STATISTICAL APPROACHES TO SENSITIVITY ANALYSIS OF MATHEMATICAL MODELS: APPLICATIONS IN ECOLOGY

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ABSTRACT

The output from mathematical models is usually dependent upon parameters in the model whose true values are not known. Uncertainty is therefore inherent in the output. This thesis is concerned with the development of statistical approaches to the problem of assessing and describing this uncertainty, by detailed consideration of the problems of analysis of two ecological models, one of the control of rats on agricultural premises, and the other of the dynamics of rabies in fox populations.

A review of the existing literature is provided, which brings together results from many different disciplines and summarises their properties, concluding with a discussion of problems that still remain.

Sensitivity analysis of the fox rabies model is then considered, and an informal graphical approach to the problem, based on a multivariate statistical technique called Biplotting, is described. The method provides an easily interpreted and informative summary of the uncertainty inherent in the output of the fox rabies model.

More objective sensitivity analysis of the rabies model is considered next. An objective numerical measure of sensitivity is defined, which has both a simple, and sometimes a practical interpretation, but which also

conveys useful general information about the degree of uncertainty in the model's predictions.

A formal decision process, arising from the farm rat control model is then discussed, and an illustration is given of the use of the sensitivity measure in this decision making process. The sensitivity measure is shown to provide not only an informal description of uncertainty, but also one which can be employed formally when making decisions based on the output of models.

Finally, methods are proposed by which the sources of uncertainty in the rabies and farm rat models can be identified and measured, using the concept of informational correlation measures.

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NOTE ON PUBLICATION

Parts of the work described in this thesis have already, at the time of submission of this thesis, been published in the scientific literature. For convenience the references are given below.

CHAPTER TWO.

L. W. HUSON (1982) A graphical aid to multivariate sensitivity analysis. Ecological Modelling 16:91-98.

CHAPTERS THREE AND FOUR.

L. W. HUSON (1984) Definition and properties of a coefficient of sensitivity for mathematical models. Ecological Modelling 21:149-159.

CHAPTER ONE

INTRODUCTION: UNCERTAINTY IN MATHEMATICAL MODELLING

1.1. The role of mathematical modelling.

Many statisticians are trained to think that the scientists and technologists with whom they work use two types of investigative technique - the observational study, such as a census or survey, and the designed and controlled experiment. These techniques are used to collect data with which hypotheses may be tested, and from which theories may gradually be developed. The theories, in turn, suggest new lines of investigation. In fact, though observational and experimental techniques are still pre-eminent, the last two decades or so have seen the rapid development of a third investigative technique, which now occupies a firm position in scientific and technological circles, even in the so-called "soft" sciences, and which is increasingly seen and accepted not just as an adjunct to more traditional research methods, but as a valid and useful technique in its own right. In short, when faced with a problem they wish to investigate, many scientists and technologists in the 1980s may neither observe nor experiment - they may, instead, build a mathematical model.

Though the roots of modelling as an investigative procedure

are hundreds of years old, the modelling approach has become commonplace only in recent years. Kiviat (1967) defines a model as "a representation of some existing or proposed system". In general, such a representation could take any form, but in scientific and technological applications, a model is usually a set of mathematical equations. Such sets of equations are constructed to represent the way a particular system is thought to operate. The equations describe relationships between particular components of the system, and it is hoped that the behaviour predicted by the equations will or may be exhibited by the real system. Mathematical models are frequently constructed by technologists on the basis of personal beliefs about the structure of the system being investigated, and about which components are important, and how they might be inter-related. It is not uncommon to find that models of this kind are constructed with little reference to real data, perhaps because it is particularly difficult or expensive to collect data, or perhaps because the system that is being modelled does not exist - that is, it may be a new physical system, such as a dam or resevoir, whose construction is being contemplated. In general terms, it is possible to say that mathematical modelling is a technique used in situations where it is difficult, expensive, or impossible to study the system by more traditional methods.

Because such difficulties are often encountered, it is not surprising that mathematical models have come to be widely

used. They have been used to represent such things as local and national economies, manufacturing processes, inventory and queueing systems, military conflicts, and animal populations, to name but a few. Reviews and bibliographies have been provided, for example, by Orcult (1960), Naylor (1969), Malcolm (1970), Greenblatt (1972), and Schultz et. al. (1976), dealing with applications of modelling in economics, industry, management sciences, social sciences, and ecology, respectively. The acceptance and use of modelling as an investigative technique reflects the view that models "have a powerful role to play in developing management strategies, and in the formulation of policy" (Conway, 1977).

1.2. Uncertainty in mathematical modelling.

The goal of all scientists and technologists who use mathematical models is to predict the behaviour of the system that their model represents. This is obvious when the output from a model is a specific numerical value, but is nonetheless true when a model is used to make rather general, often qualitative, statements about the way in which a system might behave. Sometimes such predictions are used only as aids to understanding how a system might function, but in other circumstances the predictions are intended for use in management and policy making - in other words, it is intended that some sort of action or decision should be based upon the predictions of the model.

In this sense, mathematical models are used for the same purposes as observational and experimental studies: to gain information and insight into the operation of a system. As is the case with both observational and experimental studies, however, there is a degree of uncertainty inherent in the process of mathematical modelling. In formulating the set of equations that make up a model, it is necessary to choose specific parameters and functional forms for the representation. The parameters may represent such things as rate constants or thresholds, whilst the functional forms describe relationships between certain components of the system. It is usually the case that the modeller is uncertain about the values that should be assigned to the parameters in the model, and uncertain about the adequacy of the selected functional forms. These things may have "true" values in the real system, but because the truth is not known, parameter values and functional forms are, in the terminology of Schlaifer (1967), "uncertain quantities".

Uncertainty in observational and experimental studies is conventionally taken to arise primarily from sampling variation, and it is now commonplace for the uncertainty inherent in the results of such investigations to be described and assessed using conventional techniques of statistical analysis. It is just as important, if modelling is to fulfil its potential as an aid to management, that the uncertainty inherent in the process of building and

using a mathematical model should be adequately assessed and described. Failure to describe this inherent uncertainty places the utility of a model's predictions in jeopardy, and may tend to discredit not only a particular study, but the use of mathematical models in general.

The uncertainty inherent in the output from mathematical models frequently causes controversy. A recent example of this arose from the Serpell Report on future transport policy options for the British Rail Network. In evaluating the properties of possible future rail systems the consultants retained by the Serpell Committee used a mathematical model. However, there is uncertainty about the values of some parameters in this model, and a failure to describe this uncertainty has led to criticism of some of the policy options discussed (New Scientist, 1983). Critics have argued, for example, that more accurate values of these parameters might have led to recommendation of different policy options. The use of mathematical models in transport planning has, in fact, often been criticised. Predictions from such models are not infrequently substantially different from the actual results achieved when policies are implemented (see discussion to Tanner, 1978). Small changes in the functional forms used in such models may also result in quite different forecasts (Brooks et.al., 1978).

A further example is provided by the modelling studies that have been conducted to investigate the carbon dioxide

"greenhouse effect" - the extent to which increasing carbon dioxide concentration in the atmosphere might raise global temperatures. One important parameter in such models measures atmospheric humidity, and it has been reported of the models that "a small change in the humidity parameter, still within the range of plausible values, increases the temperature effect fivefold". It has been concluded that "there is so much uncertainty surrounding the modelling that it is impossible to say whether a doubling of atmospheric carbon dioxide would increase global mean temperatures by 0.5^{o}, 10.0^{o}, or anything in between" (New Scientist, 1981; see also Kandel, 1981).

The important point about these controversies in the present context is not which of the many possible forecasts is nearest to the truth, but that the controversy might well be diminished if predictions from mathematical models were always accompanied by suitable descriptions of the uncertainty inherent in them.

1.3. Assessing and describing uncertainty in mathematical modelling.

The uncertainty inherent in model output may be of such a magnitude as to render any single unqualified prediction almost meaningless. This may even be true for parameters whose values are apparently precisely known - O'Neill (1973) has shown, using a model of a form widely employed in ecological studies, that uncertainties in parameter

values of only 10% can imply uncertainties in output of almost 100%. This may happen because of the way in which a parameter enters into the model - for example as an exponent or multiplier.

The assessment and description of uncertainty is the province of statistics. However, though countless statistical methods have been described for use in observational and experimental studies, comparatively few statistical techniques have been considered for their suitability in assessing and describing uncertainty in mathematical modelling. This is not to say that the existence of uncertainty, and its importance, have not been realised. From the earliest days of mathematical modelling there has been an awareness of the problem, and a number of techniques have been proposed by which the reliability of model predictions can be examined. These techniques are most commonly known as methods of "sensitivity analysis", "error analysis", or "uncertainty analysis". Curiously, though, such methods are not always regarded explicitly as being concerned with the assessment and description of uncertainty. Many papers on sensitivity analysis do not contain the word "uncertainty" at all. This may well be because control theorists, who developed many of the now traditional methods of sensitivity analysis, were not really concerned with uncertainties about parameter values, but rather with small variations in values of parameters in their electrical systems, induced for example by temperature variations. Nevertheless, a survey of the

methods of sensitivity analysis indicates clearly that uncertainty is the problem that motivates their development.

1.4. Definitions and conventional views of sensitivity analysis.

A number of definitions of sensitivity analysis have been presented in the literature:

"sensitivity analysis might be defined as changing results by varying input assumptions" (Durway, 1979)

"sensitivity analyses measure the rate of change of output of the model with respect to the inputs" (Blanning, 1974)

"a sensitivity analysis consists of computing the expected value and a rank ordering of contributions to the variance (of the output)" (Atherton et. al., 1974)

"sensitivity is the rate of change in one factor with respect to another factor". (McCuen, 1973)

"sensitivity analysis addresses the problem of how sensitive the solution (of the model) is to variations of or uncertainties in the parameters of the equation set" (Cukier et.al., 1978)

"sensitivity analysis is the determination of the relative

effect of small changes in various parameters - usually with the idea of devoting more resources to the refinement of critical parameters" (Miller, 1974a).

"the purpose of a sensitivity analysis is to determine the amount of variation in one or more model outputs caused by variation of a model input parameter" (Shaeffer, 1980).

"the relevant question is whether all values of a parameter within the range of plausibility produce the same overall model behaviour" (Forrester, 1975).

Definitions of sensitivity analysis typically refer to "variation" or "change" in model output. The potential for this variation lies in the fact that for a deterministic mathematical model, the output is completely determined by the numerical values assigned to parameters in the model. It is worth remarking that the concept of sensitivity analysis is frequently redundant when stochastic mathematical models are utilised, since the main feature of such models is that, by definition, representations of uncertainty are explicitly contained in the equations that constitute the model. In this thesis, therefore, attention is restricted to deterministic mathematical models - that is, models which contain no explicit representation of uncertainty. Such models are widely used in most fields of science and technology. However, when deterministic models are built, the true values for model parameters are rarely known, either because it is not possible to determine them

exactly, or because the value is, in the real system, not fixed, but subject to random variation. The implications are clear: since the parameters are uncertain quantities, any other quantity, such as model output, which depends on them, is also an uncertain quantity. Specifically, the value attained by a model output can be thought of as related directly to the value assigned to a parameter, and to have attendant uncertainties in view of the uncertainty about the true value of the parameter. If a model is to be of practical value, the nature of these uncertainties must be described, particularly to those decision makers whose job it is to utilise the predictions from models in policy formation.

Sensitivity analysis is therefore defined here as

"any technique that aids assessment and description of the uncertainty inherent in the outputs of a mathematical model"

1.5. Origins and classification of some methods of sensitivity analysis.

How are such assessments usually made in the literature? Waide and Webster (1976) place the origins of sensitivity analysis in the 1950s, roughly the time when mathematical modelling began to become firmly established. However, reference to Cruz (1973), for example, shows that many ideas brought into the realm of mathematical modelling were

originally described by engineers interested in control theory and the properties of electrical circuits. The concept of sensitivity analysis also arose, apparently independently, in the work of management scientists in the late 1950s (see e.g. Moffei, 1959; Levy, 1958, 1959; Solomon, 1959).

For convenience, existing methods of sensitivity analysis can be classified into four main categories: those based on the use of derivatives, those based on analytical or numerical error propagation, those based on the fitting of response surfaces or "metamodels", and finally a number of other techniques that have in common the property that they do not fall into one of the other three categories.

1.6. The differential sensitivity measure.

The derivative or differential sensitivity measure originates in the work of electrical engineers and control theorists (Cruz,1973), and is conveniently summarised and discussed by Tomovic (1963) and Tomovic and Vukobratovic (1970).

Denote a general mathematical model as the mapping:

$$m:(S,X,T)\longrightarrow(Y) \qquad (1.1)$$

where S is a vector of "state variables", X a vector of parameters, T a measure of time, and Y a vector of output

variables. The inclusion of T, though not strictly necessary, is helpful because many mathematical models are contructed to represent dynamic systems.

The differential sensitivity measure of an output y_i with respect to a parameter x_j is defined to be the partial derivative:

$$dy_i/dx_j \qquad (1.2)$$

It is apparent from (1.1) that (1.2) may, in general, be a function of S, X, and T. This leads Tomovic (1963) to distinguish between three types of differential sensitivity measure:

(i) the "static" sensitivity measure, where dy_i/dx_j is independent of T

(ii) the "dynamic" sensitivity measure, where dy_i/dx_j is a function of T

(iii) the "parametric" sensitivity measure, which explicitly indicates that dy_i/dx_j is a function of X.

1.7. Modifications of the differential sensitivity measure.

Both McCuen (1973) and Miller (1974b) have proposed slight modifications to the standard differential sensitivity

measure. McCuen (1973) suggested that a "relative" sensitivity coefficient would be preferable, avoiding the fact that the standard definition is dependent upon the scales of both output variate and model parameter. He proposed the measure:

$$R = (dy_i/dx_j)*(x_j/y_i) \qquad (1.3)$$

Miller (1974b), for similar reasons, defined the coefficient:

$$R' = (dD(.)/dx_j)*x'_j \qquad (1.4)$$

where x'_j is the baseline or best-estimate of the value of the parameter x_j, and D(.) is some function, selected by the modeller, of all the output variates.

1.8. Response surface methods.

Implicit in the definition of the differential sensitivity measure is the idea that the relationship between model output and parameter value is linear. In fact, the assumption usually made in control theory is that the relationship is locally linear, this assumption often being justified when the range of possible parameter values is small. A number of authors have indicated that this assumption may easily be violated in modelling applications in the "soft sciences" (e.g. McCuen, 1973: Dwyer and Kremer, 1983), and have argued that this restricts the

applicability of the differential sensitivity measure. To attempt to overcome this problem some modellers have fitted response surface models to the observed relationship between output and parameter values, and have used these response surfaces in order to study the sensitivity of the model.

Plinston (1972) provides an example of this approach, using a second order polynomial to relate output values to parameter values. To measure the sensitivity of the output with respect to a particular parameter, Plinston uses the "second derivative (of the function) with respect to each parameter". This measure has the obvious advantage that it is a constant value (since the response surface is of degree two). Plinston also suggests that useful information may be derived by plotting the response surface in the space of pairs of model parameters, and discusses how the shape of the resultant contours reflect the influence of each parameter. Blanning (1974) coined the term "metamodel" to describe response surfaces used in this way. He notes that the form of the fitted function will usually be selected empirically, following inspection of model output derived for a number of possible parameter values, and proposes that sensitivity be measured by calculating partial derivatives of the function with respect to each parameter. Kleijnen (1975), responding directly to Blanning's paper, suggests that polynomial regression models will be adequate for most applications of this kind, and goes on to suggest that the "sensitivity" of the model

to each parameter may be studied by performing a conventional statistical significance test on the coefficients attached to the parameters. This procedure is unlikely to be valid, as noted below, but is also advocated by Kohberger et.al. (1978). Iman et.al. (1981a,b) also advocate the use of simple regression metamodels, and Richels (1978) gives a typical case study of this kind of approach, applied to mathematical model of energy supply systems. A detailed discussion of the methodology, applied to a hydrological model, is provided by Arfi (1980).

Argentesi and Olivi (1978) have investigated the use of inverse polynomials as metamodels, concluding that they may be preferable, giving better fit to the observed output-parameter relationship in many cases. These authors still measure sensitivity using partial derivatives of the fitted metamodel. Valtonen (1977) suggests that the use of metamodels may provide a measure of the sensitivity of a model output to all parameters simultaneously - a kind of "global" sensitivity coefficient. Using a metamodel that is linear in all model parameters, he defines his "multiparameter sensitivity" measure as "the square root of the sum of the regression coefficients".

Kohberger et. al. (1978) and Sorooshian et. al. (1980) define rather more complex measures of sensitivity, each computed from fitted polynomial metamodels. Kohberger et. al. advocate the use of quadratic metamodels, denoted:

$$O = K + B'X + X'AX \qquad (1.5)$$

where B is the vector of coefficients of first-order terms, and A the matrix of coefficients of second order terms. X denotes the vector of model parameters, and O some function of model output. Kohberger et. al. propose that the sensitivity of the output with respect to a particular parameter may be measured, not only using the conventional partial derivative, but also by examining the eigenvalues of the matrix A. Parameters having a large influence on the model output will be associated with large eigenvalues, and hence a ranking of parameter "influence" will be possible.

1.9. Analytical propagation of uncertainty.

Derivative and response surface methods both deal with the effects of parameter uncertainty by measuring some aspect of the relationship between parameter values and output values. Output uncertainty may also be measured more conventionally - that is, by means of probabilities. If the uncertainty about a parameter's true value is described by assigning a probability density function $fx_j(.)$ to the parameter, then, in principle at least, it is possible to derive the density function of the model output, and use it to describe the uncertainty inherent in the model. Thus, if the model output $y_i = g(x_j)$, and this function has an inverse so that $x_j = h(y_i)$ is defined, then $fy_i(.)$, the probability density function of y_i, is given by:

$$fy_i(.) = | dh(y_i)/dy_i | * fx_j(h(y_i)) \qquad (1.6)$$

This result can be extended to the case of several parameters, and indeed several output variables. The necessary results are proved in many textbooks of statistical theory e.g. Mood et. al. (1974).

In practice, however, it is rarely possible to handle this expression analytically, since the function $h(y_i)$ will be quite complicated for all but the simplest mathematical models. The use of a computer algebra system (see e.g. Huson, 1983) may help. Alternatively, approximate formulae may be used to derive the mean and variance of the output distribution. Thus, for example, if the standard deviation of a parameter x_j is sx_j, then in the above notation, the standard deviation of the output y_i may be approximated by:

$$sy_i = |dg(x_j)/dx_j| * sx_j \qquad (1.7)$$

More complicated formulae are available for the case of several model parameters, including the case where parameters are believed to be correlated. A convenient summary of the necessary formulae is given by Clifford (1973).

This approach to assessing the uncertainty inherent in model output has been adopted by a number of authors, in cases where the mathematical model is relatively simple in form. Examples may be found in the work of Wright (1972),

McCuen (1973), Burns (1975), Argentesi and Olivi (1976), Reckhow and Chapra (1979), Bailey and Duppenthaler (1980), and Shaeffer (1980).

1.10. Monte Carlo propagation of uncertainty.

In cases where analytical propagation of uncertainty is not possible, and where approximations are thought to be inappropriate, perhaps the simplest way of assessing the uncertainty inherent in model output is by use of Monte Carlo techniques. The uncertainty about parameter values is described by means of probability density functions, and the output from the model calculated for various selections of parameter values, taken from the assigned distributions. The uncertainty in the output is then described by means of some function of the observed output values - the mean and variance being most commonly used for this purpose, though Reckhow and co-workers have also used box-plots and probabilities derived from the Chebyshev theorem (Reckhow, 1980: Reckhow and Simpson, 1980). Examples of this approach to uncertainty analysis may be found in the work of Burns (1975), O'Neill (1973), O'Neill et. al. (1981), Gardner et. al. (1981), and Dwyer and Kremer (1983). Kremer (1983) has discussed some alternative ways of describing the uncertainty about parameter values.

The selection of an appropriate probability density function to describe parameter uncertainty has been discussed by Tiwari and Hobbie (1976), who conclude that,

in cases of effective ignorance about the appropriate form, the use of a triangular distribution, defined on a range, is to be preferred. However, in the literature, the choice of uniform or Normal distributions is almost universal.

Parameter values are usually selected at random from the specified distributions, but in a series of papers McKay and co-workers have suggested that a more reliable method, Latin Hypercube Sampling, is available (McKay et. al., 1976, 1979: Iman et. al., 1981a,b: Iman and Conover, 1980). This requires that the parameter density function be partitioned into a number of disjoint regions, each containing the same probability mass, and that a value be selected from each of these regions during sampling. Iman and Conover (1980) have proved that, under certain conditions, such a sampling scheme gives more precise estimates of the output mean and variance than simple random sampling. However, the validity of the proof has been questioned (see discussion to Iman and Conover, 1980).

1.11. Various sensitivity coefficients.

In addition to the above techniques, which were all designed for general use in mathematical modelling, a number of ad-hoc methods of sensitivity analysis have been described in the context of specific applications. These methods are usually based on some simple function of model output and parameter values, used as a sensitivity coefficient. For example, Abouel-Nour (1967) and Jones

(1967) use the ratio:

$$y_p/y_o \qquad (1.8)$$

Here y_p is the value output by the model when a parameter value is perturbed to, say, 1.1 times its nominal value, and y_o is the baseline output value. Abouel-Nour (1967) and Ali (1968) use the coefficient:

$$(y_p-y_o)/y_o \qquad (1.9)$$

Needless to say, different conclusions can be drawn from the same model according to which sensitivity measure is used. Both Wong (1980) and Ford and Gardiner (1979) have emphasised this fact, illustrating in detail the apparently contradictory results that can be obtained by studying a given mathematical model when different sensitivity measures are used.

Brown et.al. (1978), working with a mathematical model that yields five output variates, defined as a measure of sensitivity the maximum percentage perturbation that could be applied to a parameter, such that the average perturbation of all five output variates remained below 10%.

Ibbitt (1972) proposed a sensitivity measure for mathematical models that are "fitted" to a real data set, using some optimisation routine. Ibbitt introduced "errors"

(drawn from a specified probability distribution) into the data set, and then re-fitted the model to the new data. As a measure of the sensitivity of a particular parameter, he used the standard deviation of the values of that parameter estimated at each re-fitting.

McKay et. al. (1976;1979) and Gardner et. al. (1981) used rank correlation coefficients, and Pearson correlation coefficients to measure sensitivity, or, more specifically, the extent to which each parameter in the model is related to the output.

1.12. Fourier sensitivity analysis.

Cukier et. al. (1973; 1978) and Schaibly and Shuler (1973) have proposed a method of sensitivity analysis that simplifies to some extent the measurement of sensitivity in a model that has a number of parameters. The method is essentially as follows: each parameter is varied throughout a range in repeated simulations with the model. The values selected for parameters are made periodic by utilising a transform of the form:

$$x_j = x'_j(1 + h_j(\sin w_j z)) \qquad (1.10)$$

where w_j and h_j are specified by the modeller for each parameter, and where z is varied between 0.0 and 2π. The input parameters are thus periodic, and so, usually, will be the output from the model. This permits the model output

to be represented by a fitted Fourier series, and Cukier et. al. (1973) show that certain coefficients in the Fourier representation are related to the differential sensitivity measures dy_i/dx_j. Thus, a large Fourier coefficient corresponds to a large differential sensitivity measure. The authors suggest that the main advantage of this approach is that sensitivity measures can be computed more quickly than is the case if each differential sensitivity measure is derived. There does not appear to have been any application of this technique in the literature, other than those given by Cukier and co-workers.

1.13. Discussion

The methods described above are the most frequently encountered techniques by which modellers attempt to convey something of the uncertainty inherent in model output. They can be summarised, for the purposes of discussion, as follows:

(i) differential sensitivity measures: these require no explicit description of the uncertainty inherent in model parameter values. The "sensitivity measure" is defined to be the partial derivative of the model output with respect to a particular parameter. These derivatives may be computed directly from the model, either analytically, or, more often, by numerical differentiation, or they may be derived from fitted response surfaces. Some modifications

to the basic definition exist, which alleviate the scale dependence of the original measure. The computational burden of derivation may be reduced by use of the Fourier technique.

(ii) response surfaces: these require a more explicit description of the uncertainty about parameter values. Model output is determined for various possible parameter values, and a simple model fitted to the observed output/parameter data. This "metamodel" is then analysed further to provide descriptions of output uncertainty.

(iii) error propagation: these techniques require that the uncertainty about parameter values is explicitly described by means of probability distributions. The associated distribution of model output is then derived either analytically, in exact form or using approximations, or by direct Monte Carlo simulation. Some function of the model output distribution, typically conventional measures such as the variance, are then used to describe the output uncertainty.

(iv) other methods: these do not usually involve an explicit description of parameter uncertainty; typically one or two alternative or "perturbed" parameter values are used in the sensitivity study, and model output determined for these alternatives. Some simple function of "perturbed" and "baseline" output is used to describe uncertainty.

All four categories of technique appear in the literature with roughly equal frequency, and no one method appears to be universally favoured for any particular type of model. There is only a small literature comparing alternative approaches: the papers by Burns (1975) and Gardner et. al. (1981) are examples where the differential sensitivity measure is compared with error propagation.

In order to evaluate the merits of these approaches to the assessment and description of model uncertainty, it is necessary to establish some criteria by which their properties can be judged. The following criteria are proposed here:

(i) the method used should provide a measure of at least one important aspect of output uncertainty, and should be unlikely to mislead. Exactly how uncertainty is measured is a matter for judgement in each particular application, and the objectives of the modelling exercise must be kept firmly in mind - in some applications it may be sufficient to use a variance or standard deviation, and in others it may be important to use a measure that can be related to some practical criterion

(ii) the measure of uncertainty must be easy to interpret and to use in discussing and developing the model - in particular, it must be easily understood by non-technical users, who are frequently involved with practical applications of mathematical models

(iii) the method should, either directly or by means of simple additional computations, allow the main sources of output uncertainty to be identified. In some modelling applications it may be possible to avoid the need for this, but in many the reduction of uncertainty is essential to the objectives of the model, and in such cases mere description of uncertainty is not adequate - the modeller must have measures which will help in refining and reformulating the model

(iv) the uncertainty description should not be disconnected from the practical objectives of the modelling exercise - ideally the measure should not only provide a general description of uncertainty, but it should be susceptible of utilisation when the model is applied to a specific problem

(v) whilst no method can be expected to be universally applicable or appropriate, it is desirable that the chosen method should be applicable to an acceptably wide range of models, especially models having several parameters and several output variates

(vi) the method should not present too great a computational burden - it should be easy to compute either manually, or using widely and readily available mathematical algorithms.

Considering firstly derivative sensitivity measures, it can

be seen that they do not require an explicit consideration of the uncertainty about parameter values. It is not necessary to vary a parameter throughout its entire plausible range in order to derive the differential sensitivity measure, and, indeed, unless the relationship between output and parameter is linear over this plausible range, an estimate of its slope at a single point, such as the baseline parameter value, may be very misleading. If the relationship is linear, then the differential sensitivity measure is the rate of change of output per unit change in parameter value. This may be an adequate description of the nature of output uncertainty in some cases, but without concomitant information about the range over which the parameter is distributed, it is not very helpful. Also, comparisons of sensitivity measures for different parameters and between different models are hampered by the fact that the differential sensitivity measure is scale dependent, and the values it takes may vary over several orders of magnitude, unless one of the "relative" differential sensitivity measures is used.

Furthermore, the differential sensitivity measure cannot be used to indicate the importance of each parameter in determining the total output uncertainty. For example, if the differential sensitivity measure for a parameter x_1 is 2.0 units, and that for x_2 is 3.0 units, it may appear that uncertainty caused by the second parameter is more important. However, this is clearly not the case if the uncertainty about x_2 is extremely small, whilst parameter

x_1 is known only to an order of magnitude. This underlines the fact that it will usually be necessary to be explicit about the extent of the parameter uncertainty when trying to assess the importance of a parameter's effect in model output. Put more simply, no matter what the potential influence of a parameter is, in view of its mathematical relationship with model output, it is of no interest at all from a sensitivity point of view if its value is known precisely. Yet even if this were the case, the differential sensitivity measure would still be defined, and could be computed - it might even be very large. Furthermore, the differential sensitivity measure cannot be used in any obvious way in further application of the model, for example in formal decision making. It can also be computationally burdensome to derive differential sensitivity measures, especially if numerical differentiation must be used. It therefore seems reasonable to propose that the differential sensitivity measure is of value only when the uncertainty about each parameter is such that the relationship with output is linear over the plausible range, and such that there is roughly the same degree of uncertainty about each parameter.

The use of response surfaces is an improvement in two respects. Firstly, in order to generate the data used to fit the response surface, it is necessary to be more explicit about the uncertainty in each parameter value, or at least about its plausible range. If a parameter value is known for certain, for example, it will not appear to be

related to the output in the fitted response surface. Secondly, assumptions about the relationship between model output and parameter value are explicitly described and evaluated when the response surface is fitted, and any inadequacies in the assumption should become apparent at that stage. There are therefore two main aspects of the response surface approach to be considered. The first is the extent to which the fitted model represents the actual relationship between output and parameters, and the second is how the response surface is then used to describe output uncertainty. Obviously the fitted response surface is, at best, an approximation to the actual relationship between output and parameters. The true relationship (or at least the modeller's view of this) is, after all, given by the mathematical model to which the response surface is being applied. It is for this reason that conventional statistical significance tests of the coefficients in a response surface, as proposed by Kleijnen (1975) and Kohberger et. al. (1978), are most unlikely to be appropriate. If a parameter is not believed to be related to output, it should not have been included in the original model, and, conversely, the inclusion of a parameter in the mathematical model implies a belief that it is of some importance. A test of "significance" of such a paramter would therefore seem illogical.

If differential sensitivity measures are computed from the response surface most of the disadvantages noted above will apply, except that the assumptions about the nature of the

relationship will at least be easily evaluated, by measuring the goodness-of-fit of the response surface. Whether or not some indication of the importance of each parameter is given also depends on how the response surface is used - the methods of Kohberger et. al. (1978), cited above, offer some prospect of such information.

To summarise, the use of response surface methods may be adequate when the fit of the response surface is good, and when suitable use is made of the response surface to provide descriptions of output uncertainty. In other circumstances the approach may be misleading. Generally, the fitting of response surfaces is less computationally demanding than numerical evaluation of differential sensitivity measures.

Error propagation techniques are a further improvement on both differential sensitivity techniques, and response surfaces. In the case of error propagation it is necessary to explicitly describe the uncertainty about parameter values, and, furthermore, no assumption need be made about the form of the relationship between output and parameter, except in so far as this is already implicit in the mathematical model that is being studied. The exception to this rule is when approximate error propagation formulae are utilised, since the use of these implies the assumption that nonlinear terms in the output-parameter relationship are negligible. The value of error propagation techniques therefore depends on the way in which the derived output

distribution is used to convey information about the uncertainty inherent in the model's predictions. On the debit side, no explicit measure of parameter influence is available from error propagation techniques, and there is no obvious way of employing this kind of method in practical application of the model to a specific management problem. However, error propagation is considered to be less demanding computationally than either numerical evaluation of differential sensitivity measures, or the fitting of response surfaces. It should be noted that analytical error propagation is only possible for very simple mathematical models, and this fact, together with the assumptions implicit in the use of approximate formulae, suggests that Monte Carlo error propagation is generally to be preferred.

Finally, ad-hoc approaches to sensitivity need to be considered. Perhaps the principal defect of such approaches is that they employ descriptions of parameter uncertainty that are unlikely to be adequate. For example, it is common to find that the model output may be re-evaluated only once, with a parameter set to, say, 1.1 times its baseline value. The sensitivity measures derived from such an approach, some of which are described above, may or may not be adequate as descriptions of uncertainty, depending upon the particular application. Clearly the ones described above were thought to be adequate in the contexts in which they were used. Without an adequate description of the nature of parameter uncertainty, ad-hoc approaches may

mislead; perhaps their main advantage is their simplicity, both computationally, and in interpretation. As with other techniques, it may not be possible to formulate any measure of the extent to which each parameter influences output uncertainty, except perhaps when correlation coefficients are used, as cited above.

More generally, all the above methods have some disadvantages in the case of models having several parameters and several output variables. In such cases it is usually necessary either to compute a sensitivity measure seperately for each output/parameter combination, or to formulate some compound function of all the output variables, and relate this to model parameters. Neither approach is conducive to simple description of output uncertainty.

Furthermore, none of the approaches described above provides information that can be used at later stages in the application of the model. The computational effort involved in their derivation must therefore be set solely against their value in informal sensitivity studies.

1.14 Summary.

The above introduction indicates some of the defects of presently available methods of sensitivity analysis, and accordingly suggests possibilities for the development of new techniques. The following conclusions are drawn from

the introduction; they provide the basis for the methods proposed in the remaining Chapters.

(i) the practical value of a mathematical model may be jeopardised unless a presentation of its predictions includes an appropriate description of the uncertainty induced by uncertainty about parameter values

(ii) a variety of techniques are available by which uncertainty may be assessed and by means of which uncertainty may be described to a decision-maker. The best such approaches rely on explicit description of parameter uncertainty by means of probability distributions, and the generation of a sample from the model output distribution by means of Monte Carlo simulation

(iii) uncertainty must be described by appropriate utilisation of the output distribution thus generated. To supplement methodology presently available in the literature, techniques with the following characteristics are required:

(a) a technique that provides easily interpretable information about the uncertainty inherent in model output, for models having several parameters and/or output variates

(b) a technique that provides a simple description of uncertainty, easily computable from the model output distribution, which is of both general informative value,

and also has practical interpretations, in at least some circumstances

(c) a technique that not only provides general descriptions of uncertainty, but which may also be utilised by a decision-maker when the model is employed for the purposes of practical policy formulation

(d) a technique by which the decision maker can determine which parameters in the model most influence the uncertainty in model output, and which makes as few assumptions as possible about the nature of the model, and the relationship between parameter values and output.

CHAPTER TWO

A GRAPHICAL AID TO MULTIVARIATE SENSITIVITY ANALYSIS

2.1. Introduction.

In order to clarify and to illustrate some of the proposals of Chapter One, consideration is given in this Chapter to one aspect of the uncertainty analysis of a particular ecological mathematical model - a model of the dynamics of rabies in red fox populations.

2.2 Rabies in red foxes.

Rabies is a virus disease of the central nervous system to which most, if not all, mammals are susceptible. The disease is usually fatal. Interest in rabies in Europe has increased as the current epidemic of the disease on the continent has spread - originating in Poland in the 1940s, it is now only tens of kilometres from the English Channel coastline in France. The main vector of the disease in Europe is the red fox. Many other mammals contract the disease, however, and it therefore presents three main threats: to human life, to farm animals and other livestock, and to wildlife. From a purely practical point of view, interest centres on the control of rabies, and an

essential step in planning control measures is to understand something of the way in which the disease is propagated and maintained in natural populations.

2.3. A mathematical model.

Clearly the scope for experimental work on the propagation of rabies in fox populations is limited. Studying the problem is therefore mainly restricted to observation of historical patterns, and to mathematical modelling of the process.

Anderson et.al. (1981) have developed a deterministic compartmental model of fox rabies - the fox population being represented by three classes of animal. In this model the first category, susceptible animals, have a population density D, the second category, animals that are infected by rabies but not capable of passing on the infection, have population density E, and the third category, animals that are infectious, have density F. The total population density $N = D + E + F$.

The model is constructed by considering the rates of change of these three population densities over time. Firstly, it is assumed by Anderson et. al. that the death rate of foxes is density dependent - specifically, that the per capita death rate b is linearly related to N such that the total death rate is given by the expression:

$$(b + yN)\ N \qquad (2.1)$$

Here the parameter y may be said to measure the "severity" of the density dependent increase in mortality.

Though evidence on the point is conflicting (see sources cited by Anderson et. al.), the model is further developed on the assumption that the per capita birth rate of foxes, c, is independent of total population density. The intrinsic per capita rate of increase is equal to c-b, and is represented by the symbol r.

The rate at which susceptible foxes acquire rabies is then assumed to be proportional to the density of susceptible animals and of infectious animals, according to the expression:

$$\text{rate} = BDF \qquad (2.2)$$

where B is a coefficient referred to by Anderson et. al. as a "transmission coefficient". Next it is assumed that animals pass from the infected to the infectious class at a constant per capita rate s - this parameter accordingly being the reciprocal of the "latency period" of rabies in foxes. The final assumption necessary to define the model is that rabid foxes die at constant per capita rate a. Given these assumptions, the losses and gains of each

"compartment" in the model can be written as:

changes to "susceptibles" =

+ net rate of increase

- rate of transition to "infected state"

changes to "infected" =

+ rate of transition to "infected state"

- natural death rate

- rate of transition to "infectious state"

changes to "infectious" =

+ rate of transition to "infectious state"

- natural death rate plus rabies death rate

or, as expressed mathematically by Anderson et. al., in the form of three coupled differential equations:

$$dD/dt = rD - yDN - BDF \qquad (2.3)$$

$$dE/dt = BDF - (s+b+yN)\,E \qquad (2.4)$$

$$dF/dt = sE - (a+b+yN)\,F \qquad (2.5)$$

2.4. Use of the model.

These equations may be solved numerically to show how the densities D, E, and F change over time - a graphical illustration of the changes is exemplified in Figure 2.1. To do this it is necessary to supply initial densities of

each category of animal, and numerical values for the six parameters in the model, r, y, B, s, b, and a.

What kind of information might be required from such a model, by a decision maker who is responsible for planning and organisation of rabies control? Some possibilities are:

(i) an indication of the extent to which rabies might become established in a fox population, if the population is not subject to control - this might be suitably measured for example, by the number of infected foxes present in the population after, say, a one year period following introduction of rabies

(ii) an indication of the potential persistence of rabies in such a population - measured perhaps by the number of infectious foxes in the population - those foxes that are capable of passing on the disease - at the same point in time

(iii) an indication of the maximum impact that rabies might have in such a population - measured perhaps by the maximum number of infected foxes in the population over a simulated 20 year period, and by the minimum size to which the fox population falls over the same hypothetical 20 year period.

2.5. Uncertainty analysis.

To derive single values of these model outputs, given single values of the parameters r, y, B, s, b, and a, is not difficult. The parameter values are substituted into the equations above, and these are integrated over the necessary values of t in order to generate the output. However, to give a single set of answers to a decision maker is inadequate. The "true" values of r, y, B, s, b, and a, are not known. Anderson et. al. (1981) suggest "baseline" or best-estimate values (Table 2.1.), but examination of the sources of these estimates (cited by Anderson et. al.) indicates that they cannot be considered precise. Since there is uncertainty about these parameter values, there is uncertainty inherent in values output by the model, and the decision maker must have easily assimilated and useful descriptions of this uncertainty.

2.6. First steps in the description of output uncertainty.

An appropriate first step in the sensitivity analysis of a mathematical model is to provide a general, informal description of the uncertainty inherent in the output. Such an assessment can be thought of as an example of the exploratory analysis becoming more widespread in statistics. The purpose of such a procedure is to convey, in a simple and summarised form, information about the

model that can easily be interpreted by the decision maker. Such a method is now proposed.

2.7. Sensitivity data.

Consider a model with p parameters x_i (i=1...p) and v output variates y_j (j=1...v). Where uncertainty exists about the true values of the parameters it is appropriately described by a joint probability distribution. Sensitivity data may then be generated by sampling parameter values from this joint distribution, and using these values in successive runs of the model to generate a set of possible values for the output variates. If n such sets of parameters are sampled, and the model is run with each set, the resulting data may naturally be represented as a matrix of order n x (p+v).

Often the magnitudes of the various parameters and outputs will be quite different, and such differences of scale are a notorious source of difficulty in interpreting multivariate data. It is therefore appropriate to rescale the data in some way.

In the case of output from a mathematical model, the following scheme is proposed here. For each column in the n x (p+v) matrix, there exists a "baseline" or "best estimate" value. Rescaling to express each element as a ratio of its baseline, and adjusting the mean of each

column to equal 1.0, produces a natural compatibility of scale for the matrix.

2.8. Preliminary sensitivity analysis.

A decision maker, in order to make a preliminary assessment of the uncertainty inherent in the model, is likely to require:

(i) a description of the variance-covariance structure of the model output; to assess the magnitude of the uncertainty in the outputs, and how they are related to each other

(ii) a summary that highlights any unusual or particularly deviant outputs produced by the model; such extreme results may reflect possible behaviour of the real system, or they may indicate a defect in the construction of the model

(iii) a method that permits easy diagnosis of any such extreme outputs, in terms of the parameters that most influence them, and the variates that respond most strongly.

2.9. The value of a graphical approach.

Graphical methods have in recent years become increasingly popular as aids to multivariate analysis. Since the data

generated as proposed above has the conventional multivariate matrix structure, and since the purpose of the preliminary assessment is to convey information in a summarised and easily interpreted form, it is reasonable to consider graphical approaches to the problem of describing uncertainty in model output.

Gabriel (1971) described a multivariate statistical technique that he called the Biplot, which, with the modifications described below, proves to be suitable device for evaluating multivariate sensitivity data.

2.10. The Biplot.

The Biplot is a graphical display of multivariate data based, where necessary, on a rank two approximation of the original data. The plot is formed from two sets of co-ordinates, one of which represents the rows of the matrix, and the other the columns. There are several types of Biplot, but the one most suitable for the present application, referred to by Gabriel (1971) as the GH' Biplot, assumes that the rows of the matrix are a random sample from a larger population.

The rows of the matrix are plotted so that increasing deviation of a row from the mean is shown as increasing distance from the origin of the Biplot. The distances on the Biplot in fact approximate the Mahalanobis distance

between rows of the sensitivity data matrix. The columns of the matrix are represented as projections that have lengths proportional to the variance of the column they represent. The correlation between columns is approximated by the cosine of the angle between the column projections. The position on the plot of a point representing a row of the matrix is determined by the extent to which each column contributes to the row's displacement. Gabriel (1971) has proved and described these properties in detail.

By modifying the GH' Biplot, as described below, and by using the re-scaling described above, each row of the sensitivity data matrix can be displayed in such a way that rows that are particularly displaced from the baseline - indicating a potentially deviant model response, appear distant from the Biplot origin. The position of a point representing a row, in relation to the projections that represent model parameters and outputs, allows the nature of such a response to be diagnosed. Simultaneously, the Biplot gives a clear picture of the variation in model output, and of the correlations within and between the parameters, and the output variates.

2.11. Biplot algorithm.

Let V be the variance-covariance matrix derived from the mean-centred scaled data matrix created by n runs of the model. Denote the first two latent roots and vectors of nV

by l_1 , l_2 , and c_1 , c_2 , respectively. The co-ordinates of the Biplot are given by:

$$G = \left((1.0/\sqrt{l_1})D_r c_1 \; ; \; (1.0/\sqrt{l_2})D_r c_2 \right) \sqrt{n} \qquad (2.6)$$

$$H = \left(\sqrt{l_1} c_1 \; ; \; \sqrt{l_2} c_2 \right)(1.0/\sqrt{n}) \qquad (2.7)$$

where D_r is the mean-centred sensitivity data matrix in which each value in the parameter columns has been replaced by its baseline value. Because each column is mean centred, this is equivalent to setting all elements of the first p columns to zero. This modification to the first p columns of the data matrix is non-standard, and is proposed here so that the resulting Biplot highlights only discrepant output values and not unusual parameter values. With each parameter value reset in the data matrix to its baseline value, the algorithm of (2.6) and (2.7) produces inter-point distances that reflect only differences between output on each run of the model.

Matrices G and H are, respectively, n x 2 and (p+v) x 2 matrices; each row of these matrices is treated as an x-y co-ordinate pair to produce a graphical display. The (p+v) co-ordinate pairs from matrix H are drawn as lines from the Biplot origin to the point indicated by the co-ordinates.

Occasionally a re-scaling of the sensitivity data matrix, by multiplying throughout by a scalar, may be required in

order to produce compatible scales in G and H', and so avoid a graph on which the variate vectors are too small to permit easy interpretation of the plot. An appropriate value of such a scalar is best selected by trial and error, after observing plots derived with different values. This scalar multiplication does not affect the above properties of the Biplot (though it would affect other properties not employed here - see Gabriel (1971) for details).

2.12. An application of the Biplot technique.

The Biplot technique is illustrated here by application to the mathematical model described above. The model contains six parameters, and, in this illustration, is used to predict four output variates, as described above. The baseline values of the six paramaters, from Anderson et.al. (1981), are given in Table 2.1. These parameters, and hence the outputs from the model, are uncertain quantities, and the purpose of the Biplot is to provide a concise illustration of the output uncertainty.

In order to generate a sample from the output distribution, the uncertainty about each of the parameters in the model was described by assigning to each parameter a uniform distribution, centred on the baseline value (Table 2.1) and having a range equal to 60% of the baseline. Thirty runs of the model were made, each with a selection of parameter values made randomly from these probability distributions,

and values of the four outputs calculated in each run. The output values derived in this way are given in Table 2.2.

Together with the values of the parameters used in each run, the data in Table 2.2 were used to derive the sensitivity data matrix, by re-scaling as described above, and the co-ordinates of the Biplot of the re-scaled data were derived as in (2.6) and (2.7). These co-ordinates are given in Table 2.3, and the Biplot produced from them is displayed in Figure 2.2.

2.13. Interpretation of the Biplot for the fox rabies model.

The lines on the Biplot represent the columns of the sensitivity data matrix. The line labelled P1, for instance, represents the first parameter in the model - the per capita natural rate of increase (which is a function of the per capita birth and death rates).

The lengths of these lines are proportional to the variance of the corresponding column. Thus, the Biplot conveys immediately the information that outputs labelled V1, V2, and V3 have roughly equal variance, whilst V4 varies much less. In view of the re-scaling proposed above, where each output value is expressed as a ratio of its baseline, this can be interpreted to mean that there is greater uncertainty about the outputs V1, V2 and V3, than about V4.

The variance of parameters is of less interest in the diagram, since it will usually, as in this example, be specified by the modeller in order to produce the Biplot.

The orientation of these vectors is related to the correlation structure of the data - the cosine of the angle between any two lines being an approximation to the correlation coefficient between corresponding columns.

The Biplot in Figure 2.2 therefore shows that the outputs V1, V2, and V3 are highly correlated with each other (small angles between the vectors), and rather highly negatively correlated with V4 (angles of between 145^{o} and 175^{o} between the V4 vector and the other three vectors).

Having shown that outputs V1, V2, and V3 vary the most, and that they are highly correlated with each other, and negatively correlated with V4, it is then possible to determine from the Biplot something about the causes of this uncertainty. The output vectors V1, V2 and V3 are at angles ranging between 20^{o} and 50^{o} to the vector representing the parameter P1, suggesting relatively high correlations with this parameter. It is therefore reasonable to infer, if only provisionally, that uncertainty about this parameter is quite important, in that it may be primarily responsible for the resultant uncertainty in the model outputs. Parameter P2 is at an angle of about 80^{o} to the vectors V1 and V3, suggesting

that it has only low correlation with the model output, despite the more acute angle of about 50^{o} to vector V2. Other parameters appear to have only moderate or low correlations with V1, V2 and V3, probably negative correlations in the case of P5, which points in roughly the opposite direction.

All these inferences about the uncertainty inherent in the model can be made quickly and easily once the principles of the Biplot are familiar to the modeller. The accuracy of the inferences can be confirmed by reference to Tables 2.4 and 2.5, where the actual variances and correlations of the model outputs and parameters are presented.

The numbered points 1 to 30 in Figure 2.2 represent the 30 simulation runs of the model, each with a random selection of parameter values. Most of the runs produced output clustered around the baseline (the Biplot origin). The output most markedly displaced from this baseline was that produced in the 9th run of the model: this point lies in the lower left hand corner of the figure.

The position of any point on the Biplot is determined by those columns in the sensitivity data matrix that contribute most strongly to its deviation from the baseline output. Point 9 on Figure 2.2 is displaced from the origin broadly in the direction of V1 and V3, and in the opposite direction from V4. This indicates that this particular run

of the model produced high values for outputs V1 and V3, and a rather low value for V4: overall this run was the most displaced from the baseline output.

Following from the inference made above, that parameter P1 appears to be most highly correlated with V1 and V3, it is reasonable to infer that run 9 of the model probably had a high value for parameter P1. Similar conclusions would apply to run 6 of the model, which is also displaced to some extent.

Summarising these inferences in terms of the model, the following conclusions may be reached by inspection of the Biplot:

(i) the greatest variation or uncertainty in the model output occurs for the number of infected and infectious foxes at the end of a simulated 12 month period, and for the maximum number of infected foxes in the population over a simulated 20 year period.

(ii) these outputs are highly correlated

(iii) this variation is principally attributable to uncertainty about the value of r in the differential equation model - the value of the per capita rate of natural increase.

2.14. Summary.

The Biplot, modified as described above, is a simple technique that allows a concise and easily interpretable summary of the uncertainty in model output. The decision maker can be given a single graphical summary of output variation, correlations between parameters and outputs, and of any unusual or potentially deviant model responses. The decision maker is, in the case of the red fox rabies model, shown that uncertainty about three of the four outputs in which he is interested is much the same, that he can feel more confident about the value of the fourth model output, that inability to precisely specify the intrinsic rate of increase of fox populations is the main cause of the output uncertainty, and, additionally, he is alerted to the possibility that the number of infected foxes in the population might be rather different from the values that tend to be produced by the model in repeated simulations.

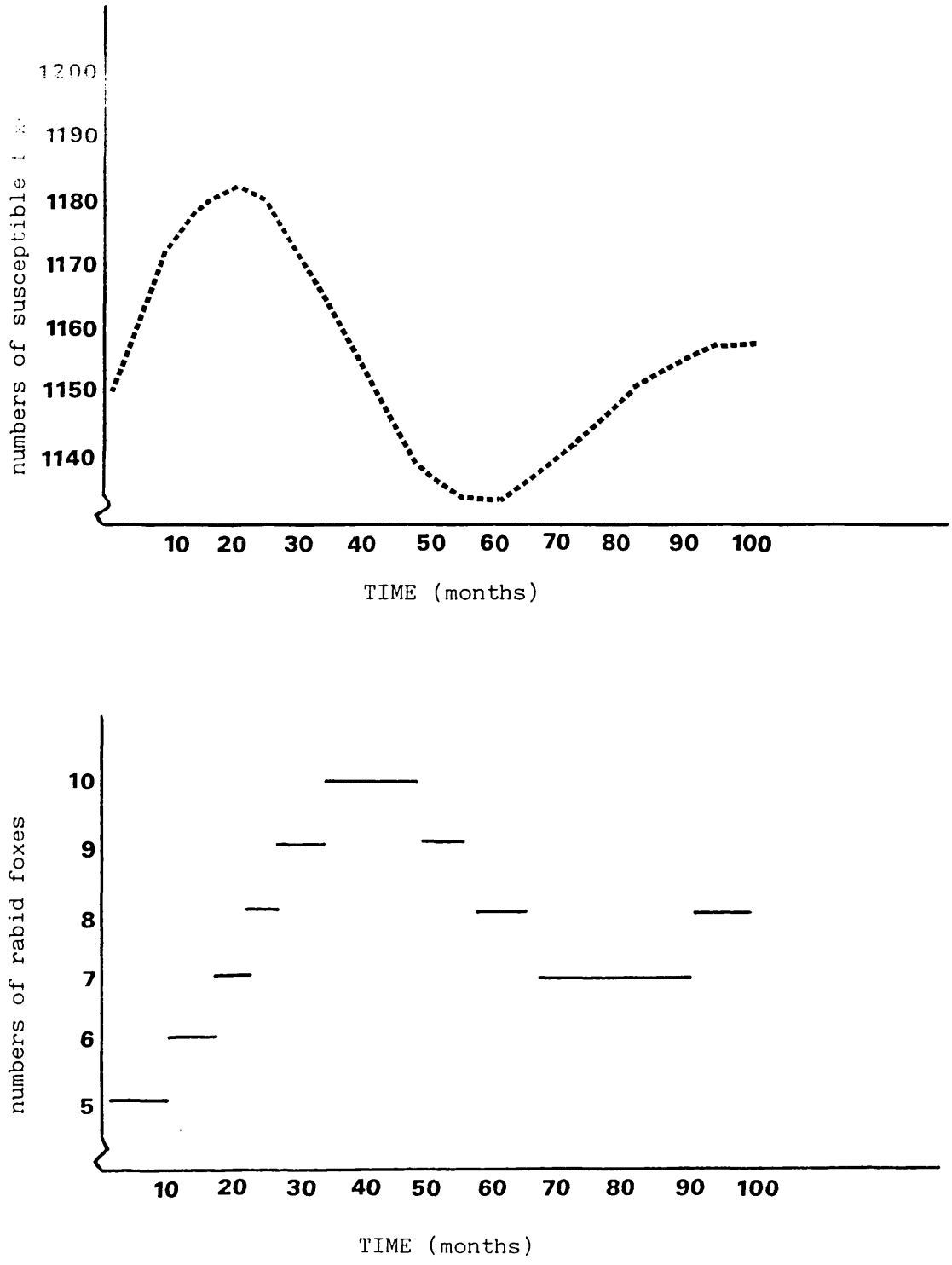

Figure 2.1. An example of a time track of fox population sizes generated by the red fox rabies model.

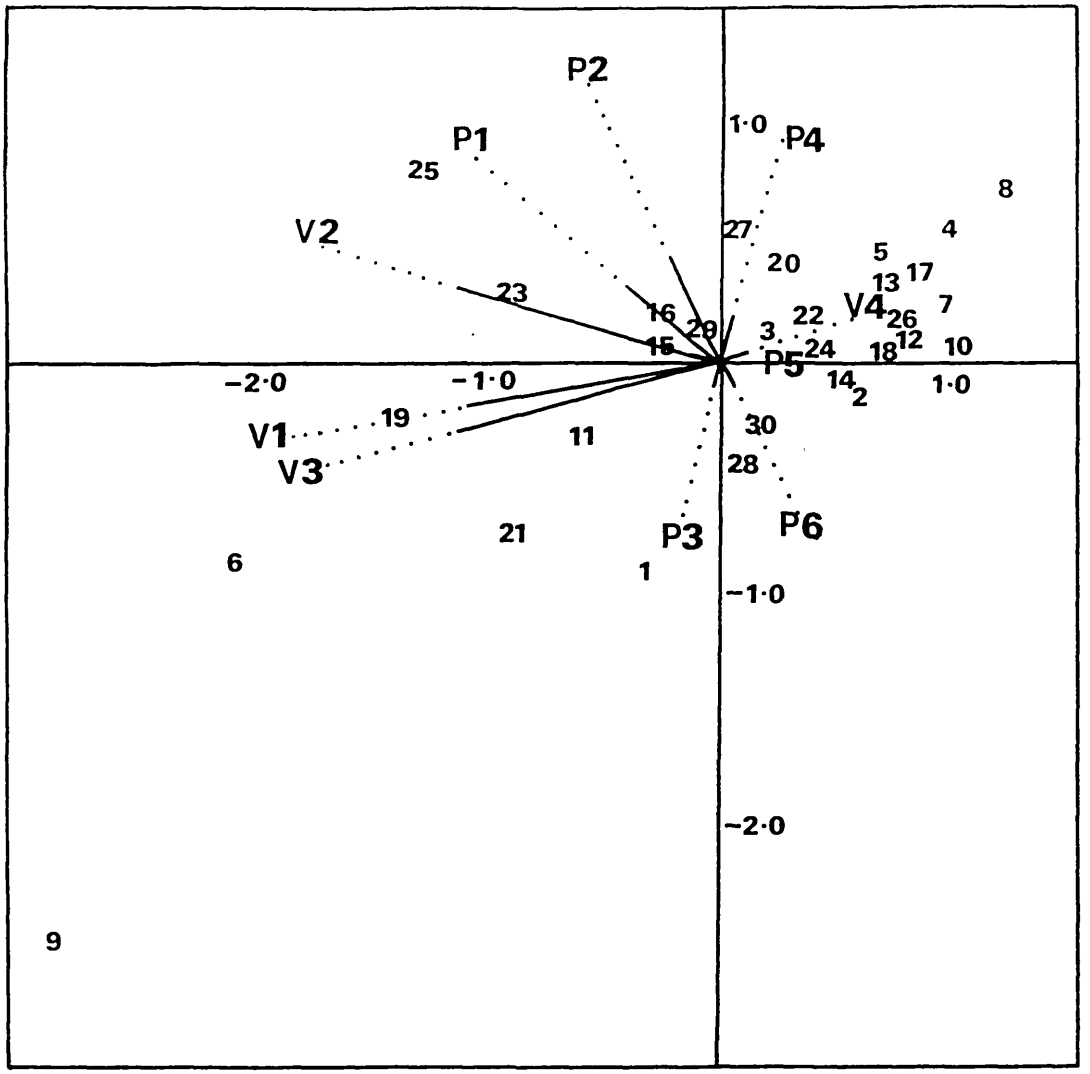

Figure 2.2. Biplot of sensitivity data from the red fox rabies model.

Table 2.1.

Baseline values of the six parameters and four output variates from the red fox rabies model.

quantity	baseline value
r	0.50
y	0.25
B	79.69
s	13.00
b	0.50
a	73.00
number of infected foxes after 12 months	23
number of infectious foxes after 12 months	4
maximum number of infected foxes over 20 yr. period	31
minimum fox population size over 20 yr. period	1138

Table 2.2.

30 values of four measures computed from the red fox rabies model, generated in successive runs of the model, each with a different set of parameter values, selected from specified probability distributions.

number of infected foxes after 12 months	number of infectious foxes after 12 months	maximum number of infected foxes over 20 yr period	minimum fox population size over 20 yr period
45.56	4.51	54.28	1000
27.72	3.61	33.25	1074
31.68	5.57	32.70	1000
16.87	3.91	17.14	1149
23.47	5.01	23.48	1068
71.74	12.12	86.52	965
19.85	3.19	20.55	1151
11.07	3.43	12.80	1104
98.40	10.35	112.97	939
19.76	2.15	23.39	1146
48.38	7.76	52.05	981
21.93	3.25	25.64	1151
23.33	4.41	23.56	1119
28.87	3.67	31.36	1062
38.50	7.53	44.62	990

Table 2.2 (continued).

number of infected foxes after 12 months	number of infectious foxes after 12 months	maximum number of infected foxes over 20 yr period	minimum fox population size over 20 yr period
37.04	8.11	45.19	1092
21.73	4.31	21.87	1124
23.05	3.48	30.00	1126
61.51	11.07	63.87	1038
30.77	6.74	32.13	1055
50.49	7.86	68.47	984
27.29	5.34	33.09	1088
48.74	10.84	50.22	1064
29.51	4.96	32.68	1108
50.11	14.16	51.60	954
23.14	3.61	23.39	1054
31.85	7.57	31.94	1019
37.94	4.39	45.18	1005
36.12	6.95	39.62	1015
34.13	5.16	42.98	1067

Table 2.3.

Co-ordinates of the Biplot for the red fox rabies model.

matrix G of the Biplot				matrix H of the Biplot	
-0.24	-0.88	0.37	0.22	-0.39	0.29
0.54	-0.12	-0.91	0.31	-0.24	0.48
0.25	0.16	0.37	0.11	-0.02	-0.06
0.97	0.58	-1.31	0.85	0.04	0.19
0.61	0.48	0.78	0.22	0.09	0.00
-2.05	-0.86	0.05	0.54	0.01	-0.05
0.93	0.30	0.07	-0.50	-1.11	-0.21
1.20	0.73	-0.08	0.14	-0.97	0.27
-2.81	-2.45	0.10	-0.21	-0.98	-0.26
1.00	0.04			0.06	0.03
-0.61	-0.30				
0.81	0.14				
0.69	0.37				
0.53	-0.09				
-0.27	0.08				
-0.30	0.21				
0.75	0.43				
0.70	0.05				
-1.40	-0.27				
0.16	0.41				
-0.90	-0.71				

Table 2.4.

Variance-covariance matrix for the parameters and output variates in the red fox rabies model.

P1	8.10					
P2	8.75	12.22				
P3	0.75	0.86	2.31			
P4	0.73	1.08	-0.41	2.52		
P5	-1.13	-0.77	-0.17	0.15	1.71	
P6	0.54	0.69	2.19	-0.13	-0.20	2.16
V1	11.58	6.17	1.80	-2.76	-2.98	0.64...
V2	11.33	6.71	-2.21	1.64	-2.26	-2.77...
V3	9.78	4.93	1.43	-3.42	-2.56	0.36...
V4	-0.31	0.08	0.04	0.23	0.07	0.15...

V1	38.94			
V2	29.24	35.78		
V3	34.55	24.47	31.57	
V4	-2.15	-1.87	-1.90	0.20

Table 2.5.

Correlation matrix for the parameters and output variates in the red fox rabies model.

P1	1.00					
P2	0.88	1.00				
P3	0.17	0.16	1.00			
P4	0.16	0.19	-0.17	1.00		
P5	-0.30	-0.17	-0.09	0.07	1.00	
P6	0.13	0.13	0.98	-0.06	-0.11...	
V1	0.65	0.28	0.19	-0.28	-0.37...	
V2	0.67	0.32	-0.24	0.17	-0.29...	
V3	0.61	0.25	0.17	-0.38	-0.35...	
V4	-0.25	0.05	0.05	0.33	0.13...	
	P1	P2	P3	P4	P5	
P6	..	1.00				
V1	..	0.07	1.00			
V2	..	-0.32	0.78	1.00		
V3	..	0.04	0.99	0.73	1.00	
V4	..	0.23	-0.77	-0.70	-0.76	1.00
		P6	V1	V2	V3	V4

CHAPTER THREE

DEFINITION AND PROPERTIES OF A COEFFICIENT OF SENSITIVITY FOR MATHEMATICAL MODELS

3.1. Introduction.

Preliminary sensitivity analysis of the fox rabies model, by means of the Biplot, conveys to the decision maker a great deal of information in concise form. The model has six parameters, and produces four output variates, yet the main features of the output uncertainty can be determined from a single graph. However, this is achieved by relying to an extent on subjective interpretation, approximation, and certain assumptions. These are discussed more fully in Chapter Six. For the purposes of the present Chapter it is sufficient to note that, in order to supplement the information gained from the Biplot, the decision maker will frequently need more objective numerical descriptions of the uncertainty inherent in the model.

As concluded in Chapter One, such descriptions are best computed from the generated sample of model output. In the literature conventional statistical measures such as ranges, standard deviations, and variances are often used for this purpose, but these measures do not describe uncertainty as such. Since the decision maker is interested in a description of uncertainty, an alternative,

probabilistic measure would seem to be preferable. Such a measure is proposed in this Chapter.

3.2. A specific measure of uncertainty.

How does a decision maker think about uncertainty in the fox rabies model? No single answer can be given to this question, but a decision maker who is responsible for planning and organising a scheme for the control of a rabies outbreak might react as follows.

A successful control scheme must be able to cope with the expected level of rabies in a fox population, and should be sufficiently flexible to cope with the range of levels thought likely. The decision maker may therefore organise a control scheme that will comfortably contain a range of levels of the disease. Given that the scheme is designed to cope with such a range, it is reasonable to suppose that, in reflecting upon the uncertainty inherent in the model output, the decision maker, rather than thinking about variances or standard deviations, will think about how likely it is that the level of rabies in a fox population will be outside the range which his control plans are designed to encompass. This line of argument suggests that a suitable measure of the uncertainty inherent in model output might well be based on the estimated probability that an actual result will fall outside a certain pre-determined range, this range being selected by the decision maker according some practical criteria. Such a measure may be defined as follows.

3.3. Generating data for sensitivity analysis.

Consider a mathematical model having p parameters x_i (i=1...p) and v output variates y_j (j=1...v). Let the vector $(x'_i ; i=1...p)$ be the vector of baseline or best estimate parameter values, and the vector $(y'_j ; j=1...v)$ be the corresponding vector of output variates. Data for sensitivity analysis may be generated by assigning a joint probability distribution to the p model parameters, and using this distribution to generate, say, n sets of parameter vectors $(x_i ; i=1...p)_r$ (r=1...n). Each of the n parameter vectors may then be used in turn in the mathematical model to generate a corresponding vector of outputs. Denote the rth parameter vector and its corresponding output vector $(x)_r$ and $(y)_r$, respectively.

3.4. Definition of a coefficient of sensitivity.

Consider in turn each of the vectors $(y)_r$ (r=1...n), generated as specified above. For each vector assign a score

$$s_r = \sum_{j=1}^{v} (d_j/v) \qquad (3.1)$$

where

$$d_j = 0 \text{ if } | y_j - y'_j | \leqslant ky'_j$$

$$d_j = 1 \text{ if } | y_j - y'_j | > ky'_j \qquad (3.2)$$

then the coefficient

$$S_k = \sum_{r=1}^{n} (s_r w_r) / \sum_{r=1}^{n} (w_r) \qquad (3.3)$$

where w_r is a weighting coefficient (see below), is defined to be the 100k% sensitivity coefficient of the v outputs for the model.

3.5. Properties of the proposed coefficient.

The above coefficient is defined as the proportion of model outputs that lie outside a range centred on the baseline output value. This conceptually simple measure of model sensitivity has several appealing properties:

(i) the coefficient is bounded by the values 0 and 1. This standardisation makes interpretation much easier and avoids the confusion that can arise with coefficients that are unbounded and may differ by several orders of magnitude.

(ii) the coefficient is defined for models having any number of parameters, and any number of output variates, and may also be calculated separately for each output produced by the model.

(iii) because the coefficient is independent of scale in the sense of (i) above, it is easy to compare the sensitivities of different models.

(iv) the derivation of the coefficient makes no assumptions about the form of the induced relationship between parameters and model outputs.

(v) the coefficient is simple to compute once sensitivity data are generated - the computation may be done manually

(vi) the coefficient has an obvious and immediate probabilistic interpretation, being based on the proportion of model outputs that lie outside a certain range.

3.6. The weighting coefficient.

The quantity w_r in (3.3) above is a weighting coefficient for the vector $(y)_r$. In some cases it may be appropriate to weight each output vector equally, in which case $w = 1 \ \forall \ r$, and (3.3) reduces to

$$S_k = \sum_{r=1}^{n} (s_r/n) \qquad (3.4)$$

However, it may sometimes be more appropriate to weight each vector of outputs according to the chance that such a vector may be realised i.e. to use as a weight w_r , a quantity proportional to the corresponding likelihood of the vector $(x)_r$.

3.7. An example.

The calculation of the sensitivity coefficient described above may be illustrated using the data output by the red

fox rabies model, and presented in Table 2.2. The first step is to choose which value of k to use in computing the coefficient. Here it is supposed, purely for the purposes of illustration, that a decision maker has designed a rabies control plan that will control an outbreak of the scale that might otherwise give rise to a situation in which the number of infected foxes after a 12 month period would be 23 (the "baseline" output from the fox rabies model), and that the plan would be adequate to cope with this level plus or minus 20%. The decision maker would then have a natural interest, when assessing his uncertainty, in the 20% sensitivity coefficient, as defined above.

The calculation of a 20% sensitivity coefficient for this model proceeds as follows. The first step is to define, for each output variate, a region centred on the baseline output and bounded by values of 0.8 and 1.2 times the baseline value. The baseline values and appropriate regions for each of the four output variates are given in Tables 2.1 and 3.1. Then, for each output vector in turn, the score s_r is computed, as described above. Thus, for the first output vector, the value for the number of infected animals in the population after a simulated 12 month period, and the value of the maximum number of infected animals in the population after a simulated 20 year period, both lie outside the corresponding 20% range. The score for this vector is therefore 2/4. The scores for other vectors are calculated similarly, and the resultant scores converted into the sensitivity coefficient defined above. Since independent uniform distributions were assigned to

the parameters used to generate the output vectors, each vector has been weighted equally. The resulting 20% sensitivity coefficients, for each output variate separately and for the model as a whole, are given in Table 3.2. From this table it can be seen that the conclusions drawn from the Biplot in Chapter Two are substantiated - the first three variates output by the model, which in the Biplot were shown to have the greatest variance, have the highest sensitivity coefficients, whereas the minimum fox population size, shown by the Biplot to have low variance, has the lowest sensitivity coefficient. The decision-maker can therefore be confident about the model's predictions concerning the minimum fox population size, but much less confident about single predicted values of the other three variates, since these are shown to have considerable uncertainty attached to them.

3.8. Summary.

Note that the use of a 20% range above was purely for the purposes of illustration, and that this should not be taken to imply that such a range would be an appropriate one to utilise in designing rabies control plans. In practice, control plans would be designed to take into account many factors in addition to those about which the rabies model provides information. Nevertheless, the principle proposed above would still apply. The fact is that, in practice, management policies are frequently designed to cope with a range of possible situations, and, given this, it is natural when describing uncertainty to think about the

probability that a result will be obtained that is outside this range.

The values of the sensitivity coefficient defined above all lie, by definition, between 0 and 1, with higher values indicating that a greater proportion of the generated output lies outside the range defined. Confining the values in this way improves the utility of the coefficient, since it becomes easy to compare the sensitivity of different outputs within a model, or between different models (see e.g. McCuen, 1973). A 20% sensitivity coefficient of, say, 0.5, immediately conveys something of the uncertainty inherent in the output - it says that 50% of the output values generated by the model lie outside the range bounded by 0.8-1.2 times the baseline. The higher the value of the coefficient the greater is the dispersion of model output values, and the greater the uncertainty in the output. Thus, for example, in the rabies model above the least sensitive output is the minimum size of the simulated population - no values of this latter output lie outside the 20% range.

A further advantage of the definition proposed above is that the choice of k permits the "severity" of sensitivity analysis to be selected by the modeller. A severe test of a model, for example, might be made by calculating the 1% sensitivity coefficient. This would indicate the proportion of model outputs that lie outside the corresponding baseline $\pm$ 1%. A low value for such a coefficient would strongly suggest that such a model is very robust to the

uncertainty that exists about its parameter values. The requirement for the modeller to choose k forces a definition to be made, at least implicitly, of the level of uncertainty in output that is considered to be important.

In the case of the red fox rabies model, the decision maker derives from the above coefficient a simple easily interpreted practical description of uncertainty - if his plans are designed to cope with ranges of problems within plus or minus 20% of the baseline model output, he can see at a glance that there is an estimated probability of about 0.5 that the true situation may lie outside these limits.

Table 3.1.

Regions centred around baseline values of model output and bounded by values ± 20% of the baseline.

output	region
no. foxes infected after 12 months	18.4 - 27.6
no. foxes infectious after 12 months	3.2 - 4.8
max. no. of infected foxes in 20 yr period	24.8 - 37.2
min. population size over 20 yr. period	910 - 1366

Table 3.2.

20% sensitivity coefficients for the fox rabies model of Anderson et.al. (1981), for the four output variates described in the text.

	20% sensitivity coefficient
no. foxes infected after 12 months	0.70
no. foxes infectious after 12 months	0.63
max. number of infected foxes in 20 yr period	0.70
min. population size over 20 yr period	0.00
whole model (all four output variates)	0.51

CHAPTER FOUR

SENSITIVITY ANALYSIS OF MATHEMATICAL MODELS IN A DECISION-THEORETIC FRAMEWORK

4.1. Introduction.

The methodology proposed in the preceeding two Chapters has been concerned with providing the decision-maker with information about the uncertainty inherent in the output from a mathematical model. A decision maker, however, not only needs to understand such uncertainty, but often needs to take it into account when using the results from a model. It would clearly be advantageous if the methods used to provide descriptions of uncertainty could also contribute to the process of working with uncertainty, but most methods of sensitivity analysis do not lend themselves to this. In this Chapter it is demonstrated that the sensitivity measure proposed in Chapter Three not only has value in general description of uncertainty, but may in some instances be directly employed in a formal decision-making process arising from the use of a mathematical model.

Consider a region of model output defined by the vectors $(y'_j + ky'_j \; ; \; j=1...v), (y'_j - ky'_j \; ; \; j=1...v)$. The proportion of model outputs that lie outside this region is the 100k% sensitivity coefficient for the model. Suppose

that this region defines a "state-of-nature" which, if it were known to be true, would lead a decision-maker to take action A. If the true state of nature lies outside this region, then suppose the decision-maker would take action B. The decision problem occurs because the true state of nature is not known. However, the 100k% sensitivity coefficient may be interpreted probabilistically, since it is the proportion of model outputs lying outside the region defined by the vectors. Similarly, the value $1-S_k$ may be taken to approximate the probability, as indicated by the model, that the first state of nature, implying action A, is true. Thus S_k , in addition to its other advantages, may be directly employed, in conjunction with a set of possible actions, and their utilities under different possible states of nature, in a conventional decision-theoretic framework (see e.g. Lindgren, 1971). The use of the coefficient in this way will be illustrated in this Chapter, by consideration of another ecological mathematical model - a model designed to aid the evaluation of methods of controlling rats on agricultural premises.

4.2. The rat control problem.

Rats are major agricultural pests in most parts of the world. Though the economic value of the damage they cause has not objectively been quantified, it is undoubtedly substantial, and much effort is expended in developing new methods of rat control. In the U.K., rat control is carried out using acute or anticoagulant rodenticides. These are

usually successful in the short term, but problems such as the ocurrence of genetic resistance to anticoagulants (Greaves and Ayres, 1967) motivate the search for new control techniques ;for example, chemosterilants. New rat control methods are investigated by field trials of candidate compounds on the farm. The results of such trials are reliable, but the process is costly and time consuming, and limited resources mean that only a small number of compounds can be tested in this way. Most are examined in the laboratory, and are only subjected to field trials if their characteristics, as revealed in the laboratory, are thought to merit this.

The use of a mathematical model allows potential control compounds to be evaluated theoretically, as well as in the laboratory, and improves the screening process by indicating whether or not the characteristics of a compound, as determined from the laboratory, are such as to make it a potential competitor with conventional rodenticides. Such a mathematical model has been developed (M.A.F.F., 1980, 1981).

4.3. The farm rat control model.

The model is a modification of the well known logistic model of population growth. Though developed many years ago, this form is still used in modern pest control modelling, and is considered to be particularly appropriate for farm rat populations (Emlen et. al., 1948; Davis,

1950).

The basic form of the logistic model may be written:

$$dN/dt = N(a-b-yN) \quad (4.1)$$

where a is the per capita birth rate, b the reciprocal of the life expectancy, and N the size (or density) of the population. The parameter y measures the severity of density dependent control of rates of increase, and is derived in practice from the observed carrying capacity of the environment.

Farm rat control methods operate by decreasing the life expectancy of a proportion of the animals (those that consume a rodenticide bait), and/or reducing the birth rate for a period of time (in the case of chemosterilants).

The basic logistic model may be modified as follows to incorporate representations of these processes:

$$dN/dt = caN - (pB+(1-p)b+yN)N \quad (4.2)$$

where p is the proportion of animals in the population that consume a rodenticide bait, B the reciprocal of the life expectancy of animals that have consumed a rodenticide bait, and c measures the reduction in average birth rate induced by chemosterilisation. These parameters can be determined in the laboratory for any particular candidate

control compound.

It is important to stress that this model was under continuous development by scientists in the Tolworth and Worplesdon Laboratories of the Agricultural Science Service during the course of the work reported here, and that, accordingly, different versions of it were used at different stages during the course of the present work. For some purposes further modifications were made, for example to incorporate time lags. However, the above form illustrates the essential features of the model.

4.4. A further example of the use of the sensitivity coefficient.

The farm rat control model is used to evaluate potential control methods by simulating population growth under controlled conditions, and measuring the success of the simulated control treatment by comparing the resultant population sizes with those generated by the model when $c = 1.0$, $p = 0.0$, (representing no population control). The result derived can be compared with that given by a conventional anticoagulant rodenticide treatment ($c = 1.0$, $p = 0.95$, $B = 30.0$). A typical result derived from the model, comparing this conventional rodenticide with a combined chemosterilant/rodenticide ($p = 0.65$, $c = 0.40$, $B = 30.0$), shows that the conventional rodenticide resulted in a total population reduction of 535 animals until recovery of the population to its pre-treatment level,

whilst the combined chemosterilant/rodenticide yields a population reduction of 542. An illustration of the output of the model is given in Figure 4.1.

4.5. Describing uncertainty in the model.

This result is strictly dependent on the values of the parameters used in the model: the values of a, b, and y. These can only be estimated from field data, and the true values are not known precisely; they are, in any case, subject to stochastic variation. The values of c, p, and B can be determined accurately in the laboratory, and are treated here as known. Because there is uncertainty about the true values of the model parameters, there is uncertainty inherent in the result of the simulation, and this must be assessed and described in some appropriate way if the results of the model are to be used to practical advantage.

Table 4.1. contains a range of values output by the model, generated by Monte Carlo error propagation, comparing the standard anticoagulant rodenticide with the experimental combined chemosterilant/rodenticide. This data was derived in repeated runs of the model, each with a randomly selected set of values for a,b, and y. The values for these parameters were selected from uniform distributions, each centred on the baseline value, and having a range equal to 60% of the baseline value. The baseline values were estimated from unpublished data held at the Tolworth

Laboratory of the Agricultural Science Service.

The 20% sensitivity measure (as defined in Chapter Three) for the chemosterilant/rodenticide output produced by the model is 8/30 = 0.27. This says simply that 27% of the runs of the model produced results that deviated from the baseline (= 542) by more than 20%.

As noted in Chapter Three, it may be possible in some applications to select a value of k that has some practical meaning. This is so in the present case. Not only does the 20% sensitivity measure convey general information in the sense described above, but the model output range associated with this definition (434-650) corresponds closely with the range of success achieved by the standard rodenticide (436-674). Thus the 20% sensitivity measure of 0.27 can be interpreted at a more practical level - as being approximately equal to the proportion of simulation results in which the chemosterilant/rodenticide produced results outside the range generated by the conventional rodenticide treatment.

4.6. The sensitivity measure and decision making.

It is also possible to carry the information contained in the sensitivity measure forward into a formal decision making context.

Consider the following decision problem: an organisation

has produced a candidate combined chemosterilant and rodenticide compound, which laboratory tests indicate has the characteristics described above. The organisation has used the mathematical model to evaluate the potential of the compound, and has produced the results of Table 4.1. The 20% sensitivity measure is relatively small, suggesting that the candidate compound has a reasonable chance of performing acceptably. However, a specific decision must be taken about its future. The following three possible actions might be considered in practice:

(i) develop the compound immediately for the pest control market

(ii) reject the candidate compound

(iii) subject the compound to field trials, and then either develop or reject, according to the performance realised in the trials (results of field trials being regarded as very reliable).

For convenience, the possible "true states" of the candidate chemosterilant/rodenticide may be restricted to three:

(i) candidate compound is less effective than rodenticide

(ii) candidate compound is as effective as rodenticide

(iii) candidate compound is more effective than rodenticide.

The financial implications of the various possible actions might be as follows:

(i) cost of developing candidate compound for the market £50000.

(ii) cost of field trials of candidate compound £10000.

(iii) income expected from compound more effective than rodenticide £300000

(iv) income expected from compound as effective as rodenticide £250000. (this would not be expected to sell as well as a compound more effective than a conventional rodenticide)

(v) income expected from compound less effective than rodenticide £0.

These figures are approximations only and are simplified to some extent; nevertheless their relative magnitudes are realistic. The decision problem may thus be summarised as in Table 4.2.

Table 4.2 has three of the components of a standard decision problem: a set of actions, a set of possible true

states of nature, and a value associated with each action/state combination. An advantage of the sensitivity measure defined above is that the definition utilises information about the probabilities of the various states, as derived from the mathematical model. State 2 in Table 4.2 - that the candidate compound is as effective as a rodenticide - has a probability that may be directly approximated by subtracting the 20% sensitivity measure from 1.0 - yielding 0.73. This is the proportion of simulation runs that produced a result comparable with the performance of the simulated rodenticide. The proportion of outputs that lie outside this range is, by definition, the 20% sensitivity measure, and this approximates the probability of the other two states. The simulation results in Table 4.1 show that 5/30 runs produced results lower than the lower bound of the 20% sensitivity range, and 3/30 produced results higher than the upper bound. The probabilities of the other two states are therefore estimated to be 5/30 = 0.17, and 3/30 = 0.10, respectively.

Given these probability estimates, the decision problem can be solved in the conventional way. The Bayes action for this problem is action 1 in Table 2: develop the candidate compound for the market. This action has expected gain £162500, compared with an expected gain of £161000 for action 3, which is next best.

4.7. Extent of model support for a decision.

Since the purpose of a sensitivity analysis is to describe uncertainty, and since uncertainty is inherent in the decision process itself, it is appropriate to continue the sensitivity analysis to the decision making stage of the modelling process. A method for doing this is now proposed.

The probabilities of each state in Table 4.2 are derived from the mathematical model, and are themselves uncertain quantities. This is because the model itself may be an inadequate representation of the farm rat population, because the results generated are only a sample, because the sample itself is based on a model of the uncertainty about parameter values, and because the range of the 20% sensitivity measure does not correspond precisely with that of the conventional rodenticide. Each one of these sources of uncertainty is acceptable in itself. Nevertheless it is appropriate to describe their possible joint implications for the decision process.

Isaacs (1963) and Fishburn et.al.(1968) suggested that an appropriate measure of the sensitivity of a decision process to errors in probability estimation, was the minimum Euclidean distance between the baseline probability set, and a set which would change the Bayes action. Thus for the above process, the baseline probabilities are (0.17,0.73,0.10) - giving action 1 as the Bayes action, whereas the set (0.21,0.69,0.10) would give action 3 in

Table 4.2 as the Bayes action.

The Euclidean distance between these sets is 0.057. Though this seems small, it is not really clear how to regard the value in the context of the decision. There is no "reference" distance to relate it to. For example, it would not be easy to compare this distance with a similar measure for a different decision process - something that might be necessary if a resources were to be allocated to either one project or another. A measure that is easily interpreted and which facilitates such comparisons is more useful.

Such a measure may be derived by relating the minimum distance to the maximum possible distance for the given process. For most conventional distance metrics this maximum will be given by replacing the smallest probability in the baseline set by 1.0, and the others by 0.0. Thus a maximum distance from (0.17,0.73,0.10) is (0.0,0.0,1.0). The ratio of the minimum distance to the greatest possible distance may be subtracted from 1.0 to give a coefficient, with values between 0 and 1, that measures the sensitivity of the decision process. Values of this coefficient near 1.0 indicate a decision that is sensitive to errors in the probabilities - a small change in these probabilities may change the Bayes action, and this would suggest that the model upon which the decision is to be based would need to be refined as much as is possible in order to support the decision. Values near 0 suggest a decision that is insensitive, and which might be taken on the basis of the

model as it stands. The value of this coefficient for the above decision process is 0.952, suggesting that this decision is very sensitive to errors in probability estimates, and indicating that the model and the parameter estimates must be refined as much as possible before taking a decision based on the results of the model.

4.8. Summary.

In the case of the farm rat control model, specific decisions may have to made about the future of certain compounds that are being evaluated. The main interest of the decision maker is in predicted performance that lies outside the range normally achieved by a well defined standard control treatment. The sensitivity measure proposed in Chapter Three thus has a natural application in the farm rat model. It provides a practical description of uncertainty that forms a natural component of the decision process that arises from the model.

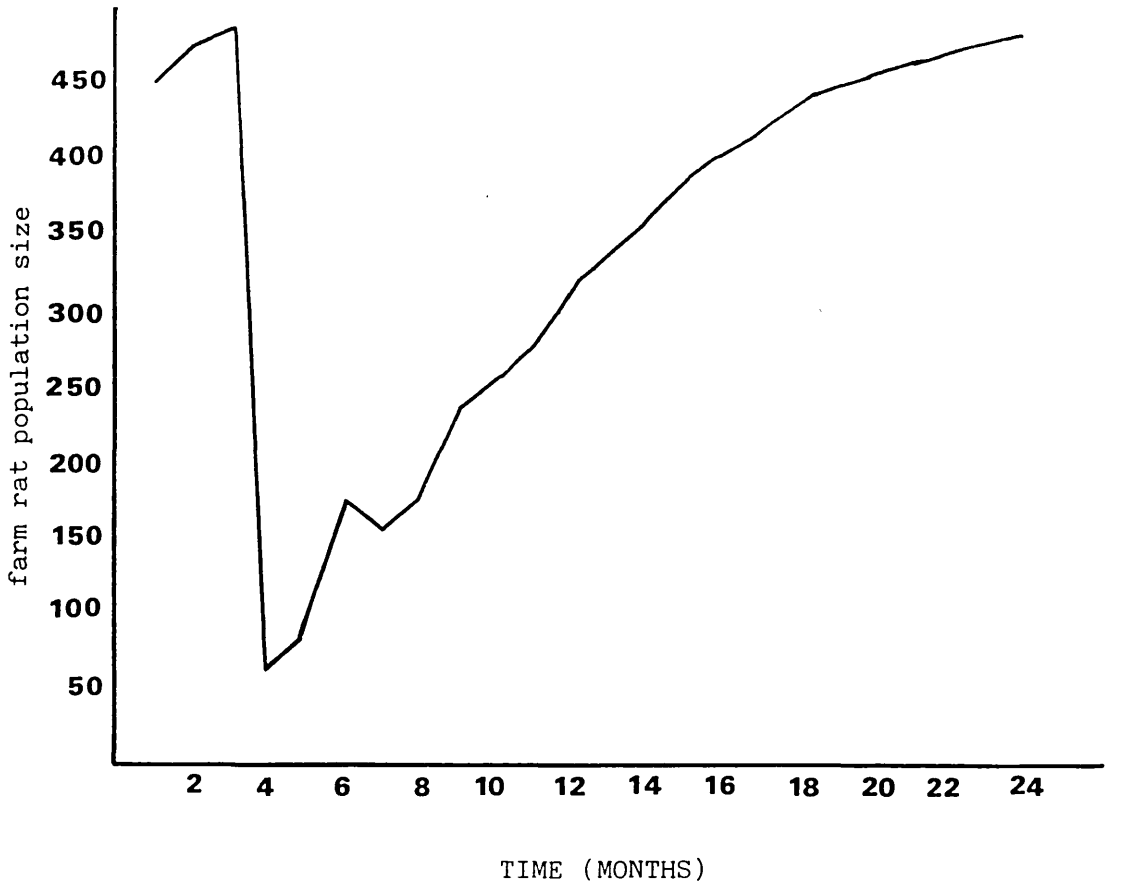

Figure 4.1. An example of a time track of rat population sizes, generated by the farm rat control model, and illustrating the effect of a simulated combined rodenticide/chemosterilant treatment implemented at month 3. The rapid decline in numbers is followed by a slow recovery, inhibited by the effects of the chemosterilant.

Table 4.1.

Measures of the reduction in farm rat population size (over a period until recovery of the population to its pre-control size) predicted by the farm rat model, for a standard rodenticide and an experimental combined chemosterilant/rodenticide. These values were derived in repeated runs of the model, each with a randomly selected set of parameter values; the first result in each case was derived with parameters set at baseline values.

simulated rodenticide treatment		simulated combined chemosterilant rodenticide treatment	
535	554	542	704
461	569	555	520
462	436	594	640
641	604	438	527
552	580	405	545
526	463	405	522
470	484	613	520
605	646	547	471
521	515	637	663
579	557	446	613
504	674	632	685
509	542	504	494
654	479	547	473
599	477	497	410
486	529	430	417

Table 4.2.

Components of a decision problem involving a potential pest control compound.

	possible states		
possible actions	(1) compound less effective than rodenticide	(2) compound as effective as rodenticide	(3) compound more effective than rodenticide
(1) develop compound for pest control market	-£50000 development costs	-£50000 development costs +£250000 expected income	-£50000 development costs +£300000 expected income
(2) reject compound	£0	-£250000 expected income	-£300000 expected income
(3) subject to field trials then develop or reject according to results	-£10000 field trial costs	-£10000 field trial costs -£50000 development costs +£250000 expected income	-£10000 field trial costs -£50000 development costs +£300000 expected income

CHAPTER FIVE

AN INFORMATIONAL MEASURE OF THE INFLUENCE OF PARAMETER UNCERTAINTY ON THE OUTPUT FROM MATHEMATICAL MODELS

5.1. Introduction.

In addition to the need to adequately describe the uncertainty in model output, it is usually necessary at some stage in the development of a mathematical model to assess the extent to which each parameter can be said to be contributing to the uncertainty. This is often done with a view to deciding which parameter estimates in a model might best be refined in further experimentation or measurement.

Ideally, any method for assessing the importance of a particular parameter should possess the following characteristics:

(i) it should require no assumption to be made about the form of the relationship between parameter and model output

(ii) it should be easy to compute

(iii) it should make use of the information already available from earlier stages of the sensitivity study.

Such a measure will be described in this Chapter.

5.2. Probability distribution of model output.

In Chapter Four it was emphasised that the sensitivity coefficient proposed in Chapter Three is a simple function of the probability distribution of model output. The value of the coefficient in both informal and formal decision making was illustrated. It was also shown that the decision made from the model output could be changed by small alterations in the output distribution.

The output probability distribution is therefore central to general description of uncertainty, to formal decision making using a mathematical model, and to sensitivity studies of the decision process itself. For these reasons, it is desirable that a method of measuring parameter influence should also make use of the model output distribution.

5.3. A further sample of output from the farm rat model.

The farm rat model was used above to provide estimates of the probabilities of each of three possible "states" of a candidate combined chemosterilant/rodenticide compound. These were as follows:

(i) candidate compound less effective than standard rodenticide (estimated probability = 0.17)
(ii) candidate compound as effective as standard rodenticide (estimated probability = 0.73)
(iii) candidate compound more effective than rodenticide (estimated probability = 0.10)

At the time that the work described in this Chapter was carried out, a new version of the farm rat control model was in use at the Tolworth Laboratory of the Agricultural Science Service, and accordingly a further sample of output from the new version of the model, again measuring the performance of the candidate chemosterilant/rodenticide compound, together with the associated values of four parameters from the mathematical model, was generated. The four model parameters considered in this illustration are:

(i) the birth rate of farm rats
(ii) the death rate of farm rats
(iii) the death rate of rats that have consumed a rodenticide bait
(iv) the duration of the effect of the candidate chemosterilant (measured in months)

Note that, at this stage in the development of the model, the parameter B is being treated as an uncertain quantity, and a new uncertain quantity, the duration of the effect of

the chemosterilant, has been introduced into the model. Uncertainty about each of these parameters was described by uniform distributions, each centred on the baseline value and having a range equal to 60% of the baseline value. A sample of 100 runs of the model, each with a random selection of parameter values, was used to generate the data in Table 5.1.

The probabilities of each of the above three possible states are estimated from this particular sample to be 0.19, 0.73, and 0.08, respectively. Note that these estimates are slightly different from those in Chapter Four, but that with these probability estimates, the 20% sensitivity coefficient is still estimated to be 0.27, and that the Bayes action in Table 4.2. would be action 1. (It should be noted that, as stated above, the farm rat model was under continuous development during the course of this work, and the differences in these probabilities is partly attributable to the fact that the model used for illustration of the work in this Chapter is a later, more developed version of that used in Chapter Four).

5.4. An informational correlation measure.

A measure of correlation which makes direct use of probabilities is available in the statistical literature, though it has been rarely used and is little known. It is proposed here as a suitable method of measuring the

influence of a parameter on model output uncertainty. The coefficient is referred to as the "information gain", or sometimes more simply as the "informational correlation measure", and a simple description of its derivation and properties is given by Linfoot (1959). For convenience in the computation of the informational correlation measure, it is helpful, though not essential, to discretise the output and parameter distributions. This is a natural step to take in view of the discretisation already employed in the decision process in Chapter Four; in fact the same categorisation as employed in the decision process is an appropriate one to use in measuring parameter influence.

Denoting the probabilities of each of the above three states of the chemosterilant/rodenticide as p_i, $i = 1...3$, and partitioning the observed distribution of a particular model parameter also into three regions, each of equal width, with probabilities q_j, $j = 1...3$, the informational correlation between the output and that parameter is defined to be:

$$I = \sqrt{1.0 - e^{-2I'}} \qquad (5.1)$$

where

$$I' = \sum_{ij} \left(r_{ij} \log r_{ij} - p_i q_j \log p_i q_j \right) \qquad (5.2)$$

and the r_{ij} are the observed joint probabilities of each of

the nine possible parameter/output groupings.

5.5. Properties of the informational correlation measure.

The informational correlation measure has a number of properties which make it particularly suitable for use in sensitivity analysis of mathematical models:

(i) it requires no assumption to be made about the form of the relationship between parameter value and output value
(ii) it is a function of the probabilities that are used to define the sensitivity coefficient above, and also used in decision-making
(iii) it is easy to compute, particularly when the computation follows earlier derivation of the probabilities for the methods described earlier in this thesis.

The informational correlation measure has been described in a number of slightly different forms in the statistical literature, the most accessible being the paper by Linfoot (1959). Despite its appealing properties, and its simplicity, it has, like many concepts from the field of information theory, not been widely employed in applied statistics. Linfoot (1959) discusses the information theoretic derivation of the measure, and its relationship with the conventional Pearson correlation coefficient.

The informational correlation measure I lies between 0 and 1, and is equal to zero only when $r_{ij} = p_i q_j$ for all i,j. The greater the "information gain", the greater the influence a particular parameter has on the model output distribution. The measure I' is, in fact, interpretable as the gain in information (in the formal sense of "information") that accrues from the knowledge that the joint distribution of output and parameter is described by the probabilities r_{ij} rather than by the products $p_i q_j$.

5.6. An example.

As an example of the calculation of an informational correlation measure, consider the model output in Table 5.1, and its relationship with the first of the model parameters - the birth rate of farm rats.

Partition the output distribution into the categories used in formulating the decision problem in Chapter Four:

(i) category one: output less than 434 (p = 0.19)
(ii) category two: output between 434 and 650 (p = 0.73)
(iii) category three: output greater than 650 (p = 0.08)

and partition the observed parameter distribution into three regions, each of equal width:

(i) category one: parameter less than 0.874 (q = 0.32)

(ii) category two: parameter between 0.874 and 0.996 (q = 0.32)

(iii) category three: parameter greater than 0.996 (q = 0.36)

The observed joint probabilities, derived from the data in Table 5.1., of each of the nine possible output/parameter category combinations are given in Table 5.2. The values of $p_i q_j$ can be computed directly and substituted into (5.1) above. The informational correlation measure is calculated to be 0.422.

The informational correlation measures for each of the four parameters in the model are given in Table 5.3., together with the conventional Pearson correlation coefficients, computed directly from the data in Table 5.1.

5.7. Interval estimates of the informational correlation measure.

For some purposes it may be valuable to model users to have interval estimates of the informational correlation coefficients, in order to aid assessment of the importance of each parameter's contribution to output uncertainty. No sampling theory is available for the informational correlation coefficient, however, and therefore, in order to derive interval estimates, the method of jack-knifing was applied to the data from Table 5.1.

A detailed description of the methodology of jack-knifing is beyond the scope of this thesis. A simple presentation of the methodology is given by Bissell and Ferguson (1975). The main purposes of the technique are (a) to enable interval estimates of quantities to be derived and (b) to reduce any bias that may exist in estimated statistics. Briefly, the method requires that the data from which the informational correlation coefficients were calculated (i.e. the data in Table 5.1) be split into a number of sub-groups, each which is, in turn, removed. A "psuedo-value" of the informational correlation coefficient is then calculated from the data remaining when a sub-group has been removed, then that sub-group is replaced, a second removed, and the process repeated. The "pseudo-values" thus derived are combined with the estimates made from the whole data set, to produce a "jack-knifed" estimate, together with its standard error.

The "jack-knifing" approach appears to work satisfactorily with estimates of informational correlation coefficients, and can be recommended for use when interval estimates of these coefficients are required. The jack-knifed estimates of the informational correlation coefficients, and their standard errors, are given in Table 5.3.

5.8. Interpretation of the informational correlation measure.

The higher the value of the informational correlation measure, the greater the influence the associated parameter has on the output distribution. Thus it can be seen from Table 5.3. that the fourth parameter in the model - the duration of the chemosterilant effect - has the greatest influence on the distribution of model output. Any attempt to reduce the uncertainty inherent in the output would therefore best be concentrated on attempts to measure the duration of the effect of the candidate chemosterilant/rodenticide with more precision.

Parameters 1, 3, and 2 have lesser influence, respectively, as the informational correlation measures decrease for these three sources of uncertainty. The Pearson correlation coefficients also suggest that parameter 4 has the greatest influence on model output, and that parameter 1 is next most important. The ranking of parameters 2 and 3, however, differs. It should be remembered that the Pearson correlation measure is only of potential value when the relationship between parameter and output is linear, or at least monotonic. The informational correlation measure may validly be applied whatever the form of this relationship.

5.9. Summary.

A natural extension of the use of probabilities in measuring sensitivity and in formal decision making is to use them directly to measure the influence that parameters have on model output uncertainty. The informational correlation measure does exactly this, and requires no restrictive assumptions to be made about the form of the parameter-output relationship. Employing this measure of parameter influence in the case of the farm rat control model allows the decision maker to make use of much of the information already computed in the earlier stages of the sensitivity analysis, and provides a reliable guide to further refinement of the model.

Table 5.1.

Sensitivity data from the farm rat control model.

success of treatment	birth rate	death rate	'B'	duration of chemosterilant effect
510	0.940	0.140	30.000	4.990
376	0.957	0.156	30.312	1.469
521	0.840	0.143	25.701	4.014
647	0.788	0.142	35.466	6.470
570	0.781	0.129	33.433	5.393
576	0.873	0.145	30.727	6.459
505	1.000	0.111	24.815	6.298
451	1.025	0.159	24.367	3.724
715	0.767	0.155	31.825	8.282
525	0.911	0.120	28.903	5.250
445	0.982	0.111	32.398	3.740
713	0.758	0.168	32.416	7.775
621	0.758	0.141	28.571	5.937
673	0.779	0.111	25.908	8.434
350	1.093	0.138	35.700	1.458
492	1.113	0.160	27.249	6.155
544	0.868	0.122	33.152	5.264
494	0.767	0.127	27.488	2.234
465	0.878	0.147	34.781	3.135
516	0.821	0.157	34.117	2.936
610	1.054	0.134	35.054	8.646

Table 5.1. (continued)

success of treatment	birth rate	death rate	'B'	duration of chemosterilant effect
452	0.794	0.114	31.789	1.929
569	0.875	0.122	27.275	6.726
365	0.975	0.144	33.794	1.314
604	0.861	0.113	27.214	7.979
535	0.774	0.118	28.748	3.824
473	0.874	0.162	24.072	3.193
659	0.917	0.168	31.679	8.195
550	1.018	0.113	35.037	7.511
734	0.755	0.141	25.803	8.595
641	0.837	0.144	28.499	7.043
527	0.847	0.147	25.159	4.055
473	0.897	0.159	26.174	2.381
370	1.080	0.128	32.458	1.936
430	1.076	0.144	26.385	2.656
313	1.031	0.150	24.362	1.051
509	1.046	0.140	27.057	6.019
350	1.074	0.159	32.169	1.481
630	0.890	0.161	25.387	7.864
547	1.062	0.147	28.893	7.227
452	1.011	0.138	33.512	3.950
578	1.066	0.160	33.989	8.185
397	1.088	0.129	28.954	2.327

Table 5.1. (continued)

success of treatment	birth rate	death rate	'B'	duration of chemosterilant effect
506	0.838	0.138	29.803	2.879
600	0.867	0.146	26.158	6.656
581	1.036	0.166	33.980	7.529
499	0.993	0.123	27.380	5.113
414	1.089	0.111	28.902	3.841
477	0.975	0.128	26.602	4.444
604	0.860	0.139	33.774	7.254
674	0.870	0.134	31.090	8.665
512	1.002	0.112	31.578	5.829
705	0.768	0.155	30.347	7.667
568	0.965	0.166	30.051	6.291
584	1.079	0.140	32.637	8.350
481	0.831	0.131	35.003	3.452
558	1.074	0.156	29.668	7.015
544	1.023	0.143	33.853	6.952
551	0.858	0.168	29.966	4.707
512	0.957	0.130	24.052	5.339
440	0.805	0.147	25.963	1.672
621	0.971	0.124	31.381	8.767
417	0.990	0.138	34.125	2.791
468	0.764	0.132	34.659	1.801
459	1.105	0.158	25.531	4.801

Table 5.1. (continued)

success of treatment	birth rate	death rate	'B'	duration of chemosterilant effect
564	1.097	0.162	29.764	8.135
310	1.110	0.130	31.409	1.262
433	1.043	0.148	31.365	2.494
410	1.099	0.142	29.210	2.457
510	0.891	0.149	29.114	4.343
567	1.028	0.133	30.820	8.041
587	0.874	0.131	25.627	6.452
661	0.970	0.165	31.721	8.719
627	0.783	0.156	24.379	5.675
594	0.902	0.121	30.539	7.783
492	0.939	0.135	34.689	4.072
498	0.920	0.154	26.349	3.002
295	1.108	0.112	29.278	1.049
495	1.108	0.131	29.298	5.693
493	1.034	0.163	31.241	5.296
562	0.781	0.124	32.392	5.130
495	1.039	0.158	34.974	5.112
597	0.814	0.162	24.115	5.506
492	0.969	0.163	27.592	3.738
475	0.902	0.133	26.450	3.032
452	0.920	0.123	25.677	3.884
575	0.813	0.135	27.139	5.567

Table 5.1. (continued)

success of treatment	birth rate	death rate	'B'	duration of chemosterilant effect
473	1.129	0.131	27.437	5.986
640	0.996	0.147	31.369	8.970
540	0.967	0.123	26.145	7.040
453	1.082	0.122	35.563	4.720
374	0.862	0.126	30.646	1.066
468	1.089	0.168	33.469	4.486
404	0.878	0.113	27.953	1.821
405	1.001	0.130	27.532	3.080
527	0.880	0.163	32.857	3.518
433	0.964	0.125	34.859	3.479
421	1.098	0.166	28.711	3.835
641	0.886	0.150	34.092	7.506
572	0.834	0.129	30.993	6.237

Table 5.2.

Joint probability distributions of output and parameters from the farm rat model.

	Output: measure of chemosterilant success		
	less than 434	434 - 650	more than 650
parameter 1			
<0.874	0.01	0.25	0.06
0.874-0.996	0.05	0.25	0.02
>0.996	0.13	0.23	0.00
parameter 2			
<0.129	0.06	0.21	0.01
0.129-0.149	0.08	0.29	0.02
>0.149	0.05	0.23	0.05
parameter 3			
<28.09	0.04	0.28	0.02
28.09-32.02	0.08	0.22	0.05
>32.02	0.07	0.23	0.01
parameter 4			
<23	0.08	0.19	0.01
23-26	0.03	0.22	0.01
>27	0.08	0.32	0.06
parameter 5			
<3.73	0.16	0.14	0.00
3.73-6.36	0.02	0.35	0.00
>6.36	0.01	0.24	0.08

Table 5.3.

Informational correlation measures, and Pearson correlation coefficients for the the model output and parameter values listed in Table 5.1.

informational correlation measure		Pearson correlation coefficient
whole data set	jacknifed (± s.e.)	
0.422	0.390(±0.078)	-0.513
0.207	0.117(±0.116)	0.187
0.217	0.130(±0.115)	0.000
0.643	0.642(±0.052)	0.878

CHAPTER SIX

DISCUSSION

6.1. The rabies and rat control models: a discussion.

The objective of the work reported in this thesis was to develop methods of uncertainty analysis, and though this has been done by considering two particular mathematical models, the development of the models themselves was outside the scope of this work. Nevertheless, some comments on them are included here.

Both models are intended as aids to the development of wildlife management policies. The rabies model can be made to produce predictions about various aspects of the dynamics of a fox population in which rabies is present. Such information is a vital part of the process of formulating plans for rabies control. The farm rat model is even more closely tied to practical pest management - it may be used directly to predict the success of particular control method.

In constructing both models, some reference to empirical data is made. However, these data are far from precise. The sources cited by Anderson et.al. (1981) provide the data on

which the rabies model is based, whilst the data used in the development of the farm rat control model is unpublished. The structure of both models is therefore strongly based on subjective belief and on the selection of convenient and simple functional forms, for which there is only limited empirical support. Nevertheless, this apparently tentative structure need not be without value, and for practical purposes it is preferable to judge a model as adequate or otherwise on the basis of the utility of its predictions, rather than the solidity of its foundations.

Both models adopt the logistic form as a basis. That is, in the absence of rabies or rat control, population growth follows the form:

$$dN/dt = N(a-b-yN) \qquad (6.1)$$

where a and b are per capita birth and death rates, respectively. The parameter y has slightly different interpretations in the two models. In the rabies model it is taken to reflect density dependent mortality, whereas in the farm rat control model it is used to measure the compound effects of density dependent mortality and changes in birth rate. The mathematical operation is, however, identical; the term yN acts to reduce the rate of increase of the population.

Anderson et.al. (1981) give no specific justification for the use of the logistic form. In the case of the farm rat control model, some support for this formulation is provided by the studies of Emlen et.al. (1948), and Davis (1950), who conclude that, in the field, rat population growth appears to follow a form that is adequately described by the logistic.

In both models, the logistic equation is developed to include representations of the specific processes involved in rabies propagation on the one hand, and in farm rat control on the other. The result of this is the appearance of extra parameters in both models. The rabies model then has six parameters: r, b, y, s, B, and a, whilst the farm rat control model has parameters a, b, and y, or, in the later version used in Chapter Five , a, b, y, B, and the duration of the chemosterilant effect. (Note that the symbols a, B, and y do not represent the same things in the two models). All these parameters are represented as temporally constant in both models - that is, they are fixed values, not subject to seasonal or cyclical fluctuations. This is not unreasonable in the case of the farm rat model, since rats on farms are known to breed more or less continuously throughout the year, and there are no known seasonal variations in mortality (see e.g. Bishop and Hartley, 1976). In the case of foxes, births are certainly seasonal, and so, probably, is mortality, but the complications that this induces may be avoided to some

extent by producing predictions at intervals of, say, 1 year or more. If the fine structure of a fox population is predicted using the rabies model, perhaps by generating population sizes at monthly intervals, then inaccuracies due to the absence of seasonal effects will be present in the predictions.

Also absent from both models is any representation of age structure, This is important only if the phenomena being studied (i.e. rabies and farm rat control), differentially affect different age classes, and, equally, if such effects can be exploited in management policies. Evidence of age-specific effects is inconclusive in both cases. Again, neither model explicitly incorporates a representation of the time lag that may be inherent in the density dependent responses of the population (though the version of the farm rat control model used in Chapter Five has such a feature built into the computer program used to solve the model). The importance of this omission again depends on the nature of the predictions that are being produced by the models - predictions of fine structure, over time scales that are short in comparison with the time lags, will be biassed by the omission, whereas longer term predictions will be less affected.

Possible improvements to the model may also be found at the more detailed level. For example, in the rabies model the rate at which susceptible animals become infected is

represented as being directly proportional to the product of the densities of the two classes concerned - the expression BDF codes this assumption. It may perhaps be more reasonable to suppose that this rabies transmission rate in fact reaches an asymptote at a given population density, and stays constant thereafter.

In the farm rat control model, it is assumed that animals either consume a quantity of rodenticide sufficient to increase their death rate from b to B, or do not, the proportions in the two categories being (1-p) and p respectively. It may be more reasonable to suppose that even a small intake of rodenticide will be enough to increase b for the animals concerned, and that a continuum of enhanced death rates is more realistic than a representation of just two possibilities.

In addition to the fact that the parameters in the model are uncertain quantities, it can therefore also be seen that there is a degree of uncertainty about the adequacy of the models as a whole. However, as was noted above, the objective of the work reported here was to develop methods for uncertainty analysis of the models as they stood, and not to develop the models themselves. The question of model adequacy is more fundamental than uncertainty analysis, though it should not be arbitrarily assumed that uncertainties about the model itself are necessarily more important than uncertainties about parameter values. The

question of model adequacy is, however, susceptible of pragmatic treatment: if a model produces a prediction which, even with its attendant uncertainty, proves to be of practical value, then the model is adequate for that purpose, regardless of its structure.

6.2. Theme of the proposed methods.

All the techniques proposed in this thesis have a common theme. Firstly, they all involve explicit description of parameter uncertainty by means of probability distributions. Next, a sample from the model output distribution is generated by repeated use of the mathematical model, with different sets of parameter values, randomly selected from the specified probability distribution. Then, some appropriate function of the output distribution is utilised to assess or describe the uncertainty in a simple way, based on as few assumptions as possible. The methods proposed have been designed so that, following explicit description of parameter uncertainty, and generation of a sample of model output, the decision maker can use this sample (i) in informal graphical sensitivity studies, (ii) to derive more objective coefficients, (iii) to aid decision making, and (iv) to measure parameter influence. The fact that all these things can be done using the methods proposed here is of great convenience and is an advantage of the methodology described above. It is shown that suitable use of the

output distribution can result in informative graphical descriptions of uncertainty for models having many parameters and output variables, and in a simple numerical measure of output uncertainty. It is further demonstrated that this measure utilises information that can be later employed in formal decision making, and in developing measures of the extent to which each parameter influences output uncertainty. It is suggested here that this general philosophy represents the most suitable approach to the study of uncertainty in mathematical modelling.

These methods constitute an addition to existing approaches to sensitivity analysis, and no claim is made that they are uniformly superior. They go some way towards alleviating the problems highlighted at the end of Chapter One, and have properties that make them appealing in some practical circumstances. However, no single method of sensitivity analysis can be guaranteed to be suitable for all types of mathematical model, since the methods adopted must be closely allied to the practical needs of the modelling problem.

6.3. The Biplot: a discussion.

The Biplot is a simple graphical display of the sample generated from the model output distribution. It was shown in Chapter Two that such a display conveys a large amount of information about the uncertainty inherent in the output

from a model, particularly in the case of a model having several outputs and parameters. The method makes no explicit assumptions about the nature of the output distribution, and should accordingly be applicable in principle to any model for which such a distribution can be derived. One implicit assumption, however, concerns the linearity of the relationship between parameter values and model output. This assumption is implicit since the Biplot uses approximations to the coefficients of correlation between vectors in the plot. Thus the use of the Biplot to make inferences about the relationships between outputs and parameters is valid only if these relationships are roughly linear. This assumption has been checked for the rabies model - the scatter diagrams shown in Figure 6.1. suggest that the inferences drawn in Chapter Two are likely to be reliable. These diagrams show the relationship between the value of the parameter r in the model, and the four output variates. In applying the Biplot technique to sensitivity studies, it will usually be advisable to plot such scattergrams as a check on the validity of the inferences.

As with all methods of sensitivity analysis, however, the Biplot has its limitations. It is, for example, impractical to use the Biplot in cases where the number of parameters and outputs is very large, since in such cases the plot would become too congested to be easily interpreted. The number of outputs and parameters that can be accomodated depends on the properties of the output distribution, since

these determine the appearance of the Biplot, but it is unlikely to be more than about 20. The sensitivity analysis of models having a large number of parameters is, in fact, a major problem, and one which this thesis has not directly addressed. In practice it is usual to work with subsets of the model in such cases, and to analyse each separately. Whilst this may not be ideal, in a practical application there may be little alternative. Steinhorst et. al. (1978) and Hosni and Doering (1979) discuss the sensitivity analysis of very large models in more detail. Additionally, the Biplot is, like most multivariate statistical techniques, only an approximation to the multivariate structure of model output, but, as Gabriel (1971) has indicated, experience suggests that the approximation will usually be good enough for reliable conclusions to be drawn. This is the case for the red fox rabies model in Chapter Two, and also for the farm rat control model, a Biplot of which is described by Huson (1982). The goodness-of-fit of the Biplot in Chapter Two, for example, measured by the ratio of the sum of first two latent roots of the matrix nV, to the trace of nV, is about 80% (the first two latent roots of the matrix nV are 100.60 and 30.96, respectively, and the trace 165.50).

Whether or not the Biplot is easy to interpret, particularly by non-technical model users, is a matter for subjective judgement. It is never easy to find simple summaries of what is essentially a complex multivariate

structure, but experience of using a particular technique generally brings with it a certain facility. The Biplot has been used in practice within the Tolworth and Worplesdon Laboratories of the Agricultural Science Service, where it has proved to be useful to model users, both technical and non-technical.

6.4. The sensitivity measure: a discussion.

The sensitivity measure defined in Chapter Three was designed to be easily computed, easily understood, to involve few assumptions, and to be capable of practical interpretation in at least some circumstances. It offers advantages over simple statistical measures such as the variance or range, since it describes something of the way in which actual output values are distributed, rather than simply their bounds or their dispersion. There is, moreover, the possibility that it may, as in the case of the farm rat model, have an interpretation that can be phrased in terms of the problem that the model is meant to elucidate.

As always, it is unlikely to be appropriate for all situations. In fact, it lends itself particularly to problems in which interest centres on system performance that lies outside some standard range. In such circumstances, the sensitivity measure may, as illustrated above, be chosen to have a practical interpretation, as

well as conveying some general information about the output. The probabilistic basis of the measure means that it lies between 0 and 1, and this often facilitates interpretation and comparisons both within and between models. The probabilistic aspect may also, as in the farm rat model, lend itself to further exploitation.

It may be seen as a disadvantage that the modeller is required to choose a value of k in order to define a sensitivity coefficient of the type proposed in Chapter Three; it is certainly something of a disadvantage that the measure in itself provides no indication of the extent to which each parameter contributes to the output uncertainty. Nevertheless, it is an alternative that should prove to be useful in further application.

6.5. Decision-making: a discussion.

An important criterion in the design of the sensitivity coefficient proposed in Chapter Three was that it should utilise information that could be carried forward to later stages in the process of using a mathematical model. As was shown in Chapter Four, the sensitivity measure itself may directly provide one of the probabilities in a decision process. This continuity of information from informal to formal use is of some value, and is not found in any other sensitivity measure. Again, this feature may not be of use in all kinds of model; it depends firstly on whether or not

a formal decision is to be made on the basis of the model, and secondly on whether the associated probabilities are related to those used in the definition of the sensitivity measure. This emphasises the fact that no sensitivity measure is universally appropriate, nor should it be expected to be. Nevertheless, the methodology described in Chapter Four may find application in real modelling problems, of a type that are not atypical of those commonly encountered. The approach is likely to be particularly appropriate for models in which a new situation is to be compared with a well defined existing standard.

Decisions based on mathematical models are inevitably hazardous, though not necessarily more so than decisions based on other investigative techniques. It is accordingly always appropriate to examine the extent to which the model "supports" a decision - that is, the extent to which the decision may safely be taken, given the uncertainty inherent in the model. Such sensitivity studies of decision processes are normal practice, but are almost always informal, and usually involve simple experimentation with alternative values of decision components. The coefficient suggested in Chapter Four assists in such assessments particularly in cases where several different decision processes are being evaluated, with a view to deciding which data base to refine. The calculation of the coefficient suggested in Chapter Four, for each decision process, allows a convenient comparison of the sensitivity

of each decision to be made.

Decision processes themselves are, of course, not universally accepted as appropriate. The difficulties of defining appropriate sets of actions, possible "states of nature", and of evaluating both probabilities and losses, are often said to be prohibitive. There is no reason why these difficulties should be any greater when a decision is based on a mathematical model. In fact, considering the complexities that are in any case involved in the construction of a formal mathematical representation of a real system, the extra work involved in formally structuring a resultant decision should not be too demanding.

6.6. Measuring parameter influence: a discussion.

The process of developing a mathematical model is iterative. Early versions of the model are studied, and in the light of the results both model structure and parameter estimates may be refined. Clearly if this is to be done efficiently a reliable guide to the relative importance of each parameter must be available. This cannot usually be obtained solely by considering the mathematical structure of the model - as noted in Chapter One, it is also necessary to consider and to realistically represent the actual uncertainty that exists about each parameter value. No matter what the mathematical relationship is between

parameter value and output, only a parameter about which there is uncertainty can be considered important in the context of sensitivity studies. The informational correlation measure proposed in Chapter Five has some valuable characteristics as a measure of parameter influence. The main advantage is that it requires no assumption to be made about the form of the mathematical relationship between parameter and output. The influence of a parameter is measured solely by examination of the joint probability distribution of parameter and output, in comparison with the probabilities that would be expected if parameter and output were independent. This technique, like all others proposed in this thesis, is based directly on the model output distribution.

Information theory in general may well prove to be a fruitful area from which to harvest ideas that may be used in mathematical modelling, and the proposal in Chapter Five is one such example. Hanna (1971) has discussed some information-theoretic concepts that might be capable of exploitation by modellers. Ideas are not hard to come by, and some of these would certainly bear further investigation. For example, many authors have noted that some overall or "global" measure of the value of a mathematical model might be useful, particularly when a situation is encountered in which it is necessary to make a comparison of several models, perhaps of quite different systems, with a view to deciding which models to attempt to

develop further, and which to abandon. The concept of information gain might help. If, prior to constructing a model, the technologist is able to describe his knowledge about a particular output quantity by saying that it may take any of n discrete values, with probabilities p_i, i = 1...n, then after construction of the model the output is observed to occupy these n classes with probabilities q_i, i = 1...n, the "information gain" from the model may be measured as:

$$\sum_i \left(q_i \log q_i - p_i \log p_i\right) \qquad (6.2)$$

This is simply a measure of the extent to which knowledge about the model output can be said to have changed following construction of the model. The computation of such a measure for a series of different models may, in some circumstances at least, aid a comparison of their merits.

6.7. Subjective probability distributions for parameters.

All the methods developed in this thesis are based on exploitation of the observed model output distribution, which is generated by describing parameter uncertainty using probability distributions. The probability distributions used here, and in most examples in the literature, are essentially subjective. They are based much

more on belief about possible values that a parameter might take than on firm empirical evidence. This is of less importance in the present context, in the sense that the distributions used above serve only to facilitate illustration of the proposed methodology. In Statistics there has been for many years a debate about whether the use of such distributions is valid, or even useful. It is not possible here to make any worthwhile contribution to that debate, and it must suffice to say that subjective probability distributions for parameters have been used here in the belief that they are both valid and useful. In view of the fact that many mathematical models are based on the modeller's essentially subjective beliefs about what constitutes an appropriate mathematical representation - views which are often only partly supported by empirical evidence - it does not seem unreasonable to propose that the modeller should extend his activities to the point of subjectively modelling his beliefs about parameter values.

6.8. Uncertainty about functional form.

In Chapter One, it was noted that uncertainty usually exists about both the true values of the parameters in a model, and also about the true nature of the mathematical relationships that are represented in the model. To make a start in the application of modelling to a particular practical problem, it is necessary to assign specific values to parameters, and to select specific forms of

relationship. Almost all the literature on sensitivity analysis deals with the problems of uncertain parameter values. This thesis is no exception; no specific techniques have been proposed for the problem of uncertainty about functional form. Some modellers feel that functional form uncertainties are more important than parameter uncertainties. This may frequently be true, and it is undoubtedly true for some models, but there is no concrete evidence on which to base such an assertion. O'Neill (1973), for example, has described a model in which parameter uncertainties were more critical than the precise form of the model. The problem of uncertainty about functional form, as many authors have noted, is inherently less tractable than that of parameter uncertainty. It is rarely impossible to say something about the plausible range of a scalar parameter, but frequently difficult to say something useful about the infinite variety of functional forms that might actually relate system components. The current state of the literature on this subject is essentially as follows. If attention is restricted to a particular family of functional forms, say second order polynomials, and it is possible to index this family by one or more parameters, then the problem reduces to that of parameter uncertainty. If a number of distinct types of function are thought plausible, then in practice this is handled by the formulation of specific alternative models. Kleijnen (1975) suggested that dependence of output on functional form might be studied by utilising different

plausible functions in the Monte Carlo generation of model output, and representing these alternatives by means of dummy variables in a response surface used to analyse the model. Burns (1973, 1975) made a similar suggestion.

This is certainly a problem that merits more detailed attention, perhaps in the first instance by specific case studies to see how often functional form is more critical than parameter uncertainty, and next to establish some general procedures for assessing the effects of uncertainty about functional forms. Two speculations may be worthwhile. Firstly, in advanced calculus the notion of the rate of change of a functional form (as opposed to the rate of change of a scalar) is defined. It may, then, be possible to utilise this concept to define measures of functional form sensitivity that are analagous to the differential sensitivity measure. Secondly, the concept of "relative curvature", from analytical geometry, may provide assistance. Measures of the "curvature" of functions of many variables are defined in some situations, and if each plausible function in a mathematical model could be indexed by some scalar measure of its curvature, then the problem even of distinct families of possible functions might be reducible to a study of parametric uncertainty - in which the "parameter" is the measured curvature of the function.

However, it is important to stress that both concepts involve relatively advanced mathematics, and might produce

results which, though valid, are not sufficiently accessible to users of mathematical models. If modelling is to be of genuine practical value, it is most important that its processes should be usable by as many people as possible. From a purely practical viewpoint, a technically inaccessible result is of little more value than a non-existent result.

6.9. The need for education.

Finally, it should be emphasised that the development of methods by which uncertainty may be assessed and described - which has been the objective of the present work - is only part of the task. There are still many models and modelling applications described in the literature of many disciplines, in which it it would appear that no attempt at all has been made to address the inherent uncertainty of the technique. Every opportunity should be taken to develop in modellers an awareness of the importance of uncertainty assessment.

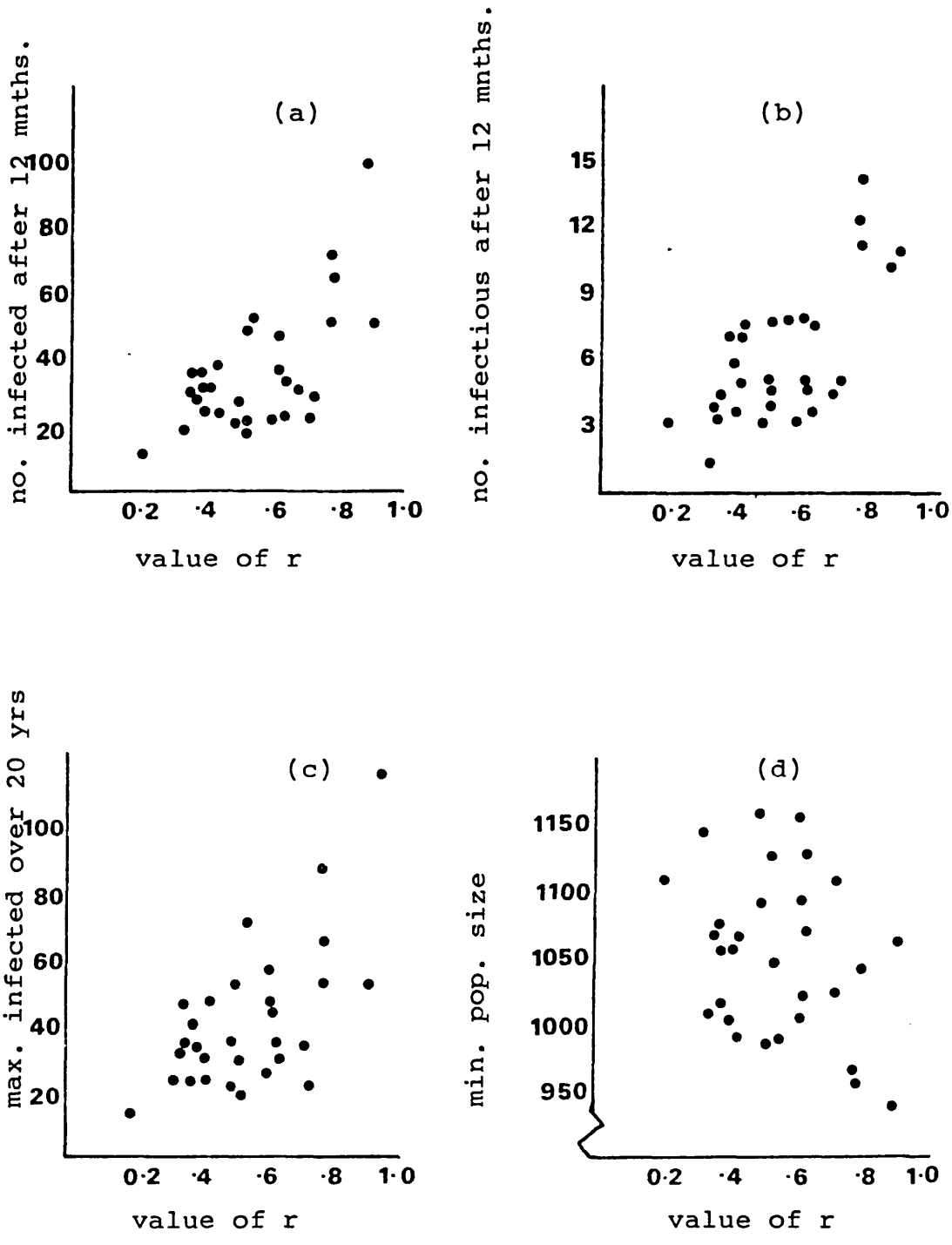

Figure 6.1. Scatter diagrams of the relationship between the value of r in the rabies model and (a) number of infected foxes after 12 months (b) number of infectious foxes after 12 months (c) maximum number of infected foxes over 20 yr. period and (d) minimum fox population size

SUMMARY OF MAIN FINDINGS

The principal contributions of this thesis are:

(i) a critical review of existing techniques of sensitivity analysis, drawing on the literature of many different disciplines

(ii) the proposal of a modified form of the Biplot for providing a concise graphical summary of the uncertainty inherent in model output, together with an indication of particularly discrepant model responses

(iii) the definition of a measure of sensitivity that has a simple and generally applicable interpretation, and which may also have a more practical meaning in some modelling applications

(iv) the demonstration that this sensitivity coefficient may be employed in a formal decision-theoretic framework, in addition to its general use

(v) a proposal for a simple measure of the sensitivity of a decision process based on a mathematical model, which may be interpreted as indicating the strength of model support for a decision

(vi) the proposal of the informational correlation measure as an appropriate method of measuring the influence that a model parameter has on the uncertainty inherent in the model output

(vii) a demonstration of the use of these techniques on two typical mathematical models

(viii) the general proposal that the most appropriate approach to sensitivity analysis will usually be to describe parameter uncertainty by means of probability distributions, to generate a sample from the model by Monte Carlo simulation, and to utilise the output distribution in various ways to provide the necessary descriptions and measures of uncertainty; it has been demonstrated that a single such output sample can be used in informal and formal description of uncertainty, can be used in formal decision-making, and also to measure parameter influence.

(ix) the suggestion of some lines of investigation for future research in both information theoretic methods, and sensitivity analysis of functional form.

APPENDIX: COMPUTER PROGRAMS

This appendix contains listings of five computer programs used in the development of the methods described in the text of the thesis. Listings are provided of the basic forms of the coded mathematical models, together with two methodological programs - one to derive co-ordinates for a Biplot, and the other to derive values of I', from which informational correlation coefficients may be calculated.

(i) PROGRAM RABIES1: this is a coding, in FORTRAN, of the red fox rabies model due to Anderson et. al. (1981). The code sets parameter values to the baseline values suggested by Anderson et. al., and then integrates the model equations numerically to produce, as output, a time track of the population density of each of the three classes of animal defined in the model. Numerical integration is carried out using the external subroutine D02EAF, available in the N.A.G. subroutine library (N.A.G., 1984).

```
      PROGRAM RABIES1
C     *
C     *
      DOUBLE PRECISION   GAMMA, BETA, XS, XE, W(3,21), Y(3)
      COMMON             /VPARS/ GAMMA, BETA
      EXTERNAL           FCN
C     *
C     *
C     *
C     *
C     *
      Y(1) = 0.95
      Y(2) = 0.04
```

```
      Y(3) = 0.01
C     *
C     *
C     *
      GAMMA = 0.50D 0
      BETA  = 79.69D 0
      XS    = 1.0D 0
C     *
C     *
C     *
C     *
C     *
      WRITE(1,30) Y(1), Y(2), Y(3)
C     *
C     *
      DO 100 I = 1,50
          XE = XS + 0.08333333D 0
          CALL D02EAF(XS,XE,3,Y,0.001D 0,FCN,W,21,0)
          XS = XE
          WRITE(1,30) Y(1), Y(2), Y(3)
30        FORMAT(3F10.4)
100   CONTINUE
      CALL EXIT
      END
      SUBROUTINE FCN(X,Y,F)
C     *
C     *
      DOUBLE PRECISION     X,Y(3),F(3),T
      DOUBLE PRECISION     R,GAMMA,BETA,SIGMA,B,ALPHA
      COMMON               /VPARS/ GAMMA, BETA
C     *
C     *
      R     = 0.50D 0
      SIGMA = 13.00D 0
      B     = 0.50D 0
      ALPHA = 73.00D 0
C     *
C     *
      T    = Y(1)+Y(2)+Y(3)
      F(1) = (R*Y(1))-(GAMMA*Y(1)*T)-(BETA*Y(1)*Y(3))
      F(2) = (BETA*Y(1)*Y(3))-((SIGMA+B+(GAMMA*T))*Y(2))
      F(3) = (SIGMA*Y(2))-((ALPHA+B+(GAMMA*T))*Y(3))
      RETURN
      END
```

(ii) PROGRAM BIPLOT: this code, written in a statistical programming language called GENSTAT, derives co-ordinates for a Biplot of the sensitivity data from the red fox rabies model.

```
"REFE" BIPLOT
"UNIT" $30
"VARI" V(1...10)
"INPU" 2
"READ/FORM=P,PRINT=E" V(1...10)
"INPU" 1
"RUN"
"CALC" V(1)=V(1)/0.500
"CALC" V(2)=V(2)/0.250
"CALC" V(3)=V(3)/79.690
"CALC" V(4)=V(4)/13.00
"CALC" V(5)=V(5)/0.500
"CALC" V(6)=V(6)/73.00
"CALC" V(7)=V(7)/23.448
"CALC" V(8)=V(8)/4.093
"CALC" V(9)=V(9)/30.804
"CALC"  V(10)=V(10)/1138.404
"RUN"
"SCAL" SC
"FOR" A=V(1...10)
"CALC" SC=MEAN(A)
"CALC" SC=1-SC
"CALC" A=A+SC
"REPE"
"RUN"
"CALC" V(1...10)=V(1...10)*1.5
"RUN"
"SET" VARS=V(1...10)
"DSSP" DISP$VARS
"SSP" DISP
"SYMMAT" COR$10
"CALC" COR=CORMAT(DISP)
"RUN"
"PRINT/LABR=1,LABC=1" DISP$10.2
"PRINT" COR$10.2
"RUN"
"SYMMAT" SIM$10
"EQUA" SIM=DISP$56,11X
"SCAL" X
"CALC" X=NVAL(V(1))
"RUN"
"MATRIX" LVECS$10,2
"DIAG" LROOTS$2
"SCAL" TRCE
"LRV" SIM;LVECS,LROOTS,TRCE
"RUN"
"SCAL" ROOT1,ROOT2
```

```
"CALC" ROOT1=ELEM(LROOTS;1)
"CALC" ROOT2=ELEM(LROOTS;2)
"CALC" ROOT1,ROOT2=SQRT(ROOT1,ROOT2)
"VARI" LVEC1,LVEC2$11
"MATR" COPY2$2,10
"CALC" COPY2=TRANS(LVECS)
"EQUA" LVEC1,LVEC2=COPY2
"RUN"
"CAPT" "" THE DATA MATRIX ""
"PRINT/FORM=P" V(1...10)$10.2
"CAPT" "" LATENT ROOTS AND VECTORS ""
"PRINT" LROOTS$6.2
"PRINT" LVECS$10.2
"PRINT" TRCE $6.2
"RUN"
"DEVA" LVECS,COPY2,SIM,DISP,LROOTS
"RUN"
"SCAL" HOLD
"FOR" ID=V(1...6)
"CALC" HOLD=MEAN(ID)
"CALC" ID=HOLD
"REPE"
"FOR" I=V(1...10)
"CALC" HOLD=MEAN(I)
"CALC" I=I-HOLD
"REPE"
"MATR" COPY1$10,X
"EQUA" COPY1=V(1...10)
"DEVA" V(1...10)
"RUN"
"MATR" DATA$X,10
"CALC" DATA=TRANS(COPY1)
"VARI" P1,P2$X
"CALC" P1=PDT(DATA;LVEC1)
"CALC" P2=PDT(DATA;LVEC2)
"CALC" P1=(1/ROOT1)*P1*SQRT(X)
"CALC" P2=(1/ROOT2)*P2*SQRT(X)
"DEVA" DATA
"RUN"
"CALC" LVEC1=LVEC1*ROOT1*(1/SQRT(X))
"CALC" LVEC2=LVEC2*ROOT2*(1/SQRT(X))
"CAPT" "" MATRIX G OF THE BIPLOT ""
"PRINT/P" P1,P2$10.2
"CAPT" "" MATRIX H OF THE BIPLOT ""
"PRINT/P" LVEC1,LVEC2$10.2
"RUN"
"CLOSE"
"STOP"
```

(iii) PROGRAM RATM1: this is a coding, in FORTRAN, of the basic form of the farm rat model used in Chapter Four. The code produces, by numerical integration, a time track of farm rat population sizes over a 24 month simulated period. Numerical integration is done by the external subroutine D02EAF (N.A.G., 1984).

```
      PROGRAM RATM1
C     *
C     *
      DOUBLE PRECISION    X,XEND,Y(1),TOL,W(1,19),PSIZE(240)
      INTEGER             N,IFAIL,IW,I
      EXTERNAL            FCN
      COMMON             /CDB/ PSIZE
C     *
C     *
      X     =   1.0D 0
      XEND  =   1.0D 0
      Y(1)  = 450.0D 0
      IFAIL =     0
      N     =     1
      TOL   = 0.001D 0
      IW    =    19
C     *
C     *
      DO 10 I = 1,240
          XEND = XEND + 0.1D 0
          CALL D02EAF(X,XEND,N,Y,TOL,FCN,W,IW,IFAIL)
          PSIZE(I) = Y(1)
          X = XEND
10    CONTINUE
C     *
C     *
      WRITE(1,30)
      DO 20 I = 1,231,10
          WRITE(1,40) I,PSIZE(I)
20    CONTINUE
30    FORMAT(16HRAT MODEL OUTPUT,//)
40    FORMAT(I10,F10.0)
      CALL EXIT
      END
      SUBROUTINE FCN(X,Y,F)
C     *
C     *
      DOUBLE PRECISION   X,Y(1),F(1),PSIZE(240)
      DOUBLE PRECISION   C,A,B,GAMMA,PHI,BETA,KCAP,PHIPR
      COMMON             /CDB/ PSIZE
C     *
C     *
```

```
      A       =   0.94D 0
      B       =   0.14D 0
      KCAP    = 500.00D 0
      BETA    =  30.00D 0
      PHI     =   0.00D 0
      PHIPR   =   0.00D 0
      C       =   1.00D 0
      GAMMA   =   (A-B)/KCAP
C     *
C     *
      IF (X .GT. 4.0D 0 .AND. X .LE. 4.5) PHI =0.70D 0
      IF (X .GT. 4.0D 0 .AND. X .LE. 7.0D 0) PHIPR = 0.70D 0
      F(1) = C*(1.0D 0-PHIPR)*A*Y(1)
      F(1) = F(1)-(((PHI*BETA)+((1.0D 0-PHI)*B)+(GAMMA*Y(1)))*Y(1))
      RETURN
      END
```

(iv) PROGRAM RATM2: a coding, in FORTRAN, of the version of the farm rat control model used in Chapter Five. This differs from the previous version in that a time-lag of density dependent responses is explicitly included in the program code. The external subroutine D02EAF (N.A.G., 1984) performs numerical integration of the model equation.

```
      PROGRAM RATM2
C     *
C     *
      DOUBLE PRECISION    X,XEND,Y(1),TOL,W(1,19),PSIZE(240)
      INTEGER             N,IFAIL,IW,I
      EXTERNAL            FCN
      COMMON             /CDB/ PSIZE
C     *
C     *
      X     =   1.0D 0
      XEND  =   1.0D 0
      Y(1)  = 450.0D 0
      IFAIL =     0
      N     =     1
      TOL   = 0.001D 0
      IW    =    19
C     *
C     *
      DO 10 I = 1,240
          XEND = XEND + 0.1D 0
          CALL D02EAF(X,XEND,N,Y,TOL,FCN,W,IW,IFAIL)
          PSIZE(I) = Y(1)
          X = XEND
10    CONTINUE
C     *
C     *
      WRITE(1,30)
      DO 20 I = 1,231,10
          WRITE(1,40) I,PSIZE(I)
20    CONTINUE
30    FORMAT(16HRAT MODEL OUTPUT,//)
40    FORMAT(I10,F10.0)
      CALL EXIT
      END
      SUBROUTINE FCN(X,Y,F)
C     *
C     *
      DOUBLE PRECISION   X,Y(1),F(1),PSIZE(240),LS
      DOUBLE PRECISION   C,A,B,GAMMA,PHI,BETA,KCAP,PHIPR
      INTEGER            I
      COMMON             /CDB/ PSIZE
C     *
```

```
C     *
      A       =   0.94D 0
      B       =   0.14D 0
      KCAP    = 500.00D 0
      BETA    =  30.00D 0
      PHI     =   0.00D 0
      PHIPR   =   0.00D 0
      C       =   1.00D 0
      GAMMA   =   (A-B)/KCAP
C     *
C     *
      I = AINT((X*10.0D 0)-9.0D 0)
      I = I-25
      IF (X .GT. 4.0D 0) LS = PSIZE(I)
      IF (X .GT. 4.0D 0 .AND. X .LE. 4.5) PHI =0.70D 0
      IF (X .GT. 4.0D 0 .AND. X .LE. 7.0D 0) PHIPR = 0.70D 0
      IF (X .GT. 4.0D 0) F(1) = C*(1.0D 0-PHIPR)*A*LS
      IF (X .LE. 4.0D 0) F(1) = C*(1.0D 0-PHIPR)*A*Y(1)
      F(1) = F(1)-(((PHI*BETA)+((1.0D 0-PHI)*B)+(GAMMA*Y(1)))*Y(1))
      RETURN
      END
```

(v) PROGRAM INFC: a coding, in FORTRAN, of an algorithm to derive values of I' from the data listed in Table 5.1.. This program is doing no more than simple arithmetic on the sensitivity data.

```
      PROGRAM INFC
C     .
      REAL ICV7,ICV8,ICV9,ICV10,ICV11,WV
C     .
      REAL V6(100),V7(100),V8(100),V9(100),V10(100)
      REAL V11(100)
C     .
      REAL B1(4),B7(4),B8(4),B9(4),B10(4),B11(4)
C     .
      REAL PD6(3),PD7(3),PD8(3),PD9(3),PD10(3)
      REAL PD11(3)
C     .
      REAL JPDV7(3,3),JPDV8(3,3),JPDV9(3,3)
      REAL JPDV10(3,3),JPDV11(3,3)
C     .
      DATA B1/300.0,434.0,650.0,900.0/
      DATA B7/0.700,0.874,0.996,1.200/
      DATA B8/0.100,0.129,0.149,0.200/
      DATA B9/24.00,28.09,32.02,36.00/
      DATA B10/19.0,23.0,26.0,30.0/
      DATA B11/1.00,3.73,6.36,10.00/
C     .
      DO 10 I = 1,100
      READ (6,*) V6(I),V7(I),
     ),V9(I),V10(I),V11(I)
10    CONTINUE
C     .
C     .
C     .
C     .
C     .
C     .
      DO 130 I = 1,3
      PD6(I) = 0.0
      DO 120 J = 1,100
      IF (V6(J) .GE. B1(I) .AND. V6(J) .LT. B1(I+1)) PD6(I)=PD6(I)+1.0
120   CONTINUE
      PD6(I) = PD6(I)/100.0
130   CONTINUE
C     .
      DO 150 I = 1,3
      PD7(I) = 0.0
      DO 140 J = 1,100
      IF (V7(J) .GE. B7(I) .AND. V7(J) .LT. B7(I+1)) PD7(I)=PD7(I)+1.0
140   CONTINUE
      PD7(I) = PD7(I)/100.0
```

```
150   CONTINUE
C     .
      DO 170 I = 1,3
      PD8(I) = 0.0
      DO 160 J = 1,100
      IF (V8(J) .GE. B8(I) .AND. V8(J) .LT. B8(I+1)) PD8(I)=PD8(I)+1.0
160   CONTINUE
      PD8(I) = PD8(I)/100.0
170   CONTINUE
C     .
      DO 190 I = 1,3
      PD9(I) = 0.0
      DO 180 J = 1,100
      IF (V9(J) .GE. B9(I) .AND. V9(J) .LT. B9(I+1)) PD9(I)=PD9(I)+1.0
180   CONTINUE
      PD9(I) = PD9(I)/100.0
190   CONTINUE
C     .
      DO 210 I = 1,3
      PD10(I) = 0.0
      DO 200 J = 1,100
      IF(V10(J).GE.B10(I).AND.V10(J).LT.B10(I+1))PD10(I)=PD10(I)+1.0
200   CONTINUE
      PD10(I) = PD10(I)/100.0
210   CONTINUE
      DO 999 I = 1,3
      PD11(I) = 0.0
      DO 998 J = 1,100
      IF (V11(J).GE.B11(I).AND.V11(J).LT.B11(I+1))PD11(I)=PD11(I)+1
998   CONTINUE
      PD11(I) = PD11(I)/100.0
999   CONTINUE
C     .
      DO 240 I = 1,3
      DO 230 K = 1,3
      JPDV7(I,K) = 0.0
      DO 220 J = 1,100
      IF (V6(J) .GE. B1(I) .AND. V6(J) .LT.
     1(I+1) .AND. V7(J) .GE. B7(K) .AND. V7(J)
     LT. B7(K+1)) JPDV7(I,K) = JPDV7(I,K)+1.0
220   CONTINUE
      JPDV7(I,K) = JPDV7(I,K)/100.0
230   CONTINUE
240   CONTINUE
C     .
      DO 270 I = 1,3
      DO 260 K = 1,3
      JPDV8(I,K) = 0.0
      DO 250 J = 1,100
      IF (V6(J) .GE. B1(I) .AND. V6(J) .LT.
     1(I+1) .AND. V8(J) .GE. B8(K) .AND. V8(J)
     LT. B8(K+1)) JPDV8(I,K) = JPDV8(I,K)+1.0
250   CONTINUE
      JPDV8(I,K) = JPDV8(I,K)/100.0
260   CONTINUE
270   CONTINUE
```

```
C     .
      DO 300 I = 1,3
      DO 290 K = 1,3
      JPDV9(I,K) = 0.0
      DO 280 J = 1,100
      IF (V6(J) .GE. B1(I) .AND. V6(J) .LT.
     1(I+1) .AND. V9(J) .GE. B9(K) .AND. V9(J)
     LT. B9(K+1)) JPDV9(I,K) = JPDV9(I,K)+1.0
280   CONTINUE
      JPDV9(I,K) = JPDV9(I,K)/100.0
290   CONTINUE
300   CONTINUE
C     .
      DO 330 I = 1,3
      DO 320 K = 1,3
      JPDV10(I,K) = 0.0
      DO 310 J = 1,100
      IF (V6(J) .GE. B1(I) .AND. V6(J) .LT.
     1(I+1) .AND.V10(J) .GE. B10(K) .AND.V10(J)
     LT. B10(K+1)) JPDV10(I,K) = JPDV10(I,K)+1.0
310   CONTINUE
      JPDV10(I,K) = JPDV10(I,K)/100.0
320   CONTINUE
330   CONTINUE
C     .
      DO 360 I = 1,3
      DO 350 K = 1,3
      JPDV11(I,K) = 0.0
      DO 340 J = 1,100
      IF (V6(J) .GE. B1(I) .AND. V6(J) .LT.
     1(I+1) .AND. V11(J) .GE. B11(K) .AND. V11(J)
     LT. B11(K+1)) JPDV11(I,K) = JPDV11(I,K)+1.0
340   CONTINUE
      JPDV11(I,K) = JPDV11(I,K)/100.0
350   CONTINUE
360   CONTINUE
C     .
      WRITE(1,401)
401   FORMAT(35HPROBABILITY DISTRIBUTIONS V1 TO V11,///)
      DO 500 I = 1,3
      WRITE(1,400) PD6(I),
     (I),PD8(I),PD9(I),PD10(I),PD11(I)
500   CONTINUE
400   FORMAT(6F7.2)
C     .
      WRITE(1,402)
402   FORMAT(20HJOINT P.D. V6 AND V7,///)
      DO 510 I = 1,3
      WRITE(1,777) JPDV7(I,1),JPDV7(I,2),JPDV7(I,3)
510   CONTINUE
C     .
C     .
777   FORMAT(3F10.4)
      WRITE(1,403)
403   FORMAT(20HJOINT P.D. V6 AND V8,///)
      DO 520 I = 1,3
```

```
      WRITE(1,777) JPDV8(I,1),JPDV8(I,2),JPDV8(I,3)
520   CONTINUE
C     .
      WRITE(1,404)
404   FORMAT(20HJOINT P.D. V6 AND V9,///)
      DO 530 I = 1,3
      WRITE(1,777) JPDV9(I,1),JPDV9(I,2),JPDV9(I,3)
530   CONTINUE
C     .
      WRITE(1,405)
405   FORMAT(21HJOINT P.D. V6 AND V10,///)
      DO 550 I = 1,3
      WRITE(1,777) JPDV10(I,1),JPDV10(I,2),JPDV10(I,3)
550   CONTINUE
C     .
      WRITE(1,406)
406   FORMAT(21HJOINT P.D. V6 AND V11,///)
      DO 570 I = 1,3
      WRITE(1,777) JPDV11(I,1),JPDV11(I,2),JPDV11(I,3)
570   CONTINUE
C     .
      ICV7 = 0.0
      DO 610 I = 1,3
      DO 600 J = 1,3
      WV = 0.0
      IF (JPDV7(I,J) .LE. 0.0) GOTO 599
      WV = JPDV7(I,J) * ALOG(JPDV7(I,J))
599   IF (PD6(I) .LE. 0.0 .OR. PD7(I) .LE. 0.0) GOTO 1001
      WV = WV - ((PD6(I)*PD7(J))*ALOG(PD6(I)*PD7(J)))
1001  ICV7 = ICV7+WV
600   CONTINUE
610   CONTINUE
C     .
      ICV8 = 0.0
      DO 630 I = 1,3
      DO 620 J = 1,3
      WV = 0.00
      IF (JPDV8(I,J) .LE. 0.0) GOTO 619
      WV = JPDV8(I,J) * ALOG(JPDV8(I,J))
619   IF (PD6(I) .LE. 0.0 .OR. PD8(I) .LE. 0.0) GOTO 1002
      WV = WV - ((PD6(I)*PD8(J))*ALOG(PD6(I)*PD8(J)))
1002  ICV8 = ICV8+WV
620   CONTINUE
630   CONTINUE
C     .
      ICV9 = 0.0
      DO 650 I = 1,3
      DO 640 J = 1,3
      WV = 0.00
      IF (JPDV9(I,J) .LE. 0.0) GOTO 639
      WV = JPDV9(I,J) * ALOG(JPDV9(I,J))
639   IF (PD6(I) .LE. 0.0 .OR. PD9(I) .LE. 0.0) GOTO 1003
      WV = WV - ((PD6(I)*PD9(J))*ALOG(PD6(I)*PD9(J)))
1003  ICV9 = ICV9+WV
640   CONTINUE
650   CONTINUE
```

```
C     .
      ICV10 = 0.0
      DO 670 I = 1,3
      DO 660 J = 1,3
      WV = 0.00
      IF (JPDV10(I,J) .LE. 0.0) GOTO 659
      WV = JPDV10(I,J) * ALOG(JPDV10(I,J))
659   IF (PD6(I) .LE. 0.0 .OR. PD10(I) .LE. 0.0) GOTO 1005
      WV = WV - ((PD6(I)*PD10(J))*ALOG(PD6(I)*PD10(J)))
1005  ICV10 = ICV10+WV
660   CONTINUE
670   CONTINUE
C     .
      ICV11 = 0.0
      DO 690 I = 1,3
      DO 680 J = 1,3
      WV = 0.0
      IF (JPDV11(I,J) .LE. 0.0) GOTO 299
      WV = JPDV11(I,J) * ALOG(JPDV11(I,J))
299   IF (PD6(I) .LE. 0.0 .OR. PD11(I) .LE. 0.0) GOTO 1006
      WV = WV - ((PD6(I)*PD11(J))*ALOG(PD6(I)*PD11(J)))
1006  ICV11 = ICV11+WV
680   CONTINUE
690   CONTINUE
691   WRITE(1,409)
409   FORMAT(18HVALUES OF I PRIME ,///)
      WRITE(1,410) ICV7
      WRITE(1,420) ICV8
      WRITE(1,430) ICV9
      WRITE(1,440) ICV10
      WRITE(1,450) ICV11
410   FORMAT(10HVARIATE 7 ,F10.5)
420   FORMAT(10HVARIATE 8 ,F10.5)
430   FORMAT(10HVARIATE 9 ,F10.5)
440   FORMAT(10HVARIATE 10,F10.5)
450   FORMAT(10HVARIATE 11,F10.5)
      STOP
      END
```

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A GRAPHICAL AID TO MULTIVARIATE SENSITIVITY ANALYSIS

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ABSTRACT

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This paper describes a graphical display designed to aid the evaluation of multivariate sensitivity data produced by a systems model. The display is a modification of the **GH′** Biplot technique. The variance and correlation structure of the model parameters and output variates is represented on the display, and a modification of the standard **GH′** Biplot is described which allows a simultaneous display of the results produced by each run of the model. Any run of the model that produces output particularly displaced from baseline output is easily detected on the display, and the nature of such a potentially sensitive reponse, in terms of the output variates and model parameters that contribute most strongly to it, can easily be diagnosed. The method is illustrated using multivariate sensitivity data from a pest control simulation model.

INTRODUCTION

The widespread use of systems modelling as a research technique in ecology and environmental biology has in recent years necessitated investigations of the extent to which predictions from models can be regarded as reliable, and of the methodology by which reliability can be assessed and described (e.g., Shaeffer, 1980; O'Neill et al., 1980).

The term 'sensitivity analysis' is now widely used to describe one of the possible approaches to this problem. Although the term is used narrowly by some authors to refer to a specific technique (e.g., Gardner et al., 1981), sensitivity analysis is best regarded as a general term for a number of techniques that are used to describe the variations in model output that occur in response to perturbations of the model parameters. Perturbation of model parameters is intended to reflect the uncertainty that inevitably exists about the true values of the parameters used. This uncertainty is propagated

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as uncertainty about model output, and must accordingly be assessed and described adequately, to permit an appropriate interpretation of existing model output, and to direct model reformulation by pinpointing the most sensitive parts of the model structure.

A number of methods of sensitivity analysis have been proposed in the literature. Tomovic (1963) suggested the use of sensitivity coefficients, which he defined as partial derivatives of the model functions, to describe the extent of model output variation. This method has recently been criticised on the grounds of both its practicality and mathematical properties (McKay et al., 1976; Steinhorst et al., 1978; Gardner et al., 1981). Plinston (1972), Miller (1974) and Argentesi and Olivi (1978) all describe extensions of this principle, using derivatives of some function of model output. For some types of model, a more direct approach is possible, by studying analytically or numerically the propagation of parameter error (Wright, 1972; O'Neill et al., 1980; Shaeffer, 1980; Gardner et al., 1981; Majkowski et al., 1981).

McKay et al. (1976) depart from the partial derivative type of sensitivity coefficient and propose instead the use of partial rank correlation coefficients to measure sensitivity. Steinhorst et al. (1978) have used ratios of mean squares, derived from an analysis of variance, for the same purpose. Plinston (1972) illustrates the advantages of graphical presentation of sensitivity data by plotting, in the space of pairs of model parameters, contours of a function that measures the change in model output induced by parameter perturbation.

All of these techniques fulfil a common purpose of describing the uncertainty inherent in model output. In practice, all will often give similar results for a given model, but all have some defects. Since it is desirable to try more than one approach to sensitivity analysis when interpreting a systems model, their individual defects are less important than their combined ability to aid the modeller to interpret and refine a model.

These techniques are, however, best suited for studying the variation of a single model output, or a function of multiple output, caused by perturbation of a single parameter. None is designed for a genuinely multivariate treatment of data generated by perturbing several parameters simultaneously and observing the variation in several output variates. A method of displaying multivariate sensitivity data is described in this paper.

THE SENSITIVITY DATA MATRIX

The type of data with which the present paper is concerned can be illustrated by consideration of a systems model with p parameters and v output variates. The purpose of the sensitivity analysis is to evaluate and describe the variation in the v output variates that is induced by perturbing

the p parameters by amounts that can be regarded as being in keeping with reality. If the perturbations that are applied to the parameters are sampled randomly from specified probability distributions, then any sample of parameter values so obtained can be regarded as a random sample from the implied population of possible parameter values (Dent and Blackie, 1979). If n such sets of parameters are sampled, and the model is run with each set, the data generated can naturally be considered as a matrix of order $n \times (p + v)$.

For each column in this matrix, there exists a baseline or best estimate value. For the first p columns, these values are the unperturbed parameter values, and for the last v columns they are the values output when the model is run using unperturbed parameter values. Rescaling the mean of each column to the baseline value and expressing each value as a ratio of the baseline value, produces a natural compatibility of scale for the matrix. Such a matrix may be referred to as a sensitivity data matrix.

A suitable method of multivariate sensitivity analysis should ideally permit:

(i) a simultaneous assessment of the variance–covariance structure of the columns of the sensitivity data matrix;

(ii) easy detection of any run of the model that results in a particularly deviant model response;

(iii) a diagnosis to be made of the nature of such a response, in terms of the parameters that most influence it, and the variates that respond most strongly.

Gabriel (1971) described a multivariate statistical technique called the Biplot, which, with appropriate modifications, proves to be a very suitable device for evaluating multivariate sensitivity data.

THE BIPLOT

The Biplot is a graphical display of multivariate data, based, where necessary, on a rank two approximation of the original data matrix. The plot is formed from two sets of co-ordinates, one of which represents the rows of the original data matrix, and the other the columns. There are several types of Biplot, but the one most suitable for the present application, referred to by Gabriel (1971) as the **GH′** Biplot, assumes that the rows of the matrix are a random sample from a larger population. The rows of the matrix are plotted so that increasing deviation of a row from the mean is shown as increasing distance from the origin of the Biplot. The distances on the Biplot in fact approximate the Mahalanobis distances between rows of the sensitivity data matrix. The columns of the matrix are represented as projections which have lengths proportional to the variance of the column they represent. The correlation between columns is approximated by the cosine of the

angle between the column projections. The orientation of a point representing a row of the matrix is determined by the extent to which each column contributes to the row's displacement. Gabriel (1971) has proved and described these properties in detail; Everitt (1978) gives a simplified description of the **GH′** Biplot.

By slightly modifying the **GH′** Biplot, as described below, each row of the sensitivity data matrix can be displayed in such a way that rows which are particularly displaced from the baseline—indicating a potentially sensitive response of the model—appear distant from the origin of the Biplot. The orientation of a point representing such a row, in relation to the projections that represent model parameters and output variates allows the nature of such a potentially sensitive response to be diagnosed. Simultaneously, the Biplot gives a clear picture of the variation in model output, and of the correlations within and between the parameters, and the output variates.

ALGORITHM

Let **S** be the variance–covariance matrix derived from the mean-centred sensitivity data matrix created by n runs of the model. Denote the first two latent roots and vectors of $n\mathbf{S}$ as l_1, l_2, and c_1, c_2, respectively. The co-ordinates of the Biplot are then given by

$$\mathbf{G} = \left[(1/\sqrt{l_1})\,\mathbf{D}_r c_1;\ (1/\sqrt{l_2})\,\mathbf{D}_r c_2\right]\sqrt{n}$$

$$\mathbf{H} = \left[\sqrt{l_1}c_1;\ \sqrt{l_2}c_2\right]1/\sqrt{n}$$

where $\mathbf{D}_r$ is the mean-centred sensitivity data matrix, in which each value in the parameter columns has been replaced by its baseline value. Because each column is mean-centred, this is equivalent to setting all elements of the first p columns to zero. The use of $\mathbf{D}_r$ is a modification of the standard **GH′** Biplot, in which the mean-centred data matrix is used without any changes in parameter columns. The modification ensures that displacements on the Biplot reflect only distances of observed model output from baseline output. Extreme values of parameters do not themselves influence the distances approximated on the Biplot.

Matrices **G** and **H** are, respectively, $n \times 2$ and $(p + v) \times 2$ matrices; each row of these matrices is treated as an x–y co-ordinate pair to produce a graphical display. The $(p + v)$ co-ordinate pairs from matrix **H** are drawn as lines from the Biplot origin to the point indicated by the co-ordinates.

Occasionally a rescaling of the sensitivity data matrix, by multiplying throughout by a scalar, may be required in order to produce compatible scales in **G** and **H′** and so avoid a graph on which the variate vectors are too small to permit easy interpretation of the plot. An appropriate value of such

a scalar is best selected by trial and error, after observing plots derived with different values. This scalar multiplication does not affect the above properties of the Biplot.

AN APPLICATION

The version of the Biplot described above has been tested with several different systems models, and its use is illustrated here with a model that simulates the growth and control of rats on agricultural premises.

The detailed structure of this model is not relevant in the present context; briefly, the model simulates the growth of a population of rats in farm buildings, by selecting birth, death, immigration and emigration rates according to certain rules. Methods of controlling the size of the population are simulated by reducing the population size to simulate a rodenticide treatment, reducing birth rate to simulate a chemosterilant treatment, and reducing immigration rate to simulate the effect of permanently baiting a rat population.

The sensitivity analysis was concerned with three output variates (denoted $V5$, $V6$, and $V7$), that measure the relative success of three types of rat control technique: chemosterilant control ($V5$), control by permanent baiting ($V6$), and control by a combination of chemosterilant and rodenticide ($V7$). Four parameters in the model (denoted $P1$, $P2$, $P3$, and $P4$) were varied: the maximum birth rate ($P1$), the turnover rate of the population at stable size ($P2$), the rate of immigration ($P3$), and the stable population size ($P4$). The baseline values of these parameters, estimated from field data, were, respectively, 1.30, 0.30, 0.20, and 250. Perturbations for the parameters were selected randomly from independent Normal distributions, with zero means and standard deviations equal to 40% of the baseline parameter value.

Thirty such sets of parameter values were selected, and model output determined for each selection. A Biplot of the resulting sensitivity data matrix, rescaled by multiplication by 1.5 to increase the absolute length of the column vectors, is given in Fig. 1.

The lines on the Biplot represent the columns of the sensitivity data matrix, the line labelled $P1$, for example, representing the maximum birth rate in the model population. The lengths of the lines are proportional to the observed variance of the corresponding column. Thus, parameter $P3$ has the greatest variance, and output variate $V7$ the smallest.

The lines representing model output are labelled $V5$, $V6$ and $V7$. Of these, variates $V5$ and $V6$ have roughly the same variance, and $V7$ shows less variation. The indication is, therefore, that the model output is primarily varying on variates $V5$ and $V6$.

The orientation of the vectors provides relevant information—the cosine

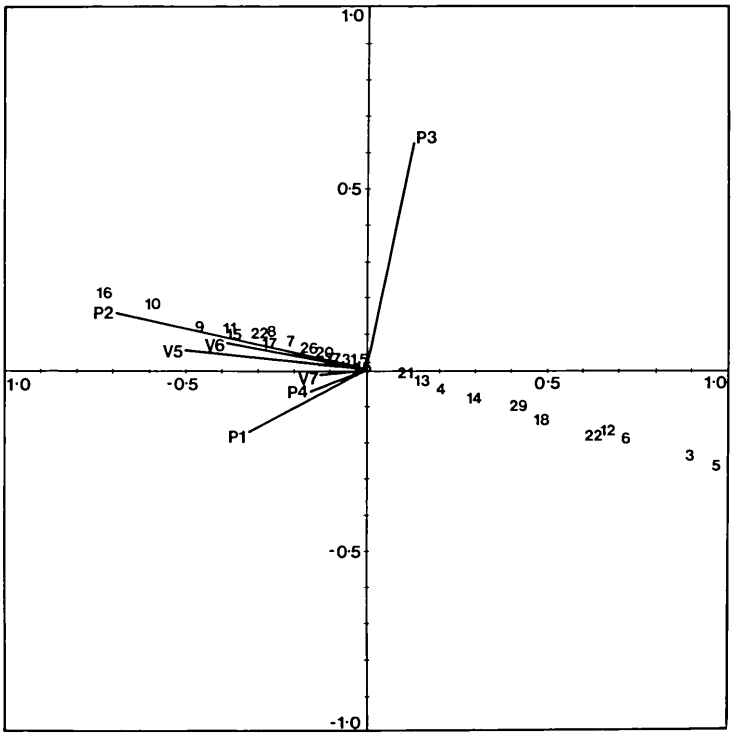

Fig. 1. A Biplot of sensitivity data generated by perturbing parameters in a model simulating the growth and control of rats on farms. Lines labelled *P*1, *P*2, *P*3 and *P*4 represent, respectively, parameters measuring the maximum birth rate in the model population, the turnover rate of the model population at stable size, the rate of immigration, and the stable population size. Lines labelled *V*5, *V*6, and *V*7 represent quantities output by the model: measures of success of chemosterilant control, permanent baiting control, and combined rodenticide and chemosterilant control, respectively. Numbered points represent different runs of the simulation model, each with a different selection of parameter values.

of the angle between any two lines being an approximation to the correlation coefficient between the corresponding columns. Thus variates *V*5 and *V*6 are highly correlated with each other, since the angle between the vectors is about 5 degrees and the cosine about 1. Both variates *V*5 and *V*6 are also quite highly correlated with *V*7.

Having established that *V*5 and *V*6 are the model outputs that vary most in response to parameter perturbation, the next step is to examine the apparent correlations between parameter vectors and these two output variates, to see which parameters appear to be related most strongly to the output variates.

The Biplot shows clearly that parameter *P*2 is most highly correlated with output variates *V*5 and *V*6. The greater angles between vectors *P*1, *P*3 and *P*4, and variates *V*5 and *V*6 indicate progressively lower correlations. The

vector representing $P3$, for example, is at an angle of about 90° to both $V5$ and $V6$, indicating a very low correlation.

The general conclusion is, therefore, that the model output is most sensitive to parameter $P2$, and that this parameter induces variation primarily in output variates $V5$ and $V6$.

The numbered points 1–30 in Fig. 1 represent the 30 runs of the model, each with a random selection of parameter values. The points lie in a reasonably straight line in the same direction as vectors $V5$ and $V6$. The orientation of any point on the Biplot is determined by those columns in the sensitivity data matrix that contribute most strongly to its deviation from the mean output. Thus, point 16 on Fig. 1, lying outermost in the direction of $V5$ and $V6$, has the highest overall values of these two variates, while point 5, lying in the opposite direction, has the lowest overall values.

For the model used in the present illustration, there is no real indication of a sensitive response. The output in Fig. 1 forms a smooth continuum from high output values of $V5$ and $V6$, to low output values, with no response deviating markedly from this general trend. Any response likely to be judged sensitive would be expected to be markedly displaced from the other points on the Biplot.

To summarise, the results show that the population turnover rate (parameter $P2$) is the parameter to which the model is most sensitive, and that this parameter induces variation mainly in chemosterilant success ($V5$) and permanent baiting success ($V6$).

CONCLUSION

If systems models are to achieve their full potential as aids to environmental management, it is important that their predictions should be properly understood and placed in perspective. Sensitivity analysis has an important role to play in this process, by clarifying the nature and extent of the dependence of output on parameter values. All methods of sensitivity analysis are designed to assist the systems modeller to describe adequately the extent to which model output varies in response to changes in model parameters. The method described in this paper fulfils the same function, but extends the sensitivity analysis techniques available to the systems modeller in that it is specifically designed to aid the assessment of multivariate sensitivity data. When used in conjunction with existing univariate techniques, the method offers the prospect of a clear description of model sensitivity.

Sensitivity analysis techniques, including the one presented here, serve only to describe the extent to which model output varies in response to parameter perturbations, and are only one approach to the problem of evaluating a system model (Naylor and Finger, 1967; Shaeffer, 1980).

Regardless of the extent of the output variation which is apparent, a sensitivity analysis itself cannot indicate whether or not a model is useful. In order to reach such a conclusion the apparent variation must be considered with regard to the purpose for which the model was constructed (see e.g. Naylor and Finger, 1967; Van Horn, 1971).

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DEFINITION AND PROPERTIES OF A COEFFICIENT OF SENSITIVITY FOR MATHEMATICAL MODELS

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ABSTRACT

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This paper contains a definition of a measure of the uncertainty inherent in the output of a mathematical model. The definition is essentially a formalisation of the kind of uncertainty measures that have been previously used in various modelling applications in many disciplines. The characteristics of the measure are described, and an example is given of its use with a particular mathematical model.

INTRODUCTION

As the use of mathematical models becomes more widespread in all fields of science and technology, the need to use the predictions from models in an efficient and rational way becomes more pressing. Such predictions are strictly conditional upon the specific structure of the model, and also upon the values assigned to certain parameters within the model. Generally, there is uncertainty about the true values of such parameters, and therefore uncertainty is inherent in the predictions of the model. Clearly, if mathematical models are to achieve their full potential as aids to management, this uncertainty must be properly assessed and described. This has been recognised since the early days of practical mathematical modelling (see e.g. Moffei, 1959), and consequently a variety of techniques for assessing and describing this uncertainty have been proposed in the literature of many disciplines. Such techniques are variously referred to as methods of "sensitivity analysis", "error analysis", or "uncertainty analysis". They vary considerably in complexity and applicability. No single method is suitable for all models, and many techniques have been developed in the context of specific applications.

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The most extensively investigated methods are those deriving from the control theory and electrical engineering literature (Brayton and Spence, 1980). Here, sensitivity is defined in terms of differential coefficients (see e.g. Tomovic, 1963; Tomovic and Vukobratovic, 1970; Cruz, 1973). Such sensitivity measures have been advocated for use in ecological applications (e.g. McCuen, 1973; Hudetz, 1975), but have also been criticised on the grounds that the implicit assumption of linearity (e.g. Dwyer and Kremer, 1983), and the computational burden required to obtain solutions for large models (e.g. Steinhorst et al., 1978), restrict their applicability. Computational techniques and analytic methods abound in the literature; for a recent example see Kercher (1983).

Propagation of error, either analytically or by means of Monte Carlo simulation, is a popular technique. This requires that the uncertainty about parameter values be described by means of probability distributions. Methods from basic statistical theory, or approximations to such methods, may then be used to derive distributional properties, such as means and variances, of the model output. Examples of this approach may be found in the work of Wright (1972), McCuen (1973), Burns (1975), Argentesi and Olivi (1976), Reckhow and Chapra (1979), Shaeffer (1980), Bailey and Duppenthaler (1980), and many others. A convenient summary of the necessary formulae is given by Clifford (1973). However, analytical methods can only be applied to relatively simple mathematical models, and the alternative—direct error propagation by Monte Carlo simulation—is often pursued. Examples may be found in the work of Burns (1975), O'Neill et al. (1980), Gardner et al. (1981), and Dwyer and Kremer (1983). Kremer (1983) has drawn attention to the theoretical implications of exactly how uncertainty is modelled in this approach, and McKay and co-workers have extensively discussed techniques of selecting parameter values from the specified probability distributions in the case of sensitivity analysis (McKay et al., 1976, 1979; Iman and Conover, 1980).

Because of the complexity of many mathematical models a number of authors have suggested that sensitivity analyses may conveniently be carried out on response surfaces fitted to the empirically determined relationship between parameter values and model output. Blanning (1974) coined the term "metamodel" to describe such response surfaces. Examples of this approach are described by Plinston (1972), Richels (1978), Argentesi and Olivi (1978) and Kohberger et al. (1978). The value of this technique is dependent upon the goodness-of-fit of the response surface.

In addition to the above techniques, many other methods of sensitivity or uncertainty analysis have been described, in the literature of many disciplines. These include a variety of simple sensitivity coefficients (e.g. Abouel-Nour, 1967; Jones, 1967; Ali, 1968; Mann, 1971; Brown et al., 1978;

Summers and McKellar, 1981), some simple uncertainty measures (e.g. Moffei, 1959; Reckhow, 1980; Reckhow and Simpson, 1980), Fourier transform techniques (e.g. Cukier et al., 1978), and graphical methods (Plinston, 1972; Huson, 1982).

The purpose of this paper is to give a formal definition of a simple measure of the uncertainty in a model output. The method has not been described in this form before, but is essentially a formalisation of the kind of approach that has been used in the past in specific applications (e.g. Moffei, 1959; Reckhow, 1980; Wong, 1980). It may prove to be a useful addition to the many techniques of sensitivity analysis that already exist in the literature.

GENERATING DATA

Consider a mathematical model having p parameters x_i $(i = 1 \ldots p)$ and v output variates y_j $(j = 1 \ldots v)$. Let the vector $(\tilde{x}_i;\ i = 1 \ldots p)$ be the vector of baseline or best estimate parameter values, and the vector $(\tilde{y}_j;\ j = 1 \ldots v)$ be the corresponding vector of output variates. Data for sensitivity analysis are commonly generated by assigning a joint probability distribution to the p model parameters, and using this distribution to generate, say, n sets of parameter vectors $(x_i;\ i = 1 \ldots p)_r$ $(r = 1 \ldots n)$. Each of the n parameter vectors may then be used in turn in the mathematical model to generate a corresponding vector of outputs. Denote the rth parameter vector and its corresponding output vector by $(x_i)_r$ and $(y_j)_r$, respectively.

DEFINITION OF A MEASURE OF SENSITIVITY

Consider in turn each of the vectors $(y_j)_r$ $(r = 1 \ldots n)$, generated as specified above. For each vector assign a score

$$s_r = \sum_{j=1}^{v} d_j / v \tag{1}$$

where

$$\begin{aligned} d_j &= 0 \quad \text{if } |y_j - \tilde{y}_j| \leqslant \alpha \tilde{y}_j \\ d_j &= 1 \quad \text{if } |y_j - \tilde{y}_j| > \alpha \tilde{y}_i \end{aligned} \tag{2}$$

then the coefficient

$$S_\alpha = \sum_{r=1}^{n} s_r w_r / \sum_{r=1}^{n} w_r \tag{3}$$

where w_r is a weighting coefficient, is defined to be the 100 α% sensitivity coefficient of the v outputs for the model.

PROPERTIES OF THE PROPOSED COEFFICIENT

The above coefficient is defined as the proportion of model outputs that lie outside a boundary centred on the baseline output value. This conceptually simple measure of model sensitivity has several appealing properties:

(i) The coefficient is bounded by the values 0 and 1. This standardisation makes interpretation much easier and avoids the confusion that can arise with coefficients that are unbounded and may differ by several orders of magnitude.

(ii) The coefficient is defined for models having any number of parameters, and any number of output variates, and may also be calculated separately for each output produced by the model.

(iii) Because the coefficient is independent of scale in the sense of (i) above, it is easy to compare the sensitivities of different models.

(iv) The derivation of the coefficient makes no assumptions about the form of the induced relationship between parameters and model outputs.

(v) The coefficient is simple to compute once sensitivity data are generated—the computation may be done manually.

(vi) The coefficient has an obvious and immediate probabilistic interpretation, being based on the proportion of model outputs that lie outside a certain range.

THE WEIGHTING COEFFICIENT

The quantity w_r in (3) above is a weighting coefficient for the vector $(y_j)_r$. In some cases it may be appropriate to weight each output vector equally, in which case $w = 1$, $\forall\, r$, and (3) reduces to

$$S_\alpha = \sum_{r=1}^{n} s_r/n \tag{4}$$

However, it will often be more appropriate to weight each vector of outputs according to the chance that such a vector may be realised, i.e. to use as a weight w, a quantity proportional to the corresponding likelihood of the vector $(x_i)_r$.

AN EXAMPLE

The calculation of the sensitivity coefficient described above may be illustrated using the model proposed by Anderson et al. (1981). The model consists of a set of three differential equations containing six parameters. These parameters are (i) the per capita birth rate of the red fox (r), (ii) a parameter measuring the severity of density dependent mortality control

(gamma), (iii) a parameter measuring the rate of transition of foxes from the susceptible to the infected state (beta), (iv) the reciprocal of the average latency period of rabies in the red fox (sigma), (v) the per capita death rate of the red fox (b), and (vi) the reciprocal of the death rate of rabid foxes (alpha). This model has been solved numerically to yield four output variates: (i) the number of infected foxes in the population after a simulated 12 month period, (ii) the number of infectious foxes in the population after a simulated 12 month period, (iii) the maximum number of infected foxes in the population over a simulated 20 year period, and (iv) the minimum size of the population over a simulated 20 year period. All simulations were carried out for an initial population size of 1200 animals, with 15 rabid foxes in the population. For more details of the model, the description by Anderson et al. (1981) should be consulted.

The baseline values of the six parameters described above are given in Table I, together with the output values they generate when used in the model. There is, of course, some uncertainty about the true values of these parameters, and the effects of this on the uncertainty in model output should be described in some appropriate way. For the present purposes of illustration only, the uncertainty about these parameter values has been represented by uniform distributions, each centred on the baseline parameter value, and having a range equal to 30% of the baseline value. The results of repeated

TABLE I

Baseline values of the six parameters and four output variates from the red fox rabies model

Quantity	Baseline value
r	0.50
gamma	0.25
beta	79.69
sigma	13.00
b	0.50
alpha	73.00
Number of infected foxes after 12 months	23
Number of infectious foxes after 12 months	4
Maximum number of infected foxes over 20 yr. period	31
Minimum fox population size over 20 yr. period	1138

TABLE II

Sensitivity data from the fox rabies model

Number of infected foxes after 12 months	Number of infectious foxes after 12 months	Maximum number of infected foxes over 20 yr. period	Minimum fox population size over 20 yr. period
45.56	4.51	54.28	999
27.72	3.61	33.25	1074
31.68	5.57	32.90	999
16.87	3.91	17.14	1149
23.47	5.01	23.48	1068
71.74	12.12	86.52	965
19.85	3.19	20.55	1151
11.07	3.43	12.80	1104
98.40	10.35	112.97	939
19.76	2.15	23.39	1146
48.38	7.76	52.05	981
21.93	3.25	25.64	1151
23.33	4.41	23.56	1119
28.87	3.67	31.36	1062
38.50	7.53	44.62	990
37.04	8.11	45.19	1092
21.73	4.31	21.87	1124
23.05	3.48	30.00	1126
61.51	11.07	63.87	1038
30.77	6.74	32.13	1055
50.49	7.86	68.47	984
27.29	5.34	33.09	1088
48.74	10.84	50.22	1064
29.51	4.96	32.68	1108
50.11	14.16	51.60	954
23.14	3.61	23.39	1054
31.85	7.57	31.94	1019
37.94	4.39	45.18	1005
36.12	6.95	39.62	1015
34.13	5.16	42.98	1067

runs of the model with parameter values selected from these distributions are given in Table II.

The calculation of a 20% sensitivity coefficient for this model proceeds as follows. The first step is to define, for each output variate, a region centred on the baseline output and bounded by values of 0.8 and 1.2 times the baseline value. The appropriate regions for each of the four output variates are given in Table III. Then, for each output vector in turn, the score s_r is

TABLE III

Regions centred around baseline values of model output and bounded by values ±20% of the baseline

Output	Region
No. foxes infected after 12 months	18.4–27.6
No. foxes infectious after 12 months	3.2–4.8
Max. no. of infected foxes in 20 yr. period	24.8–37.2
Min. population size over 20-year period	910–1366

computed, as described above. Thus, for the first output vector, the value for the number of infected animals in the population after a simulated 12 month period, and the value of the maximum number of infected animals in the population after a simulated 20-year period, both lie outside the corresponding 20% range. The score for this vector is therefore 2/4. The scores for other vectors are calculated similarly, and the resultant scores converted into the sensitivity coefficient defined above. Since independent uniform distributions were assigned to the parameters used to generate the output vectors, each vector has been weighted equally. The resulting 20% sensitivity coefficients, for each output variate separately and for the model as a whole, are given in Table IV.

TABLE IV

20% sensitivity coefficients for the fox rabies model of Anderson et. al. (1981), for the four output variates described in the text

	20% sensitivity coefficient
No. foxes infected after 12 months	0.70
No. foxes infectious after 12 months	0.63
Max. number of infected foxes in 20 yr. period	0.70
Min. population size over 20 yr. period	0.00
Whole model (all four output variates)	0.51

USES OF THE COEFFICIENT

The values of the sensitivity coefficient defined above all lie, by definition, between 0 and 1, with higher values indicating that a greater proportion of the generated output lies outside the range defined. Confining the coefficient values in this way improves the utility of the coefficient, since it becomes easy to compare the sensitivity of different outputs within a model, or between different models (see e.g. McCuen, 1973). A 20% sensitivity coefficient of, say, 0.5, immediately conveys something of the uncertainty inherent in the output—it says that 50% of the output values generated by the model lie outside the range bounded by 0.8–1.2 times the baseline. The higher the value of the coefficient the greater is the dispersion of model output values, and the greater the uncertainty in the output. Thus, for example, in the rabies model above, the least sensitive output is the minimum size of the simulated population—no values of this latter output lie outside the 20% range.

A further advantage of the definition proposed above is that the choice of α permits the "severity" of sensitivity analysis to be selected by the modeller. A severe test of a model, for example, might be made by calculating the 1% sensitivity coefficient. This would indicate the proportion of model outputs that lie outside the corresponding baseline $\pm 1\%$. A low value for such a coefficient would strongly suggest that such a model is very robust to the uncertainty that exists about its parameter values. The requirement for the modeller to choose α forces a definition to be made, at least implicitly, of the level of uncertainty in output that is considered to be important.

USE OF THE COEFFICIENT IN FORMAL DECISION MAKING

The sensitivity coefficient proposed above may therefore be used in informal assessment of the level of uncertainty inherent in the output from a mathematical model. However, the use of the coefficient also extends to more formal decision making. Consider a region of model output defined by the vectors $(\tilde{y}_j + \alpha\tilde{y}_j;\ j = 1 \ldots v)$, $(\tilde{y}_j - \alpha\tilde{y}_j;\ j = 1 \ldots v)$. The proportion of model outputs that lie outside this region is the 100 α% sensitivity coefficient for the model. Suppose that this region defines a "state-of-nature" which, if it were known to be true, would lead a decision-maker to take action A. If the true state of nature lies outside this region, then suppose the decision-maker would take action B. The decision problem occurs because the true state of nature is not known. However, the 100 α% sensitivity coefficient may be interpreted probabilistically, since it is the proportion of model outputs lying outside the region defined by the vectors. Similarly, the value $1 - S_\alpha$ may be taken to approximate the probability, as indicated by the

model, that the first state of nature, implying action A, is true. Thus S_α, in addition to its other advantages, may be directly employed, in conjunction with a set of possible actions and their utilities under different possible states of nature, in a conventional decision–theoretic framework (see e.g. Lindgren, 1971). The use of the coefficient in this way will be illustrated elsewhere.

CONCLUSION

In many modelling applications in the literature, simple uncertainty measures have been used. The definition given here has not been explicitly used in the past, though many essentially similar concepts have been applied. The measure defined above has a number of useful characteristics, including simplicity, ease of computation, and probabilistic interpretation, and may prove to be a useful addition to the many techniques already described both in the ecological literature and elsewhere.

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