NON COMPACT SYMMETRIC SPACES AND THE TODA MOLECULE EQUATIONS

by

Luiz Agostinho Ferreira

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Department of Physics
Blackett Laboratory
Imperial College
London SW7 2BZ

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Abstract

It has been shown by Olshanetsky and Perelomov that the Toda Molecule (TM) equations associated with any simple Lie group G describe special geodesic motions on the Riemannian non compact symmetric space which is the quotient of the normal real form of G, G^N, by its maximal compact subgroup. We explain this in more detail and show that the "fundamental Poisson bracket relation" involving the Lax operator A and leading to the Yang-Baxter equation and integrability properties is a direct consequence of the fact that the Iwasawa decomposition for G^N endows the symmetric space with a hidden group theoretic structure. We extend this geometric picture to the quantum level by implementing a quantum reduction procedure where the solutions to the Schroedinger equation for the Toda Molecule systems are seen as projections from the free wave functions on the symmetric spaces G^N/K. The algebraic structure of the classical model holds true at the quantum level and leads to the "fundamental commutation relation" and the quantum Yang-Baxter equation. We then show the quantum integrability of the TM systems by constructing the quantum conserved quantities in involution. Our analysis of the classical and quantum geodesic motions applies uniformly to all the non-compact Riemannian globally symmetric spaces.

To my wife FATIMA

com muito amor

Preface

The work presented in this thesis was carried out in the Department of Physics, Imperial College, between October 1981 and June 1985 under the supervision of Prof. David I. Olive. The thesis is based in part on a paper written by the author in collaboration with his supervisor and except where stated in the text the work is original, and has not been submitted in this or any other University for any other degree.

I am very grateful to Prof. David I. Olive for his frequent helpful advice throughout this work and for all I have learnt from him in the past years. I would like to thank the Physics and Mathematics Departments of the University of Virgina (USA) for the hospitality during the period Sep/82 - Jul/83 when the early part of this work was done. I would also like to thank Dr. John Gipson for sending us a copy of his thesis. Many thanks to J. F. Gomes for helpful discussions, and to other members of the Theory Group (IC).

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CONTENTS

1. Introduction

In recent years there has been an increasing interest in the non-linear aspects of gauge theories. The reason is that several features of these theories, such as the classical soliton solutions, have been indicated that they possess symmetries much richer and deeper than those ones that can be directly inferred from the lagrangian. The study of these non-linear phenomena is of crucial importance for a better understanding of the structure of gauge theories and it can help clarifying the role of solitons in the quantum theory. It is believed that solitions, like the magnetic monopoles, correspond to particles in the spectrum of the quantized theory. These ideas are based in the early work of Skyrme (1) and some other results in two dimensional field theory (2). Electromagnetic duality conjectures (3) have already been proposed to explain how this would work in four dimensional field theory and their validity seem to be most favoured in the N=4 supersymmetric gauge theories (4).

The techniques used in their studies have attracted the attention not only of theoretical physicists but of mathematicians too. The reason is that several topics in physics and mathematics that have developed in a quite independent way have now been shown to be intrinsically related. Much attention has been given to the study of integrable systems (5,6) because it is believed that integrability is in some way connected to the existence of solitons solutions in a given theory. We say a system is integrable (7) if it possess a number of conserved quantities in involution, i.e. with vanishing Poisson bracket, that is equal to the number of degrees of freedom. When that happens it is possible to find a canonical transformation to a new set of canonical variables (the action-angle variables) where the conserved quantities are the momenta and the Hamiltonian can be written in terms of them only, with coordinates eliminated. The Hamilton equations can then be easily integrated. The relation between integrability and solitions is not well established yet and effort has been made to understand the structures and symmetries underlying the integrability properties that are common to most integrable systems. Field theory models in four dimensions are in general very complex and it is sensible to carry out these studies with simpler models in one and two dimensions since they present features very similar to those of gauge theories.

In this thesis we consider the Toda molecule models in one dimension. The story of these models go back to the 50's when Fermi, Pasta and Ulam (8) did a computer experiment to study the ergodic behaviour of a system of several particles in a line interacting with their nearest neighbours via non-harmonic springs. If the system were ergodic the trajectories on the phase space would eventually fill it entirely. But instead they noticed that after a finite time the initial configuration was repeated indicating the existence of conserved quantities that constrained the phase space. Later the Japonese physicist Toda(9) noticed that the system could be solved analytically if the spring tension were proportional to the exponential of the distance between the particles. When written in some suitable coordinates the potential energy for such a system takes the form $\exp(K_{ab}\phi_b)$, where K_{ab} is the Cartan matrix (10) for the Lie algebra of SU(r+1) (r being the number of particles). The occurrence of this matrix is not accidental an it was later realized that the algebra of SU(r+1) does play a role in these models. Such facts motivated the generalization of this model to the cases where K is the Cartan matrix of any simple Lie group. These models are now called Toda Molecule (TM).[11] The equations of motion for the Toda Molecule systems are given by:

$$\frac{d^2\phi_a}{dt^2} = - \exp(\sum_b K_{ab}\phi_b) \qquad a,b = 1,2 \ldots r \qquad (1.1)$$

where the non-singular matrix K is the Cartan matrix for a simple Lie group G (of rank r).

The solutions to these models have already been constructed [12,13] and they have also been shown to be completely integrable [14,15]. Eq. (1.1) (with t replaced by i times radius) is known to govern the radial dependence of certain spherically symmetric monopole solutions [16,17] in the Bogomolny-Prasad-Sommerfield limit[18,19]. No solitons solutions have been found to the Toda Molecule equations and one reason is that the ground state correspond to divergent values of ϕ.

If we replace the matrix K in (1.1) by the extended Cartan matrix [20] for an affine Kac-Moody algebra [21] we obtain the so called Toda Lattice TL models. These are also completely integrable systems [14]. The Cartan matrix for these algebras is singular and therefore it has a null vector. This means we can find certain linear combination of fields which satisfies the free equation of motion (with vanishing potential). This combination can consistently be set zero and so the Toda Lattice equations have a unique constant solution defining a ground state. Perhaps this is the most striking difference between the TM and TL models. The Toda Lattice models possess solitons solutions and the Sine-Gordon and Bollough-Dodd (22) models are examples of TL models.

In this thesis we explore an interesting feature of the Toda Molecule models which some authors (12) believe may underly all integrable systems. The solutions of these models, as it was first observed by Olshanetsky and Perelomov (12), can be viewed as projection of certain geodesic motions on a symmetric space G^N/K explained below. We develop this geometric picture further to study the integrability properties of the TM models. We also extend this picture to the quantum level by showing how to construct solutions to the Schroedinger equation for the TM models from free wave functions on the symmetric space G^N/K.

Our analysis of the classical and quantum geodesic motion works in a similar way for all non-compact Riemannian globally symmetric spaces (23) and we think other integrable models can be obtained by using a reduction procedure similar to that one used for the spaces G^N/K.

The symmetric spaces we consider are coset spaces F/K where F is a real non-compact simple Lie group furnished with a Cartan involution σ ($\sigma^2 = 1$, $\sigma \neq 1$). K is the subgroup of F invariant under σ and it is a maximal compact subgroup (23) (see section 2). These spaces are non-compact, Riemannian and globally symmetric. They have very special properties due to the Iwasawa decomposition of F as we now explain. The Lie algebra $\underset{\sim}{f}$ of F is decomposed in even and odd subspaces under σ

$$\underset{\sim}{f} = \underset{\sim}{k} + \underset{\sim}{p} , \quad \sigma(\underset{\sim}{k}) = \underset{\sim}{k} , \quad \sigma(\underset{\sim}{p}) = - \underset{\sim}{p} \tag{1.2}$$

with $\underset{\sim}{k}$ being the Lie algebra of K. Let $\underset{\sim}{a}$ denote a maximal abelian subspace of $\underset{\sim}{p}$. According to Helgason [(23)] the algebra of $\underset{\sim}{f}$ is decomposed as

$$\underset{\sim}{f} = \underset{\sim}{n} + \underset{\sim}{a} + \underset{\sim}{k} \tag{1.3}$$

where $\underset{\sim}{n}$ is a maximal nilpotent subalgebra of $\underset{\sim}{f}$ and it is a direct sum of the subspaces $\underset{\sim}{f}_\lambda = \{X \varepsilon \underset{\sim}{f} \ : \ [H,X] = \lambda(H)X \text{ for } H \varepsilon \underset{\sim}{a}\}$ forpositive roots λ's. $(\underset{\sim}{n} = \sum_{\lambda > o} \underset{\sim}{f}_\lambda)$. Accordingly the elements of the group F decompose as (Iwasawa decomposition)

$$g = n \ a \ k \tag{1.4}$$

where $a \ \varepsilon \exp \underset{\sim}{a}$, n is an element of the connected subgroup of F corresponding to $\underset{\sim}{n}$ and $k \ \varepsilon \ K$. It then follows that the quantities na can be used to parametrize the cosets of F which constitute the points of the symmetric space F/K. These quantities na form a group $\mathbb{B}$ which is solvable and whose Lie algebra will be relevant for the integrability. The spaces F/K are then endowed with a hidden group theoretic structure given by the group $\mathbb{B}$.

The symmetric spaces relevant for the Toda molecule models are the coset spaces G^N/K and they are examples of the symmetric spaces F/K we just described. G^N is the "normal" form (real and non-compact) of the complex Lie group G whose Cartan matrix appear in (1.1). Its generators are the Cartan subalgebra generators H_i (i=1,2...rank G^N) and the step operators E_α (see section 5). The relevant Cartan involution here is

$$\sigma(H_i) = - H_i \quad , \quad \sigma(E_\alpha) = -E_{-\alpha} \tag{1.5}$$

Therefore the generators of the maximal compact subgroup K, in this case, are $E_\alpha - E_{-\alpha}$ (α any root) and the maximal abelian subspace $\underset{\sim}{a}$ of $\underset{\sim}{p}$ is the Cartan subalgebra of G^N whose generators are the H_i's. Th nilpotent subgroup n is obtained by exponentiating a real linear combination of the positive root step operators E_α (α>o). Here, the group $\mathbb{B}$ is the Borel subgroup of G^N which by definition is the maximal solvable subgroup of G^N.

The geodesic motion on the Riemannian non-compact symmetric spaces F/K is described in section 2. By defining

$$g^{-1}\frac{dg}{dt} = A + iB \tag{1.6}$$

where A and B are odd and even respectively under σ, we show that the Lagrangian for such motion is:

$$\mathcal{L} = \mathrm{Tr}(A^2)/2 \tag{1.7}$$

and that the corresponding equation of motion

$$\frac{dA}{dt} + i[B,A] = 0 \tag{1.8}$$

takes the form of a Lax pair equation [24].

The Lagrangian (1.7) and, more importantly, the corresponding Hamiltonian are automatically positive. This is a reflection of the Riemannian nature of the symmetric space. If we choose K = 1 in (1.4) which means that we are parametrizing the point of the symmetric space by an element of $\mathbb{B}$ then (1.6) can be expanded in terms of the generators of $\mathbb{B}$ with coefficients which are functions of the coordinates and momenta.

In section 3 we explain how to evaluate the Poisson bracket algebra of these coefficients by using a refined version of Noether's Theorem (proved in appendix I). Since the coefficients turn out to be Noether charges for right transformations

$$b \rightarrow b'' = b\, b' \qquad b,\, b' \in \mathbb{B} \tag{1.9}$$

their Poisson bracket algebra is the Lie algebra of $\mathbb{B}$ (even though these quantities are only partially conserved by virtue of (1.8)).

The Lagrangian (1.7) is invariant under left transformations

$$g \rightarrow g'' = g'\, g \qquad g,\, g' \in F \tag{1.10}$$

and using Noether's Theorem we show that the conserved charges associated with this symmetry are the coefficients of $X = \dot{x}x^{-1}/2$ when it is expanded in terms of the generators of F. x is the 'principal variable" [12,25,26] introduced in (2.6).

We also show that the Poisson bracket algebra of these charges is the Lie algebra of F.

The results of section 3 are used in section 4 to construct what Faddeev [5] calls the Fundamental Poisson Relations (FPR). By using the expression for the variation of the right charges under right transformations we construct, in a quite simple way, the FPR for the Lax operators A and B. The FPR for the left charges $X = \dot{x}\, x^{-1}/2$ is also constructed using the same method.

In section 5 we show that for the particular case of the symmetric spaces G^N/K the Fundamental Poisson Relation for A and B takes the form

$$\{A + iB \overset{\otimes}{,} A + iB\} = - \left[\mathbf{P},\ (A + iB) \otimes 1 + 1 \otimes (A + iB)\right] \tag{1.11}$$

where $\mathbf{P}$ is the quantity previously constructed by Turok and Olive [14]. We also explain that because of a similarity in structure between $\mathbf{P}$ and $A + iB$ the equation (1.11) can be written in the form of a classical Yang-Baxter equation [27,28]

We then explain, in section 6, how to restrict in a consistent way the geodesic motion on the symmetric spaces G^N/K to obtain the Toda Molecule equations (1.1). Essentially this is done by choosing some particular values for the conserved left charges corresponding to the positive root step operators E_α ($\alpha > 0$) which is consistent with the algebra of these charges. The solutions to the TM equations are those geodesics which satisfy these initial conditions. We also show how to reduce the Lax and FPR equations (1.8) and (1.11) to their known forms appropriate to (1.1). We believe that a similar reduction procedure can be applied to the symmetric spaces F/K to obtain other integrable systems.

We then turn our attention to the quantum free motion of a particle on non-compact Riemannian symmetric spaces. The Hamiltonian for such motion, as it is explained in section 7, is the Laplace-Beltrami operator, which is the generalization to curved space of the ordinary Laplacian for scalar functions on flat space.

Our motivation for this study is that the geometric picture of Olshanetsky and Perelomov is also valid at the quantum level. By using a reduction procedure [(26)] we show, in section 11, that the solutions of the Schroedinger equation for the Toda Molecule systems can be obtained from the free wave functions on the symmetric spaces G^N/K.

As we explain in section 7, there is no ambiguity in the construction of quantum operators for classical quantities which are linear in the momenta. In addition if these quantities satisfy some algebra under the Poisson bracket then their corresponding quantum operators satisfy the same algebra under the quantum commutator. Since the Noether charges corresponding to the left and right transformations are linear in the momenta t then follows that the agebraic structure underlying the classical geodesic motion on the symmetric spaces F/K holds true at the quantum level. We then explore these symmetries to study the quantum integrability properties of the Toda Molecule systems.

In section 8 we construct the Laplace-Beltrami operator for the non-compact symmetric spaces F/K. In section 9 we define the quantum right and left transformations and construct the quantum operators for the corresponding Noether charges. We then show that the Fundamental Poisson Relations (FPR) remain true in the quantum theory when we replace the Poisson bracket by the quantum commutator as a consequence of the fact that the classical and quantum algebras of the Noether charges are the same. These relations are then called the Fundamental Commutation Relations (FCR). We also construct the quantum analog of the Lax pair equation by introducing modified Lax operators which contain terms which are quadratic in the generators of the Lie algebra of the solvable subgroup $\mathbb{B}$.

In section 10 we specialize to the symmetric spaces G^N/K. and show that the quantum version of the relation (1.11) lead us to the quantum Yang-Baxter equation. We then explain, in section 11, the quantum reduction procedure. We show that the solutions of the quantum Toda Molecule systems can be obtained by projecting the free wave functions on G^N/K onto eigenfunctions of the quantum operators for the left charges corresponding to the positive root step operators.

In section 12 we construct the quantum conserved quantities in involution and show that the Toda Molecule systems are integrable at the quantum level.

2. Geodesic Motion

Much of our work applies to the geodesic motion of a particle moving on any Riemannian non-compact symmetric space and we shall therefore postpone specialization to the special type described in the introduction to section 5. It is this type which leads to the Toda equations (1.1) when appropriate constraints are made. Let F denote any non compact simple Lie group furnished with a Cartan involution σ[23]. That σ is an involution means that it is an automorphism of F satisfying

$$\sigma^2 = 1 \qquad (2.1)$$

The generators of the Lie algebra $\underset{\sim}{f}$ of F can be split into two subspaces even or odd under σ (by 2.1):

$$\underset{\sim}{f} = \underset{\sim}{k} + \underset{\sim}{p} \ , \quad \sigma(\underset{\sim}{p}) = -\underset{\sim}{p} \ , \quad \sigma(\underset{\sim}{k}) = \underset{\sim}{k} \qquad (2.2)$$

It follows that as σ is also an automorphism of $\underset{\sim}{f}$

$$[\underset{\sim}{k},\underset{\sim}{k}] \subset \underset{\sim}{k} \ , \quad [\underset{\sim}{k},\underset{\sim}{p}] \subset \underset{\sim}{p} \ , \quad [\underset{\sim}{p},\underset{\sim}{p}] \subset \underset{\sim}{k} \ . \qquad (2.3)$$

If T and S are elements of $\underset{\sim}{f}$ we have a σ invariant Killing form which we write as

$$\mathrm{Tr}(TS) = \mathrm{Tr}(\sigma(T)\sigma(S))$$

The Cartan property is that (in our notation)

$Tr(T\sigma(S))$ is negative definite. (2.4)

So, by (2.2)

$$Tr(TS) \begin{cases} \text{is positive definite if } T,S \in \underset{\sim}{p} \\ \text{is negative definite if } T,S \in \underset{\sim}{k} \\ \text{is zero if } T \in \underset{\sim}{p} \text{ and } S \in \underset{\sim}{k} . \end{cases} \quad (2.5)$$

$\underset{\sim}{k}$ generates a subgroup $K \subset F$ which is a maximal compact subgroup. We think of the remaining generators $\underset{\sim}{p}$ as "non compact" generators. K is also the subgroup of F invariant with respect to σ. For any element $g \in F$ we define a "principal variable"[12,25,26] (see appendix V):

$$x(g) = g\sigma(g)^{-1} \quad \text{so } \sigma(x) = x^{-1} \quad . \quad (2.6)$$

As x (gk) = x (g) if $k \in K$, we see that x is actually defined on the cosets F/K which constitute the points of the symmetric space F/K. In fact, there is a unique correspondence between the cosets and the point x and we are going to regard x as one way of labelling a point of the symmetric space. The geodesic equation of motion for a particle on the symmetric space is

$$\frac{d}{dt}(x^{-1}\dot{x}) = 0 \quad . \quad (2.7)$$

We shall show that the Lagrangian leading to this can be written

$$\mathcal{L} = Tr(x^{-1}\dot{x})^2/8 \quad . \quad (2.8)$$

In order to do this it is useful to define

$$g^{-1}\dot{g} = A + iB \quad , \quad A \in \underset{\sim}{p} \; , \; B \in \underset{\sim}{k} \; , \tag{2.9}$$

so that

$$A = (g^{-1}\dot{g} - \sigma(g^{-1}\dot{g}))/2 \quad , iB = (g^{-1}\dot{g} + \sigma(g^{-1}\dot{g}))/2 \tag{2.10}$$

Then it is easy to check that

$$x^{-1}\dot{x} \quad = \quad 2\sigma(g)A\sigma(g)^{-1} \; . \tag{2.11}$$

Hence by (2.8) we can write the Lagrangian in the alternative form

$$\mathcal{L} = \mathrm{Tr}(A^2)/2 \tag{2.12}$$

According to (2.5) and (2.9) the Lagrangian is therefore positive, a reflection of the Riemannian nature of the space. The Hamiltonian will likewise be positive thus assuring a desirable physical feature. Differentiating (2.11)

$$\frac{d}{dt}\,(x^{-1}\dot{x}) \;=\; 2\;\sigma(g)(\dot{A} + i[B,A])\sigma(g)^{-1} \tag{2.13}$$

This shows that the geodesic equation of motion (2.7) is equivalent to a "Lax pair" equation [24]

$$\dot{A} + i[B,A] = 0. \tag{2.14}$$

As $A \in \underset{\sim}{p}$ (equation 2.9) this equation has precisely as many components as the particle has degrees of freedom, namely dim $\underset{\sim}{p}$ = dim (F/K) and is therefore the natural way of writing the equation of motion in preference to (2.7) the components of which contain redundancies as they number dim $\underset{\sim}{f}$.

So far we have not assigned a unique element g to each coset and if we change our choice :

$$g \rightarrow g' = g\,k$$

we find that A and B change accordingly;

$$\begin{aligned} A &\rightarrow A' = k^{-1}Ak \\ B &\rightarrow B' = k^{-1}Bk + k^{-1}\dot{k} \ . \end{aligned} \qquad (2.15)$$

Thus B transforms as a K gauge potential and the Lax equation (2.14) states that A is covariantly constant. Notice that the Lagrangian (2.8) and (2.12) is K gauge invariant.

Let us vary g infinitesimally by a right translation

$$\delta g = g\,R \qquad (2.16)$$

where for the time being R ε $\underset{\sim}{f}$ and is small. Then

$$\delta(g^{-1}\dot{g}) = \dot{R} - [R, g^{-1}\dot{g}] \ , \qquad (2.17)$$

and the response of the Lagrangian (2.12) is

$$\begin{aligned} \delta\mathcal{L} &= \mathrm{Tr}(A\delta(g^{-1}\dot{g})) = \mathrm{Tr}(\dot{R}A - iR[B,A]) \\ &= \frac{d}{dt}\mathrm{Tr}(RA) - \mathrm{Tr}R(\dot{A} + i[B,A]) \ . \end{aligned} \qquad (2.18)$$

If we consider R as an Euler Lagrange variation we conclude that $\mathcal{L}$ is indeed the correct Lagrangian as it yields the correct equation of motion (2.14).

Now vary g infinitesimally by a left variation

$$\delta g = L g \qquad (2.19)$$

where for the time being $L \in \underset{\sim}{f}$ and is small. We find from (2.6) that

$$\delta x = Lx - x\sigma(L) , \qquad (2.20)$$

$$\delta(x^{-1}\dot{x}) = [\sigma(L), x^{-1}\dot{x}] + x^{-1}\dot{L}x - \sigma(\dot{L}) . \qquad (2.21)$$

As

$$\sigma(x^{-1}\dot{x}) = - \dot{x}x^{-1}, \qquad (2.22)$$

we find that the response of the Lagrangian in the form (2.8) is

$$\delta \mathcal{L} = \mathrm{Tr}(\dot{x}x^{-1}\dot{L})/2 = -\mathrm{Tr}(x^{-1}\dot{x}\,\sigma(\dot{L}))/2 . \qquad (2.23)$$

Thus if L is an Euler-Lagrange variation we find the Euler-Lagrange equation of motion in the form (2.7).

We remark that many of the equations derived so far, for example the Lax equation (2.14) hold for any coset space F/K without assuming the symmetric space or other properties. We now come to the crucial point where the special nature of the symmetric space described matters.

It is explained in Helgason [23] how, given a non compact simple group F equipped with a Cartan involution σ, it is possible to construct a maximal abelian subalgebra $\underset{\sim}{a}$ of $\underset{\sim}{p}$ and a maximal nilpotent subalgebra $\underset{\sim}{n}$ such that according to a result of Iwasawa there exists a unique decomposition of any element of F into three factors

$$g = nak \ , \tag{2.24}$$

where n and a are obtained by exponentiating $\underset{\sim}{n}$ and $\underset{\sim}{a}$ respectively with real coefficients and $k \in K$. If σ is the involution (1.5) mentioned in the introduction $\underset{\sim}{a}$ and $\underset{\sim}{n}$ are as specified in (4.3).

Thus given the Iwasawa decomposition (2.24) there exists a "natural" choice of representative from each coset corresponding to each point of the symmetric space, simply $k = 1$. There is no analogue of this for the compact symmetric spaces more commonly studied in physics.

In this "gauge" $k = 1$, $g = na$. Since $\underset{\sim}{n} + \underset{\sim}{a}$ forms a subalgebra of $\underset{\sim}{f}$ this means that the representative points g all belong to the group $\mathbb{B} \subset F$ obtained by exponentiating $\underset{\sim}{b} = \underset{\sim}{n} + \underset{\sim}{a}$. Thus the symmetric space F/K is endowed with a hidden group theoretical structure whose influence on the dynamics of our particles is studied in the next sections.

Given this gauge our Euler Lagrange variations (2.16) should respect it. Thus in (2.16) we should restrict $R \in \underset{\sim}{b}$.

As dim $\underset{\sim}{b}$ = dim $\underset{\sim}{p}$ this still yields the same Euler Lagrange equation (2.14).

3. Exact and Broken Symmetries of the Lagrangian

We now show that the Lagrangian (2.8) or (2.12) exhibits at least three kinds of symmetry, and evaluate the corresponding Noether charges. For certain of these symmetries the Poisson bracket algebra of the associated Noether charges is isomorphic to that of the original infinitesimal variations, despite the fact that in one case the symmetries are not exact but broken. Later we show how these resultant algebras lead to the integrability properties and in particular the Yang-Baxter equations, both for the geodesic motion problem and the Toda molecule equations (1.1)

The analysis depends on a refined version of Noether's Theorem stated in detail and proven in appendix I. We consider the response of a Lagrangian $\mathcal{L}(q,\dot{q})$ to infinitesimal variations of the coordinates q_i,$(i = 1...N)$.

$$\delta q_i = \varepsilon(t) F_i(q,\dot{q})$$

$$\text{and} \quad \delta\mathcal{L} = \frac{d}{dt}(\varepsilon X) + \dot{\varepsilon} Q + \varepsilon D \qquad (3.1)$$

(with X, Q and D depending on q and $\dot{q}$ only and <u>not</u> $\ddot{q}$)

Q, the Noether charge is read off as the coefficient of $\dot{\varepsilon}$ in (3.1) and it satisfies the partial conservation equation

$$\dot{Q} = D$$

We distinguish two interesting special cases which we denote cases α and β.

Case α is the circumstance that D vanishes in (3.1) so that Q is conserved. Q can then be regarded as the canonical generator generating the original transformation with constant ε (as this is now canonical).

The second circumstance, case β, is that the variations F are velocity independent. Then it can be shown that for three variations δ, δ' and δ'' of this type which satisfy

$$[\delta, \delta'] q_i = \delta'' q_i \qquad (\varepsilon'' = \varepsilon \varepsilon') \tag{3.2}$$

then

$$\{Q, Q'\}_{PB} = -Q'' . \tag{3.3}$$

Thus, the Poisson bracket algebra of the Noether charges coincides (up to a sign) with the algebra of the original infinitesimal variations, even though the charges may be only partially conserved.

A more complete statement of these results and their proofs appears in appendix I.

The three types of variation we shall consider in turn are respectively global right translations of g (case β), global left translations of g (case α and β) and time translations (case α).

The most important symmetry in what follows is the most unexpected one, that due to right translations of g. The reason any such symmetry is unexpected is that in general right action on g does not have an unambiguous action on the cosets which are the points of this symmetric space. That is if $g \to gr$, $r \in F$, gr and g k r do not usually lie in the same coset F/K. However for this special sort of symmetric space we are considering we can use the Iwasawa decomposition (2.24) as explained in the preceding section to choose a "gauge" k = 1. This leaves g an element of the group $\mathbb{B}$ whose points are thus in precise correspondence with the points of the symmetric space. Right action on g by another element of $\mathbb{B}$ leaves g in $\mathbb{B}$ as $\mathbb{B}$ is a group thus transporting one point of this symmetric space, g, to another, gb.

We shall now consider an infinitesimal global version of this

$$\delta_{R_o} g = \varepsilon(t) g R_o \quad , \; R_o \in \underset{\sim}{b} \; , \; \dot{R}_o = 0 \tag{3.4}$$

By equation (2.18) the response of the Lagrangian is, as $R = \varepsilon R_o$, $\delta\mathcal{L} = \dot{\varepsilon}\, Tr(R_o A) - i\, \varepsilon\, Tr\,(R_o[B,A])$ as the variation (3.4) is velocity independent the conditions of case β are satisfied. Comparing with (3.1) we can read off the Noether charge as the coefficient of $\dot{\varepsilon}$ (notice that X=0):

$$Q(R_o) = Tr(R_o A) \tag{3.5}$$

As $[\delta_{R_o}, \delta_{R_o'}]\, g = g\, [R_o, R_o'] = \delta_{[R_o, R_o']}\, g$

when $\varepsilon = 1$ we have from (3.2), (3.3) and (3.5)

$$\{ \mathrm{Tr}(AR_o) \ , \ \mathrm{Tr}(AR_o') \}_{PB} = - \mathrm{Tr}(A[R_o, R_o']) \ . \tag{3.6}$$

This specifies the Poisson brackets of the components of A and is essentially the so-called "fundamental Poisson relation" which will lead to the classical Yang-Baxter equations as explained in the section 5. We emphasize that equation (3.6) holds even though these quantities are not conserved. The situation resembles that in particle physics where current algebra relations can be derived even though the currents are not conserved.

Now consider infinitesimal global left translations of g, initially preserving the gauge $k = 1$ so that

$$\delta_{L_o} g = \varepsilon(t) L_o g \quad , \quad L_o \in \underset{\sim}{b} \ , \ \dot{L}_o = 0 \ . \tag{3.7}$$

By equation (2.23) the response of the Lagrangian is

$$\delta \mathcal{L} = (1/2)\dot{\varepsilon} \ \mathrm{Tr}(\dot{x}x^{-1}L_o) = -(1/2)\dot{\varepsilon} \ \mathrm{Tr}(x^{-1}\dot{x}\sigma(L_o)) \tag{3.8}$$

Thus if $\dot{\varepsilon}$ vanishes the Lagrangian is invariant. Conditions (α) and (β) of our Noether's theorem are both satisfied. Comparing with eq. (3.1) we see that X and D both vanish so that we can read off from (3.8) the coefficient of $\dot{\varepsilon}$ as the conserved charge

$$X(L_o) = Tr(\dot{x}x^{-1}L_o)/2 = - Tr(x^{-1}\dot{x}\sigma(L_o))/2 \qquad (3.9)$$

Since when $\varepsilon = 1$

$$[\delta_{L_o} , \delta_{L'_o}]g = - [L_o , L'_o]g = - \delta_{[L_o, L'_o]}g$$

we derive from (3.2) and (3.3) that

$$\{X(L_o) , X(L'_o)\}_{PB} = X([L_o, L'_o]) \qquad (3.10)$$

where in the first instance L_o and L_o' are both elements of $\underset{\sim}{b}$
Obviously $[\delta_{L_o} , \delta_{R_o}] g = 0$
when L_o and R_o are both elements of $\underset{\sim}{b}$. Hence by (3.2) and (3.3)

$$\{X(L_o) , Tr(AR_o)\}_{PB} = 0 \quad . \qquad (3.11)$$

Now let us consider the effect of enlarging the class of left and right transformations from $\mathbb{B}$ to F by including elements of K. As we commented before x and the Lagrangian are invariant under right translations $g \to gk$ even if k depends on time, as long as it is an element of K. Hence there is no Noether charge associated with infinitesimal global variations of this kind and indeed the expression (3.5) vanishes by virtue of (2.5) and (2.9).

On the other hand if we consider global left transformations $g \to g' = \ell g = n' a' k'$ and then "gauge" k' to unity we see from (3.8) that we have a Noether symmetry whenever $\dot{\varepsilon} = 0$ and L_o is any generator of $\underset{\sim}{f}$, not just $\underset{\sim}{b}$.

Thus in equation (3.10) the range of allowed L_0 and L_0' can be extended from $\underset{\sim}{b}$ to $\underset{\sim}{f}$. Nevertheless left and right transformations of g only commute when both are constrained to $\mathbb{B}$. Hence in equation (3.11) the range of L_0 cannot be extended from $\underset{\sim}{b}$ to $\underset{\sim}{f}$.

Finally let us consider time translations of $\delta g = \varepsilon \dot{g}$

$$\delta g = g\,(\varepsilon\, g^{-1}\dot{g}) = (\varepsilon\, \dot{g}\, g^{-1})\, g$$

thus applying equation (2.23) with $L = \varepsilon\, \dot{g}\, g^{-1}$ we find

$$\delta \mathcal{L} = \mathrm{Tr}\ (\dot{x}\ x^{-1} \frac{d}{dt}\ (\varepsilon\ \dot{g}\ g^{-1}))/2$$

Since

$$\dot{x}x^{-1} = 2gAg^{-1} \quad \text{and} \quad \dot{g}g^{-1} = g(g^{-1}\dot{g})g^{-1}$$

$$\delta \mathcal{L} = \dot{\varepsilon}\ \mathrm{Tr}(A(A+iB)) + \varepsilon\ \mathrm{Tr}(A(\dot{A}+i\dot{B}))$$

$$= \frac{d}{dt}\ (\varepsilon \mathrm{Tr}A^2/2) + \dot{\varepsilon}\ \mathrm{Tr}A^2/2$$

using (2.5) and (2.9). Comparing with eq. (3.1) we see that D vanishes. Thus condition α of Noether's Theorem is satisfied so that the generator of the time translations, namely the Hamiltonian, H, is conserved and given by the coefficient of $\dot{\varepsilon}$ as

$$H = \mathrm{Tr}A^2/2 \qquad (3.12)$$

when canonical variables are used. This is indeed positive by equations (2.5) and (2.9) as mentioned earlier.

Obviously it is conserved by equations (2.7) and (2.11).
So are the quantities

$$H_n = \mathrm{Tr}(A^n)/n \tag{3.13}$$

4. The Fundamental Poisson Relation

We now explore the symmetries of the geodesic motion on F/K, discussed in the last section, to construct what Faddeev[5] calls the Fundamental Poisson Relation (FPR). This relation enables us to construct the conserved quantities and to show their involution, and therefore it plays a central role in the integrability of our model.

Let T_s (s=1,2... dim $\mathbb{B}$) denote the generators of the solvable subgroup $\mathbb{B}$. According to (2.18) the variation of the Lagrangian (2.12) under a right translation

$$\delta_s b = \varepsilon(t) b T_s \qquad T_s \in \underset{\sim}{b},\ \dot{T}_s = 0 \tag{4.1}$$

is given by (as $R=\varepsilon T_s$)

$$\delta_s \mathcal{L} = \dot{\varepsilon} \mathrm{Tr}(T_s A) + \varepsilon\, \mathrm{Tr}(T_s [A, iB]) \tag{4.2}$$

Then, according to theorem A of Appendix I, we can read off the charge $Q(T_s)$ as the coefficient of $\dot{\varepsilon}$

$$Q(T_s) = \mathrm{Tr}(T_s A) \tag{4.3}$$

and the quantity D as the coefficient of ε (note that X = 0).

$$D(T_s) = \mathrm{Tr}(T_s [A, iB]) \tag{4.4}$$

According to the same theorem, the variation of $Q(T_r)$ under the transformation (4.1) is:

$$\delta_s Q(T_r) = \varepsilon\{Q(T_r),\ Q(T_s)\}_{PB} + \varepsilon \frac{\partial D(T_s)}{\partial \dot{\eta}^u} \frac{\partial Q(T_r)}{\partial p_u} \tag{4.5}$$

where η^r (r=1,2...dim $\mathbb{B}$) are the parameters of $\mathbb{B}$ and are used as coordinates on F/K. p_r is the canonical momentum ($p_r = \frac{\partial \mathcal{L}}{\partial \dot{\eta}^r}$). In order to evaluate this further we make use of the fact that

$$b^{-1} \frac{\partial b}{\partial \eta^r} = M_r^{\ s}(\eta) T_s \tag{4.6}$$

where the function $M_r^{\ s}(\eta)$ and its properties are considered in appendix III. Then by (2.10) and (4.6), the Lax operators in the "gauge K=1" can be written as:

$$A = \dot{\eta}^r M_r^{\ s}(\eta) P_s \qquad \text{and} \qquad iB = \dot{\eta}^r M_r^{\ s} k_s \tag{4.7}$$

where $P_s = \frac{1}{2}(T_s - \sigma(T_s))$ and $k_s = \frac{1}{2}(T_s + \sigma(T_s))$. Note that $\{P_s, s=1...\dim \mathbb{B}\}$ forms a basis for the odd subspace $\underset{\sim}{p}$ of $\underset{\sim}{f}$ (see (2.2)). Calculating the canonical momenta by using (2.12) and (4.7) we get that $Q(T_s)$, given by (4.3), is:

$$Q(T_s) = M_s^{-1 r}\ p_r \tag{4.8}$$

where $M_s^{-1 r}$ is the inverse of $M_r^{\ s}$.
In addition we get that

$$A = G^{rs} Q(T_r) P_s \qquad \text{and} \qquad iB = G^{rs} Q(T_r) k_s \tag{4.9}$$

where

$$G_{rs} = Tr(P_r P_s) = Tr(T_r P_s) \tag{4.10}$$

and $G^{ru}G_{us} = \delta_s{}^r$. Note that G_{rs} is the Killing form of $\underset{\sim}{f}$ restricted to the odd subspace $\underset{\sim}{p}$, and according to (2.5) it is positive definite.

Using eq. (IV.13) of appendix IV one can easily check that:

$$\frac{\partial D(T_s)}{\partial \dot{\eta}^u} = -M_u{}^r \; G_{sv} \; I^v_{rt} \; G^{tw} \; Q(T_w) \tag{4.11}$$

where (see appendix IV)

$$I^v_{rt} = G^{vu} \, (G_{rs} f^s_{ut} + G_{ts} f^s_{ur}) \tag{4.12}$$

and so it is symmetric $I^v_{rt} = I^v_{tr}$. And f^u_{rs} are the structure constants of the subgroup $\mathbb{B}$. $([T_r, T_s] = f^u_{rs} T_u)$.
According to (IV.12) I^v_{rt} satisfies

$$\frac{1}{2} \, [\sigma(T_r), \, T_t] + \frac{1}{2} \, [\sigma(T_t), T_r] = I^v_{rt} \, P_v \tag{4.13}$$

Then from (4.5), (4.8) and (4.11)

$$\delta_s \, Q(T_r) = \varepsilon \{Q(T_r), \, Q(T_s)\}_{PB} - \varepsilon \, G_{sv} I^v_{rt} \, G^{tu} \, Q(T_u) \tag{4.14}$$

This relation contains all the information we need to construct the Fundamental Poisson Relation, since it gives the Poisson bracket for the charges $Q(T_r)$ which, according to (4.9), form the Lax operator A.

Multiplying both sides of (4.14) by the tensor product of two odd generators of the subspace $\underset{\sim}{p}$, namely $G^{rz}G^{sw}P_z \times P_w$, we get, using (4.9):

$$\{A \overset{\otimes}{,} A\}_{PB} = \frac{1}{\varepsilon} G^{sw} (\delta_s A) \otimes P_w + G^{rz} G^{tu} Q(T_u) P_z \otimes (I^v_{rt} P_v)$$

Under the right translation (4.1) the Lax operator $A = (b^{-1}\dot{b} - \sigma(b^{-1}\dot{b}))/2$ transforms as (with ε constant)

$$\delta_s A = -\frac{1}{2} \varepsilon [T_s, b^{-1}\dot{b}] + \frac{1}{2} \varepsilon[\sigma(T_s), \sigma(b^{-1}\dot{b}]$$

$$= - \varepsilon [k_s, A] - \varepsilon [P_s, iB] \tag{4.15}$$

Therefore using (4.13), (4.15) and (4.9)

$$\{A \overset{\otimes}{,} A\}_{PB} = G^{rs}(P_r \otimes [k_s,A] - [k_s,A] \otimes P_r) -$$

$$- G^{rs}(P_r \otimes [P_s,iB] + [P_s, iB] \otimes P_r) \tag{4.16}$$

But, using eq. (IV. 13)

$$G^{rs}(P_r \otimes [P_s, k_u] + [P_s,k_u] \otimes P_r) =$$

$$= \frac{1}{2} (G^{rs}(f^t_{su} - I^t_{su}) + G^{ts} (f^r_{su} - I^r_{su})) P_r \otimes P_t$$

and by (4.12) we see this vanishes.

Then the operator

$$\mathbb{S} = G^{rs} P_r \otimes P_s \tag{4.17}$$

commutes with any generator k_s, in the sense that:

$$[\mathbb{S}, 1 \otimes k_s + k_s \otimes 1] = 0 \tag{4.18}$$

and the last term of (4.16) vanishes.

Define the operators:

$$\mathbb{P} = - \frac{1}{2} G^{rs}(T_r \otimes \sigma(T_s) - \sigma (T_r) \otimes T_s) \tag{4.19}$$

and

$$\mathbb{R} = \sigma_L \mathbb{P} = -\sigma_R \mathbb{P} = \frac{1}{2} G^{rs}(T_r \otimes T_s - \sigma(T_r) \otimes \sigma(T_s)) \tag{4.20}$$

where the subindices L and R mean that the automorphism σ is acting respectively on the left and right entries of the tensor product.

And so:

$$\frac{\mathbb{P} + \mathbb{R}}{2} = G^{rs} k_r \otimes P_s \quad \text{and} \quad \frac{\mathbb{P} - \mathbb{R}}{2} = -G^{rs} P_r \otimes k_s \tag{4.21}$$

Therefore (4.16) can be written as:

$$\{A \overset{\otimes}{,} A\}_{PB} = -\left[\frac{\mathbb{P}+\mathbb{R}}{2}, A \otimes 1\right] - \left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1 \otimes A\right]$$

$$= -\frac{1}{2}\left[\mathbb{P}, A \otimes 1 + 1 \otimes A\right] - \frac{1}{2}\left[\mathbb{R}, A \otimes 1 - 1 \otimes A\right] \qquad (4.22)$$

and a relation like this is what Faddeev calls a Fundamental Poisson relation. As we mentioned before we can use it to construct conserved quantities in involution. Indeed, from (4.22) and (4.21) we have:

$$\{A, TrA^N/_N\}_{PB} = -G^{rs}\{[k_s, A]Tr(P_rA^{N-1}) - P_rTr([k_s, A^N])\}$$

$$= [A, iB_N] \qquad (4.23)$$

where we have defined

$$iB_N = Tr_R\{(1 \otimes A^{N-1})(\frac{\mathbb{P}+\mathbb{R}}{2})\} \qquad (4.24)$$

and where the subindex R means we are taking the trace of the right entry of the tensor product.

From (4.23) it follows that the quantities TrA^N are in involution, i.e.

$$\{TrA^N, TrA^M\}_{PB} = 0 \qquad (4.25)$$

and since, according to (3.12), TrA^2 is the hamiltonian, it also follows that they are constants of motion.

In fact any of the quantities $H_N = \frac{1}{N} TrA^N$ can be used as a new hamiltonian with the corresponding Lax pair being A and iB_N. Of course $B_2 = B$.

Analogously we can calculate the Poisson bracket between the entries of the Lax operators A and iB. Multiplying (4.14) by $G^{rz} G^{sw} k_z \otimes P_w$ we get, using (4.9)

$$\{iB \overset{\otimes}{,} A\}_{PB} = \frac{1}{\varepsilon} G^{sw} (\delta_s iB) \otimes P_w + G^{rz}G^{tu}Q(T_u) k_z \otimes (I^v_{rt} P_v)$$

Since $iB = \frac{1}{2} (b^{-1}\dot{b} + \sigma(b^{-1}\dot{b}))$, we see that under the right translation (4.1) iB transforms as:

$$\delta_s iB = - \varepsilon[P_s, A] - \varepsilon[k_s, iB] \qquad (4.26)$$

and therefore using (4.13), 4.26), (4.21) and 4.17)

$$\{iB \overset{\otimes}{,} A\}_{PB} = - [\frac{\mathbb{P} + \mathbb{R}}{2}, 1 \otimes iB + iB \otimes 1] -$$

$$- [\mathbb{S}, A \otimes 1] + [\mathbb{Q}, 1 \otimes A] \qquad (4.27)$$

where

$$\mathbb{Q} = G^{rs} k_r \otimes k_s \qquad (4.28)$$

And interchanging the left and right entries of (4.27)

$$\{A \overset{\otimes}{,} iB\}_{PB} = - \left[\frac{\mathbb{P} - \mathbb{R}}{2}, 1 \otimes iB + iB \otimes 1\right] +$$

$$+ [\mathbb{S}, 1 \otimes A] - [\mathbb{Q}, A \otimes 1] \qquad (4.29)$$

A similar, but much simpler, analysis can be done for the left translations discussed in the last section. If we denote the generators of F by L_i (i=1,2...dim F), the variation of the Lagrangian (2.8) under the left translation

$$\delta_i g = \varepsilon(t) L_i g \qquad L_i \in \underset{\sim}{f}, \ \dot{L}_i = 0, \ g \in F \qquad (4.30)$$

is, according to (2.23) (as $L = \varepsilon(t)L_i$)

$$\delta_i \mathcal{L} = \frac{\dot{\varepsilon}}{2} \mathrm{Tr} \ (\dot{x}x^{-1} L_i) \qquad (4.31)$$

Then according to theorem A of Appendix I this is a symmetry of the Lagrangian, and the conserved charge can be read off as the coefficient of $\dot{\varepsilon}$, i.e.

$$X(L_i) = \frac{1}{2} \mathrm{Tr}(\dot{x}x^{-1} L_i) \qquad (4.32)$$

Note that we are using X to denote the Noether's charge to agree with the notation of the last section, and there should be no confusion with the X which appear in appendix I. So, again, according to theorem A the variation of $X(L_j)$ under the translation (4.30) is:

$$\delta_i X(L_j) = \varepsilon \{X(L_j), X(L_i)\}_{PB} \tag{4.33}$$

$\dot{x}x^{-1}$ is an element of the Lie algebra of F, and expanding it in terms of the basis L_i, we get, from (4.32), that

$$X \equiv \frac{1}{2} \dot{x}x^{-1} = g^{ij} X(L_i)L_j \tag{4.34}$$

where

$$g_{ij} = Tr(L_i L_j)$$

is the Killing form of $\underset{\sim}{f}$ and g^{ij} its inverse.

From (2.21) and (2.22) we get that the variation of X, given by (4.34), under the left translation (4.30), with ε constant, is (as $L = \varepsilon L_i$)

$$\delta_i X = \varepsilon [L_i, X] \tag{4.35}$$

Therefore multiplying (4.33) by $g^{ik} g^{jl} L_k \otimes L_l$ and then by $g^{ik} g^{jl} L_l \otimes L_k$, we get, using (4.34) and (4.35)

$$\{X \overset{\otimes}{,} X\}_{PB} = - [\mathbb{C}, 1 \otimes X]$$

$$= [\mathbb{C}, X \otimes 1] \tag{4.36}$$

where

$$\mathbb{C} = g^{ij} L_i \otimes L_j \tag{4.37}$$

is a Casimir like operator, since it commutes with any generator of the Lie algebra of $\underset{\sim}{f}$:

$$[\mathbb{C}, 1 \otimes L_i + L_i \otimes 1] = 0 \tag{4.38}$$

From (4.36) we get

$$\{TrX^N, X\}_{PB} = 0 \tag{4.39}$$

and therefore, the quantities Tr X^N are in involution, i.e.

$$\{Tr\ X^N, TrX^M\}_{PB} = 0 \tag{4.40}$$

In fact, according to (2.11), they are the same quantitites as TrA^N, and (4.40) reproduces the result already obtained in (4.25). However, the reason why these quantities are in involution is more clear here because, since the charges $X(L_i)$ satisfy the algebra of F under the Poisson bracket, we see that the quantities TrX^N are like the Casimir operators of the algebra of F.

Since TrX^2 is the hamiltonian, eq. (4.39) is saying that X is a constant of motion, and this is just a consequence of the fact that the left translations are a symmetry of the Lagrangian.

5. The fundamental Poisson relation and the Yang-Baxter equations.

So far we have worked with any simple non-compact Lie group F equipped with a Cartan involution. Now we shall specialise F to G^N, the normal real form of the complex simple Lie group whose Cartan matrix occurs in the Toda molecule equation (1.1). g^N is the real Lie algebra generated by the usual Cartan subalgebra generators H_i and the step operators E_α.

$$\left.\begin{aligned} &[H_i , H_j] = 0 \qquad i=1...r \\ &[H_i , E_{\pm\alpha}] = \pm\alpha^i E_{\pm\alpha} \\ &[E_\alpha , E_{-\alpha}] = 2\alpha.H/\alpha^2 \\ &[E_\alpha , E_\beta] = N_{\alpha\beta}E_{\alpha+\beta} \qquad \alpha,\beta > 0 \end{aligned}\right\} \tag{5.1}$$

together with other equations not needed explicitly. Notice that all structure constants are real confirming that this is indeed a real Lie algebra. The Cartan involution is

$$\sigma(H_i) = - H_i \quad , \quad \sigma(E_\alpha) = - E_{-\alpha} \quad . \tag{5.2}$$

Then the various subspaces and subalgebras of g^N have the following bases

$$\underset{\sim}{k} = \{E_\alpha - E_{-\alpha}\} \quad \alpha > 0$$

$$\underset{\sim}{p} = \{E_\alpha + E_{-\alpha}\ ,\ H_i\} \quad \alpha > 0\ ,\ i=1,2\ldots r$$

$$\underset{\sim}{a} = \{H_i\} \quad i=1\ldots r \tag{5.3}$$

$$\underset{\sim}{n} = \{E_\alpha\} \quad \alpha > 0 \quad .$$

If we normalize our Killing form by $Tr(H_iH_j) = \delta_{ij}$ it follows that

$$Tr(H_iE_\alpha) = 0 \quad \text{and} \quad Tr(E_\alpha E_{-\beta}) = 2\delta_{\alpha\beta}/\alpha^2 . \tag{5.4}$$

It is easy now to check the Cartan property (2.4). We suspect that the work of this section can be generalized to any choice of σ but we have not checked this completely. In fact, in the last section, we have shown how to obtain the "fundamental Poisson relation" involving the Lax operator A for any choice of σ. But our analysis there did not lead us to the Yang-Baxter equation, which we now show how to obtain for the particular case of the normal real form G^N.

In section 3 we saw that the components of $x^{-1}\dot{x}/2$ and $g^{-1}\dot{g}$ (in the k = 1 gauge) constituted Noether charges with definite algebraic properties. Our first aim is to develop a new notation which expresses this clearly.

In the gauge k = 1, g is obtained by exponentiating the H_i and E_α's ($\alpha > 0$) (eqs. 2.24, 5.3) and so we can expand

$$g^{-1}\dot{g} = A + iB = \sum_i D_iH_i + \sum_{\alpha>0} (\alpha^2/2)D_\alpha E_\alpha$$

for some coefficients to be determined. As A is the odd part of $g^{-1}\dot{g}$ under σ (2.10) we have by (5.2)

$$A = \sum_i D_i H_i + \sum_{\alpha>0} (\alpha^2/2) D_\alpha (E_\alpha + E_{-\alpha})/2$$

By (5.4)

$$D_i = Tr(AH_i) = - Tr(A\sigma(H_i))$$

$$D_\alpha = 2\ Tr(AE_\alpha) = -2\ Tr(A\sigma(E_{-\alpha}))$$

Let us define

$$D(T) = - Tr(A\sigma(T)) \quad , \quad T \in \sigma(\underset{\sim}{b}) \ . \qquad (5.5)$$

Then by the Noether charge algebra (3.6) and the fact that σ is an automorphism of the Lie algebra

$$\{D(T) \ , \ D(T')\}_{PB} = D([T,T']) \quad , \quad T,T' \in \sigma(\underset{\sim}{b}) . \qquad (5.6)$$

Thus we have found

$$g^{-1}\dot{g} = A + iB = D = \sum_i H_i D(H_i) + 2 \sum_{\alpha>0} (\alpha^2/2) E_\alpha D(E_{-\alpha}) \qquad (5.7)$$

This combination will be called D, for short. Notice the factor 2 in the second term of (5.7).

Similarly we find that

$$\dot{x}x^{-1}/2 = -\sigma(x^{-1}\dot{x})/2 =$$

$$= \sum_i H_i X(H_i) + \sum_{\alpha>0} (\alpha^2/2)(E_\alpha X(E_{-\alpha}) + E_{-\alpha} X(E_\alpha)) \quad (5.8)$$

when the coefficients X(L) satisfy equations (3.9) and (3.10)

We see that the Hamiltonian H (3.12) can be expressed in two alternative ways

$$H = [\sum_i D(H_i)^2 + \sum_{\alpha>0} \alpha^2 D(E_{-\alpha})^2]/2 \quad (5.9)$$

$$= [\sum_i X(H_i)^2 + \sum_{\alpha>0} (\alpha^2/2) X(E_\alpha) X(E_{-\alpha})]/2$$

Following a notation introduced in statistical physics[28,5] equation (5.6) can be written in another way.

$$\{D \overset{\otimes}{,} D\}_{PB} = -[\mathbb{P}, D \otimes 1 + 1 \otimes D] \quad (5.10)$$

Faddeev calls a relation like this a "fundamental Poisson relation"[5]. Apart from a factor -2 (a 2 to compensate for a different definition of Lagrangian and a minus for convenience), $\mathbb{P}$ is the operator constructed by Olive and Turok[14].

$$\mathbb{P} = \sum_{\alpha>0} (\alpha^2/2)(E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha) = \mathbb{C}_+ - \mathbb{C}_- \quad (5.11)$$

Equation (5.10) is proven in appendix II following the methods of Olive and Turok[14] and using properties of root systems of Lie algebras. In that work it is also useful to define an operator

$$\mathbb{C} = \sum_i H_i \otimes H_i + \sum_{\alpha>0} (\alpha^2/2)(E_\alpha \otimes E_{-\alpha} + E_{-\alpha} \otimes E_\alpha) \qquad (5.12)$$

$$= \mathbb{C}_o + \mathbb{C}_+ + \mathbb{C}_-$$

It is possible to rewrite (5.10) as

$$\{D \otimes D\}_{PB} = - [\mathbb{P} + \mathbb{C} , D \otimes 1 + 1 \otimes D] \qquad (5.13)$$

as the $\mathbb{C}$ contribution vanishes identically. We now show that this equation has the same structure as the Yang Baxter equation. This is because

$$\mathbb{P} + \mathbb{C} = \mathbb{C}_o + 2 \mathbb{C}_+ = \sum_i H_i \otimes H_i + 2 \sum_{\alpha>0} (\alpha^2/2) E_\alpha \otimes E_{-\alpha} \qquad (5.14)$$

and the structure of this expression is very similar to that of D in equation (5.7). We can develop this resemblance by introducing a triple notation with three spaces. The first and third spaces are occupied by the left and right Lie algebra generators in (5.10) and (5.13). The middle entry is the space of dynamical variables in which the bracket operation is the Poisson bracket. With the suffices referring to these three spaces we define

$$D_{12} = D \otimes 1 = 2 \sum_{\alpha>0} (\alpha^2/2) E_\alpha \otimes D(E_{-\alpha}) \otimes 1 + \sum_i H_i \otimes D(H_i) \otimes 1$$

$$D_{32} = 1 \otimes D = 2 \sum_{\alpha>0} (\alpha^2/2) 1 \otimes D(E_{-\alpha}) \otimes E_\alpha + \sum_i 1 \otimes D(H_i) \otimes H_i$$

$$D_{13} = \mathbb{P} + \mathbb{C} = 2\sum_{\alpha>0}(\alpha^2/2)E_\alpha \otimes 1 \otimes E_{-\alpha} + \sum_i H_i \otimes 1 \otimes H_i$$

Then equation (5.13) can be written

$$\{D_{12}\ ,\ D_{32}\}_{PB} = -\ [D_{13}\ ,\ D_{12} + D_{32}]$$

or

$$\{D_{12}\ ,\ D_{32}\}_{PB} + [D_{13}\ ,\ D_{12}] + [D_{13}\ ,\ D_{32}] = 0\ . \qquad (5.15)$$

The quantum version of this is the Yang-Baxter equation (27,28) which is the infinitesimal version of the triangle equation. We have learnt how to obtain this from something rather geometrical namely the geodesic motion of a particle on a non-compact Riemannian symmetric space.

Finally we see how to extract from (5.10) the Poisson brackets of the individual components of A and B by using the involution σ. Let us define σ_L and σ_R as σ acting respectively on the left and right entries of $\{D \overset{\otimes}{,} D\}$. By considering $(1 - \sigma_L - \sigma_R + \sigma_L\ \sigma_R)$ acting on equation (5.10) we find

$$\{A \overset{\otimes}{,} A\} = -\left[\frac{\mathbb{P}+\mathbb{R}}{2}, A\otimes 1\right] - \left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1\otimes A\right] \qquad (5.16)$$

$$= -(1/2)[\mathbb{P}\ ,\ A\otimes 1 + 1\otimes A] - (1/2)[\mathbb{R}\ ,\ A\otimes 1 - 1\otimes A]$$

where

$$\mathbb{R} = \sigma_L \mathbb{P} = -\ \sigma_R \mathbb{P} = \sum_{\alpha>0}(\alpha^2/2)(E_\alpha \otimes E_\alpha - E_{-\alpha}\otimes E_{-\alpha}) \qquad (5.17)$$

similarly we find

$$\{A \overset{\otimes}{,} B\}_{PB} = - \left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1 \otimes B + B \otimes 1\right] \qquad (5.18)$$

$$\{iB \overset{\otimes}{,} iB\}_{PB} = - \left[\frac{\mathbb{P}-\mathbb{R}}{2}, A \otimes 1\right] - \left[\frac{\mathbb{P}+\mathbb{R}}{2}, 1 \otimes A\right] \qquad (5.19)$$

Using (5.2), (5.3) and (5.4) one can check that the operators $\mathbb{P}$ and $\mathbb{R}$, defined in the last section (see (4.19) and (4.20)), correspond, in the case of the symmetric space $G^N/_K$, to those operators given by (5.11) and (5.17). Therefore the relations (4.22) and (5.16) are indeed the same. The operators $\mathbb{S}$ and $\mathbb{Q}$, defined respectively by (4.17) and (4.28), are given, in the case of $G^N/_K$, by:

$$\mathbb{S} = \sum_i H_i \otimes H_i + \frac{1}{2} \sum_{\alpha>0} \frac{\alpha^2}{2} (E_\alpha + E_{-\alpha}) \otimes (E_\alpha + E_{-\alpha}) \qquad (5.20)$$

and

$$\mathbb{Q} = \frac{1}{2} \sum_{\alpha>0} \frac{\alpha^2}{2} (E_\alpha - E_{-\alpha}) \otimes (E_\alpha - E_{-\alpha}) \qquad (5.21)$$

Comparing with (5.12) we see that $\mathbb{S} - \mathbb{Q}$ is a Casimir like operator

$$\mathbb{S} - \mathbb{Q} = \mathbb{C} \qquad (5.22)$$

and therefore it commutes with any generator of the Lie algebra of G^N. In particular, if T_s is a generator of the Borel subgroup $\mathbb{B}$, we have

$$[\,\mathbb{S} - \mathbb{Q},\ 1 \otimes T_s + T_s \otimes 1\,] = 0 \qquad (5.23)$$

Acting with σ on the left entry, and since

$$\sigma_L \mathbb{S} = -\mathbb{S} = \sigma_R \quad \text{and} \quad \sigma_L \mathbb{Q} = \mathbb{Q} = \sigma_R \mathbb{Q} \qquad (5.24)$$

we get

$$[\,\mathbb{S} + \mathbb{Q},\ 1 \otimes T_s + \sigma(T_s) \otimes 1\,] = 0 \qquad (5.25)$$

Adding up (5.23) and (5.25) and using (4.18) we get

$$[\,\mathbb{S},\ 1 \otimes P_s\,] - [\,\mathbb{Q},\ P_s \otimes 1\,] = 0 \qquad (5.26)$$

and therefore the relation (4.29), in the case of G^N/K, is indeed the same as (5.18).

6. Reduction to the Toda Molecule System

The final step is to find the precise constraints on the geodesic trajectories necessary to produce the Toda molecule equation (1.1) with the correspondingly constrained quantities A and B satisfying the usual Lax pair and fundamental Poisson relations of that system. The constraints have to be self consistent and one possibility is to constrain

$$X(E_\alpha) = D(E_{-\alpha}) = 0 \qquad \alpha \text{ any positive non simple root} \qquad (6.1)$$

These equations are self consistent since the quantities put to zero in (6.1) form a closed subalgebra under the Poisson brackets by equations (3.10), (3.11), (6.5) and (6.6). The quantities $X(E_\alpha)$ are conserved and so by putting them equal to zero when α is a positive but non-simple root we are choosing initial conditions on the trajectory. It remains to see how these conditions imply the vanishing of the $D(E_{-\alpha})$ in (6.1)and how this leads to the Toda equations (1.1). To do this we introduce explicit (horospheric) coordinates (ϕ_a, ρ_α) for the symmetric space;

$$g = na \ , \ a = \exp(\sum_a \phi_a H_a/2) \ , \qquad n = \exp(\sum_{\alpha>0} \rho_\alpha E_\alpha)$$

$$n^{-1}\dot{n} = \sum_{\alpha>0} E_\alpha V_\alpha \qquad , \qquad V_\alpha = \sum_{\beta>0} \dot{\rho}_\beta m_{\beta\alpha}(\rho) \qquad (6.2)$$

It is understood that the greek indices refer to positive roots whereas the Latin indices refer to the r simple roots. Thus

$$H_a = 2a.H/a^2 \quad \text{where } a \text{ is a simple root.}$$

The variables ϕ_a will be the same as those appearing in (1.1). Up to now we have managed to keep our notation relatively simple by not introducing specific coordinates. More concrete proofs of our previous Poisson bracket relations can be obtained using these coordinates (6.2) but we leave this as an exercise for the reader.

In terms of the new variables (6.2)

$$g^{-1}\dot{g} = A + iB = a^{-1}\dot{a} + a^{-1}n^{-1}\dot{n}a$$

$$=(1/2)\sum_a \dot{\phi}_a H_a + \sum_{\alpha>0} \exp(-K_{\alpha b}\phi_b/2)V_\alpha E_\alpha \qquad (6.3)$$

using the commutators (5.1) and introducing

$$K_{\alpha b} = 2\ \alpha.b/_{b^2}$$

When α as well as b is a simple root this forms the Cartan matrix occurring in (1.1).

Also we see

$$x^{-1}\dot{x} = (n^+)^{-1}[a^{-2}n^{-1}\dot{n}a^2 + 2a^{-1}\dot{a} + \dot{n}^+(n^+)^{-1}]n^+ \qquad (6.4)$$

It is difficult to evaluate this further but it is easy to see that if γ is one of the highest positive roots whose step operator E_γ occurs in $n^{-1}\dot{n}$ in the sense that $E_{\gamma+\alpha}$ does not occur for any positive root α, then the coefficient of E_γ in $x^{-1}\dot{x}$ is simply $V_\gamma \exp(-K_{\gamma b}\phi_b)$ (as it is the same as the coefficient of E_γ in $a^{-2}n^{-1}\dot{n}a^2$ since the n^+'s do not affect this term). As $x^{-1}\dot{x}$ is conserved, so is this coefficient. Hence, if V_γ vanishes initially, it does so for all time. So therefore do the coefficients of E_γ in $n^{-1}\dot{n}$ and $g^{-1}\dot{g}$ since they are, respectively by (5.7), (6.2) and (6.3), V_γ and $V_\gamma \exp(-K_{\gamma b}\phi_b/2)$. In particular $D(E_{-\gamma})$ vanishes. This argument can now be repeated for any of the remaining highest roots until only steps operators for simple roots a remain in $n^{-1}\dot{n}$ and $g^{-1}\dot{g}$. The coefficient of E_a in $x^{-1}\dot{x}$ is then $V_a \exp(-K_{ab}\phi_b)$ and constant. By equation (5.8) we have

$$a^2X(E_a) = V_a\exp(-K_{ab}\phi_b) \qquad , \qquad a \text{ simple}$$

But by equations (6.3) and (5.7) the coefficient of E_a in $g^{-1}\dot{g}$ is

$$a^2D(E_{-a}) = V_a\exp(-K_{ab}\phi_b/2) = \qquad (6.5)$$

$$= a^2X(E_a)\exp(K_{ab}\phi_b/2)$$

by the preceding result. In this way we have integrated the ρ equations of motion by choosing special initial conditions leaving as the only degrees of freedom the ϕ variables. By (6.5) we now have

$$\bar{A} + i\bar{B} = (1/2)\sum_a \dot{\phi}_a H_a + \sum_a E_a a^2 X(E_a)\exp(K_{ab}\phi_b/2) \qquad (6.6)$$

where now both sums extend over simple roots only. The resultant $\bar{A}$ and $\bar{B}$ form the Lax pair for the Toda molecule equation

$$\frac{d^2}{dt^2}\phi_a = - a^4 X(E_a)^2 \exp(K_{ab}\phi_b)$$

Thus the constants $a^2 X(E_a)$ constitute coupling constants. To obtain precisely (1.1) we assign these constants the value unity. Then (6.6) yields the usual Lax pair for equations (1.1) (apart from a minus sign in B when compared to ref. (14)owing to a different sign in 2.14). These substitutions for the integrals of motion $X(E_{-\alpha})$ can be made directly into the equations of motion or the Hamiltonians (but not the Lagrangian). All the Poisson bracket relations remain valid with these substitutions.

We now show how the fundamental Poisson relations (5.10), (5.16) and (5.19) reduce to the ones obtained by Olive and Turok[(14)].

In Appendix II it is shown that for any positive root β

$$[\mathbb{P}\ ,\ \frac{(E_\beta+E_{-\beta})}{2}\otimes 1 + 1\otimes\frac{(E_\beta+E_{-\beta})}{2}\,] = (\beta^2/2)(H_\beta\otimes\frac{(E_\beta+E_{-\beta})}{2} -$$

$$- \frac{(E_\beta+E_{-\beta})}{2}\otimes H_\beta) + (1/2)\sum_{\substack{\alpha,\gamma>0\\ \alpha+\gamma=\beta}} (\alpha^2\gamma^2/\beta^2)N_{\alpha\gamma}(E_\alpha\otimes E_\gamma + E_{-\alpha}\otimes E_{-\gamma})\,.$$

When A is reduced by (6.5) only step operators for simple roots occur. When β is simple it cannot be expressed as a sum of two positive roots α and γ and hence the last term in the equation vanishes. The remaining term is odd under either σ_L or σ_R by (5.2). Hence so is the left hand side. Thus by (5.17)

$$[\mathbb{P}\ ,\frac{(E_\beta+E_{-\beta})}{2}\otimes 1 + 1\otimes\frac{(E_\beta+E_{-\beta})}{2}\,] = [\mathbb{R}\ ,\frac{(E_\beta+E_{-\beta})}{2}\otimes 1 - 1\otimes\frac{(E_\beta+E_{-\beta})}{2}\,]$$

This leads to the fact that after the reduction the two terms on the right hand side of (5.16) become equal yielding

$$\{\bar{A} \overset{\otimes}{,} \bar{A}\} = -\ [\ \mathbb{P}\ ,\ \bar{A}\otimes 1 + 1\otimes\bar{A}\,]$$

which is a result of Olive and Turok. Similarly

$$\{\ \bar{B} \overset{\otimes}{,} \bar{B}\ \} = 0\quad .$$

7. Some Comments about Quantum Mechanics on Curved Spaces

The quantization of physical systems on curved spaces as well as of non-linear systems, has been the subject of a great deal of research in theoretical physics in the last decades, and a complete understanding of these matters is still lacking. In this section we do not want to consider, in detail, the difficulties facing this subject, but instead we want to discuss some basic facts one should take into account when quantizing the motion of a free particle on a non-compact Riemannian manifold . In fact, these will prove to be sufficient for our purposes in the next section when we study the quantum geodesic motion on the non-compact symmetric spaces F/K.
Much of our discussion is based on ref. (29).

We will use the canonical quantization procedure and since we believe that one system of coordinates is as good as any other, we will require our rules of quantization, as well as our quantum mechanical equations, to be covariant under general coordinate transformations. Together with the requirement of hermiticity, this will help us in solving some of the ambiguities in the ordering of the quantum operators in the hamiltonian and will lead us to the Laplace-Beltrami operator.

In our analysis we will be considering wave functions for spinless particles only, and therefore the first thing we have to do is to define an inner product for scalar functions which is invariant under general coordinate transformation, since we want the expectation values of physical quantities to be so.

Then, since the volume element

$$dV = g^{1/2}\, d^n x, \qquad g = \det g_{ij}$$

where g_{ij} is the metric of the space, is invariant, we define the invariant inner product as:

$$\langle\psi|\phi\rangle = \int dv \ \psi^* \phi \tag{7.1}$$

where the integration is carried out over the entire range of the coordinates values.

Under this inner product, the operator $- i\hbar \frac{\partial}{\partial x^i}$ is not hermitian, since(*)

$$\int dv \ \psi^*(-i\hbar\partial_i \phi) = \int dv(-i\hbar\partial_i \psi)^* \phi + \frac{i\hbar}{2} \int dv \ (\partial_i \ln g)\psi^* \phi.$$

However, it can be made hermitian by adding to it the term $\frac{i\hbar}{4} \partial_i(\ln g)$. Therefore the quantum operators for the canonical variables x_i and p_i in the coordinate representation are given by:

$$\hat{x}_i = x_i, \qquad \hat{p}_i = - i\hbar \left(\frac{\partial}{\partial x^i} + \frac{1}{4} \partial_i(\ln g)\right) \tag{7.2}$$

(*) When performing the integration by parts we are supposing that the boundary conditions satisfied by the wave functions are such that the surface term vanishes. If the wave functions are normalizable, i.e. $\int dV \ \psi^* \psi$ is finite, then it follows, since the space is non compact, that $g^{1/2} \psi^* \psi$ has to vanish at infinity. This is sufficient to make the surface term to vanish. The same difficulties arise in ordinary quantum mechanics and similar assumptions are made there.

Since it can be easily checked that they indeed satisfy the canonical commutation relations:

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\, \delta_{ij} \qquad (7.3)$$

Notice that $\partial_i(\ln g)/2$ is the contracted Christoffel symbol Γ^j_{ji}. (This can be checked by using e.q. (IV.8) of appendix IV and eq. (8.8))

Now let's see how the operators $\hat{x}_i$ and $\hat{p}_i$ transform under a general coordinate transformation. In classical mechanics, for every transformation of coordinates of the form:

$$x_i \rightarrow x_i' = x_i'(x) \qquad (7.4)$$

there exist a corresponding transformation of the conjugate momenta of the form:

$$p_i \rightarrow p_i' = \frac{\partial x^j}{\partial x'^i} p_j \qquad (7.5)$$

In quantum mechanics the product of two hermitian opertors is only hermitian if they commute. If they do not commute we have to take the symmetric product to get an hermitian operator. Then, we see there is no ambiguity in defining the transformation (7.4) in the quantum theory, because it depends only upon the coordinates, which commute with each other, and therefore $\hat{x}_i'$ is hermitian. The transformation (7.5) can also be defined in the quantum theory in an unambiguous way because, since it is linear in the momenta, all ways of symmetrizing the non-commuting parts of it are equivalent. One can easily see that this is indeed true for any quantity of the form f(x)p, by expanding f in powers series of the x's and then inserting p between the x's in any symmetrical fashion in each term of the series like:

$$\frac{1}{2n} \sum_{m=1}^{n} (\hat{x}^{n-m}\, \hat{p}\, \hat{x}^{m} + \hat{x}^{m}\, \hat{p}\, \hat{x}^{n-m})$$

One sees that by commuting $\hat{p}$ with the $\hat{x}$'s symmetrically to the right and to the left one gets two terms of order $\hbar$ which cancel each other due to the fact that the commutator of $\hat{p}$ with $\hat{x}$ is a c-number. Thus, all methods of symmetrization are equivalent to:

$$\frac{1}{2}\,(\hat{f}(x)\hat{p} + \hat{p}\hat{f}(x)) \tag{7.6}$$

Therefore the quantum analogue of the transformations (7.4) and (7.5) are:

$$\hat{x}_i \rightarrow \hat{x}'_i = x'_i\,(x) \tag{7.7a}$$

$$\hat{p}_i \rightarrow \hat{p}'_i = \frac{1}{2}\,\{\frac{\partial x^j}{\partial x'^i},\ \hat{p}_j\}$$

$$= \frac{\partial x^j}{\partial x'^i}\,\hat{p}_j + \frac{1}{2}\,[\hat{p}_j,\ \frac{\partial x^j}{\partial x'^i}] \tag{7.7b}$$

Using (7.2) and the fact that $\Gamma^j_{ji} = \partial_i(\ln g)/2$ we get

$$\hat{p}'_i = - i\hbar \frac{\partial}{\partial x'^i} - \frac{i\hbar}{2} [\frac{\partial x^j}{\partial x'^i} \Gamma^k_{kj} + \frac{\partial}{\partial x^j} (\frac{\partial x^j}{\partial x'^i})]$$

And the second term is exactly the transformation for the contracted Christoffel symbol, and therefore the canonical momenta transform in a covariant way under coordinate transformations:

$$\hat{p}_i \rightarrow \hat{p}'_i = - i\hbar (\frac{\partial}{\partial x'^i} + \frac{1}{2} \Gamma'^j_{ji}) = - i\hbar (\frac{\partial}{\partial x'^i} + \frac{1}{4} \frac{\partial}{\partial x'^i}(\ln g'))$$

(7.8)

The next step in our analysis is the construction of the quantum operator for the free hamiltonian on a curved space. Classically this hamiltonian is just:

$$H = \frac{1}{2m} g^{ij} p_i p_j \qquad (7.9)$$

and since, according to (7.5), the classical canonical momentum transforms like a covariant vector, we see that H is invariant under general coordinate transformations.

The problem we face in constructing a quantum operator for H is that, since it is quadratic in the momenta, there are several inequivalent ways of symmetrizing it. The simplest one:

$$\hat{H}_1 = \frac{1}{2m} \hat{p}_i \hat{g}^{ij} \hat{p}_j \qquad (7.10)$$

is not covariant under general coordinate transformations, and the reason is that, according to (7.7b), the quantity $\hat{p}_i \psi$ (with ψ a scalar) does not transform like a vector.

Therefore, the requirement of covariance imposes restrictions on the possible ways of symmetrizing H and, as we will see, it will lead us to the correct quantum operator for H in a quite unambiguous way.

From (7.2) we see that

$$\hat{g}^{1/4}\, \hat{p}_i \hat{g}^{-1/4} = -i\hbar \frac{\partial}{\partial x^i} \qquad (7.11)$$

Therefore the quantity $\hat{g}^{1/4}\, \hat{p}_i \hat{g}^{-1/4}\psi$ transforms like a covariant vector. Its hermitian conjugate is (since g_{ij} is real):

$$(\hat{g}^{1/4}\, \hat{p}_i \hat{g}^{-1/4})^+ = \hat{g}^{-1/4}\, \hat{p}_i \hat{g}^{1/4} = -i\hbar \left(\frac{\partial}{\partial x^i} + \frac{1}{2} \frac{\partial}{\partial x^i}(\ln g)\right)$$

$$= -\, i\hbar \left(\frac{\partial}{\partial x^i} + \Gamma^j_{ji}\right) \qquad (7.12)$$

Then, if V^i is a contravariant sector, $\hat{g}^{-1/4}\, \hat{p}_i\, \hat{g}^{1/4}\, V^i$ is its covariant divergence, which is a scalar. So, since $\hat{g}^{ij}\, \hat{g}^{1/4}\, \hat{p}_j\, \hat{g}^{-1/4}\, \psi$ is a contravariant vector, we conclude that the operator

$$\hat{H} = \frac{1}{2m}\, \hat{g}^{-1/4}\, \hat{p}_i \hat{g}^{1/4}\, \hat{g}^{ij}\, \hat{g}^{1/4}\, \hat{p}_j\, \hat{g}^{-1/4} \qquad (7.13)$$

is hermitian and covariant under general coordinate transformations. Since

$$(\partial_i + \Gamma^j_{ji})\, V^i = (\partial_i + g^{-1/2}\, \partial_i g^{1/2})\, V^i = g^{-1/2}\partial_i (g^{1/2} V^i)$$

we can write it as

$$\hat{H} = -\frac{\hbar^2}{2m}\, \Delta \tag{7.14}$$

where

$$\Delta = g^{-1/2}\partial_i g^{1/2} g^{ij}\, \partial_j \tag{7.15}$$

is the Laplace-Beltrami operator, which is the generalization of the Laplacian for scalar functions on curved space.

Therefore the Schrödinger equation for the free motion for a particle on a curved space is:

$$\hat{H}\, \psi = -\frac{\hbar^2}{2m}\, \Delta\psi = i\hbar\, \frac{\partial\psi}{\partial t} \tag{7.16}$$

One could add to the hamiltonian (7.14) a term proportional to the curvature scalar

$$\hat{H}_c = \frac{-\hbar^2}{2m}\, (\Delta + \lambda R) \tag{7.17}$$

with λ a dimensionless constant, since this operator is also hermitian and covariant, and in addition has the advantage of being conformal invariant for some values of λ. (For more details see problem 4 of Chapter V of ref. (30)). Other terms can also be added to (7.14) without destroying its hermiticity and covariance like R^2, $R_{ij}R^{ij}$ etc (R_{ij} being the Ricci tensor defined in appendix IV) but for these terms the constant λ would not be dimensionless and we would be introducing a scale in the theory.

The Hamiltonian (7.14) is positive definite if the metric of the space is so. Consider

$$\langle\phi|(-\Delta)|\phi\rangle = -\int dv\,\phi^*\Delta\phi = -\int d^nx\ \phi^*\partial_i(g^{1/2}g^{ij}\partial_j\phi)$$

and integrating by parts

$$\langle\phi|(-\Delta)|\phi\rangle = \int dv\ (\partial_i\phi)^* g^{ij}(\partial_j\phi)$$

So

$$\langle\phi|(-\Delta)|\phi\rangle > 0 \quad \text{if } g_{ij} \text{ is positive definite} \qquad (7.18)$$

8. The Quantum geodesic Motion on Riemannian non-compact symmetric spaces

We now discuss the quantum geodesic motion on the Riemannian non-compact symmetric spaces F/K described in section 2.

The hamiltonian for such motion, according to the discussion of the last section, is the Laplace-Beltrami operator. Since we will be exploring the hidden group theoretic structure of these spaces endowed by the solvable subgroup $\mathbb{B}$=na, our main aim in this section is to write the Laplace-Beltrami operator on F/K in terms of the parameters of that subgroup.

The metric for these spaces can be read off from the Lagrangian (2.8) as:

$$g_{rs}(\zeta) = \frac{1}{4}\,\mathrm{Tr}(x^{-1}\frac{\partial x}{\partial \zeta^r}\,x^{-1}\frac{\partial x}{\partial \zeta^s}) \tag{8.1}$$

where $x = g\sigma(g)^{-1}$ ($g \in F$) is the principal variable defined in (2.6) (see appendix V) and ζ^r (r=1,2...dimF/K) is some set of coordinates on the symmetric space F/K. (The factor 1/4 is for convenience). Although the coordinates ζ^r can be written in terms of the parameters of the group F, the converse is not true since all the elements of F belonging to a given coset correspond to the same values of ζ^r. However according to (2.24) and the discussion thereafter, there is a one to one correspondence between elements of $\mathbb{B}$ and points on F/K due to the Iwasawa decomposition of F. Therefore the parameters of the solvable subgroup $\mathbb{B}$ can be used as coordinates on F/K.

If we denote the parameters of $\mathbb{B}$ by η^r (r=1,2..dim $\mathbb{B}$), it is easy to check that, without fixing the gauge K = 1 we get (we are going to use the same indices for ζ and η since dim $\mathbb{B}$= dim F/K):

$$x^{-1}\frac{\partial x}{\partial \zeta^r} = \frac{\partial \eta^s}{\partial \zeta^r}\,\sigma(b)\left[b^{-1}\frac{\partial b}{\partial \eta^s} - \sigma(b^{-1}\frac{\partial b}{\partial \eta^s})\right]\sigma(b)^{-1}$$

$$= \frac{\partial \eta^s}{\partial \zeta^r} M_s{}^u(\eta)\,\sigma(b)\left[T_u - \sigma(T_u)\right]\sigma(b)^{-1} \qquad (8.2)$$

where we have used (4.6) (see appendix III also) and T_u are the generators of $\mathbb{B}$.

Thus the metric (8.1) can be written as:

$$g_{rs}(\zeta) = \frac{\partial \eta^u}{\partial \zeta^r}\frac{\partial \eta^v}{\partial \zeta^s} M_u{}^t(\eta) M_v{}^w(\eta) G_{tw} \qquad (8.3)$$

where G_{tw} is defined in (4.10). Performing a change of coordinates, from ζ to η, we see we can write g_{rs} in terms of the parameters of $\mathbb{B}$ as:

$$g_{rs}(\eta) = \frac{\partial \zeta^u}{\partial \eta^r}\frac{\partial \zeta^v}{\partial \eta^s}\; g_{uv}(\zeta)$$

$$= M_r{}^u(\eta) M_s{}^v(\eta) G_{uv} \equiv (M\;G\;M^T)_{rs} \qquad (8.4)$$

We notice that the quantities $M_r{}^s(\eta)$ are like tetrad on vierbein that relate the metric G_{rs} on the odd subspace $\underset{\sim}{p}$ of $\underset{\sim}{f}$ under σ (which is the tangent plane to F/K at the unit) to the metric g_{rs} on the symmetric space F/K.

According to (2.5), G_{rs} is positive definite and from (8.4) we see that g_{rs} is also positive definite. Therefore, according to (7.18), we conclude that the hamiltonian (7.17) in the case of the symmetric spaces F/K is positive definite.

Using (7.15) and (8.4) we calculate the Laplace-Beltrami operator for F/K:

$$\Delta = (\det M)^{-1} \frac{\partial}{\partial \eta^r} M^{-1}{}_u{}^r\, G^{ut}\, M^{-1}{}_t{}^s \det M \frac{\partial}{\partial \eta^s}$$

$$= G^{rs}\, \nabla_r \nabla_s + G^{ut} (\det M)^{-1} \frac{\partial}{\partial \eta^r} (\det M\, M^{-1}{}_u{}^r) \nabla_t \qquad (8.5)$$

where ∇_r is the right shift operator (see appendix III)

$$b^{-1} \nabla_r b = T_r, \qquad \nabla_r = M^{-1}{}_r{}^s \frac{\partial}{\partial \eta^s}, \qquad T_r \in \underset{\sim}{b} \qquad (8.6)$$

But

$$(\det M)^{-1} \partial_r (\det M\, M^{-1}{}_u{}^r) = M^{-1}{}_u{}^r \partial_r \ln(\det M) - M^{-1}{}_u{}^s (\partial_r M_s{}^t) M^{-1}{}_t{}^r$$

$$= M^{-1}{}_u{}^r\, M^{-1}{}_s{}^t (\partial_r M_t{}^s - \partial_t M_r{}^s)$$

$$= f^s_{su} \qquad (8.7)$$

where we have used eq. (III.4) of appendix III (f^u_{rs} are the structure constants of the subgroup $\mathbb{B}$) and the fact that for any non singular matrix M we have (see sec. 7, chap.4 of ref. (31))

$$\mathrm{Tr}(M^{-1} \partial_r M) = \partial_r \ln(\det M) \qquad (8.8)$$

Notice that for a semisimple group (8.7) would vanish since f^s_{su} is the trace of the generator T_u in the adjoint representation. However, the group B is solvable and therefore not semisimple.

Therefore from (8.5) and (8.7)

$$\Delta = G^{rs} \nabla_r \nabla_s + G^{rs} f^u_{ur} \nabla_s \tag{8.9}$$

The operators ∇_r defined in (8.6) are not hermitian, since, using (8.7):

$$\int dv\ \psi^* \nabla_r \phi = - \int dv\ (\nabla_r \psi)^* \phi - f^u_{ur} \int dv \psi^* \phi \tag{8.10}$$

where, according to the last section, dv is the invariant volume element $dv = (\det g_{rs})^{1/2}\, d^n \eta^r$.

We have shown in the last section that the Laplace Beltrami operator is hermitian, and in order to see this more clearly in (8.9) we define the operators:

$$\nabla_r' = \nabla_r + \frac{1}{2} f^u_{ur} \tag{8.11}$$

which according to (8.10) are antihermitian.

Note that these operators satisfy the same commutation relations as ∇_r (see appendix III):

$$[\nabla_r', \nabla_s'] = [\nabla_r, \nabla_s] = f^u_{rs} \nabla_u = f^u_{rs} \nabla_u' \tag{8.12}$$

Since the term we have to add to get the last equality, namely

$f^u_{rs}\, f^v_{vu}$, vanishes according to the Jacobi identity for the structure constants. Then the Laplace-Beltrami operator for F/K, given by (8.9), becomes

$$\Delta = G^{rs}\nabla'_r\, \nabla'_s - \frac{1}{4}\, G^{rs} f^u_{ur} f^v_{vs} \tag{8.13}$$

which is clearly hermitian, since the structure constants f^u_{rs} of the subgroup $\mathbb{B}$ are real.

Since G^{rs} is positive definite (see (2.5)) the second term of (8.13), namely $G^{rs}f^u_{ur}f^v_{vs}$, is a non negative constant. And for the same reason the operator $-\, G^{rs}\nabla'_r\nabla'_s$ is positive definite, since:

$$\langle\psi|(-G^{rs}\nabla'_r\nabla'_s)|\psi\rangle = -\int dv\;\; \psi^* G^{rs}\nabla'_r\nabla'_s\psi$$

$$= \int dv\;\; (\nabla'_r\psi)^* G^{rs}(\nabla'_s\psi) \geqslant 0 \tag{8.14}$$

Therefore, the hamiltonian for the geodesic motion on F/K:

$$\hat{H} = -\frac{\hbar^2}{2}\,\Delta = -\frac{\hbar^2}{2}\,(G^{rs}\nabla'_r\nabla'_s - \frac{1}{4}\, G^{rs}f^u_{ur}f^v_{vs}) \tag{8.15}$$

is bounded below by a non negative constant:

$$\langle\psi|\hat{H}|\psi\rangle \geqslant \frac{\hbar^2}{8}\, G^{rs}f^u_{ur}f^v_{vs} \geqslant 0 \tag{8.16}$$

This constant is invariant under general coordinate transformations since it is related to the covariant divergence of the vector Π_r defined by eq. (IV.3) of appendix IV. Using the relations (IV.18), (IV.13) and (IV.11) we have:

$$D_r \Pi^r = G^{rs} f^u_{ur} P_s, \qquad P_s = \frac{1}{2} (T_s - \sigma(T_s)), \qquad T_s \in \underset{\sim}{b} \tag{8.17}$$

and therefore, using (4.10):

$$\mathrm{Tr}(D_r \Pi^r)^2 = G^{rs} f^u_{ur} f^v_{vs} \tag{8.18}$$

which is clearly a scalar.

9. The quantum left and right charges and the Fundamental Commutation Relations

In section 3 we have considered the exact and broken symmetries of the Lagrangian for the classical motion on the Riemannian non compact symmetric spaces F/K. In that analysis we have made use of the Noether's theorem (proved in appendix I) to calculate the charges (conserved or not) corresponding to those symmetries and also their algebra under the Poisson bracket.

At the quantum level a similar analysis is more difficult because we do not know the quantum analog of Noether's theorem. In fact the difficulties already appear in the definition of the quantum transformations since in general, due to ordering problems of quantum operators, they can not be defined in an unambiguous way.

However, the transformations we are interested in, namely the global left and right translations of elements of F, belong to a class for which it is relatively easy to construct a quantum version. They are velocity independent transformations and all the relevant quantities involved, including the Noether's charges, are at most linear in the momenta. As we have shown in section 7 (see discussion leading to eq. (7.6)) there is no ambiguity in constructing quantum operators for classical quantities which depend linearly upon the momenta.

In theorem A of appendix I we see that if the transformation is velocity independent so is X, since the variation of the Lagrangian in this case can not depend upon the accelerations. Then, according to corollary A, we can put X to zero by a suitable choice of G, and so, the Noether's charges have the form:

$$Q = \sum_i F^i(q)p_i \tag{9.1}$$

In addition, if the Lagrangian is quadratic in the velocities, the quantity $D_i \equiv \frac{\partial D}{\partial \dot{q}^i}$ depends linearly in the velocities. In the case of right translations by elements of $\mathbb{B}$, we can eliminate the velocities from D_i and write it as a linear function of the momenta (see eqs. (4.8) and (4.11)). For the left translations by elements of F, D vanishes. Therefore, any velocity independent transformation which has such property (i.e. that $\frac{\partial D}{\partial \dot{q}^i}$ is at most a linear function of the momenta with velocities eliminated), can be defined in an unambiguous way in the quantum theory. The variations of the coordinates and momenta, following theorem A of appendix I, are given by:

$$\delta\hat{q}^i = \frac{\varepsilon}{i\hbar}\,[\hat{q}^i, \hat{Q}]_{QM} = \varepsilon\,\hat{F}^i(q) \tag{9.2}$$

$$\delta\hat{p}_i = \frac{\varepsilon}{i\hbar}\,[\hat{p}_i, \hat{Q}]_{QM} + \varepsilon\,\hat{D}_i \tag{9.3}$$

where $\hat{Q}$ and $\hat{D}_i$ are respectively the quantum operators for Q given by (9.1) and $D_i = \frac{\partial D}{\partial \dot{q}^i}$, and they are constructed following the arguments leading to (7.6). In particular

$$\hat{Q} = \frac{1}{2} \sum_i (\hat{p}_i \hat{F}^i(q) + \hat{F}^i(q)\hat{p}_i) \tag{9.4}$$

Clearly the variations (9.2) and (9.3) are hermitian operators.

We now discuss the algebra of these quantum transformations and of their corresponding quantum charges. According to theorem B and corollary B of appendix I, if we have three velocity independent variations δ, δ' and δ'' of the type described in theorem A, satisfying: ($\varepsilon\varepsilon' = \varepsilon''$)

$$[\delta, \delta']q^i = \delta'' q^i \tag{9.5}$$

then the corresponding variation of the momenta satisfy the same relation:

$$[\delta, \delta']p_i = \delta'' p_i \tag{9.6}$$

and the Poisson bracket of the Noether's charges satisfy

$$\{Q, Q'\}_{PB} = - Q'' \tag{9.7}$$

When we replace the classical variations by their corresponding quantum versions (9.2) and (9.3), the relation (9.5) obviously remains true, since the variations of the coordinates depend upon the coordinates only.

In order to see that the relation (9.6) also remains true in the quantum theory we notice that, for the type of transformations we are considering, the variation of the momenta is a linear function of the momenta and has the form:

$$\delta p_i = f_i^{\ j}(q) p_j \tag{9.8}$$

where $f_i^{\ j}(q)$ can be calculated from the classical version of (9.3). Therefore it follows from (9.6) that the functions f, f'and f" have to satisfy:

$$f_i^{"j}(q) = F^k \frac{\partial f_i^{'j}}{\partial q^k} - F'^k \frac{\partial f_i^{\ j}}{\partial q^k} + f_i^{'k} f_k^{\ j} - f_i^{\ k} f_k^{'j} \tag{9.9}$$

Since (9.3) was constructed by using the arguments leading to (7.6), the quantum version of (9.8) is: (which is the same as (9.3))

$$\delta\hat{p}_i = \frac{1}{2} (\hat{f}_i^{\ j} \hat{p}_j + \hat{p}_j \hat{f}_i^{\ j}) \tag{9.10}$$

Using (9.9) one can easily check that

$$[\delta, \delta'] \hat{p}_i = \frac{1}{2} (\hat{f}_i^{"\,j} \hat{p}_j + \hat{p}_j \hat{f}_i^{"j}) = \delta'' \hat{p}_i \tag{9.11}$$

where we have used the fact that the commutator of the momentum with a function of the coordinates commutes with any other function of the coordinates. So, this shows that (9.6) is also true at the quantum level.

Using similar arguments we can show that the commutator of the quantum Noether's charges satisfy the same relations as (9.7). From (9.1) and (9.7) we have:

$$F''^i(q) = F^j(q)\ \{F'^i(q), p_j\}_{PB} - F'^j(q)\ \{F^i(q), p_j\}_{PB} \quad (9.12)$$

Using (9.4) we can calculate the commutator between the quantum charges. However when doing that attention should be paid to the fact that the commutator of $p_i F^i$ with $F'^j p_j$ can be decomposed in two different, but equivalent, ways. The result we get is:

$$[\hat{Q}, \hat{Q}']_{QM} = -\frac{1}{2}(\hat{F}^j[\hat{F}'^i, \hat{p}_j]_{QM} - \hat{F}'^j[\hat{F}^i, \hat{p}_j]_{QM})\hat{p}_i -$$

$$-\frac{1}{2}\hat{p}_i(\hat{F}^j[\hat{F}'^i, \hat{p}_j]_{QM} - \hat{F}'^j[\hat{F}^i, \hat{p}_j]_{QM}) \quad (9.13)$$

But, since $\frac{1}{i\hbar}[\hat{F}^j, \hat{p}_i]_{QM} \equiv \{F^j, p_i\}_{PB}$, we see from (9.12) and (9.4) that the r.h.s. of (9.13) is the quantum operator for Q". Then:

$$[\hat{Q}, \hat{Q}']_{QM} = -i\hbar\, \hat{Q}'' \quad (9.14)$$

Therefore for those velocity independent transformations which satisfy the conditions of theorem A and for which $\frac{\partial D}{\partial \dot{q}^i}$ is at most linear in the momenta (with velocities eliminated) the results obtained in appendix I at the classical level remain true at the quantum level. Since the left and right translations discussed in section 3 are examples of these transformations, the results of that section hold true for the quantum geodesic motion on the symmetric spaces F/K. We now discuss this in more detail.

According to the arguments leading to eq. (7.6) the quantum operator for the right charges, given by (4.8), are:

($\partial_s \equiv \frac{\partial}{\partial \eta^s}$, η^s parameters of $\mathbf{B}$)

$$\hat{Q}(T_r) = -\frac{i\hbar}{2}\{M_r^{-1s}(\partial_s + \frac{1}{4}\partial_s \ln(\det g)) + (\partial_s + \frac{1}{4}\partial_s \ln(\det g))M^{-1s}_{\ r}\}$$

where we have used the coordinate representation for the momenta given by (7.2). From (8.4) we have

$$\partial_s \ln(\det g) = 2(\det M)^{-1}\partial_s \det M \tag{9.15}$$

and using (8.7)

$$\hat{Q}(T_r) = -i\hbar(M^{-1s}_{\ r}\partial_s + \frac{1}{2}f^s_{sr}) = -i\hbar\, \nabla'_r \tag{9.16}$$

where the operators ∇'_r are defined in (8.11) and from (8.10) we see that $\hat{Q}(T_r)$ is indeed hermitian. According to (9.14) these operator satisfy the same algebra as the classical charges and so:

$$[\hat{Q}(T_r), \hat{Q}(T_s)]_{QM} = -i\hbar\, \hat{Q}([T_r, T_s]) \quad T_r, T_s \in \underset{\sim}{b} \tag{9.17}$$

This relation can also be obtained from (8.12). Since the parameters η^r of the subgroup $\mathbf{B}$ are being used as coordinates on F/K, they become quantum operators in the quantum theory, and therefore from (8.6) and (8.11), we notice that;

$$\hat{Q}(T_r)\,\hat{b} = -i\hbar\,\hat{b}\,T_r + \hat{b}\,\hat{Q}(T_r) \tag{9.18}$$

where the group element $b \in \mathbb{B}$ is now considered as a quantum operator, and obviously has to lie in some representation of the abstract group $\mathbb{B}$. There is no ordering problems in defining it, since it depends upon the coordinates only which commute with each other.

Therefore:

$$[\hat{Q}(T_r), \hat{b}]_{QM} = - i\hbar \hat{b} T_r \tag{9.19}$$

This relation will play an important role in our dicusssion, because it is responsible for the main differences between the classical and quantum geodesic motion on F/K. At the classical level the functions of the canonical variables commute with group elements and therefore the algebra of Noether's charges under the Poisson bracket does not "get mixed" with the Lie algebra of the group generators. At the quantum level, however, the same is not true since quantum operators do not commute with group elements. This, in fact, is the origin of the difficulties we face when constructing quantum conserved quantities in involution for the Toda Molecule models (see section 12).

In analogy with eq. (4.9) we define the quantum Lax operators by:

$$\hat{A} = G^{rs} \hat{Q}(T_r)P_s = - i\hbar G^{rs} \nabla_r' P_s \tag{9.20}$$

$$i\hat{B} = G^{rs} \hat{Q}(T_r)k_s = - i\hbar G^{rs} \nabla_r' k_s \tag{9.21}$$

From eq. (8.15) we see that the Lax operator A is related to the quantum hamiltonian by:

$$\hat{H} = \frac{1}{2} \mathrm{Tr}\, \hat{A}^2 + \frac{\hbar^2}{8} G^{rs} f^u_{ur} f^v_{vs} \tag{9.22}$$

At the classical level the hamiltonian and $\frac{1}{2}\mathrm{Tr}A^2$ coincide (see (3.12)), but at the quantum level we see they differ by a real constant. According to (8.18), this constant is a scalar and so $\mathrm{Tr}\hat{A}^2$ is also an acceptable hamiltonian since it transforms covariantly under general coordinate transformation and it is hermitian. In addition it is a positive definite operator and unlike $\hat{H}$, it can have zero energy ground state since from (8.16):

$$\langle\psi|\mathrm{Tr}\hat{A}^2|\psi\rangle \geq 0 \tag{9.23}$$

The Fundamental Poisson Relations derived in section 4, namely eqs. (4.22), (4.27) and (4.29), remain true when we replace the classical Lax operators by their corresponding quantum version and the Poisson bracket by the quantum commutator since the algebras of the classical and quantum right charges are the same and since the l.h.s. of those equations involve Lie brackets only.

Therefore we have:

$$[\hat{A} \overset{\otimes}{,} \hat{A}]_{QM} = - i\hbar \left[\frac{\mathbb{P}+\mathbb{R}}{2}, \hat{A} \otimes 1\right] - i\hbar \left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1 \otimes \hat{A}\right] \tag{9.24}$$

$$[i\hat{B} \overset{\otimes}{,} \hat{A}]_{QM} = -\, i\hbar \left[\frac{\mathbb{P}+\mathbb{R}}{2},\ 1 \otimes i\hat{B} + i\hat{B} \otimes 1\right] - i\hbar\, [\mathbb{S},\ \hat{A} \otimes 1] +$$

$$+\, i\hbar [\mathbb{Q},\ 1 \otimes \hat{A}] \qquad (9.25)$$

$$[\hat{A} \overset{\otimes}{,} i\hat{B}]_{QM} = -i\hbar\left[\frac{\mathbb{P}-\mathbb{R}}{2}, 1 \otimes i\hat{B} + i\hat{B} \otimes 1\right] + i\hbar[\mathbb{S}, 1 \otimes \hat{A}] -$$

$$-\, i\hbar[\mathbb{Q}, \hat{A} \times 1] \qquad (9.26)$$

and relations like these are called in the literature Fundamental Commutation Relations(5).

The quantum operators for the left charges, given by (4.32), are also constructed using the arguments leading to eq. (7.6) since, like the right charges, they are linear in momenta. However, since $\dot{x}\, x^{-1}$ can not be easily written in terms of the canonical variables, writing these operators in a way as explicit as we did for the right charges would be complicated. But, if we restrict the generators L_i in (4.32) to the Lie algebra of $\mathbb{B}$ we can write the corresponding operators in an explicit form.

Using eq. (2.11) in the "gauge K=1" and since $\sigma(x) = x^{-1}$ we get from (4.32):

$$X(T_r) = \mathrm{Tr}\ (A\ b^{-1} T_r b) = A_r^{\ s}(\eta)\ Q(T_s) \qquad (9.27)$$

when $A_r^{\ s}(\eta)$ is the adjoint representation of B and $Q(T_s)$ are the right charges defined in (4.3). Since this expression is linear in the momenta all methods of symmetrization give the same result (see sec. 7). Then, by symmetrizing $Q(T_s)$ first we obtain the operator (9.16) and then by symmetrizing the rest we get:

$$\hat{X}(T_r) = - \frac{i\hbar}{2} (\hat{A}_r{}^s(\eta)\nabla'_s + \nabla'_s \hat{A}_r{}^s(\eta))$$

Using eq. (III.13) of appendix III we see that this is just the operator given by (III.8) which generates left translation on the group B:

$$\hat{X}(T_r) = - i\hbar\, \hat{A}_r{}^s \nabla'_s = i\hbar \nabla^L_r \tag{9.28}$$

Using eqs. (8.10) and (III.13) one can easily check that these operators are indeed hermitian. According to (III.12) they commute with the right charges:

$$[\hat{X}(T_r), \hat{Q}(T_s)]_{QM} = 0 \qquad T_r, T_s \in \underset{\sim}{b} \tag{9.29}$$

which is the quantum analogy of eq. (3.11).

The quantum operators for the remaining left charges, i.e. those corresponding to the generators of K, can be constructed in a similar way although we do not do it explicitly here. Together with the operators (9.38) and according to the arguments leading to (9.14) they satisfy the same commutation relations as the classical charges.

So we have

$$[\hat{X}(L_i), \hat{X}(L_j)]_{QM} = i\hbar\, \hat{X}([L_i, L_j]) \qquad L_i, L_j \in \underset{\sim}{f} \tag{9.30}$$

As a consequence of that the relation (4.36) remains true when we replace the Poisson bracket by the quantum commutator and the classical operator X, given by (4.34), by its quantum version:

$$\hat{X} = g^{ij}\hat{X}(L_i)L_j \tag{9.31}$$

Thus we have

$$[\hat{X} \overset{\otimes}{,} \hat{X}]_{QM} = -i\hbar\ [\mathbb{C},\ 1 \otimes \hat{X}]$$

$$= i\hbar\ [\mathbb{C},\ \hat{X} \otimes 1] \tag{9.32}$$

This relation will play an important role when we construct the quantum conserved quantities in involution in section 12 because, unlike eq. (9.24), its r.h.s. contains quantum operators either on the left or right entry only and therefore we do not have ordering problems when calculating the commutator of powers of $\hat{X}$. In fact it is easy to check, using (9.32), that:

$$[\mathrm{Tr}\hat{X}^n,\ \hat{X}]_{QM} = 0 \tag{9.33}$$

and therefore

$$[\mathrm{Tr}\hat{X}^n,\ \mathrm{Tr}\hat{X}^m]_{QM} = 0 \tag{9.34}$$

The operators $\mathrm{Tr}\hat{X}^n$ are manifestly hermitian for n = 2 only, and therefore they can not be taken as the quantum operators for the conserved quantities. (See discussion in section 12).

Classically we know that, as a consequence of the relation (2.11), the quantities $\mathrm{Tr}X^n$ and $\mathrm{Tr}A^n$ are proportional. At the quantum level this relation is not true anymore and the reason is that eq (2.11) suffers some quantum corrections. We now discuss that. Using eqs. (2.11) and (4.9) we can write (4.34) : as

$$X = -\frac{1}{2}\sigma(x^{-1}\dot{x}) = bAb^{-1}$$

$$= G^{rs}\, Q(T_r)\; bP_s b^{-1} \tag{9.35}$$

Replacing the classical quantities by their corresponding quantum operators and symmetrizing the non-commuting parts we obtain an hermitian operator for X, which is the same operator as (9.31) since, according to the discussion in section 7, all methods of symmetrizing a quantity linear in momenta are equivalent. Then:

$$\hat{X} = \frac{1}{2}\, G^{rs}(\hat{Q}(T_r)\hat{b}P_s\hat{b}^{-1} + \hat{b}P_s\hat{b}^{-1}\hat{Q}(T_r)) \tag{9.36}$$

Using eqs. (9.19) and (9.20) we then obtain the quantum analog of the relation (2.11):

$$\hat{X} = \hat{b}(\hat{A} - \frac{i\hbar}{2}\, G^{rs}\{T_r, P_s\})\hat{b}^{-1}$$

$$= \hat{b}\;\tilde{A}\;\hat{b}^{-1} \tag{9.37}$$

where we have defined the operator

$$\tilde{A} = -\frac{i\hbar}{2}\;G^{rs}\{\nabla_r' + T_r,\; P_s\} \tag{9.38}$$

Obviously the definition of $\tilde{A}$ only makes sense in a given representation of the Lie algebra of $\mathbb{B}$, since it contains terms quadratic in the generators. Operators with such terms have already been used by Mansfield [32] in order to obtain quantum zero curvature conditions for Toda systems. The origin of these terms, however, is not well understood yet, and we believe it could be clarified by a better understanding of the "gauge transformations" of the Lax operators at the quantum level. The interesting fact about $\tilde{A}$ is that the matrix and differential operator representations of the Lie algebra of $\mathbb{B}$ are in some sense "mixed" and, according to (8.12), we have:

$$[\![\nabla_r' + T_r, \nabla_s' + T_s]\!] = f^u_{rs} (\nabla_u' + T_u)$$

where the bracket $[\![,]\!]$ means we are taking the quantum commutator and the Lie bracket at the same time.

We can use relation (9.37) to find the relation between the quantities $\mathrm{Tr}\hat{X}^n$ and $\mathrm{Tr}\hat{A}^n$, and for n = 2 this is quite simple to do. From eq. (9.19) we have

$$[\hat{Q}(T_r), \hat{b}P_s\hat{b}^{-1}]_{QM} = - i\hbar\, \hat{b}[T_r, P_s]\hat{b}^{-1} \tag{9.39}$$

and therefore form (9.36) and (9.20)

$$\begin{aligned} \mathrm{Tr}\hat{X}^2 &= \mathrm{Tr}\hat{A}^2 + \frac{\hbar^2}{4} G^{rs}G^{uv}\mathrm{Tr}([T_r, P_s][T_u, P_v]) \\ &= \mathrm{Tr}\hat{A}^2 + \frac{\hbar^2}{4} G^{rs} f^u_{ur} f^v_{vs} \end{aligned} \tag{9.40}$$

where we have used eqs. (IV.11), (IV.12) and (IV.32) of appendix

IV to get the last equality.

Therefore we see from (9.22) that

$$\hat{H} = \frac{1}{2} \mathrm{Tr}\hat{X}^2 \tag{9.41}$$

and from (9.33)

$$[\hat{H}, \hat{X}]_{QM} = 0 \tag{9.42}$$

so the left charges are conserved. Since, according to (9.30), the left charges generate the algebra of F under the quantum commutator we see from (9.31) and (9.41) that the hamiltonian, and consequently the Laplace-Beltrami operator, is the quadratic Casimir operator for F, and (9.42) is just a consequence of that.

We now consider the quantum time evolution of the Lax operator A. From eqs. (8.6), (8.11) and (8.15) we have

$$\hat{b}^{-1}\hat{H}\hat{b} = \hat{H} - \frac{\hbar^2}{2} G^{rs}(T_r T_s + 2\, T_r \nabla'_s)$$

$$= \tilde{H} - i\hbar\,(\tilde{A} + i\tilde{B}) \tag{9.43}$$

where we have defined the operators

$$i\tilde{B} = i\hat{B} - \frac{i\hbar}{2} G^{rs}\{T_r, k_s\}$$

$$= - \frac{i\hbar}{2} G^{rs}\{\nabla'_r + T_r, k_s\} \tag{9.44}$$

and

$$\tilde{H} = \hat{H} + \frac{\hbar^2}{2} G^{rs} T_r T_s \tag{9.45}$$

and the operator $\tilde{A}$ is defined by (9.38)

Notice that the term quadratic in the generators in (9.45) has the same form as $Tr\hat{A}^2$, with ∇'_r replaced by T_r, and therefore apart from a constant it has the same form as the hamiltonian. Using (9.37) and (9.43) we can write (9.42) as:

$$\frac{1}{i\hbar} [\tilde{H}, \tilde{A}] = [i\tilde{B}, \tilde{A}] \tag{9.46}$$

and this is the quantum analog of the Lax pair equation (2.14)(*).

(*) We are grateful to N. Ganoulis for helpful discussions on the quantum Lax pair equation.

10. The Fundamental Commutation Relations and the quantum Yang-Baxter equation for the symmetric spaces G^N/K.

In section 5 we have shown how to obtain the Yang-Baxter equation for the symmetric spaces G^N/K from the Fundamental Poisson Relation (5.10). We now show that those equations remain true at the quantum level as a consequence of the fact that the algebras of the classical and quantum right charges are the same as we have shown in the last section.

In analogy with eq. (5.5) we define

$$\hat{D}(T) = -\,\mathrm{Tr}(\hat{A}\sigma(T)) = -\,\hat{Q}(\sigma(T)) \quad , \; T \in \sigma(\underset{\sim}{b}) \qquad (10.1)$$

where σ is the Cartan involution of G^N defined in (5.2) and $\hat{Q}(\sigma(T))$ is the quantum operator for the right charges, given by (9.16), for the case of the symmetric spaces G^N/K.

Thus it follows from (9.17) that

$$[\hat{D}(T), \hat{D}(T')]_{QM} = i\hbar\, \hat{D}([T, T']) \quad , \quad T, T' \in J(\underset{\sim}{b}) \qquad (10.2)$$

Define, in analogy with (5.7)

$$\hat{D} = \hat{A} + i\hat{B} = \sum_i H_i \hat{D}(H_i) + 2 \sum_{\alpha > o} \frac{\alpha^2}{2} E_\alpha \hat{D}(E_{-\alpha}) \qquad (10.3)$$

where $\hat{A}$ and $i\hat{B}$ are respectively the odd and even parts of $\hat{D}$ under σ. We see that the relation (5.10) remains true when we replace the classical D by its quantum operator (10.3) and the Poisson bracket by the quantum commutator, since its r.h.s. involves Lie brackets only and since the classical and quantum algebras satisfied by the components of D are the same.

So

$$[\hat{D} \overset{\otimes}{,} \hat{D}]_{QM} = -i\hbar[\mathbb{P}, \hat{D} \otimes 1 + 1 \otimes \hat{D}] \tag{10.4}$$

For the same reasons the Yang-Baxter equation (5.15) also holds true at the quantum level

$$[\hat{D}_{12}, \hat{D}_{32}]_{QM} + [\hat{D}_{13}, \hat{D}_{12}] + [\hat{D}_{13}, \hat{D}_{32}] = 0 \tag{10.5}$$

where $\hat{D}_{12}$ and $\hat{D}_{32}$ have the same form as their classical analog with D(T) replace by $\hat{D}(T)$ and $\hat{D}_{13} = i\hbar\, D_{13}$.

By acting with the automorphism σ on the left and right entries of (10.4) in a suitable way (see sec. 5) we obtain the Fundamental Commutation relations for the Lax pair operators:

$$[\hat{A} \overset{\otimes}{,} \hat{A}]_{QM} = -\, i\hbar\, [\frac{\mathbb{P} + \mathbb{R}}{2}, \hat{A} \otimes 1] - i\hbar\, [\frac{\mathbb{P} - \mathbb{R}}{2}, 1 \otimes \hat{A}] \tag{10.6}$$

$$[\hat{A} \overset{\otimes}{,} \hat{B}]_{QM} = -\, i\hbar\, [\frac{\mathbb{P} - \mathbb{R}}{2}, 1 \otimes \hat{B} + \hat{B} \otimes 1] \tag{10.7}$$

$$[i\hat{B} \overset{\otimes}{,} i\hat{B}]_{QM} = -\, i\hbar\, [\frac{\mathbb{P} - \mathbb{R}}{2}, \hat{A} \otimes 1] - i\hbar\, [\frac{\mathbb{P} + \mathbb{R}}{2}, 1 \otimes \hat{A}] \tag{10.8}$$

11. The quantum reduction to the Toda Molecule system

We have seen in section 6 that the solutions of the Toda Molecule equations correspond to some geodesics on the symmetric spaces G^N/K. We now show that at the quantum level this geometrical picture is also true. By using a reduction procedure [26] we prove that solutions to the Schroedinger equation for the Toda molecule system can be obtained from free wave functions on G^N/K by projecting them on eigenfunctions of the quantum operators for the left charges corresponding to the positive root step operators. This reduction is possible because the Laplace-Beltrami operator for G^N/K, and so the hamiltonian, when written in terms of the horospheric coordinates (ϕ_a, ρ_α) introduced in (6.2), decomposes in radial and angular parts, each of them containing derivatives w.r.t. either ϕ_a or ρ_α only. We believe this reduction procedure works in a similar way for all non-compact symmetric spaces F/K described in section 2, but we have not checked it. We will use the Chevalley basis for G^N and so, according to (5.3), the generators of the Borel subgroup $\mathbb{B}$ are $H_a = 2a.\,H/a^2$ (a=1,2...r) (a simple root and r = rank G^N) , and E_α (α>o). Then, using the definition (5.2) of the Cartan involution σ of G^N and the normalization of the Killing form given by (5.4) we see that G_{rs} defined in (4.10), in the case of G^N/K, has the form:

$$G_{ab} = 4a.b/a^2b^2 = (2/a^2)K_{ab}\ , \quad G_{a\alpha} = 0\ , \quad G_{\alpha\beta} = \delta_{\alpha\beta}/\alpha^2 \qquad (11.1)$$

where K_{ab} is the Cartan matrix which apppears in (1.1). Using

(6.2) we see that $M_r^{\ s}$ defined in (4.6) has the form:

$$M_a^{\ b} = \frac{1}{2}\,\delta_a^{\ b} \quad , \; M_a^{\ \alpha} = M_\alpha^{\ a} = 0 \; , \quad M_\alpha^{\ \beta} = m_\alpha^{\ \beta}(\rho)\,\exp(-K_{\beta a}\,\phi_a/2) \tag{11.2}$$

where $$n^{-1}\frac{\partial n}{\partial \rho^\alpha} = m_\alpha^{\ \beta}(\rho) E_\beta \quad , \; K_{\alpha a} = 2\;\alpha.a/a^2 \tag{11.3}$$

Therefore, the metric (8.4) for the symmetric spaces G^N/K written in horospheric coordinates (ϕ_a, ρ_α) has a block diagonal form:

$$g_{ab} = \frac{1}{4}\,G_{ab} = K_{ab}/2a^2 \;, \; g_{a\alpha} = 0 \quad , \; g_{\alpha\beta} = (mJm^T)_{\alpha\beta} \tag{11.4}$$

where we have introduced

$$J_{\alpha\beta}(\phi) = \exp(-K_{\alpha a}\,\phi_a)\;\delta_{\alpha\beta/\alpha^2} \tag{11.5}$$

From the commutation relation (5.1) we have

$$f^u_{\ au} = \sum_{\alpha > o} K_{\alpha a} = 2 \quad , \quad f^u_{\ \alpha u} = 0 \tag{11.6}$$

since [25] $2\;\delta.a/a^2 = 1$, where a is a simple root and $\delta = \frac{1}{2}\sum_{\alpha > o}\alpha.$

Using (10.1), (9.16) and (11.2) we obtain the quantum operators for the right charges:

$$\hat{D}(H_a) = -\,i\hbar\;\nabla_a' = -\,i\hbar\;2\;(\frac{\partial}{\partial\phi_a} - \frac{1}{2}) = 2\;\hat{p}_a \tag{11.7}$$

where p_a is the canonical momentum conjugate to ϕ_a and it can be calculated from (7.2) using (11.4) and (11.5)

$$\hat{D}(E_{-\alpha}) = -i\hbar\nabla'_\alpha = -i\hbar \exp(K_{\alpha a} \phi_a/2)\nabla_\alpha \tag{11.8}$$

where ∇_α are the generators of right transformations on the nilpotent subgroup in (see(11.3))

$$\nabla_\alpha = m^{-1\beta}_{\alpha} \frac{\partial}{\partial\rho^\beta} \quad , \qquad n^{-1}\nabla_\alpha n = E_\alpha \tag{11.9}$$

Therefore, using (11.1), (11.7) and (11.8) we see that the hamiltonian (8.15) decomposes in two parts:

$$\hat{H} = -\frac{\hbar^2}{2}(\Delta^{(a)} + \Delta^{(n)}) \tag{11.10}$$

where $\Delta^{(a)}$ is called the radial part ($\partial_a \equiv \frac{\partial}{\partial\phi^a}$)

$$\Delta^{(a)} = g^{ab}(\partial_a \partial_b + \frac{1}{2} f^u_{ua}\partial_b) \tag{11.11}$$

and $\Delta^{(n)}$ is the angular part

$$\Delta^{(n)} = \sum_{\alpha>o} \alpha^2 \exp(K_{\alpha a} \phi_a)\nabla_\alpha\nabla_\alpha \tag{11.12}$$

When we performed the classical reduction in section 6, we identified the solutions of the Toda Molecule equations with those geodesics on G^N/K where the left charges $X(E_\alpha)$ $(\alpha>o)$ assumed the values (6.1). In the quantum reduction we have to look for eigenvalues of the quantum operators $\hat{X}(E_\alpha)$ whose eigenvalues are the same as (6.1), i.e.

$$\hat{X}(E_\alpha)\Theta_\lambda = \lambda_\alpha\Theta_\lambda \tag{11.13}$$

where

$$\lambda_\alpha = \begin{cases} \lambda_a & \text{if } \alpha \text{ is a simple root } a \\ 0 & \text{if } \alpha \text{ is a positive non simple root} \end{cases} \tag{11.14}$$

Notice that, in general, the operators $\hat{X}(E_\alpha)$ can not have simultaneous eigenvalues because they do not commute. Indeed, from (9.30) and (5.1) we have:

$$[\hat{X}(E_\alpha), \hat{X}(E_\beta)]_{QM} = i\hbar\, N_{\alpha\beta}\hat{X}(E_{\alpha+\beta}) \quad , \ \alpha,\beta>o \tag{11.15}$$

However, we see that the r.h.s. of (11.15) does not contain operators corresponding to simple roots and therefore it is possible for the operators $\hat{X}(E_\alpha)$ to have simultaneous eigenfunctions with eigenvalues satisfying (11.14). Since these operators are hermitian and since they commute with the hamiltonian (see (9.42)) their eigenvalues are real and constant in time.

Using (9.28) and (11.2) we have:

$$\hat{X}(E_\alpha) = -\, i\hbar\, a_\alpha{}^\beta(\rho)\nabla_\beta = -\, i\hbar\, \nabla_\alpha{}^L \tag{11.16}$$

where ∇_β is defined in (11.9) and $a_\alpha{}^\beta(\rho)$ is the adjoint representation of the nilpotent subgroup n:

$$n^{-1} E_\alpha n = a_\alpha{}^\beta(\rho)\, E_\beta \tag{11.17}$$

From eq. (III.9) of appendix III we see $\nabla_\alpha{}^L$ are the generators of left translation on the subgroup n.

Since n is the exponentiation of the positive root step operators we see from (11.17) that $a_\alpha{}^\beta$ is an upper triangular matrix, i.e.(*)

$$a_\alpha{}^\beta = \begin{cases} 1 \text{ if } \alpha=\beta \\ o \text{ if } \beta-\alpha \text{ is not a positive root} \end{cases} \tag{11.18}$$

and consequently $a_a{}^b = \delta_a{}^b$ if a and b are simple roots.

Therefore from (11.16) and (11.18) we conclude that if Θ_λ is an eigenfunction of $\hat{X}(E_\alpha)$ with eigenvalues (11.14) then it is also an eigenfunction of ∇_α with the same eigenvalues, since:

$$-i\hbar\nabla_\alpha\Theta_\lambda = -i\hbar\; a^{-1}_\alpha{}^\beta \nabla_\beta{}^L\Theta_\lambda = \lambda_\alpha\Theta_\lambda \tag{11.19}$$

Since the inverse of $a_\alpha{}^\beta$ also satisfies (11.18) ($a^{-1}_\alpha{}^\beta(n) = a_\alpha{}^\beta(n^{-1})$. The operators $X(E_\alpha)$, according to (11.16), do not

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(*) From eq. (III.3) of appendix III and (11.18) we see that $m_\alpha{}^\beta$ is also a upper triangular matrix with 1's on the diagonal. Therefore detm = deta = 1 and using (11.4),(11.5) and (11.6) we conclude that detg ~ $\exp(-2\sum_a \phi_a)$ and so it is independent of the angle variables ρ_α.

depend upon the radial variables ϕ_a and therefore these eigenfunctions Θ_λ are functions of the angular variables ρ_α only.

From (8.7), (11.2) and (11.6) we have

$$\sum_{\alpha>0} \frac{\partial}{\partial\rho^\alpha}((\det m){m^{-1}}_\beta{}^\alpha) = 0 \tag{11.20}$$

and therefore using this and (11.9) one can show that

$$\int dN\, \psi^* \nabla_\alpha \phi = - \int dN (\nabla_\alpha \psi)^* \phi \tag{11.21}$$

where dN is the integration measure for the angle variables

$$dN = (\det g_{\alpha\beta})^{1/2} d^n \rho \tag{11.22}$$

Using (11.6) and eq. (III.13) of appendix III we get from (11.21) that

$$\int dN\, \psi^* \nabla_\alpha{}^L \phi = - \int dN (\nabla_\alpha^L \psi)^* \phi \tag{11.23}$$

Therefore if there are two eigenfunctions Θ_λ and $\Theta_\lambda{}'$ of the operator $\hat{X}(E_\alpha)$ corresponding to the eigenvalues λ and λ' respectively, we have

$$(\lambda_\alpha' - \lambda_\alpha) \int dN\; \Theta_\lambda^* \; \Theta_\lambda{}' = 0 \tag{11.24}$$

and so, they are orthogonal if the eigenvalues λ and λ' are distinct.

The radial part of the Laplace-Beltrami operator, given by (11.11), contains a term which is linear in derivatives w.r.t. ϕ_a, and since the quantum hamiltonian for the Toda Molecule system does not contain such a term it is useful, in the reduction procedure, to define the operator

$$\hat{H}_o = -\frac{\hbar^2}{2} J^{1/4} \Delta(a) J^{-1/4}$$

$$= -\frac{h^2}{2} [g^{ab} \partial_a \partial_b - \delta^2] \qquad (11.25)$$

where

$$J = \det J_{\alpha\beta} \qquad (11.26)$$

and where we have used the fact that

$$\frac{1}{4} G^{ab} f^u_{au} f^v_{bv} = \delta^2, \qquad \delta = \frac{1}{2} \sum_{\alpha>o} \alpha \qquad (11.27)$$

We now show how to obtain solutions for the quantum Toda systems from the free wave functions on the symmetric space G^N/K.

<u>Theorem</u> "If $\psi(\phi,\rho,t)$ is a solution of the Schroedinger equation for the free motion on the symmetric spaces G^N/K:

$$\hat{H}\psi = i\hbar \frac{\partial\psi}{\partial t} \tag{11.28}$$

with $\hat{H}$ given by (11.10), then the reduced wave function

$$\psi_R(\phi,t) = J^{-1/4}\int dN\ \psi(\phi,\rho,t)\ \Theta_\lambda^*(\rho) \quad , \ dN = (\det g_{\alpha\beta})^{1/2} d^n\rho \tag{11.29}$$

with $\Theta_\lambda(\rho)$ obeying (11.13) and (11.14), satisfy

$$(\hat{H}_o + V_\lambda(\phi))\psi_R(\phi,\ t) = i\hbar \frac{\partial\psi_R(\phi,t)}{\partial t} \tag{11.30}$$

where $V_\lambda(\phi)$ is the Toda potential

$$V_\lambda(\phi) = \sum_a \frac{a^2}{2} \lambda_a^2 \ \ \exp(K_{ab}\phi_b) \tag{11.31}$$

and $\hat{H}_o$ is given by (11.25)".

Proof

The r.h.s. of (11.29) depends explicitly upon time through $\psi(\phi,\rho,t)$ only, and therefore using (11.28), (11.10) and (11.4) we have

$$i\hbar \frac{\partial\psi_R}{\partial t} = J^{-1/4} \int dN\ (i\hbar \frac{\partial\psi}{\partial t})\ \Theta_\lambda^* =$$

$$= -\frac{\hbar^2}{2}\{J^{1/4}\int d^n\rho(\det m)(\Delta^{(a)}\psi)\Theta_\lambda^* + \sum_{\alpha>o} \alpha^2\exp(K_{\alpha a}\phi_a)J^{-1/4}\int dN(\nabla_\alpha\nabla_\alpha\psi)\Theta_\lambda^*\}$$

Using (11.25) the first term becomes:

$$\frac{-\hbar^2}{2} J^{1/4} \int d^n\rho(\det m)\ (\Delta^{(a)}\psi)\Theta_\lambda^* = \hat{H}_o\psi_R(\phi,t)$$

Using (11.14), (11.19) and (11.21) the second term becomes:

$$\frac{-\hbar^2}{2} \sum_{\alpha>o} \alpha^2 e^{K_{\alpha a}\phi_a} J^{-1/4}\int dN(\nabla_\alpha\nabla_\alpha\ \psi)\Theta_\lambda^* = \sum_a \frac{a^2}{2} \lambda_a^2\ e^{K_{ab}\phi_b}\ \psi_R(\phi,t)$$

and so the theorem is proven.

As we mentioned above the eigenvalues λ_a of $\hat{X}(E_\alpha)$ are real and constant in time and, according to (11.31), we see that λ_a^2 play the role of non negative coupling constants for the Toda Molecule systems.

The reduction of quantum operators is performed by the same method used for the hamiltonian. Let $\hat{O}$ be a quantum operator and consider its action on an arbitrary wave function $\psi(\phi,\rho)$:

$$\hat{O}\ \psi(\phi,\rho) = \psi'(\phi,\rho) \tag{11.32}$$

Using (11.29) we calculate the reduced wave function $\psi_R(\phi)$ and $\psi'_R(\phi)$ corresponding to ψ and ψ' respectively. We define the reduced form of $\hat{O}$, if it exists, by the operator $\hat{O}_R$ that maps ψ_R into ψ'_R

$$\hat{O}_R\psi_R(\phi) = \psi'_R(\phi) = J^{-1/4}\int dN(\hat{O}\psi)\Theta_\lambda^* \tag{11.33}$$

We do not discuss here the conditions under which this operator exists and if it is unique or not. However we show below how to construct the reduced form of the operators relevant for our discussion.

Notice that if there are three operators $\hat{O}$, $\hat{O}'$ and $\hat{O}''$ satisfying

$$[\hat{O} , \hat{O}']_{QM} = \hat{O}''$$

and they all possess a reduced form, then

$$[\hat{O}_R, \hat{O}'_R]_{QM} = \hat{O}''_R$$

The reason why this is so, it is because the reduced form of the product of two operators is the product of their reduced forms. Therefore the reduction procedure preserves the algebra of the quantum operators.

According to (11.7) the quantum operator for the canonical momentum associate to ϕ is:

$$\hat{p}_a = - i\hbar \, (\partial_a - \frac{1}{2}) \tag{11.34}$$

Its reduced form, according to (11.33), is:

$$\begin{aligned} \hat{p}^R_a \, \psi_R &= - i\hbar \; J^{1/4} \int dN \; (\det m) [(\partial_a - \frac{1}{2}) \psi] \Theta^*_\lambda \\ &= - i\hbar \; \partial_a \psi_R \end{aligned} \tag{11.35}$$

Since Θ_λ is independent of ϕ_a and since (see(11.5) and (11.6))

$$[\partial_a, J^P] = -2PJ^P$$

Notice that $\hat{p}_a^R$ is hermitian under the measure $dV = (\det g_{ab})^{1/2} d^r\phi$ and therefore is the correct momentum operator for the Toda Molecule systems. The reduced form of the operators $\hat{X}(E_\alpha) = -i\hbar\nabla_\alpha^L$ are just the eigenvalues (11.14) since using (11.23)

$$\hat{X}(E_\alpha)_R\phi_R = J^{-1/4}\int dN(-i\hbar\nabla_\alpha^L\phi)\Theta_\lambda^* = \begin{cases} \lambda_a\phi_R & \text{if } \alpha \text{ is simple} \\ 0 & \text{if } \alpha \text{ is not simple} \end{cases} \tag{11.36}$$

Similarly, using (11.19), we see that the reduced form of the operators $-i\hbar\nabla_\alpha$ are also the eigenvalues (11.14). So

$$-i\hbar\,\nabla_\alpha^{(R)} \equiv \hat{X}(E_\alpha)_R \tag{11.37}$$

Obviously the reduced form of the operators for the radial variables ϕ_a are the ϕ_a's themselves. Therefore we have shown that any quantum operator constructed out of ϕ_a, p_a, ∇_α and ∇_α^L have a reduced form. In particular, the reduced form of the quantum Lax pair operators, given by (10.3), is

$$\hat{D}_R = \hat{A}_R + i\hat{B}_R = \frac{1}{2}\sum_{a,b} g^{ab}\,\hat{p}_a^R\,H_b + \sum_a a^2\lambda_a\,\exp(K_{ab}\phi_b/2)E_a \tag{11.38}$$

where we have used (11.7), (11.8), (11.35) and (11.37).

If we set $a^2 \lambda_a = 1$ we obtain the usual expression for the Lax Pair operators. The quantum hamiltonian for the Toda Molecule systems then becomes, according to (11.25), (11,30) and (11.31)

$$\hat{H}_R = \frac{1}{2} \sum_{a,b} g^{ab} \hat{p}_a^R \hat{p}_b^R + \sum_a \frac{1}{2a^2} \exp(K_{ab}\phi_b) + \frac{\hbar^2}{2} \delta^2 \qquad (11.39)$$

Using the same arguments of section t one can check that the reduced Lax operators satisfy

$$[\hat{A}_R \overset{\otimes}{,} \hat{A}_R]_{QM} = -i\hbar \; [\mathbb{P}, \hat{A}_R \otimes 1 + 1 \otimes \hat{A}_R]$$

and

$$[\hat{B}_R \overset{\otimes}{,} \hat{B}_R]_{QM} = 0$$

12. The quantum conserved quantities and integrability

We now construct the quantum conserved quantities in involution for the geodesic motion on the non-compact symmetric spaces F/K and show that the Toda Molecule systems are integrable at the quantum level. In sectin 4 we have showm that the classical conserved quantities could be constructed using the Fundamental Poisson Relations (FPR) either for the Lax operator A, eq. (4.22), or for the left charges X eq. (4.36).

At the quantum level the use of the Fundamental Commutation Relations (FCR) to construct conserved quantities is cumbersome because the usual multiplication law on the tensor product space, namely $(A \otimes B)(C \otimes D) = AC \otimes BD$, is not valid when the left and right entries contain non commuting quantum operators. For this reason the FCR for the quantum Lax operator $\hat{A}$, eq. (9.24), does not provide us with a simple way of constructing quantum conserved quantities .

However, the FCR for the quantum left charges $\hat{X}$, eq. (9.32), possess a very useful and nice property. Its r.h.s. contains quantum operators either on the left or right entry only and therefore when we calculate the commutator of powers of $\hat{X}$ we do not have the sort of problems we described above. Indeed, using that relation one can easily check that the operators $\mathrm{Tr}\hat{X}^n$ are in involution (see (9.34)).

Although the quantum operators occurring in $\hat{X}$, namely $\hat{X}(L_i)$ (see (9.31)), are hermitian, $\hat{X}$ itself is not guaranteed to be so. The generators of the non compact real form F of G are not, in general, hermitian. Therefore the operators $\mathrm{Tr}\hat{X}^n$ are not hermitian (except for $n = 2$ since the Killing form of F is real) and so, can not be taken as the quantum conserved quantities.

However, the trace of the symmetric product of n generators of the Lie algebra of F is a real number whenever the classical quantity TrX^n is real$^{(*)}$. The reason is that TrX^n is a polynomial of degree n in the left charges $X(L_i)$ which are real functions of the classical canonical variables. These charges are functionally independent and since they obviously commute the coefficient of each independent term has the form $g^{i_1j_1} g^{i_2j_2} \ldots g^{i_nj_n} Tr^{(s)}(T_{j_1}T_{j_2}\ldots T_{j_n})$ (see (4.34)), where $Tr^{(s)}$ means the trace of the symmetric product. Thus these coefficients are real if TrX^n is so.

Therefore, we conclude that the quantum operators

$$\hat{I}_n = \frac{1}{n!}Tr^{(s)}\hat{X}^n = \frac{1}{n!}g^{i_1j_1}\ldots g^{i_nj_n}\hat{X}(L_{i_1})\ldots\hat{X}(L_{i_n})Tr^{(s)}(L_{j_1}L_{j_2}\ldots L_{j_n}) \tag{12.1}$$

are hermitian whenever TrX^n is real.

(*)We know that classically $TrA^n = TrX^n$ and in general TrA^n is real. In fact for the normal real form G^N of G the Lax operator A is hermitian, since the generators of the odd subspace $\underset{\sim}{p}$ of $\underset{\sim}{g}^N$, namely H_i and $(E_\alpha+E_{-\alpha})$ are hermitian $(H_i^+=H_i,\ E_\alpha^+ = E_{-\alpha})$. According to Helgason [(23)] (Chapter X, sec. 2) for all involutive automorphism of the non-compact (compact) classical Lie algebras the generators of the odd subspace $\underset{\sim}{p}$ of $\underset{\sim}{f}$ are hermitian (anti-hermitian) in the defining representation. We believe this is true in general.

Using (9.32) one can check that

$$[\hat{I}_n, \hat{X}]_{QM} = 0 \tag{12.2}$$

therefore

$$[\hat{I}_n, \hat{I}_m]_{QM} = 0 \tag{12.3}$$

So, the quantities $\hat{I}_n$ are in involution and are conserved since $\hat{I}_2$ is the Hamiltonian (see (9.41)). We notice the operators (12.1) correspond in fact to the invariant Casimir operators for $\underset{\sim}{f}$ since the left charges $\hat{X}(L_i)$ generate the algebra of F under the quantum commutator.

Having the conserved quantities $\hat{I}_n$ written in terms of the left charges is not very useful for our purposes because we do not know how to write these operators in terms of the canonical variables of the Toda Molecule systems by using the reduction procedure explained in the last section. However, one may convince oneself that by using the relations (9.31), (9.36) and (9.39) one can eliminate the left charges from (12.1) and write the operators $\hat{I}_n$ in terms of the quantum Lax operator $\hat{A}$ only, which can be reduced. Therefore when that is done the reduced form of $\hat{I}_n$ become the quantum conserved quantities for the Toda Molecule systems, and since the commutation relation (12.3) is preserved under reduction we see that these systems are integrable at the quantum level.

We have not found a proper way of writing $\hat{I}_n$ explicity in terms of the Lax operator $\hat{A}$. We think this requires a better understanding of the richer algebraic structure underlying these systems at the quantum level.

13. Conclusions

We have developed further the geometric picture of Olshanetsky and Perelomov[12] where the solutions of the Toda Molecule models are seen as some special geodesics on the symmetric spaces G^N/K and have shown how to extend this picture to the quantum level. This has proven to be of great importance in clarifying the algebraic structure underlying the classical and quantum integrability of these models.

The non-compact Riemannian symmetric spaces F/K we considered have a hidden group theoretic structure due to the Iwasawa decomposition of F ($g = n\ a\ k$) which enables us to parametrize the points of the symmetric space by the elements na of the solvable subgroup $\mathbb{B}$. These facts are responsible for the existence of a broken symmetry which is the right transformations by elements of $\mathbb{B}$. The non conserved charges associated with this symmetry are the components of the Lax pair operators A and B, and their Poisson bracket algebra is the Lie algebra of $\mathbb{B}$. This algebraic structure holds true at the quantum level simply because these charges are linear in momenta. Then it follows that the classical and quantum algebras of these charges lead to the "Fundamental Poisson Relation" and "Fundamental Commutation Relation" [5] respectively and, in the case of the symmetric spaces G^N/K, these can be written in the form of the Yang-Baxter equation [27, 28].

Classically the operators A and B satisfy the Lax pair equation which describes the geodesic motion on the symmetric space. One of the main differences between the classical and quantum theories appears when we try to construct the quantum analog of that equation. In order to do that we have to add to the quantum Lax pair operators and to the quantum Hamiltonian terms which are quadratic in the generators of the Lie algebra of $\mathbb{B}$. The interesting fact about these terms is that they have the same structure as the quantity to which they are added with the quantum right charge associated to a given generator of $\mathbb{B}$ being replaced by the generator itself. We think an elucidation of the role of such terms is crucial to understand the richer algebraic structure which underlies these models at the quantum level. In addition we believe this will provide us with a clearer method of constructing the quantum conserved quantities in involution. The method we presented in section 12, although enables us to show the quantum integrability of the Toda Molecule Models, does not teach us much about the other structures responsible for that integrability and about other properties that could be useful in the study of integrable systems.

Our analysis of the classical and quantum geodesic motions applies uniformly to all non-compact Riemannian symmetric spaces and some results we obtained for the spaces G^N/K could perhaps be generalized. We believe that a classical and quantum reduction procedures similar to those ones used for G^N/K could be applied to obtain other integrable models from the geodesic motion on the symmetric spaces F/K. This would be important to check the conjecture by Olshanetsky and Perelomov [12] that this geometric picture of geodesic motions on symmetric spaces underly all integrable systems. The method we used to construct the Fundamental Poisson Relations have considerably elucidated their origin and perhaps it could be used to check if the Yang Baxter equation also exists in the case of the spaces F/K.

The two dimensional field theory version of the Toda Molecule models are also believed to be integrable and it would be interesting to check if the solutions of these models correspond to some type of motion on the symmetric spaces G^N/K. In appendix IV we make some comments about how these ideas could perhaps be implemented.

It would be very interesting to apply the ideas of this thesis to the Toda Lattice systems. They are associated with the Kac-Moody algebras instead of ordinary Lie algebras and their root systems are infinite dimensional and this accounts for the appearance of a spectral parameter in the Toda Lattice models. For that reason these models have an infinite family of Lax pairs labelled by this parameter. The integrability properties and the fundamental Poisson Relation of the TL models are very similar to those of the TM models (14) and perhaps this could be understood in terms of the Iwasawa decomposition for the Kac-Moody group. However, as far as we know, the mathematical theory for these groups does not exist yet.

Appendix I : Noether's Theorem

This is a standard theorem yet we have not succeeded in finding in the literature a statement and proof sufficiently explicit for the use made in the text. Here we present the version we use followed by an outline proof. We consider a nonsingular dynamical system with a finite number of coordinates $q_1 .. q_N$.

Theorem A

Consider the variation

$$\delta q_i = \varepsilon(t)\, F_i(q, \dot{q}, t) \qquad i = i \,..\, N$$

(with $\delta\dot{q}_i = \frac{d}{dt}(\delta q_i)$ as is usual in Lagrangian theory) and suppose that without using any equations of motion the corresponding response of the Langrangian is found to be

$$\delta \mathcal{L} = \frac{d}{dt}(\varepsilon X) + \dot{\varepsilon} Q + \varepsilon D \qquad (I.1)$$

where X, Q and D may depend on q, $\dot{q}$ and t (but not accelerations). Then

(i) $\frac{dQ}{dt} = D$ by virtue of the equation of motion

(ii) $Q = \sum_{i=1}^{N} p_i F_i - X$

(iii) Q can be expressed in terms of canonical variables q_i and p_i (with velocities eliminated) and

$$\{q_i, \varepsilon Q\}_{PB} = \delta q_i$$

$$\{p_i, \varepsilon Q\}_{PB} = \delta p_i - \varepsilon \frac{\partial D}{\partial \dot{q}_i}$$

where δp_i is calculated directly from the variations of the coordinates and velocities using the equations of motion if necessary, and ε is taken to be constant.

Corollary A

There is an arbitrariness in the definition of Q, X and D. If we replace

$$X \to X - G, \quad Q \to Q + G, \quad D \to D + \dot{G} \tag{I.2}$$

where $G = G(q,t)$, the original definition and all the subsequent properties still hold good. If we can choose G so that D becomes velocity independent then the variation (with $\dot{\varepsilon}=0$) is seen to be canonical with Q the infinitesimal generator. It still has an ambiguity which is an additive function of time. If D can be chosen to be zero Q is conserved and certainly generates the infinitesimal transformation. Its ambiguity is just an additive constant.

Theorem B

If there are three variations, δ, δ' and δ'', of the type considered in Theorem A, satisfying (with ε and ε' constants, and $\varepsilon'' = \varepsilon\ \varepsilon'$):

$$[\delta', \delta]q_i = \delta'' q_i \tag{I.3a}$$

and

$$[\delta', \delta]\dot{q}_i = \delta'' \dot{q}_i \tag{I.3b}$$

and if the coefficients of the accelerations in $(\varepsilon\delta' D - \varepsilon'\delta D')$ vanish, then

(i) $\{Q', Q\}_{PB} = -Q''$

(ii) $[\delta', \delta]p_i = \delta'' p_i$

Corollary B

For velocity independent variations, the condition (I.3b) is a consequence of (I.3a). And for canonical and velocity independent variations, the quantity $(\varepsilon\delta' D - \varepsilon'\delta D')$ is acceleration independent.

Proof of Theorem A

(i) follows immediately from Hamilton's action principle. Alternatively if we evaluate δ directly by calculation we get

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\delta\dot{q}_i = \frac{d}{dt}(\varepsilon p_i F_i) + \left(\frac{\partial\mathcal{L}}{\partial q_i} - \dot{p}_i\right)F_i$$

Comparing with (I.1) result (ii) follows by equating coefficients of $\dot{\varepsilon}$ and (i) by equating coefficients of ε.

We now obtain an important relation by comparing the occurrence of the acceleration $\ddot{q}_i$ in the two expressions for $\delta\mathcal{L}$. It occurs only once, linearly, in the terms $\varepsilon p_j \dfrac{\partial F_j}{\partial \dot{q}_i} \ddot{q}_i$ and $\varepsilon \dfrac{\partial X}{\partial \dot{q}_i} \ddot{q}_i$ respectively. Since no equations of motion are used these terms are equal for all accelerations. Hence

$$\frac{\partial X}{\partial \dot{q}_i} = \sum_j p_j \frac{\partial F_j}{\partial \dot{q}_i} \tag{I.4}$$

Now we examine the structure of Q. By (ii)

$$dQ = \sum_i dp_i\, F_i + \sum_{i,j} \left(p_j \frac{\partial F_j}{\partial q_i} - \frac{\partial X}{\partial q_i}\right) dq_i + \sum_{i,j} \left(p_j \frac{\partial F_j}{\partial \dot{q}_i} - \frac{\partial X}{\partial \dot{q}_i}\right) d\dot{q}_i +$$

$$+ \sum_i \left(p_i \frac{\partial F_i}{\partial t} - \frac{\partial X}{\partial t}\right) dt$$

and by (I.4) we see that the coefficient of $d\dot{q}$ vanishes thereby confirming the first assertion in (iii). Further $\dfrac{\partial Q}{\partial p_i} = F_i$ yielding the second part of (iii), while

$$\frac{\partial Q}{\partial q_i} = \sum_j p_j \frac{\partial F_j}{\partial q_i} - \frac{\partial X}{\partial q_i} = - \{p_i,\ Q\}_{PB} \tag{I.5}$$

Finally we calculate δp_i where $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}(q,\dot{q},t)$. Using (I.1) and taking ε constant we get

$$\delta p_i = \varepsilon \frac{\partial}{\partial \dot{q}_i}(\dot{X}+D) - \varepsilon \frac{\partial \mathcal{L}}{\partial q_j} \frac{\partial F_j}{\partial \dot{q}_i} - \varepsilon \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{\partial \dot{F}_j}{\partial \dot{q}_i}$$

Using (I.4), (I.5) and the fact that for any function $f(q,\dot{q},t)$

$$\frac{\partial \dot{f}}{\partial \dot{q}_i} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) + \frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial \dot{q}_j} \frac{\partial \ddot{q}_j}{\partial \dot{q}_i} \qquad (I.6)$$

we obtain

$$\delta p_i = \varepsilon \{p_i, Q\}_{PB} + \varepsilon \frac{\partial D}{\partial \dot{q}_i} + \varepsilon \sum_j \frac{\partial F_j}{\partial \dot{q}_i} \left(\dot{p}_j - \frac{\partial \mathcal{L}}{\partial q_j}\right) \qquad (I.7)$$

This yields the last part of (iii) on using Lagrange's equations of motion. If the variation of j^{th} coordinate is velocity independent it is unnecessary to use the corresponding equation of motion.

Proof of Corollary A

The corollary A is self evident once it is recognized G must be velocity independent in order to keep D acceleration independent.

Proof of Theorem B

From (I.3) we see that the variations of any function $f(q,\dot{q},t)$ satisfy $[\delta', \delta]f = \delta'' f$. Since the canonical momentum, $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, can be written as a function of q, $\dot{q}$ and t the part (ii) is proven.

Analogously the variations of the Lagrangian satisfy (using (I.1) with $\dot{\varepsilon} = \dot{\varepsilon}' = 0$ and $\varepsilon'' = \varepsilon\, \varepsilon'$):

$$\delta''\mathcal{L} = [\delta', \delta]\mathcal{L} = \varepsilon\, \delta'\dot{X} - \varepsilon'\delta\dot{X}' + \varepsilon\, \delta' D - \varepsilon'\delta D'$$
$$= \varepsilon''\dot{X}'' + \varepsilon'' D'' \qquad (I.8)$$

Using (I.6) and the fact that for any function $f(q,\dot{q},t)$

$$\frac{\partial \dot{f}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial f}{\partial q_i}\right) + \frac{\partial f}{\partial \dot{q}_j}\frac{\partial \ddot{q}_j}{\partial q_i}$$

we get

$$\delta\dot{f} = \frac{d}{dt}(\delta f) + \frac{\partial f}{\partial \dot{q}_i}\left(\delta\ddot{q}_i - \frac{d}{dt}(\delta\dot{q}_i)\right)$$

and therefore from (I.3) (since $\delta'\delta\dot{q}_i = \varepsilon\delta'\dot{F}_i$)

$$\frac{\partial F_i}{\partial \dot{q}_j}\left(\delta'\ddot{q}_j - \frac{d}{dt}(\delta'\dot{q}_j)\right) - \frac{\partial F'_i}{\partial \dot{q}_j}\left(\delta\ddot{q}_j - \frac{d}{dt}(\delta\dot{q}_j)\right) = 0 \qquad (I.9)$$

Using this relation and (I.4) we conclude that:

$$\varepsilon\, \delta'\dot{X} - \varepsilon'\delta\dot{X}' = \frac{d}{dt}(\varepsilon\, \delta' X - \varepsilon'\delta X') \qquad (I.10)$$

From (I.3a) we see that the functions F_i, F_i' and F_i'', which define the variations of the coordinates (see theorem A), have to satisfy:

$$F''_i = F'_j \frac{\partial F_i}{\partial q_j} - F_j \frac{\partial F_i'}{\partial q_j} + \dot{F}'_j \frac{\partial F_i}{\partial \dot{q}_j} - \dot{F}_j \frac{\partial F_i'}{\partial \dot{q}_j} \qquad (I.11)$$

But since the variation δ'' satisfy the conditions of theorem A, F''_i is acceleration independent, and the coefficients of the accelerations on the r.h.s. of (I.11) must vanish

$$\frac{\partial F_i}{\partial \dot{q}_j} \frac{\partial F'_j}{\partial \dot{q}_k} - \frac{\partial F'_i}{\partial \dot{q}_j} \frac{\partial F_j}{\partial \dot{q}_k} = 0$$

So, using (I.4)

$$\frac{\partial X}{\partial \dot{q}_j} \frac{\partial F'_j}{\partial \dot{q}_k} - \frac{\partial X'}{\partial \dot{q}_j} \frac{\partial F_j}{\partial \dot{q}_k} = 0$$

and therefore the quantity $(\varepsilon\delta'X - \varepsilon'\delta X')$ is acceleration independent.

Since no equations of motion are used, the coefficients of the accelerations on both sides of (I.8) have to be equal.Therefore, using (I.10), we get from (I.8) that, if $(\varepsilon\delta'D - \varepsilon'\delta D')$ is acceleration independent

$$\varepsilon'' \frac{\partial X''}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} (\varepsilon\delta'X - \varepsilon'\delta X')$$

and so, $\varepsilon''X''$ and $(\varepsilon\delta'X - \varepsilon'\delta X')$ differ by a function of q and t. But according to corollary A this is exactly the arbitrariness of X". Therefore we can set

$$\varepsilon''X'' = \varepsilon\delta'X - \varepsilon'\delta X' \tag{I.12}$$

According to theorem A, $Q'' = \sum_i p_i F_i - X''$, and so from (I.11), (I.12) and (I.4)

$$Q'' = \sum_i (F_i' \frac{\partial Q}{\partial q_i} - F_i \frac{\partial Q'}{\partial q_i})$$

but again from theorem A, $\frac{\partial Q}{\partial p_i} = F_i$, and so the theorem is proven.

Proof of corollary B

For velocity independent variations the relation (I.9) is automatically satisfied, and therefore it is clear that (I. 3b) is a direct consequence of (I. 3a).

The coefficient of the acceleration $\ddot{q}_i$ in $(\varepsilon\delta'D - \varepsilon'\delta D')$ is

$$\frac{\partial D}{\partial \dot{q}_j} \frac{\partial F_j'}{\partial \dot{q}_i} - \frac{\partial D'}{\partial \dot{q}_j} \frac{\partial F_j}{\partial \dot{q}_i}$$

and so, it vanishes for velocity independent variations as well as for canonical variations since for these, according to corollary A, D can be made velocity independent.

We then see that, for velocity independent variations, the results of theorem B can be obtained by imposing (I.3a) only.

Appendix II

In this appendix we prove relation (5.10) following the methods of reference [14].

Using the commutation relations (5.1) it is easy to check that the operator $\mathbb{P}$ defined by (5.11) satisfies:

$$[\mathbb{P} \,,\, 1\otimes H_a + H_a\otimes 1] = 0 \qquad \text{(II.1)}$$

In order to evaluate the commutator of $\mathbb{P}$ with step operators we make use of the Casimir like operator $\mathbb{C}$ defined by (5.12). It has the property of commuting with any generator T of the algebra,[14].

$$[\mathbb{C} \,,\, 1\otimes T + T\otimes 1] = 0 \qquad \text{(II.2)}$$

Using (5.1) we have, for any positive root β, that(r=rank of G^N):

$$[\mathbb{C}_o, 1\otimes E_\beta + E_\beta\otimes 1] = \sum_{i=1}^{r}(H_i\otimes \beta_i E_\beta + \beta_i E_\beta\otimes H_i)$$

$$= (\beta^2/2)(H_\beta\otimes E_\beta + E_\beta\otimes H_\beta) \qquad \text{(II.3)}$$

Where $\mathbb{C}_o = \sum_i H_i\otimes H_i$ and $H_\beta = 2\beta.H/\beta^2$

Since $\mathbb{C} = \mathbb{C}_o + \mathbb{C}_+ + \mathbb{C}_-$ we get from (II.2) and (II.3) that:

$$[\mathbb{C}_+ + \mathbb{C}_-, 1\otimes E_\beta + E_\beta\otimes 1] = -(\beta^2/2)(H_\beta\otimes E_\beta + E_\beta\otimes H_\beta) \quad (II.4)$$

But we also have

$$[\mathbb{C}_+, 1\otimes E_\beta + E_\beta\otimes 1] = \sum_{\alpha>0}(\alpha^2/2)(E_\alpha\otimes [E_{-\alpha}, E_\beta] + [E_\alpha, E_\beta]\otimes E_{-\alpha}) \quad (II.5)$$

and

$$[\mathbb{C}_-, 1\otimes E_\beta + E_\beta\otimes 1] = \sum_{\alpha>0}(\alpha^2/2)(E_{-\alpha}\otimes [E_\alpha, E_\beta] + [E_{-\alpha}, E_\beta]\otimes E_\alpha) \quad (II.6)$$

Since (II.4) contains terms proportional to the product of Cartan subalgebra generators with positive step operators only, we conclude that the sum of terms proportional to $E_\gamma\otimes E_{-\gamma'}$, $(\gamma, \gamma'>0)$ in (II.5) must vanish since we do not have such contribution in (II.6). Similarly the sum of terms proportional to $E_{-\gamma}\otimes E_{\gamma'}$, $(\gamma, \gamma'>0)$ in (II.6) must also vanish since we do not have such contribution in (II.5). In addition we conclude that the sum of terms proportional to the product of positive step operators in (II.5) and (II.6) must vanish, i.e. ,

$$\sum_{0<\alpha<\beta}(\alpha^2/2)(E_\alpha\otimes [E_{-\alpha}, E_\beta] + [E_{-\alpha}, E_\beta]\otimes E_\alpha) = 0 \quad (II.7)$$

Therefore we get that

$$[\mathbb{C}_+ - \mathbb{C}_-,\ 1 \otimes E_\beta + E_\beta \otimes 1] = (\beta^2/2)(H_\beta \otimes E_\beta - E_\beta \otimes H_\beta) +$$

$$+ 2 \sum_{0<\alpha<\beta} (\alpha^2/2) E_\alpha \otimes [E_{-\alpha}, E_\beta] \qquad (II.8)$$

Now, suppose α, β and γ are positive roots and $\beta = \gamma+\alpha$, then:

$$[E_{-\alpha}, E_\beta] = a\ E_\gamma$$

and

$$Tr([E_{-\alpha}, E_\beta] E_{-\gamma}) = 2a/\gamma^2 = Tr([E_{-\gamma}, E_{-\alpha}] E_\gamma) = (2/\beta^2) N_{\alpha\gamma}$$

and so $\quad a = (\gamma^2/\beta^2) N_{\alpha\gamma} \qquad (II.9)$

where we have used (5.4), and the commutator $[E_{-\gamma}, E_{-\alpha}]$ was evaluated by applying the automorphism (5.2) to the last equation in (5.1).

Since $\mathbb{P} = \mathbb{C}_+ - \mathbb{C}_-$ we have:

$$[\mathbb{P},\ 1 \otimes E_\beta + E_\beta \otimes 1] = (\beta^2/2)(H_\beta \otimes E_\beta - E_\beta \otimes H_\beta) +$$

$$+ \sum_{\substack{\alpha,\gamma>0 \\ \beta=\alpha+\gamma}} (\alpha^2 \gamma^2/\beta^2) N_{\alpha\gamma}\ E_\alpha \otimes E_\gamma \qquad (II.10)$$

From (5.7), (II.1) and (II.10) we get

$$[\mathbb{P},\ 1 \otimes D + D \otimes 1] = 2 \sum_{\beta>0} (\beta^2/2)^2 D(E_{-\beta})(H_\beta \otimes E_\beta -$$

$$- E_\beta \otimes H_\beta) - \sum_{\beta>0} \sum_{\substack{\alpha,\gamma>0 \\ \beta=\alpha+\gamma}} \alpha^2 \gamma^2 D([E_{-\alpha}, E_{-\gamma}]) E_\alpha \otimes E_\gamma \qquad (II.11)$$

Using (5.6) and (5.7) we conclude (5.10) is true.

The commutator between $\mathbb{P}$ and step operators corresponding to negative roots can be easily obtained from (II.10) by making use of the involutive automorphism σ. From (5.2) and the fact that

$$\sigma_R \sigma_L \mathbb{P} = - \mathbb{P}$$

we get from (II.10) that

$$[\mathbb{P} , 1 \otimes E_{-\beta} + E_{-\beta} \otimes 1] = (\beta^2/2)(H_\beta \otimes E_{-\beta} - E_{-\beta} \otimes H_\beta) +$$

$$+ \sum_{\substack{\alpha, \gamma > 0 \\ \beta = \alpha + \gamma}} (\alpha^2 \gamma^2 / \beta^2) N_{\alpha\gamma} E_{-\alpha} \otimes E_{-\gamma} \qquad \text{(II.12)}$$

Appendix III

Some useful relations in Lie group theory

Let G be a Lie group with generators $T_a (a = 1, 2 \ldots \dim G)$. The adjoint representation, $A_a{}^b(g)$, of an element $g \varepsilon G$ is defined by:

$$g^{-1} T_a \, g = A_a{}^b(g) \; T_b \qquad \text{(III.1)}$$

In the text we make use of the quantity $M_a{}^b(g)$ defined by:

$$g^{-1} \partial_a g = M_a{}^b(g) T_b \qquad \text{(III.2)}$$

where $\partial_a \equiv \frac{\partial}{\partial \eta^a}$, and η^a are the parameters of G.

We now show that $M_a{}^b$ is related to the adjoint representation of G by:

$$M_a{}^b(\eta) = \int_o^1 d\tau \; A_a{}^b(\tau\eta) \qquad \text{(III.3)}$$

Suppose we can write g as $g(T) = \exp T$, with $T = \eta^a T_a$. Then, if τ is an arbitrary continuous parameter, we have:

$$\frac{\partial}{\partial \tau} g(\tau T) = T \; g(\tau T)$$

and so $\left(\frac{\partial}{\partial \tau} g(\tau T)\right) g^{-1}(\tau T) = T$

Taking the derivative of both sides w.r.t. η^a and conjugating by $g(\tau T)$ (since $\partial g^{-1} = - g^{-1}(\partial g)g^{-1}$)

$$g^{-1}(\tau T)\partial a\{(\partial_\tau g(\tau T))g^{-1}(\tau T)\}g(\tau T) = g^{-1}(\tau T)T_a g(\tau T) = A_a{}^b(\tau\eta)T_b$$

$$= \partial_\tau\{g^{-1}(\tau T)\partial_a g(\tau T)\}$$

Then, integrating over τ, we prove (III.3) since the generators T_a are linearly independent.

Using eq. (III.2) one can easily check that $M_a{}^b$ satisfies:

$$\partial_b M_a{}^c - \partial_a M_b{}^c = f^c_{de}\, M_a{}^d\, M_b{}^e \qquad \text{(III.4)}$$

where f^c_{ab} are the structure constants of the Lie algebra of G ($[T_a, T_b] = f^c_{ab}\, T_c$).

Multiplying (III.4) by $M^{-1}{}_d{}^a\, M^{-1}{}_e{}^b\, M^{-1}{}_c{}^f$ we get

$$M^{-1}{}_a{}^d\partial_d M^{-1}{}_b{}^c - M^{-1}{}_b{}^d\partial_d M^{-1}{}_a{}^c = f_a{}^d{}_b\, \bar{M}^1{}_d^c \qquad \text{(III.5)}$$

where $M_a{}^{-1b}$ is the inverse of $M_a{}^b$.

From (III.2) we see that the differential operators defined by:

$$\nabla_a{}^R = M^{-1}{}_a{}^b\, \partial_b \qquad \text{(III.6)}$$

are generators of right translations:

$$g^{-1}\nabla_a{}^R\, g = T_a \quad \text{and} \quad \nabla_a{}^R g = g\, T_a \qquad \text{(III.7)}$$

Similarly, the operators

$$\nabla_a^{\ L} = A_a^{\ b}(\eta)\ \nabla_b^{\ R} \tag{III.8}$$

are generators of left translations

$$\nabla_a^{\ L}\ g = T_a g \tag{III.9}$$

Using (III.7) we see the operators $\nabla_a^{\ R}$ satisfy

$$[\nabla_a^{\ R},\ \nabla_b^{\ R}] = f^c_{ab}\ \nabla_c^{\ R} \tag{III.10}$$

and using (III.9)

$$[\nabla_a^{\ L},\ \nabla_b^{\ L}] = -f^c_{ab}\ \nabla_c^{\ L} \tag{III.11}$$

Since the left and right translations commute we get from (III.7) and (III.9) that

$$[\nabla_a^{\ R},\ \nabla_b^{\ L}] = 0 \tag{III.12}$$

In addition we get from (III.1) and (III.7) that the action of $\nabla_a^{\ R}$ on the adjoint representatioin is:

$$\nabla_c^{\ R}\ A_a^{\ b}(g) = A_a^{\ d}\ f^b_{dc} \tag{III.13}$$

Appendix IV

Some geometrical aspects of the Riemannian non-compact symmetric space

According to (8.4) the metric for the Riemannian non compact symmetric spaces F/K, written in terms of the parameters η^r of the solvable subgroup $\mathbb{B}$ = na, is given by:

$$g_{rs}(\eta)=M_r{}^u(\eta)M_s{}^t(\eta)G_{ut} \equiv (M\ G\ M^T)_{rs}, \quad r,s=1,\ 2\ldots \dim F/K \tag{IV.1}$$

where $M_r{}^s$ was introduced in (4.6) and G_{rs} in (4.10).

Under a general coordinate transformation on F/K, (which according to the discussion in section 8 is equivalent to a reparametrization of the subgroup $\mathbb{B}$) we see from (4.16) that $M_r{}^s$ transforms like a covariant vector (in the lower index)

$$M'{}_r{}^s = \frac{\partial\eta^u}{\partial\eta'^r} M_u{}^s \tag{IV.2}$$

Therefore, if we define ($\partial_r \equiv \frac{\partial}{\partial\eta^r}$)

$$\Pi_r = \left(b^{-1}\partial_r b - \sigma\,(b^{-1}\partial_r b)\right)\!\big/2 = M_r{}^s\ P_s \tag{IV.3}$$

with

$$P_s = \frac{1}{2}\,(T_s-\sigma(T_s)) \quad , \quad T_s \in \underset{\sim}{b} \tag{IV.4}$$

we see that Π_r is an element of the odd subspace $\underset{\sim}{p}$ under σ of the Lie algebra of F and at the same time a covariant vector. Due to these properties Π_r will play an important role in relating the algebraic and geometric aspects of F/K.

Its contravariant form is

$$\Pi^r = g^{rs}\Pi_s = G^{us}\, M^{-1}{}_u{}^r\, P_s \qquad \text{(IV.5)}$$

The metric (IV.1), in terms of Π_r, can be written as:

$$g_{rs} = \mathrm{Tr}\,(\Pi_r \Pi_s) \qquad \text{(IV.6)}$$

and therefore

$$\mathrm{Tr}(\Pi_r\; \Pi^s) = \delta_r{}^s \qquad \text{(IV.7)}$$

i.e., the covariant and contravariant forms of Π are orthogonal w.r.t. the Killing form of F.

The Christoffel symbol [31] (or affine connection) defined by:

$$\Gamma^u_{rv} = \frac{1}{2}\, g^{us}(\partial_r g_{sv} + \partial_v g_{sr} - \partial_s g_{rv}) \qquad \text{(IV.8)}$$

can be easily calculated from (IV.1) using eq. (III.4), to give

$$\Gamma^u_{rv} = \frac{1}{2}\, M^{-1}{}_t{}^u(\partial_r M_v{}^t + \partial_v M_r{}^t) + \frac{1}{2}\, W^u_{rv} \qquad \text{(IV.9)}$$

where

$$W^{u}_{rv} = I^{t}_{sw} \, M^{-1}{}_{t}{}^{u} \, M_{r}{}^{s} \, M_{v}{}^{w} \qquad \text{(IV.10)}$$

and

$$I^{u}_{rv} = G^{ut} \, (G_{rw} \, f^{w}_{tv} + G_{vw} \, f^{w}_{tr}) \qquad \text{(IV.11)}$$

where f^{u}_{rs} are the structure constants of the subgroup $\mathbb{B}$ = na. Note that I^{u}_{rv} is symmetric, $I^{u}_{rv} = I^{u}_{vr}$.

In fact, I^{u}_{rv} satisfies:

$$\frac{1}{2} \, [\sigma(T_r), T_v] - \frac{1}{2} \, [T_r, \, \sigma(T_v)] = I^{u}_{rv} \, P_u \; , \; T_s \in \underset{\sim}{b} \qquad \text{(IV.12)}$$

To prove this, we see that the l.h.s. is odd under σ and therefore is a linear combination of the P_u, which is a basis for the odd subspace $\underset{\sim}{p}$. Since

$$G_{rs} = Tr(P_r P_s) = Tr(P_r T_s) = \frac{1}{2} \, \{Tr(T_r T_s) - Tr(\sigma(T_r) T_s)\}$$

we see that the first term of the r.h.s. of the above expression does not contribute to (IV.11). Since it is the Killing form of $\underset{\sim}{b}$. Therefore multiplying both sides of (IV.12) by T_t, and taking the trace we see that I^{u}_{rv} in (IV.12) is the same as the one given by (IV.11)

In addition we have from (IV.12)

$$[P_r, \, k_s] = \frac{1}{2} \, (f^{u}_{rs} - I^{u}_{rs}) \, P_u \qquad \text{(IV.13)}$$

where $k_s = \frac{1}{2}(T_s + \sigma(T_s))$

Using eq (III.4) of appendix III one can check that Π_r, given by (IV.3), satisfies:

$$\partial_s \Pi_r - \partial_r \Pi_s = F^u_{rs} \Pi_u \qquad \text{(IV.14)}$$

where

$$F^u_{rs} = M^{-1}{}_v{}^u f^v_{tw} M_r{}^t M_s{}^w \qquad . \qquad \text{(IV.15)}$$

Therefore, from (IV.9) and (IV.14) we see that the covariant derivative of Π_r is:

$$D_s \Pi_r = \partial_s \Pi_r - \Gamma^u_{sr} \Pi_u$$

$$= \frac{1}{2}(F^u_{rs} - W^u_{rs})\Pi_u \qquad \text{(IV.16)}$$

Defining

$$\kappa_s = b^{-1}\partial_s b + \sigma(b^{-1}\partial_s b) = M_s{}^u k_u \qquad \text{(IV.17)}$$

and using (IV.13) we see (IV.16) can be written as:

$$D_s \Pi_r - [\Pi_r, \kappa_s] = 0 \qquad \text{(IV.18)}$$

so, the covariant derivative of Π_r can be written as a Lie bracket. This equation, as we show below, is related to the Lax pair equation and is responsible for the vanishing of the covariant derivative of the curvature tensor, which is the property that characterizes the symmetric spaces [23].

Using eq. (III.4) we have, (since the covariant curl is just the ordinary curl).

$$D_r \kappa_s - D_s \kappa_r = \partial_r \kappa_s - \partial_s \kappa_r = F^u_{sr}\ \kappa_u \qquad \text{(IV.19)}$$

and since

$$[P_r,\ P_s] + [k_r,\ k_s] = f^u_{rs}\ k_u \qquad \text{(IV.20)}$$

we get

$$D_r \kappa_s - D_s \kappa_r + [\Pi_r,\ \Pi_s] + [\kappa_r,\ \kappa_s] = 0 \qquad \text{(IV.21)}$$

For any covariant vector V_r we have [31]

$$D_v D_s V_r - D_s D_v V_r = - R^u_{rsv} V_u \qquad \text{(IV.22)}$$

where R^u_{rsv} is the Riemann-Christoffel curvature tensor:

$$R^u_{rsv} = \partial_v \Gamma^u_{rs} - \partial_s \Gamma^u_{rv} + \Gamma^t_{rs}\ \Gamma^u_{vt} - \Gamma^t_{rv}\ \Gamma^u_{st} \qquad \text{(IV.23)}$$

Using (IV.18) and (IV.21) we can show that

$$D_v D_s \Pi_r - D_s D_v \Pi_r = [\Pi_r [\Pi_s,\ \Pi_v]] \qquad \text{(IV.24)}$$

and so, from (IV.22), we obtain the curvature tensor for the symmetric spaces F/K:

$$R^u_{rsv} \Pi_u = - [\Pi_r, [\Pi_s, \Pi_v]] \qquad (IV.25)$$

This result, in fact, is true for all symmetric spaces which are coset spaces (see ref. (23)). However, for the symmetric spaces we are considering Π_r assumes the simple form (IV.3) due to the fact that the Iwasawa decomposition endows these spaces with a hidden group theoretic structure, i.e., the solvable subgroup $\mathbb{B}$ = na. It is also true that for those spaces the curvature tensor has vanishing covariant derivative, and in our case this can be explicit shown by using (IV.18).

As a consequence of that equation the covariant derivative of the commutator of two Π's is also a commutator

$$D_t[\Pi_s, \Pi_v] = [[\Pi_s, \Pi_v], \kappa_t]$$

and similarly

$$D_t[\Pi_r, [\Pi_s, \Pi_v]] = [[\Pi_r, [\Pi_s, \Pi_v]], \kappa_t]$$

Therefore from (IV.25) we get

$$D_t R^u_{rsv} = 0 \qquad (IV.26)$$

Since the covariant derivative of the metric vanishes, the operations of covariant differentiation and rising and lowering indices commute. It then follows that the Ricci tensor:

$$R_{rs} = R^u_{rus} \qquad (IV.27)$$

and the curvature scalar

$$R = g^{rs} R_{rs} \tag{IV.28}$$

have both vanishing covariant derivative. And since the covariant derivative of a scalar is just the ordinary derivative it follows that these symmetric spaces have constant curvature scalars. Let us then evaluate the curvature scalar for the Riemannian non-compact symmetric spaces F/K. From (IV.7) and (IV.25):

$$R^u_{rsv} = - \mathrm{Tr}([\Pi_r[\Pi_s, \Pi_v]] \Pi^u)$$

$$= - g^{ut} \mathrm{Tr}([\Pi_r,[\Pi_s,\Pi_v]] \Pi_t) \tag{IV.29}$$

and from (IV.28), (IV.1) and (IV.3):

$$R = - g^{rv} g^{st} \mathrm{Tr}([\Pi_r,[\Pi_s, \Pi_v]] \Pi_t)$$

$$= - G^{rv} G^{st} \mathrm{Tr}([P_r, [P_s, P_v]]P_t) \tag{IV.30}$$

Since $P_r = (T_r - \sigma(T_r))/2$ and $k_r = (T_r + \sigma(T_r)/2$, we have

$$[P_s, P_v] = \frac{1}{2} f^u_{sv} k_u - T_{sv} \tag{IV.31a}$$

and

$$[k_s, k_v,] = \frac{1}{2} f^u_{sv} k_u + T_{sv} \tag{IV.31b}$$

where

$$T_{sv} = \frac{1}{4}\,[T_s,\ \sigma(T_v)] + \frac{1}{4}\,[\sigma(T_s),\ T_v] \tag{IV.32}$$

But, from the Jacobi identity and from (IV.31b)

$$[[P_r,\ k_s],\ k_v] - [[P_r,\ k_v],\ k_s] = [P_r[k_s,\ k_v]]$$

$$= \frac{1}{2}\, f^u_{sv}\ [P_r,\ k_u] + [P_r,\ T_{sv}] \tag{IV.33}$$

Therefore from (IV.31a) and (IV.33):

$$[P_r,[P_s,\ P_v]] = f^u_{sv}\ [P_r,\ k_u] - [[P_r,k_s],\ k_v] + [[P_r,\ k_v],\ k_s] \tag{IV.34}$$

and so, from (IV.30), (IV.34) and (IV.13)

$$R = -\ G^{rv}\{\frac{1}{2}\ f^u_{sv}\ (f^s_{ru} - I^s_{ru}) - \frac{1}{4}\,(f^u_{rs} - I^u_{rs})(f^s_{uv} - I^s_{uv}) +$$

$$+ \frac{1}{4}\,(f^u_{rv} - I^u_{rv})(f^s_{us} - I^s_{us})\}$$

$$= -G^{rv}\{\frac{1}{4}\ f^u_{sv}f^s_{ru} + \frac{1}{2}\ f^u_{rs}I^s_{uv} - \frac{1}{4}\ I^u_{rv}(f^s_{us}-I^s_{us}) - \frac{1}{4}\ I^u_{rs}I^s_{uv}\}$$

Using (IV.11) we get

$$I^u_{ru} = f^u_{ur},\qquad G^{rv}\ I^u_{rv} = 2\ G^{ut}\ f^v_{tv}$$

$$G^{rv}\ I^u_{rs}\ I^s_{uv} = G^{rv}\ f^u_{rs}\ I^s_{uv}$$

$$= G^{ut}\ f^v_{tr}\ f^r_{vu} - G^{rv}\ G^{st}\ G_{uw}\ f^u_{rs}\ f^w_{vt}$$

Therefore:

$$R = G^{rs} f^u_{ur} f^v_{vs} + \frac{1}{2} G^{rv} f^u_{rs} f^s_{vu} + \frac{1}{4} G^{rv} G^{st} G_{uw} f^u_{rs} f^w_{vt} \tag{IV.35}$$

Using (5.1), (5.2), (5.3) we observe that for the symmetric spaces $G\ /K$:

$$G^{rs} f^u_{ur} f^v_{vs} = 4\delta^2 \quad , \quad G^{rv} f^u_{rs} f^s_{vu} = \sum_{\alpha>o} \alpha^2$$

$$G^{rv} G^{st} G_{uw} f^u_{rs} f^w_{vt} = 2 \sum_{\alpha>o} \alpha^2 + \sum_{\alpha,\beta>o} \frac{\alpha^2\beta^2}{(\alpha+\beta)^2} N^2_{\alpha\beta}$$

where $\delta = \frac{1}{2} \sum_{\alpha>o} \alpha$

Using the fact that, for any positive root γ

$$1 - 2\ \delta.\gamma/\gamma^2 + \frac{1}{2} \sum_{\substack{\alpha,\beta>o \\ \alpha+\beta=\gamma}} \alpha^2\beta^2/(\gamma^2)^2 N^2_{\alpha\beta} = 0$$

we obtain the result that the curvature scalar for G^N/K is:

$$R_{G^N/_K} = 6\ \delta^2 + \sum_{\alpha>o} \frac{\alpha^2}{2}$$

$$= \frac{1}{2} \sum_{\alpha>o} (6\delta + \alpha).\alpha \tag{IV.36}$$

We now want to show the relation between the Lax pair equation and eq. (IV.18). In order to do this we introduce a parameter t, which by definition is invariant under general coordinate transformation and therefore it is a scalar. We then consider the coordinates η^r of the symmetric space as functions of t, and consequently t parametrize curves on this space.

Define the operators A and B by:

$$\begin{aligned} A dt &= \Pi_r d\eta^r \\ iB dt &= \kappa_r d\eta^r \end{aligned} \tag{IV.37}$$

The r.h.s. of these equations are clearly scalars, and since dt is a scalar, it follows that the operators A and B are also scalars. In fact if t is time we see from (IV.3) and (IV.17) that A and B defined by (IV.37) coincide with the Lax operators defined in (2.10) (up to a K gauge transformation (2.15)).

The covariant derivative of a covariant vector V_r along the curve $\eta^r(t)$ is defined by: (see chap. 4, sec. 9 of ref. (31)).

$$\frac{DV_r}{Dt} = \frac{dV_r}{dt} - \Gamma^u_{rs} \frac{d\eta^s}{dt} V_u = \frac{d\eta^s}{dt} D_s V_r \tag{IV.38}$$

and it is a vector:

$$\frac{DV'_r}{dt} = \frac{\partial\eta^s}{\partial\eta'^r} \frac{DV_s}{Dt} \tag{IV.39}$$

Then, we have (since A is a scalar)

$$\frac{DA}{Dt} = \frac{dA}{dt} = \frac{d\Pi_r}{dt}\frac{d\eta^r}{dt} + \Pi_r \frac{d^2\eta^r}{dt^2}$$

$$= \frac{d\eta^s}{dt}\frac{d\eta^r}{dt} D_s \Pi_r + \left(\frac{d^2\eta^r}{dt^2} + \Gamma^r_{uv}\frac{d\eta^u}{dt}\frac{d\eta^v}{dt}\right)\Pi_r$$

and therefore from eq. (IV.18):

$$\frac{dA}{dt} - i[A,B] = \left(\frac{d^2\eta^r}{dt^2} + \Gamma^r_{uv}\frac{d\eta^u}{dt}\frac{d\eta^v}{dt}\right)\Pi_r \qquad \text{(IV.40)}$$

The r.h.s. of this relation is just the geodesic equation, and the path taken by a particle on the symmetric space that satisfies it will be such that the time elapsed is in extremum.

A similar analysis can be done if we introduce several parameters, namely, x^μ (μ=1,2...d) which can be seen as space-time coordinates. Again, by definition, these parameters will be scalars w.r.t. general coordinate transformations on the symmetric space. In analogy with the one parameter case we define the operators A_μ and iB_μ by:

$$A_\mu dx^\mu = \Pi_r d\eta^r$$
$$iB_\mu dx^\mu = \kappa_r d\eta^r \qquad \text{(IV.41)}$$

These operations are vectors w.r.t. to space-time coordinate transformations and scalars w.r.t. to coordinate transformations on the symmetric space.

The curves on the symmetric space will be parametrized by the d parameters x^{μ}, and according to (IV.38) the covariant derivative along a curve where all parameters but one are kept fixed is given by:

$$D_{\mu}V_r \equiv \frac{DV_r}{Dx^{\mu}} = \frac{\partial V_r}{\partial x^{\mu}} - \Gamma^{v}_{rs} \frac{\partial \eta^s}{\partial x^{\mu}} V_v = \frac{\partial \eta^s}{\partial x^{\mu}} D_s V_r \tag{IV.42}$$

However, these covariant derivatives do not commute. Using (IV.22)

$$[D_{\mu}, D_{\nu}]V_r = -\frac{\partial \eta^s}{\partial x^{\nu}} \frac{\partial \eta^t}{\partial x^{\mu}} R^{u}_{rst} V_u \tag{IV.43}$$

Analogously to the one parameter case we have (since $d\eta^r = \frac{\partial \eta^r dx^{\mu}}{\partial x^{\mu}}$)

$$D_{\mu} A_{\nu} \equiv \frac{DA_{\nu}}{Dx^{\mu}} = \partial_{\mu} A_{\nu} = \frac{\partial \eta^r}{\partial x^{\nu}} \frac{\partial \eta^s}{\partial x^{\mu}} D_s \Pi_r + \left(\frac{\partial^2 \eta^r}{\partial x^{\mu} \partial x^{\nu}} + \Gamma^{r}_{st} \frac{\partial \eta^s}{\partial x^{\mu}} \frac{\partial \eta^t}{\partial x^{\nu}} \right) \Pi_r$$

and from (IV.18)

$$\partial_{\mu} A_{\nu} - i\, [A_{\nu}, B_{\mu}] = \left(\frac{\partial^2 \eta^r}{\partial x^{\nu} \partial x^{\mu}} + \Gamma^{r}_{st} \frac{\partial \eta^s}{\partial x^{\nu}} \frac{\partial \eta^t}{\partial x^{\mu}} \right) \Pi_r \tag{IV.44}$$

If we contract the space-time indices we get:

$$\partial^{\mu} A_{\mu} - i [A_{\mu}, B^{\mu}] = \left(\frac{d^2 \eta^r}{d\tau^2} + \Gamma^{r}_{st} \frac{d\eta^s}{d\tau} \frac{d\eta^t}{d\tau} \right) \Pi_r \tag{IV.45}$$

where $d\tau^2 = dx_\mu dx^\mu$ is the proper time. Again, the r.h.s. of (IV.44) is a geodesic equation and the path, taken by a particle on the symmetric space that satisfies it, will be such that the proper time elapsed is in extremum.

One can check that the Lagrangian for such motion is

$$= \mathrm{Tr}(A_\mu\ A^\mu)/2 = \mathrm{Tr}(x^{-1}\partial_\mu x)^2/8 \tag{IV.46}$$

where x is the principal variable defined in (2.6) and the last equality is obtained by similar calculations leading to (2.11). But whatever equation of motion we choose for the particle on the symmetric space, the operators A_μ and B_μ have to satisfy some constraints that are purely geometrical. Since the r.h.s. of (IV.44) is symmetrical in μ and ν we have

$$\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, B_\nu] - i[A_\nu, B_\mu] = 0 \tag{IV.47}$$

In addition eq. (IV.21) introduces another constraint, which by using the covariant derivative of B_μ

$$iD_\nu B_\mu \equiv \frac{iDB_\mu}{Dx^\nu} = i\partial_\nu B_\mu = \frac{\partial\eta^r}{\partial x^\mu}\frac{\partial\eta^s}{\partial x^\nu} D_s\kappa_r + \left(\frac{\partial^2\eta^r}{\partial x^\nu\partial x^\mu} + \Gamma^r_{st}\frac{\partial\eta^s}{\partial x^\mu}\frac{\partial\eta^t}{\partial x^\nu}\right)\kappa_r$$

can be written as:

$$i\partial_\mu B_\nu - i\partial_\nu B_\mu + [A_\nu, A_\mu] + [iB_\nu, iB_\mu] = 0 \tag{IV.48}$$

Defining $D_\mu = A_\mu + iB_\mu = b^{-1}\partial_\mu b$ we get, by adding (IV.47) and (IV.48)

$$\partial_\mu D_\nu - \partial_\nu D_\mu + [\sigma(D_\nu), \sigma(D_\mu)] = 0 \qquad \text{(IV.49)}$$

since $\sigma(A_\mu) = - A_\mu$ and $\sigma(B_\mu) = B_\mu$.

But since $D_\mu = b^{-1} \partial_\mu b$, it satisfies

$$\partial_\mu D_\nu - \partial_\nu D_\mu + [D_\mu, D_\nu] = 0 \qquad \text{(IV.50)}$$

Therefore, from (IV.49) and (IV.50) we see that the curl of B_μ vanishes

$$\partial_\mu B_\nu - \partial_\nu B_\mu = 0 \qquad \text{(IV.51)}$$

Since the solutions of the one dimensional Toda Molecule systems are geodesics on the symmetric spaces G^N/K it would be interesting to check if the solutions of the two dimensional version of these systems correspond to some type of motion on the same symmetric spaces. However, if there is such a motion, it is certainly not governed by the Lagrangian (IV.46). One can check that the reduction conditions necessary to obtain the two dimensional Toda molecule equations from (IV.45) are not compatible with the constraints (IV.47), (IV.48) and (IV.51). Perhaps this could be overcome by introducing the "gauge potentials" in a way different from (IV.41).

Appendix V. Matrix realization of symmetric spaces.

Let $X = G/K$ be a symmetric space and σ be the involutive automorphism of G ($\sigma^2 = 1$, $\sigma \neq 1$) under which K is invariant ($\sigma(K) = K$).

Consider the subspace S of G defined by:

$$S = \{ g \in G: \sigma(g) = g^{-1} \} \qquad (V.1)$$

Obviously the unit element of G belongs to S since $\sigma(I) = I = I^{-1}$.

The group G acts as a transformation group on S by:

$$s \to g\, s\, \sigma(g)^{-1} \qquad s \in S, \qquad g \in G \qquad (V.2)$$

G acts transitively on the component S_I of S connected to the identity (which means S_I is homogenous) since for any s and $s' \in S_I$, the element $s'^{1/2} s^{-1/2}$ of G maps s to s'. The existence of the element $s^{1/2}$ for the component non-connected to the identity is not guaranteed because we do not have the exponential map.

The isotropy group of the unit element I is the subgroup K and since G acts transitively on S_I, the isotropy group of any element of S_I is isomorphic to K.

Any homogenous space is locally isomorphic to its transformation group modulo its isotropy group, but if the action of the transformation group is transitive this isomorphism is global.

Therefore S_I is globally isomorphic to the symmetric space $X = G/K$.

And since the identity element can be mapped to any element of S_I by the action of G, $I \to g\ \sigma(g^{-1})$, we get a matrix realization for X, where any element x of G/K can be written as:

$$x = g\ \sigma(g^{-1}) \qquad g \in G, \qquad x \in X = G/K \qquad \text{(V.3)}$$

We should notice that S_I is obtained by the exponential map from the subspace of the algebra of G which is odd under the involutive automorphism σ.

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