GAUGE PROPERTIES AND CONVEXITY OF THE EFFECTIVE POTENTIAL

by

DESMOND ALEXANDER JOHNSTON

A thesis presented for the Degree of Doctor of Philosophy of the University of London and the Diploma of Membership of Imperial College

Department of Physics
The Blackett Laboratory
Imperial College
London SW7 2BZ

September 1986

ABSTRACT

We investigate the gauge dependence and convexity properties of the effective potential using the Nielsen Identities which control the gauge-parameter dependence of physical quantities. We show that it is perfectly feasible to use the 't Hooft gauge in effective potential calculations, contrary to some earlier claims. In addition to Nielsen's original derivation we show how the identities may be derived by extending the set of B.R.S. transformations which act on the theory and we demonstrate the utility of the method by rederiving our results for the 't Hooft gauge and deriving the identities in the planar gauge, which is a variant of the axial gauge. We also show that it is possible to derive Nielsen Identities for an effective action which contains composite operators.

In the chapter on convexity we extend the results of Fujimoto et al. on producing a convex effective potential from the loop expansion for $\lambda\phi^4$, to more general scalar Higgs representations, making use of a condition on admissible vacua that arises from the Nielsen identities when we choose an 't Hooft gauge. We also consider the constraint effective potential and show, using a computer simulation, that this too gives rise to a convex result.

The last chapter deals with the calculation of a convex effective potential at finite temperature. After giving a brief outline of the two complementary finite temperature formalisms and a brief reprise of Rivers' imaginary time formalism work we show how to produce a convex effective potential at finite temperature using the tadpole method of calculation in a real time formalism. We show that this, as in the imaginary time case, gives a result which breaks down at a temperature which is below the supposed critical temperature of the theory, invalidating the loop expansion results for critical temperatures in spontaneously broken gauge theories.

PREFACE

The work presented in this thesis was carried out in the Theoretical Physics group of the Department of Physics, Imperial College, London between October 1983 and September 1986, under the supervision of Dr. H.F. Jones. Unless otherwise stated, the work is original and has not been submitted for a degree of this or any other university.

I would like to thank my supervisor Hugh Jones for help during the course of this work and Professors T.W.B Kibble and L. O'Raifeartaigh for useful discussions. A special word of thanks is also due to Dr. R. Rivers for his explanations of finite temperature quantum field theory. Some of the work in section 3 was done in collaboration with M. Hindmarsh.

Financial support during the course of the work was provided by the Department of Education for Northern Ireland.

CONTENTS

LIST OF FIGURES

CHAPTER 1 INTRODUCTION

1. INTRODUCTION.

The notion of spontaneous symmetry breaking has been integral to many of the recent developments in quantum field theory. The Salam-Weinberg SU(2)×U(1) [1] theory of the weak interactions, although now relegated to the role of a low energy effective theory, relies upon the symmetry breaking effect of the Higgs scalars to provide a gauge invariant mass term for the gauge bosons that carry the short range weak force, the W's and the Z, which were observed recently for the first time at CERN [2]. In a spontaneously broken theory the possible vacua are labelled by some order parameter (for instance the vacuum expectation value of the physical Higgs scalar in the case of the Salam-Weinberg model) and it may happen that the vacuum energy density is lower for some non-zero value of this parameter than for the zero case. The true vacuum of the theory is then taken to be that with the asymmetric, non-zero order parameter and the theory is said to be spontaneously broken.

An essential tool in the study of spontaneous symmetry breaking is the effective action and its constant field limit, the effective potential. These were introduced by Schwinger [3] and later exploited by Jona-Lasinio [4] and Nambu [5], who was the first person to use the loop expansion in a perturbative evaluation of the effective action. We follow below the exposition of Coleman [6] in order to get a feel for the physical meaning of the effective action and demonstrate why it is the correct object to consider in a determination of the vacuum of a spontaneously broken theory.

In a classical field theory the ordinary potential $U(\phi)$ is the energy density per unit volume for the state in which the field has the constant value ϕ. The situation in the quantum theory is similar: $V(\bar{\phi})$, the effective potential, is the energy density per unit volume of the theory in which the expectation value of the quantum field is $\bar{\phi}$, defined as

$$\frac{\langle 0|\phi|0\rangle_J}{\langle 0|0\rangle_J} = \bar{\phi} = \frac{\delta W[J]}{\delta J} \tag{1.1}$$

where $W[J]$ is the connected generating functional.

We observe that for a slowly varying source J we could write the following expansion for $W[J]$

$$W[J]=\int d^4x(-\xi(J)+\frac{1}{2}X(J)(\partial_\mu J)^2+\ldots) \tag{1.2}$$

The effective potential may be defined by using a similar expansion of the effective action Γ, which one can show to be the generating functional for one-particle-irreducible (1PI) diagrams [7].

$$\Gamma[\bar{\phi}]=\int d^4x(-V(\bar{\phi})+\frac{1}{2}Z(\bar{\phi})(\partial_\mu\bar{\phi})^2+\ldots) \tag{1.3}$$

If we consider a source that has a constant value J throughout a box of side L for a duration of time T and is switched on and off slowly we find the following expression for W (we have set $\hbar=1$)

$$\langle 0^+|0^-\rangle=\exp(iW[J])=\exp(-iL^3T\xi(J)) \tag{1.4}$$

Throughout the box we have changed the Hamiltonian density from H to $H-J\bar{\phi}$, and as the change was adiabatic we should still be in the ground state of the new Hamiltonian. If we turn off the perturbation the ground state will go back to the ground state of the original theory but it will retain its phase. We expect states to develop in time as exp(-iET), so we can identify $\xi(J)$ from (1.4) as the energy per unit volume of the perturbed Hamiltonian.

Now consider the problem of constructing a state $|0\rangle$ which is a stationary state of $\langle 0|H|0\rangle$ subject to $\langle 0|0\rangle=1$ and $\langle 0|\phi|0\rangle=\bar{\phi}$. This may be solved by using Lagrange multipliers E and J for the first and second constraints respectively. We are therefore considering minimizing

$$\langle 0|H-E-J\bar{\phi}|0\rangle \tag{1.5}$$

From (1.5) we can see that

$$\bar{\phi}=\langle 0|\phi|0\rangle= -\frac{\delta E}{\delta J} \tag{1.6}$$

and

$$\langle 0|H|0\rangle=E - J\frac{\delta E}{\delta J} \tag{1.7}$$

At this point we recall that the effective action may be defined as the Legendre transform of the connected generating functional

$$\Gamma[\bar{\phi}]=W[J]-\int J\bar{\phi} \tag{1.8}$$

which gives, on substituting in from (1.2) and (1.3)

$$-V(\bar{\phi})=-\xi(J)-J\bar{\phi} \tag{1.9}$$

or alternatively

$$V(\bar{\phi})=\xi(J) - J \frac{\delta\xi(J)}{\delta J} \tag{1.10}$$

which, on identifying $\xi(J)$ with E, agrees with the definition of $\langle 0|H|0\rangle$, the energy density of the system subject to the constraint $\langle 0|\phi|0\rangle=\bar{\phi}$. The effective potential is thus the energy density of the system and source subject to this constraint. We also note that at J=0 the value of the effective potential gives the vacuum energy density of the system, a physical quantity which should therefore be gauge invariant in gauge theories.

After this general discussion we now consider perturbative evaluation of the effective potential, and in this the definition (1.8) and the expansion (1.3) prove to be of value. As $\Gamma[\bar{\phi}]$ is the generating functional for 1PI diagrams [7] we can write

$$\Gamma[\bar{\phi}]=\sum\frac{1}{n!}\int d^4x_1 ..d^4x_n\Gamma(x_1 ...x_n)(\bar{\phi}(x_1)-v)...(\bar{\phi}(x_n)-v) \tag{1.11}$$

where $\Gamma(x_1 ...x_n)$ is the sum of all 1PI diagrams with n external legs and v is the vacuum expectation value of ϕ. We now expand $\Gamma(x_1 ...x_n)$ in momentum space to get

$$\Gamma(x_1 ...x_n)=\int d^4k_1 ...d^4k_n\delta(k_1 +...+k_n)\exp(ik_ix_i)\tilde{\Gamma}(k_1 ...k_n) \tag{1.12}$$

If we now substitute this into (1.11) we find the series

$$\Gamma[\bar{\phi}]=\int d^4x\sum\frac{1}{n!}\tilde{\Gamma}(0...0)(\bar{\phi}(x)-v)^n \tag{1.13}$$

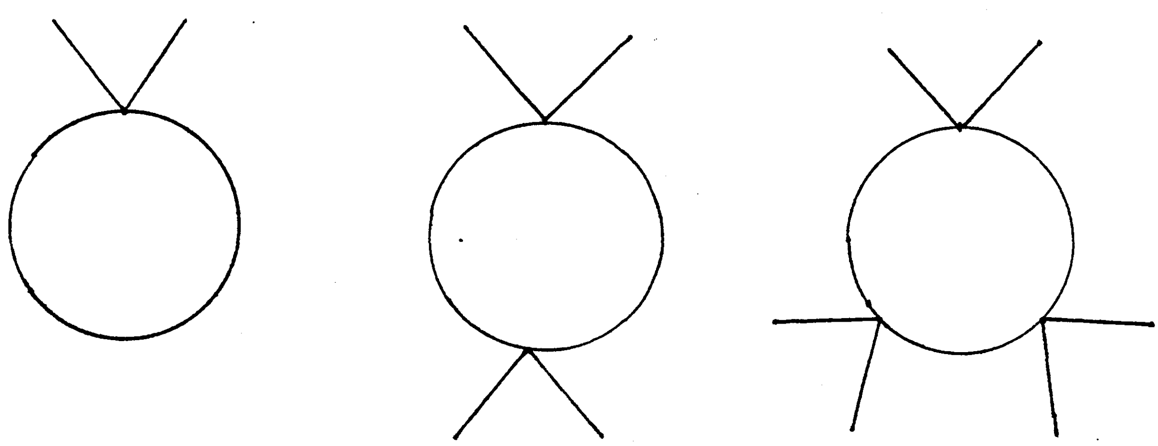
where v is the vacuum expectation value of ϕ

To get the effective potential version of this we let the field $\bar{\phi}(x)$ tend to a constant and find

$$V(\bar{\phi})=-\Sigma\frac{1}{n!}\tilde{\Gamma}(0_1\ldots0_n)(\bar{\phi}-v)^n \qquad (1.14)$$

Formula (1.14) generates two of the standard methods of calculating the effective potential. In both cases one works in the context of a loop, or equivalently h, expansion. One could choose to take (1.14) as it stands and perform the infinite sum to a given loop order to calculate V. In the canonical example of a scalar $\lambda\phi^4$ theory one could evaluate the one-loop effective potential by summing over graphs of the form

<u>(Fig.1)</u>



This would give for the zero-and one-loop contributions to V

$$V(\bar{\phi})=\frac{1}{2}\mu^2\bar{\phi}^2+\frac{\lambda}{4!}\phi^4+i\int d^4k\Sigma\frac{1}{2n}\left(\frac{\lambda\bar{\phi}^2}{2(k^2-\mu^2+i\varepsilon)}\right)^n \qquad (1.15)$$

which gives, performing the sum,

$$V(\bar{\phi})=U(\bar{\phi})+\frac{i}{2}\int d^4k\ \ln\left(1+\frac{\lambda\phi^2}{2(k^2-\mu^2+i\varepsilon)}\right) \qquad (1.16).$$

Alternatively, instead of performing the sum, we could differentiate (1.14) w.r.t $\bar{\phi}$ and then set $\bar{\phi}$=v. This gives

$$\left.\frac{\partial V}{\partial \bar{\phi}}\right|_{\bar{\phi}=v}=-\tilde{\Gamma}^{1}(0) \qquad \text{or} \qquad V(\bar{\phi})=-\int^{\bar{\phi}}dv\tilde{\Gamma}^{1}(0) \tag{1.17}$$

where $\tilde{\Gamma}^{1}(0)$ is the sum of all 1PI Feynman diagrams at zero momentum with one external leg, the so-called tadpole graphs. The method was first suggested by Weinberg [8] and elaborated by Lee and Sciaccaluga [9] in the context of dimensional regularization, where it proves to be especially convenient. We shall find in a later chapter that the technical quirks of calculating an effective potential in real-time finite temperature quantum field theory make it useful there also.

Another method of calculating V by using background fields is rather more indirect but it is perhaps the most convenient in practice. It was first introduced by Jackiw [10], but we follow here the more lucid exposition of Fujimoto et al. [11]. If we have the classical potential U

$$U(\phi)=\frac{1}{2}M_{ab}\phi_a\phi_b+\frac{1}{3!}F_{abc}\phi_a\phi_b\phi_c+\frac{1}{4!}G_{abcd}\phi_a\phi_b\phi_c\phi_d \tag{1.18}$$

then $V(\bar{\phi})$ will be given by

$$V(\bar{\phi})=U(\bar{\phi})+\Sigma\left(M(\bar{\phi}),F(\bar{\phi}),G\right) \tag{1.19}$$

where Σ denotes the sum of all vacuum graphs in a theory where we have made the substitutions $M_{ab}\to M_{ab}(\bar{\phi})$ and $F_{abc}\to F_{abc}(\bar{\phi})$ with the functions given by

$$M_{ab}(\bar{\phi})=M_{ab}+F_{abc}\bar{\phi}_c+\frac{1}{2}G_{abcd}\bar{\phi}_c\bar{\phi}_d$$

$$F_{abc}(\bar{\phi})=F_{abc}+G_{abcd}\bar{\phi}_d \tag{1.20}$$

To prove this consider the effect of a c-number shift in the argument of the path integral. The measure will be unchanged and the Lagrangian will behave as

$$L(\phi)+J\phi \rightarrow L(\phi+c)+J(\phi+c) \tag{1.21}$$

One could rewrite the right hand side as

$$L(c)+Jc+\phi(\frac{\partial L}{\partial c}+J)+\Delta L(\phi) \tag{1.22}$$

where $\Delta L(\phi)=L(\phi+c)-L(c)-\phi\frac{\partial L}{\partial c}$

We can absorb the shift in ΔL into a shift in the parameters by writing

$$\Delta L(\phi,p)=L(\phi,p(c)) \tag{1.23}$$

where the notation $p(c)$ means we have made the replacements (1.20) in the parameters of the theory. We now translate this into the following identity for the connected generating functional $W[J,p]$, where we have explicitly displayed the parameter dependence in the path integral:

$$W[J,p]=W'[J',p(c)]+\Omega\left(\left(L(c)-c\frac{\partial L}{\partial c}\right)+c\int J'\right) \qquad (1.24)$$

where Ω is the spacetime volume and

$$J'=J+\frac{\partial L}{\partial c}$$

We now consider the Legendre transform definition of the effective action from both W and W'

$$\Gamma[\bar{\phi},p]=W[J,p]-\int\bar{\phi}J$$

$$(1.25)$$

$$\Gamma'[\bar{\phi}',p(c)]=W'[J',p(c)]-\int\bar{\phi}'J'$$

where, as usual, $\bar{\phi}=\frac{\delta W}{\delta J}$ and $\bar{\phi}'=\frac{\delta W'}{\delta J'}$. We now note from (1.24) that

$$\bar{\phi}=\frac{\delta W'}{\delta J'}\frac{\delta J'}{\delta J}+c = \bar{\phi}'+c \qquad (1.26)$$

If we substitute the definitions above back into (1.24) we find the following identity for the effective action

$$\Gamma[\bar{\phi},p]=\Gamma[\bar{\phi}',p(c)]+L(c)\Omega+\frac{\partial L}{\partial c}\int\bar{\phi}' \qquad (1.27)$$

The constant field limit of this gives the required identity for the effective potential

$$V(\bar{\phi},p)=V'(\bar{\phi}-c,p(c))-L(c)-(\bar{\phi}-c)\frac{\partial L}{\partial c} \qquad (1.28)$$

With our sleight of hand in shifting the field we have succeeded in expressing the original V in terms of a V' calculated in a theory with shifted parameters along with functions of the original Lagrangian.

In particular for $\bar{\phi}=c$ we find that

$$V(c,p)=V'(0,p(c))-L(c)$$

or (1.29)

$$V(c,p)=V'(0,p(c))+U(c)$$

We thus see that the sum of all the diagrams with no external legs (the vacuum graphs) added to the classical potential in the shifted theory gives the effective potential.

As a final method of calculating the effective potential we can consider a saddle point expansion, although this becomes progressively more tedious as one proceeds to higher orders. The one-loop result is, however, fairly easy to obtain and we give a derivation below following [12],[13]. By analogy with finite dimensional integrals we look for extrema of of the exponent in the path integral, that is fields which satisfy the classical equations of motion.

$$(\nabla^2+m^2)\phi_0+U'(\phi_0)=J \tag{1.30}$$

If we have a convex classical potential U we assume that we have the trivial solution $\phi_0=0$ for J=0 (the case of a non-convex potential is considered in chapter 3 in detail). We now shift the integration variable in the path integral $\phi\rightarrow\phi_0+\phi'$, keep the quadratic portion in ϕ' explicitly and expand the higher order terms perturbatively to get

$$Z[J]=\exp(iS[\phi_0,J])\int[D\phi']\exp(i\int d^4x\frac{1}{2}(\nabla^2\phi'-(m^2+U''(\phi_0))\phi'^2) - \sum_{n\geq 3}\frac{\phi'^n}{n!}V^n(\phi_0)) \tag{1.31}$$

To obtain the required loop (or h) expansion we rescale the field $\phi' \to \sqrt{\hbar}\phi'$. As we shall only be interested in the one-loop result we drop the higher order terms, giving

$$Z[J]=\exp(\frac{i}{\hbar}S[\phi_0,J])\int[D\phi']\exp(i\int d^4x\frac{1}{2}(\nabla^2\phi'^2-\phi'^2(m^2+U''(\phi_0)))) \quad (1.32)$$

With W[J] defined in the usual manner by

$$W[J]=-i\hbar\ln Z[J] \quad (1.33)$$

we see that the leading term in W[J], to order h^0, is just $S[\phi_0,J]$. The order $\hbar$ term is given by performing the Gaussian integration

$$\int[D\phi']\exp(-i\int d^4x\ \frac{1}{2}\phi'(\nabla^2+m^2+U''(\phi_0))\phi') \quad (1.34)$$

which gives

$$\exp(-\frac{1}{2}Tr.\ln(\nabla^2+m^2+U''(\phi_0))) \quad (1.35)$$

where the trace operates on all the indices (spacetime, group ..). We thus find

$$W[J]=S[\phi_0,J]+\frac{i\hbar}{2}Tr.\ln(\nabla^2+m^2+U''(\phi_0)) \quad (1.36)$$

We can now use the standard Legendre transform definition to find $\Gamma[\bar{\phi}]$. We recall that

$$\Gamma[\bar{\phi}]=W[J]-\int J\bar{\phi} \quad (1.37)$$

and note that $\frac{\delta W}{\delta J}=\frac{\delta S}{\delta J}+O(\hbar)=\phi_0+O(\hbar)$, so $\bar{\phi}$ is given by ϕ_0 to lowest order. If we now expand $S[\bar{\phi},J]$ order by order in h we find that

$$S[\bar{\phi},J]= S[\phi_0,J] + \frac{1}{2}\left.\frac{\delta^2 S}{\delta^2\phi}\right|_{\phi=\phi_0}$$

because $\left.\frac{\delta S}{\delta\phi}\right|_{\phi=\phi_0}= 0$. We thus find that

$$S[\bar{\phi},J]=S[\phi_0,J] + O(\hbar^2) \tag{1.38}$$

We can also substitute $\bar{\phi}$ for ϕ_0 in the trace, as the difference here will also be of $O(\hbar^2)$. This gives us finally

$$W[J]=S[\bar{\phi}] + J\bar{\phi} + \frac{i\hbar}{2}\mathrm{Tr}\ \ln(\nabla^2+m^2+U''(\bar{\phi})) \tag{1.40}$$

and

$$\Gamma[\bar{\phi}]=S[\bar{\phi}] + \frac{i\hbar}{2}\mathrm{Tr}\ \ln(\nabla^2+m^2+U''(\bar{\phi})) \tag{1.41}$$

This is the required result for the one-loop effective potential.

There are two subtleties hidden in the formalism that we have developed so far, the problems of the gauge dependence and convexity of the effective potential. In a gauge theory an injudicious choice of gauge, even one that might be useful in other contexts, may render the effective potential apparently gauge dependent, which contradicts the interpretation of the minimum value of the potential as the vacuum energy density. This apparent paradox was resolved by Nielsen and is the subject matter of chapter 2. The other problem is that of the convexity of the effective potential. In a spontaneously broken theory one may choose to start with a classical potential which is non-convex, but one can derive a condition to show that the effective potential is convex, which contradicts the result of the loop expansion in such cases. The difficulty may be circumvented by a more careful treatment of the extrema in the path integral, and this is the subject of chapter 3.

One might also enquire as to whether the effective potential (in its convexified form) remains a viable tool at finite temperature, and we examine this in chapter 4. The answer, as to so many questions, is "Up to a point" (in this case a point of inflection!) [14]. We shall find all of the methods of calculating the effective potential that we have outlined so far of some utility in our investigations. We examine the Nielsen identities by using Jackiw's field shifting method to verify them at one-loop order. The proof of convexity at zero temperature relies upon the saddle point approach, as does an imaginary time formalism calculation of a finite temperature effective potential. We shall find, however, that the tadpole method is the most convenient for calculating a real-time formalism finite temperature effective potential.

Apart from the chapter on convexity we shall take the Abelian Higgs model (charged scalar electrodynamics) as our prototypical gauge theory. We choose to ignore the probable triviality of $\lambda\phi^4$ theory in 4 dimensions, a non-perturbative effect, as the calculations are typical of non- trivial theories.

CHAPTER 2 GAUGE DEPENDENCE

2.1 GAUGE DEPENDENCE: GENERAL CONSIDERATIONS

The treatment of the Higgs [15] mechanism using the effective potential at tree level (i.e. the classical potential)is a standard exercise in most quantum field theory text books. For instance, if we consider the Abelian Higgs model with the Lagrangian

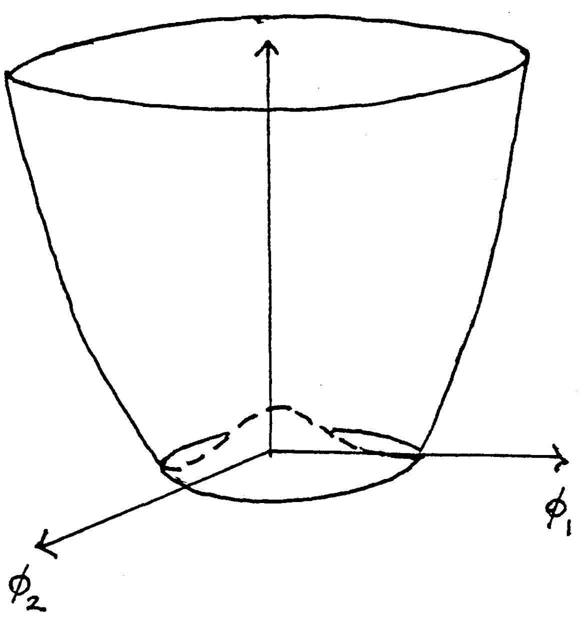
$$L=(D_\mu\phi)(D^\mu\phi) + m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{2.1}$$

where

$$D_\mu\phi = \partial_\mu\phi - igA_\mu\phi \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

we find that the potential U has a minimum at $\|\phi\|=v/\sqrt{2}$, where v is $\sqrt{(m^2/\lambda)}$. If we now write $\phi=(\phi_1+ i\phi_2)/\sqrt{2}$, with ϕ_1 and ϕ_2 real fields, we could choose $\langle 0|\phi_1|0\rangle=v$ and $\langle 0|\phi_2|0\rangle=0$. Picking out one of the possible vacua on the ring of minima as the physical vacuum has broken the original U(1) symmetry.

(Fig.2) The classical potential in (2.1)



Our initial parametrization of ϕ proves to be rather inconvenient from the point of view of a physical interpretation. If we choose instead

$$\phi(x)= \frac{1}{\sqrt{2}}(v+\eta(x))\exp(i\xi(x)/v) \qquad (2.2)$$

and carry out the gauge transformation to the unitary gauge

$$\phi'(x)=\exp(-i\xi(x)/v)\phi(x) \quad , \quad B_\mu(x)=A_\mu(x)-\frac{1}{gv}\partial_\mu\xi(x)$$

we find that the Lagrangian (2.1) is given in terms of the gauge transformed variables as

$$L=\frac{1}{2}(\partial_\mu\eta)^2 - \frac{1}{2}m^2\eta^2-\frac{1}{4}(\partial_\mu B_\nu -\partial_\nu B_\mu)^2 +\frac{1}{2}(gv)^2 B_\mu B^\mu$$
$$+\frac{1}{2}g^2 B_\mu B^\mu\eta(2v+\eta)- \lambda v^2\eta^3 - \frac{1}{4}\lambda\eta^4 \qquad (2.4)$$

The unphysical Higgs mode $\xi(x)$ has been gobbled up to give a mass term to the original U(1) gauge field.

This seemingly transparent classical argument does not carry over directly to the quantum theory because there one must work with a Lagrangian that includes a gauge-fixing term. The effective potential as defined in (1.14)

$$V(\bar{\phi}) = -\Sigma\frac{1}{n!} \tilde{\Gamma}(0,\ldots 0)[\bar{\phi}-v]^n \qquad (1.14)$$

is an off-shell quantity and one would not expect, a priori, such an object to be gauge independent.

Some explicit calculations (after Jackiw [16]) for massless scalar QED confirm these suspicions. Indeed, if we choose the gauge fixing term for (2.1) to be of the form

$$-\frac{1}{2\xi}\,(\partial_\mu A^\mu + e v_i \phi_i)^2 \qquad (2.5)$$

we would find that even the tree level potential contained the unphysical quantities ξ and v [17], [18].

$$V(\bar{\phi}) = \frac{1}{2\xi}(v_i \phi_i)^2 - \frac{1}{2}m^2\bar{\phi}^2 + \frac{\lambda}{4!}\phi^4 \qquad (2.6)$$

One possible solution to the problem is to ignore it! One could choose to shift the Higgs fields by their tree level vacuum expectation values (provided we did not choose a gauge such as (2.5)) and retain tadpole graphs in higher order calculations [19]. This would only work in cases where the vacuum is already determined at the tree level, which would preclude the consideration of the Coleman-Weinberg type symmetry breaking, where it is the radiative corrections that lead to the spontaneous symmetry breakdown. It would also prevent one examining models in which the classical Higgs potential has a larger symmetry than the rest of the Lagrangian (which gives rise to the so-called pseudo-Goldstone bosons [20]). These, too , require the inclusion of one-loop effects to determine the true vacuum. Finally, one might also like to check that radiative corrections do not change the minima even in the standard case.

Other solutions that have been advanced are that only in the "physical" unitary gauge does the effective potential have any significance [18] and that expressing V in terms of renormalized quantities rather than bare ones resolves the problem [21]. The correct approach was, however, first proposed by Nielsen [22], and we examine the identities he derived in the rest of this chapter, drawing heavily on the later work of Aitchison and Fraser [23] and Fukuda and Kugo [24].

If one considers the effective potential for a gauge theory such as (2.1) it will depend explicitly on ξ, so we write it as $V(\bar{\phi},\xi)$. The vacuum is determined by the condition

$$\frac{\partial V}{\partial \bar{\phi}} = 0 \tag{2.7}$$

and spontaneous symmetry breaking occurs when (2.7) has a non-zero solution, say $\bar{\phi}_0(\xi)$. This situation would be gauge invariant if, under a small change in ξ, the value of V at the minimum (which is after all a physical quantity) remained constant.

$$V(\bar{\phi}_0+\delta\bar{\phi}_0,\xi+\delta\xi) = V(\bar{\phi}_0,\xi) = V_{min} \tag{2.8}$$

We could write this as

$$\left.\frac{\partial V}{\partial \bar{\phi}}\right|_{\xi} \cdot \frac{d\bar{\phi}}{d\xi} + \frac{\partial V}{\partial \xi} = 0 \tag{2.9}$$

which states that the total differential of V wrt to ξ at the minimum is zero.

In a similar manner the masses of the Higgs particles will depend upon both ϕ and ξ.

$$m^2 \equiv m^2(\bar{\phi}_0,\xi) \tag{2.10}$$

If we demand that these too be gauge invariant under a change in ξ we find, by analogy with (2.9)

$$\left.\frac{\partial m^2}{\partial \bar{\phi}}\right|_{\xi} \cdot \frac{d\bar{\phi}}{d\xi} + \frac{\partial m^2}{\partial \bar{\phi}} = 0 \qquad (2.11)$$

Nielsen's remarkable result was to derive, using the B.R.S. [24] invariance of the theory, a set of identities of precisely the form above, thus guaranteeing the gauge invariance of physical quantities. They were

$$\xi\frac{\partial V}{\partial \xi} + C(\bar{\phi},\xi)\frac{\partial V}{\partial \bar{\phi}} = 0 \qquad \text{(a)}$$

(2.12)

$$\xi\frac{\partial m^2}{\partial \xi} + C(\bar{\phi},\xi)\frac{\partial m^2}{\partial \bar{\phi}} = 0 \qquad \text{if } \frac{\partial V}{\partial \bar{\phi}} = 0 \qquad \text{(b)}$$

The object $C(\phi,\xi)$ is obtained as an explicit field-theoretic expression and could be calculated in some expansion scheme. We note that equation (2.12a) exceeds our requirements, as it does not just apply at the minimum of V. Along the characteristics of V, the curves in the $\bar{\phi}$, ξ plane for which

$$\frac{d\bar{\phi}}{d\xi} = \frac{C(\bar{\phi},\xi)}{\xi} \quad , \qquad (2.13)$$

V is a constant (see Fig.3 overleaf)

We might also observe this by differentiating (2.12a) wrt to $\bar{\phi}$.

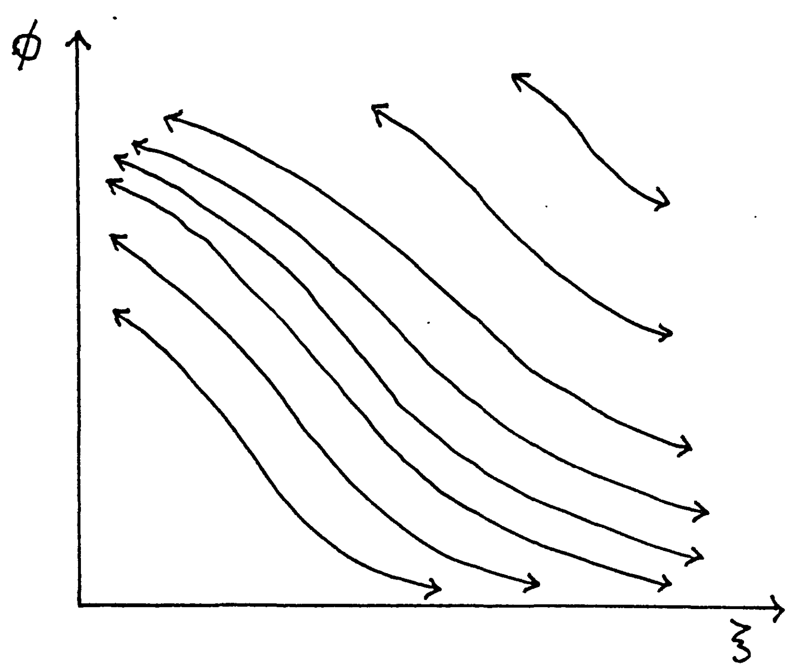
$$\left(\frac{\partial}{\partial\xi} + \frac{C(\bar{\phi},\xi)}{\xi}\frac{\partial}{\partial\bar{\phi}}\right)\frac{\partial V}{\partial\bar{\phi}} + \frac{1}{\xi}\frac{\partial C(\bar{\phi},\xi)}{\partial\bar{\phi}}\frac{\partial V}{\partial\bar{\phi}} = 0 \tag{2.14}$$

At the minimum of V the last term vanishes and we find that

$$\left.\frac{\partial V(\bar{\phi},\xi+\delta\xi)}{\delta\bar{\phi}}\right|_{\phi_0+\frac{C(\phi_0,\xi)}{\xi}\delta\xi} = 0 \tag{2.15}$$

which means that an infinitesimal change $\xi\to\xi+\delta\xi$, compensated by a change $\phi_0\to\phi_0+(C(\phi_0,\xi)/\xi)\delta\xi$, keeps V at its minimum value. The beauty of Nielsen's results is that they enable one to calculate in closed form the change in $\bar{\phi}_0$ that compensates for the gauge variation.

(Fig.3) The characteristics of (2.12a)



2.2 THE NIELSEN IDENTITIES.

Nielsen's original work was with an Abelian Higgs model in the Fermi gauge.

$$L_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \qquad (2.16)$$

but explicit calculations are plagued by infrared divergences except in the Landau ($\xi \to 0$) gauge. One must therefore choose to regulate these divergences in a suitable manner or to work in a gauge such as (2.5) or the 't Hooft gauge proper

$$L_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu + e\xi v_i \phi_i)^2 \qquad (2.17)$$

where $v_i = \varepsilon_{ij}\langle 0|\phi_j|0\rangle$ with $\varepsilon_{12}=-1$ and $\varepsilon_{21}=1$, in which the divergences do not appear.

There are two objections to the choice (2.17), which are mooted in the paper by Aitchison and Fraser. They are, firstly, that it is inconvenient and possibly inconsistent to introduce into the Lagrangian a quantity that one is supposed to be calculating and, secondly, that it is impossible to derive the requisite Nielsen identity for this case. This clashes with the complementary work of Fukuda and Kugo [24] who used an old-fashioned approach to the derivation of Nielsen identities and who saw no impediment to the use of 't Hooft gauges. The derivation of the Nielsen identity also removes the objections of Taylor and Dolan and Jackiw [17],[18] that are encapsulated in equation (2.6). A correct choice of vacuum will remove the gauge dependent term that is present at the tree level.

As a prelude to Nielsen's derivation we sketch the approach of Fukuda and Kugo, which bears a similar relationship to his work as the original derivation of the Ward-Takahashi identities does to a derivation based on B.R.S invariance [25].

We use ϕ_i to stand for all the fields, with the index i being a generic one that labels all their attributes. We shall also use Greek letters for group indices. The generating functional is written as

$$\exp(\frac{i}{\hbar}W_f[J])=\int[D\phi]\Delta_f[\phi]\exp(\frac{i}{\hbar}(S[\phi]-\frac{1}{2}F^2+J\phi)) \qquad (2.18)$$

where $\Delta_f[\phi]=\mathrm{Det}M_f$ and M_f is defined by observing the behaviour of the gauge fixing F under a gauge transformation

$$\begin{aligned} F_\alpha(\phi)&\to F_\alpha(\phi)+M_{\alpha\beta}u_\beta \\ \phi_i&\to\phi_i+(\Lambda_i^\alpha+t^\alpha{}_{ij}\phi_j)u_\alpha \end{aligned} \qquad (2.19)$$

where $\Lambda_i^\alpha=\frac{1}{g}\partial_\mu\delta^4(x-x_\alpha)$ for the gauge fields and is zero otherwise.

If we now subject (2.18) to the transformation [7]

$$\phi_i\to\phi_i+(\Lambda_i^\alpha+t^\alpha{}_{ij}\phi_j)(M_f^{-1})_{\alpha\beta}u^\beta \qquad (2.20)$$

we find

$$[-F_\alpha(\frac{\hbar}{i}\frac{\delta}{\delta J})+J_i(\Lambda_i^\beta+t^\beta{}_{ij}(\frac{\hbar}{i}\frac{\delta}{\delta J}))(M_f^{-1})_{\beta\alpha}]\exp(\frac{i}{\hbar}W[J]) = 0 \qquad (2.21)$$

which allows us to estimate the change in $W_f[J]$ under a variation in the gauge fixing from F to $F+\Delta F$.

$$\exp(\tfrac{i}{\hbar}W_{f+\delta f}[J]) - \exp(\tfrac{i}{\hbar}W_f[J]) =$$
$$\int[D\phi]\Delta_f[\phi]\exp(\tfrac{i}{\hbar}(S[\phi]-\tfrac{1}{2}F^2 + J\phi))\tfrac{i}{\hbar}J_i(\Lambda_i^\beta + t^\beta{}_{ij}\phi_j)\cdot [M_f^{-1}]_{\beta\alpha}\cdot\Delta F_\beta(\phi) \tag{2.22}$$

We can write this as [7]

$$W_{f+\delta f}[J] - W_f[J] = J_i\langle f_i\rangle \tag{2.23}$$

where $f_i = (\Lambda_i^\alpha + t^\alpha{}_{ij}\phi_j)[M_f^{-1}]_{\alpha\beta}\,\Delta F_\beta(\phi)$ and the expectation value is defined in the usual manner. This may be translated into the change in the effective action by using the Legendre transforms

$$\begin{aligned}\Gamma_{f+\delta f}[\phi^{f+\delta f}] &= W_{f+\delta f}[J] - J\,\phi^{f+\delta f}\\ \Gamma_{f+\delta f}[\phi^f] &= W_{f+\delta f}[J] - J\,\phi^f\\ \Gamma_f[\phi^f] &= W_f[J] - J\,\phi^f\end{aligned} \tag{2.24}$$

to give

$$\Gamma_{f+\delta f}[\phi^{f+\delta f}] - \Gamma_f[\phi^f] = -\tfrac{i}{\hbar}\,J_i\langle\phi_i f_j\rangle J_j \tag{2.25}$$

If we work to order ΔF it is easier to consider

$$\Gamma_{f+\delta f}[\phi^f] - \Gamma_f[\phi^f] = J_i\langle f_i\rangle \tag{2.26}$$

which is the precursor of the Nielsen identity. The differential version of (2.26) for an Abelian Higgs model with gauge fixing of the form (2.5) is

$$\frac{\delta\Gamma}{\delta\xi} = \langle(J_1\phi_2 - J_2\phi_1 - \tfrac{1}{e}\,\partial.J)_x\left(\frac{e}{-\nabla^2 - ev\phi_1}\right)_{xy}(\partial.A - v\phi_2)_y\rangle \tag{2.27}$$

If we now choose $J_1 = \frac{\partial V}{\partial \phi}$, $J_2 = J_\mu = 0$ we find

$$\xi\frac{\partial\Gamma}{\partial\xi} + \frac{1}{2}\frac{\partial V}{\partial\phi} \langle \phi_{2x} \left(\frac{-e}{-\nabla^2 - ev\phi_1}\right)_{xy} (\partial.A - v\phi_2)_y \rangle = 0 \qquad (2.28)$$

which is the effective action version of the first Nielsen identity.

Having seen in outline how the Fukuda and Kugo approach works we now move on to consider Nielsen's work itself, closely following the approch of Aitchison and Fraser in [23]. Instead of keeping the determinantal factor in the path integral we introduce the usual ghost fields and make use of the B.R.S. invariance of the gauge fixed Lagrangian to derive our results. The essence of the method is to append an extra term to the action in the generating functional and to arrange for the B.R.S variation of this new term to be equal (to within a term that vanishes when one considers the effective potential) to $\xi\frac{\partial L}{\partial\xi}$. This will become the $\xi\frac{\partial V}{\partial\xi}$ term in the Nielsen identity. We first give a general discussion before examining the case of a 't Hooft gauge fixing in detail. Consider the generating functional

$$\tilde{Z}_k[J] = \int[D\phi] \exp \frac{i}{h}(S[\phi] + J_i\phi_i + K_iQ_i + hO) \qquad (2.29)$$

where we have included the standard source terms for the fields and a source term for the B.R.S. variations (or charges) that are non-linear in the fields ($Q_i = \delta\phi_i$, where δ is the B.R.S variation). The subscript k denotes their presence and the twiddle denotes the presence of the operator O.

The KQ term does not affect the B.R.S. invariance since, for the fields that require it, $\delta^2\phi = 0$. It is inserted in order to linearize the resulting identities. We also try to pick an operator O such that $\delta O = \bar{O} = \xi\frac{\partial L}{\partial\xi}$ to give the required term in the Nielsen identities. This works because $\frac{\partial\Gamma}{\partial\xi} = \frac{\partial W}{\partial\xi}$ (see appendix A, courtesy of [23])

Bearing this in mind we now rewrite (2.29) explicitly for an Abelian Higgs model to get

$$\tilde{Z}_k[J] = \int[DA_\mu][D\psi][D\psi^*][D\phi_i]\ \exp\frac{i}{\hbar}(\tilde{S}_k) \qquad (2.30)$$

where the action is given by

$$\tilde{S}_k = \int d^4x(L + K_i e\psi\varepsilon_{ij}\phi_j + J_i\phi_i + J_\mu A^\mu + \eta^*\psi + \psi^*\eta + hO) \qquad (2.31)$$

and we use the full gauge-fixed Lagrangian for L

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - e\varepsilon_{ij}(\partial_\mu\phi_i)\phi_j A^\mu$$
$$+\frac{1}{2}e^2A^2\phi^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{1}{2\xi}F^2 + \psi^*M\psi \qquad (2.32)$$

The B.R.S. transforms for this Lagrangian are given by

$$\delta A_\mu = \varepsilon\partial_\mu\psi \quad , \quad \delta\psi = 0 \quad , \quad \delta\phi_i = \varepsilon e\varepsilon_{ij}\psi\phi_j$$
$$\delta\psi^* = -\frac{\varepsilon}{\xi}F \qquad (2.33)$$

where ε is an anticommuting parameter and ψ and $\bar{\psi}^*$ are the ghost and anti-ghost respectively. We now carry out a B.R.S. transform on $\tilde{Z}_k$ to find that

$$\int d^4z[DA_\mu]\ldots[D\phi_i](J_\mu\partial^\mu\psi - \eta\tfrac{1}{\xi}F(A_\mu,\phi_i) + j_i e\psi\varepsilon_{ij}\phi_j)\tilde{Z}_k = \tag{2.34}$$

$$= -i\int d^4z[DA_\mu]\ldots[D\phi_i]h(z)\delta O(z)\exp\tfrac{i}{\hbar}(\tilde{S}_k)$$

We can rewrite this as an operator identity on $\tilde{Z}_k$.

$$\int d^4z(J_\mu\partial^\mu \frac{\delta}{\delta\eta^*} - \eta\frac{1}{\xi}\, F(\frac{\delta}{\delta J_\mu}, \frac{\delta}{\delta J_i}) + J_i\, \frac{\delta}{\delta K_i})\tilde{Z}_k = \text{R.H.S.} \tag{2.35}$$

We now perform a Legendre transform to obtain an identity on the effective action.

$$\tilde{\Gamma}_k = \tilde{W}_k - \int d^4x(J_\mu\bar{A}^\mu + \eta^*\bar{\psi} + \bar{\psi}^*\eta + J_i\bar{\phi}_i)$$

where (2.36)

$$\tilde{W}_k = -ih\ \ln\ \tilde{Z}_k \quad \text{and} \quad \frac{\delta\tilde{W}_k}{\delta J_\mu} = \bar{A}_\mu \quad \text{etc.}$$

This gives

$$\int d^4z(-\frac{\delta\tilde{\Gamma}_k}{\delta\bar{A}_\mu}\,\partial_\mu\bar{\psi} + \frac{\delta\tilde{\Gamma}_k}{\delta\bar{\psi}^*}\,\frac{1}{\xi}\,F(\bar{A}_\mu,\phi_i) - \frac{\delta\tilde{\Gamma}_k}{\delta\bar{\phi}_i}\,\frac{\delta\tilde{\Gamma}_k}{\delta K_i}) = \tag{2.37}$$

$$-\frac{1}{\tilde{Z}_k}\int d^4z[DA_\mu]\ldots[D\phi_i]h(z)\delta O(z)\exp\tfrac{i}{\hbar}(\tilde{S}_k)$$

To obtain the effective action precursor of the Nielsen identity we differentiate (2.37) w.r.t. h and then set h to zero, which removes the tildes. We note that $\frac{\delta\Gamma}{\delta h} = \Gamma(O)$, where we denote an insertion of the operator O in Γ by $\Gamma(O)$. We find (dropping k)

$$\int d^4x\int d^4z(-\frac{\delta\Gamma(O(x))}{\delta\bar{A}_\mu(z)}\,\partial_\mu\bar{\psi}(z) + \frac{\delta\Gamma(O(x))}{\delta\bar{\psi}^*(z)}\,\frac{1}{\xi}\,F - \frac{\delta\Gamma(O(x))}{\delta\bar{\phi}_i(z)}\,\frac{\delta\Gamma}{\delta K_i(z)} \tag{2.38}$$

$$-\frac{\delta\Gamma(O(x))}{\delta K_i(z)}\,\frac{\delta\Gamma}{\delta\bar{\phi}_i(z)}) = \frac{1}{Z}\int d^4x\int[DA_\mu]..[D\phi_i]\delta O(x)\exp\tfrac{i}{\hbar}S$$

The l.h.s. of (2.38) is already in the required form, so it only remains to transform the r.h.s. We shall find in explicit calculations that

$$\frac{1}{Z}\int d^4x\int[DA_\mu]..[D\phi_i]\delta O(x)\exp\frac{i}{\hbar}S = \frac{\partial W}{\partial\xi}\xi + \frac{1}{2}\int d^4x\eta(x)\frac{\delta W}{\delta\eta} \quad (2.39)$$

From appendix A this equals

$$\xi\frac{\partial\Gamma}{\partial\xi} + \frac{1}{2}\int d^4x \frac{\delta\Gamma}{\delta\bar{\phi}^*(x)}\bar{\phi}^*(x) \quad (2.40)$$

The second term, of the form $\frac{1}{2}\eta\bar{\phi}^*$, will vanish when we consider the effective potential. If we specialize to constant $\bar{\phi}$ and set the other classical fields to zero, we find

$$\xi\frac{\partial V}{\partial\xi} - \int d^4x \frac{\delta\Gamma(O(x))}{\delta K_i(0)}\frac{\partial V}{\partial\bar{\phi}} = \frac{-F(0,\bar{\phi})}{\Omega}\int d^4z\int d^4x \frac{\delta\Gamma(O(x))}{\delta\bar{\phi}^*(z)} \quad (2.41)$$

Thus, provided that $F(0,\bar{\phi})=0$, a point that we discuss in detail in the next section, we find

$$\xi\frac{\partial V}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial V}{\partial\bar{\phi}} = 0 \quad (2.12a)$$

with $C(\bar{\phi},\xi)$ given by the expression

$$-\int d^4x \frac{\delta\Gamma(O(x))}{\delta K_j(0)} \quad (2.42)$$

To obtain the Nielsen identity for the mass of the Higgs particle we observe that the masses are given by the poles of the propagator or, alternatively, by the zeros of the inverse propagator

$$\frac{\delta^2\Gamma}{\delta\bar{\phi}_i(x)\delta\bar{\phi}_j(y)} = iG^{-1}(x-y)_{ij} \tag{2.43}$$

We are interested in the mass of the physical Higgs particle which does not couple to the gauge boson. For our Abelian Higgs model we can span the space in which the Higgs fields live by the vectors $\eta=(1,0)$ and $e=(0,1)$. If we arrange for the η direction to be the physical Higgs (see the next section) we can then split G^{-1} into

$$[G^{-1}]_{ij} = G^{-1}_{phys.}\eta_i\eta_j + (\delta_{ij} - \eta_i\eta_j).\text{rest} \tag{2.44}$$

The physical Higgs mass is then given by

$$G^{-1}_{phys}(p^2=m^2) = 0 \tag{2.45}$$

We have, in the Abelian Higgs model,

$$G^{-1}(x-y) = \left.\frac{\delta^2\Gamma}{\delta\bar{\phi}_1(x)\delta\bar{\phi}_1(y)}\right|_{\bar{\phi}=\bar{\phi}_0} \tag{2.46}$$

We now proceed by differentiating (2.37) twice w.r.t. ϕ_1 to get the following equation, where we have dropped terms that vanish in the effective potential.

$$\xi\frac{\partial}{\partial\xi}\frac{\delta^2\Gamma}{\delta\bar{\phi}_1(y)\delta\bar{\phi}_1(w)} = \int d^4x d^4z\Big(\frac{\delta\Gamma}{\delta\bar{\phi}_1}\frac{\delta^3\Gamma(O(x))}{\delta\bar{\phi}_1\delta K_1(z)\delta\phi_1(y)\delta\phi_1(w)}$$

$$+\frac{\delta^2\Gamma}{\delta\bar{\phi}_1(x)\delta\bar{\phi}_1(w)}\frac{\delta^2\Gamma(O(x))}{\delta K_1(z)\delta\phi_1(y)} + \frac{\delta^2\Gamma}{\delta\phi_1(x)\delta\phi_1(y)}\frac{\delta^2\Gamma(O(x))}{\delta K_1(z)\delta\phi_1(w)} \tag{2.47}$$

$$+\frac{\delta^3\Gamma}{\delta\phi_1(x)\delta\phi_1(y)\delta\phi_1(w)}\frac{\delta\Gamma(O(x))}{\delta K_1(z)}\Big)$$

If we evaluate the expression at $\frac{\delta\Gamma}{\delta\bar{\phi}} = 0$ and use translational invariance to see that the second terms in the products on the second line can be written as functions of x-y and x-w we can write (2.47) as

$$\left(\xi\frac{\partial}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial}{\partial\bar{\phi}}\right) G^{-1}{}_{phys}\Big|_{\bar{\phi}=\bar{\phi}_0} =$$

$$\int d^4x\int d^4z[G^{-1}{}_{phys}(x-w)F(z,x-y) + G^{-1}{}_{phys}(x-y)F(z,x-w)] \tag{2.48}$$

If we Fourier transform this we find, using the convolution theorem,

$$\left(\xi\frac{\partial}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial}{\partial\bar{\phi}}\right) G^{-1}{}_{phys}(p^2)=2G^{-1}{}_{phys}(p^2)\int d^4r\int d^4z\exp(ip.r)F(z,r) \tag{2.49}$$

From this we see that if G^{-1} vanishes at a particular value of p^2 it will also do so under the transformations $\xi\to\xi+\delta\xi$ and $\bar{\phi}\to\bar{\phi}+(C(\bar{\phi},\xi)/\xi)\delta\xi$. We therefore have the second Nielsen identity (2.12b).

2.3 THE NIELSEN IDENTITY IN THE 't HOOFT GAUGE

We now give a careful derivation of the effective potential Nielsen identity in the 't Hooft gauge proper, following in detail the prescription of the preceding section for the Abelian Higgs model. The gauge fixing and ghost part of the Lagrangian is now given by (c.f. 2.32)

$$-\frac{1}{2\xi}(\partial_\mu A^\mu + e\xi v_i\phi_i)^2 + \partial_\mu\psi^*\partial^\mu\psi - e^2\psi^*\psi\varepsilon_{ij}\xi v_i\phi_j \qquad (2.50)$$

and the B.R.S. transforms are given by

$$\delta A_\mu = \varepsilon\partial_\mu\psi \quad , \quad \delta\psi=0 \quad , \quad \delta\phi_i = \varepsilon e\varepsilon_{ij}\psi\phi_j$$
$$\delta\psi^* = \frac{\varepsilon}{\xi}(\partial_\mu A^\mu + e\xi v_i\phi_i) \qquad (2.51)$$

We now choose the operator O that was introduced in the last section to be

$$O = \frac{1}{2}\psi^*(\partial_\mu A^\mu - e\xi v_i\phi_i) \qquad (2.52)$$

(the change in sign inside the brackets w.r.t. the gauge fixing term is correct, see section 2.7)
The B.R.S. transform of the operator O is given by

$$\delta O = \varepsilon\left[\frac{1}{2\xi}((\partial_\mu A^\mu)^2 - e^2\xi^2(v_i\phi_i)^2) + \frac{1}{2}\psi^*\nabla^2\psi - \frac{1}{2}e^2\xi\psi^*\psi\varepsilon_{ij}v_i\phi_j\right] \qquad (2.53)$$

If we use the equation of motion for the ghost field $\frac{\delta L}{\delta\psi^*} = \eta$ we can write this as

$$\delta O = \varepsilon\left[\frac{1}{2\xi}\left((\partial_\mu A^\mu)^2 - e^2\xi^2(v_i\phi_i)^2\right) + \frac{1}{2}\,\psi^*\eta - e^2\xi\psi^*\psi\varepsilon_{ij}v_i\phi_j\right] \tag{2.54}$$

We now compare this with

$$\xi\frac{\partial L}{\partial\xi} = \left[\frac{1}{2\xi}\left(\partial_\mu A^\mu)^2 - e^2\xi^2(v_i\phi_i)^2\right) - e^2\xi\psi^*\psi\varepsilon_{ij}v_i\phi_j\right] \tag{2.55}$$

and we see that, to within a term that vanishes when we consider the effective potential $(\frac{1}{2}\psi^*\eta)$, $\xi\frac{\delta L}{\delta\xi}$ has been expressed in terms of a B.R.S. transformed operator. We now follow the argument of the preceding section through to obtain the first Nielsen identity

$$\xi\frac{\partial V(\bar{\phi}_i,\xi)}{\partial\xi} - \int d^4x\frac{\delta\Gamma(O(x))}{\delta K_j(0)}\,\frac{\partial V}{\partial\bar{\phi}_j} = -\frac{e\xi v_i\bar{\phi}_i}{\Omega}\int d^4x d^4z\frac{\delta\Gamma(O(x))}{\delta\bar{\psi}^*(z)} \tag{2.56}$$

We note at this point that (2.56) does not quite have the form required of the first Nielsen identity because of the inhomogeneous term on the r.h.s. We prefer to take a different view from that espoused in both [23] and [24] in dealing with this. These say that, having predetermined a direction of symmetry breaking, one then chooses the direction of v to be perpendicular to this in ϕ-space, thus eliminating the inhomogeneous term. To us this seems inconsistent, because one is supposed to be calculating the minima using the effective potential calculated from the gauge-fixed Lagrangian, so one should not prejudge the direction of symmetry breaking. The correct approach is to regard the homogeneity of equation (2.56) (i.e. the condition $v_i\bar{\phi}_i=0$) as a condition on the solutions of the equation $\frac{\partial V}{\partial \bar{\phi}} = 0$. For instance at the tree level with an 't Hooft style gauge fixing we find two possible solutions for the minima of the classical potential. They are

$$\bar{\phi}_{i0} = \varepsilon_{ij}\frac{v_j}{\|v\|}\left(\frac{6m^2}{\lambda}\right)^{1/2}$$

$$\bar{\phi}_{i0} = \frac{v_i}{\|v\|}\left(\frac{6}{\lambda}(m^2-\xi e^2 v^2)\right)^{1/2} \tag{2.57}$$

The second of these solutions is gauge variant, but if we impose the condition $v_i\bar{\phi}_i= 0$ on the solutions we find that it is not allowed. To reiterate, the homogeneity of the Nielsen identity ensures the absence of possible gauge dependent minima and is to be regarded as a constraint on the allowed directions of symmetry breaking in ϕ-space.

The $v_i\bar{\phi}_i= 0$ condition does not just arise in connection with the Nielsen identities; it is involved in the breaking of B.R.S. symmetry, which is hardly surprising if we consider the means of derivation of the Nielsen identity.

If we consider, after de Wit [26], the Noether current for a B.R.S. transform, the so-called Taylor-Slavnov current

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu\phi_i)}\,\delta\phi_i + \frac{\partial L}{\partial(\partial_\mu A_\sigma)}\,\delta A_\sigma + \frac{\partial L}{\partial(\partial_\mu\phi^*)}\,\delta\phi^* \tag{2.58}$$

We find that

$$\partial_\mu \langle T(J^\mu(x)\ X\ )\rangle = i\ \langle T(\partial_\mu \phi \frac{\delta}{\delta A_\mu} - e\phi\varepsilon_{ij}\phi_j \frac{\delta}{\delta\phi_i} - F\frac{\delta}{\delta\phi^*})\ X\ \rangle \quad (2.59)$$

where X is any combination of fields and F is the gauge fixing term. If we now choose X to be $\psi^*(0)$ we find

$$\partial_\mu \langle T(J_\mu(x)\ \psi^*(0)\ )\rangle = -i\ \langle T\ F\ \frac{\delta\psi^*(0)}{\delta\psi^*(x)}\rangle$$

$$= -i\delta(x)\langle F\rangle \quad (2.60)$$

The integrated form of (2.60) will give the standard Slavnov identity iff the r.h.s. is zero, i.e. $\langle F\rangle=0$. In an 't Hooft like gauge this may be written as

$$\langle \partial_\mu A^\mu + e\xi v_i \phi_i\rangle = 0 \quad (2.61)$$

or $v_i \bar{\phi}_{i0} = 0$, which is slightly weaker than the $v_i \bar{\phi}_i = 0$ condition that follows from the Nielsen identities (but is still sufficient to maintain the gauge invariance of physical quantities).

If we bear the preceding discussion in mind we can choose $v_i = v e_i \quad e=(0,1)$, which gives $\bar{\phi}_i = \bar{\phi}\eta_i \quad \eta_i=(1,0)$. We can thus consider the effective potential as a function of $\bar{\phi}_1$ only.

$$\xi\frac{\partial V(\bar{\phi},\xi)}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial V}{\partial\bar{\phi}} = 0 \quad (2.12a)$$

2.4 ONE LOOP VERIFICATION OF THE EFFECTIVE POTENTIAL IDENTITY

Verifying the Nielsen identities to one-loop order is not only an instructive exercise in its own right but it also throws light (in the mass identity) on the viability of the 't Hooft gauge proper (one with $v_i = \varepsilon_{ij}\langle 0|\phi_j|0\rangle = \varepsilon_{ij}\bar{\Phi}_{j0}$) in effective potential calculations. Our results agree with those of Fukuda and Kugo who stated (rather opaquely) that the correct way to handle the 't Hooft gauge proper was to exclude the $\bar{\Phi}_{j0}$ from any of the differentiations on the effective potential, avoiding any possible circular calculations. We shall belabour this point in our verification of the mass identity to one-loop order. All our calculations are carried out in the framework of a loop expansion, using Jackiw's field-shifting method for calculating V (and its relatives such as $C(\bar{\phi},\xi)$). We expand (2.12a) order by order in h, noting that $C(\bar{\phi},\xi)$ receives its first contributions at one-loop order (as one may see from carrying out the standard field rescaling $\phi \to \sqrt{\hbar}\phi$ in (2.63))

$$\xi\frac{\partial V^1}{\partial \xi} + C^1(\bar{\phi},\xi)\frac{\partial V^0}{\partial \bar{\phi}} = 0 \tag{2.62}$$

where the superscripts denote the order in h. The function $C(\bar{\phi},\xi)$ is calculated from

$$C(\bar{\phi},\xi) = -ih\int d^4x\langle 0|T(\frac{i}{\hbar})^2[-\frac{1}{2}\phi^*(\partial_\mu A^\mu(x) - e\xi v_i\Phi_i(x)) \cdot$$
$$e\phi(0)\phi(0)\exp(\frac{i}{\hbar}S_{eff}[\bar{\phi},\Phi])]|0\rangle \tag{2.63}$$

where we have used Φ for the quantum field and $\bar{\phi}$ for the c-number shift. The effective Lagrangian in the shifted theory is given by

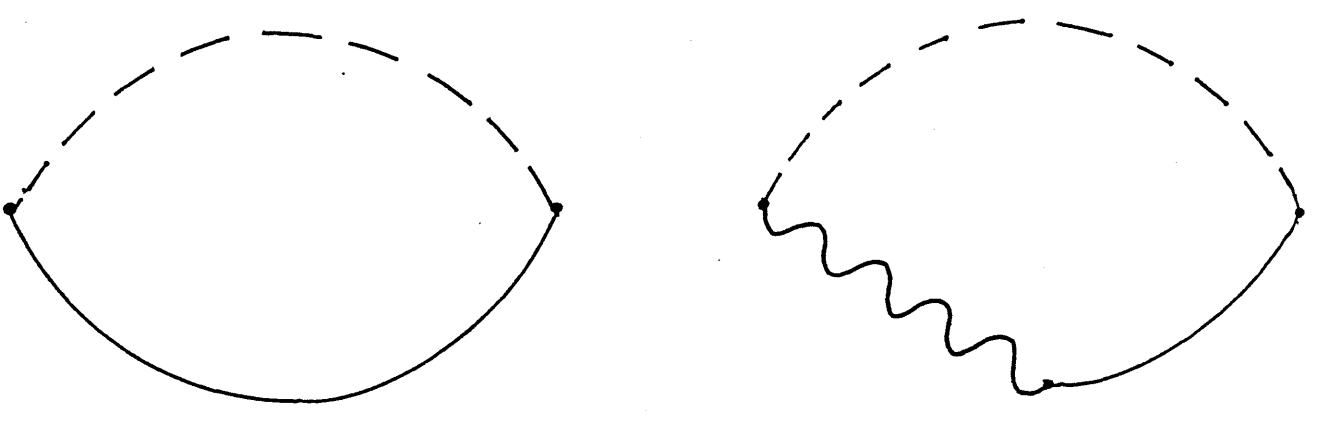
$$S_{eff}[\bar{\phi},\Phi] = S[\Phi+\bar{\phi}] - S[\bar{\phi}] - \int d^4x\phi(x)\frac{\partial L}{\partial\Phi}\Bigg|_{\Phi=\bar{\phi}} \qquad (2.64)$$

The one-loop term comes from dropping the interactions from the exponent and is

$$C^1(\bar{\phi},\xi) = i\hbar\int d^4x\langle 0\left|T(\frac{i}{\hbar})^2\frac{1}{2}\ \phi^*(x)(\partial_\mu A^\mu(x) - e\xi v\ \Phi_2(x))e\phi(0)\Phi_2(0)\right|0\rangle \qquad (2.65)$$

To evaluate this we use Wick's theorem to contract out the fields, giving a ghost propagator multiplying either a Higgs propagator or a mixed gauge boson/Higgs propagator. These give the graphs

(Fig.4) The graphs for $C^1(\phi,\xi)$



which we evaluate using the vertices and propagators in appendix B to get

$$C^1(\bar{\phi},\xi) = \frac{ie}{2}\int d^4k \; \frac{i}{k^2+e^2\xi v\bar{\phi}} \; \frac{-ik^2(\xi\bar{\phi} + \xi v)}{D_n}$$

$$\frac{-ie^2\xi v}{(k^2+e^2\xi v\bar{\phi})} \; \frac{i(k^2-\xi e^2\bar{\phi}^2)}{D_n} \tag{2.66}$$

where

$$D_n = k^4 - k^2(m_2^2 - 2e^2\xi v\bar{\phi}) + e^2\phi^2(e^2\xi^2 v^2+\xi m_2^2)$$

and (2.67)

$$m_2^2 = \frac{1}{6}\lambda\bar{\phi}^2 - m^2 \quad , \quad m_1^2 = \frac{1}{2}\lambda\bar{\phi}^2 - m^2$$

We can simplify (2.66) to

$$C^1(\bar{\phi},\xi) = \frac{ie^2\xi}{2}\int d^4k \; \frac{(2v+\bar{\phi})k^2 - e^2\xi v\bar{\phi}^2}{(k^2+e^2\xi v\bar{\phi})\, D_n} \tag{2.68}$$

We shall save ourselves the labour of evaluating (2.68) explicitly by considering the integral expresion for the one-loop effective potential, which in our gauge is

$$V^1(\bar{\phi},\xi) = i\int d^4k\Big(\ln(k^2+e^2\xi v\bar{\phi}) - \frac{3}{2}\ln(-k^2+e^2\bar{\phi}^2)$$

$$-\frac{1}{2}\ln(k^2-m_1^2) - \frac{1}{2}\ln D_n \Big) \tag{2.69}$$

Differentiating w.r.t ξ we find

$$\xi\frac{\partial V^1}{\partial\xi} = i\int d^4k \left(\frac{e^2\xi v\bar{\phi}}{k^2+e^2\xi v\bar{\phi}} - \frac{1}{2}\,\frac{2k^2e^2\xi v\bar{\phi} + 2e^4\bar{\phi}^2\xi^2v^2 + e^2\bar{\phi}^2m_2^2\xi}{D_n}\right) \tag{2.70}$$

We can simplify this to

$$\xi\frac{\partial V^1}{\partial \xi} = -\frac{ie\xi\bar{\phi}m_2^2}{2}\int d^4k\,\frac{(2v+\bar{\phi})k^2 - v\xi e^2\bar{\phi}^2}{(k^2+e^2\xi v\bar{\phi})\,D_n} \tag{2.71}$$

To complete our evaluation of the identity we note that

$$\frac{\partial V^0}{\partial \bar{\phi}} = m_2^2\,\bar{\phi} \tag{2.72}$$

So, multiplying this by (2.68) and adding the result to (2.71) we obtain the required one-loop Nielsen identity.

$$\xi\frac{\partial V^1}{\partial \xi} + C^1(\bar{\phi},\xi)\frac{\partial V^0}{\partial \bar{\phi}} = 0 \tag{2.73}$$

2.5 ONE-LOOP VERIFICATION OF THE MASS IDENTITY IN THE 't HOOFT GAUGE PROPER

In verifying this identity we must exercise some care in the treatment of the $\varepsilon_{ij}\bar{\phi}_{j0}$ term that we introduce in the gauge fixing. We note that the argument of the effective potential $\bar{\phi}$, which we generate by the field shift used to perform the calculation is NOT the same as the $\bar{\phi}_0$ in the gauge fixing except at the minima of the potential. We should therefore not include the $\bar{\phi}_0$ in the functional dependence when we are performing differentiations such as $\frac{\partial V}{\partial \bar{\phi}}$ or $\frac{\partial^2 V}{\partial \bar{\phi}^2}$. Earlier calculations by Weinberg vindicate this viewpoint [20], as will our following calculation of the mass identity. Weinberg evaluated one-loop tadpole graphs in a 't Hooft gauge without distinguishing between $\bar{\phi}$ and $\bar{\phi}_0$ and found that

$$T = -i\frac{\partial V}{\partial \bar{\phi}}\bigg|_{\bar{\phi}=\bar{\phi}_0} - m^2 e\bar{\phi}_0 \int d^4k \frac{1}{k^2(\xi k^2 - e^2\bar{\phi}_0^2)} \tag{2.74}$$

where his potential was equivalent to (2.69) with v set equal to $-\bar{\phi}$ before the differentiation was performed. From the presence of the non-derivative term in (2.74) Weinberg concluded that the effective potential was valid in the 'tHooft gauge only when $\xi \to \infty$ However, one can obtain a derivative expression for the one-loop tadpole graphs by differentiating (2.69) w.r.t. $\bar{\phi}$ and then setting $v = -\bar{\phi}_0$, which is consistent with our interpretation of the gauge fixing v_i being $\varepsilon_{ij}\phi_{j0}$.

If we exercise a similar degree of care in the verification of the mass identity at one-loop level we obtain the correct result. Equation (2.45) which defines the pole of the propagator gives the physical mass as

$$m^2 - m_1^2 - \Sigma(m^2) = 0 \tag{2.75}$$

where $\Sigma(p^2)$ is the sum of the physical Higgs self energy graphs at momentum p^2. As we are working to one-loop level we can write this as

$$m^2 - m_1^2 - \Sigma(m_1^2) + O(\hbar^2) = 0 \tag{2.76}$$

Now, following [23], we choose $\lambda \sim O(e^4)$ to simplify our calculations (one makes a similar choice in the Coleman-Weinberg model). This allows us to expand Σ^1 to order $e^2\lambda$, which is where gauge dependence first enters in the one-loop effective potential. We rewrite (2.76) as

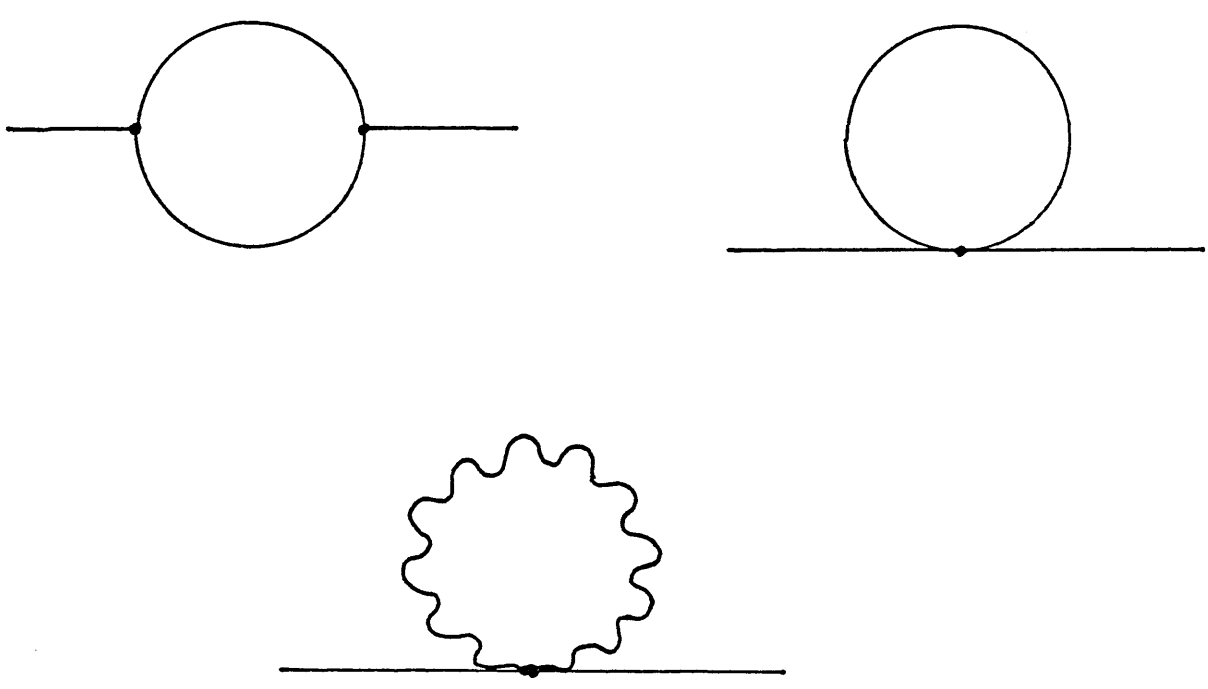
$$m^2 = m_1^2 + \Sigma^1(o) + m_1^2 \left.\frac{\partial \Sigma^1(p^2)}{\partial p^2}\right|_{p^2=0} \tag{2.77}$$

where the superscript on the the Σ denotes the loop order, which is not included on the m for notational convenience. We can rewrite this as

$$m^2 = \frac{\partial^2 V^1}{\partial \bar{\phi}^2} + m_1^2 \left.\frac{\partial \Sigma(p^2)}{\partial p^2}\right|_{p^2=0} . \tag{2.78}$$

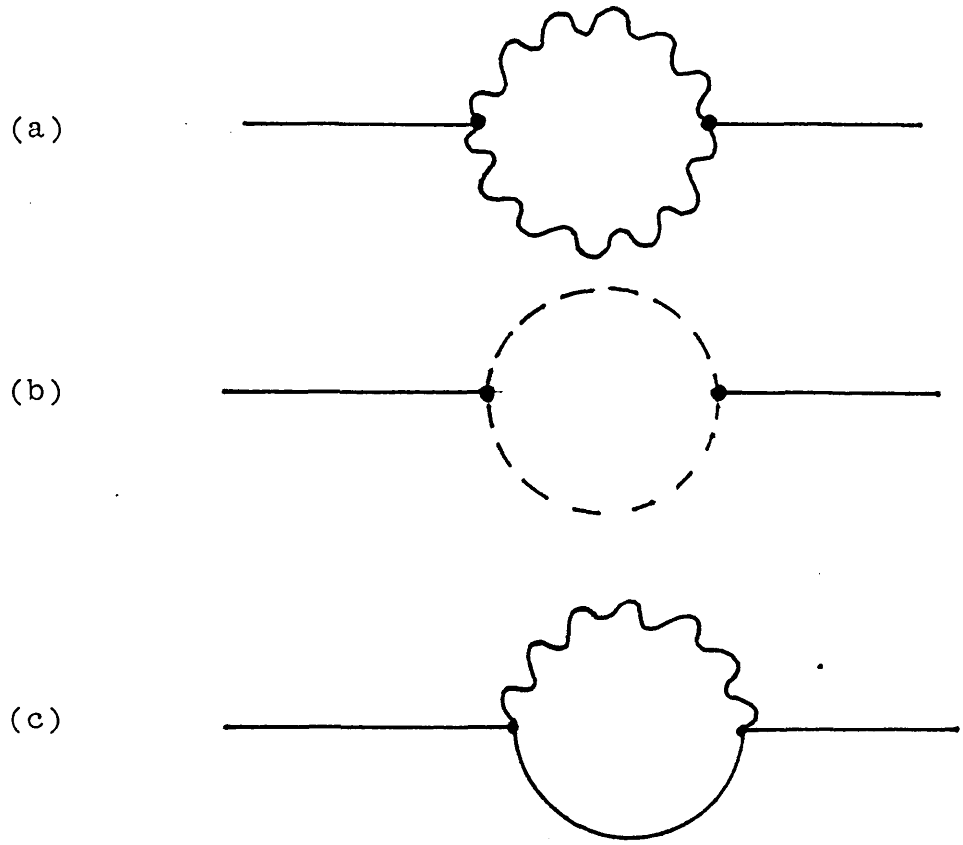
Choosing $v_i = \varepsilon_{ij}\phi_{j0}$ eliminates the Higgs-photon mixing graphs and the expansion scheme eliminates those shown below from a calculation of Σ.

(Fig.5) Graphs that are not necessary



We are left with the three graphs shown below in order to evaluate Σ^1.

(Fig.6) Graphs to be calculated



We evaluate them using dimensional regularization and the MS subtraction scheme to get the following results

$$\frac{\partial \Sigma^1(p^2)}{\partial p^2} = \frac{e^2\xi}{16\pi^2}\left(\frac{3}{\xi - 1}\ln \xi - \frac{11}{6}\right) \quad \text{fig 6a} \tag{2.79}$$

$$\frac{\partial \Sigma^1(p^2)}{\partial p^2} = \frac{e^2\xi}{16\pi^2}\left(\frac{1}{6}\right) \quad \text{fig 6b} \tag{2.80}$$

$$\frac{\partial \Sigma^1(p^2)}{\partial p^2} = \frac{e^2\xi}{16\pi^2}\left(\frac{2}{3} + \ln\frac{e^2\xi\bar{\phi}^2}{M^2} - \frac{3}{\xi - 1}\ln\xi\right) \quad \text{fig 6c} \tag{2.81}$$

where, in the above, M is the arbitrary renormalization mass

As we are working to $O(\hbar)$ we can evaluate Σ^1 at the classical minimum of the potential, which we shall call c. We note that

$$m_1^2(c) = \frac{1}{3}\lambda c^2 \qquad \text{and} \qquad \left.\frac{\partial m_1^2}{\partial \bar{\phi}}\right|_{\bar{\phi}=c} = \lambda c \qquad (2.82)$$

Adding up the terms from the graphs evaluated at c gives

$$\left.\frac{\partial \Sigma^1(p^2)}{\partial p^2}\right|_{\bar{\phi}=c} = \frac{e^2\xi}{16\pi^2}\left(\ln \frac{e^2\xi c^2}{M^2} - 1\right) \qquad (2.83)$$

We now carefully evaluate the $\frac{\partial^2 V}{\partial \bar{\phi}^2}$ term in (2.78), taking care not to include the $\bar{\phi}_0$ term from the gauge fixing in the differentiation.

$$\left.\frac{\partial^2 V^1}{\partial \bar{\phi}^2}\right|_{\bar{\phi}=c} = \frac{e^2\xi\lambda c^2}{32\pi^2}\left(\frac{1}{3}\ln \frac{e^2\xi c^2}{M^2} - \frac{1}{3}\right) \qquad (2.84)$$

If we multiply (2.83) by (2.82a) and add the result to (2.84) we find, from (2.78), that

$$\left.\xi\frac{\partial m^2}{\partial \xi}\right|_{\bar{\phi}=c} = \frac{e^2\xi\lambda c^2}{32\pi^2}\left(\ln \frac{e^2\xi c^2}{M^2}\right) \qquad (2.85)$$

In our expansion scheme and gauge we find that

$$\left.C^1(\bar{\phi},\xi)\right|_{\bar{\phi}=c} = -\frac{e^2\xi c}{32\pi^2}\ln \frac{e^2\xi c^2}{M^2} \qquad (2.86)$$

So from (2.82b), (2.85) and (2.86) we can see that the second Nielsen identity is verified.

2.6 THE GAUGE BOSON MASS IDENTITY.

For the sake of completeness we give a brief discussion of Nielsen identity for the gauge boson mass, which is defined as the zero of the transverse part of

$$\frac{\delta^2\Gamma}{\delta\bar{A}_\mu(y)\delta\bar{A}_\nu(w)} \tag{2.87}$$

We proceed in a similar manner to the derivation of the Nielsen identity for the Higgs boson mass, starting with (2.37) and differentating it twice w.r.t. A. This gives, after dropping terms which vanish because of the conservation of ghost number or the setting of the classical fields to zero or because we demand $v_i\bar{\Phi}_i = 0$

$$\xi\frac{\partial}{\partial\xi}\frac{\delta^2\Gamma}{\delta\bar{A}_\mu(y)\delta\bar{A}_\nu(w)} = \int d^4x d^4z \frac{\delta^2\Gamma}{\delta\bar{A}_\mu(y)\delta\bar{\phi}_1(v)}\frac{\delta^2\Gamma(O(x))}{\delta K(z)^1\delta A_\nu(w)}$$

$$+\frac{\delta^2\Gamma}{\delta\bar{A}_\nu(w)\delta\bar{\phi}_1(v)}\frac{\delta^2\Gamma(O(x))}{\delta K_1(z)\delta\bar{A}_\mu(y)} + \frac{\delta^3\Gamma}{\delta\bar{A}_\mu(y)\delta\bar{A}_\nu(w)\delta\bar{\phi}_1(v)}\frac{\delta\Gamma(O(x))}{\delta K_1(z)} \tag{2.88}$$

$$+\frac{\delta\Gamma}{\delta\bar{\phi}_1}\frac{\delta^3\Gamma(O(x))}{\delta K_1(z)\delta A_\mu(y)\delta A_\nu(w)}$$

The third term on the r.h.s. may be taken across to give the required identity and the last term will vanish at the minimum of the effective potential. To dispose of the remaining first two terms we note that the mixed Higgs/gauge boson inverse propagators

$$\frac{\delta^2\Gamma}{\delta\bar{A}_\mu(w)\delta\bar{\phi}_1(v)} \tag{2.89}$$

and its companion term are proportional to k_μ and k_ν respectively in momentum space. As we are interested in the transverse part of the gauge boson propagator, applying the transverse projector, $g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$, will give zero from these terms. (2.88) will then become

$$\left(\xi\frac{\partial}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial}{\partial\bar{\phi}} \right)\frac{\delta^2\Gamma}{\delta A_\mu(y)\delta A_\nu(w)}\Bigg|_{\text{trans.}} = 0 \tag{2.90}$$

which is the effective action precursor of the relevant Nielsen identity. In a similar manner to the mass identity for the Higgs scalar we argue that if (2.90) is zero at some m^2 it will remain zero under the transformations $\xi\to\xi + \delta\xi$ and $\bar{\phi} \to \bar{\phi} + (C(\bar{\phi},\xi)/\xi)\delta\xi$ so we obtain

$$\left(\xi\frac{\partial}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial}{\partial\bar{\phi}} \right) m^2_{\text{vector}} = 0 \tag{2.91}$$

The proof does not quite follow the lines of that for the Higgs boson mass, where the vanishing of the inverse propagator on shell disposes of the terms similar to (2.89). In the present case the mixed propagator has a gauge dependent pole, so we cannot use this argument.

2.7 AN ALTERNATIVE DERIVATION OF THE NIELSEN IDENTITIES

We present in this section an alternative derivation of the Nielsen identities which is based on some work by Piguet and Sibold [27]. It does not raise any new points but it does remove the element of guesswork in choosing the operator O and it sets the Nielsen identities in their proper context in a group of identities that control the gauge dependence of the generating functionals in a gauge theory. The trick in this case is to enlarge the B.R.S. transforms to act on the gauge parameter as well and to use the auxiliary field method of gauge-fixing that was first promoted by Kugo and Ojima [28]. The auxiliary field approach has the great advantage (to mathematicians!) of making all the B.R.S. transformations nilpotent, which suggests a possible geometric interpretation for the B.R.S. variation in terms of exterior derivations.

Piguet and Sibold introduce a B.R.S. variation on the gauge parameter ξ

$$\delta\xi = \varepsilon\,\chi \quad , \quad \chi \text{ Grassmannian} \tag{2.92}$$

and show that under this extended set of B.R.S. transformations the Slavnov-Taylor identity becomes (in a Yang-Mills theory)

$$S(\Gamma) + \chi \frac{\partial\Gamma}{\partial\xi} = 0 \tag{2.93}$$

where S is the usual Slavnov operator

$$S(\Gamma) = \mathrm{Tr}\int d^4x \left[\frac{\delta\Gamma}{\delta\rho^{\mu}} \frac{\delta\Gamma}{\delta\bar{A}_{\mu}} + \frac{\delta\Gamma}{\delta\sigma} \frac{\delta\Gamma}{\delta\bar{\phi}} + \bar{B} \frac{\delta\Gamma}{\delta\bar{\phi}^{*}} \right] \tag{2.94}$$

In the above σ and ρ^μ are sources for the B.R.S. variations of ϕ and A_μ respectively and B is the auxiliary field which allows us to write the gauge-fixing part of the Lagrangian in the form

$$L_{gf} = \frac{\xi}{2} B^2 + B(\partial_\mu A^\mu) + \frac{1}{2} \chi\phi^* B + \partial_\mu \phi^* D^\mu \phi \qquad (2.95)$$

One can easily check that this is invariant under the B.R.S. variations

$$\delta A_\mu = \varepsilon\partial_\mu\phi \quad , \quad \delta\phi = 0 \quad , \quad \delta\phi^* = \varepsilon B \quad , \quad \delta\xi = \varepsilon\chi \qquad (2.96)$$

We also note that in the absence of the last transformation and the corresponding $\frac{1}{2}\chi\phi^* B$ term in the Lagrangian, eliminating B by Gaussian integration in the path integral (it has no kinetic terms) gives the standard Fermi gauge-fixing.

$$L_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \partial_\mu\phi^* D^\mu\phi \qquad (2.97)$$

The effective action precursor of the Nielsen identity is then simply obtained by differentating (2.93) w.r.t. χ and then setting $\chi = 0$ (taking care with the sign of anticommuting quantities in the process)

$$S\left(-\frac{\partial\Gamma}{\partial\chi}\right) + \frac{\partial\Gamma}{\partial\xi} = 0 \qquad (2.98)$$

We can now recast the gauge-fixing term in our Abelian Higgs model into a form which is similar to (2.95) and which is invariant under the new B.R.S. variation introduced in (2.96). We then perform explicitly the steps leading to (2.98). The gauge fixing is transmuted into

$$L_{gf} = \frac{\xi}{2} B^2 + B(\partial_\mu A^\mu + e\xi v_i \phi_i) + \partial_\mu \psi^* D^\mu \psi$$

(2.99)

$$-e^2 \xi \psi^* \psi \varepsilon_{ij} v_i \phi_j + \frac{1}{2} \chi \psi^* B + e\chi \psi^* v_i \phi_i$$

If we denote,as before the insertion of an operator O in Γ by $\Gamma(O)$ we find

$$\frac{\partial \Gamma}{\partial \chi} = \Gamma\left(\frac{1}{2} \psi^* B + e\psi^* v_i \phi_i \right) \qquad (2.100)$$

If we integrate out the auxiliary field we will replace it by its minimum value in the exponent of the path integral, which is given by the solution to its equation of motion

$$\frac{\partial L}{\partial B} = \xi B + (\partial_\mu A^\mu + e\xi v_i \phi_i) + \frac{1}{2} \chi \psi^* = 0 \qquad (2.101)$$

Using (2.101) we can substitute for B in (2.100) and we find that at $\chi = 0$

$$\frac{\partial \Gamma}{\partial \chi} = \Gamma\left(-\frac{1}{2\xi} (\partial_\mu A^\mu - e\xi v_i \phi_i) \right) = \Gamma(P) \qquad (2.102)$$

We see that the operator insertion is precisely $\frac{1}{\xi}$ times the operator O that we had to construct in the previous derivation

The B.R.S. variations are given by

$$\delta\psi^* = \varepsilon B \ , \ \delta\psi = 0 \ , \ \delta B = 0 \ , \ \delta A_\mu = \varepsilon \partial_\mu \psi \ , \ \delta \phi_i = \varepsilon e \varepsilon_{ij} \psi \phi_j$$

(2.103)

$$\delta\xi = \varepsilon\chi$$

We note that, as advertised, $\delta^2 = 0$ on all the fields. Eq. (2.93) becomes

$$\int d^4x\left(\frac{\delta\Gamma}{\delta\bar{A}_\mu(x)} \partial_\mu\bar{\psi}(x) + \frac{\delta\Gamma}{\delta K_i(x)} \frac{\delta\Gamma}{\delta\bar{\phi}_i(x)} + \bar{B}(x) \frac{\delta\Gamma}{\delta\bar{\psi}^*(x)} \right)$$

$$+ \chi\frac{\delta\Gamma}{\delta\xi} = 0 \tag{2.104}$$

We can substitute for B in this from (2.101) to get

$$\int d^4x\left(\frac{\delta\Gamma}{\delta\bar{A}_\mu(x)} \partial_\mu\bar{\psi}(x) + \frac{\delta\Gamma}{\delta K_i(x)} \frac{\delta\Gamma}{\delta\bar{\phi}_i(x)} \right.$$

$$\left. -\frac{1}{\xi} \left(\partial_\mu\bar{A}^\mu(x) + e\xi v_i\bar{\phi}_i(x)\right)\frac{\delta\Gamma}{\delta\bar{\psi}^*(x)} - \frac{1}{2\xi} \chi \bar{\psi}^* \frac{\delta\Gamma}{\delta\bar{\psi}^*(x)} \right)$$

$$+ \chi\frac{\partial\Gamma}{\partial\xi} = 0 \tag{2.105}$$

We now differentiate w.r.t. χ and set $\chi = 0$ to get the equivalent of (2.98)

$$\int d^4x d^4z\left(\frac{\delta\Gamma(P(z))}{\delta\bar{A}_\mu(x)} \partial_\mu\bar{\psi}(x) + \frac{\delta\Gamma(P(z))}{\delta K_i(x)}\frac{\delta\Gamma}{\delta\bar{\phi}_i(x)} + \frac{\delta\Gamma}{\delta K_i(x)} \frac{\delta\Gamma(P(z))}{\delta\bar{\phi}_i(x)} \right.$$

$$\left. - \frac{1}{\xi} \left(\partial_\mu\bar{A}^\mu(x) + e\xi v_i\bar{\phi}_i(x)\right) \frac{\delta\Gamma(P(z))}{\delta\bar{\psi}^*(x)} - \frac{1}{2\xi} \frac{\delta\Gamma(P(z))}{\delta\bar{\psi}^*(x)} \bar{\psi}^*(x)\right)$$

$$- \frac{\partial\Gamma}{\partial\xi} = 0 \tag{2.106}$$

Multiplying through by ξ, specializing to x-independent $\bar{\phi}$ and setting the other fields to zero gives

$$\xi\frac{\partial V}{\partial \xi} - \int d^4x \; \frac{\delta\Gamma(O(x))}{\delta K_i(0)} \; \frac{\partial V}{\partial \bar{\phi}_i(x)} = -\frac{e\xi v_i \bar{\phi}_i \xi}{\Omega} \int d^4x d^4z \; \frac{\delta\Gamma(O(x))}{\delta\phi^*(z)} \tag{2.107}$$

which is identical to (2.56) in our previous derivation of the Nielsen identities.

We could also demonstrate the independence of physical quantities from the gauge-fixing vector v_i using these techniques. In the older approach we would have found that

$$v_i\frac{\partial L}{\partial v_i} = -ev_i(\partial_\mu A^\mu + e\xi v_i\phi_i) - e^2\phi^*\phi\varepsilon_{ij}v_i\phi_j \tag{2.108}$$

which can be generated from the B.R.S. transform of the operator

$$O = e\phi^*\phi_i\xi v_i \tag{2.109}$$

In our approach we could introduce the new B.R.S. transform

$$\delta v_i = \varepsilon\rho_i \tag{2.110}$$

where the ρ_i have the appropriate group properties but are anticommuting objects. We would modify the gauge fixing term to be invariant under this new transformation

$$L_{gf} = \frac{\xi}{2} B^2 + B(\partial_\mu A^\mu + e\xi v_i\phi_i) + \partial_\mu\phi^*\partial^\mu\phi - e\psi^*\xi\rho_i\phi_i$$
$$-e^2\xi\phi^*\phi\varepsilon_{ij}v_i\phi_j \tag{2.111}$$

The Nielsen identity would then be, symbolically

$$S\left(-\frac{\partial\Gamma}{\partial\rho_i}\right) + \frac{\partial\Gamma}{\partial v_i} = 0 \tag{2.112}$$

where

$$-\frac{\partial\Gamma}{\partial\rho_i} = \Gamma(\ e\psi^*\xi\phi_i) \tag{2.113}$$

and corresponds, to within a multiplying factor of v_i, to the result in the previous approach.

As a footnote we observe that the use of extended B.R.S. identities clears up a small puzzle in the choice of the operator O. With a gauge-fixing of the form

$$-\frac{1}{2\xi}\,(\partial_\mu A^\mu + ev_i\phi_i) \tag{2.114}$$

O is given by

$$O = -\frac{1}{2}\,\psi^*(\partial_\mu A^\mu + ev_i\phi_i) \tag{2.115}$$

whereas, with a gauge-fixing of the form

$$-\frac{1}{2\xi}\,(\partial_\mu A^\mu + e\xi v_i\phi_i) \tag{2.116}$$

it is given by

$$O = -\frac{1}{2}\,\psi^*(\partial_\mu A^\mu - e\xi v_i\phi_i) \tag{2.117}$$

The difference in sign can be seen to arise from the extra $\frac{1}{2}\chi\psi^* v_i\phi_i$ term in (2.99) which preserves B.R.S. invariance under

$\delta\xi \to \varepsilon\chi$.

We have made no attempt at discussing the renormalizability of the extended actions. However, Piguet and Sibold give an extensive discussion of the renormalizability properties of a Yang-Mills theory with the extra $\frac{1}{2}\chi\phi^*B$ term in the gauge fixing and show that the theory is essentially unchanged in comparison with the usual case. Similar considerations would apply to the extended actions in our Abelian Higgs model.

2.8 NIELSEN IDENTITIES IN THE AXIAL GAUGE

In order to further demonstrate the utility of the Piguet-Sibold approach we shall give a derivation of the Nielsen identities in the axial gauge. There are two points to bear in mind for such a derivation: the first is that there are various possible ways of implementing an axial gauge in the path integral with gauge fixings of the form.

$$\frac{\xi}{2} B^a (f^{ab})^{-1} B^b + B^a (n_\mu A^{\mu a}) \tag{2.118}$$

The most obvious choice with $f^{ab} = -1$ is pathological for $\xi \neq 0$ because the propagator is of the form

$$\frac{1}{ip^2} \left[g_{\mu\nu} - \frac{n_\mu p_\nu + n_\nu p_\mu}{(n_\mu p^\mu)} + \frac{p_\mu p_\nu n^2}{(n_\mu p^\mu)^2} + \xi \frac{p_\nu p_\nu p^2}{(n_\mu p^\mu)^2} \right] \tag{2.119}$$

which goes as O(1) for large p^2. This invalidates the usual power counting arguments used in demonstrating the renormalizability of the theory [29]. Another possible choice is the planar gauge with

$$f^{ab} = \left(\frac{\partial^2}{n^2} \right)^{ab} \tag{2.120}$$

The propagator takes a particularly simple form in this gauge

$$\frac{1}{ip^2} \left[g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{n_\mu p^\mu} \right] \tag{2.121}$$

We note that the auxiliary field method of gauge-fixing used in (2.118) would obviate the need for Nielsen-Kallosh ghosts if one were working in a background field formalism where f^{ab} would be equal to $\frac{D^2(A)^{ab}}{n^2}$, where D(A) is the background field covariant derivative. If we had written the gauge fixing in the usual form

$$- \frac{1}{2\xi} (n_\mu a^{\mu a}) f^{ab} (n_\mu a^{\mu a}) \tag{2.122}$$

where a_μ is the quantum field, we would have needed to add the terms

$$\omega^a f^{ab} \omega^b + \frac{1}{2} \gamma^a f^{ab} \gamma^b \tag{2.123}$$

where ω is a complex anticommuting ghost and γ is a real commuting ghost to reproduce the required $\sqrt{\det f}$ factor that ensures the invariance of the measure. This is automatically reproduced upon the integration of the B fields.

The second point to note is that the axial gauges, like the Fermi gauges, will suffer from infrared divergences. However, it is argued by Thompson and Yu [29] that these may be regulated in the context of dimensional regularization and do not affect the veracity of the identities. With the preceding provisos in mind we write the gauge-fixing term in the planar gauge for an Abelian Higgs model as

$$L_{gf} = \frac{\xi}{2} B \left(\frac{n^2}{\partial^2}\right) B + B(n_\mu A^\mu) - \phi^*(n_\mu \partial^\mu)\phi \tag{2.124}$$

We now choose to extend the set of B.R.S. transforms acting on our Lagrangian by including

$$\delta\xi = \varepsilon\chi \qquad \text{and} \qquad \delta n_\mu = \varepsilon\rho_\mu \tag{2.125}$$

where both χ and the components of ρ_μ are anticommuting objects. In order to maintain the B.R.S. invariance of the gauge fixing term under these new transformations we must add the following

$$\frac{1}{2}\chi\phi^*\left(\frac{n^2}{\partial^2}\right)B - \phi^*(\rho_\mu A^\mu) - \xi\phi^*\left(\frac{n_\mu\rho^\mu}{\partial^2}\right)B \tag{2.126}$$

We now consider separately the change in Γ under a change in ξ and n_μ. In the first case we have the equation

$$S\left(\frac{\partial\Gamma}{\partial\chi}\right) + \frac{\partial\Gamma}{\partial\xi} = 0 \qquad \text{with } \delta n_\mu = 0 \tag{2.127}$$

and in the second we have

$$S\left(\frac{\partial\Gamma}{\partial\rho_\mu}\right) + \frac{\partial\Gamma}{\partial n_\mu} = 0 \qquad \text{with} \quad \delta\xi = 0 \tag{2.128}$$

We find that the corresponding operator insertions are given by

$$\left.\frac{\partial\Gamma}{\partial\chi}\right|_{\chi=0} = \Gamma\left(\frac{1}{2}\,\phi^*\left(\frac{n^2}{\partial^2}\right) B\right) \tag{2.129}$$

and

$$\left.\frac{\partial\Gamma}{\partial\rho_\mu}\right|_{\rho_\mu=0} = \Gamma\left(\phi^* A^\mu + \xi\phi^*\left(\frac{n^\mu}{\partial^2}\right)B\right) \tag{2.130}$$

If we now eliminate the auxiliary field B in the time-honoured manner we can see that

$$\left.\frac{\partial\Gamma}{\partial\chi}\right|_{\chi=0} = \Gamma\left(-\frac{1}{2\xi}\phi^*(n_\mu A^\mu)\right) \tag{2.131}$$

and

$$\left.\frac{\partial\Gamma}{\partial\rho_\mu}\right|_{\rho_\mu=0} = \Gamma\left(\phi^*\left(g^{\mu\sigma} - \frac{n^\mu n^\sigma}{n^2}\right) A^\sigma\right) \tag{2.132}$$

This allows us to obtain the Nielsen identity in its usual form for ξ

$$\xi\frac{\partial V}{\partial\xi} + C(\bar{\phi},\xi)\frac{\partial V}{\partial\bar{\phi}} = 0 \tag{2.12a}$$

C is now given by

$$C(\bar{\phi},\xi) = i\hbar\int d^4x\langle 0\left|T(\frac{i}{\hbar})^2[-\frac{1}{2}\psi^*(x)(n_\mu A^\mu(x))e\psi(0)\phi_2(0).\right.$$

$$\exp(\frac{i}{\hbar}S_{eff}[\Phi,\phi])]\left|0\right\rangle \tag{2.133}$$

In a similar manner we can write the equation for n_μ dependence

$$\frac{\partial V}{\partial n_\mu} + D^\mu(\bar{\phi},\xi)\frac{\partial V}{\partial\bar{\phi}} = 0 \tag{2.134}$$

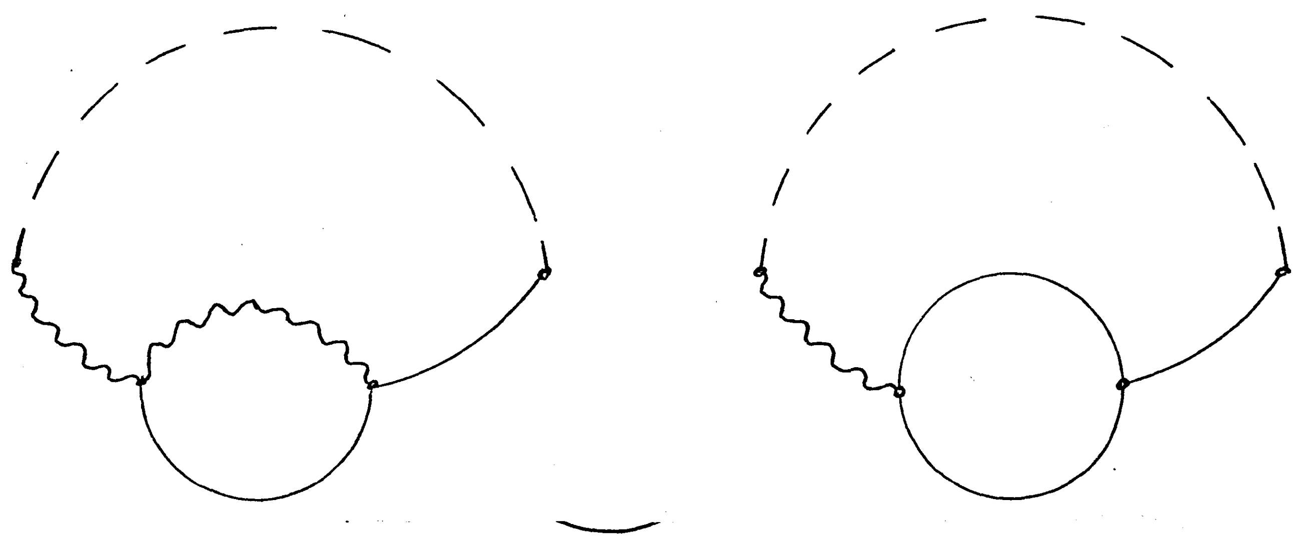
with D^μ given by

$$D^\mu(\bar{\phi},\xi) = i\hbar\int d^4x\langle 0\left|T(\frac{i}{\hbar})^2[\qquad (g^{\mu\sigma}-\frac{n^\mu n^\sigma}{n^2})\psi^*(x)A^\sigma(x)e\psi(0)\phi_2(0)\right.$$

$$\exp(\frac{i}{\hbar}S_{eff}[\Phi,\phi])\left|0\right\rangle \tag{2.135}$$

We note that C and D^μ do not receive any contributions at the one-loop level because there is no mixed A/ϕ propagator in the planar gauge. The two-loop contribution is given by a graph of the form

(Fig.7) Graph for $C^2(\bar{\phi},\xi)$



We thus see that, formally at any rate, deriving Nielsen identities presents no problems in a planar gauge, though explicit calculations will have to overcome infrared divergences.

2.9 A NIELSEN IDENTITY FOR AN EFFECTIVE ACTION CONTAINING COMPOSITE OPERATORS

As well as the usual effective action which has a scalar field as its argument one might also like to consider a generalization which depends not only on $\bar{\phi}(x)$ but also on a possible expectation value for $T\phi(x)\phi(y)$ which we shall call $G(x,y)$. The physical vacuum for such an action would then be given by

$$\frac{\delta\Gamma[\bar{\phi},G]}{\delta\bar{\phi}} = \frac{\delta\Gamma[\bar{\phi},G]}{\delta G} = 0 \qquad (2.136)$$

Such a formalism is useful in the study of dynamical symmetry breaking, where an object such as $T\phi(x)\phi(y)$ may develop a vacuum expectation value [30].

To develop the formalism, consider (after [30]) the following action with a compound source.

$$\tilde{Z}_k[J,L] = \int[D\phi]\exp\frac{i}{\hbar}(S[\phi] + J_i\phi_i + K_iQ_i + \frac{1}{2}\phi_iL_{ij}\phi_j) \qquad (2.137)$$

where we have again lapsed into condensed notation

More explicitly

$$\frac{1}{2}\,\phi_i L_{ij}\phi_j = \frac{1}{2}\int d^4x d^4y\;\phi_i(x)\;L_{ij}(x,y)\;\phi_j(y) \tag{2.138}$$

We now define $\widetilde{W}_k[J,L]$ by

$$\widetilde{W}_k[J,L] = -ih\;\ln\;\widetilde{Z}_k[J,L] \tag{2.139}$$

and the classical field $\bar{\phi}$ as

$$\frac{\delta\widetilde{W}_k[J,L]}{\delta J_i(x)} = \bar{\phi}_i(x) \tag{2.140}$$

We also have the new relation

$$\frac{\delta\widetilde{W}_k[J,L]}{\delta L_{ij}(x,y)} = \frac{1}{2}\left[\bar{\phi}_i(x)\bar{\phi}_j(y) + \hbar G_{ij}(x,y)\right] \tag{2.141}$$

To obtain the effective action we perform the Legendre transform

$$\widetilde{\Gamma}_k[\bar{\phi},G] = \widetilde{W}_k[J,L] - \int d^4x\;\bar{\phi}_i(x)J_i(x) - \frac{1}{2}\int d^4x d^4y\,\bar{\phi}_i(x)L_{ij}(x,y)\bar{\phi}_j(y)$$
$$-\frac{1}{2}\int d^4x d^4y\;G_{ij}(x,y)L_{ji}(y,x) \tag{2.142}$$

We observe from the above definition that

$$\frac{\delta\widetilde{\Gamma}_k}{\delta\bar{\phi}_i(x)} = -J_i(x) - \int d^4x\;L_{ij}(x,y)\bar{\phi}_j(y) \tag{2.143}$$

and

$$\frac{\delta\widetilde{\Gamma}_k}{\delta G_{ij}(x,y)} = -\frac{1}{2}\hbar\;L_{ij}(x,y) \tag{2.144}$$

The standard effective action corresponds to $\Gamma[\bar{\phi},G]$ evaluated at $L = 0$, or equivalently $\Gamma[\bar{\phi},G]$ for the values of G which satisfy $\frac{\delta\Gamma}{\delta G} = 0$. One may show that $\Gamma[\bar{\phi},G]$ is the generating functional for 2-particle-irreducible diagrams with lines representing $hG(x,y)$ [31] and it can be computed in a similar manner to $\Gamma[\bar{\phi}]$ by considering the vacuum graphs for the shifted Lagrangian, but this time retaining only the 2PI graphs.

We now consider our canonical example, the Abelian Higgs model, for which we can write the generating functional more explicitly as

$$\tilde{Z}_k[J,L] = \int[DA_\mu][D\phi][D\phi^*][D\phi_i] \exp \frac{i}{\hbar} \int d^4x[L + K_i e\phi\varepsilon_{ij}\phi_j + J_\mu A^\mu$$
$$+ J_i\phi_i + \phi^*\eta + \eta^*\phi + hO + \frac{1}{2}\int d^4y\phi_i(x)L_{ij}(x,y)\phi_j(y)] \tag{2.145}$$

We have again chosen O so that $\delta O = \xi\frac{\partial L}{\partial\xi}$. We now carry out a B.R.S. transform and use the invariance of the measure and gauge-fixed Lagrangian to obtain

$$\int d^4x\int[D\phi](J_\mu\partial^\mu\phi - \eta\, F(A_\mu,\phi) + J_i e\phi\varepsilon_{ij}\phi_j + h\delta O$$
$$+ \frac{1}{2}\int d^4y e\phi(x)\varepsilon_{il}\phi_l(x)L_{ij}(x,y)\phi_j(y)$$
$$+ \frac{1}{2}\int d^4y e\phi(y)\varepsilon_{jl}\phi_i(x)L_{ij}(x,y)\phi_l(y))\exp\frac{i}{h} S_{eff}[\Phi,\phi] = 0 \tag{2.146}$$

where F is the gauge-fixing term.

To express the terms containing L more succinctly we write them as

$$e\int d^4x \left(\varepsilon_{il}\psi(x)\phi_l(x)f_i^1(x)\right) + e\int d^4y \left(\varepsilon_{jl}\psi(y)\phi_l(y)f_j^1(y)\right) \tag{2.147}$$

where

$$f_i^1(x) = \int d^4y\, L_{ij}(x,y)\, \phi_j(y)$$
$$f_j^2(y) = \int d^4x\, L_{ij}(x,y)\, \phi_i(x) \tag{2.148}$$

We now note that the invariance of the source term containing L_{ij} under the exchange of x and y implies that $L_{ij}(x,y) = L_{ji}(y,x)$, so f^1 and f^2 are equal, as are both the terms in (2.147). We can thus rewrite (2.146) as

$$\int d^4x \int [D\phi]\ \Big(J_\mu \partial^\mu \psi - \frac{\eta}{\xi} F(A_\mu,\phi) + J_i e\psi\varepsilon_{ij}\phi_j + h\delta O$$
$$+ \int d^4y e\psi(x)\varepsilon_{il}\phi_l(x)L_{ij}(x,y)\phi_j(y)\Big)\exp\frac{i}{\hbar}\, S_{eff}[\Phi,\phi] = 0 \tag{2.149}$$

This may be transformed, as in the standard case, to an operator identity acting on $\tilde{Z}_k$

$$\int d^4x\Big(J^\mu\partial_\mu \frac{\delta}{\delta\eta} - \frac{\eta}{\xi} F\Big(\frac{\delta}{\delta J_\mu}, \frac{\delta}{\delta J_i}\Big) + J_i \frac{\delta}{\delta K_i} + \int d^4y \frac{\delta}{\delta K_i} L_{ij} \frac{\delta}{\delta J_j}\Big)$$
$$\cdot\, \tilde{Z}_k[J,L] = -i\int d^4x \int [D\phi] h(x)\delta O(x)\exp\frac{i}{\hbar}\, S_{eff}[\Phi,\phi] \tag{2.150}$$

We now proceed to write this as an identity on the 2PI (in ϕ only) generating functional $\Gamma[\bar{\phi},G]$. The fields other than ϕ are Legendre transformed in the standard manner but (2.140) and (2.141) apply for ϕ. We find that (2.150) is transformed into

$$\int d^4x \left(- \frac{\delta\tilde{\Gamma}_k}{\delta\bar{A}_\mu} \partial_\mu\bar{\phi} - \frac{\delta\tilde{\Gamma}_k}{\delta\bar{\phi}^*} F(\bar{A}_\mu, \bar{\phi}_i) - \frac{\delta\tilde{\Gamma}_k}{\delta\bar{\phi}_i} \frac{\delta\tilde{\Gamma}_k}{\delta K_i} - 2\hbar\int d^4y\, d^4z \frac{\delta\Gamma}{\delta G_j(x,y)} D^{-1}_{jk}(y-z) \frac{\delta^2\Gamma}{\delta K_j(x)\delta\phi_k(z)} \right) \tag{2.151}$$

$$\frac{1}{\tilde{Z}_k} \int d^4x \int [D\phi]\ h\delta O(x) \exp\frac{i}{\hbar} S_{eff}[\Phi,\phi]$$

where we have dropped the x dependence in the l.h.s. for the sake of compactness.

Remarkably, this is of exactly the same form as the Nielsen identity on the standard effective potential, so we can follow the proof through to obtain an identity on the effective potential which is defined by

$$V(\bar{\phi},G)\int d^4x = - \Gamma[\bar{\phi},G] \Big|_{\text{translationally invariant}} \tag{2.152}$$

Subject to the usual constraint on the solutions arising from the gauge-fixing we find (SYMBOLICALLY)

$$\xi\frac{\partial V}{\partial \xi} + D(\bar{\phi},G,\xi)\frac{\partial V}{\partial \bar{\phi}} + \int E(\bar{\phi},G,\xi)\frac{\delta V}{\delta G} = 0 \tag{2.153}$$

where V, D and E are to be calculated from 2PI vacuum graphs with internal lines set equal to $\hbar G$.

The identity states that under a change in the gauge parameter $\xi \to \xi + \delta\xi$ the value of V is preserved by the compensating change $\bar{\phi} \to \bar{\phi} + (D(\bar{\phi},\xi)/\xi)\delta\xi$ and that there is a similar change in G.

2.10 CONCLUSIONS

What lessons may be learnt from the preceding somewhat involved calculations? The most important is that it is, indeed, possible to use the 't Hooft gauge in effective potential calculations. There is no breakdown in the Nielsen identities and there is no circular argument involved in introducing ϕ_{j0} into the gauge-fixing. This is reassuring, as the gauge is not only convenient from the point of view of doing calculations because of the absence of mixed gauge boson Higgs propagators, but is also supposed to be a finite ξ version of the Unitary gauge in which only physical particles appear (the $\xi\to\infty$ limit of the 't Hooft gauge is the unitary gauge). Our discussion shows that this limit is not singular.

The other main conclusion is that the Nielsen identity sits in a broad class of identities derived by Piguet and Sibold which govern the gauge dependence of the generating functionals and we have used this relationship in our alternative derivation. As a final observation we note that the $v_i\bar{\phi}_i = 0$ condition that emerges from the Nielsen identity will be of some use in our investigations of convexity.

CHAPTER 3 CONVEXITY

3.1 CONVEXITY: INTRODUCTION

If we consider the example of a spontaneously broken $\lambda\phi^4$ theory we find that the one-loop effective potential is given by (using dimensional regularization and the MS subtraction scheme)

$$V = \frac{1}{64\pi^2}(U'')^2\left(\ln\frac{U''}{M^2} - \frac{1}{2}\right) \tag{3.1}$$

where U" is the second derivative of the classical potential

$$U'' = -m^2 + \frac{\lambda}{2}\phi^2 \tag{3.2}$$

The logarithmic term in (3.1) becomes complex inside the points of inflection of the classical potential, $\phi = \pm\surd(2m^2/\lambda)$. This behaviour is not simply a one-loop artifact; at higher orders one still finds $\ln^n U''$ terms (see, for example, Appendix F). The effective potential measures an energy density, so such complexity would seem to signal some kind of instability in our choice of the vacuum configuration. One might also observe that

the one-loop effective potential which we have calculated is non-convex in form (one of the standard definitions of convexity for a function $f(\phi)$ is that $f''(\phi) > 0$ for all ϕ, which is manifestly incorrect for our potential). However, one can show quite easily, for a Euclidean field theory, that the effective potential should be convex [31]. If we consider the expansion for W[J] with a slowly varying source

$$W[J] = \int d^4x \left[-\xi(J) + \frac{1}{2} X(J)(\partial_\mu J)^2 + \dots \right] \tag{3.3}$$

we have, for a constant source

$$W[J] \approx - \Omega\xi(J) \tag{3.4}$$

Now $Z[J] = \exp\frac{1}{\hbar} W[J]$, so we find that, in the constant source limit

$$Z(J) = \exp\left(-\frac{1}{\hbar}\Omega\xi(J)\right) \tag{3.5}$$

where

$$Z(J) = \int[D\phi]\exp\left(-\frac{1}{\hbar}\left(S[\phi] - J\int\phi\right)\right) \tag{3.6}$$

If we now calculate $\frac{\partial^2\xi}{\partial J^2}$, using (3.5) and (3.6) we find that it is equal to

$$-\frac{1}{\Omega\hbar}\left(\langle (\int\phi)^2 \rangle - \langle\int\phi\rangle^2 \right) \tag{3.8}$$

where we have defined $\langle O \rangle$ by

$$\frac{\int[D\phi]\; O \exp-\frac{1}{\hbar}\left(S[\phi] - J\int\phi\right)}{\int[D\phi]\exp-\frac{1}{\hbar}\left(S[\phi] - J\int\phi\right)} \tag{3.9}$$

For a well-defined measure, which we do have for a Euclidean field theory (but not for a Minkowski one), the term in the brackets in (3.6) is a variance and hence a positive quantity.

Hence $\frac{\partial^2 \xi}{\partial J^2} < 0$. If we now recall the functional relation

$$\int d^4z \frac{\delta^2 \Gamma}{\delta\bar{\phi}(x)\delta\bar{\phi}(z)} \frac{\delta^2 W}{\delta J(z)\delta J(y)} = - \delta^4(x-y) \quad (3.8)$$

whose constant source limit is given by

$$\frac{\partial^2 \xi}{\partial J^2} \frac{\partial^2 V}{\partial \bar{\phi}^2} = -1 \quad (3.9)$$

we can see that $\frac{\partial^2 V}{\partial \bar{\phi}^2} > 0$, or that V is convex. This is obviously at odds with our one-loop result.

A further hint of possible trouble with the formalism comes from observing some of the general properties of Legendre transforms [11]. For simplicity consider the function of a single variable f(x). Its Legendre transform is defined to be

$$f^1(y) = f(x) - xy \qquad \text{where} \qquad y = \frac{df}{dx} \quad (3.10)$$

Graphically, this means that we are plotting the intersection of the tangent to f against its slope. The transform is involutive, provided that the original function is convex

$$(f^1)^1(x) = f(x) \quad (3.11)$$

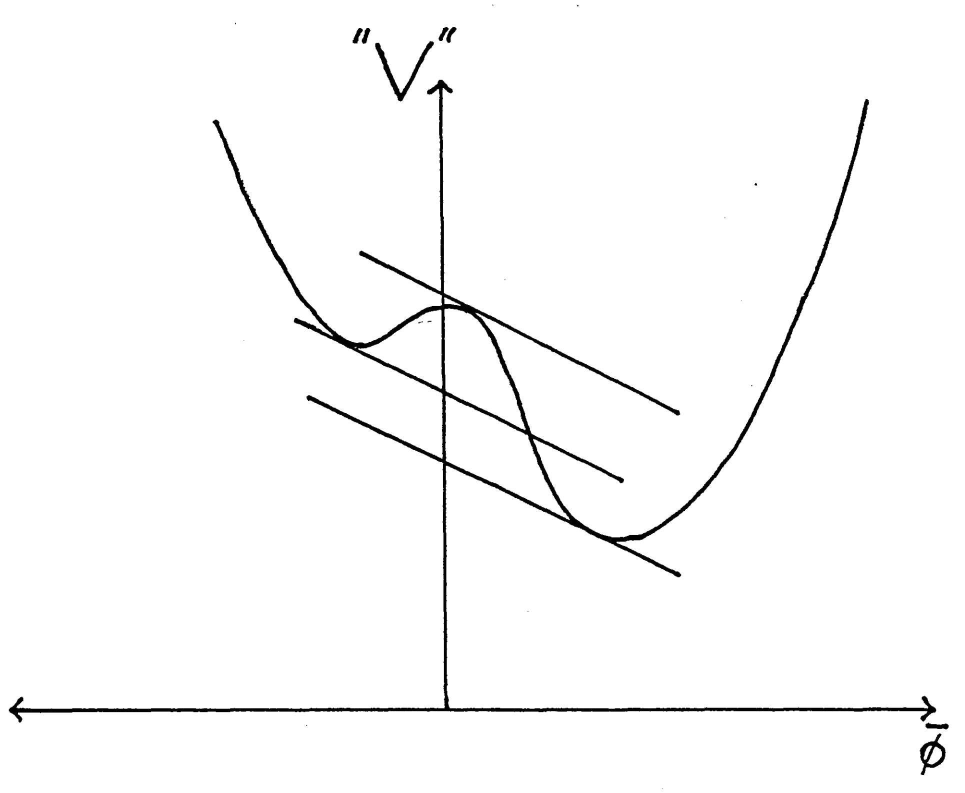
and it satisfies (c.f. (3.8),(3.9))

$$\frac{d^2 f^1(y)}{dy^2} \frac{d^2 f(x)}{dx^2} = -1 \quad (3.12)$$

The transformation will thus break down if either of the two terms in (3.12) is zero, which will happen unless both are strictly convex or concave. As our effective action and hence effective potential are defined by a Legendre transform this may give rise to problems.

One can raise further questions about the standard loop expansion result, which we will henceforward call "V" to indicate its dubiety, by examining the analogy between Euclidean field theory and statistical mechanics, after Haymaker and Perez-Mercader [32]. Bearing in mind the expansion (3.3) and the arguments of Coleman [6] sketched in the introduction, one notes that the reverse Legendre transform on "V" gives $\xi(J)$, the vacuum energy density with a source J. To find $\xi(J)$ for a given J we can make use of the graphical interpretation of the Legendre transform. We search for a tangent to $V(\phi)$ of slope J and its intercept will give us $\xi(J)$. For a non-convex "V" there may be more than one tangent of a given slope.

(Fig.8) Tangents to "V"



However, the values of $\xi(J)$ that are generated by the tangents to the non-convex portion of "V" are higher than those from tangents of identical slope outside the region of non-convexity. This suggests that the non-convex region corresponds to an unphysical, higher energy branch of the curve and that the correct V should be taken to be the convex hull of "V".

A similar conclusion may be reached by examining the quantum-mechanical arguments of Evans and McCarthy [33]. Consider a spontaneously broken $\lambda\phi^4$ potential with minima at $\pm\phi_0$, giving two degenerate vacua, say $|+\rangle$ and $|-\rangle$. For a finite spacetime volume there will be matrix elements of observables connecting the two, as it requires a finite amount of energy to cross the hump in the potential. In the infinite volune limit, however, the matrix elements will vanish and we may write

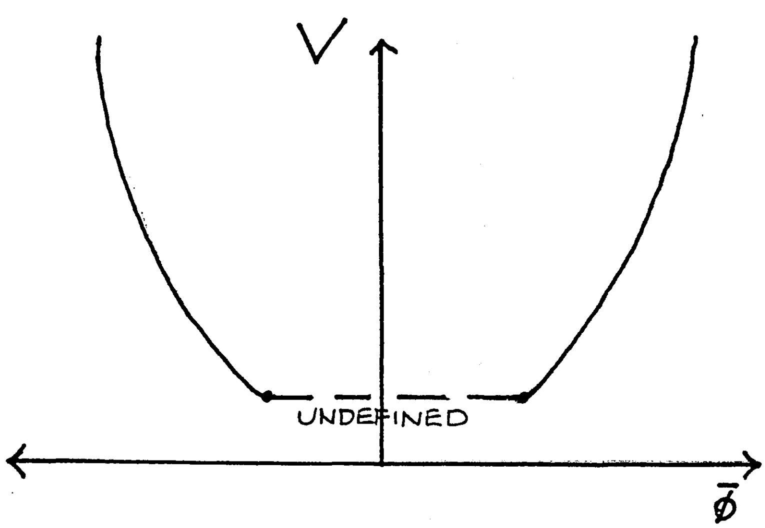
$$\langle\pm|H|\pm\rangle = E_0\ \delta_{\pm\pm}$$
$$\langle\pm|\Phi|\pm\rangle = \pm\phi_0\delta_{\pm\pm} \qquad (3.13)$$

where H is the Hamiltonian density and Φ is the quantum field operator. Any linear combination of the two vacua will be an equally good ground state with energy E_0 , as we may see from the following

$$|\lambda\rangle = \lambda_+|+\rangle + \lambda_-|-\rangle \quad \text{where } \|\lambda_+\|^2 + \|\lambda_-\|^2 = 1$$
$$\langle\lambda|H|\lambda\rangle = E_0 \quad \text{and} \quad \langle\lambda|\Phi|\lambda\rangle = (\|\lambda_+\|^2 - \|\lambda_-\|^2)\phi_0 \qquad (3.14)$$

Note that the state $|\lambda\rangle$ has a field vacuum expectation value lying between the two minima. Since the effective potential may be defined as the minimum of $\langle H\rangle$ subject to $\langle\Phi\rangle = \bar{\phi}$ we have found that its true value between $-\bar{\phi}_0$ and $+\bar{\phi}_0$ is E_0, i.e. a linear interpolation. If the spacetime volume is finite there will be, as we have already stated, transitions between the two vacua giving a single minimum at $\bar{\phi} = 0$, which shows up in both the path integral approach [11] and lattice calculations [34]. For a single system we would have to be in one or other of the vacua so it is perhaps best to think of the effective potential as being undefined between the two minima

(Fig.9) The convex effective potential V



The result generalizes easily if we have discrete minima, say $|i\rangle$, i=1,n each with energy E_0.

$$\langle i|H|j\rangle = E_0\delta_{ij}$$
$$\langle i|\Phi|j\rangle = \phi_0^i\ \delta_{ij} \quad \text{(no sum)} \tag{3.15}$$

If we take a state $|\lambda\rangle$

$$|\lambda\rangle = \sum_{i=1}^{n} \lambda_i |i\rangle \qquad \sum_{i=1}^{n} \|\lambda_i\|^2 = 1 \tag{3.16}$$

we find, in a similar manner to (3.15)

$$\langle\lambda|H|\lambda\rangle = E_0$$
$$\langle\lambda|\Phi|\lambda\rangle = \sum_{i=1}^{n} \|\lambda_i\|^2 \bar{\phi}_0^i \tag{3.17}$$

which is just the convex hull of the set of minimum values $\{\bar{\phi}_0\}$. Once again the linearly interpolated regions correspond to a mixed vacuum state.

Both the work of Haymaker and Perez-Mercader and that of Evans and McCarthy are a patch-up on the incorrectly calculated "V" to excise the non-convex regions. It would obviously be more satisfying to calculate the correct, linearly interpolated V from scratch. This was done by Fujimoto et al. for the case of spontaneously broken $\lambda\phi^4$ in [11], where they showed that a careful treatment of the saddle-points in the path integral (we are doing a saddle-point calculation in making a loop expansion for the effective potential, as noted in the introduction) leads to a linear interpolation over the non convex portions of "V". Before examining this approach in section 2.3 we make an aside on an alternative definition of the effective potential which is useful in non-perturbative calculations and which provides further evidence for convexity.

2.2 THE CONSTRAINT EFFECTIVE POTENTIAL

Retaining the volume of spacetime explicitly we define the function $\omega(\bar{\phi})$ by [35],[36]

$$\exp(- \Omega\omega(\bar{\phi})) = \int[D\phi]\ \delta(c - \bar{\phi})\ \exp(-S[\phi]) \tag{3.18}$$

where we have set h = 1 and $c = \frac{1}{\Omega} \int d^4x\ \phi(x)$. If we now multiply both sides of (3.18) by $\exp(\Omega J\bar{\phi})$ and integrate over $\bar{\phi}$ we find

$$\int d\bar{\phi}\ \exp\ \Omega(J\bar{\phi} - \omega(\bar{\phi})) = \int[D\phi]\ \exp\ (-S[\phi]+J\int\phi) = \exp(-\Omega\xi(J)) \tag{3.19}$$

In the infinite volume limit one may take the saddle pont expansion for the l.h.s. to be exact and find

$$\sup{}_J \Omega(J\bar{\phi} - \omega(\bar{\phi})) = -\Omega\xi(J) \tag{3.20}$$

which is, in effect a Legendre transform (the sup/inf may be inserted into the Legendre transform in order to handle non-convex functions). We shall denote this modified transform by a superscript l'. It agrees with l for a convex function. We thus have

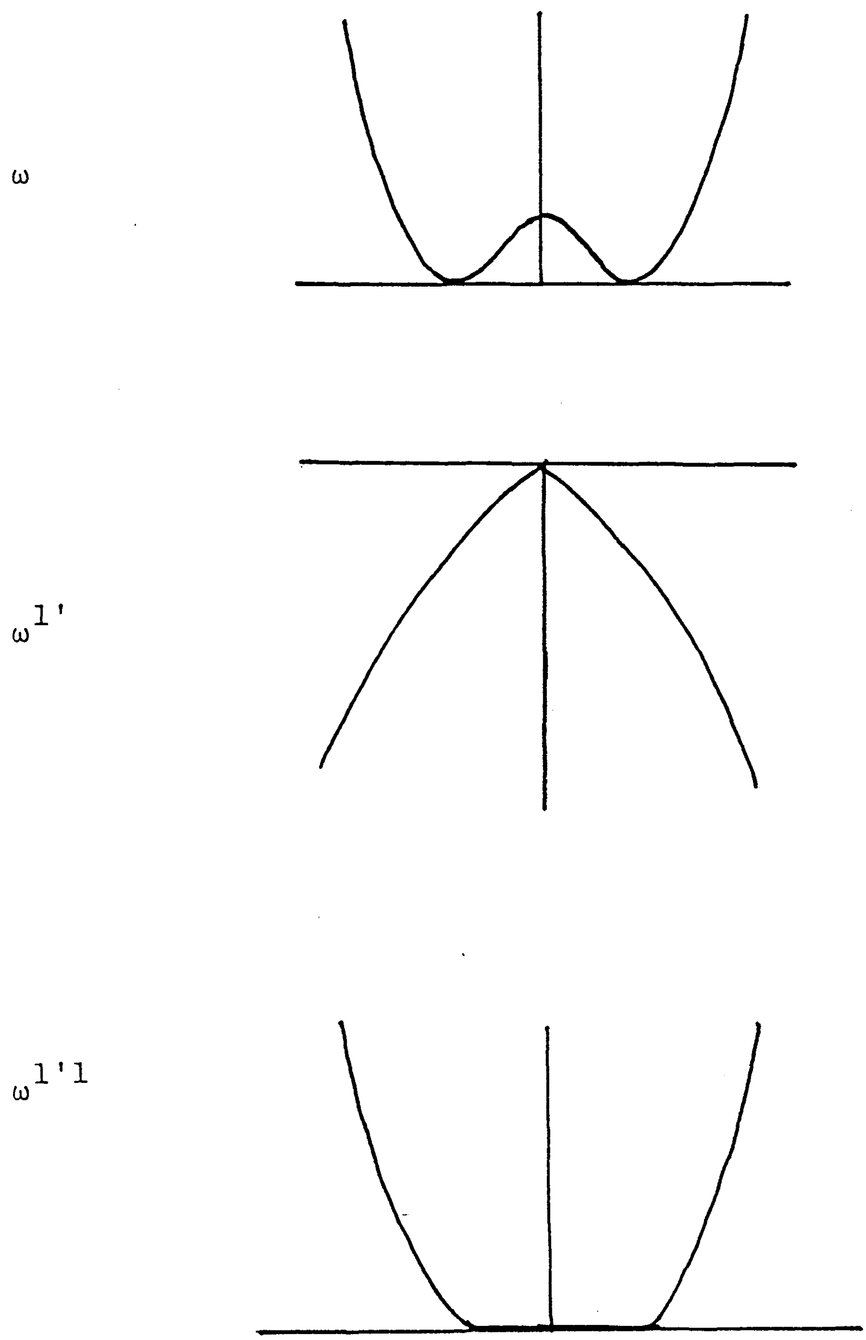
$$\omega^{l'}(\bar{\phi}) = \xi(J) \tag{3.21}$$

We also know that $V(\bar{\phi})$, the standard effective potential, is given by $\xi^1(J)$ so

$$\omega^{1'1}(\bar{\phi}) = \xi^1(J) = V(\bar{\phi}) \tag{3.22}$$

Now $\omega^{1'}$ will be single valued by virtue of the sup/inf definition so $\omega^{1'1}$ will be the convex hull of ω and a V derived in this manner will be convex.

<u>(Fig.10)</u> The double Legendre transform of ω



We shall see, however, that a computer evaluation of ω (not V) using Wilson recursion relations [36] produces a convex result so that, by virtue of 1 equalling 1' for convex functions and the involutive property of the Legendre transform, the constraint effective potential ω is actually identical to the standard V.

The idea behind the recursion relations is quite simple. One considers the Fourier decomposition of the fields in the Lagrangian

$$\phi(x) = \phi_0(x) + \phi_1(x) \tag{3.23}$$

where we regularize the theory with a cutoff Λ and where we split the k integration so that

$$\phi_0(x) = \int_{\Lambda/2 \,<\, \|k\| \,<\, \Lambda} \exp(ik.x)\phi(k)\, d\!\!{}^-{}^d k \tag{3.24}$$

with the ϕ_1 being the remainder. We can consider the field variables as sitting on a lattice of spacing Λ^{-1} (the inverse highest momentum scale in the theory). This allows us to decompose the action and integrate out the higher modes in the fields contained in ϕ_0.

$$\begin{aligned} \frac{1}{2}\int (\partial_\mu \phi)^2 &\simeq \Lambda^{2-d} \sum_n \phi_{0_n}^2 + \frac{1}{2}\int (\partial_\mu \phi_1(x))^2 \\ \int U(\phi(x)) &\simeq \Lambda^{-d} \sum_n \frac{1}{2} U(\phi_{0_n} + \phi_{1_n}) + \frac{1}{2} U(-\phi_{0_n} + \phi_{1_n}) \end{aligned} \tag{3.25}$$

We have very crudely represented the wavelike nature of the ϕ_0 we are integrating over by a step function inside each lattice cell and we note that this does not contain any zero component, which is consistent with our constraint effective potential calculation where we integrate over everything except the zero mode, and we have used d to denote the dimension of spacetime. With the above approximations the action becomes

$$S[\phi] = \int \frac{1}{2} (\partial_\mu \phi_1(x))^2 + \sum_n \Lambda^{2-d} \phi_{0_n}^2 + \frac{1}{2} \Lambda^{-d} U(\phi_{0_n} + \phi_{1_n}) + \frac{1}{2} \Lambda^{-d} U(-\phi_{0_n} + \phi_{1_n}) \tag{3.26}$$

One now integrates over the ϕ_0 in the path integral and then repeats the process with $\Lambda/4 < \|k\| < \Lambda/2$ (and so on). In this manner one derives a recursion relation for the constraint effective potential.

$$U_{\lambda+1}(\bar{\phi}) = -2^{-d\lambda} \ln \left(F[U_\lambda(\bar{\phi})]/F[U_\lambda(0)] \right) \tag{3.27}$$

where

$$F[U_\lambda(\phi)] = \int_{-\infty}^{\infty} dy \exp \left(-2^{(d-2)\lambda} y^2 - 2^{d\lambda-1} \left(U_\lambda(y+\phi) - U_\lambda(-y+\phi) \right) \right) \tag{3.28}$$

and U_1 is given by the classical potential. As $\lambda \to \infty$, $U_\lambda \to \omega$.

A Fortran program to perform the above process is given in Appendix C, and it gives results identical to Fukuda, who found that starting with a non-convex classical potential the iteration served to convexify the potential. For sufficiently large values of (m^2/λ) the effective potential was flat bottomed, whereas for smaller values it had a true minimum at $\bar{\phi} = 0$. With our parameters $(m^2/\lambda) > 2.5$ gave a flat potential.

Lattice field theory calculations [37] give similar results, with a flat bottomed or genuinely convex effective potential emerging even when the the classical potential is non-convex. We conclude that explicit evaluations of the path integral, insofar as they are possible, support the notion of a convex effective potential. Where, then, is the perturbative expansion breaking down?

3.3 THE INTERPOLATED LOOP EXPANSION

If we have a non-convex classical potential U the minima of $U(\bar{\phi})-J\bar{\phi}$ are not uniquely specified for all J. For certain critical values of J, say J_c, a small variation in J will cause the absolute minimum of $U(\bar{\phi})-J\bar{\phi}$, $\bar{\phi}_0(J)$, to jump discontinuously between the local minima. For instance, in a spontaneously broken $\lambda\phi^4$ theory, J_c is zero and a small variation either side of this causes the absolute minimum to jump between the two local minima. To deal with this we subdivide the path integral into regions R_i, each of which contains one of the local minima, which we shall denote by $\bar{\phi}_o^i$ [11].

$$\exp\left(\frac{1}{\hbar}W[J]\right)=\sum_i \int_{R_i}[D\phi]\ \exp\left(-\frac{1}{\hbar}\left(S[\phi]-\int J\phi\right)\right)$$

$$=\sum_i \exp\left(\frac{1}{\hbar}W_i[J]\right) \tag{3.29}$$

where

$$W_i[J] = \exp(-\frac{\Omega}{\hbar}(U(\phi_0^i) - J\phi_0^i))\int_{R_i}[D\phi]\ \exp(-\frac{1}{\hbar}\int\Delta^i L)$$

$$\Delta^i L = L(\phi + \phi_0^i) - L(\phi_0^i) - J\phi \qquad (3.30)$$

$$\Omega = \int d^4x$$

The approximation necessary to evaluate (3.29) is the extension of the regions R_i to cover the full range of ϕ. We would expect the overlap error thus introduced to be exponentially small, as the $\int\Delta^i L$ are large and positive in the region of overlap. With each R_i replaced by the full range of integration (3.29) can be thought of as defining a Jackiw-style expansion about each ϕ_0^i

$$\exp(\frac{1}{\hbar}W[J]) = \sum_i \exp(-\frac{\Omega}{\hbar}(\text{"V"}(\bar{\phi}^i) - J\bar{\phi}^i)) \qquad (3.31)$$

The "V" is calculated in the standard loop expansion about the appropriate ϕ_0^i to any desired order. Outside a critical range, $J-J_c > O(h)$ one of the terms in summation (3.31) will dominate and we recover the standard loop expansion, which is acceptable as we would be outside the non convex portion of "V". Inside the critical range we need to consider the full summation.

Thus to evaluate $\bar{\phi}$ in this interval we would consider

$$\bar{\phi} = \frac{\delta W}{\delta J} = \frac{\sum_i \bar{\phi}^i \exp(-\frac{\Omega}{\hbar} (\text{"V"}(\bar{\phi}^i) - J\bar{\phi}^i))}{\sum_i \exp(-\frac{\Omega}{\hbar} (\text{"V}(\bar{\phi}^i) - J\bar{\phi}^i))} \tag{3.32}$$

Writing $J = J' + J_c$ we obtain

$$\bar{\phi} = \frac{\sum_i \bar{\phi}^i \exp(-\frac{\Omega}{\hbar} (\text{"V"}(\bar{\phi}^i) - J'\bar{\phi}^i - J_c\bar{\phi}^i))}{\sum_i \exp(-\frac{\Omega}{\hbar} (\text{"V"}(\bar{\phi}^i) - J'\bar{\phi}^i - J_c\bar{\phi}^i))} \tag{3.33}$$

Now $-J_c + \text{"V"}(\bar{\phi}^i)$ is the same for all the terms in the summation, so we can divide it out to obtain

$$\bar{\phi} = \frac{\sum_i \bar{\phi}^i \exp(-\frac{\Omega}{\hbar} J'\bar{\phi})}{\sum_i \exp(-\frac{\Omega}{\hbar} J'\bar{\phi})} \tag{3.34}$$

In these summations the element, or elements, with the largest component in the J' direction will dominate in the infinite volume limit. If $J'\bar{\phi}^i$ is maximized for a set of $\bar{\phi}^i$, then $\bar{\phi}$, as defined by (3.34), will lie at the centroid of the figure defined by the $\bar{\phi}^i$. Although this might apparently lie in a non-convex portion of "V" it has resulted from a linear combination of states of equal energy which are orthogonal in the Hilbert space in the infinite volume limit (if $J \simeq 0$ they would be vacua), and is therefore part of a linearly interpolated region.

We have thus shown that $\bar{\phi}$ will lie at a $\bar{\phi}^i$, in which case it will be outside a non-convex region, or it will lie on a linearly interpolated section of the graph. The non-convex portions of "V" have been excised in the manner suggested by the arguments of Haymaker and Perez-Mercader.

To make these general considerations a little more concrete consider the canonical example of a spontaneously broken $\lambda\phi^4$ theory. We have, at the one-loop level for $J \sim 0$

$$\exp\left(\frac{1}{\hbar} W(J)\right) = \exp\left(-\frac{\Omega}{\hbar}\left(\text{"V"}^1(\bar{\phi}_0) - J\bar{\phi}_0\right)\right) + \exp\left(-\frac{\Omega}{\hbar}\left(\text{"V"}^1(-\bar{\phi}_0) + J\bar{\phi}_0\right)\right) \tag{3.32}$$

or, as $\text{"V"}^1(\bar{\phi}) = \text{"V"}^1(-\bar{\phi})$

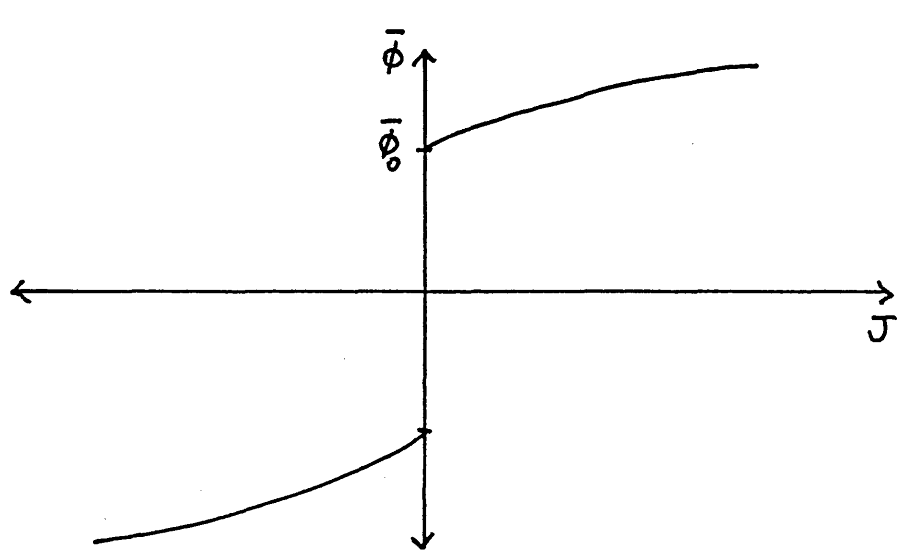
$$\frac{1}{\hbar} W(J) = \ln 2 - \frac{\Omega}{\hbar} \text{"V"}^1(\bar{\phi}_0) + \ln \cosh\left(\frac{\Omega J \bar{\phi}_0}{\hbar}\right) \tag{3.33}$$

We thus find that

$$\bar{\phi} = \frac{1}{\Omega}\frac{\partial W}{\partial J} = \bar{\phi}_0 \tanh\left(\frac{\Omega J \bar{\phi}_0}{\hbar}\right) \tag{3.34}$$

which exhibits precisely the step-function-like behaviour in the $\Omega \to \infty$ limit that we have taken as a signal of convexity

(Fig.11) Step-function-like behaviour of $\bar{\phi}$



In this simple example we may proceed further and calculate V itself, instead of demonstrating the linear interpolation by more indirect means. We have, for J~0

$$\Gamma^1(\bar\phi) = W^1 - \Omega J\bar\phi^1 = -\Omega \text{"}V\text{"}^1(\bar\phi_0) + h \ln 2\cosh(\frac{\Omega J\bar\phi_0}{\hbar})$$

$$-\Omega J\bar\phi_0 \tanh(\frac{\Omega J\bar\phi_0}{\hbar}) \tag{3.35}$$

Now $\frac{\bar\phi}{\bar\phi_0} = \tanh(\frac{\Omega J\bar\phi_0}{\hbar})$, which allows us to eliminate the ln cosh term and J in (3.35) to get

$$\Gamma^1(\bar\phi) = -\Omega\text{"}V\text{"}^1(\bar\phi_0) + \hbar \ln[(\frac{\bar\phi_0+\bar\phi}{\bar\phi_0-\bar\phi})^{1/2} + (\frac{\bar\phi_0-\bar\phi}{\bar\phi_0+\bar\phi})^{1/2}]$$

$$-\frac{\hbar}{2} \bar\phi \ln(\frac{\bar\phi_0+\bar\phi}{\bar\phi_0-\bar\phi}) \tag{3.36}$$

Using V= $-\Omega\Gamma$ we can rewrite this as

$$V^1(\bar\phi) = \text{"}V\text{"}^1(\phi_0) + \frac{\hbar}{2\Omega}(\frac{\bar\phi_0+\bar\phi}{\bar\phi_0} \ln \frac{\bar\phi_0+\bar\phi}{\bar\phi_0} + \frac{\bar\phi_0-\bar\phi}{\bar\phi_0} \ln \frac{\bar\phi_0-\bar\phi}{\bar\phi_0}) \tag{3.37}$$

which is valid for $\|\phi\| < \phi_0$. This exhibits both the linear interpolation and finite volume correction we have mentioned previously.

3.4 GAUGE FIXING AND EXPLICIT EXAMPLES OF CONVEXIFICATION

The demonstration of the convexity of the effective potential from the properties of $\bar{\phi}$ depended crucially on the various saddle point contributions being separated. In general with a group G being broken to a (maximal, unless we are sufficiently perverse in our choice of Higgs representation) subgroup H, the manifold of possible minima will be given by G/H and we do not achieve this separation. One could ignore the problem and simply extend the sum in (3.34) to an integration over the manifold of minima as advocated by Fujimoto et al. [11] and 'O Raifeartaigh et al. [35]. This would then give

$$\bar{\phi} = \frac{\int d\mu(\phi)\ \phi\ \exp(-\frac{\Omega}{h} J'\phi)}{\int d\mu(\phi)\ \exp(-\frac{\Omega}{h} J'\phi)} \tag{3.38}$$

where the integration is understood to be over the manifold of minima and $d\mu(\phi)$ is a suitable measure.

For example consider the case of SO(2) → 0 as in the Abelian Higgs model. If we let J tend to zero in the ϕ_1 direction we find

$$\bar{\phi}_1 = \frac{\rho\int_0^\pi \cos\theta\ \exp(-\lambda\rho\cos\theta)d\theta}{\int_0^\pi \exp(-\lambda\rho\cos\theta)d\theta} \tag{3.39}$$

where we have parametrized $\bar{\phi}_1$ as $\rho\cos\theta$ and $\lambda = \frac{\Omega J}{h}$. This gives

$$\bar{\phi}_1 = \rho\ \frac{I_1(\lambda\rho)}{I_0(\lambda\rho)} \tag{3.40}$$

This has the required step function like behaviour. Instead of choosing an ad-hoc parametrization we could confine the region of integration to the manifold of minima using δ-functions (or distributions!). This would give

$$\bar{\phi}_1 = \frac{\int d\phi_1 \int d\phi_2 \delta(\surd(\phi_1^2+\phi_2^2)-\rho)\ \phi_1 \exp(-\lambda\phi_1)}{\int d\phi_1 \int d\phi_2 \delta(\surd(\phi_1^2+\phi_2^2)-\rho)\ \exp(-\lambda\phi_1)} \tag{3.41}$$

which again integrates out to give (3.40). As another example consider SO(3) $\rightarrow$ SO(2) where the manifold of minima is S^2. If we again take J to zero in the ϕ_1 direction we have

$$\bar{\phi}_1 = \frac{\int d\phi_1 \int d\phi_2 \int d\phi_3 \delta(\surd(\phi_1^2 + \phi_2^2 + \phi_3^2)-\rho)\ \phi_1\ \exp(-\lambda\phi_1)}{\int d\phi_1 \int d\phi_2 \int d\phi_3 \delta(\surd(\phi_1^2 + \phi_2^2 + \phi_3^2)-\rho)\ \exp(-\lambda\phi_1)} \tag{3.42}$$

Although the specific function this produces is different

$$\bar{\phi}_1 = \rho(\frac{1}{\lambda} - \coth\lambda) \tag{3.43}$$

the step function like behaviour is still present.

Despite the above results the initial replacement $\Sigma \rightarrow \int d\mu(\phi)$ is dubious and it would be preferable to avoid it in a more rigorous approach. If we are dealing with the Higgs sector in a gauge theory we could make use of our freedom of choice in choosing the gauge fixing to discretize the minima. The gauge to choose is just the 't Hooft gauge, which provides the required discretization because of the $v_i\bar{\phi}_i = 0$ condition arising from the Nielsen identities.

We note first that the $v_i \bar{\phi}_i$ condition reqires some modification for more complicated gauge theories. It should actually read $v_i^\alpha \, \eta_i^r = 0$ where the η^r span the physical Higgs space and the α is a group index (see [23]), which is saying that the v_i^α lie in the Goldstone boson subspace. The easiest way to achieve this is just to choose the 't Hooft gauge proper

$$v_i^\alpha = -i\xi \; T_{ij}^\alpha \langle 0 | \Phi_j | 0 \rangle = -i\xi T_{ij}^\alpha \bar{\phi}_{j0} \tag{3.44}$$

where the T^α are generators in the appropriate representation, and we shall assume in the rest of this section that this has been done.

If we consider first a real vector representation, our condition with the Higgs fields in the N of SO(N) reads

$$T_{ij}^\alpha \bar{\phi}_{j0} \bar{\phi}_i = 0 \tag{3.45}$$

which, in particular, on the manifold of minima will be

$$T_{ij}^\alpha \bar{\phi}_{j0} \bar{\chi}_i = 0 \tag{3.46}$$

where χ_i lies on the manifold. We can write any χ_i as the group rotation of a $\bar{\phi}'_{i0}$.

$$\chi_i = \exp(i\theta^\alpha \; T_{ij}^\alpha) \; \bar{\phi}'_{j0} \tag{3.47}$$

So (3.47) reads

$$\frac{\partial}{\partial \theta^\alpha} \left(\bar{\phi}_{i0} \cdot \bar{\phi}'_{i0} \right) = 0 \tag{3.48}$$

If we are considering the breaking SO(N)→SO(n-1) only one of the Higgs fields will acquire a vacuum expectation value, so (3.48) will be satisfied for a direction that is either parallel or antiparallel to it. The manifold of minima for the above breaking will be a hypersphere, so this picks out, as allowed minima, two antipodal points where the axis along the direction of symmetry breaking intersects the hypersphere.

For a complex vector representation we would write the Higgs fields as the sum of two real fields $\phi_i = \phi_{i1} + i\phi_{i2}$ and consider the gauge-fixing

$$-\frac{1}{2\xi}\left(\partial_\mu A^{\mu\alpha} + (v_i\phi_i^*) + (v_i^*\ \phi_i)^*\right)^2 \tag{3.48}$$

The argument for the real case is now repeated with the number of components doubled.

For a Higgs field in the adjoint we can write (3.45) as

$$f^\alpha_{ij}\bar{\Phi}_{j0}\bar{\Phi}_i = 0 \tag{3.49}$$

because the T^α_{ij} are represented by the structure constants f^α_{ij}. If we make the definitions $\Theta = \bar{\Phi}_{j0}R_j$ and $\Phi = \bar{\Phi}_i R_i$, where the R_i are the elements of the appropriate Lie algebra (3.49), becomes

$$[\Theta,\Phi] = 0 \tag{3.50}$$

This means that, choosing Θ to lie in the Cartan sub-algebra, Φ is constrained to lie in the Cartan subalgebra also. All the equivalent minima in U − Jϕ are disconnected because, with Φ in the Cartan subalgebra, they are taken into one another by the elements of the Weyl group which leave the projection of J onto the Cartan subalgebra invariant. This is a discrete group.

For the vector representations the only critical value of the current is zero and we just consider convexifying between the absolute minima. This is not the case for the adjoint representation, where there may be solutions to $\frac{\partial U}{\partial \phi} = -J_c$ and hence to $\frac{\partial "V"}{\partial \phi} = -J_c$ for values of J_c other than zero.

As an example of this one could consider an SU(3) adjoint potential, though this is not representative in that the relation $(\mathrm{Tr}\Phi^2)^2 = 2\,\mathrm{Tr}\Phi^4$ ensures that the non-convexity vanishes when the quartic terms become dominant. This is not necessarily true in other cases, as one can see from consideration of a general adjoint potential at large $\|\phi\|$.

$$U(\bar{\phi}) = A\left(\Sigma\bar{\phi}_i^2\right) + \frac{1}{2}A_1\left(\Sigma\bar{\phi}_i^2\right) \tag{3.51}$$

The matrix of second derivatives for this,

$$\frac{\partial^2 U}{\partial\bar{\phi}_j\,\partial\bar{\phi}_k} = \left(4A(\Sigma\bar{\phi}^2) + 6A_1\bar{\phi}_j^2\right)\delta_{jk} + 8A\,\bar{\phi}_j\bar{\phi}_k\,, \tag{3.52}$$

would be positive semi-definite (which would ensure convexity) iff $v_i U_{ij} v_j \geq 0$ for all v and $\bar{\phi}$. If A were negative, which is allowed by the positivity constraint on the potential, choosing $v = (0,1,0\ldots0)$ and $\bar{\phi} = (\|\phi\|,0,0\ldots0)$ would give $v_i U_{ij} v_j < 0$, thus showing the potential was non-convex at this point.

Returning to the SU(3) case, we can write the SU(3) adjoint potential as [38].

$$U(\bar{\phi}) = p(1-\bar{\phi}^2)^2 + q(\bar{\phi}_i + \sqrt{3}d_{ijk}\bar{\phi}_j\,\bar{\phi}_k)^2 \tag{3.53}$$

where p,q > 0 and the d_{ijk} are defined by

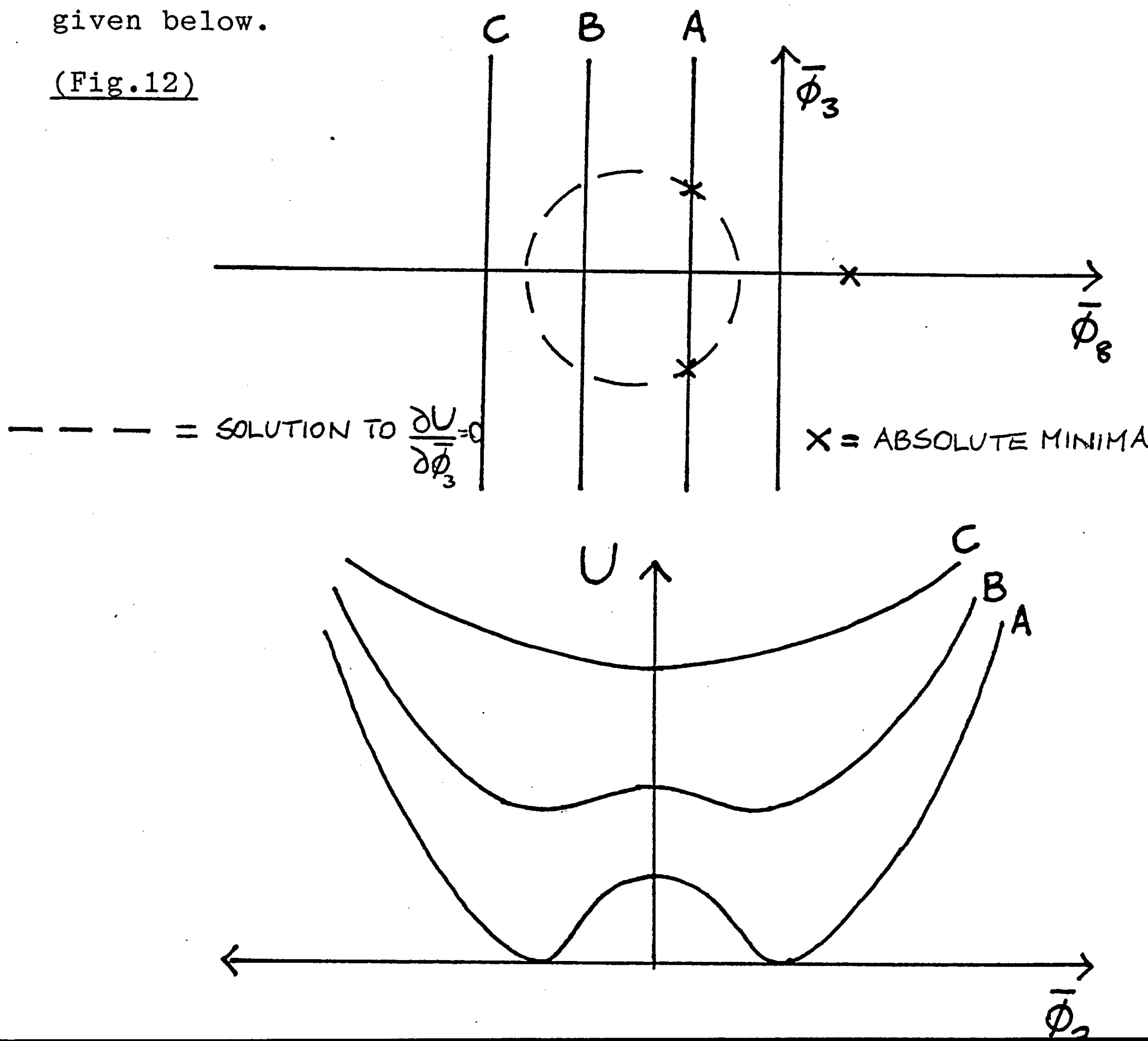
$$[R_i \ , \ R_j]_+ = \frac{4}{3}\,\delta_{ij} + 2d_{ijk}R_k \tag{3.54}$$

If we now restrict $\bar{\phi}$ to lie in the Cartan subalgebra, spanned by λ_3 and λ_8 in the Gell-Mann basis, the potential may be written as

$$U(\bar{\phi}_3,\bar{\phi}_8) = p(1 - \bar{\phi}_3^2 - \bar{\phi}_8^2)^2 + q[\ \bar{\phi}_3^2(1 + 2\bar{\phi}_8)^2 + (\bar{\phi}_8 - \bar{\phi}_8^2 + \bar{\phi}_3^2)\ ] \tag{3.55}$$

A plot of $U(\bar{\phi}_3,\bar{\phi}_8)$ shows that there are now other solutions for J_c. The form of $U(\bar{\phi}_3)$ on the three sections A,B,C is also given below.

(Fig.12)



In section B, for instance, the minima 1,2 are solutions to $\frac{\partial U}{\partial \phi}$ = J for some J in the 8 direction and we would have to convexify between points 1 and 2. By the time we have reached C the quartic terms are dominant and the non-convexity has dissappeared.

In the general adjoint case the shape of the region to be convexified can be determined for any given direction of J by using the correspondence between the Dynkin diagrams that are used to classify the symmetry breaking in the adjoint representation and the Coxeter graphs that are used to describe the symmetry of regular polytopes [38].

In both the vector and the adjoint case we have essentially picked out a direction of allowed symmetry breaking by our choice of gauge. We have not, however, determined our orientation, leaving a residual inversion or Weyl symmetry, and it is this which provides the convexification.

3.5 DISCUSSION

The convexification of the effective potential which we have described does not cancel out the spontaneous symmetry breaking displayed by the classical potential. If the Lagrangian admits a Higgs mechanism

$$\frac{1}{2}\,(D_\mu\phi)^2 \rightarrow \frac{1}{2}\,(A^\alpha_\mu T^\alpha\phi)^2 = \frac{1}{2}\,M_{\alpha\beta}A_{\mu\alpha}A^{\mu\beta} \qquad (3.56)$$

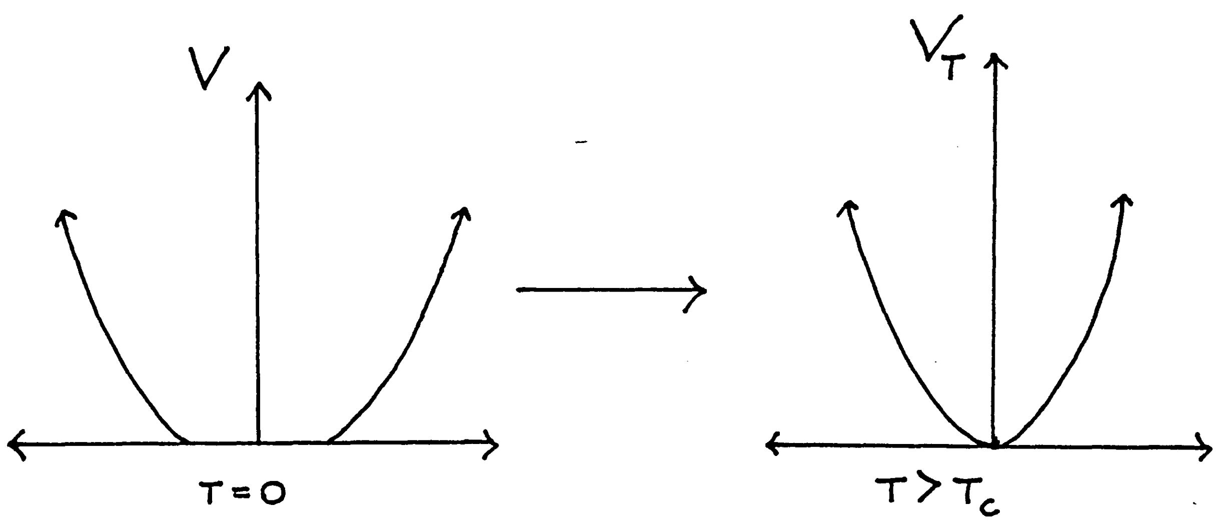
where $M_{\alpha\beta} = (\phi, T_\alpha T_\beta\phi)$ and the T's are the group generators, this is unchanged and gives the same result when ϕ is in the neighbourhood of a classical minimum. The role of V is to select between the various possible minima. A value of $\bar{\phi}$ lying in the linearly interpolated region arises from a mixed vacuum state and we would not expect this in a static situation. The convexification merely serves to remind us that our assumption of a constant background field to expand about in the effective potential is invalid for some values of J.

CHAPTER 4 CONVEXITY AT FINITE T

4.1 CONVEXITY AT FINITE T

It is one's general intuition that symmetries ought to be restored with increasing temperature. This would suggest the qualitative behaviour shown below for a spontaneously broken $\lambda\phi^4$ potential.

(Fig.13) Behaviour of V with increasing temperature.



At some temperature, say T_c, the theory will undergo a transition from its spontaneously broken state to a symmetric vacuum ($\langle 0|\Phi|0\rangle = 0$). In order to try and observe these effects perturbatively in the calculation of the effective potential one needs to know how to deal with quantum field theories at finite temperature. We present a short review of two complementary formalisms in the next section and consider a convex effective potential evaluated in both. Unfortunately both our attempts at producing a finite temperature,convex effective potential come unstuck at a temperature which is lower than the supposed critical temperature of the theory (this may not simply be a fault in the formalism; see section 4.8).

4.2 FINITE TEMPERATURE CALCULATIONS

The initial approach to finite temperature calculations in quantum field theory was in the so-called imaginary time formalism. We follow the exposition of Bernard [39]. We consider a quantum field theory described by the Hamiltonian density $H(\pi,\Phi)$ where $\Phi(x,t)$ is the Heisenberg field operator and $\pi(x,t)$ is the conjugate momentum. The Schroedinger picture operator is then $\Phi(x,0)$. Let $|\phi_0\rangle$ and $|\phi_1\rangle$ be eigenstates of $\Phi(x,0)$ with eigenvalues $\phi_0(x)$ and $\phi_1(x)$ respectively. The transition amplitude for going from $|\phi_0\rangle$ at time 0 to $|\phi_1\rangle$ at time T is just given by

$$\langle\phi_1|\exp(-iHT)|\phi_0\rangle = N \int[D\phi][D\pi] \exp i\left(\int_0^T dt\int d^3x \left(\pi\frac{\partial\phi}{\partial t} - H\right)\right) \tag{4.1}$$

where the field integral runs over all configurations that start at $\phi_0(x)$ at time 0 and end at $\phi_1(x)$ at time $t = T$. The momentum integral is unrestricted. If we now let $iT=\beta$ and make the variable change $it=\tau$

$$\langle\phi_1|\exp(-\beta H)|\phi_0\rangle = N \int[D\phi][D\pi] \exp\left(\int_0^\beta d\tau\int d^3x \left(i\pi\frac{\partial\phi}{\partial\tau} - H\right)\right) \tag{4.2}$$

The finite temperature partition function $Z = \mathrm{Tr}\,(\exp(-\beta H))$, where β is now identified with the inverse temperature, can be calculated from (4.2) by restricting ourselves to periodic field configurations. This establishes the formal analogy between Euclidean quantum field theory (which we are doing by virtue of the Wick rotation on t) and statistical mechanics.

$$Z = \sum_{\phi} \langle\phi\left|\exp(-\beta H)\right|\phi\rangle = N \int_{\phi\ \text{periodic}} [D\phi][D\pi] \exp\left(\int_0^\beta dt \int d^3x \left(i\pi\frac{\partial\phi}{\partial\tau} - H\right)\right) \tag{4.3}$$

We can now integrate out the π's provided that H is no more than quadratic in π. This replaces π by its value at the stationary point and adds a determinant if the term in π is ϕ dependent. This gives

$$Z = N'(\beta)\int_{\text{periodic}}[D\phi] \exp\left(\int d\tau \int d^3x\ L_{eff}(\phi, i\dot{\phi})\right) \tag{4.8}$$

We note that the normalization may contain temperature dependence. As the fields are periodic we may consider their Fourier expansion

$$\phi(x,\tau) = \frac{1}{\beta} \sum_n \int d^3\bar{k}\ \exp(i\bar{k}.x)\exp(i\omega_n\tau)\,\phi_n(\bar{k}) \tag{4.9}$$

where $\omega_n = \dfrac{2\pi n}{\beta}$. If we now use the identity

$$\int_0^\beta d\tau\ \exp i(\omega_n - \omega_m)\tau = \beta\delta_{nm} \tag{4.10}$$

and substitute the Fourier decomposition of ϕ into the Lagrangian we find for the quadratic part

$$S = \frac{1}{2\beta} \sum_n \int d^3\bar{k}\ \phi_n(\bar{k}) \left(\omega_n^2 + \bar{k}^2 + m^2\right) \phi_{-n}(-\bar{k}) \tag{4.11}$$

The propagator is thus

$$\frac{1}{\omega_n^2 + \bar{k}^2 + m^2} \tag{4.12}$$

and the Feynman rules are obtained from the usual ones by the replacements

$$\int d^4k \rightarrow \frac{i}{\beta} \sum_n \int d^3\bar{k} \qquad \text{and} \qquad k_0 \rightarrow i\omega_n \tag{4.13}$$

$$\delta^4(k_1 + k_2 + \ldots) \rightarrow -i\beta\, \delta_{\omega_1 + \omega_2 + ..}\, \delta^3(\bar{k}_1 + \bar{k}_2 + ..) \tag{4.14}$$

To determine the temperature dependent normalization one must do a Feynman style discretization of the path integral to find, symbolically [40]

$$\ln N'(\beta) = -(\ln\beta) \sum_n \int d^3\bar{k} \tag{4.15}$$

One further sublety in the formalism ocurs in the treatment of gauge theories. Tr $(\exp(-\beta H))$ must be evaluated in a physical gauge to avoid introducing specious states in the trace or one must modify the definition of the finite temperature partition function to include a projector onto physical states.

$$Z = \mathrm{Tr}\,(P \exp(-\beta H)) \tag{4.16}$$

It is shown in [40] by Hata and Kugo that this may be written as

$$Z = \mathrm{Tr}\,(\exp(-\beta H - \pi Q_c)) \tag{4.17}$$

where Q_c is the Fadeev-Popov ghost charge, generated by a scale transformation on the ghost and anti-ghost fields.

Because of the presence of infinite sums for the energy modes explicit calculations in the imaginary time formalism tend to be rather unwieldy, especially at more than one-loop level. An alternative approach is the real time formalism, which is also known as thermofield dynamics (T.F.D.). This has been developed largely by Umezawa and his collaborators at Alberta, who have explored both the basic formalism and the technical aspects of calculations in some detail. We shall give a brief review below; for a fuller treatment see, for example Umezawa's book [41] or the forthcoming book by Rivers [42]. Both the imaginary time formalism and T.F.D. are, in fact, special applications of a more general complex time method that was first proposed by Mills as long ago as 1955 [43]. T.F.D. has the conceptual advantage of having a clear physical picture behind it, instead of being based on a purely formal analogy between path integrals. We think of our universe as being divided into a large box in which we do our calculations and a remainder, which we shall treat as a thermal bath at temperature T. At this stage we can see heuristically that there is likely to be some form of field doubling in such a theory because energy may be absorbed in our box by either the excitation of new quanta in the box or the annihilation of holes in the thermal bath. We would like to express the thermal equilibrium values in terms of expectation values between pure states $|\beta\rangle$ i.e.

$$\langle\beta|O|\beta\rangle = \frac{\sum_n \langle n|O|n\rangle \exp(-\beta E_n)}{\sum_n \exp(-\beta E_n)} \qquad (4.18)$$

With a single Fock space we would find it impossible to define a state to satisfy the above. If we defined

$$|\lambda\rangle = \sum_n |n\rangle \exp(-\frac{1}{2}\beta E_n)/\sum_n \exp(-\frac{1}{2}\beta E_n) \qquad (4.19)$$

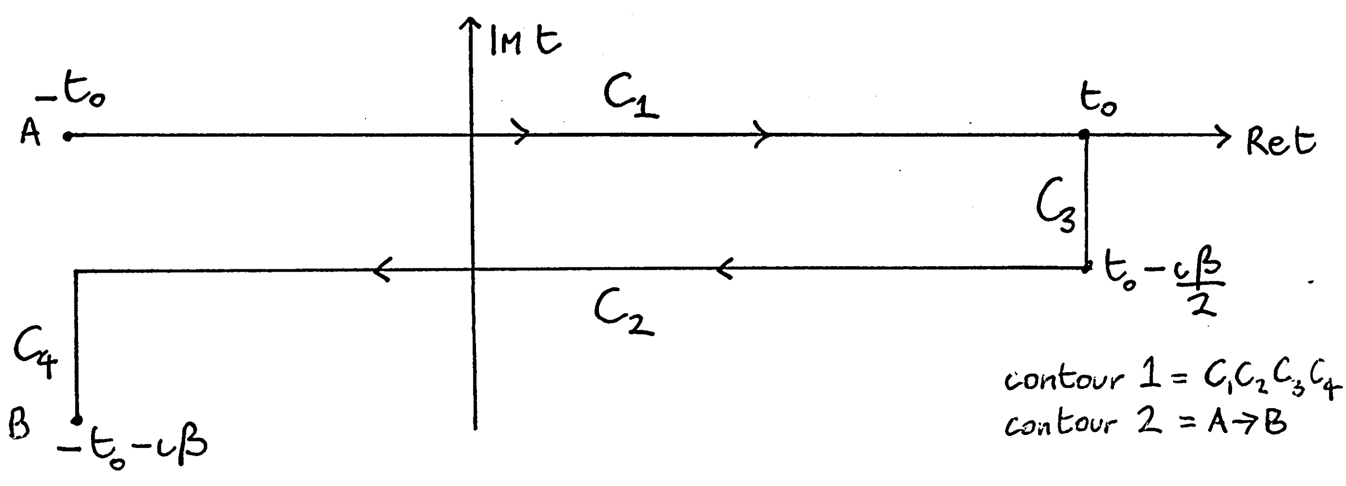
$\langle\lambda|O|\lambda\rangle$ would contain off diagonal terms coming from the matrix elements $\langle m|O|n\rangle$. To remove these we consider the product space $|n\rangle \times |\tilde{m}\rangle$ where we adjoin a new Fock space to the first and label its elements with tildes. Operators would become $O \times 1$ in the old space and $1 \times O$ in the new. We can now define

$$|\beta\rangle = \sum_n |n\rangle \times |\tilde{n}\rangle \exp(-\frac{1}{2}\beta E_n)/\sum_n \exp(-\frac{1}{2}\beta E_n) \qquad (4.20)$$

such that (4.18) is satisfied. The $\langle\tilde{m}|\tilde{n}\rangle$ product in the new space ensures that the matrix elements of O are diagonal.

To see what this approach means in a path integral context consider the path of integration we use in the complex t plane at finite temperature. In the imaginary time formalism we are integrating from $t = 0$ to $t = -i\beta$, which we can translate to $t = -t_0$ to $t = -t_0 - i\beta$ by the periodicity of the fields. The real time formalism corresponds to choosing the contour 2 to integrate along instead of 1.

(Fig.14) Contours in the complex time plane



The required doubling of the fields comes from fields defined on the contours C_1 and C_2 (the portions C_3 and C_4 decouple in the $t_0 \rightarrow \infty$ limit, which we take at the end of any calculation). In fact, any contour that starts at A and ends at B would be just as valid provided that ImT is monotonically decreasing along the contour, which is demanded by causality. We have chosen contour 2 for calculational convenience and because it gives a Hermitian Lagrangian [44],[45]. Fields defined on C_2 can be thought of as thermal ghosts and correspond to the holes in the thermal bath.

$$\phi_2(x,t) = \phi(x,t - i\beta) \tag{4.21}$$

We can write, formally for the path integral

$$Z[J] = N'(\beta) \int_{periodic} [D\phi] \quad \exp(i\int_c (L + J\phi)) \tag{4.22}$$

where $\int_c$ means that the time integral is taken over the contour 2
The Lagrangian is given by

$$L = \frac{1}{2} \phi \left(\partial_c^2 - \bar{\nabla}^2 + m^2 \right) \phi \quad + U(\phi) \tag{4.23}$$

where the time derivative ∂_c is along the contour and the fields are defined on the contour. This can be recast into an integral with fields involving real time arguments (see for example [44]). We find that we obtain a matrix propagator Δ_{ij} $i,j = 1,2$, where the 1 field is defined on the contour C_1 and the 2 field is defined on the contour C_2. The mixing between the fields occurs only in the propagator and not at any vertices. The matrix propagator is given by

$$\Delta_{ij} = \begin{pmatrix} \dfrac{1}{k^2 - m^2 - i\varepsilon} & 0 \\ 0 & \dfrac{-1}{k^2 - m^2 - i\varepsilon} \end{pmatrix}$$

$$- \frac{2\pi i \delta(k^2-m^2)}{\exp(-\beta \| k_0 \|)-1} \begin{pmatrix} 1 & \exp(-\frac{1}{2}\beta \| k_0 \|) \\ \exp(-\frac{1}{2}\beta \| k_0 \|) & 1 \end{pmatrix} \tag{4.24}$$

The potential term in the theory becomes $U(\phi_1) - U(\phi_2)$. The Green's functions are defined as the vacuum expectation values of the physical 1 fields, so only these appear on the external legs of any diagrams that we are calculating. We have traded off a simplification in the individual graphs (integrals now instead of both integrals and sums in the imaginary time formalism) against an increase in the number of graphs. From the point of view of effective potential calculations there are, however, some technical difficulties lurking in the new formalism, particularly as the presence of the heat bath complicates Lorentz covariance (the $k_0 \to 0$ and $\bar{k} \to 0$ limits do not necessarily commute [45]). We shall also note that the most obvious definition of the effective potential is identically zero in T.F.D.

4.3 IMAGINARY TIME EFFECTIVE POTENTIAL AT FINITE T

We follow the careful treatment of Rivers [46] in this section to show that it is possible to produce a convex effective potential at finite temperature, but only up to a temperature at which the minima have crept in to the classical points of inflection, at least for a classically spontaneously broken potential. The same problem does not arise in Coleman-Weinberg symmetry breaking, so calculations of the critical temperature performed in this context are still reliable (in so far as perturbative calculations ever are). The failure of the calculation in the spontaneously broken case means that much of the work of Dolan and Jackiw [18] is incorrect. We discuss briefly the criteria for the restoration of symmetry at finite temperature to show why this is so.

They note that for a reflection invariant effective potential

$$\frac{\partial V(\bar{\phi}^2)}{\partial \bar{\phi}} = 2\,\bar{\phi}\,\frac{\partial V}{\partial \bar{\phi}^2} \qquad (4.25)$$

so symmetry breaking will be absent when $\frac{\partial V}{\partial \bar{\phi}^2} \neq 0$ for $\bar{\phi} \neq 0$. We now decompose the effective potential into a zero temperature part V^0 and a finite temperature part V^β . One can show that finite temperature effects do not introduce any new infinities into the theory [47], so one may still choose to perform renormalization at zero temperature and regard the contribution of V^β as a finite temperature correction. The mass parameter of the theory can be defined as

$$m^2 = \frac{\partial^2 V^0}{\partial \bar{\phi}^2} \Bigg|_{\bar{\phi}=0} \tag{4.26}$$

To see whether we are sitting at a minimum or a maximum at the origin Dolan and Jackiw suggest that we consider

$$\frac{\partial^2 V}{\partial \bar{\phi}^2} \Bigg|_{\bar{\phi}=0} = m^2 + \frac{\partial^2 V^\beta}{\partial \bar{\phi}^2} \Bigg|_{\bar{\phi}=0} \tag{4.27}$$

If the r.h.s. > 0 we are supposedly at a minimum and the the critical temperature is defined to be that for which the r.h.s. = 0. However, we know that this is too simplistic; because of convexity one cannot simply look at the origin of the effective potential to see whether one is rolling up or downhill. Even if one ignored the problem with convexity one would have to deal with the complexity of the higher order corrections at the origin (at one loop and high temperature there is a fortuitous cancellation which does not persist at higher orders).

We now consider explicitly the calculation of a convex one-loop effective potential at finite temperature to see whether it is possible to obtain some other criterion for the restoration of symmetry. We first decompose unity as

$$\int dc \; \delta(c - \frac{1}{\Omega}\int d^4x \; \phi(x)) \tag{4.28}$$

and insert this into the path integral expression for the generating functional Z

$$N \int [D\phi] \int dc \; \delta(c - \frac{1}{\Omega} \int d^4x \; \phi(x)) \; \exp(-\frac{1}{a} (S[\phi] - \int J\phi)) \tag{4.29}$$

where we have inroduced the artificial loop expansion parameter a because h appears in the limits of integration (via β) in the action. If we now write $\phi(x)$ as $c + \sqrt{a}\eta(x)$, insert it into the path integral and exponentiate the delta fuction in (4.30) we find

$$Z = \int dc d\alpha \int [D\phi] \exp(-\frac{1}{a} (S[c+\sqrt{a}\eta] - Jc\Omega)) \exp(i\alpha \int d^4x \eta) \tag{4.30}$$

which we can expand in η to get

$$Z = N \int dc\ A(c) \exp\ (-\frac{\Omega}{a} (U(c) - Jc)) \tag{4.31}$$

where, dropping terms of $O(\eta^3)$ and higher

$$A(c) = \int d\alpha \int [D\eta] \exp(i\alpha \int d^4x \eta) \exp(-\frac{1}{2} \int d^4x \eta (-\nabla^2 + U''(c)) \eta \tag{4.32}$$

If $U''(c) > 0$ we may perform the Gaussian integration to get

$$A(c) = (U''(c))^{1/2} \exp\ (-\frac{1}{2}\Omega\ \Sigma \int dk\ \ln\ (k^2 + U''(c)) \tag{4.33}$$

Thus we get finally

$$Z = N \int dc\ (U''(c))^{1/2} \exp(-\Omega(\text{"}V\text{"}^1(c) - Jc)) \tag{4.34}$$

In a similar manner to our discussion of the constraint effective potential the $\Omega \to \infty$ limit will enforce the convexity. If we now enquire what happens to the expression at finite temperature we encounter difficulties. The high temperature limit of the one loop effective potential is given by [18]

$$V(\bar{\phi}) = U(\bar{\phi}) + \frac{U''(\bar{\phi})}{24\beta^2} + O(\frac{1}{\beta}) \tag{4.35}$$

From the above we find that the minima creep in with increasing temperature

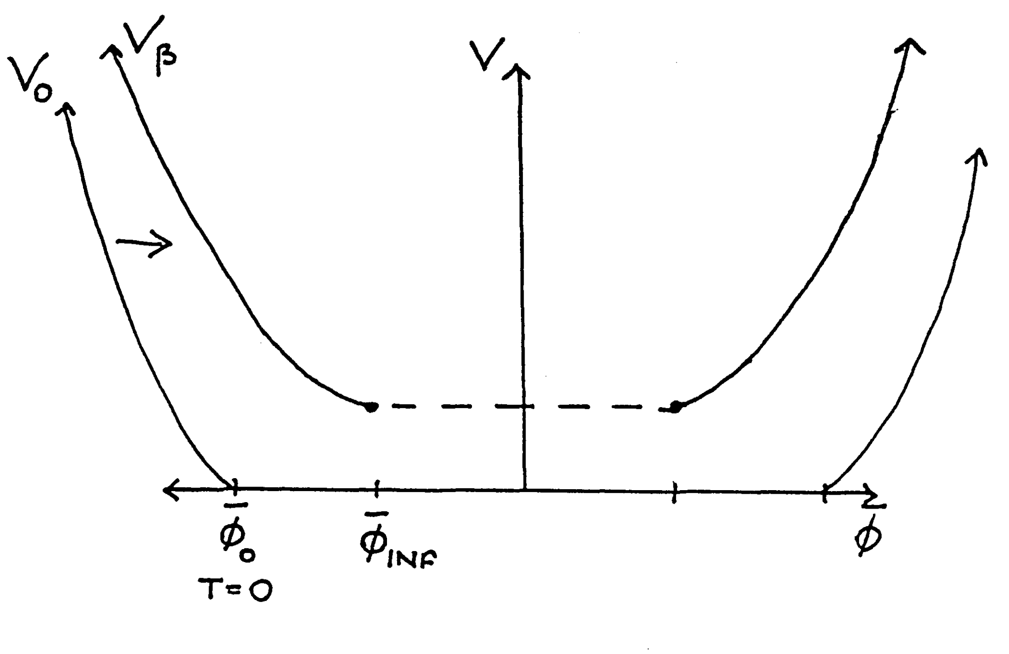
$$\bar{\phi}_0 = \pm \left(\frac{6}{\lambda}\left(m^2 - \frac{\lambda}{24\beta^2}\right)\right)^{1/2} \tag{4.36}$$

At some temperature they will reach the classical points of inflection

$$\bar{\phi}_{inf} = \pm \left(\frac{2m^2}{\lambda}\right)^{1/2} \tag{4.37}$$

U" will become negative and the whole expansion scheme will break down.

<u>(Fig.15)</u> Minima creep into the classical points of inflection



The breakdown does not occur in the Coleman-Weinberg model, where a similar expression to (4.34) may be derived, but in this case U"(c) is positive for all values of c because the spontaneous symmetry breaking is induced by the radiative corrections. In the spontaneously broken case we have here, however, we have not reached the supposed phase transition when our calculation breaks down. This might conceivably be a quirk of the imaginary time formalism, so in the next section we will consider the calculation of a finite temperature convex effective potential using the real time formalism.

4.4 REAL TIME CONVEX EFFECTIVE POTENTIAL AT FINITE T

We can exploit the presence of the source term for the 2 field in the real time formalism generating functional to exhibit an exchange symmetry between the 1 and 2 quantities which will imply that the effective potential defined in the most natural manner is actually zero [45]. If we consider the finite temperature Lagrangian

$$L = \frac{1}{2} \phi_i \Delta_{ij} \phi_j - U(\phi_1) + U(\phi_2) \qquad (4.38)$$

We find that it has the following symmetry, where we denote complex conjugation by an asterisk

$$L[\phi_1, \phi_2] = -L^*[\phi_2^*, \phi_1^*] \qquad (4.39)$$

As a consequence of this the generating functional Z has the symmetry

$$Z[J_1,J_2] = Z^*[-J_2,-J_1] \tag{4.40}$$

where we have assumed real sources for simplicity. This gives the following symmetry for W

$$W[J_1,J_2] = - W^*[-J_2,-J_1] \tag{4.41}$$

If we now define $\bar{\phi}_i$ by

$$\bar{\phi}_i = \frac{\delta W}{\delta J_i} \tag{4.42}$$

we find that

$$\bar{\phi}_1[J_1,J_2] = \bar{\phi}_2^*[-J_2,-J_1] \tag{4.43}$$

With an effective action defined by

$$\Gamma[\bar{\phi}_1,\bar{\phi}_2] = W[J_1,J_2] - J_1\bar{\phi}_1 - J_2\bar{\phi}_2 \tag{4.44}$$

our symmetry becomes

$$\Gamma[\bar{\phi}_1 ,\bar{\phi}_2] = - \Gamma[\bar{\phi}_2^* ,\bar{\phi}_1^*] \tag{4.45}$$

with a similar identity applying to the effective potential. The ground state of Γ is located by solving the equations $\frac{\partial V}{\partial \phi_i} = 0$, i.e. $J_i = 0$, and we can see that this implies from (4.43) that the ground state values of $\bar{\phi}_1$ and $\bar{\phi}_2$ are identical.

However, if we consider the effective potential version of (4.45)

$$V(\bar{\phi}_1, \bar{\phi}_2) = -V^*(\bar{\phi}_2^*, \bar{\phi}_1^*) \tag{4.46}$$

we see that this must be zero for identical field arguments. Although this is true only at the minimum of the potential, explicit calculations show that the one-loop term in V is ill defined unless $\bar{\phi}_1 = \bar{\phi}_2$, in which case it is zero! We can sidestep the problem by considering the expansion for $\Gamma[\bar{\phi}_1, \bar{\phi}_2]$

$$\Gamma = \sum_{n,m} \int d^4x_1 \ldots d^4x_n \int d^4y_1 \ldots d^4y_m \, \Gamma^{n,m}(x_1 \ldots y_n)$$
$$[\phi_1(x_1) - v] \ldots\ldots [\phi_2(y_m) - v] \tag{4.47}$$

where the superscripts denote the number of 1 and 2 field external legs (the symmetry we have discussed means that v is the same for both $\bar{\phi}_1$ and $\bar{\phi}_2$). We can insert the momentum expansion into this

$$\Gamma[\bar{\phi}_1, \bar{\phi}_2] = \int d^4x \, (-V(\bar{\phi}_1, \bar{\phi}_2) + ..) \tag{4.48}$$

to obtain

$$V = -\sum_{n,m} \frac{1}{n!\, m!} \Gamma^{n,m}(p=0) \, [\bar{\phi}_1 - v]^n \, [\bar{\phi}_2 - v]^m \tag{4.49}$$

If we now differentiate this w.r.t $\bar{\phi}_1$ and set $\bar{\phi}_1 = \bar{\phi}_2 = v$ we find

$$\left.\frac{\partial V}{\partial \bar{\phi}_1}\right|_{\bar{\phi}_1 = \bar{\phi}_2 = v} = -\Gamma^{1,0}(p=0) \tag{4.50}$$

The l.h.s. equals the proper ϕ_1 tadpole in the finite temperature quantum field theory defined by the shifted Lagrangian $L(\phi_1 + v, \phi_2 + v)$. For a constant field the result can trivially be continued to the imaginary time axis (by translational invariance) so we have

$$\left.\frac{\partial V}{\partial \bar{\phi}_1}\right|_{\bar{\phi}_1 = \bar{\phi}_2 = v} = \left.\frac{\partial V_{\text{imaginary time}}}{\partial \bar{\phi}}\right|_{\bar{\phi} = v} \tag{4.51}$$

We can thus integrate the sum of our tadpole graphs in the real time formalism to obtain the imaginary time effective potential. If we want to obtain a convex effective potential, however, we must find out how to incorporate convexity into our tadpole calculations and this is the subject of the next section.

4.5 ZERO TEMPERATURE TADPOLES

We attempt to use the suggestion of Fujimoto et al., [11] that the effects of subsidiary minima in the path integral should be taken into account. As in the saddle point case, for certain values of J more than one minima will contribute to the path integral. If we consider $\lambda\phi^4$ we find that at $J \simeq 0$

$$\ln\left(\exp(\Gamma_1) + \exp(\Gamma_2) \right) = \tilde{\Gamma} \tag{4.52}$$

where $\Gamma_1 = \Gamma[\bar{\phi}_0]$ and $\Gamma_2 = \Gamma[-\bar{\phi}_0]$ and the $\tilde{\Gamma}$ on the r.h.s. is the total convex effective action. If we differentiate Γ w.r.t. $\bar{\phi}_0$ we find

$$\frac{\partial\tilde{\Gamma}}{\partial\bar{\phi}_0} = \frac{\frac{\partial\Gamma_1}{\partial\bar{\phi}_0}\exp\Gamma_1 + \frac{\partial\Gamma_2}{\partial\bar{\phi}_0}\exp\Gamma_2}{\exp\Gamma_1 + \exp\Gamma_2} \tag{4.53}$$

If we now recall that $\Gamma_1 = \Gamma_2$ we find

$$\frac{\partial\tilde{\Gamma}}{\partial\bar{\phi}_0} = \frac{1}{2}\left(\frac{\partial\Gamma_1}{\partial\bar{\phi}_0} + \frac{\partial\Gamma_2}{\partial\bar{\phi}_0} \right) \tag{4.54}$$

which we shall call the total tadpole, or $\tilde{\Gamma}^1$ for brevity. In general, if the Lagrangian has some symmetry group G the various minima, say $\langle\phi\rangle$, can all be written as the group transform of one particular minimum, say $\langle a\rangle$, so $\tilde{\Gamma}^1$ can be thought of as a function of $\langle a\rangle$ for the purpose of integrating the tadpole to obtain V.

To see the implications of (4.54) for the form of the effective potential consider again a spontaneously broken $\lambda\phi^4$ theory. For convenience in calculation we use dimensional regularization and follow the methods of Lee and Sciaccaluga [9]. To calculate the tadpole we must work order by order in h, to any desired degree of accuracy. The "zero-loop tadpole" is just the differential of the classical potential U. If we choose

$$U = -\frac{1}{2} m \bar{\phi}^2 + \frac{1}{4!} \lambda \bar{\phi}^4 \tag{4.55}$$

then the potential in the shifted theory is given by

$$U = -\frac{1}{2} M^2 \bar{\phi}^2 + \frac{1}{3!} F \bar{\phi}^3 + \frac{\lambda}{4!} \bar{\phi}^4 \tag{4.56}$$

where

$$M = \left.\frac{\partial^2 U}{\partial \bar{\phi}^2}\right|_{\bar{\phi} = \bar{\phi}_0} \qquad F = \left.\frac{\partial^3 U}{\partial \bar{\phi}^3}\right|_{\bar{\phi} = \bar{\phi}_0} \tag{4.57}$$

The tadpole is given by

$$U' = -M^2 \bar{\phi} + \frac{1}{2} F \bar{\phi}^2 + \frac{\lambda}{6} \bar{\phi}^3 \tag{4.58}$$

For $J \neq 0$ only one of the minima contributes and, upon integration, we recover U. However, at J=0, (4.54) applies and we have

$$\tilde{\Gamma}^1 = U'\Big|_{\bar{\phi}_0} + U'\Big|_{-\bar{\phi}_0} = 0 \tag{4.59}$$

Thus our zero-loop effective potential at J=0, from the integration of (4.59) is just a constant, giving the desired linear interpolation.

If we now consider the one-loop tadpole we find it is given by

$$\Gamma^1 = -\frac{iF}{32\pi^4} \int d^4k \frac{1}{k^2 - M^2} \tag{4.60}$$

where $F=U'''$ and $M^2=U''$. After dimensional regularization and minimal subtraction this becomes

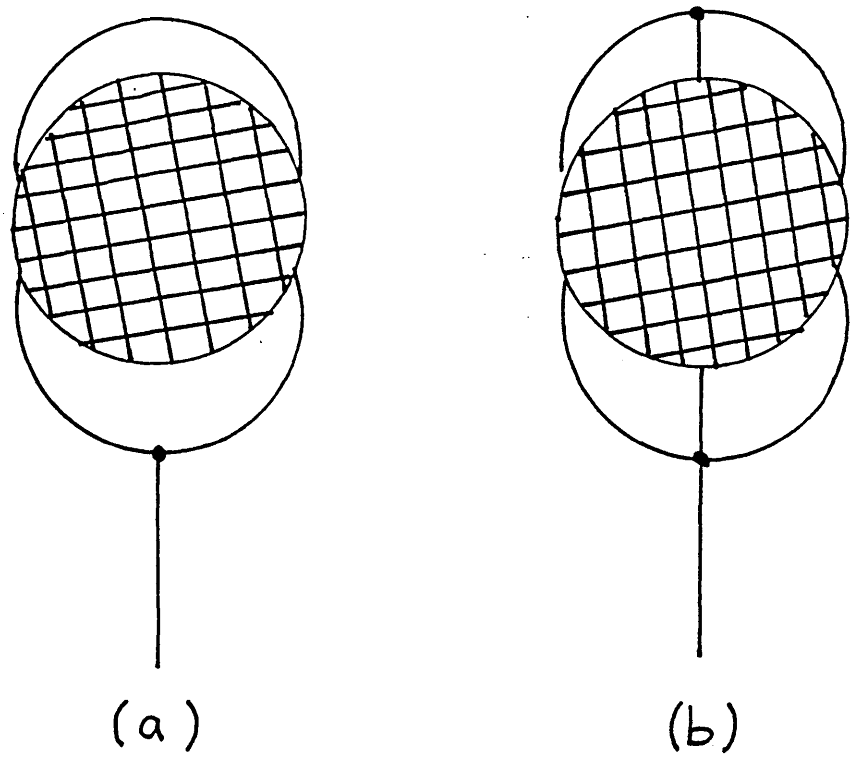
$$\Gamma^1 = \frac{F}{32\pi^2} \ln \frac{M^2}{\mu^2} \qquad (4.61)$$

where μ^2 is the arbitrary mass-squared introduced in the renormalization. Because $F(-\bar{\phi})=-F(\bar{\phi})$ and $M^2(-\bar{\phi})=M^2(\bar{\phi})$ the contributions from ϕ_0 and $-\phi_0$ when J=0 to the total tadpole will cancel out, giving a constant contribution to V and preserving the flat-bottomed bucket shape. When J≠0 we recover the standard one-loop effective potential upon integration of (8)

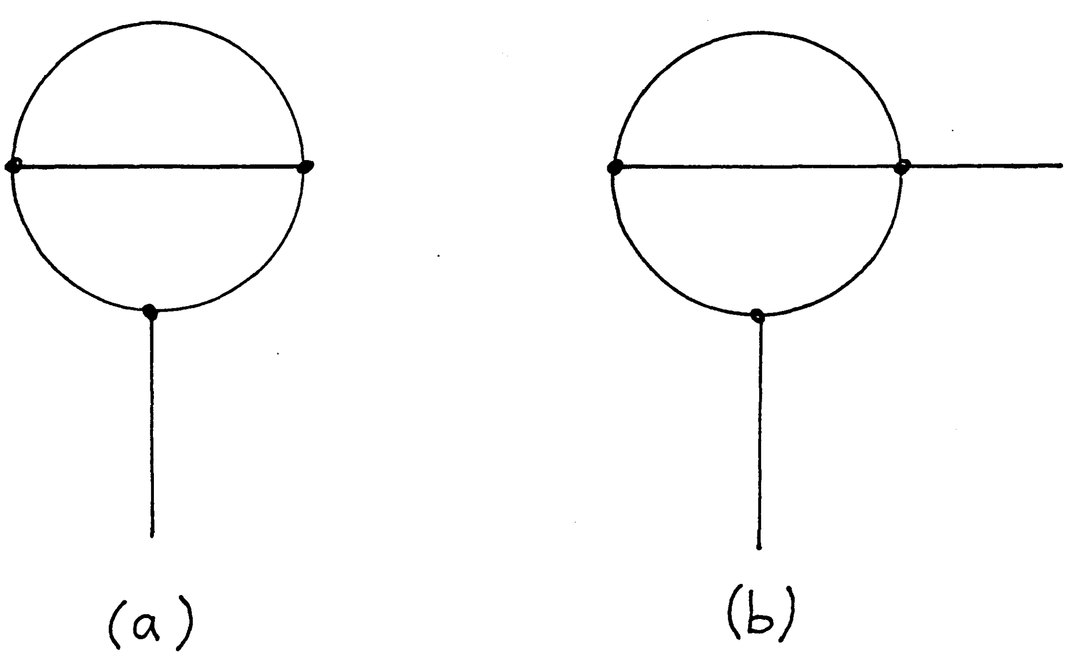
$$V=\frac{1}{64\pi^2} M^4 (\ln\frac{M^2}{\mu^2} - \frac{1}{2}) \qquad (4.62)$$

We can see on general grounds that this behaviour will be maintained to all orders. The tadpoles must be of the form shown overleaf. The tadpole in Fig.16(a) is proportional to F and that in Fig.16(b) to Fλ. Inside the blobs the F's must always occur in pairs; otherwise we would have more than one external leg, as we can see in Fig.17(b). All the other elements occurring in the tadpole, such as M^2, are symmetric under $\bar{\phi}_0 \leftrightarrow -\bar{\phi}_0$ interchange so, with a tadpole proportional to f, a sum over the two minima will give zero whatever the order in the loop expansion.

(Fig.16) General form of tadpole graphs



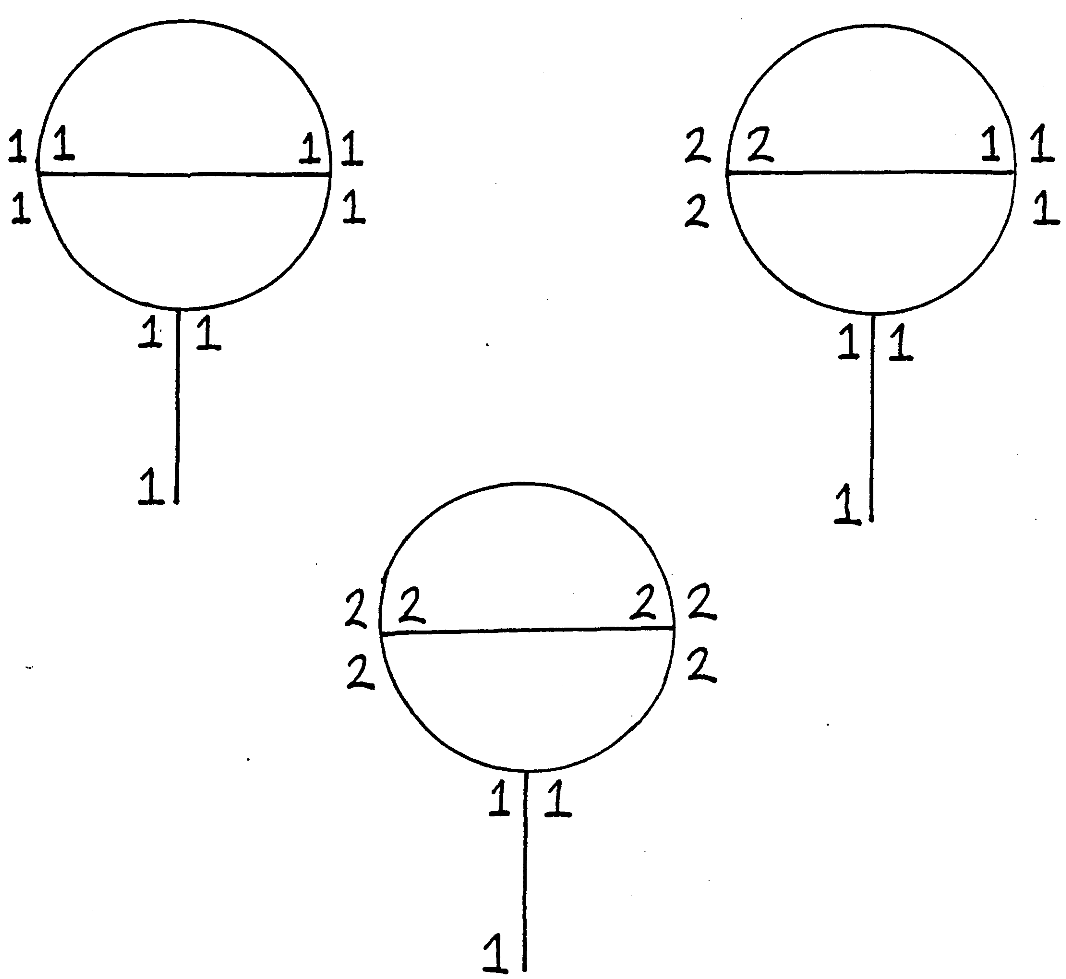
(Fig.17) Two-loop graphs



4.6 TADPOLES AT FINITE TEMPERATURE AND FOR GAUGE THEORIES

If we consider the calculation of a finite temperature effective potential using T.F.D. we obtain a similar convexification. The only mixing between the physical "1" fields and the thermal doublet "2" fields in T.F.D. occurs in the propagators and the vertices differ only in sign (not in structure). To obtain the required graphs for a finite temperature loop calculation of the tadpole one takes the zero temperature graph, fixes the external leg to be a "1" field and then distributes "1" and "2" labels over the ends of the propagators in as many ways as possible. For example some of the two-loop tadpoles in the $\lambda\phi^4$ theory are shown below.

(Fig.18) Two-loop tadpoles



Although the form of the propagators is different at finite temperature

$$\Delta_{11}= \frac{1}{k^2-M^2+i\varepsilon} - 2\pi i \frac{\delta(k^2-M^2)}{\exp(\beta\|k_0\|)-1} \tag{4.63}$$

$$\Delta_{12}=\Delta_{21}= -2\pi i\delta(k^2-M^2)\frac{\exp(\beta\|k_0\|/2)}{\exp(\beta\|k_0\|)-1} \tag{4.64}$$

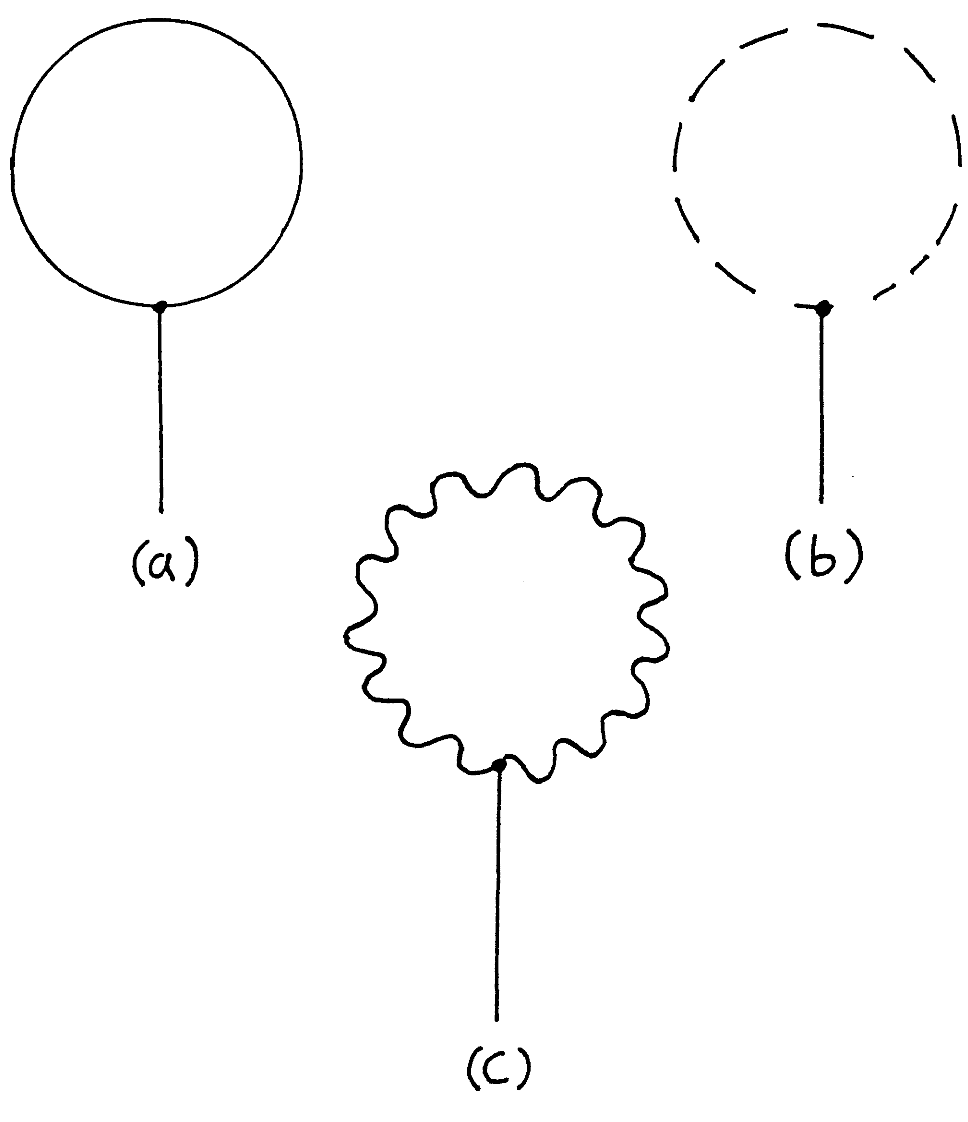
$$\Delta_{22}= \frac{1}{k^2-M^2-i\varepsilon} -2\pi i\frac{\delta(k^2-M^2)}{\exp(\beta\|k_0\|)-1} \tag{4.65}$$

they are still $\phi_0 \to -\phi_0$ symmetric. We can therefore apply the diagrammatic arguments of the previous section to each tadpole Fig. 18 in turn and observe that the J=0 contributions from both minima cancel. The result generalizes in the same manner as the zero temperature case to all orders because we can take each distribution of "1" and "2" labels in turn and show that the contributions of the two minima in each cancel separately by virtue of the $F(\bar{\phi}) \to -F(-\bar{\phi})$ reflection antisymmetry (we remember that the form of the vertices is the same for "1" and "2" fields and that the field shift we perform to evaluate the potential is the same for both "1" and "2" fields).

As a simple example of a spontaneously broken gauge theory we consider an Abelian Higgs model. In order to use the same cancellation mechanism as in the $\lambda\phi^4$ case we work in an 't Hooft gauge. The Lagrangian will be that given by (2.32) and the gauge fixing given by (2.50). With the given gauge fixing the allowed directions of symmetry breaking are constrained by the Nielsen identities to be parallel or antiparallel to $\bar{\phi}_{i0}$ (if we are in the 't Hooft gauge proper), picking out two points on the ring of minima of the potential as we discussed in chapter 3. As in the $\lambda\phi^4$ theory only one of the minima will contribute to an evaluation of the effective potential when $J\neq 0$, whereas for $J=0$ both points contribute, giving a constant effective potential.

We can see this at the one-loop level (the zero-loop is identical to the pure scalar) by considering the diagrams below

<u>(Fig.19)</u> One loop-tadpoles in the Abelian Higgs model



The propagators and vertices necessary to evaluate these are listed in appendix B and the results are

$$\text{Fig.19(a)} = \frac{\lambda}{3}\,\bar{\phi}_{i0}\int d^4k\left(\frac{1}{k^2-m_1^2} + \frac{k^2-\xi^2 e^2\,\bar{\phi}_0^{\,2}}{D_n}\right) \tag{4.66}$$

$$\text{Fig.19(b)} = e^2\xi\,\bar{\phi}_{i0}\varepsilon_{ij}\int d^4k\,\frac{1}{k^2} \tag{4.67}$$

$$\text{Fig.19(c)} = 2e^2\,\bar{\phi}_{i0}\int d^4k\left(\frac{3}{k^2-e^2\,\bar{\phi}_0^{\,2}} + \frac{\xi(k^2-m_2^2-e^2\xi\,\bar{\phi}_0^{\,2})}{D_n}\right) \tag{4.68}$$

The proportionality to $\bar{\phi}_0$ which ensures the cancellation of tadpoles from the two minima is still present and the generalization of the one-loop result to all orders and finite temperature follows the same path as the scalar case.

4.7 DISCUSSION

We have seen that it is possible to produce a finite-temperature, convex effective potential using T.F.D. tadpoles. However, as the temperature increases the minima of the effective potential creep inwards. At some temperature $\bar{\phi}_0$ will reach the points of inflection of the classical potential where $M^2=U''=0$, and the loop expansion will break down. This is almost obvious from the presence of M^2 as the mass term in the propagators of T.F.D., and explicit calculations provide confirmation. For instance the two-loop effective potential of the pure scalar theory in the $T\to\infty$, $M^2 \to 0$ limit is given by (see appendix D)

$$V = \frac{1}{2} m^2 \bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 + \frac{\lambda \phi^2}{48} T^2 + \frac{\lambda^2 \bar{\phi}^2}{256\pi^4} \ln \frac{M^2}{\mu^2} T^2 \qquad (4.69)$$

which not only diverges but also emphasizes terms in the expansion of higher order in h. As we noted before, in the imaginary time formalism, this problem will not arise for a theory with Coleman-Weinberg type symmetry breaking because there $M^2 > 0$ for all values of $\bar{\phi}$.

Our attempt at producing a finite temperature, convex effective potential in the real time formalism has thus run into exactly the same problem as the imaginary time calculation. When the minima of the potential creep in to the points of inflection of the classical potential the expansion breaks down. That there may be something more to this breakdown than purely formal difficulties is sugested by some recent calculations that were carried out by de Carvalho et al. [48] who considered a one-loop effective action which, instead of being expanded about a uniform background, was expanded about the one-dimensional kink solution to the classical equations of motion.

$$\phi_k(x) = \left(\frac{6m^2}{\lambda}\right) \tanh \left(\frac{mx}{\sqrt{2}}\right) \qquad (4.70)$$

Like a Block wall in a spin system the solution interpolates between the two possible vacua and could be thought of as a domain wall. One can now compare the free energy (given by the effective potential) at a given temperature for this solution with the uniform backround using the imaginary time formalism. This gives, on dividing by the area of the wall, a free energy per unit area.

$$f(T) = \frac{1}{A} \left(\Gamma(T,\phi_k(x)) - \Gamma(T,\phi) \right) \qquad (4.71)$$

If we evaluate the one-loop approximation to (4.70) we find that, at sufficiently high temperature

$$f(T) = 4\sqrt{2}\frac{m^3}{\lambda} - \frac{\sqrt{2}mT^2}{4} \qquad (4.72)$$

This will vanish, and hence it will become energetically favourable to have a kink solution rather than a constant solution, when $T^2 = \frac{16m^2}{\lambda}$, which is known as the percolation temperature. If we now recall the one-loop result for the minima at finite temperature

$$\bar{\phi}_0 = \pm \left(\frac{6}{\lambda} \left(m^2 - \frac{\lambda}{24\beta^2}\right)\right)^{1/2} \qquad (4.36)$$

We see that this is precisely the temperature at which the minima reach the points of inflection $\bar{\phi}_{inf} = \pm \left(\frac{2m^2}{\lambda}\right)$. The suggestion is that the breakdown in the loop expansion might signal the formation of domain walls. It might prove illuminating to examine the Coleman-Weinberg model where such a breakdown does not occur to see whether a percolation temperature appears here also, which would cast doubt on the interpretation above. The problem here is that the classical potential is really of order h because we impose $\lambda \sim e^4$ so we must include the one-loop terms from the gauge fields in our equations of motion if we are attempting to find a kink solution similar to (4.70) to expand about. These will contain logarithmic parts and hence may not be very amenable to solution.

APPENDIX A: SOURCES WHICH ARE NOT LEGENDRE TRANSFORMED

Consider differentiating $\tilde{\Gamma}$ w.r.t. a source which we do not Legendre transform, say h.

$$\frac{\delta\tilde{\Gamma}}{\delta h}\Bigg|_{\text{total}} = \frac{\delta\tilde{\Gamma}}{\delta h}\Bigg|_{\bar{\phi}\,=\,\text{constant}} + \frac{\delta\tilde{\Gamma}}{\delta\bar{\phi}}\frac{\delta\bar{\phi}}{\delta h} \qquad \text{(A1)}$$

If we now use the standard Legendre transform definition of $\tilde{\Gamma}$

$$\tilde{\Gamma}[\bar{\phi}] = \tilde{W}[J] - J\bar{\phi} \qquad \text{(A2)}$$

We find

$$\frac{\delta\tilde{\Gamma}}{\delta h}\,\text{total} = \frac{\delta\tilde{W}}{\delta h}\Bigg|_{\bar{\phi}\,=\,\text{constant}} + \frac{\delta\tilde{W}}{\delta J}\frac{\delta J}{\delta\bar{\phi}}\frac{\delta\bar{\phi}}{\delta h} - \frac{\delta J}{\delta\bar{\phi}}\frac{\delta\bar{\phi}}{\delta h}\bar{\phi} - J\frac{\delta\bar{\phi}}{\delta h} \qquad \text{(A3)}$$

The second and third terms cancel by observing that $\bar{\phi} = \frac{\delta\tilde{W}}{\delta J}$ and we also note that $J = -\frac{\delta\tilde{\Gamma}}{\delta\bar{\phi}}$. This gives us

$$\frac{\delta\tilde{\Gamma}}{\delta h}\Bigg|_{\bar{\phi}\,=\,\text{constant}} = \frac{\delta\tilde{W}}{\delta h}\Bigg|_{\bar{\phi}\,=\,\text{constant}} \qquad \text{(A3)}$$

Or, as we can express J as a function of $\bar{\phi}$.

$$\frac{\delta\tilde{\Gamma}}{\delta h}\Bigg|_{\bar{\phi}\,=\,\text{constant}} = \frac{\delta\tilde{W}}{\delta h}\Bigg|_{J\,=\,J(\bar{\phi})} \qquad \text{(A4)}$$

Now $\frac{\delta\tilde{W}}{\delta h}$ is just equal to

$$\frac{\int[D\phi]\ O\ \exp \frac{i}{h}\left(\int d^4x\ L + K_i Q_i + hO + J_i\phi_i \right)}{\tilde{Z}} \tag{A5}$$

and we evaluate this at $J = J(\bar{\phi})$, which is exactly what we would have done in the standard Legendre transform definition. We therefore write.

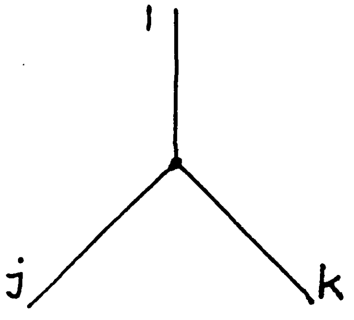
$$\frac{\delta\tilde{\Gamma}}{\delta h} = \frac{\delta\tilde{W}}{\delta h} \tag{A6}$$

$\frac{1}{\xi}$ can be regarded as an x-independent source to give, in a similar manner,

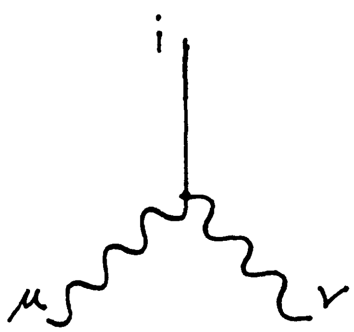
$$\frac{\delta\tilde{\Gamma}}{\delta \xi} = \frac{\delta\tilde{W}}{\delta \xi} \tag{A7}$$

APPENDIX B: PROPAGATORS AND VERTICES FOR THE ABELIAN HIGGS MODEL

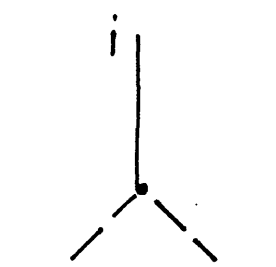
The vertices in the shifted theory that we actually make use of in our calculations are



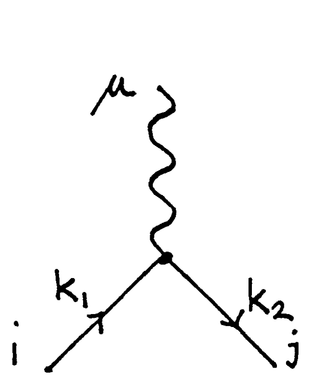
$$= -\, i\frac{\lambda}{3}\left(\delta_{ij}\bar{\phi}_k + \delta_{jk}\bar{\phi}_i + \delta_{ki}\bar{\phi}_j\right) \quad \text{(B1)}$$



$$= i2e^2\bar{\phi}_i g_{\mu\nu} \quad \text{(B2)}$$



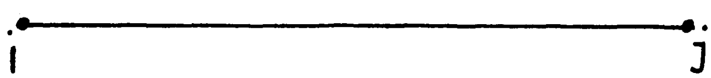
$$= -ie^2\xi\bar{\phi}_j\varepsilon_{ji} \quad \text{(B3)}$$



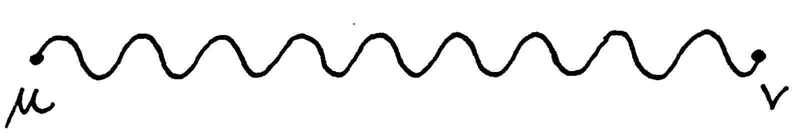
$$= -e\varepsilon_{ij}(k_1 + k_2)^{\mu} \quad \text{(B4)}$$

The propagators we use are

$$\frac{i}{k^2 - e^2 \varepsilon_{ij} \xi v_i \bar{\phi}_j} \tag{B5}$$



$$i\frac{(k^2 - \xi e^2 \bar{\phi}^2)}{D_n} (\delta_{ij} - \eta_i \eta_j) \quad + \quad i\frac{\eta_i \eta_j}{k^2 - m_1^2} \tag{B6}$$



$$-i\frac{1}{k^2 - e^2 \bar{\phi}^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \quad - \quad i\frac{\xi(k^2 - m_2^2 - e^2 \xi v^2)}{D_n} \frac{k_\mu k_\nu}{k^2} \tag{B7}$$

APPENDIX C: FORTRAN PROGRAM USED FOR WILSON RECURSION RELATIONS

We used the following Fortran 77 program running on the CDC machines at Imperial College computing centre to evaluate the Wilson recursion relations. The infinite integral in (3.28) is approximated by a finite range and evaluated numerically, using a NAG library routine. The results were sent to tape4=out and a graph plotted on the line-printer. Two typical graphs are shown after the program.

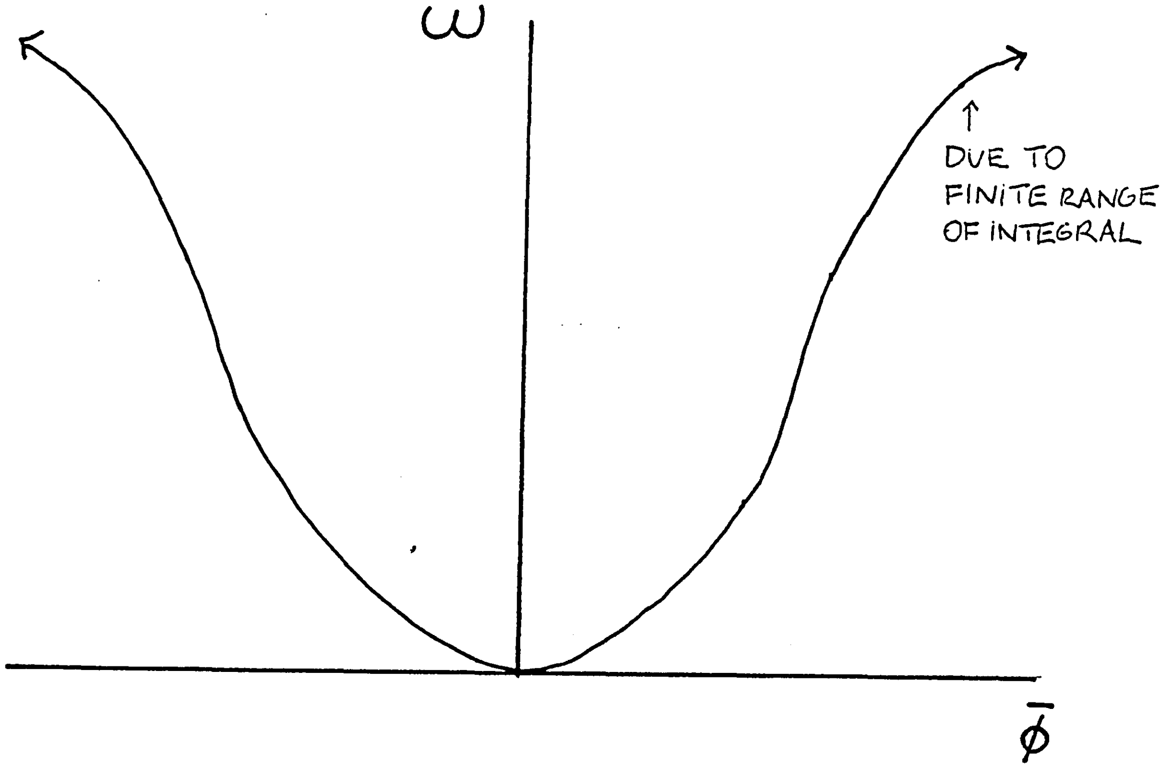
```
PROGRAM BFUK(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE4=OUT)

C LIST OF VARIABLES
C
C      M2 = MASS-SQUARED / LAMBDA = QUARTIC COUPLING /
C      A = INTEGRATION RANGE / D = SPACETIME DIMENSION /
C      DY = INCREMENT FOR INTEGRATION / C = CONSTANT IN
C      CLASSICAL POTENTIAL (INSERTED FOR NUMERICAL REASONS)
C
C SETUP PARAMETERS
C
      COMMON/USEFUL/A,D,DY,U(0:11,110),NYM
      INTEGER NPHIM,MAXINT,IFAIL,NGRAPH
      REAL DPHI,LAMBDA,M2,TEMP(0:11,110),X(110),Y(110),Z,ERROR,
     +UMID,C
      DATA A/2.0/,D/3.0/,DY/0.05/,M2/.1/,LAMBDA/.1/,IFAIL/1/
      DPHI = DY
      NYM = IFIX(2.*A/DY)
      NPHIM = NYM
      MAXINT = 2
      C = 2.
C
C FILL AN INTIAL ARRAY WITH CLASSICAL POTENTIAL VALUES
C
      DO 10 I=1,NYM
        Z = -A + FLOAT(I)*DY
        U(0,I) = -M2*(Z**2) + LAMBDA*(Z**4) + C
        TEMP(0,I) = U(0,I)
 10   CONTINUE
C
C INTEGRATION LOOP
C
      DO 20 I = 1,MAXINT
         DO 30 J = 1,NPHIM
            DO 40 K = 1,NYM
               Y(K) = YINT(K,J,I)
               X(K) = -A + K*DY
 40         CONTINUE
         CALL D01GAF(X,Y,NYM,ANS,ERROR,IFAIL)
C
C ERROR HANDLING
C
      IF(IFAIL)9,9,8
 8    IF(IFAIL.EQ.1)WRITE(6,995)
      IF(IFAIL.EQ.2)WRITE(6,996)
      IF(IFAIL.EQ.3)WRITE(6,997)
```

```
C
C ERROR ABORT
C
      STOP
C
C IF NOT CARRY ON WITH INTEGRATION
C
 9       U(I,J) = ANS
 30   CONTINUE
C
C RESCALE THE U VALUES
C
      UMID = U(I,IFIX(NYM/2.))
      DO 35 L = 1,NYM
         TEMP(I,L) = U(I,L)
         U(I,L) = -2.**(-D*FLOAT(I-1))*ALOG(TEMP(I,L)/UMID)
 35   CONTINUE
 20   CONTINUE
C
C FORMATS FOR ERRORS
C
 995 FORMAT(1X,"<4 POINTS")
 996 FORMAT(1X,"WRONG ORDER")
 997 FORMAT(1X,"NOT DISTINCT")
C
C PRINTOUT RESULTS
C
      DO 60 I=0,MAXINT
         WRITE(4,998)I,A,M2,LAMBDA
         WRITE(4,898)
         DO 50 J = 1,NYM,2
            Z = -A + FLOAT(J)*DY
            WRITE(4,999)U(I,J),TEMP(I,J),Z,J
 50      CONTINUE
 60   CONTINUE
C
C GRAPHICAL OUTPUT
C
      DO 70 I = 0,MAXINT
         WRITE(4,998)I,A,M2,LAMBDA
         WRITE(4,898)
         NGRAPH = NYM/4
         DO 80 J = NGRAPH,NYM-NGRAPH
            X(J) = U(I,J)
 80      CONTINUE
         CALL PLOT(X,NYM)
 70   CONTINUE
C
C FORMATS FOR PRINTOUT
C
 998 FORMAT (1X//"ITERATION",I3,1X,"A",F6.2,"M2",F6.2,"LAMBDA"
    +,F6.2/)
 898 FORMAT (1X,8X,"U VALUES",12X,"T VALUES",12X,"X VALUES",12X
    +,"N"/)
 898 FORMAT (1X,"GRAPHICAL REPRESENTATION OF OUTPUT"//)
 999 FORMAT (1X,1PE20.6,1PE20.6,1PE20.6,I10)
```

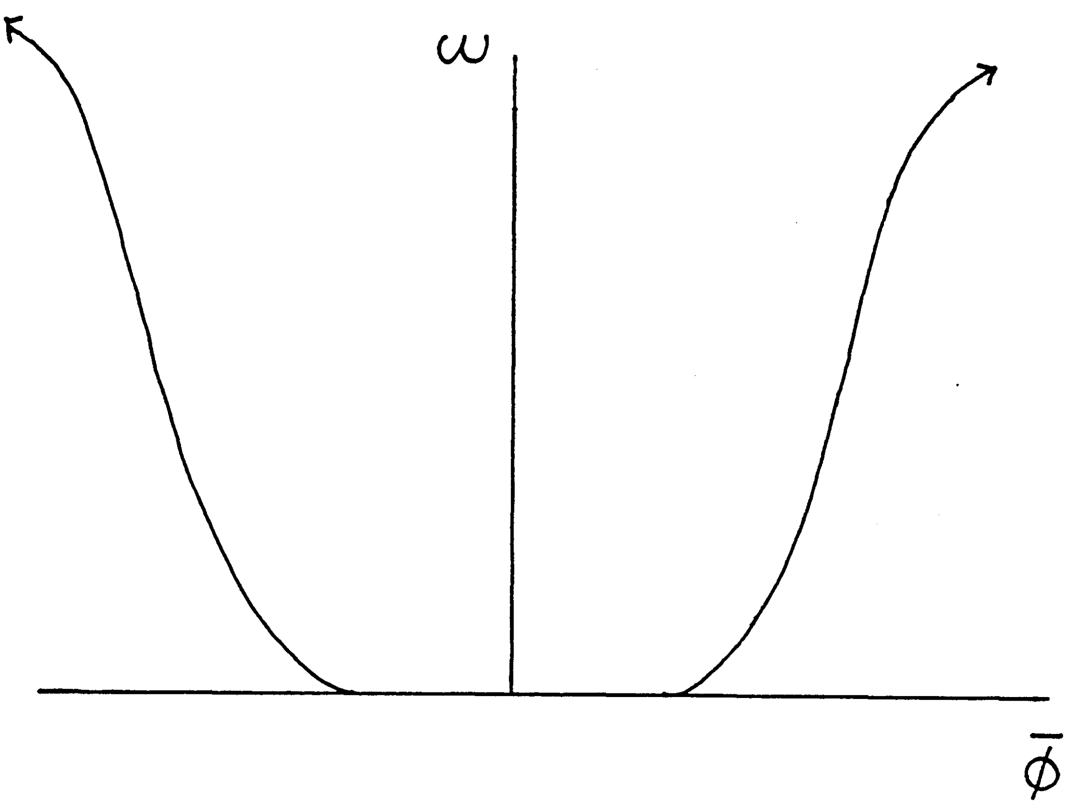
```
C
      STOP
      END
C
C FUNCTION TO CALCULATE THE VALUES OF THE INTEGRAND
C
       FUNCTION YINT(NY,NPHI,NMINT)
       COMMON/USEFUL/A,D,DY,U(0:11,110),NYM
       REAL W
       INTEGER NPLUS,NMINUS,NINT
C
C LET NINT = NINT - 1 TO GET THE CORRECT INDEX VALUES
C
       NINT = NMINT - 1
       W = -A + NY*DY
       YINT = EXP (-2**((D-2.)*FLOAT(NINT))*(W**2))
C
C IS -Y + PHI < -A OR > A.  IF SO FORGET U(-Y+PHI)
C
       IF((NPHI-NY+IFIX(NYM/2.)).LE.0)GOTO 1
       IF((NPHI-NY-IFIX(NYM/2.)).GT.0)GOTO 1
       NMINUS = NPHI - NY + IFIX(NYM/2.)
C
C CALCULATE THE CONTRIBUTION OF U(-Y+PHI)
C
       YINT = YINT + EXP(-2**(D-1.)*FLOAT(NINT))*U(NINT,NMINUS))
C
C IS Y+PHI < -A OR > A.  IF SO FORGET U(Y+PHI)
C
 1     IF((NPHI+NY+IFIX(NYM/2.)).LE.0)GOTO 2
       IF((NPHI+NY-IFIX(NYM/2.)).LE.0)GOTO 2
       NPLUS = NPHI +NY -IFIX(NYM/2.)
C
C CALCULATE THE CONTRIBUTION OF U(Y+PHI)
C
       YINT = YINT + EXP(-2**(D-1.)*FLOAT(NINT))*U(NINT,NPLUS))
 2     CONTINUE
       RETURN
       END
```

The subroutine PLOT called from the main program is concerned with producing a graph on the line printer and we have not reproduced it here.

Typical graphs for the effective potential produced by the program



$(m^2/\lambda) > 2.5$



APPENDIX D: THE TWO LOOP FINITE TEMPERATURE EFFECTIVE POTENTIAL

The two-loop zero temperature effective potential for a $\lambda\phi^4$ theory is given by [9]

$$V_0 = \frac{1}{256\pi^4}\left[F^2M^2\left(\frac{1}{4}\ln^2\frac{M^2}{\mu^2} - \frac{9}{7}\left(\ln\frac{M^2}{\mu^2} - 1\right)\right)\right] + \frac{\lambda M^4}{256\pi^4}\left[\frac{1}{4}\ln^2\frac{M^2}{\mu^2} + \frac{1}{4}\ln\frac{M^2}{\mu^2} - \frac{79}{28}\right] \tag{D1}$$

and the two loop finite temperature effective potential by

$$V_\beta = \frac{\lambda M^4}{32\pi^4}F_1^2(\beta M) + \frac{F^2M^2}{128\pi^4}\int dxdy\ B(\beta M)G(x,y) + \frac{\lambda M^4}{128\pi^4}\left(\frac{1}{2} + \ln\frac{M^2}{\mu^2}\right)F_1(\beta M) + \frac{F^2M^2}{128\pi^2}\left(\frac{1}{2} + \frac{\pi}{\surd 3} + \ln\frac{M^2}{\mu^2}\right) F_1(\beta M) \tag{D2}$$

where

$$F_1(\beta M) = \int dx\ \surd(x^2-1)\ \frac{1}{\exp(\beta Mx) - 1}$$

$$B = \frac{1}{(\exp(\beta Mx)-1)\ (\exp(\beta My)-1)} \tag{D3}$$

and

$$G(x,y) = \ln \frac{[\ (1+2\surd(x^2-1)(y^2-1))^2\ -\ 4x^2y^2]}{[\ (1-2\surd(x^2-1)(y^2-1))^2\ -\ 4x^2y^2]} \tag{D4}$$

Now as $\beta, M^2 \rightarrow 0$ $F_1(\beta M) \rightarrow \dfrac{\pi^2}{6\beta M^2}$,,so in this limit the leading contribution comes from (D2) and is given by

$$V \sim \frac{F^2}{128\pi^4} \ln \frac{M^2}{\mu^2} \frac{\pi^2}{6\beta^2} \tag{D5}$$

which, on substituting $F = \lambda\bar{\phi}$ and adding the one-loop result gives (4.69)

REFERENCES

[1] Weinberg S. : Phys. Rev. Lett. 19 ,1264 (1967)

Salam A. : Elementary Particle Physics (Nobel symposium No. 8) ed. N. Svartholm. Almqvist and Wilsell, Stockholm (1968)

[2] CERN UA1 and UA2 collaboration results

[3] Schwinger J. : Phys. Rev. 82 664,914 (1951)

[4] Jona-Lasinio G. : Nuovo Cimento 34 1790 (1964)

[5] Nambu Y. : Phys. Lett. 26B (1968)

[6] Coleman S. : Erice Lectures 1973 "Secret Symmetry" published as Laws of Hadronic Matter (ed. A. Zichichi) Plenum Press, New York

[7] Abers E.S. and Lee B.W. : Physics Reports 9C No.1 (1973)

[8] Coleman S. and Weinberg E. : Phys. Rev. D7 1888 (1973)

[9] Lee S.Y. and Sciaccalluga A.M. : Nucl. Phys. B96 435 (1975)

[10] Jackiw R. : Phys Rev. D9 1686 (1974)

[11] Fujimoto Y. , O'Raifeartaigh L. and Parravicini G. : Nucl. Phys. B212 268 (1983)

[12] Itzykson C. and Zuber J.B. : Quantum Field Theory p448 McGraw-Hill 1980, New York

[13] Iliopoulous J. Itzykson C. and Martin A. : Rev. Mod. Phys. 47 165 (1975)

[14] Waugh E. : Scoop ,Penguin Books, London

[15] Higgs P.W. : Phys. Rev. Lett. 12 132 (1964)

[16] Jackiw R. : Phys. Rev. D9 1686 (1974)

[17] Taylor J.C. : Gauge Theories of the Weak Interactions Cambridge University Press (1976)

[18] Dolan L. and Jackiw R. : Phys. Rev. D9 2904 (1974)

[19] Appleqvist T. , Carraz one J. , Goldman T. , Quinn H.R. : Phys. Rev. D8 1747 (1973)

[20] Weinberg S. : Phys. Rev. D7 2887 (1973)

[21] Frère J.M. and Nicoletopolous : Phys. Rev. D11 2332 (1975)

[22] Nielsen N.K. : Nucl. Phys. B101 173 (1975)

[23] Aitchison I.J.R. and Fraser C.M. : Annals of Physics (NY) 156 No.1 1 (1984)

[24] Fukuda R. and Kugo T. : Phys. Rev. D13 3469 (1976)

[25] Becchi C. Rouet A. and Stora R. : Phys. Lett. 52B 344 (1974)

[26] de Wit B. and Papanicolaou : Nucl. Phys. B113 261 (1976)

[27] Piguet O. and Sibold K. : Nucl. Phys. B253 517 (1985)

[28] Kugo T. and Ojima I. : Supplement of Rep. Prog. Phys. 66 1 (1979)

[29] Thompson G. and Wu H. : Phys. Rev. D31 2141 (1985)

[30] Cornwall J. Jackiw R. and Tomboulis E. : Phys. Rev. D10 2428 (1974)

[31] Rivers R. : lecture notes Imperial College D.I.C. course

[32] Haymaker R.W. and Perez-Mercader J. : Phys. Rev. D27 1948 (1983)

[33] Evans M. and McCarthy J. : Phys. Rev. D31 1799 (1985)

[34] O'Raifeartaigh L. , Wipf A. and Yoneyama H. : D.I.A.S. preprint D.I.A.S. STP-85-29 (1985)

[35] Fukuda R. : Prog. Theo. Phys. 56 258 (1978)

[36] Wilson K.G. : Phys. Rev. D6 419 (1972)

[37] Do Amaral H.G. , De Carvalho C.A.A. , Shellard P.C. : Revista Brasiliera de Physica (1984)

[38] Hindmarsh M. and Johnston D. : J. Phys. A 19 141 (1986)

[39] Bernard C.W. : Phys. Rev. D9 3311 (1974)

[40] Hata H. and Kugo T. : Phys. Rev. D21 3333 (197)

[41] Umezawa H. , Matsumoto H. and Tachiki M. : Thermo-Field Dynamics and Condensed Matter States, North Holland, Amsterdam (1982)

[42] Rivers R. : Functional Methods, forthcoming Cambridge University Press book

[43] Mills R. : Propagators for Many Particle Systems, Gordon and Breach, New York 1969

[44] Niemi A. and Semenoff G. : Nucl. Phys. B230 181 (1984)

[45] Niemi A. and Semenoff G. : Annals of Physics (NY) 152 (1984)

[46] Rivers R. : Zeit. für Phy. C 22 137 (1984)

[47] Collins J.C. : Renormalization, Cambridge University Press 1984

[48] De Carvalho C.A.A. , Bazeia D. , Eboli O.J.P. , Marques G.C. , da Silva A.J. and Ventura I. : Phys. Rev. D31 1411 (1985)