2}\, I_{on} \qquad (3.5.3)$$

where $J_\perp^{(n)} = \bar{u}\,\partial_\perp^2\, z.\gamma\,\gamma_5 (z.\overleftrightarrow{D})^n d\ , \ z^2 = 0$

The $\partial_\perp^2$ factor in the current definition has two effects; firstly we are sampling the transverse momentum at the vertex and secondly on a practical level this implies that tree diagrams do not contribute due to the vacuum averaging of $\partial_\perp^2$ at the vertices; thus only loop diagrams need to be taken into account. Usually in the sum rules of two-point functions the leading correction, in the chiral limit, is the G^2 operator; however, because of the typical loop suppression factor of $1/4\pi$ associated with this the , tree level, four-fermion operator is equally significant. Here, however, the G^2 diagram is alone the most important correction and thus we need only consider this diagram to achieve a reasonable accuracy.

A suitable framework for calculations where one probes the transverse momentum is the Sudakov variables technique, where we decompose a momentum k as:

$$k = \alpha p + \beta\eta + k_\perp \qquad (3.5.4)$$

where η and p are both considered lightlike. $k_\perp$ is space-

like and of dimension n-2; it satisfies $k_\perp \cdot p = k_\perp \cdot \eta = 0$
The momentum integral is thus transformed as:

$$\int d^n p \rightarrow \int \frac{q\cdot\eta}{(2\pi)^2}\, d\alpha\, d\beta\, d^{n-2}p_\perp \tag{3.5.5}$$

Using this framework the Born diagram and the G^2 coefficient (using the Fock-Schwinger gauge) are straight-forwardly calculated. The results are:

$$I_{n0} = \frac{3}{8\pi^2(n+1)(n+3)(n+5)(n+7)}\left[1 + \langle 0|\frac{\alpha_s}{\pi} G^2|0\rangle \frac{2(n+7)\pi^2}{M^4}\right] \tag{3.5.6}$$

The phenomenological side requires the use of the matrix elements corresponding to pion saturation of the correlator:

$$\langle 0|\bar{d}\, z\cdot\gamma\, \gamma_5\, \overrightarrow{\partial}^{z}_{\perp} (z\cdot\overleftrightarrow{D})^n u|\pi^+(p)\rangle_{\mu^2} = i\sqrt{2} f_\pi\, (z\cdot p)^{n+1} C \langle \xi^n \rangle_{\varphi_2} \tag{3.5.7}$$

Adding the standard continuum term to (3.5.7), using a dispersion relation and taking the Borel transform gives the phenomenological side of the sum rules:

$$\frac{2\,C^2 f_\pi^2 \langle \xi^n \rangle_{\varphi_2}}{M^6} + \frac{3\exp(-S_0/M^2)}{8\pi^2(n+1)(n+3)(n+5)(n+7)} = I_{n0}$$

(3.5.8)

Equating (3.5.6) and (3.5.8) gives our set of sum rules. The parameters f_π^2, C and $\langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle$ are already known, ($f_\pi^2 = 0{\cdot}0177$, $C^2 = 0{\cdot}0031$, $\langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle = 0{\cdot}012\ \mathrm{GeV}^2$) and we can thus write:

$$n=0: \quad \frac{\langle \xi^0 \rangle_{\varphi_2}}{M^6} + 3{\cdot}30 \exp\left(-\frac{S_0}{M^2}\right) = 3{\cdot}30\left(1 + \frac{0{\cdot}182}{M^4}\right) \tag{3.5.9}$$

$$n=2: \quad \frac{\langle \xi^2 \rangle_{\varphi_2}}{M^6} + 0{\cdot}367 \exp\left(-\frac{S_0}{M^2}\right) = 0{\cdot}367\left(1 + \frac{0{\cdot}234}{M^4}\right) \tag{3.5.10}$$

$$n=4: \quad \frac{\langle \xi^4 \rangle_{\varphi_2}}{M^6} + 0{\cdot}100 \exp\left(-\frac{S_0}{M^2}\right) = 0{\cdot}100\left(1 + \frac{0{\cdot}286}{M^4}\right) \tag{3.5.11}$$

These sum rules were treated in the standard fashion, varying S_0 and $\langle \xi^n \rangle_{\varphi_2}$ over a range where the power corrections were between 5% and 30% of the right hand side of the sum rule. Now, φ_2 was defined such that $\langle \xi^0 \rangle_{\varphi_2}$ must be unity and so the variation of this is a check of the trustworthiness of our results. Such a procedure yields:

$$\langle \xi^0 \rangle_{\varphi_2 \,\bar{M}^2 = 1.1\,\mathrm{GeV}^2} = 0.98 \,, \quad S_0 = 0.65\,\mathrm{GeV}^2$$

$$\langle \xi^2 \rangle_{\varphi_2 \,\bar{M}^2 = 1.5\,\mathrm{GeV}^2} = 0.28 \,, \quad S_0 = 0.65\,\mathrm{GeV}^2$$

$$\langle \xi^4 \rangle_{\varphi_2 \,\bar{M}^2 = 1.7\,\mathrm{GeV}^2} = 0.10 \,, \quad S_0 = 0.65\,\mathrm{GeV}^2 \qquad (3.5.12)$$

The value of $\langle \xi^0 \rangle_{\varphi_2}$ gives us confidence in our results for $\langle \xi^2 \rangle_{\varphi_2}$ and $\langle \xi^4 \rangle_{\varphi_2}$. We should now compare the moments of (3.5.12) with those of the asymptotic wavefunction (3.5.1), which are:

$$\langle \xi^0 \rangle_{\varphi_2^{as}} = 1 \,, \quad \langle \xi^2 \rangle_{\varphi_2^{as}} = \frac{1}{9} \simeq 0.11 \,, \quad \langle \xi^4 \rangle_{\varphi_2^{as}} = \frac{1}{33} \simeq 0.03 \qquad (3.5.13)$$

Evidently the two wavefunctions differ greatly.

We now want to propose a model wavefunction which should fit the moments found in (3.5.12). To do this we first consider some physical expectations; we believe $\varphi(\xi)$ is positive and also that the probability of the whole momentum being carried by one quark is very small, i.e. $\varphi(\xi) \to 0$ as $\xi \to \pm 1$. With this in mind we propose the following wavefunction:

$$\varphi_2(\xi, \mu_0^2 \simeq 1.5\,GeV^2) = \frac{315}{16}\,\xi^2(1-\xi^2)^3 \tag{3.5.14}$$

The moments of this form are very close to those predicted in (3.5.12):

$$\langle \xi^0 \rangle = 1, \quad \langle \xi^2 \rangle = 0.27, \quad \langle \xi^4 \rangle = 0.11 \tag{3.5.15}$$

This model (3.5.14) has not been proven to be the non-asymptotic form of the wavefunction, but it can be hoped to duplicate the true higher twist pion wavefunction if used in the calculation of power corrrections to exclusive processes. It should be stressed that in the vertex sum rules we are considering,only the moment $\langle \xi^2_{\varphi_2} \rangle$ is taken into account.

3.6 Analysis of the Vertex Function Sum Rules

Using the results of the last section we can reduce the uncertainty in our vertex function sum rules (equations (3.4.14) and (3.4.15)). The low energy input required comes in three forms; that for which experiment gives the value with good accuracy (f_π , m, f_ρ), that which we have just obtained from two-point sum rules ($\langle \xi^2_{\varphi_2} \rangle$) and that which experiment gives us with a potentially significant error (g). The experimentally allowed range of g is (48):

$$1 \cdot 36 \leq g \leq 2 \cdot 50 \; (GeV^{-1}) \tag{3.6.1}$$

and we now use the ω^{0} sum rule (3.4.14) to specify a value of g. This consistency check for the whole approach thus gives the g value we use in our analysis of (3.4.15); although we will also consider how the variation of g inside the range (3.6.1) would affect our results.

The treatment of (3.4.14) follows the standard lines for two-point sum rules. The difference between the resonance and QCD sides of the rule is minimized over a range of M^2 by varying the values of g and S_0 used. This was done over the range $0 \cdot 5 \leq M^2 \leq 4 \cdot 0 \, (GeV^2)$ and gave a very good fit, with the final results:

$$S_0 = 1 \cdot 39 \; GeV^2, \quad g = 1 \cdot 4 \; GeV^{-1} \tag{3.6.2}$$

The value of g thus extracted can be put into our second sum rule to give the prediction for $\langle \xi^2 \rangle$. First, however, we want to improve our treatment by including the development of $\langle \xi^2 \rangle$ over the range of M^2, i.e. we insert the anomalous dimension effects on the wavefunction. To do this we consider the definition of the wavefunction:

$$\varphi(\xi, \mu^2) = (1-\xi^2) \sum_{n=0}^{\infty} a_n C_n(\xi) \left\{ \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right\}^{d_n}$$

$$\frac{\alpha_s(\mu^2)}{4\pi} = \frac{1}{b \ln \frac{\mu^2}{\Lambda^2}} \quad , \; b = 11 - \frac{2}{3} n_f \tag{3.6.3}$$

$$d_n = \frac{C_f}{b} \left[1 - \frac{2}{n(n+1)} + 4 \sum_{s=2}^{n} \frac{1}{s} \right]$$

the $C_n(\xi)$ are the Gegenbauer polynomials of order 3/2 (these are orthogonal with the measure $(1-\xi^2)$: see appendix C for details). The lower order polynomials are:

$$C_0(\xi) = 1$$

$$C_2(\xi) = \frac{3}{2}(5\xi^2-1) \tag{3.6.4}$$

which can be re-arranged as:

$$\xi^2 = \frac{1}{5}\left[\frac{2}{3}\, C_2(\xi) + C_0(\xi)\right] \tag{3.6.5}$$

Taking into account the orthogonality property implies:

$$\langle \xi^2 \rangle_{\mu_0^2} = \frac{1}{5}\left(\frac{2}{3}\,\eta_2 a_2 + \eta_0 a_0\right) \tag{3.6.6}$$

$$\eta_n = \int_{-1}^{1} d\xi\,(1-\xi^2)\, C_n^2(\xi) = \frac{2(n+1)(n+2)}{(2n+3)} \tag{3.6.7}$$

i.e. $\eta_0 = \frac{4}{3}$, $\eta_2 = \frac{24}{7}$

(Note that asymptotically $\eta_2 a_2 \to 0$, $\eta_0 a_0 \to 1$ and $\langle \xi^2 \rangle \to \frac{1}{5}$ the value associated with the asymptotic wave function $\varphi_{as} = \frac{3}{4}(1-\xi^2)$). We can now rewrite (3.6.6) with the help of (3.6.3) to get:

$$\langle \xi^2 \rangle_{\mu^2} = \frac{1}{5}\left[\frac{2}{3}\eta_2(a_2)_{\mu_0^2}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{d_2} + \eta_0(a_0)_{\mu_0^2}\right]$$

$$= \langle \xi^2 \rangle_{\mu_0^2}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{d_2} + \frac{1}{5}\left(1 - \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{d_2}\right) \quad (3.6.8)$$

If we take $n_f = 4$ then $d_2 = \frac{32}{75}$ $(\simeq 0.427)$. The equation can be rewritten as:

$$\langle \xi^2 \rangle_{\mu^2} = \langle \xi^2 \rangle_{\mu_0^2}\left(\frac{\ln \mu_0^2/\Lambda^2}{\ln \mu^2/\Lambda^2}\right)^{32/75} + \frac{1}{5}\langle \xi^2 \rangle_{\mu_0^2}\left[1 - \left(\frac{\ln \mu_0^2/\Lambda^2}{\ln \mu^2/\Lambda^2}\right)^{32/75}\right] \quad (3.6.9)$$

where we have used

$$\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} = \frac{\ln \mu_0^2/\Lambda^2}{\ln \mu^2/\Lambda^2} \quad (3.6.10)$$

Evidently (3.6.9) has the correct asymptotic behaviour; we can now use it in our second vertex sum rule, the new version of which we give below:

$$\frac{m^3 f_\rho g}{6 M^2 f_\pi}\exp\left\{-\frac{m^2}{M^2}\right\} + \left[\langle \xi^2 \rangle_{\mu^2} + \frac{5}{4}\left(\langle \xi^2 \rangle_{M^2} - 1\right)\frac{\alpha_s}{3\pi}\right] \cdot$$

$$\cdot \exp\left\{-\frac{s_0}{M^2}\right\} =$$

$$= \langle \xi^2 \rangle_{M^2} + \frac{5}{4} \left(\langle \xi^2 \rangle_{M^2} - 1 \right) \frac{\alpha_s}{3\pi} - \Delta^{\omega^2} \tag{3.6.11}$$

We are now familiar with the treatment of (3.6.11). The parameters we fix are: $\Lambda = 150$ MeV, g from (3.6.2), $\langle n_1 n_2 n_3 \rangle$ from the asymptotic wavefunction, the low energy moments of φ_2 found from the two-point sum rules of section 3.5, (3.5.12) and the remaining (well known) experimental data, taken from Ref. 48. We fit over a range of M^2 by varying $\langle \xi^2 \rangle$ and S_0 (so as to minimize the difference between the two sides of the sum rule); a good fit is obtained with the values:

$$\langle \xi^2 \rangle_{\bar{M}^2 = 2.25\ \text{GeV}^2} = 0.39 \, , \quad S_0 = 0.76\ \text{GeV}^2 \tag{3.6.12}$$

To compare our result directly with that of Chernyak and Zhitnitsky we have to renormalize our value to where they calculated . With the help of (3.6.9) we predict $\langle \xi^2 \rangle_{\bar{M}^2 = 1.5\ \text{GeV}^2} = 0.40$! Although the identity of the two results is doubtless fortuitous, it is certainly a strong confirmation of their result. The moment should be compared with that of the model wavefunction (3.1.7) proposed by Chernyak and Zhitnitsky, $\langle \xi^2 \rangle_{\text{model}} = 0.43$.

3.7 Discussion

It might naively be thought that the remaining source of uncertainty in the sum rules, the value of g to be used, could allow us to make a prediction of $\langle \xi^2 \rangle$ more in accord with the asymptotic wavefunction; such, however, is not the case. The value of g predicted from our first sum rule is, essentially, at the bottom end of the experimentally allowed range and any change in g will tend to increase the $\langle \xi^2 \rangle$ prediction from (3.6.9). For example, should we maximize g we would predict $\langle \xi^2 \rangle_{\bar{M}^2 = 2.25} = 0.45$.

We are thus forced to the conclusion that the low-energy moment, $\langle \xi^2 \rangle$, of the pion wavefunction is indeed $\gtrsim$ 0.40. The limiting case of convexity, $\varphi(\xi) = \frac{1}{2}$, has a value of $\langle \xi^2 \rangle = \frac{1}{3} \simeq 0.33$ and so we are forced unambiguously to the conclusion that the low energy pion wavefunction is non-convex (fig. 10).

The non-asymptotic form of the higher twist wavefunction φ_2 evaluated earlier (3.5.14) is also non-convex, although in this case this is not forced upon us by the moments ($\langle \xi^2 \rangle < \frac{1}{3}$, $\langle \xi^4 \rangle < \frac{1}{5}$.) However, the probability of finding a constituent with more than half the pion longitudinal momentum must certanly be considered greater than that predicted by the asymptotic wavefunction. The wavefunction given above should have similar properties to the true non-asymptotic form and can be used in calculations of the power corrections to exclusive processes involving the pion.

It would of course be interesting to have a prediction for the $\langle \xi^4 \rangle$ moment of the lowest twist pion wavefunction. We have not presented such a calculation here because of the uncertainty associated with extending our calculation to ω^4. Although, unlike previous work on vertex sum rules (19, 56, 57), we have here incorporated the effects of power corrections and radiative corrections and attempted to model continuum effects, the major difficulty lies in the power corrections for an ω^4 sum rule: this would incorporate of the order of twenty moments of the higher twist wavefunction, φ_3 , which is unknown in the low-energy region. Therefore we would have little control over the power corrections to such a sum rule and so we have not attempted to construct one.

It is perhaps worth commenting on the differences between the vertex sum rules for the pion and those considered in the second chapter for the proton. The most obvious difference, the existence of sum rules for odd powers of ω , is caused by the mass difference between the proton and the ρ -meson. (Although, as we have seen, the neglect of this mass gap can yield valuable information about the baryon wavefunction). This gives us more constraints upon the baryon wavefunction; but this is offset by the greater complexity of the wavefunction, $V(x_i) - A(x_i)$, which we are unable to specify completely from our sum rules, unlike the situation for the mesonic wavefunction which it is possible to predict moment by moment from the sum rules. Considerations of this type may be taken into account when attempting to construct vertex sum rules in other channels.

CHAPTER FOUR

REMARKS AND OUTLOOK

4.1 Extensions to Vertex Sum Rules

The possibility of extending the use of power corrections, radiative corrections and continuum effects is undoubtedly appealing. One option would be to extend the work of Ref. 57, in an attempt to explain the A_1 width. Another interesting idea would be to try to improve the earlier analysis of baryons (56). However, a new problem appears here; the uncertainty in the choice of currents used to represent the proton (47) will lead to still worse complications in the power corrections. Also here the extra non-perturbative constants required for such an analysis (the equivalents of C and C´ earlier) would have to be extracted. Such a calculation would, however, be extremely useful as it would give information on the constituent transverse momentum in the proton.

Vertex function sum rules could also be applied to other, new, vertices. Interesting possibilities are the vertices which correspond to η and η' states (and might be hoped to provide some information on the gluonic components of such states) and vertices involving mesons containing one or two heavy quarks.

4.2 Remark on the Choice of the Proton Current

The problems in choosing a satisfactory current to represent the proton have been stressed throughout the second chapter of this thesis. It is indeed satisfactory that the results we obtained in that chapter agree with those of Chernyak and Zhitnitsky despite the fact that the two analyses use different currents. However, in general analyses of two-point functions using different currents for the proton do not so agree. If we take a linear combination of two currents to represent the proton and vary the extent of their admixture some stability can be found, but in the region of the current suggested by Ioffe this is not the case (58).

This is surprising, as Ioffe has strongly argued that background effects should be strongly suppressed for his choice of current and and so we should expect the results to correspond to just the lowest lying resonance with the correct quantum numbers - the proton.

Consideration of the wavefunction structure of the proton may explain this instability and furthermore explain why Ioffe´s current gives reasonable results for three-point functions.

The difficulty comes from a consideration of the proton interpolating directly with a current:

$$\langle 0|\eta_\alpha|p\rangle = \text{const.}\, N_\alpha \sum_R C_R R$$

$$\eta_\alpha = (u^{a\,T} C\, \Gamma_\eta u^b)(\Gamma'_\eta d^c)_\alpha\, \varepsilon^{abc} \tag{4.2.1}$$

where N_α is the proton spinor and R runs over the three parts of the proton wavefunction (V, A and T). Using the notation of (2.7.5) for the general form of the wavefunction (2.3.1):

$$\langle 0|u^a_\alpha(z_1)\, u^b_\beta(z_2)\, d^c_\gamma(z_3)\,\varepsilon^{abc}|p\rangle_{\mu^2} = -\frac{1}{4}\sum_R (\Gamma_R C)_{\alpha\beta}\cdot (\Gamma'_R N)_\gamma R \tag{4.2.2}$$

Evidently the factors C_R of (4.2.1) are dependent on the trace:

$$\mathrm{tr}\left(\overline{C\Gamma_R}\, C\Gamma_\eta\right) \sim \mathrm{tr}\left(\Gamma_R \Gamma_\eta\right) \tag{4.2.3}$$

For the current due to Ioffe we have $\Gamma_\eta = \gamma_\mu$; using the Γ_R given in (2.3.1) we see that the trace (4.2.3.) is zero unless R = V. For other (mixtures of) currents this is not the case and a mixture of wavefunctions will contribute to (4.2.1).

It may thus be argued that in the study of baryonic two-point functions, where we use currents like that of Ioffe we only consider the interpolation of some part of the fields of the baryon with the current. As we move away from such a simple current other fields (e.g. A) will contribute and when we have a mixture which involves both V and A in sizeable proportions it may indeed be hoped that some stability may be found in changing the current slightly. By contrast in the vertex function considered above, however, all the fields contribute due to the three-point nature of the vertex. It is amusing to consider the resemblance of this to our hypothesis that it is the similar neglect of antisymmetric components which is responsible for the poor agreement between theory and experimental in the study of exclusive processes.

4.3 General Outlook

In this penultimate section we consider the following areas: the theoretical status of low-energy hadronic wavefunctions from various sum rules, the experimental situation (with the use of such wavefunctions), and the current situation with regard to lattice investigations of this area.

The consistency between the various sum rules is certainly impressive; for baryons there is the internal consistency between the three sets of sum rules for the V, A and T wavefunctions and both for mesons and baryons the vertex sum rules formalism strongly supports the results of the two-point function sum rules. This evidence for an appreciable antisymmetric component in the nuclear wavefunction is very strong, and similarly the sum rules strongly suggest that the momentum is shared between the quarks in a manner qualitatively different from the asymptotic form, the effect being so strong that the momentum is, most probably, distributed very unevenly between the quarks (for the lowest twist wavefunction the most likely configuration has one quark carrying eighty-five percent of the momentum).

That these predictions contain some new physics is best seen by comparing them with the two previous models generally used, the perturbation theory predictions and the non-relativistic quark model results. The results of perturbation theory are highly symmetric; for the nucleon we only retain the totally symmetric wavefunction, $120 x_1 x_2 x_3$, and there is no discrimination between the quarks; in the case of the pion we still retain the symmetry between the quarks, the asymptotic wavefunction being $\frac{3}{4}(1-\xi^2)$ where $x_1 = (1+\xi)/2$, $x_2 = (1-\xi)/2$. It seems intuitively reasonable to approach such symmetries assymptotically; however, as we approach the energy scales typical of resonance physics and non-perturbative effects start to come into play this highly symmetric state of affairs need no longer prevail.

The other end of the energy scale, in a sense, is the non-relativistic quark model, where the momentum is again evenly shared out. Here the proton wavefunction is of the form $\delta(x_1-\frac{1}{3})\,\delta(x_2-\frac{1}{3})$ and that of the pion is $\delta(\xi)$; the constituents all carry the same fraction of the momentum with unit probability.

Somewhere between these two limits lies the realistic situation: the momentum typical of exclusive processes is not such that the asymptotic theory may be expected to work (and, of course, the resonances, being confined objects, are immediately beyond perturbation theory) and the quark masses are so small that non-relativistic theory cannot be applicable. The results of the various sum rules which indicate that this intermediate position is so different from the extreme cases, also point to the importance of physics that we do not yet understand in the construction of resonances. The sum rules, with their semi-phenomenological approach, do not explain their predictions, and whilst the experimental data continue to agree with the sum-rule-based wavefunctions the mechanics behind these effects and a better understanding of them must be an important task for QCD.

The vast number of diagrams required in such calculations may in a more indirect manner test QCD. If we consider meson-nucleon scattering (figure 14) and then consider that to each of the topologies in figure 14 we must associate of the order of ten thousand diagrams

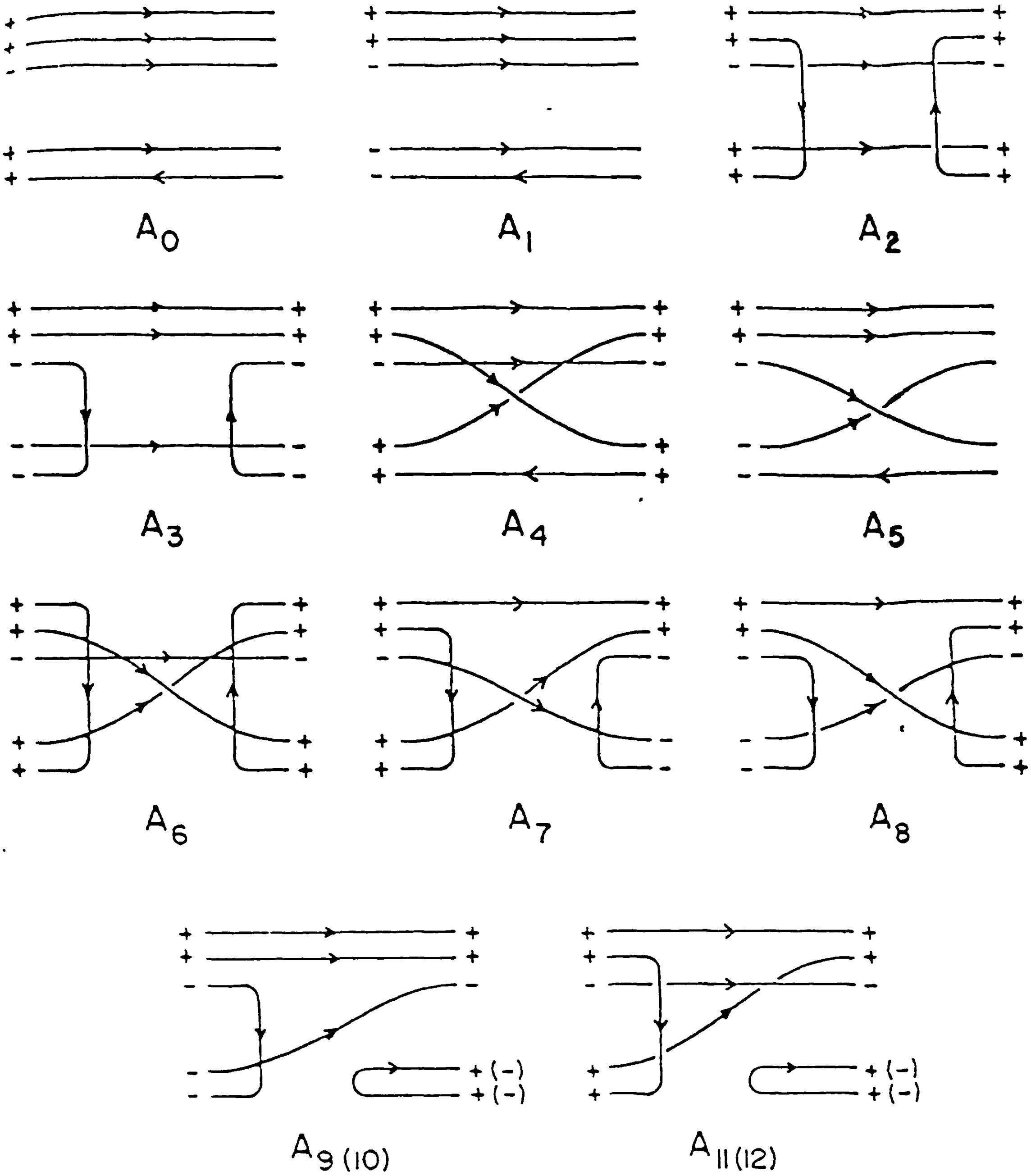


Figure 14: Meson-nucleon scattering quark amplitudes- the 13 fundamental topologies.

corresponding to the various ways of attaching the four gluon lines required to compute the scattering in Born approximation alone we may realize that the study of such processes can give us information upon the convergence properties of perturbation theory.

The theoretical difficulties in the evaluation of exclusive processes are not confined to the length of the calculation. After one has evaluated the underlying quark amplitudes (of, say, figure 14) as an algebraic function of the hadronic energy, scattering angle and the constituent momentum fractions, x_i, one must convolute these with the hadronic wavefunctions in terms of the quark and gluon constituents. The difficulties here are of two kinds; firstly there is the practical difficulty associated with the dimensionality of the inegrations (note that in figure 14 we have six independent variables to integrate over) and secondly there are often found to be singularities in the integrations. The treatment of the singularities is not understood, but current practice (17, 59) is to regulate them. It is fortunate that many processes and predictions are independent of this difficulty, but the correct treatment of these singularities is an open topic. For example, in the calculation of the power corrections to the $\pi\rho\gamma$ form factor (59) the integral $J = \int_0^1 dx/x$ occurs. Gorsky (59) chooses to regulate this integral such that J = 3, and expects uncertainties resulting from this to change his final answer by a factor of roughly two. Similarly Chernyak and Zhitnitsky (17) include logarithmically divergent integrals and expect this to lead to uncertainties of the order of 50% (They, for example, use $\frac{1}{2}\int_{-1}^{1} d\xi/(1-\xi) \simeq 1\cdot 3$. The origin of these singularities is believed to be the incorrect treatment of confinement in perturbation theory and more work is required here.

As mentioned above,many predictions should be independent of these difficulties. Because there are many more processes than there are wavefunctions one can attempt to link these processes in a manner independent of the details of the calculation of the amplitude, thus avoiding, one hopes, the difficulty raised above in the exact evaluation of quark amplitudes. For example Farrar has predicted (60) the following relationship:

$$\mathrm{Amp}\left[\gamma\gamma \to p\bar{\Delta}^{-}\right] = -\sqrt{2}\,\mathrm{Amp}\left[\gamma\gamma \to \left(p\bar{p} + n\bar{n} - 2\Lambda\bar{\Lambda} - \Sigma^{-}\bar{\Sigma}^{-}\right)\right] \quad (4.3.1)$$

where this is true for each helicity combination separately. The validity of (4.3.1) depends on being able to neglect power corrections and on the use of asymptotic SU(6) symmetry for the flavour wavefunctions. Similar work has been done by Lipkin (61), who produces relationships between the spin flip transitions concerning the ratio $\sigma(\pi^{-}p \to \rho^{-}p)/\sigma(\pi^{+}p \to \rho^{+}p)$. This he predicts to be unity if the mechanism for the reactions is gluon exchange, sixteen if it is quark-antiquark annihilation and one-sixteenth if it is quark exchange. The work of Lipkin rests upon slightly more general assumptions than SU(6) symmetry, but still requires that the wavefunction can be taken to be totally symmetric among the valence quarks and that higher twist effects are also negligible. This, of course, also goes against the results from the various sum rules for the proton and it would therefore be interesting to see if these predictions are not experimentally fulfilled; it is also of interest to see whether modified relationships of the (60, 61) can be derived with more realistic wavefunctions, incorporating some antisymmetric component.

Both Farrar and Lipkin assumed, as is usual, that higher twist effects can be neglected. The application of power corrections to exclusive processes has been confined to simple processes so far: $\pi \rightarrow \gamma^{*}\gamma^{*}$ ((54) and chapter 2 of this thesis), the π-meson electromagnetic form factor (62) and the $\pi\rho\gamma$ form factor (59)). The major problems in these considerations lie in (the difficulty in) normalizing the wavefunction and giving the non-asymptotic wavefunction form. With regard to the first difficulty (17, 45) we can use suitable sum rules to estimate the overall normalization (see section 3.4 of this thesis). As for the non-asymptotic forms of the higher-twist wavefunctions, so far only that of the $\bar{q}\,\overleftarrow{\partial}_{\perp}^{2}\,\gamma_{\mu}\gamma_{5}\,q$ component of the pion has been evaluated ((63), chapter 3), and, as with the lowest-twist components, a significant deviation from the asymptotic form was found. The first experimental information upon such form factor processes has recently come from the decay, $\psi \rightarrow \pi^0 \omega$ (64). The theoretical contributions give an estimate for

$$Br \quad \frac{\psi \rightarrow \pi^0 \omega}{\psi \rightarrow e^+ e^-} \tag{4.3.2}$$

which is quite close to the experimental value (64). It is thus to be hoped that, in general, the higher-twist, power corrections do not play an essential role in the study of form factors at intermediate energies . However, this may not be the case for all processes and investigation of such effects is required. The determination of the non-asymptotic wavefunctions of the various higher twist components is also desirable; there are many such components (two-particle and four-particle power corrections have already been considered by Gorsky (54, 59, 62)) and they may be found, in principle, via the sum rule methods.

Comparison of purely perturbative QCD with experiment seems a long way off (consideration of equation (3.6.9) to see how the $\langle \xi^2 \rangle$ moment of the lowest-twist pion wavefunction runs, together with the results from the sum rules, shows that the values of $\langle \xi^2 \rangle$ such that the wavefunction may be convex are still a long way off), so what is the experimental situation with respect to the mixture of perturbative QCD and the non-perturbative wavefunctions?

Many results have been quoted above and many ideas for testing these theories put forward. An important recent result is that of Farrar, Maina and Neri (65) for the process $\gamma\gamma \longrightarrow p\bar{p}$, which they have calculated using the wavefunction proposed by Chernyak and Zhitnitsky (39). The importance of this is that apart from the nucleon form factors (calculated by Chernyak and Zhitnitsky in (39)) it was not totally obvious that processes would, in general, be sharply dependent on the shape of the wavefunction. The prediction is greatly improved over that of the symmetric wavefunction, but still somewhat smaller than the data (factors of two to ten). However, the experimental data are clearly poor with large errors and so it is difficult to speculate as to the source of any discrepancies. (For instance until the data are better we cannot rule out the effects of other resonances).

Experimental data are generally in qualitative agreement with the results of the sum rules. Reasonable results have been obtained for heavy mesonic decays, mesonic and baryonic form factors,and many predictions exist which remain to be tested experimentally. Another recent experimental result is somewhat more puzzling (66). A study of πN scattering was made and the $\mu^+ \mu^-$ pairs produced observed. The mechanism for these pairs was assumed to be quark-antiquark annihilation, with the virtual photon so produced decaying into the muons. From this an exploration was made of the distribution of quarks in the pion. The result of their data analysis was that there was a finite chance of finding a quark carrying all the pion's longitudinal momentum. This possibility has not been taken into account in any of the above work, where we have forced $\varphi(\xi) \to 0$ as $\xi \to \pm 1$. (Similarly in the case of the nucleon, the model wavefunction is such that at $x_i \to 1$, it approaches the asymptotic form). An argument for such a constraint to be applied to models of hadronic wavefunctions is due to Chernyak and Zhitnitsky (39). It is based upon observation of the sum rules for moments with large n values (or large $\sum n$ in the baryonic case); in such sum rules the non-perturbative corrections become more important, with respect to the perturbative term. For example, in the sum rules for the pion wavefunction (26), the ratio of the coefficients of the unit operator to the G^2 operator is (modulo a constant factor) given by (n +3)/M , and this implies that the corresponding duality interval will be very big for large n. Of course, the low-lying resonances that the sum rules are trying to investigate will not fill this large duality interval but only a smaller one, S^{∞} , which will be independent of n for large n. Duality thus implies (26), for large n:

$$\int_0^{S^{\infty}} ds\, \mathrm{Im}\, I^{(n)}_{res}(s) = \int_0^{S^{\infty}} ds\, \mathrm{Im}\, I^{(n)}_{pert.\,th.}(s) \qquad (4.3.3)$$

where $I^{(n)}_{res}$ is the lowest resonance contribution to the relevant correlator and $I^{(n)}_{pert.\,th.}$ is the contribution from perturbation theory. From (4.3.3) we evidently get that the moments of the lowest resonance:

$$f_{res} \langle \xi^n \rangle \sim \int_0^{S^\infty} ds \; \mathrm{Im}\, I_{res}^{(n)}(s) \tag{4.3.4}$$

have the same n-dependence as the asymptotic form (which is also determined by the perturbative contribution, Appendix C). The higher moments evidently correspond to the $\xi \to \pm 1$ behaviour, and so Chernyak and Zhitnitsky conclude that the true hadronic wavefunctions have the same behaviour as their asymptotic form in such a limit, .

Evidently the aymptotic wavefunction of the lowest twist pion component goes to zero as $\xi \longrightarrow \pm 1$ and so the recent data (66) are seemingly at odds with the argument of Chernyak and Zhitnitsky. If the conclusion of (66) is correct then it will obviously be more difficult to construct model wavefunctions and to make reliable predictions of exclusive processes, in that we will have lost a valuable constraint on the form of any such model. Of course, the data may change, but it is perhaps worth commenting on their parametrization of the pion structure function:

$$F_\pi(x_\pi) \propto x_\pi^\alpha \left[(1 - x_\pi)^\beta + \gamma \right] \tag{4.3.5}$$

It is possible that a more sophisticated parametrization involving an attempt to incorporate the physical picture of the pion suggested by the sum rules could give different results. Indeed it may be hoped that the conclusion of (66) that there is a non-zero possibility of finding a quark carrying all the momentum is merely due to the pion momentum being in reality, on average, split unevenly between the quark and the antiquark (although never entirely apportioned to one constituent).

We should also consider the application of lattice gauge theories to the areas discussed in this thesis. This is, however, an area which has not yet been properly explored. A proposal (67) to calculate the matrix elements of the form $\overline{\psi}\Gamma\overleftrightarrow{D}\overleftrightarrow{D}\cdots\overleftrightarrow{D}\psi$ has been made by Kronfeld and Photiadis, and Wilcox and Woloshyn have tried to model the meson electromagnetic form factor on the lattice with SU(2) colour.

Kronfeld and Photiadis considered currents relevant to mesonic currents of twist two and say that this can be extended to baryons, gluonia and higher twist operators in a straightforward way. They consider operators:

$$\mathcal{O}^{(n)}_{\mu_0\cdots\mu_n} = i^n\,\overline{\psi}\,\Gamma_{\mu_0}\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n}\psi - \text{traces} \qquad (4.3.6)$$

and proceed to calculate the effects of mixing on these operators. The operators of lower dimension in mass produce power corrections and associated divergences, which necessitates the subtraction of counter terms. The analysis of the mixing of the counterterms was carried out using the charge conjugation, parity and hypercubic symmetries of the theory. As n (4.3.6) increases, the mixing of the operators becomes rapidly more complex. The renormalised operators are, schematically, given by:

$$O_R^{(n)} = O_u^{(n)} - C_F \frac{\alpha_s}{4\pi} \sum_i f_i(\alpha_s)\, O_i \qquad (4.3.7)$$

where R, U stand for renormalised and unrenormalised respectively. The O_i are the mixing operators, with the same quantum numbers as O_u and the f_i have been calculated to one-loop order in (67).

The feasibility of carrying out such an analysis via standard Monte-Carlo techniques is, to some extent, dependent on the hadronic model one favours. Considering the example of Kronfeld and Photiadis for a correctly renormalised $O^{(2)}$ and only retaining the most important counter term, $\bar{\psi}\Gamma\psi$ one has the result:

$$\frac{f_i O_i}{\langle O^{(2)} \rangle_R} = 2.5\, C_F \frac{\alpha_s}{4\pi} \frac{a^{-2}\langle \bar{\psi}\Gamma\psi \rangle}{\langle \bar{\psi}\Gamma \overleftrightarrow{D} \overleftrightarrow{D} \psi \rangle} \simeq 2.5 \frac{C_F \alpha_s \lambda^2}{4\pi \quad a^2} \qquad (4.3.8).$$

where a is the lattice spacing and l is a measure of the valence quark separation. The ratio of (4.3.8) must be small for the signal of the renormalised operator to be potentially extractable. If we consider a bag picture with $l \sim 1$ fm and a about $(1.0 \text{ GeV})^{-1}$ then the ratio is about 2 or 3; for a model with a core surrounded by a cloud of pions with l, say, 5 times smaller much smaller "a" values can obviously be used. Unless this latter core picture is correct it seems unlikely that values of n greater than or equal to four can be considered, because the suppression of the power low divergences would require ridiculously large lattice spacings. Another source of errors, the neglected higher order corrections, is also dependent on the l /a ratio. It requires an initial study to investigate the correct value of l, to discover whether the analysis can be extended beyond $O^{(2)}$ or whether higher dimensional operators are beyond the reach of the analysis described above.

Should the latter be the case it is, in principle, possible to improve the situation by using a different lattice action. Improving the action to extend the scaling region to larger values of a would allow a decrease in the ratio l /a, which would thus diminish the ratio (4.3.8). Similarly an action with added non-renormalizable interactions should improve the short distance behaviour and diminish the f_i coefficients.

If all these difficulties can be surmounted it is evidently of the greatest interest to have lattice predictions for such matrix elements (and for those from baryonic and higher-twist operators.) Comparison of the new results with the sum rule predictions should give clear QCD predictions that could be compared with the data from experiment. The results (69) of Gottlieb and Kronfeld unfortunately disagree with the sum rule predictions for the pion and -meson wavefunction moments, predicting (for example) $\langle \xi_\pi^2 \rangle = 1{\cdot}68$. Clearly this requires a wavefunction with negative values somewhere. We are dubious, however, of their result and expect the true errors to be extremely large. Further work in this area would be very useful.

4.4 Evaluation of Vacuum Condensates

The non-perturbative input to sum rules for vacuum to vacuum functions consists primarily of vacuum condensates. As remarked earlier, uncertainties in the values of these condensates (21) lead to uncertainties in the conclusions of such sum rules. We would like to conclude this thesis by describing an attempt at evaluating the condensates directly, in a manner rather different to the usual technique of choosing them to optimize an ordinary sum rule (20).

Essentially, the approach of Launer (21) is to apply an operator to both sides of a sum rule which projects out condensates of a given dimensionality. The use of the simple "pole plus continuum" model for the phenomenological side of the two-point function with the I = 1 electromagnetic current already led to a value for the G^2 condensate nearly twice the standard (20) value. However, it is important to take into account both radiative corrections to the gluon condensate and the realistic nature of the hadronic spectrum in the channel concerned.

The calculation of the radiative correction (70) necessitates the use of computer algebra, since there are so many terms contributing. Quark and gluon propagators in an external field (71) are used and these must be expanded in terms of gluonic fields. After vacuum averaging one is left with a very large number of terms. To evaluate these integrals separately would be extremely tedious, and instead we apply the operators $q \cdot \partial/\partial q$ and $\partial/\partial q_\mu \cdot \partial/\partial q_\mu$ to a general integral to obtain relations between the integrals. With the help of these relations any integral required can be related to some mixture of three types that we know. Using the pattern matching facility of the computer algebra system this gives a final expression for the radiative correction:

$$C = \frac{T}{6} \frac{\alpha_s}{\pi} \left\{ 1 + \frac{\alpha_s}{\pi} \left(\frac{1}{2} C_A - \frac{1}{4} C_F \right) + \cdots \right\} \qquad (4.4.1)$$

where C is the coefficient of $\langle G^2 \rangle / Q^4$ and T, C_A and C_F are colour weights (1/2, 3 and 4/3 respectively for SU(3)). Both our calculations and that of Loladze et al. (70) agree, giving us confidence in the result.

The result can be substituted into the set of sum rules, but we also need to substitute the experimental data into the programme to give the most realistic representation of the hadronic spectrum. This work is currently in progress (21) and requires the choice of a suitable parametrization of the data. Such fits can readily be performed with the MINUIT program of the CERN library. It is hoped that the gluonic condensate $\langle \alpha_s G^2 \rangle$ will soon be accurately calculated by this method. We also hope to check the accuracy of the factorization hypothesis as applied to four-fermion operators.

APPENDIX A: The Fock-Schwinger Gauge

Although any gauge condition must give the same answer for a correlation function of colourless sources, the calculation of coefficients in the O.P.E. is most economically performed using a formalism which retains the gauge invariance explicitly. Such a technique is the Fock-Schwinger gauge (22) where the potential $A_\mu^a(x)$ is expressible directly in terms of the gluon field strength tensor.

This gauge is as follows:

$$(x - x_0)_\mu A_\mu^a(x) = 0 \tag{A1}$$

where A_μ^a is the four potential and X_0 is an arbitrary point in the space; it plays the role of a gauge parameter. There follows for (A.1) the symmetry requirement that final physical answers must be independent of X_0 . In what follows, however, we shall merely put $X_0 = 0$.

From the identity:

$$A_\mu(y) = \frac{\partial}{\partial y_\mu}\left(A_\rho(y)\, y_\rho\right) - y_\rho \frac{\partial A_\rho(y)}{\partial y_\mu} \tag{A.2}$$

by applying (A.1), with $X_0 = 0$ we get:

$$A_\mu(y) = y_\rho G_{\rho\mu}(y) - y_\rho \frac{\partial A_\mu(y)}{\partial y_\rho} \tag{A.3}$$

Substituting $y = \alpha X$ we see that the left hand side of (A.3) is a full derivative and so integrating over α from 0 to 1 we get:

$$A^a_\mu(x) = \int_0^1 \alpha \, d\alpha \, G^a_{\rho\mu}(\alpha x) x_\rho \tag{A.4}$$

A more convenient means for expressing $A^a_\mu(x)$ can be obtained by induction and the use of the equation:

$$x_{\alpha_1} \cdots x_{\alpha_N} \left(\partial_{\alpha_1} \cdots \partial_{\alpha_N}\right) G_{\rho\mu}\Big|_{x=0} = x_{\alpha_1} \cdots x_{\alpha_N} \left(\mathcal{D}_{\alpha_1} \cdots \mathcal{D}_{\alpha_N}\right) G_{\rho\mu}\Big|_{x=0} \tag{A.5}$$

Expanding (A.4) using (A.5) and integrating over α gives:

$$A_\mu(x) = \frac{1}{2.0!} x_\rho G_{\rho\mu}(0) + \frac{1}{3.1!} x_\alpha x_\rho \left(\mathcal{D}_\alpha G_{\rho\mu}(0)\right)$$

$$+ \frac{1}{4.2!} x_\alpha x_\beta x_\rho \left(\mathcal{D}_\alpha \mathcal{D}_\beta \, G_{\rho\mu}(0)\right) + \cdots \tag{A.6}$$

Analogously we obtain:

$$\psi(x) = \psi(0) + x_\alpha D_\alpha \psi(0) + \frac{1}{2} x_\alpha x_\beta D_\alpha D_\beta \psi(0) + \dots \quad (A.7)$$

For small $A_\mu(x)$ fields we can express the quark propagator as the standard series:

$$i S(x,y) = i S^{(0)}(x-y) + g \int d^4z \, i S^{(0)}(x-z) \, i \not{A}(z) i \cdot S^{(0)}(z-y) + g^2 \int d^4z \, d^4z' \, i S^{(0)}(x-z') \cdot i \not{A}(z') \, i S^{(0)}(z'-z) \, i \not{A}(z) \, i S^{(0)}(z-y) + \dots \quad (A.8)$$

where $S^{(0)}(x-y)$ is the free quark propagator:

$$S^{(0)}(x-y) = \frac{1}{(2\pi)^2} \frac{\not{x} - \not{y}}{(x-y)^4} \quad (A.9)$$

Limiting ourselve, for simplicity, to consideration of the operator $G^a_{\mu\nu} G^a_{\mu\nu}$ we can merely keep the first term of (A.6). Substituting this and (A.9) into (A.8) we can re-express S(x, y) and after performing the integrations obtain:

$$S(x,y) = \frac{\not{r}}{(2\pi)^2 r^4} - \frac{1}{8\pi^2} \frac{r_\alpha}{r^2} \tilde{G}_{\alpha\varphi}(0) \gamma_\varphi \gamma_5 +$$

$$+\left\{\frac{i}{4\pi^2}\frac{\not{r}}{r^4}y_\rho x_\mu G_{\rho\mu}(0) - \frac{1}{192\pi^2}\frac{\not{r}}{r^4}\left(x^2y^2-(x\cdot y)^2\right)\cdot\right.$$

$$\left.\cdot\, G_{\varphi\chi}(0)\, G_{\varphi\chi}(0)\right\} + \cdots \qquad \text{(A.10)}$$

where

$$r = x - y,\quad G_{\alpha\beta} = \frac{g}{2}\lambda^a G^a_{\alpha\beta},\quad \tilde{G}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}G_{\mu\nu}$$

Note that for y = 0 the term in the braces in (A.10) vanishes; however, for currents involving derivatives or for three-point functions such a simplification may not be used. An analogous expression to (A.10) for a quark propagator may be readily obtained, and full expressions for quark and gluonic propagators up to dimension 6 operators may be found in (22). The use of such propagators in calculations enormously simplifies the computation of coefficients. For example we give the calculation of the G^2 coefficient in the vector channel of massless quarks, $j_\mu = \bar{q}\gamma_\mu q$.

$$\Pi_{\mu\nu} = i\int e^{iq\cdot x}d^4x\,\langle 0|T\left(j_\mu(x)\,j_\nu(0)\right)|0\rangle$$

$$= i\int e^{iq\cdot x}d^4x\;\mathrm{tr}\left\{\gamma_\mu S(x,0)\gamma_\nu S(0,x)\right\} \qquad \text{(A.11)}$$

The G^2 term in the trace, using (A.10) with y = 0, is thus

$$-\frac{1}{8\pi^2}\, tr\left(\gamma_\mu \gamma_\beta \gamma_5 \gamma_\nu \gamma_\delta \gamma_5\right) \tilde{G}_{\alpha\beta}\, \tilde{G}_{\gamma\delta}\, \frac{x_\alpha x_\gamma}{x^4}$$

$$= -\frac{\langle g^2 G^2 \rangle}{384\,\pi^4}\, \frac{2x_\mu x_\nu + x^2 g_{\mu\nu}}{x^4} \tag{A.12}$$

where we have used the vacuum average:

$$\langle G^a_{\mu\nu}(0)\; G^b_{\alpha\beta}(0)\rangle = \frac{1}{96}\, \delta^{ab}\left(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}\right)\langle G^2\rangle \tag{A.13}$$

Noting that (A.12) is transverse, we can contract indices and by the use of the equation (22):

$$\int \frac{d^4x\; e^{ip\cdot x}}{(x^2)^n} = \frac{i\,(-1)^n\, 2^{4-2n}\, \pi^2}{\Gamma(n-1)\,\Gamma(n)} \cdot (p^2)^{n-2}\, \ln(-p^2) \tag{A.14}$$

proceed to momentum space. The result is

$$-\frac{1}{16\pi^2}\,\frac{1}{q^2}\langle g^2 G^2\rangle \tag{A.15}$$

which is the result first obtained in (20), but derived now with much less effort.

Appendix B: The Borel Transform

The Borel, or Laplace, transformation converts the weighting of the sum rules into a form which is much more sensitive to the lowest resonance parameters and also less sensitive to higher power corrections. This appendix contains some useful definitions, relations and comments.

For a function f(x) the Borel transform gives:

$$\tilde{f}(\lambda) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{\lambda/x} f(x)\, x\, d\left(\frac{1}{x}\right) \tag{B.1}$$

where the integration contour runs to the right of all singularities. The inverse transformation is thus:

$$f(x) = \int_0^{\infty} \tilde{f}(\lambda)\, e^{-\lambda/x}\, \frac{d\lambda}{x} \tag{B.2}$$

The effects of the transform can be seen by considering the series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_k x^k + \dots \tag{B.3}$$

The transform yields:

$$\tilde{f}(\lambda) = a_0 + \frac{a_1\lambda}{1!} + \frac{a_2\lambda^2}{2!} + \cdots \frac{a_k\lambda^k}{k!} + \cdots \tag{B.4}$$

the factorial suppression of higher power corrections mentioned in section 2.1.

A differential operator equivalent to the Borel transform was introduced in (20):

$$\tilde{\Pi}(M^2) = \lim_{\substack{n, Q^2 \to \infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!} (Q^2)^n \left(-\frac{d}{dQ^2}\right)^n \Pi(Q^2) \tag{B.5}$$

For brevity we write:

$$\tilde{\Pi}(M^2) = \hat{L}_M \Pi(Q^2) \tag{B.6}$$

That $\hat{L}_M$ provides the factorial suppression is easy to see; however, we must show the generality of the equivalence of $\hat{L}_M$ to the Borel transform, since, as noted in (20), the expansion in Q^{-2} may break down at some, high, power. If we assume only that the polarization operator satisfies the standard dispersion relations:

$$\Pi(Q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\,\Pi(s)\,ds}{s+Q^2} + \text{possible subtractions} \tag{B.7}$$

then we can show that $\hat{L}_M$ is indeed equivalent to the Borel transform (20). The effect of $\hat{L}_M$ on the dispersion relation is easily seen to be:

$$\hat{L}_M \Pi(Q^2) = \frac{1}{\pi M^2}\int \mathrm{Im}\,\Pi(s)\, e^{-s/M^2} ds = \tilde{\Pi}(M^2) \tag{B.8}$$

(Note that for k positive; $\hat{L}_M (Q^2)^k = 0$).
The inverse transform, (B.2), of (B.8) can be shown to give the original polarization operator up to the, undefined, constant term, i.e. we obtain:

$$\int_0^\infty \tilde{\Pi}(M^2)\, e^{-Q^2/M^2}\, Q^2\, d\left(\frac{1}{M^2}\right) = Q^2\left(-\frac{d}{dQ^2}\right)\Pi(Q^2) \tag{B.9}$$

thus showing the coincidence of $\hat{L}_M$ and the Borel transform.

The above is usually sufficient, but one sometimes finds functions of the form $\left(\frac{1}{Q^2}\right)^k \ln\left(Q^2/\mu^2\right)^{-\gamma}$, where γ is, generally, non-integer. Such terms appear from the anomalous dimensions of operators and can be shown to transform as (20, 22):

$$\hat{L}_M\left[\left(\frac{1}{Q^2}\right)^k\left(\ln\frac{Q^2}{\mu^2}\right)^{-\gamma}\right]=\frac{1}{\Gamma(k)}\left(\frac{1}{M^2}\right)^k\left(\ln\frac{M^2}{\mu^2}\right)^{-\gamma}\cdot$$

$$\cdot\left[1+O\left(\ln\frac{M^2}{\mu^2}\right)^{-1}+\ldots\right] \tag{B.10}$$

The above is accurate enough generally, but exact transformations can be found in the literature (43). There also exists a non-relativistic limit of the operator (44). Further Borel transforms (20) have been shown to be counter-productive, as they introduce oscillating weight functions which are difficult to control.

Appendix C: The Form of Asymptotic Wave Functions

In this appendix we describe the simple method for finding the asymptotic form of various wave functions (45) and give a list of them for various channels. It should be emphasized that these asymptotic forms are purely perturbative objects determined solely by the quantum numbers of the channel, so they can be found merely by the computation of simple Feynman diagrams. Firstly we shall consider the lowest twist pion wavefunction in some detail as an example, then we shall give results for a variety of mesonic, gluonic and baryonic cases.

The leading twist pion wave function $\varphi_\pi(\xi,\mu)$ is defined by the bilocal matrix element:

$$\langle 0|\bar{d}(z)\not{z}\gamma_5 \exp\left(ig\int_{-z}^{z} d\sigma_\nu B_\nu(\sigma)\right)u(-z)|\pi^+(q)\rangle_{\mu^2}$$

$$= if_\pi\, z\cdot q \int_{-1}^{1} d\xi\, e^{i\xi(z\cdot q)}\varphi_\pi(\xi)$$

$$+ \cdots \tag{C.1}$$

where $z^2=0$ and μ is the renormalization point. The system of multiplicativily renormalised operators $\{O_n\}$ where $O_n(0)=\bar{d}(0)\not{z}\gamma_5 P_n(iz\cdot\overleftrightarrow{D}/z\cdot q)u(0)$, $P_n(\xi)$ are unknown polynomials and $\overleftrightarrow{D}=\overrightarrow{D}-\overleftarrow{D}$, is now used in the decomposition of the bilocal operator into the local forms O_n . The decomposition is as follows:

$$\varphi_\pi(\xi,\mu) = \varphi_{as}(\xi)\sum_{n=0}^{\infty} f_\pi^n(\mu)P_n(\xi) \tag{C.2}$$

and has the properties :

$$\int_{-1}^{1} d\xi\, \varphi_{as}(\xi)P_n(\xi)P_m(\xi) = \delta_{nm}. \tag{C.3}$$

$$f_\pi^n(\mu) = \int_{-1}^{1} d\xi\, \varphi_\pi(\xi,\mu)P_n(\xi)$$

The constants $f_\pi^n(\mu)$ depend on μ via the equation:

$$f_\pi^n(\mu_2) = f_\pi^n(\mu_1)\exp\left\{-\int_{\alpha_s(\mu_1)}^{\alpha_s(\mu_2)} \frac{d\alpha}{\beta(\alpha)}\gamma_n(\alpha)\right\} \quad (C.4)$$

where $\gamma_n(\alpha)$ are the corresponding anomalous dimensions, $\gamma_{n+1} > \gamma_n$, and $\beta(\alpha)$ is the Gell-Mann-Low function $\{P_n(\xi)\}$ and $\varphi_{as}(\xi)$ are, as previously stated, determined by perturbation theory, while the $f_\pi^n(\mu)$ are given by non-perturbative physics and are truly hadronic properties.

We now want to find the form of $\varphi_{as}(\xi)$. To do this we make use of the conformal invariance of QCD at short distances. If we consider two local operators O_2 and which correspond to different representations of the conformal group, the the two-point function, in the Born approximation, is:

$$T_{12} = i\int d^4x\, e^{iq\cdot x}\langle 0|T(O_1(x)O_2(0))|0\rangle = 0 \quad (C.5)$$

due to the conformal invariance. (Note that this is true in the $q^2 \longrightarrow \infty$ limit even if quark masses are non-zero as the conformal symmetry breaking effects die off in the limit). In general (46), (C.5) remains true in the leading logarithm approximation. We now use (C.5) to calculate $\varphi_{as}(\xi)$ and $P_n(\xi)$. Consider the correlator:

$$T_n(q,z) = i\int d^4x\, e^{iq\cdot x} \langle 0| T\left(\bar{d}(x)\not{z}\gamma_5(iz\cdot\overleftrightarrow{D})^n \cdot\right. \tag{C.6}$$

$$\left.\cdot\, u(x), \bar{u}(0)\not{z}\gamma_5\, d(0)\right)|0\rangle$$

$$= (z\cdot q)^{n+2}\, T_n(q^2)$$

where $z^2 = 0$.

Evidently $\bar{d}(x)\not{z}\gamma_5(iz\cdot\overleftrightarrow{D})^n u(x)$ corresponds to a mixture of $O_m(x)$ for $m \leq n$, and $\bar{u}(0)\not{z}\gamma_5 d(0)$ is $O_0(0)$. Then equation (C.5) implies that the Born approximation is purely proportional to the $O_0(x)\,O_0(0)$ term. Thus the correlators must depend upon $\varphi_{as}(\xi)$ and a weighting factor. In this case calculating the correlator gives:

$$T_n(q^2) = -\frac{1}{4\pi^2}\ln(-q^2)\int_{-1}^{1} d\xi\, \frac{3}{4}(1-\xi^2)\,\xi^n \tag{C.7}$$

Thus we see firstly that $\varphi_{as}(\xi) = \frac{3}{4}(1-\xi^2)$, where the factor of 3/4 normalizes the wave function and secondly, from the requirement of orthogonality of the polynomials with the measure $\varphi_{as}(\xi)$, we see that the Gegenbauer polynomials $C_n^{3/2}(\xi)$ are the system of polynomials that give the multiplicatively renormalised operators:

$$\{O_n\} = \{\bar{d}\, \not{z}\, \gamma_5\, C_n^{3/2}\left(\frac{z\cdot\overleftrightarrow{D}}{z\cdot\partial}\right) u\} \tag{C.8}$$

Below we give the results for various currents of the asymptotic wavefunction and the associated polynomial system:

$$\begin{array}{ll}
\bar{\psi}(1\pm\gamma_5)\psi & \varphi_{as}\sim 1 \longrightarrow C_n^{1/2}(\xi) \\
\left.\begin{array}{l}\bar{\psi}\gamma_\mu(1\pm\gamma_5)\psi \\ \bar{\psi}\,\sigma_{\mu\nu}\psi\end{array}\right\} & \varphi_{as}\sim(1-\xi^2)\longrightarrow C_n^{3/2}(\xi) \\
G_{\mu\nu}\,G_{\alpha\beta} & \varphi_{as}\sim(1-\xi^2)^2\longrightarrow C_n^{5/2}(\xi) \\
\bar{\psi}\gamma_\mu(D_\perp^2)^k\psi & \varphi_{as}\sim(1-\xi^2)^{2k+1}\longrightarrow C_n^{2k+3/2}(\xi) \\
\bar{\psi}(D_\perp^2)^k\psi & \varphi_{as}\sim(1-\xi^2)^{2k}\longrightarrow C_n^{2k+1/2}(\xi)
\end{array} \tag{C.9}$$

This approach can be, trivially, extended to the calculation of three-particle operators for baryons and higher twist mesonic operators. However, although $\varphi_{as}(x_1 x_2 x_3)$ can be simply determined (where x_i are the momentum fractions carried by the constituents) the orthogonality conditions are insufficient now to uniquely fix the polynomials. The relevant three-particle Born diagrams yield:

$$\left.\begin{array}{l}\psi\, C\gamma_\mu(1\pm\gamma_5)\psi\psi \\ \psi\, C\sigma_{\mu\nu}\psi\psi\end{array}\right\} \varphi_{as}\sim x_1x_2x_3$$

$$\bar{\Psi}\gamma_\mu(1\pm\gamma_5)\psi G_{\alpha\beta} \qquad \varphi_{as}\sim x_1x_2x_3^2$$

$$G_{\mu\nu}G_{\lambda\sigma}G_{\alpha\beta} \qquad \varphi_{as}\sim(x_1x_2x_3)^2 \tag{C.10}$$

The above results include every case used in this thesis.

Appendix D: Light Cone Dispersion Relations

In this appendix we briefly sketch the proof of light-cone dispersion relations. To render the proof more transparent we neglect all indices (Lorentz and otherwise) and consider two scalar local operators $J_1(x)$ and $J_2(x)$ such that the matrix element:

$$\langle 0|J_1\left(\tfrac{z}{2}\right)J_2\left(-\tfrac{z}{2}\right)|p\rangle \tag{D.1}$$

is no more singular at $z^2\to 0$ than $1/z^2$. We now consider the retarded amplitude R(q):

$$R(q)=i\int d^4z\, e^{iq\cdot z}\langle 0|\,\theta(z^0)\left[J_1\left(\tfrac{z}{2}\right)J_2\left(-\tfrac{z}{2}\right)\right]|p\rangle \tag{D.2}$$

in which the local commutativity of J_1 and J_2 ensures that we can replace $\theta(z^0)$ by $\theta(n.z)$, where n is a positive light-like vector, $n^2 = 0$, $n^0 > 0$. Standard dispersion theory then implies that we can write, for all real k:

$$R(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dE}{E - i\varepsilon} V(k + En) \tag{D.3}$$

where $V = \frac{1}{2}(V_+ - V_-)$

$$V_+(q) = \int d^4z \, e^{iq.z} \langle 0| J_1\left(\tfrac{z}{2}\right) J_2\left(-\tfrac{z}{2}\right)|p\rangle$$
$$V_-(q) = \int d^4z \, e^{iq.z} \langle 0| J_2\left(-\tfrac{z}{2}\right) J_1\left(\tfrac{z}{2}\right)|p\rangle \tag{D.4}$$

If the dispersion integral (D.3) is no more divergent than $1/z^2$ the final dispersion relation is unsubtracted.

We can now use two scalar parameters to give the q dependence of $V_\pm$. and R:

$$q_1^2 = \left(\tfrac{1}{2}p + q\right)^2 , \quad q_2^2 = \left(\tfrac{1}{2}p - q\right)^2 \tag{D.5}$$

Now using, q = k + En, with $n^2 = 0$, we can express q_1^2, and q_2^2 in terms of new variables (using $p^2 = m^2$):

$$q_1^2 = (1+\omega)s + \lambda + m^2/4$$

$$q_2^2 = (1-\omega)s - \lambda + m^2/4 \qquad \text{(D.6)}$$

where:

$$q^2 = s = 2(k.n)E + k^2$$

$$\omega = \frac{1}{2}\frac{p.n}{k.n} \quad , \quad \lambda = k.p - \omega k^2 \qquad \text{(D.7)}$$

From this we see that if we vary E (but hold k and n fixed) q_1^2 and q_2^2 vary along a straight line parametrised by (D.6). (D.3) can thus be rewritten:

$$R(q^2;\omega,\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ds\,(\mathrm{sign}\,\omega)}{s-q^2-(\mathrm{sign}\,\omega)i\varepsilon}V(s;\omega,\lambda) \qquad \text{(D.8)}$$

To keep the vector q real, one requires that:

$$\omega\lambda \geqslant \frac{m^2}{4} \qquad \text{(D.9)}$$

which follows from requiring $(p.q)^2 \geqslant p^2q^2$. This is necessary because the amplitudes $V_\pm(q)$ are generally not defined for complex q, and (D.9) requires the straight line to avoid regions where p and/or q are complex.

We are also interested in choosing $|\omega| < 1$, so as to expand in powers of ω below. This case has the property that there exists $S_0(\omega, \lambda)$ such that:

$$V_{\pm}(s, \omega, \lambda) = 0 \quad , \quad s < S_0 \tag{D.10}$$

Also, since $V_{\pm}(q)$ vanish if $\pm q + \frac{p}{2}$ is outside of the forward light cone and, in particular, vanish if:

$$\pm q \cdot n + \tfrac{1}{2} p \cdot n < 0 \quad , \; n^2 = 0 , \; n^0 > 0 \tag{D.11}$$

then we have $V_{\pm}(q)$ vanishing if:

$$\pm \frac{1}{\omega} + 1 < 0 \tag{D.12}$$

which implies:

$$\theta(\mp \omega) V_{\pm}(s; \omega, \lambda) = 0 \tag{D.13}$$

and consequently we have:

$$\begin{aligned}(\text{sign}\,\omega)\, V(s; \omega, \lambda) &= \tfrac{1}{2} \{ V_+(s; \omega, \lambda) + V_-(s; \omega, \lambda) \} \\ &:= W(s, \omega, \lambda)\end{aligned} \tag{D.14}$$

We now use the principal value prescription:

$$\int \frac{f(q')\,dq'}{q'-q-i\varepsilon} = P\int \frac{f(q')\,dq'}{q'-q} + \pi i f(q') \tag{D.15}$$

to re-express (D.8) as:

$$R(q^2,\omega,\lambda) = \frac{1}{\pi} P\int_{s_0}^{\infty} \frac{ds\, W(s,\omega,\lambda)}{s-q^2} + i\, V(q^2;\omega,\lambda) \tag{D.16}$$

This result can be used to describe the time-ordered product amplitude:

$$\begin{aligned} T(q) &= i\int d^4z\, e^{iq\cdot z} \langle 0|T(J_1(\tfrac{z}{2})J_2(-\tfrac{z}{2}))|p\rangle \\ &= R(q) + iV_-(q) \end{aligned} \tag{D.17}$$

in terms of the dispersion relation (using (D.15, 16, 17)):

$$T(q^2;\omega,\lambda) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds}{s-q^2-i\varepsilon}\, W(s;\omega,\lambda) \tag{D.18}$$

Equation (D.18) is our light-cone dispersion relation and is valid under the assumptions above.

Now, equations (D.6) define the straight line over which we take the dispersion relationship (for fixed ω). The plane is shown in figure 15.

If we define new variables X, Y such that:

$$X = \tfrac{1}{2}(q_1^2 + q_2^2) \quad , \quad Y = \tfrac{1}{2}(q_1^2 - q_2^2) \tag{D.19}$$

then we can rewrite (D.9) as:

$$\left(X - \tfrac{1}{4}m^2\right) \leq Y^2 \tag{D.20}$$

which defines a parabola inside which the dispersion relation is not defined. The fixed ω (solid) line is defined as:

$$q^2 = \left(\frac{1}{\omega}\right) P.q - \frac{\lambda}{\omega} \tag{D.21}$$

or, equivalently:

$$q_2^2 = \left(\frac{1-\omega}{1+\omega}\right) q_1^2 - \frac{2\lambda}{1+\omega} \tag{D.22}$$

and so in the $\lambda \to 0$ limit we have:

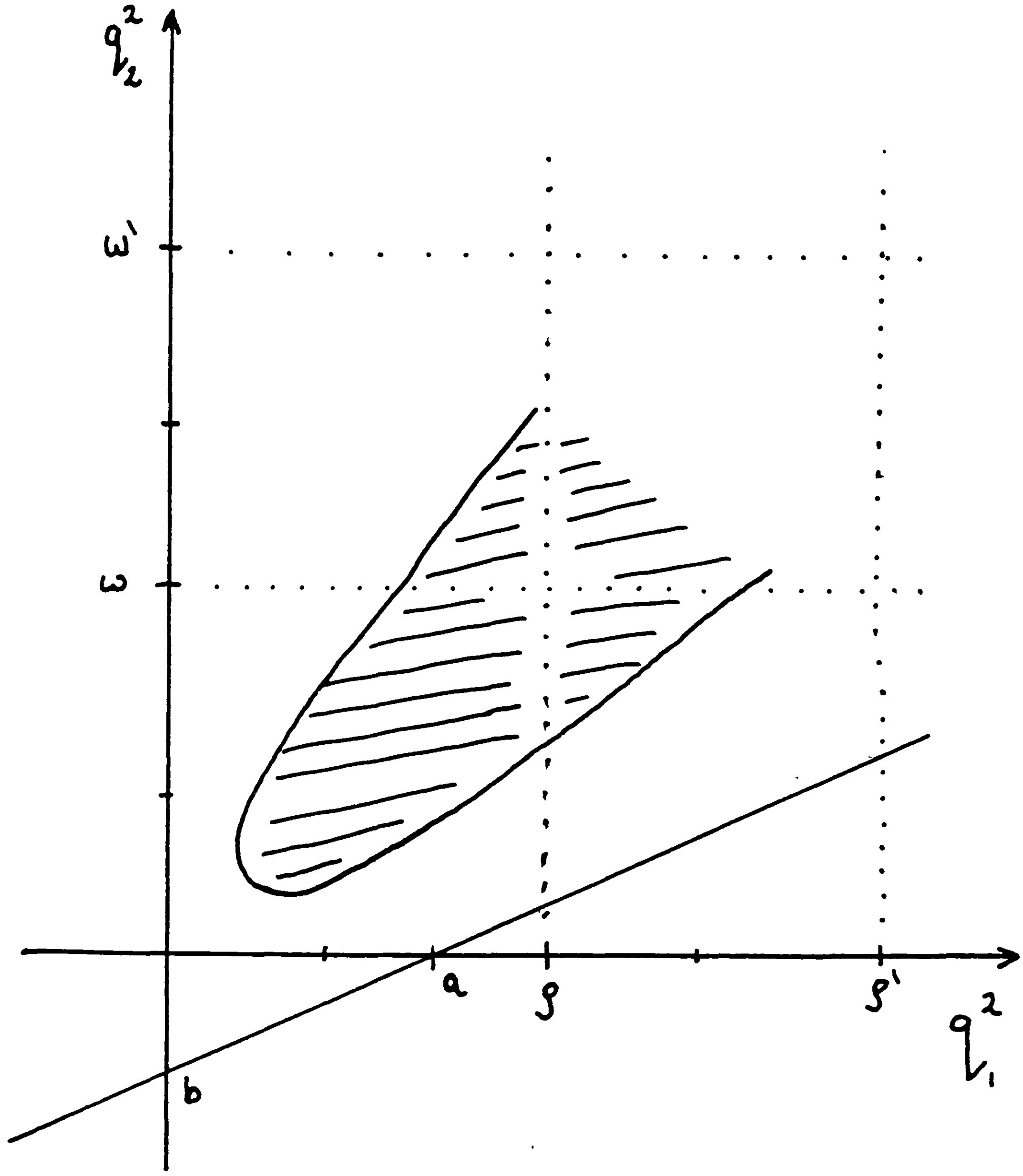


Figure 15: The q_1^2-q_2^2 plane for the $q_1 q_2 \rho$ vertex. The shaded area denotes the forbidden parabola (D.20).

$$\omega = \left(\frac{q_1^2 - q_2^2}{q_1^2 + q_2^2} \right) \tag{D.23}$$

In this thesis we always consider the $\lambda = 0$, zero-width limit.

Appendix E: Radiative Corrections to the Pion Vertex Function

The radiative corrections (figure 12) give contributions (in units of $\frac{g^2 C_F}{(4\pi)^2} \Gamma\left(1+\frac{\varepsilon}{2}\right)\left(\frac{4\pi\mu^2}{-q^2}\right)^{\varepsilon/2}$, where $C_F = 4/3$):

Diagrams of Type (i):

$$-8 - 4\ln(1-\omega\xi) + 2\ln(1-\omega^2) + \frac{4}{\varepsilon_{UV}} - \frac{8}{\varepsilon_{IR}}$$

$$+ \left\{ \frac{2(1-\omega)}{\omega(1-\xi)} \left[\ln\left(\frac{1-\omega\xi}{1-\omega}\right)\frac{2}{\varepsilon_{IR}} + \frac{1}{2}\left(\ln^2(1-\omega) - \ln^2(1-\omega\xi)\right)\right] \right.$$

$$+ \frac{2(1-\omega\xi)}{\omega(1-\xi)} \ln\left(\frac{(1-\omega\xi)}{(1-\omega)}\right) + \frac{1}{\omega(1-\xi)}\Big((1-\omega\xi)\ln(1-\omega\xi)$$

$$\left. - (1-\omega)\ln(1-\omega)\Big) + \begin{pmatrix} \omega \to -\omega \\ \xi \to -\xi \end{pmatrix} \right\}$$

The Box Diagram (Type (ii):

$$\frac{4(1+\omega\xi)}{\omega^2(1-\xi)\,\varepsilon_{IR}}\left[\frac{(1+\omega\xi)}{(1+\xi)}\ln\left(\frac{1+\omega\xi}{1-\xi}\right)-\frac{1+\omega}{2}\ln\left(\frac{1+\omega}{1-\omega}\right)\right.$$

$$+\frac{\varepsilon}{2}\left\{\frac{1+\omega\xi}{1+\xi}\ln(1+\omega\xi)-\frac{1+\omega}{1+\xi}\ln(1-\omega)-\frac{1+\omega\xi}{2(1+\xi)}\left(\ln^2(1+\omega\right.\right.$$

$$\left.-\ln^2(1-\omega)\right)+\frac{1-\omega}{2}\ln(1-\omega)-\frac{1+\omega}{2}\ln(1+\omega)$$

$$\left.\left.-\frac{1+\omega}{2}\left(\ln^2(1-\omega)-\ln^2(1+\omega)\right)\right\}\right]$$

Self Energy Diagrams (Type (iii)):

$$-\frac{4}{\varepsilon_{UV}}+\frac{2}{\varepsilon_{IR}}+\left(\ln(1-\omega\xi)-1\right)$$

REFERENCES

1. R.P. Feynman : Photon Hadron Interactions (W.A. Benjamin Co., New York 1972)

2. A.J. Buras: Rev.Mod. Phys. 52 (1980) 199

3. H.D. Politzer: Phys. Rev. Lett. 30 (1973) 1346
D.J. Gross and F. Wilczek: Phys. Rev. Lett. 30 (1973) 1343

4. K.G. Wilson: Phys. Rev. 179 (1979) 1499
D.J. Gross and F. Wilczek: Phys. Rev. D8 (1973) 3633, D9 (1974) 980

5. G. Altarelli and G. Parisi: Nucl. Phys. B126 (1977) 298

6. Yu. L. Dokshitzer, D.I. Dyakonov and S.I. Troyan: Phys. Rep. 58 (1980) 269
N.S. Craigie and H.F. Jones: Nucl. Phys. B172 (1980) 59

7. G. Parisi and R. Petronzio: Nucl. Phys. B154 (1979) 427
H.F. Jones and J. Wyndham: Nucl. Phys. B172 (1980) 59

8. S.J. Brodsky and G.P. Lepage: Phys. Rev. D22 (1980) 2157

9. S.J. Brodsky and G.P. Lepage: Phys. Rev. D24 (1981) 1808

10. S.J. Brodsky and G.R. Farrar: Phys. Rev. Lett. 31 (1973) 1153
V.A. Matveev, R.M. Muradyan and A.V. Tavkhelidze: Lett.Nuovo Cimento 7 (1973) 719

11. J.R. Smith et al.: Phys. Rev. D30 (1984) 851

12. P.H. Damgaard: Nucl. Phys. B211 (1983) 435
G.R. Farrar, E. Maina and F. Neri: Rutgers preprint RU-84-13

13. N.Isgur and C. Llewellyn-Smith: Phys. Rev. Lett. 52 (1984) 1080

14. G.R. Farrar: Proc. VIth International Workshop on Photon-Photon Collisions, Lake Tahoe, California, 1984

15. V.N. Baier and A.G. Grozin: Phys. Lett. 96B (1980) 181

16. M. Fontannaz and H.F. Jones: Z. Phys. C28 (1985) 371

17. V.L. Chernyak and A.R. Zhitnitsky: Phys. Rep. 112 (1984) 173

18. M.K. Chase: Nucl. Phys. B167 (1980) 125
F. del Aguila and M.K. Chase: Nucl. Phys. B193 (1981) 517

19. N.S. Craigie and J. Stern: Nucl. Phys. B216 (1983) 209

20. M.A. Shifman, A.I. Vainshtein and V.I. Zakharov: Nucl. Phys. B147 (1979) 385, 448, 519

21. E.V. Shuryak: Phys. Rep. 115 (1984) 151
G. Launer: Imperial College preprint ICPT/84-85/10
S.Generalis, H.F. Jones, G. Launer and M.J. Lavelle: in preparation

22. V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov: Fortschritte der Physik 32 (1984) 585

23. B.L. Ioffe: Nucl. Phys. B188 (1981) 317 (E: B191 (1981) 591)

24. N.V. Krasnikov, A.A. Pivovarov and N.N. Tavkhelidze: Z. Phys. C19 (1983) 301

25. R.A. Bertlmann, G. Launer and E. de Rafael: Marseille preprint CPT 84/P. 1586 (1984)

26. V.L. Chernyak and A.R. Zhitnitsky: Nucl. Phys. B201 (1982) 492 (E: B214 (1983) 547)

27. B.L. Ioffe: Proc. XXII International Conference on High Energy Physics, Leipzig 1984

28. L.J. Reinders, H.R. Rubinstein and S. Yazaki: Nucl. Phys. B196 (1982) 125

29. M.B. Voloshin: Sov. J. Nucl. Phys. 29 (1979) 703

30. V.M. Belyaev and B. Yu. Blok: Moscow preprint 1984, ITEP-161

31. J. Govaerts, F. de Viron, D. Gusbin and J. Weyers: Phys. Lett. 128B 262

32. V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov: Nucl. Phys. B191 (1981) 301

33. V.I. Eletsky, B.L. Ioffe and Y.I. Kogan: Moscow preprint 1982, ITEP-98

34. V.A. Novikov et al.: Nucl. Phys. B237 (1984) 525

35. J.S. Bell and R.A. Bertlmann: CERN preprint 1983, TH-3769

36. K. Zalewski: Cracow preprint, TPJU-21/83

37. A. Andrikopoulou: Z. Phys. C22 (1984) 63

38. A. Andrikopoulou: Imperial College preprint ICTP/82-83/19

39. I.R. Zhitnitsky: Novisibirsk prepient 1982, 82-155
V.L. Chernyak and I.R. Zhitnitsky: Nucl. Phys. B236 (1984) 52

40. M.A. Avdeenko, V.L. Chernyak and S.A. Korenblit: Sov. J. Nucl. Phys. 33 (1981) 252

41. A.B. Henriques, B.H. Kellett and R.G. Moorhouse:

Ann. Phys. 73 (1975) 125

42. G.P. Lepage and S.J. Brodsky: Phys. Rev. Lett. 43 (1979) 545 (E: 43 (1979) 1625)
43. S. Narison and E. de Rafael: Phys. Lett. 103B (1981) 57
44. M.B. Voloshin: Nucl. Phys. B154 (1979(365
45. A.R. Zhitnitsky, I.R. Zhitnitsky and V.L. Chernyak: Sov. J. Nucl. Phys. 38 (1983) 775
46. Yu. M. Makeenko: Sov. J. Nucl. Phys. 33 (1981) 440
47. Y. Chung, H.G. Dosch, M. Kremer and D. Schell: Z. Phys. C15 (1982)367

    B.L. Ioffe: Moscow preprint 1982, ITEP-131
48. O. Dumbrajs et al.: Nucl. Phys. B216 (1983) 277
49. S. Narison and N. Paver: CERN preprint Th-3608
50. A.S. Gorsky: Yad. Fiz. 41 (1985) 430
51. S.L. Adler: Phys. Rev. 177 (1969) 26

    J.S. Bell and R. Jackiw: Nuovo Cim. 60A (1969) 47
52. M.B. Voloshin: Moscow preprint (1982), ITEP-8
53. E. Braaten: Gainesville preprint, UFTP-82-11
54. A.S. Gorsky: Moscow preprint (1984), ITEP-102
55. A.R. Zhitnitsky, I.R. Zhitnitsky and V.L. Chernyak: Sov. J. Nucl. Phys. 38 (1983) 645
56. M.J. Lavelle: Nucl. Phys. B to be published
57. N.S. Craigie, N. Paver and Riazuddin: Trieste preprint ICTP/84/89
58. H.G. Dosch: Proceedings,Workshop on Non-Perturbative Methods, Montpellier 1985
59. A.S. Gorsky: Moscow preprint 1985, ITEP-71
60. G.R. Farrar: Phys. Rev. Lett. 53 (1984) 28

61. H.J. Lipkin: Phys. Rev. Lett. 53 (1984) 2075

62. A.S. Gorsky: Moscow preprint 1984, ITEP-168

63. M.J. Lavelle: Z. Phys. C. to be published

64. J. Perrier: SLAC-Pub-3436, 1984

65. G.R. Farrar, E. Maina and F. Neri: Rutgers preprint, RU-85-08

66. S. Palestini et al.: Princeton preprint DOE/ER/3072-23

67. A.S. Kronfeld and D.M. Photiadis: Cornell preprint, CLNS-84/633

68. W. Wilcox and R.M. Woloskys: TRIUMF preprint, TRI-PP-85-4

69. S. Gottlieb and A.S. Kronfeld: Cornell, CLNS-85/656

70. G.T. Loladze, L.R. Surguladze and F.V. Tkachov: INR preprint 1985

D. Broadhurst, S. Generalis, G. Launer and M.J. Lavelle: unpublished

71 E.V. Shuryak and A.I. Vainshtein: Nucl. Phys. B199 (1982) 451, B201 (1982) 141