COSMIC STRINGS AND BEADS

by

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ABSTRACT

Spontaneously broken gauge theories are now thought to be the first step in a fully unified theory of fundamantal processes. The occurrence of topologically stable solutions to the classical field equations in a large class of these theories, such as domain walls, strings, and monopoles, is of great interest to cosmologists because these objects will appear after phase transitions in the early universe. Of particular interest are strings, for they provide a promising way of seeding galaxy formation.

Just after the phase transition at which they are formed, the motion of strings is strongly affected by friction with the surrounding medium. This period is investigated, and a mechanism for the generation of baryon asymmetry by the decay of small loops of string into heavy bosons is examined. A lower bound on the scale of the phase transition is derived.

A new type of stable solution to the field equations of the Yang-Mills-Higgs system is presented, the 'bead', which can be thought of as a monopole on a string. Such beads are shown to exist in a large class of Grand Unified Theories, and their properties and a few of their cosmological implications discussed.

When we take fermions into account, it is well known that there exist solutions to the Dirac equation localised around the string, corresponding to bound fermions moving along it at the speed of light. The circumstances under which these fermions can make the string behave like a superconducting wire are investigated, and it is shown that when these 'zero modes' encounter a bead they undergo a process closely analogous to the Callan-Rubakov effect, whereby the fermions exchange charge with the bead. The thesis is concluded with some general remarks about zero modes in cosmology.

PREFACE

The work presented in this thesis was carried out in the Theoretical Particle Physics group at the Department of Physics, Imperial College, London between October 1983 and September 1986 under the supervision of Professor T.W.B. Kibble, with the financial support of an SERC Research Studentship. Unless otherwise stated, the work is original, and it has not been submitted before for a degree of this or any other university. The material of section 3.1 and part of section 3.3 was carried out in collaboration with T.W.B. Kibble.

I am deeply indebted to my supervisor, Tom Kibble, for his constant interest and help throughout. I am also grateful to Andy Albrecht, David Bennett, Ed Copeland, Richard Davis, Josh Friemann, David Olive, John Preskill and Neil Turok for many useful discussions.

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"I got no strings to hold me down,
to make me fret or make me frown..."

- Pinocchio (Walt Disney Pictures)

CONTENTS

CHAPTER 1: INTRODUCTION.

1.1 Cosmic strings and the earky universe.

The success of the Salam-Weinberg unification of electromagnetic and weak interactions into a spontaneously broken SU(2)×U(1) symmetric gauge theory [1], and that of SU(3) as the gauge group of strong interactions [2], leads naturally to the supposition that these symmetries are subgroups of a larger group such as SU(5) [3], SO(10) [4], or E_6 [5]. The full symmetry is broken by the Higgs-Kibble mechanism [6] at a scale of 10^{15} or 10^{16} GeV. These Grand Unified Theories (GUTs) can provide good values for the Weinberg angle, although some appear in trouble because of the refusal of the proton to decay quickly enough. At sufficiently high temperatures the gauge symmetries are restored [7] (this is similar to the restoration of rotational symmetry to the ground state of a ferromagnet as it is heated past its Curie point). In the hot big bang cosmology we would therefore expect the full unifying symmetry to become manifest at early times, and as the universe expands and cools there occur a series of transitions to phases of lower symmetry until the presently observed $SU(3)_c \times U(1)_{em}$ symmetry is reached. The consequences of these phase transitions are an extremely fertile area of interaction between cosmology and particle physics. For example, a phase transition with supercooling leads to a state where the energy density of the universe is dominated by the constant vacuum energy of a scalar field involved in the symmetry breaking, and it expands exponentially or 'inflates' to 10^{28} times its original size. This inflationary scenario [9] neatly solves the horizon and flatness [10] problems of traditional cosmology, and can generate density perturbations sufficient for initiating galaxy formation [11]. Furthermore, topologically stable objects can appear after the phase transitions, such as domain walls, strings, and monopoles [12,13,14], and their existence (which for the monopole is a necessary consequence of unification) can produce observable effects. For example, strings provide an attractive scenario for seeding galaxy formation [15-20] which in many ways is more attractive than one based solely on inflationary density perturbations. When combined with 30 eV neutrinos to satisfy the theoretical prejudice towards $\Omega = 1$, based on the success of inflation, cosmic strings can seed small scale mass concentrations such as galaxies and clusters, while the pancaking [21] of neutrinos supplies the large scale structure in superclusters and voids [22]. In view of the promise that cosmic strings hold it is important to ascertain their detailed properties in GUT models. Moreover, in the recent plethora of superstring inspired models [23], there are always extra U(1) symmetries that need to be broken somewhere between the electroweak scale 10^2 GeV and the GUT scale 10^{16} GeV, producing cosmic strings which are not necessarily important for galaxy formation but nevertheless have observable properties [24], and in Chapter 2, by re-examining the work of Bhattacharjee et al. [25], we find that intermediate scale strings can account for the observed preponderance of matter over antimatter.

Recently, the possibility that cosmic strings could behave like superconducting wires has been raised [26], with intriguing consequences for astrophysics and cosmology [24,26]. The current is carried by charged particles trapped on the string, and questions raised about the direction of travel when these so-called zero modes are fermions have led to the idea of the 'bead' [27]. In Chapter 3 the bead, which can be thought of as a monopole on a string, is introduced and discussed, while in Chapter 4 the fermionic zero modes on cosmic strings are examined in more general cases than those considered by Witten [26]. The existence of an effect analogous to the Callan-Rubakov effect [28] for these modes when they encounter a bead is demonstrated. Lastly, some general remarks about bead cosmology are made.

The rest of this chapter is concerned with reviewing some aspects of cosmic strings and introducing some of the ideas and techniques necessary for later sections.

1.2 The Nielsen-Olesen string.

In this section we will discuss the string solutions to a (classical) gauge field theory. The term 'string' suggests a solution to the equations of motion which departs from the ground state in a small region around a line in space. The Nielsen-Olesen string [29] is such a solution to the Abelian Higgs model, and is in effect a relativistic generalisation of the vortex lines or flux tubes that crop up in superconductors.

Consider the Abelian Higgs Lagrangian

$$= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D_{\mu}\phi)^{*}D^{\mu}\phi + m^2\phi^{*}\phi - \lambda(\phi^{*}\phi)^2 \qquad (1.2.1)$$

where, as usual

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu = \partial_\mu + ieA_\mu$$

This is U(1) symmetric, and the vacuum states of the theory, defined by

$$|\phi| = \surd(m^2/2\lambda)\ , \qquad A_\mu = 0 \tag{1.2.2}$$

forms a manifold isomorphic to S^1 - the bottom of the 'wine bottle' Higgs potential. We shall look for a solution of finite energy per unit length with cylindrical symmetry about the z axis, clearly a reasonable ansatz for a string. Finite energy per unit length means that the covariant derivative of ϕ should vanish at large distances from the z axis:

$$eA_\mu\phi = i(\partial_\mu\phi)$$

With the trial solution

$$\begin{aligned} \phi(x) &= \surd(m^2/2\lambda)f(\rho)\exp(in\theta) \\ A_\mu(x) &= \delta_{\mu\phi}A(\rho) \end{aligned} \tag{1.2.3}$$

where ρ and θ are cylindrical polar coordinates, we see that at large distances f = 1 and A = $(-n/e\rho)$, while f and A vanish as $\rho \to 0$. The flux Φ through a large loop c in the x-y plane is given by

$$\Phi = \int B_z dxdy = \int_c A^\mu dx_\mu = 2\pi n/e$$

Hence we see that the condition of the single-valuedness of ϕ imposes quantisation on the flux. If equations (1.2.3) are substituted into the equations of motion the following asymptotes as $\rho \to \infty$ are found [29,78]

$$A \to -n/e\rho + O(\exp(-m_v\rho)) \tag{1.2.4}$$

$$f \to 1 - O(\exp(-m\rho)) \tag{1.2.5}$$

where $m = \min(2m_v, m_s)$ and m_v and m_s are the vector and scalar boson masses, equal to $(e|\phi(\infty)|)$ and $(\sqrt{2\lambda}|\phi(\infty)|$ respectively. From (1.2.4) and (1.2.5) we see that the magnetic field is confined to a tube of width m_v^{-1} while the potential energy departs from its vacuum value inside a tube of width m^{-1}. The energy per unit length is [29] approximately $|\phi(\infty)|^2$. This, then, is a string. It is not clear how this survives quantisation: it is possible to quantise small oscillations around this classical configuration [30,78] but whether the string is stable in the full quantum theory is an open question. The answer is probably yes, because the string is topologically stable: in order to deform the solution to the trivial vacuum ϕ must at some point become discontinuous, and hence we would have to pass through a state of infinite energy density. Such a tunnelling process would take an infinite time.

We now turn to a discussion of the classical motion of such a string in spacetime [31,32], and we shall see that for motions in which the thickness of the string can be ignored the string behaves exactly like the massless relativistic string of Nambu [33]. The zeros of the Higgs field trace out a sheet in spacetime, and this suggests that we define an new orthonormal set of coordinates λ^α, such that $x^\mu(\lambda^0,\lambda^1,0,0)$ are the coordinates of this world sheet. Hence λ^0 and λ^1 are coordinates in the sheet and λ^2 and λ^3 are space coordinates normal to the string. The action is

$$S = \int d^4x\sqrt{-g}\,\mathcal{L} \tag{1.2.6}$$

where g is the determinant of the 4-metric and $\mathcal{L}$, the Lagrangian density, is non-vanishing only in a small region w $\sim m_s{}^{-1}$ around the world sheet. Rewriting the action in the new coordinate system we have

$$S = \int d\lambda^0 d\lambda^1 \int d\lambda^2 d\lambda^3 (\sqrt{-\gamma} + \ldots)\,\mathcal{L} \tag{1.2.7}$$

where γ is the induced metric on the world sheet and the dots indicate terms which vanish as λ^2 and λ^3 tend to zero. In the limit $w \to 0$ we may integrate over the transverse coordinates to obtain

$$S = -\mu \int d\lambda^0 d\lambda^1 \sqrt{-\gamma} \tag{1.2.8}$$

where $\mu \sim |\phi(\infty)|^2$ and

$$\gamma = \left(\frac{\partial x}{\partial \lambda^0}\right)^2 \left(\frac{\partial x}{\partial \lambda^1}\right)^2 - \left(\frac{\partial x}{\partial \lambda^0} \cdot \frac{\partial x}{\partial \lambda^1}\right)^2 \tag{1.2.9}$$

This is exactly the action for a relativistic string of tension μ [31]. Denoting differentiation with respect to the timelike coordinate λ^0 by a dot and the spacelike coordinate λ^1 by a prime we can define a Lagrangian

$$L(\dot{x},x') = -\mu[(\dot{x}.x')^2 - (x')^2(\dot{x})^2] \tag{1.2.10}$$

The relevant 4-metric for cosmology is the Friedmann-Robertson-Walker (FRW) metric which describes an expanding space of constant curvature [8]. The square of the invariant line element is

$$ds^2 = (d\tau^2 - d\underset{\sim}{x}^2)R^2(\tau) \tag{1.2.11}$$

where $d\underset{\sim}{x}^2$ is the square of the line element of the spacelike hypersurfaces, $R(\tau)$ is the scale factor, and τ is the so-called conformal time defined from the usual time coordinate t by $dt^2 = R^2(\tau)d\tau^2$. To simplify the equations of motion we can choose a gauge [34]

$$\lambda^0 = \tau \qquad\qquad \dot{\underset{\sim}{x}}\cdot\underset{\sim}{x}' = 0 \tag{1.2.12}$$

In this gauge the equations of motion can be shown to be [35,36]

$$\dot{\underset{\sim}{x}}^{\boldsymbol{\cdot}} + 2\dot{\underset{\sim}{x}}(1 - \dot{\underset{\sim}{x}}^2)\dot{R}/R = \frac{1}{\varepsilon}\frac{\partial}{\partial\sigma}\left(\frac{\underset{\sim}{x}'}{\varepsilon}\right) \tag{1.2.13}$$

where

$$\varepsilon = \sqrt{(\underset{\sim}{x}'^2/(1 - \dot{\underset{\sim}{x}}^2))} \tag{1.2.14}$$

From $E = \int R^3 d^3 x T^{00} = \mu \int d\sigma\varepsilon$ we see that $\mu\varepsilon$ is the energy per unit length of the string [36]. The second term on the left hand side is like a damping term proportional to $1/\tau$, caused by the expansion of the universe and not to be confused with damping caused by friction with the surrounding medium, while the magnitude of the right hand side is essentially the inverse of the curvature radius of the string. When damping is small we can rescale $\sigma \rightarrow \varepsilon\sigma$ so that $\dot{\underset{\sim}{x}}^2 + \underset{\sim}{x}'^2 = 1$. It is therefore consistent to assume that the damping is small when [35]

$$\left|2\dot{\underset{\sim}{x}}/\tau\right| << \left|x''\right| \tag{1.2.15}$$

Thus for a wave of coordinate amplitude a_0 and coordinate wavelength λ_0

$$\tau >> \lambda_0^2/a_0 \tag{1.2.16}$$

so in terms of the proper amplitude and wavelength a and λ

$$t >> \lambda^2/a \tag{1.2.17}$$

The strings are Brownian and so $\lambda \sim a$ on scales larger than the persistence length. We can therefore conclude that damping is negligible on scales less than the horizon size, except for small amplitude oscillations. On larger scales the strings move very slowly with respect to the surrounding matter and are conformally stretched, that is, both the wavelength and the amplitude go as R(t) (~ $t^{1/2}$ in a radiation dominated universe). The horizon size goes as t and eventually catches up with λ, after which time the damping term may be ignored and the equation of motion is

$$\ddot{\underset{\sim}{x}} - \underset{\sim}{x}'' = 0 \qquad (1.2.18)$$

with constraints $\dot{\underset{\sim}{x}}^2 + \underset{\sim}{x}'^2 = 1$ and $\dot{\underset{\sim}{x}}.\underset{\sim}{x}' = 0$, which is the equation of motion of a free string. Hence well inside the horizon strings move freely [30,35]. This picture has been extensively confirmed by simulations [37].

1.3 Strings in non-Abelian gauge theories.

The above discussion of the string solution to the Abelian Higgs model may be extended to a general gauge theory [13]. The important thing to note is that in tracing a loop in space well away from the string the vacuum manifold M is covered n times. This is the winding number of the map $\phi:S^1 \rightarrow M$ and is a topologically conserved quantity [38]. Field configurations with different values of n are not smoothly deformable into each other, and so n labels equivalence classes of maps. These can be given a group structure to produce the first homotopy group π_1, which essentially tells us about the 'holes' in a manifold. If $\pi_1(M)$ is non-trivial, (i.e. contains more than just the identity) then there are string solutions to the theory, and in the Abelian Higgs model π_1 is just the group of integers under addition, Z. For a general spontaneously broken theory where a gauge group G is broken to a subgroup H, M $\simeq$ G/H, and if G is connected and simply connected (i.e. if $\pi_0(G) \simeq 0 \simeq \pi_1(G)$) then we can use the relation

$$\pi_1(G/H) \simeq \pi_0(H) \simeq H/H_c \tag{1.3.1}$$

where H_c is the component of H connected to the identity. What this means is that non-contractible loops in G/H are images of paths in G joining disconnected pieces of H.

Let us be a little more concrete and represent the Higgs field by a vector in weight space, $\Phi(\rho,\phi)$, where ϕ is now the azimuthal angle. The string solution to a non-Abelian theory may be written in a form where rotations about the z axis are induced by gauge rotations

$$\Phi(\rho,\phi) = g(\phi)\Phi(\rho) = e^{i\phi Q}\,\Phi(\rho) \tag{1.3.2}$$

such that $g(2\pi)$ is in the disconnected component of H. It is not possible to deform this to a trivial solution with $g(\phi)$ in H_c for all ϕ without moving $g(2\pi)$ out of H, and so this is a string solution, with $\Phi(\rho = 0) = 0$. It need not necessarily be the lowest energy solution: indeed, if Q has eigenvectors with zero eigenvalue it will save potential energy to have at least one of them non zero at the core of the string [27].

A large class of symmetry breaking patterns with strings have been found by Olive and Turok [39]. Let G be a simple Lie group with simple roots α_i ($i = 1, \ldots,$ rank G). Its fundamental weights λ_i are defined by

$$2\lambda_i \cdot \alpha_j/(\alpha_j)^2 = \delta_{ij} \tag{1.3.3}$$

Suppose that the Higgs field is in a representation obtained by symmetrising the product of n fundamental representations with highest weight λ_Φ, and that the parameters of the potential are chosen such that the manifold of minima is the gauge orbit through a standard Φ_0 aligned with the highest weight $|n\lambda_\Phi\rangle$. (This is actually impossible for a single Higgs field in real representation; in that case we must form a complex representation out of a pair of such fields).

We can now describe the little group H of Φ_0 [39]. Let α_Φ be the unique simple root that is not orthogonal to λ_Φ. Then the continuous part of H is the subgroup K of G obtained by exponentiating the algebra whose Dynkin diagram is given by removing the dot corresponding to α_Φ from that of G. The fundamental weight λ_Φ generates a discrete unbroken subgroup Z_{nk} which coincides with K at k points. The full little group H is therefore [39]

$$H = K \times Z_{nk}/\, Z_k$$

where Z_k is a cyclic subgroup of the centre of K. From (1.3.1) we see that $\pi_1(G/H)$ is Z_n, so strings labelled by a winding number conserved modulo n result. A particular case of this type of symmetry breaking is presented by Nielsen and

Olesen [29], where SU(2) is broken by a pair of adjoint Higgs fields Φ_1, Φ_2 to Z_2, although they did not present the lowest energy solution. The stable solution can be written so that the gauge rotation that acts on the Higgs field is generated by T_3, the third isospin component [27]

$$\Phi_1 = f_1(\rho)(0,0,1) \qquad \Phi_2 = f_2(\rho)(\cos\phi,\sin\phi,0) \tag{1.3.4}$$

$$A_\mu = \delta_{\mu\phi}A(\rho)(0,0,1)$$

where $f_1(0) \neq 0$, $f_2(0) = 0 = A(0)$, and f_1 and f_2 tend to constants at infinity, with A vanishing as $(e\rho)^{-1}$. Note that the Higgs field at the core of the string is non zero because it is parallel to the zero eigenvector of T_3. In Nielsen and Olesen's version, Φ_1 also vanished at the core as follows

$$\Phi_1 = f_1(\rho)(-\sin\phi,\cos\phi,0) \tag{1.3.5}$$

According to Everett and Aryal [76], (1.3.4) is energetically preferred. The difference between the two types of solution will be important in Chapter 3 when we discuss the bead.

1.4 Formation and evolution of cosmic strings.

Let us now suppose that there is a phase transition occuring in the early universe in which the gauge symmetry is broken in such a way that $\pi_1(G/H) \neq 0$: in other words, strings may be formed. Kibble [13] has discussed the mechanism by which they appear. The essential point is that just after the phase transition, which we shall take to occur at time t_c, the direction of the Higgs field in group space is uncorrelated beyond some distance $\xi(t_c)$. This means we may divide space up into domains of size ξ in which the direction of the Higgs field is more or less constant, but between which there is no correlation. If the Higgs field traverses a non-contractible loop in G/H when going around an edge where three or more domains meet then a string will be trapped at the junction. We therefore expect there to be of order one string in each volume of size ξ^{-3} [13]. We can estimate the initial correlation length by reference to an O(N) model which exhibits a second order phase transition. The effective potential, including the finite temperature corrections [13], takes the form

$$V(\phi) = \lambda(\phi^2 - \eta^2)^2 + (\surd\lambda)AT^2\phi^2 \qquad (1.4.1)$$

where T is the temperature and A is of order 1. Above the critical temperature

$$T_c \simeq \eta \qquad (1.4.2)$$

the minimum is at $\phi = 0$, while well below this temperature ϕ reaches its zero temperature value of η, and the original O(N) symmetry is broken to O(N-1). The correlation length of the Higgs field is essentially the inverse of its mass

$$\xi \simeq m_s^{-1} \simeq \surd\lambda\eta \simeq \surd\lambda T_c \qquad (1.4.3)$$

Initially, therefore, there is approximately one segment of string of length ξ per volume of ξ^3, and so with μ being the mass per unit length the energy density in string is

$$\rho_S(t_c) \sim \mu\xi^{-2} \tag{1.4.4}$$

Vachaspati and Vilenkin [40] have performed a Monte Carlo simulation to study the fractal dimension and size distribution of strings and loops just after the phase transition. Their method is to assign phases $2\pi n/3$ $(n = 0,1,2)$ at random to each site of a cubic lattice, and when n changes around a face in one of three ways - (0,0,1,2), (0,1,1,2), or (0,1,2,2) - a string is assigned to that face. They found that ~80% of the length of string was in the form of 'infinite' string, while the rest was contained in a scale free distribution of loops. The length l of string between two points a distance R apart was

$$l \sim R^2/\xi \tag{1.4.5}$$

which is the same as that for a random walk, although the long distance correlations are different. Simulations on different lattices [41] give essentially the same results. Kibble has done the same thing for Z_2 strings [42], approximating SU(2) by the tetrahedral group, and found the proportion in long strings to be higher, about 94%, a figure supported by Aryal et al. [43].

With regard to the loop distribution we mean by scale invariant that it contains no dimensionful constants, and so the number density dn of loops of sizes between R and R+dR must be

$$dn \sim dR/R^4 \tag{1.4.5}$$

Initially, the strings are heavily damped by friction [13,44], by a force per unit length of $\sigma\rho v$, where σ is the total cross section per unit length for the particles constituting the medium, which has energy density ρ, through which the strings are moving with velocity v. A section of string with radius of curvature L experiences an acceleration of μ/L towards the centre of curvature and soon reaches its terminal velocity v_t determined by [13,44]

$$\rho\sigma v_t = \mu/L \tag{1.4.6}$$

Kinks on a scale L will be damped in a characteristic time $t_d = \mu/\rho\sigma$ and straightened in a time L/v_t, so that the correlation length goes as [13,45,46]

$$\frac{d\xi}{dt} \simeq \frac{\xi}{\xi/v_t} = t_d/\xi \tag{1.4.7}$$

Everett has shown that [44]

$$\sigma \simeq \frac{\pi^2}{T}\ln^2(T/T_c) \tag{1.4.8}$$

so, ignoring the logarithm and using the approximate expression for the energy density in a radiation dominated universe $\rho \simeq 0.03m_p^2/t^2$, where $m_p = G^{-1/2}$ is the Planck mass, with a value of about 10^{19} GeV, we find that [46]

$$t_d \simeq (G\mu)m_p^{1/2}t^{3/2} \tag{1.4.9}$$

$$\xi^2 \simeq (G\mu)m_p^{1/2}t^{5/2} \tag{1.4.10}$$

Friction damping is important until time t_* when the scale below which kinks in the string move freely becomes equal to the horizon size, given by

$$t_* = t_d = \xi = (G\mu)^{-2}m_p \tag{1.4.11}$$

This is about 10^{-31} seconds for GUT scale strings. For light strings it can be quite late: for example t_* can reach the nucleosynthesis era for $(G\mu) \sim 10^{-22}$. We will examine the evolution of a system of strings between the phase transition and t_* in more detail in Chapter 2. After t_* strings move freely inside the horizon size t, and we move into an era where this is the only scale in the process. Waves on the string above this size are conformally stretched with the expansion of the universe as we saw in section 1.2. Since the horizon size grows as t while the wavelength grows as $t^{1/2}$ or $t^{2/3}$ according to whether the universe is radiation or matter dominated sooner or later any particular wave will fall inside the horizon and start to move freely. The wave oscillates with constant proper amplitude [34] and the wavelength is redshifted with expansion, and so the energy in a comoving volume remains very nearly constant [37]. At some point during its motion the string can intersect itself, and what happens then is a problem which has been tackled numerically for global strings (i.e. ones formed by the breaking of a global symmetry) by Shellard [47] and analytically by Copeland and Turok [48]. It seems that when strings intersect it is very probable that they will reconnect the other way: here we use the term 'probable' in the sense of an average over intersection angles and velocities. This is plausible because the total length of string, and hence the energy, is decreased this way. For very high transverse velocities and for nearly parallel configurations the strings pass through each other, in the first case because the time scale for reconnection is longer than the time taken to pass though, and in the second because little enrgy is saved. However, in the above sense, the reconnection probability p is approximately unity [47,48].

If a string intersects itself then a loop is formed, leaving the long string somewhat straighter. This process of loop formation is crucial to the string picture, because the network of long strings must lose energy at a rate sufficient to stop its energy density from dominating the universe. Vilenkin [49] has discussed a possible string dominated universe in which $p = 0$, which is filled with strings of persistence length greater than the horizon size. In this universe the energy in a comoving volume R^3 is proportional to μR so the energy density goes as μR^{-2}, and therefore $R \sim t$. The success of the current ideas about nucleosynthesis requires that the universe be radiation dominated when the temperature is a few MeV, so the evolution towards string domination would have to be very slow and the strings formed very late [49]. However, it is difficult to see how causality can allow the strings to ever straighten out on scales much greater than t. Even if $p \approx 1$ it is possible that the universe becomes string dominated, as discussed by Kibble [46]. Here the network has a small persistence length, kept so by a dynamic equilibrium in which small loops are chopped off from and reconnect to the network. Small oscillations on long strings and loops have constant energy in a comoving volume [37] and hence behave like matter, with a scale factor going as $t^{2/3}$ [46].

Numerical simulations [37], however, confirm the following picture [34-37] in which strings form a small constant fraction of the energy density. Soon after a wave falls within the horizon and starts to move at relativistic speed a loop can be chopped off. This loop will fission into a number (of order 10) of daughter loops, some of which will fall into the class of non-self-intersecting solutions found by Kibble and Turok [55] and survive. Thus we expect that one loop of size t will be formed per horizon volume per expansion time, so if n(t) is the number density of loops at time t

$$dn(t) = \frac{\nu}{t^4}dt \tag{1.4.12}$$

where $\nu \approx 0.01$ [37]. The number density will decrease as the cube of the scale factor, so the number density at time t of loops formed between t' and t' + dt' is

$$n(t,t')dt' = \nu \left(\frac{R(t')}{R(t)}\right)^3 \frac{dt'}{t'^4} \tag{1.4.13}$$

In a radiation dominated universe

$$n(t,t')dt' = \nu \frac{dt'}{t^{3/2}t'^{5/2}} \tag{1.4.14}$$

If the loops are formed with size t' this can be translated into a formula for the number density of loops in a size interval dL

$$n(t,L)dL = \nu \frac{dL}{t^{3/2}L^{5/2}} \tag{1.4.15}$$

The whole system of long strings and loops thus evolves in a scale free manner; that is, there are no dimensionful constants in the above. There is one long string per horizon volume belonging to the network of conformally stretched brownian string with persistence length t, and a set of loops distributed in size according to (1.4.15). There is a lower cut off in size caused by loss of energy in the form of gravitational radiation [50-54]. A loop of size L once well inside the horizon will oscillate freely, obeying the flat space equations of motion (1.2.18), and the radiation rate can be estimated from the quadrupole formula $\dot{E} \simeq -GM^2R^4\omega^6$ [51]. A loop of size L oscillates with frequency L/2 [55] so that

$$\dot{E} = -\gamma G\mu^2 \tag{1.4.16}$$

Numerical calculations by Vachaspati and Vilenkin [51] indicate that $\gamma \simeq 50$, a figure confirmed analytically by Burton [53]. The lifetime τ of a loop of size L is therefore about $L/(\gamma G\mu)$, and the smallest loops at time t are of size $(\gamma G\mu)t$.

The energy density of the long strings, ρ_∞, can be obtained by considering the network to be a set of string segments of length t separated by a distance t. In that case

$$\rho_\infty \simeq \mu/t^2 \tag{1.4.17}$$

Relative to the radiation energy density $\rho_r \simeq 0.03m_p^2/t^2$ this is the fraction

$$\rho_\infty/\rho_r \simeq 30G\mu \tag{1.4.18}$$

The energy density of the loops ρ_l is easily calculated:

$$\rho_l = \int_{(\gamma G\mu t)}^{t} \frac{\beta\mu\nu}{t^{3/2}L^{5/2}} dL \tag{1.4.19}$$

where the mass of a loop size L is $\beta\mu L$, with $\beta \simeq 9$ [37]. Hence

$$\rho_l/\rho_r \simeq 30\beta\nu(G\mu)^{1/2}/\gamma \tag{1.4.20}$$

For GUT scale strings, $G\mu \simeq 10^{-6}$ and this ratio is about 10^{-4}, just the size of perturbation needed to initiate galaxy formation [56]; this and the scale free nature of the distribution make cosmic strings a prominent candidate for seeding galaxy formation, as will be briefly discussed in the next section.

1.5 Cosmic strings and galaxy formation.

Zel'dovich [15] first suggested that linear structures with a mass per unit length of about 10^{34} GeV^2 could provide the necessary spectrum of density perturbations for initiating galaxy formation. In Vilenkin's scenario [15], it the oscillating loops chopped off inside the horizon during the evolution of the network that provide the scale invariant spectrum of density perturbations. It can be shown that the time averaged field of an oscillating loop is identical to that of a surface with mass density proportional to $\dot{\underset{\sim}{x}}^2$, and with total mass equal to that of the loop [16], so that the perturbations are in the form of effectively point-like seed masses. The number density at time t of loops formed between time t' and t' + dt' is

$$dn(t,t') \sim (R(t)/R(t'))^3 t'^{-4} dt' \qquad (1.5.1)$$

When the loops are formed, they are about t' in size, because they have been chopped off a network of long strings which have straightened out below the horizon size. Hence the mass M of the loops when formed is $\mu t'$, and in a radiation dominated universe the number density of loops with masses between M and M + dM is

$$dn(t,M) \sim t^{-3/2}(M/\mu)^{-5/2}\ d(M/\mu) \qquad (1.5.2)$$

Before decoupling the loops cause small acoustic adiabatic perturbations [56], oscillating at constant amplitude until t_{dec}(the time of decoupling of matter and radiation), at which point damping by photon diffusion occurs of all adiabatic perturbations below the Silk mass [57]

$$M_{Silk} \sim 1.3 \times 10^{12} (\Omega h^2)^{-3/2} M_{\odot}$$

where Ω is the ratio of the mass density of the universe to the critical density, and h expresses the uncertainty in measurements of the Hubble constant H_0:

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$$

At t_{dec} the loops themselves start to collect baryons around them, in a manner described by the spherical collapse model [58]. According to this model, the mass collapsed and virialised around a seed mass δM at time t is

$$M \simeq \delta M (t/t_{dec})^{2/3} \tag{1.5.3}$$

The virialised mass of the accreted object is proportional to the seed mass, so if a certain size loop forms a galaxy, we might expect larger loops to form clusters of galaxies, because large loops will accrete small loops as well. The cosmic string theory accounts for the observed galaxy-galaxy and cluster-cluster correlation functions in a natural way. These functions are defined from the number density of the objects (galaxies, clusters) by

$$\xi_{g,c}(r) = \left(n_{g,c}(r)/\overline{n}_{g,c} - 1 \right) \tag{1.5.4}$$

where the barred quantity is the average. When expressed in scale-invariant way in terms of the mean separation of the objects, d,

$$\xi_{g,c}(r) \simeq \beta_{g,c}(r/d)^{-1.8} \qquad (1.5.5)$$

It is found that $\beta_g \simeq 1.1$ [59] and $\beta_c \simeq 0.27$ [59] for Abell clusters [60], defined as regions containing more than 50 bright galaxies inside a radius of $1.5h^{-1}$Mpc. Their mean separation is $55h^{-1}$ Mpc, whereas for galaxies it is about $5h^{-1}$ Mpc. In the string picture the perturbations that seed the formation of galaxies and clusters are produced in a scale invariant manner by loops, and the stronger correlation of galaxies is a result of gravitational clustering. Turok [19] has found a remarkable agreement between $\xi(r/d)$ from numerical simulations and (1.4.5), and this must be counted as one of the succeses of cosmic strings. Furthermore, by asking that the loops that today have the Abell separation are massive enough to accrete enough matter for an Abell cluster, one finds that $G\mu \sim 5\times10^{-6}$[19,18], which fits in with the idea that strings should be formed at the GUT scale. Cosmic strings can also give good values for the peculiar velocities of galaxies [61]. This figure for $G\mu$ is consistent with bounds on gravitational radiation from observations of the timing of the millisecond pulsar [62], from nucleosynthesis [63], and from variations in the temperature of the microwave background [63,64]. The best evidence for cosmic strings would be an observation of a linear discontinuity of $\delta T/T \sim 10^{-5}$ in the background, caused by the peculiar 'missing angle' in the space around the string [65,66]. This could be observed if current sensitivities are increased by a factor of about 5 [63]. Of course, as Peebles has pointed out [67], there are problems with this theory. One of the most puzzling is the overabundance of loops under the size which is thought to accrete a galaxy. Since the virialised mass is proportional to the seed mass, and mass is held to trace luminosity, the luminosity distribution function for galaxies $\phi(L)$, that is the number density of galaxies with luminosities between L and L + dL, should go like $L^{-5/2}$. Unfortunately, the exponent is measured to be more like -1.3 [68]. A better understanding of the details of collapse around a cosmic string is clearly required.

1.6 Superconducting strings.

The discovery by Witten [26] that under certain circumstances cosmic strings can behave like superconducting wires suggests the possibility that strings might be observable through their electromagnetic interactions as well as their gravitational ones. Chudnovsky et al. [24] have analysed the interactions of superconducting strings with plasmas and they show that they are synchrotron sources, and go so far as to suggest that a recently observed radio source showing filamentary structures [69] may be a cosmic string, although there are adequate conventional explanations [70]. In this section we shall see how a string may be superconducting, and introduce some of the ideas that will be explored in more detail in Chapter 4.

Let us consider two chiral spinors Ψ_l and Ψ_r, where $\gamma_5\Psi_l = -\Psi_l$ and $\gamma_5\Psi_r = \Psi_r$, coupled to gauge and Higgs fields in the usual way

$$\mathcal{L}_f = \bar{\Psi}_{l,r}\gamma\cdot(i\partial + eA)\Psi_{l,r} - g(\bar{\Psi}_l\Phi\,\Psi_r + \bar{\Psi}_r\Phi^+\Psi_l) \qquad (1.6.1)$$

The fields $\Psi_{l,r}$ and Φ transform under a gauge group $U(1)\times\overline{U(1)}$ with charges $(q_l,\bar{q})$, $(q_r,-\bar{q})$, and $(-q_l-q_r,0)$, so that when Φ has a non zero vacuum expectation value the unbroken subgroup is $\overline{U(1)}$ and string solutions exist. Suppose we have such a string along the z axis, so that the Higgs and gauge fields can be written

$$\Phi(\rho,\phi) = e^{i\phi}\Phi_0(\rho) \qquad A_\mu = \delta_{\mu\phi}A(\rho) \qquad (1.6.2)$$

where Φ_0 vanishes at the origin and tends exponentially to a constant, η say, at infinity. The Dirac equation in the string background can now be solved, and we shall find that there are some solutions which are localised around the string and propagate along it at the speed of light [26,72,73]. These are the so-called zero modes. We make the assumption that in a z independent background the spinors are separable as follows

$$\Psi_{l,r} = \alpha(t,z)\psi_{l,r}(\rho,\phi) \qquad (1.6.3)$$

where α is a function and $\psi_{l,r}$ are spinors, and take $(q_l + q_r)$ to be +1 for simplicity. The Dirac equation for ψ_l then reduces to

$$i\gamma^{i}D_{i}(\alpha\psi_{l}) = \alpha\left((i\gamma^{\rho}\partial_{\rho} - q_{l}A\gamma^{\phi})\psi_{l} + ge^{i\phi}\Phi_{0}(\rho)\psi_{r}\right) \quad (1.6.4)$$

where i = 0,3 and D_μ is the covariant derivative. Solutions exist in which the right hand side vanishes [72,73]; these are the zero modes found by Caroli et al. for vortices in a type II superconductor [91] and by Jackiw and Rossi for the Nielsen-Olesen vortex in 2+1 dimensions [72], which are normalisable in the directions transverse to the string and have the properties [73]

$$i\gamma_1\gamma_2\psi_l = \psi_l \qquad i\gamma_1\gamma_2\psi_r = -\psi_r \qquad \gamma_1\psi_l = i\psi_r \quad (1.6.5)$$

It follows that $\gamma_0\gamma_3\psi_{l,r} = -\psi_{l,r}$ and so α obeys

$$(\partial_t + \partial_z)\alpha = 0 \quad (1.6.6)$$

Therefore the solution is $\alpha = \alpha(t - z)$, which corresponds to a particle trapped on the string travelling at the speed of light in the +z direction. Replacing the string by an antistring is equivalent to rotating the string through 180^0 about the x axis, and so the particle travels in the -z direction on an antistring. If instead $(q_l + q_r) = -1$ the phase of the Higgs field in (1.6.4) changes in the opposite sense, and so these fermions effectively 'see' an antistring and the direction of travel is towards $z = -\infty$.

The cancellation of anomalies in the 3+1 dimensional theory has important consequences for the physics of the zero modes [26]. Let the generators of the $U(1)\times\overline{U(1)}$ symmetry be Q and $\overline{Q}$ respectively. The interesting anomaly is the $\overline{Q}Q\overline{Q}$: for each pair (ϕ_l^i, ϕ_r^i) bound to the string the coefficient of the anomaly is proportional to $(\overline{q}^i)^2 q_l^i + (-\overline{q}^i)^2 q_r^i = (\overline{q}^i)^2 (q_l^i + q_r^i)$. Since they are all coupling to the same Higgs, $(q_l^i + q_r^i)$ is equal to the same quantity, q say, for all +z movers and equal to -q for all -z movers, and the anomaly cancellation condition gives [26]

$$\sum_{\substack{+z \\ \text{movers}}} (\overline{q}^i)^2 = \sum_{\substack{-z \\ \text{movers}}} (\overline{q}^i)^2 \tag{1.6.7}$$

The simplest anomaly free theory useful for our purposes is one with a chiral fermion of $\overline{Q}$ charge +e travelling in the +z direction and another of $\overline{Q}$ charge +e travelling the other way. The fermionic part of the 3+1 dimensional action is

$$S_f = \int d^4x \sum_{\overline{Q}} \overline{\phi}^{\overline{Q}}_{l,r} i\gamma . D\phi^{\overline{Q}}_{l,r} - g(\overline{\phi}^{\overline{Q}}_l \Phi \; \phi^{\overline{Q}}_r + \text{h.c.}) \tag{1.6.8}$$

Using the relations (1.6.5) we find that $\overline{\phi}_l \phi_r$, $\overline{\phi}_l \gamma_a \phi_l$ and $\overline{\phi}_r \gamma_a \phi_r$ $(a = 1,2)$ all vanish, and after integrating over the directions transverse to the string we are left with

$$S_f = \int dtdz \; \alpha_1^+ i(D_t + D_z)\alpha_1 + \alpha_2^+ i(D_t - D_z)\alpha_2 \tag{1.6.9}$$

where α_1 and α_2 are the +z and -z moving modes respectively. From them we may form a two component Dirac spinor α and the action becomes

$$S_f = \int dtdz\ \bar{\alpha} i \gamma^i D_i \alpha \qquad \alpha = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \tag{1.6.10}$$

where the γ^i are two-dimensional Dirac matrices appropriate to the world sheet of the string. This equation can be solved by bosonising the field theory, as Witten did [26], but the essential physics of the interaction with an electromagnetic field can also be found by solving the Dirac equation. In the gauge $A_0 = 0$, an electric field with component tangential to the string equal to E can be described by a gauge potential $A_z = Et$. This would obtain if, for example, the string were moving steadily through a constant magnetic field. The solutions to the Dirac equation are then

$$\begin{aligned} \alpha_1 &\sim \exp[-ik(t-z) - ieEt^2/2] \\ \alpha_2 &\sim \exp[-ik(t+z) + ieEt^2/2] \end{aligned} \tag{1.6.11}$$

The energy and momentum operators are $E = i\partial_t$ and $p_z = -i\partial_z + eA_z$, so the 2-momenta (E,p_z) of the +z and -z moving modes are (k + eEt, k+eEt) and (k - eEt, -k + eEt) respectively. This means that the modes gain or lose energy according to their direction of motion: if e is positive, +z movers gain energy and momentum from the electric field, while -z movers lose it. If the initial ground states for the zero modes are Dirac 'seas' filled to the k = 0 level, then the $\pm z$ movers are all shifted up (or down) at a rate $\pm eE$. The density of states per unit length in two dimensions is $1/2\pi$, so after time t there will be $k/2\pi$ holes of charge $-e$ travelling at the speed of light in the $-z$ direction and the same number of particles of charge $+e$ travelling in the $+z$ direction. Hence the total current J is [26]

$$J = e^2 Et/\pi \tag{1.6.12}$$

If there are several $\pm z$ moving modes trapped on the string each with the sum of the square of their charges being Σe^2, the total current is

$$J = J_l + J_r = \Sigma e^2 Et/\pi \tag{1.6.13}$$

The energy of each mode changes at a rate $e\partial_t A_z = eE$, so that in this approximation the current obeys the equation

$$dJ/dt = e^2 E/\pi \tag{1.6.14}$$

which is characteristic of a superconductor. When the field is turned off the current continues to flow.

There is an upper limit to this current, due to the fact that fermions can leave the string if they have high enough energy. An unbound fermion has mass $m_f = g\eta$ at infinity, so when the energy of a zero modes is bigger than this it becomes energetically favourable for it to make a transition to a lower energy but unbound state, of which there are plenty available. The number of states per unit length for a single mode with Fermi momentum m_f is $m_f/2\pi$: this energy will eventually be reached by the top of the Fermi sea if the field is applied for long enough, at which point the field energy goes into creating unbound particles rather than increasing the current. The maximum current for a single mode of charge e is therefore

$$J_{max} = em_f/2\pi = em_f c^2/h \tag{1.6.15}$$

The maximum current for electrons is about 20 amps, while for superheavy fermions of mass about 10^{15} GeV it can reach 10^{20} amps. If the electric field is not switched off when J_{max} is reached, particles with charge e_i will leave the string at a rate per unit length of

$$d^2N/dtdz = |e_i|E/h \tag{1.6.16}$$

The anomaly cancellation condition means that there is no build-up of charge on the string since particles of equal and opposite charge are created at the same rate. This situation does not hold for axion strings [74] where the effective 2-d theory on the string is anomalous. Here the non conservation of charge on the string is compensated by a radial current carried by the axion field [74].

In a realistic situation the time varying currents induced on the string by an electromagnetic wave will produce a back e.m.f., thus modifying the right hand side of (1.6.14) by a factor L. Witten finds this inductance to be [26]

$$L = (1 + \Sigma e^2 \ln(\eta/\omega)/2\pi^2)^{-1} \tag{1.6.17}$$

where η is the mass scale of the string and ω the frequency of the incident wave. For GUT scale strings of galactic size this is about 10^{-1}, but for most purposes it can be neglected.

There are also bosonic charge-carrying zero modes [26]. The Higgs potential may be such that it is energetically favourable for a charged massless component to have an expectation value at the core of the string, which is possible for components with $\overline{Q} = 0$. Indeed, we shall see in Chapters 3 and 4 that it is natural in SO(10) to suppose that this happens. The result of this core v.e.v. is that the scalar field can have modes of the form $e^{i\theta(t,z)}\phi(x,y)$. The effective two dimensional action of the modes, after the transverse coordinates have been integrated out, can be shown to be equivalent to the bosonised version of the fermionic zero modes [26]. If σ is the coordinate along a loop of string the persistence of the current (i.e. its superconducting property) is guaranteed by the fact that $N = \int d\sigma . d\theta / d\sigma 2\pi$ must be an integer, and in the absence of a field $J \sim d\sigma / d\theta$. The maximum current is determined by the fact that N is only well defined if $\phi \neq 0$ everywhere at the core of the string. When the current approaches $|\phi(0,0)|$ then the probability for tunnelling processes in which N changes by an integer presumably becomes appreciable. Hence the maximum current for bosonic zero modes is

$$J_{max} \simeq e\eta \qquad (1.6.18)$$

which is bigger than the maximum current for fermionic zero modes by a factor g^{-1} .

Now we turn to a few aspects of the astrophysics of superconducting strings, as discussed by Chudnovsky et al. [24]. An oscillating loop of size R in a constant magnetic field B_0 will see an alternating electric field $\underset{\sim}{E} = \underset{\sim}{v}\times\underset{\sim}{B}_0/c$ and develop an alternating current with period R/c. From (1.6.19) we see that for typical loop velocities of order c the typical current in the loop will be

$$J = ce^2 B_0 R/h \qquad \text{(SI units)} \qquad (1.6.19)$$

This is about $3.10^{12}(R/10^{18}\ \text{m})(B_0/10^{-10}\ \text{T})$ amperes, where 10^{18} m is a typical size of a loop in a galaxy and 10^{-10} Tesla a typical value for a galactic magnetic field. The value can only be reached for strings with heavy charge carriers; if the charge carriers are light the current will be limited by J_{max} and the loop will turn kinetic energy into particles at a rate of about

$$dN/dt \simeq 2.10^{31}\ (R/10^{18}\ \text{m})(B_0/10^{-10}\ \text{T})\ \text{s}^{-1} \qquad (1.6.20)$$

If these particles are quarks and leptons this means that mass is created at a rate of about 3.10^4 kg s^{-1} or about 10^{14} kg per period, over a region of about 30 pc. This is clearly insignificant compared with the mass of the loop itself: over one galactic revolution (10^{15} s) only about 10^{20} kg is produced - not even enough to make a small moon.

The oscillating current-carrying loops, if they were still around, are sitting in an ionised plasma, which means that as the string sweeps through it the charged particles can produce radiation by shock heating and synchrotron emission as they spiral in the very intense magnetic fields near the string. Chudnovsky et al. estimate the rate of energy loss due to synchrotron radiation in a plasma of number density n to be [24]

$$\dot{E}_{sy} = 10^{24}(B_0/10^{-10}\ \mathrm{T})^{3/2}(R/10^{18}\ \mathrm{m})^{5/2}(n/10^{-6}\ \mathrm{m}^{-3})\ \mathrm{Js}^{-1} \tag{1.6.21}$$

at a typical wavelength of

$$\lambda \sim 0.2(n/10^{-6}\ \mathrm{m}^{-3})^{-1/2}\ \mathrm{m} \tag{1.6.22}$$

This is much greater than the width of the string because the radiation occurs at the so-called tangential discontinuity, where the ambient lines of force pile up against those produced by the current on the string. Shock heating of the plasma dissipates much more energy for large loops: the energy loss due to this effect is [24]

$$\dot{E}_{sh} = 10^{33}(B_0/10^{-10}\ \mathrm{T})(R/10^{18}\ \mathrm{m})^2(n/10^{-6}\ \mathrm{m}^{-3})^{1/2}\ \mathrm{Js}^{-1} \tag{1.6.23}$$

Now, 10^{33} Js^{-1} is about 10^{28} GeV^2, so this will be comparable to gravitational radiation if $50G\mu^2 < 10^{28}$ GeV^2 (see equatiom (1.4.16)), that is, if $G\mu < 10^{-12}$ [24].

In Chapter 4 we shall look at zero modes on strings in non-Abelian gauge theories, and find significant differences in their current carrying ability.

CHAPTER 2: FRICTION DOMINATED STRINGS AND BARYOGENESIS.

2.1 Cosmic strings during the friction dominated era.

In this chapter, the evolution of a system of cosmic strings during the period when their motion is strongly affected by friction is discussed. This lasts from the time the network of strings is formed, $t_c \simeq (G\mu)^{-1} t_p$, until a time $t_* \simeq (G\mu)^{-1} t_c$, when friction is unimportant for all curvature scales below the horizon [13,34]. In some superstring inspired models [23] $(G\mu)$ can be as small as 10^{-34}, corresponding to the electroweak scale $\mu \sim 10^2$GeV, so t_* could be as large as 10^{25}s. The evolution of light strings is thus dominated by frictional forces until comparatively late times. We shall see that friction changes the details of the evolution of the string system, and indeed may just conceivably cause the universe to become string dominated shortly after the phase transition. At any rate, the number density of loops formed during this time with sizes between R and R + dR is not the scale free distribution

$$n(t,R)dR = \nu t^{-3/2} R^{-5/2} dR \qquad (2.1.1)$$

What happens when small loops decay into heavy 'X' bosons will be investigated, and finally we shall re-examine the mechanism for baryon number production proposed by Bhattacharjee et al. [25], and show that loops decaying during the heavily damped period can produce realistic amounts of baryon asymmetry. An approximately Brownian network of infinite strings and loops is produced at the phase transition [13,40] with correlation length $\xi \simeq (\lambda\eta)^{-1}$, where λ is the quartic self-coupling of the relevant Higgs field and η its vacuum expectation value. The critical temperature T_c is essentially η. Using the relation between time and temperature $t \simeq 0.3 m_p/\sqrt{N_*}T^2$, (where N_* is the effective number of relativistic degrees of freedom, equal to the number of bosonic plus 7/8 times the number of fermionic degrees of freedom), we find that this happens at a time t_c given by $t_c \simeq 0.3\, m_p/\sqrt{N_*}\eta^2$. The strings begin to move under tension $\mu \simeq \eta^2$, and a segment of string curved on a scale r experiences a straightening force of μ/r. However, the network is immersed in radiation with density $\rho = \pi^2 N_* T^4/30 = 3m_p^2/32\pi t^2$ and so a segment of string moving with velocity v will experience a retarding force per unit length of $\sigma\rho v$, where σ is the cross-section per unit length. The smallest scales of the string will thus move at a terminal velocity $v_t \simeq \mu/\sigma\rho\xi$. Everett [44] has calculated the cross-section for particles scattering off the string; for particles of momentum $\sim$ T it is

$$\sigma(T) \simeq \pi^2/T^2 \ln^2(T/T_c) \tag{2.1.2}$$

Hence we can define a damping time $t_d = \mu/\sigma\rho \simeq G\mu m_p^{1/2} t^{3/2}$, so that $v_t \simeq t_d/\xi$. We may define the scale below which damping is negligible as that for which the terminal velocity is 1; hence this scale, $r_f(t)$, is

$$r_f(t) \simeq (G\mu) m_p^{1/2} t^{3/2} \tag{2.1.3}$$

From this the evolution of the correlation length of the string network can be found [13,94]. The rate of increase of the correlation length will be given essentially by the terminal velocity, so that

$$\frac{d\xi}{dt} \simeq \frac{t_d}{\xi} \tag{2.1.4}$$

This may be integrated to give

$$\xi^2(t) = \xi^2(t_c) + (G\mu)m_p^{1/2}(t^{5/2}-t_c^{5/2}) \tag{2.1.5}$$

Comparing $r_f(t_c)$ from (2.1.3) with $\xi(t_c) \simeq (\lambda\eta)^{-1}$ we see that initially, $r_f \ll \xi$, and the network is essentially "stuck" by friction in the surrounding radiation and conformally stretched until $\xi(t_c)a(t)/a(t_c) = r_f(t)$, where a is the Friedmann-Robertson-Walker scale factor. At this stage he universe is radiation dominated with $a(t) \sim t^{1/2}$, so using the expressions for t_c and $\xi(t_c)$, and equation (2.1.3) we find $t = t_c/\lambda$. At this point the ratio of the energy density in string ρ_s, to that in relativistic particles, ρ_r, is

$$\frac{\rho_s}{\rho_r}(t) = \frac{\mu}{\xi^2(t_c)} \frac{a^2(t_c)}{a^2(t)} \left(\frac{3}{32\pi} \frac{m_p^2}{t^2}\right)^{-1} = 30\lambda^2 \frac{t}{t_c} = 30\lambda \tag{2.1.6}$$

so if λ is more than about 0.03, there is a chance that the energy density of the universe will become dominated by strings with a very small correlation length. It is straightforward to see that this string dominated universe, as discussed by Kibble [94], evolves almost like a matter dominated one, with the scale factor proportional to t^β, where β is between 2/3 and 1, but much closer to 2/3. The energy in a comoving length of infinite string with

persistence length much less than horizon size remains almost constant because redshifting just straightens the string [37], while the number of strings per unit volume just goes as a^{-3}. Hence
the energy density goes as a^{-3} when $\xi \ll t$. The exponent of t cannot quite be equal to 2/3, because if $\xi \simeq t$ the whole network of string is conformally stretched so that the energy density, $\mu\xi^{-2}$, becomes proportional to a^{-2}. In this case, discussed by Vilenkin for non-intercommuting strings [49], the scale factor goes as t. Albrecht and Turok however have simulated such strings [37], and found that $\xi \ll t$ and that the energy density went as a^{-3}. It would seem that the smaller ξ/t, the closer an infinite string dominated universe is to a matter dominated one. If the strings do reconnect the other way when they intersect, loops will be chopped off the straightening network, but since their energy also scales as a^{-3}, a mixture of loops and long string will still behave like matter. Whatever the relative proportion of long strings and loops this is still a disaster: a universe dominated by string from early times is clearly not ours. However, in view of the uncertainties in the calculation, especially in the estimate of the force due to friction, few firm conclusions on the value of λ can be drawn. Anyway, let us suppose that λ is sufficiently small that the universe remains radiation dominated. Eventually, the constant terms in (2.1.5) will become negligible and $\xi(t)$ will obey

$$\xi^2(t) \simeq (G\mu)m^{1/2}t^{5/2} \qquad (2.1.7)$$

The network will continue to evolve under friction until the persistence length ξ catches up with the horizon, at a time t_* given by $\xi(t_*) \simeq t_*$. After t_*, which is approximately $(G\mu)^{-1}t_c$, ξ can grow no faster than t, by causality, and the usual string scenario begins to take effect.

Between t_* and t_c we can expect loops to be chopped off the network, and in a volume ξ^3 one loop will be produced in a time v_t/ξ, so that the loop production rate per unit volume is

$$\frac{dn}{dt} = \nu \frac{1}{\xi^4}\frac{d\xi}{dt} \tag{2.1.8}$$

where $\nu \simeq 10^{-2}$ [37]. (Note that in the scaling solution we have $\xi \simeq t$ and $dn/dt = \nu t^{-4}$.) The loops thus produced will initially shrink at their terminal velocity: if r is the radius,

$$\dot{r} \simeq -t_d/r \simeq (G\mu)m_p{}^{1/2}t^{3/2}/r \tag{2.1.9}$$

So that if t' is the time of formation (neglecting numerical factors of order unity)

$$r^2(t) = r^2(t') - (G\mu)m_p{}^{1/2}(t^{5/2} - t'^{5/2}) \tag{2.1.10}$$

Therefore the loops will begin to move freely, i.e. friction will be unimportant, when

$$r^2(t) < r_f{}^2(t) \tag{2.1.11}$$

The loops will be formed with size $r(t') \sim \xi(t')$, and using equations (2.1.3) and (2.1.7) we obtain

$$2t'^{5/2} - t^{5/2} < (G\mu)m_p^{1/2}t^3 \qquad (2.1.12)$$

Thus the loop starts to move freely between t' and $2^{2/5}t'$, almost immediately in fact, and we can take its size to be $r_f(t')$. The loop will be fairly circular when it starts to move freely because irregularities will be smoothed out during the damped collapse, but there is no danger of it shrinking to a point and annihilating immediately because to do that the loop would have to be circular to within a string thickness. This is highly unlikely. The energy in the loops will go into heating up the radiation, but it is easy to see that the rate of energy loss as heat is insignificant, except for very large values of $(G\mu)$: let $\rho_s \simeq \mu\xi^{-2}$ be the energy density in the network and $\rho_r \simeq 0.03m_p^2t^{-2}$ be the energy density in radiation. Now ρ , which goes as $t^{-7/2}$, is the maximum rate that energy density can appear as loops. We shall shortly see that the loop energy density goes as $t^{-3/2}$ to begin with, and then as t^{-2}, which is always a slower decrease. Hence $\dot{\rho}_s$ is an upper bound for the rate of heating. Using (2.1.5)

$$\dot{\rho}_s/\dot{\rho}_r \simeq 10(G\mu)^{1/2}(t_c/t)^{1/2} \qquad (2.1.13)$$

so even if the upper bound were saturated and all the energy lost by the network were to go into particles, the effect on the radiation density would be negligible.

The loops, then, are produced at a rate $\nu\xi(t)^{-4}\dot{\xi}(t)$ (equation (2.1.8)) with size $r_f(t)$, whereupon they radiate gravitationally [50-54] at a rate $\gamma'(G\mu)\mu$, with $\gamma' \simeq 50$. Their initial mass is $\beta\mu r_f(t)$, where β is close to 2π, so the lifetime of the loop will be about $(\gamma G\mu)^{-1} r_f(t)$ where $\gamma \simeq 10$. The loop number density will decrease as the FRW scale factor cubed, so if $n(t,t')dt'$ is the number density at time t of loops formed between t' and t' + dt' in a radiation dominated universe, then using (2.1.7) we have

$$n(t,t')dt' = \nu(G\mu)^{-3/2} m_p^{-3/4} t^{-3/2} t'^{-13/4} dt' \qquad (2.1.14)$$

Note that this is not scale free. Let us now see how the mass density evolves towards its scaling value. The density in infinite string is $\sim \mu\xi^{-2}$, so

$$\rho_\infty/\rho_r \simeq 30(G\mu)^{1/2}(t_c/t)^{1/2} \qquad (t_c \ll t < t_*) \qquad (2.1.15)$$

The density in loops is

$$\rho_1(t) \simeq \int_{t_s}^{t} \beta\mu r_f(t') \frac{\nu}{\xi(t')^4}\frac{d\xi}{dt'}\left(\frac{a(t')}{a(t)}\right)^3 dt' \qquad (2.1.16)$$

Any loop formed before t_s will have decayed by gravitational radiation to a size of order of its width, $\alpha\eta^{-1}$ (where α is the inverse of a coupling constant), whereupon it will have decayed into massive bosons. Taking this into account we can find t_s by solving

$$(\gamma G\mu)^{-1}(r_f(t_s) - \alpha\eta^{-1}) = t \qquad (2.1.17)$$

so defining t^+ to be $(G\mu)^{-1/2} t_c$ we find from (2.1.3)

$$t_s = \alpha^{2/3} t_c (1 + \gamma t/\alpha t^+)^{2/3} \tag{2.1.18}$$

Performing the integration, we have

$$\rho_l(t) = \beta\nu\mu(G\mu)^{-1/2} m_p^{-1/4} t^{-3/2} t_s^{-3/4} \tag{2.1.19}$$

Hence for $t < t^+$ ρ_l goes as $t^{-3/2}$ and

$$\rho_l/\rho_r \simeq 30(\beta\nu/\alpha^{1/2})(G\mu)^{3/4}(t/t_c)^{1/2} \tag{2.1.20}$$

and for $t > t^+$ ρ_l goes as t^{-2} and

$$\rho_l/\rho_r \simeq 30(\beta\nu/\gamma^{1/2})(G\mu)^{1/2} \tag{2.1.21}$$

which is the value obtained from the scaling solution. Thus the oscillating loops come to dominate the energy density of the string network at $t \simeq (\alpha^{1/2}/\beta\nu)(G\mu)^{-1/4} t_c$. The loops are initially very much smaller than the persistence length of the string network because $\xi/r_f \simeq (G\mu)^{-1/4}(t/t_c)^{1/4}$, and so we might expect that the reconnection probability [95] is very small. At this stage, therefore, the system probably could not start to evolve towards the string dominated universe discussed by Kibble [94]

2.2 The decay of small loops and baryon asymmetry.

In this section we shall investigate the final stages of a loop's life, when it disappears into a burst of heavy bosons. The hope was originally to try and get a lower bound on the string tension, because for low values of μ there are lots of small loops about, and some observational constraints might have been found on the flux of particles resulting from their decays. Unfortunately, as we will see, this flux is too low to have any significant effects, but the decays of small loops are a mechanism for providing out of equilibrium decays of massive bosons, an essential ingredient for the generation of baryon asymmetry n_B/s [96]. Thus, by re-examining the work of Bhattarcharjee et al. [25], we will find that reasonable values for n_B/s can be obtained.

First let us calculate the decay rate of small loops. Recall that the decay time of a loop formed at t_s is given by (2.1.18). Hence the decay rate per unit volume, $d\Gamma_l/dV$, is

$$\frac{d\Gamma_l}{dV} = \nu[(G\mu)^{3/2}m_p{}^{3/4}\alpha^{13/6}t^{3/2}t_c{}^{13/4}(1 + \gamma t/\alpha t^+)^{13/6}]^{-1} \tag{2.2.1}$$

For $t < t^+$ this is

$$\frac{d\Gamma_l}{dV} = (\nu/\alpha^{13/6})(G\mu)^4 m_p{}^4(t^+/t)^{3/2} \tag{2.2.2}$$

and for $t > t^+$

$$\frac{d\Gamma_l}{dV} = (\nu/\gamma^{13/6})(G\mu)^4 m_p{}^4(t^+/t)^{11/3} \tag{2.2.3}$$

The loops are assumed to decay when their radii are $\alpha\eta^{-1}$, which means that they will release energy of about $2\pi\alpha\eta$ in the form of heavy gauge and Higgs bosons, which decay sooner or later (depending on their couplings to fermions) into relativistic particles. Let us first consider the case where the particles decay in much less than an expansion time. The rate at which energy appears as radiation, $\dot{\rho}_{l\to\gamma}$, is then

$$\dot{\rho}_{l\to\gamma} = \left(\frac{2\pi\nu}{\alpha^{7/6}}\right)(G\mu)^{9/2}m_p^5\left(\frac{t^+}{t}\right)^{3/2} \qquad (t < t^+) \qquad (2.2.4)$$

At $t \simeq t^+$ this will be important compared to $\dot{\rho}_r \simeq 0.06 m_p^5 (G\mu)^{9/2}(t^+/t)^3$ if $(2\pi\nu/\alpha^{7/6}) > 0.06$, so it just conceivable that the universe undergoes a brief period of reheating around $t = t^+ = (G\mu)^{-1/2}t_c$. At other times the energy from the decays will be rapidly thermalised and have little effect. The decay rate Γ_x of heavy bosons, which have mass m_x, will be on dimensional grounds approximately given by [97] $\Gamma_x \sim g^2 m_x$, where g is either the Higgs or gauge coupling to fermions. Hence the bosons will decay in less than expansion time if

$$g^{-2}m_x^{-1} < t(T) \simeq 0.3 m_p/N_*^{1/2}T^2 \qquad (2.2.5)$$

This is only possible for phase transitions occuring at a scale η such that

$$g^{-2}m_x^{-1} < t(T_c \simeq \eta) \qquad (2.2.6)$$

or

$$\eta < g^2(0.3/N_*^{1/2})m_p \tag{2.2.7}$$

For Higgs bosons this could be as low as 10^{13} GeV.

Let us now consider what happens when $\Gamma_X << t^{-1}$ for $t > t_c$. After the phase transition the X-bosons acquire a mass, but because their decay rate is so slow their number density and hence their energy density just decreases like a^{-3}. Therefore the universe becomes matter dominated at a time t_X given by

$$\frac{\rho_X}{\rho_r}(t_X) \simeq \frac{\rho_X}{\rho_r}(t_c)\left(\frac{a(t_X)}{a(t_c)}\right)^3 \simeq \frac{1}{N_*}\left(\frac{t_X}{t_c}\right)^{1/2} = 1 \tag{2.2.8}$$

This continues until the massive bosons decay at $\tau \simeq g^{-3}\eta^{-1}$ which is about $g^{-3}(G\mu)^{1/2}t_c$.

If the interactions of these bosons violate C and CP, their out of equilibrium decays can generate a non-zero baryon number density [96]. Suppose that on average the decay of an $(X,\overline{X})$ pair produces baryon number ε. Before t_X the number density of X bosons n_X is equal to the number density of any relativistic species, so that the ratio of n_X to the entropy density s is just N_*^{-1}. This means that if $\tau < t_X$, the baryon asymmetry produced is $n_B/s \simeq 10^{-2}\varepsilon$. If $\tau > t_X$, most of the photons we see today came from the decays of X bosons, in which case $n_B/s \simeq \varepsilon$.

The evaporation of loops of string into heavy bosons also produces an out of equilibrium decay, as pointed out by Bhattarcharjee et al. [25]. They used the fact that an initially static loop of length l collapses to a doubled loop in a time 1/4 [55], at which point, if it is a $\underline{Z_2}$ string [29], it will annihilate into bosons, and their decay produce the baryon asymmetry. They estimated that the fraction f of loops that would collapse and annihilate immediately would be about 10^{-1}. In view of numerical simulations [37] this now seems highly optimistic; the loop is being chopped off a moving string so it appears very unlikely that it would form an initially static configuration. What is more likely is that the loops start to oscillate, after an initial period of friction damped shrinking, and radiate away their energy gravitationally until their radii become comparable to their width. At this point their field theoretic origin becomes important. Presumably they decay fairly quickly into bosons, and Shellard's global string simulations [47] lend support to this. Let us assume that the average number of heavy bosons per loop decay is N_1, and let us also assume that the decay rate of bosons into light particles is sufficiently fast so that there is no era of matter domination after the phase transition. If the average baryon number per decay is ε, then during the friction dominated period

$$dn_B/dt \simeq A\alpha^{-13/6}(t^+/t)^{3/2} \qquad (t < t^+) \tag{2.2.9}$$

$$dn_B/dt \simeq A\gamma^{-13/6}(t^+/t)^{11/3} \qquad (t > t^+) \tag{2.2.10}$$

where $A = \varepsilon\nu N_1(G\mu)^4 m_p{}^4$. To a first approximation we may ignore the contribution of decaying loops to the entropy density s, which is approximately

$$s \simeq N_* T^3 \simeq (G\mu)^{9/4} m_p{}^3 (t^+/t)^{3/2} \tag{2.2.11}$$

so that the baryon number asymmetry generated is about

$$n_B/s \simeq \int (dn_B/dt)s^{-1}dt \simeq (\varepsilon\nu N_1/\gamma^{13/6})(G\mu)^{7/4}m_p t^+$$
$$\simeq (\nu N_1/\gamma^{13/6})(G\mu)^{1/4}\varepsilon \qquad (2.2.12)$$

Using $\nu \simeq 10^{-2}$ and $\gamma \simeq 10$ we have

$$n_B/s \simeq 10^{-4}(G\mu)^{1/4}N_1\varepsilon \qquad (2.2.13)$$

Hence we see that for GUT scale strings, where $G\mu \simeq 10^{-6}$, this mechanism is insignificant compared to the usual scenario unless $N_1 \simeq 10^3$, which seems hard to obtain. However, if C and CP are not broken until a string-producing phase transition at 10^{13} GeV, the decay rate of bosons will be fast enough so that they are always in equilibrium, and little or no baryon asymmetry will be generated in this way. However, the decay of string loops can easily generate $n_B/s \simeq 10^{-9}$ with $N_1 \simeq 10$ and $\varepsilon \simeq 10^{-3}$. The quantity ε can never be greater than 10^{-2} [97] so realistic amounts of baryon asymmetry cannot be produced by string loops if $(G\mu) < 10^{-16}$

In conclusion, we have seen how in the friction dominated period the initial network of string evolves towards the scaling solution. In order that the persistence length goes from $(G\mu)^{1/2}t_c$ to t loops must be formed faster than the scaling rate per unit volume of νt^{-4}, in fact from (2.1.8) it is

$$dn/dt = \nu(G\mu)^{-3/2}m_p^{-3/4}t^{-19/4} \qquad (2.2.14)$$

These loops initially shrink under friction until the acceleration due to curvature in the string is large enough to overcome the damping forces, and then they radiate gravitationally in the usual way before annihilating into massive bosons. The out of equilibrium decay of these bosons can generate realistic amounts of baryon asymmetry if $G\mu > 10^{-16}$. If $G\mu > 10^{-13}$ the decay rate of the bosons can be slow enough for the usual out of equilibrium decay scenario to work, in which case the loop contribution is swamped.

CHAPTER 3: BEADS ON STRINGS

3.1 The SU(2) bead.

In section 1.3 we saw that an SU(2) gauge theory broken by a pair of adjoint Higgs to Z_2 had string solutions with first homotopy group Z_2, which are often called Z_2 strings, with the form (1.3.4) carrying flux $2\pi/e$. The antistring is obtained by replacing ϕ, the azimuthal angle, by $-\phi$, and A_ϕ by $-A_\phi$, so that after a gauge transformation by $\exp(i\pi T^3)$ the Higgs fields may be written

$$\Phi_1 = f_1(\rho)(0,0,-1)$$
$$\Phi_2 = f_2(\rho)(\cos\phi,\sin\phi,0) \tag{3.1.1}$$

The appellation Z_2 means that string and antistring are smoothly deformable into each other through states of finite energy per unit length; in fact, all we do is smoothly change the sign of Φ_2. This does not mean, however, that string and antistring are identical up to a gauge transformation, although their asymptotic values at infinity are related by a rotation through π about Φ_2. This gauge transformation is singular at the origin, so that the two solutions are indeed distinct [27]. In order to get from one to the other it is clear that if Φ_1 and Φ_2 are to remain everywhere perpendicular then $\Phi_1(\rho = 0)$ must vanish somewhere during the deformation, At that point it is energetically favourable for Φ_1 to be in the Nielsen-Olesen configuration (1.3.5), which at large ρ is just a $\pi/2$ rotation of (1.3.4) about Φ_2. Recall that this is a higher energy configuration. We are now able to present the bead: in its simplest form for a string centred on the z axis it is just a solution which interpolates between string at $z = +\infty$ and antistring at $z = -\infty$. In view of the above discussion for $SU(2) \to Z_2$ it is clear that there is a region of higher energy per unit length where Φ_1 vanishes. Minimising the energy of the solution will, in the competition between gradient energy in the z direction and the 'potential' energy per unit length (all the other pieces in the expression for the energy), produce a configuration in which the departure from the ground states will be confined to a region of order m_s^{-1}, the inverse of a scalar boson mass. We shall show that this is true under a reasonable set of assumptions in the next section.

We can also see why the bead may be called a monopole on a string. Suppose that Φ_1 and Φ_2 gain expectation values at two different scales, so that we can imagine an intermediate U(1) theory:

$$SU(2) \xrightarrow{\Phi_1} U(1) \xrightarrow{\Phi_2} Z_2 \tag{3.1.2}$$

After the first stage, 't Hooft-Polyakov monopoles can appear [14,73]. For a monopole centred on the origin $\Phi_1(r = 0)$ vanishes and takes up a 'hedgehog' configuration everywhere else, that is to say that if we identify space and isospin axes Φ_1 points away from the origin and reaches its vacuum value outside a spherical region of size m_s^{-1}. The direction of Φ_1 in group space defines the generator of charge rotations, and we can define a U(1) flux by

$$\text{Flux} = \int dS_i (\Phi_1 . B^i) \tag{3.1.3}$$

where the integral is taken over the sphere at infinity. This is equal to $4\pi/e$ [14]. Following Bais [75] we may ask what happens to this flux after the second stage of symmetry breaking when Φ_2 acquires an expectation value orthogonal to Φ_1 (this can be arranged by having a $(\Phi_1 . \Phi_2)^2$ term in the potential). Clearly, it is confined to a tube extending to infinity, and since the stable flux tubes - strings - of the Z_2 theory carry a flux $2\pi/e$, we need two of them attached to the monopole. We can also see that there must be two flux tubes by the following argument: in this gauge if Φ_2 is to be orthogonal to a radial vector it must be a tangent vector to spheres centred on the monopole, and so by a well-known theorem [38] it must vanish at two points at least. These define the positions of the flux tubes, and arranging them in opposite directions along the z axis gives us exactly the bead solution described above, but globally rotated by $\pi/2$ around the T^3 isospin axis.

We can now write down the form of the Higgs fields at infinity for the SU(2) bead solution, which will exhibit a form useful for later generalisation. Let us form a complex representation $\Phi = \Phi_1 + i\Phi_2$, and take $\Phi_0 \sim (i,0,1)^T$. In the gauge we have been working in, the solution at large distances from the z axis for a bead at the origin is

$$\Phi(\phi, z) = e^{i\phi T^3} \, e^{i\chi(z)T^1} \, \Phi_0 \tag{3.1.4}$$

where $\chi(+\infty) = 0$, $\chi(0) = \pi/2$, and $\chi(-\infty) = \pi$. The solution at $z = -\infty$ is gauge equivalent to the antistring, $\overline{\Phi} = \exp(-i\phi T^3)\Phi_0$ by the gauge transformation $\exp(-i\pi T^1)$.

In section 3.3 we shall investigate the question of whether this can be generalised to the more general symmetry breaking schemes, in particular $G \to K\times Z_2$ [39,27].

3.2 Solving for the bead configuration.

Having written down a form for the bead solution in (3.1.4), it now remains to substitute into the equations of motion to find a static solution satisfying the bead boundary conditions. The solution may be written as follows :

$$\Phi_1 = f_1(\rho,z)\begin{pmatrix}\sin\phi\ \sin\chi\\ \cos\phi\ \sin\chi\\ \cos\chi\end{pmatrix} + a(\rho,z)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Phi_2 = f_2(\rho,z)\begin{pmatrix}\cos\phi\\ -\sin\phi\\ 0\end{pmatrix} \tag{3.2.1}$$

$$A_\phi = -\frac{1}{e\rho}(1 - F(\rho,z))\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad A_z = -\frac{\chi'}{e}(1 - G(\rho,z))\begin{pmatrix}\cos\phi\\ -\sin\phi\\ 0\end{pmatrix}$$

where f_1, f_2, A_ϕ, and A_z all vanish at the origin and a, F, and G all vanish at infinity. The boundary conditions on $\chi'(z)$ are $\chi'(\pm\infty) = 0$. The form for the gauge potentials follows from the condition that the covariant derivative $(\partial_\mu + ieA_\mu)\Phi_{1,2}$ vanishes at infinity. The solution could be found by minimising the energy of this configuration which would give six coupled nonlinear partial differential equations. In principle they are numerically solvable, but we can extract much information without going to all this trouble, for we know roughly what the solution will look like anyway. The transition between string and antistring will occupy only a small distance, probably of order m_s^{-1}, around the origin, where χ will change from 0 to π, and $a(\rho)$ will change sign. The functions f_1, f_2, F, and G will not depend very much on z since they are determined mainly by the shape of the solution in the transverse direction. In order to find the size of the bead all we need to is look at the equations of motion near the origin, where f_1 and f_2 vanish and F and G are equal to one. The equations of motion for the Higgs field are

$$-D^{\mu}D_{\mu}\Phi_a = \frac{\partial V(\Phi_a)}{\partial \Phi_a} \qquad (a = 1,2) \qquad (3.2.2)$$

where

$$V = \lambda(\Phi_1^2 + \Phi_2^2 - \eta^2)^2 + \delta(\Phi_1^2 - \Phi_2^2)^2 + \varepsilon(\Phi_1 . \Phi_2)^2 \qquad (3.2.3)$$

(This is not the most general potential.) When the two Higgs fields are orthogonal the last term vanishes, as is the case here. In the limit $\rho \to 0$ equation (3.2.2) reduces to

$$\nabla^2 \begin{pmatrix} f_1 \sin\phi \sin\chi \\ f_1 \cos\phi \sin\chi \\ f_1 \cos\chi + a \end{pmatrix} = \left[4\lambda(a^2 - \eta^2) + 4\delta a^2\right] a \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (3.2.4)$$

By contracting both sides with $(\sin\phi, \cos\phi, 0)$ and $(0,0,1)$ and using $f_1(0,z) = 0$, we see that

$$\sin\chi\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\right)f_1 = 0 \qquad (3.2.5)$$

$$\frac{\partial^2}{\partial z^2}a - [4\lambda(a^2 - \eta^2) + 4\delta a^2]a = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}(f_1\cos\chi + a) \qquad (3.2.6)$$

Near the z axis, f_1 can be expanded in powers of ρ

$$f_1 = c_1(z)\rho + c_2(z)\rho^2/2 \qquad (3.2.7)$$

Substituting into (3.2.5) we find that the left hand side is equal to $3\sin(\chi)c_2/2$, and so $c_2(z)$ vanishes for all z. Furthermore, the right hand side of (3.2.6) will have a piece equal to $c_1\cos(\chi)\rho^{-1}$ unless it is cancelled by the derivatives of $a(\rho,z)$, and since the left hand side is non-singular at the origin we may therefore conclude that the coefficient of ρ in the expansion of $a(\rho,z)$ about the z axis is $-c_1(z)\cos\chi$. In order to proceed further it is necessary to make the assumption that $a(\rho,z)$ is separable near $\rho = 0$, at least up to $O(\rho^2)$: we write $a(\rho,z) = R(\rho)Z(z)$ with $Z(z) = -\cos(\chi)c_1(z)/c_1(\infty)$ so that $Z(\pm\infty) = \pm 1$, and we obtain

$$\frac{d^2Z}{dz^2} - [4\lambda(R(0)^2Z^2 - \eta^2) - 4\delta R(0)^2Z^2]Z = 2R''(0) \qquad (3.2.8)$$

The derivatives of Z vanish at $z = \pm\infty$, so

$$R''(0) = [2\lambda\eta^2 - 2(\lambda+\delta)R(0)^2]R(0) \qquad (3.2.9)$$

Substituting back into (3.2.8) a simple equation for Z results:

$$\frac{d^2 Z}{dz^2} = 4(\lambda+\delta)R(0)^2(Z^2 - 1)Z \qquad (3.2.10)$$

This has the well-known 'kink' solution

$$Z = \tanh\left[2R(0)\surd(\lambda + \delta)(z - z_0)\right] \qquad (3.2.11)$$

In order to estimate R(0) we must examine the Higgs potential at $z = +\infty$

$$V = \lambda\left((f_1 + a)^2 + f_2^2 - \eta^2\right)^2 + \delta\left((f_1 + a)^2 - f_2\right)^2 \qquad (3.2.12)$$

At the core where f_1 and f_2 vanish the potential is minimised when $a = R(0) = \surd\left(\lambda/(\lambda + \delta)\right)$, so we might expect the true value of R(0) to be slightly larger in order to save radial gradient energy. If we take this difference to be small, we find that the bead is confined to a region of order $(\eta\surd\lambda)^{-1} = m_s^{-1}$ in size, as was expected. If Φ_1 and Φ_2 have different expectation values η_1 and η_2 ($\eta_1 > \eta_2$), which we could arrange by having $\eta^2 = \eta_1^2 + \eta_2^2$ and changing the second term in (3.2.3) to $\delta(\Phi_1^2\eta_2 - \Phi_2^2\eta_1)^2/\eta^2$, the calculation goes through as before, but with δ replaced by $\delta(\eta_2/\eta)^2$. If the two scales are very different the bead size is determined by the larger scale, which controls the symmetry breaking SU(2) $\rightarrow$ U(1), that is, the bead is similar in size to the monopole of that theory. This is in keeping with the idea of a bead being a monopole on a string.

3.3 Beads and non-Abelian strings.

The question now arises whether it is possible to generalise the bead solution to other symmetry breaking schemes with Z_2 strings. It is not possible to answer this question in general, but we shall address two particularly relevant classes of symmetry breaking, $G \to K \times Z_2$ [39] which contains $SU(2) \to Z_2$ as the simplest example, and also $SO(n) \to S[O(p) \times O(n-p)]$. However, SU(2) is a special case for there is no continuous symmetry in the unbroken subgroup. In all other cases we are faced with the possibility that the string is gauge equivalent to the antistring, by which we mean that there exists an element k in the connected component of the unbroken subgroup such that if the string solution is

$$\Phi(\rho,\phi) = e^{i\phi Q}\Phi(\rho) \tag{3.3.1}$$

then

$$ke^{i\phi Q}\Phi(\rho) = e^{-i\phi Q}\Phi(\rho) \tag{3.3.2}$$

We shall see that there are two sorts of string, one in which this is the case [76], and one in which the string is homotopic but not gauge equivalent to the antistring, on which beads can exist. The issue of the conditions under which each is the lowest energy solution has not yet been resolved [77].

In general we would expect their energies to depend on the dimensionless parameters of the Higgs potential, but a full answer can only be obtained by accurate numerical solutions to the equations of motion.

In section 1.3 we saw that a compact simple Lie group G could be broken to $K \times Z_n$ by a Higgs field Φ in a symmetric n^{th} rank tensor representation aligned with the highest weight $|\lambda_\Phi\rangle$, where λ_Φ is a fundamental weight [39]. The first homotopy group π_1 of $G/(K \times Z_n)$ is Z_n, so n-1 topologically inequivalent strings result, and we may write the solution for a string on the z axis in a form in which the fields are independent of z and are a gauge rotation at ∞, namely (3.3.1) in which $\Phi(\rho) \to \Phi_0 = |n\lambda_\Phi\rangle\eta$ as $\rho \to \infty$. We shall consider two possibilities, exemplifying the two types of string_n mentioned earlier, one where Q is in the Cartan subalgebra and one where it is not. Let us consider the first possibility, in which case we may write $Q = q.H$, where H^i are the generators of the Cartan subalgebra ($i = 1,\ldots,\text{rank } G$). At infinity, single-valuedness of the Higgs field imposes a quantisation condition on Q

$$n\lambda_\Phi . q = m \tag{3.3.3}$$

where m is an integer, so that the Higgs field changes by a phase at infinity

$$\Phi(\infty,\phi) = \Phi_0 e^{im\phi} \tag{3.3.4}$$

We shall call these strings 'phase' strings for this reason. From here on we confine our remarks to the most relevant case $n = 2$, for which the string has $m = 1$ and is homotopically equivalent to the antistring with $m = -1$. However, these solutions are not gauge equivalent, for that would require an element k of the unbroken subgroup K such that

$$kQk^{-1} = -Q \tag{3.3.5}$$

and this is not possible because of the component of q in the direction of λ_Φ. This means that the deformation which takes the string to the antistring will in general take the field configuration through states of higher energy.

This is the basis of the bead solution, which interpolates between string at $z = \infty$ and antistring at $z = -\infty$. At large ρ it may be written as

$$\Phi(\phi,z) = e^{i\phi Q}e^{i\chi(z)T}\Phi_0 = g(\phi,z)\Phi_0 \tag{3.3.6}$$

where $\chi(+\infty) = 0$ and $e^{i\chi(-\infty)T}$ anticommutes with Q, so that under the gauge transformation $e^{-i\chi(-\infty)T}$ the solution as $z \to -\infty$ is $e^{-i\phi Q}\Phi_0$, the antistring. One way of arranging this is to have q parallel to the simple root not orthogonal to λ_Φ which we may call α_Φ, and the quantisation condition (3.3.3) means that

$$Q = \alpha_\Phi.H/(\alpha_\Phi)^2 \tag{3.3.7}$$

There are good energetic reasons for this, as will be explained shortly. This Q and complex combinations of the corresponding step operators, E_{α_Φ} and $E_{-\alpha_\Phi}$, form an SU(2) subalgebra generated by (T^1, T^2, T^3) with $T^3 = \alpha_\Phi/(\alpha_\Phi)^2$, $T^2 = -i(E_{\alpha_\Phi} - E_{-\alpha_\Phi})/\sqrt{2}$ and $T^1 = (E_{\alpha_\Phi} + E_{-\alpha_\Phi})/\sqrt{2}$. A suitable T is clearly T^2.

A U(1) flux, the analogue of the Φ_1 component of flux in the SU(2) case can be defined by having the generator X be

$$X = g(\phi, z)(\lambda_\Phi.H)g^{-1}(\phi, z) \tag{3.3.8}$$

In going from $z = \infty$ to $z = -\infty$ tr(QX) changes sign, and hence so does the U(1) flux. In this sense we can justify the idea of a bead as a monopole on a string, for it is a source of this flux which is trapped inside oppositely oriented strings.

We now consider the case where Q is equal not to the diagonal generator T^3 but to one of the other two, say T^1, which is not in the Cartan subalgebra of G. In this case Φ does not change by a phase at infinity beause Φ_0 is not an eigenvector of Q, so we can call these strings 'non-phase' strings, and we shall now see that there is a set of gauge transformations generated by an element of the Lie algebra of K which takes Q to -Q. Consider a generator $B = \beta.H$ such that

$\beta.\alpha \neq 0$, and $\beta.\lambda_\Phi = 0$. Then

$$[B,T^1] = i(\beta.\alpha)T^2$$
$$[B,T^2] = -i(\beta.\alpha)T^1 \qquad (3.3.9)$$

and B generates the required transformation

$$e^{i\theta B}T^1 e^{-i\theta B} = \cos((\beta.\alpha)\theta)T^1 - \sin((\beta.\alpha)\theta)T^2 \qquad (3.3.10)$$

taking T^1 to $-T^1$ when $\theta = \pi/(\beta.\alpha)$. Hence the string solution $e^{i\phi Q}\Phi(\rho)$ is gauge equivalent to the antistring solution by $k = e^{i\pi B/(\beta.\alpha)}$. Such a B always exists except for the trivial case of SU(2). Clearly, if the non-phase string is the stable configuration stable beads cannot exist without additional Higgs fields to break the string-antistring symmetry [76], and so it is important to find under what conditions which of the two types of solution is the stable one. This question cannot be settled without numerical calculations of the energies of the solutions, which Everett and Aryal have done for SU(2) $\to$ Z_2 [76]. However, this does not fully model the situation for larger groups so to conclude as they have done that beads do not exist for SO(10) is not neccessarily correct [77].

Let us see how the SU(2) strings fit into the above discussion. The Nielsen-Olesen Z_2 string is a phase string, an assertion which can be demonstrated as follows: by forming a complex representation $\Phi = \Phi_1 + i\Phi_2$ this string solution (equation (1.2.9) with Φ_1 replaced by (1.2.10)) can be written at infinity as

$$\Phi(\infty,\phi) = e^{i\phi T^3}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\eta/\sqrt{2} \qquad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3.11}$$

If $\Phi_0^T = (i,1,0)\eta/\sqrt{2}$, this is equal to $e^{i\phi}\Phi_0$ and (3.3.11) describes a phase string. On the other hand the string (3.3.4) may be written, after a gauge transformation by $e^{i\pi T^2/2}$, as $e^{i\phi T^1}\Phi_0$ which is an example of a non-phase string. We may decompose Φ_0 into eigenvectors ϕ_λ of T^1, where $\lambda = \pm 1$ and 0 [76]

$$\Phi(\infty,\phi) = e^{i\phi T^1}((\phi_1 + \phi_{-1})/2 + \phi_0/\sqrt{2})\eta \tag{3.3.12}$$

making clear that the behaviour of Φ as a function of the azimuthal angle is not a phase change. This solution can have a piece proportional to ϕ_0 non-vanishing at the origin because its azimuthal covariant derivative can vanish everywhere, and so potential energy is saved, which makes it likely that the non-phase string is the lower energy configuration of the two. The computer calculations of Everett and Aryal confirm this [76]. However, they did not take into account the fact that the Higgs fields of the Nielsen-Olesen string could also have a piece in the T^3 direction which need not vanish as $\rho \to 0$.

As mentioned earlier, $SU(2) \to Z_2$ is a special case because it has no unbroken continuous symmetries. Let us address the general case $G \to K\times Z_2$. First consider phase strings, for which Q in (3.3.1) is an element of the Cartan subalgebra of G. From the quantisation condition (3.3.3) we know that

$$Q = \lambda_\Phi . H/2(\lambda_\Phi)^2 + \mu . K \tag{3.3.13}$$

where the K^i are the generators of the Cartan subalgebra of K. The energy per unit length of the string is made up of three pieces

$$E = \int d^2x[\,tr(\underset{\sim}{B}^2)/2 + tr(\underset{\sim}{D}\Phi^+ . \underset{\sim}{D}\Phi) + V(\Phi)] \tag{3.3.14}$$

magnetic, gradient, and potential energies respectively. Since B_z is proportional to Q making $tr(Q^2)$ as small as possible will save magnetic energy, which means that there is a tendency for μ to vanish in (3.3.13), that is by having Q entirely orthogonal to the Lie algebra of K. However, if the Higgs field is in a representation which has no weights with zero eigenvalues under $\lambda_\Phi . H$, it may be energetically favourable for $\mu \neq 0$ so that Q can have zero eigenvalues. For example, if α_Φ is the simple root not orthogonal to λ_Φ, then $Q = \alpha_\Phi/(\alpha_\Phi)^2$ is just such a Q. Magnetic energy will be saved if we choose the basis of K such that α_Φ is a short root. In general it is to be expected that the direction of Q depends on the ratios of the dimensionless couplings in the Higgs potential to the square of the gauge coupling e. An interesting point to notice is that if $\mu \neq 0$, the string is carrying some flux of the unbroken subalgebra: this is rather intriguing because we usually expect colour magnetic fields of an unbroken symmetry group to be unconfined. In the next section the long range magnetic fields of the two types of string will be investigated.

Comparing the phase string with the non-phase string is more difficult, for in general both can have Φ non-vanishing at the origin, and both have the same value of $\mathrm{tr}(Q^2)$. Let us write down the equations of motion, defining the function F by

$$A_\phi = \frac{1}{e\rho}\big(1 - F(\rho)\big) \tag{3.3.15}$$

and using (3.3.1). They are

$$\Big(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{F^2}{\rho^2}Q^2\Big)\Phi = \frac{\partial V}{\partial\Phi^+} \tag{3.3.16}$$

$$F'' - \frac{F'}{\rho} - 2e^2\frac{F}{\rho^2}(\Phi^+Q^2\Phi) = 0 \tag{3.3.17}$$

where the potential has the form

$$V = \lambda_1\big(\mathrm{tr}(\Phi^+\Phi) - \eta^2\big)^2 + \lambda_2\,\mathrm{tr}(\Phi^+\Phi\Phi^+\Phi) \tag{3.3.18}$$

Everett and Aryal [76] have argued that since, at infinity, $\Phi^+Q^2\Phi = \eta^2$ for the phase string and $\eta^2/2$ for the non-phase string that the latter is energetically favoured. While it is true that the energy contains a piece $\mathrm{tr}(D_\phi\Phi^+D_\phi\Phi) = F^2(\Phi^+Q^2\Phi)/\rho^2$, we see from (3.3.16) after multiplying by Φ^+ and integrating by parts that the radial gradient energy contains a piece which cancels it, and by the equations of motion we find

$$E = \int d^2x\Big[\mathrm{tr}(B^2)/2 + V(\Phi) - \mathrm{tr}\Big(\Phi^+\frac{\partial V}{\partial\Phi^+}\Big)\Big] \tag{3.3.19}$$

Written in this form, it is not clear whether the phase or non-phase string has lower energy. The determination of the conditions for which each is the stable string must await an accurate numerical solution to (3.3.16) and (3.3.17).

We conclude this section with a couple of examples, one of which fits into the $G \to K\times Z_2$ class of strings, and the other serves to illustrate the class $SO(n) \to S[O(n-p)\times O(p)]$ [80,81].

Consider a Higgs field transforming under the six dimensional representation of SU(3) formed by symmetric complex 3×3 matrices transforming under an element g of SU(3) as $\Phi \to \Phi' = g\Phi g^T$. If the potential is such that Φ_0 is zero except in the bottom right hand entry then SU(3) is broken to $SU(2)\times Z_2$, where the SU(2) is generated by the Gell-Mann matrices λ^1, λ^2, and λ^3 and the Z_2 by λ^8. The phase string solution may be written as [79]

$$\begin{aligned} \Phi(\rho,\phi) &= e^{i\phi Q}\Big(\Phi(\rho) + \sum_i M_i a_i(\rho)\Big)e^{i\phi Q} \qquad (i=1,2) \\ A_\mu &= -\delta_{\mu\phi}\frac{Q}{e\rho}(1 - F(\rho)) \end{aligned} \tag{3.3.20}$$

where

$$\begin{aligned} Q &= -\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \Phi(\rho) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} f(\rho) \\ M_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & M_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.3.21}$$

The M_i are invariant under Q and so the functions a_i need not vanish as $\rho \to 0$. The generator Q could also be parallel to a weight

$$Q = \mathrm{diag}(-1,-1,\ 2)/4 \qquad\qquad a_i(\rho) = 0 \qquad\qquad (3.3.22)$$

which has smaller $\mathrm{tr}(Q^2)$ but vanishing $\Phi(\rho = 0)$.

The non-phase string may be written as (3.3.20) but with Q equal to the matrix with $\sigma^1/2$ in the bottom right hand corner and $M_1 = \mathrm{diag}(0,1,-1)/\surd 2$, and this is gauge equivalent to the antistring under $k = \mathrm{diag}(1,-1,1) = e^{i\pi\lambda_3}$.

Finally, consider the beaking $SO(3) \to O(2)$ by a Higgs field in the 5 dimensional representation of traceless symmetric 3×3 matrices. This is achieved by having the Higgs at infinity be $\Phi_0 = \mathrm{diag}(1,1,-2)\eta$, in which case the generator of the O(2) is T^3. The string solution may be written

$$\Phi(\rho,\phi) = e^{i\phi T^1}\Phi_0 e^{-i\phi T^1} \qquad\qquad (3.3.23)$$

which is clearly a non-phase string gauge equivalent to the antistring by a rotation $e^{i\pi T^3}$. This is the simplest example of the general form $SO(n) \to S[O(n-p)\times O(p)]$ which has been discussed by Lazarides et al. [80,81]. The rank of the subgroup is unchanged so that the Cartan subalgebra is unbroken, and consequently the strings cannot be phase strings. By embedding the above in traceless symmetric n×n matrices we see that in this scheme strings are always gauge equivalent to antistrings.

In summary, then, we have found that in non-Abelian gauge theories with string solutions there are two kinds of string, the non-phase string which is gauge equivalent to the antistring in the sense (3.3.2) and the phase string which is not. If $\pi_1(G/H) = Z_2$, when a phase string is deformed into the antistring it passes through states which in general have higher energy per unit length. The bead is just a solution which interpolates in space between string and antistring and so cannot exist on the phase string without additional Higgs fields. We have found that beads can exist in a large class of theories of the form $G \to K \times Z_2$ although not without extra Higgs in $SO(n) \to S[O(n-p) \times O(p)]$.

3.4 Long range fields of beads.

All along we have been working with the idea that beads are like monopoles on strings, and in this section it is pursued to see whether, like the monopole, the bead has long range magnetic fields decaying as r^{-2} at infinity. This actually depends on the direction of the generator of rotations around the string, Q, but let us first recall some facts about monopoles in non-Abelian gauge theories [14,73,82-29].

When a gauge symmetry G is broken to a subgroup H by some Higgs field Φ gaining a vacuum expectation value $\|\Phi\| = \eta$ we may look for finite energy solutions in three space dimensions by requiring that the Higgs field approaches the vacuum manifold M sufficiently fast as $r \to \infty$. This provides a map $\Phi : S^2_\infty \to M \simeq G/H$. Now, suppose that there exist equivalence classes of these maps which are non trivial, that is, $\pi_2(G/H) \neq 0$, then by continuity $\Phi(S^2_r)$, where S^2_r is a sphere of radius r, must contract to a point in the Higgs field space as $r \to 0$. However, unless Φ is a deformation of the constant map, it cannot do this while remaining in the vacuum manifold, and so there must be a region in space where $V(\Phi)$ is greater than its vacuum value. When spherical symmetry is imposed and the energy minimised there are finite energy configurations with magnetic fields decreasing as r^{-2} at infinity, which are non-Abelian generalisations of the Dirac monopole [88]. When $\pi_2(G) \simeq 0 \simeq \pi_1(G)$ we may use an analogous formula to (1.3.1)

$$\pi_2(G/H) \simeq \pi_1(H) \tag{3.4.1}$$

The Dirac monopole has a string singularity in the gauge potential [89] which is unobservable and can be removed by defing the gauge potential on two patches, the upper and lower hemispheres of a large sphere centred on the monopole, in which case the fields may be written

$$\begin{aligned} A^U_\phi &= \ \ g(1 - \cos\theta)/r\sin\theta \\ A^L_\phi &= -g(1 + \cos\theta)/r\sin\theta \end{aligned} \tag{3.4.2}$$

Both are nonsingular on their respective hemispheres and both have the curl

$$\underset{\sim}{B} = g\frac{\underset{\sim}{r}}{r^3} \tag{3.4.3}$$

Where the two patches intersect we must demand that there be no physical difference between the two potentials: that is, they must differ by a gauge transformation

$$A_\phi^U(\theta = \pi/2) - A_\phi^L(\theta = \pi/2) = \frac{i}{er}\left(\frac{\partial}{\partial\phi}\Omega\right)\Omega^{-1} \tag{3.4.4}$$

where $\Omega = e^{i2eg\phi}$. Single valuedness of Ω requires

$$eg = n/2 \tag{3.4.5}$$

which is the Dirac quantisation condition for Abelian monopoles. For a non-Abelian monopole which appears in a symmetry breaking $G \to H$ with $\pi_2(G/H) \neq 0$, the gauge transformation in (3.4.4) is

$$\Omega(\phi) = \exp(i2egM\phi) \tag{3.4.6}$$

where M is in the Lie algebra of H. We can always conjugate this into the Cartan subalgebra [90], and the single valuedness condition becomes that $\Omega(2\pi)$ must act trivially on any weight of the root lattice of G, $\Lambda(G)$:

$$e^{4\pi iegM}|\lambda\rangle = |\lambda\rangle \tag{3.4.7}$$

If we write $M = m.H$, where the H^i are the generators of the Cartan subalgebra ($i = 1,\ldots,\text{rank } G$), this can be expressed as [87]

$$2egm.\lambda \in Z \qquad \forall\ \lambda \in \Lambda(G) \tag{3.4.8}$$

We see that eg just affects the normalisation of M; in particular if we choose eg = 1 we have that $2m.\lambda$ is an integer for all elements of the weight lattice. Therefore m is an element of the coroot lattice $\Lambda_r(G^V)$, which is the lattice generated by the coroots α^V of the dual group G^V of G. The coroots are defined by [84] $\alpha^V = \alpha/(\alpha)^2$ for each $\alpha \in \Phi(G)$, the root system of G. For simply laced groups the coroot lattice is isomorphic to the root lattice, and with the conventional normalisation $\alpha^V = \alpha/2$.

The non-Abelian monopole fields are, analogous to (3.4.2),

$$A_\phi^U = m.H(1 - \cos\theta)/r\sin\theta$$

$$A_\phi^L = -m.H(1 + \cos\theta)/r\sin\phi \qquad (3.4.9)$$

$$\underset{\sim}{B} = m.H\frac{\underset{\sim}{r}}{r^3}$$

A particularly relevant class of theories with monopoles is one in which $G \rightarrow K\times U(1)$ (locally) by an adjoint Higgs with components parallel to a fundamental weight λ_Φ [86]. The generators of K are those which commute with $\lambda_\Phi.H$, not including $\lambda_\Phi.H$ which generates the U(1) subgroup of H. It can be further shown [85] that the stable monopoles are those for which m has component $\lambda_\Phi/2(\lambda_\Phi)^2$ in the direction of the U(1) generator (this gives $2m.\lambda_\Phi = 1$ which is the lowest non-trivial U(1) charge), and is in addition a short root of

G^V, short because the magnetic energy goes as $\int(m^2B^2)d^3x$. Note that these are identical to the conditions on the generator Q of the phase of the Higgs field in the phase string.

Let us now turn to the bead which appears in the symmetry breaking scheme $G \to K\times Z_2$, where the Higgs is parallel to the weight $|2\lambda_\Phi\rangle$ (see sections 1.3 and 3.3, and ref. [39]). Let the bead be centred at the origin on a string lying along the z axis. Since we are looking for monopole-type fields the appropriate coordinate system to use is spherical polars, notwithstanding the complications that appear at $\cos\theta = \pm 1$. When θ is small at sufficiently large r we may write

$$\Phi(\phi) = e^{i\phi Q}|2\lambda_\Phi\rangle\eta \qquad (3.4.11)$$

and recalling the results of sections 1.3 and 3.3 there are two cases for q in $Q = q.H$ if α_Φ is not parallel to λ_Φ. One is where $q = \alpha_\Phi/(\alpha_\Phi)^2$, and so there is some component on the Lie algebra of K, and one in which $q = \lambda_\Phi/2(\lambda_\Phi)^2$. Let us consider the first case.

At large r the bead may be written

$$\Phi(r,\theta,\phi) = e^{i\phi Q}e^{i\Theta(\theta)T^2}\Phi_0(r,\theta) = g(\phi,\theta)\Phi_0 \qquad (3.4.12)$$

where $T^2 = -i(E_{\alpha_\Phi} - E_{-\alpha_\Phi})/\sqrt{2}$ so that $e^{-i\pi T^2}Qe^{i\pi T^2} = -Q$. The $|2\lambda_\Phi\rangle$ component of Φ_0 must vanish at $\theta = 0,\pi$. Presumably it will be energetically favourable to have Θ be as smooth a function of θ as possible, so that $\Theta(\theta) = \theta$. For finite energy the covariant derivatives of Φ must vanish at large r, except around $\theta = 0$ and π where the string is, so if we write T^3 for Q to facilitate later calculations,

$$A_\phi = -T^3(1 - F(r,\theta))/er\sin\theta$$
$$A_\theta = \underset{\sim}{\phi}\cdot\underset{\sim}{T}(1 - G(r,\theta))/er \qquad (3.4.13)$$

where $\underset{\sim}{\phi}$ is a unit azimuthal vector in the space spanned by the SU(2) subalgebra generated by T^3, T^2, and $T^1 = (E_{\alpha_\Phi} + E_{-\alpha_\Phi})/\surd 2$. The functions F and G differ from zero only in a small region of order $m_v{}^{-1}$ from the string, so that as $r \to \infty$ they become 'spikes' with unit amplitude at $\theta = 0$ and π and their θ derivatives become

$$\lim_{r\to\infty} \frac{\partial F}{\partial\theta},\frac{\partial G}{\partial\theta} = \delta(\theta - \pi) - \delta(\theta) \qquad (3.4.14)$$

The magnetic field is radial and given by

$$B_r = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial}{\partial\phi} A_\theta\right) + ie[A_\theta, A_\phi] \qquad (3.4.15)$$

$$= \left(\frac{1}{er^2\sin\theta}\,\frac{\partial F}{\partial\theta}\right)T^3 \qquad (3.4.16)$$

This is just the field for two oppositely oriented strings along the z axis. Note the similarity of A_ϕ and B_r to the monopole solution (3.4.9): in fact we may identify Q with -M, so that this bead is like a $G \to K\times U(1)$ monopole with all its flux confined to the strings.

Now let us consider the case where the generator of the phase of the stable string is parallel to $\lambda_\Phi.H$, and the group is such that α_Φ is in a different direction in the weight space. As indicated in section 3.2, in the bead solution q must be parallel to a root in order that there exist a group element $e^{i\pi T^1}$ that conjugates q to -q. At $z = \pm\infty$ we expect the string to reach its ground state which has no component of non-Abelian flux in K at the core, so that $A_\phi(\theta \to 0,\pi) \sim \lambda_\Phi.H$. We might therefore expect that the difference between q and λ_Φ 'escapes' from the core and that there are long range fields away from the string. It will turn out that there is such a solution. Let us define κ, the component of q in the Lie algebra of K, by

$$q = \lambda_\Phi/2(\lambda_\Phi)^2 + \kappa \tag{3.4.17}$$

We can always add a term $A(\theta)g(\kappa.H)g^{-1}/er\sin\theta$ to A_ϕ without affecting the covariant derivative of Φ, with a view to finding a solution with the correct A_ϕ near the string. We can calculate $g(\kappa.H)g^{-1}$ as follows: first note that

$$\begin{aligned}[\kappa.H,T^1] &= i\kappa.\alpha_\Phi T^2 \\ [\kappa.H,T^2] &= -i\kappa.\alpha_\Phi T^1\end{aligned} \tag{3.4.18}$$

From this it follows that

$$e^{i\theta T^2}\kappa.He^{-i\theta T^2} = \kappa.H + (\kappa.\alpha_\Phi)T^3 - \kappa.\alpha_\Phi(T^3\cos\theta + T^1\sin\theta)$$

and therefore

$$g(\kappa.H)g^{-1} = \kappa.H + (\kappa.\alpha_\Phi)(T^3 - \underset{\sim}{r}.\underset{\sim}{T}) \qquad (3.4.19)$$

Now we can calculate the magnetic field of the bead which has gauge potentials

$$A_\phi = -\frac{(1 - F)}{er\sin\theta} T^3 + \frac{A(\theta)(\kappa.\alpha_\Phi)}{er\sin\theta}\left(T^3 - r.T + \frac{\kappa.H}{\kappa.\alpha_\Phi}\right)$$

$$(3.4.20)$$

$$A_\theta = \frac{\underset{\sim}{\phi}.\underset{\sim}{T}}{er}(1 - G)$$

After a straightforward piece of algebra the result is

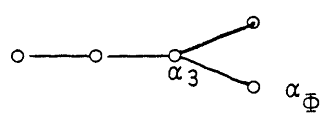
$$B_r = \frac{1}{er^2\sin\theta}\left(\frac{dA}{d\theta}\, g(\kappa.H)g^{-1} + \frac{\partial F}{\partial\theta}T^3\right) \qquad (3.4.21)$$

From the requirement that at large r and as θ tends to 0 and π the solution must look like a string with $A_\phi = -\lambda_\Phi.H/2(\lambda_\Phi)^2 er\sin\theta$, we have

$$\lambda_\Phi/2(\lambda_\Phi)^2 = q + A(0)\kappa \qquad (3.4.22)$$

$$\lambda_\Phi/2(\lambda_\Phi)^2 = -q - A(\pi)(\kappa.\alpha_\Phi)[2q + \kappa/(\kappa.\alpha_\Phi)] \qquad (3.4.23)$$

Equation (3.4.17) requires that A(0) = 1 in (3.4.22), and that $A(\pi) = -1$ and $\kappa.\alpha_\Phi = 1/2$ in (3.4.23). The contribution to the energy of the long range radial magnetic field is proportional to $\int d^3x(\frac{dA}{d\theta})^2/\sin^2\theta$, and this is minimised and the boundary conditions satisfied for $A(\theta) = \cos\theta$, and substituting this into (3.4.21) we see there really is a monopole on the string. The condition $2(\kappa.\alpha_\Phi) = 1$ gives us some information about κ: there are at most three simple roots of K which are not orthogonal to α_Φ, and we can find the coefficients k_i in an expansion of κ in these roots by minimising $\sum_i k_i^2$, which appears linearly in the magnetic energy. In particular, in $SO(10) \to SU(5)\times Z_2$ with a Higgs in a $\underline{126}$, α_Φ corresponds to one of the fishtails in the Dynkin diagram of SO(10):



Hence if α_3 is the simple root of the fork of the fishtail, $\kappa = -\alpha_3/2$, which is a coroot of κ.

In conclusion to this section, we have seen that there are two answers to the question of the long range fields of the bead in $G \to K\times Z_2$. If the stable string has $q = \lambda_\Phi/2(\lambda_\Phi)^2$ then there is no non-Abelian flux down the core of the string, and if λ_Φ is not parallel to a root κ will not be zero, and the bead will look like a monopole of charge $-\kappa.H$ with two strings attached. If, on the other hand, q is the shortest simple coroot not orthogonal to λ_Φ, then the κ component of flux will be confined to the string and there is no r^{-2} monopole-type field. This has relevance to a situation where there is an intermediate stage of symmetry breaking with an adjoint Higgs

$$G \to K\times U(1) \to K\times Z_2$$

Then the stable monopoles after the first stage will have magnetic charges with some U(1) component and also a component in the Lie algebra of K, which is actually a root of the dual group K^{V} [87]. When the U(1) symmetry is broken to Z_2 the Abelian flux must be confined in some way, either with the non-Abelian flux is left behind, leaving a monopole with charge κ on the string, or with the non-Abelian flux taken along the strings as well. Which of the two actually is the case depends on the relative importance of the magnetic and Higgs potential energy; we see from (3.4.21) that the magnetic energy, $E_M = tr(B^2)/2$, is proportional to e^{-2}, so in the limit $e \to 0$ we expect the minimising of E_M to be the deciding factor. In that case any 'unbroken' flux will escape from the string and take up a spherically symmetric monopole-type configuration. On the other hand the Higgs potential energy of the solution goes as λ^{-1}, so for sufficiently small λ, q will align itself with a root and all the magnetic flux will be confined.

Finally, we note that the bead has a dyonic degree of freedom [92,73], exactly as might be expected from treating it as a monopole on a string. First, a brief summary of the properties of the dyon is in order. The dyon is a paticle with both magnetic and electric charge, and in a monopole solution the electric charge arises because it has a degenerate set of solutions related by global charge rotations, and motions in this manifold give the monopole a charge. This is entirely analogous to spatial translations, which produce a degenerate set of solutions parameterised by collective coordinates which are the position of the centre of the monopole, and motions in this manifold produce states which have non-zero 'charge' under the generators of the translation group, namely momentum.

Consider the 't Hooft-Polyakov monopole, which appears when SU(2) (generators T^1, T^2, T^3) is broken by an adjoint Higgs Φ to U(1), and whose fields can be written [14,89]

$$\Phi = (\hat{\underset{\sim}{r}}\cdot\underset{\sim}{T})f(r)$$

$$A_i = (\hat{\underset{\sim}{r}}\times\underset{\sim}{T})_i(1 - F(r))/er \tag{3.4.24}$$

$$\underset{\sim}{B} = (\hat{\underset{\sim}{r}}\cdot\underset{\sim}{T})\frac{\hat{\underset{\sim}{r}}}{r^2}(1 - K(r))$$

The unbroken generator is $Q_{em} = \hat{\underset{\sim}{r}}\cdot\underset{\sim}{T}$, and if we make a gauge transformation which at large r is $e^{i\varepsilon Q_{em}}$ we see that the Higgs and magnetic fields at infinity are invariant,but the gauge potentials are rotated. This is a sensible collective coordinate for the monopole, because if we started with a widely separated monopole-antimonopole pair and cut the solution in two, rotated the monopole relative to the antimonopole, and glued it back together again, we would obtain a new solution which is not merely a gauge transformation of the old one [73]. If the monopoles have a long range non-Abelian magnetic field, the only H rotations that can be considered as a collective coordinate are those which leave the long range magnetic field invariant.

Let us suppose that the monopole is moving in this degenerate manifold of globally gauge rotated solutions so that $\varepsilon = \varepsilon(t)$. We may write the fields in a gauge where all the time dependence is in A_0, so that

$$A_0 = \frac{i}{e}(\partial_t \varepsilon) Q_{em} \tag{3.4.25}$$

Since Q_{em} depends on θ and ϕ, to avoid singularities at $r = 0$ ε must be a function of r and vanish at the origin. Hence there is a radial electric field, and the minimum energy solution for a particular charge is the dyon. When these motions are quantised a discrete spectrum of charged states results because the motion is in a compact manifold [73].

Exactly the same considerations apply to both sorts of bead, (3.4.13) and (3.4.20), which have a degenerate set of solutions generated by those generators of K which leave $\underset{\sim}{B}$ invariant at large r but which act on $\underset{\sim}{A}$ non-trivially. When all the flux is confined, as in (3.4.13), there may be complications with dyonic charge carried along the string, carried by massless bosonic excitations [26], but in the second case (3.4.20) where there are no bosonic zero modes we do genuinely obtain something that looks like a dyon on a string. We will see in the next chapter the importance this has for the fermion zero modes.

3.5 Beads in cosmology.

We have seen that beads can have magnetic charges, and the presence of these charges on an oscillating loop will produce dipole radiation. Let us assume as a best estimate that all the beads act as independently oscillating magnetic charges, so that if there are N beads of charge g on a loop size R, this radiation will dominate if

$$Ng^2R^2\omega^4 > 100G\mu^2 \tag{3.5.1}$$

or, using $g \sim e^{-1}$

$$N > e^2(m_{loop}/m_p)^2 \tag{3.5.2}$$

Hence this is significant only for very small loops.

The presence of a bead on a string will not affect waves on it above a certain scale, as the forces on the bead will change slowly enough for its inertia not to matter. If the frequency of the string motion is R^{-1}, then the bead will be undrgoing typical accelerations of R^{-1}. If the angle the strings make at the bead is θ, then θ will be small and the wave on the string little affected if

$$\mu/m_b >> R^{-1} \tag{3.5.3}$$

where m_b is the mass of the bead. If the bead is a monopole formed at an earlier phase transition then this equation gives the scale above which waves on the string are affected negligibly by the the bead. If $m_b \sim \mu^{1/2}$, as for beads formed with the string, then this scale is just the width of the string, are the waves are never bothered by beads. There is, of course, always some interaction between beads and waves: a travelling wave will tend to carry a bead along with it, as one can easily convince oneself with a real bead and a real string.

How many beads do we expect per unit length of string? Suppose the beads are formed with the string. Then there will be of order one per persistence length along the string, as the Higgs field is uncorrelated beyond this distance. They will not necessarily be stationary, so they will move towards each other and annihilate, rather like two kinks. This process will be encouraged by waves on the string moving in opposite directions carrying the beads along with them, so a reasonable guess would be that there is always of order one bead per persistence length of string. Thus when loops are chopped off the network there will be a small (even) number of beads on each one. We have already seen that fairly soon after the phase transition the presence of beads does not affect the subsequent motion of the string network, and radiation from beads is insignificant. Thus the existence of beads on cosmic string will not radically affect their evolution and their ability to seed galaxy formation remains.

CHAPTER 4: ZERO MODES ON NON-ABELIAN STRINGS

4.1 Zero modes on strings.

In section 1.6 the existence of fermion zero modes on Abelian strings was demonstrated. As we have seen, the structure of a string in a non-Abelian theory is more complicated, involving more than one component of the Higgs field, and it is by no means obvious that they should also possess zero modes, nor is it obvious that such solutions, should they exist, be superconducting. In this chapter zero modes on both phase and non-phase strings occuring in the breaking scheme $G \rightarrow K \times Z_2$ will be investigated. This is an important class, because the Higgs that causes the breaking also gives Majorana masses to unobserved fermions, which is a vital ingredient in realistic chiral theories [3,4,5]. If we want to put chiral fermions in a fundamental representation of some compact simple Lie group there is always at least one extra fermion that must be given a large mass (for example, in SO(10) it is the right-handed neutrino), so the Higgs representation must be contained within the symmetric product of two fundamental representations. Just such a Higgs produces the symmetry breaking under discussion. Recalling the results of previous sections and ref. [39], the Higgs field Φ lines up parallel to the weight $|2\lambda_\Phi\rangle$, where λ_Φ is a fundamental weight, and the string solution may be written

$$\Phi(\rho,\phi) = e^{i\phi Q}\left(|2\lambda_\Phi\rangle f(\rho) + |0\rangle a(\rho)\right) \qquad (4.1.1)$$

where $Q|0\rangle = 0$, $f(0) = 0$, $f(\infty) = \eta$, and $a(\rho)$ vanishes exponentially outside the string. The simple roots of the continuous part of the unbroken subgroup K are those simple roots of G for which $\alpha.\lambda_\Phi = 0$. The unique simple root not orthogonal to λ_Φ is denoted α_Φ, and satisfies $2\alpha_\Phi.\lambda_\Phi/(\alpha_\Phi)^2 = 1$. The SU(2) subgroup generated by $T^3 = \alpha_\Phi.H/(\alpha_\Phi)^2$, $T^2 = -i(E_{\alpha_\Phi} - E_{\alpha_\Phi})/\sqrt{2}$, and $T^1 = (E_{\alpha_\Phi} + E_{\alpha_\Phi})/\sqrt{2}$, where the H^i are the generators of the Cartan subalgebra of G and E_α are step operators satisfying $[H^i, E_\alpha] = \alpha^i E_\alpha$, does not leave $|2\lambda_\Phi\rangle$ invariant and hence Q can be chosen from them. There are essentially two choices; Q can be in the Cartan subalgebra, so that if $Q = q.H = T^3$ then $q = \alpha_\Phi/(\alpha_\Phi)^2$, or Q can be chosen to be one of the other two generators, T^1 say. If $Q = T^3$ the Higgs field changes by a phase at infinity and the string is said to be a phase string

$$\Phi(\rho,\phi) = |2\lambda_\Phi\rangle f(\rho)e^{i\phi} + |0\rangle a(\rho) \qquad (4.1.2)$$

If $Q = T^1$ the behaviour at infinity is not a phase change, and we have the non-phase string

$$\Phi(\rho,\phi) = e^{i\phi T^1}|2\lambda_\Phi\rangle f'(\rho) + |0\rangle' a'(\rho) \qquad (4.1.3)$$

where $T^1|0\rangle = 0$. This is gauge equivalent to

$$\Phi^g(\rho,\phi) = e^{i\phi T^3}e^{i\pi T^2/2}(|2\lambda_\Phi\rangle f'(\rho) + |0\rangle' a'(\rho)) \qquad (4.1.4)$$

It will be convenient to rewrite this in terms of eigenstates of T^3, for which we need to know the effect of T^2 on the weights in (4.1.4). The generator T^2 is composed of step operators, and we know that

$$E_{-\alpha_\Phi}|2\lambda_\Phi\rangle = N_1|2\lambda_\Phi - \alpha_\Phi\rangle$$

$$(E_{-\alpha_\Phi})^2|2\lambda_\Phi\rangle = N_2N_1|2\lambda_\Phi - 2\alpha_\Phi\rangle \tag{4.1.5}$$

$$(E_{-\alpha})^3|2\lambda_\Phi\rangle = 0$$

This last follows because the length of the α_Φ string through $2\lambda_\Phi$ is $2\alpha_\Phi.(2\lambda_\Phi)/(\alpha_\Phi)^2 = 2$. Hence the three weights $|2\lambda_\Phi\rangle$, $|2\lambda_\Phi - \alpha_\Phi\rangle$, and $|2\lambda_\Phi - 2\alpha_\Phi\rangle$ form a triplet representation of the SU(2) generated by the T^i (i = 1,2,3), with T^3 eigenvalues +1, 0, -1 respectively. In a notation following Everett and Aryal [76] we can write them as symmetric matrices ϕ_1, ϕ_0, and ϕ_{-1}, in which case

$$\Phi(\infty,0) = [(\phi_1 + \phi_{-1})/2 + \phi_0/\surd 2]\eta \tag{4.1.6}$$

Thus the phase string in this notation is

$$\Phi(\rho,\phi) = e^{i\phi}\phi_1 f(\rho) + \phi_0 a(\rho) \tag{4.1.7}$$

and the non-phase string is

$$\Phi(\rho,\phi) = e^{i\phi T^3}[(\phi_1 + \phi_{-1})/2 + \phi_0/\surd 2]f' + \phi_0 a' \tag{4.1.8}$$

with $f'(0) = f(0)$; $f'(\infty) = \eta = f(\infty)$; $a'(0)$, $a(0) \neq 0$, $a'(\infty) = 0 = a(\infty)$.

In section 4.2 we shall consider the phase string, which is not gauge equivalent to the antistring and which can therefore support beads, and we shall find that the zero mode solutions are more complicated than in the Abelian string [26] because of the role played by the zero eigenvector component $a(\rho)$. In particular we will see that particles and antiparticles can travel in opposite directions when excited by an electric field, and the consequent annihilations effectively cause the string to have a resistance. In section 4.3 non-phase strings will be considered, where there is a set of gauge transformations taking the string continuously to the antistring, and it will emerge that although there are zero modes they cannot support an electric current. In section 4.4 we will discuss what happens when fermion zero modes encounter a bead, and it will be shown that there is an effect analogous to the Callan-Rubakov effect [28,73] in which the fermions can change quantum numbers and leave the string. In the final section some remarks about the cosmological implications of superconducting zero modes and beads will be made.

4.2 Fermions and phase strings.

In this section the discussion of section 1.6 will be generalised to fermions in a non-Abelian string background, in particular we consider the phase strings in the class of theories $G \rightarrow K\times Z_2$. The fermions are in a fundamental representation R with highest weight λ_Φ, while the Higgs field is in the symmetrised product of two R's and is parallel to the highest weight $|2\lambda_\Phi\rangle$. The fermionic part of the Lagrangian is then

$$\mathcal{L}_f = \bar{\Psi}^{l,r}\gamma.(\partial - eA)\Psi^{l,r} - g(\bar{\Psi}^{l}\Phi\Psi^{r} + h.c.) \qquad (4.2.1)$$

where $\gamma^5\Psi^{l,r} = \mp\Psi^{l,r}$. The group transformation laws are $\Phi \to {}^{g}\Phi = g\Phi g^T$, $\Psi^l \to {}^{g}(\Psi^l) = g\Psi^l$, and $\Psi^r \to {}^{g}(\Psi^r) = (g^T)^{-1}\Psi^r$. In order for there to exist a right handed spinor transforming this way, either R is real or pseudoreal, or if R is complex then $\Psi^r = (\Psi^l)^c$. The conjugate spinor is defined from Ψ by $\Psi^c = C\bar{\Psi}^T$, where C is the charge conjugation matrix. In fact the supposed chirality of nature indicates that in realistic models R will be complex, but we shall not impose this condition yet.

As usual we will take the string along the z axis so that $\Phi(\rho,\phi) = \phi_1 f(\rho)e^{i\phi} + \phi_0 a(\rho)$ and $A_\mu = \delta_{\mu\phi}Q\big(F(\rho) - 1\big)/e\rho$. The fermi fields have expansions in weights of G

$$\Psi^{l,r} = \sum_\lambda \phi_\lambda^{l,r}|\lambda\rangle \qquad (4.2.2)$$

We only need consider those weights for which $\langle\lambda'|\phi_{1,0}|\lambda\rangle$ does not vanish, for it is these components of the fermi field which are confined to the vicinity of the string. The highest weight of the fermion representation is $|\lambda_\Phi\rangle$, the next is $|\lambda_\Phi-\alpha_\Phi\rangle$. These are the only two weights which can be combined to give non-zero inner products with $|2\lambda_\Phi\rangle$ and $|2\lambda_\Phi- \alpha_\Phi\rangle$, and they form an SU(2) doublet with eigenvalues ±1/2 under $\alpha_\Phi . H/(\alpha_\Phi)^2$. The Lagrangian for these two components is then

$$\mathcal{L}_f = \bar{\Phi}^{l,r}_{\pm}\gamma.(i\partial \mp eA/2)\phi^{l,r}_{\pm}$$

$$- g\left(f(\rho)e^{i\phi}\bar{\Phi}^{l}_{+}\phi^{r}_{+} + a(\rho)\bar{\Phi}^{l}_{+}\phi^{r}_{-} + a^{*}(\rho)\bar{\Phi}^{l}_{-}\phi^{r}{}_{+} + \text{h.c}\right) \quad (4.2.3)$$

The equations we have to solve to find the transverse zero modes are, for ϕ^l

$$\gamma^i(i\partial_i - eA_i/2)\phi^l_+ + g\left(f(\rho)e^{i\phi}\phi^r_+ + a(\rho)\phi^r_-\right) = 0$$

$$\gamma^i(i\partial_i - eA_i/2)\phi^l_- + ga^*(\rho)\phi^r_+ = 0 \quad (4.2.4)$$

where i = 1,2, and there is another pair for ϕ^r with ϕ replaced by $-\phi$, and a by a^*. Note that applying $\gamma^0\gamma^3$ leaves the form of the equations unaltered, so that if $(\phi^{l,r}_+,\phi^{l,r}_-)$ is a solution then so is $\gamma^0\gamma^3(\phi^{l,r}_+,\phi^{l,r}_-)$. We may therefore resolve the solutions into eigenstates of $\gamma^0\gamma^3$, which means for fermions with $\gamma^5 = \lambda$ that $i\gamma^1\gamma^2 = \lambda\gamma^0\gamma^3$. Let the eigenvalues of $i\gamma^1\gamma^2$ on Ψ^l be μ ($\mu = \pm 1$), in which case equations (4.2.4) become

$$i\gamma^1 e^{i\mu\phi}\left(\frac{\partial}{\partial\rho} + i\frac{\mu}{\rho}\frac{\partial}{\partial\phi} - \frac{\mu}{2}eA_\phi\right)\phi^l_+ + g\left(fe^{i\phi}\phi^r_+ + a\phi^r_-\right) = 0$$

$$i\gamma^1 e^{i\mu\phi}\left(\frac{\partial}{\partial\rho} + i\frac{\mu}{\rho}\frac{\partial}{\partial\phi} + \frac{\mu}{2}eA_\phi\right)\phi^l_- + ga^*\phi^r_+ = 0 \quad (4.2.5)$$

These can be further simplified by defining spinors $\chi_{\pm}^{l,r}$ as follows

$$\phi_{\pm}^{l} = \exp\left(\pm\frac{\mu e}{2}\int_0^{\rho} d\rho' A_{\phi}(\rho')\right) e^{il_{\pm}\phi}\ \chi_{\pm}^{l}$$

$$\phi_{\pm}^{r} = \exp\left(\pm\frac{\mu e}{2}\int d\rho' A_{\phi}(\rho')\right) e^{ir_{\pm}\phi}\ \chi_{\pm}^{l} \tag{4.2.6}$$

where $l_{\pm}$ and $r_{\pm}$ are integers to be determined. Equations (4.2.5) then become

$$i\gamma^{1} e^{i(\mu + l_{+})\phi}\left(\frac{\partial}{\partial\rho} - \mu\frac{l_{+}}{\rho}\right)\chi_{+}^{l}$$

$$+ g\left(f(\rho)e^{i(1 + r_{+})\phi}\chi_{+}^{r} + a(\rho)e^{\mu R(\rho)}\chi_{-}^{r}e^{ir_{-}\phi}\right) = 0$$

$$i\gamma^{1} e^{i(\mu + l_{-})\phi}\left(\frac{\partial}{\partial\rho} - \mu\frac{l_{-}}{\rho}\right)\chi_{-}^{l} + g a(\rho)e^{-\mu R(\rho)}\chi_{+}^{r}e^{ir_{+}\phi} = 0 \tag{4.2.7}$$

where

$$R(\rho) = \int_0^{\rho} d\rho' e A_{\phi}(\rho') \tag{4.2.8}$$

At large ρ, $R(\rho) \to -\ln\rho$ and at the core, since $F(\rho) \sim 1 - c\rho^{2}$, $R(\rho) \sim -\rho^{2}(ec/2)$. If we take into account the other two equations we obtain 6 equations in the 4 unknowns $l_{\pm}$, $r_{\pm}$:

$$\mu + l_{+} = 1 + r_{+} = r_{-}$$

$$\mu + l_{-} = r_{+}$$

$$- \mu + r_{+} = -1 + l_{+} = l_{-}$$

$$- \mu + r_{-} = l_{+} \tag{4.2.9}$$

If $\mu = +1$, then $l_+ = r_+ = 0$ and $-l_- = r_- = 1$, whereas if $\mu = -1$, then $l_+ = -r_+ = 1$ and $l_- = r_- = 0$. Since $\chi^l_\pm$ and $\chi^r_\pm$ are eigenvectors of $i\gamma^1\gamma^2$ with eigenvalues $+\mu$ and $-\mu$ respectively it is clear that $i\gamma^1\chi^l_\pm \propto \chi^r_\pm$. In order that χ^l_+ be an exponentially decreasing (rather than increasing) function at large ρ the constant of proportionality must be unity, and equations (4.2.7) become

$$\left(\frac{\partial}{\partial\rho} + \frac{(1-\mu)}{2\rho}\right)\chi^l_+ + g\left(f(\rho)\chi^l_+ + a(\rho)e^{\mu R(\rho)}\chi^l_-\right) = 0$$

$$\left(\frac{\partial}{\partial\rho} + \frac{(1+\mu)}{2\rho}\right)\chi^l_- + ga^*(\rho)e^{-\mu R(\rho)}\chi^l_+ = 0 \qquad (4.2.10)$$

At large ρ, $\psi^l_- \sim \rho^{-1/2}$, so that as they stand these solutions are not normalisable. We can interpret this as meaning that the ψ_- fermion is not bound to the string, although this state of affairs can be remedied with another Higgs field, as we shall shortly demonstrate.

Let us therefore consider the behaviour of the solutions as $\rho \to 0$. We may drop the labels l,r so that

$$\lim_{\rho\to 0} \chi_\pm = A_\pm \rho^{n_\pm} \qquad (4.2.11)$$

Substituting into (4.2.10) we conclude that in order to avoid singularities at the origin in the solutions

$$n_- - 1 = n_+ = 0 \qquad (\mu = +1)$$

$$n_+ - 1 = n_- = 0 \qquad (\mu = -1)$$

We can also obtain relations between A_+ and A_-

$$A_-(n_- + 1) + ga^*(0)A_+ = 0 \qquad (\mu = +1)$$

$$A_+(n_+ + 1) + ga(0)A_- = 0 \qquad (\mu = -1)$$

All this information is summarised in Table 4.1 below.

Table 4.1

Form of solution	$\rho \to 0$ behaviour	$\rho \to \infty$ behaviour
$(\mu = +1)$		
$\phi_+(\rho,\phi) \sim e^{R/2}\chi_+(\rho)$	A	$\rho^{-1/2}e^{-g\eta\rho}$
$\phi_-(\rho,\phi) \sim e^{-R/2}\chi_-(\rho)e^{-i\phi}$	$-(\frac{ga^*(0)A}{2})\rho$	$\rho^{-1/2}$
$(\mu = -1)$		
$\phi_+(\rho,\phi) \sim e^{-R/2}\chi_+(\rho)e^{i\phi}$	$-(\frac{ga(0)A}{2})\rho$	$\rho^{-1/2}e^{-g\eta\rho}$
$\phi_-(\rho,\phi) \sim e^{R/2}\chi_-(\rho)$	A	$\rho^{-1/2}$

Note the importance of the non-vanishing component of Φ at the origin: if $a(\rho)$ were to be everywhere zero, as would be the case if $Q \propto \lambda_{\Phi}.H$ (the one considered by Witten [26]) the equations for ψ_+ and ψ_- would decouple. For $\mu = +1$, ψ_- would obey $i\gamma^1(\partial_\rho + \rho^{-1})\psi_- = 0$ as $\rho \to 0$ and so go as ρ^{-1}, and similarly ψ_+ for $\mu = -1$. The normalisable mode has $\mu = +1$ and has a ψ_+ component only: this is Witten's result that there is only one zero mode for a pair of chiral fermions gaining mass from the string Higgs field. On the other hand, when there is a classical Higgs field at the core of the string coupling the zero mode pair $\psi_+^{l,r}$ to another pair $\psi_-^{l,r}$ which are massless at the normalisability of the solution is destroyed. This may be interpreted as the large value of the Higgs field at the core promoting decays of the form $\psi_+ \to \Phi\psi_-$ and the ψ_- escaping to infinity. Thus true zero modes in this case require an additional Higgs to bind ψ_- to the string. This can be done by arranging a coupling with constant g' to a Higgs which gains a vacuum expectation value η' in a component transforming as $\psi_-\overline{\psi}_-$. This then modifies the second of the zero mode equations (4.2.4) by the addition of a term $g'f'(\rho)e^{-i\phi}\psi_-^r$, so that the large ρ behaviour of ψ_- becomes $\rho^{-1/2}e^{-g'\eta'\rho}$ which is clearly normalisable.

Given that we have two normalisable zero modes with opposite values of μ, let us now turn to their motion in the z direction. They obey

$$i(\gamma^0\partial_t - \gamma^3\partial_z)\psi_\pm = 0 \qquad (4.2.12)$$

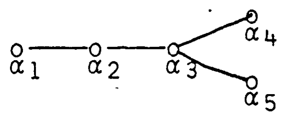
so bearing in mind that $\psi_\pm$ are eigenstates of $\gamma^0\gamma^3$

$$(\partial_t + \mu\partial_z)\psi_\pm = 0 \qquad (4.2.13)$$

Hence the two zero modes travel in opposite directions at the speed of light. When there is no Q = 0 Higgs coupling ψ_+ to ψ_- at the core the zero modes are pure ψ_+ (with $\mu = +1$) travelling in the +z direction and pure ψ_- (with $\mu = -1$) travelling in the -z direction. If there is such a coupling, i.e. $a(0) \neq 0$, then the travelling modes must be a mixture of ψ_+, ψ_- and Φ.

Note that under certain circumstances, for example in SO(10) where ψ_+^l is the left handed antineutrino, ψ_+^r is actually the conjugate of ψ_+^l, so that this component gains a Majorana mass from the Higgs field Φ. The relation between the spinor and its complex conjugate means that it must be a real function of $(\omega t - \mu z)$, but since in realistic theories ψ_+ is uncharged this has no bearing on the superconducting modes.

This discussion is best illustrated with an example such as SO(10), based on one considered by Witten but with a different Q [26]. The left handed fermions lie in a <u>16</u> and the Higgs's in a <u>126</u>, a <u>45</u>, and a <u>10</u>. The Φ_{126} breaks SO(10) to SU(5)$\times Z_2$ giving the right handed neutrino a mass, and the broken SU(2) is the right handed isospin subgroup with generators T_r^i (i = 1,2,3). We can therefore identify ψ_+^l with the left handed antineutrino and ψ_-^l with the left handed positron. In addition there are three other right handed isospin doublets consisting of the left handed anti-up and anti-down quarks, but these do not couple to the component of the Φ_{126} which gains a vacuum expectation value and so at this stage they are not bound to the string. Let us label the simple roots $\alpha_1, \ldots, \alpha_5$ and identify them on the Dynkin diagram as follows



Thus $T^3_r = \alpha_5 . H/(\alpha_5)^2$ and $\Phi_{126} = |2\lambda_5\rangle\eta$. The Dynkin diagram of the SU(5) subgroup is just that made up of α_1 to α_4. Conventionally, SU(5) is broken to $SU(3) \times SU(2)_L \times U(1)_Y$ by the adjoint Higgs parallel to $\lambda_3 . H$, where λ_i are defined by (1.3.3), so that the left handed isospin group is generated by $T^3_l = \alpha_4 . H/(\alpha_4)^2$, $T^2 = -i(E_{\alpha_4} - E_{\alpha_4})/\sqrt{2}$, and $T^1 = (E_{\alpha_4} + E_{-\alpha_4})/\sqrt{2}$. No fermions get masses at this stage, and since $[T^3_r, \lambda_3 . H] = 0$ the Φ_{45} does not change around the string. Finally the Higgs $SU(2)_L$ doublet that gives masses to the electrons and quarks is contained in the <u>10</u> of SO(10). This doublet has $T^3_r = +1/2$ which can be seen as follows. The elements of the Cartan subalgebra of SO(10) can be written as $(\sigma^2/2) \times M$, where M is a diagonal 5×5 hermitian matrix. The generators $\alpha_i . H$ correspond to M = diag(1,-1,0,0,0), diag(0,1,-1,0,0), diag(0,0,1,-1,0), diag(0,0,0,1,-1), and diag(0,0,0,1,1) respectively, and so the $SU(2)_L$ Higgs doublet in question is $(\phi^+)^T = (1,i) \times (0,0,0,1,0)$ and $(\phi^0)^T = (1,i) \times (0,0,0,0,1)$ which clearly has $T^3_l = \pm 1/2$ and $T^3_r = +1/2$. The component ϕ^0 has the SU(5) quantum numbers of the left handed neutrino, and gains a vacuum expectation value of η' at the electroweak scale. However, if there were a string present, this component of the ϕ_{10} would change phase by $e^{i\phi/2}$ around it and hence not be single valued. Clearly, some modification to Q is required in order to make the phase change around the string an integer multiple of 2π, subject to the condition that $tr(Q^2)$ be as small as possible to minimise magnetic energy. As it stands the lowest energy solution is to make ϕ^0 invariant under the new phase generator, which we can call Q', in which case $tr(Q'^2)$ is minimised for

$$Q' = (\sigma^2/2)\times[(1,1,1,1,0)/2] = 3Q_{em}/2 + T^3_l + T^3_r \quad (4.2.14)$$

In this case there is no phase change in ϕ_{10} around the string and it remains at its vacuum value all the way to the core. Hence there can be no quark and lepton zero modes. Interestingly though, there is some electromagnetic flux running along the string even though electromagnetism is unbroken. In order to get quark and lepton zero modes we could follow Witten [26] by including an additional Higgs in the $\underline{210}$ and arranging the couplings so that a neutral component with $T^3_l = T^3_r = 1/2$ gains a large expectation value at the core of the string. Alternatively, we could arrange couplings between the $\underline{126}$ and the $\underline{10}$ such that the charged component of the $SU(2)_L$ Higgs doublet, ϕ^+, gains a large expectation value at the core. Either way, it is energetically favourable for

$$Q' = (\sigma^2/2)\times(0,0,0,0,2) = T^3_r - T^3_l \quad (4.2.15)$$

and so the neutral component ϕ^0 changes phase by 2π around the string. Hence in an obvious notation $(\phi^+,\phi^0) = (a'(\rho),f'(\rho)e^{i\phi})$, and from equations (4.2.12) we can write down the equations for the analogues of the χ's.

$$\left(\frac{\partial}{\partial\rho} + \frac{(1-\mu)}{\rho}\right) u + g'\left(f'(\rho)u + a'(\rho)e^{\mu R'(\rho)}d\right) = 0$$
$$\left(\frac{\partial}{\partial\rho} + \frac{(1+\mu)}{\rho}\right) d + g'\left(f'(\rho)d + a'^{*}(\rho)e^{-\mu R'(\rho)}u\right) = 0 \qquad (4.2.16)$$

where for convenience we have ignored the fact that u and d have different couplings to the Higgs field. In view of the above discussion we see that these equations possess normalisable solutions with both $\mu = +1$ and $\mu = -1$, behaving as $\rho^{-1/2}\exp(-m_q\rho)$ at large ρ, so there are u and d quark modes travelling in both directions on the string.

Now we turn to the question of whether these modes are superconducting. If the discussion of section 1.6 were to apply, we would expect an electric field to excite u and $\bar{d}$ quarks travelling one way and d and $\bar{u}$ travelling the other. However, there is now the possibility of particle-antiparticle annihilations which will make the current relax to zero. Hence the string is not superconducting and in effect has a resistance, a conclusion which generalises to any phase string with a Higgs field at the core coupling the zero modes together.

We can estimate the energy loss per unit length due to this process as follows. The annihilation cross-section for two massless particles heading towards each other on the string is given roughly by

$$\sigma(k + k') \sim \left|\int d^2x\phi^{+}(\mu{=}1)\phi(\mu{=}-1)\right|^2 e^4/(k + k')^2 \qquad (4.2.17)$$

where the integral represents the overlap between the oppositely travelling modes, which couple to a radiation field with z component of angular momentum equal to 1. From Table 4.1 we can estimate its value to be about $g'a'(0)w$, where w is the width of zero mode. Let us define $\zeta = g'^2 w^2 |a'(0)|^2$. If dk and dk' are the number densities per unit length and wave number interval then the annihilation rate per unit length is

$$d^2N/dtdz \sim \iint dk dk' \sigma(k + k')/w^2 \qquad (4.2.18)$$

If the fermi seas are filled to a momentum k_f, the rate of energy loss per unit length is

$$\begin{aligned} d\varepsilon/dtdz &\sim \iint dk dk'(k + k')\sigma(k + k')/w^2 \\ &\sim e^4 g'^2 |a'(0)|^2 k_f \qquad (4.2.19) \end{aligned}$$

The energy per unit length in the zero modes is about k_f^2, so there is a characteristic time τ for the decay of any current $J \sim ek_f$ of

$$\tau \sim k_f/e^4 g'^2 |a'(0)|^2 \qquad (4.2.20)$$

Now, $|a'(0)|$ is typically $O(\eta')$ so even for a large current with $k_f = O(\eta')$ the relaxation time is about 10^{-17} seconds. We can estimate what the fermi momentum is by equating the rate of increase of energy in a loop of size R ($\gg \eta'^{-1}$), which is eE, to the energy loss. This implies

$$k_f \sim E/Re^3\eta'^2 \tag{4.2.21}$$

For realistic electric fields generated by motions through galactic magnetic fields, $E << \eta^2$, and this and therefore the energy loss are clearly negligible. Hence we conclude when there is component of Higgs at the core of the string coupling oppositely travelling fermions together, there are no significant currents on the string. Recalling (4.2.15), we see that in SO(10) with quarks and leptons bound to the string, the charged component of the Higgs elctroweak doublet must be non-zero at the core, so these $SO(10) \to SU(5)\times Z_2$ strings are not superconducting.

Note that if quarks and leptons are not bound to the string it can be superconducting because of the charged component of the Φ_{126} at the core, but the maximum current will be limited to about 0.5 MeV by decays into electrons and neutrinos.

This has been a rather complicated section, but it is possible to draw some conclusions together. A simple phase string in $G \to K\times Z_2$ with just one Higgs field does not have normalisable zero modes if the Higgs field does not vanish at the origin, because a component of the fermion field behaves as $\rho^{-1/2}$ at infinity. If there is a further stage of symmetry breaking in which this fermion gets a Dirac mass there two possibilities. Firstly, if it energetically favoured that the second Higgs field does not change phase around the string, there may be a bosonic superconducting mode carried by the first Higgs at the core of the string if this component is charged. The current carried by this mode is limited by the decays of Higgs into fermions. On the other hand, if the second Higgs field does change phase around the string, the fermion ψ_- (and some others consistent with anomaly cancellation in the effective two dimensional theory on the string) will be trapped on the string. The string will only be superconducting if there is no possibility of particles and antiparticles travelling in opposite directions on the string so that they annihilate. We have seen that when there is a component of the Higgs field at the core which couples oppositely travelling modes together, then just this situation results. A particular example was the string in $SO(10) \rightarrow SU(5) \times Z_2$ with quark and lepton zero modes.

4.3 Fermions and non-phase strings.

In this section we shall repeat the clculations of the last section to show that there are travelling zero modes on the non-phase string. However, when we go on to consider the response of the solutions to the application of an electric field, we shall see that the situation is complicated by the fact that Q and Q_{em} need not commute, and this destroys the superconductivity. Let us again consider the string along the z axis, and for a non-phase string the background fields can be written (4.1.8)

$$\Phi(\rho,\phi) = \left[\left(e^{i\phi}\phi_1 + e^{-i\phi}\phi_{-1}\right)f(\rho)/2 + \phi_0\left(f(\rho)/\sqrt{2} + a(\rho)\right)\right] \tag{4.3.1}$$

$$A_\mu = \delta_{\mu\phi} T^3 \left(F(\rho) - 1 \right)/e\rho$$

Note that the functions f, a, and F will be in general different than those in the last section, although the boundary conditions are the same. Writing $\phi_\pm^{l,r}$ for the fermion SU(2) doublets under T^3, the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_f &= \overline{\phi}_\pm^{l,r}\gamma.(i\partial \mp eA/2)\phi_\pm^{l,r} \\ &- \tfrac{1}{2}gf(\rho)(e^{i\phi}\overline{\phi}_+^l\phi_+^r + e^{-i\phi}\overline{\phi}_-^l\phi_-^r) \\ &- g((a(\rho) + f(\rho)/\sqrt{2})\overline{\phi}_+^l\phi_-^r + (a^*(\rho) + f(\rho)/\sqrt{2})\overline{\phi}_-^l\phi_+^r) - h.c.\end{aligned} \tag{4.3.2}$$

The equations of motion for the transverse zero modes are (for ϕ_+^l and ϕ_-^r)

$$\begin{aligned}&\gamma^i(i\partial_i - eA_i/2)\phi_+^l + \tfrac{1}{2}gf(\rho)e^{i\phi}\phi_+^r + g(a(\rho) + \tfrac{f(\rho)}{\sqrt{2}})\phi_-^r = 0 \\ &\gamma^i(i\partial_i + eA_i/2)\phi_-^l + \tfrac{1}{2}gf(\rho)e^{-i\phi}\phi_-^r + (a^*(\rho) + \tfrac{f(\rho)}{\sqrt{2}})\phi_+^r = 0\end{aligned} \tag{4.3.3}$$

The other two are obtained by the replacements $l \to r$, $a \to a^*$, and $\phi \to -\phi$. Repeating the steps (4.2.4) to (4.2.10) we arrive at

$$\begin{aligned}&(\frac{\partial}{\partial\rho} + \frac{(1-\mu)}{2\rho})\chi_+^l + \tfrac{1}{2}gf\chi_+^r + g(a + f/\sqrt{2})e^{\mu R}\chi_-^r = 0 \\ &(\frac{\partial}{\partial\rho} + \frac{(1-\mu)}{2\rho})\chi_-^l + \tfrac{1}{2}gf\chi_-^r + g(a^* + f/\sqrt{2})e^{-\mu R}\chi_+^r = 0\end{aligned} \tag{4.3.4}$$

It is clear that as $\rho \to \infty$ $\chi_+^{l,r}$ and $\chi_-^{l,r}$ are equal and vanish as $\rho^{-1/2}\exp(-g\eta\rho)$ for both $\mu = +1$ and $\mu = -1$, and since f vanishes as ρ or faster at the origin, the $\rho \to 0$ behaviour is the same as in the last section. Hence we have two normalisable zero modes, with $\mu = \pm 1$, as summarised in Table 4.2 below. Note that $N = 1 + 1/\sqrt{2}$.

Table 4.2

Form of solution	$\rho \to 0$ behaviour	$\rho \to \infty$ behaviour
$(\mu = +1)$		
$\psi_+(\rho,\phi) \sim e^{R/2}\chi_+(\rho)$	A	$\rho^{-1/2}e^{-Ng\eta\rho}$
$\psi_-(\rho,\phi) \sim e^{-R/2}\chi_-(\rho)e^{-i\phi}$	$-(\frac{ga^*(0)A}{2})\rho$	$\rho^{-1/2}e^{-Ng\eta\rho}$
$(\mu = -1)$		
$\psi_+(\rho,\phi) \sim e^{-R/2}\chi_+(\rho)e^{i\phi}$	$-(\frac{ga(0)A}{2})\rho$	$\rho^{-1/2}e^{-Ng\eta\rho}$
$\psi_-(\rho,\phi) \sim e^{R/2}\chi_-(\rho)$	A	$\rho^{-1/2}e^{-Ng\eta\rho}$

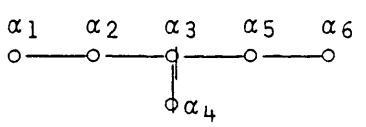
Now let us consider the t,z dependence. In the absence of any other fields we just have

$$i(\gamma^0\partial_t - \gamma^3\partial_z)\psi_\pm = 0 \qquad (4.3.5)$$

or

$$(\partial_t + \mu\partial_z)\,\phi_\pm = 0 \qquad (4.3.6)$$

and there are zero modes travelling in opposite directions. Now we ask if these modes are superconducting. There is a real difference to the phase string, because the generator of the azimuthal dependence of the string Higgs field, Q, does not necessarily commute with the electromagnetic charge generator Q_{em}. A non-commuting example is $SO(10) \to SU(5)\times Z_2$ in the gauge where $Q = T^1 = (E_{\alpha_5} + E_{-\alpha_5})/\surd 2$ and Q_{em} is diagonal. Here, $Q_{em} = (\alpha_1 + 2\alpha_2 + 3\alpha_3).H/3$, so $[Q_{em}, Q] = -iT^2$. A commuting example is $E_6 \to SO(10)\times Z_2$ with the simple root assignment



In this case $Q = T^1 = (E_{\alpha_6} + E_{-\alpha_6})/\surd 2$ and $[Q_{em}, Q] = 0$.

Now T^1 is not diagonal, so if $[Q_{em}, Q]$ vanishes then Q_{em} is zero in the blocks where T^1 is non-zero. Hence the fermions, which fall into doublets under T^1, are uncharged and the zero modes are not superconducting. If Q_{em} does not commute with Q, then the bound doublets are not eigenstates of the electromagnetic charge operator, and Q_{em} is also a function of ϕ. In the gauge $A_0 = 0$ the t,z equations of motion are therefore

$$(\partial_t + \mu\partial_z)\phi_\pm = -i\mu e A_z Q_{em}(\phi)\phi_\pm \qquad (4.3.7)$$

However, $Q_{em}\psi_{\pm}$ are now linear combinations of eigenstates of T^1, which are different functions of ρ and ϕ, with coefficients depending on ϕ, and equations (4.3.6) cannot be satisfied for all ρ and ϕ. Hence the separation into transverse coordinates and t,z is no longer valid when an electromagnetic field is applied; in effect the zero modes are destroyed by the field. Hence we conclude that non-phase strings are not superconducting.

4.4 Fermion zero modes and beads.

In this section we investigate what happens when a fermion trapped in a superconducting zero mode encounters a bead. Recall that these modes can occur only when oppositely travelling fermions are not coupled by a Higgs field at the core of the string. We may write the bead fields at large ρ as

$$\Phi(\phi,z) = e^{i\phi Q} e^{i\chi(z)T} |2\lambda_{\Phi}\rangle\eta \tag{4.4.1}$$

The generators Q and T can be identified with the generators T^3 and T^2 of the broken SU(2) subalgebra in $G \to K\times Z_2$, and this may be rewritten in terms of the eigenvalues ϕ_{+1}, ϕ_0, and ϕ_{-1} of T^3 as

$$\Phi(\phi,z) = [\tfrac{1}{2}(e^{i\phi}(\cos\chi + 1)\phi_1 + e^{-i\phi}(\cos\chi - 1)\phi_{-1}) \\ -i\frac{\sin\chi}{\sqrt{2}}\phi_0]\eta \tag{4.4.2}$$

We can see explicitly how the string turns into an antistring as $\cos\chi$ goes from 1 to -1. Bearing in mind the results of the last two sections the equations of motion at large ρ for the two spinors $\chi_{\pm}^{1}$ defined in equation (4.2.6) can be written down immediately.

$$\left(\frac{\partial}{\partial\rho} + \frac{(1-\mu)}{2\rho}\right)\chi_{+}^{1} + \tfrac{1}{2}g\eta(\cos\chi + 1)\chi_{+}^{1} = 0$$
$$\left(\frac{\partial}{\partial\rho} + \frac{(1+\mu)}{2\rho}\right)\chi_{-}^{1} + \tfrac{1}{2}g\eta(\cos\chi - 1)\chi_{-}^{1} = 0 \qquad (4.4.3)$$

Hence the large ρ behaviour for both $\mu = +1$ and -1 is, neglecting powers of ρ,

$$\chi_{\pm}^{1} \sim \exp(-g\eta(\cos\chi \pm 1)\rho/2) \qquad (4.4.4)$$

The results of section 3.2 indicate that we can expect $\cos\chi$ to behave like $\tanh(m_s z)$: at any rate, there will be a region of order m_s^{-1} outside which $\cos\chi$ differs very little from +1 or -1. Thus well away from the bead in the +z direction ψ_- is effectively unbound (but still normalisable) while ψ_+ is confined to the string, whereas the situation is reversed for negative z, where ψ_- is bound to the antistring. Conversely, for the antibead, which interpolates between antistring at $z = +\infty$ and string at $z = -\infty$, with Higgs field $\bar{\Phi}$ given by

$$\bar{\Phi} = e^{-i\phi Q}e^{i\chi T}|2\lambda_{\bar{\Phi}}\rangle\eta \qquad (4.4.5)$$

ψ_+ is bound to the string when z is negative. Now, the t,z dependence is determined by equation (4.2.13), $(\partial_t + \mu\partial_z)\psi_+ = 0$, so the $\mu = +1$ solution moves in the +z direction. Let there be a ψ_+ zero mode travelling along the string from negative z towards the antibead at z = 0. At large ρ the fermion wavefunction behaves as

$$\psi_+ \sim e^{-ik(t - z)}\rho^{-1/2}\exp(g\eta(\cos\chi - 1)\rho/2) \qquad (4.4.6)$$

so that upon passing through the bead $\cos\chi \to 1$ and the fermion is effectively unconfined, and can be scattered off the string. Note that ψ_+ is no longer massive at positive z, because the direction of symmetry breaking has been rotated by $e^{i\pi T}$: it is now the ψ_- component which is bound to the string, with $\mu = -1$. In fact, the whole Cartan subalgebra has been conjugated by $e^{i\pi T}$, because if $\Phi = g(\phi,z)|2\lambda_\Phi\rangle\eta$ its little group is the little group of $|2\lambda_\Phi\rangle$ conjugated by $g(\phi,z)$. This means that $\psi_+(z \to +\infty)$ has the same quantum numbers as $\psi_-(z \to -\infty)$ (and vice versa); suppose Y is some generator such that at $z = +\infty$, $Y\psi_\pm = y_\pm\psi_\pm$, then at $z = -\infty$ the quantum numbers are given by the eigenvalues of

$$Y(\phi,-\infty) = (e^{i\phi Q}e^{i\pi T})Y(e^{-i\pi T}e^{-i\phi Q}) \qquad (4.4.7)$$

Now $\psi_\pm$ have eigenvalues $\pm 1/2$ under Q, and using the SU(2) property $e^{i\pi T^2}\psi_+ \propto \psi_-$, we find that at $z = -\infty$

$$Y(z = -\infty)\psi_\pm = y_\mp\psi_\pm \qquad (4.4.8)$$

Thus we can interpret the wavefunction (4.4.6) as being a particle which changes its quantum numbers at the bead and becomes unbound. This is reminiscent of the Callan-Rubakov effect [28,73], whereby fermions scattering in a $j = 0$ channel off a monopole exchange charge with its dyon degree of freedom [92].

Now suppose there is another stage of symmetry breaking caused by another Higgs Φ' gaining a vacuum expectation value of η', at which ψ_- gains a mass $g'\eta'$ and behaves as $e^{-g'\eta'\rho}$ at large ρ and large z. (Note that for the modes to be superconducting Φ' must also vanish at the core.) This Higgs will be a function of z as in equation (4.4.1),

$$\Phi'(\phi,z) = \left[\tfrac{1}{2}\left(e^{-i\phi}(\cos\chi + 1)\phi'_{-1} + e^{i\phi}(\cos\chi - 1)\phi'_{1}\right) - i\frac{\sin\chi}{\sqrt{2}}\,\phi'_0\right]\eta' \tag{4.4.9}$$

(The $Q = -1$ part is non-zero at $z = +\infty$ because it is giving ψ_-, with $Q = -1/2$, a mass.) Taking into account the coupling of the fermi fields to both Higgs's the behaviour of the $\mu = +1$ and -1 solutions at large ρ in the bead background will be

$$\psi_+ \sim e^{-ik(t-z)}\rho^{-1/2}\exp\left[-\left(g\eta(\cos\chi+1) - g'\eta'(\cos\chi-1)\right)\rho\right]$$

$$\tag{4.4.10}$$

$$\psi_- \sim e^{-ik(t+z)}\rho^{-1/2}\exp\left[-\left(g'\eta'(\cos\chi+1) - g\eta(\cos\chi-1)\right)\rho\right]$$

respectively. Recalling that the quantum numbers of ψ_+ and ψ_- are interchanged either side of the bead, it makes sense that the coefficients in the exponents should also be interchanged.

The $\mu = -1$ solution might be reflected by the bead into a $\mu = +1$ solution but the amplitude for this process is down by $O(e^2)$ on the transmission amplitude. Thus we can conclude that when ϕ_+ and ϕ_- zero modes interact with a bead they are most likely to pass through and interchange quantum numbers. There is a small probability of reflection, but this must also change a ϕ_+ mode into a ϕ_- mode (or vice versa). This may lead to one of them leaving the string if it has momentum greater than $g'\eta'$.

Now, conserved charges cannot just disappear. If, for example, ϕ_+ and ϕ_- have different electric charge then the difference must be transferred somewhere when the mode passes through the bead. We saw in the last section that the bead may indeed have charge degrees of freedom, so in order for charge to be conserved the interaction of the zero mode with the bead must excite it into a charged state. We might call this a Callan-Rubakov effect for beads on strings.

4.5 Superconducting zero modes in cosmology.

In the work by Chudnovsky et al. [24], it was shown how a superconducting string might become observable through shock heating of the surrounding plasma or perhaps by synchrotron emission. Another electromagnetic process which might lead to observable effects is radiation by large currents flowing in the string. Here we investigate under what conditions this might become important. An oscillating loop of radius R carrying a current J will emit magnetic dipole radiation at a rate given roughly by the dipole formula

$$\dot{E} \sim -(\pi J R^2)^2 \omega^4 \qquad (4.5.1)$$

where $(\pi J R^2)$ is the magnetic dipole moment if the loop were circular. Since $\omega \sim R^{-1}$ we find that this will be more important than gravitational radiation if

$$J^2 > 100 G\mu^2 \qquad (4.5.2)$$

Given that J is limited by a mass $g\eta$, and $\mu \sim \eta^2$, we find

$$g > 10(G\mu)^{1/2} \qquad (4.5.3)$$

If the current carriers are fermions, a typical value for g would be 10^{-3}. Hence if $G\mu < 10^{-8}$ magnetic dipole could conceivably dominate gravitational radiation if the current could somehow attain its maximum value. Of course, motion through a magnetic field is required to create a current in the first place. Little is known about primordial fields, but it is usual to assume that large scale fields appeared with the formation of galaxies. If a string moves a distance d through a coherent galactic magnetic field B_0 it will pick up a current [26]

$$J \sim c e^2 B_0 d/h \qquad (4.5.4)$$

which for a typical galactic distance of 10^4 pc ~ 10^{20}m is about 10^{14} amps. It is difficult to imagine how larger currents could be generated, so realistically dipole radiation can only dominate for

$$G\mu < 10^{-1}(J/m_p) \sim 10^{-11} \qquad (4.5.5)$$

This field will dominate the galactic magnetic field out to a distance given by

$$\frac{\mu_0 J}{4\pi r} > 10^{-10}\ \mathrm{T} \qquad (4.5.6)$$

A loop of size R can pick up a current J ~ $10^{12}(R/10^{18}$ m) amp, (see equation (1.6.19)) so the magnetic fields of the loop dominate out to

$$r/R < 10^{-3} \qquad (4.5.7)$$

so even for a 10^4 pc loop the effects due to its magnetic field could only be observed out to about 10pc from the string [26].

4.6 Conclusions.

In Chapter 2 the evolution of a system of strings between the time of its formation at a second order phase transition and the time when friction in the surrounding medium becomes unimportant was investigated. It was shown that if the initial correlation length of the Higgs field were too small, i.e. if the quartic coupling constant were too big, then the universe could become string dominated soon after the phase transition. The decays of small loops into heavy bosons during this time could provide the out of equilibrium decays necessary to generate baryon asymmetry if $G\mu > 10^{-16}$. Once $G\mu$ reaches 10^{-12}, though, the bosons can be stable enough for the bosons to last a few expansion times after the phase transition before decaying, and this mechanism is more effective.

Chapter 3 presented the bead, which appears as an interpolation between string and antistring in a theory with Z_2 strings such as $G \rightarrow K \times Z_2$. The bead can be thought of as a monopole on a string, and, depending on the theory, some or all of the monopole's flux can be confined to the string. It was shown that beads need not radically affect the conventional string scenario.

Finally, in Chapter 4, fermion zero modes on non-Abelian strings were examined, taking into account the possibility of having a Higgs field at the core of the string coupling oppositely moving fermions. It was found that superconductivity was only possible if this core field vanished. If superconducting zero modes encounter a bead they can exchange charge in a process similar to the Callan-Rubakov effect. A brief discuusion of some cosmological consequences of beads and zero modes was given.

An important question remains to be answered: under what conditions do beads appear on a string? It has been pointed out [76] that there are string solutions which are gauge equivalent to antistrings, but it remains to solve the coupled non-linear partial differential equations in order to find the values of the coupling constants for which this is the most stable string.

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