DOMINANT FEEDBACK:
CONCEPTS, ALGORITHMS AND CONVERGENCE

by

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To Guiomar Scerni,
My Mother,
For me the most important of all.

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ABSTRACT

This thesis considers optimal choice of output feedbacks F for linear systems $\dot{x}(t) = Ax(t) + Bu(t)$: $x(0) = x_0$, $y(t) = Cx(t)$, $u(t) = Fy(t)$ using gradient methods based on Allwright's approach of dominant feedback. Let the standard infinite-time quadratic cost be written as $x_0'K(F)x_0$. Then, a feedback $\bar{F}$ is said to dominate F if $x_0'K(\bar{F})x_0 \leqq x_0'K(F)x_0$, for all x_0, with the inequality holding strictly for at least one x_0, i.e. if $K(\bar{F}) \underline{<} K(F)$ for the ordering $\underline{<}$ for positive semidefinite matrices defined by $K(\bar{F}) \leqq K(F)$, $K(\bar{F}) \neq K(F)$. The methods are extensions of gradient methods for scalar optimization to the matrix-valued function K(F). A sequence $\{F^j\}$ is yielded such that $K(F^{j+1}) \underline{<} K(F^j)$, for F^{j+1} which makes the first-order approximation to $\Delta K^j = K(F^{j+1}) - K(F^j)$, denoted by dK^j, negative definite or semi-definite. The former case yields actually $\Delta K^j < 0$. The latter may generate $\Delta K^j \underline{<} 0$ or not, depending on the contributions of the higher order terms of the Taylor expansion of ΔK^j. Here an implementable version of Allwright's algorithm is given, and convergence of the sequence $\{\lambda\max dK^j\}$ to zero when $\{F^j\}$ is infinite is proved. Three extensions of the algorithm are presented, which solve the problem when constraints are imposed on the feedback matrix. The feasible set defined by the constraint functions is characterized, respectively, by (i) a convex compact set, with nonempty interior, defined by continuous functions (ii) a linear variety (iii) a particular case of (i), where the functions are linear. Implementable versions for the algorithms are given. Convergence proofs are provided for the first two.

In connection with the above three types of results have been developed, namely: (i) derivation of the r-th order derivatives of the solution of the Lyapunov equation $(A+BFC)'K(F) + K(F)(A + BFC) = -(Q + C'F'RFC)$ (ii) a method for obtaining the solution of $\min\{\|x-y\| : x \in X, y \in Y\}$ where X is a convex set and Y is an orthant (iii) discussion of the

non-differentiable problem of optimizing the largest eigenvalue of the Fréchet-differential of K(F).

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CHAPTER 1

INTRODUCTION

1. INTRODUCTION

Consider the linear, time-invariant, multivariable system with output feedback, given by the equations:

(1) $\dot{x}(t) = Ax(t) + Bu(t) : x(0) = x_0$

(2) $y(t) = Cx(t)$

(3) $u(t) = Fy(t)$,

where:

$x \in \mathbb{R}^n$ is the state vector,

$x_0 \in \mathbb{R}^n$ is the initial state vector,

$y \in \mathbb{R}^r$ is the output vector,

$u \in \mathbb{R}^m$ is the control vector,

$A \in \mathbb{R}^{nxn}$ is the plant matrix, which is assumed to be asymptotically stable,

$B \in \mathbb{R}^{nxm}$ is the control matrix,

$C \in \mathbb{R}^{rxn}$ is the output matrix,

$F \in \mathbb{R}^{mxr}$ is the feedback matrix, which asymptotically stabilizes the closed loop system.

Suppose the system has the standard infinite-time quadratic performance criterion

(4) $V(x, u) = \int_0^\infty \{x(t)'Qx(t) + u(t)'Ru(t)\}dt$

with both matrices $Q \in \mathbb{R}^{nxn}$ and $R \in \mathbb{R}^{mxm}$ symmetric and positive definite.

Now write V in terms of F and x_0 using the closed loop dynamic equation

$$\dot{x}(t) = (A + BFC)\, x(t) : x(0) = x_0,$$

which has the solution

$$x(t) = \exp[(A + BFC)t]x_0.$$

and, from (4),

$$V(x_0, F) = \int_0^\infty x_0'\{\exp[(A + BFC)t]'(Q + C'F'RFC)\exp[(A+BFC)t]\}x_0\, dt$$

It is convenient to represent V as the quadratic form

(5) $$V(x_0, F) = x_0' K(F)x_0$$

where

(6) $$K(F) \triangleq \int_0^\infty \exp[(A+BFC)t]'(Q+C'F'RFC)\exp[(A+BFC)t]dt$$

In practice it is often desirable to find an optimal feedback F* which minimizes $V(x_0, F)$. However $V(x_0, F)$ is also a function of the initial state, and it is not practical to choose a different F* for each initial condition; therefore it is desirable to find a way to overcome this problem, so that an F* is obtained which is "good" for a set of initial conditions. Levine and Athans in their well-known paper [2.16] eliminated the dependence of V on the initial state by averaging the performance obtained when x_0 is a random variable uniformly distributed on the surface of the n-dimensional unit sphere. Therefore the optimal F* is that which minimizes the expected value of $V(x_0, F)$,

(7) $$\hat{V}(F) = E\{V(x_0, F)\}$$
$$= tr[K(F)X_0],$$

where $X_0 = E\{x_0x_0'\}$ and tr is the trace function. This identity is shown in (2.38).

Allwright and Mao in [6.1] proposed another approach to optimal output feedback. It is based on the fact that

$$x_0' K(F) x_0 \leqq \| K(F) \|_2 \; \| x_0 \|^2 ,$$

where $\| \; \|_2$ denotes the 2-norm (see (2.45)). Thus, the largest cost value among all those obtained when x_0 and F vary is at most $\| K(F) \| \; \| x_0 \|^2$. Therefore a reasonable cost to minimize becomes:

$$\text{(8)} \qquad \hat{V}(F) = \| K(F) \|_2$$

$$= \lambda\max K(F) ,$$

where λmax is the largest eigenvalue. The second equality in (8) is a consequence of K(F) being positive semidefinite (from (6)).

In the two approaches described, the cost function $\hat{V}(F)$ is a scalar function of the matrix F. A method which has a cost which is a matricial function of F has been proposed by Allwright in [2.3]. He puts

$$\text{(9)} \qquad \hat{V}(F) = K(F)$$

Matricial-function optimization requires definition of the order relationship to be used. An obvious choice is the usual ordering for positive matrices. Then a matrix A is said to be greater than B if and only if A-B is positive definite, i.e.

$$A > B \Longleftrightarrow A-B > 0.$$

Let us consider that, for two feedbacks F^1 and F^2, $K(F^1) > K(F^2)$. Then,

$$x_0' K(F^1) x_0 > x_0' K(F^2) x_0, \text{ for all } x_0,$$

and therefore,

$$V(x_0, F^1) > V(x_0, F^2), \text{ for all } x_0.$$

Thus, if $K(F^2)$ is "smaller" than $K(F^1)$ the cost at F^2 will be smaller than at F^1, for all initial conditions x_0. The order relationship $<$ when used in the feedback context will be called "strict dominance". In the above example we say that F^2 strictly dominates F^1.

Another choice is the "dominance" ordering, defined as follows. For any two matrices A and B,

$$A \geq B \iff \begin{cases} x'Ax \geqq x'Bx, & \text{for all } x \\ \bar{x}'A\bar{x} > \bar{x}'B\bar{x}, & \text{for some } \bar{x}. \end{cases}$$

In particular, if $K(F^1) \geq K(F^2)$, then

$$V(x_0, F^1) \geqq V(x_0, F^2), \quad \text{for all } x_0$$

$$V(\bar{x}_0, F^1) > V(\bar{x}_0, F^2), \quad \text{for some } \bar{x}_0.$$

Hence, if either F^2 dominates or strictly dominates F^1, the cost will be reduced at least for some initial condition, and will never be increased for any initial condition.

Note that neither $<$ nor $\leq$ are total order relationships, since for two matrices A and B it might be that neither $A > B$ nor $A < B$ ($A \geq B$ nor $A \leq B$). Nevertheless an F^* will be defined as a strict local minimizer, with respect to $<$, for $K(F)$ when, for all comparable F in a neighbourhood of F^*, with $F \neq F^*$,

$$K(F^*) < K(F).$$

Or, using $\leq$, F^* is a strict local minimizer for $K(F)$ when, for all comparable F near F^*,

$$K(F^*) \leq K(F).$$

In the former case F* is "strictly locally dominant" and in the latter "locally dominant".

One might have a non-strict minimum in the sense that for all F near F*, either $K(F^*) < K(F)$ or $K(F^*) = K(F)$. Similarly for $\leq$.

Allwright has proposed two iterative procedures in [2.3]. Both give a sequence of dominating feedbacks, i.e., they generate sequences $\{F^j\}$ such that

(10) $K(F^0) \geq K(F^1) \geq \ldots \geq K(F^j) \geq \ldots$

or else

(11) $K(F^0) > K(F^1) > \ldots > K(F^j) > \ldots$

The first procedure requires the assumption that the system is such that rank[B] > dim ker[C]. The second is an algorithm of the gradient type for minimizing K(F), and it was the basis for the development of this thesis. Assume that F^j is a nondominant feedback, i.e., there exist some F satisfying $K(F) \leq (<) K(F^j)$. An iteration consists of (i) determining a normalized matrix S^j such that it minimizes (with respect to either $\leq$ or $<$) the first order approximation to $K(F^j + S^j) - K(F^j)$
(ii) determining a scalar λ^j such that $K(F^j + \lambda^j S^j) \leq (<) K(F^j)$.
S^j is called the <u>search direction</u> and λ^j the <u>step length</u> or <u>step size</u>. Step (i) is the <u>search direction subproblem</u> and step (ii) is the <u>line search</u> (as varying the parameter λ^j varies F along a line in the direction S^j). There are two versions for the algorithm generating sequences of the type (10) or (11).

Consider, given a normalized mxr matrix S, the Taylor expansion of $K(F + S) - K(F)$ about F,

(12) $K(F+S) - K(F) = dK(F; S) + d^2K(F; S, S) + \ldots,$

where the right-hand side terms will be defined in Chapter 2. The first-order term $dK(F; S)$ is minimized, in Allwright's sense, when its largest eigenvalue is minimized. Suppose $dK(F; S) < 0$. Then, in a neighbourhood of F, $K(F + \lambda S) < K(F)$ since the higher-order terms are insignificant for small λ. If $dK(F; S) = 0$, and so K is stationary along S, then the higher order terms will define the behaviour of K along S. The case $dK(F; S) > 0$ gives $K(F + \lambda S) > K(F)$ obviously. Finally, if $dK(F; S)$ is indefinite, for some set of x_0s, $x_0' K(F + \lambda S)x_0 > x_0 K(F)x_0$, and for another set of x_0s, the inequality is reversed. Hence, if $\lambda\max dK(F; S) < 0$, S is a descent direction for K, if $\lambda\max dK(F; S) = 0$, there may be a reduction in K along S, and if $\lambda\max dK(F; S) > 0$, K does not decrease along S. Therefore the search direction subproblem reduces to that of minimizing $\lambda\max dK(F^j; S)$. The minimizer S^j is the "steepest" descent direction.

The line search to choose λ^j is done by minimizing appropriate scalar functions of $K(F^j + \lambda S^j)$.

The subject of Chapter 2 is the study of this algorithm. An implementable version, for which the line search can be implemented computationally, is described. It is proved that, for the implementation, if $\{F^j\}$ is an infinite sequence such that $dK(F^j; S^j) < 0$ then, as $j \to \infty$, $\lambda\max dK(F^j; S^j) \to 0$. The interpretation of this property is that, if it is assumed that F^* is some accumulation point of $\{F^j\}$ which stabilizes the system, then $\lambda\max dK(F^*, S^*) = 0$ for the steepest descent direction S^*, and so F^* is potentially a locally dominant feedback. It is shown that $K(F^j)$ converges to some $K^* \geqq 0$. It is also shown that, if $\tilde{S}^j$ is a descent direction, not necessarily the steepest, giving $\lambda\max dK(F^j; \tilde{S}^j)$ "sufficiently negative" then $\lambda\max dK(F^j; \tilde{S}^j) \to 0$ as well.

The three following chapters deal with optimal constrained output feedback. Allwright's algorithm has been adapted to accept the constraints considered. The basic modification is in the determination of the search direction, which now must be feasible, i.e. it is such that all feedbacks along it, in the vicinity of the point, are feasible (in that they belong to the constraint set).

In Chapter 3 the feasible set for F is a compact convex set, with nonempty interior, defined by a finite number of continuous functions. For the constrained algorithm, convergence of the sequence of λmaxs is proved using the theory of closedness of algorithms. An implementation is developed and convergence is proved for that in a direct way.

Chapter 4 first describes the geometric-based method developed by Allwright for computing the unconstrained steepest descent direction. Then it considers the constrained problem where the feasible set is a linear variety of the space of feedbacks. This means considering entries of F satisfying a set of linear equations, which has applications to decentralized control. The method proposed is based on projecting the problem onto the linear variety, which permits the problem to be viewed as unconstrained, allowing Allwright's method for computing the unconstrained search direction to be applied.

The third constrained problem is studied in Chapter 5 and is a particular case of the first, where the constraint set is a rectangle. This arises when the entries of F are desired to take values only in certain intervals. This type of constraint might be useful to help limit the instantaneous size of u and hence make the feedback of more practical use. An implementable feasible direction algorithm is developed; for it convergence is not guaranteed since the search direction is defined now in a "non-closed" form, and thus the theory of Chapter 3 does not apply. However computing it seems to be simpler than computing a "closed" search direction. To help find search directions in this case, a method is proposed for obtaining the minimum distance between a convex set and an orthant.

Chapter 6 suggests two methods for solving the search direction subproblem when the feedback matrix is constrained to any given convex set. The first requires solving a sequence of problems of the kind: minimize tr $f(S)^{\ell}$, for $\ell = 1,2,\ldots$. The limit of the sequence of solutions, when $\ell \to \infty$, is the solution to the subproblem. The second is to minimize the (nondifferentiable) function $dK(F; \cdot)$ on the variable S, using subgradient-type algorithms.

CONTRIBUTIONS OF THE THESIS

(1) Proof of the differentiability of K(F) using the theory of Kronecker products and the derivation of its r-th order derivatives: Lemma (2.11).

(2) Evaluation of an upper-bound for the norm of the r-th Frechet-differential of K assuming that K(F) is bounded: Lemmas (2.45) and (2.62).

(3) Determination of lower bound quadratic functions for $g(\lambda) = \lambda\min[K(F) - K(F + \lambda S)]$ and $\phi(\lambda) = tr[x_0'(K(F) - K(F + \lambda S))x_0]$ and their application to the line search for Allwright's algorithm: Section 2.4.

(4) Proof of the convergence of the sequence $\{\lambda\max dK(F^j; S^j)\}$ for Allwright's algorithm : Theorem (2.96).

(5) Proof of the convergence of the sequence $\{\lambda\max dK(F^j; S^j)\}$ for the conceptual constrained algorithm (first case) using the theory of closedness of algorithms: Theorems (3.10), (3.17).

(6) A procedure for the constrained feedback problem (first case) with an implementable line search: Theorem (3.23) and Section 3.4.

(7) Proof of the convergence of the sequence $\{\lambda\max dK(F^j; S^j)\}$ for the constrained algorithm (first case): Theorem (3.33).

(8) Proof that the minimization of the support function of a set over a subspace can be equivalently restated as the minimization of the support function of the set projected onto that subspace, and application of it to the constrained feedback problem (second case): Theorem (4.18).

(9) Properties of the function χ: $y \longrightarrow \| \theta(y) - y\|^2$, where $\theta(y)$: $y \longrightarrow \arg\min\{\| x - y\| : x \in P\}$, for an orthant P: Propositions (5. 7) and (5.18), (5.22) and Theorem (5.39).

(10) A procedure for obtaining the minimum distance between a general convex set and an orthant: Algorithms (5.48), (5.64).

(11) Proof of the equivalence of the two problems: minimizing the distance between a convex set and an orthant (which are disjoint) and minimizing the support function of the convex set over the orthant: Theorem (5.98).

(12) A necessary and sufficient condition for optimality of a locally constrained dominant feedback (third case): Theorem (5.101).

(13) An implementable procedure for obtaining the constrained dominant feedback (third case): Algorithms (5.107), (5.113).

(14) Determination of the expression for the gradient of $tr(dK(F;S) + \alpha I)^{\ell}$ with respect to the variable S: Theorem (6.12).

(15) Evaluation of the steepest descent direction at S of the function $\lambda max\ dK(F; S)$: Proposition (6.28).

CHAPTER 2

CONVERGENCE PROPERTIES OF THE ALLWRIGHT'S ALGORITHM

As has been said, Allwright's method for finding unconstrained dominant output feedbacks uses the descent direction approach of algorithms for nonlinear function optimization. At the iteration j, a descent direction is selected along which the feedback is varied. It is chosen to be the matrix S^j which makes the first order approximation to $K(F^j + S) - K(F^j)$ as "negative" as possible. A line search rule is then applied to improve K along S^j, giving F^{j+1}. Then, provided $K(F^j + S^j) - K(F^j)$ is negative definite, the new point F^{j+1} will dominate F^j. Thus, the algorithm constructs a sequence of dominating output feedbacks. The main question that arises here is: does the sequence F^j converge to a locally dominant feedback?

2.1 THE FRECHET-DIFFERENTIAL OF K

The following lemma (11) proves that K(F) is a Fréchet-differentiable function, of class C^∞. Let dK(F; S) be the first order change in K caused by changing F to F+S. The matrix dK(F; S) is then called the F-differential of K at F with increment S. The first order change in dK(F; S), caused by changing F to F+D is the second order F-differential of K at F with increments S and D, and is denoted by $d^2K(F; S, D)$. In general, for the k-th F-differential of K at F, with k increments $S^1, S^2, \ldots, S^k$, the notation is $d^kK(F; S^1, S^2, \ldots, S^k)$. The functions d^kK are all k-linear and symmetric in the arguments $S^1, \ldots, S^k$. (see Ref. [8]).

The first F-differential of K, dK(F; S) can be written in terms of the matrices $\partial K(F)/\partial f_{ij}$, as

(1) $$dK(F; S) = \sum_{ij} \frac{\partial K(F)}{\partial f_{ij}} s_{ij}$$

in which $\sum_{ij}$ denotes $\sum_{i=1}^{m} \sum_{j=1}^{r}$. Thus, the second differential is

$$d^2K(F;\ S,\ D) \triangleq d[dK(F;\ S);\ D]$$

$$= \sum_{k\ell} \left(\frac{\partial dK(F;\ S)}{\partial f_{k\ell}} \right) d_{k\ell}$$

$$= \sum_{k\ell} \frac{\partial}{\partial f_{k\ell}} \left(\sum_{ij} \frac{\partial K(F)}{\partial f_{ij}} s_{ij} \right) d_{k\ell}$$

$$= \sum_{k\ell} \sum_{ij} \frac{\partial^2 K(F)}{\partial f_{k\ell} \partial f_{ij}} s_{ij}\, d_{k\ell},$$

and the k-th order differential is

$$\text{(2)} \quad d^k K(F;\ S^1,\ldots,S^k) = \sum_{i_k j_k} \cdots \sum_{i_1 j_1} \frac{\partial^k K(F)}{\partial f_{i_k j_k} \cdots \partial f_{i_1 j_1}} s^1_{i_1 j_1} \cdots s^k_{i_k j_k}$$

For the sake of simplicity the first partial derivatives will be denoted by

$$\text{(3)} \quad \Gamma_{ij}(F) = \frac{\partial K(F)}{\partial f_{ij}} .$$

It must be emphasized that the subscripts on this symbol do not refer to the entries in the matrix, but to the different matrices. The second partial derivatives are denoted by

$$\Gamma^{ij}_{k\ell}(F) = \frac{\partial^2 K(F)}{\partial f_{k\ell} \partial f_{ij}}$$

and, in general,

$$\Gamma^{i_1 j_1}_{\vdots\; i_k j_k}(F) = \frac{\partial^k K(F)}{\partial f_{i_k j_k} \cdots \partial f_{i_1 j_1}} .$$

We shall give the following definitions and state some propositions that will be useful when proving lemma (11).

(4) <u>DEFINITION</u>. The "Kronecker Product", $C = (c_{rs})$, of two matrices A, $\ell \times m$, and B, $p \times q$, is defined to be the $\ell p \times mq$ matrix $C = A \otimes B$, in which

$$c_{rs} = a_{i_1 j_1} b_{i_2 j_2}$$

with

$$r = p(i_1 - 1) + i_2$$
$$s = q(j_1 - 1) + j_2$$
$$i_1 = 1, \ldots, \ell;\ i_2 = 1, \ldots, p$$
$$j_1 = 1, \ldots, m;\ j_2 = 1, \ldots, q\ .$$

For example,

$$C = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \cdot & & & \\ a_{\ell 1}B & a_{\ell 2}B & \cdots & a_{\ell m}B \end{bmatrix}$$

(5) <u>PROPOSITION</u>. Let $A(x)$ be a $\ell \times m$ differentiable, matrix-valued function of a scalar x, and let B be a $p \times q$ constant matrix. Define the functions $F(x) = A(x) \otimes B$ and $G(x) = B \otimes A(x)$. Then $F(x)$ and $G(x)$ are differentiable and $F'(x) = A'(x) \otimes B$ and $G'(x) = B \otimes A'(x)$.

<u>PROOF</u>. By definition,

$$f_{rs}(x) = a_{i_1 j_1}(x) b_{i_2 j_2}$$

for i_1, j_1, i_2, j_2 indices satisfying the rules in (4). Then

$$F'(x) = (f'_{rs}(x))$$
$$= (a'_{i_1 j_1}(x) b_{i_2 j_2})$$
$$= A'(x) \otimes B$$

Since $A'(x) = (a'_{i_1 j_1}(x))$. Similarly, $G'(x) = B \otimes A'(x)$. ∇

(6) PROPOSITION. (Graham [11] page 63) . Let $A(X)$, $\ell \times m$, and $B(X)$, $m \times n$, be two differentiable, matrix-valued functions of a matrix X. Then,

$$\frac{\partial (AB)}{\partial x_{ij}} = \frac{\partial A}{\partial x_{ij}} B + A \frac{\partial B}{\partial x_{ij}} \quad . \qquad \nabla$$

(7) DEFINITION. The "Kronecker Sum" of two square matrices A, $m \times m$, and B, $n \times n$, is defined to be the $mn \times mn$ square matrix $A \oplus B$ given by the expression

$$A \oplus B = A \otimes I_n + I_m \otimes B$$

in which I_n and I_m are the identity matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{m\times m}$, respectively. ∇

(8) PROPOSITION. (Graham [11]). Let $\{\lambda_i\}$ and $\{x^i\}$ be the eigenvalues and the corresponding eigenvectors for a $m \times m$ matrix A, and let $\{\mu_j\}$ and $\{y^j\}$ be the eigenvalues and the corresponding eigenvectors for a $n \times n$ matrix B. Then $A \oplus B$ has eigenvalues $\{\lambda_i + \mu_j\}$, with corresponding eigenvectors

$$\{x^i \otimes y^j\} = \begin{bmatrix} x_1^i \begin{bmatrix} y_1^j \\ \vdots \\ y_n^j \end{bmatrix} \\ \vdots \\ x_m^i \begin{bmatrix} y_1^j \\ \vdots \\ y_n^j \end{bmatrix} \end{bmatrix}$$

PROOF. Using the basic properties of the K-product such as mixed product and scalar multiplication rules, as defined by

$$(M \otimes N)(P \otimes Q) = MP \otimes NQ$$

$$a(M \otimes N) = (aM \otimes N) = (M \otimes aN),$$

we obtain the following equality

$$(A \oplus B)(x \otimes y) = (A \otimes I_n)(x \otimes y) + (I_m \otimes B)(x \otimes y)$$

$$= (Ax \otimes y) + (x \otimes By)$$

$$= \lambda(x \otimes y) + \mu(x \otimes y)$$

$$= (\lambda + \mu)(x \otimes y).$$

The result follows. ∇

It is useful to introduce the basis matrices E_{ij}, the components of which vanish except for the one labelled i,j, which is unity. The arbitrary matrix $A = (a_{ij})$ can then be expressed as the double sum

(9) $$A = \sum_{ij} a_{ij} E_{ij}$$

in analogy with the expression for vectors.

Also we introduce the following vec notation for matrices.

(10) DEFINITION. Let X be a $\ell \times m$ matrix. Define vec X by the ℓm-vector x which is the column elongation of X by columns, i.e.,

$$\text{vec } X = x = (x_{11}, \ldots, x_{\ell 1}, x_{12}, \ldots, x_{\ell 2}, \ldots, x_{1m}, \ldots, x_{\ell m})'. \qquad \nabla$$

We may now state the main lemma of this section.

(11) LEMMA. Let $A(F) = A + BFC$ and define

$$\mathcal{F} = \{F \in \mathbb{R}^{m \times r} : A(F) \text{ is asymptotically stable}\}.$$

Then the function $K(\cdot): \mathcal{F} \to \mathbb{R}^{n \times n}$ defined by

(12) $$K(F) = \int_0^\infty \exp[A(F)'t][Q + C'F'RFC] \exp[A(F)t]dt$$

is of class C^∞. The partial F-derivative matrices, $\Gamma_{ij}(F)$, satisfy the Lyapunov equation,

$$(13)\quad \Gamma_{ij}(F)A(F)+A(F)'\Gamma_{ij}(F) = -C'E'_{ij}[B'K(F) + RFC] - [B'K(F) + RFC]'E_{ij}C.$$

The matrices of partial F-derivatives of higher order also satisfy different Lyapunov equations.

PROOF. In order to show that $\mathcal{F}$ is an open set, which is necessary when talking about differentiability on $\mathcal{F}$, consider the function

$$h: \mathcal{F} \longrightarrow \mathcal{H}$$
$$F \longmapsto \lambda_i(F),$$

that associates to a matrix $F \in \mathcal{F}$, its i-th eigenvalue $\lambda_i(F)$, which lies on the open left halfplane denoted by $\mathcal{H}$. Then, since h is continuous, $\mathcal{F}$ must be open.

It is well known that the matrix K(F) defined by (12) is the unique symmetric solution to the matrix equation

$$(14)\quad K(F)A(F) + A(F)'K(F) = -Q - C'F'RFC.$$

Refer for example to [5] , page 175 . This equation can be equivalently represented by using the Kronecker product between appropriate matrices. In fact, let

$$D(F) = C'F'RFC + Q$$
$$d(F) = \text{vec}\ D(F)$$
$$k(F) = \text{vec}\ K(F),$$

and note that

$$(I_n \otimes A(F)')k(F) = \begin{bmatrix} A(F)' & & & \bigcirc \\ & A(F)' & & \\ & & \ddots & \\ \bigcirc & & & A(F)' \end{bmatrix} \begin{bmatrix} k^1(F) \\ k^2(F) \\ \vdots \\ k^n(F) \end{bmatrix} = \begin{bmatrix} A(F)'k^1(F) \\ A(F)'k^2(F) \\ \vdots \\ A(F)'k^n(F) \end{bmatrix} =$$

$$= \text{vec}\ (A(F)'K(F))$$

$$(A(F)' \otimes I_n)k(F) = \begin{bmatrix} a_{11}(F)I_n & \cdots & a_{n1}(F)I_n \\ a_{12}(F)I_n & & a_{n2}(F)I_n \\ \vdots & & \vdots \\ a_{1n}(F)I_n & \cdots & a_{nn}(F)I_n \end{bmatrix} \begin{bmatrix} k^1(F) \\ k^2(F) \\ \vdots \\ k^n(F) \end{bmatrix} =$$

$$= [\textstyle\sum k_{1i}a_{i1}, \sum k_{2i}a_{i1}, \ldots, \sum k_{ni}a_{i1}, \ldots, \sum k_{1i}a_{in}, \sum k_{2i}a_{in}, \ldots$$

$$\ldots, \textstyle\sum k_{ni}a_{in}]'$$

$$= [k_1(F)a^1(F), \ldots, k_n(F)a^1(F), \ldots, k_1(F)a^n(F), \ldots, k_n(F)a^n(F)]'$$

$$= \text{vec}\ (K(F)A(F)).$$

(15) $[(A(F)' \otimes I_n) + (I_n \otimes A(F)')]k(F) = -d(F)$,

which, by using definition (7), can be written as

(16) $(A(F)' \oplus A(F)')k(F) = -d(F)$.

However, $F \in \mathcal{F}$ implies that $A(F)$ is nonsingular. Therefore, according to Proposition (8), the eigenvalues of the sum $P(F) = A(F)' \oplus A(F)'$ are $\{2\lambda_i\}$, in which λ_i are the eigenvalues of $A(F)'$ (= eigenvalues of $A(F)$), thereby making $P(F)$ nonsingular. Thus, the inverse $P^{-1}(F)$ exists, and

(17) $k(F) = -P^{-1}(F)d(F)$,

in which

(18) $P(F) = (A(F)' \otimes I_n) + (I_n \otimes A(F)')$

Note however that A(F)' is a differentiable function of F. Thus proposition (5) applies and the partial derivatives of P(F) are

(19) $$\frac{\partial P(F)}{\partial f_{ij}} = \left(\frac{\partial A(F)'}{\partial f_{ij}} \otimes I_n \right) + \left(I_n \otimes \frac{\partial A(F)'}{\partial f_{ij}} \right) .$$

The partial derivatives of the inverse, $P^{-1}(F)$, can be proved to be

(20) $$\frac{\partial P^{-1}(F)}{\partial f_{ij}} = -P^{-1}(F) \frac{\partial P(F)}{\partial f_{ij}} P^{-1}(F).$$

In fact, Proposition (6) applied to

$$P^{-1}(F)P(F) = I_{n^2}$$

gives

$$\frac{\partial P^{-1}(F)}{\partial f_{ij}} P(F) + P^{-1}(F) \frac{\partial P(F)}{\partial f_{ij}} = 0,$$

from which (20) follows. From this, and the fact that d(F) is differentiable, (17) can be differentiated with respect to f_{ij}, yielding

(21) $$\frac{\partial k(F)}{\partial f_{ij}} = P^{-1}(F) \left[\frac{\partial P(F)}{\partial f_{ij}} P^{-1}(F)d(F) - \frac{\partial d(F)}{\partial f_{ij}} \right] .$$

The expression for the partial derivatives of P(F) in (19) is easily seen to be continuous for all f_{ij}, Further, $P^{-1}(F)$ is continuous. This is guaranteed by the Banach inverse theorem (it says that, if A is a continuous linear operator and A^{-1} exists then A^{-1} is continuous - see [17]), and the fact that P(F) is linear (it is a linear function of A(F) = A + BFC). Hence, the vectors $\partial k(F)/\partial f_{ij}$ in (21) are continuous in f_{ij}. It follows from this that k(F), and therefore K(F), is a continuously differentiable function of F (refer to [8], page 167).

In order to determine the higher derivatives, it must be noted first that d(F) and A(F), and therefore P(F), are infinitely differentiable (all the derivatives, from the second onwards, vanish). Differentiating

(18) r times gives

$$(22)\quad \frac{\partial^r P(F)}{\partial \ldots} = \frac{\partial^r A(F)'}{\partial \ldots} \oplus \frac{\partial^r A(F)'}{\partial \ldots}$$

where the dots have an obvious meaning. The r-th derivative of $P^{-1}(F)$, $\partial^r P^{-1}(F)/\partial\ldots$, then is derived from (20) and (22). In fact, if we expand $\partial^r P^{-1}(F)/\partial\ldots$ using (20), we see that we shall need the derivatives of P(F) of order up to r, which comes from (22), plus the derivatives of $P^{-1}(F)$ of order up to r-1, which comes from (20) itself used recursively. Hence, since it is clear that, for all r, $\partial^r P^{-1}(F)/\partial\ldots$ is continuous, k(F) (and K(F)) is of class C^∞.

Equally (13) can be shown to hold as follows. Multiplying (21) by P(F) gives

$$(23)\quad P(F)\frac{\partial k(F)}{\partial f_{ij}} = -\frac{\partial P(F)}{\partial f_{ij}} k(F) - \frac{\partial d(F)}{\partial f_{ij}} .$$

Then, substituting from (19) and (20), (23) becomes

$$(24)\quad [A(F)'\otimes I_n) + (I_n \otimes A(F)')]\frac{\partial k(F)}{\partial f_{ij}} = -[(\frac{\partial A(F)'}{\partial f_{ij}} \otimes I_n) + (I_n \otimes \frac{\partial A(F)'}{\partial f_{ij}})]k(F) - \frac{\partial d(F)}{\partial f_{ij}}$$

Now, it is obvious that $\partial k(F)/\partial f_{ij}$ (resp. $\partial d(F)/\partial f_{ij}$) is the column elongation of $\partial K(F)/\partial f_{ij}$ (resp. $\partial D(F)/\partial f_{ij}$). Then, using the same reasoning as when we showed (14) was equivalent to (15), (24) becomes, in matrix notation,

$$(25)\quad \frac{\partial K(F)}{\partial f_{ij}} A(F) + A(F)'\frac{\partial K(F)}{\partial f_{ij}} = -K(F)\frac{\partial A(F)}{\partial f_{ij}} - \frac{\partial A(F)'}{\partial f_{ij}} K(F) - \frac{\partial D(F)}{\partial f_{ij}} .$$

Note however that

$$E_{ij} = \frac{\partial F}{\partial f_{ij}}$$

and so,

(26) $\frac{\partial A(F)}{\partial f_{ij}} = BE_{ij}C,$

and

(27) $\frac{\partial D(F)}{\partial f_{ij}} = C'E_{ij}'RFC + C'F'RE_{ij}C.$

Finally equation (13) is obtained by substituting (26) and (27) into (25), and by using the notation introduced in (2) for the partial derivatives of K.

For the partial derivatives of higher order, we differentiate (25) with respect to the other entries of F, which yields Lyapunov equations of the type

(28) $\Gamma^{i_1 j_1}_{\vdots \atop i_k j_k}(F)A(F) + A(F)'\,\Gamma^{i_1 j_1}_{\vdots \atop i_k j_k}(F) = -T^{i_1 j_1}_{\vdots \atop i_k j_k}(F).$

The expressions for the matrices $T^{i_1 j_1}_{\vdots \atop i_k j_k}(F)$ will be developed in the proof of lemma (45) in Section 3. ∇

Allwright proved the differentiability of K in [2] using a different approach. He used a perturbation method, which involved evaluating the Lyapunov equation due to a small change in the feedback. The Lyapunov method was used since it enabled estimates for the error between $K(F + \Delta)$ and its tangent at F to be found. In fact, an estimate for ε such that,

$$\| K(F + \Delta) - K(F) - dK(F; \Delta) \| < \varepsilon \| \Delta \|$$

for all Δ such that $F + \Delta \in \mathcal{F}$ and $\|\Delta\| < \delta(\varepsilon)$, for some $\delta(\varepsilon)$, was found. In the above expression

(29) $dK(F; \Delta) = \int_0^\infty \exp[A(F)'t]H \exp[A(F)t]dt,$

where

(30) $H = C'\Delta'[B'K(F) + RFC] + [B'K(F) + RFC]'\Delta C,$

which will be shown to be true straightforwardly in Section 3.

An alternative, simpler proof has been obtained here by using the theory of the Kronecker product between matrices. By transforming the (matricial) Lyapunov equation in K into a vector equation of the type $k = -P^{-1}d$, in which P^{-1} and d are continuously differentiable functions, the differential of K immediately followed. This approach has been suggested by Brockett [6] and Barnett [4] in order to solve Lyapunov equations.

2.2 DESCRIPTION OF THE METHOD

The purpose of this section is to describe the two procedures due to Allwright [3] for the dominant feedback problem, both of which are for the case when there are no constraints on F involved, except that $F \in \mathcal{F}$. The distinction between them is basically that in the first the dominance is strict, whereas in the second this is not guaranteed. In addition, the selection of the step length is done differently, although this is not the basic difference between them.

(31) DOMINANT FEEDBACK ALGORITHM [3]

1. Set $j = 0$

   Choose an intial feedback $F^0 \in \mathcal{F}$.

2. Set

$$\pi(F^j) = \min\{\lambda\max\, dK(F^j;\, S) : S \in \mathcal{S}\}$$

3. If $\pi(F^j) \geq 0$ terminate; else continue.

4. Choose the search direction S^j such that

$$S^j \in \arg\min\{\lambda\max\, dK(F^j;\, S) : S \in \mathcal{S}\}$$

5. Choose the step length λ^j such that

$$\lambda^j \in \arg\max\{\lambda\min[K(F^j) - K(F^j + \lambda S^j)]: \lambda \geqq 0\}.$$

6. Set $F^{j+1} = F^j + \lambda^j S^j$, $j = j+1$ and go to step 1. ∇

Here $\lambda\max[D]$ ($\lambda\min[D]$) denotes the maximal (minimal) eigenvalue of the symmetric matrix D, $\arg\min\{\phi(x): x \in X\}$ (arg max) denotes the set of all minimizers (maximizers) by the function ϕ on the set X, and S denotes the unit Frobenius-sphere in $\mathbb{R}^{mxr}$, i.e.

$$S = \{S \in \mathbb{R}^{mxr}: \sum_{ij} s_{ij}^2 = 1\} = \{S \in \mathbb{R}^{mxr}: \|S\|_F = 1\}$$

(32) <u>DOMINANT FEEDBACK ALGORITHM</u> [3]

1. Set $j = 0$.

   Choose $F^0 \in F$.

2. Set

$$\pi(F^j) = \min\{\lambda\max dK(F^j; S): S \in S\}.$$

3. If $\pi(F^j) > 0$ terminate; else continue.

4. Choose the search direction S^j such that

$$S^j \in \arg\min\{\lambda\max dK(F^j; S): S \in S\}.$$

5. Choose the step length λ^j such that

$$\lambda^j \in \arg\max\{tr[(K(F^j)-K(F^j+\lambda S^j))X_0]:\lambda\min[K(F^j)-K(F^j+\lambda S^j)]\geqq 0,\ \lambda \geqq 0\}.$$

6. If $tr[(K(F^j) - K(F^j + \lambda^j S^j))X_0] = 0$ terminate; else continue.

7. Set $F^{j+1} = F^j + \lambda^j S^j$, $j = j+1$ and go to 2. ∇

Here $X_0 = E\{x_0 x_0'\}$ and $tr[\cdot]$ denotes the trace.

In Algorithm (31) the search direction S^j is selected so that the first order approximation to $K(F^j + S^j) - K(F^j)$, $dK(F^j; S^j)$, is negative definite. In fact, since it is chosen so that $\pi(F^j) = \lambda\max\, dK(F^j;\, S^j)$ is negative, then $dK(F^j;\, S^j) < 0$. In algorithm (32), S^j is chosen so as to make $dK(F^j;\, S^j)$ negative semi-definite.

We must now specialize the definition of a descent direction for the problem of minimizing (9):

(33) <u>DEFINITION</u> A direction $S \neq 0$ from F will be called a <u>descent direction at F</u> for K if S is such that $K(F + \lambda S) \leq K(F)$ for all $\lambda \in (0, \bar{\lambda})$, for some $\bar{\lambda} > 0$. A normalized descent direction S is such that $\| S \|_F = 1$. ∇

The next lemma proves that S^j is a (normalized) descent direction, yielding a strict reduction in the cost K along it, when $dK(F^j;\, S^j) < 0$. It also shows that F^j is locally dominant (there are no descent directions emanating from F^j) when $dK(F^j;\, S^j) \nleq 0$, i.e., when $dK(F^j;\, S^j)$ is positive definite or it is indefinite. No information is given when $dK(F^j;\, S^j) \leqq 0$ with $\lambda\max\, dK(F^j;\, S^j) = 0$, and then there may be descent directions at F^j or not, depending on the contributions of the higher order terms.

(34) <u>LEMMA</u> [3]. Suppose $\pi(F) = \min\{\lambda\max\, dK(F;\, S) : S \in \mathcal{S}\}$ and $\tilde{S} \in \arg \pi(F)$. Then, for $F \in \mathcal{F}$,

(a) If $\pi(F) < 0$, there exists a real $\bar{\lambda} > 0$ such that

$$K(F + \lambda\tilde{S}) < K(F), \text{ for all } \lambda \in (0, \bar{\lambda}).$$

(b) If $\pi(F) > 0$, there exists a real $\alpha > 0$ such that $K(F') \nleq K(F)$, for every F' satisfying $\| F - F' \| < \alpha$. ∇

Therefore we may establish the following definitions:

(35) <u>DEFINITION</u> S is said to be a <u>first-order descent direction</u> if and only if $\lambda\max\, dK(F;\, S)$ is negative. ∇

(36) DEFINITION F is said to be first-order locally dominant feedback if and only if $\pi(F) \geqq 0$, i.e. if and only if there are no first-order descent directions at F. ∇

The structure of the algorithms is based on the information given by lemma (34). In fact, if $\pi(F^j) < 0$, algorithm (31) proceeds iteratively in order to find a feedback F^{j+1} that strictly dominates F^j $(K(F^{j+1}) < K(F^j))$, which is guaranteed to exist by (34a). It must be noted that the step length λ^j found by maximizing $\lambda \min[K(F^j)-K(F^j+\lambda S^j)]$ will provide us with such a dominating $F^{j+1} = F^j + \lambda^j S^j$. It terminates when the first-order locally-dominant feedback $(\pi(F^j) \geqq 0)$ is discovered, relying partly on (34b), by deciding to stop when $\pi(F^j) > 0$.

Algorithm (32) goes further, iterating when $\pi(F^j) = 0$, if this is possible. When $\pi(F^j) = 0$, S^j is either descent (not necessarily yielding a strictly dominating feedback) or else it is such that $K(F^{j+1}) = K(F^j)$. In both cases, any other direction S is non-descent, since for them it will be the case that $\lambda \max dK(F^j; S) > 0$. In the former case an iteration is carried out whereas in the latter termination is set, since we will then have a locally dominant F^j. Theorem (37) shows that, due to the way λ^j has been defined, F^{j+1} dominates (not necessarily strictly) F^j, by giving the largest possible decrease in the expected value of the cost $V(x_0, F) = x_0'K(F)x_0$ along S^j:

(37) THEOREM [3]. If algorithm (32) generates a sequence $F^0,\dots,F^q$, then

$$K(F^q) \leq \dots \leq K(F^0)$$

PROOF. For any matrix E, the expected value of x'Ex, when x is randomly distributed with zero mean and covariance X, is equal to the trace of EX. Indeed, using properties of the trace function,

(38) $E\{x'Ex\}$

$= E\{tr[x'Ex]\}$

$= E\{tr[Exx']\}$

$= tr[EX]$.

Therefore,

(39) $tr[(K(F^j) - K(F^{j+1}))X_0] = E\{x_0'(K(F^j) - K(F^{j+1}))x_0\}$.

But since F^{j+1} is chosen to give strictly positive $tr[(K(F^j)-K(F^{j+1}))X_0]$, there must be at least one x_0 for which $x_0'(K(F^j) - K(F^{j+1}))x_0 > 0$. Now, since $\lambda min[K(F^j) - K(F^{j+1})]$ is kept non-negative, $K(F^j) \geqq K(F^{j+1})$. This shows that $K(F^j) \geq K(F^{j+1})$. ∇

When $\pi(F^j) = 0$, the situation of S^j not being a descent direction is detected by checking whether $tr[(K(F^j) - K(F^j + \lambda^j S^j))X_0] = 0$ holds. The reason for this is that λ^j maximizes the trace. Therefore, for all $\lambda \geqq 0$ such that $\lambda min[K(F^j) - K(F^j + \lambda S^j)] \geqq 0$, we must have

(i) $tr[(K(F^j) - K(F^j + \lambda S^j))X_0] = E\{x_0'(K(F^j) - K(F^j + \lambda S^j))x_0\} \leqq 0$.

On the other hand the requirement $\lambda min[K(F^j) - K(F^j + \lambda S^j)] \geqq 0$ implies that $x_0'(K(F^j) - K(F^j + \lambda S^j))x_0 \geqq 0$ for all x_0, which gives

(ii) $E\{x_0'(K(F^j) - K(F^j + \lambda S^j))x_0\} \geqq 0$.

Then, combining (i) and (ii),

$$E\{x_0'(K(F^j) - K(F^j + \lambda S^j))x_0\} = 0.$$

However $x_0'(K(F^j) - K(F^j + \lambda S^j))x_0$ is a continuous function of x_0. Therefore, by the above,

$$x_0'(K(F^j) - K(F^j + \lambda S^j))x_0 = 0$$

for all x_0, which means that $K(F^j) = K(F^j + \lambda S^j)$. In summary, if the condition $tr[(K(F^j) - K(F^j + \lambda^j S^j))X_0] = 0$ holds then, for all $\lambda \geqq 0$ such that $\lambda min[K(F^j) - K(F^j + \lambda S^j)] \geqq 0$, $K(F^j) = K(F^j + \lambda S^j)$ (i.e., $\lambda = 0$ is the only $\lambda \geqq 0$ such that $\lambda min[K(F^j)-K(F^j+\lambda S^j)]\geqq 0)$. Thus, Algorithm (32) terminates with a first-order locally-dominant feadback too ($\pi(F^j) \geqq 0$).

It is worthwhile noting that the initial matrix F^0 may result in no iteration at all for either algorithm.

The rest of this section is concerned with a geometrical analysis of the step length procedure (Step 5).

Let

$$E(\lambda) = K(F^j) - K(F^j + \lambda S^j)$$

$$g(\lambda) = \lambda min[E(\lambda)]$$

$$h^\lambda(x) = x'E(\lambda)x$$

Then, by definition of the smallest eigenvalue,

$$g(\lambda) = min\{h^\lambda(x) : x \in B_n\}$$

where B_n is the Euclidean unit sphere in $\mathbb{R}^n$. Therefore we can formulate the step length subproblem, in (31) and (32) respectively, as

(40) $\max\{g(\lambda) : \lambda \geqq 0\}$

and

(41) $\max\{E\{h^\lambda(x_0)\} : g(\lambda) \geqq 0,\ \lambda \geqq 0\}$,

where fact (39) was taken into account to write (41). For the sake of geometrical interpretation, consider $x_0 \in B_2 \subset \mathbb{R}^2$. Then linearize B_2 into a segment of line L. In this way the quadratic $h^{\lambda}(x_0)$ (in the variable x_0) can be represented as a function on $L \subset \mathbb{R}$. Three examples are depicted in Figure (42). Obviously $h^{\lambda}(x_0)$ is symmetric about $x_0 = 0$, and its shape depends on the parameter λ. Its maximum and minimum values are the largest and smallest eigenvalue, respectively, of $E(\lambda)$.

Consider algorithm (31) with step length subproblem (40). Because $g(\lambda^j) > 0$, the quadratic $h^{\lambda^j}(x_0)$ is positive for all x_0, as shown in Figure (42b). For Algorithm (32), subproblem (41) requires that the expected value of $h^{\lambda^j}(x_0)$ be made as large as possible, while this curve lies on or above the real axis.

Another criterion could be chosen to determine λ^j. Optimizing the largest eigenvalue of $E(\lambda)$ subject to $g(\lambda) \geq 0$ is possible, but it may happen that it yields big cost reduction for only a few values of x_0, and little or no reduction for most values, which would be undesirable, as for example, in figure (42c).

2.3 AN UPPER BOUND FOR THE F-DIFFERENTIAL OF K

In this section we shall prove that, for all matrices F such that $K(F) \leq K(F^0)$, for some $F^0 \in \mathcal{F}$, then the norm of the Fréchet-differential of K (of any order) is bounded, and we shall determine such a bound. As a consequence, if $\{F^j\}$ is a sequence of feedback matrices generated by either of the algorithms described in the previous section, and so $K(F^j) \leq K(F^0)$, for all j and initial $F^0 \in \mathcal{F}$, that result will hold for all F^j. This result will be needed in the last section of this chapter.

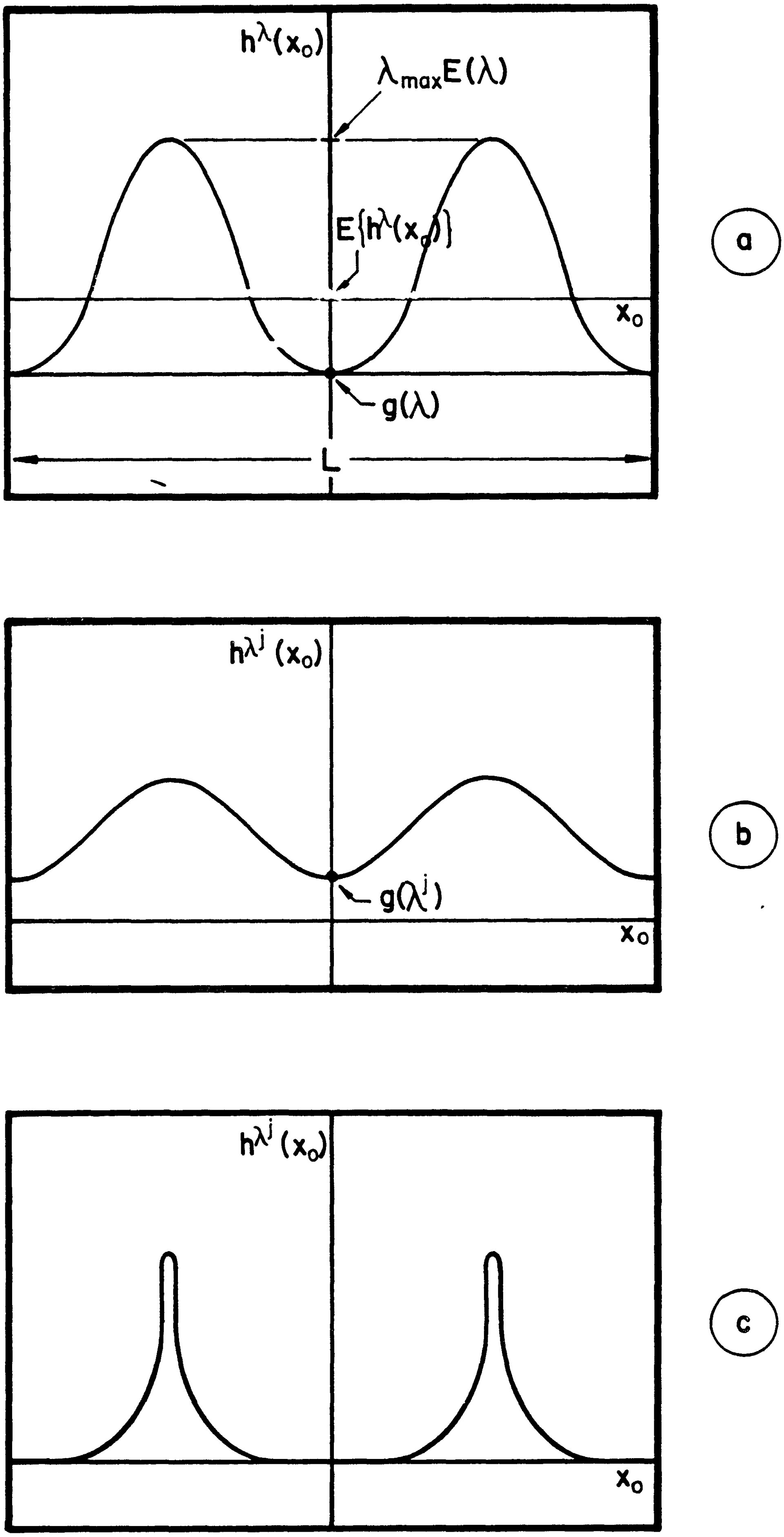
$h^{\lambda}(x_o)$
$\lambda_{max}E(\lambda)$
$E\{h^{\lambda}(x_o)\}$
x_o
$g(\lambda)$
L
a
$h^{\lambda^j}(x_o)$
$g(\lambda^j)$
x_o
b
$h^{\lambda^j}(x_o)$
x_o
c

Figure 42.

In order to do that, the continuity of the function π must be proved. It is facilitated by the

(43) THEOREM. [20], Thm. B3.20]. Let ψ: $\mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}$ be a continuous function and let H be a compact subset of $\mathbb{R}^q$. Then the functions θ: $\mathbb{R}^p \longrightarrow \mathbb{R}$ and ϕ: $\mathbb{R}^q \longrightarrow \mathbb{R}$ defined by

$$\theta(z) = \min\{\psi(z, h) : h \in H\},$$

$$\phi(z) = \max\{\psi(z, h) : h \in H\}$$

are also continuous. ∇

(44) LEMMA. The function

$$\pi: \mathcal{F} \longrightarrow \mathbb{R}$$

$$F \longmapsto \min\{\lambda\max\, dK(F; S); S \in \mathcal{S}\}$$

is continuous.

PROOF. Lemma (11) says that K is continuously differentiable on $\mathcal{F}$. The first F-differential $dK(F; S): \mathcal{F} \times \mathcal{S} \longrightarrow \mathbb{R}^{n\times n}$ is therefore continuous in the variable F. It is also continuous in the variable S because it is linear in S (see [8], page 167). The continuity of it comes out after applying Theorem (43) twice. Indeed, first consider the function defined on $\mathbb{R}^n \times \mathcal{F} \times \mathcal{S}$ by

$$\chi(z, F, S) = z'\, dK(F; S) z$$

and maximize it in z over B_n. The theorem then says that

$$\lambda\max\, dK(F; S) = \max\{\chi(z, F, S) : z \in B_n\}$$

is continuous in F and S. Then since $\mathcal{S}$ is compact, the result follows from a second application of the theorem identifying ψ with $\lambda\max\, dK(F; S)$. ∇

The main result of this section can now be stated.

(45) LEMMA. Let $F^0 \in \mathcal{F}$ be the initial output feedback matrix for Algorithm (31) or Algorithm (32), and suppose that it is generated a sequence $\{F^\ell\}$. Then, for all numbers $k \geq 1$, and for any matrices $S_1,\ldots,S_k \in \mathbb{R}^{mxr}$,

$$\| d^k K(F^\ell;\ S_1,\ldots,S_k) \| \leqq \beta_k(F_0) \| S_1 \| \ldots \| S_k \| ,$$

in which $\beta_k(F_0)$ is a finite positive number that depends on F^0 and the parameters of the system, and $\| \cdot \|$ denotes the 2-norm.

NOTE. For a symmetric nxn matrix A, the 2-norm $\| A \|_2$ is defined by

$$\| A \|_2 = \max\{ \| Ax \|_2 : \| x \|_2 = 1\} = \max\{ |\lambda_i| : i=1,\ldots,n\},$$

where $\| \cdot \|_2$ denotes the Euclidean norm for vectors.

PROOF. Let $F = F^\ell$ for some ℓ. The first step is to establish the expressions for the matrices $T^{i_1 j_1}_{\vdots\ i_k j_k}$ in the Lyapunov equations (28). Basically this is accomplished by differentiating equation (14) up to k times and rearranging the terms of the equation obtained. Differentiating it once gives equation (13), and so we have already $T_{i_1 j_1}$ (k = 1), which is given by the right-hand side of (13). It can be rearranged as:

$$(46)\quad T_{i_1 j_1}(F) = K(F)(BE_{i_1 j_1}C)+(BE_{i_1 j_1}C)'K(F)+(RFC)'E_{i_1 j_1}C+C'E'_{i_1 j_1}(RFC).$$

The second differentiation of the Lyapunov equation (k=2), with respect to the entry $f_{i_2 j_2}$ (possibly $i_1 = i_2$, $j_1 = j_2$) gives

$$(47)\quad T^{i_1 j_1}_{i_2 j_2}(F) = \Gamma_{i_1 j_1}(F)(BE_{i_2 j_2}C) + (BE_{i_2 j_2}C)'\Gamma_{i_1 j_1}(F) +$$

$$\Gamma_{i_2 j_2}(F)(BE_{i_1 j_1}C)+(BE_{i_1 j_1}C)'\Gamma_{i_2 j_2}(F)+(RE_{i_2 j_2}C)'E_{i_1 j_1}C +$$

$$C'E'_{i_1 j_1}(RE_{i_2 j_2}C)$$

We remark that the matrices E are constant. Another differentiation gives the case k = 3, i.e.,

$$(48)\quad T^{i_1j_1}_{\substack{i_2j_2\\ i_3j_3}}(F) = \Gamma^{i_1j_1}_{i_2j_2}(F)(BE_{i_3j_3}C)+(BE_{i_3j_3}C)'\Gamma^{i_1j_1}_{i_2j_2}(F) + \Gamma^{i_1j_1}_{i_3j_3}(F)(BE_{i_2j_2}C)+(BE_{i_2j_2}C)'\Gamma^{i_1j_1}_{i_3j_3}(F) + \Gamma^{i_2j_2}_{i_3j_3}(F)(BE_{i_1j_1}C)+(BE_{i_1j_1}C)'\Gamma^{i_2j_2}_{i_3j_3}(F).$$

The expression for the general case k > 3 can then be induced, and hence

$$(49)\quad T^{i_1j_1}_{\substack{\vdots\\ i_kj_k}}(F) = \Gamma^{i_1j_1}_{\substack{\vdots\\ i_{k-1}j_{k-1}}}(F)(BE_{i_kj_k}C)+(BE_{i_kj_k}C)'\Gamma^{i_1j_1}_{\substack{\vdots\\ i_{k-1}j_{k-1}}}(F) + \Gamma^{i_1j_2}_{\substack{\vdots\\ i_{k-2}j_{k-2}\\ i_kj_k}}(F)(BE_{i_{k-1}j_{k-1}}C)+(BE_{i_{k-1}j_{k-1}}C)'\Gamma^{i_1j_1}_{\substack{\vdots\\ i_{k-2}j_{k-2}\\ i_kj_k}}(F) + \dots + \Gamma^{i_2j_2}_{\substack{\vdots\\ i_kj_k}}(F)(BE_{i_1j_1}C)+(BE_{i_1j_1}C)'\Gamma^{i_2j_2}_{\substack{\vdots\\ i_kj_k}}(F).$$

Now it is possible to obtain an analytical formula for the partial derivative matrices Γ. Taking into account that the above matrices T are symmetric, we can say that the unique symmetric solution to the Lyapunov equation (28) is

$$(50)\quad \Gamma^{i_1j_1}_{\substack{\vdots\\ i_kj_k}}(F) = \int_0^\infty \exp[A(F)'t]\; T^{i_1j_1}_{\substack{\vdots\\ i_kj_k}}(F)\exp[A(F)t]dt.$$

(see [5] page 175).

We must digress at this point and prove (29)-(30). The total first-order differential $dK(F;\, S)$ can be written, using (1), (46) and (50), as

$$dK(F;\, S) = \int_0^\infty \exp[A(F)'t] \sum_{ij} s_{ij}\, T_{ij}(F)\, \exp[A(F)t]dt$$

$$= \int_0^\infty \exp[A(F)'t]\{K(F)(B\sum s_{ij}\, E_{ij}C) + (B\sum s_{ij}\, E_{ij}C)'K(F) +$$

$$(RFC)'\sum s_{ij}\, E_{ij}C + C'(\sum s_{ij}\, E_{ij})'(RFC)\}\, \exp[A(F)t]dt.$$

Since $\sum s_{ij}\, E_{ij} = S$, this proves what we wanted for $S = \Delta$.

The next step is to obtain upper bounds for the norm of partial derivatives matrices Γ, using (46), (47), (49) and (50), and therefore for the norm of the total derivatives, using (2).

Upper bounds on $\|T\|$ are obtained taking the norms in equalities (46), (47) and (49). Therefore, since $\|E_{ij}\| = 1$,

(51) $$\|T_{i_1j_1}(F)\| \leqq 2\|B\|\,\|C\|\,\|K(F)\| + 2\|C\|\,\|R\|\,\|FC\|$$

(52) $$\|T^{i_1j_1}_{i_2j_2}(F)\| \leqq 2\|B\|\,\|C\|\{\|\Gamma_{i_1j_1}(F)\| + \|\Gamma_{i_2j_2}(F)\|\} + 2\|C\|^2\,\|R\|$$

and for $k \geqq 3$,

(53) $$\|T^{i_1j_1}_{\vdots\; i_kj_k}(F)\| \leqq 2\|B\|\,\|C\|\{\|\Gamma^{i_1j_1}_{\vdots\; i_{k-1}j_{k-1}}(F)\| + \|\Gamma^{i_1j_1}_{\vdots\; i_{k-2}j_{k-2}\, i_kj_k}(F)\| + \ldots$$

$$\ldots + \|\Gamma^{i_2j_2}_{\vdots\; i_kj_k}(F)\|\}$$

Taking norms in equation (50) gives

(54) $$\|\Gamma^{i_1j_1}_{\vdots\; i_kj_k}(F)\| \leqq \|T^{i_1j_1}_{\vdots\; i_kj_k}(F)\|\, J(F)$$

in which

(55) $$\mathcal{J}(F) = \int_0^\infty \| \exp[A(F)t] \|^2 dt.$$

By observing the norm inequalities (51) - (54), it is immediate that, if we find upper bounds for $\mathcal{J}(F)$, $\| K(F) \|$ and $\| FC \|$, then bounds for $\| \Gamma^{i_1 j_1}_{\vdots\, i_k j_k}(F) \|$, for all k, will be obtained using recursively those inequalities.

That $\| K(F) \| \leqq \| K(F^0) \|$ is immediate from the dominance property of the algorithms.

Since $C'F'RFC \geqq 0$, equation (14) gives

$$K(F) = \int_0^\infty \exp[A(F)'t][Q + C'F'RFC] \exp[A(F)t]dt$$
$$\geqq \int_0^\infty \exp[A(F)'t]Q \exp[A(F)t]dt,$$

and therefore

$$\| K(F) \| \geqq (\lambda\min Q)\mathcal{J}(F),$$

which, since $Q > 0$, gives the bound

(56) $$\mathcal{J}(F) \leqq (\lambda\min Q)^{-1} \| K(F^0) \| .$$

A bound for $\| FC \|$ may be obtained by using the following result due to Allwright [1]: for the system $\dot{x} = (A + BF)x$, if F is such that $\| K(F) \| \leqq \| K(F^0) \|$, then

$$\| F \| \leqq \frac{\| B \| \, \| K(F^0) \|}{\rho} + \left\{ \frac{\| B \|^2 \| K(F^0) \|^2}{\rho^2} + \frac{2\| A \| \, \| K(F^0) \|}{\rho} \right\}^{\frac{1}{2}},$$

in which ρ is such that $R \geqq \rho I > 0$. For the output feedback problem $\dot{x} = (A + BFCx)$, the above inequality becomes

(57) $$\| FC \| \leqq \frac{\| B \| \, \| K(F^0) \|}{\rho} + \left\{ \frac{\| B \|^2 \| K(F^0) \|^2}{\rho^2} + \frac{2\| A \| \, \| K(F^0) \|}{\rho} \right\}^{\frac{1}{2}} \triangleq F_c$$

with $\rho = \lambda\min R$.

Having established the bounds for $J(F)$, $\| K(F) \|$ and $\| FC \|$, we have proved that bounds for all norms $\| \Gamma \|$ exist. In the lemma that follows we will develop the expressions for them. For the moment we only know that

$$(58) \quad \left\| \Gamma^{\substack{i_1 j_1 \\ \vdots \\ i_k j_k}}(F) \right\| \leqq \gamma_k(F^0),$$

for some numbers $\gamma_k(F^0)$, for all $i_1,\ldots,i_k$, $j_1,\ldots j_k$ and $k \geqq 1$.

Finally to obtain the last result, observe that ([22], page 183):

$$(59) \quad \sum_{ij} |s_{ij}| \leqq r \max_j \sum_i |s_{ij}| \triangleq r\| s \|_1$$

and

$$(60) \quad \| s \|_1 \leqq r^{\frac{1}{2}} \| s \|_2 .$$

Hence, taking the norms in (2), it follows from (2) and (58) - (60) that

$$(61) \quad \| d^j K(F;\ s_1,\ldots,s_k) \| \leqq \gamma_k(F^0) \sum_{i_1 j_1} |s^1_{i_1 j_1}| \ldots \sum_{i_k j_k} |s^k_{i_k j_k}|$$

$$\leqq \gamma_k(F^0) r^{3k/2} \| s_1 \| \ldots \| s_k \|$$

$$= \beta_k(F^0) \| s_1 \| \ldots \| s_k \| ,$$

where $\beta_k(F^0) = \gamma_k(F^0) r^{3k/2}$ and $\gamma_k(F^0)$ are given by (63) - (66). ∇

(62) LEMMA. The numbers $\gamma_k(F^0)$, $k \geqq 1$, for (58), are given recursively by the expressions

$$(63) \quad \gamma_1(F^0) = 2\| C \| \, \| K(F^0) \| \, (\lambda\min Q)^{-1} (\| B \| \, \| K(F^0) \| + \| R \| F_c)$$

$$(64) \quad \gamma_k(F^0) = \frac{k!}{2} \left(\frac{a}{2} \right)^{k-2} (a\gamma_1(F^0) + b), \ k > 1$$

in which

(65) $a = 4\| K(F^0)\| \, \| B\| \, \| C\| \, (\lambda\min Q)^{-1}$,

(66) $b = 2\| K(F^0)\| \, \| C\|^2 \, \| R\| (\lambda\min Q)^{-1}$,

and F_c is given in (57).

PROOF. It follows immediately from (51), (54), (56) and (57) that

$$\begin{aligned} \| \Gamma_{i_1j_1}(F)\| &\leqq \| T_{i_1j_1}(F)\| \, J(F) \\ &\leqq (2\| B\| \, \| C\| \, \| K(F)\| + 2\| C\| \, \| R\| \, \| FC\| \,)J(F) \\ &\leqq (2\| B\| \, \| C\| \, \| K(F^0)\| + 2\| C\| \, \| R\| \, F_c) \, \| K(F^0)\| \, (\lambda\min Q)^{-1} \end{aligned}$$

proving (63). Similarly, combining inequalities (52), (54), (56) and (58), we have

$$\begin{aligned} \| \Gamma^{i_1j_1}_{i_2j_2}(F)\| &\leqq \| T^{i_1j_1}_{i_2j_2}(F)\| \, J(F) \\ &\leqq (2\| B\| \, \| C\| \{\| \Gamma_{i_1j_1}(F)\| + \| \Gamma_{i_1j_2}(F)\| \} + 2\| C\|^2 \, \| R\|) \, \| K(F^0)\| \\ &\qquad (\lambda\min Q)^{-1} \\ &\leqq 4\| B\| \, \| C\| \, \| K(F^0)\| \, (\lambda\min Q)^{-1} \gamma_1(F^0) + 2\| C\|^2 \, \| R\| \, \| K(F^0)\| \\ &\qquad (\lambda\min Q)^{-1} \\ &= a\gamma_1(F^0) + b \end{aligned}$$

in which a and b are defined by (65) and (66). Hence,

(67) $\gamma_2(F^0) = a\gamma_1(F^0) + b$

In the same way, the combination of (53), (54), (56) and (67) gives

$$\| \Gamma^{i_1j_1}_{\substack{i_2j_2 \\ i_3j_3}}(F)\| \leqq \frac{3a}{2} (a\gamma_1(F^0) + b)$$

and we therefore may write that

$$\gamma_3(F^0) = \frac{3!}{2} \cdot \frac{a}{2} \cdot (a\gamma_1(F^0) + b).$$

By observing the next cases (k > 3), we can establish an induction on k. Then, for all $k \geqq 2$,

$$\| \Gamma^{i_1 j_1}_{\vdots \; i_k j_k}(F) \| \leqq \frac{k!}{2} \left(\frac{a}{2} \right)^{k-2} (a\gamma_1(F^0) + b),$$

which proves (64). ∇

2.4 IMPLEMENTABLE VERSION OF THE ALGORITHM

Algorithms (31) and (32) are both non-implementable since to evaluate the search direction and step length exactly is inadmissible in practice, **as such optimization problems cannot be solved normally in a finite** number of operations. Throughout this chapter we shall assume that the search direction can be evaluated and shall concentrate on determining a computable approximation for the step length. Recall that the step lengths λ^j for the two algorithms are defined by the respective solutions of the optimization problems:

(40) $\max\{g(\lambda) : \lambda \geqq 0\}$ and

(41a) $\max\{\phi(\lambda) : g(\lambda) \geqq 0, \lambda \geqq 0\}$ where

$$g(\lambda) = \lambda \min[K(F^j) - K(F^j + \lambda S^j)],$$

$$\phi(\lambda) = E\, \mathrm{tr}[x_0'(K(F^j) - K(F^j + \lambda S^j))x_0].$$

Here will be suggested implementable procedures to replace conceptual rules (40) and (41a), based on Armijo's method for the case when $\pi(F^j) < 0$. The property to be proved in Section 5 assumes that such implementations are used. An implementation for optimization problem (40) will generate a λ^j such that $g(\lambda^j) > 0$, and for problem (41a), $\phi(\lambda^j) > 0$. By implementing the step length procedure we do not spoil the dominance property of either algorithm. First, $g(\lambda^j) > 0$ implies that $K(F^{j+1}) < K(F^j)$ directly. Also, as shown in Theorem (37),

$\phi(\lambda^j) > 0$ and $g(\lambda^j) \geqq 0$ yield $K(F^{j+1}) \leq K(F^j)$.

Among a number of implementable algorithms for scalar optimization, we favour Armijo's method because it is suitable for non-convex functions, and on account of its simplicity. In Armijo's method, at each iteration, one gradient evaluation and several function evaluations are needed. In the method we will present, gradient information is not needed.

When constraints are not considered, Armijo's method is (see Figure (68a) and refer to [20] pp 36, 169): select α and β in (0, 1) and $\rho > 0$ (recommended values are $\alpha = 0.5$, $\beta \in (0.5, 0.8)$ and $\rho = 1$); then $\tilde{\lambda}$ is chosen to be $\rho\beta^q$, for the smallest integer $q \geqq 0$ such that

$$f(\rho\beta^q) \geqq \alpha f'(0)\rho\beta^q,$$

i.e. such that the function $f(\lambda)$ lies above or on the line $y = \alpha f'(0)\lambda$.

Here we alter Armijo's method by:

(i) estimating (from below) the gradient of the unidimensional objective function (g or ϕ) at the origin, by determining a lower bound quadratic form for it in either case, and using the gradient of the quadratic for which a formula is available.

(ii) including the constraint in situation (41a), by defining the initial step size ρ to be such that g is non-negative for all $\lambda \leqq \rho$. If the optimal step length is $\lambda^j = \rho\beta^q$ for some $q \geqq 0$, since $\rho\beta^q \leqq \rho$, then $g(\lambda^j) \geqq 0$.

In order to do this, consider iteration j. For the sake of simplicity in what follows F, S, γ_k and π denote F^j, S^j, $\gamma_k(F^0)$ and $\pi(F^j)$, respectively.

Problem (40) is considered first. A quadratic approximation for

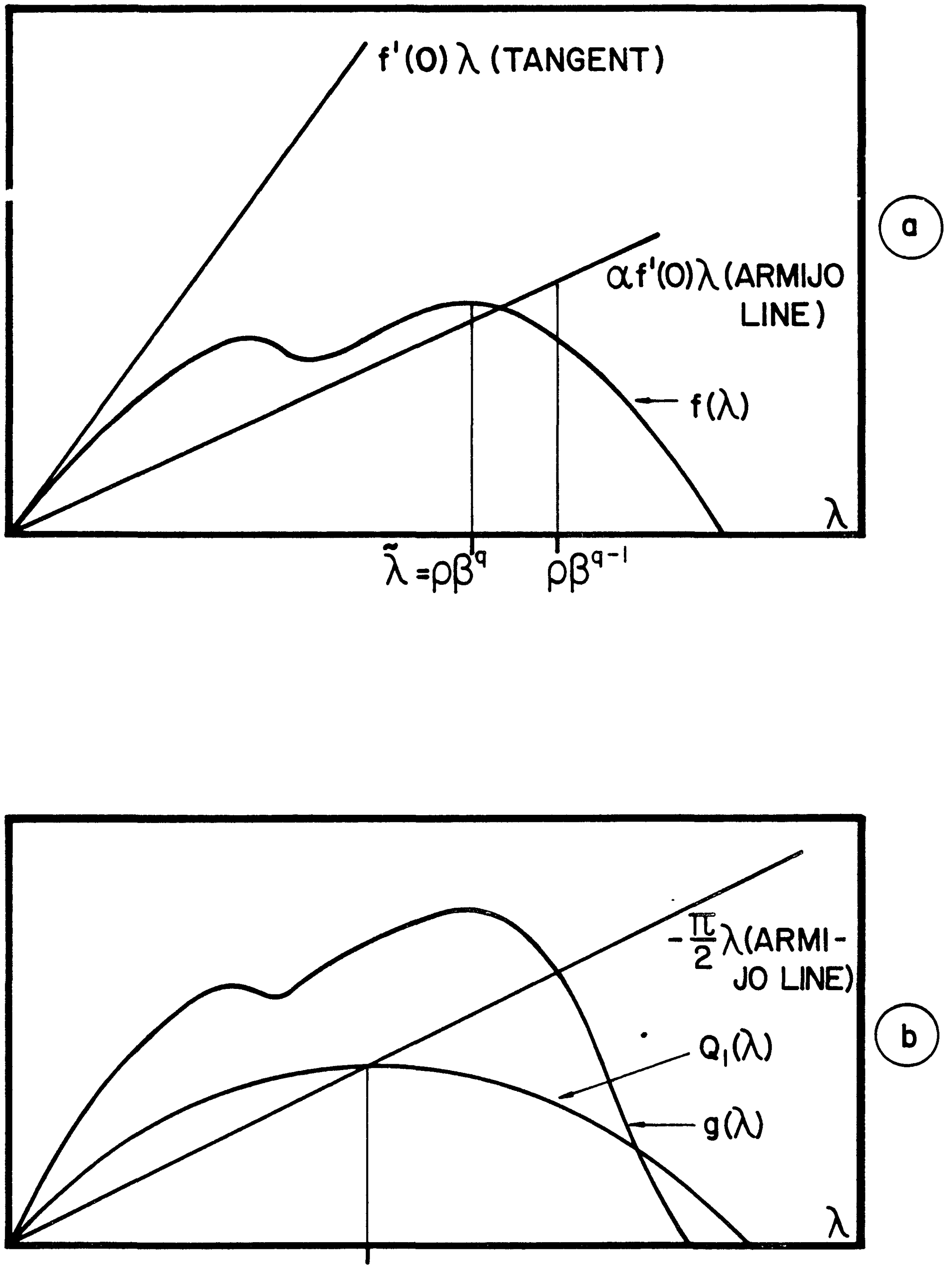
f'(O) λ (TANGENT)
αf'(O)λ (ARMIJO LINE)
f(λ)
λ
λ̃ =ρβ^q
ρβ^{q-1}
a
-π/2 λ(ARMI-JO LINE)
Q₁(λ)
g(λ)
λ
b

Figure 68.

g, Q_1, will be proved to exist, so that $g(\lambda) \geqq Q_1(\lambda)$ for small λ. One part of the proof concerns showing that, given $\delta \in (0, 1/3ar^{3/2})$, for all λ in $(0, \delta)$,

$$(69) \quad \| d^2K(F + \lambda S;\ S,\ S) \| \leqq \frac{r^3\gamma_2}{2} \left\{ 1 + \frac{1}{1 - 3ar^{3/2}\delta} \right\}$$

and, for $Z(\lambda) = K(F + \lambda S) - K(F) - dK(F;\ \lambda S)$,

$$(70) \quad \| Z(\lambda) \| \leqq \frac{r^3\gamma_2}{4} \left\{ 1 + \frac{1}{1 - 3ar^{3/2}\delta} \right\} \lambda^2.$$

The Taylor expansion of $d^2K(F + \lambda S;\ S,\ S)$, as a function of its first argument, gives

$$d^2K(F+\lambda S;S,S) = d^2K(F;S,S) + \lambda d^3K(F;S,S,S) + \frac{1}{2!}\lambda^2 d^4K(F;S,S,S,S) + \ldots$$

Taking norms, we obtain:

$$\| d^2K(F+\lambda S;S,S) \| \leqq \| d^2K(F;S,S) \| + \lambda \| d^3K(F;S,S,S) \|$$

$$+ \frac{1}{2!}\lambda^2 \| d^4K(F;S,S,S,S) \| + \ldots$$

$$\leqq r^3\gamma_2 \| S \|^2 + \lambda r^{9/2}\gamma_3 \| S \|^3 + \frac{1}{2!}\lambda^2 r^{12/2}\gamma_4 \| S \|^4 + \ldots$$

(using lemma (45) and (61))

$$\leqq r^3\gamma_2 + \lambda r^{9/2}\frac{3!}{2}\left(\frac{a}{2}\right)\gamma_2 + \frac{1}{2!}\lambda^2 r^{12/2}\frac{4!}{2}\left(\frac{a}{2}\right)^2\gamma_2 + \ldots$$

(using lemma (62) and $\| S \| \leqq \| S \|_F = 1$)

$$= \frac{r^3\gamma_2}{2}\left\{2 + r^{3/2}\frac{3!}{1!}\left(\frac{a}{2}\right)\lambda + (r^{3/2})^2\frac{4!}{2!}\left(\frac{a}{2}\right)^2\lambda^2 + \ldots\right\}$$

$$\leqq \frac{r^3\gamma_2}{2}\left\{2 + \sum_{i=1}^{\infty}(3ar^{3/2}\lambda)^i\right\}$$

(since $(i+2)!/i! \leqq 6^i$ for all $i \geqq 1$)

(71) $$= \frac{r^3\gamma_2}{2}\{1 + \sum_{i=0}^{\infty}(3ar^{3/2}\lambda)^i\}$$

Now, if we assume that $\lambda \leqq \delta$ for some number $\delta < 1/3ar^{3/2}$, the above sum has a limit, and then

$$\| d^2K(F + \lambda S;\ S,\ S) \| \leqq \frac{r^3\gamma_2}{2}\left\{1 + \frac{1}{1 - 3ar^{3/2}\lambda}\right\}.$$

Hence inequality (69) follows. Now, using the Taylor formula for the second-order expansion of K, we have

$$K(F + \lambda S) = K(F) + dK(F;\ \lambda S) + Z(\lambda)$$

where

$$Z(\lambda) = \int_0^1 (1-t)\, d^2K(F + t\lambda S;\ \lambda S,\ \lambda S)\, dt.$$

Therefore, using (69),

(72) $$\begin{aligned} \| Z(\lambda) \| &\leqq \lambda^2 \int_0^1 (1-t) \| d^2K(F + t\lambda S;\ S,\ S) \| \, dt \\ &\leqq \lambda^2 \int_0^1 (1-t)\, dt\, \frac{r^3\gamma_2}{2}\left\{1 + \frac{1}{1 - 3ar^{3/2}\delta}\right\} \\ &\leqq \frac{r^3\gamma_2}{4}\left\{1 + \frac{1}{1 - 3ar^{3/2}\delta}\right\}\lambda^2 . \end{aligned}$$

and (70) is proved.

Before going further we need the next fact:

(73) <u>FACT</u> ([22] page 316). For two symmetric matrices A and B, if $\{\alpha_i\}$ are the eigenvalues of A and $\{\beta_i\}$ are the eigenvalues of the sum A+B, then for all i,

$$\beta_i \leqq \alpha_i + \| B \|_2 \qquad \nabla$$

It follows from all the above considerations that, for all $\lambda \in (0,\ \delta)$,

$$g(\lambda) = \lambda\min[K(F) - K(F + \lambda S)]$$

$$= \lambda\min[-dK(F; \lambda S) - Z(\lambda)]$$

$$\geqq \lambda\min[-dK(F; \lambda S)] - \| - Z(\lambda) \| \qquad \text{(by (73))}$$

$$= -\lambda\max\ dK(F; \lambda S) - \| Z(\lambda) \|$$

$$= -(\lambda\max\ dK(F; S))\lambda - \| Z(\lambda) \|$$

(74) $$\triangleq -\pi\lambda - \| Z(\lambda) \|$$

$$\geqq -\pi\lambda - \frac{r^3\gamma_2}{4}\left\{1 + \frac{1}{1 - 3ar^{3/2}\delta}\right\}\lambda^2 \qquad \text{(by (70))}$$

Hence, for $\lambda \in (0, \delta)$

(75) $g(\lambda) \geqq Q_1(\lambda)$

for

(76) $$Q_1(\lambda) = -\pi\lambda - \frac{r^3\gamma_2}{4}\left\{1 + \frac{1}{1 - 3ar^{3/2}\delta}\right\}\lambda^2$$

i.e., for small λ, the function g is lower bounded by the quadratic Q_1. A consequence of this fact is that the slope of g at the origin is higher than that of Q_1, thereby an Armijo line for Q_1, i.e. any linear variety defined by $y = \alpha Q_1'(0)\lambda$, with $\alpha < 1$, is an Armijo line for g as well, as we can see in Figure (68b). Here we shall use the Armijo line that intercepts the quadratic at its maximum point. The equation that defines it is:

$$y = -\frac{\pi}{2}\lambda$$

Hence, the Armijo-based algorithm for determining the solution to problem (40) is:

(77) <u>STEP LENGTH SUBALGORITHM FOR ALGORITHM (31)</u>.

1. Choose $\beta \in (0.5, 0.8)$;

Set $q = 0$.

2. Set $\lambda = \beta^q$.

3. Compute

$$\theta(\lambda) = g(\lambda) + \frac{\pi}{2}\lambda .$$

4. If $\theta(\lambda) < 0$ set $q = q+1$ and go to 2; else continue.

5. Set $\lambda^j = \lambda$ and stop. ∇

Problem (41a) is slightly more complicated since we must guarantee that λ^j satisfies the constraint. First we will modify the statement of the problem, to specialize it to the situation when $\pi(F^j) < 0$. Note that this assumption guarantees that there exists a strictly dominating feedback along S^j, so there exists a λ such that $g(\lambda) > 0$. This permits stating (41a) equivalently as

(41b) $\max\{\phi(\lambda): g(\lambda) > 0, \lambda > 0\}$.

Then, the initial step size ρ for the Armijo's based subprocedure will be chosen so that $g(\lambda) > 0$ for all $\lambda \leqq \rho$. So, by setting $\lambda^j = \rho\beta^q$, we will make sure that $g(\lambda^j) > 0$ since, for any $q \geqq 0$, $\rho\beta^q \leqq \rho$. We shall prove that there exists a quadratic form approximation Q_2 for ϕ, for small λ, such that $\phi(\lambda) \geqq Q_2(\lambda)$, which will enable the use of the tangent of Q_2 for generating an Armijo line for ϕ.

Let ε be any positive number greater than 2. Then define

(79) $$\delta(\varepsilon) = \frac{\varepsilon - 2}{3ar^{3/2}(\varepsilon - 1)} > 0 .$$

Similarly to what has been done for problem (40), we will show that, given ε satisfying (79), there exists a $\delta(\varepsilon)$, which is given in (79), such that, for all $\lambda \in (0, \delta(\varepsilon)]$,

(80) $$\| d^2K(F + \lambda S; S, S) \| \leqq \frac{r^3\gamma_2}{2} \varepsilon$$

and, for $Z(\lambda) = K(F + \lambda S) - K(F) - dK(F; \lambda S)$,

(81) $$\| Z(\lambda) \| \leqq \frac{r^3\gamma_2}{4} \varepsilon \lambda^2 .$$

In fact, consider inequality (71)

$$\| d^2K(F + \lambda S; S, S) \| \leqq \frac{r^3\gamma_2}{2} \{1 + \sum_{i=0}^{\infty} (3ar^{3/2}\lambda)^i\},$$

and assume that $\lambda \in (0, \delta(\varepsilon)]$. Then, from (79)

$$\lambda \leqq \frac{\varepsilon - 2}{3ar^{3/2}(\varepsilon - 1)} < \frac{1}{3ar^{3/2}}$$

and inequality (71) becomes, as before,

(82) $$\| d^2K(F + \lambda S; S, S) \| \leqq \frac{r^3\gamma_2}{2} \left\{ 1 + \frac{1}{1 - 3ar^{3/2}\lambda} \right\} .$$

However, since $\lambda \leqq \delta(\varepsilon)$,

(83) $$1 + \frac{1}{1 - 3ar^{3/2}\lambda} \leqq 1 + \frac{1}{1 - 3ar^{3/2}\delta(\varepsilon)} = \varepsilon .$$

Therefore, combining inequalities (82) and (83), we get (80). Also from (72) and (80) we obtain (81), i.e.,

$$\begin{aligned} \| Z(\lambda) \| &\leqq \lambda^2 \int_0^1 (1-t) \| d^2K(F + t\lambda S; S, S) \| dt \\ &\leqq \frac{r^3\gamma_2}{4} \varepsilon \lambda^2 \end{aligned}$$

Now two facts must be remarked:

(84) <u>FACT</u>. For two matrices $E \geqq 0$ and $X > 0$, $\operatorname{tr} EX \geqq \lambda\min X \operatorname{tr} E$.

<u>PROOF</u>. $X - (\lambda\min X) I \geqq 0$

$$\Rightarrow \quad E^{\frac{1}{2}}(X - (\lambda\min X) I)E^{\frac{1}{2}} \geqq 0$$

$\Rightarrow \quad tr[E^{\frac{1}{2}}(X - (\lambda min\, X)\, I)E^{\frac{1}{2}}] \geqq 0$

$\Rightarrow \quad tr[E^{\frac{1}{2}}\, X\, E^{\frac{1}{2}}] \geqq \lambda min\, X\, tr[E]$

and since tr[ABC] = tr[BCA], the result follows. ∇

(85) <u>FACT</u>. For a matrix E with eigenvalues $\{\lambda_i\}$,

$$tr[E] = \sum_i \lambda_i \qquad \nabla$$

Hence, noting that $\delta(\varepsilon) < 1/3ar^{3/2}$, we have that, for all $\lambda \in (0, \delta(\varepsilon)]$,

$$\phi(\lambda) = tr[(K(F) - K(F + \lambda S))X_0]$$

$$\geqq \lambda min\ X_0\ tr[K(F) - K(F + \lambda S)] \quad \text{(by (84))}$$

$$\geqq n\ \lambda min\ X_0\ \lambda min[K(F) - K(F + \lambda S)]$$

(86) $$= n\ \lambda min\ X_0\ g(\lambda)$$

$$\geqq -n\ \lambda min\ X_0\, \pi\, \lambda - n\ \lambda min\ X_0 \| Z(\lambda) \| \quad \text{(by (74))}$$

(87) $$\geqq -n\ \lambda min\ X_0\, \pi\, \lambda - \frac{n\ \lambda min\ X_0 r^3 \gamma_2 \varepsilon}{4} \lambda^2 \quad \text{(by (81))}$$

In summary, for all $\lambda \in (0, \delta(\varepsilon)]$,

(88) $\phi(\lambda) \geqq Q_2(\lambda)$

for

(89) $$Q_2(\lambda) = -n\ \lambda min\ X_0\, \pi\, \lambda - \frac{n\ \lambda min\ X_0 r^3 \gamma_2 \varepsilon}{4} \lambda^2$$

We remark that the above quadratic is a function of the parameter ε. This means that there exists a family of quadratics which approximate ϕ from below in a neighbourhood of the origin. Any member of the family could provide an Armijo line for ϕ, in the same way as an Armijo line for g was obtained from Q_1. In whatever case the line equation is $y = -\frac{n\ \lambda min\ X_0\, \pi}{2} \lambda$. However the one we will select will automatically give a feasible initial step size.

Consider the family of quadratics $\{Q_2^\varepsilon\}$ for different values of ε shown in diagram (90). Note that, as ε increases, $\delta(\varepsilon)$, the number up to which $Q_2^\varepsilon \leqq \phi$, increases too. Take the first one (for the smallest ε) such that its positive root, say ρ_ε, is smaller than $\delta(\varepsilon)$. The formula that gives ρ_ε is obtained from (89). Then

$$(90) \quad \rho_\varepsilon = -\frac{4\pi}{r^3 \gamma_2 \varepsilon}$$

We shall prove that ε satisfying

$$(91) \quad \frac{\varepsilon(\varepsilon - 2)}{\varepsilon - 1} \geqq \frac{12a\gamma_1}{\gamma_2},$$

for a, γ_1 and γ_2 of (63) - (65), implies that $\rho_\varepsilon \leqq \delta(\varepsilon)$. In fact, since $\pi < 0$, and $\| S \| \leqq 1$, it follows from (61) that

$$-\pi = -\lambda\max\, dK(F;\, S) = \| dK(F;\, S) \| \leqq r^{3/2}\gamma_1.$$

This, combined with (90) - (91), yields

$$\rho_\varepsilon = -\frac{4\pi}{r^3\gamma_2\varepsilon} \leqq \frac{4\gamma_1}{r^{3/2}\gamma_2\varepsilon} \leqq \frac{\varepsilon - 2}{3ar^{3/2}(\varepsilon - 1)} = \delta(\varepsilon)$$

Now consider an ε satisfying the condition $\rho_\varepsilon \leqq \delta(\varepsilon)$. Then we can prove that ρ_ε is eligible for the initial step size in the sense that $g(\lambda) \geqq 0$ for all λ in the interval $[0, \rho_\varepsilon]$. In fact, since $Q_2^\varepsilon(\lambda) \geqq 0$ for $\lambda \in [0, \rho_\varepsilon]$, then

$$(n\, \lambda\min\, X_0)^{-1} Q_2^\varepsilon(\lambda) \geqq 0.$$

However, from (86) - (87) - (89), we know that for $\lambda \in (0, \delta(\varepsilon)]$,

$$g(\lambda) \geqq (n\, \lambda\min\, X_0)^{-1} Q_2^\varepsilon(\lambda).$$

So, for $\lambda \leqq \rho_\varepsilon \leqq \delta(\varepsilon)$

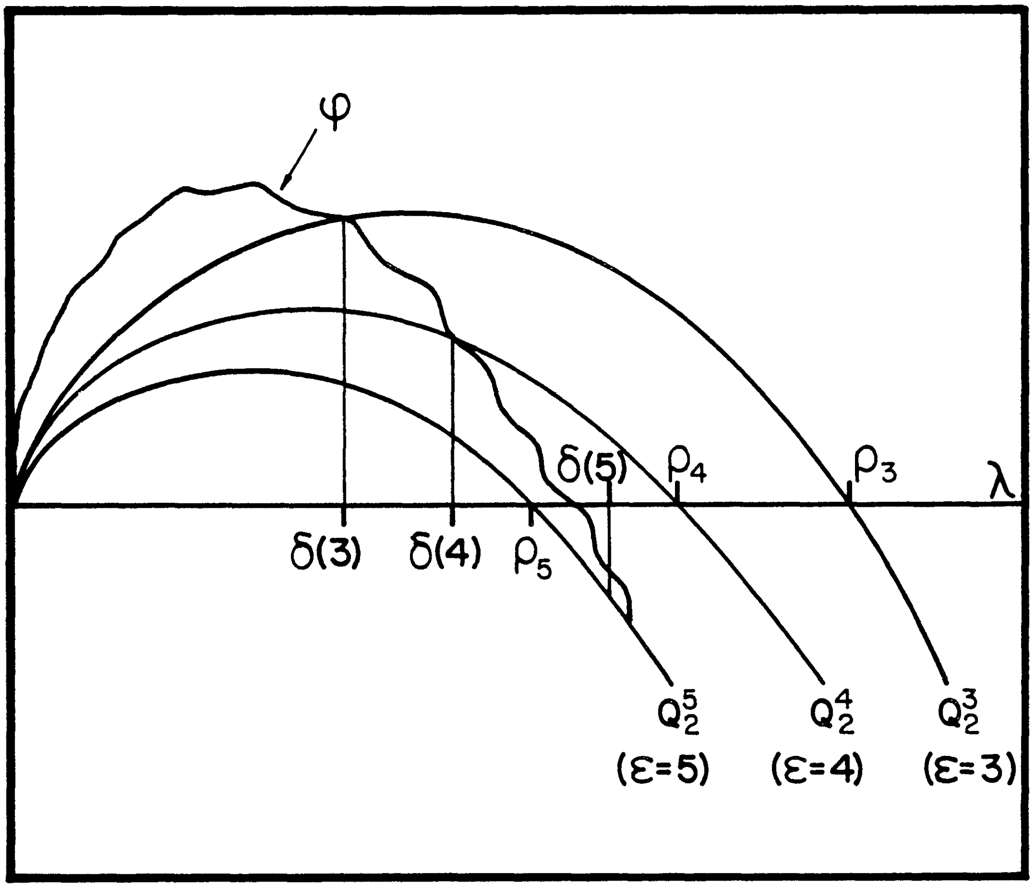

Figure 90.

$g(\lambda) \geqq 0.$

This means that either ρ_ε or $\delta(\varepsilon)$ could be used as an initial step size, provided it is the minimum of the two. We shall choose ρ_ε since it is a function of π and therefore convenient for us in the next section. Thus we need ρ_ε to be smaller than $\delta(\varepsilon)$, and therefore we choose ε satisfying (91). Finally note that ρ_ε may be such that $g(\rho_\varepsilon) = 0$, which, as mentioned before, is undesirable. Then, the initial step size will be defined as $\rho_\varepsilon\beta$. Hence, the algorithm for obtaining the solution to problem (41b) is:

(92) <u>STEP LENGTH SUBALGORITHM FOR ALGORITHM (32)</u>.

1. Choose $\beta \in (0.5, 0.8)$ and the smallest $\varepsilon > 2$ satisfying

$$\frac{\varepsilon(\varepsilon - 2)}{\varepsilon - 1} \geqq \frac{12a\gamma_1}{\gamma_2}$$

where a, γ_1 and γ_2 are given in (63) - (65); set $q = 1$.

2. Compute the initial step size

$$\rho = -\frac{4\pi}{r^3\gamma_2\varepsilon} \quad .$$

3. Set $\lambda = \rho\beta^q$

4. Compute

$$\theta(\lambda) = \phi(\lambda) + \frac{n\ \lambda\text{min}\ X_0\,\pi}{2}\lambda \quad .$$

5. If $\theta(\lambda) < 0$ set $q = q+1$ and go to 3; else continue.

6. Set $\lambda^j = \lambda$ and stop. ∇

(93) <u>REMARK</u>. As inequality (91) is equivalent to

$$\varepsilon^2 - \left(2 + \frac{12a\gamma_1}{\gamma_2}\right)\varepsilon + \frac{12a\gamma_1}{\gamma_2} \geqq 0$$

for $\varepsilon > 2$, and this quadratic is positive for large ε, them the choice for ε required in step 1 is always possible. ∇

(94) REMARK. An alternative choice for the initial step size is $\rho = 1$. In this case however one must evaluate $g(\lambda)$ and check if $g(\lambda) \geqq 0$.

2.5 A SPECIAL PROPERTY FOR INFINITE SEQUENCES OF DOMINATING FEEDBACKS

In this section we shall be concerned with infinite sequences of feedbacks, with elements F^j such that $\pi(F^j) < 0$, which are not necessarily convergent and are generated using algorithms (31) and (32). Assuming that $\{F^j\}$ is such a sequence, it will be proved that the sequence $\{\pi(F^j)\}$ tends to zero as j tends to infinity. In the case when $\{F^j\}$ has accumulation points in $\mathcal{F}$ or a limit, this fact has the interesting interpretation that the limit or accumulation point F^* is first-order locally-dominant.

(95) LEMMA [23]. Let $\{P_j\}$, $j \in \{1,2,...\}$, be a sequence of $n \times n$ symmetric matrices such that $P_1 \leqq P_2 \leqq \dots$ and $P_j \leqq P$, for all j and some P. Then $P^* = \lim_{j\to\infty} P_j$ exists and $P^* \leqq P$. ∇

The main result is stated in:

(96) THEOREM. Consider the implementable versions of algorithm (31) and (32), in which the step lengths are computed using subalgorithms (77) and (92), respectively, and assume that $\{F^j\}$ is an infinite sequence of dominating feedbacks constructed by either algorithm, such that $\pi(F^j) < 0$. Then, as $j \to \infty$,

$$K(F^j) \to K^*$$

for some $K^* \geqq 0$, and

$$\pi(F^j) \to 0.$$

PROOF. The first result of the theorem comes from the monotone non-increasing property of the cost sequence, the fact that $K(F^j) \geqq 0$

for all j, and from lemma (95).

The second result will be shown to hold for Algorithm (32) only, although it is true for Algorithm (31) also. The proof for Algorithm (31) can be easily obtained by replacing ϕ by g and dropping $n\ \lambda\min X_0$ in the following proof. Follow the proof using diagram (97).

The first thing to be shown is that

(98) $\lambda^j = \rho\beta^q > \frac{\beta\rho}{2}$

where q is the smallest number for which $\theta(\rho\beta^q) \geqq 0$, where θ is given in step 4, and ρ is defined in step 2. Indeed, consider

$$\theta(\lambda) = \phi(\lambda) + \frac{n\ \lambda\min X_0\ \pi(F^j)}{2}\lambda$$

and denote by ξ the first positive root of $\theta(\lambda) = 0$, i.e. the first point at which ϕ encounters the Armijo line. (If ϕ does not intercept the line along the positive axis it means that ϕ is always above it, and so $\lambda^j = \rho$, therefore satisfying (98)). Thus

(99) $\phi(\xi) = -\frac{n\ \lambda\min X_0\ \pi(F^j)}{2}\xi > 0$

However note that, by the way the line was defined, crossing Q_2 at its top,

(100) $-\frac{n\ \lambda\min X_0\ \pi(F^j)}{2}\lambda \geqq Q_2(\lambda) \iff \lambda \in [\rho/2, \infty)$

Also since $\phi(\xi) > 0$, it must be true that

(101) $\phi(\xi) \geqq Q_2(\xi)$,

which can easily be proved by contradiction using the facts that $\rho < \delta(\varepsilon)$, $Q_2(\lambda) \geqq 0 \iff \lambda \in [0, \rho]$ and $\phi(\lambda) \geqq Q_2(\lambda)$, for $\lambda \in (0, \delta(\varepsilon)]$. Therefore it follows from (99) - (101) that

(102) $\xi \in [\rho/2, \infty)$.

In order to obtain (98), the next task will be to show that

(103) $\lambda^j > \min\{\rho\beta^2, \xi\beta\}$.

In fact, since $\lambda^j = \rho\beta^q$ for the smallest $q \geqq 1$ such that $\theta(\lambda^j) \geqq 0$, $\lambda^j \leqq \rho\beta$ and $\lambda^j \leqq \xi$, i.e.

(104) $\lambda^j = \rho\beta^q \leqq \min\{\rho\beta, \xi\}$.

Now suppose that

(105) $\lambda^j = \rho\beta^q \leqq \beta\min\{\rho\beta, \xi\}$

then

$$\rho\beta^{q-1} \leqq \min\{\rho\beta, \xi\}$$

which contradicts (104) for the smallest $q \geqq 1$. Thus (105) is false and (103) holds.

Note that from (103) and (104), λ^j belongs to the interval $(\min\{\rho\beta^2, \xi\beta\}, \min\{\rho\beta, \xi\})$ (see figure (97)).

Finally results (102) and (103) will prove (98). Indeed, (102) gives

$$\xi\beta \geqq \frac{\rho\beta}{2}$$

and $\beta \in (0.5, 0.8)$ implies that

$$\rho\beta^2 > \frac{\rho\beta}{2},$$

so that

$$\min\{\xi\beta, \rho\beta^2\} \geqq \frac{\rho\beta}{2}.$$

Combining the above with (103) yields (98).

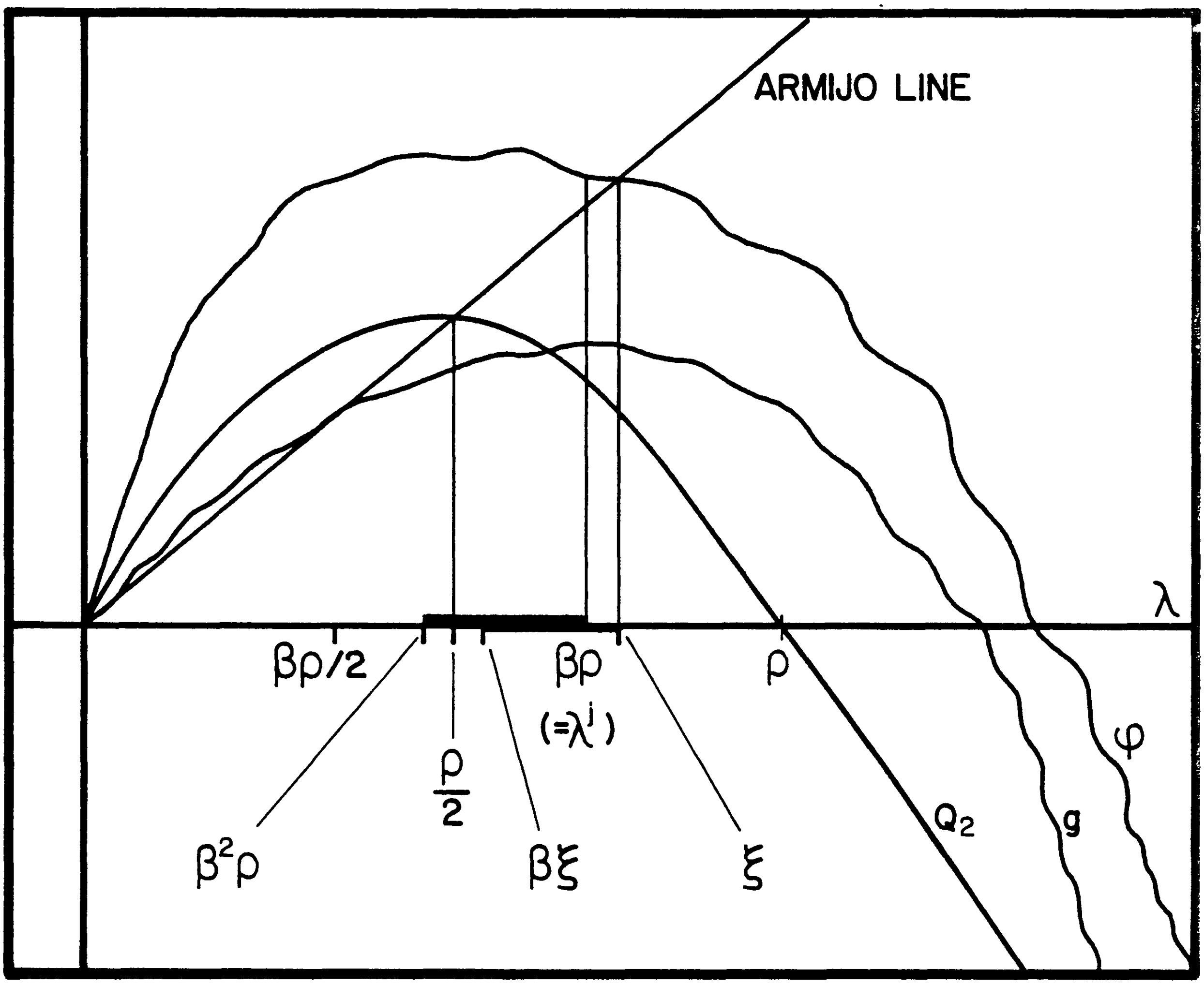

Figure 97.

The convergence of the sequence $\{\pi(F^j)\}$ can now be shown. Since $\theta(\lambda^j) \geqq 0$,

$$\phi(\lambda^j) \geqq - \frac{n\ \lambda\min\ X_0\ \pi(F^j)}{2}\ \lambda^j$$

$$> - \frac{n\ \lambda\min\ X_0\ \pi(F^j)}{2}\ \frac{\beta\rho}{2} \quad \text{(using (98))}$$

(106) $$= \frac{n\ \lambda\min\ X_0\ \beta}{r^3\ \gamma_2\ \varepsilon}\ \pi(F^j)^2 \quad \text{(using (90))}$$

However the convergence of the cost sequence $\{K(F^j)\}$ ensures that, for the zero matrix $\mathbb{O}$,

$$K(F^j) - K(F^j + \lambda^j S^j) \to \mathbb{O}$$

and then

$$\phi(\lambda^j) = \mathrm{tr}[(K(F^j) - K(F^j + \lambda^j S^j))X_0] \to 0.$$

Consequently, from the above and (106),

$$\pi(F^j) \to 0. \qquad \nabla$$

This result, together with the continuity of the function π, leads us to the conclusion that, if F^* is an accumulation point of $\{F^j\}$, then $\pi(F^*) = 0$ (observe that we are not saying that there exist accumulation points for the sequence). It is of interest to see how inequality (106) can be used to prove $\pi(F^j) \to 0$, by means of a partly different approach. Our next task will be to describe it.

We shall use the theory of computational algorithms developed by Polak in [20].

Consider the problem of minimizing a continuously differentiable function $f: \mathbb{R}^n \to \mathbb{R}$. Let us call points in $\mathbb{R}^n$ desirable if they are e.g. local minimizers. Let $a: \mathbb{R}^n \to \mathbb{R}^n$ be a search function. The following is an abstract algorithm for computing desirable points.

(107) <u>ALGORITHM MODEL</u> [20]

1. Compute a $z_0 \in \mathbb{R}^n$.

2. Set $i = 0$.

3. Compute $a(z_i)$.

4. Set $z_{i+1} = a(z_i)$.

5. Determine whether z_{i+1} is desirable using f.

6. If z_{i+1} is desirable, stop; else set $i = i+1$ and go to 3. ∇

(108) <u>THEOREM</u> [20]. Suppose that

(i) f is either continuous at all nondesirable points $z \in \mathbb{R}^n$ or else f is bounded from below;

(ii) for every nondesirable $z \in \mathbb{R}^n$, there exists an $\varepsilon(z) > 0$ and a $\delta(z) > 0$ such that

$$f(z') - f(a(z')) \geqq \varepsilon(z)$$

for all z' such that $\| z - z' \| \leqq \delta(z)$.

Then, either the sequence $\{z_i\}$ constructed by algorithm (107) is finite and its penultimate element is desirable, or else it is infinite and every accumulation point of $\{z_i\}$ is desirable. ∇

We shall call a feedback F^j desirable, if it is first-order locally-dominant, i.e. if $\pi(F^j) \geqq 0$. The function f will be identified with the continuously differentiable function $tr[K(F)X_0]$. The algorithm model to be considered is algorithm (32) with the line search rule given in (92). Since this algorithm was designed to optimize K(F), we must show that it also optimizes f, i.e., at each iteration j, $f(F^{j+1}) < f(F^j)$. This is obviously true for it has been proved that F^{j+1} dominates F^j, thus, using (39),

$$E\{x_0'K(F^j)x_0 - x_0'K(F^{j+1})x_0\}$$

$$= tr[K(F^j)X_0] - tr[K(F^{j+1})X_0]$$

$$= f(F^j) - f(F^{j+1}) > 0.$$

Therefore, algorithm (32) - (92) fits model (107) with $f = tr[K(F)X_0]$.

It is simple to prove that theorem (108) applies here. Since f is continuous, we must only prove that assumption (ii) holds. So, suppose that F^j is nondesirable, i.e. $\pi(F^j) < 0$, and let

$$\varepsilon(F^j) = -\frac{\pi(F^j)}{2} > 0.$$

Then, by continuity of π, there exists a $\delta(F^j) > 0$, so that, for all F' with $\|F^j - F'\| \leqq \delta(F^j)$,

$$|\pi(F^j) - \pi(F')| \leqq \varepsilon(F^j) = -\frac{\pi(F^j)}{2},$$

and so,

$$-\pi(F') \geqq -\frac{\pi(F^j)}{2}$$

which implies

$$\pi(F')^2 \geqq \frac{\pi(F^j)^2}{4}.$$

Now, using (106) with F' and $S' \in \arg \pi(F')$, and the above,

$$\phi(\lambda) = tr[(K(F') - K(F' + \lambda S'))X_0]$$

$$= tr[K(F')X_0] - tr[K(F' + \lambda S')X_0]$$

$$\geqq \frac{n \lambda\min X_0 \beta}{r^3 \gamma_2 \varepsilon} \pi(F')^2$$

$$\geqq \frac{n \lambda\min X_0 \beta}{4r^3 \gamma_2 \varepsilon} \pi(F^j)^2 > 0$$

proving (ii). Thus, theorem (108) holds. Then, for F^* an accumulation point of $\{F^j\}$, $\pi(F^*) \geqq 0$. However, we cannot have $\pi(F^*) > 0$ by continuity of π and since $\pi(F^j) < 0$ for all j. Hence $\pi(F^*) = 0$.

We finish this chapter with two remarks. First, suppose we consider the search direction is the matrix $\tilde{S}^j$ which minimizes approximately $\lambda\max\, dK(F^j;\, \cdot)$ over $\mathcal{S}$. Then all the results of this and the previous sections are valid, if we replace $\pi(F^j)$ $(= \lambda\max\, dK(F^j;\, S^j))$ by $\lambda\max\, dK(F^j;\, \tilde{S}^j)$. Second, it is worthwhile noting that, although the algorithms studied do not always give convergence of feedbacks, this does not matter as convergence of $\{K(F^j)\}$ to K^* is all that is needed practically rather than convergence of $\{F^j\}$ to some F^*.

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CHAPTER 3

CONSTRAINED DOMINANT FEEDBACK

3. CONSTRAINED OUTPUT FEEDBACK

This chapter is concerned with a constrained output feedback problem. We shall consider the feasible set to be a compact convex set M, having a nonempty interior, and defined by continuous functions g^i:

$$M \triangleq \{X \in \mathbb{R}^{mxr} : g^i(X) \leq 0, \ i = 1,\ldots,q\}$$

An extension of the method described in Chapter 2 will be developed here. It is well known that convergence of a sequence of non-desirable points generated by a feasible direction algorithm, to a desirable point, in constrained optimization, may not be achievable if the algorithm is not "closed". In this chapter it will be proved that, by choosing the search direction as the vector S which minimizes the largest eigenvalue of $\lambda max(F; S)$, when F+S lies in M, our algorithm is closed. Moreoever, when the search direction is determined approximately, in the sense that S is chosen so that $\lambda max \; dK(F; S)$ is a fraction of the minimum of $\lambda max \; dK(F; \cdot)$ over M, closedness is maintained. Therefore convergence of a non-desirable sequence of feedbacks to a desirable feedback is guaranteed. A desirable feedback will be an F satisfying $\min\{\lambda max \; dK(F; S) : F+S \in M\} = 0$. As before, convergence means only function valued convergence since K is not convex. So, (function-valued) convergence of an (infinite) sequence $\{F^j\}$ to F^*, within this context, implies that F^* is an accumulation point of $\{F^j\}$, and it is desirable. We shall not be concerned in this chapter with implementable procedures for determining the search direction. In the chapters that follow we shall draw attention to that point. Finally, convergence is also proved for the algorithm having a computable step length analogous to that of Chapter 2, using a direct approach.

3.1 STATEMENT OF THE OPTIMIZATION PROBLEM

Given an initial condition x_0 for the system (1.1-3), consider the:

(1) CONSTRAINED PROBLEM

Minimize $\{V(x_0,F): F \in M\}$ in the dominance sense,

where M is a compact convex set with nonempty interior, defined by continuous functions $g^i: \mathbb{R}^{m \times r} \to \mathbb{R}$ as

$$M = \{X \in \mathbb{R}^{m \times r} : g^i(X) \leqq 0,\ i = 1,\dots,q, \text{ and } X \text{ is stabilizing, i. e. } X \in \mathcal{F}\}$$

and for which an initial output feedback $F^0 \in M$ is provided. ∇

Given a feedback $F \in \mathcal{J}$, denote by $M(F)$ the set

$$M(F) \triangleq \{Y \in \mathbb{R}^{m \times r} : Y = X - F,\ X \in M\}$$

thus

(2) $F + S \in M \Longleftrightarrow S \in M(F)$,

i.e. $M(F)$ is the set of points S such that $F + S$ is feasible.

(3) DEFINITION. A direction $S \neq 0$ from F will be called a feasible direction at F, for problem (1), if S is such that $F + \lambda S \in M$ whenever $0 < \lambda \leqq \bar{\lambda}$, for some $\bar{\lambda} > 0$. The cone of the feasible directions is defined as the set of all feasible directions from F. ∇

In a feasible direction algorithm, a descent direction is selected from the cone of the feasible directions. Here we shall optimize the largest eigenvalue of $dK(F; S)$, as we have done before in Chapter 2, with S varying over $M(F)$, and shall use the optimizing S as the search direction, if it is descent. The next important lemma gives a framework for the feasible direction algorithm:

(4) LEMMA. Let $F \in M$ and suppose

$$\bar{S} \in \arg\min\{\lambda\max dK(F; S) : S \in M(F)\}$$

$$\bar{\pi}(F) = \lambda\max dK(F; \bar{S})$$

Then

(a) If $\bar{\pi}(F) < 0$, there exists a real $\bar{\lambda} > 0$ so that $K(F + \lambda\bar{S}) < K(F)$, for all $\lambda \in (0, \bar{\lambda})$.

(b) If $\bar{\pi}(F) = 0$ and $\bar{S} = 0$ is the only global minimizer, then there exists a $\delta > 0$ so that $K(F') \nleq K(F)$ for all $F' \in M: \|F - F'\| < \delta$.

∇

The above lemma will be proved in Chapter 5 (see (5.85)), for convenience. The implication of this lemma is that when $\bar{\pi}(F)$ is negative, $\bar{S}$ will be chosen as the search direction, because it ensures a decrease in the cost along it. When $\bar{\pi}(F) = 0$ and $\bar{S} = 0$ is the only minimizer, termination will occur since this means that F is locally dominant. For the case $\bar{\pi}(F)$ equals zero with a nonzero argument $\bar{S}$, as no information can be obtained from the lemma, it will be necessary to search along $\bar{S}$, to determine whether a reduction in the cost is achievable.

The conceptual algorithm for problem (1) is:

(5) FEASIBLE DIRECTION ALGORITHM FOR CONSTRAINED OUTPUT FEEDBACK

1. Select an initial stabilizing matrix F^0 in M; set $j = 0$.

2. Compute

$$\bar{\pi}(F^j) = \min\{\lambda\max dK(F^j; S) : S \in M(F^j)\}$$

and

$$S^j \in \arg\min\{\lambda\max dK(F^j; S) : S \in M(F^j)\}.$$

3. If $\bar{\pi}(F^j) = 0$ and $S^j = 0$ is the only minimizer, set $F^* = F^j$ and stop; else select a nonzero minimizer S^j and continue.

4. Compute the step length λ^j

$$\lambda^j \in \arg\max\{tr[K(F^j)-K(F^j+\lambda S^j))X_0] : \lambda min[K(F^j)-K(F^j+\lambda S^j)] \geqq 0, \lambda \in [0,1]\}.$$

5. If $tr[(K(F^j)-K(F^j+\lambda S^j))X_0] = 0$ stop; else continue.

6. Set $F^{j+1} = F^j + \lambda^j S^j$, $j = j+1$, and go to Step 2. ∇

As we can see this algorithm follows the same pattern as unconstrained algorithm (2.32). However two obvious differences appear: the first concerns the stop condition, which is due to the change of part (b) in Lemma 4. The other is a consequence of the inclusion of constraints. In order to ensure all the points be inside M, we require that S^j belongs to $M(F^j)$ and λ^j to the interval [0, 1]. Then since $M(F^j)$ is convex, all feedbacks of the form $F^j + \lambda S^j$, $\lambda \in [0, 1]$, and therefore F^{j+1}, will belong to M. Fortunately these requirements will not spoil the dominance property of the generated sequence. For, consider Theorem (2.37) characterizing dominance for algorithm (2.32). The assumptions used there are $tr[(K(F^j) - K(F^{j+1}))X_0] > 0$ and $\lambda min[K(F^j) - K(F^{j+1})] \geqq 0$. Since these conditions are imposed by the step length procedure only, which is basically the same for this algorithm, then Theorem (2.37) also applies here, and therefore we restate it:

(6) <u>THEOREM</u>. Let $\{F^j\} \subset M$ be a sequence of feedback matrices generated by Algorithm (5). Then, for all j,

$$K(F^{j+1}) \leq K(F^j). \qquad \nabla$$

It turns out then that convergence of the cost sequence $\{K(F^j)\}$ to some $K^* \geqq 0$ is implied, as was shown in Chapter 2.

Although this algorithm has the same structure as the unconstrained algorithm, the property of convergence proved for that one (namely, $\pi(F^j) \to 0$) does not hold automatically here, with constraints included. Even though there is a dominating feedback created at each iteration, the sequence might jam at some point on the boundary of the feasible set, a phenomenon that may apppear in constrained optimization, and not get any closer to a first order locally-dominant feedback. Thus, convergence must not be assumed for the above algorithm, but must be proved. This is the subject of the next section.

3.2 CLOSEDNESS PROPERTY OF THE ALGORITHM

An important fact arises when we deal with constrained optimization and feasible direction methods. That is, there exists a relationship between convergence to a desirable point and the way the search direction is chosen. Here, as already mentioned, convergence of a sequence to a desirable point means that the sequence has desirable accumulation points (not necessarily, but occasionally, a limit point).

We need to generalize the concept of continuity for point-to-point mappings to continuity for point-to-set mappings. In the theory of algorithms, this continuity property is known as "closedness".

(7) DEFINITION. A point-to-set mapping $A: X \to Y$ is said to be closed at $x \in X$ if the assumptions

(i) $x_k \to x, \quad x_k \in X$

(ii) $y_k \to y, \quad y_k \in A(x_k)$

imply

(iii) $y \in A(x)$.

The point-to-set mapping A is said to be closed on X if it is closed at each point of X. By a closed algorithm we mean an algorithm having x^{j+1} an element of $A(x^j)$, with A closed . ∇

(8) DEFINITION. Let $X \subset \mathbb{R}^p$ be a given feasible set. A set $\Gamma \subset \mathbb{R}^{2p}$ consisting of pairs (x, d), with $x \in X$ and d a feasible direction at x, is said to be a set of uniformly feasible direction vectors if there exists a $\delta > 0$ such that $(x, d) \in \Gamma$ implies that $x + \alpha d$ is feasible for all α, $0 \leqq \alpha \leqq \delta$. The number δ is referred to as the feasibility constant of the set Γ. ∇

The closedness property of algorithms is crucial to establish convergence. Refer to [6 , pages 125, 143], where it is shown that, if a feasible direction algorithm uses a closed feasible direction selection map and generates uniformly feasible directions, convergence is guaranteed, irrespective of how the line search is performed.

In what follows it is shown that Algorithm (5) is closed. Moreover, closedness is proved for the algorithm with an approximate search direction. Then, in view of the above considerations, convergence to a desirable point is achieved, for both cases. Lemma (4) yields the necessary condition of optimality for an F, $\bar{\pi}(F) = 0$. Thus, as before, F will be called desirable if $\bar{\pi}(F) = 0$. Consequently, for any accumulation point F^* of $\{F^j\}$, $\bar{\pi}(F^*) = 0$ (since $\mathcal{M}$ is compact, $F^* \in \mathcal{M}$, and so $\bar{\pi}$ is defined at F^*).

For the sake of simplicity, define the function $\sigma(F,.) : \mathbb{R}^{mxr} \rightarrow \mathbb{R}$ by

$$\sigma(F, S) = \lambda\max\, dK(F; S)$$

Thus, Step 2 of Algorithm (5) requires solving the optimization problem:

(9) SEARCH DIRECTION PROBLEM. Given $F \in M$, solve the problem

$$\min\{\sigma(F, S) : S \in M(F)\}$$

where

$$M(F) = \{Y \in \mathbb{R}^{mxr} : Y = X - F, \quad X \in M\} \qquad \nabla$$

Thus, a search direction is selected from the set of arguments that minimize $\sigma(F, \cdot)$ over $M(F)$.

Closedness of Algorithm (5) is demonstrated in two steps:

First, it should be noted that any sequence $\{(F^j, S^j)\}$ of ordered pairs generated by Algorithm (5), is uniformly feasible. Indeed, since S^j must be an element of $M(F^j)$, which is convex, $F^j + \alpha S^j$ is feasible for all α in $[0, 1]$. The feasibility constant is therefore $\delta = 1$.

Then , the proof of closedness of the search direction map is shown in the theorem:

(10) THEOREM. The point-to-set mapping

$$d: M \to C(M)$$

$$F \mapsto \arg\{\min\sigma(F, S) : S \in M(F)\},$$

is closed, where $C(M)$ is the set of subsets of M.

PROOF. Let F be an arbitrary point in M and $\{F^j\}$ be a sequence of points in M such that $F^j \to F$. Assume that, for $S^j \in d(F^j)$, $S^j \to S$. First it will be shown that $S \in M(F)$. By definition of $M(F)$,

$$M(F) = \{X - F: g^i(X) \leqq 0, \quad i = 1,\ldots,q\}$$

$$= \{Y: g^i(F + Y) \leqq 0 \quad , \quad i = 1,\ldots,q\}$$

Therefore, the constraints that define $M(F)$ are functions of two variables, F and Y, and we shall denote them by

$$g^i(F + Y) = G^i(F, Y).$$

Suppose, for proof by contradiction, that for some constraint G^i,

$$G^i(F, S) > 0$$

and set $\varepsilon = G^i(F, S)$. Then, by the continuity of G^i, there exists a $N > 0$ such that, for $j > N$,

$$|G^i(F^j, S^j) - G^i(F, S)| < \varepsilon$$

and therefore

$$G^i(F^j, S^j) > G^i(F, S) - \varepsilon = G^i(F, S) - G^i(F, S) = 0$$

which is a contradiction. Then, for all i,

$$G^i(F, S) \leqq 0,$$

and so $S \in M(F)$.

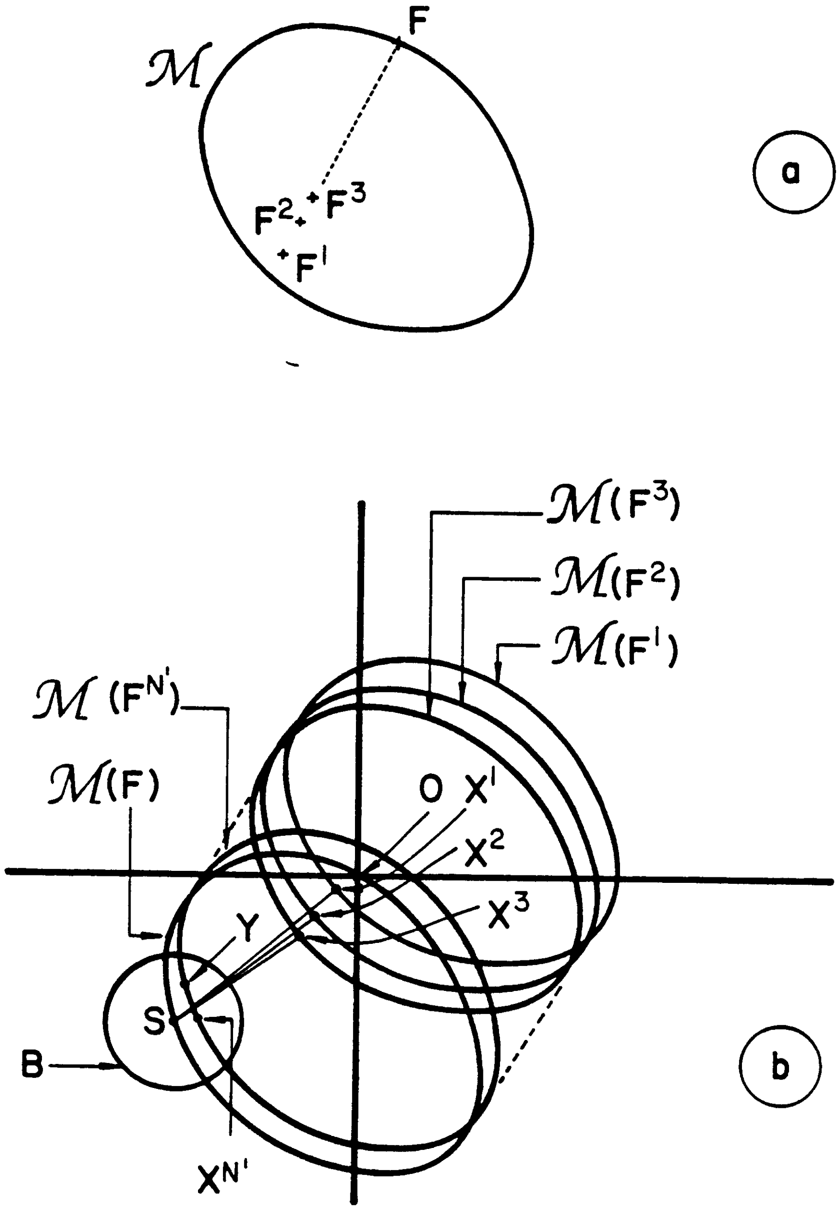
Next we shall prove that there exists a sequence of matrices $\{X^j\}$, with $X^j \in M(F^j)$, convergent to any given $S \in M(F)$. Refer to figure (11) for the geometrical interpretation. Let, for a given $\varepsilon > 0$,

$$B \triangleq \{Y: \|Y - S\| \leqq \varepsilon\}.$$

Because $M(F)$ has a nonempty interior, and $S \in M(F)$, there is intersection between B and $\mathring{M}(F)$. Consider a $Y \in B \cap \mathring{M}(F)$. Then, $G^i(F, Y) < 0, \forall i$, and thus, there exists a $N' > 0$ such that, for $j \geqq N'$,

$$|G^i(F^j, Y) - G^i(F, Y)| \leqq -G^i(F, Y),$$

i.e.



𝓜
F
F^2
F^3
F^1
a
$\mathcal{M}(F^3)$
$\mathcal{M}(F^2)$
$\mathcal{M}(F^1)$
$\mathcal{M}(F^{N'})$
$\mathcal{M}(F)$
O
X^1
X^2
X^3
Y
S
B
$X^{N'}$
b

Figure II.

$$G^i(F^j, Y) \leqq 0,$$

and so $Y \in M(F^j)$. Now, let $\{X^j\}$ be the sequence defined by

$$X^j \in \arg\min\{\| X - S\| : X \in M(F^j)\}$$

(note that convexity of $M(F^j)$ implies uniqueness of X^j). Thus, for $j \geqq N'$, since $Y \in B \cap M(F^j)$,

$$\| X^j - S\| \leqq \| Y - S\| \leqq \varepsilon,$$

i.e. $X^j \in B$. Finally, as ε can be made as small as we want, this shows that $X^j \to S$.

To prove that S minimizes $\sigma(F, S)$ over $M(F)$, suppose that there exists a $S' \in M(F)$ such that

$$\sigma(F, S') < \sigma(F, S),$$

and choose $\varepsilon = \dfrac{\sigma(F, S) - \sigma(F, S')}{2} > 0$.

Recall that σ is continuous in both arguments. Then $\sigma(F^j, S^j) \to \sigma(F, S)$, and so there exists a $N'' > 0$ so that for $j > N''$,

$$|\sigma(F^j, S^j) - \sigma(F, S)| < \varepsilon$$

and consequently,

$$(12) \quad \sigma(F^j, S^j) > \frac{\sigma(F, S) + \sigma(F, S')}{2} .$$

On the other hand, if $\{X^j\}$ is a sequence of vectors $X^j \in M(F^j)$ convergent to S', then $\sigma(F^j, X^j) \to \sigma(F, S')$, and therefore there exists an integer N''' so that for $j > N'''$

$$|\sigma(F^j, X^j) - \sigma(F, S')| < \varepsilon.$$

Thus,

(13) $\sigma(F^j, X^j) < \dfrac{\sigma(F, S) + \sigma(F, S')}{2}$

It follows from (12), (13) that, for $j > \max\{N'', N'''\}$,

$$\sigma(F^j, X^j) < \sigma(F^j, S^j),$$

which is a contradiction by the definition of S^j, and thereby

$$S \in \arg\min\{\sigma(F, S) : S \in M(F)\}.$$

Hence, d is closed. ∇

Figure (14) depicts a unidimensional example where, for a sequence $\{F^j\}$ with $F^j \to F$, $d(F^j) = \arg\min\{\sigma(F^j, S) : S \in M(F^j)\}$ is a singleton, and d(F) is a set. The closedness property of d means that the limit point of $\{S^j\}$ is an element of d(F).

If we modify the algorithm and use for the search direction a computable vector S^j, which does not necessarily minimize $\sigma(F^j, \cdot)$ over $M(F^j)$ exactly, the closedness property may still be preserved. For example, define S^j as the vector which yields $\sigma(F^j, S^j)$ at most a δ-fraction of the minimum of σ over $M(F^j)$, i.e.,

(15) $\sigma(F^j, S^j) \leqq \delta\min\{\sigma(F^j, S) : S \in M(F^j)\}$.

(Figure (16) shows the set $\hat{d}(F^j)$, with elements S^j satisfying (15). The meaning and significance of the set $\Omega(F^j)$ will be seen in Chapter 4.) We have then the following result:

(17) THEOREM. Let $\delta \in (0, 1)$. Then the point-to-set mapping

$$\hat{d}: M \to C(M)$$

$$F \to \{S \in M(F) : \sigma(F, S) \leqq \delta\min\{\sigma(F, X) : X \in M(F)\}$$

is closed.

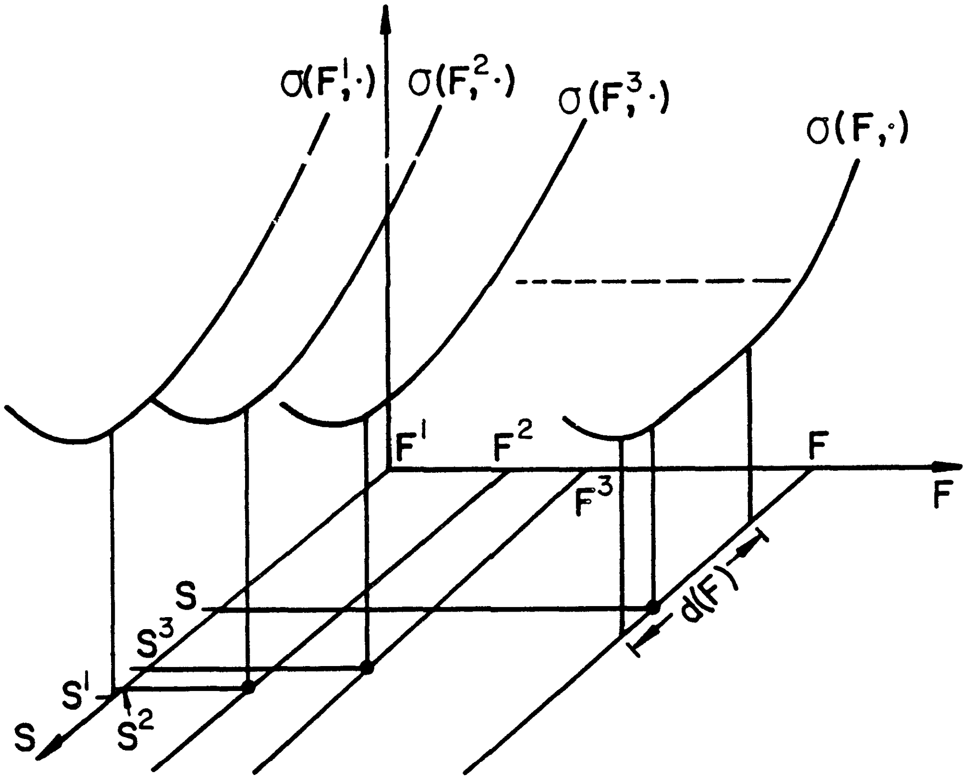


Figure 14.

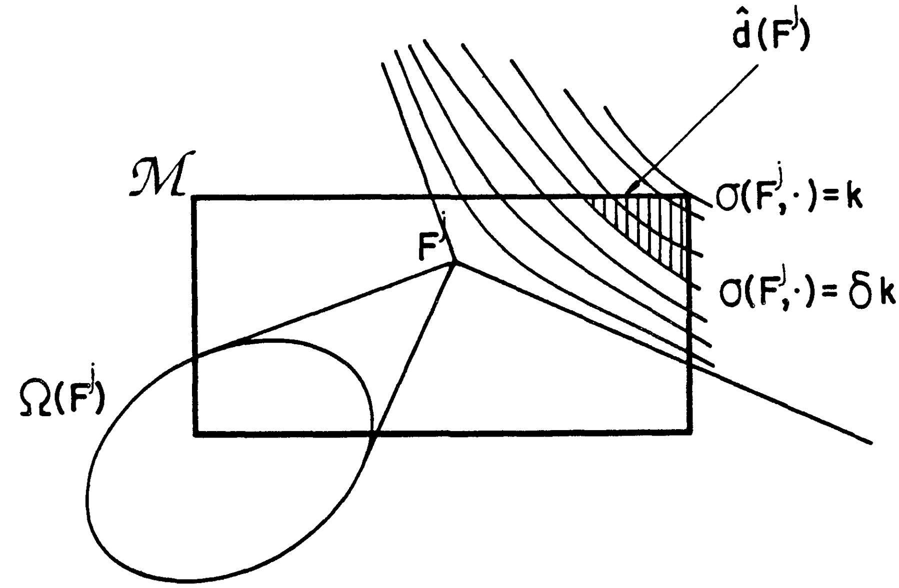


Figure 16.

PROOF. Let F be an arbitrary point in M and $\{F^j\}$ be a sequence of points in M such that $F^j \to F$. Assume that $\{S^j\}$ is a sequence of points $S^j \in \hat{d}(F^j)$, i.e.

$$\sigma(F^j, S^j) \leqq \delta \min\{\sigma(F^j, X) : X \in M(F^j)\}$$

and that

$$S^j \to S.$$

The assumption $S^j \to S$ alone leads to $S \in M(F)$ (as proved in Theorem (10)). In order to show that $S \in \hat{d}(F)$, we suppose, for proof by contradiction, that

$$\sigma(F, S) > \delta \min\{\sigma(F, X) : X \in M(F)\}.$$

It follows from this then that we can define

$$\varepsilon = \frac{\sigma(F, S) - \delta \min\{\sigma(F, X) : X \in M(F)\}}{2} > 0.$$

As before, the continuity of σ will be used. Since $F^j \to F$ and $S^j \to S$, there exists $N > 0$ such that, for $j > N$,

$$|\sigma(F^j, S^j) - \sigma(F, S)| < \varepsilon$$

and so

$$(18) \quad \sigma(F^j, S^j) > \frac{\sigma(F, S) + \delta \min\{\sigma(F, X) : X \in M(F)\}}{2}$$

However we can show that d being closed implies that

$$(19) \quad \min\{\sigma(F^j, X) : X \in M(F^j)\} \to \min\{\sigma(F, X) : X \in M(F)\}.$$

In fact, assume that $Y^j \in d(F^j)$ and $Y^j \to Y$. Then $Y \in d(F)$, i.e.

$$Y \in \arg\min\{\sigma(F, X) : X \in M(F)\}.$$

This, together with the fact that $\sigma(F^j, Y^j) \to \sigma(F, Y)$, proves (19).

Thus, we can guarantee that for $j > N'$, for some $N' > 0$,

$$\left|\min\{\sigma(F^j, X) : X \in M(F^j)\} - \min\{\sigma(F, X) : X \in M(F)\}\right| < \frac{\varepsilon}{\delta}$$

and hence,

$$(20) \quad \delta\min\{\sigma(F^j, X) : X \in M(F^j)\} < \frac{\sigma(F, S) + \delta\min\{\sigma(F, X) : X \in M(F)\}}{2}$$

Hence, by (18) and (20), if $j > \max\{N, N'\}$,

$$\sigma(F^j, S^j) > \delta\min\{\sigma(F^j, X) : X \in M(F^j)\},$$

which contradicts the assumption that $S^j \in \hat{d}(F^j)$, and that proves that $S \in \hat{d}(F)$. ∇

Before finishing this section we need to add a remark concerning the size of the step length interval. Assume that the search direction problem is solved exactly. Then it can be shown that there is always a solution S^j lying in the boundary of $M(F^j)$. In order to see this, consider S^j with $\sigma(F^j, S^j) < 0$ and suppose by contradiction that $S^j \notin \partial M(F^j)$. Then there exists $\lambda > 1$ such that $\lambda S^j \in \partial M(F^j)$, and so

$$\sigma(F^j, \lambda S^j) = \lambda \max dK(F^j; \lambda S^j) = \lambda\lambda \max dK(F^j; S^j) < \lambda \max dK(F^j; S^j) = \sigma(F^j; S^j).$$

This contradicts the assumption that S^j is the minimizer of σ, therefore $S^j \in \partial M(F^j)$. Now consider S^j with $\sigma(F^j, S^j) = 0$. Then, if $S^j \notin \partial M(F^j)$, there exists a λ such that $\lambda S^j \in \partial M(F^j)$, and $\sigma(F^j, \lambda S^j) = 0$. So, λS^j is a minimizer too. Hence, since by definition of S^j, $\sigma(F^j, S^j) \leqq 0$ always occur, we have proved what we wanted. Figure (21) illustrates a situation for the case where M is defined by linear constraints. In some cases S^j may not be unique, for example when the contours of $\sigma(F^j, \cdot)$ are linear. The importance of the fact that S^j can always be selected in the boundary of $M(F^j)$, is that, by doing this, $F^j + \lambda S^j$, when λ ranges over [0, 1], will span the entire feasible segment of

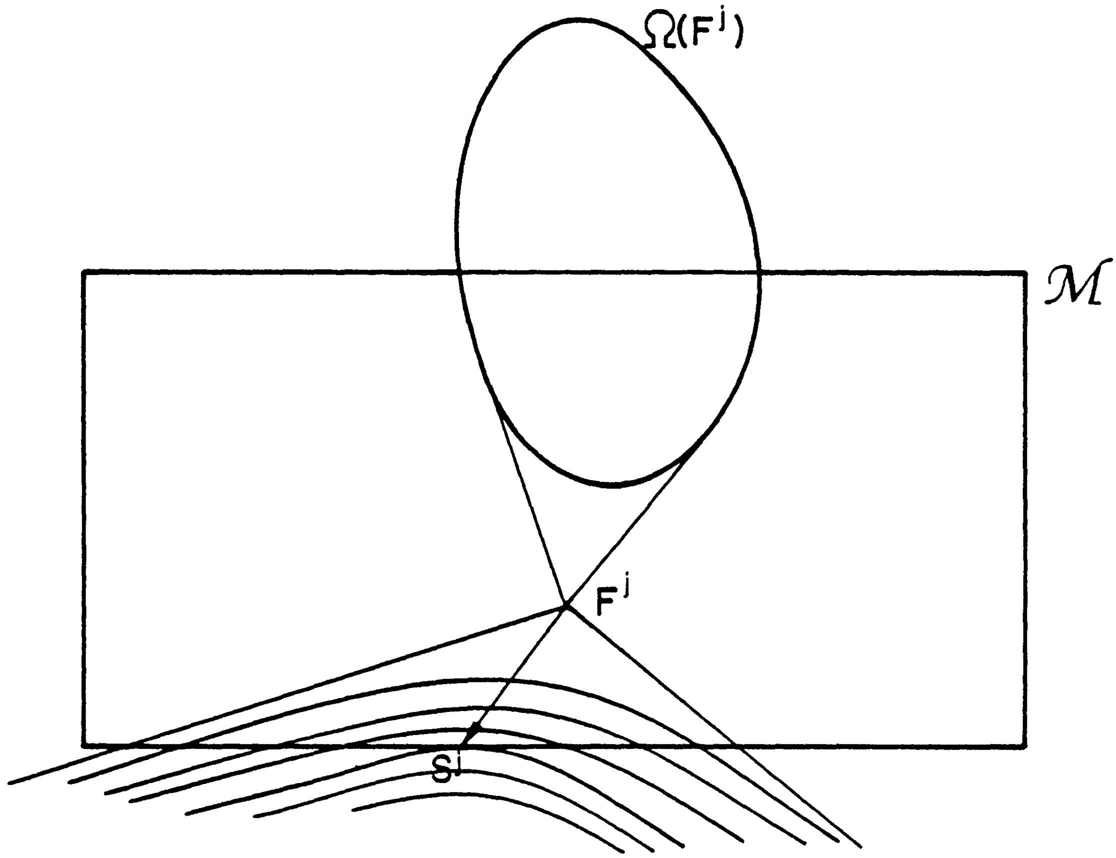


Figure 21.

line along S^j, and so the step length interval becomes simply [0, 1].

3.3 A COMPUTABLE STEP LENGTH

In this section an implementation for the line search of Algorithm (5) (Step 4) is given, for the case when $\sigma(F^j, S^j) < 0$. An expression for the step length will be obtained by considering a lower bound quadratic approximation for σ along the search direction, the idea we have introduced in Chapter 2.

Let $F = F^j$ and $S \in \hat{d}(F)$. The exact step length solves

(22) $\max\{tr[(K(F) - K(F+\lambda S))X_0]: \lambda\min[K(F) - K(F+\lambda S)] \geqq 0,\ \lambda \in [0, 1]\}$.

An implementation will be defined by a $\hat{\lambda}$ such that

$$\phi(\hat{\lambda}) = tr[(K(F) - K(F + \hat{\lambda}S))X_0] > 0$$

and the two constraints of (22) are satisfied. An implementable step length is expressed in the following theorem.

(23) THEOREM. Let F and S generated in Alg.(5), with $\sigma(F,S)<0$. Then, for

$$\hat{\lambda} \triangleq \begin{cases} \dfrac{-2\sigma(F, S)}{r^3\gamma_2\varepsilon}, & \text{if } \dfrac{-2\sigma(F, S)}{r^3\gamma_2\varepsilon} < 1 \\ 1, & \text{otherwise} \end{cases}$$

where $\varepsilon > 2$ is chosen so as to satisfy

$$\frac{\varepsilon(\varepsilon - 2)}{\varepsilon - 1} \geqq \frac{12ad\gamma_1}{\gamma_2},$$

d is the diameter of M, and a, γ_1 and γ_2 are given in (2.63) - (2.65),

$$tr[(K(F) - K(F + \hat{\lambda}S))X_0] > 0$$

$$\lambda\min[K(F) - K(F + \hat{\lambda}S)] > 0$$

$$\hat{\lambda} \in [0, 1],$$

and so $\hat{\lambda}$ is a computable step length for Algorithm (5).

PROOF. Assume that M has diameter d. Then $\dot{M}(F)$, because it is a displacement of M, has also diameter d, i.e., for $S \in M(F)$, $\|S\| \leqq d$. Some of the results of Section 2.4 will be used here, and we devote the next paragraph to recall them (there d=1).

Given an $\varepsilon > 2$, it is proved in (2.79) - (2.80) that

$$\| d^2K(F + \lambda S; S, S)\| \leqq \frac{r^3\gamma_2\varepsilon}{2}$$

for all $\lambda \in [0, \delta(\varepsilon)]$, where

(24) $$\delta(\varepsilon) = \frac{\varepsilon - 2}{3\,adr^{3/2}(\varepsilon - 1)}.$$

Consequently, as stated in (2.87),

$$\phi(\lambda) = tr[(K(F) - K(F + \lambda S))X_0]$$

(25) $$\geqq -n\lambda \min X_0 \ \sigma(F, S)\lambda - \frac{1}{4} n\lambda \min X_0 r^3\gamma_2\varepsilon\lambda^2$$

$$\triangleq Q^{\varepsilon}(\lambda).$$

This means that, for each $\varepsilon > 2$, there exists a quadratic Q^{ε} (function of ε) that is a lower bound approximation for ϕ on the nonempty interval $[0, \delta(\varepsilon)]$. The positive root of Q^{ε} is

(26) $$\rho_{\varepsilon} = - \frac{4\sigma(F, S)}{r^3\gamma_2\varepsilon}$$

It can be easily verified that, as ε increases, $\delta(\varepsilon)$ increases too, whereas ρ_{ε} gets smaller. So, it might be expected that, for sufficiently large ε, the condition

(27) $$\rho_{\varepsilon} \leqq \delta(\varepsilon)$$

holds. This is actually true, as can be seen in the following. First, substitute the expressions for $\delta(\varepsilon)$ and ρ_ε, from (24) and (26), into (27). This gives

$$(28) \quad \frac{\varepsilon(\varepsilon - 2)}{\varepsilon - 1} \geqq - \frac{12ad\sigma(F, S)}{r^{3/2}\gamma_2}$$

Then, consider the fact, derived from Lemma (2.45),

$$- \sigma(F, S) = - \lambda\max dK(F; S) = \| dK(F; S) \| \leqq r^{3/2}\gamma_1 .$$

This reveals that, if ε is chosen so as to satisfy

$$(29) \quad \frac{\varepsilon(\varepsilon - 2)}{\varepsilon - 1} \geqq - \frac{12ad\gamma_1}{\gamma_2} ,$$

Then (28) is valid. It has already been remarked (see (2.93)) that $\varepsilon > 2$ satisfying (29) can always be found, therefore condition (27) can be achieved for some $\varepsilon > 2$. Figure (30) illustrates an example showing two curves determined by numbers ε and ε', with $\varepsilon > \varepsilon'$, and condition (27) holding for Q^ε. Now, consider an $\varepsilon > 2$ satisfying (27). Since the quadratic form Q^ε is positive over $(0, \rho_\varepsilon)$, fact (25) ensures that ϕ is positive over $(0, \rho_\varepsilon)$ as well. Furthermore, (27) guarantees that $g(\lambda) = \lambda\min[K(F) - K(F + \lambda S)] > 0$ for $\lambda \in (0, \rho_\varepsilon)$. In summary, for such an ε and all λ in $(0, \rho_\varepsilon)$,

$$(31) \quad \begin{cases} \phi(\lambda) > 0 \\ g(\lambda) > 0 \end{cases}$$

Let the number $\hat{\lambda}$ be defined as the point at which Q^ε attains its maximum on [0, 1]. The unconstrained maximum of Q^ε is achieved for $\lambda = \rho_\varepsilon/2$, therefore

$$\hat{\lambda} = \begin{cases} \rho_\varepsilon/2, & \text{if } \rho_\varepsilon < 2 \\ 1, & \text{otherwise.} \end{cases}$$

Finally, as $\hat{\lambda}$ satisfies (31) and belongs to the interval [0, 1], and

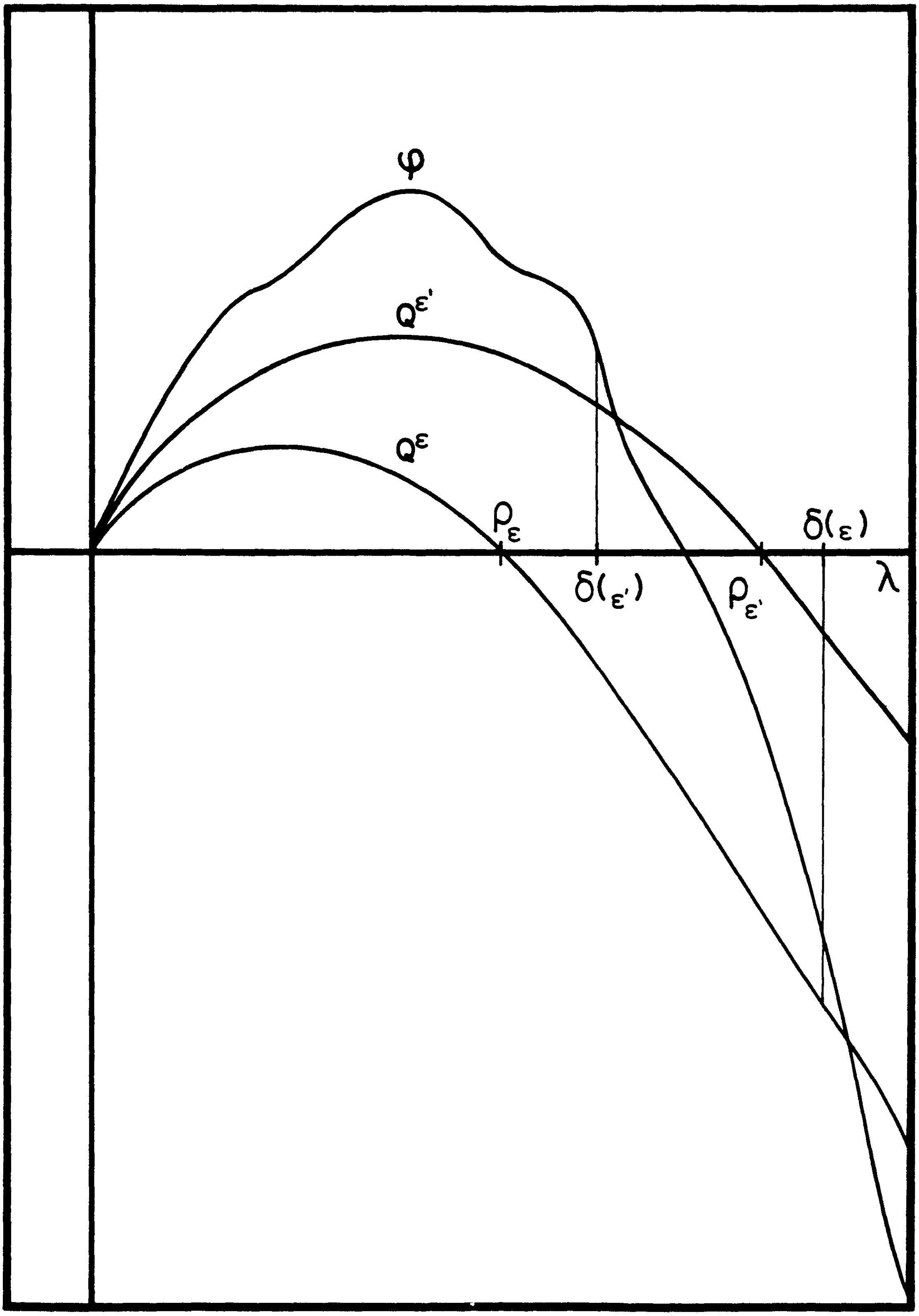

φ
$Q^{\varepsilon'}$
Q^{ε}
ρ_ε
$\delta(\varepsilon')$
$\rho_{\varepsilon'}$
$\delta(\varepsilon)$
λ

Figure 30.

expressions (2.63) - (2.65) accounts for the computability of the parameters a, γ_1 and γ_2, $\hat{\lambda}$ is a computable step length for algorithm (5). ∇

3.4 AN IMPLEMENTABLE ALGORITHM

This is the implementable version of Algorithm (5), using the search direction S^j and line search $\lambda^j = \hat{\lambda}$ presented in Sections 3.2 and 3.3 repectively, and where we assume that, for all j, $\sigma(F^j, S^j) < 0$. We assume also that a search direction satisfying (15) can be determined numerically, and thus we refer to such a search direction as being computable.

(32) AN IMPLEMENTABLE ALGORITHM FOR CONSTRAINED OUTPUT FEEDBACK.

1. Select an initial $F^0 \in M$; select δ and δ' in (0, 1); select a number $\varepsilon > 2$ satisfying

$$\frac{\varepsilon(\varepsilon - 2)}{\varepsilon - 1} \geqq \frac{12ad\gamma_1}{\gamma_2}$$

where a, γ_1 and γ_2 are given in (2.63) - (2.65), and d is the diameter of M; set j = 0.

2. Compute an S^j such that

$$\sigma(F^j, S^j) \leqq \delta\min\{\sigma(F^j, S) : S \in M(F^j)\}$$

where

$$\sigma(F, S) \triangleq \lambda\max\, dK(F; S)$$

3. If $-\sigma(F^j, S^j) < \delta'$ then set $F^* = F^j$ and stop; else continue.

4. Compute the step length λ^j using

$$\lambda^j = \begin{cases} -\dfrac{2\sigma(F^j, S^j)}{r^3\gamma_2\varepsilon}, & \text{if } -2\sigma(F^j, S^j) < r^3\gamma_2\varepsilon \\ 1, & \text{otherwise} \end{cases}$$

5. Set $F^{j+1} = F^j + \lambda^j S^j$, $j = j+1$, and go to Step 2. ∇

Convergence for the abstract algorithm (5) ($\bar{\pi}(F^*) = 0$) was proved in Section 2. Convergence for the implementable version above will be proved in the next section, and that will justify the stop condition encountered in Step 3.

3.5 CONVERGENCE PROOF FOR THE IMPLEMENTABLE ALGORITHM

In Section 2 we have demonstrated, using the theory of closedness of algorithms, that if the conceptual algorithm (5) generates an infinite sequence of output feedbacks $\{F^j\}$ such that $\bar{\pi}(F^j) < 0$, then any accumulation point F^* of the sequence satisfies $\bar{\pi}(F^*) = 0$. In this section we shall be concerned with proving the same fact for the implementable algorithm (32), using a direct approach.

(33) THEOREM. Assume that $\{F^j\}$ is an infinite sequence generated by implementable algorithm (32), with the initial feedback $F^0 \in M$, and elements F^j satisfying $\bar{\pi}(F^j) < 0$. Then, for any accumulation point $F^* \in M$ of $\{F^j\}$,

$$\bar{\pi}(F^*) = 0$$

PROOF. It follows from Theorem (6) which applies also for Algorithm (32), and the fact that $K(F^j) \geqq 0$ for all j, that, as $j \to \infty$,

(34) $$\phi(\lambda^j) = tr[(K(F^j) - K(F^j + \lambda^j S^j))X_0] \to 0$$

(see Theorem (2.96)). Also, for the step length λ^j of Theorem (23), for all iterations j,

$$\phi(\lambda^j) \geqq Q^\varepsilon(\lambda^j)$$

and thus, using (25),

$$\phi(\lambda^j) \geqq Q^{\varepsilon}(\lambda^j) = \begin{cases} \dfrac{n\lambda\min X_0\, \sigma(F^j, S^j)^2}{r^3\gamma_2\varepsilon}, \text{ if } -\dfrac{2\sigma(F^j, S^j)}{r^3\gamma_2\varepsilon} < 1 & (35a) \\ -n\lambda\min X_0\, (\sigma(F^j, S^j) + \dfrac{r^3\gamma_2\varepsilon}{4}) \text{ otherwise} & (35b) \end{cases}$$

However it is possible to show that situation (35b) does not occur for very large j, $-\dfrac{2\sigma(F^j, S^j)}{r^3\gamma_2\varepsilon} \geqq 1$ is true only for a finite number of iterations. In fact, suppose by contradiction that $-\dfrac{2\sigma(F^j, S^j)}{r^3\lambda_2\varepsilon} \geqq 1$ for an infinite number of iterations. Then, it follows from (34) and (35b) that

$$\sigma(F^j, S^j) + \frac{r^3\gamma_2\varepsilon}{4} \to 0,$$

as $j \to \infty$, and therefore

$$\sigma(F^j, S^j) \to -\frac{r^3\gamma_2\varepsilon}{4} .$$

Hence,

$$-\frac{2\sigma(F^j, S^j)}{r^3\gamma_2\varepsilon} \to \frac{1}{2} .$$

Consequently, for a given $\delta \in (0, 1/2)$, there exists a position N such that, for all $j > N$,

$$\left| -\frac{2\sigma(F^j, S^j)}{r^3\gamma_2\varepsilon} - \frac{1}{2} \right| < \delta,$$

and thus,

$$-\frac{2\sigma(F^j, S^j)}{r^3\gamma_2\varepsilon} < 1,$$

contradicting the assumption. So, from some point of the sequence onwards, case (35a) always applies, and then (34) implies that

$\sigma(F^j, S^j) \to 0$. From this fact however, and since $\sigma(F^j, S^j) \leqq \delta\, \bar{\pi}(F^j) < 0$, it follows that $\bar{\pi}(F^j) \to 0$. Compactness of M implies that any accumulation point of $\{F^j\}$ lies in M, hence $\bar{\pi}(F^*) = 0$. ∇

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CHAPTER 4

DOMINANT FEEDBACK WITH LINEAR EQUALITY CONSTRAINTS

4. DOMINANT FEEDBACK WITH LINEAR EQUALITY CONSTRAINTS

In Section 1 we shall demonstrate that the search direction problem for the constrained feedback algorithm, namely, the minimization of the largest eigenvalue of dK(F; S), with S ranging over the unit sphere $\mathcal{S}$, is equivalent to solving a minimum norm problem involving a set defined by the partial derivatives of K(F). This development is based on Allwright's work in [2] and [1]. The subject of Section 2 is the definition and analysis of a dominant output feedback problem, subject to a set of linear equality constraints. We shall prove that a search direction for the feasible direction algorithm associated with the optimization problem can also be obtained by means of a minimum norm problem solution, as in the unconstrained case, using projections onto the subspace defined by the constraints.

Throughout this chapter we shall employ the vec notation for the matrices of $\mathbb{R}^{mxr}$ and transform them into vectors of $\mathbb{R}^p$, p = mr. As before, we shall denote f = vec F, and by $f \in \mathcal{F}$ we mean f = vec F, where $F \in \mathcal{F}$.

4.1 MINIMIZATION OF λmax dK(f; s) OVER THE UNIT FROBENIUS-SPHERE

We start by presenting the following well-known concept of convex analysis:

(1) DEFINITION. Let Ω be a convex set in $\mathbb{R}^p$, the support function of Ω is defined to be the function

$$\sigma^{\Omega} : \mathbb{R}^p \to \bar{\mathbb{R}}$$

$$s \mapsto \sup\{<s, \omega> : \omega \in \Omega\} \qquad \nabla$$

In general, $\sigma^{\Omega}(s)$ may be infinite. If Ω is compact, $\sigma^{\Omega}(s) = \max\{<s, \omega>: \omega \in \Omega\}$ is finite. The support function is convex (see, for example ['C], page 36).

The problem we shall be concerned with here is:

(2) SEARCH DIRECTION PROBLEM

minimize$\{\lambda\max dK(f; s): s \in S\}$

where

$$S = \{s \in \mathbb{R}^p : \|s\|_F = 1\} \qquad \nabla$$

The reformulation of the above problem needs the definition of the following abstract set:

$$(3) \quad \Omega(f) \triangleq \text{co} \left\{ \omega \in \mathbb{R}^p : \omega_i = x' \frac{\partial K(f)}{\partial f_i} x,\ x \in Bn \right\}$$

where Bn is the unit Frobenius-sphere in $\mathbb{R}^n$, and co{X} denotes the convex hull of X (i.e., the set consisting of all the convex combinations of the elements of X).

Then, the definition of the largest eigenvalue of a matrix, and definition (2.1) of dK(f; s), allow us to write:

(4) $\lambda\max dK(f; s)$

$$\triangleq \max\{x' dK(f; s) x : x \in Bn\}$$

$$= \max\{x' (\textstyle\sum s_i \frac{\partial K(f)}{\partial f_i}) x : x \in Bn\}$$

$$= \max\{\textstyle\sum s_i (x' \frac{\partial K(f)}{\partial f_i} x) : x \in Bn\}$$

$$= \max\{\langle s, \omega \rangle : \omega \in \Omega(f)\}$$

$$\triangleq \sigma^{\Omega(f)}(s) \ .$$

Throughout most of this thesis the support function will be referred to the set $\Omega(f)$, with the only exception occurring in the following section. Therefore, for the sake of simplicity, we shall omit the reference to $\Omega(f)$ (except in Section 4.2, obviously). The variable f to which $\Omega(f)$ refers, however, sometimes needs to be specified. When this is the case, $\sigma(f, s)$ will be used for $\sigma^{\Omega(f)}(s)$. This is consistent with the notation employed in previous chapters, and we may summarize it as follows:

$$\lambda\max dK(f; s) \equiv \sigma^{\Omega(f)}(s) \equiv \sigma(f, s) \equiv \sigma(s)$$

It should be noted that a maximizing x in Bn for $x'dK(f; s)x$ is an eigenvector of $dK(f; s)$ associated with its largest eigenvalue. On the other hand, observe that a maximizing ω for $\langle s, \omega\rangle$ in $\Omega(f)$ is a contact point between $\Omega(f)$ and its supporting hyperplane with outward normal s. Hence, result (4) implies that, corresponding to any normalized eigenvector x associated with $\lambda\max dK(f; s)$, $\omega = (\omega_i) = \left(x' \frac{\partial K(f)}{\partial f_i} x \right)$ is a contact point

Thus, an equivalent formulation for problem (2) is:

(5) <u>SEARCH DIRECTION PROBLEM</u>.

$$\text{minimize}\{\sigma(s) : s \in S\}$$

where σ is the support function to the set $\Omega(f)$ given in (3). ∇

Obviously a solution for the above problem exists, since σ is convex and S is compact. Allwright has pointed out a procedure for solving it partially, which we shall describe next. Let a global minimizer be called $\hat{s}$. The approach is based on the following three theorems [1]:

(6) <u>THEOREM</u>. Consider a set $\Omega \subset \mathbb{R}^p$ and let σ be the support function of Ω. Then, if $0 \notin \Omega$:

(i) $\arg\min\{\|\omega\| : \omega \in \Omega\}$ is a singleton and, for $\hat{\omega} \in \arg\min\{\|\omega\| : \omega \in \Omega\}$,

$\min\{\sigma(s) : s \in S\} = -\|\hat{\omega}\| < 0$

(ii) $\arg\min\{\sigma(s) : s \in S\} = \{\hat{s}\}$, for $\hat{s} = -\hat{\omega}/\|\hat{\omega}\|$ ∇

(7) THEOREM. For Ω and σ as defined in Theorem (6), if $0 \in \partial\Omega$:

(i) $\min\{\sigma(s) : s \in S\} = 0$

(ii) $\arg\min\{\sigma(s) : s \in S\} = \{s \in S$: s is an outward normal to a hyperplane supporting Ω at $0\}$ ∇

(8) THEOREM. For Ω and σ as defined in Theorem (6), if $0 \in \mathring{\Omega}$:

(i) For $\hat{\omega} \in \arg\min\{\|\omega\| : \omega \in \partial\Omega\}$,

$$\min\{\sigma(s) : s \in S\} = \|\hat{\omega}\| > 0$$

(ii) $\arg\min\{\sigma(s) : s \in S\} = \{\hat{s}\}$, for $\hat{s} = \hat{\omega}/\|\hat{\omega}\|$ ∇

Let, for the support function associated with $\Omega(f)$,

$$\pi = \min\{\sigma(s) : s \in S\}.$$

Then $\pi = \pi(f)$, for $\pi(f)$ defined in previous chapters, since

$$\pi(f) \triangleq \lambda\max dK(f; \hat{s}) = \sigma(\hat{s}) \triangleq \pi.$$

Part (i) of all the above theorems suggest the methods for determining the sign of π, which are to investigate whether the origin is outside, in the boundary or in the interior of $\Omega(f)$, i.e., they give the following "if and only if" criteria:

(9a) $0 \notin \Omega(f) \iff \pi < 0$

(9b) $0 \in \partial\Omega(f) \iff \pi = 0$

(9c) $0 \in \mathring{\Omega}(f) \iff \pi > 0$

Part (ii) is concerned about finding the solution $\hat{s}$. Since there is no need to actually find a solution to (5) when $\pi > 0$, we only should have to seek methods for

(i) determining the minimum norm point of $\Omega(f)$ when $0 \notin \Omega(f)$.

(ii) determining the outward normal to the supporting hyperplane to $\Omega(f)$ at the minimum norm point when $0 \in \partial\Omega(f)$.

Unfortunately the latter seems to be a difficult task to do and we shall not be concerned with it here. For the minimum norm problem, when $0 \notin \Omega(f)$, several algorithms are available. Allwright proposes the use of one due to Y.C. Ho [4], which is of practical implementation for the feedback problem. Others can be found in [3], [5], [6], [7], [8] and [9], for example. Since either $0 \notin \Omega(f)$ or $0 \in \Omega(f)$ holds, then, for the solution of the minimum norm problem, either $\hat{s} \neq 0$ or $\hat{s} = 0$. Consequently, by solving the minimum norm problem, information is obtained that permits us to decide whether $\pi < 0$ or $\pi \geq 0$, and to find the search direction when $\pi < 0$. Fortunately, this is all that is required for an implementation of Algorithm (2.31), as far as the search direction is concerned. For the implementation of Ho's Algorithm, the particular thing needed is the evaluation of a contact point of the supporting hyperplane to $\Omega(f)$, normal to a given vector s. This can be done nicely, as has been discussed, provided an eigenvector associated with dK(f; s) is computed. In Section 5.7 we shall describe this algorithm, which will be proved to be a particular case of an algorithm due to Allwright.

4.2 THE EQUALITY CONSTRAINED DOMINANT FEEDBACK PROBLEM

The problem we shall be concerned with here is:

(10) EQUALITY CONSTRAINED PROBLEM. Given an initial condition $x_0 \in \mathbb{R}^n$ for the system (1.1-3)

$$\text{minimize}\{V(x_0,F): Tf=d,\ f \in \mathcal{F}\} \text{ in the dominance sense}$$

where $p = mr$, $\ell < p$, $f \in \mathbb{R}^p$, $d \in \mathbb{R}^\ell$ and $T \in \mathbb{R}^{\ell x p}$ and is of full rank. Besides, an initial feedback $f^0 \in \mathcal{F}$ satisfying $Tf^0 = d$ must be provided. ∇

Consider a point f in the feasible set for (10), the affine set of dimension $\ell < p$,

$$M \triangleq \{f \in \mathcal{F}: Tf = d\}.$$

A feasible direction from f must satisfy $T(f + \lambda s) = d$ for small $\lambda > 0$, therefore the vector s must belong to the subspace of $\mathbb{R}^p$

$$L = \{x \in \mathbb{R}^p : Tx = 0\},$$

and so, not only for some $\lambda > 0$, but for all $\lambda \in \mathbb{R}$, $f + \lambda s \in M$. Consequently, the search direction for a feasible direction algorithm for (10) should be chosen as the solution of

(11) SEARCH DIRECTION PROBLEM

$$\text{minimize}\{\lambda\max dK(f;\ s): s \in S \cap L\}$$

where

$$L = \{x \in \mathbb{R}^p : Tx = 0\}$$ ∇

if the minimum is nonpositive and allows the cost to decrease along it. In fact, we shall demonstrate that Lemma (2.34) is valid when the constraint $s \in S$ is generalized to $s \in S \cap \mathcal{D}$, where $\mathcal{D}$ is a cone (Lemma (5.74)).

Owing to the structure of the feasible set, we shall be able to transform the above constrained optimization problem into a simpler problem, constrained only to a unit sphere, much like problem (2). By using (4) we can restate (11) as follows:

(12) SEARCH DIRECTION PROBLEM.

$$\text{minimize}\{\max\{\langle s, \omega\rangle : \omega \in \Omega(f)\} : s \in S \cap L\}$$

for $\Omega(f)$ of (3). ∇

We shall prove some general results concerning an arbitrary convex set Ω and an ℓ-dimensional subspace P of $\mathbb{R}^p$, with $\ell < p$.

(13) NOTATION. For any point $\omega \in \mathbb{R}^p$ and subspace P of $\mathbb{R}^p$, we denote the orthogonal projection ω^+ of ω onto P by $\pi_P(\omega)$. For a set $\Omega \subset \mathbb{R}^p$,

$$\pi_P(\Omega) \triangleq \{\omega^+ = \pi_P(\omega) : \omega \in \Omega\} .$$ ∇

(14) LEMMA.

Let $s \in \mathbb{R}^p$ fixed, and let P be a subspace of $\mathbb{R}^p$ with dimension $\ell < p$ containing s. Then, if $\Omega \subset \mathbb{R}^p$,

$$\langle s, \omega\rangle = \langle s, \omega^+\rangle$$

for any $\omega \in \Omega$, where $\omega^+ = \pi_P(\omega)$.

PROOF. Let n be a vector normal to P, of unit norm. Note that the orthogonal projection of ω onto P, $\omega^+ = \pi_P(\omega)$, satisfies

$$\omega^+ = \omega - \langle\omega, n\rangle n ,$$

and therefore, taking the inner product with s,

$$\langle s, \omega^+\rangle = \langle s ,\omega\rangle - \langle\omega, n\rangle\langle s, n\rangle.$$

However,.since $s \in \mathcal{P}$, $\langle s, n\rangle = 0$. Thus, the result is proved. ∇

This lemma gives rise to the following corollary:

(15) COROLLARY. Let $s \in \mathbb{R}^p$ and $\mathcal{P} \subset \mathbb{R}^p$ a subspace of dimension $\ell < p$ containing s. Then, if $\Omega \subset \mathbb{R}^p$ is compact and $\Omega^+ = \pi_{\mathcal{P}}(\Omega)$,

$$\max\{\langle s, \omega\rangle: \omega \in \Omega\} = \max\{\langle s, \omega^+\rangle: \omega^+ \in \Omega^+\},$$

i.e.,

$$\sigma^{\Omega}(s) = \sigma^{\Omega^+}(s),$$

and, for a maximizer

$$\hat{\omega} \in \arg\max\{\langle s, \omega\rangle: \omega \in \Omega\},$$

the projected point $\hat{\omega}^+ = \pi_{\mathcal{P}}(\hat{\omega})$ satisfies

$$\hat{\omega}^+ \in \arg\max\{\langle s, \omega^+\rangle: \omega^+ \in \Omega^+\}.$$

PROOF. Let $\hat{\omega} \in \arg\max\{\langle s, \omega\rangle: \omega \in \Omega\}$. Then, by Lemma (14),

(16) $\langle s, \hat{\omega}\rangle = \langle s, \hat{\omega}^+\rangle,$

where $\hat{\omega}^+ = \pi_{\mathcal{P}}(\hat{\omega})$. The result will be proved if we show that

$$\langle s, \hat{\omega}^+\rangle = \max\{\langle s, \omega^+\rangle: \omega^+ \in \Omega^+\}.$$

In fact, this is true. For, suppose that there exists a $\bar{\omega}^+ \in \Omega^+$ satisfying

$$<s, \bar{\omega}^+> \; > \; <s, \hat{\omega}^+>.$$

Then, using the above and (16),

(17) $<s, \bar{\omega}^+> \; > \; <s, \hat{\omega}>$

But, by the definition of Ω^+, $\bar{\omega}^+ = \pi_P(\bar{\omega})$ for some $\bar{\omega} \in \Omega$, and the following relationship holds (using Lemma (14) once more):

$$<s, \bar{\omega}^+> = <s, \bar{\omega}>.$$

Therefore, from this and (17)

$$<s, \bar{\omega}> \; > \; <s, \hat{\omega}>,$$

which contradicts the assumption that $\hat{\omega}$ minimizes $<s, \omega>$ over Ω. ∇

Finally we can prove:

(18) <u>THEOREM</u>. Let $P \subset \mathbb{R}^p$ be a subspace of dimension $\ell < p$ and let $\Omega \subset \mathbb{R}$ be compact. Then $\hat{s}$ is the solution vector to the problem

(19) $\min\{\sigma^{\Omega}(s) : s \in S \cap P\}.$

iff it solves

(20) $\min\{\sigma^{\Omega^+}(s) : s \in S \cap P\}$

for $\Omega^+ = \pi_P(\Omega)$.

<u>PROOF</u>. (⇒) Since $\hat{s}$ solves (19) it must belong to P. Therefore it follows from Corollary (15) that

(21) $\sigma^{\Omega}(\hat{s}) = \sigma^{\Omega^+}(\hat{s})$.

Suppose there is a vector $\hat{s}^+ \in S \cap P$ such that

(22) $\sigma^{\Omega^+}(\hat{s}^+) < \sigma^{\Omega^+}(\hat{s})$.

Then Corollary (15) says that

(23) $\sigma^{\Omega^+}(\hat{s}^+) = \sigma^{\Omega}(\hat{s}^+)$.

Hence, combining (21), (22) and (23), we have

$$\sigma^{\Omega}(\hat{s}^+) < \sigma^{\Omega}(\hat{s}),$$

which contradicts the assumption that $\hat{s}$ minimizes (19). This proves that (22) is false and then, $\hat{s}$ minimizes σ^{Ω^+} over $S \cap P$, i.e., we have proved what we wanted. (This proof is completed on page 102) ∇

Now, setting $\Omega = \Omega(f)$ and $P = L$ in Theorem (18), Problem (19) becomes the search direction problem (12), and problem (20), the desired transformed search direction problem, which we restate below:

(24) <u>SEARCH DIRECTION PROBLEM</u>

$$\text{minimize}\{\max\{\langle s, \omega^+\rangle : \omega^+ \in \Omega^+(f)\} : s \in S_\ell\}$$

where

$$S_\ell = \{x \in L : \|x\|_F = 1\}$$ ∇

Formulation (24) reveals an optimization problem in $\mathbb{R}^\ell$, similar to the optimization problem corresponding to the unconstrained dominant feedback problem described in Section 1,

$$\text{minimize}\{\sigma^{\Omega(f)}(s) : s \in S\},$$

and thus, the theory developed for that can be applied to this situation.

Let

$$\pi^+ = \min\{\sigma^{\Omega^+(f)}(s) : s \in S_\ell\} = \min\{\sigma^{\Omega(f)}(s) : s \in S_\ell\}$$

Then (9) reveals that

$$0 \notin \Omega^+(f) \Longleftrightarrow \pi^+ < 0$$

$$0 \in \partial\Omega^+(f) \Longleftrightarrow \pi^+ = 0$$

$$0 \in \mathring{\Omega}^+(f) \Longleftrightarrow \pi^+ > 0.$$

Diagram (25) illustrates examples of two situations: $0 \notin \Omega^+(f)$, and the corresponding negative solution $\hat{s}$ of (24) (i.e., $\pi^+ = \sigma^{\Omega^+(f)}(\hat{s}) < 0$), and $0 \in \mathring{\Omega}^+(f)$. The shadowed region indicates the polar cone of $\Omega(f)$, which is the cone consisting of the vectors $s \in \mathbb{R}^2$ such that $\overset{\Omega}{\sigma}(s) < 0$, i.e., the descent directions at f. In the second example there is no intersection between this cone and the subspace L, as expected, since $0 \in \mathring{\Omega}^+(f)$ implies that $\pi^+ > 0$, and therefore absence of descent directions at f belonging to L.

Much as in Section 1, solution of (24) using Ho's algorithm reveals that either $0 \notin \Omega^+(f)$ or $0 \in \Omega^+(f)$, i.e., that either $\pi^+ < 0$ or $\pi^+ \geqq 0$ holds, and, in the case $\pi^+ < 0$, it computes the required solution.

The implementation of the algorithm needs the evaluation of the contact point of a supporting hyperplane to $\Omega^+(f)$, with normal $s \in S_\ell$. However, it can be easily proved, using Corollary (15), that a supporting hyperplane to $\Omega(f)$ supports $\Omega^+(f)$ as well, and any contact point to $\Omega^+(f)$ is the projection of a contact point to $\Omega(f)$. Since this is the vector $\hat{\omega}$ with components

$$\hat{\omega}_i = x' \frac{\partial K(f)}{\partial f_i} x,$$

for some normalized eigenvector x of $dK(f; s)$ associated with its largest eigenvalue, the contact point to $\Omega^+(f)$ is computed by

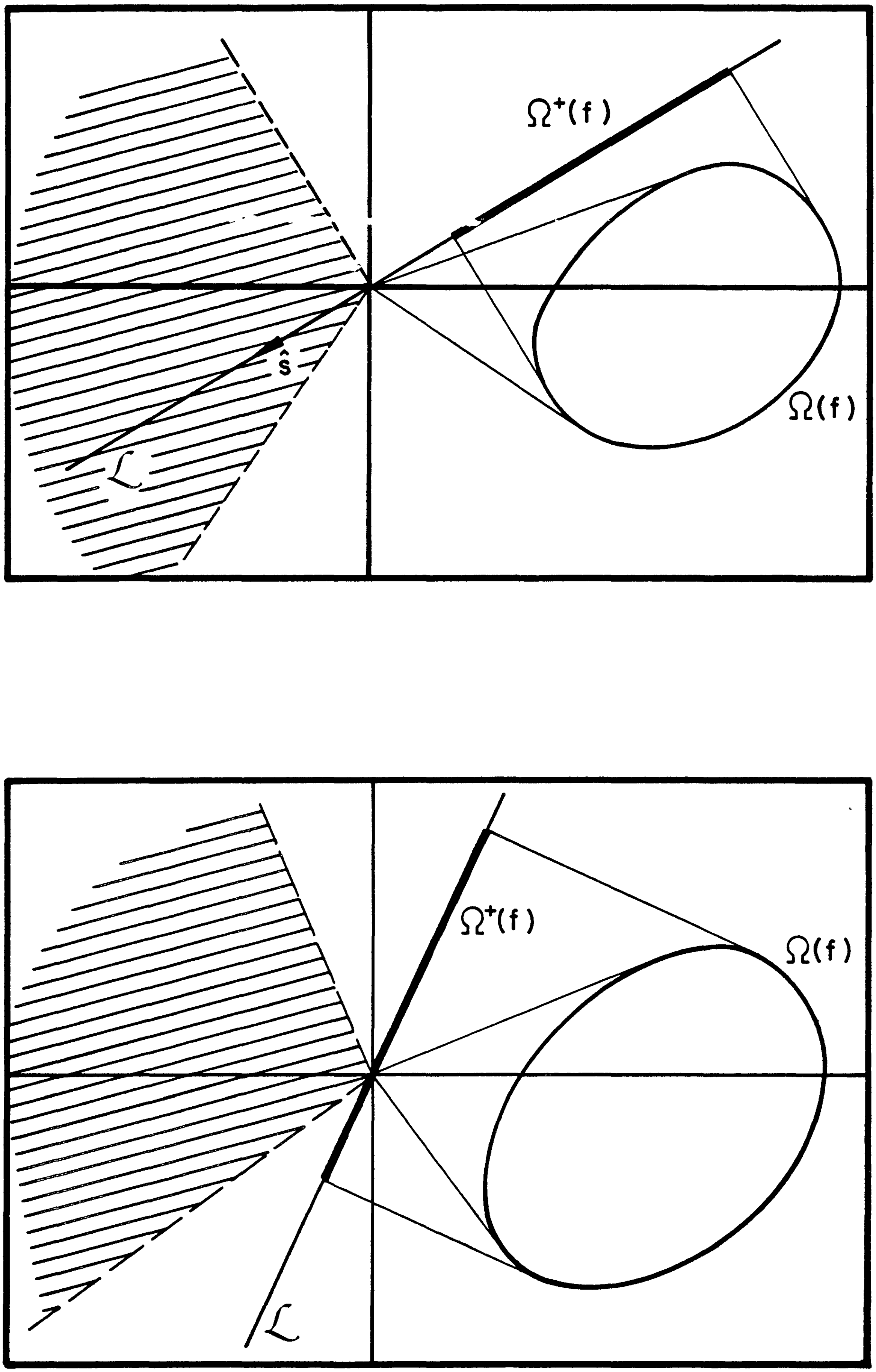

Figure 25.

$$\hat{\omega}^{+} = \hat{\omega} - \langle\hat{\omega}, n\rangle n ,$$

for n the normal to L with unit norm.

PROOF OF THEOREM (18) (cont.).

($\Leftarrow$) Let $\hat{s}^{+}$ be the solution to problem (20) and assume there is a vector $\hat{s} \in S \cap P$ such that

$$\sigma^{\Omega}(\hat{s}) < \sigma^{\Omega}(\hat{s}^{+}) .$$

Since, by Corollary (15),

$$\sigma^{\Omega}(\hat{s}) = \sigma^{\Omega^{+}}(\hat{s})$$

and

$$\sigma^{\Omega}(\hat{s}^{+}) = \sigma^{\Omega^{+}}(\hat{s}^{+}) ,$$

the above inequality implies

$$\sigma^{\Omega^{+}}(\hat{s}) < \sigma^{\Omega^{+}}(\hat{s}^{+}) .$$

This contradicts the assumption that $\hat{s}^{+}$ solves (20), therefore the first inequality is false and then $\hat{s}^{+}$ solves (19). ∇

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CHAPTER 5

ALGORITHM FOR FINDING THE MINIMUM DISTANCE BETWEEN AN ORTHANT AND A CONVEX SET AND APPLICATION OF IT TO THE DOMINANT FEEDBACK PROBLEM

5. ALGORITHM FOR FINDING THE MINIMUM DISTANCE BETWEEN AN ORTHANT AND A CONVEX SET AND APPLICATION OF IT TO THE DOMINANT FEEDBACK PROBLEM

This chapter is concerned first with solving a convex optimization problem, in which we have a convex function and the objective is to minimize it over a given convex set. Then, the methodology developed for that will be applied to the dominant feedback theory, giving a solution for the search direction problem (3.12), when a particular class of linear constraints is present.

(1) NOTATION. In most parts of this chapter we shall use the vec notation for matrices. Corresponding to a mxn matrix A:

- a_{ij} refers to the (i,j)-th element of the matrix A
- a refers to the vector vec A
- a_i refers to the i-th element of the vector a
- a^i refers to the i-th column of the matrix A

Also, if $\mathcal{A}$ is a set of matrices, by $a \in \mathcal{A}$ we mean $a = \text{vec } A$, where $A \in \mathcal{A}$.

5.1 PROBLEM FORMULATION

(2) DEFINITION. An orthant in $\mathbb{R}^p$ can be defined as the set of points x such that $Ax \leqq 0$, where A is a non-singular diagonal matrix with diagonal elements either 1 or -1.

The problem we will be concerned with here can be stated as follows:

(3) MINIMUM DISTANCE PROBLEM.

$$\min\{\|c\|^2 : c = x-y,\ x \in \mathcal{P},\ y \in \Omega\}$$

where $\Omega \subset \mathbb{R}^p$ is a convex set and $\mathcal{P} \subset \mathbb{R}^p$ an orthant, and $\Omega \cap \mathcal{P} = \phi$. ∇

In order to reduce the dimension of problem (3), which is 2p, we can minimize with respect to x, leaving the minimization with respect to y to be done. Define

$$\theta(y) \triangleq \arg\min\{\|x-y\| : x \in P\}$$

and minimize $\|\theta(y) - y\|^2$ over Ω. We remark that $\theta(y)$ is unique. Hence, problem (3) can be equivalently stated, with the number of variables halved, as

(4) MINIMUM DISTANCE PROBLEM.

$$\min\{\chi(y) = \|\theta(y) - y\|^2 : y \in \Omega\}$$

where $\Omega \subset \mathbb{R}^p$ is a convex set. ∇

(5) DEFINITION. A function f is K-Lipschitzian if, for some $K > 0$,

$$\|f(x_1) - f(x_2)\| \leqq K\|x_1 - x_2\| ,$$

for all x_1 and x_2 in the domain. ∇

Note that a Lipschitzian function is also continuous.

(6) PROPOSITION. Consider the orthant

$$P \triangleq \{x \in \mathbb{R}^p : Ax \leqq 0\}$$

where A is a nonsingular diagonal matrix with diagonal entries 1 and/or -1. Then, for any $y \notin P$

$$\langle\theta(y) - y, y\rangle < 0$$

and, for all $x \in P$,

$$\langle\theta(y) - y, x\rangle \geqq 0.$$

PROOF. Suppose $y \notin P$. The closest point in P to y is $\theta(y)$ therefore the vector $c = \theta(y) - y$ is normal to a supporting hyperplane to P at $\theta(y)$. If we denote it by H, for some $b \in \mathbb{R}$, H is defined by $\{x \in \mathbb{R}^p : \langle c, x\rangle = b\}$. But H passes through the origin since it meets P in one entire face containing $\theta(y)$ (see [6], page 101) and all faces contain the origin (a face is defined by the intersection of hyperplanes of the kind $\{x: \langle a^i, x\rangle = 0\}$, where a^i is a column of A. The vertex is a zero-dimensional face.) Hence, $b = 0$ and

$$H \triangleq \{x \in \mathbb{R}^p : \langle c, x\rangle = 0\}$$

Because H separates (not strictly) P from the point y, then for all $x \in P$, either

(i) $\langle c, x\rangle \geqq 0$ and $\langle c, y\rangle < 0$

or

(ii) $\langle c, x\rangle \leqq 0$ and $\langle c, y\rangle > 0$.

Note however that $\langle c, \theta(y)\rangle > \langle c, y\rangle$. In fact, this is a consequence of $\|\theta(y) - y\|^2 > 0$. Therefore (i) applies, and the proposition is proved. ∇

We can now prove the following fact:

(7) PROPOSITION. The function $\theta: \Omega \to P$

$$y \mapsto \arg\min\{\|x - y\| : x \in P\}$$

is 1-Lipschitzian.

PROOF. Consider two points in Ω, y_1 and y_2, and let $x_1 = \theta(y_1)$, $x_2 = \theta(y_2)$. Define the two halflines emerging from x_1 and x_2,

$$L_1 = \{y \in \mathbb{R}^p : y = x_1 - \lambda(x_1 - y_1),\ \lambda \geqq 0\}$$

$$L_2 = \{y \in \mathbb{R}^p : y = x_2 - \lambda(x_2 - y_2),\ \lambda \geqq 0\}.$$

Take two arbitrary points in L_1 and L_2,

$$v_1 = x_1 - \lambda_1(x_1 - y_1)$$
$$v_2 = x_2 - \lambda_2(x_2 - y_2)$$

and consider the difference vector

(8) $$v_1 - v_2 = x_1 - x_2 - \lambda_1(x_1 - y_1) + \lambda_2(x_2 - y_2)$$

It can be noted that

$$<x_1 - x_2, - \lambda_1(x_1 - y_1) + \lambda_2(x_2 - y_2)> \geqq 0.$$

In fact, first note that

(9) $$<x_1 - x_2, - \lambda_1(x_1 - y_1) + \lambda_2(x_2 - y_2)>$$
$$= - <x_1, \lambda_1(x_1 - y_1)> + <x_2, \lambda_1(x_1 - y_1)> + <x_1, \lambda_2(x_2 - y_2)>$$
$$- <x_2, \lambda_2(x_2 - y_2)>$$
$$= - \lambda_1<x_1, x_1 - y_1> + \lambda_1<x_2, x_1 - y_1> + \lambda_2<x_1, x_2 - y_2>$$
$$- \lambda_2<x_2, x_2 - y_2>$$

Also, we have that $x_1 - y_1$ is orthogonal to x_1, which is a result from the projection theorem (see for example [3], page 64). Then,

(10) $$<x_1 - y_1, x_1> = 0.$$

Similarly,

(11) $$<x_2 - y_2, x_2> = 0$$

and, from proposition (6), because x_1 and x_2 belong to P,

(12) $$<x_1 - y_1, x_2> \geqq 0,$$

(13) $$<x_2 - y_2, x_1> \geqq 0.$$

It follows from (9) - (13) that, for λ_1 and λ_2 non-negative, (9) is non-negative too. For the sake of simplicity let $a = x_1 - x_2$ and $b = -\lambda_1(x_1 - y_1) + \lambda_2(x_2 - y_2)$. So, (8) can be written as

(14) $v_1 - v_2 = a + b$

and the non-negativeness of (9) means that

(15) $\langle a, b\rangle \geqq 0$.

In addition, we have that

$$\| a + b\|^2 = \| a\|^2 + \|b\|^2 + 2\langle a, b\rangle$$

which implies, by using (15), that

(16) $\| a + b\| \geqq \| a\|$.

So, from (14),

$$\| v_1 - v_2\| \geqq \| x_1 - x_2\| .$$

Now, since v_1 and v_2 are two arbitrary points in each halfline, we set $v_1 = y_1$ and $v_2 = y_2$ ($\lambda_1 = \lambda_2 = 1$ in (8)) and the result follows, i.e.

$$\| y_1 - y_2\| \leqq \| x_1 - x_2\| . \qquad \nabla$$

In Propositions (17) and (22) the function χ is proved to be convex and continuously differentiable.

(17) PROPOSITION. The function

$$\chi: \Omega \longrightarrow \mathbb{R}$$
$$y \longmapsto \| \theta(y) - y\|^2$$

is convex.

PROOF. The distance function between an arbitrary point $y \in \Omega$ and P is convex (see [5], page 34) and non-negative. Since the square function is convex and increasing on the positive axis, its square is convex too. ∇

Proposition (22) needs the formula for the projection of a point $y \notin P$ onto P, which is given next.

(18) PROPOSITION. Consider $y \notin P$. Then

$$\arg\min\{\|x - y\|^2 : x \in P\} = \theta(y),$$

where

$$(\theta(y))_i = \begin{cases} y_i & \text{if } \langle y, a^i\rangle < 0 \\ 0, & \text{otherwise} \end{cases}$$

PROOF. Recall that the matrix that defines P, A, is unitary. Therefore, its columns form an orthonormal basis for $\mathbb{R}^p$. Let

$$y = \Sigma\alpha_i a^i.$$

Therefore,

$$\langle y, a^i\rangle = (\Sigma\alpha_j a^j)'a^i = \Sigma\alpha_j\langle a^j, a^i\rangle = \alpha_i$$

However,

$$\chi(y) = \|\theta(y) - y\|^2 = \min\{\|x-y\|^2 : x \in P\}$$

(19) $$= \min\{\Sigma(\beta_i - \alpha_i)^2 : \beta_i \leqq 0\}$$

where β_i is the component of x for the i-th axis with respect to the basis $\{a^i\}$. Note that $x \in P$ is equivalent to $\beta_i = \langle x, a^i\rangle \leqq 0$ by definition of P. Now consider the two cases:

(i) $\alpha_i = \langle y, a^i\rangle < 0.$

In this case the minimizer of (19) is $\beta_i = \alpha_i$

(ii) $\alpha_i = \langle y, a^i\rangle \geqq 0$

This implies that $\beta_i = 0$ minimizes (19).

So, the solution for (19) is

(20) $x = \Sigma\beta_i a^i$

where

$$\beta_i = \begin{cases} \alpha_i & \text{if } \langle y, a^i \rangle < 0 \\ 0, & \text{otherwise} \end{cases}$$

However, the original basis for $\mathbb{R}^p$, $\{e^i\}$, is such that

$$e^i = \pm a^i.$$

Therefore, if $x_i(y_i)$ is the component of $x(y)$ along the i-th axis with respect to the basis $\{e^i\}$, then

(21a) $\alpha_i = \beta_i \Longleftrightarrow x_i = y_i$

(21b) $\beta_i = 0 \Longleftrightarrow x_i = 0$

Thus (20) - (21) imply that the minimizer $x \in P$ for $\|x - y\|^2$, $\theta(y)$, is defined by

$$(\theta(y))_i = \begin{cases} y_i, & \text{if } \langle y, a^i \rangle < 0 \\ 0, & \text{otherwise} \end{cases}$$

as we wanted to show. ∇

We can now demonstrate the following:

(22) PROPOSITION. χ is a C^1-function and $\nabla\chi(y) = 2(y - \theta(y))$.

PROOF. Consider $y \in \Omega$. The directional derivative of χ at y with respect to a vector $h \in \mathbb{R}^p$ (assuming h is of unit norm) is defined to be the limit

(23) $$\chi'(y; h) \triangleq \lim_{\lambda \to 0} \frac{\|\theta(y+\lambda h) - (y+\lambda h)\|^2 - \|\theta(y) - y\|^2}{\lambda}$$

when it exists. In order to find the expression for $\chi'(y; h)$ define the following set of indices,

(24a) $J(y) \triangleq \{i: \alpha_i > 0\}$

(24b) $\bar{J}(y) \triangleq \{i: \alpha_i < 0\}$

(24c) $K(y) \triangleq \{i: \alpha_i = 0, \langle h, a^i\rangle < 0\}$

(24d) $\bar{K}(y) \triangleq \{i: \alpha_i = 0, \langle h, a^i\rangle \geq 0\}$

where $\alpha_i = \langle y, a^i\rangle$. Denote by $V(y, \rho)$ the open ball centered at y with radius

$$\rho = \min\{|\alpha_i|: \alpha_i \neq 0, \quad i = 1,\ldots,p\},$$

i.e.

$$V(y, \rho) \triangleq \{y' \in \mathbb{R}^p : \|y - y'\| < \rho\}.$$

As in the previous proposition, let

(25) $y = \Sigma\alpha_i a^i$

and

(26) $y + \lambda h = \Sigma\beta_i a^i.$

Since by assumption $\Omega \cap P = \phi$, $y \notin P$, therefore we can suppose $\alpha_i \neq 0$ for some i (since $\alpha_i = 0$ for all i implies $y = 0$, which belongs to P). Refer to Figure (27) which shows an example with $\alpha_1 \neq 0$ and $\alpha_2 = 0$. Note that α_i and β_i are the projections of y and $y + \lambda h$, respectively, onto the i-th axis i.e., $\alpha_i=\langle y,a^i\rangle$, $\beta_i=\langle y+\lambda h,a^i\rangle$. Thus we have, for some α_i, $\rho=|\alpha_i|$ and, if we consider λ small enough so that $y + \lambda h$ belongs to $V(y, \rho)$, then β_i has the same sign as α_i (in the example both are positive).

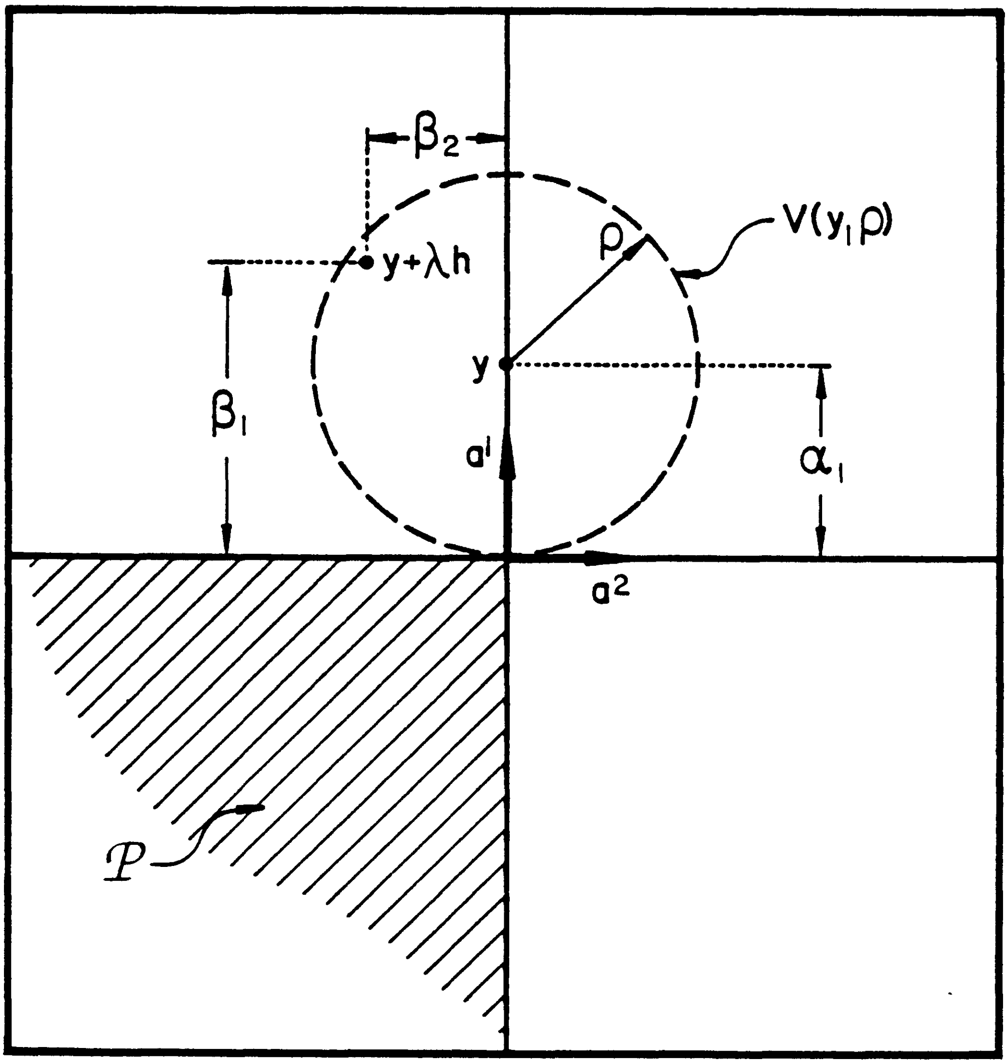


Figure 27.

In order to evaluate the limit (23) consider the following facts separately, where Proposition (18) and Definition (24) have been used, plus the fact that α_i and β_i have both the same sign:

$$(28)\quad \sum_{i\in\bar{J}(y)} (\theta(y) - y)_i^2 = \sum_{i\in\bar{J}(y)} (y_i - y_i)^2 = 0$$

$$(29)\quad \sum_{i\in J(y)} (\theta(y) - y)_i^2 = \sum_{i\in J(y)} (0 - y_i)^2 = \sum_{i\in J(y)} y_i^2$$

$$(30)\quad \sum_{i\in\bar{J}(y)} (\theta(y + \lambda h) - (y + \lambda h))_i^2 = \sum_{i\in\bar{J}(y)} ((y + \lambda h)_i - (y + \lambda h)_i)^2 = 0$$

$$(31)\quad \sum_{i\in J(y)} (\theta(y + \lambda h) - (y + \lambda h))_i^2 = \sum_{i\in J(y)} (y + \lambda h)_i^2$$

Also note that, from (25) - (26),

$$(32)\quad h = \frac{1}{\lambda} \sum(\beta_i - \alpha_i)a^i$$

and therefore, for $i \in K(y)$, where $\alpha_i = 0$ and $\langle h, a^i\rangle < 0$, (32) implies that $\beta_i < 0$. Therefore $\langle y + \lambda h, a^i\rangle = \beta_i < 0$, and so Proposition (18) says that

$$(\theta(y + \lambda h))_i = (y + \lambda h)_i$$

Thus,

$$(33)\quad \sum_{i\in K(y)} (\theta(y + \lambda h) - (y + \lambda h))_i^2 = 0$$

Similarly, for $i \in \bar{K}(y)$, (32) implies that $\beta_i \geqq 0$, i.e., $\langle y + \lambda h, a^i\rangle \geqq 0$. So,

$$(\theta(y + \lambda h))_i = 0,$$

and therefore, because $\alpha_i = 0$ implies $y_i = 0$,

$$(34)\quad \sum_{i\in\bar{K}(y)} (\theta(y + \lambda h) - (y + \lambda h))_i^2 = \sum_{i\in\bar{K}(y)} (y + \lambda h)_i^2 = \sum_{i\in\bar{K}(y)} (\lambda h_i)^2.$$

Finally,

$$(35)\quad \sum_{i\in K(y)\cup\bar{K}(y)} (\theta(y) - y)_i^2 = \sum_{i\in K(y)\cup\bar{K}(y)} (0 - 0) = 0$$

It follows from (28) - (31), (33) - (35) then that the limit (23) is:

$$\chi'(y;\, h) = \lim_{\lambda\to 0} \frac{1}{\lambda}\left\{ \sum_i (\theta(y + \lambda h) - (y + \lambda h))_i^2 - \sum_i (\theta(y) - y)_i^2 \right\}$$

$$= \lim_{\lambda\to 0} \frac{1}{\lambda}\left\{ \sum_{i\in J(y)\cup\bar{K}(y)} (\theta(y + \lambda h) - (y + \lambda h))_i^2 - \sum_{i\in J(y)} (\theta(y) - y)_i^2 \right\}$$

$$= \lim_{\lambda\to 0} \frac{1}{\lambda}\left\{ \sum_{i\in J(y)} (y_i + \lambda h_i)^2 + \sum_{i\in\bar{\bar{K}}(y)} (\lambda h_i)^2 - \sum_{i\in J(y)} y_i^2 \right\}$$

$$= \lim_{\lambda\to 0} \frac{1}{\lambda}\left\{ \sum_{i\in J(y)} (2\lambda y_i h_i + \lambda^2 h_i^2) + \sum_{i\in\bar{\bar{K}}(y)} \lambda^2 h_i^2 \right\}$$

$$(36)\qquad = \sum_{i\in J(y)} 2y_i h_i$$

By observing the above expression, which is linear and continuous in h, we come to the conclusion that χ is differentiable at y (see, for example, [3] page 172). For $\nabla\chi(y)$, the gradient of χ at y,

$$\nabla\chi(y)'h = \chi'(y;\, h)$$

and so, the components of $\nabla\chi(y)$ can be obtained by calculating the directional derivatives of χ along the axis directions. Hence, using (36),

$$(37)\quad (\nabla\chi(y))_i = \begin{cases} 2y_i, & \text{if } \langle y, a^i\rangle > 0 \\ 0, & \text{otherwise} \end{cases}$$

Note that for $\langle y, a^i\rangle = 0$, $y_i = 0$, and thereby $(\nabla\chi(y))_i$ is continuous. Consequently χ is continuously differentiable. Moreoever, $\nabla\chi$ can be written in terms of $\theta(y)$. From Proposition (18),

$$\nabla\chi(y) = 2(y - \theta(y)). \qquad \nabla$$

Figure (38) shows the diagram of the contours of χ in a two-dimensional example. This function is not twice differentiable, since it can be observed from (37) that $(\nabla\chi(y))_i$ is not differentiable at a point y such that $\langle y, a^i\rangle = 0$.

To finish this section we prove the following property concerning the function χ:

(39) THEOREM. Consider the function

$$\chi: \Omega \longrightarrow \mathbb{R}$$
$$y \longrightarrow \|\theta(y) - y\|^2$$

and let $y \in \Omega$, $h \in \mathbb{R}^p$ with $y + h \in \Omega$. Then

$$\chi(y + h) \leq \chi(y) + \nabla\chi(y)'h + 2\|h\|^2$$

PROOF. The Taylor series expansion for χ is

$$\chi(y + h) = \chi(y) + \int_0^1 \nabla\chi(y + th)'h \, dt$$

$$= \chi(y) + \nabla\chi(y)'h + \int_0^1 (\nabla\chi(y + th) - \nabla\chi(y))'h \, dt$$

$$\leq \chi(y) + \nabla\chi(y)'h + \left\|\int_0^1 (\nabla\chi(y + th) - \nabla\chi(y))dt\right\|\|h\|$$

(40) $$\leq \chi(y) + \nabla\chi(y)'h + \int_0^1 \|\nabla\chi(y + th) - \nabla\chi(y)\| \, dt\|h\|$$

However, Propositions (7) and (22) give,

$$\|\nabla\chi(y + th) - \nabla\chi(y)\|$$

$$= 2\|th - \theta(y + th) + \theta(y)\|$$

$$\leq 2\|th\| + 2\|\theta(y + th) - \theta(y)\|$$

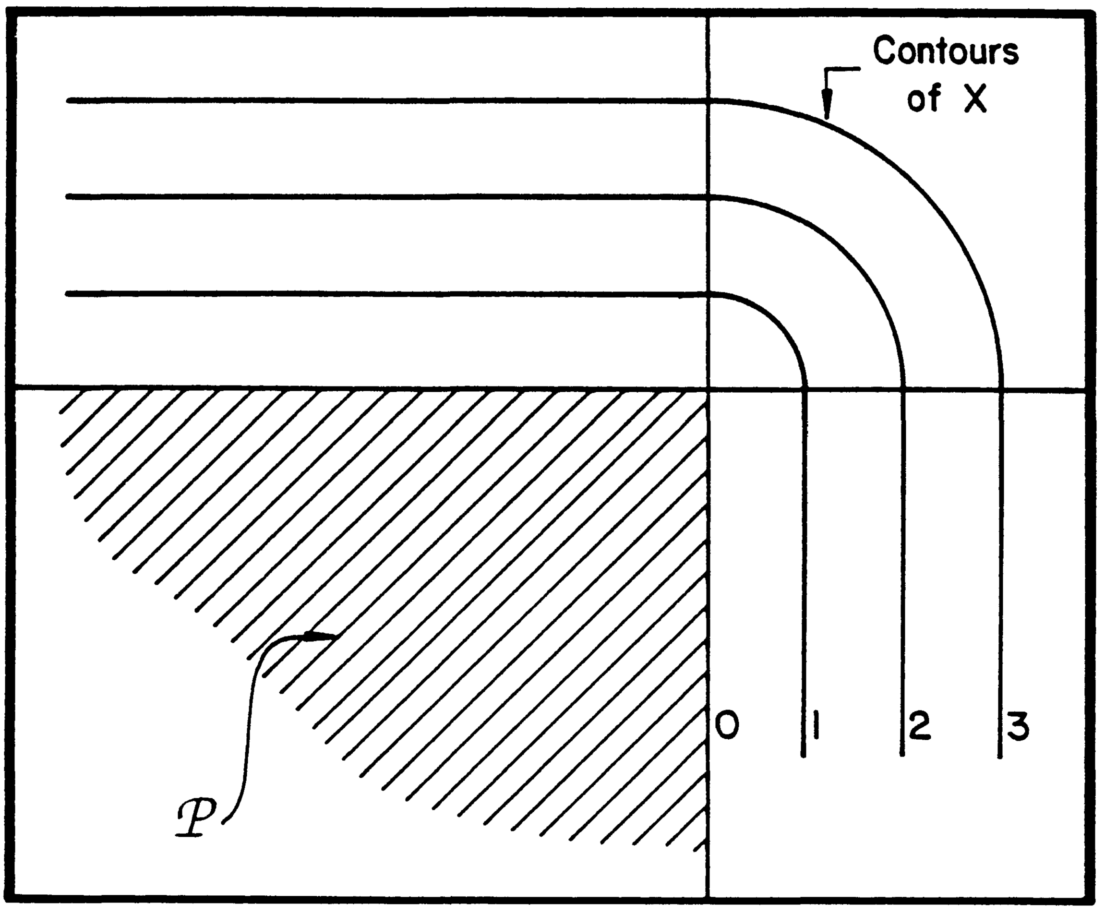


Figure 38.

$$\leqq 2\|th\| + 2\|th\|$$

$$= 4|t|\,\|h\|$$

Hence, from (40),

$$\chi(y + \lambda h) \leqq \chi(y) + \nabla\chi(y)'h + 4\int_0^1 t\,dt\|h\|^2$$

$$= \chi(y) + \nabla\chi(y)'h + 2\|h\|^2 \qquad \nabla$$

5.2 DESCRIPTION OF THE ALGORITHM

Different procedures for solving minimum norm problems are found in the literature. It is a common practice to minimize the norm function over a convex set using a feasible direction algorithm. Bearing this approach in mind, a feasible direction algorithm studied by Allwright [1] is discussed next.

Consider the problem of minimizing a convex function $f: Y \longrightarrow \mathbb{R}$ on a compact convex subset Y of a Hilbert space, where f is twice continuously differentiable on Y with second derivative $\nabla^2 f$ satisfying

$$(41) \quad G \leqq \nabla^2 f(y) \leqq G + mI$$

for some positive semidefinite matrix G and some finite positive number m.

At each iteration j, the algorithm minimizes with respect to $z \in Y$ the following quadratic approximation (at a point y_j) to f,

$$(42) \quad f^s(y_j, z) = f(y_j) + \langle\nabla f(y_j), z - y_j\rangle + \frac{1}{2}\langle z - y_j, G(z - y_j)\rangle$$

A minimizing z is denoted z_j. The feasible descent direction $z_j - y_j$ is then used as a search direction from y_j. The algorithm, in its non-implementable form, is described by:

(43) ALGORITHM FOR CONVEX OPTIMIZATION

1. Choose $\varepsilon \in (0, 1)$, $y_0 \in Y$

   Set $j = 0$.

2. Find the search direction $h_j = z_j - y_j$, where

   $$z_j \in \arg\min\{f^s(y_j, z) : z \in Y\}$$

   and $f^s(y_j, z)$ is given by (42).

3. If $f(y_j) - f^s(y_j, z_j) < \varepsilon$ stop; else continue.

4. Determine the line search parameter

   $$\lambda_j \in \arg\min\{f(y_j + \lambda h_j) : \lambda \in [0, 1]\}.$$

5. Set $y_{j+1} = y_j + \lambda_j h_j$, $j = j+1$ and go to 2.

The idea upon which the algorithm is based is given by the following argument. If $f^s(y_j, \cdot)$ is a reasonably good approximation to $f(\cdot)$, then z_j should be a good approximation to the minimizer $\hat{y}$ for f, and so the search along the feasible direction $z_j - y_j$ from y_j should give a fairly good approximation to $\hat{y}$. Since in a neighbourhood of $\hat{y}$, f can be approximated by $f^s(y_j, \cdot)$ quite well, the algorithm is likely to converge to $\hat{y}$. In fact, convergence rate information is obtained in [1]. When $G > 0$, a rate is obtained which is independent of the geometry of Y and of the number of the constraints that define the feasible set. This does not always happen when other feasible direction algorithms are employed. When $G \geqq 0$, $f^s(y, \cdot)$ is a linear approximation to f at y, i.e. it is the tangent plane at y. In this case, only the diameter of the feasible set affects the convergence rate information.

This algorithm has a nice feature that, when f^s and Y together have a suitably simple structure, the minimization in Step 2 can be carried out fairly simply. This is the case for problem (4) when considering a linear approximation to the cost function, as will be done.

We shall say that a satisfactory point y_j is obtained if, for ε small, an ε-approximation to $\hat{y}$ has been achieved, in that $f(y_j) - f(\hat{y}) < \varepsilon$. This is ensured by the stopping condition of Step 2 of Algorithm (43). For, from [1 , Lemma 3.1], it turns out that

$$f^s(y_j, z_j) \leqq f(\hat{y}).$$

Since $f(y_j) \geqq f(\hat{y})$ then

$$0 \leqq f(y_j) - f(\hat{y}) \leqq f(y_j) - f^s(y_j, z_j).$$

So, a satisfactory point is reached when

$$f(y_j) - f^s(y_j, z_j) < \varepsilon,$$

and, since $\{f(y_j)\} \to f(\hat{y})$ and $z_j \to \hat{y}$ (by the Lemma), this condition will definitely be satisfied for some j.

Although for the proof of convergence of Algorithm (43) ([1], Theorem 3.1) the objective function f is required to be twice continuously differentiable, that is not essential for the convergence study there since what is in fact needed, is that

(I) For some $M > 0$ and for all h such that $y + h \in Y$,

(44) $$f(y + h) \leqq f(y) + \langle \nabla f(y), h\rangle + \frac{1}{2}\langle h, Gh\rangle + \frac{1}{2} M\|h\|^2$$

and that

(45) (II) $f^s(y, y+\hat{h}) \leqq f(\hat{y})$,

where $\hat{h}$ minimizes $f^s(y, y+h)$ with respect to h such that $y + h \in Y$.

In order to use Algorithm (43) to minimize χ, which is not a C^2-function, we draw attention to two facts: (i) Theorem (39) proves inequality (44) for χ, with $G = 0$ and $M = 4$; (ii) inequality (45) also holds for $f = \chi$. In fact, since χ is convex and differentiable

$$\chi(y + h) \geqq \chi(y) + \langle\nabla\chi(y), h\rangle$$

$$= \chi^s(y, y+h), \tag{46}$$

and then

$$\chi^s(y, y+\hat{h}) = \min\{\chi^s(y, y+h) : y + h \in \Omega\}$$

$$\leqq \min\{\chi(y + h) : y + h \in \Omega\}$$

$$= \chi(\hat{y}).$$

Hence, the convergence (rate) information of Theorem 3.1 [1] holds for Algorithm (43), when applied to the function χ. We need only to remark that, although the proof presented there assumes that an approximate line search is used, the proof can easily be modified to accept an exact line search, as should be expected, and which will be the case when applying Algorithm (43) to χ. The search parameter considered in [1] is of the Armijo type, i.e. the step length λ is chosen to be β^ℓ, for some $\beta < 1$ and for the smallest integer $\ell > 0$ such that

$$f(y + \beta^\ell h) \leqq f(y) + \frac{1}{2}\beta^\ell\langle\nabla f(y), h\rangle. \tag{47}$$

However the exact line parameter is chosen to be λ so that

$$f(y + \lambda h) = \min\{f(y + \lambda h) : \lambda \in [0, 1]\}$$

and hence

$$f(y + \lambda h) \leqq f(y + \beta^\ell h)$$

for the smallest $\ell > 0$ satisfying (47).

These considerations lead us to apply algorithm (43) to minimize χ, using its linear approximation χ^s of (46). The description of the algorithm follows, and diagram (49) sketches the situation at a given iteration.

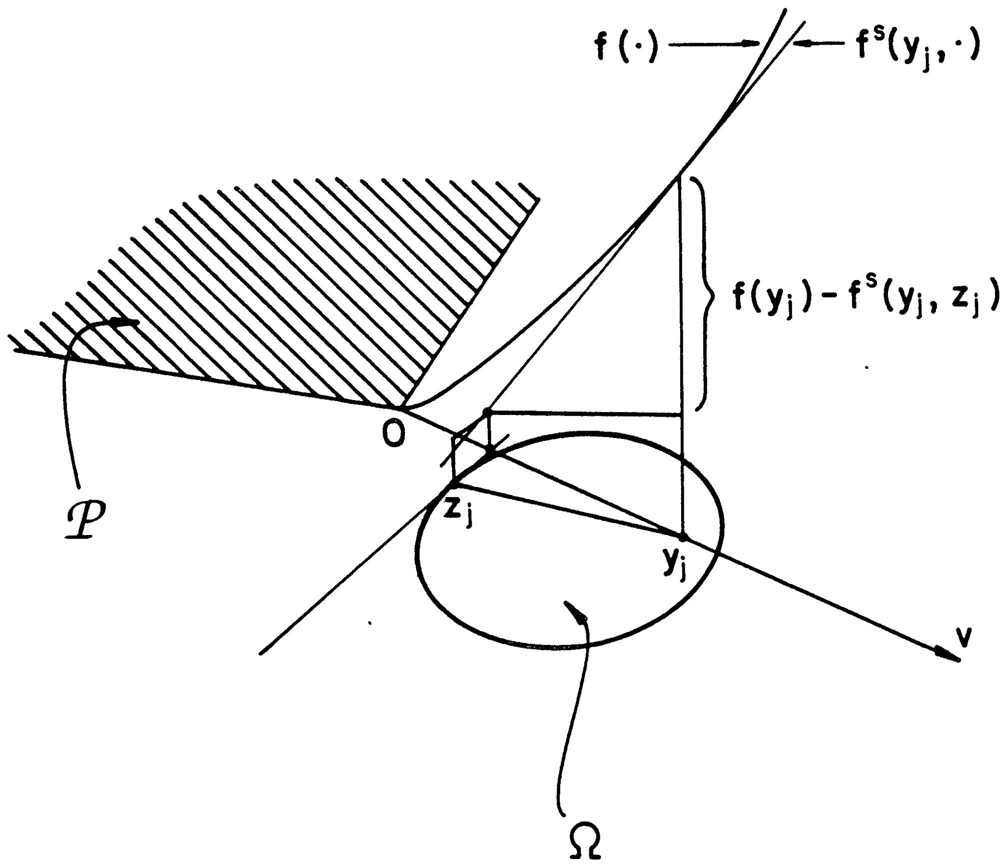


Figure 49.

(48) ALGORITHM FOR THE MINIMUM DISTANCE PROBLEM

1. Choose $\varepsilon \in (0, 1)$, $y^0 \in \Omega$

   Set $j = 0$.

2. Calculate the gradient v by

   $$v = 2(y^j - \theta(y^j)).$$

3. Find a point

   $$z^j \in \arg\min\{\langle v, z\rangle : z \in \Omega\}.$$

4. Find the search direction

   $$h^j = z^j - y^j.$$

5. If $\langle v, -h^j\rangle < \varepsilon$, set $\tilde{y} = y^j$ and stop;

   else continue.

6. Determine the line search parameter

   $$\lambda^j \in \arg\min\{\chi(y^j + \lambda h^j) : \lambda \in [0, 1]\}.$$

7. Set $y^{j+1} = y^j + \lambda^j h^j$, $j = j+1$ and go to 2.

(50) REMARK. The determination of z^j in Step 3 is done by minimizing $\langle \nabla\chi(y^j), z\rangle$ on Ω since this is equivalent to minimizing $\chi^s(y^j, z)$ on Ω, where

(51) $$\chi^s(y^j, z) \triangleq \chi(y^j) + \langle \nabla\chi(y^j), z - y^j\rangle$$ ∇

(52) REMARK. The stop condition of Step 5 comes from (51), since

$$\chi(y^j) - \chi^s(y^j, z^j) = \langle \nabla\chi(y^j), y^j - z^j\rangle = \langle v, -h^j\rangle,$$

and Step 3 of Algorithm (43). ∇

An illustration of the application of Algorithm (48) is shown in the two examples of Figure (53). The sequence $\{y^i\}$ actually achieves

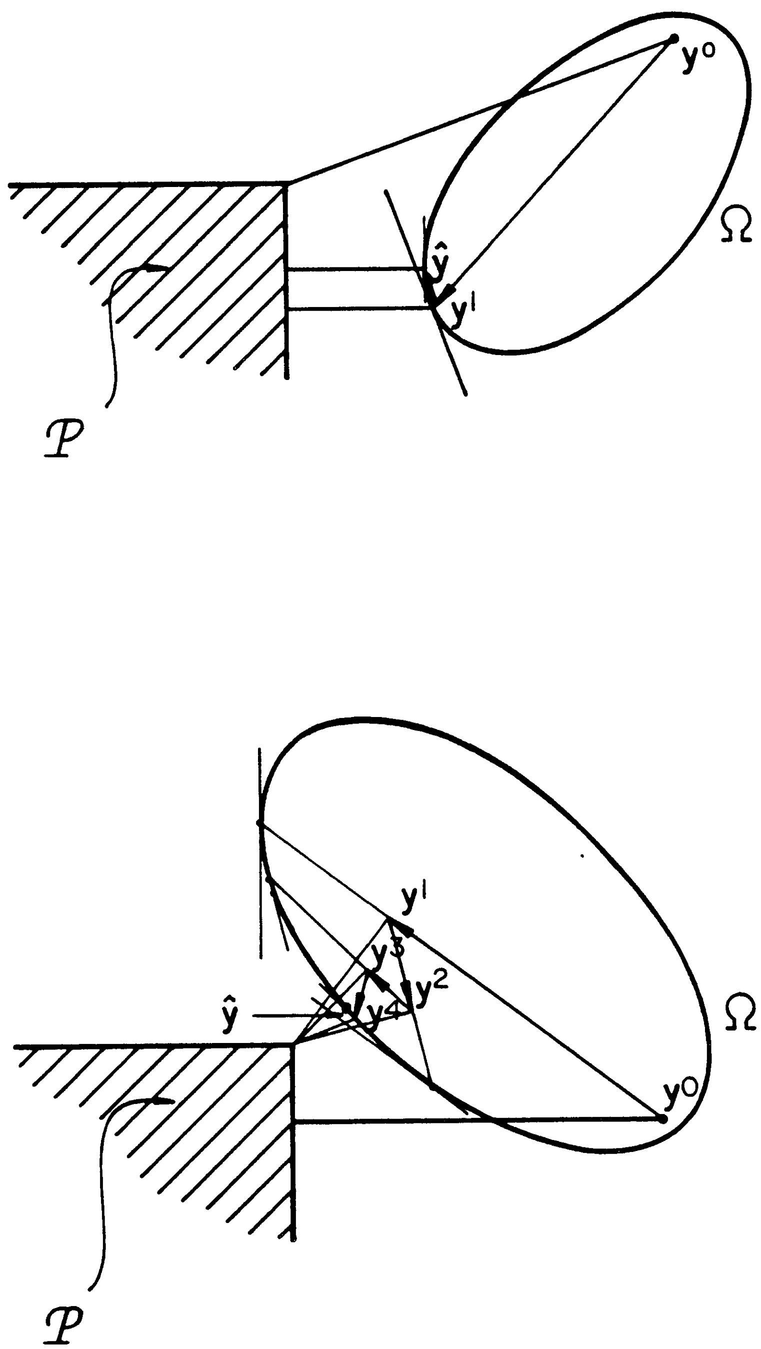


Figure 53.

$\hat{y}$, in the first example, after two iterations.

5.3 THE LINE SEARCH SUBALGORITHM

This section is concerned with the determination of an exact solution to the line search problem described in Step 6 of Algorithm (48). The fact that the minimum of χ along the segment of line $[y^j, y^j + \lambda h^j]$, $\lambda \in [0, 1]$ can be found exactly has an obvious advantage over the Armijo implementation since it provides a faster rate of convergence. This can be seen by using $\beta = 1$ in the formula given in Theorem 3.1 [1].

At some iteration j, let $y = y^j$, $h = h^j$ and the optimizing $\hat{\lambda} = \lambda^j$. Then, for $\hat{\lambda}$ we have either $\hat{\lambda} \in (0, 1)$ or $\hat{\lambda} \in \{0, 1\}$. In what follows we shall assume that $\hat{\lambda} \in (0, 1)$ and attempt to obtain $\hat{\lambda}$ by finding the value of λ that makes zero the derivative $\partial\chi(y + \lambda h)/\partial\lambda$. The procedure for doing that will reveal whether $\hat{\lambda}$ is actually in (0, 1) or in $\{0, 1\}$.

Let $f(\lambda) = \chi(y + \lambda h)$. This function is piecewise quadratic, as can be shown by writing it as

$$f(\lambda) = \chi(y + \lambda h) = \sum_i (\theta(y + \lambda h) - (y + \lambda h))_i^2 .$$

Note that for $i \in \bar{J}(y + \lambda h) = \{i: \langle y+\lambda h, a^i\rangle < 0\}$, $(\theta(y+\lambda h))_i = (y+\lambda h)_i$, hence the corresponding term in the sum vanishes. For the remaining indices $i \in \{i: \langle y+\lambda h, a^i\rangle \geqq 0\}$, $\theta(y + \lambda h))_i = 0$. For $i \in K(y + \lambda h) \cup \bar{K}(y + \lambda h) = \{i: \langle y+\lambda h, a^i\rangle = 0\}$, $(y + \lambda h)_i = 0$, consequently

$$f(\lambda) = \sum_{i \in J(y+\lambda h)} (y + \lambda h)_i^2 =$$

$$= \sum_{i \in J(y+\lambda h)} (y_i^2 + 2\lambda y_i h_i + \lambda^2 h_i^2)$$

Thus, as λ varies on the interval [0, 1], the point $y + \lambda h$ varies on the segment [y, z], and therefore the product $\langle y + \lambda h, a^i \rangle$, for fixed a^i also varies. Since $i \in J(y + \lambda h)$, if and only if $\langle y + \lambda h, a^i \rangle$ is positive, the set $J(y + \lambda h)$ may change as λ varies. Hence we arrive at the conclusion that there exists a partition of [0, 1], formed by an ordered set of roots of $\langle y + \lambda h, a^i \rangle = 0$, for all i. Inside each piece of the partition a certain number of terms is summed up and thereby different quadratics are obtained. Thus, f is piecewise quadratic, say, on the partition $\{\lambda_1, \ldots, \lambda_N\}$ of [0, 1]. It is continuous because χ is continuous.

(54) EXAMPLE. Diagram (55) shows a two-dimensional example. The line segment [y, z], the function f and its derivative are depicted. The line segments A, B and C are defined by:

$$A = [\lambda_1, \lambda_2] \triangleq \{\lambda: \langle y + \lambda h, a^1 \rangle \leq 0, \langle y + \lambda h, a^2 \rangle \geq 0\}$$

$$B = [\lambda_2, \lambda_3] \triangleq \{\lambda: \langle y + \lambda h, a^1 \rangle \geq 0, \langle y + \lambda h, a^2 \rangle \geq 0\}$$

$$C = [\lambda_3, \lambda_4] \triangleq \{\lambda: \langle y + \lambda h, a^1 \rangle \geq 0, \langle y + \lambda h, a^2 \rangle \leq 0\}$$

The index sets are:

$$J(y + \lambda h) = \{2\}, \lambda \in A$$

$$J(y + \lambda h) = \{1, 2\}, \lambda \in B$$

$$J(y + \lambda h) = \{1\}, \lambda \in C$$

It follows then that f is defined by:

$$f(\lambda) = \begin{cases} h_2^2\lambda^2 + 2y_2h_2\lambda + y_2^2, & \lambda \in A \\ (h_1^2 + h_2^2)\lambda^2 + 2(y_1h_1 + y_2h_2)\lambda + (y_1^2 + y_2^2), & \lambda \in B \\ h_1^2\lambda^2 + 2y_1h_1\lambda + y_1^2, & \lambda \in C \end{cases}$$

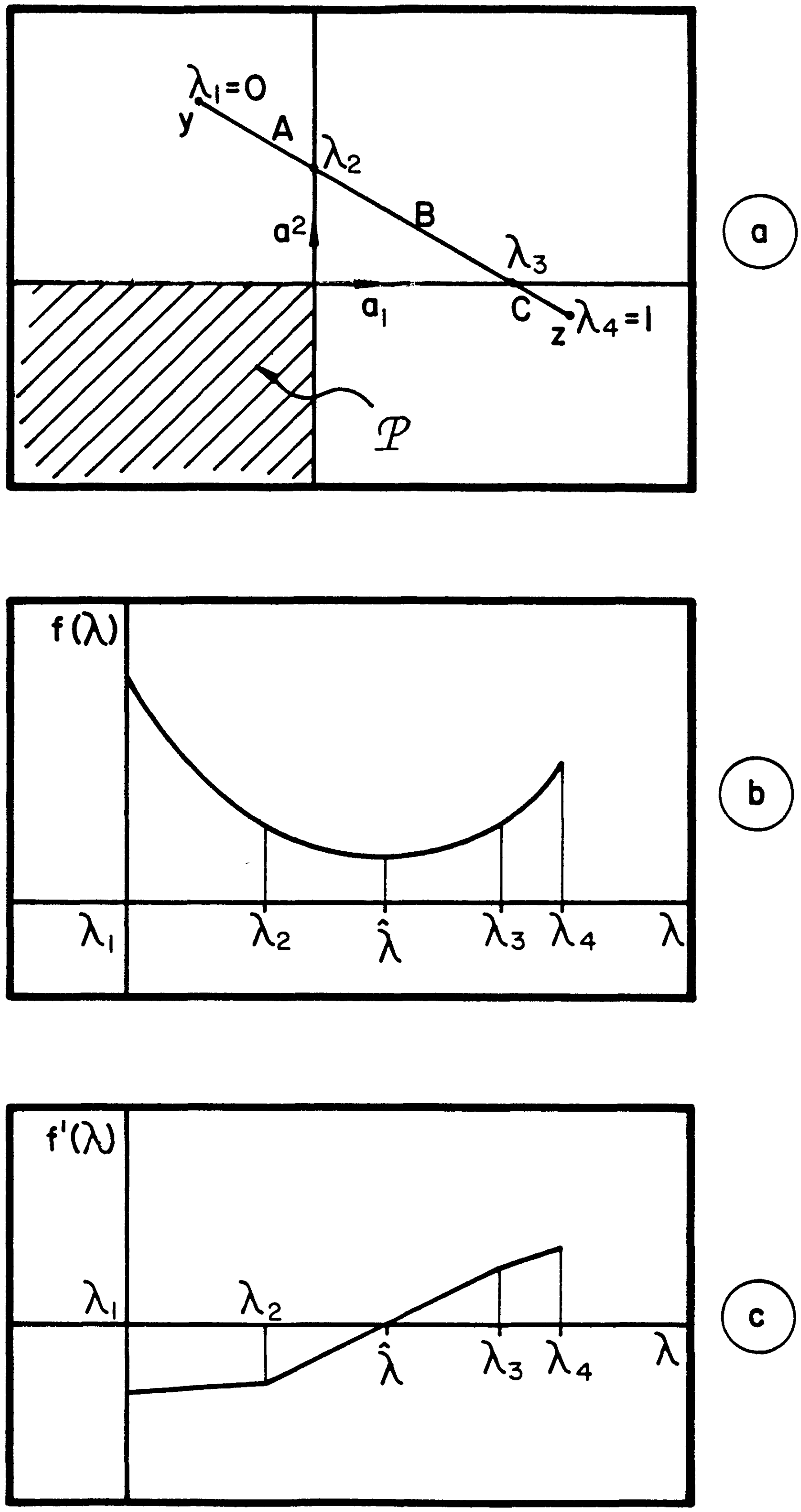


Figure 55.

and its piecewise linear derivative, by:

$$f'(\lambda) = \begin{cases} 2h_2^2\lambda + 2y_2h_2, & \lambda \in A \\ 2(h_1^2 + h_2^2)\lambda + 2(y_1h_1 + y_2h_2), & \lambda \in B \\ 2h_1^2\lambda + 2y_1h_1, & \lambda \in C \end{cases}$$ ∇

(56) EXAMPLE. Another example, three-dimensional, is depicted in Figure (57). ∇

In order to determine the break points of the piecewise function $\tilde{f}(\lambda)$ on [0, 1], $\lambda_1, \ldots, \lambda_N$, we must solve each of the set of equations

(58) $\langle y + \lambda h, a^i\rangle = 0, \quad i = 1, \ldots, p$

and consider those solutions λ_i, $i = 2, \ldots, N-1$, which belong to the interval (0, 1). The extremes are $\lambda_1 = 0$, $\lambda_N = 1$. Obviously $N \leqq p$, where p is the dimension of the space of the variable y. In order to solve (58) we assume that $\langle h, a^i\rangle \neq 0$. In fact, the indices i so that $\langle h_1 a^i\rangle = 0$ are not considered for we are attempting to find the values of λ such that $\langle y + \lambda h, a^i\rangle$ changes sign, and

$$\langle y + \lambda h, a^i\rangle = \langle y, a^i\rangle + \lambda\langle h, a^i\rangle,$$

so, $\langle h, a^i\rangle = 0$ implies that $\langle y + \lambda h, a^i\rangle$ does not vary with λ. The equations of (58) have therefore the solutions

(59) $$\lambda_i = -\frac{\langle y, a^i\rangle}{\langle h, a^i\rangle} = \frac{1}{1 - z_i/y_i}$$

A criterion to investigate whether $\lambda_i \in (0, 1)$, is easily seen to be given by

$$\lambda_i \in (0, 1) \iff \frac{z_i}{y_i} < 0.$$

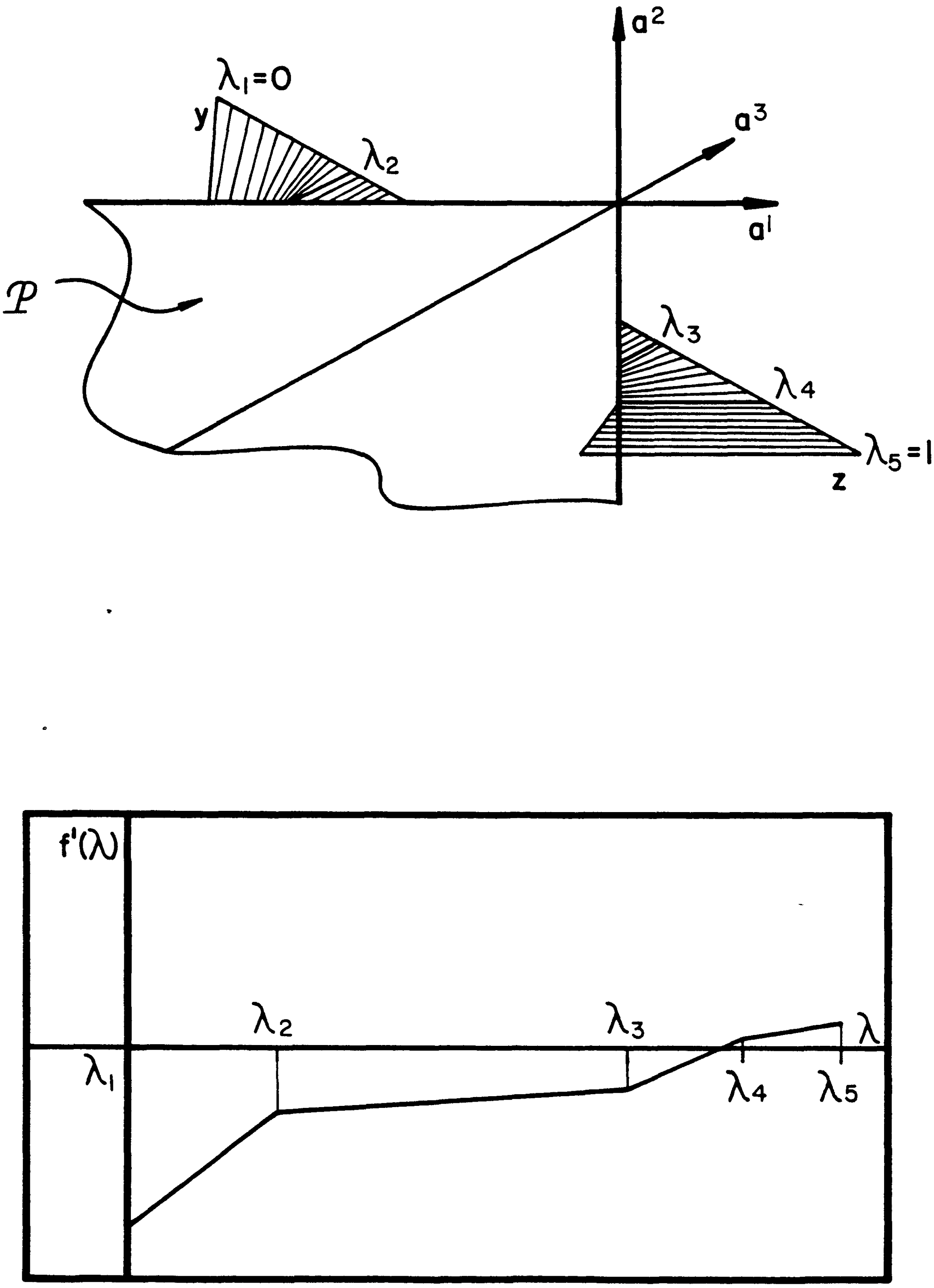


Figure 57.

To form the partition these roots must be ordered. For that, note from (59) that

(60) $\lambda_i < \lambda_j \Leftrightarrow \frac{z_i}{y_i} < \frac{z_j}{y_i}$

We can summarize what has been said as: the partition $\{\lambda_1, \ldots, \lambda_N\}$, $N \leqq p$ is determined by

(61) $$\begin{cases} \lambda_1 = 0 \\ \lambda_i = \dfrac{1}{1 - z_i/y_i}, \; i \in \{2,\ldots,N-1\} = \{k = 1,\ldots,p: \dfrac{z_k}{y_k} < 0\} \\ \lambda_N = 1 \end{cases}$$

and the ordering may be done using (60). Two procedures are suggested here for the determination of the minimum of $f(\lambda)$:

(i) The derivative $f'(\lambda)$ is piecewise linear, so, once the segment containing the optimal $\hat{\lambda}$, i.e. the point that makes zero f', has been found, $\hat{\lambda}$ may be obtained exactly by linear interpolation. Then, the optimization procedure reduces to finding that segment, which can be done using the bisection method, which is quite effective for large N.

The bisection method for finding a zero of a function $g: \mathbb{R} \to \mathbb{R}$ is restricted to the case when the function is defined over a bounded interval and changes sign at the zero point. Since it is unrealistic in practice to expect to find a point $\bar{x}$ such that $g(\bar{x})$ is exactly zero, the algorithm provides an interval [a, b] such that

$$g(a)g(b) < 0 \text{ and } |a - b| < \delta,$$

where δ is some small tolerance. Such an interval [a, b] is called an interval of uncertainty when we know the zero lies in it. The bisection method reduces the uncertainty interval by comparing function values. Suppose that an interval [a, b] has been specified in which $g(a)g(b) < 0$. We test $g((a + b)/2)$. If it is zero then the algorithm terminates; otherwise a new interval of uncertainty is produced by discarding the value of a or b, depending on whether g(a) or g(b) agrees in sign with g((a + b)/2.

The search for the segment containing a point at which f' equals zero will be done using the idea of the bisection method, by seeking a segment [a, b] for which $f'(a)f'(b) < 0$. We shall not be concerned with the condition $|a - b| < \delta$. Instead, a new condition is specified ensuring that a and b belong to the same piece of the partition. For that, we shall check whether

$$J(a) = J(b).$$

Summarizing: a segment containing the zero is discovered if we find a subinterval [a, b] of [0, 1] which satisfies

$$f'(a)f'(b) < 0 \text{ and } J(a) = J(b).$$

Assume that the segment [a, b] has been found. Then,

$$f'(a) = -\frac{f'(b) - f'(a)}{b - a}(\hat{\lambda} - a)$$

and hence

$$(62) \quad \hat{\lambda} = a - f'(a)\frac{b - a}{f'(b) - f'(a)}.$$

Diagram (63) shows an example in which $\hat{\lambda}$ is found in five steps. We start with the uncertainty interval $[a_1, b_1]$. Condition $f'(a_1)f'(b_1) < 0$ holds but not $J(a_1) = J(b_1)$ (a_1 and b_1 do not belong to the same piece of partition). Then, $[a_1, b_1]$ is halved and a_1 discarded, since sign $(f'(a_1))$ = sign $(f'(a_2))$. Therefore $[a_2, b_1]$ is the new uncertainty interval. b_1 is renamed b_2 and the same procedure is

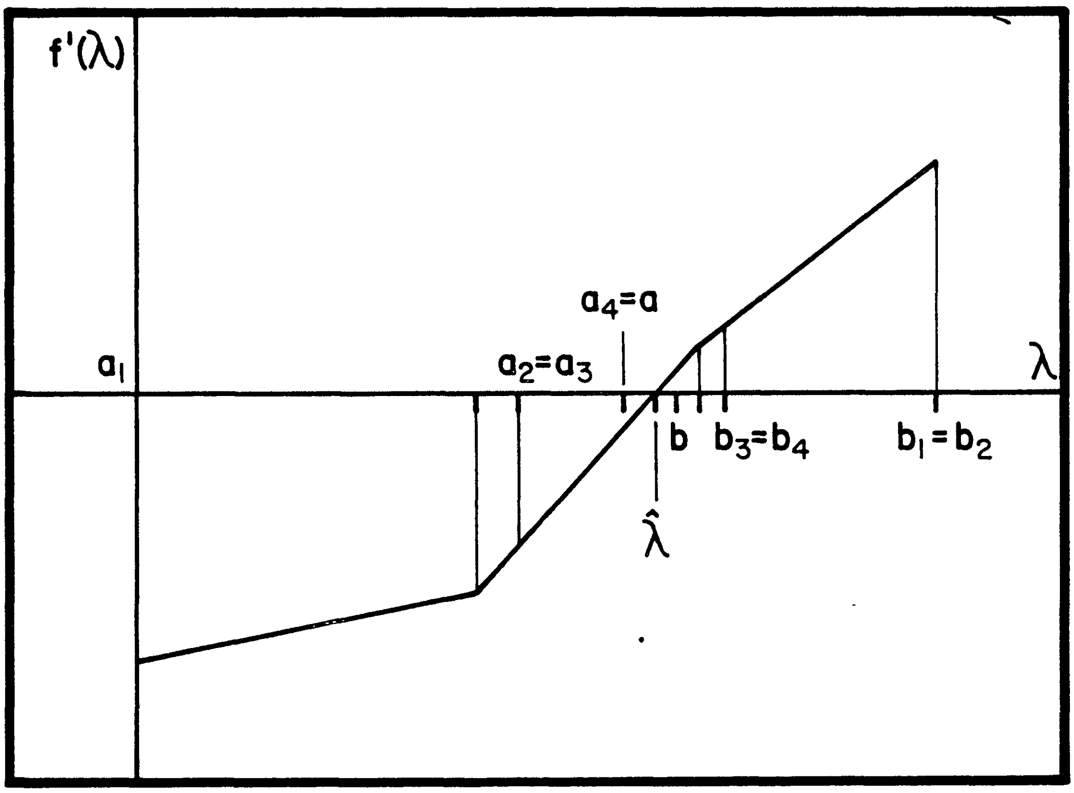


Figure 63.

done again. It stops when the uncertainty interval [a, b] is obtained. Finally observe that, by the convexity of $f(\lambda)$, we conclude that $f'(0) \geqq 0$ implies that $\hat{\lambda} = 0$, whereas $f'(1) \leqq 0$ implies $\hat{\lambda} = 1$. Since in either case we have $f'(0)f'(1) \geqq 0$, the test $f'(0)f'(1) < 0$ should inform us that formula (62) is to be used. Hence, the line search algorithm is:

(64) <u>LINE SEARCH ALGORITHM</u> (Determination of λ^j for step 6 of Algorithm (48). Here y, z, h and λ refer to y^j, z^j, h^j and λ^j.)

1. Set $a = 0$, $b = 1$.

2. Determine

$$I = \{i \in \{1, \dots, p\}: z_i/y_i < 0\}$$

$$J(y) \triangleq \{i \in I: \langle y, a^i \rangle > 0\}$$

$$J(z) \triangleq \{i \in I: \langle z, a^i \rangle > 0\}.$$

3. Evaluate

$$f'(0) = 2\sum_{i \in J(y)} y_i h_i$$

$$f'(1) = 2\sum_{i \in J(z)} z_i h_i$$

4. If $f'(0)f'(1) < 0$ then go to 6 ([0, 1] is the uncertainty interval); else continue.

5. If $f'(0) \geqq 0$ set $\lambda = 0$ and stop; else set $\lambda = 1$ and stop.

6. If $J(a) = J(b)$ go to 11; else continue.

7. Set $m = (a + b)/2$.

8. Determine

$$J(y + mh) = \{i \in I: \langle y + mh, a^i \rangle > 0\}.$$

9. Evaluate

$$f'(m) = \sum_{i \in J(y+mh)} 2h_i^2 m + 2y_i h_i$$

10. If $f'(m) < 0$ then set $a = m$ ($[m, b]$ is the new uncertainty interval) and go to 6; else, if $f'(m) > 0$ then set $b = m$ ($[a, m]$ is the new uncertainty interval) and go to 6; else set $\lambda = m$ and stop.

11. Evaluate

$$\lambda = a - f'(a) \frac{b - a}{f'(b) - f'(a)} \qquad \nabla$$

(ii) A second procedure to determine the step length is proposed next. Here we shall investigate the root of the equation

$$f'(\lambda) = 0.$$

Now

$$f'(\lambda) = \frac{\partial}{\partial \lambda} \chi(y + \lambda h)$$

$$= \frac{\partial \chi(z)}{\partial z} \frac{\partial z}{\partial \lambda} \Bigg|_{z = y+\lambda h}$$

$$= 2(y + \lambda h - \theta(y + \lambda h))'h$$

$$= 2\sum_i (y + \lambda h - \theta(y + \lambda h))_i h_i$$

Evaluating the term $(y + \lambda h - \theta(y + \lambda h))_i$, using proposition (18), we have

$$(y + \lambda h - \theta(y + \lambda h))_i = \begin{cases} (y + \lambda h)_i, & \text{if } \langle y + \lambda h, a^i \rangle > 0 \\ 0 & \text{, otherwise} \end{cases}$$

Consequently,

$$f'(\lambda) = 2 \sum_{i \in J(y+\lambda h)} (y + \lambda h)_i h_i$$

Hence, $f'(\lambda) = 0$ if and only if

$$(65)\quad \lambda = - \frac{\sum_{i \in J(y+\lambda h)} y_i h_i}{\sum_{i \in J(y+\lambda h)} h_i^2}$$

A trial and error method that yields the solution to (65) is sketched:

(a) Determine the breakpoints $\lambda_1, \ldots, \lambda_N$ for $f(\lambda)$, using (61).

(b) Determine the index set $\bar{J}(y + \lambda h)$ from (24b), for λ varying inside each interval $[\lambda_i, \lambda_{i+1}]$.

(c) Evaluate the expression (65) for each interval $[\lambda_i, \lambda_{i+1}]$, remembering that $J(y + \lambda h)$ is the same for each $\lambda \in [\lambda_i, \lambda_{i+1}]$.

(d) If, for an interval $[\lambda_i, \lambda_{i+1}]$, λ given by formula (65) belongs to the interval, then λ is the solution of (65), and therefore of $f'(\lambda) = 0$. This happens because the right hand side of (65) is constant whenever $\lambda \in [\lambda_i, \lambda_{i+1}]$.

(e) If $f'(\lambda)$ has no roots in [0, 1] then the minimum of $f(\lambda)$ on [0, 1] is achieved at one of the ends, 0 or 1. The relevant end is 0 when $f'(0) \geq 0$, and is 1 when $f'(1) \leq 0$.

5.4 DESCRIPTION OF THE LINEARLY CONSTRAINED DOMINANT OUTPUT FEEDBACK PROBLEM

In this and the subsequent sections, we shall study a particular linearly constrained optimal feedback problem. An algorithm will be devised for that, which is a variation of Algorithm (2.32), with the search direction and line search adapted to the constrained situation. Unlike in Algorithm (3.5), the search direction will be defined in a way that it does not ensure convergence. Nevertheless there is an advantage gained from the fact that the method presented in the previous sections can be applied to this problem, providing an implementable algorithm.

(66) LINEARLY CONSTRAINED PROBLEM. Given an initial output feedback matrix $F_0 \in \mathcal{F}$ and an initial condition $x_0 \in \mathbb{R}^n$, for the system (1.1-3)

minimize $\{V(x_0,F) : F \in \mathcal{M}\}$ in the dominance sense

where

$$\mathcal{M} \triangleq \{F = (f_{ij}) \in \mathcal{F} : a_{ij} \leqq f_{ij} \leqq b_{ij},\ i \in \{1,\ldots,m\},\ j \in \{1,\ldots,r\}\}$$

for some real numbers a_{ij} and b_{ij}, with $a_{ij} \neq b_{ij}$. ∇

Then a feasible direction from F will be a matrix S, such that $F + S \in \mathcal{M}$, which (by (3.2)) implies that $S \in \mathcal{M}(F)$, for

(67) $$\mathcal{M}(F) \triangleq \{S \in \mathbb{R}^{m\times r} : S = X - F,\ X \in \mathcal{M}\}$$

$$= \{S \in \mathbb{R}^{m\times r} : a_{ij} - f_{ij} \leqq s_{ij} \leqq b_{ij} - f_{ij}\}$$

$$= \{S \in \mathbb{R}^{m\times r} : g_{ij}(F) \leqq s_{ij} \leqq h_{ij}(F)\},$$

where

$$g_{ij}(F) \triangleq a_{ij} - f_{ij}$$

$$h_{ij}(F) \triangleq b_{ij} - f_{ij}.$$

The cone of the feasible directions is given by

(68) $$\mathcal{D} = \{S \in \mathbb{R}^{m\times r} : \lambda S \in \mathcal{M}(F),\ \lambda \in (0, \bar{\lambda}],\ \text{some } \bar{\lambda} > 0\}$$

(69) EXAMPLE. (Refer to Figure (70)). Consider $F = [f_{11}, f_{12}] \in \mathbb{R}^{1\times 2}$ ($\equiv \mathbb{R}^2$), and assume F lies on the boundary of $\mathcal{M}$, the active constraint being g_{12}, i.e. $g_{12}(F) = 0$. Then the set $\mathcal{M}(F)$ is defined by

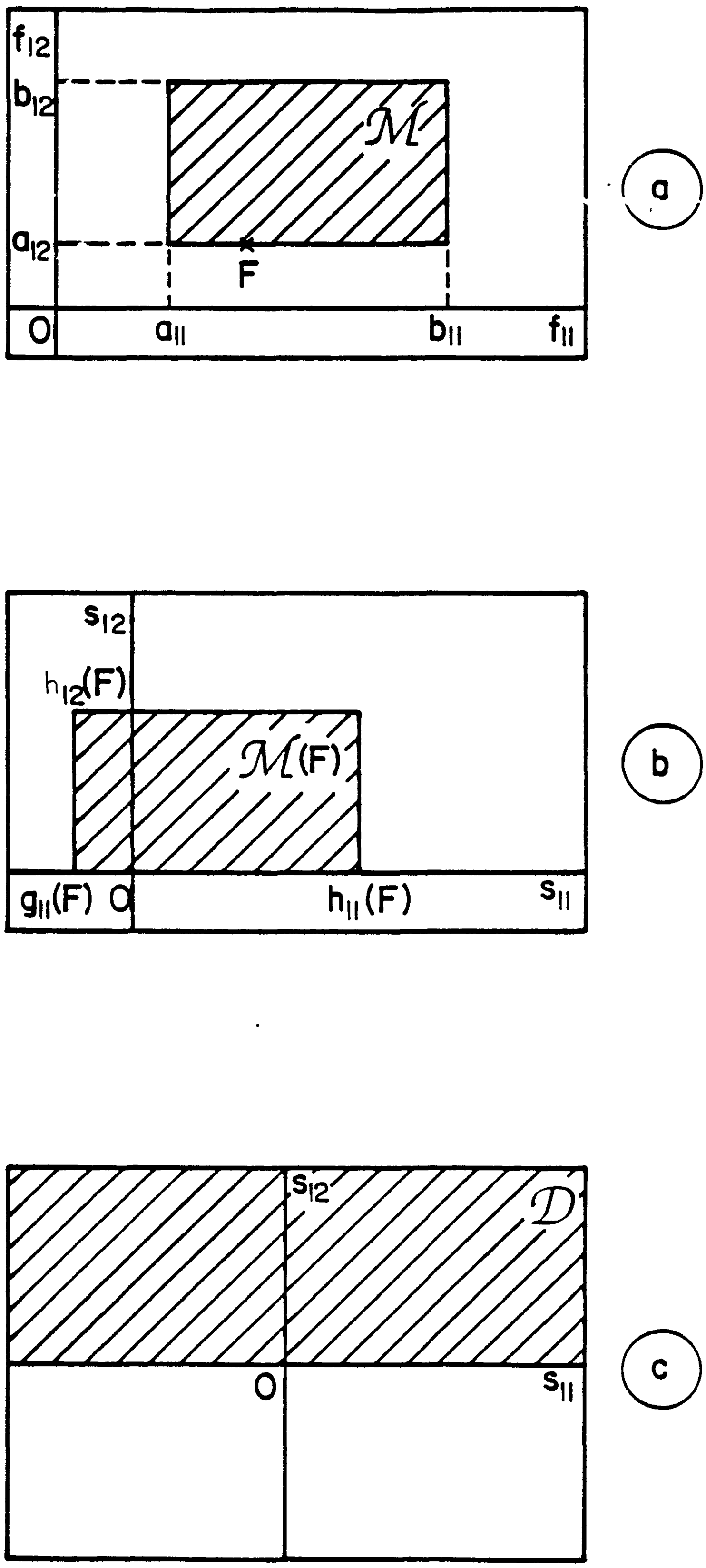

f12
b12
a12
M
F
O
a11
b11
f11
a
s12
h12(F)
M(F)
g11(F)
O
h11(F)
s11
b
s12
D
O
s11
c

Figure 70.

$$g_{11}(F) \leqq s_{11} \leqq h_{11}(F)$$

$$0 \leqq s_{12} \leqq h_{12}(F)$$

with $g_{11}(F) < 0$, $h_{11}(F) > 0$, and $h_{12}(F) > 0$. Since there is absence of freedom for s_{12}, in the sense that it cannot take negative values, $\mathcal{D}$ will be defined simply as:

$$\mathcal{D} = \{S \in \mathbb{R}^{1x2} : s_{12} \geqq 0\}$$

or, denoting S in vec notation, we can write

$$\mathcal{D} = \{s \in \mathbb{R}^2 : \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \leqq 0\}$$

$$= \{s \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \leqq 0\} \cup \{s \in \mathbb{R}^2 : \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \leqq 0\}$$

i.e., $\mathcal{D}$ is a union of two orthants, as is clear from the figure. ∇

Because of the geometry of the set M, in a general space $\mathbb{R}^{mxr}$, it is clear that, whatever is the relative position of F in M, $\mathcal{D}$ is a union of orthants, the number of which depends on that relative position. In the particular case of an interior point, $\mathcal{D} = \mathbb{R}^{mxr}$.

The algorithm for solving problem (66) shall use the search direction which coincides with Allwright's unconstrained direction when F is an interior point of the feasible set M. For, it will be chosen as the normalized S such that it minimizes the largest eigenvalue of dK(F; S) inside the cone of the feasible directions at F. The statement of the optimization problem is

(71) <u>SEARCH DIRECTION PROBLEM</u>.

Minimize $\{\sigma(F, S) : S \in \mathcal{S} \cap \mathcal{D}\}$

where

$\mathcal{S}$ is the unit sphere in $\mathbb{R}^{mxr}$

$\sigma(F, S) = \lambda \max dK(F; S)$,

$\mathcal{D} \triangleq \{S = (s_{ij}) \in \mathbb{R}^{mxr}; s_{ij} \leq 0, (i,j) \in I_1 x J_1; s_{ij} \geq 0, (i,j) \in I_2 x J_2\}$,

$I_1 \subseteq \{1,\ldots,m\}, I_2 \subseteq \{1,\ldots,m\}, J_1 \subseteq \{1,\ldots,p\}, J_2 \subseteq \{1,\ldots,p\}$. ∇

Note that the index sets I_1, I_2, J_1 and J_2 are all functions of F, and that $I_1 \cap I_2 = \phi$, $J_1 \cap J_2 = \phi$ since $s_{ij} \geq 0$ and $s_{ij} \leq 0$ cannot hold both for the same pair (i,j) (the reason for this is that equality constraints are not considered, i.e. $a_{ij} \neq b_{ij}$).

We have seen that the mapping that defines the search direction must satisfy certain properties in order to guarantee convergence of the resulting algorithm. We shall demonstrate that the search direction of (71) does not. However, instead of showing that those properties are not valid directly, we shall prove that it does not satisfy a necessary condition for convergence. There are examples showing that the jamming phenomenon takes place when inactive constraints are ignored in the definition of the search direction, due to the fact that not using them may force the step length to go to zero, even far from the optimum (see [4]). To remedy this, we may ignore only the "sufficiently" negative constraints and include those "close" to being active. Hence, using inactive constraints to compute the search direction is fundamental to guarantee convergence of the algorithm. So, it is important to find out whether the definition for the search direction given in (71) takes into account the inactive constraints at F. Clearly it does not, since $\mathcal{D}$ has been defined by means of the active constraints only. This shows that convergence will not be necessarily achieved for the algorithm to be presented here. The results have, however, possible applications elsewhere, which is why they are included here.

For any function g with domain $X = \bigcup_{k=1}^{K} X^k$, it is a fact that

$$\min\{g(x): x \in X\} = \min\{\min\{g(x): x \in X^k\}: k = 1,\dots,K\}.$$

Then, the original minimization problem is equivalent to solving all the K subproblems $\min\{g(x): x \in X^k\}$ and then taking the solution which gives minimum g. In the same way we shall divide problem (71). Assume that $\mathcal{D} = \bigcup_k \mathcal{P}^k$, for some orthants $\mathcal{P}^k$. Then, a subproblem will be to minimize σ on a particular orthant, and there will be as many subproblems as the number of orthants. The solution will be chosen among all the solutions to the subproblems as the one that gives minimum σ. Denoting a fixed orthant $\mathcal{P}^k$ simply by $\mathcal{P}$, we have therefore the

(72) <u>SEARCH DIRECTION PROBLEM FOR ONE ORTHANT</u>.

$$\text{minimize } \{\sigma(F, s): s \in \mathcal{S} \cap \mathcal{P}\}$$

where

$$\mathcal{P} = \{s = (s_{ij}) \in \mathbb{R}^{m\times r} : s_{ij} \leqq 0, (i,j) \in I_1 \times J_1; s_{ij} \geqq 0, (i,j) \in I/I_1 \times J/J_1\}$$

$$= \{s \in \mathbb{R}^p : As \leqq 0\}$$

where A is a diagonal matrix with diagonal elements $a_{ij} \in \{1, -1\}$ $I = \{1,\dots,m\}$, $J = \{1,\dots,p\}$. ∇

Here I/I_1 denotes the set $\{i \in I: i \notin I_1\}$.

The algorithm using rule (71) for the search direction has an analog to Lemma (3.4). So, for the pair

(73a) $\hat{s} \in \arg\min\{\lambda\max\, dK(F; s): s \in \mathcal{S} \cap \mathcal{D}\}$

(73b) $\hat{\pi}(F) = \lambda\max\, dK(F; \hat{s})$, we can prove

(74) LEMMA. Let $F \in M$, $\hat{S}$ and $\hat{\pi}(F)$ given in (73), then:

(a) If $\hat{\pi}(F) < 0$, there exists a real $\hat{\lambda} > 0$ so that

$$K(F + \lambda\hat{S}) < K(F), \text{ for all } \lambda \in (0, \hat{\lambda}].$$

(b) If $\hat{\pi}(F) > 0$, there exists a $\delta > 0$, so that $K(F') \nleq K(F)$, for all $F' \in M$: $\| F - F' \| < \delta$.

PROOF.

(a) As K is Fréchet-differentiable at F (Section 2.1), given any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for all S such that $\| S \| < \delta(\varepsilon)$,

(75) $K(F + S) = K(F) + dK(F, S) + Z$

where

(76) $\| Z \| < \varepsilon \| S \|$.

Assume that $\hat{\pi}(F)$ is negative and choose $\varepsilon \in (0, -\hat{\pi}(F))$, then there exists $\delta(\varepsilon)$ so that (75) - (76) hold for all S such that $\| S \| < \delta(\varepsilon)$. Let $\hat{\lambda} = \delta(\varepsilon)/\| \hat{S} \|$, and consider the point $F + \lambda\hat{S}$, $\lambda \in (0, \hat{\lambda}]$. Then,

$$\| \lambda\hat{S} \| = \lambda \| \hat{S} \| \leqq \hat{\lambda} \| \hat{S} \| = \delta(\varepsilon),$$

$$K(F + \lambda\hat{S}) = K(F) + dK(F; \lambda\hat{S}) + Z,$$

with

$$\| Z \| < \varepsilon \| \lambda\hat{S} \| .$$

Hence, for all $x \neq 0$,

$$x'K(F+\lambda\hat{S})x = x'K(F)x + x'dK(F;\ \lambda\hat{S})x + x'Zx$$

$$< x'K(F)x + x'dK(F;\ \lambda\hat{S})x + \varepsilon\|\lambda\hat{S}\|\ x'x$$

$$\leq x'K(F)x + x'dK(F;\ \lambda\hat{S})x + \varepsilon\lambda x'x$$

since $\|\hat{S}\|_2 \leq \|\hat{S}\|_F = \sum_{ij}(\hat{s}_{ij}^{\,2})^{1/2} = 1$. Then,

$$x'K(F+\lambda\hat{S})x < x'K(F)x + (\max_{y\neq 0} \frac{y'dK(F;\ \lambda\hat{S})y}{y'y})x'x$$

$$+ \varepsilon\lambda\ x'x$$

$$= x'K(F)x + \lambda\max dK(F;\ \lambda\hat{S})x'x +$$

$$+ \varepsilon\lambda\ x'x$$

$$= x'K(F)x + \lambda\hat{\pi}(F)x'x + \varepsilon\lambda\ x'x$$

$$= x'K(F)x + \lambda(\hat{\pi}(F) + \varepsilon)x'x$$

$$< x'K(F)x,$$

since $\hat{\pi}(F) + \varepsilon < 0$. Consequently, $F + \lambda\hat{S}$ strictly dominates F for $\lambda \in (0, \hat{\lambda}]$.

(b) Assume now that $\hat{\pi}(F) > 0$ and choose $\varepsilon \in (0, \hat{\pi}(F))$. Then, whenever $\|S\| < \delta(\varepsilon)$,

$$K(F+S) = K(F) + dK(F;\ S) + Z$$

for some Z with $\|Z\| < \varepsilon\|S\|$. Consider any $S \neq 0$, with $\|S\| < \delta(\varepsilon)$. Then, for all x,

$$x'K(F+S)x \geq x'K(F)x + x'dK(F;\ S)x - x'x\|Z\|$$

(77) $$> x'K(F)x + \|S\|_F\, x'dK(F;\ \frac{S}{\|S\|_F})x - x'x\varepsilon\|S\|$$

Assume now that $S \in \mathcal{D}$. Consequently $S/\|S\|_F \in \mathcal{D}$ since $\mathcal{D}$ is a cone. Thus because $\| S/\|S\|_F \|_F = 1$, we have

$$\lambda\max\, dK(F; \frac{S}{\|S\|_F}) \geqq \min\{\lambda\max\, dK(F; X) : \|X\|_F = 1,\ X \in \mathcal{D}\}$$
$$= \hat{\pi}(F)$$

and then, for $\bar{x}$ an eigenvector of $dK(F; \frac{S}{\|S\|_F})$ associated with $\lambda\max\, dK(F; S/\|S\|_F)$,

$$\frac{\bar{x}'dK(F; S/\|S\|_F)\bar{x}}{\bar{x}'\bar{x}} \geqq \hat{\pi}(F),$$

i.e.,

$$(78) \quad \bar{x}'dK(F; \frac{S}{\|S\|_F})\bar{x} \geqq \hat{\pi}(F)\bar{x}'\bar{x}$$

So, for that $\bar{x}$, if S is such that $S \in \mathcal{D}$ with $\|S\| < \delta(\varepsilon)$, (77) - (78) hold, and hence, since $\|S\| \triangleq \|S\|_2 \leqq \|S\|_F$,

$$\bar{x}'K(F + S)\bar{x} > \bar{x}'K(F)\bar{x} + \|S\|_F \bar{x}'dK(F; \frac{S}{\|S\|_F})\bar{x} - \bar{x}'\bar{x}\varepsilon\|S\|$$
$$\geqq \bar{x}'K(F)\bar{x} + \|S\|_F \hat{\pi}(F)\bar{x}'\bar{x} - \bar{x}'\bar{x}\varepsilon\|S\|$$
$$\geqq \bar{x}'K(F)\bar{x} + \|S\|_F \bar{x}'\bar{x}(\hat{\pi}(F) - \varepsilon)$$
$$> \bar{x}'K(F)\bar{x}$$

Thus, F + S does not dominate F. Now, since the cone $\mathcal{D}$ contains $\mathcal{M}(F)$, we have shown that, for all sufficiently small S in $\mathcal{M}(F)$, F + S does not dominate F, or, in other words, for all $F' \in \mathcal{M}$ sufficiently close to F, F' does not dominate F. ∇

If $\mathcal{D} = \mathbb{R}^p$ in the above lemma, then

$$\hat{S} \in \arg\min\{\lambda\max\, dK(F; S) : S \in \mathcal{S}\},$$

$\hat{\pi}(F) = \lambda\max\, dK(F;\, \hat{S})$.

Lemma (74) then proves Lemma (2.34), for the unconstrained feedback case (proved originally by Allwright in [2]).

The information given in Theorem (3.6), that each iteration yields a dominating feedback, also holds for the algorithm, which is:

(79) <u>BASIC ALGORITHM FOR A LINEARLY CONSTRAINED DOMINANT OUTPUT FEEDBACK</u>

1. Select a $F^0 \in M$; set $j = 0$.

2. Compute

$$\hat{\pi}(F^j) = \min\{\lambda\max\, dK(F^j;\, S) : S \in \mathcal{S} \cap \mathcal{D}\}$$

where $\mathcal{D}$ is given in (68).

3. If $\hat{\pi}(F^j) > 0$ set $F^* = F^j$ and stop; else continue.

4. Define the search direction by

$$S^j \in \arg\min\{\lambda\max\, dK(F^j;\, S) : S \in \mathcal{S} \cap \mathcal{D}\}.$$

5. Compute the upper bound $\hat{\lambda}^j$ for the step length, which is such that, for all $\lambda \in [0, \hat{\lambda}^j]$,

$$F^j + \lambda S^j \in M.$$

6. Compute the step length λ^j

$$\lambda^j \in \arg\max\{\mathrm{tr}[(K(F^j)-K(F^j+\lambda S^j))X_0] : \lambda\min[K(F^j)-K(F^j+\lambda S^j)] \geqq 0, \lambda \in [0,\hat{\lambda}^j]\}.$$

7. If $\mathrm{tr}[(K(F^j) - K(F^j + \lambda^j S^j))X_0] = 0$ stop; else continue.

8. Set $F^{j+1} = F^j + \lambda^j S^j$, $j = j+1$, and go to Step 2. ∇

(80) <u>REMARK</u>. An implementable version of this algorithm will be given in (113). ∇

The situation here is similar to the unconstrained feedback case. The condition $\hat{\pi}(F) < 0$ is sufficient for the existence of feasible descent directions, along which stands $\hat{S} \in \arg \hat{\pi}(F)$ as the steepest. The situation $\hat{\pi}(F) = 0$ may also provide feasible descent directions, in which case they would be the arguments of $\hat{\pi}(F)$.

We shall complete this section by proving Lemma (3.4) of Chapter 3, which had been left, using Lemma (74) and the following result (only part (ii) is needed but we prove part (i) for its own sake):

(81) LEMMA. Let $F \in \mathcal{M}$ and

$$\bar{S} \in \arg\min\{\lambda\max dK(F; S) : S \in \mathcal{M}(F)\}$$

$$\bar{\pi}(F) = \lambda\max dK(F; \bar{S})$$

$$\hat{S} \in \arg\min\{\lambda\max dK(F; S) : S \in \mathcal{S} \cap \mathcal{D}\}$$

$$\hat{\pi}(F) = \lambda\max dK(F; \hat{S}).$$

Then

(i) $\hat{\pi}(F) < 0 \iff \bar{\pi}(F) < 0$

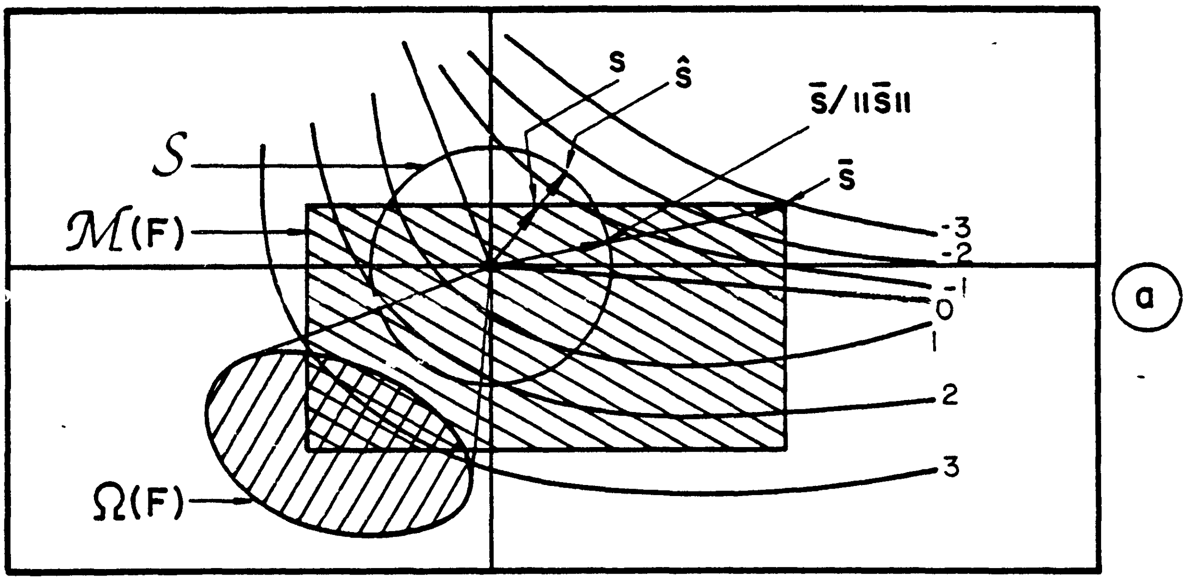
(ii) $\hat{\pi}(F) > 0 \iff \bar{\pi}(F) = 0$ and $\bar{S}$ is unique.

PROOF. Consider diagram (82).

(i) ($\Rightarrow$) Assume $\hat{\pi}(F) < 0$, and consider the vector $S = \lambda\hat{S}$. By the definition of $\mathcal{D}$, there exists a $\lambda > 0$ so that $S \in \mathcal{M}(F)$. For that λ,

$$\sigma(F, S) = \sigma(F, \lambda\hat{S}) = \lambda\sigma(F, \hat{S}) = \lambda\hat{\pi}(F) < 0.$$

Hence, since $S \in \mathcal{M}(F)$,


S
$\mathcal{M}$(F)
Ω(F)
s
ŝ
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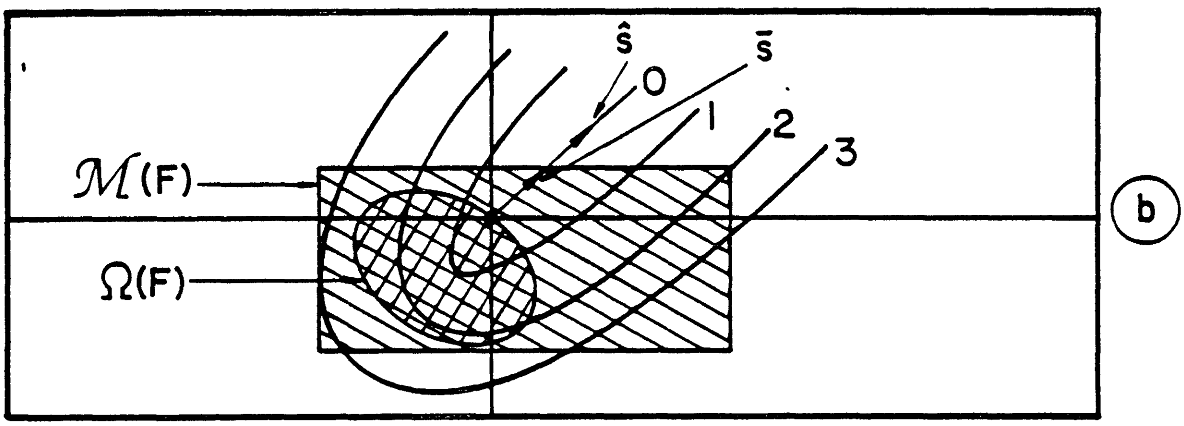
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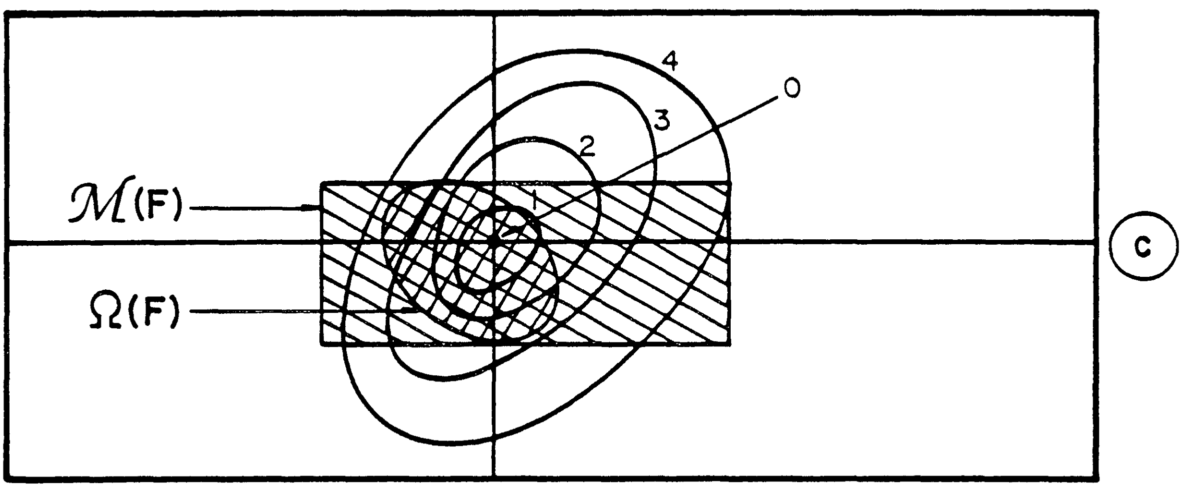

$\mathcal{M}$(F)
Ω(F)
4
0
3
2
1
c

Figure 82.

$$\bar{\pi}(F) = \min\{\sigma(F, X) : X \in \mathcal{M}(F)\} \leqq \sigma(F, S) < 0.$$

(⇐) Now $\|\bar{S}\| \neq 0$, since $\bar{\pi}(F) < 0$ (other wise $\bar{\pi}(F) = 0$).

Hence

$$(83) \quad \sigma(F, \frac{\bar{S}}{\|\bar{S}\|}) = \frac{1}{\|\bar{S}\|}\sigma(F, \bar{S}) \triangleq \frac{1}{\|\bar{S}\|}\bar{\pi}(F) < 0.$$

Further, since $\bar{S} \in \mathcal{M}(F)$ and $\mathcal{D}$ is defined by

$$\mathcal{D} = \{X \in \mathbb{R}^{m\times r} : \lambda X \in \mathcal{M}(F),\ \lambda \in [0, \bar{\lambda}],\ \bar{\lambda} > 0\},$$

then $\frac{\bar{S}}{\|\bar{S}\|} \in \mathcal{D}$. It follows from this fact and (83) that

$$\hat{\pi}(F) = \min\{\sigma(F, S) : S \in \mathcal{S} \cap \mathcal{D}\} \leqq \sigma(F, \bar{S}/\|\bar{S}\|) < 0.$$

(ii) (⇒) Assume that $\hat{\pi}(F) > 0$. Observe that $\bar{\pi}(F)$ can never be positive, since $0 \in \mathcal{M}(F)$ for any $F \in \mathcal{M}$, and $\sigma(F, 0) = 0$. Hence, the result $\bar{\pi}(F) = 0$, as well as $\bar{S} = 0 \in \arg\bar{\pi}(F)$, is an immediate consequence of part (i). Now, consider the uniqueness of $\bar{S}$. Assume that there exists a nonzero $\bar{S} \in \arg\bar{\pi}(F)$. Then, for that $\bar{S}$

$$\sigma(F, \bar{S}) = \|\bar{S}\|\,\sigma(F, \frac{\bar{S}}{\|\bar{S}\|}) = 0$$

and thus, since $\bar{S}/\|\bar{S}\| \in \mathcal{S} \cap \mathcal{D}$ (it belongs to $\mathcal{D}$ because $\bar{S} \in \mathcal{M}(F)$),

$$\hat{\pi}(F) = \min\{\sigma(F, S) : S \in \mathcal{S} \cap \mathcal{D}\} \leqq \sigma(F, \frac{\bar{S}}{\|\bar{S}\|}) = 0,$$

which is a contradiction.

(⇐) Assume that $\bar{\pi}(F) = 0$ and that $\bar{S} = 0$ is the only argument for $\bar{\pi}(F)$. Suppose, for proof by contradiction, that $\hat{\pi}(F) = 0$. Then, for $\hat{S} \in \mathcal{S} \cap \mathcal{D}$

$$(84) \quad \sigma(F, \hat{S}) = 0$$

Consider $S = \lambda\hat{S}$ for some $\lambda > 0$ such that $S \in M(F)$. Then, it follows from (84) that $\sigma(F, S) = 0$, with $S \neq 0$. Thus there exists a nonzero S in $M(F)$ which makes σ zero, and so it is an argument for $\bar{\pi}(F) = 0$, contradicting the assumed uniqueness of $\bar{S}$. Hence, $\hat{\pi}(F) \neq 0$. Since, by part (i), $\hat{\pi}(F) < 0$ cannot hold, it must be true that $\bar{\pi}(F) > 0$. ∇

Lemma (3.4) is an immediate consequence of Lemmas (74) and (81), as follows:

(85) PROOF OF LEMMA (3.4).

(a) Suppose $\bar{\pi}(F) < 0$. In part (a) of Lemma (74) it is proved that $\hat{\pi}(F) = \sigma(F, \hat{S}) < 0$ implies that, for small λ, $F + \lambda\hat{S}$ dominates F. Since only the negativeness of σ at $\hat{S}$ was used to show that, it is also true for $\bar{S}$ and $\bar{\pi}(F) = \sigma(F, \bar{S})$, i.e. there exists a $\bar{\lambda} > 0$ so that, for $\lambda \in (0, \bar{\lambda}]$, $K(F + \lambda\bar{S}) < K(F)$.

(b) Suppose $\bar{\pi}(F) = 0$ and $\bar{S} = 0$ is the only argument of $\bar{\pi}(F)$. Then Lemma (74) part (b) and Lemma (81) part (ii) give the result. ∇

5.5 EQUIVALENCE OF PROBLEMS

In this section we shall formulate a theory relating problems (4) and (72). It will be demonstrated that solving the former problem provides useful information about the solution to the latter, and, in certain cases, the solution itself. The main results are given in Theorems (98) and (101). They generalize some of the results of Section 4.1.

For that, we need to prove some facts and introduce some concepts:

(87) LEMMA. Let P be an orthant in $\mathbb{R}^p$ and $q \in \mathbb{R}^p$. Then $q \in P$, if and only if $\langle x, q\rangle \geq 0$ for all elements x of P.

PROOF.

($\Rightarrow$) Let the orthant be defined by

$$P = \{x \in \mathbb{R}^n : Ax \leqq 0\},$$

where A is a diagonal matrix with entries $a_{ij} \in \{1, -1\}$, and suppose $q \in P$. Let x be an arbitrary element of P. Then we have $Aq \leqq 0$ and $Ax \leqq 0$, and thus,

$$\langle Ax, Aq \rangle = x'A'Aq \geqq 0.$$

But $A'A = A^2 = I$, since A is orthogonal. Thus $\langle x, q \rangle \geqq 0$.

($\Leftarrow$) Let $q \in \mathbb{R}^p$ and assume that, for all $x \in P$, $\langle x, q \rangle \geqq 0$. Note however that, for the j-th column of A, a^j

$$A(-a^j) = A'(-a^j) = \begin{bmatrix} \langle a^1, -a^j \rangle \\ \vdots \\ \langle a^j, -a^j \rangle \\ \vdots \\ \langle a^p, -a^j \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} \leqq 0$$

This implies that $-a^j \in P$, and so $\langle -a^j, q \rangle \geqq 0$ by assumption. Since this holds for all j,

$$-Aq = \begin{bmatrix} \langle -a^1, q \rangle \\ \vdots \\ \langle -a^p, q \rangle \end{bmatrix} \geqq 0$$

and hence $Aq \leqq 0$, proving what we wanted. ∇

(88) PROPOSITION. Let P be an orthant in $\mathbb{R}^p$ given by

$$P = \{x \in \mathbb{R}^p : Ax \leqq 0\},$$

where A is a diagonal matrix with diagonal elements in $\{1, -1\}$. If, for some $y \notin \mathcal{P}$,

$$\hat{s} \in -\arg\min\{\{\max\langle s, x-y\rangle : x \in \mathcal{P}\} : s \in \mathcal{S}\}$$

then $\hat{s} \in \mathcal{P}$.

PROOF. Consider the closest point in $\mathcal{P}$ to y, $\theta(y)$, and let

$$c = \theta(y) - y.$$

It follows from Proposition (6) that, for all $x \in \mathcal{P}$,

$$\langle c, x\rangle \geqq 0$$

Lemma (87) then says that $c \in \mathcal{P}$.

On the other hand define

$$Z \triangleq \{z = x-y : x \in \mathcal{P}\}$$

Thus, changing variables, the expression

$$\hat{s} \in -\arg\min\{\{\max\langle s, x-y\rangle : x \in \mathcal{P}\} : s \in \mathcal{S}\}$$

becomes

$$\hat{s} \in -\arg\min\{\{\max\langle s, z\rangle : z \in Z\} : s \in \mathcal{S}\}$$

(Refer to diagram (89) for the geometrical interpretation.)

Since $0 \notin Z$ (otherwise $y \in \mathcal{P}$) and

$$\max\{\langle s, z\rangle : z \in Z\} = \sigma(s),$$

for $\sigma(s)$ the support function to Z, we can use Theorem (4.6) in order to say that

$$\arg\min\{\sigma(s) : s \in \mathcal{S}\} = \{-\frac{\hat{\gamma}}{\|\hat{\gamma}\|}\},$$

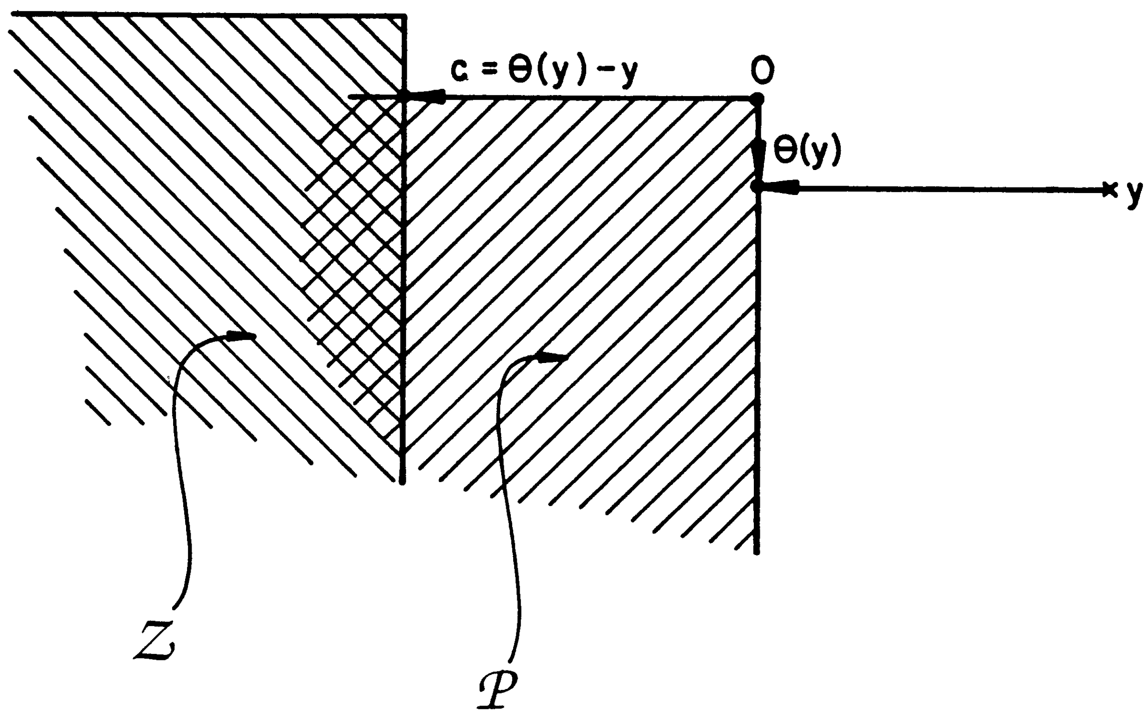


Figure 89.

where $\hat{\gamma}$ is the minimum norm point of Z. However $\theta(y)$ is the closest point to y in P, which implies that $c = \theta(y) - y$ is the minimum norm point of Z. Hence,

$$\arg\min\{\sigma(s) : s \in S\} = - \frac{c}{\|c\|}$$

and so, since $c \in P$ and P is a cone,

$$\hat{s} = -\arg\min\{\sigma(s) : s \in S\} = \frac{c}{\|c\|} \in P. \qquad \nabla$$

(90) <u>COROLLARY</u>. Suppose $P \cap \Omega = \phi$ and let $\hat{c}$ be the (nonzero) solution to Problem (3). Then $\hat{s} = \hat{c}/\|\hat{c}\|$ belongs to P.

We must now state the following definitions:

(91) <u>DEFINITION</u>. For a convex function f defined on a convex set C in a vector space X, we define the convex set [f, C], called the epigraph of f, by

$$[f, C] = \{(r, x) \in \mathbb{R} \times X : x \in C,\ f(x) \leqq r\}. \qquad \nabla$$

(92) <u>DEFINITION</u>. Given a concave function g defined on a convex subset D of X, we define the hypograph of g as

$$[g, D] = \{(r, x) \in \mathbb{R} \times X : x \in D,\ r \leqq g(x)\}. \qquad \nabla$$

Theorem (98) is an application of the important Fenchel Duality Theorem for conjugate functions. Before stating it, we shall introduce some concepts of the dual optimization theory (see [3]).

(93) <u>DEFINITION</u>. Let f be a convex function defined on a convex set C in a normed space X. The conjugate set C* is defined as

$$C^* = \{x^* \in X^* : \sup_{x \in C} [\langle x, x^* \rangle - f(x)] < \infty\}$$

and the function f* conjugate to f is defined on C* as

$$f^*(x^*) = \sup_{x \in C} [\langle x, x^*\rangle - f(x)]. \qquad \nabla$$

Here X* denotes the (normed) dual space of X, the space of all bounded linear functions on X. If x* is an element of X*, by $\langle x, x^*\rangle$ we mean x*(x), i.e. the value of the linear function $x^* \in X^*$ at $x \in X$. For the geometrical interpretation of the conjugate function in $\mathbb{R}$ refer to Figure (94). The dual space of $\mathbb{R}^n$ is itself $\mathbb{R}^n$ in the sense that the function x* can be represented by a vector in $\mathbb{R}^n$, normal to the hyperplane $\langle x, x^*\rangle = 0$. It is proved that the number f*(x*) is such that the hyperplane

$$r = \langle x, x^*\rangle - f^*(x^*)$$

is a support hyperplane of [f, C].

The minimum vertical separation of the sets [f, C] and [g, D] is given in terms of the conjugate functions f* and g* as follows:

(95) <u>THEOREM</u> (Fenchel Duality Theorem) [3]. Assume that f and g are, respectively, convex and concave functions on the convex sets C and D in a normed space X. Assume that $C \cap D$ contains points in the relative interior of C and D and that either [f, C] or [g, D] has nonempty interior. Suppose further that $\mu = \min\{f(x) - g(x) : x \in C \cap D\}$ is finite. Then

$$\mu = \min\{f(x)-g(x) : x \in C \cap D\} = \max\{g^*(x^*) - f^*(x^*) : x^* \in C^* \cap D^*\}$$

where the maximum on the right is achieved by some $x_0^* \in C^* \cap D^*$. If the minimum on the left is achieved by some $x_0 \in C \cap D$, then

$$\max\{\langle x, x_0^*\rangle - f(x) : x \in C\} = \langle x_0, x_0^*\rangle - f(x_0)$$

and

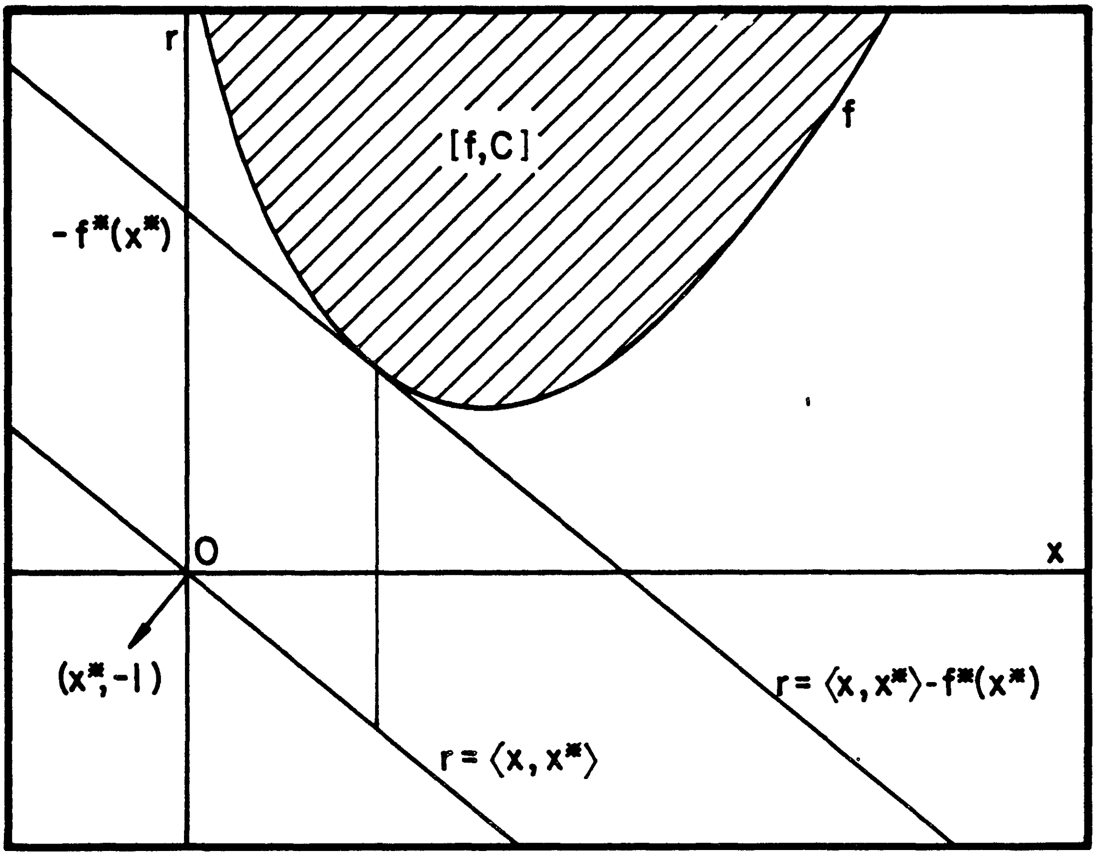


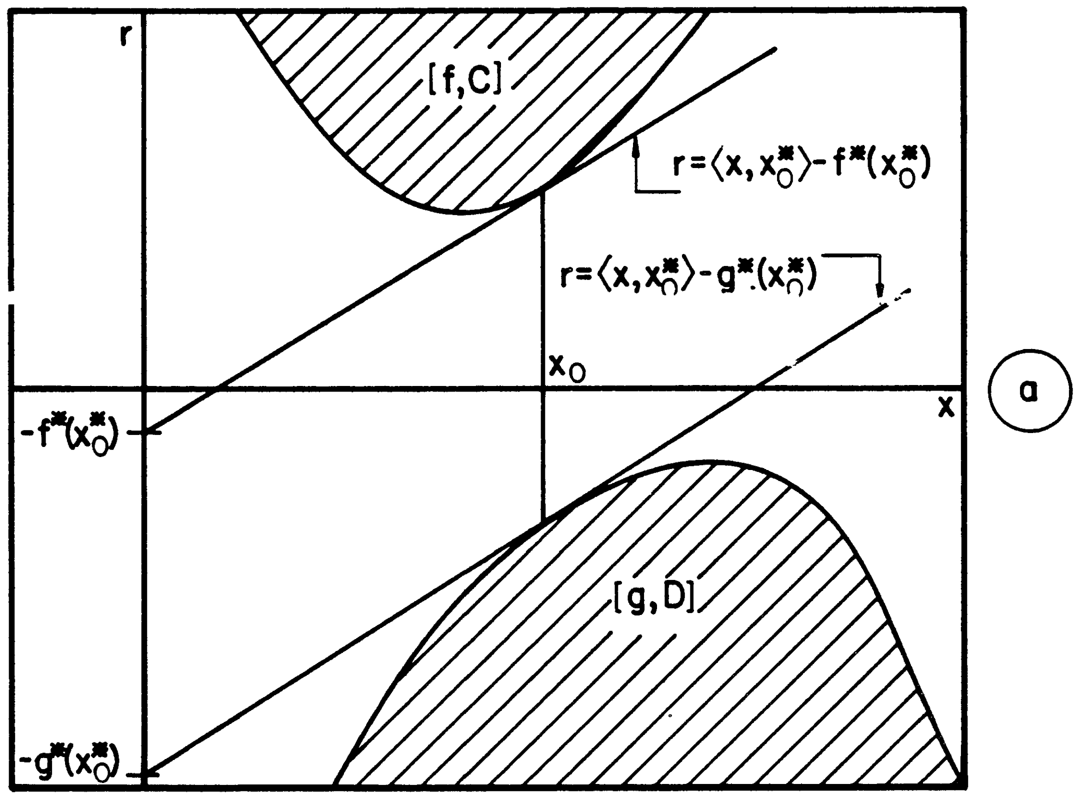
Figure 94.

$$\min\{\langle x, x_0^*\rangle - g(x) : x \in D\} = \langle x_0, x_0^*\rangle - g(x_0). \qquad \nabla$$

This theorem says that the minimal vertical separation of the sets [f, C] and [g, D] is equal to the maximal vertical separation of any two parallel supporting hyperplanes separating these sets, provided they are not vertical. Refer to diagram (96a) for the geometrical interpretation.

In what follows we shall define a situation in which the assumption concerning $C \cap D$ will not hold. However it can be seen that this assumption is too strong to prove the result. In fact, it can be checked from [3], page 202, that $C \cap D$ is required to contain points in the relative interior of C and D in order to guarantee the existence of a nonvertical separating hyperplane between [g, D] and [f - μ, C], the vertical displacement of [f, C] tangent to [g, D] at x_0. Therefore, if we prove that there exists such a separating hyperplane, then Fenchel's Theorem can be applied in our case, provided the other assumptions are valid. This will be done in Theorem (98).

In what follows, the sets $\mathcal{P}$ and $\Omega(f)$ (considered as subsets of $\mathbb{R}^p$, p = mr, when the vec notation is used) will be regarded as part of the epigraph and the hypograph, respectively, of two functions on $\mathbb{R}^{p-1}$ (see illustration (96b)). To characterize those functions we must rotate the basis of the coordinate system of $\mathbb{R}^p$. We define the direction of the vertical axis to be the direction of $\hat{c} = \hat{x} - \hat{y}$, where $\hat{x} = \theta(\hat{y}) \in \mathcal{P}$ and $\hat{y} \in \Omega(f)$, i.e., where $\hat{x}$, $\hat{y}$ solve the problem $\min\{\|x-y\| : x \in \mathcal{P}, y \in \Omega(f)\}$. So, the vertical unit vector is $\hat{c}/\|\hat{c}\|$. The remaining coordinate directions are not relevant for the proof and need not be specified. Define C as the orthogonal projection of $\mathcal{P}$ onto the subspace normal to $\hat{c}$ and let $f: C \to \mathbb{R}$ so that f(c) is the smallest number for which (c, f(c)) belongs to $\mathcal{P}$, i.e.,



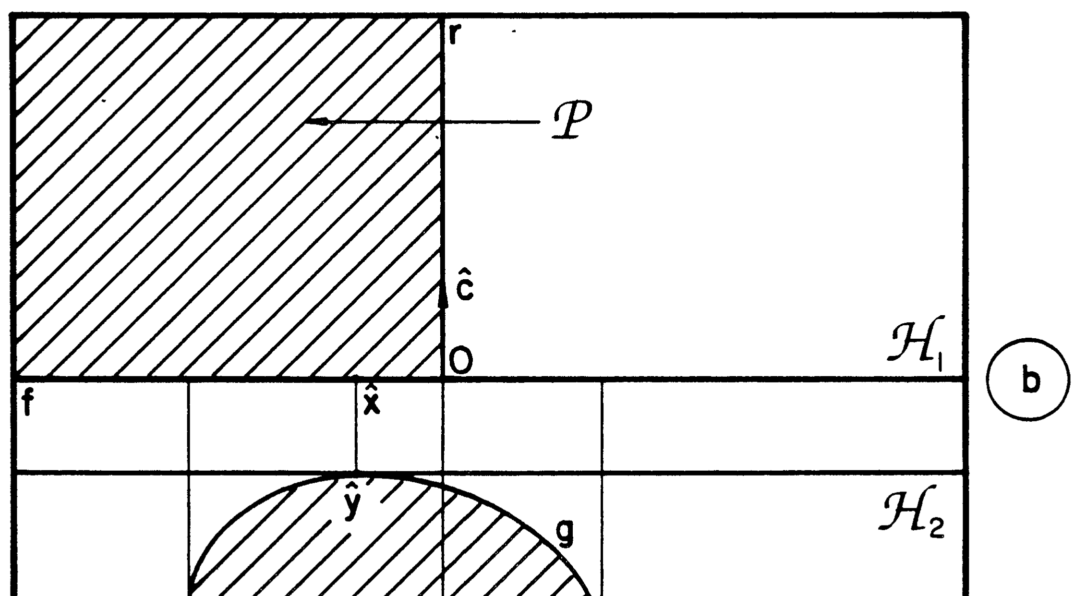


Figure 96.

$$f(c) \triangleq \min\{r \in \mathbb{R} : (c, r) \in P\}.$$

Note that the epigraph [f, C] coincides with P, and that C is convex. In order to prove that f is convex, take two elements of C, c_1 and c_2, and consider the convex combination $\alpha c_1 + (1 - \alpha)c_2$, $\alpha \in [0, 1]$. Let

(97) $r = \alpha f(c_1) + (1 - \alpha)f(c_2)$.

Then

$$(\alpha c_1 + (1 - \alpha)c_2, r) = \alpha(c_1, f(c_1)) + (1 - \alpha)(c_2, f(c_2)),$$

i.e., the point $(\alpha c_1 + (1 - \alpha)c_2, r)$ of $\mathbb{R}^p$ equals a convex combination of two points of P and thereby belongs to P. It follows from the definition of f that, for r of (97),

$$f(\alpha c_1 + (1 - \alpha)c_2) \leqq r,$$

showing that f is convex.

Similarly define the concave functional on a convex set $g: D \to \mathbb{R}$ such that for all elements $d \in D$

$$g(d) = \max\{r \in \mathbb{R} : (d, r) \in \Omega(f)\}.$$

Finally, we can state

(98) <u>THEOREM</u>. If $\hat{y}$ solves problem (4) and $\hat{c} = \theta(\hat{y}) - \hat{y} \neq 0$, then $\hat{c}/\|\hat{c}\|$ is the only solution for problem (72).

<u>PROOF</u>. Consider the situation shown in Figure (96b) with the convex and concave functions, f and g respectively, defined by the two sets P and $\Omega(f)$. As before, let $\hat{c} = \hat{x} - \hat{y}$, where $\hat{y} \in \Omega(f)$ and $\hat{x} = \theta(\hat{y}) \in P$ are the closest points in the sets. Let H_1 and H_2 be the supporting hyperplanes, at $\hat{x}$ and $\hat{y}$ respectively,

$$H_1 = \{x: < \frac{\hat{c}}{\|\hat{c}\|}, x> = 0\}$$

$$H_2 = \{x: < \frac{\hat{c}}{\|\hat{c}\|}, x> = -\|\hat{c}\|\}.$$

Recall that if H_1 supports P it must contain the origin. H_2 is defined as the hyperplane parallel to H_1 at a distance of $\|\hat{c}\|$, which passes through $\hat{y}$. The sign of $< \frac{\hat{c}}{\|\hat{c}\|}, \hat{y}>$ (negative, by Proposition (6)) gives the exact expression. The halfspace defined by H_2 that contains $\Omega(f)$ is

$$\{x: < \frac{\hat{c}}{\|\hat{c}\|}, x> \leqq -\|\hat{c}\|\},$$

since H_2 must separate $\Omega(f)$ from P and we have (Proposition (6)) that, for all $x \in P$, $< \frac{\hat{c}}{\|\hat{c}\|}, x> \geqq 0 > -\|\hat{c}\|$. It is clear from the figure that a supporting hyperplane to $\Omega(f)$ can be seen as supporting [g, D] as well. Thus, the above halfspace contains [g, D] and so, for all $x \in [g, D]$,

$$(99) \quad < \frac{\hat{c}}{\|\hat{c}\|}, x> \leqq -\|\hat{c}\|.$$

Now, consider a point $x \in [f, C]$ and define the vertical displacement of the set [f, C],

$$\{z = x - \hat{c}: x \in [f, C]\}.$$

Since by definition of [f, C], $x = (r, y)$, for some $y \in C$ and $r \geqq f(y)$, an element of the above set can be written as $z = (r, y) - (\|\hat{c}\|, 0) = (r - \|\hat{c}\|, y)$, and the set as

$$\{(r - \|\hat{c}\|, y): y \in C, r \geqq f(y)\}$$

$$= \{(r - \|\hat{c}\|, y): y \in C, r - \|\hat{c}\| \geqq f(y) - \|\hat{c}\|\}$$

$$= \{(r', y): y \in C, r' \geqq f(y) - \|\hat{c}\|\}$$

$$\triangleq [f - \|\hat{c}\|, C].$$

Now, the fact that we have $< \frac{\hat{c}}{\|\hat{c}\|}, x> \geqq 0$ for all $x \in P$ implies that, for all $z \in [f - \|\hat{c}\|, C]$,

$$< \frac{\hat{c}}{\|\hat{c}\|}, z + \hat{c} > \geqq 0,$$

and thus

$$< \frac{\hat{c}}{\|\hat{c}\|}, z> \geqq - \|\hat{c}\| .$$

Consequently, this fact and (99) proves that H_2 separates $[f - \|\hat{c}\|, C]$ from [g, D]. As we have remarked this fact substitutes for one of the assumptions of Fenchel's Theorem. The set P (≡ [f, C]) has nonempty interior, therefore the theorem applies for the pair [f, C] and [g, D], and we can say that $\|\hat{c}\|$, the minimal vertical separation of [f, C] and [g, D], is equal to the largest of the vertical distances between any supporting hyperplane below [f, C] and above [g, D]#. Consequently, for any other pair of parallel supporting hyperplanes, the vertical distance between them is not greater than $\|\hat{c}\|$. Consider now the (orthogonal) distance between any two parallel hyperplanes. Obviously it is not greater than the vertical distance, so the situation is that the largest distance between any pair of supporting hyperplanes is $\|\hat{c}\|$. In order to write down this algebraically, which will lead us to the final result, consider two arbitrary supporting hyperplanes to [f, C] and [g, D],

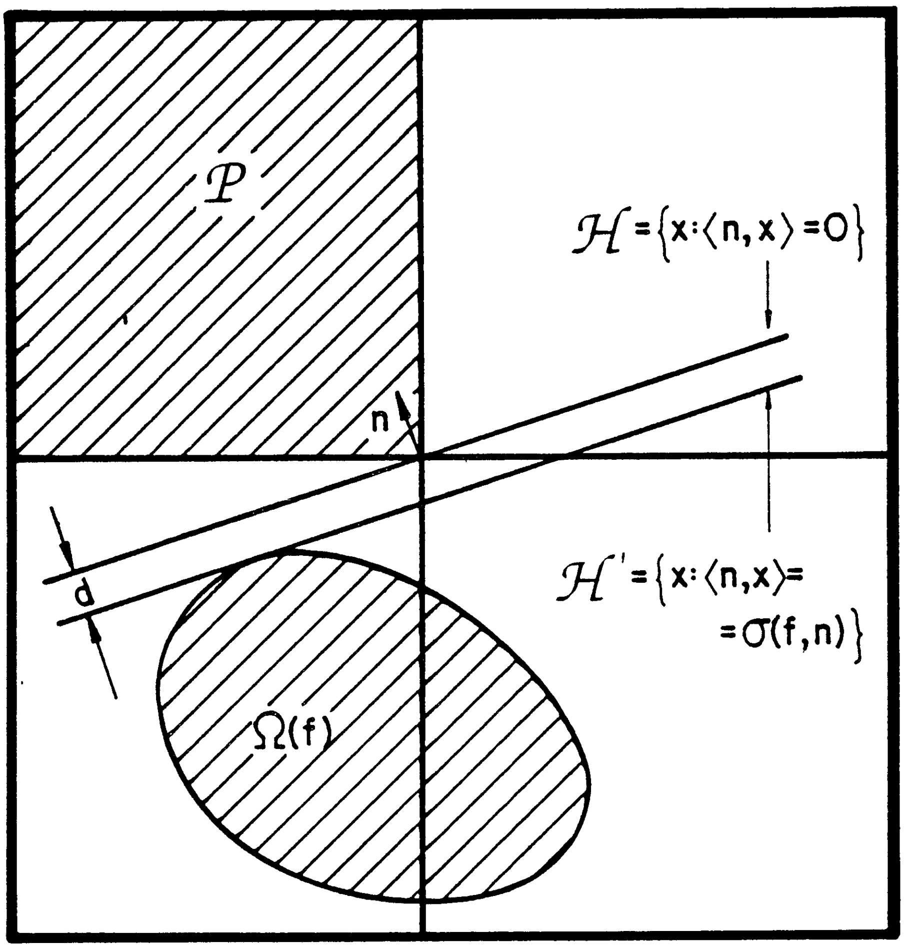
$$H = \{x: <n, x> = 0\}$$

and

$$H' = \{x: <n, x> = -d\},$$

for some $d > 0$, with n chosen with unit norm and in such a way that the halfspace $\{x: <n, x> \geqq 0\}$ contains [f, C] (see Figure (100)). Lemma (87) says that a vector n satisfies $<n, x> \geqq 0$ for all elements x of P if and only if n is itself an element of P. Thus, for the normal

#which are parallel



$\mathcal{P}$
$\mathcal{H} = \{x : \langle n, x \rangle = 0\}$
n
d
$\mathcal{H}' = \{x : \langle n, x \rangle = \sigma(f, n)\}$
$\Omega(f)$

Figure 100.

n to H and H', $n \in S \cap P$. The distance between the hyperplanes is d and, since $0 \in H$, d can be written in terms of the supporting function to $\Omega(f)$ (see [3], page 136), as

$$d = -\max\{<n, x>: x \in \Omega(f)\} = -\sigma(f, n).$$

Therefore, as $\|\hat{c}\|$ is the largest of those distances, for all $n \in S \cap P$,

$$\|\hat{c}\| = -\sigma(f, \frac{\hat{c}}{\|\hat{c}\|}) \geqq -\sigma(f, n),$$

i.e.

$$\sigma(f, \frac{\hat{c}}{\|\hat{c}\|}) \leqq \sigma(f, n).$$

Hence, since by corollary (90) the unit vector $\frac{\hat{c}}{\|\hat{c}\|} \in P$,

$$\frac{\hat{c}}{\|\hat{c}\|} \in \arg\min\{\sigma(f, n): n \in S \cap P\},$$

and on account of the fact that $\hat{c}/\|\hat{c}\|$ is the only unit normal to H and H' with support function $-\|\hat{c}\|$, the result follows. ∇

Another important result is given by:

(101) <u>THEOREM</u>. $P \cap \Omega(f) = \phi$ if and only if $\sigma(f, \hat{s}) < 0$, for $\hat{s}$ the solution to problem (72) (i.e., if and only if $\hat{s}$ is a first-order descent direction).

<u>PROOF</u>. (⇒) Suppose $\Omega(f) \cap P = \phi$. Then $\hat{c} \neq 0$ and, for $\hat{s} = \hat{c}/\|\hat{c}\|$, $\sigma(f, \hat{s}) = -\|\hat{c}\| < 0$. (from proof of Theorem (98)).

Alternatively, this can be proved directly as follows:

(⇒) Suppose $\Omega(f) \cap P = \phi$. Then there exists a strongly separating hyperplane H between $\Omega(f)$ and P, which implies that there exists a unit normal n to the hyperplane such that

$$\min\{<n, x>: x \in P\} > \max\{<n, \omega>: \omega \in \Omega(f)\}.$$

However, $0 \in P$, therefore the left hand side of the above inequality is nonpositive, and so

(102) $\max\{<n, \omega>: \omega \in \Omega(f)\} < 0.$

Now consider the supporting hyperplane to $\mathcal{P}$, parallel to $\mathcal{H}$, i.e., with normal n as well. In the proof of proposition (6) it is shown that any hyperplane supporting $\mathcal{P}$ is of the form $\{x \in \mathbb{R}^p : <n, x> = 0\}$. Obviously the halfspace $\{x \in \mathbb{R}^p : <n, x> \geq 0\}$ contains $\mathcal{P}$. Thus, for all $x \in \mathcal{P}$, $<n, x> \geq 0$. However from this and Lemma (87), $n \in \mathcal{P}$. Thus, (102) gives

$$\sigma(f, \hat{s}) = \min\{\{\max<s, \omega>: \omega \in \Omega(f)\}: s \in \mathcal{S} \cap \mathcal{P}\}$$

$$\leq \max\{<n, \omega>: \omega \in \Omega(f)\} < 0$$

($\Leftarrow$) Assume that

$$\sigma(f, \hat{s}) = \max\{<\hat{s}, \omega>: \omega \in \Omega(f)\} < 0.$$

As a consequence, for all $\omega \in \Omega(f)$, $<\hat{s}, \omega> < 0$. However $\hat{s}$ is a solution to problem (72), and so it must belong to $\mathcal{P}$. It follows from Lemma (87) therefore that $\omega \notin \mathcal{P}$. This proves that $\mathcal{P} \cap \Omega(f) = \phi$. ∇

The practical significance of Theorems (98) and (101) is summarized next. Theorem (101) says that $\mathcal{P} \cap \Omega(f) = \phi$ if and only if the solution to the search direction problem is first-order descent ($\sigma(f, \hat{s}) < 0$). Observe however that saying $\mathcal{P} \cap \Omega(f) = \phi$ is equivalent to saying that the solution to the minimum distance problem between $\mathcal{P}$ and $\Omega(f)$ is nonzero. Thus, a necessary and sufficient condition for a first-order descent solution to the search direction problem is that the minimum distance solution be nonzero. If this is true then the minimum distance vector defines uniquely the search direction solution, which is what Theorem (98) demonstrates.

Thus, the minimum distance solution defines the solution to the constrained search direction problem. Theorems (98) and (101) are the equivalent of some of the results of Section 4.1 (summarized in Theorem (4.6) and result (4.9a)). Those showed the relationship between the minimum norm problem and the support function minimization problem for a convex set. Since the search direction problem is the minimization of the support function of $\Omega(f)$, they gave the relationship between the minimum norm and search direction problems. Here we have related the minimum distance with the constrained search direction problems. We end this section by stating the following two results, equivalent to (4.9b) and (4.9c):

$$-P \cap \Omega(f) \subset \partial\Omega(f) \iff \sigma(f, \hat{s}) = 0$$

$$-P \cap \mathring{\Omega}(f) \neq \phi \iff \sigma(f, \hat{s}) > 0$$

5.6 ALGORITHM FOR THE SEARCH DIRECTION PROBLEM

The fact that the original problem of seeking a search direction for the constrained dominant feedback algorithm may be converted into a minimum distance optimization problem, enables the use of the theory developed in Sections 5.1, 5.2 and 5.3, to be applied for solving the former problem. Thus, the algorithm for determining the minimum distance between an orthant and a convex set, algorithm (48), will be applied to P and $\Omega(f)$. In order to adapt it a few remarks must be made.

The minimum distance problem between the orthant P and the convex set $\Omega(f)$ reduces to minimizing $\chi = \|\theta(y) - y\|^2$ when $y \in \Omega(f)$ and $\theta(y)$ is the projection of y onto P. A point $y \in \Omega(f)$ is defined as an ε-approximation for $\chi(\hat{y})$ when $|\chi(y) - \chi(\hat{y})| < \varepsilon$. Although Theorem (98) says that any minimizer $\hat{y}$ for χ such that $\chi(\hat{y}) \neq 0$ gives the minimizer $\hat{s}$ for σ through the formula

$$\hat{s} = \frac{\theta(\hat{y}) - \hat{y}}{\| \theta(\hat{y}) - \hat{y} \|},$$

an ε-approximation to $\chi(\hat{y})$ will not necessarily give an ε-approximation to $\sigma(\hat{s})$ ($= \sigma(f, \hat{s})$, where the variable f is omitted). This implies that the stopping condition given by Step 5 of Algorithm (48) needs to be changed in order to provide a solution that ε-approximates $\sigma(\hat{s})$ instead of $\chi(\hat{y})$.

Diagram (104) depicts the values of both functions $\chi(y^j)$ and $\sigma(s^j)$, for a few iterations of Algorithm (48) on the example of Figure (53b). The function which is being minimized is χ not σ, and thus there is no reduction of σ at each iteration. Since, unlike χ, σ depends on the geometry of $\Omega(f)$, they have quite different behaviour, even near the minimum point. Despite that, we know that the minimum of both coincides (in the sense that $\hat{y}$ gives $\hat{s}$) and that σ is continuous. So, Algorithm (48) will eventually approach the minimal value of σ and this will ensure a negative $\sigma(y^j)$ after a finite number of iterations, which is basically our objective.

The change in the stopping condition must be such that the algorithm terminates when a s^j satisfying

(103) $\quad |\sigma(s^j) - \sigma(\hat{s})| < \varepsilon$

is reached. However, from Theorem (98),

$$\sigma(\hat{s}) = - \| \theta(\hat{y}) - \hat{y} \| ,$$

and since $\| \theta(\hat{y}) - \hat{y} \| \leqq \| \theta(y^j) - y^j \|$,

$$-\sigma(\hat{s}) \leqq \| \theta(y^j) - y^j \|$$

which yields

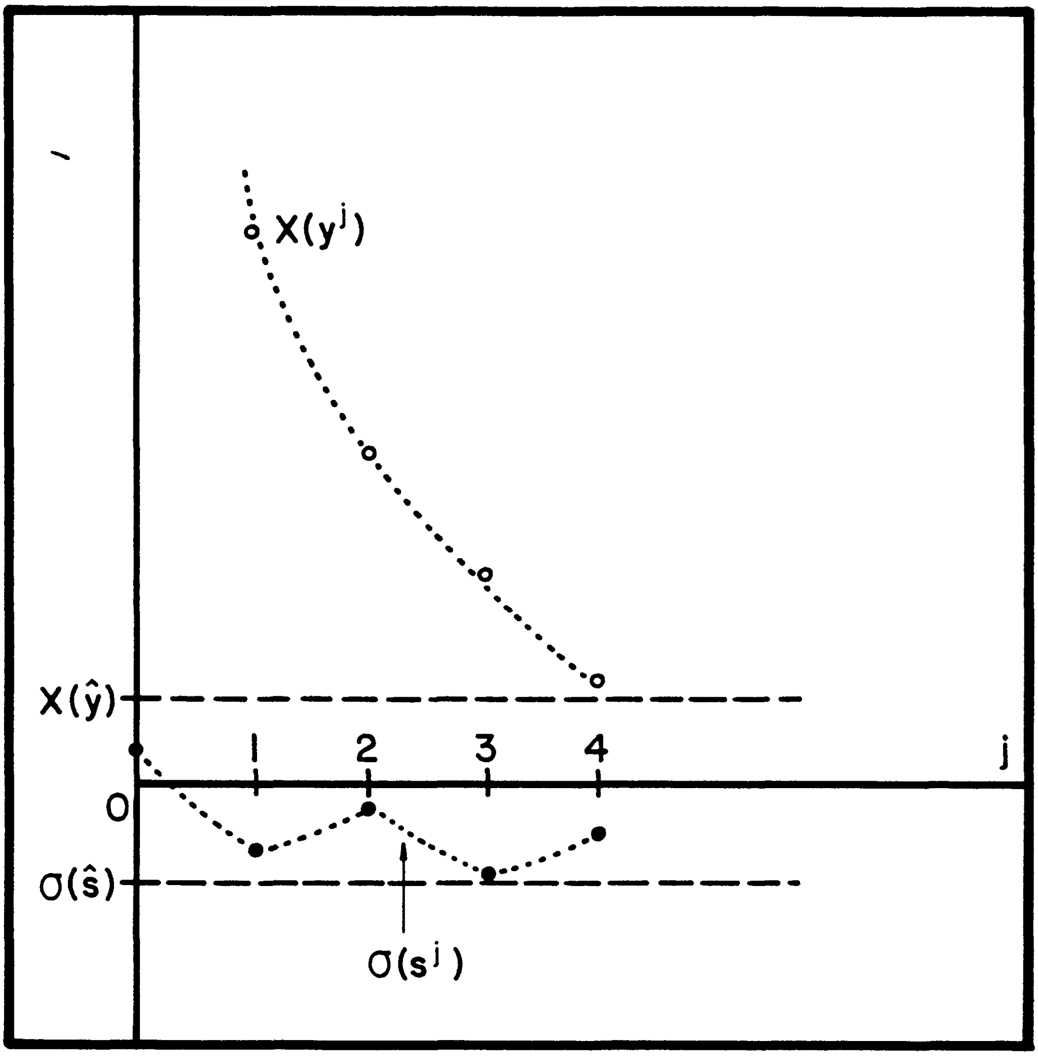


Figure 104.

$$\sigma(s^j) - \sigma(\hat{s}) \leqq \sigma(s^j) + \| \theta(y^j) - y^j \| .$$

Thus, it follows from the above that, if at the j-th iteration

$$\sigma(s^j) + \| \theta(y^j) - y^j \| < \varepsilon$$

then condition (103) is satisfied. Hence the above inequality guarantees an ε-approximation to $\sigma(\hat{s})$, and since (see Figure (105)),

(106) $$d = \sigma(s^j) + \| \theta(y^j) - y^j \| = < \frac{v}{\|v\|}, y^j - z^j >$$

the stopping condition will be:

$$< \frac{v}{\|v\|}, y^j - z^j > < \varepsilon.$$

Besides this, another test examining the negativeness of σ must be inserted. A computable formula for $\sigma(s^j)$ follows from (106), which is

$$\sigma(s^j) = < \frac{v}{\|v\|}, y^j - z^j > - \| \theta(y^j) - y^j \| .$$

Obviously it is impossible to prespecify ε and guarantee the termination of the algorithm with $\sigma(s^j) < 0$. A practical procedure is to re-estimate ε if that does not occur. We finish by pointing out that the stopping criteria that tests convergence of χ in Algorithm (48) is $d\|v\| < \varepsilon$, which for $\|v\|$ small, does not imply that $d < \varepsilon$, as expected.

The algorithm devised to give a negative solutuon to the search direction problem (72) therefore becomes:

(107) <u>ALGORITHM FOR THE SEARCH DIRECTION PROBLEM</u>

1. Choose $\varepsilon \in (0, 1)$, $y^0 \in \Omega(f)$, $\varepsilon' \in (0,\varepsilon)$.

   Set $j = 0$.

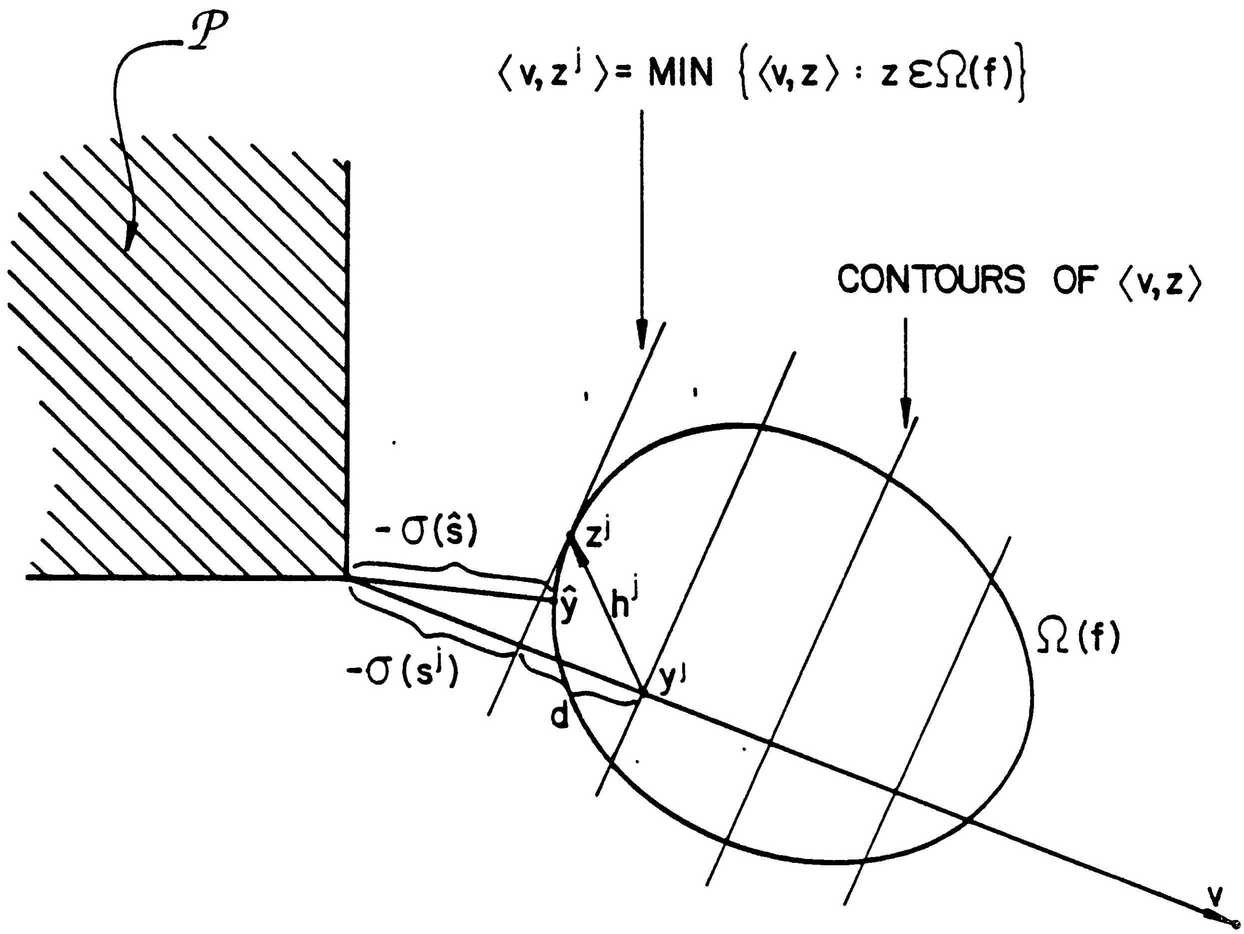


Figure 105.

2. Calculate the gradient v by

$$v_i = (\nabla\chi(y^j))_i = \begin{cases} 2y_i^j, & \text{if } \langle y^j, a^i\rangle > 0 \\ 0, & \text{otherwise} \end{cases}$$

where a^i are the columns of A.

3. (See Remark (108)). Find an eigenvector x^j associated with the smallest eigenvalue of dK(f; v), then

$$x^j \in \arg\min\{x'\, dK(f;\, v)x:\ x \in Bn\}.$$

4. (See Remark (108). Find the contact point z^j between $\Omega(f)$ and its supporting hyperplane normal to v, where

$$z^j = \begin{bmatrix} x^{j'} \dfrac{\partial K(f)}{\partial f_1} x^j \\ \vdots \\ x^{j'} \dfrac{\partial K(f)}{\partial f_p} x^j \end{bmatrix}.$$

5. Find the search direction

$$h^j = z^j - y^j.$$

6. If $\langle \dfrac{v}{\|v\|}, -h^j\rangle < \varepsilon$ then go to 7; else go to 8.

7. If $\sigma(s^j) = \langle \dfrac{v}{\|v\|}, -h^j\rangle - \|\theta(y^j) - y^j\| < 0$ then set

$$\hat{s} = \frac{\theta(y^j) - y^j}{\|\theta(y^j) - y^j\|},$$

where $\theta(y^j)$ is given in (18), and stop: else set $\varepsilon = \varepsilon/2$. If $\varepsilon < \varepsilon'$ stop; else continue.

8. Determine the step length parameter λ^j using for example Algorithm (64).

9. Set $y^{j+1} = y^j + \lambda^j h^j$, $j = j+1$, and go to 2. ∇

(108) REMARK. Steps 2 and 3 find the minimizer of $\langle v, \cdot\rangle$ over $\Omega(f)$. This is done in the same way as to obtain the maximizer (i.e., to evaluate the support function of $\Omega(f)$), which was shown in Section 4.1. Although the changes are obvious, we shall describe the procedure for them here. Recall that

$$\Omega(f) = \mathrm{co}\{z \in \mathbb{R}^p : z_i = x' \frac{\partial K(f)}{\partial f_i} x,\ x \in Bn\}$$

and so, the minimum of $\langle v, z\rangle$, as z ranges over $\Omega(f)$, is achieved for some z^j of the type

$$z^j = \left[x^{j'} \frac{\partial K(f)}{\partial f_1} x^j, \ldots, x^{j'} \frac{\partial K(f)}{\partial f_p} x^j \right]',$$

for some $x^j \in Bn$. Then

$$\langle v, z^j\rangle = \sum v_i z_i^j = \sum v_i x^{j'} \frac{\partial K(f)}{\partial f_i} x^j$$

$$= x^{j'} (\sum v_i \frac{\partial K(f)}{\partial f_i}) x^j = x^{j'} dK(f; v) x^j .$$

i.e. optimizing $\langle v, z\rangle$ with respect to $z \in \Omega(f)$ is equivalent to optimizing $x' dK(f; v) x$ with respect to $x \in Bn$. An x minimizing $x' dK(f; v) x$ on Bn is an eigenvector of $dK(f; v)$ associated with the minimal eigenvalue of $dK(f; v)$. Hence x^j is such an eigenvector. Furthermore, since the minimum of $\langle v, z\rangle$ is achieved at any contact point of $\Omega(f)$ with its supporting hyperplane normal to v (see Figure (105)), then z^j must be a contact point. Steps 3 and 4 find x^j and z^j. ∇

5.7 THE MINIMUM NORM PROBLEM AND THE UNCONSTRAINED SEARCH DIRECTION

This section is concerned with the particular case in which $P = \{0\}$, and problem (3) reduces to the:

(109) MINIMUM NORM PROBLEM.

$$\min\{\|y\|^2 : y \in \Omega\},$$

where $\Omega \subset \mathbb{R}^p$ is a convex set. ∇

The algorithm of [1] for convex optimization applied to the norm function

$$\chi(y) = \|y\|^2,$$

produces an algorithm, having only two points in which it differs from Algorithm (48), which optimizes $\|\theta(y) - y\|^2$. First, the expression for the gradient of the function χ becomes

$$\nabla\chi(y) = 2y.$$

Also, the line search involves minimizing the quadratic

$$f(\lambda) = \chi(y + \lambda h) = \|y + \lambda h\|^2 = \sum(y + \lambda h)_i^2,$$

and so, the minimization over the interval [0, 1] is immediate.

The obvious application of this in the dominant feedback theory is for the case when the search direction is defined by

$$\hat{s} \in \arg\min\{\lambda\max dK(f; s) : s \in S\},$$

which happens in the unconstrained dominant feedback algorithm. In fact, since $\lambda\max dK(f; s)$ is the support function to the set $\Omega(f)$, it follows from Theorem (4.6) that $\hat{s} = \hat{\omega}/\|\hat{\omega}\|$, where $\hat{\omega}$ is the minimum norm of $\Omega(f)$.

The corresponding algorithm is therefore a slight variation of Algorithm (107):

(110) ALGORITHM FOR THE UNCONSTRAINED SEARCH DIRECTION

1. Choose $\varepsilon \in (0, 1)$, $y^0 \in \Omega(f)$.

   Set $j = 0$.

2. Calculate the gradient of the norm function

   $v = \nabla\chi(y^j) = 2y^j$.

3. Find an eigenvector x^j associated with the smallest eigevalue of $dK(f; v)$. Then

   $$x^j \in \arg\min\{x'dK(f; v)x : x \in Bn\}.$$

4. Find the contact point z^j between $\Omega(f)$ and its supporting hyperplane normal to v, where

   $$z^j = \begin{bmatrix} x^{j'} \dfrac{\partial K(F)}{\partial f_1} x^j \\ \vdots \\ x^{j'} \dfrac{\partial K(f)}{\partial f_p} x^j \end{bmatrix}.$$

5. Find the search direction

   $$h^j = z^j - y^j.$$

6. If $< \dfrac{v}{\|v\|}, -h^j > < \varepsilon$ then go to 7; else go to 8.

7. If $\sigma(s^j) = < \dfrac{v}{\|v\|}, -h^j > - \|y^j\| < 0$ then set $\hat{s} = -\dfrac{y^j}{\|y^j\|}$

   and stop; else set $\varepsilon = \varepsilon/2$ and continue.

8. Determine the step length parameter λ^j by

   $$\lambda^j = \text{sat}\left(- \frac{\sum y_i^j h_i^j}{(\sum h_i^j)^2} \right)$$

   where the function sat is defined by

$$\text{sat}(\alpha) \triangleq \begin{cases} 0, & \alpha < 0 \\ \alpha, & \alpha \in [0, 1] \\ 1, & \alpha > 1 \end{cases} .$$

9. Set $y^{j+1} = y^j + \lambda^j h^j$, $j = j+1$, and go to 1. ∇

(111) REMARK. The step length is chosen in Step 8 as the scalar λ^j such that it minimizes the quadratic

$$f(\lambda) = \sum((y_i^j)^2 + 2y_i^j h_i^j \lambda + h_i^j \lambda^2)$$

over [0, 1], since the unconstrained minimum of f is $-\dfrac{\sum y_i^j h_i^j}{\sum (y_i^j)^2}$.
Actually, since χ is the norm function, $y^j+\lambda^j h^j$ is the closest point to the origin of the segment of line $[y^j, y^j+h^j]$. ∇

Algorithm (110) is essentially Y.C. Ho's algorithm used in [2].

5.8 THE IMPLEMENTABLE ALGORITHM FOR THE LINEARLY CONSTRAINED DOMINANT FEEDBACK PROBLEM

Before describing an overall algorithm that finds a dominant output feedback with the entries of the feedback matrix ranging over given intervals (Problem (66)), there is still one detail to be discussed: the limit imposed by the constraints on the unidimensional search along the search direction. In other words, the problem that remains to be solved is, given a search direction, to find the upper bound for the step length. This is very simple to do owing to the type of constraints considered.

It is helpful to start with the two-dimensional example shown in Diagram (112). Suppose the search direction $\hat{s}$ has been found, by finding the closest point in $\Omega(f)$, as shown in the figure. The search interval $[0, \hat{\lambda}]$ will be determined by finding the first point at which

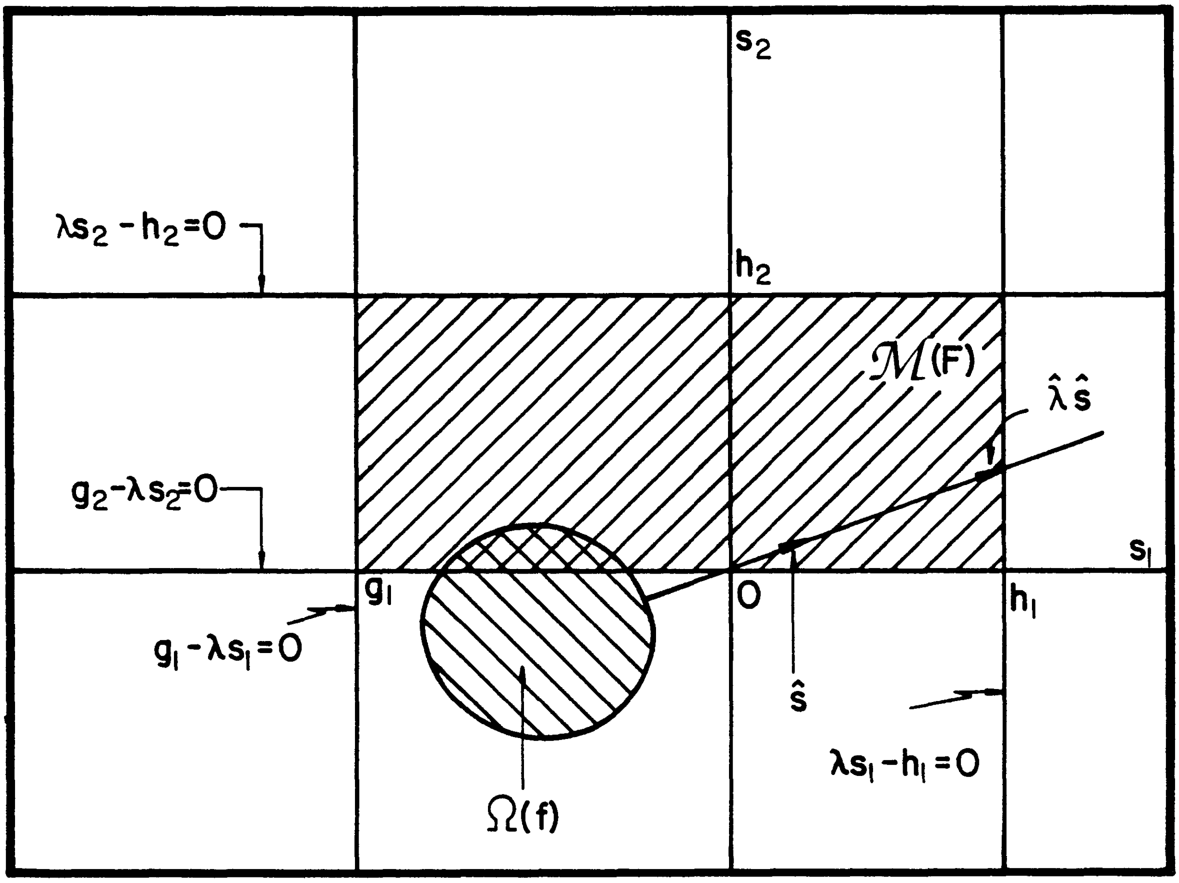

s_2
$\lambda s_2 - h_2 = 0$
h_2
$\mathcal{M}(F)$
$\hat{\lambda}\hat{s}$
$g_2 - \lambda s_2 = 0$
s_1
g_1
0
h_1
$g_1 - \lambda s_1 = 0$
$\hat{s}$
$\lambda s_1 - h_1 = 0$
$\Omega(f)$

Figure 112.

a constraint becomes active, as we move along $\hat{s}$. Recall from (67) that

$$M(f) = \{s \in \mathbb{R}^p : g_i(f) \leqq s_i \leqq h_i(f)\}.$$

This set can be equivalently defined as

$$M(f) = \{\lambda s \in \mathbb{R}^p : \lambda \in \mathbb{R}\,,\ s \in S,\ g_i(f) \leqq \lambda s_i \leqq h_i(f)\}.$$

In the example the constraint $\lambda\hat{s}_1 - h_1 \leqq 0$ is the first to become active, therefore $\hat{\lambda} = h_1/\hat{s}_1$.

In order to develop a general procedure for finding the upper bound for the step length along a direction s, consider the following three cases

(i) $\hat{s}_i > 0$

$\lambda\hat{s}_i - h_i(f) \leqq 0$ gives the solution $\hat{\lambda} = h_i(f)/\hat{s}_i$

$g_i(f) - \lambda\hat{s}_i \leqq 0$ holds for all $\lambda \geqq 0$ since $g_i(f) = a_i - f_i \leqq 0$.

(ii) $\hat{s}_i < 0$

$\lambda\hat{s}_i - h_i(f) \leqq 0$ holds for all $\lambda \geqq 0$, since $h_i(f) = b_i - f_i \geqq 0$

$g_i(f) - \lambda\hat{s}_i \leqq 0$ gives the solution $\hat{\lambda} = g_i(f)/\hat{s}_i$.

(iii) $\hat{s}_i = 0$

Here both $\lambda\hat{s}_i - h_i(f) \leqq 0$ and $g_i(f) - \lambda\hat{s}_i \leqq 0$ hold for all $\lambda \geqq 0$.

Thus, in view of the results for each case, the step length interval $[0, \hat{\lambda}]$ is chosen by

$$\hat{\lambda} = \min\left\{ \frac{h_i(f)}{\hat{s}_i}\,,\ \frac{g_j(f)}{\hat{s}_j} : i,j \in \{1,\dots,p\} \text{ such that } \hat{s}_i > 0,\ \hat{s}_j < 0 \right\}$$

We may now describe the implementation of Algorithm (79).

(113) IMPLEMENTABLE ALGORITHM FOR LINEARLY CONSTRAINED DOMINANT OUTPUT FEEDBACK

1. Select $F^0 \in M$, $\varepsilon \in (0, 1)$, $\beta \in (0.5, 0.8)$.

   Set $j = 0$.

2. Choose the search direction s^j by following Steps 2.1 to 2.8 below, where vec notation is used, and $p = mr$.

2.1. Find the active index sets at f^j

$$I(f^j) = \{i = 1,\dots,p:\ f_i = a_i\}$$

$$J(f^j) = \{i = 1,\dots,p:\ f_i = b_i\}.$$

2.2. If $I(f^j) \cup J(f^j) \neq \phi$ then go to Step 2.5 (f^j lies in the boundary of M); else continue.

2.3. Perform Algorithm (110), using f^j and the tolerance ε.

2.4. If $\sigma(\hat{s}) = \lambda\max\, dK(f^j;\ s) \geq 0$ (obtained from (110)) then set $f^* = f^j$ and stop; else set the search direction $s^j = \hat{s}$ and go to Step 3.

2.5. Construct the $p \times p$ diagonal matrix $A = (a_{ij})$ with diagonal elements

$$a_{ii} = \begin{cases} -1 & \text{if } i \in I(f^j) \\ 1 & \text{if } i \in J(f^j) \\ 0 & \text{otherwise} \end{cases} .$$

2.6. Construct the nonsingular diagonal $p \times p$ matrices A^k, $k = 1,\dots,2^K$, where K is the number of zeros in the diagonal of A, obtained from A by substituting the zeros of the diagonal with all possible combinations of 1 and -1, i.e., a matrix A^k will be such that

$$\begin{cases} a_{ii}^k = a_{ii}, & \text{if } a_{ii} \neq 0 \\ a_{ii}^k = \pm 1, & \text{if } a_{ii} = 0 \\ a_{ij}^k = 0, & \text{if } i \neq j \end{cases}$$

(Note: each A^k defines an orthant. The union of them is P.)

2.7. For $k = 1,\ldots,2^K$, follow Steps 2.7.1 to 2.7.2.

2.7.1 Perform Algorithm (107), using f^j, the tolerance ε, and the orthant

$$P^k = \{x \in \mathbb{R}^p : A^k x \leqq 0\}.$$

2.7.2 Set $s(k) = \hat{s}$.

2.8. If $\min\{\sigma(s(k)) : k = 1,\ldots,2^K\} \geqq 0$ then set $f^* = f^j$ and stop; else set the search direction

$$s^j \in \arg\min\{\sigma(s(k)) : k = 1,\ldots,2^K\}.$$

3. Compute the step length interval $[0, \hat{\lambda}^j]$ along s^j by

$$\hat{\lambda}^j = \min\left\{ \frac{b_{hi} - f_{hi}}{s_{hi}^j}, \frac{f_{k\ell} - a_{k\ell}}{s_{k\ell}^j} : h,k \in \{1,\ldots,m\},\ i,\ell \in \{1,\ldots,r\}, \text{ such that } s_{hi}^j > 0,\ s_{k\ell}^j < 0 \right\}.$$

4. Compute the step length λ^j by following Steps 4.1 to 4.6 below.

4.1. Set $q = 1$.

4.2. Compute a $k > 2$ such that

$$\frac{k(k-2)}{k-1} \geqq \frac{12a\gamma_1}{\gamma_2}$$

where

$$a = 4\|K(F_0)\|\,\|B\|\,\|C\|\,(\lambda\min Q)^{-1}$$

$$\gamma_1 = 2\|C\|\,\|K(F_0)\|\,(\lambda\min Q)^{-1}\,(\|B\|\,\|K(F_0)\| + \|R\|\,F_c)$$

$$\gamma_2 = a\gamma_1 + b$$

$$b = 2\|K(F_0)\|\,\|C\|^2\,\|R\|\,(\lambda\min Q)^{-1}$$

$$F_c = \frac{\|B\|\,\|K(F_0)\|}{\lambda\min R} + \left\{ \frac{\|B\|^2\,\|K(F_0)\|^2}{(\lambda\min R)^2} + \frac{2\|A\|\,\|K(F_0)\|}{\lambda\min R} \right\}^{\frac{1}{2}} .$$

4.3. Compute

$$\rho = -\frac{4\sigma(S^j)}{kr^3\gamma_2} .$$

4.4. Set $\lambda = \rho\beta^q$.

4.5. Compute

$$\theta(\lambda) = tr[(K(F^j) - K(F^j+\lambda S^j))X_0] + \frac{1}{2}n\ \lambda\min X_0\ \sigma(S^j)\lambda .$$

4.6. If $\theta(\lambda) < 0$ or $\lambda > \hat{\lambda}^j$ set $q = q+1$ and go to 4.4; else set $\lambda^j = \lambda$ and continue.

6. Set $F^j = F^j + \lambda^j S^j$, $j = j+1$, and go to Step 2. ∇

(114) REMARK. The parameter λ^j is chosen in Step 4 using the Armijo rule described in Section 2.4. The Armijo line function here, $\frac{1}{2}n\ \lambda\min X_0\ \sigma(S^j)\lambda$, uses (obviously) $\sigma(S^j)$ instead of $\pi(F^j)$. ∇

(115) REMARK. It is not possible to prove convergence for the above algorithm, as already discussed. One way to overcome this problem is to include the "almost" active constraints in the definition of the search direction. This is done, for example, by defining the ε-active constraints at f, and taking them into account in the definition of the orthants. The ε-active index sets at f would be:

$$I_{\varepsilon}(f) = \{i = 1,\ldots,p : a_i - f_i + \varepsilon \geq 0\}$$

$$J_{\varepsilon}(f) = \{i = 1,\ldots,p : f_i - b_i + \varepsilon \geq 0\} \qquad \nabla$$

It is worthwhile summarizing what has been done in this chapter. In the first three sections we described a convex optimization problem and developed a method for solving it. In the rest of the chapter we discussed two applications for this study. First, how it leads to a practical procedure to compute a search direction for an algorithm for determining a dominant feedback, subject to linear constraints. Then, how it provides an optimality test for the case in which the feedback happens to lie on the boundary of the feasible set, which may be done irrespective of the algorithm used. Besides this we may add, as another justification for the mathematical work in the first three sections, that the proposition and solution of a problem in convex optimization theory is of interest in its own right. One advantage of the methodology of solution is that the number of variables of the original problem is halved. Another nice feature of it, as far as the dominant feedback problem is concerned, is that, by minimizing $\|\theta(y) - y\|^2$ over $\Omega(f)$, it yields a feasible vector $s^j = \theta(y^j) - y^j$ at each iteration. In contrast, the standard formulation to the problem, which is to minimize $\|x-y\|^2$, $x \in P$, $y \in \Omega(f)$, would only necessarily produce a feasible vector at the limit point of the resulting sequence. Since $\theta(y)$ can be evaluated very simply, there is no disadvantage in the approach used.

Another important point concerns Lemma (3.4). Although the result obtained there is based on the premise that the matrix $\hat{S}$ is an exact minimizer for $\sigma(S)$, it can be easily seen, following the proof, that part (a) applies also when the optimization is carried out approximately. In practical terms this implies that an accurate minimization to find a descent direction is not needed. Part (b) however is not true for an approximate solution $\tilde{S}$. In fact, assuming that $\sigma(\tilde{S}) > 0$, it is not necessarily true that the minimal value $\sigma(\hat{S})$ is positive, i.e., that F is locally dominant. We must emphasize the significance of part (a), in that it justifies naming a matrix S such that $\lambda max\ dK(F; S) < 0$ as a first-order descent direction.

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CHAPTER 6

MINIMIZATION OF THE LARGEST EIGENVALUE OF dK(F; S)

6. MINIMIZATION OF THE LARGEST EIGENVALUE OF dK(F; S)

The aim of this chapter is to describe two methods for minimizing, with respect to S, the largest eigenvalue of the Frechet-differential dK(F; S). The first method involves a matrix-norm minimization problem, and was suggested for solving this problem by Allwright in [2]. We have added the development of the expression for a gradient matrix, which is computable, and is needed for the implementation of the method. The procedure is reliable in the sense that convergence is achievable, provided some subproblems are solved (by any feasible direction algorithm) accurately. The second method relies on the theory of subdifferentials. The subdifferential of a convex, not necessarily differentiable function, is a point-to-set map. Its codomain consists of subgradients. The ideas of subdifferentials and subgradients generalize the concept of differential and gradient of a differentiable function. These concepts were introduced by Rockafellar in [13] and were later extended by Clarke in [5] for Lipschitzian function, giving rise to the generalized gradients. Subgradients and generalized gradients are important tools when dealing with nondifferentiable optimization. A simple feasible direction algorithm for that uses the generalized gradient instead of the gradient. Unfortunately convergence may fail for this algorithm - a fact which is explained by the lack of continuity of the generalized gradient. To ensure convergence it is necessary to replace gradients with the so-called smeared generalized gradients. For the details, refer to [12]. Here we shall describe how to determine the steepest descent direction for the convex function λmax dK(F; S) in terms of its subgradient so that a simple algorithm for minimizing this function may be obtained. It is hoped that this will be helpful in the task of developing a complete algorithm using smeared subgradients, which has not been done yet.

6.1 OPTIMIZING THE 2-NORM OF $dK(F;\, S) + \alpha I$

An algorithm for minimizing the 2-norm of a general symmetric-matrix-valued function when the parameter ranges over any nonempty set is studied in [2]. This algorithm can be applied to minimizing the maximum eigenvalue of $dK(F;\, S)$ when S ranges over $\mathcal{M}(F)$.

Theorem (2) gives the basis for an algorithm for solving the general problem (the infimum is used since the minimum may not exist):

(1) MATRIX-NORM MINIMIZATION PROBLEM

Find $\inf\{\|G(x)\| : x \in C\}$

where $G(x) = G(x)' \in \mathbb{R}^{nxn}$ for all $x \in C$, and $C \subset \mathbb{R}^p$ ∇

(2) THEOREM. Suppose

(i) $G(x) = G(x)' \in \mathbb{R}^{nxn}$, for all $x \in C$, C a nonempty subset of $\mathbb{R}^p$.

(ii) $\theta > 0$.

(iii) For each even non-negative integer ℓ, x^ℓ is chosen to infimize $tr[G(x)^\ell]$ approximately on C. In the sense that

(3) $$tr[G(x^\ell)^\ell] \leqq (1+\theta)\ \inf\{tr[G(x)^\ell] : x \in C\}$$

is satisfied.

Then, as $\ell \to \infty$,

$$\|G(x^\ell)\| \to \inf\{\|G(x)\| : x \in C\}.$$

Further, if for all $x \in C$, $G(x) \geqq 0$, then the non-negative integers ℓ can be used instead of the even non-negative integers.

PROOF. See [2] and Appendix. We conclude that an algorithm for minimizing $\|G(x)\|$ would be to solve approximately the sequence of optimization problems

(4) $\quad \text{minimize}\{\text{tr}[G(x)^{\ell}]: x \in C\}$

where $\ell = 2,4,\ldots,$ using a standard algorithm for differentiable functions The minimization must be carried out as accurately as possible, since satisfaction of (3) cannot be checked as the minimum of $\text{tr}[G(x)^{\ell}]$ on C is not known and a sharp lower bound does not seem to be available.

Theorem (5) shows that it is possible to minimize the largest eigenvalue of a symmetric matrix by solving a matrix-norm minimization problem:

(5) THEOREM. Suppose

(i) $L(x) = L(x)' \in \mathbb{R}^{n\times n}$ with $L(x) \geqq -\alpha I$, for all $x \in C$, for some real $\alpha > 0$.

(ii) $G(x) = L(x) + \alpha I$.

Then

$$\inf\{\lambda\max L(x): x \in C\} = \inf\{\lambda\max G(x): x \in C\} - \alpha$$

and, if $\tilde{x}$ gives

$$\|G(\tilde{x})\| \leqq \inf\{\|G(x)\|: x \in C\} + \varepsilon$$

Then

$$\lambda\max L(\tilde{x}) \leqq \inf\{\lambda\max L(x) : x \in C\} + \varepsilon$$ ∇

PROOF. Since $G(x) = L(x) + \alpha I$ and $L(x) \geqq -\alpha I$, for all $x \in C$,

$$G(x) \geqq 0.$$

Hence

(6) $\lambda\max G(x) = \| G(x) \|$.

However assumption (ii) implies that each eigenvalue $\lambda_i L(x)$ is equal to $\lambda_i\, G(x) - \alpha$, and so,

(7) $\lambda\max L(x) = \lambda\max G(x) - \alpha$.

Consequently,

$$\inf\{\lambda\max L(x) : x \in C\} = \inf\{\lambda\max G(x) : x \in C\} - \alpha$$

which with (6) proves the first assertion of the theorem. Now, suppose $\tilde{x}$ is such that

$$\| G(\tilde{x}) \| \leqq \inf\{ \| G(x) \| : x \in C\} + \varepsilon$$

Then, from this, (ii), (6) and (7),

$$\begin{aligned} \lambda\max L(\tilde{x}) &= \lambda\max G(\tilde{x}) - \alpha \\ &= \| G(\tilde{x}) \| - \alpha \\ &\leqq \inf\{\|G(x)\| : x \in C\} - \alpha + \varepsilon \\ &= \inf\{\lambda\max L(x) : x \in C\} + \varepsilon \end{aligned}$$

so that the second assertion is proved. ∇

Theorem (5) reveals that the (approximate) infimum of $\lambda\max L(x)$, when $x \in C$ and $L(x)$ is symmetric, can be obtained by infimizing (approximately) the norm of another symmetric matrix, as long as a lower bound for $L(x)$, for all x in C, is known. Therefore

(5) reveals that the (approximate) infimum of $\lambda\max L(x)$, when $x \in C$ and $L(x)$ is symmetric, can be obtained by infinimizing (approximately) the norm of another symmetric matrix, as long as a lower bound bound for $L(x)$, for all x in C, is known. Therefore Theorems (2) and (5) provide a method of solution for the following problem:

(8) λmax MINIMIZATION PROBLEM

$$\text{infimize}\{\lambda\max L(x) : x \in C\}$$

for $L(x) = L(x)' \in \mathbb{R}^{nxn}$ with $L(x) \geqq \alpha I$, for a given $\alpha > 0$ and for all $x \in C$, and $C \subset \mathbb{R}^p$. ∇

If x is identified with the search direction matrix S, $\mathbb{R}^p$ with $\mathbb{R}^{mxr}$, C with the compact feasible set $M(F)$, and $L(x)$ with the Frechet-differential $dK(F; S)$, then problem (8) is just the search direction problem (3.9) for the constrained dominant output feedback problem.

A lower bound α for $dK(F; S)$, when $S \in M(F)$, is obtained as follows: for all $S \in M(F)$, and $z \in \mathbb{R}^n$,

$$\begin{aligned} z'dK(F; S)z &= z'(\sum_{ij} s_{ij}\Gamma_{ij})z \qquad \text{(see (2.1))} \\ &= \sum_{ij} s_{ij} z'\Gamma_{ij} z \\ &\geqq -\sum_{ij} s_{ij} \|\Gamma_{ij}\|_F \|z\|_F^2 \quad . \end{aligned}$$

Hence

$$\text{(9)} \qquad \begin{aligned} dK(F; S) &\geqq -\sum_{ij} s_{ij} \|\Gamma_{ij}\|_F I \\ &\geqq -mrd \sum_{ij} \|\Gamma_{ij}\|_F I \end{aligned}$$

for d the diamter of $M(F)$. However, for any matrix A,

$$\|A\|_F \triangleq (\sum_{ij} a_{ij}^2)^{\frac{1}{2}} = (\mathrm{tr}[A'A])^{\frac{1}{2}}.$$

Taking this into account, (9) yields,

$$dK(F;\ S) \geqq -\ mrd \sum_{ij} (\mathrm{tr}[\Gamma_{ij}^2])^{\frac{1}{2}} I$$

and so we can use

$$(10) \quad \alpha = mrd \sum_{ij} (\mathrm{tr}[\Gamma_{ij}^2])^{\frac{1}{2}}.$$

Now, let us turn our attention to trace subproblem (4) specialized to the dominant feedback problem, namely,

$$(11) \quad \text{minimize}\{\mathrm{tr}(dK(F;\ S) + \alpha I)^{\ell} : S \in M(F)\},$$

for fixed $\ell \geqq 2$ and α of (10).

This objective function is differentiable and an expression for the gradient will be derived in Theorem (12). The proof is based on the works of Hutcheson in [9], where he develops an expression for $\partial \mathrm{tr}\, K(F)/\partial F$, and of Allwright/Mao in [1], where $\partial \mathrm{tr} K(F)^{\ell}/\partial F$ was obtained. In that paper the trace approach to the norm-minimization problem was used. However the authors' objective there was to minimize $\| K(F) \|$, attempting to obtain an optimal output feedback.

(12) <u>THEOREM</u>. If $F \in \mathcal{F}$, then

$$\frac{\partial \mathrm{tr}(dK(F;\ S) + \alpha I)^{\ell}}{\partial S} = 2(B'K(F) + RFC)LC'$$

where $L = L' \in \mathbb{R}^{n\times n}$ satisfies

$$(A + BFC)L + L(A + BFC)' = -\ell(dK(F;\ S) + \alpha I)^{\ell-1}$$

<u>PROOF</u>. Let

$$J(S) = \mathrm{tr}(dK(F;\ S) + \alpha I)^{\ell}$$

It follows from definition (2.1) - (2.3) that

$$\frac{\partial J(S)}{\partial s_{ij}} = \frac{\partial}{\partial s_{ij}} \; tr(\sum_{pq} \Gamma_{pq}(F)s_{pq} + \alpha I)^{\ell}$$

$$\triangleq \lim_{\delta s_{ij}\to 0} \frac{tr(\sum_{pq}\Gamma_{pq}(F)s_{pq} + \Gamma_{ij}(F)\delta s_{ij} + \alpha I)^{\ell} - tr(\sum_{pq}\Gamma_{pq}(F)s_{pq} + \alpha I)^{\ell}}{\delta s_{ij}}$$

$$\triangleq \lim_{\delta s_{ij}\to 0} \frac{tr((dK(F;S) + \alpha I) + \Gamma_{ij}(F)\delta s_{ij})^{\ell} - tr(dK(F;S) + \alpha I)^{\ell}}{\delta s_{ij}}$$

However the first-order approximation to the numerator is

$$tr[(dK(F;S)+\alpha I)^{\ell-1}\Gamma_{ij}(F)\delta s_{ij}] + tr[(dK(F;S)+\alpha I)^{\ell-2}\Gamma_{ij}(F)\delta s_{ij}(dK(F;S)+\alpha I)] + ..$$

$$+ \; tr[\Gamma_{ij}(F)\delta s_{ij}(dK(F;S) + \alpha I)^{\ell-1}]$$

Now since $tr[A^iBA^j] = tr[A^{i+j}B]$ for A,B symmetric, the above expression is equal to $\ell tr[(dK(F;\, S) + \alpha I)^{\ell-1}\Gamma_{ij}(F)\delta s_{ij}]$. Hence,

$$(13) \quad \frac{\partial J(S)}{\partial s_{ij}} = \ell tr[(dK(F;\, S) + \alpha I)^{\ell-1}\Gamma_{ij}(F)].$$

In order to find the expression for the gradient (which is easier to compute), let $L = L'$ be the solution to the Lyapunov equation

$$(14) \quad (A + BFC)L + L(A + BFC)' = -\ell(dK(F;\, S) + \alpha I)^{\ell-1},$$

and recall the Lypunov equation for the first partial derivatives of K(F) (from (2.13)),

$$(15) \quad (A + BFC)'\Gamma_{ij}(F) + \Gamma_{ij}(F)(A + BFC) = -C'E_{ij}'(B'K(F) + RFC) - (B'K(F)+RFC)'E_{ij}C$$

Post-multiplying (14) by $\Gamma_{ij}(F)$ and taking traces,

$$(16) \quad 2tr[L(A + BFC)'\Gamma_{ij}(F)] = -\ell tr[(dK(F;\, S) + \alpha I)^{\ell-1}\Gamma_{ij}(F)]$$

Pre-multiplying (15) by L and taking traces,

(17) $\mathrm{tr}[L(A+BFC)'\Gamma_{ij}(F)] = -\mathrm{tr}[LC'E_{ij}'(B'K(F) + RFC)]$

Combining (16) and (17) gives

$$\ell\,\mathrm{tr}[(dK(F;\,S)+\alpha I)^{\ell-1}\Gamma_{ij}(F)] = 2\mathrm{tr}[LC'E_{ij}'(B'K(F) + RFC)]$$

which can be substituted in (13), giving

$$\frac{\partial J(S)}{\partial s_{ij}} = 2\mathrm{tr}[LC'E_{ij}'(B'K(F)+RFC)]$$

The gradient matrix of J(S) is obtained by writing the partial derivatives as $e^{i'}\frac{\partial J(S)}{\partial S}\tilde{e}^{j}$, where $\{e^{i}\}$ and $\{\tilde{e}^{j}\}$ are the standard orthonormal basis for $\mathbb{R}^m$ and $\mathbb{R}^r$, respectively, writing the matrices E_{ij} as $e^{i}\tilde{e}^{j'}$, and using the trace property tr[ab'] = b'a. Hence

$$\begin{aligned} e^{i'}\frac{\partial J(S)}{\partial S}\tilde{e}^{j} &= 2\mathrm{tr}[LC'\tilde{e}^{j}e^{i'}(B'K(F) + RFC)] \\ &= 2e^{i'}(B'K(F) + RFC)LC'\tilde{e}^{j} \end{aligned}$$

As this holds for all i,j,

$$\frac{\partial J(S)}{\partial S} = 2(B'K(F) + RFC)LC' \qquad \nabla$$

Thus, a computable expression for the gradient of $\mathrm{tr}(dK(F;\,S) + \alpha I)^{\ell}$ is available, and hence, an optimization algorithm for constrained optimization, that requires function and gradient evaluation, could be used for solving the sequence of subproblems (11). These would lead us to the solution of (8). It is worthwhile mentioning that Henry in his thesis [8] gives a detailed description a feasible direction algorithm for minimizing $\|K(F)\|$, using the trace approach. It is possible to extend if for the case of the minimization of $\| dK(F;S) + \alpha I\|$.

6.2 SUBGRADIENT-BASED OPTIMIZATION OF λmax dK(F; S)

In order to extend methods of feasible directions for problems with locally Lipschitz functions, the extension of some concepts of calculus are necessary. In what follows we shall review some of these extended concepts, which form the basis of the theory of generalized gradients.

(18) DEFINITION. Consider a sequence of real numbers $\{x_k\}$. We define

$$\limsup_{k\to\infty} x_k = \sup\{x \in \bar{\mathbb{R}} : x_{n_k} \to x, \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_k\}\}$$

∇

(19) DEFINITION. Let X be a banach space and $f: X \to \mathbb{R}$ locally Lipschitz (i.e. Lipshitz on any bounded subset of X), and let $v \in X$. Then the generalized directional derivative of f at x, in the direction v, is given by

$$df^0(x;\ v) = \limsup_{\substack{h\to 0\\ \lambda\to 0^+}} \frac{f(x+h+\lambda v) - f(x+h)}{\lambda}$$ ∇

Recall that the one-sided directional derivative of f at x is

$$df(x;\ v) = \lim_{\lambda\to 0^+} \frac{f(x+\lambda v) - f(x)}{\lambda}$$

so, $df^0(x;\ v)$ can be seen as the supremum of the limits

$$\lim_{h_i\to 0} df(x+h_i;\ v)$$

for all sequences $\{h_i\}$.

(20) DEFINITION. The generalized gradient of f at x, denoted by $\partial f(x)$, is the nonempty set of all ξ in X^* satisfying

$$df^0(x;\ v) \geqq \langle v,\ \xi\rangle$$

for all v in X. ∇

X^* is the dual space of X, hence ξ is a linear functional on X.

(21) DEFINITION. A function f is said to be regular at x if, for every v in X, $df(x;\ v)$ exists and satisfies $df(x;\ v) = df^0(x;\ v)$. ∇

(22) FACT. Let $f: X \to \mathbb{R}$ be locally Lipschitz. Then

(i) $df^0(x; \cdot)$ is the support function of $\partial f(x)$. Hence, for all $v \in X$, we have

$$df^0(x; v) = \max\{\langle v, \xi\rangle : \xi \in \partial f(x)\}$$

(ii) A necessary condition of optimality is given by $0 \in \partial(x)$.

(iii) If f is continuously differentiable then it is regular and $\partial f(x) = \{\nabla f(x)\}$.

(iv) If f is convex then it is regular and $\partial f(x)$ coincides with the subgradient in the sense of convex analysis. ∇

The characterization of the generalized gradient and directional derivative of a max function is given below:

(23) THEOREM. Let U be a compact subset of $\mathbb{R}^n$ and let $g: \mathbb{R}^p \times U \to \mathbb{R}$ have the following properties:

(i) $g(x, u)$ is continuous in (x, u)

(ii) $\nabla_x g(x, u)$ exists and is continuous in (x, u)

Then, if we let $f(x) = \max\{g(x, u) : u \in U\}$,

$$df(x; v) = df^0(x; v) = \max\{\langle v, \nabla_x g(x, u)\rangle : u \in M(x)\}$$

where

$$M(x) = \{u \in U: g(x, u) = f(x)\}$$

is the set of maximizers of $g(x, u)$ in U and

$$\partial f(x) = \text{co}\{\nabla_x g(x, u) : u \in M(x)\} \qquad \nabla$$

This theorem is a simplified version of Theorem (2.1) of [5].

Now we are ready to obtain the expression for $d\sigma(s;\ v)$ and $\partial\sigma(s)$:

(24) THEOREM. The generalized gradient of the function $\sigma(s) \triangleq \lambda \max dK(f;\ s)$ is

$$\partial\sigma(s) = \arg\max \{\langle s, \omega\rangle : \omega \in \Omega(f)\}$$

and the directional derivative in the direction v is

$$d\sigma(s;\ v) = \max\{\langle v, \omega\rangle : \omega \in \partial\sigma(s)\}$$

PROOF. Identifying x with s, u with ω, U with $\Omega(f)$ and $g(x, u)$ with $\langle s, \omega\rangle$ in Theorem (23), the function $f(x)$ becomes

$$\max\{\langle s, \omega\rangle : \omega \in \Omega(f)\} = \lambda \max dK(f;\ s) = \sigma(s).$$

Since $\nabla_s g\langle s, \omega\rangle = \omega$, the assumptions of the theorem hold. Note that the set of maximizers,

$$M(s) = \arg\max \{\langle s, \omega\rangle : \omega \in \Omega(f)\},$$

is convex. Then, the theorem claims that

$$\partial\sigma(s) = M(s).$$

Also, that

$$d\sigma(s;\ v) = \max\{\langle v, \omega\rangle : \omega \in M(s)\}$$

(The last expression can be obtained directly using (22.i) and (22.iv)).

∇

For the points at which σ is differentiable, $\partial\sigma(s) = \{\nabla\sigma(s)\}$. We note that the expression for the gradient then can be determined directly, using the following fact:

(25) FACT. [4] Let $A(x)$ be a differentiable matrix function of a parameter x, and suppose $\lambda(A(x))$ is a simple eigenvalue of $A(x)$.

Then, if v(A(x)) is the eigenvector associated with $\lambda(A(x))$,

$$\frac{\partial\lambda(A(x))}{\partial x} = v(A(x))' \frac{\partial A(x)}{\partial x} v(A(x)) . \qquad \nabla$$

So, when the maximum eigenvalue

$$\sigma(s) = \lambda\max \sum_i s_i \frac{\partial K(f)}{\partial f_i} ,$$

is simple,

$$\frac{\partial\sigma(s)}{\partial s_i} = x' \frac{\partial K(f)}{\partial f_i} x,$$

where x denotes the normalized eigenvector corresponding to $\sigma(s)$. The gradient is therefore

$$\nabla\sigma(s) = \begin{bmatrix} x' \frac{\partial K(f)}{\partial f_1} x \\ \vdots \\ x' \frac{\partial K(f)}{\partial f_p} x \end{bmatrix}$$

Now, since $x \in B_n$, that vector is an element of $\Omega(f)$, by definition of thisset, i.e. $\nabla\sigma(s) \equiv \omega(x)$. However we have seen earlier (see (4.4)) that $\omega(x)$ is the maximizer of <s; ·> on $\Omega(f)$. Hence,

$$\nabla\sigma(s) = \arg\max \{<s, \omega>: \omega \in \Omega(F)\} .$$

We conclude from this that a simple eigenvalue $\sigma(s)$ implies differentiability of σ at s, and if σ is nondifferentiable at s, then $\sigma(s)$ is a multiple eigenvalue. The converse is not always true.

A descent direction for σ at s will be a normalized vector v along which $d\sigma(s; v)$ is negative. It is not true, as might be thought, that any vector with opposite direction to an element of the generalized gradient is a descent direction, but only some elements. The steepest descent direction is the one along which $d\sigma(s; v)$ is most negative, and is in the opposite direction to the minimal-norm element of the generalized gradient, as shown next.

(26) THEOREM. The steepest descent direction for σ at a point s is the vector $\hat{v} = -\hat{\omega}/\|\hat{\omega}\|$, for $\hat{\omega}$ the solution of

(27) $\min\{\|\omega\| : \omega \in M(s)\}$

where

$$M(s) = \arg\max\{<s, \omega>: \omega \in \Omega(f)\},$$

assuming $0 \notin \Omega(f)$.

PROOF. The steepest descent direction is found by minimizing $d\sigma(s; v)$ when v ranges over the unit sphere in $\mathbb{R}^p$. Then, it is the solution of

$$\min\{\max\{<v, \omega>: \omega \in M(s)\}: v \in S\}$$

Recall that the search direction problem for the dominant feedback problem (minimize $\sigma(s)$ on $M(f)$) assumes that $0 \notin \Omega(f)$. Therefore, since $M(s) \subset \Omega(f)$, $0 \notin M(s)$. Consequently, the theorem follows from the results of Section 4.1. ∇

Finally we shall turn to the problem of the implementation of problem (27). As we have a norm minimization problem on a convex set, it is natural to think of applying Y.C. Ho's algorithm. Recall that, for this to be possible, a contact point of a supporting hyperplane to M(s) normal to be given vector v must be evaluated, i.e. it must be possible to compute a vector in

$$\arg\max\{<v, \omega>: \omega \in M(s)\}.$$

That can be done, as explained next.

(28) PROPOSITION. $\arg\max\{<v, \omega(x)>$: x is a normalized eigenvector associated with $\lambda\max dK(f; s)\} \subset \arg\max\{<v, \omega>: \omega \in M(s)\}$.

PROOF. Let

$$\rho(f) = \{\omega(x) \in \mathbb{R}^p : \omega_i(x) = x' \frac{\partial K(f)}{\partial f_i} x,\ x \in B_n\}.$$

It can be proved (using linearity of the function $\langle s, \cdot\rangle$) that

$$\mathrm{co}\{\arg\max\{\langle s, \omega\rangle : \omega \in \rho(f)\}\}$$

$$= \arg\max\{\langle s, \omega\rangle : \omega \in \mathrm{co}\ \rho(f)\}$$

$$\triangleq \arg\max\{\langle s, \omega\rangle : \omega \in \Omega(f)\}$$

$$= M(s)$$

However we have seen in (4.4) that

$$\arg\max\{\langle s, \omega\rangle : \omega \in \rho(f)\}$$

$$= \{\omega(x) : x \text{ is a normalized eigenvector associated with } \lambda\max\ dK(f;\ s)\}$$

therefore we conclude that

(29) $M(s) = \mathrm{co}\{\omega(x) : x$ is a normalized eigenvector associated with $\lambda\max\ dK(f;\ s)\}$.

Now, consider any vector $v \in \mathbb{R}^p$ and consider an element $\hat{\omega}$ of $\arg\max\{\langle v, \omega\rangle : \omega \in M(s)\}$. Then, since it belongs to $M(s)$, it follows from (29) that $\hat{\omega}$ is a convex combination of points $\omega(x^i)$, for x^i normalized eigenvectors associated with $\lambda\max\ dK(f;\ s)$, i.e.

$$\hat{\omega} = \sum_{i=1}^{\ell} a_i \omega(x^i);\ a_i \in (0,\ 1);\ \sum_i a_i = 1$$

Since $\hat{\omega}$ maximizes $\langle v, \cdot\rangle$ on $M(s)$, and for $i = 1,..,\ell$, $\omega(x^i) \in M(s)$, we have that

(30) $\langle v, \hat{\omega}\rangle = \sum_i a_i \langle v, \omega(x^i)\rangle \geqq \langle v, \omega(x^j)\rangle$

for $j = 1,\ldots,\ell$, i.e.

$$\sum_{i \neq j} a_i \langle v, \omega(x^i)\rangle \geqq (1 - a_j)\langle v, \omega(x^j)\rangle.$$

Suppose that for some j, say j = 1, the inequality holds strictly. Then

$$(31)\quad \sum_{i \neq 1} a_i \langle v, \omega(x^i)\rangle > (1 - a_1)\ \langle v, \omega(x^1)\rangle$$

which implies that

$$a_1 \langle v, \omega(x^1)\rangle < \frac{a_1}{1-a_1} \sum_{i \neq 1} a_i \langle v, \omega(x^i)\rangle .$$

Therefore, using the above,

$$\langle v, \hat{\omega}\rangle = \sum_i a_i \langle v, \omega(x^i)\rangle < \frac{a_1}{1-a_1} \sum_{i \neq 1} a_i \langle v,\omega(x^i)\rangle + \sum_{i \neq 1} a_i \langle v, \omega(x^i)\rangle$$

$$= \frac{1}{1-a_1} \sum_{i \neq 1} a_i \langle v, \omega(x^i)\rangle$$

Note however that $\sum_{i \neq 1} \frac{a_i}{1-a_1} = 1$ and therefore $\sum_{i \neq 1} \frac{a_i}{1-a_1}\, \omega(x^i)$ belongs to M(s). Hence, the above inequality is a contradiction and so we have proved that (31) is false. It follows from (30) then that, for all j,

$$\langle v, \hat{\omega}\rangle = \langle v, \omega(x^j)\rangle,$$

and so all $\omega(x^j)$, $j = 1,\ldots,\ell$, are maximizers of $\langle v, \cdot\rangle$ on M(s). Then the proposition is proved. ∇

Thus, in order to find a maximizer for $\langle v,\cdot\rangle$ over M(s), the proposition suggests to seek it amongst the points of the form $\omega(x)$, for x a normalized eigenvector associated with λmax dK(f; s). Let us write such an eigenvector as

$$x = L\alpha,\ \|\alpha\| = 1,$$

where L projects a normalized $\alpha \in \mathbb{R}^n$ onto the subspace spanned by the eigenvectors associated with λmax dK(f; s). Then a $\omega(x)$ can be represented in terms of α as

$$(32) \quad \omega(\alpha) = \begin{bmatrix} \alpha'L'H_1L\alpha \\ \vdots \\ \alpha'L'H_pL\alpha \end{bmatrix} .$$

A maximizer for $\langle v, \cdot\rangle$ over the set of such points $\omega(\alpha)$, i.e., a point in

$$(33) \quad \arg\max\{\langle v, \omega(\alpha)\rangle : \|\alpha\| = 1\}$$

can be obtained as follows: write (33) as

$$\arg\max\{\alpha'(\sum_i v_i L'H_iL)\alpha : \|\alpha\| = 1\}.$$

A maximizer for that is a normalized eigenvector associated with $\lambda\max \sum_i v_i L'H_iL$. So, choose such an eigenvector $\hat{\alpha}$ and use formula (32) in order to evaluate $\hat{\omega}$, i.e. $\hat{\omega} = \omega(\hat{\alpha})$. This completes the proof that the minimum-norm point of M(s), and therefore the steepest descent direction of $\sigma(s)$ at s, can be evaluated.

The steepest descent direction could be used to form a feasible direction algorithm for optimising σ. Unfortunately the algorithm does not necessarily converge to an optimal point, as mentioned earlier. Nevertheless it could be the basis for a convergent algorithm: an algorithm that defines the search direction using ε-smeared generalized gradients, as defined below, and uses an ε-reduction scheme in order to drive ε to zero as a stationary point is approached.

(33) <u>DEFINITION</u>. For any $\varepsilon > 0$, the ε-smeared generalized gradient of a Lipschitz function f(x) is defined by

$$\partial_\varepsilon f(x) \triangleq \mathrm{co}\{ \bigcup_{x' \in B(x,\varepsilon)} \partial f(x') \}$$

where $B(x, \varepsilon)$ is the closed ball centered in x and with radius ε. ∇

The search direction is defined as the vector $\hat{v} = -\hat{\omega}/\|\hat{\omega}\|$, where now $\hat{\omega}$ solves

(34) $\min\{\|\omega\| : \omega \in \partial_{\varepsilon}\, \sigma(s)\}$

instead of Problem (27). It is know that when functions are "semismooth" it is possible to get a good approximation to the minimum norm point of the ε-smeared generalized gradient in a finite number of operations (see [12]). It is also a fact that a convex function is semismooth (see [10]). Hence, it seems possible to obtain a good approximation to the solution of (34).

In this section we proposed a non-convergent steepest direction algorithm for optimizing the non-differentiable function σ, and we pointed out that ε-smeared generalized gradients can be used in order to develop a convergent algorithm. This is a topic which requires more work. The paper [11] is referred to as a possible starting point. There an ε-smeared descent direction algorithm, using an Armijo-type line search, was developed for a particular example.

APPENDIX

PROOF OF THEOREM (2) [2]. Since $G(x)$ is symmetric, it can be written as $G(x) = V \Lambda V^{-1}$, where $\Lambda = \text{diag}\{\lambda_1(x) \ldots \lambda_n(x)\}$ and $\lambda_i(x)$ are the (real) eigenvalues of $G(x)$. Hence $G(x)^{\ell} = V \Lambda^{\ell} V^{-1}$ and $\text{tr}[G(x)^{\ell}] = \Sigma\lambda_i(x)^{\ell}$.

Let $\lambda(x) = [\lambda_1(x) \ldots \lambda_n(x)]'$. Then for even integers $\ell \geqq 0$

(35) $$\text{tr}[G(x)^{\ell}] = \Sigma\lambda_i(x)^{\ell} = \Sigma|\lambda_i(x)|^{\ell} = (\|\lambda(x)\|_{\ell})^{\ell}$$

This is also true for all integers $\ell \geqq 0$ if $G(x) \geqq 0$, $\forall x \in C$, since then $\lambda_i(x) \geqq 0$, $\forall x \in C$. From now on it will be assumed that even integers ($\geqq 0$) are considered for general J and that all integers ($\geqq 0$) are allowed if $G(x)$ is positive semi-definite on C. From (3) and (5)

(36) $$\|\lambda(x^{\ell})\|_{\ell}^{\ell} \leqq (1 + \theta) \inf\{\|\lambda(x)\|_{\ell}^{\ell} : x \in C\}.$$

For $\varepsilon > 0$, consider $x_{\varepsilon} \in C$ chosen so

(37) $$\|\lambda(x_{\varepsilon})\|_{\infty} \leqq \inf\{\|\lambda(x)\|_{\infty} : x \in C\} + \varepsilon/4$$

Now

(38) $$\|\lambda(x)\|_{\infty} = \max_{i} |\lambda_i(x)| = \|G(x)\|$$

Subsequently

(39) $$\hat{\lambda} \triangleq \inf\{\|G(x)\| : x \in C\} = \inf\{\|\lambda(x)\|_{\infty} : x \in C\}.$$

Hence, from (37) and (39)

(40) $$\|\lambda(x_{\varepsilon})\|_{\infty} \leqq \hat{\lambda} + \varepsilon/4$$

By Jensen's inequality

$$\| \lambda(x^{\ell}) \|_{\infty}^{\ell} \leqq \| \lambda(x^{\ell}) \|_{\ell}^{\ell}$$

$$\leqq (1 + \theta) \inf\{\| \lambda(x) \|_{\ell}^{\ell} : x \in C\} \text{ (from (36))}$$

$$\leqq (1 + \theta) \ \| \lambda(x_{\varepsilon}) \|_{\ell}^{\ell}$$

so

(41) $\| \lambda(x^{\ell}) \|_{\infty} \leqq (1 + \theta)^{1/\ell} \ \| \lambda(x_{\varepsilon}) \|_{\ell}$

Clearly

(42) $\| \lambda(x^{\ell}) \|_{\infty} \geqq \inf\{\| \lambda(x) \|_{\infty} : x \in C\} = \hat{\lambda} \geqq 0$

and by Jensen's inequality, as $\ell \to \infty$

$$\| \lambda(x_{\varepsilon}) \|_{\ell} \to \| \lambda(x_{\varepsilon}) \|_{\infty}$$

so that there exists ℓ_{ε} such that, for all $\ell > \ell_{\varepsilon}$,

$$\| \lambda(x_{\varepsilon}) \|_{\ell} - \| \lambda(x_{\varepsilon}) \|_{\infty} < \varepsilon/4.$$

Consequently

$$\| \lambda(x_{\varepsilon}) \|_{\ell} < \| \lambda(x_{\varepsilon}) \|_{\infty} + \varepsilon/4$$

(43) $\qquad < \hat{\lambda} + \varepsilon/2, \ \forall \ell > \ell_{\varepsilon}$ (from (40))

Hence

$$0 \leqq \| \lambda(x^{\ell}) \|_{\infty} - \hat{\lambda} \quad \text{(from (42))}$$

$$\leqq (1 + \theta)^{1/\ell} \ \| \lambda(x_{\varepsilon}) \|_{\ell} - \hat{\lambda} \text{ (from (41))}$$

$$\leqq (1 + \theta)^{1/\ell} \ (\hat{\lambda} + \varepsilon/2) - \hat{\lambda}, \ \forall \ell > \ell_{\varepsilon} \text{ (from (43))}$$

i.e.

(44) $0 \leqq \| \lambda(x^{\ell}) \|_{\infty} - \hat{\lambda} \leqq ((1 + \theta)^{1/\ell} - 1)\hat{\lambda} + (1+\theta)^{1/\ell} \varepsilon/2, \ \forall \ell > \ell_{\varepsilon}.$

Since $(1 + \theta)^{1/\ell} \to 1$ as $\ell \to \infty$, ℓ'_ε can be selected so that $\ell'_\varepsilon > \ell_\varepsilon$ and so that the following hold.

Whenever $\ell > \ell'_\varepsilon$

(a) $(1 + \theta)^{1/\ell} < 4/3.$

(b) $((1 + \theta)^{1/\ell} - 1)\hat{\lambda} < \varepsilon/3$

Then, from (44)

$$0 \leqq \| \lambda(x^\ell) \|_\infty - \hat{\lambda} < \varepsilon, \ \forall \ell > \ell'_\varepsilon$$

This can be done for every real $\varepsilon > 0$; so, as $\ell \to \infty$,

$$\| \lambda(x^\ell) \|_\infty \to \hat{\lambda}.$$

In view of (38) and (39), as $\ell \to \infty$

$$\| G(x^\ell) \|_\infty \to \inf\{\| G(x) \| : x \in C\}. \qquad \nabla$$

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CHAPTER 7

CONCLUSIONS

7. CONCLUSIONS

This work was based on Allwright's approach for optimal unconstrained output feedback, which used a feasible direction algorithm for minimizing the matricial function $K(F)$ on the space of stabilizing feedbacks. He has proposed two versions, given in Algorithm 2.31 and 2.32. An iteration considers a search direction emerging from F, that is the steepest descent direction of the cost K, in the sense that it makes the first order term of the Taylor expansion of $K(F + S) - K(F)$ to be as negative as possible. The matrix $dK(F, S)$ is most negative, in Allwright's sense, when its largest eigenvalue is most negative. Along the search direction, for one version, the line search is done that makes $K(F + \lambda S) - K(F)$ the most negative in the above sense. The other version minimizes Levine-Athans' objective function, $tr(K(F + \lambda S) - K(F))X_0$, subject to $K(F + \lambda S) - K(F) \leqq 0$. Let $\hat{S}$ be a normalized vector that minimizes $dK(F; S)$. If $dK(F; \hat{S}) < 0$ then $\hat{S}$ is the steepest descent direction, and a line search is performed along it. Otherwise (i.e. $dK(F; \hat{S})$ positive, nonnegative definite or indefinite - implying that $\lambda max\ dK(F; \hat{S}) \geqq 0$), termination is set for algorithm 2.31. When $\lambda max\ dK(F; \hat{S}) > 0$, F is locally dominant, but for the case $\lambda max\ dK(F; \hat{S}) = 0$ (F stationary), depending on the other terms of the expansion, K may still be reduced along the direction $\hat{S}$. This possibility is investigated in Algorithm 2.32. If reduction is not possible, it is because either K increases or else there may not exist order relationship between feedbacks along $\hat{S}$, and F. The three cases are a consequence of $dK(F; \hat{S})$ being negative definite, positive definite and indefinite, respectively.

In Chapter 2 those algorithms have been described and analysed. Implementations for the line searches have been devised, based on Armijo's method. The slope of the Armijo line is defined as a fraction of the tangent at the origin, of a quadratic approximation to the unidimensional objective function. For the determination of the quadratic, the main tools were the Lyapunov equations involving the Fréchet - differentials of K. A variable initial step size was used for Algorithm 2.32 due to the fact that at each iteration the line search is constrained. For those implementations proofs are given for convergence of a sequence $\{F^j\}$, such that $\lambda\max\, dK(F^j, \hat{S}^j) < 0$, which meant that $\lambda\max dK(F^j;\hat{S}^j) \to 0$ and $K(F^j) \to K^* \geqq 0$, as $j \to \infty$, and not that the sequence $\{F^j\}$ converges itself. As a consequence, an accumulation point of the sequence, if it exists, is first-order locally dominant. It is shown that convergence also occurs when $\hat{S}^j$ is not the steepest descent direction, but is such that it makes $\lambda\max dK(F^j; \hat{S}^j)$ a fraction of its minimal value. For evaluating the steepest descent direction at F^j, i.e. the normalised vector that minimizes $dK(F^j; S)$, Allwright has proposed a partial solution, that suits Algorithm 2.31, as follows. He has shown that minimizing $\lambda\max\, dK(F^j; S)$ on $\mathcal{S}$ is equivalent to finding the minimum norm point of $\partial\Omega(F^j)$. Then he presents a practical method for computing the minimum norm point of a convex set. Since when $0 \notin \mathring{\Omega}(F^j)$ the minimum norm point of both $\Omega(F^j)$ and $\partial\Omega(F^j)$ coincides, it is possible to find the minimum norm point of $\partial\Omega(F^j)$, when $0 \notin \mathring{\Omega}(F^j)$, using that method. However it is only possible to evaluate the minimizer of $\lambda\max\, dK(F^j; S)$ in terms of the minimum norm point in a simple way, when $0 \notin \Omega(F^j)$. It is shown that $0 \notin \Omega(F^j)$ corresponds to the case $\lambda\max\, dK(F^j; \hat{S}^j) < 0$. The condition $0 \notin \Omega(F^j)$ obviously can be detected if the solution to the minimum norm problem is nonzero. Hence, for Algorithm 2.31, the steepest descent direction can always be evaluated.

In the rest of the thesis constraints have been considered. The algorithms for the case of constrained output feedback have been devised.

Three cases have been studied.

The first, studied in Chapter 3, considers the feasible set as a compact convex set, with nonempty interior, defined by a finite number of continuous functions. The procedure is based on Algorithm 2.32, of feasible direction type. The search direction subproblem becomes that of minimizing λmax dK(F; S) over $M(F)$, which is a displacement of the feasible set. Convergence has been proved using the theory of closedness of algorithms. The meaning of convergence is that of above. Then, on account of the compactness of $M(F)$, the subsequences of $\{F^j\}$ converge to first-order locally dominant feedbacks in $M(F)$. Convergence is also proved for the algorithm using the search direction S^j that makes λmax dK(F; S) a fraction of its minimum value. The implementation of the line search, when $dK(F^j; S^j) < 0$ is based on a quadratic approximation of $tr[(K(F^j) - K(F^j + \lambda S^j))X_0]$ along the line, without the help of an Armijo line. Although probably less accurate, this led to the convergence proof. The search direction subproblem is dealt with in Chapter 6. Two procedures have been suggested. The first, due to Allwright, requires solving a sequence of problems of the type: minimize $tr(dK(F; S) + \alpha I)^{\ell}$, where S ranges over $M(F)$. The limit of the sequence of solutions, when $\ell \to \infty$, is the solution to the search direction subproblem. A second method is outlined and its development is suggested for future work. It has been described a procedure for determining the steepest descent direction at S, of the nondifferentiable function λmax dK(F; ·), and shown that it can be evaluated. The suggestion is that a feasible direction algorithm using ε-smeared subgradients be developed.

The second constrained feedback problem considers the feasible set as a linear variety of the space of feedbacks (Chapter 4). A descent direction algorithm is suggested for which the steepest descent direction at F^j is the vector that minimizes λmax $dK(F^j; S)$ over L, a displacement of the linear variety. The methodology proposed is based on projecting the problem onto L. This consists of projecting the set $\Omega(F^j)$ onto L and then, since on L the problem can be viewed as unconstrained, applying Allwright's minimum norm procedure for determining the normalized minimizer of λmax $dK(F^j; S)$, using the projected set. As in the unconstrained situation, convergence occurs.

The third problem, studied in Chapter 5, is a particular case of the first. The feasible set is defined by linear inequalities generated when the entries of the feedback matrix are to be kept within certain intervals. An implementable feasible direction algorithm is developed. The search direction is defined as the steepest feasible direction, i.e., the vector that minimizes λmax $dK(F^j; S)$ on the cone generated by $M(F)$. The line search is that of Algorithm 2.32. Since the search direction as it has been defined does not permit that the close active constraints be anticipated, there is no guarantee of convergence. The procedure for obtaining the search direction is as follows. When F^j is an interior point of the feasible set, it is the steepest descent direction, and thus, Ho's algorithm should be used. For F^j on the boundary, a geometric-based method has been developed, following the idea of Allwright's minimum norm problem equivalence. First we propose an algorithm for finding the minimum distance between a convex set and an orthant. This problem has been formulated as a norm minimization problem and it is shown that Allwright's algorithm for convex optimization for twice-differentiable functions can be used to minimize it. Then it is proved that, if $\hat{C}$ is a nonzero solution to the minimum distance problem between $\Omega(F)$ and an orthant, then $\hat{S} = \hat{C}/\|\hat{C}\|$ is the steepest descent direction at F on the orthant. Since the cone generated by

$M(F)$ is a union of orthants, the most negative among such vectors $\hat{S}$, for all orthants, is the steepest feasible direction. If for all orthants $\hat{C} = 0$, F is first-order locally dominant. Although there exists a relationship between the solutions to the minimum distance problem and the search direction problem, there does not exist a relationship between their ε-approximations, since the two problems optimize distinct functions. Therefore the algorithm only terminates after ensuring that an ε-approximation to the steepest feasible direction has been found.

Finally, if one considers the third problem accepting also equality constraints, a combination of the second and third methods for obtaining the search direction is possible. Note that the feasible set could then be viewed as defined by linear inequalities only, on the linear variety generated by the equality constraints. Therefore the method would involve projecting $\Omega(F)$ onto L, and determining the minimum distance between the projected set and the orthants of L generated by $M(F)$.