STABILITY, STABILIZATION AND DESIGN OF MULTIDIMENSIONAL RECURSIVE DIGITAL FILTERS

by

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To my parents

ABSTRACT

The thesis consists of an investigation of the stability, stabilization and design of multidimensional recursive digital filters.

The stability conditions as well as tests for checking them are studied. A novel stability test for two-dimensional recursive filters is proposed. A recently introduced testing method is then extended to multidimensions. The extension of Lyapunov's test to higher dimensions is critically examined. The Lyapunov technique is shown not to be extendable to multidimensional systems.

New stabilization techniques are proposed for two-dimensional recursive filters. An algorithm is given for stabilization of digital filters in the cepstrum domain.

Design techniques in the frequency domain are studied with particular reference to techniques involving spectral transformation methods. A two-variable reactance function is given for designing filters with circular symmetry. Complex transformations are developed for the design of fan and quadrant fan filters having guaranteed stability.

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TABLE OF CONTENTS

Page No.

CHAPTER 3 (cont.)

Page No.

CHAPTER 4 (cont.)

Page No.

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

$\sum_{i=1}^{h} a_i$	$a_1 + a_2 + \ldots + a_n$
$\prod_{i=1}^{n} a_i$	$a_1 a_2 \ldots a_n$
$\in$	belongs to
$\notin$	does not belong to
$\cap$	the intersection of two sets
$\cup$	the union of two sets
$\forall$	for all
$\hat{q}$	cepstrum of q
DFT	discrete Fourier transform
IDFT	inverse discrete Fourier transform
FFT	Fast Fourier transform
ℓn	Natural logarithm
Im	imaginary part
Re	real part

CHAPTER 1

INTRODUCTION

Digital signal processing has been a growing and dynamic field for more than a decade. Depending on the type of the input and output sequences, digital signal processing can be classified into two broad groups: one-dimensional signal processing and multidimensional signal processing. In the first case, the data is given as a function of a single integer variable, such as that obtained by sampling a time function. In the latter case, the data is a function of several integer variables such as that obtained by sampling a two-dimensional picture. There has been a considerable amount of research and development concerning theory and design of one-dimensional signal processors. Most of the theory of multidimensional systemsis similar to one-dimensional systems. However, some important concepts and design techniques can not be routinely extended to deal with multidimensional problems. As a result, digital processing of multidimensional data has required separate treatment leading to the development of its separate theory and design technique.

In the following Section, we will briefly mention some of the most important applications of multidimensional digital filters. We also review preliminaries related to the mathematical representation of such signals. Then, a comprehensive survey of multidimensional recursive digital filters is presented. Finally, the outline of the thesis is given in the last Section.

1.1 APPLICATIONS OF MULTIDIMENSIONAL DIGITAL FILTERS

Certain signals are inherently two- or multi-dimensional, and it appears advantagous to develop two- and multidimensional techniques for the processing of such signals. Major emphasis in this effort has been directed to the processing of two-dimensional data because of its widespread applications in the following areas;

a) Biomedicine [1]: Biomedicine is an application area, in which the use of digital signal processing techniques has had great impact. The most important reason for this, is from the clinical viewpoint to have better quality images, from which a better diognosis can be carried out. For example, Two- dimensional digital filters are used to reduce spatial low frequency components in an X-ray image making features with large high frequency components such as fracture easier to identify.

b) Nuclear Physics [2]: Digital filtering of radiographs is used in nondestructive testing, such as for the measurement of the internal dimensions of nuclear fuel rods.

c) Space Imagery [3]: Digital processing of satellite images has been used in monitoring enviromental effects, earth resources, and urban land use. In these applications two-dimensional digital filters have been used to enhance, or reduce boundaries, remove low-frequency shading effects, reduce noise, and correct for distortion in the imaging system.

d) <u>Seismic Prospecting [4]</u>: In order to gather data about subsurface structure, a number of seismic detectors are placed at stations along a line passing through a shallow bore hole. The digitized outputs of the detectors after an explosion form a two-dimensional array with time along one axis and distance along the other. Reflected energy at the detectors provides information about the depth and nature of subsurface features.

A two-dimensional processing reduces the noise and separates signals from different sources for evaluation.

e) <u>Geophysics [5]</u>: Atmosperic temperature and pressure data may be smoothed by digital means before plotting on weather maps. Similarly, magnetic and gravity measurements can be processed to reduce the effect of surface anomalies in order to identify large subsurface features.

1.2 REPRESENTATION OF MULTIDIMENSIONAL SIGNALS

A multidimensional system can be characterized by an operator transforming an n-dimensional input sequence $\{x(m_1,m_2,\ldots,m_n)\}$ to an n-dimensional output sequence $\{y(m_1,m_2,\ldots,m_n)\}$. We can indicate this fact notationally as

$$\{y(m_1,m_2,\ldots,m_n)\} = \mathcal{L}\Big[\{x(m_1,m_2,\ldots,m_n)\}\Big] \tag{1.1}$$

where $\mathcal{L}$ is an operator.

Like other signal processing systems, multidimensional digital filters can be classified as time invariant or time dependent, casual or noncasual, linear or nonlinear [6]. A linear time invariant and casual filter can be defined with the following three properties.

A. Space Invariance:

A multidimensional filter is said to be space invariant if its internal parameters do not change with space. This means that a specific excitation will always produce the same response independently of the space of application. This means that the output is independent of the position of input.

A multidimensional system $\mathcal{L}$ is space invariant if and only if an input array $\{x(m_1,m_2,\ldots,m_n)\}$ produces an output array $\{y(m_1,m_2,\ldots,m_n)\}$ then $\{x(m_1-m_{10},m_2-m_{20},\ldots,m_n-m_{no})\}$ produces an output array $\{y(m_1-m_{10},m_2-m_{20},\ldots,m_n-m_{no})\}$ for all $(m_{10},m_{20},\ldots,m_{no})$.

B. Linearity:

A multidimensional system is linear if and only if it satisfies the following conditions;

$$\mathcal{L}\Big[\alpha\{x(m_1,m_2,\ldots,m_n)\}\Big] = \alpha\mathcal{L}\Big[\{x(m_1,m_2,\ldots,m_n)\}\Big] \tag{1.2}$$

and

$$\mathcal{L}\left[\{x_1(m_1,m_2,\ldots,m_n)\} + \{x_2(m_1,m_2,\ldots,m_n)\}\right]$$

$$= \mathcal{L}\left[\{x_1(m_1,m_2,\ldots,m_n)\}\right] + \mathcal{L}\left[\{x_2(m_1,m_2,\ldots,m_n)\}\right] \qquad (1.3)$$

for all possible values of α and all possible values of exitation $\{x_1(m_1,m_2,\ldots,m_n)\}$ and $\{x_2(m_1,m_2,\ldots,m_n)\}$.

C. Causality:

A causal digital multidimensional filter is one whose response at a specific instant is independent of subsequent values of the exitation. More precisely, a multidimensional digital filter is causal if and only if the impulse response, $\{h(m_1,m_2,\ldots,m_n)\}$ has nonzero values only for $\bigcap_{i=1}^{n}\{m_i\}>0$.

The definition of causality as above, when considered in one-dimension, is identical with the definition of a causal array in time.

Multidimensional arrays having this property are sometimes termed 'first quadrant array' because causality is meaningless outside the time dimension.

A linear, time-invariant, causal recursive multidimensional filter is represented by the following difference equation;

$$y(m_1,m_2,\ldots,m_n)= \sum_{k_1=0}^{K_1} \sum_{k_2=0}^{K_2} \cdots \sum_{k_n=0}^{K_n} a(k_1,k_2,\ldots,k_n)x(m_1-k_1,m_2-k_2,\ldots m_n-k_n)$$

$$-\sum_{l_1=0}^{L_1} \sum_{l_2=0}^{L_2} \cdots \sum_{l_n=0}^{L_n} b(l_1,l_2,\ldots,l_n)y(m_1-l_1,m_2-l_2,\ldots,m_n-l_n) \tag{1.4}$$

$$l_1+ l_2 +\ldots+ l_n \neq 0$$

with $b(0,0,\ldots,0)=1$. $x(m_1,m_2,\ldots,m_n)$ and $y(m_1,m_2,\ldots,m_n)$ denote the input and output signals. The corresponding z-transfer function is

$$H(z_1,z_2,\ldots,z_n) = \frac{\sum_{k_1=0}^{K_1} \sum_{k_2=0}^{K_2} \cdots \sum_{k_n=0}^{K_n} a(k_1,k_2,\ldots,k_n) z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}}{\sum_{l_1=0}^{L_1} \sum_{l_2=0}^{L_2} \cdots \sum_{l_n=0}^{L_n} b(l_1,l_2,\ldots,l_n) z_1^{l_1} z_2^{l_2} \ldots z_n^{l_n}} \tag{1.5}$$

1.3 SURVEY OF THE PREVIOUS WORK ON MULTIDIMENSIONAL RECURSIVE FILTERS

Phenomenal advances in digital integrated circuit technology during the last decade have made the digital signal processing approach economically practical and often a more convenient method of signal processing. As a consequence of this development, most of the work on multidimensional digital filters have been reported in the last ten years, with more than three-fourths of these appearing in the last five years.

Historically, the inital work on multidimensional digital filtering was performed by researchers in geophysical industries for processing of seismic, gravitational, and magnetic data. Shanks et al [7] in 1972 published the first technique for designing two-dimensional recursive filters. Here, they consider a one-dimensional recursive filter as a special case of two-dimensional filter. By rotating the frequency axes of the resultant filter, they arrived at a two-dimensional transfer function. There are two problems associated with this technique. First, in general, the rotated two-dimensional filter may not be stable even though its one-dimensional anology prototype is stable. Second, the frequency characteristics of the rotated version is not simply related to that its parent one-dimensional filter. This makes it difficult to design two-dimensional filters with prescribed frequency response characteristics.

In 1974, Costa and Venetsanopoulos [8] used the Shanks rotated filters to design two-dimensional low-pass filters. In this method, it is shown that the rotated filters can be used in designing circularly symmetric two-dimensional recursive filters. A stability criterion was developed which showed that angles of rotation of the designed filter from 0^{o} to -90^{o} resulted in stable filters. Therefore, the design technique could not achieve the required total angular span and the cutoff boundary was not circular.

Ahmadi et al [9] in 1976 suggested a simple first order two-dimensional reactance function to transform a one-dimensional continuous low-pass filter function to a two-dimensional continuous low-pass function. Later, in a review paper, Chakrabarti and Mitra [10] in 1977 generalized Ahmadi's approach as a unified qualitative theory of designing two-dimensional filters via spectral transformations.

In the computer aided optimization approaches, a nonlinear optimization procedure is used to adjust iteratively the filter coefficients to minimize the error criterion. A major difficulty is ensuring the stability of the resultant two-dimensional transfer function. In 1974, Maria and Fahmy [11] used a l_p-optimization technique and avoided this problem by constraining the filters they designed to have transfer functions which are products of simple first and second order terms. This facilitates testing stability of the approximation at each step of the optimization as these low order terms can be tested using a set of inequalities associated with the filter coefficients. However, Pendergrass [12] in 1975 pointed out that l_p-algorithm may converge to an unstable solution.

Alternative optimization approaches are studied by Bedner [13] and Ramamoorthy and Bruton [14] . In the first case, the differential correction optimization algorithm is used to approximate a given frequency response of desired two-dimensional recursive filter. Stability is checked after each iteration of the optimization using a stability testing algorithm. In the latter case, the denominator of the two-variable anolog transfer function is algebrically expressed in a suitable form which is always guaranteed to be realizable by a passive network thus ensuring stability.

The stability testing problem can also be avoided by designing a separable two-dimensional recursive filter approximating the frequency response characteristics. In this case, the stability testing reduces to that of checking the stability of one-dimensional filters. Moreover, a separable filter is also more economical to implement. In 1975, Twogood and Mitra [15] described a computer-aided method for designing separable filters.

In the spatial domain design problem, a filter transfer function is chosen to approximate a finite extent of the two-dimensional impulse response. Shanks et al [7] suggested the first technique in addressing this problem. They developed a least-squares approach with the spatial error criterion used to arrive at a best approximation of the spatial response. Unfortunately, their recursive filter design technique, in general, does not lead to stable filters. To overcome this problem, the authors suggested using a planar least squares inverse (PLSI) stabilization technique to arrive at a stable approximation. However, Genin and Kamp [16] showed that the PLSI technique for two-dimensional filter functions is not valid in general. Therefore, this approach may not lead to a stable approximation.

In order to avoid the stability problem in the spatial design procedure Abramatic et al [17] have presented a design technique for filters with separable denominator functions from the impulse response of the prototype filter. Another notable contribution in spatial domain designs is due to Parker and Souchon [18] .

From the above discussion, it is clear that a major concern in the design of multidimensional recursive digital filters is ensuring the stability of the filter. The earliest statement of a theorem for the BIBO stability of such filters was presented by Shanks [7] . However, this theorem requires an infinite algorithm to test the stability. An alternative stability criterion which offers a finite algorithm was stated by Huang [19] .

A number of authors attempted to implement Huang's criterion. To this end, notable contributions have been made by Anderson and Jury [20] , and Maria and Fahmy [21] . These contributions are essentially based on a generalization of Schur-Cohn test [22] developed for checking stability of one-dimensional filter functions.

Recently, Strintzis [23] , DeCarlo et al [24] and, O'Connor [25] obtained several equivalent conditions for stability. These conditions were also implemented in the form of a Nyquist-like stability test [26] , the phase unwrapping technique [27] and, the complex cepstrum test [28] .

In 1977, in a prize winning paper, Goodman [29] showed that the Shanks theorem [7] is only sufficient (necessity does not hold). This is due to the effect of the numerator on stability.

Some of the design methods for two-dimensional recursive filters produce inherently unstable filters. Bernabo et al [30] used the well-known McClellan's transformation to design approximate circular symmetry frequency responses. The resulting filter is unstable. They used the Pistor [31] decomposition technique in order to obtain four one-quadrant recursive digital filters each recursing in a different direction. The obtained filter is zero phase and stable. A similar approach was also used by King [32] for fan filter design with complex transformations.

1.4 OUTLINE OF THE THESIS

In this thesis the problems of stability, stabilization, and design of multidimensional recursive digital filters are considered.

In Chapter 2, the concept of stability is defined for two-dimensional recursive filters. Stability criteria for these filters are discussed and tests for determining the stability are reviewed. A novel stability test is developed. This test is based on the properties of the two-variable inner determinants. The problem of the extension of Lyapunov's test into two-dimensional cases are discussed. Some difficulties relating to this extension are pointed out.

Stability problems of polynomials of dimensions higher than two are discussed in Chapter 3. In the main part of this chapter, a cepstral stability test for multi-dimensional digital filters is introduced.

Three stabilization techniques are reviewed in Chapter 3. One of these methods is then modified to include a more general class of two-dimensional recursive filters. Next, a new spectral factorization method is suggested as an alternative procedure. An algorithm is also developed for the stabilization of digital filters by Pistor Method.

In Chapter 5, frequency domain design techniques for two-dimensional digital filters are considered. A new two-dimensional reactance function is proposed for the design of recursive filters with circular symmetry. Two different design techniques are also presented in this chapter. The first of them is obtained by using the well-known Ahmadi transformation and the second with complex transformations. It is shown that the latter method gives the complete solution for the fan filter and quadrant fan filter design.

The final chapter summarizes the work of the thesis and a number of problems are suggested in which further research may be conducted.

CHAPTER 2

STABILITY OF TWO-DIMENSIONAL DISCRETE SYSTEMS

We will begin this chapter with a brief review of the stability property of quarter-plane (causal) filters. The stability conditions as well as the tests for checking them will be thoroughly discussed. These various stability criteria are well developed in the literature; consequently in order to keep within the limits of the thesis, most of the known proofs are not repeated and only new ones will be considered.

In the main part of this chapter, a novel stability test will be introduced. This test is based on properties of two-variable inner determinants. In the last section, the extension of Lyapunov's test for two-dimensional digital filters will be examined. It is shown that the direct extension of the Lyapunov test to higher dimensions is not valid in general.

2.1 STABILITY PROPERTY OF QUARTER PLANE DIGITAL FILTERS

A difference equation which describes the input-output relationships of a spatially causal (first quadrant) digital filter is presented as:

$$y(m,n) = \sum_{k=0}^{K} \sum_{l=0}^{L} p(k,l)x(m-k,n-l) - \sum_{\substack{i=0 \\ (i,j)\neq(0,0)}}^{I} \sum_{j=0}^{J} q(i,j)y(m-i,n-j) \qquad (2.1)$$

where $\{x(m,n)\}$ and $\{y(m,n)\}$ denote the input and output sequences, respectively. The two-dimensional z-transform of the above linear equation leads to the transfer function:

$$H(z_1,z_2) = \frac{P(z_1,z_2)}{Q(z_1,z_2)} \qquad (2.2)$$

where

$$P(z_1,z_2) = \sum_{k=0}^{K} \sum_{l=0}^{L} p(k,l)\ z_1^k z_2^l$$

$$Q(z_1,z_2) = \sum_{i=0}^{I} \sum_{j=0}^{J} q(i,j)\ z_1^i z_2^j$$

In the first quadrant case, since $q(0,0)=1$ is assumed, $Q(z_1,z_2)\neq 0$ in some open neighbourhood U_ε^2 of $(0,0)$;

where

$$U_\varepsilon^2 = \{(z_1,z_2): |z_1| < \varepsilon, |z_2| < \varepsilon\}$$

Hence in U_ε^2 the transfer function $H(z_1,z_2)$ is analytic and has the power series expansion [42] - [44]:

$$H(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h(m,n)\; z_1^m z_2^n \tag{2.3}$$

$h(m,n)$ is the unit sample response of the causal filter.

A widely used stability criterion is bounded input, bounded output (BIBO) stability.

<u>Definition 2.1</u>: Just as in the one-dimensional case, we will say that the system with the transfer function $H(z_1,z_2)$ is BIBO stable if any bounded input sequences produce a bounded output sequence, that is, if there exists a finite real number λ such that for any bounded input sequence $\{x(m,n)\}_{m,n=o}^{\infty}$ the zero state response of the system as given by (2.1) satisfies

$$||y||_\infty < \lambda ||x||_\infty \tag{2.4}$$

where the norm, $||\ \ ||_\infty$, over the space of bounded sequences is defined as:

$$||u||_\infty \triangleq \left\{\max \sup_m \sup_n |u(m,n)| , \sup_n \sup_m |u(m,n)|\right\}$$

It readily follows that the ensuring theorem is valid.

<u>Theorem 2.1 [75]</u>:

A two-dimensional linear system described by (2.2) is BIBO stable if and only if there exists a real λ such that for all positive integers m,n

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} | h(m,n) | \leqslant \lambda < \infty \tag{2.5}$$

<u>Note</u>:

$$h(m,n) = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_2} H(z_1,z_2) z_1^{-m+1} z_2^{-n+1} \, dz_1 dz_2$$

where C_1 and C_2 are boundaries of the unit bidisc.

Consider equation (2.2) where $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime (i.e. the polynomials have no irreducible factors in common):

i) A 2-tuple (z_1,z_2) such that $Q(z_1,z_2)=0$ but $P(z_1,z_2) \neq 0$ will be called a pole or a nonessential singularity of the first kind (such singularities are not isolated points and they are analogous to a pole in the one-dimensional case).

ii) A 2-tuple (z_1,z_2) such that $Q(z_1,z_2) = P(z_1,z_2) = 0$ will be called a nonessential singularity of the second kind (such points have no one-dimensional analogs).

Clearly, if (z_1,z_2) is a pole, $H(z_1,z_2) = \infty$. If (z_1,z_2) is a nonessential singularity of the second kind, the value of $H(z_1,z_2)$ is undefined.

2.2 STABILITY CONDITIONS

A stability theorem due to Shanks [48] states:

Theorem 2.2:

The transfer function $H(z_1,z_2)$ is BIBO stable if and only if

$$Q(z_1,z_2) \neq 0 \text{ for all } (z_1,z_2) \in \{(z_1,z_2): |z_1| \leq 1 , |z_2| \leq 1\} \tag{2.6}$$

provided $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime.

Since the theorem requires the primeness of P and Q, all irreducible factors common to $P(z_1,z_2)$ and $Q(z_1,z_2)$ should first be cancelled (mutually prime polynomials). A test for the existence of common factors is given in [76], and an algorithm for the extraction of the greatest common factor is given in [77]. A similar theorem with some generalization for the case when $P(z_1,z_2) = 1$ was given by Farmer and Badner [75].

Shanks' theorem was used by many authors as the necessary and sufficient condition for stability. However, Goodman [29] has shown that Shanks' theorem is only sufficient (necessity does not hold). This is due to the effect of the numerator on stability (which has no analog in the one-dimensional case).

2.2.1 Effect of Numerator on Stability

In some cases $H(z_1,z_2)$ has nonessential singularities of the second kind on the distinguished boundary (i.e. $\{(z_1,z_2);\ |z_1| = 1$ and $|z_2| = 1\}$) but $\{h(m,n)\}$ is absolutely summable [29]. The following two examples illustrate this point;

$$H_1(z_1,z_2) = \frac{(1-z_1)^8(1-z_2)^8}{2-z_1-z_2} = \frac{P_1(z_1,z_2)}{Q(z_1,z_2)} \qquad (2.7)$$

$$H_2(z_1,z_2) = \frac{(1-z_1)(1-z_2)}{2-z_1-z_2} = \frac{P_2(z_1,z_2)}{Q(z_1,z_2)} \qquad (2.8)$$

The above transfer functions have mutually prime numerator and denominator. The denominator $Q(z_1, z_2) \neq 0$ on $\{(z_1, z_2): |z_1| \leq 1, |z_2| \leq 1\}$ except at $|z_1| = |z_2| = 1$. Both $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$ have nonessential singularities of the second kind at $z_1 = z_2 = 1$. Goodman showed that $H_2(z_1, z_2)$ is BIBO unstable; and $H_1(z_1, z_2)$ is BIBO stable. Hence, Shanks' theorem is sufficient for BIBO stability.

A mention of such type of singularities was also noted by Humes and Jury [59], and Bose and Newcomb [79]. The test for the presence or absence of nonessential singularities of the second kind on the unit bidisc was studied by Anderson, Bose and Jury [80], and Rajan et al [81]. Later, it has been shown by Goodman [68], that a double bilinear transformation approach, for designing a two-dimensional recursive digital filter from a predetermined two-dimensional anolog transfer function, may in certain cases lead to unstable solutions.

2.2.2 Remarks

For effective design of two-dimensional digital filters nonessential singularities of the second kind must be avoided. Hence, for consideration of design which avoids such singularities, the BIBO stability is refered to as "structural stability". Therefore, equation (2.6) gives the necessary and sufficient condition for structural stability.

By assuming that $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime and $Q(0,0) \neq 0$. Then the following relationships hold:

a) BIBO Stability $\Leftrightarrow$ $\sum_{m=o}^{\infty}\sum_{n=o}^{\infty} |h(m,n)| < \infty$

b) $Q(z_1,z_2) \neq 0$ in $\bar{U}^2$ $\overset{\rightarrow}{\not\leftarrow}$ BIBO Stability

c) $Q(z_1,z_2) \neq 0$ in $\bar{U}^2$ $\overset{\not\rightarrow}{\leftarrow}$ BIBO Stability
except at $|z_1|=|z_2|=1$

d) $\sum_{m=o}^{\infty}\sum_{n=o}^{\infty} |h(m,n)|^2 < \infty$ $\overset{\not\rightarrow}{\leftarrow}$ BIBO Stability

e) $\lim_{m,n\to\infty} \{h(m,n)\} = 0$ $\overset{\not\rightarrow}{\leftarrow}$

$$\sum_{m=o}^{\infty}\sum_{n=o}^{\infty} |h(m,n)| < \infty \text{ or } \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} |h(m,n)|^2 < \infty$$

f) $Q(z_1,z_2) \neq 0$ in U^2 $\overset{\not\rightarrow}{\leftarrow}$ $|h(m,n)| \leqslant M < \infty$
for all m,n

g) $|H(z_1,z_2)| \leq N < \infty$ in $U^2 \longrightarrow \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} |h(m,n)|^2 < \infty$

h) $Q(z_1,0) \neq 0$ in $U'^2 \longrightarrow \sum_{m=o}^{\infty} |h(m,n)| < \infty$ for all n

where $\bar{U}^2$ denotes the closed unit bidisc:

$$\bar{U}^2 = \{(z_1,z_2): |z_1| \leq 1 \; |z_2| \leq 1\}$$

and U^2 is the open unit bidisc.

2.3 CRITERIA FOR STABILITY TEST

Given,

$$H(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{2.9}$$

We assume that $A(z_1,z_2)$ and $B(z_1,z_2)$ are coprime (no common factor). Furthermore, we also assume that $H(z_1,z_2)$ has no nonessential singularity of the second kind on the unit bidisc. Hence the stability condition (which is called structural stability) is both necessary and sufficient.

According to Shanks' theorem [7] the condition for stability is:

$$B(z_1, z_2) \neq 0 \quad \text{if} \quad |z_1| \leq 1 \;;\; |z_2| \leq 1 \tag{2.10}$$

All the zeroes of $B(z_1, z_2)$ are outside the unit bidisc. The test of that requires infinite mapping from all values $|z_1| \leq 1$ on $B(z_1, z_2) = 0$. If the image of the map lies completely outside $|z_2| = 1$, the filter is stable, otherwise it is not. This test is computationally involved and does not lead to a finite algorithm.

In a later work, Huang [19] has obtained another criterion which simplifies the stability test.

Theorem 2.3:

A causal filter with a z-transform function $H(z_1, z_2)$ is stable if:

(1) The map of $\bar{d}_1 = (z_1; |z_1| = 1)$ in the z_2 - plane according to $B(z_1, z_2) = 0$ lies outside $d_2 \equiv (z_2; |z_2| \leq 1)$ and,

(2) No point in $d_1 = (z_1; |z_1| \leq 1)$ maps into $z_2 = 0$, by the relationship $B(z_1, z_2) = 0$.

Huang's theorem is based on the earlier work of Ansell [65] on the stability of two-dimensional Hurwitz polynomials. Its rigorous proof has been supplied by Goodman [33], Davis [34], and Murray [35].

The conditions are equivalent to;

$$\text{i)} \quad B(z_1,z_2) \neq 0 \quad |z_1| = 1 \quad |z_2| \leq 1$$

$$\text{ii)} \quad B(z_1,0) \neq 0 \quad |z_1| \leq 1 \qquad (2.11)$$

The testing of conditions in eqn. (2.11) can be performed by a finite algorithm. Various forms of such an algorithm will be presented later in this chapter.

In recent years, it has been shown by several investigators that the conditions (2.11) can be replaced by new ones.

a) Strintzis [23] showed that the second condition of (2.11) can be replaced by $B(z_1,a) \neq 0$ for all $|a| \leq 1$ and $|z_1| \leq 1$. Hence the equivalent criterion will be

$$B(z_1,a) \neq 0 \quad |a| \leq 1, \ |z_1| \leq 1$$

$$B(z_1,z_2) \neq 0 \quad |z_1| = 1 \quad |z_2| \leq 1 \qquad (2.12)$$

b) DeCarlo et al [24] and Strintzis [23] showed another criterion which is equvalent to (2.11). This is given as follows.

$$B(a, z_2) \neq 0 \quad \text{for some } a, \ |a| \leq 1 \quad \text{when } |z_2| \leq 1$$

$$B(z_1, b) \neq 0 \quad \text{for some } b, \ |b| \leq 1 \quad \text{when } |z_1| \leq 1$$

$$B(z_1,z_2) \neq 0 \quad |z_1| = |z_2| = 1 \qquad (2.13)$$

In particular with the choice of a=b=1, the above condition becomes

$$B(1\ ,z_2) \neq 0 \qquad |z_2| \leqslant 1$$

$$B(z_1,1\ ) \neq 0 \qquad |z_1| \leqslant 1$$

$$B(z_1,z_2) \neq 0 \qquad |z_1| = |z_2| = 1 \tag{2.14}$$

c) Another criterion was developed by deCarlo et al [49] and it is presented as follows:

$$B(z\ ,z\ ) \neq 0 \qquad |\ z| \leqslant 1$$

$$B(z_1,z_2) \neq 0 \qquad |z_1| = |z_2| = 1 \tag{2.15}$$

d) The following criterion was obtained by Jury [45].

$$B(z^s,z^t) \neq 0 \qquad |z\ | \leqslant 1$$

$$B(z_1,z_2) \neq 0 \qquad |z_1| = |z_2| = 1 \tag{2.16}$$

s,t integers

g) $$B(z_1^s,z_2^t) \underset{\text{unstable}}{\overset{\text{stable}}{\Leftrightarrow}} B(z_1,z_2) \qquad s,t \text{ integers} \tag{2.17}$$

h) $$B(z_1^{k_1}z_2^{k_2},z_1^{k_4}z_2^{k_3}) \underset{\text{unstable}}{\overset{\text{stable}}{\Leftrightarrow}} B(z_1,z_2) \tag{2.18}$$

provided $k_1k_4 \neq k_3k_2$ and k_i's integers

2.4 VARIOUS STABILITY TESTS

The object of this section is to review procedures for checking the stability conditions discussed in preceding sections. We will be mainly concerned with various implementations which have been developed for stability testing of causal two-dimensional recursive filters.

2.4.1 Symmetric Matrix Form

Anderson and Jury [20] used the Schur- Cohn matrix to test the conditons of Huang's theorem [19]. If we recall the conditions of this criterion;

1) $B(z_1,z_2) \neq 0$ when $|z_1| = 1$, $|z_2| \leq 1$

   $B(z_1,z_2)$ should have all its zeroes (when $|z_1| = 1$) outside the unit circle in the z_2 plane.

2) $B(z_1,0) \neq 0$ for all values $|z_1| \leq 1$

   A one-dimensional polynomial having all its zeroes outside the unit circle in the z_1 - plane.

Condition (2) reduces to the stability test of one-dimensional polynomial.

To test condition (1), we write $B(z_1,z_2)$ as a polynomial in z_2 but with coefficients in z_1. Since $|z_1| = 1$, these coefficients are complex. Hence the problem reduces to determining the Routh distribution of a polynomial with complex coefficients.

The theorem of Schur-Cohn [45] deals with the root distribution with respect to unit circle of a polynomial with complex coefficients.

Let

$$F(z) = \sum_{i=o}^{N} a(i) z^{i} \tag{2.19}$$

The matrix $C = \{\gamma(i,j)\}$ is a "Hermitian Matrix" in which

$$\gamma(i,j) = \sum_{p=1}^{i} \left[a(n-i+p)\bar{a}(n-j+p)-\bar{a}(i-p)a(j-p)\right] \ , \ i \leq j \tag{2.20}$$

where

$$\gamma(j,i) = \bar{\gamma}(i,j) \quad , \quad i > j \quad ; \ i,j = 1,2,\ldots,N$$

The Schur-Cohn matrix is positive definite and symmetric if, and only if, all the roots of $F(z) = 0$, are inside the unit circle; it is negative definite if, and only if, all roots are outside the unit circle. When $|z_1| = 1$, the minors of the Schur-Cohn matrix are of the following form:

$$f_i(z_1) = \sum_{j=o}^{N} c_j(z_1^j + z_1^{-j}) \tag{2.21}$$

and have to be positive for all $|z_1| = 1$.

These polynomials are reciprocal ones, they have the same number of roots inside the unit circle as outside the unit circle. The condition for a reciprocal polynomial to be positive for all $|z_1| = 1$, is that it is positive at one point, such as $z_1=1$, and has no roots on the unit circle. This reduces to the root distribution of a real polynomial.

2.4.2 Resultant Method

A recently introduced new stability test [86] for one-dimensional digital filters is extended to apply to two-dimensional case. The method is based on the following theorem.

Theorem 2.4 [86]:

Let D(z) be polynomial of degree n, having real coefficients, and let

$$D(z) = \left[D_1(z) + D_2(z)\right]$$

where

$$D_1(z) = \tfrac{1}{2}\left[D(z) + z^n D(z^{-1})\right] \tag{2.22}$$

$$D_2(z) = \tfrac{1}{2}\left[D(z) - z^n D(z^{-1})\right] \tag{2.23}$$

Then $D(z) \neq 0$ in $|z| \geq 1$ if and only if all zeroes of $D_1(z)$ and $D_2(z)$ are simple, are located on the unit circle $|z| = 1$, and also separate each other on the unit circle.

The two-dimensional filter function is assumed to have no non-essential singularities of the second kind on the unit bidisc. Then the denominator polynomial $B(z_1,z_2)$ can be rewritten as:

$$B_1(z_1,z_2) = z_2^{n_2} B(z_1,z_2^{-1}) \tag{2.24}$$

n_2 is the degree of z_2 in $B(z_1,z_2)$. Let

$$B_1(z_1,z_2) = \sum_{k=0}^{n_2} b_k(z_1) z_2^k \tag{2.25}$$

$$B_1^*(z_1,z_2) = \sum_{k=0}^{n_2} \overline{b_k(z_1)} z_2^k \tag{2.26}$$

on $|z_1| = 1$, $\bar{z}_1 = z_1^{-1}$, and let

$$D^o(z_1,z_2)\Big|_{|z_1|=1} = B_1(z_1,z_2) B_1^*(z_1,z_2) \tag{2.27}$$

$$= \sum_{k=0}^{2n_2} \left\{ \sum_{\ell} c_\ell (z_1^\ell + z_1^{-\ell}) \right\} z_2^k \tag{2.28}$$

where the c_ℓ's in (2.28) are constant. Substituting $z_1 = e^{j\theta}$ in (2.28), we obtain:

$$D^o(z_1,z_2)\Big|_{|z_1|=1} = \sum_{k=0}^{2n_2} \left\{ \sum_{\ell} 2c_\ell \cos \ell\theta \right\} z_2^k \tag{2,29}$$

Using the trigonometric identity,

$$\text{Cos}\, n\theta = \sum_{s=0}^{m} \binom{n}{2k} (-1)^s \ \text{Cos}^{(n-2s)}\theta \sin^{2s}\theta \tag{2.30}$$

where $m=n/2$ for n even and $(n-1)/2$ for n odd. Equation (2.28) can be written as

$$D(x,z_2) = D^o(z_1,z_2)\Big|_{|z_1|=1}$$

$$= \sum_{k=0}^{2n_2} d_k(x) z_2^k \tag{2.31}$$

where $d_k(x)$ are polynomials in $x = \text{Cos}\theta$. Then $D_1(x,z_2)$ and $D_2(x,z_2)$ are defined as

$$D_1(x,z_2) = \tfrac{1}{2}\left[D(x,z_2) + z^{2n_2} D(x,z_2^{-1}) \right] \tag{2.32}$$

$$D_2(x,z_2) = \tfrac{1}{2}\left[D(x,z_2) - z_2^{2n_2} D(x,z_2^{-1}) \right] \qquad (2.33)$$

Finally, the second condition in (2.11) can be tested by using the following theorem which is based on theorem 2.4.

Theorem 2.5 [38]:

$B(z_1,z_2) \neq 0$ for $|z_1| = 1$, and $|z_2| \geq 1$ if and only if:

1) the zeroes of $D_1(0,z_2)$ and $D_2(0,z_2)$ are located on the unit circle $|z_2|=1$,
2) the zeroes of $D_1(0,z_2)$ and $D_2(0,z_2)$ are simple and alternate on the unit circle $|z_2|$, and
3) the resultant [22] $R(x)$ of $D_1(x)$ and $D_2(x)$ has no real roots in the interval $-1 \leq x \leq 1$.

Example 2.1:

We will use the resultant technique to test the criterion in equation (2.11); Given

$$B(z_1,z_2) = (12 + 10z_1 + 2z_1^2) + (6 + 5z_1 + z_1^2)z_2 \qquad (2.34)$$

from the second condition in equation (2.11)

$$B(z_1,0) = 12 + 10z_1 + 2z_1^2 \quad \text{in} \quad |z_1| \leq 1 \qquad (2.35)$$

Then the first condition will determine the stability.

The resultant R(x) can be obtained from the resultant matrix (appendix-A) . R(x) is the resultant of $D_1(x,z_2)$ and $D_2(x,z_2)$.

$$R(x) = -1296(20736x^8 + 241920x^7 + 1231200x^6 + 3570000x^5 + 7437500x^3 + 53433750x^2 + 2187500x + 398625) \qquad (2.36)$$

Since $R(x) \neq 0$ in $-1 \leqslant x \leqslant 1$, $B(z_1,z_2) \neq 0$ in $|z_1| = 1$, $|z_2| \geqslant 1$. This with the other result in equation (2.35), imply that $B(z_1,z_2) \neq 0$ in $|z_1| \leqslant 1$, $|z_2| \leqslant 1$.

2.4.3 Nyquist-Like Test

An extension of Nyquist stability test to two-dimensions has been provided by DeCarlo, Murray, and Seaks in a series of papers [24], [26], [49].

Let a polydisc D_α in C^2 be defined by

$$D_\alpha = \left\{ (e^{j\alpha}, z_2) : |z_2| \leqslant 1 \right\} \qquad (2.37)$$

where α is real and such that $0 \leqslant \alpha \leqslant 2\pi$, and let a disc D be defined by

$$D = \left\{ (z_1, 0) : |z_1| \leqslant 1 \right\} \qquad (2.38)$$

D_α and D correspond to the region of analyticity in equation

(2.11). Then DeCarlo, Murray, and Seaks established the following results:

Theorem 2.6 [49]:

A causal recursive digital filter characterized by the rational two-dimensional transfer function

$$G(z_1,z_2) = \frac{P(z_1,z_2)}{Q(z_1,z_2)} \tag{2.39}$$

where any irreducible common factors of $P(z_1,z_2)$ and $Q(z_1,z_2)$ have been cancelled and where $G(z_1,z_2)$ has no nonessential singularities of the second kind on the boundary of the unit bidisc.

$G(z_1,z_2)$ is stable in BIBO sense if and only if the Nyquist plots of the one-dimensional functions

$$Q(e^{j\alpha}, z_2) \ , \quad 0 \leqslant \alpha \leqslant 2\pi \tag{2.40}$$

and

$$Q(z_1, 0) \tag{2.41}$$

do not equal or encircle zero in the complex plane. We can obtain other graphical tests by involving the equivalent criteria of (2.12) - (2.14). This leads to the following theorems.

Theorem 2.7 [49]:

The two-dimensional digital filter described in theorem 2.6 is BIBO stable if and only if

i) $Q(z_1,z_2)$ has no zeroes on $|z_1| = |z_2| = 1$;

ii) The Nyquist plots for the one-dimensional function $Q(1,z_2)$ and $Q(z_1,0)$ do not encircle zero.

Theorem 2.8 [49]:

The two-dimensional digital filter described in theorem 2.6 is BIBO stable if and only if

i) $Q(z_1,z_2)$ has no zeroes on $|z_1|=|z_2|=1$,

ii) The Nyquist plots for one-dimensional functions $Q(1,z_2)$ and $Q(z_1,1)$ do not encircle zero.

Theorem 2.9 [49]:

The two-dimensional filter described in theorem 2.6 is BIBO stable if and only if

i) $Q(z_1,z_2)$ has no zeroes on $|z_1|=|z_2|= 1$.

ii) The Nyquist plot of the single variable function $Q(z,z)$ does not encircle zero.

Example 2.2:

Let the transfer function of a digital filter be

$$G(z_1,z_2) = \frac{1}{1 + 0.25z_1 + 0.25z_2} = \frac{P(z_1,z_2)}{Q(z_1,z_2)} \tag{2.42}$$

Step 1: Draw the Nyquist plot for $B(z_1,0)$. This curve is shown in Fig.2.1; as it does not encircle zero we proceed to the next step as outlined in theorem 2.6.

Step 2: Now consider the family of Nyquist plots for the function

$$B(e^{j\alpha}, z_2) \; : \quad 0 \leqslant \alpha \leqslant 2\pi \tag{2.43}$$

This family of curves does not encircle "0" as indicated in Fig. 2.2. Hence the filter is stable.

2.4.4 Table Method

Routh, in 1877 in his Adams prize paper (at Cambridge) has suggested a stability test or table. The Routh table checks the root-distributions of a one-dimensional polynomial with respect to the imaginary axis in the s-plane. A similar table form exists for the root-distribution of real or complex polynomials with respect to unit circle. In 1961, Jury [89] suggested a table to check the stability of one-dimensional digital filters. Later, Maria and Fahmy [21] modified the Jury table and used it to check the second condition of Huang's theorem [19].

A computer program for stability testing, based on Mariaand Fahmy method is given in [90].

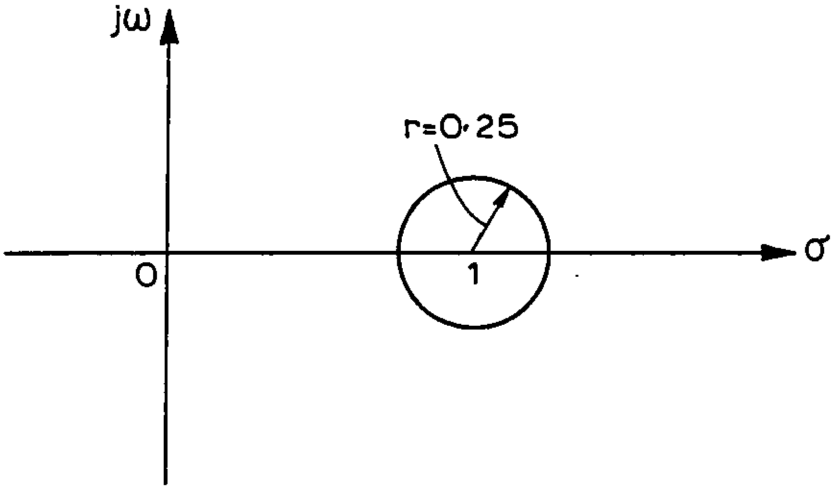


Fig. 2·1 Nyquist plot of $B(z_1,0)$ in (2·41)

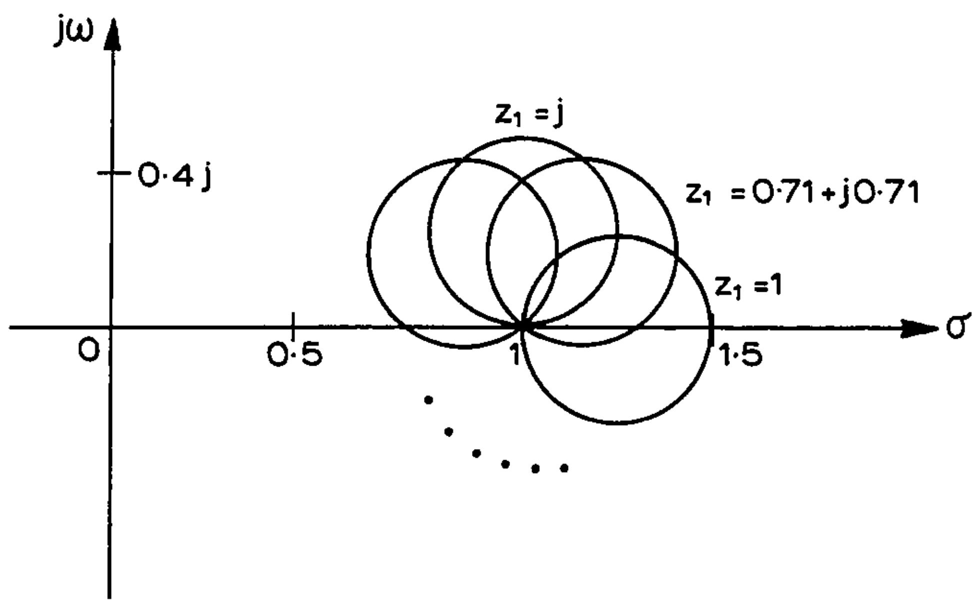


Fig. 2·2 Nyquist plot of $B(e^{j\alpha},z_2)$ in (2·41)

2.4.5 Impulse Response Test

Application of the stability test based on the one-dimensional polynomials was proposed by Kirshnamurthy [87]. Recently, impluse response test for two-dimensional digital filters was studied by Strintzis [23], [93], and Vidyasagar and Bose [92]. The stability test for two-dimensional filters is based on the following theorem:

Theorem 2.10 [93]:

Let H be the upper limit of the double sequence $\left\{ |h(m,n)|^{1/m+n} \right\}$:

$$H = \overline{\lim_{m \text{ and/or } n}} \; |h(m,n)|^{1/m+n} \tag{2.44}$$

If $H(z_1,z_2)$ is rational in z_1 and z_2,

(i) $H < 1$ is necessary and sufficient for convergence of

$$H(z_1,z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h(m,n) z_1^m z_2^n \tag{2.45}$$

in $\{|z_1| \leqslant 1 \, , \, |z_2| \leqslant 1\}$ and for BIBO stability of the filter.

(ii) The following condition is also necessary and sufficient for the convergence of (2.45) in $\{|z_1| \leqslant 1 \, , \, |z_2| \leqslant 1\}$

and for BIBO stability of the filter:

$$|h(m,n)| < K\mu^{m+n} \quad 0 \leq K < \infty\,, \quad |\mu| < 1 \tag{2.46}$$

It is clear that if $H > 1$, the filter is unstable. The case where $H=1$, is discussed in the following lemma.

Lemma 2.1 [93]:

If $H(z_1,z_2)$ is rational and $H=1$, then the unstable singularities may only occur in one of the following three regions:

1) $|z_1|=1$, z_2 arbitrary
2) z_1 arbitrary, $|z_2|=1$
3) along the perimeter (but not the interior) of the set $\{|z_1| \leq 1\ ,\ |z_2| \leq 1\}$, i.e. when

$$H(z_1,z_2) = \infty \quad \text{for some} \quad |z_1| = |z_2| = 1 \tag{2.47}$$

$$H(z_1,z_2) \neq \infty \quad \text{if either} \ |z_1| < 1 \ \text{or} \ |z_2| < 1 \tag{2.48}$$

As a direct consequence of the lemma 2.1 and the theorem 2.10, we have the following theorem.

Theorem 2.11 [93]:

If $H(z_1,z_2)$ is rational and not in the class of functions described by (2.47) and (2.48), the following conditions are all equivalent; each is necessary and sufficient for BIBO stability of the filter.

(iii) $h(m,n) \to 0$ when $m \to +\infty$, or $n \to +\infty$ or both (2.49)

$$\text{(iv)} \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |h(m,n)|^{p} < +\infty \quad p > 1 \tag{2.50}$$

Conditions (i)-(iv) of theorem 2.10 and 2.11 are different from the one-dimensional case. In particular (iii) and (iv) are not equal to (i) and (ii) because of Lemma 2.1. In order to illustrate this point, the following example is given by Goodman [91];

$$H(z_1,z_2) = \frac{2}{2-z_1-z_2} \tag{2.51}$$

The above filter is BIBO unstable, but the unit sample response g(m,n) is such that

$$\lim_{m,n\to\infty} \{|h(m,n)|\} = 0 \tag{2.52}$$

However, following corresponding conditions for the one-dimensional case are equivalent.

$$\overline{\lim} \quad (|h(n)|)^{1/n} < 1$$

$$|h(n)| \leqslant K\mu^n , \qquad 0 \leqslant K < +\infty , \qquad |\mu| < 1$$

$$|h(n)| \leqslant P_n \qquad \lim_{n\to\infty} \quad P_n = 0$$

$$\sum_{n=0}^{\infty} |h(n)|^p < +\infty \quad \text{for any } p > 1$$

2.4.6 Cepstral Test

Complex cepstrum was used for stabilization of two-dimensional recursive filters by Pistor [31]. He did not present any algorithm for testing the stability. Such a test was later obtained by Ekstrom and Woods [94], as an application of two-dimensional spectral factorization. It is based on a two-dimensional factorization operation involving the autocorrelation function of the filter which covers both quarter - and half-planes. However, recently, the existence of a two-dimensional complex cepstrum has been shown by Dudgeon [39]. Based on such existence, Ekstrom and Twogood [28] have obtained an alternate test which removes the earlier complexity and is computationally attractive.

Dudgeon [39] has shown that essential singularities and zeroes of a transfer function $H(z_1,z_2)$ map into essential singularities of

$$\hat{H}(z_1,z_2) = \ln\left[H(z_1,z_2)\right] \tag{2.53}$$

Now if $H(z_1,z_2)$ is stable filter, it can be written in a power series with $m,n \in \mathcal{R}$ (where $\mathcal{R}$ is the region of support of the filter). Hence $\hat{H}(z_1,z_2)$ can be similarly expanded as

$$\hat{H}(z_1,z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{h}(m,n)\ z_1^m z_2^n \tag{2.54}$$

From this fact the following theorem can be derived;

<u>Theorem 2.12 [28]:</u>

The asymmetric half-plane recursive filter, $H(z_1,z_2) = 1/Q(z_1,z_2)$ is stable if and only if its cepstrum $\hat{h}(m,n)$ has support on $\mathcal{R}$.

The implementation of this theorem into a stability test is as follows:

Step 1) Form $Q(z_1,z_2)$ for $q(m,n)$ of the filter to be tested.

Step 2) Calculate $\hat{Q}(z_1,z_2)$ and its inverse z-transform to obtain the cepstrum $\hat{q}(m,n)$.

Step 3) $\hat{q}(m,n)=0$ for $m,n \notin \mathcal{R}$ then the filter is stable. If $\hat{q}(m,n)\neq 0$ for $m,n \notin \mathcal{R}$ then the filter is unstable.

However, there are some difficulties in performing the second step. Indeed, in order to ensure the analyticity of $\hat{Q}(w_1,w_2)$ which is equal to:

$$\hat{Q}(w_1,w_2) = \ln\left[Q(w_1,w_2)\right]$$

$$= \ln\left|Q(w_1,w_2)\right| + j\arg\left[Q(w_1,w_2)\right] \qquad (2.55)$$

The phase, arg $\left[Q(w_1,w_2)\right]$ must be periodic and continuous [39]. To ensure the continuity, one can use phase unwrapping [28] , and to ensure the periodicity (with period 2π), one uses the method of linear phase removal.

The cepstral method is mainly applicable for numerical testing. Therefore, it is not possible to obtain stability inequality conditions.

2.5 COMPARISON OF STABILITY TESTS

Since the size of the Schur-Cohn matrix is equal to the degree of the denominator polynomial, the calculation of principal minors becomes rather tedious with increasing size of the matrix. Hence, it is difficult to show the positive definiteness of the symmetric matrix of Schur-Cohn for higher orders.

Resultant method reduces the stability testing problem to checking polynomials for positivity over the local region. The presence or absence of real zeroes in this region can be determined by Sturm's theorem. However, the real zero determination might be computationally complicated for higher orders. The table method of Maria and Fahmy is more practical than the symmetric matrix method of Anderson and Jury and the resultant method of Bose. All matrices involved are second order only. The Nyquist-like test is a graphical approach to test the stability. Because the Nyquist plot is related to the frequency response, it appears that graphical tests are useful not only for checking the stability but also for design purposes where certain changes in the frequency response are required.

The computational comparison between the cepstral test of Ekstrom and Woods with table form of Maria and Fahmy showed that the cepstral method to be more efficient. However, a drawback of the cepstral method is the assumption that, to avoid problems in carrying out the logarithm, the transfer function should not have any singularities on the distinguished boundary. Therefore, it is less reliable than the other tests when the zeroes of the polynomial are near the unit polydisc.

2.6 TESTING THE STABILITY WITH INNER DETERMINANTS

In this section a new method is proposed to check the stability of two-dimensional filters. In this test two-variable inner determinants are used to examine the roots of polynomials with real coefficients. The amount of computation needed for this method is comparable to that needed when using the other procedures discussed in the previous section.

2.6.1 Introduction to Inners

The term "inners" is given to certain square submatrices that arise within a square NxN matrix. This term as well as other definitions connected with it, was first proposed by Jury [95] in 1970. Since that time many articles on the theory and applications of this concept have been published [83] , [84]. Most of this work is also discussed in a recently published book [22].

The importance of inner approach lies mainly in the theoretical unification of both continuous time and discrete time theories (especially stability theory), as well as in the computational unification obtained by utilizing only one algorithm for a number of different applications.

2.6.2 Definitions of Inners

Definition 2.2: Let Δ be an N x N matrix. Form the matrix Δ_{N-2}, of dimension N-2 x N-2, by deleting the first and last rows and first and last columns of Δ ; then Δ_{N-2} is called inner. Now repeat this process on Δ_{N-2} to form Δ_{N-4}. Continue this process until it ends thus forming $\Delta_1, \Delta_3, \Delta_5, \ldots, \Delta_{N-2}$ for N odd and $\Delta_2, \Delta_4, \ldots, \Delta_{N-2}$ for N even. The appropriate set is called the inners of the matrix [22].

Remark:

If N is even (larger than or equal to four), the number of inners is (N-2)/2. The inners $\Delta_2, \Delta_4, \ldots$ are designated as first, second, ... inners, respectively. If

N is odd, the number of inners is (N-1)/2. The first, second, ..., inners are Δ_1, Δ_3, ..., respectively. Note that in this case the first inner, Δ_1, is a one element matrix, that is, a scalar.

Example 2.3:

Let N=6, the inners of a 6 x 6 matrix are formed as follows:

$$\left[\Delta_6\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \tag{2.56}$$

The inners are Δ_2 and Δ_4.

<u>Definition 2.3:</u> If the determinants of all the inners, as well as that of the matrix itself, are positive, we designate the matrix as positive <u>innerwise</u>. If all the determinants are negative, we designate the matrix as <u>negative innerwise</u>.

<u>Definition 2.4:</u> If the determinants of the inners of Δ_N as well as that of Δ_N are zero, we designate Δ_N as <u>null innerwise</u>. If none of the determinants is zero, we designate it <u>as nonnull</u> innerwise.

2.6.1 <u>The Main Result</u>

The sufficient conditions for the filter function $1/B(z_1,z_2)$ to be stable are

$$B(1,z_2) \neq 0 \quad |z_2| \leq 1 \tag{2.57}$$

$$B(z_1,1) \neq 0 \quad |z_1| \leq 1 \tag{2.58}$$

$$B(z_1,z_2) \neq 0 \quad |z_1| = |z_2| = 1 \tag{2.59}$$

The proof of the above stability criterion can be found in [23] . The first two conditions (2.57)-(2.59) reduce to a one-dimensional stability test, and are computationally trivial to implement. However, the last condition (2.59) is rather difficult to test. In order to test the third condition, we transfer the zeros of a complex variable polynomial on the unit polydisc to zeros of the real variable polynomial in a region.

$B(z_1, z_2)$ can be written in recursive canonical form as a polynomial in z_2 with coefficients which are polynomials in z_1

$$B(z_1, z_2) = \sum_{j=0}^{q} b_j(z_1) z_2^j \tag{2.60}$$

Let $D(z_1, z_2) = B(z_1, z_2) B(z_1^{-1}, z_2)$. (2.61)

$D(z_1, z_2)$ can be rewritten as:

$$D(z_1, z_2) = \sum_{j=0}^{2q} \sum_{i=0}^{p} c_i (z_1^i + z_1^{-i}) z_2^j \tag{2.62}$$

The following substitutions for $(z_1^i + z_1^{-i})$ may now be made

$$(z_1 + z_1^{-1}) = 2x$$

$$(z_1^2 + z_1^{-2}) = 4x^2 - 2$$

$$(z_1^3 + z_1^{-3}) = 8x^3 - 6x \tag{2.63}$$

$$\vdots$$

On $|z_1| = 1$, x is real and $D(z_1, z_2)$ becomes

$$D(x, z_2) = D(z_1, z_2)\Big|_{|z_1| = 1} \qquad (2.64)$$

$$= \sum_{j=0}^{2q} d_j(x) z_2^j. \qquad (2.65)$$

Let $G(x, z_2) = D(x, z_2) D(x, z_2^{-1})$

$G(x, z_2)$ can be rewritten as:

$$G(x, z_2) = \sum_{j=0}^{4q} h_j(x)(z_2^j + z_2^{-j}) \qquad (2.66)$$

Then by substitution of (2.63) for $(z_2^j + z_2^{-j})$, $G(x, z_2)$ can be written as:

$$G(x, y) = G(x, z_2)\Big|_{|z_2|=1} = \sum_{j=0}^{4q} q_j(x) y^j \qquad (2.67)$$

Theorem 2.13:

$B(z_1,z_2) \neq 0$ for $|z_1| = |z_2| = 1$ if and only if $G(x,y) \neq 0$ for $x \in [-1,+1]$ and $y \in [-1,+1]$

The proof of this theorem can be found in Appendix B, the following lemma is obtained from theorem 2.13.

Lemma 2.2:

$B(z_1,z_2) \neq 0$ for $|z_1| = |z_2| = 1$ if

a) $G(x,y) \neq 0$ for $x \in [-1,+1]$ and $y \in [-\infty,+\infty]$

b) $G(x,y) \neq 0$ for $y \in [-1,+1]$ and $x \in [-\infty,+\infty]$

The proof of this lemma is simple and based on theorem 2.13. After obtaining the above results, the one-variable inner determinants [46], [50] can be used to test the third condition (2.59).

Theorem 2.14:

The number of real roots, N, of

$F(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_o,\ a_n > 0$ is

$$N = \mathrm{var}\left[1, -|\Delta_1^1|,\ +|\Delta_3^1|, \ldots\ldots, (-1)^n|\Delta_{2n-1}^1|\right]$$

$$- \mathrm{var}\left[1,\ |\Delta_1^1|\ ,\ |\Delta_3^1|, \ldots\ldots, |\Delta_{2n-1}^1|\ \right] \qquad (2.68)$$

Where 'var' denotes the number of variations of signs, and

$|\Delta_i^1|$, $i = 1, 2, .., 2n-1$ are the inner determinants in the matrix $[\Delta']$ in (2.69).

$$
[\Delta'] = \begin{bmatrix}
a_n & a_{n-1} & a_{n-2} \cdots & a_o & \cdots 0 & \cdots & \cdots 0 \\
 & a_n & a_{n-1} \cdots & a_1 & \cdot\cdot\ a_o & \cdots & \cdots 0 \\
 & & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
 & & a_n & a_{n-1} & a_{n-2} & \cdots & \cdots\ a_o \\
 & & 0 & na_n & (n-1)a_n & \cdots & \cdots\ a_1 \\
 & & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
0 & na_n & (n-1)a_{n-1} & \cdots a_2 & a_1 & \cdots & \cdots 0 \\
na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \cdots a_1 & 0 & \cdots & \cdots 0
\end{bmatrix}
\qquad (2.69)
$$

The proof of theorem 2.14 can be found in [22]. In our case, the coefficients, a_n, are functions of x.

2.6.4 Examples:

The purpose of the following examples is to illustrate the procedure described above, step by step. In particular, Example 2.5 is also used by Bose [38] and Anderson and Jury [20] to illustrate the resultant technique and symmetric matrix form, respectively. It is found that the proposed test is much simpler than both techniques.

Example 2.4:

Given $B(z_1,z_2) = 2 + 2z_1 + z_2$. Clearly first and second conditions (2.57), (2.58) are satisfied. Therefore, the third condition (2.59) will determine whether or not the filter is structurally stable. We obtain $G(x,y)$ as:

$$G(x,y) = 16(5 + 4y)x^2 + 8(18 + 17y + 4y^2)x + (65 + 72y + 32y^2)$$

and one-variable inner determinants [46] are found.

$$\Delta_1^1(y) = 32(5 + 4y)$$

$$\Delta_3^1(y) = 1024\ (5 + 4y)(16y^4 + 8y^3 - 15y^2 - 8y - 1)$$

Table 2.1 shows zeros and sign distribution of inner determinants in the region of $y \varepsilon [-1, +1]$.

Due to theorem 2.14, we find

$$N_{(-1,-\frac{1}{4})} = \mathrm{var}[+,-,-] - \mathrm{var}[+,+,-] = 0$$

$$N_{(-\frac{1}{4},+1)} = \mathrm{var}[+,-,-] - \mathrm{var}[+,+,-] = 0$$

where $N_{(-1,-\frac{1}{4})}$ and $N_{(-\frac{1}{4},+1)}$ denotes the number of real zeros in x of G(x,y) for any fixed value of y in $(-1,-\frac{1}{4})$ and $(-\frac{1}{4},+1)$, respectively. However, we can show that $N_{-\frac{1}{4}} = 2$, where $N_{-\frac{1}{4}}$ is the number of real zeros of $G(x,-\frac{1}{4})$. Taking $y = -\frac{1}{4}$, $G(x,-\frac{1}{4}) = 64x^2 + 112x + 49$, which has a real root in $-1 \leqslant x \leqslant 1$. From theorem 2.13 we conclude that $B(z_1,z_2)$ has zero(s) on the unit polydisc. Hence the filter function is unstable.

Example 2.5:

Given $B(z_1,z_2) = (12 + 10z_1 + 2z_1^2) + (6 + 5z_1 + z^2)z_2$.

The first and second conditions are satisfied, the third condition (2.59) will determine the stability. We obtain G(x,y) as:

$$G(x,y) = (4y + 5)^2(24x^2 + 70x + 50)^2$$

one-variable inner determinants are:

$$\Delta_1^1(x) = 32(24x^2 + 70x + 50)^2$$

$$\Delta_3^1(x) \equiv 0$$

Both Δ_1^1 and Δ_3^1 do not have any real root in the interval $I=[-1,+1]$, $N_I = 0$. Since the number of real roots, N_I, is equal to zero, the lemma 2.2 guarantees that $B(z_1,z_2)$ has no zero on the distinguished boundary, i.e.,

Table 2·I

Sign of Inners in Example 2·4

	-1		$-0{\cdot}25$		$+1$
		I_1		I_2	
$\Delta_1'(y)$		+ + +		+ + + + + +	
$\Delta_3'(y)$	0	− − −	0	− − − − − −	0

Table 2·II

Sign of Inners in Example 2·5

	-1		$+1$
		I	
$\Delta_1'(x)$		+ + + + + + + + +	
$\Delta_3'(x)$		− − − − − − − − −	

$B(e^{j\omega_1}, e^{j\omega_2}) \neq 0$. The filter function is stable. Since this is an example of a separable system simpler techniques for assessing stability are available.

2.6.5 Remarks:

In the second example above, $\Delta_3^1(x) \equiv 0$ implies the presence of a common factor in $G(x,y)$ and $G'(x,y) = \delta G(x,y)/\delta y$, and this can be extracted via rational operations [46], [76].

2.7 DIFFICULTIES WITH THE EXTENSION OF LYAPUNOV'S TEST FOR TWO-DIMENSIONAL FILTER FUNCTIONS

2.7.1 Introduction

The extension of Lyapunov's function to two-dimensional digital recursive filters has been studied by several investigators [97] - [99]. Algizi and Fahmy [97] derived a criterion which sufficiently guarantees the absence of overflow oscillations. Most recently, Agathoklis [99] used the two-dimensional Lyapunov's test for estimation of the stability margin.

The two-dimensional Lyapunov's test is given as follows [97] - [99]; $\det[I - z_1 A_2 - z_2 A_1] \neq 0$ in the domain $\bar{U}^2$ if and only if there exists a block diagonal matrix P such that

$$Q = [A_1 + A_2]^T P [A_1 + A_2] - P \tag{2.70}$$

Q is negative definite,

where A_1 and A_2 matrices are obtained from the following state-space model given by Fornasini and Marchesini [96];

$$x(m+1,\ n+1) = A_1x(m+1,\ n) + A_2x(m,n+1) \tag{2.71}$$

and

$$\bar{U}^2 = \{(z_1,z_2) : |z_1| \leqslant 1,\ |z_2| \leqslant 1\} \tag{2.72}$$

The system in equation (2.71) represents the zero-input condition of the state behaviour. The z-transform of this system is given by

$$X(z_1,z_2)\ [I-z_2A_1- z_1A_2]= 0 \tag{2.73}$$

It can be shown that the system (2.71) is bounded-input bounded-output (BIBO) stable if and only if

$$\det [I-z_2A_1- z_1A_2] \neq 0 \quad \text{in } \bar{U}^2 \tag{2.74}$$

Since eigenvalues of two-dimensional digital filter functions are not singular points but two-dimensional manifolds, one should expect that finite order Lyapunov's functions should not carry all the information for the singularities of the given system. The following counter example shows that the extension of Lyapunov's test to the two-dimensional case is not valid in general.

2.7.2 A Counter Example for the Extension

Consider the following state-space system

$$x_1(m+1,\ n+1) = 0.5x_1(m+1,\ n) + 2x_2(m+1,\ n)$$

$$x_2(m+1,\ n+1) = 0.5x_1(m,n+1) \tag{2.75}$$

From this model, A_1 and A_2 can be obtained as:

$$A_1 = \begin{bmatrix} 0.5 & 2 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} \tag{2.76}$$

and $\det[I-z_2A_1-z_1A_2]$ can be written as

$$\det[I-z_2A_1-z_1A_2] = 1 - 0.5z_2 - z_1z_2 \tag{2.77}$$

It can be shown that the determined function in equation (2.77) has a zero at $(z_1,z_2) = (0.5 \mp j0.5,\ 0.8 \pm j0.4)$. Hence the system given by (2.76) is unstable [87]. However, if one considers the Lyapunov's test with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.78}$$

as a positive definite matrix. Then, by applying equation (2.70), we obtain:

$$Q = [A_1 + A_2]^T P [A_1 + A_2] - P$$

$$= \begin{bmatrix} 0.5 & 2 \\ 0.5 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 2 \\ 0.5 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 1 \\ 1 & 3 \end{bmatrix} \tag{2.79}$$

where Q is negative definite. Therefore, the above criterion guarantees that the det $[I - z_2 A_1 - z_1 A_2]$ has no zeroes in the closed unit polydisc ($\bar{U}^2$). However, it has been shown that this determinant function has a zero at $(z_1, z_2) = (0.5 \mp j0.5,\ 0.8 \pm j0.4)$.

2.7.3 Remarks

The above counter example shows that the direct extension of Lyapunov's test to the two-dimensional case is not true in general. However, one may show the validity of the extension for separable systems (i.e. $H(z_1, z_2) = H_1(z_1) H_2(z_2)$). Hence the extended test is necessary but not sufficient.

2.8 EXTENSION OF LYAPUNOV'S METHOD FOR TESTING THE STABILITY OF ROESSER'S MODEL

All the stability tests reviewed in Section 2.4, are concerned with BIBO stability. However, recently Fornasini and Marchesini [101] introduced a frequency dependent Lyapunov equation for their models [103]. The application of Lyapunov tests to Roesser's model [57] has not yet been developed. In the following section, a similar approach is used to generalize Lyapunov's method for testing Roesser's model which is a more general model than that of Fornasini and Marchesini.

2.8.1 Formulation of Roesser's Model

In the following formulation, i,j are integer valued vertical and horizontal coordinates, and $\{R\} \in \mathcal{R}^{n_1}, \{S\} \in \mathcal{R}^{n_2}$ are sets which convey information vertically and horizontally, respectively. The input and output of the system are $\{u\} \in \mathcal{R}^{p}$, $\{y\} \in \mathcal{R}^{n}$. The system to be considered is discrete, causal, and its state and output functions are described by

$$R(i+1,j) = A_1R(i,j) + A_2S(i,j) + B_1u(i,j)$$

$$S(i,j+1) = A_3R(i,j) + A_4S(i,j) + B_2u(i,j)$$

$$y(i,j) = C_1R(i,j) + C_2S(i,j) + D_4u(i,j) \tag{2.80}$$

For zero input conditions, the state behaviour of the system is given by

$$R(i+1,j) = A_1R(i,j) + A_2S(i,j)$$

$$S(i,j+1) = A_3R(i,j) + A_4S(i,j) \tag{2.81}$$

If we define the z-transform of $x(i,j)$ to be

$$X(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x(m,n)z_1^m z_2^n ,$$

equations (2.81) can be written in the matrix z-transformed form,

$$\begin{bmatrix} R(z_1,z_2) \\ \\ S(z_1,z_2) \end{bmatrix} = \begin{bmatrix} A_{10}z_1 + A_{01}z_2 \end{bmatrix} \begin{bmatrix} R(z_1,z_2) \\ \\ S(z_1,z_2) \end{bmatrix} \tag{2.82}$$

where

$$A_{10} = \begin{bmatrix} A_1 & A_2 \\ \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \quad ; \quad A_{01} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \\ A_3 & A_4 \end{bmatrix} \tag{2.83}$$

where $\underset{\sim}{0}$ is a null matrix of appropriate dimensions.

Then the following theorem can be derived

Theorem 2.15:

A 2-dimensional discrete system which is described by (2.80) is internally stable [101] if and only if the polynomial

$$A(z_1,z_2) = \det \left[I - A_{10}z_1 - A_{01}z_2 \right] \tag{2.84}$$

has no zeroes in the closed unit polydisc.

The proof of theorem 2.15 can be found in [102]. The following corollary can be obtained from Huang's theorem [19].

Corollary 2.1:

A 2-dimensional discrete system which is described by (2.80) is internally stable if and only if the complex matrix

$$A_{10} + e^{jw} A_{01} \tag{2.85}$$

is stable (i.e. magnitudes of its eigenvalues are less than 1) for real w.

2.8.2 Lyapunov Equation for Roesser's Model

From the above corollary, it can be shown that the system (2.80) is internally stable if and only if the Lyapunov equation

$$P(w) = I + \left[A_{10} + e^{-jw} A_{01}\right]^T P(w) \left[A_{10} + e^{jw} A_{01}\right] \qquad (2.86)$$

admits a positive definite Hermitian solution P(w) for every real w. The positive definite character of P(w) can be checked by applying Sturm's test to the principal minors of P(w) [101] . If we assume that system (2.80) is internally stable,

$$P(w) = \sum_{k=-\infty}^{\infty} P_k e^{jwk} \qquad (2.87)$$

is the solution of the frequency dependent Lyapunov equation (2.86). It can be shown that the Fourier coefficients P_k satisfy following properties [101] :

1) For any integer k, $P_k = P^T_{-k}$ and

$$P_0 = \sum_{r,s=0}^{\infty} \left[A^{r,s}\right]^T A^{r,s}$$

$$P_1 = \sum_{r,s=0}^{\infty} \left[A^{r+1,s}\right]^T A^{r,s+1}$$

$$\vdots$$

$$P_k = \sum_{r,s=0}^{\infty} \left[A^{r+k,s}\right]^T A^{r,s+k} \qquad (2.88)$$

and so on.

Where

$$A = \begin{bmatrix} A_1 & A_2 \\ & \\ A_3 & A_4 \end{bmatrix} \tag{2.89}$$

and

$$A^{0,0} = 1 \ , \ A^{r,0} = A_{10}^{r} \ , \ A^{0,s} = A_{01}^{s}$$

$$A^{-r,s} = A^{r,-s} = 0 \quad \text{for} \quad r \geqslant 1 \ , \ s \geqslant 1$$

$$A^{r,s} = A_{10}A^{r-1,s} + A_{01}\, A^{r,s-1} \tag{2.90}$$

2) The doubly infinite block Teoplitz matrix

$$\mathscr{P} = \begin{bmatrix} \ddots\, P_{-1} & \ddots\, P_0 & \ddots\, P_1 & \ddots & & \\ \ddots & P_{-1} & P_0 & P_1 & \ddots & \\ & \ddots & P_{-1} & P_0 & P_1 & \ddots \\ & & \ddots & \ddots & \ddots & \end{bmatrix} \tag{2.91}$$

induces positive definite scalar product in the space $\ell_2(C^n)$.

It is interesting to note that the Toeplitz matrix $\mathscr{P}$ satisfies the equation,

$$\mathscr{P} = \mathscr{I} + \mathscr{A}^{*}\mathscr{P}\mathscr{A} \tag{2.92}$$

where $\mathscr{I}$ is the (infinite) identity matrix and $\mathscr{A}$ and $\mathscr{A}^{*}$ are the doubly infinite block Toeplitz matrices:

$$\mathscr{A} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ 0 & A_{10} & A_{01} & 0 & & \\ & 0 & A_{10} & A_{01} & 0 & \\ & & 0 & A_{10} & A_{01} & 0 \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.93}$$

$$\mathscr{A}^{*} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ 0 & A_{01}^{T} & A_{01}^{T} & 0 & & \\ & 0 & A_{01}^{T} & A_{10}^{T} & 0 & \\ & & 0 & A_{01}^{T} & A_{10}^{T} & 0 \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.94}$$

Equations (2.92)-(2.94) are the generalization of one-dimensional Lyapunov's equation for Roesser's model. Equation (2.86) depends on the parameter w, so that the check of the positive definitness of its solution requires to test the variations in sign of the polynomial principal minors of P(w). On the other hand, (2.92) is infinite dimensional, and does not give any finite procedure for checking the stability.

CHAPTER 3

STABILITY OF MULTIDIMENSIONAL DESCRETE SYSTEMS

Stability problems of polynomials of dimensions higher than two arise in several applications of system theory-including, but not restricted to, multidimensional digital filtering and automatic control. The mathematical basis of multidimensional stability problems lies in the theory of complex function of several variables [42]- [44].

Conditions for stability have recently been formulated for discrete systems characterized by multi-variable rational functions. All these conditions as well as tests for checking them will be discussed.

In the main part of this chapter, a cepstral stability test for multidimensional digital filters will be introduced. The test is based on the properties of N-dimensional complex cepstrum of filters with rational transfer functions.

3.1 STABILITY OF MULTIDIMENSIONAL DIGITAL FILTERS

Let the z-transform of an N-dimensional first quadrant filter function be:

$$H(z_1,z_2,\ldots,z_n) = \frac{A(z_1,z_2,\ldots,z_n)}{B(z_1,z_2,\ldots,z_n)} \tag{3.1}$$

where

$$A(z_1,z_2,\ldots,z_n) = \sum_{m_1=0}^{M_1} \sum_{m_2=0}^{M_2} \cdots \sum_{m_n=0}^{M_n} a(m_1,m_2,\ldots,m_n) z_1^{n_1} z_2^{n_2} \ldots z_n^{n_n}$$

$$B(z_1,z_2,\ldots,z_n) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \cdots \sum_{n_n=0}^{N_n} b(n_1,n_2,\ldots,n_n) z_1^{n_1} z_2^{n_2} \ldots z_n^{n_n}$$

with $b(0,0,\ldots,0) \neq 0$. Without loss of generality, we assume $b(0,0,\ldots,0) = 1$, the coefficients $a(m_1,m_2,\ldots,m_n)$ and $b(n_1,n_2,\ldots,n_n)$ are real constants, not necessarily all non-zero. The transfer function $H(z_1,z_2,\ldots,z_n)$ can be expanded in a power series of $z_1,z_2,\ldots,z_n$ as:

$$H(z_1,z_2,\ldots,z_n) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} h(k_1,k_2,\ldots,k_n) z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} \tag{3.2}$$

where $h(k_1,k_2,\ldots,k_2)$ is the unit sample response of the transfer function in equation (3.1).

In Chapter 2, it has been shown that a two-dimensional system is BIBO stable if the output sequence is bounded for any bounded input sequence. Therefore, for a multidimensional system, the input function $\{x(m_1,m_2,\ldots m_n)\}$ is bounded if:

$$|x(m_1,m_2,\ldots,m_n)| \leq M < \infty \quad \forall (m_1,m_2,\ldots,m_n) \tag{3.3}$$

The output sequence, $\{y(m_1,m_2,\ldots,m_n)\}$ must, for a stable system, also be bounded,

$$|y(m_1,m_2,\ldots,m_n)| \leq N < \infty \quad \forall (m_1,m_2,\ldots,m_n) \tag{3.4}$$

One may directly obtain the following theorem for a N-dimensional discrete system to be BIBO stable.

<u>Theorem 3.1 [75]:</u>

A N-dimensional system described by (3.1) is BIBO stable if and only if there exist a real $\lambda < \infty$ such that for all positive integers $(m_1,m_2,\ldots,m_n)$

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} |h(m_1,m_2,\ldots,m_n)| \leq \lambda < \infty \tag{3.5}$$

3.2 CONDITIONS FOR STABILITY

The first stability theorem for N-dimensional digital transfer functions is given by Justice and Shanks [48]:

Theorem 3.2:

The transfer function described by (3.1) is BIBO stable if and only if

$$B(z_1,z_2,\ldots,z_n) \neq 0 \quad \bigcap_{i=1}^{n} \{|z_i| \leqslant 1\} \qquad (3.6)$$

provided $A(z_1,z_2,\ldots,z_n)$ and $B(z_1,z_2,\ldots,z_n)$ are mutually prime.

However, due to the effect of numerator on the stability, the above condition (3.6) is sufficient but not necessary [29]. If we consider the nonessential singulaties of the second kind on the distinguished boundary the following theorem can be proven.

Theorem 3.3:

If $H(z_1,z_2,\ldots,z_n)$ represents a BIBO stable filter then $H(z_1,z_2,\ldots,z_n)$ has no nonessential singularities of the first kind in the unit polydisc and no nonessential singularities of the second kind in the unit polydisc except possibly on the distinguished boundary.

From the above theorems, we conclude that when A and B are mutually prime and no nonessential singularities of the second kind on the distinguished boundary of the unit polydisc exist, the condition (3.6) is necessary and sufficient for structural stability.

3.3 CRITERIA FOR STABILITY TESTS

In the preceding section, it has been shown that the necessary and sufficient condition for structural stability of N-dimensional digital transfer function is:

$$B(z_1, z_2, \ldots, z_n) \neq 0 \qquad \bigcap_{i=0}^{n} |z_i| \leqslant 1 \tag{3.7}$$

Recently, various criteria have been suggested to simplify the stability test by several investigators.

Equation (3.1) can be tested using any one of the following criteria;

a) Criterion of Anderson and Jury [52]:

$$B(z_1, 0, \ldots, 0) \neq 0 \qquad |z_1| \leqslant 1 \tag{3.8}$$

$$B(z_1, z_2, 0, \ldots, 0) \neq 0 \qquad \{|z_1| = 1\} \cap \{|z_2| \leqslant 1\} \tag{3.9}$$

$$\vdots$$

$$B(z_1, z_2, \ldots, z_{n-1}, 0) \neq 0 \qquad \left\{ \bigcap_{i=1}^{n-2} |z_i| = 1 \right\} \cap \{|z_{n-1}| \leqslant 1\} \tag{3.10}$$

$$B(z_1, z_2, \ldots\ldots, z_{n-1}, z_n) \neq 0 \qquad \left\{ \bigcap_{i=1}^{n-1} |z_i| = 1 \right\} \cap \{|z_n| \leqslant 1\} \tag{3.11}$$

This criterion is the generalization of Huang's condition [19].

b) Criterion of Strintzis [23]:

i) For some $b_1, \ldots, b_n$ such that $|b_r| = 1$, $r = 1, 2, \ldots, n$ and for all i, $i = 1, 2, \ldots, n$

$$B(z_1, z_2, \ldots, z_n) \neq 0 \quad \text{when} \quad z_r = b_r,\ r \neq i \quad \text{and} \quad |z_i| \leq 1$$

ii) $B(z_1, z_2, \ldots, z_n) \neq 0$ when $|z_1| = |z_2| = \ldots = |z_n| = 1$

(3.12)

For simplicity one can choose $b_r = 1$;

Example: Let $n=3$, stability condition:

i) $B(z_1, 1, 1) \neq 0 \qquad |z_1| \leq 1$

$B(1, z_2, 1) \neq 0 \qquad |z_2| \leq 1$

$B(1, 1, z_3) \neq 0 \qquad |z_3| \leq 1$

$$B(z_1, z_2, z_3) \neq 0 \qquad |z_1| = |z_2| = |z_3| = 1 \qquad (3.13)$$

c) Criterion of DeCarlo et al [49], [88]:

i) $B(z, z, \ldots, z) \neq 0 \qquad |z| \leq 1$

ii) $B(z_1, z_2, \ldots, z_n) \neq 0 \qquad |z_1| = |z_2| = \ldots = |z_n| = 1 \qquad (3.14)$

3.4 STABILITY TESTS

3.4.1 Nyquist-Like Test

DeCarlo, Murray, and Seaks [49],[88] generalized the theorems (2.6)-(2.9) which are given for two-dimensional digital systems to the multidimensional case.

Theorem 3.4 [49]:

The multidimensional filter in (3.1) is structurally stable if and only if:

i) $B(z_1, z_2, \ldots, z_n)$ has no zeroes on $\bigcap_{i=1}^{n} |z_i| = 1$

ii) the Nyquist plots for the one-dimensional function

$B(1, \ldots, 1, z_k, 0, \ldots, 0)$, $k = 1, 2, \ldots, n$

do not encircle zero.

Theorem 3.5 [49]:

Let B be as in (3.1). The filter is structurally stable if and only if:

i) $B(z_1, z_2, \ldots, z_n)$ has no zeroes on $\bigcap_{i=1}^{n} |z_i| = 1$

ii) the Nyquist plots for the one-dimensional function

$B(1, 1, \ldots, 1, z_k, 1, \ldots, 1)$, $k = 1, 2, \ldots, n$

do not encircle zero.

Theorem 3.6 [49]:

Let B be described as in (3.1). The filter is structurally stable if and only if

i) $B(z_1, z_2, \ldots, z_n) \neq 0$ for $\bigcap_{i=1}^{n} |z_i| = 1$

ii) the Nyquist plot for the one dimensional function

$$B(z_1, z_2, \ldots, z_n) \qquad z_1 = z_2 \ldots = z_n = z$$

does not encircle zero.

3.4.2 Table Form

The generalization of the table form from two-dimensional digital filters to the multidimensional case has been studied by Anderson, Bose and Jury [80] and Bose and Kamat [54]. In the former work, the use of the table form for the discrete case for N=4 was introduced. Bose and Kamat [54] suggested an algorithm for the computer implementation. The algorithm is based on the generation of a number of multidimensional polynomials, reduction of each of these into several single dimensional polynomials by a finite dimensional rational operations. A detailed discussion of the table form for multidimensional digital filters can be found in [95].

3.4.3 Impulse Response Test

Strintzis [93] generilazed the impulse response test from two-dimensional to multidimensional digital filters. It can be shown that the following condition is necessary and sufficient for BIBO stability of $H(z_1, z_2, \ldots, z_n)$.

$$\overline{\lim}\ |h(k_1, k_2, \ldots, k_n)|^{1/(k_1+k_2+\ldots+k_n)} \qquad (3.15)$$

for all but a finite number of values of $(k_1,k_2,\ldots,k_n)$ [93]. Where $h(k_1,k_2,\ldots,k_n)$ is the impulse response of the filter.

All the other theorems of Section 2.4.5 can be readily generalized from two-dimensional to multidimensional case.

3.4.4 Symetrix Matrix Form

Bose and Jury [55] applied the Schur-Cohn matrix to test the stability of 3-dimensional digital filters. We write the 3-dimensional polynomials as follows:

$$B(z_1,z_2,z_3) = \sum_{i=0}^{n} b_i(z_1,z_2)\, z_3^i \tag{3.16}$$

Using the same procedure as for the 2-dimensional case, we obtain an innerwise hermitian matrix as function of $|z_1|=|z_2|=1$. This matrix has to be positive innerwise for all $|z_1|=|z_2|= 1$.

Since it is innerwise Hermitian, we require that it be positive innerwise at one point on the bidisc, usually $z_1= z_2=1$ and the determinant of the matrix, which is real function of two-variables, z_1 and z_2, be positive for all $|z_1|=|z_2|= 1$. This can be carried out in terms of root distribution with respect to unit circle as shown by Bose and Jury [55].

3.5 CEPSTRAL STABILITY TEST FOR N-DIMENSIONAL RATIONAL POLYNOMIALS

All stability tests for multidimensional digital filters summarized in Section 3.4 are applicable for only causal N-D digital filters. Our aim in this section, is to introduce an alternative stability test for a general class of N-dimensional digital filters. The test is based on the existance of the cepstra for N-dimensional rational polynomials (N>2). Another objective of this work is to demonstrate the extension of the Ekstrom-Twogood [28] cepstral method of testing to the multidimensional case.

3.5.1 The Existence of Cepstra for N-Dimensional Rational Polynomials

Oppehneim et al [38] showed that one class of signals for which cepstra are defined are those whose z-transforms are rational polynomials and non-zero on the unit circle. Later, Dudgeon [39] extended this result to 2-D signals. He [39] has shown that any 2-D signals s(m,n) whose z-transform is a ratio of 2-D polynomials, that is,

$$S(z_1,z_2)=\frac{\sum_m\sum_n a(m,n)\, z_1^m z_2^n}{\sum_m\sum_n b(m,n)\, z_1^m z_2^n} \tag{3.17}$$

will have a well defined 2-D complex cepstrum provided that

$$S(e^{j\omega_1}, e^{j\omega_2}) \neq 0 \quad \text{for } -\pi \leq \omega_1, \omega_2 \leq \pi \tag{3.18}$$

and provided that the origin of the signal has been adjusted to ensure that the phase is continuous and periodic in both frequency variables.

In an exactly analogous manner, we can demonstrate that any N-D array having a rational z-transform will also have a well-defined N-D complex cepstrum provided, 1) the transfer function does not have any singularity or zero on the N-D unit polydisc; 2) we are careful to eliminate any linear phase components by an appropriate shift of the original array.

Consider a N-D signal whose z-transform is a ratio of N-D polynomials, that is,

$$S(z_1, z_2, \ldots, z_n) = \sum_{p_1} \sum_{p_2} \cdots \sum_{p_n} s(p_1, p_2, \ldots, p_n) z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}$$

$$= \frac{\sum_{m_1} \sum_{m_2} \cdots \sum_{m_n} a(m_1, m_2, \ldots, m_n) z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}}{\sum_{m_1} \sum_{m_2} \cdots \sum_{m_n} b(m_1, m_2, \ldots, m_n) z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}}$$

$$= \frac{A(z_1, z_2, \ldots, z_n)}{B(z_1, z_2, \ldots, z_n)} \tag{3.19}$$

Where sums on $m_1, m_2, \ldots, m_n$ have a finite number of terms. The filter function (3.19) has a Fourier transform which is an N-D polynomial in $\exp(j\omega_1), \ldots, \exp(j\omega_n)$. We shall now show that if a signal of finite extent, $s(m_1, m_2, \ldots, m_n)$, has a Fourier transform $S(\omega_1, \omega_2, \ldots, \omega_n)$, this will be the sum of a linear component plus a continuous, centrosymmetric (odd), and periodic component.

Let $z_1 = \exp(j\omega_1)$; $z_2 = \exp(j\omega_2); \ldots; z_{n-1} = \exp(j\omega_{n-1})$ and consider

$$S(\omega_1, \omega_2, \ldots, \omega_{n-1}, z_n) = \sum_{m_n} \left[\sum_{m_1} \sum_{m_2} \cdots \sum_{m_{n-1}} s(m_1, m_2, \ldots, m_{n-1}, m_n) \right.$$

$$\left. \cdot \exp\{-j(\omega_1 m_1 + \omega_2 m_2 + \ldots + \omega_{n-1} m_{n-1})\} \right] z^{m_n} \tag{3.20}$$

as a 1-D polynomial in z_n with parameters $\omega_1, \omega_2, \ldots, \omega_{n-1}$. Since $s(m_1, m_2, \ldots, m_{n-1}, m_n)$ is of finite extent $s(\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n)$ will have poles inside the unit circle only at $z_n = 0$. The phase function is defined as:

$$\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n) = \mathrm{Im}\left\{ \oint_{|z_n|=1} \frac{S'(\omega_1,\omega_2,\ldots,\omega_{n-1},z_n)}{S(\omega_1,\omega_2,\ldots,\omega_{n-1},z_n)}\, dz_n \right\} +$$

$$\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},0) \qquad (3.21)$$

where the prime denotes differentiation with respect to z_n. The contour of integration starts at $z_n=1$ and proceeds around the unit circle to $z_n = \exp(j\omega_n)$. It is necessary to define a constant $\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},0)$ to be the phase as a function of ω_n for $\omega_n = 0$.

By constructing the phase $\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n)$ in this manner, we are assured that $\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n)$ is continuous and odd, that is,

$$\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n) = \Phi(-\omega_1,-\omega_2,\ldots,-\omega_{n-1},-\omega_n) \qquad (3.22)$$

When $\omega_n = 2\pi$, the contour integration in (3.21) is a closed curve. Using Cauchy's residue theorem we can write

$$\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},2\pi) = 2\pi r_n - 2\pi N_n \qquad (3.23)$$

where r_n is the number of roots of $B(\omega_1,\omega_2,\ldots,\omega_{n-1},z_n)$ inside the unit circle and N_n is the number of poles at $z_n=0$. If we let $k_n = r_n - N_n$, then it is clear that:

$$\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n+2\pi) = \Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n)+ 2\pi k_n \qquad (3.24)$$

Similarly, we can write:

$$\Phi(\omega_1,\omega_2,\ldots,\omega_i+2\pi,\ldots,\omega_n)= \Phi(\omega_1,\omega_2,\ldots,\omega_i,\ldots,\omega_n)+ 2\pi k_i \qquad (3.25)$$

Now we will show that k_n is not a function of $\omega_1,\omega_2,\ldots,\omega_{n-1}$. Indeed, if we examine the roots of $S(\omega_1,\omega_2,\ldots,\omega_{n-1},z_n)$ as we continuously vary parameters $\omega_1,\omega_2,\ldots,\omega_{n-1}$ from zero to 2π, we discover that the roots move about in continuous manner. Therefore, for a root to move from inside to outside the unit circle (or vice versa), it must lie on the unit circle for some values of $\omega_1,\omega_2,\ldots,\omega_{n-1}$. This however violates our assumption, that $S(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n) \neq 0$. Hence the number of roots inside the unit circle (z_n=1) is not a function of $\omega_1,\omega_2,\ldots,\omega_{n-2},\omega_{n-1}$. similar argument can be made to show that k_i is not a function of $\omega_1,\omega_2,\ldots,\omega_{i-1},\omega_{i+1},\ldots,\omega_{n-1},\omega_n$ for $i = 1,2,3,\ldots,n-2, n-1$.

Then the given phase function, $\Phi(\omega_1,\omega_2,\ldots,\omega_{n-1},\omega_n)$ which is continuous and odd, can be written as the sum of linear and periodic phase components;

$$\Phi(\omega_1,\omega_2,\ldots,\omega_n)= \Phi_p(\omega_1,\omega_2,\ldots,\omega_n)+ \Phi_L(\omega_1,\omega_2,\ldots,\omega_n) \qquad (3.26)$$

where

$$\Phi_L(\omega_1,\omega_2,\ldots,\omega_n)= k_1\omega_1 + k_2\omega_2+ \ldots + k_n\omega_n \qquad (3.27)$$

and the term $\Phi_p(\omega_1,\omega_2,\ldots,\omega_n)$ is continuous, odd, and periodic. In order to eliminate the linear phase terms and leave only the periodic part, $\Phi_p(\omega_1,\omega_2,\ldots,\omega_n)$ we can define a new signal by shifting the origin, defining

$$s_p(m_1,m_2,\ldots,m_n) = s(m_1-k_1,m_2-k_2,\ldots,m_n-k_n) \qquad (3.28)$$

The signal $s_p(m_1,m_2,\ldots,m_n)$ will have a continuous, odd, and periodic phase function $\Phi_p(\omega_1,\omega_2,\ldots,\omega_n)$. We may then form the function

$$\hat{S}_p(\omega_1,\omega_2,\ldots,\omega_n) = \ell n\left[|S_p(\omega_1,\omega_2,\ldots,\omega_n)|\right] + j\Phi_p(\omega_1,\omega_2,\ldots,\omega_n) \qquad (3.29)$$

which has a real inverse Fourier transform denoted by $\hat{s}_p(m_1,m_2,\ldots m_n)$ and is called the cepstrum of $s_p(m_1,m_2,\ldots,m_n)$

<u>Remark 3.1:</u>

If the phase is computed using the complex logarithm or arctangent function, only the principle value of Φ_p will be obtained. The principle value is a number between π and $-\pi$, which can still exhibit discontinuities of 2π in some cases, despite the subtraction of the linear phase component. To ensure continuity, an algorithm is required, which examines the phase at each point in the transform array and removes the jumps of 2π which are present in the principle value. This removal of the discontinuities is achieved by adding an appropriate multiple of 2π and is called 'phase unwrapping'. Both elimination of the linear phase and N-D phase unwrapping technique will be discussed in Section 3.5.5 and Section 3.5.4, respectively.

3.5.2 Stability of Weakly-Causal N-Dimensional Recursive Filters

A causal N-D recursive filter function $1/B(z_1, z_2, \ldots, z_n)$ is defined as one in which

$$b(m_1, m_2, \ldots, m_n) = 0 \quad \bigcup_{i=1}^{n} m_i < 0 \qquad (3.30)$$

This is also a definition of a first quadrant function.

We may, however, define a weakly-causal filter over a sector of N-D space. An example of such a filter is given by the following recursive form

$$y(p_1, p_2, \ldots, p_n) = x(m_1, m_2, \ldots m_n) - \sum_{m_1=0}^{M_1} \sum_{m_2=0}^{M_2} \cdots \sum_{m_n=0}^{M_n} b(m_1, m_2, \ldots, m_n) y(p_1 - m_., p_2 - m_2, \ldots, p_n - m_n)$$

$$m_1 + m_2 + \ldots + m_n \neq 0$$

$$-\sum_{m_1=0}^{-M_1} \sum_{m_2=0}^{M_2} \cdots \sum_{m_n=0}^{M_n} b(m_1, m_2, \ldots, m_n) y(p_1 - m_2, p_2 - m_2, \ldots, p_n - m_n) \qquad (3.31)$$

We define the weakly-causal region, R_w to be of the form

$$R_w = R_c \bigcup R_s$$

where $R_c = \left\{ (m_1, m_2, \ldots, m_n) \ : \ M_i \geq m_i \geq 0, \quad \forall i = 1,2,\ldots n \right\}$

and $R_s = \left\{ (m_1, m_2, \ldots, m_n) \ : -M_i \leq m_1 < 0 \quad \text{and} \right.$

$$\left. M_i \geq m_i \geq 0 \quad \forall i = 2,3,\ldots,n \right\} \tag{3.32}$$

In general we can say that for the weakly-causal functions some of the subscripts of the N-D sequence have negative values and the remaining subscripts have positive values.

Using the defined notation we have

$$y(p_1,p_2,\ldots,p_n) = x(m_1,m_2,\ldots,m_n) - \sum\sum_{R'_w}\cdots\sum b(m_1,m_2,\ldots,m_n)\, y(p_1-m_1, p_2-m_2,\ldots, p_n-m_n) \tag{3.33}$$

where $R_w = R'_w \bigcup (0,0,\ldots 0)$ (3.34)

and the filter function

$$H(z_1,z_2,\ldots,z_n) = \frac{1}{B(z_1,z_2,\ldots,z_n)}$$

$$= \frac{1}{\sum\sum_{R_w}\cdots\sum b(m_1,m_2,\ldots,m_n)\, z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}} \tag{3.35}$$

Then the stability of the causal and the weakly-causal N-D recursive filter is defined by the location of the essential singularities of $H(z_1,z_2,\ldots,z_n)$ in the complex $z_1,z_2,\ldots,z_n$ space.

Theorem 3.7:

The causal recursive filter $H(z_1,z_2,\ldots,z_n) = 1/B(z_1,z_2,\ldots,z_n)$ is stable if and only if:

$$B(z_1,z_2,\ldots,z_n) \neq 0 \quad \text{for all } (z_i,\ i=1,2,\ldots,n)\in D_1 \tag{3.36}$$

$$\text{where } D_1 = \left\{ (z_i, i = 1,2,\ldots,n) : \bigcap_{i=1}^{n} |z_i| \leq 1 \right\}$$

The proof of theorem 3.7 is in [48] . This theorem can be applied to weakly-causal filters by transformations.

Theorem 3.7 is difficult to implement numerically and requires an infinite algorithm [45] . In [52], it was shown that this criterion can be replaced by a more flexible and considerably simpler stability test.

Theorem 3.8:

Let $B(z_1,z_2,\ldots,z_n)$ be a polynomial in n variables. Then condition (3.36) is equivalent to:

$$B(z_1,z_2,\ldots z_n) \neq 0 \left\{ \bigcap_{i=1}^{n-1} |z_i|=1 \right\} \bigcap \left\{ |z_n| \leq 1 \right\} \tag{3.37}$$

$$B(z_1,z_2,\ldots,z_{n-1},0)\neq 0 \left\{\bigcap_{i=1}^{n-2} |z_i| = 1\right\} \bigcap \left\{|z_{n-1}| \leq 1\right\} \tag{3.38}$$

$$B(z_1,z_2,\ldots,z_{n-2},o,o)\neq 0 \left\{\bigcap_{i=1}^{n-3} |z_i| = 1\right\} \bigcap \left\{|z_{n-2}| \leq 1\right\} \tag{3.39}$$

$$B(z_1,z_2,o,\ldots,o) \neq 0 \left\{|z_1| = 1\right\} \bigcap \left\{|z_2| \leq 1\right\} \tag{3.40}$$

$$B(z_1,o,o,\ldots,o) \neq 0 \quad \left\{|z_1| \leq 1\right\} \tag{3.41}$$

Theorem 3.8 will be used to develop an N-D stability criterion using the complex cepstrum.

Remark 3.2:

In the definition (3.31), given for the weakly causal systems, only one of the subscripts of the N-D sequence has negative values and others have positive values. The stability test presented in this paper is applicable to the more general class of non-causal N-D recursive filters, in which the number of negative subscripts may be up to N-1 for an N-D digital recursive filter.

Remark 3.3:

The unit sample response of a stable, weakly-causal filter is non zero not only over the first quadrant, R_c, but also over some other section of the N-D sphere, in our case, R_s. In order to implement the stability criterion (theorem 3.8), we must find the location of the zeroes of $B(z_1, z_2, \ldots, z_n)$. However, this is not possible with available practical testing procedures [23], [26],[54], [55], [62]. Therefore, one of the unique advantages of the given stability test is that the non-causal N-D recursive filters can be tested using the complex cepstrum properties.

Remark 3.4:

The above classification subdivides n-dimensional recursive filters into the following categories

(a) Causal filters comprising one member in which

$$h(m_1, m_2, \ldots m_n) = 0 \qquad \bigcup_{i=1}^{n} m_i < 0$$

(b) Non causal filters comprising one member in which the support of $h(m_1, m_2, \ldots m_n)$ is unrestricted.

(c) Weakly causal filters comprising $(2^n - 2)$ members in which the support of $h(m_1, m_2, \ldots m_n)$ is constrained to be positive for at least one element of m_i.

3.5.3 Cepstral Stability Test for Causual and Weakly-Causal N-Dimenensional Recursive Filters

In Section 3.5.1, it was shown that the essential singularities and zeroes of $H(z_1,z_2,\ldots,z_n)$ map into essential singularities of its cepstrum, $\hat{H}(z_1,z_2,\ldots,z_n)$. Our stability criterion is based on the simple property of the N-D cepstral transformation. From theorem 3.7, recall that if $H(z_1,z_2,\ldots,z_n)$ is stable, it can be written in power series for $m_1,m_2,\ldots,m_n \varepsilon R_c$.

Since the regions of analyticity of $H(z_1,z_2,\ldots,z_n)$ and $\hat{H}(z_1,z_2,\ldots,z_n)$ are identical, it must also follow that $\hat{H}(z_1,z_2,\ldots,z_n)$ can be similarly expanded as:

$$\hat{H}(z_1,z_2,\ldots,z_n) = \Sigma\Sigma\ldots\Sigma\hat{h}(m_1,m_2,\ldots,m_n)\, z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \quad (3.42)$$

Theorem 3.9 summarizes the previous results.

Theorem 3.9:

The causal N-D recursive filter $H(z_1,z_2,\ldots,z_n)$ is stable if and only if its cepstrum $\hat{h}(m_1,m_2,\ldots,m_n)$ has a support on the first quadrant, R_c.

Proof

'If' part. By the existence of N-D cepstra: because $\hat{H}(z_1,z_2,\ldots,z_n)$ is analytic on $\left\{|z_1|=|z_2|=\ldots=|z_{n-1}|=1, \text{ and } |z_n| \leq 1\right\}$, $\hat{h}(m_1,m_2,\ldots,m_n)$ takes support on the entire

$\{m_1, m_2, \ldots, m_{n-1}\}$ subscripts. The additional region of analyticity for $\hat{H}(z_1, z_2, \ldots, z_{n-1}, 0)$ on $\left\{ |z_1| = |z_2| = \ldots = |z_{n-2}| = 1, \right.$ and $\left. |z_{n-1}| \leq 1 \right\}$ ensures that

$$\hat{h}(m_1, m_2, \ldots, m_n) = 0 \quad \text{for } m_n < 0$$

And by continuing this argument, for the n sets of variables we obtain

$$\hat{h}(m_1, m_2, \ldots, m_n) = 0 \quad \text{for } \bigcap_{i=0}^{n} m_i < 0$$

This shows that the cepstrum of a causal N-D stable recursive filter function has only support in the first quadrant,

$$R_c = \left\{ \bigcap_{i=1}^{n} m_i \geqslant 0 \right\}$$

'Only if" part. By contradiction, suppose $H(z_1, z_2, \ldots, z_n)$ is unstable. In Section 3.5.1, it was proved that the region of analyticity is identical for both the cepstrum and the original N-D signal. However, from theorem 3.8, it can be seen that at least, one of the stability conditions must be violated. For example, if the first condition is not satisfied; $H(z_1, z_2, \ldots, z_n) = 0$ on $\left\{ |z_1| = |z_2| = \ldots = |z_{n-1}| = 1 \text{ and } |z_n| \leqslant 1 \right\}$ ensures that

$$\hat{h}(m_1, m_2, \ldots, m_n) \neq 0 \quad \text{for } m_1 < 0 \tag{3.43}$$

This contradicts the assumption that the cepstrum has support only on the first quadrant. Hence, the causal N-D recursive filter function is stable if and only if its cepstrum has support on the first quadrant.

Theorem 3.10:

The weakly-causal N-D recursive filter $H(z_1, z_2, \ldots, z_n)$ is stable if and only if its cepstrum $\hat{h}(m_1, m_2, \ldots, m_n)$ has support on R_w where

R_w is the support of $h(m_1, m_2, \ldots .. m_n)$.

Proof

The proof of this theorem is similar to theorem 3.9. 'If' part can be proved by the existence of the N-D cepstra and theorem 3.8. 'Only if' part is proved by contradiction.

3.5.4 Phase Unwrapping Theory for N-Dimensional Recursive Filters:

The computation of the complex cepstrum is complicated by the fact that the complex logarithm is multivalued. Indeed, the complex logarithm can be expressed as:

$$\hat{S}(\omega_1, \omega_2, \ldots, \omega_n) = \ln[|S(\omega_1, \omega_2, \ldots, \omega_n)|] + j\Phi(\omega_1, \omega_2, \ldots, \omega_n) \tag{3.44}$$

where

$$S(\omega_1,\omega_2,\ldots,\omega_n) = \sum_{m_1=0}^{M_1-1} \sum_{m_2=0}^{M_2-1} \cdots \sum_{m_n=0}^{M_n-1} s(m_1,m_2,\ldots,m_n).$$

$$\exp\left[-j\left(\frac{2\pi\omega_1 m_1}{M_1}\right)\right]\exp\left[-j\left(\frac{2\pi\omega_2 m_2}{M_2}\right)\right]\ldots\exp\left[-j\left(\frac{2\pi\omega_n m_n}{M_n}\right)\right]$$

$$0 \leqslant \omega_1,\omega_2,\ldots,\omega_n \leqslant N \tag{3.45}$$

The standard computer complex logarithm function uses the principle value for the phase, and consequently 'jumps' of 2π in the value of the phase may be seen.

Various phase unwrapping algorithms have been discussed by several investigators [28],[69],[70],[138]. Tribolet [70] proposed a phase unwrapping algorithm which combines the information contained in both phase derivative and the principle value of the phase. Later, Dudgeon [73] used Tribolet's method for computation of the 2-D complex cepstrum. And recently, Bhanu and McClellan [69] have suggested a new phase unwrapping technique which is based on fitting splines to the phase derivative curve.

However, we have found that the phase unwrapping technique used by Ekstrom and Twogood [28] is the most successful one for the N-D application. This algorithm proceeds as follows; first $\mathrm{Arg}[B(\omega_1,0,\ldots,0)]$, $\mathrm{Arg}[B(0,\omega_2,0\ldots,0)]$.., and $\mathrm{Arg}[B(0,0,\ldots,\omega_n)]$ are calculated by simply taking $\omega_1,\omega_2,\ldots,\omega_n$, respectively, and adding or subtracting 2π whenever a phase discontinuity is encountered. Following these 1-D unwrapping on the axis, 2-D unwrapping is required for checking the discontinuities between the present phase value and previously unwrapped neighbouring phase values. This algorithm will continue until the phase of the N-D signal is unwrapped.

3.5.5 Linear Phase Removel of N-Dimensional Signal

After the phase unwrapping of the N-D signal, the removal of the linear phase component is the final step of computation of the complex cepstrum. In Section 3.5.1, it was shown that we can write the phase function as

$$\Phi(\omega_1,\omega_2,\ldots,\omega_n) = \Phi_p(\omega_1,\omega_2,\ldots,\omega_n) + k_1\omega_1+k_2\omega_2+\ldots+k_n\omega_n \qquad (3.46)$$

To eliminate the linear-phase terms leaving only the periodic part $\Phi_p(\omega_1,\omega_2,\ldots,\omega_n)$ we can define a new signal by shifting the origin of the original signal, to generate

$$s_p(m_1,m_2,\ldots,m_n) = s(m_1-k_1,m_2-k_2,\ldots,m_n-k_n) \qquad (3.47)$$

The signal $s_p(m_1,m_2,\ldots,m_n)$ will have continuous, odd, and periodic phase.

It is interesting to note that the determination of the coefficients of the linear-phase component, $(k_1,k_2,\ldots,k_n)$

is N separate 1-D problems.

The linear portions of $Arg[B(0,\ldots,\omega_i,0,\ldots,0)]$ of the unwrapped phase can be obtained from the original phase curve as:

$$\Phi(0,..0,\omega_i+2\pi,0...0) = \Phi(0,..0,\omega_i,0...0) + 2\pi k_i \tag{3.48}$$

for all $i = 1,2,\ldots n$

It can be shown that this problem is identical to determining the coefficient of the linear-phase component,

$$x_i(m_i) = \sum_{m_i} \cdots \sum_{m_i-1} \sum_{m_i+1} \cdots \sum_{m_n} s(m_1,m_2,\ldots m_n) \tag{3.49}$$

Hence, the parameter k_i can be determined by observing the degree of the phase linearity of the 1-D signal in equation (3.49).

<u>Remark 3.5:</u>

From the above analysis, we conclude that any methods which are developed to facilitate the determination of the coefficients of the linear-phase component of 1-D signal can be used for an N-D signal.

3.5.6 IMPLEMENTATION OF THE CEPSTRAL TEST

A block diagram for the numerical implementation of the stability test is shown in Fig.3.1. First, we obtain the discrete Fourier transform (DFT) of the given array in the spatial time domain. Then, by taking the complex logarithm of $B_a(\omega_1,\omega_2,\ldots,\omega_n)$, we get

$$\hat{B}_a(\omega_1,\omega_2,\ldots,\omega_n) = \ln\left[B_a(\omega_1,\omega_2,\ldots,\omega_n)\right] \tag{3.50}$$

where $\hat{B}_a(\omega_1,\omega_2,\ldots,\omega_n)$ is the aliased version of $\hat{B}(\omega_1,\omega_2,\ldots,\omega_n)$. The cepstrum of $b(m_1,m_2,\ldots,m_n)$ is dentoted by $\hat{B}(z_1,z_2,\ldots,z_n)$.

The third step is used to obtain $\hat{b}_a(m_1,m_2,\ldots,m_n)$. The error in this approximation is given by:

$$\hat{b}_a(m_1,m_2,\ldots,m_n)-\hat{b}(m_1,m_2,\ldots,m_n)= \sum_{R_\infty}\sum\ldots\sum b(m_1+i_1Q_1,m_2+i_2Q_2,\ldots,m_n+i_nQ_n) \tag{3.51}$$

for the DFT size $Q_1 \times Q_2 \times \ldots \times Q_n$.

where $R_\infty = \bigcap_{i=1}^{n} (-\infty < m_i < \infty)$.

Clearly, the approximation

$$\hat{b}_a(m_1,m_2,\ldots,m_n) \approx \hat{b}(m_1,m_2,\ldots,m_n) \tag{3.52}$$

is a good one only if $b(m_1,m_2,\ldots m_n)$ decays rapidly. A good approximation can be obtained by using moderate sized FFT's.

Finally, the support of the cepstrum is determined and checked that it coincides with the support of $b(m_1,m_2,\ldots m_n)$.

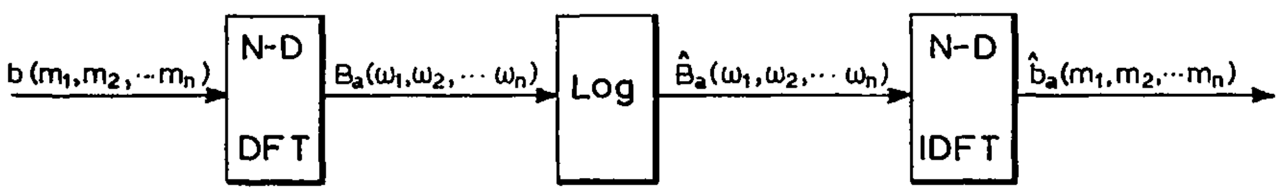


Fig. 3·1 Block diagram of the stability test using N-D DFT

CHAPTER 4

STABILIZATION TECHNIQUES

In Chapters II and III, several methods for testing the stability of 2-D and N-D digital filters were discussed. Since many design procedures available for designing 2-D recursive filters [32], [33], [40] fail to produce stable ones, a stabilization technique is used to process the unstable transfer function in order to obtain a new and stable function which has approximately the same amplitude response.

Three stabilization techniques are now reviewed. Two of them rely on varying the phase of the filter without affecting its amplitude. The third method is applicable to zero-phase functions.

One of these methods is then modified to include a more useful class of 2-D recursive digital filters which has great practical importance in the area of image enhancement. Next, a new spectral factorization technique is suggested as an alternative stabilization procedure. After that, an algorithm is developed for the stabilization of digital filters by the Pistor method [26].

4.1 PLANAR LEAST SQUARE INVERSE TECHNIQUE (PLSI):

It is known in the literature that the least square inverse of a 1-D polynomial which represents the denominator of a discrete function is always stable. Because of this, synthesis of one dimensional digital filter can be designed with guaranteed stability. Shanks et al [7] proposed an extension of this technique to the two-dimensional case. Before reviewing the method in detail, some useful definitions and preliminary theorems to its understanding are given.

<u>Definition 4.1:</u> A sequence, $\{b(n)\}$, is a minimum phase sequence when its z-transform $B(z)$ has no zeroes inside the unit circle in the z-plane.

If the roots of $B(z)$ are available, we can simply replace a pole inside the unit circle with polar coordinates (r,θ) by a pole outside the unit circle with polar coordinates $(1/r,\theta)$ [36]. The amplitude response of the filter is left unchanged by this procedure, since a root is replaced by its mirror image. However, if the roots are not available, we can use the following theorem.

<u>Theorem 4.1:</u> Given a real finite polynomial $B(z)$, any least squares inverse of $B(z)$ is the z-transform of a minimum phase sequence.

The polynomial $P(z)$ is the least square inverse of $B(z)$ if the convolution of the corresponding sequence $\{p(n)\}$ and $\{b(n)\}$ approximate the unit impulse sequence with the least square error criterion. Thus, an unstable filter with denominator $B(z)$ can be approximated by a stable filter with the same numerator, and denominator $P(z)$.

The least square inverse stabilisation technique for 2-D functions is based on the following definition and the conjecture nowadays known as "Shanks' Conjecture" which extends the 1-D procedure to the 2-D case.

Definition 4.2: A 2-D sequence $\{b(m,n)\}$ is a mininum phase sequence when its z-transform $B(z_1,z_2)$ has no zeroes in the unit polydisc.

4.1.1 Shanks' Conjecture [73]:

Given a 2-D real finite polynomial $B(z_1,z_2)$, any least square inverse of $B(z_1,z_2)$ is the z-transform of a minimum phase sequence.

This is a very important conjecture because it implies that the filter $F(z_1,z_2) = 1/P(z_1,z_2)$ must be stable if $P(z_1,z_2)$ is PLSI. However, Genin and Kamp [16] found the following counter example for disproving Shanks' conjecture.

Counterexample 4.1 [16]:

Let

$$P(z_1,z_2) = \sum_{m=0}^{M} \sum_{n=0}^{N} p(m,n) z_1^m z_2^n \qquad (4.1)$$

by the PLSI of a given polynomial

$$B(z_1,z_2) = \sum_{k=0}^{K} \sum_{l=0}^{L} b(k,l) z_1^k z_2^l \qquad (4.2)$$

in the sense that the coefficients p(m,n) minimize the quadratic error norm

$$Q = (1-p(o,o)b(o,o))^2 + \sum_{i=0}^{I} \sum_{j=0}^{J} g^2(i,j) \qquad (4.3)$$

where $\{g(i,j)\} = \{b(k,l)\} * \{p(m,n)\}$. $*$ denotes the convolution.

The polynomial $B(z_1,z_2)$ of the degree K=L=3 and with the coefficients

$b(0,0) = b(3,3) = 1$

$b(0,1) = b(1,0) = b(2,3) = b(3,2) = -1.15$

$b(0,2) = b(2,0) = b(1,3) = b(3,1) = -0.902$

$$b(0,3) = b(3,0) = 1.75$$

$$b(1,1) = b(2,2) = 3.72$$

$$b(1,2) = b(2,1) = -2.23 \tag{4.4}$$

admits the following PLSI of the degree M=N=1:

$$P(z_1,z_2) = (4.13 + 2.12z_1 + 2.12z_2 - 0.606z_1z_2)10^{-2} \tag{4.5}$$

It can readily be seen that the first condition of theorem 2.3 in Chapter II is satisfied, but the second one is not, because for $z_1 = -1$, the PLSI has a zero at $z_2 = -0.736$, i.e. inside the unit circle.

4.1.2 Jury's Conjecture:

The counterexample of Genin and Kamp relates to an inverse polynomial of lower degree than the original polynomial. They did not present a counterexample for the same degree. However, all the examples of Shanks' are related to the same degree inverse as the original polynomial Jury [112] therefore, introduced a new conjecture with additional constraints.

Jury's Conjecture [112]: If the original 2-dimensional polynomial and inverse are of the same degree, then the reciprocal of the PLSI is a stable filter.

This conjecture [112] has been verified for low degree poynomials.

Jury-Anderson Verification [113] : In this work, the verification of Jury's conjecture for special low order polynomials was presented. The key to the verification lies in utilizing the centro-symmetric properties of a particular Toeplitz matrix, which arises in the equations of the approximate inverse. The prescribed poynomial is:

$$A(z_1,z_2) = a(0,0) + a(1,0)z_1 + a(0,1)z_2 + a(1,1)z_1z_2 \tag{4.6}$$

and its inverse

$$B(z_1,z_2) = b(0,0) + b(1,0)z_1 + b(0,1)z_2 + b(1,1)z_1z_2 \tag{4.7}$$

$$A(z_1,z_2) \doteq \frac{1}{B(z_1,z_2)} \tag{4.8}$$

Using the 2-D convolution;

$$\begin{aligned} C(z_1,z_2) &= A(z_1,z_2)\ B(z_1,z_2) \\ &= c(0,0) + c(0,1)z_1 + c(0,1)z_2 + c(1,1)z_1z_2 \\ &\quad + c(2,0)z_1^2 + c(0,2)z_2^2 + c(2,1)z_1^2z_2 \\ &\quad + c(1,2)z_1z_2^2 + c(2,2)z_1^2z_2^2 \end{aligned} \tag{4.9}$$

c(i,j)'s are computed from the given polynomial a(i,j)'s and b(i,j)'s to obtain the approximate inverse form,

$$Q = (1-c(0,0))^2 + \sum_{i,j} c^2(i,j)$$

$$= (1-a(0,0)b(0,0))^2 + \sum_{i,j} c^2(i,j) \qquad (4.10)$$

and we seek to minimize the quantity Q with respect to b(i,j)'s. This can be expressed in the matrix form

$$\begin{bmatrix} \Gamma_{00} & \Gamma_{10} & \Gamma_{01} & \Gamma_{11} \\ \Gamma_{10} & \Gamma_{00} & \Gamma'_{11} & \Gamma_{01} \\ \Gamma_{01} & \Gamma'_{11} & \Gamma_{00} & \Gamma_{10} \\ \Gamma_{11} & \Gamma_{01} & \Gamma_{10} & \Gamma_{00} \end{bmatrix} \begin{bmatrix} b(0,0) \\ b(1,0) \\ b(0,1) \\ b(1,1) \end{bmatrix} = \begin{bmatrix} a(0,0) \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.11)$$

where

$$\begin{aligned} \Gamma_{00} &= a(0,0) + a(1,0) + a(0,1) + a(1,1) \\ \Gamma_{11} &= a(1,1)\ a(0,0) \\ \Gamma'_{11} &= a(1,1)\ a(1,0) \\ \Gamma_{01} &= a(0,1)\ a(0,0) + a(1,1)\ a(1,0) \\ \Gamma_{10} &= a(1,0)\ a(0,0) + a(1,1)\ a(0,1) \end{aligned} \qquad (4.12)$$

or more compactly as:

$$\underset{\sim}{\Gamma}\, \underset{\sim}{b} = \underset{\sim}{a} \tag{4.13}$$

'$\underset{\sim}{\Gamma}$' is the block Toeplitz matrix. This kind of matrices appears often in 2-D problems. Utilizing the centrosymmetry of the above equation yields,

$$\begin{bmatrix} \Gamma_{00}+\Gamma_{11} & \Gamma_{01}+\Gamma_{10} & 0 & 0 \\ \Gamma_{10}+\Gamma_{01} & \Gamma_{00}+\Gamma_{11}' & 0 & 0 \\ 0 & 0 & \Gamma_{00}-\Gamma_{11} & \Gamma_{01}-\Gamma_{10} \\ 0 & 0 & \Gamma_{10}-\Gamma_{01} & \Gamma_{00}-\Gamma_{11}' \end{bmatrix} \begin{bmatrix} b(0,0)-b(1,1) \\ b(1,0)+b(0,1) \\ -b(0,0)+b(1,1) \\ -b(1,0)+b(1,1) \end{bmatrix} = \begin{bmatrix} a(0,0) \\ 0 \\ -a(0,0) \\ 0 \end{bmatrix} \tag{4.14}$$

Denoting the top left 2x2 submatrix by A, and lower right submatrix by C. If $\Delta_1 = \det[A]$, $\Delta_2 = \det[C]$, and A_{22}, A_{21}, C_{22}, C_{21} are the corresponding cofactors, then it can be shown that:

$$|A_{21}| < |A_{22}| \quad ; \quad |C_{21}| < |C_{22}| \tag{4.15}$$

$$\Delta_1 > 0 \qquad \Delta_2 > 0 \tag{4.16}$$

Conditions in (4.16) imply that Toeplitz matrix is positive definite. In order to test the stability of $B(z_1,z_2)$, it is sufficient to check the following conditions (Huang Theorem [19]).

$$\left|\frac{b(1,0)}{b(0,0)}\right| < 1 \tag{4.17}$$

$$\left|1 \pm \frac{b(1,0)}{b(0,0)}\right| > \left|\frac{b(1,1) \pm b(0,1)}{b(0,0)}\right| \tag{4.18}$$

By solving the matrix equation (4.14), we get:

$$b(0,0) = \tfrac{1}{2}\, a(0,0) \left[\frac{A_{22}}{\Delta_1} + \frac{C_{22}}{\Delta_2} \right]$$

$$b(1,1) = \tfrac{1}{2}\, a(0,0) \left[\frac{A_{22}}{\Delta_1} - \frac{C_{22}}{\Delta_2} \right]$$

$$b(0,1) = \tfrac{1}{2}\, a(0,0) \left[-\frac{A_{22}}{\Delta_1} + \frac{C_{22}}{\Delta_2} \right]$$

$$b(1,0) = \tfrac{1}{2}\, a(0,0) \left[-\frac{A_{21}}{\Delta_1} - \frac{C_{21}}{\Delta_2} \right] \tag{4.19}$$

Using the above identities one can show that the PLSI is indeed stable. Let us check the first condition which requires:

$$\left|\frac{b(1,0)}{b(0,0)}\right| < 1 \tag{4.20}$$

$$\left|\frac{b(1,0)}{b(0,0)}\right| = \left|\frac{C_{21}\Delta_1 + A_{21}\Delta_2}{C_{22}\Delta_1 + A_{22}\Delta_2}\right| < 1 \tag{4.21}$$

Similarly the second condition (4.18) can be satisfied.

From equations (4.6) and (4.7), it is evident that the polynomial and its inverse are of the same degree.

Jury-Anderson verification has been extended for the following polynomials and their inverses;

a) Jury, Kolavannu and Anderson Extension [116]: In this work, the conjecture has been verified for the following polynomial and its inverse,

$$A_k(z_1,z_2) = a(0,0) + a(0,1)z_2 + a(k,0)z_1^k + a(k,1)z_2z_1^k \tag{4.22}$$

$$B_k(z_1,z_2) = b(0,0) + b(0,1)z_2 + b(k,0)z_1^k + b(k,1)z_2z_1^k \tag{4.23}$$

b) Delsarte, Genin and Kamp Extension [114] : The conjecture has been verified for the following polynomial and its inverse,

$$A_{sk}(z_1,z_2) = a(0,0) + a(0,s)z_2^s + a(k,0)z_1^k + a(k,s)z_1^k z_2^s \tag{4.24}$$

$$B_{sk}(z_1,z_2) = b(0,0) + b(0,s)z_2^s + b(k,0)z_1^k + b(k,s)z_1^k z_2^s \tag{4.25}$$

where s,k are integers larger than one.

c) Jury and Choppora Extension [115] : The conjecture has been verified for the following polynomial and its inverse,

$$A_{m+n,s+t}(z_1,z_2) = a(0,0) + a(m,t)z_1^m z_2^t + a(n,s)z_1^n z_2^s + a(m+n,s+t)z_1^{m+n} z_2^{s+t} \tag{4.26}$$

$$B_{m+n,s+t}(z_1,z_2) = b(0,0) + b(m,t)z_1^m z_2^t + b(n,s)z_1^n z_2^s + b(m+n,s+t)z_1^{m+n} z_2^{s+t} \tag{4.27}$$

with $(mt - ns) \neq 0$.

Despite the fact that Jury's conjecture has been verified for low degree polynomials, it has not been proved in general. The difficulty lies on the condition of the positivity of polynomials with literal coefficients which can be obtained up to fourth degree [22]. Further work on this problem was published by Bednar [120]. In this work, Bednar commented on the mathematical difficulties in verifying the conjecture (If it is possible) in general. In a later survey by Merserau and Dudgeon [131] further numerical verification of the conjecture is mentioned.

Recent investigations [104] , however, led to the following simple counterexample to disprove the Jury's conjecture.

Counterexample 4.2:

Let,

$$C(z_1,z_2) = \sum_{m=0}^{3} \sum_{n=0}^{3} c(m,n) z_1^m z_2^n \tag{4.28}$$

$$\text{where } \left\{c(m,n)\right\} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 4 & 5 & 2 \\ 2 & 5 & 4 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \tag{4.29}$$

The planar-least squares inverse of the same degree of $C(z_1,z_2)$

$$P(z_1,z_2) = \sum_{k=0}^{3} \sum_{\ell=0}^{3} p(k,\ell) z_1^k z_2^\ell \tag{4.30}$$

$$P(k,\ell) = \begin{bmatrix} 1.0 & -0.66150676 & 0.43829224 & -0.24508737 \\ -0.66150676 & 0.11920286 & -0.11038349 & 0.24198055 \\ 0.43829224 & -0.11038349 & -0.02373516 & -0.12054714 \\ -0.24508737 & 0.24198055 & -0.12054714 & 0.08504455 \end{bmatrix} \tag{4.31}$$

The above PLSI form was obtained in the sense that $p(k,\ell)$ minimizes the quadratic error norm,

$$Q = \left(1-g(0,0)\right)^2 + \sum_{i,j} g^2(i,j)$$

$$= \left(1-c(0,0)p(0,0)\right)^2 + \sum_{i,j} g^2(i,j) \tag{4.32}$$

Where

$$g(i,j) = \sum_{m=0}^{3} \sum_{n=0}^{3} c(m,n)p(i-m,j-n) \tag{4.33}$$

According to Huang's theorem [19], the obtained PLSI of the same degree polynomial, $P(z_1,z_2)$ is unstable. This can readily be seen from the failure of the second condition of Huang's theorem. Fig.4.1 shows the mapping of z_1- unit circle onto z_2 plane.

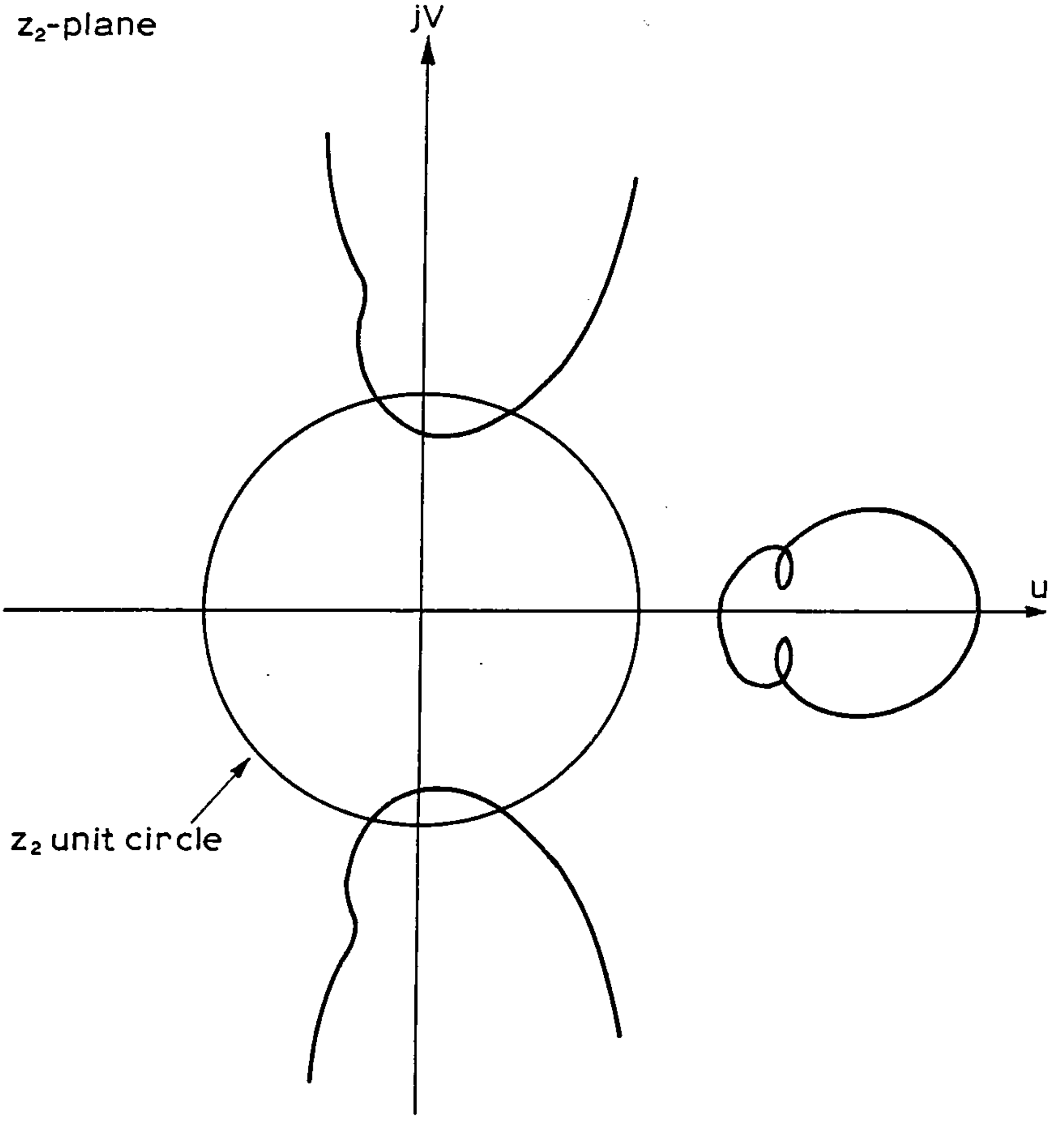


Fig. 4·1 Mapping of z_1 unit circle onto z_2 plane

Comments on Jury's Conjecture: The above counterexample shows that Jury's conjecture is not valid in general. However, it is still desirable to exhaust all possible cases where the conjecture is valid. Generally, this problem is similar to the well-known "Aizerman's Conjecture" [132] in control literature.

4.2 DISCRETE HILBERT TRANSFORMATION TECHNIQUE

For 1-D causal sequences the stabilization technique via discrete Hilbert transform is based on the definition 4.1 and the following theorem [71].

Theorem 4.2: A sequence $\{b(n)\}$ is a minimum phase sequence if and only if the logarithm of its amplitude spectrum $\log |B(e^{jw})|$ and its phase spectrum $\phi(w)$ are related by the Hilbert transform:

$$\phi(w) = \frac{1}{2\pi} \int_0^{2\pi} \log |B(e^{jw})| \cot(\frac{u-w}{2}) du \qquad (4.34)$$

The theorem suggests a stabilization procedure for an unstable filter with the denominator $B(z)$ corresponding to a non-minimum phase sequence $\{b(n)\}$ as follows:

Step 1. Calculate the amplitude spectrum $|B(e^{jw})|$

Step 2. Replace the original phase spectrum by the phase spectrum evaluated by (4.34).

Step 3. From $|B(e^{jw})|$ and the new phase spectrum evaluate the corresponding sequence $\{b(n)\}$ which is minimum phase according to theorem 4.2.

To implement the above procedure on a digital computer, an approximate discrete version of equation (4.35) can be used.

$$\phi(k) = \frac{1}{N} \sum_{i=0}^{N} \log|H(i)| \left[1-(-1)^{k-i}\right] \cot \frac{\pi}{N}(k-i) \tag{4.35}$$

The expression (4.35) can be further simplified to:

$$\phi(k) = \frac{2}{N} \sum_{i=0,2,4,.}^{N-1} \log|B(i)| \cot \frac{\pi}{N}(k-i), \quad k \text{ odd} \tag{4.36}$$

$$\phi(k) = \frac{2}{N} \sum_{i=1,3,5,.}^{N-1} \log|B(i)| \cot \frac{\pi}{N}(k-i), \quad k \text{ even} \tag{4.37}$$

Similarly, DFT and IDFT relations are used to evaluate the amplitude spectrum at discrete values of w and corresponding sequences. The relation (4.35) is equivalent to:

$$\phi(k) = -j\,\mathrm{DFT}\left\{\mathrm{sgn}(k)\,\mathrm{IDFT}\left[\log|B(k)|\right]\right\} \tag{4.38}$$

The relation (4.38) is called the discrete Hilbert transform (DHT).

It has been shown that the approximation involved in using the DHT form, rather than the exact one, namely relation (4.34), is the same as that which results when the continuous integral is replaced by trapezoidal rule of numerical integration [121].

2-D Discrete Hilbert Transform Technique [121]:

Read and Treitel [121] extended 1-D DHT techniques to two-dimensions. Before applying the procedure for the 2-D case some new functions are defined.

A finite discrete impulse response is causal if;

$$b(n_1,n_2) = 0 \qquad \text{for} \quad n_1 > \frac{N_1}{2}, \quad n_2 > \frac{N_2}{2} \tag{4.39}$$

Where n_1 varies on the discrete set $\{0,1,2,...,N_1-1\}$ and n_2 varies over the set $\{0,1,2,...,N_2-1\}$. The even and odd parts of a such sequence are defined as:

$$b_e(n_1,n_2) = \frac{1}{2}\left[b(n_1,n_2) + b(N_1-n_1,N_2-n_2)\right] \tag{4.40}$$

$$b_o(n_1,n_2) = \frac{1}{2}\left[b(n_1,n_2) - b(N_1-n_1,N_2-n_2)\right] \tag{4.41}$$

respectively. The odd and even parts of a causal periodic sequence is related by:

$$b_o(n_1,n_2) = \left[\mathrm{sgn}(n_1,n_2) + \mathrm{bdy}(n_1,n_2)\right] b_e(n_1,n_2) \tag{4.42}$$

Where the sgn function is a finite (2-D) version of the (1-D) signum function and it is defined by:

$$\mathrm{sgn}(n_1,n_2) = \begin{cases} 1 & 0 < n_1 < N_1/2 \text{ and } \quad 0 < n_2 < N_2/2 \\ 1 & N_1/2 < n_1 < N_1 \text{ and } \quad N_2/2 < n_2 < N_2 \\ 0 & \text{elsewhere} \end{cases} \tag{4.43}$$

The bdy function makes boundary adjustments and is defined by:

$$\mathrm{bdy}(n_1,n_2) = \begin{cases} 1 & n_2=0 \text{ and } \quad 0 < n_1 < N_1/2 \\ -1 & n_2=0 \text{ and } \quad N_1/2 < n_1 < N_1 \\ 1 & n_1=0 \text{ and } \quad 0 < n_2 < N_2/2 \\ -1 & n_1=0 \text{ and } \quad N_2/2 < n_2 < N_2 \\ 0 & \text{elsewhere} \end{cases} \tag{4.44}$$

The causal periodic sequence $b(n_1,n_2)$ is the sum of its odd and even parts:

$$b(n_1,n_2) = \left[b_e(n_1,n_2)\right] + \left[b_o(n_1,n_2)\right] \tag{4.45}$$

Taking the DFT of both sides yields:

$$\mathrm{DFT}\left[b(n_1,n_2)\right] = \mathrm{DFT}\left[b_e(n_1,n_2)\right] + \mathrm{DFT}\left[b_o(n_1,n_2)\right] \tag{4.46}$$

using the well-known properties of the DFT for even and odd functions results in:

$$B_r(n_1,n_2) = \text{DFT}\left[b_e(n_1,n_2)\right] \tag{4.47}$$

$$B_i(n_1,n_2) = \text{DFT}\left[b_o(n_1,n_2)\right] \tag{4.48}$$

Where

$$B(n_1,n_2) = B_r(n_1,n_2) + jB_i(n_1,n_2) = \text{DFT}\left[b(n_1,n_2)\right] \tag{4.49}$$

Taking the IDFT of both sides of (4.47) and substituting into (4.42) and using equation (4.48) results in;

$$B_i(n_1,n_2) = -j\text{DFT}\left\{\left[\text{sgn}(n_1,n_2) + \text{bdy}(n_1,n_2)\right]\bullet\text{IDFT}\left[B_r(n_1,n_2)\right]\right\} \tag{4.50}$$

This last relation defines the (2-D) discrete Hilbert transform. It clearly corresponds to the continuous transform given in equation (4.34).

If the sequence $b(n_1,n_2)$ is a minimum phase sequence than (4.50) becomes:

$$\phi(n_1,n_2) = -j\left\{\left[\text{sgn}(n_1,n_2) + \text{bdy}(n_1,n_2)\right]\bullet\text{IDFT}\left[\log|B(n_1,n_2)|\right]\right\} \tag{4.51}$$

The expression (4.51) gives an approximate minimum phase for a (2-D) amplitude response $B(n_1,n_2)$. Forming a minimum-phase version of an array by using the equation (4.51) can be summarized by the following steps below.

Step (1): Given a finite discrete (2-D) array, the coefficient array should be augmented with zeroes to satisfy the condition for causality. The added zeroes increase the size of the array, so that it becomes amenable to Fast Fourier Transform analysis.

Step (2): The natural length of the amplitude spectrum of the augmented (2-D) array should be calculated.

Step (3): The (2-D) discrete Hilbert transform must be applied to this (2-D) array. Thus the log of the magnitude is treated as the real and the discrete Hilbert transform then yields the imaginary part.

Step (4): The imaginary part is used as the phase spectrum corresponding to the given amplitude spectrum. These two spectral characteristics completely describe the transform of the minimum-phase array.

Step (5): After conversion from amplitude and phase to real and imaginary parts, the inverse transform is determined and truncated to obtain the same dimensions as the original array. This yields the minimum-phase version of the original array.

Fig. 4.2 shows the block diagram of stabilizing a filter function by discrete Hilbert transform.

Although the discrete Hilbert transform procedure works for most examples, it has been shown in [122] that there exist some cases where it proves to be of no value.

<u>Counterexample 4.3</u>:

Woods [122] pointed out that it was not in general possible to achieve the stabilization and at the same time require the amplitudes to be same.

Consider the following causal, first-quadrant filter function

$$F(z_1, z_2) = \frac{1}{A(z_1, z_2)} = \frac{1}{\sum_{m=0}^{1} \sum_{n=0}^{1} a(m,n) z_1^m z_2^n} \tag{4.52}$$

where

$$a(0,0) = 1/4 \qquad a(1,0) = 1$$

$$a(0,1) = 0 \qquad a(1,1) = 1/4 \tag{4.53}$$

It can be shown that the filter function $F(z_1, z_2)$ is unstable. The denominator function, $A(z_1, z_2)$ becomes zero in the unit polydisc. Letting $A(u,v)$ be the DFT of $\{a(m,n)\}$, we obtain for the magnitude,

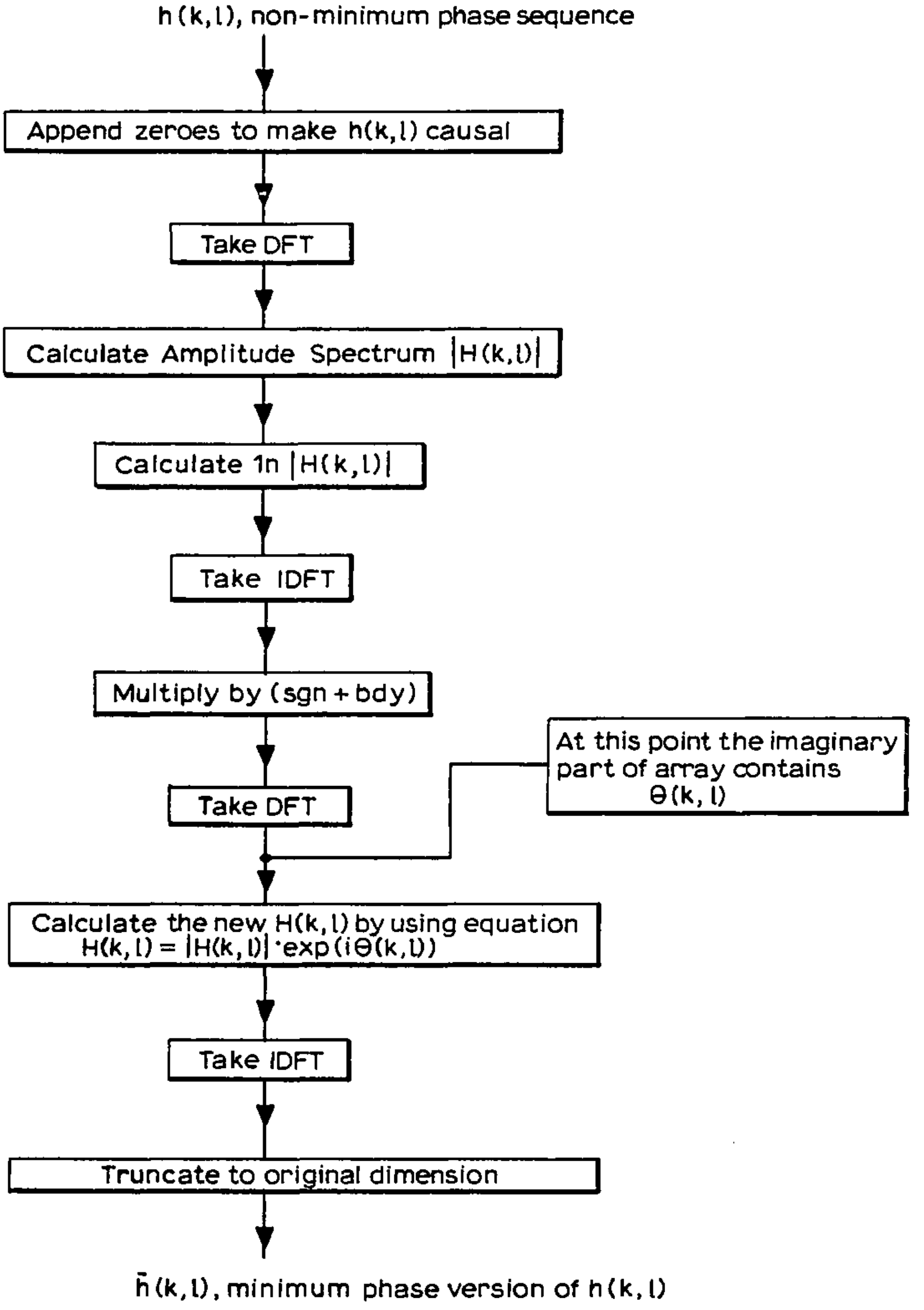


Fig. 4·2 Procedure for obtaining a minimum phase version of a (2-D) sequence.

$$|A|^2 = 9/8 + 1/2\ (\cos 2\pi u + \cos 2\pi v) + 1/8\ \cos 2\pi(u+v)$$

$$\text{on} \quad [-\tfrac{1}{2}, +\tfrac{1}{2}] \times [-\tfrac{1}{2}, +\tfrac{1}{2}] \tag{4.54}$$

Now, we assume that there exists a 2x2 array b(k,l) corresponding to a recursive filter stable in the first quadrant. Its magnitude is given by:

$$\begin{aligned} |B|^2 = \ & b^2(0,0) + b^2(0,1) + b^2(1,0) + b^2(1,1) \\ & + 2\{b(0,0)b(1,0) + b(0,1)b(1,1)\}\cos(2\pi u) \\ & + 2\{b(0,0)b(0,1) + b(1,0)b(1,1)\}\cos(2\pi v) \\ & + 2b(0,0)b(1,1)\cos 2\pi(u+v) + 2b(1,0)b(0,1)\cos 2\pi(u-v) \end{aligned} \tag{4.55}$$

Equating coefficients of $|A|^2$ and $|B|^2$ to match the amplitude and obtain either

$$\{b(k,l)\} = \pm \ \{a(k,l)\} \tag{4.56}$$

or

$$\{b(k,l)\} = \pm \ \{a(k,l)\}^T \tag{4.57}$$

where the suprscript T refers to matrix transposition about k=l diagonal. In both cases {b(k,l)} is unstable.

The above counterexample shows that it is not always possible to stabilize a 2-D unstable filter by using DHT. The main difficulty is due to the truncation of infinite series and approximation of integral by finite summations. In a separate and independent work, Bose [124] gave other counterexamples for the continuous Hilbert transform and explained why stabilization via Hilbert transformation without appreciable change in the frequency response cannot, in general, be implemented in multi-dimensional filters. In a recent work by Murray [135], further elaboration of this method of stabilization is discussed.

4.3 PISTOR STABILIZATION METHOD

The two stabilization techniques mentioned above, PLSI and DHT, cause a modification of the unstable transfer function of the filter in such a manner the amplitude response is kept approximately unchanged, while the phase response is adjusted to ensure the stability of the modified transfer function. Therefore, these procedures [7], [121], are not applicable to zero-phase functions which do not permit phase modifications in any manner which will improve the stability.

A zero-phase filter has the particular property that its unit sample response is symmetric about (m,n) origin along any radius through the origin. Because of this symmetry, the unit sample response of a zero-phase filter is not a one-quadrant function. By the way, the filter function can be transformed by translation to one of the single quadrant functions. Hence, the z-transform of this filter could be associated with four different filters. However, none of them would be stable [31].

Pistor [31], [126] showed that unstable two-dimensional recursive filters having zero-phase can be decomposed into four stable filters that recurse in four different directions. This decomposition is based on the relationship of stability of recursive filters to the absolute summability of certain operators called cepstra [138]. We consider a real-valued discrete function c with a limited number of sample points, in which

$$c = \left\{c(m,n)\right\} \qquad \begin{aligned} |m| &\leq 2M_c = \alpha \\ |n| &\leq 2N_c = \beta \end{aligned} \tag{4.58}$$

z-transform of this array is of zero phase and non-negative for all $(z_1, z_2) \varepsilon R$

$$\mathrm{Im}\left\{C(u,v)\right\} \equiv 0 \tag{4.59}$$

$$\mathrm{Re}\left\{C(u,v)\right\} > 0 \tag{4.60}$$

where

$$C(u,v) = \sum_{m=-\alpha}^{\alpha} \sum_{n=-\beta}^{\beta} c(m,n)\, e^{-j2\pi(um+vn)} \tag{4.61}$$

$$R = \left\{(z_1, z_2) : |z_1| = |z_2| = 1\}\right\} \tag{4.62}$$

Equations (4.59) and (4.61) imply central symmetry of c,

$$\check{c}(m,n) = c(-m,-n) \tag{4.63}$$

Pistor showed that an unstable filter $1/C(z_1,z_2)$ can be be decomposed into stable filters that recurse in four different directions shown in Fig. 4.3.

$$\frac{1}{C(z_1,z_2)} = \prod_{\ell=1}^{4} \frac{1}{K_\ell^{-1}(z_1,z_2)} \tag{4.64}$$

or, in the cepstrum domain

$$\hat{c} = \sum_{\ell=1}^{4} {}^{\ell}\hat{k} \tag{4.65}$$

where $\hat{c}$ is the cepstrum of c.

4.3.1 Determination of Approximate Cepstrum

The cepstrum of a given array can be determined by DFT techniques as indicated in Fig.4.4. This can be done by using the fast Fourier transform. Since c(m,n) is not a casual function, some shifting operations will be necessary.

$$C(k_1,k_2) = \sum_{m=-\alpha}^{\alpha} \sum_{n=-\beta}^{\beta} \check{c}(m,n) W_M^{mk_1} W_N^{nk_2}$$
$$k_1 \varepsilon [-\alpha,\alpha]$$
$$k_2 \varepsilon [-\beta,\beta] \tag{4.66}$$

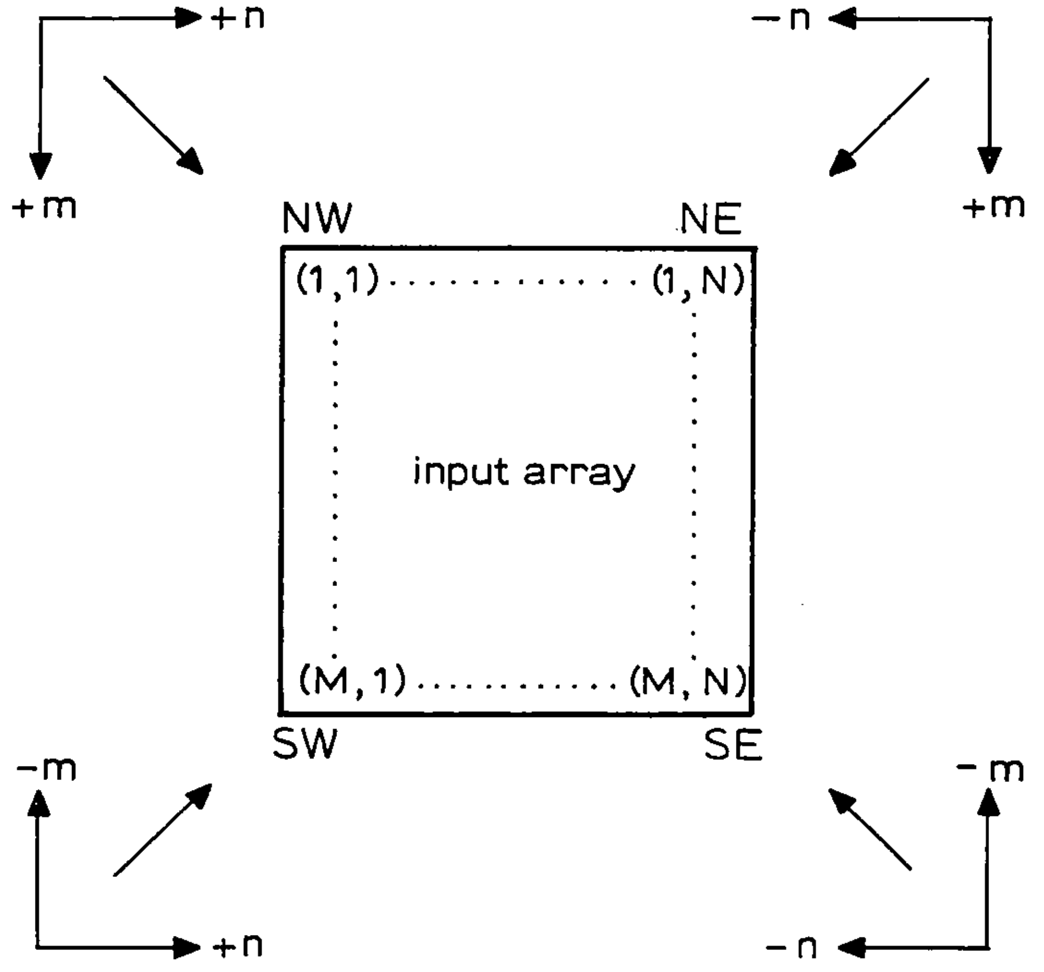


Fig. 4·3 Pistor's decomposed single-quadrant filters convolved recursively with an input array.

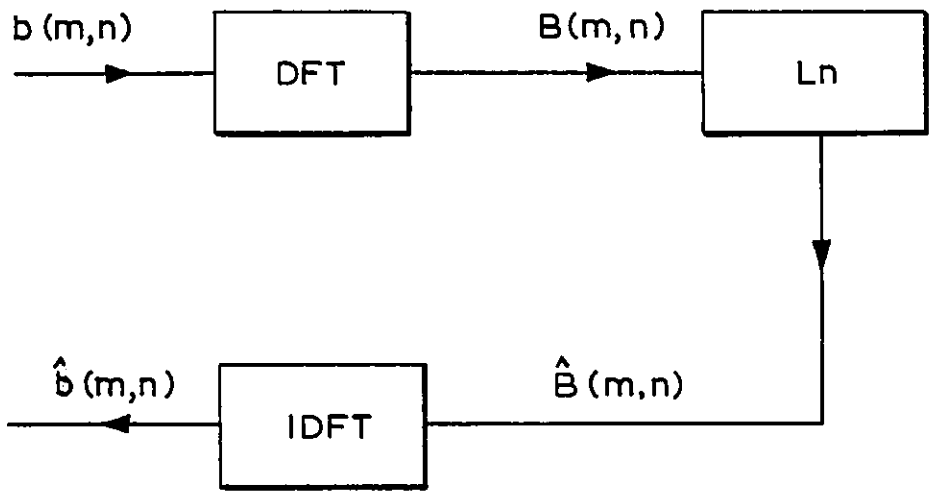


Fig. 4·4 Block diagram of determination of the approximate cepstrum transform

where

$$M = 2\alpha+1 \quad , \quad N = 2\beta+1$$

$$W_M = e^{-\frac{2\pi}{M}} \quad , \quad W_N = e^{-\frac{2\pi}{N}}$$

In order to use FFT algorithms, equation (4.66) can be transformed

$$C(k_1-\alpha,k_2-\beta) = \sum_{m=0}^{2\alpha} \sum_{n=0}^{2\beta} c(m-\alpha,n-\beta) W_M^{(m-\alpha)(k_1-\alpha)} W_N^{(n-\beta)(k_2-\beta)} \qquad (4.67)$$

$$k_1 \varepsilon [0,2\alpha]$$
$$k_2 \varepsilon [0,2\beta]$$

However, FFT algorithms are applicable for one-dimensional arrays.

Therefore, it would be necessary to modify the two-dimensional DFT equation.

$$C(k_1-\alpha,k_2-\beta) = K_{k_1k_2} \sum_{m=0}^{2\alpha} K_m W_M^{k_1 m} \sum_{n=0}^{2\beta} c(m-\alpha,n-\beta) K_n W_N^{k_2 n}$$

$$k_1 \varepsilon [0,2\alpha]$$
$$k_2 \varepsilon [0,2\beta] \qquad (4.68)$$

where:

$$K_m = W_M^{-\alpha m} \quad , \quad K_n = W_N^{-\beta n} \quad , \quad K_{k_1k_2} = W_M^{-\alpha(k_1-\alpha)} \; W_N^{-\beta(k_2-\beta)}$$

Since we use logarithm in the process, it is essential that FFT values must be real positive. Hence, the symmetry property must be considered at each step. Inverse FFT of logarithm array can be obtained similarly. Due to the aliasing error, it is helpful to insert zeros around the original array. The degree of aliasing can be controlled by sampling rate. Adding more zeros will result in less aliased versions in the frequency domain. It has been found that the aliasing greatly affected the value of the decomposed arrays, if FFT is used on a matrix less than 32x32 points. This number is the minimum acceptable size of FFT.

3.2 DECOMPOSITION BY QUADRANTS IN THE CEPSTRUM DOMAIN

Because $\{c(m,n)\}$ is centrally symmetric, so is its spectrum $C(u,v)$, and consequently its cepstrum $\widehat{C}(u,v)$ must be also of central symmetry. Due to this symmetry in the cepstrum domain the following relations must hold:

$$^{1}\hat{b}(k,o) = {}^{2}\hat{b}(k,o) = \frac{1}{2}\,\hat{c}(k,o) \tag{4.69}$$

$$^{1}\hat{b}(o,\ell) = {}^{3}\hat{b}(o,\ell) = \frac{1}{2}\,\hat{c}(k,\ell) \tag{4.70}$$

$$^{3}\hat{b}(k,o) = {}^{4}\hat{b}(k,o) = \frac{1}{2}\,\hat{c}(k,o) \tag{4.71}$$

$$^{4}\hat{b}(o,\ell) = {}^{1}\hat{b}(o,\ell) = \frac{1}{2}\,\hat{c}(k,\ell) \tag{4.72}$$

$$^{1}\hat{b}(o,o) = {}^{2}\hat{b}(o,o) = {}^{3}\hat{b}(o,o) = {}^{4}\hat{b}(o,o) = \frac{1}{4}\,\hat{c}(o,o) \tag{4.73}$$

Thus, four one-quadrant sequences $\{{}^{\ell}\hat{b}(m,n)\}, \ell = 1,2,3,4$, will satisfy the equation (4.74).

$$\hat{c}(m,n) = {}^{1}\hat{b}(m,n) + {}^{2}\hat{b}(m,n) + {}^{3}\hat{b}(m,n) + {}^{4}\hat{b}(m,n) \qquad (4.74)$$

4.3.3 Determination of Decomposed Arrays in Time Domain

First, we calculate the first quadrant arrays in spatial (time) domain. Next, the fourth quadrant array will be obtained. Then, we get the third and second quadrant arrays by rotating through 180° the first and fourth quadrants respectively.

In the calculation process, the first entry ${}^{1}b(o,o)$ can be obtained by definition of cepstrum; indeed,

$$\hat{B}_1(z_1,z_2) = \ell n\left[B_1(z_1,z_2)\right] \qquad (4.75)$$

for $(z_1,z_2) = (0,0)$:

$${}^{1}\hat{b}(o,o) = \ell n\left[{}^{1}b(o,o)\right] \qquad (4.76)$$

$${}^{1}b(o,o) = \exp\left[{}^{1}\hat{b}(o,o)\right] \qquad (4.77)$$

Form equation (4.75)

$$B_1(z_1,z_2) = \exp\left[\widehat{B}(z_1,z_2)\right] \qquad (4.78)$$

In order to find other coefficients of the array, we differentiate equation (4.78) with respect to z_1 and z_2.

$$z_1 \frac{\partial}{\partial z_1} B_1(z_1,z_2) = B_1(z_1,z_2)\ z_1 \frac{\partial}{\partial z_1} \hat{B}_1(z_1,z_2) \qquad (4.79)$$

$$z_2 \frac{\partial}{\partial z_2} B_1(z_1,z_2) = B_1(z_1,z_2)\ z_2 \frac{\partial}{\partial z_2} \hat{B}_1(z_1,z_2) \qquad (4.80)$$

Equations (4.77) and (4.78) correspond to:

$$\{m\ {}^1b(m,n)\} = \{{}^1\hat{b}(m,n)\} * \{m\ {}^1b(m,n)\} \qquad (4.81)$$

$$\{n\ {}^1b(m,n)\} = \{{}^1\hat{b}(m,n)\} * \{n\ {}^1b(m,n)\} \qquad (4.82)$$

These identifies yield following relations

$${}^1b(p,q) = \sum_{m=1}^{p} \sum_{n=0}^{q} \left(\frac{m}{p}\right) {}^1\hat{b}(m,n)\, {}^1b(p-m,q-n) \quad p \neq 0 \qquad (4.83)$$

$${}^1b(p,q) = \sum_{m=0}^{p} \sum_{n=1}^{q} \left(\frac{n}{q}\right) {}^1\hat{b}(m,n)\, {}^1b(p-m,q-n) \quad q \neq 0 \qquad (4.84)$$

4.3.4 Problem of Truncation:

Pistor's stabilization theorem [31] guarantees that the power series $B_1(z_1,z_2)$ is absolutely convergent for all $(z_1,z_2) \varepsilon R^1$

$$B_1(z_1,z_2) = \exp\left[\hat{B}_1(z_1,z_2)\right]$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} {}^1b(m,n)\, z_1^m\, z_2^n \qquad (4.85)$$

$$R^1 = \{(z_1,z_2) : |z_1| \leq 1 \quad |z_2| \leq 1\}$$

where ${}^1b$ is defined in equations (4.77),(4.83),(4.84).

However, in practice, it is not possible to consider an infinite number of coefficients. If we want to implement numerically, some truncation becomes mandatory. This truncation means not only that decomposition becomes approximate, but also that recursive stability of the decomposed one quadrant functions may be affected.

In the next section, a recursive computational algorithm is presented for the computation of spectral factors of unstable digital filter functions having prescribed bound on the error in the amplititude response and with assured stability.

4.4 AN ALGORITHM FOR STABILIZATION OF 2-D RECURSIVE FILTERS

4.4.1 Introduction

In Section 4.3, we showed Pistor's technique to decompose unstable two-dimensional recursive filters having non-zero, non-imaginary frequency response into four stable filters, each of which recurses in a different direction.

It is shown that an unstable filter $1/C(z_1,z_2)$ can be decomposed into four stable filters,

$$\frac{1}{C(z_1,z_2)} = \prod_{\ell=1}^{4} \frac{1}{B_\ell(z_1,z_2)} \tag{4.86}$$

where

$$B_\ell(z_1,z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} {}^{\ell}b(m,n)\, z_1^m z_2^n \tag{4.87}$$

$$\ell=1,2,3,4$$

Equation (4.87) shows that for exact factorization an infinite number of coefficients are needed. However, in practice this is not possible and some form of truncation is mandatory. Recent investigations [90], [109] have shown that when the factors are truncated, the decomposed filters do not preserve the original amplitude response.

4.4.2 Determination of stability in cepstrum domain:

The existence of a two-dimensional complex cepstrum has been proved by Dudgeon [39]. He has shown that essential singularities of a transfer function $H(z_1,z_2)$ map into essential singularities of $\hat{H}(z_1,z_2) = \ell n\left[H(z_1,z_2)\right]$. From this Ekstrom and Twogood [28] derived the theorem 2.12, stated in Chapter 2, for half-plane recursive filters.

However, in the following we shall consider the stability of decomposed, one-quadrant, recursive filter functions and for this purpose we shall re-state theorem 2.12 as follows.

Theorem 4.3:

A causal recursive filter function $H(z_1, z_2)$ is stable if and only if its cepstrum $\hat{h}(m,n)$ has support on the first quadrant.

Since Pistor's method is based on the spectral factorization in the cepstrum domain, it is simple to incorporate the above stability condition in the computation process without additional cost.

4.4.3 Determination of frequency performance:

Let $F_{m,n}$ be the desired magnitude response at a frequency $(m\delta\theta,\ n\delta\theta)$ where

$$F_{m,n} = \frac{1}{C(e^{jm\delta\theta}, e^{jn\delta\theta})} \tag{4.88}$$

and $D_{m,n}$, the approximated magnitude response of the cascade decomposed arrays where

$$D_{m,n} = \prod_{\ell=1}^{4} \frac{1}{B_\ell(e^{jm\delta\theta}, e^{jn\delta\theta})} \tag{4.89}$$

and $\delta\theta$ is the frequency increment between samples. We may define the error index

$$E(\lambda,\gamma) = \sum_{m=1}^{\lambda} \sum_{n=1}^{\gamma} \left[F_{m,n} - |D_{m,n}| \right]^{\rho} \tag{4.90}$$

where ρ is a positive integer. In our design we consider the least square error in which $\rho = 2$. For a given error E_o, the proposed method will find the minimum number of coefficients in the truncated frequency array needed to meet the frequency specification. If the error $E(\lambda,\gamma) > E_o$, the number of coefficients will be increased until the desired first truncation is reached when $E(\lambda,\gamma) \leqslant E_o$. Subsequently the stability is tested and further increase of array size is implemented until stability is achieved or the process is terminated.

4.4.4 Algorithm

Step 1 : Given the denominator of the zero phase filter. Set the truncation parameters $M=M_{min}$, $N=N_{min}$. Set a parameter S=0 to index the decomposition part of the program.

Step 2 : Find the complex cepstrum $\hat{c}(m,n)$ from the relation

$$\{\hat{c}(m,n)\} \Leftrightarrow \hat{C}(z_1,z_2) = \ln[C(z_1,z_2)]$$

Step 3 : If S=1 go to step 9.

Step 4 : Decompose the cepstrum into four quadrants [31].

Step 5 : Calculate the spatial domain arrays up to a size M x N.

Step 6 : Calculate the error index from equation (4.90).

Step 7 : If $E(\lambda,\gamma) > E_o$, set N = N+1, M = M+1 and go to step 5.

Step 8 : Set S=1 to initiate the stability test and go to step 2.

Step 9 : If the complex cepstrum has support in the first quadrant, exit .

Step 10: Set M = M+1, N = N+1, and go to step 2.

The implementation of the algorithm is shown by the flow chart of Fig. 4.5.

4.4.5 Example

The application of the above technique was illustrated using the second example of reference [31]. The transfer function of the filter is:

$$F_2(z_1,z_2) = \frac{1}{C_2(z_1,z_2)} \tag{4.91}$$

where

$$C_2(z_1,z_2) = \sum_{m=-2}^{2} \sum_{n=-2}^{2} c(m,n) z_1^m z_2^n \tag{4.92}$$

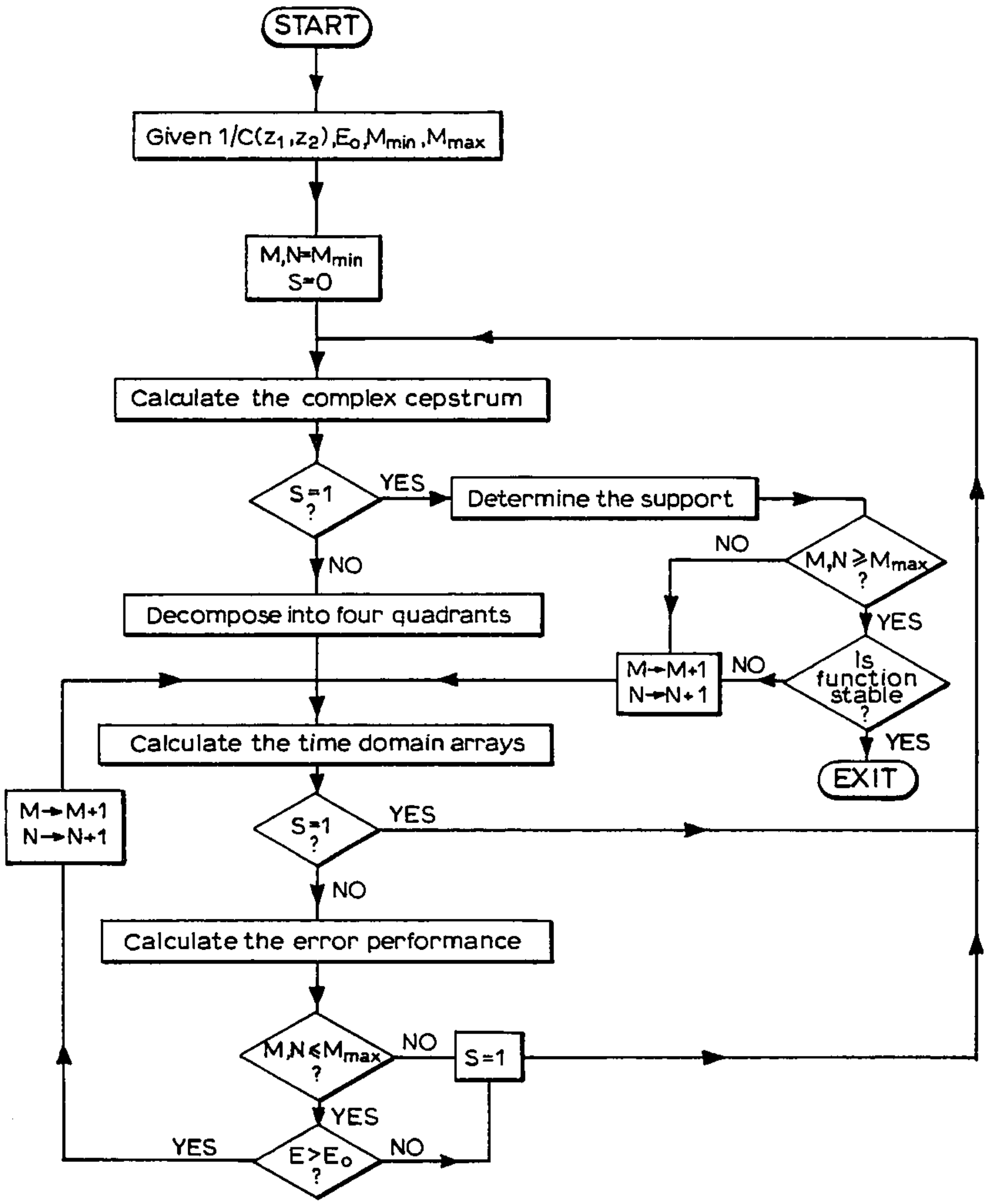


Fig. 4·5 Algorithm for stabilization of 2-D recursive digital filters.

$$\{c(m,n)\} = \begin{bmatrix} 0.68850 & 2.7639 & 4.15082 & 2.7639 & 0.68850 \\ 2.7639 & 11.0930 & 16.65830 & 11.0930 & 2.76390 \\ 4.15082 & 16.6580 & 25.2903 & 16.6583 & 4.15082 \\ 2.7639 & 11.0930 & 16.65830 & 11.0930 & 2.76390 \\ 0.68850 & 2.7639 & 4.15082 & 2.7639 & 0.68850 \end{bmatrix} \qquad (4.93)$$

The amplitude response of the given zero-phase filter is shown in Fig. 4.6. Fig. 4.7 shows the amplitude response of the cascaded decomposed filter functions. The stabilized filter has 64 coefficients in each quadrant and the error index $E(\lambda,\gamma) = 3.21320$.

4.4.6 General Remarks:

The use of Pistor's stability criterion for the design of stable two-dimensional recursive digital filters has been critically examined. A novel algorithm has been proposed in order to control the effect of truncation and frequency response. The stability of the decomposed filters has been tested in the cepstrum domain without introducing additional computational effort.

4.5 A MODIFIED STABILIZATION TECHNIQUE FOR 2-D RECURSIVE FILTERS

Although the two-stabilization techniques, namely the planar least-squares inverse of Shanks [7] and discrete Hilbert transform of Read and Treitel [121] both work in most practical situations, there are some cases where these

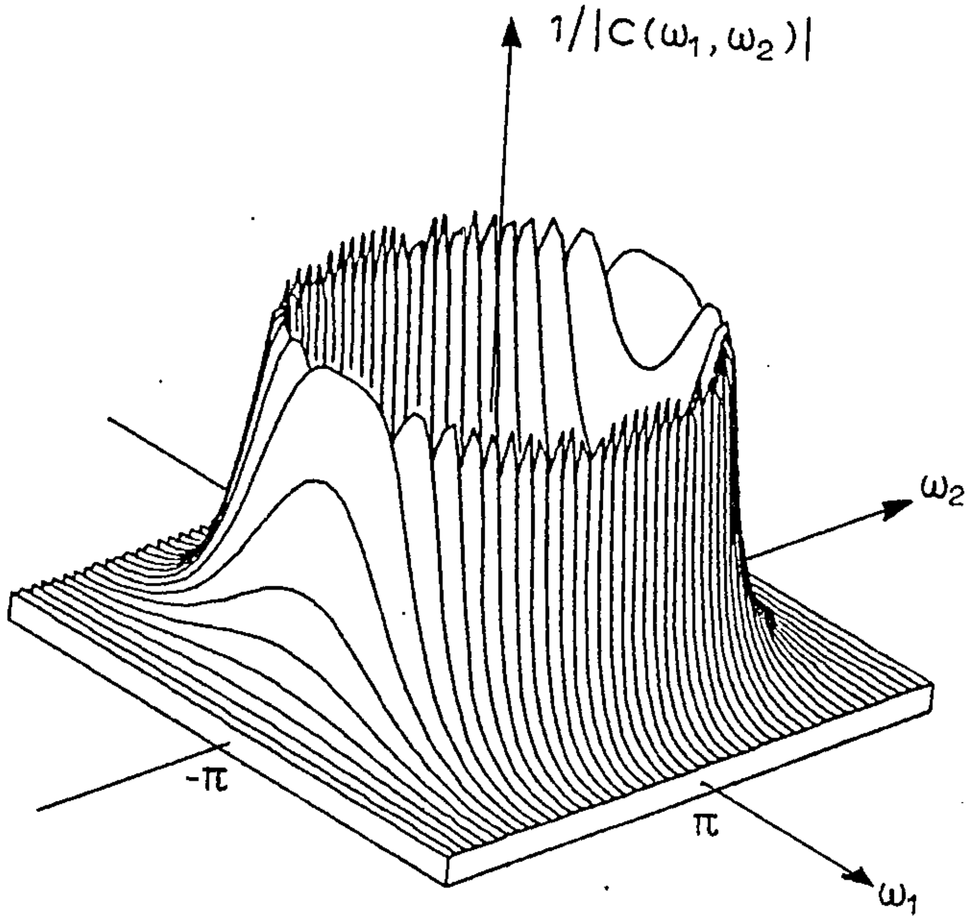


Fig. 4·6 Frequency response of a zero-phase filter $1/C(z_1, z_2)$

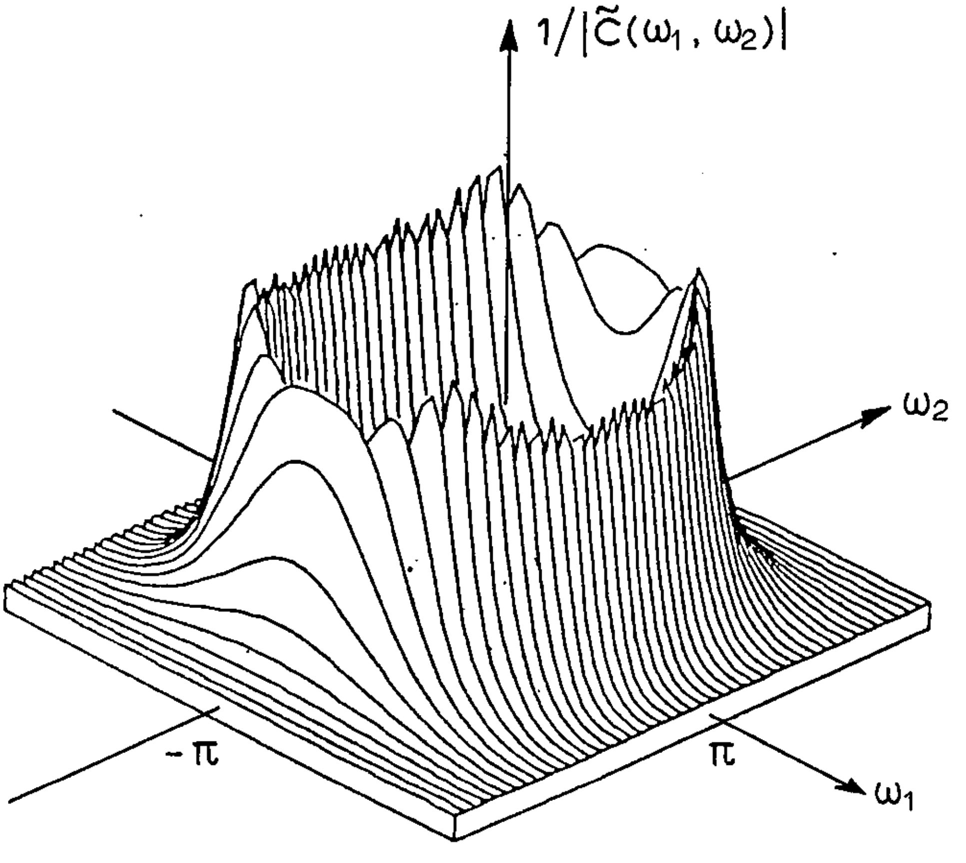


Fig. 4·7 Frequency response of a filter obtained by cascading four decomposed filters of 64 coefficients each.

methods fail to produce stable results. Counterexample 3.1 - 3.3. In this Section, we will present a new stabilization procedure based on the spectral factorization in the cepstrum domain.

Pistor's stability criterion [31] will be used for the stability of the resulting filter. It is shown that any causal 2-dimensional filter function can be decomposed into stable filters that recurse in the half-plane.

The new procedure is then compared with the planar least-squares inverse technique and discrete Hilbert transform method. It is found that the present stabilization technique has a better frequency response approximation than the existing ones. The method is therefore an attractive alternative to the least-square procedure and the discrete Hilbert transform method.

4.5.1 Decomposition of Unstable Filters into Half-Plane

A technique for the stabilization of two-dimensional (2-D) zero phase recursive digital filters has been propounded by Pistor [31]. It is here shown how this technique may be applied to non zero-phase systems.

A 2-D causal first quadrant recursive filter having transfer function

$$F_1(z_1,z_2) = \frac{1}{B_1(z_1,z_2)} = \frac{1}{\sum_{m=0}^{\alpha}\sum_{n=0}^{\beta} {}^1b(m,n)\, z_1^m z_2^n} \qquad (4.94)$$

can not have a zero phase response.

However, by a suitable combination of two such recursive filters we can produce the zero phase filter having transfer function:

$$F(z_1,z_2) = \frac{1}{C(z_1,z_2)} \qquad (4,95)$$

where

$$C(z_1,z_2) = B_1(z_1,z_2)B_1(z_1^{-1},z_2^{-1}) = \sum_{m=-\alpha}^{\alpha}\sum_{n=-\beta}^{\beta} c(m,n)z_1^m z_2^n \qquad (4.96)$$

The amplitude response of $F(z_1,z_2)$ will be:

$$F(e^{j\omega_1},e^{j\omega_2}) = \frac{1}{\left|C(e^{j\omega_1},e^{j\omega_2})\right|}$$

$$= \frac{1}{\left|B_1(e^{j\omega_1},e^{j\omega_2})\right|^2} \qquad (4.97)$$

$B_1(z_1^{-1},z_2^{-1})$ is obviously the denominator of a third quadrant filter and may be written:

$$B_1(z_1^{-1},z_2^{-1}) = B_3(z_1,z_2) = \sum_{m=-\alpha}^{0}\sum_{n=-\beta}^{0} {}^3b(m,n)z_1^m z_2^n \qquad (4.98)$$

Again from (4.94) and (4.96) c(m,n) may be written as the convolution of ^{1}b and ^{3}b
as:

$$c(m,n) = \sum_{k=0}^{\alpha} \sum_{\ell=0}^{\beta} {}^{1}b(k,\ell)\,{}^{3}b(k-m,\ell-n) = {}^{1}b(m,n) * {}^{3}b(m,n) \qquad (4.99)$$

Since $C(z_1,z_2)$ is the denominator function of a zero phase system we may apply Pistor's stabilization technique to c(m,n). As a first step we obtain the cepstrum as:

$$\hat{c}(m,n) = \text{IDFT}\left\{\ln\left[\text{DFT}[c(m,n)]\right]\right\} \qquad (4.100)$$

Rahter than use (4,99) for obtaining c(m,n) it is simpler to evaluate the DFT of c(m,n) directly from

$$\text{DFT}\left[c(m,n)\right] = \text{DFT}\left[{}^{1}\tilde{b}(m,n)\right] . \ \text{DFT}\left[{}^{3}\tilde{b}(m,n)\right] \qquad (4.101)$$

where

$$\{{}^{1}\tilde{b}(m,n)\} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ & \\ \underset{\sim}{0} & \{{}^{1}b(m,n)\} \end{bmatrix} \text{and } \{{}^{3}\tilde{b}(m,n)\} = \begin{bmatrix} \{{}^{3}b(m,n)\} & \underset{\sim}{0} \\ & \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}$$

the size of both these augmented matrices being $(2\alpha-1) \times (2\beta-1)$.

On completion of the Pistor procedure we are able to decompose $C(z_1,z_2)$ into four stable filters each recursing in a different direction as:

$$C(z_1,z_2) = \prod_{\ell=1}^{4} C_\ell(z_1,z_2) \qquad (4.102)$$

Since $C(z_1,z_2)$ is zero phase

$$C_3(z_1,z_2) = C_1(z_1^{-1},z_2^{-1}) \tag{4.103}$$

and $$C_4(z_1,z_2) = C_2(z_1^{-1},z_2^{-1}). \tag{4.104}$$

Hence

$$C(z_1,z_2) = C_1(z_1,z_2)C_2(z_1,z_2)C_1(z_1^{-1},z_2^{-1})C_2(z_1^{-1},z_2^{-1}) \tag{4.105}$$

and the frequency response is:

$$C(e^{j\omega_1},e^{j\omega_2}) = |C_1(e^{j\omega_1},e^{j\omega_2})|^2 \tag{4.106}$$

Comparison of (4.106) with (4.97) shows that:

$$|B_1(e^{j\omega_1},e^{j\omega_2})| = |C_1(e^{j\omega_1},e^{j\omega_2})C_2(e^{j\omega_1},e^{j\omega_2})| \tag{4.107}$$

Equation (4.107) shows that the given unstable arbitrary phase filter $B_1(z_1,z_2)$ can be stabilized using only two of the decomposed filters which recurse in only the first two quadrants.

The proposed methods can be compared with other stabilization techniques [7], [97] ,[121]. Except for filters with only a few samples these methods are difficult to apply and the first method has been shown in general to be based on a fallacy. Finally the proposed method had the advantage of being constructive.

The outlined method may be extended in a straight forward manner to multidimensional filters having arbitrary phase.

4.5.2 Example

The unstable denominator array is:

$$\begin{bmatrix} 1.0 & -0.75 & 0.9 \\ 1.5 & -1.2 & 1.3 \\ 1.2 & 0.9 & 0.5 \end{bmatrix} \tag{4.108}$$

The root map of this array, Fig.4.8 shows that the system is unstable.

Application of the outlined method results in the two quadrant filters

$$^1b = \begin{bmatrix} 1.54345 & -0.151975 & 0.220596 \\ 0.360485 & -0.104939 & 0.364880 \\ -0.002245 & 0.298473 & 0.076050 \end{bmatrix} \tag{4.109}$$

$$^2b = \begin{bmatrix} 0.220596 & -0.151975 & 1.54345 \\ 0.458488 & -0.381673 & 0.360485 \\ 0.175024 & -0.053412 & -0.002245 \end{bmatrix} \tag{4.110}$$

These filters are stable in the first and second quadrants respectively. The frequency response of the original filter is shown in Fig.4.9 and of the cascade of the two decomposed filters in Fig. 4.10.

This may be compared with the filter as stabilized by the Hilbert Transform technique and the planar least squares method. The frequency responses after stabilization are shown in Fig.4.11 for the Hilbert method and Fig.4.12 for the PLSI.

The error between the frequency response of the original and stabilized network is measured by:

$$e(B,D) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \left[H_B(i,j) - H_D(i,j) \right]^2$$

where H_B and H_D are the magnitude of the original unstable frequency array and of the stabilized array respectively.

Considering $N_1 = N_2 = 59$ the errors for the three methods denoted by subscripts H (Hilbert), PLSI (Planar Least Square Inverse), MP (New Modified Pistor) are

$$\begin{aligned} e_{PLSI}(B,D) &= 0.79541 \times 10^5 \\ e_H(B,D) &= 0.10463 \times 10^4 \\ e_{MP}(B,D) &= 0.35887 \times 10^3 \end{aligned} \qquad (4.112)$$

The new technique thus shows a slight reduction of error over the Hilbert technique and an enormous advantage over the PLSI method. In addition the frequency responses of the filter stabilized by the new technique is closer to that of the original unstable filter.

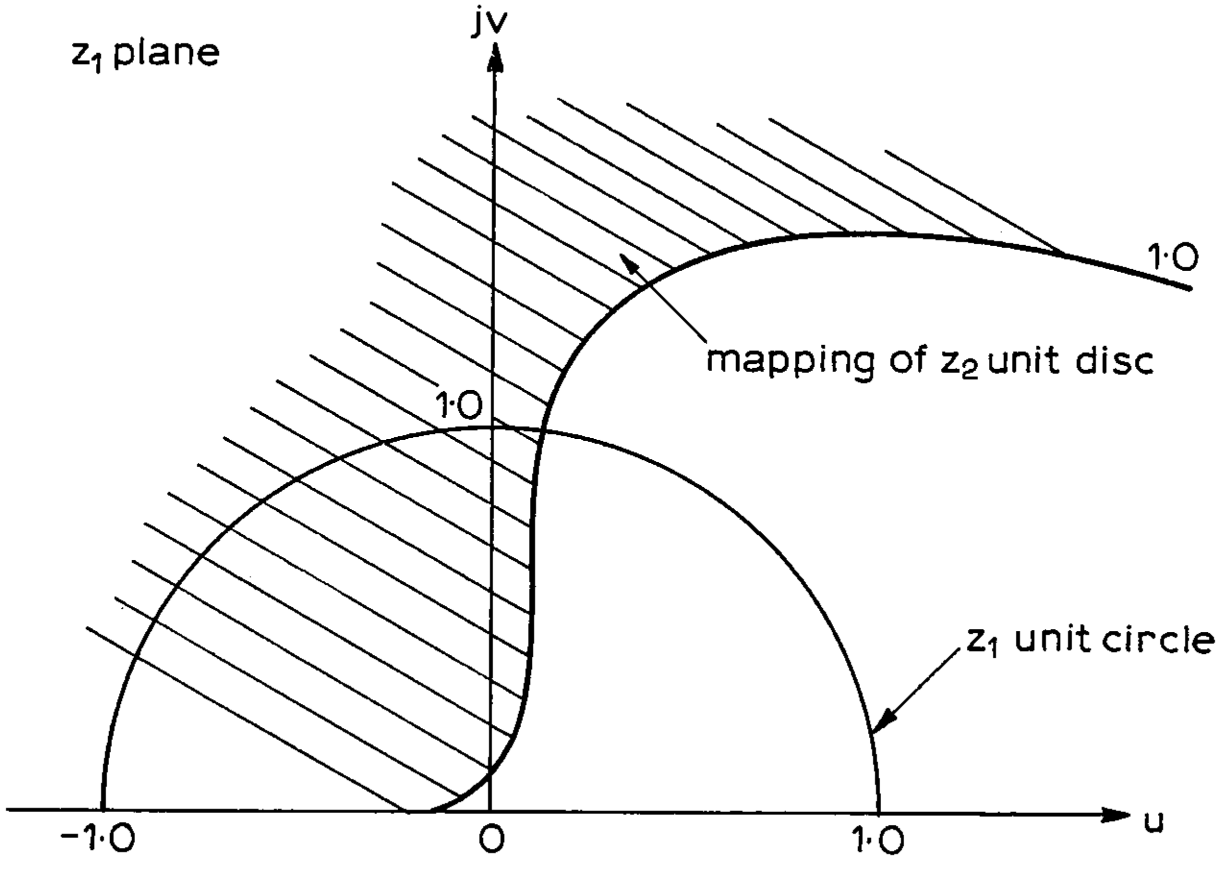


Fig. 4·8 Mapping of z_2 unit circle onto z_1 plane for unstable filter.

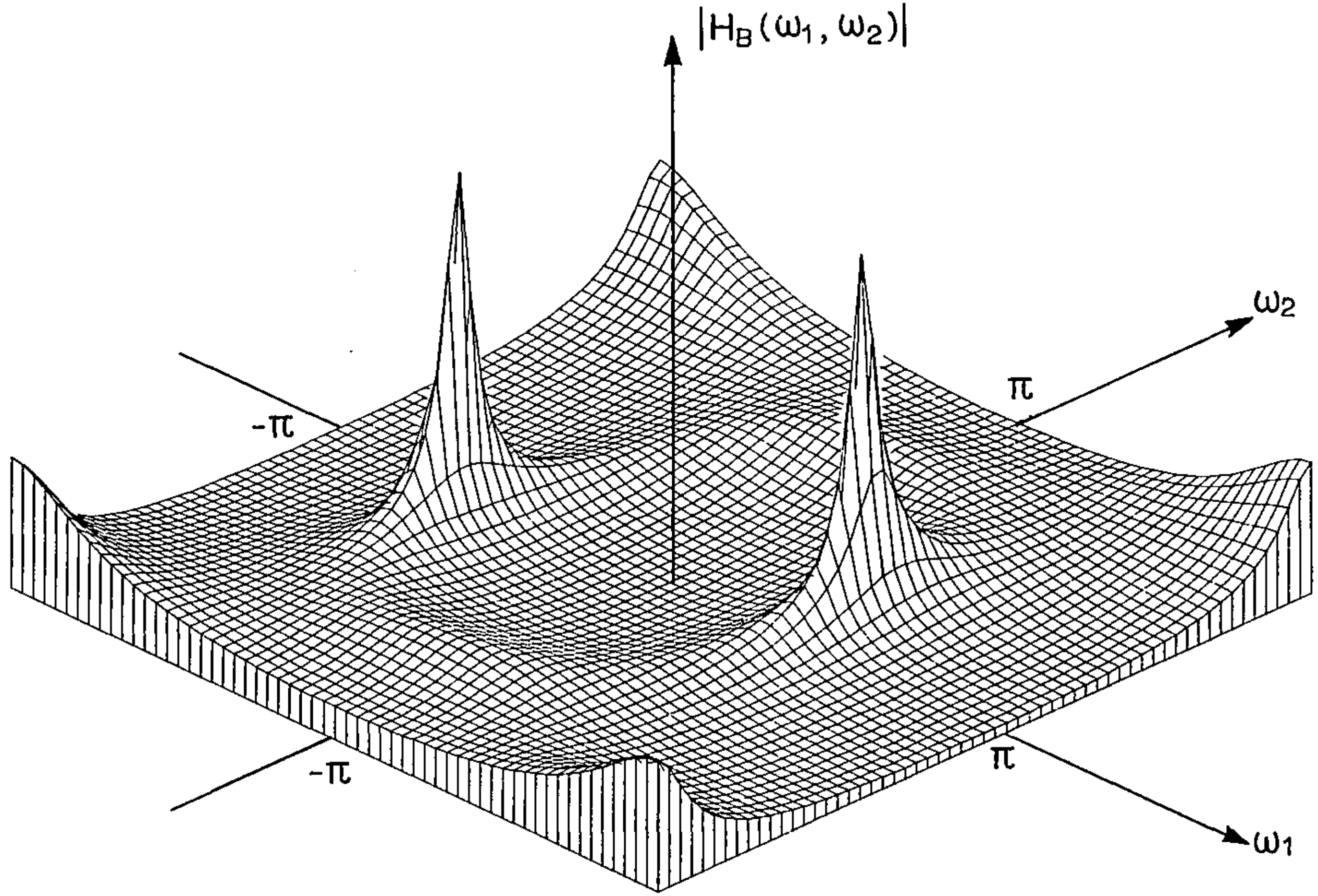


Fig. 4·9(a) Amplitude response of the original unstable filter.

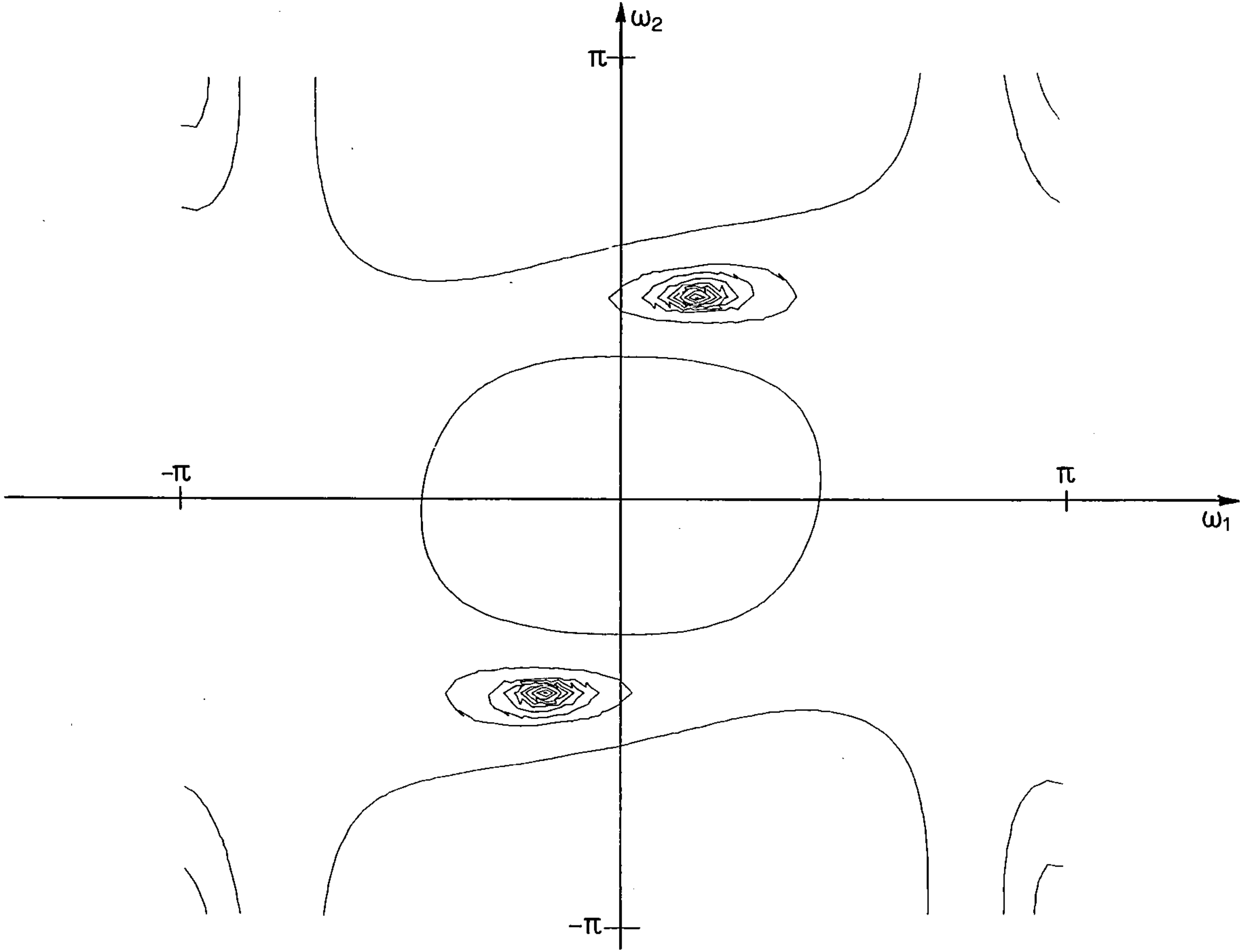


Fig. 4·9(b) Contour plot of the response of the filter of Fig. 4·9(a)

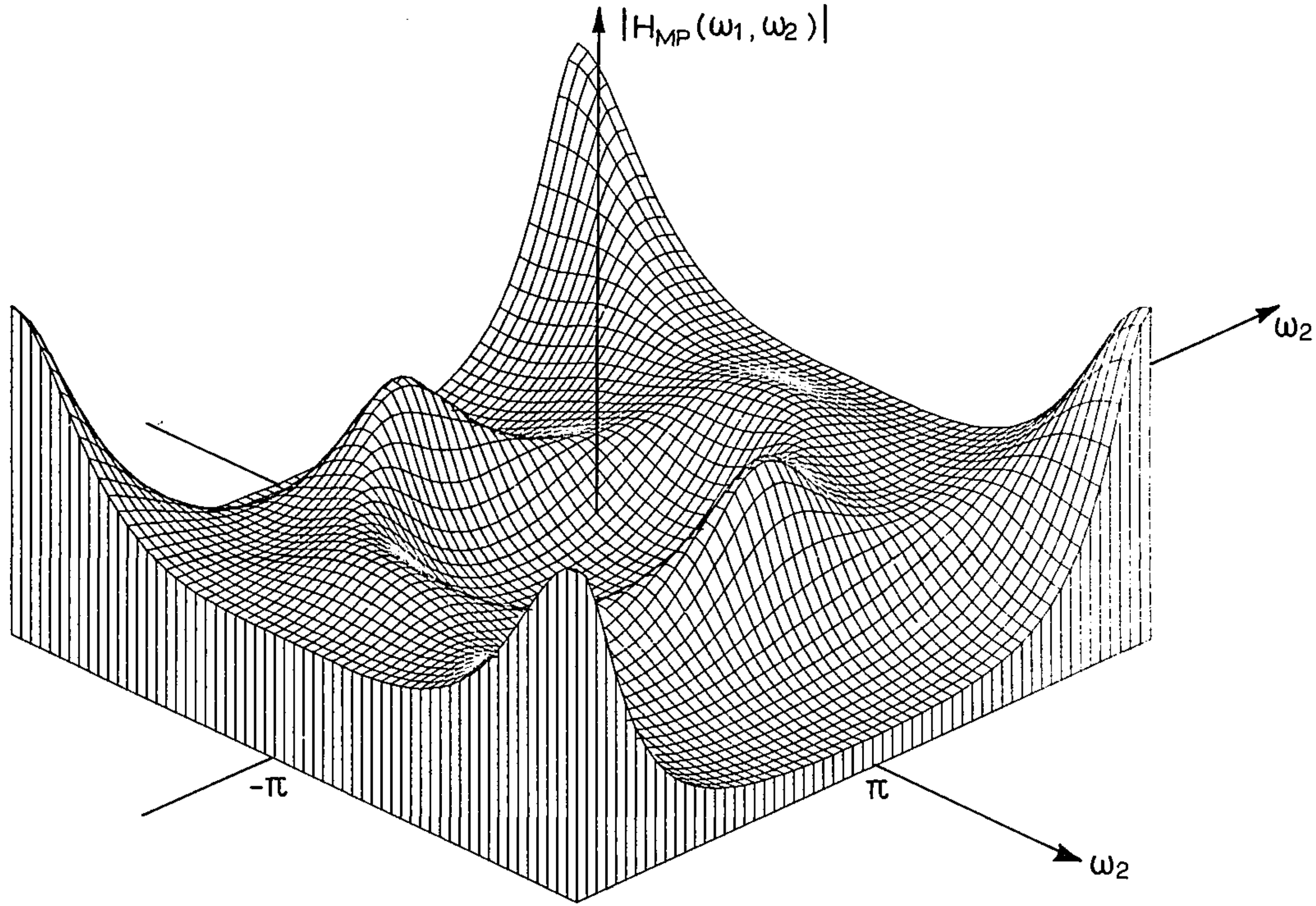


Fig. 4·10 (a) Amplitude response of the stabilized filter by spectral factorization.

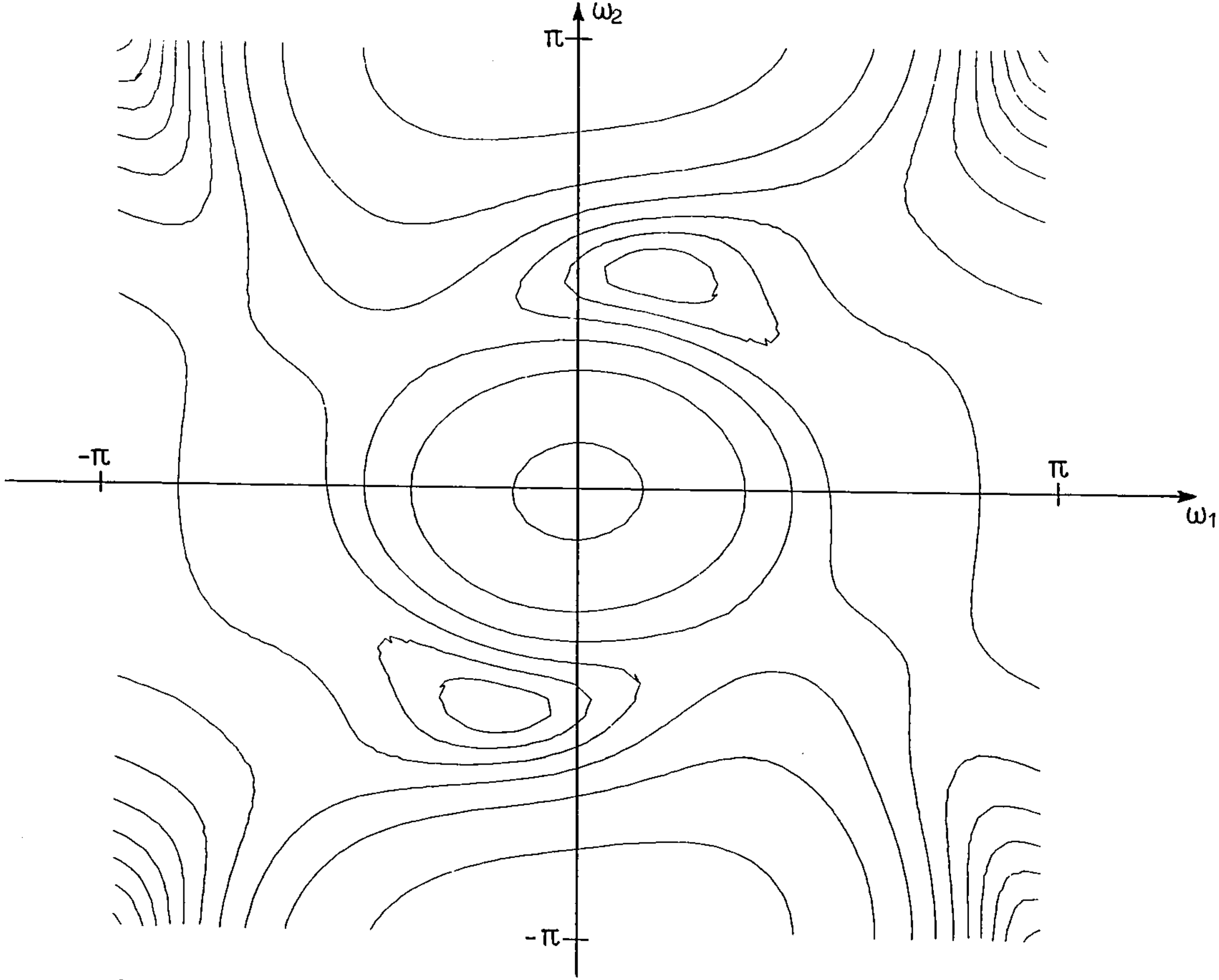


Fig. 4·10 (b) Contour plot of the response of filter of Fig. 4·10 (a)

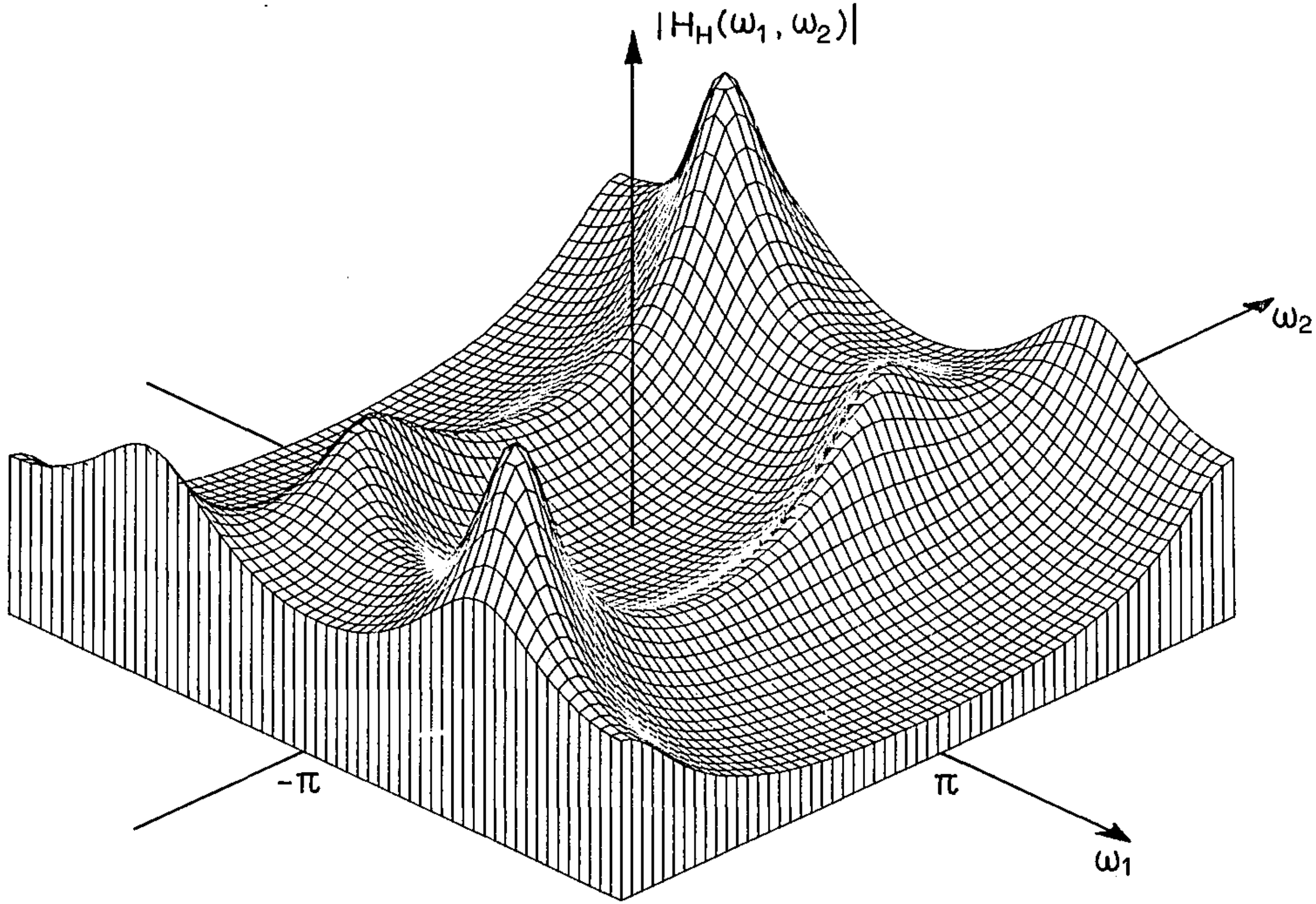


Fig. 4·11 (a) Amplitude response of the stabilized filter by Hilbert transform technique.

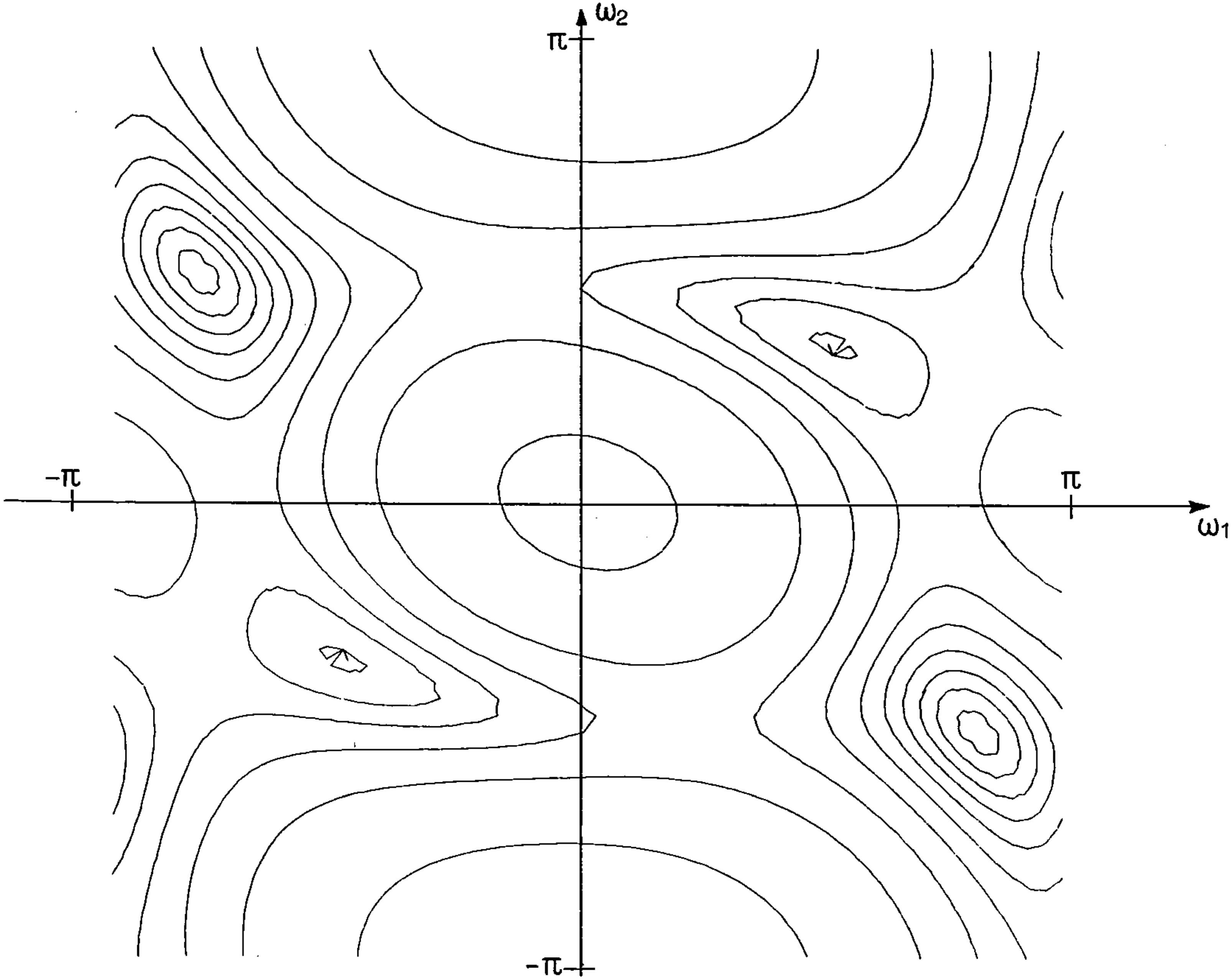


Fig. 4·11(b) Contour plot of the response of the filter of Fig. 4·11(a)

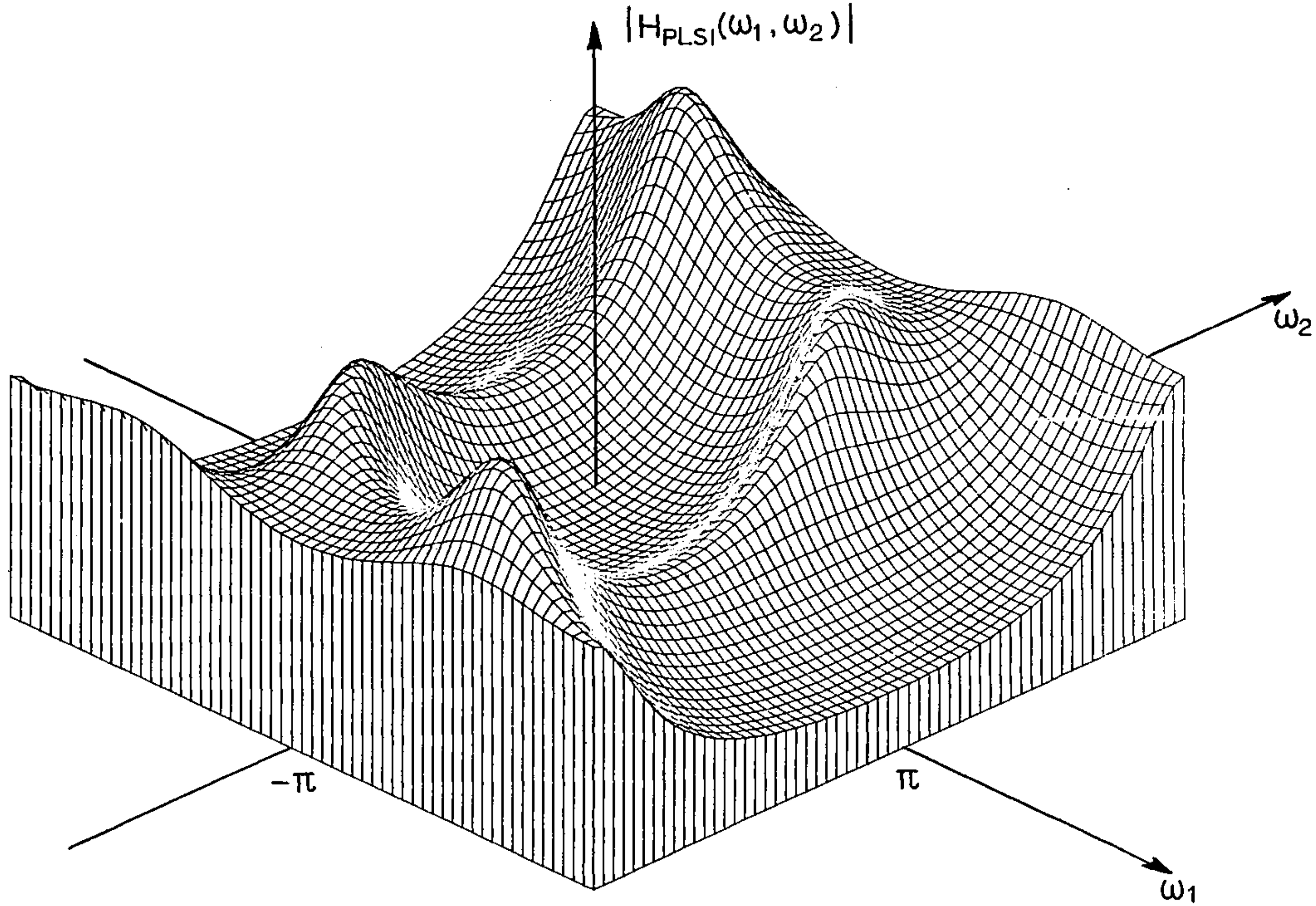


Fig. 4·12 (a) Amplitude response of the stabilized filter by the PLSI technique.

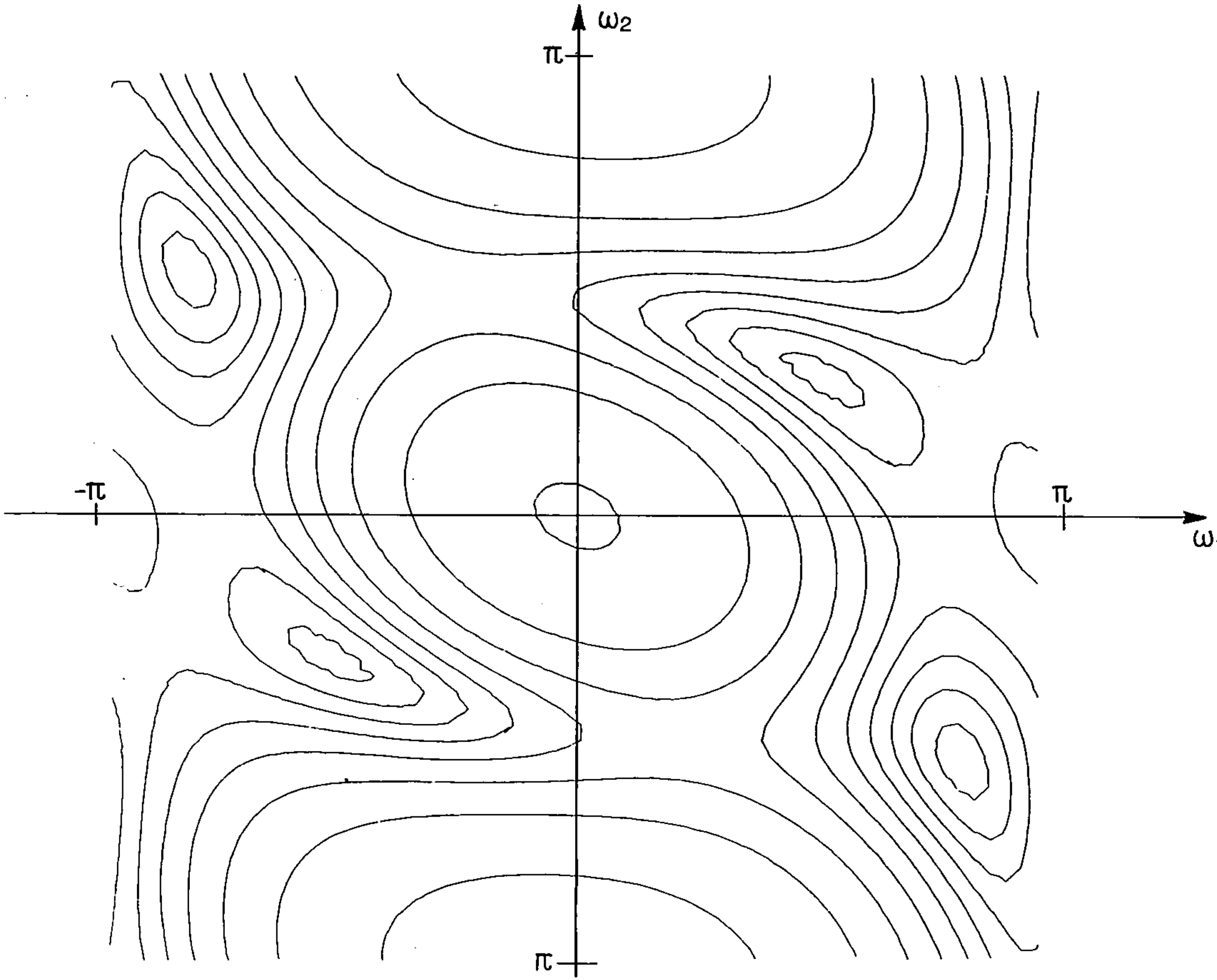


Fig. 4·12 (b) Contour plot of the response of the filter of Fig. 4·12 (a)

4.5.3 Remarks:

The Pistor technique has been shown to be capable of extension to the stabilization of any two dimensional digital filter. It has the advantage over other methods in that it is always valid and provides a constructive technique compared with the methods of references [7] and [121]. Furthermore no counter examples as in [16] and [122] have yet been found.

4.6 A NEW TWO-DIMENSIONAL SPECTRAL FACTORIZATION TECHNIQUE WITH APPLICATION IN RECURSIVE FILTERING

This section indicates an alternative stabilization technique for 2-dimensional recursive filters. The method involves the spectral factorization of the unstable arrays in the cepstrum domain. It is shown that if the factorization is made row by row or column by column in each quadrant, then the stability of the resulting 2-D recursive filter can be examined with ease.

The stability is determined by an assessment of the number of zeroes within the unit circle of non-reciprocal one dimensional polynomials. Since the stability testing is computationally straight forward a large number of coefficients can be considered for better approximation of the given frequency characteristics using this decomposition technique.

Details of the computation required in the implementation of the procedure is presented. An example is given to show the effectiveness of the procedure.

4.6.1 Problem Introduction:

We consider in this section, linear shift-invariant digital filters for which the input-output sequences are related by a linear constant coefficient difference equation of the form

$$\sum_{k}\sum_{\ell} b(k,\ell)y(m-k,n-\ell) = \sum_{k}\sum_{\ell} a(k,\ell)x(m-k,n-\ell) \tag{4.113}$$

or equivalently, in the frequency domain by the digital transfer function $H(z_1,z_2)$ defined as

$$H(z_1,z_2) = \frac{\sum_{k}\sum_{\ell} a(k,\ell)z_1^k z_2^\ell}{\sum_{k}\sum_{\ell} b(k,\ell)z_1^k z_2^\ell} = \frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{4.114}$$

Pistor's stabilization technique [31] decomposes an unstable filter $1/B(z_1,z_2)$ which is zero phase and is nonnegative for all values of z_1 and z_2 into stable filters that recurse into four directions such that

$$B(z_1,z_2) = \prod_{i=1}^{4} {}^{i}B(z_1,z_2) \tag{4.115}$$

where ${}^{i}B$, i=1,2,3,4 denotes the denominators of the decomposed filters that recurse in one-quadrant.

Each of the decomposed one-quadrant filters is stable if it has an infinite number of coefficients [31]. However, in practice, in order to implement numerically, some truncation becomes mandatory [109]. This truncation means not only that the decomposition becomes approximate, but also that the recursive stability of the decomposed one-quadrant functions may be affected. Hence, the stability of truncated one-quadrant filters must be tested. It has recently been shown that the Pistor stabilization technique requires a large number of coefficients in order to obtain a satisfactory frequency characteristic. Thus considerable computation time is necessary for testing the stability of the truncated one-quadrant array.

However, it can be shown that the redecomposition of one-quadrant filters in the cepstrum domain will reduce the two-dimensional stability problem to a one-dimensional one. It is shown that the decomposition can be done row by row or column by column.

4.6.2 Two Dimensional Spectral Factorization

The two-dimensional cepstrum of $B(z_1,z_2)$ is simply defined as:

$$\hat{B}(z_1,z_2) = \ell n\left[B(z_1,z_2)\right] \tag{4.116}$$

where ln denotes the natural logarithm. The cepstrum

$b(m,n)$, i.e. $\hat{b}(m,n)$ is a function whose z-transform is given by:

$$\hat{B}(z_1,z_2) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{b}(m,n) z_1^m z_2^n \tag{4.117}$$

Let $\hat{C}(z_1,z_2)$ be the first quadrant filter function in the cepstrum domain defined as

$$\hat{C}(z_1,z_2) = \sum_{m=0}^{M} \sum_{n=0}^{N} \hat{c}(m,n) z_1^m z_2^n \tag{4.118}$$

where

$$\hat{c}(m,n) = \begin{cases} \hat{b}(m,n) & m > 0 \text{ and } n > 0 \\ \frac{1}{2}\hat{b}(m,n) & m = 0 \text{ and } n > 0 \text{ or } m > 0 \text{ and } n = 0 \\ \frac{1}{4}\hat{b}(m,n) & m = 0 \text{ and } n = 0 \\ 0 & \text{elsewhere} \end{cases} \tag{4.119}$$

The redecomposition can be stated as follows:

$$\hat{C}(z_1,z_2) = \sum_{j=0}^{N} z_1^j \hat{C}_j(z_2) \tag{4.120}$$

or

$$\hat{C}(z_1,z_2) = \sum_{i=0}^{M} z_2^i \hat{R}_i(z_1) \tag{4.121}$$

where $\widehat{C}_j(z_2)$'s and $\widehat{R}_i(z_1)$'s are the z-transform of the decomposed column and row vectors respectively.

In order to find the spatial domain transfer function of $C(z_1,z_2)$, we can apply the inverse transform:

$$C(z_1,z_2) = \exp\left[\hat{C}(z_1,z_2)\right] \tag{4.122}$$

From equations (4.120) and (4.121),

$$C(z_1,z_2) = \exp\left[\sum_{j=0}^{N} z_1^j \hat{C}_j(z_2)\right] = \prod_{j=0}^{N} \exp\left[z_1^j \hat{C}_j(z_2)\right] \tag{4.123}$$

or

$$C(z_1,z_2) = \exp\left[\sum_{i=0}^{M} z_2^i \hat{R}_i(z_1)\right] = \prod_{i=0}^{M} \exp\left[z_2^i \hat{R}_i(z_1)\right] \tag{4.124}$$

It can be shown that if all terms in the multiplication are absolutely convergent in the unit bidisc, then $C(z_1,z_2)$ is absolutely convergent, i.e. the first quadrant filter function is recursively stable [31].

In Section 4.3, it was shown that from any unstable filter function (non-zero, non-imaginary), it is always possible to obtain recursively stable functions. We shall use equations (4.77), (4.83), and (4.84) derived in Chapter lV, to calculate the spatial domain function of the filter.

For this purpose, $\hat{C}(z_1,z_2)$ can be written as:

$$\hat{C}(z_1,z_2) = \sum_{j=0}^{\gamma} \left[\sum_{m=0}^{M} \sum_{n=0}^{N} \hat{c}_j(m,n) z_1^m z_2^n \right] \tag{4.125}$$

where the suffix j identifies the row or column of the decomposition and γ is the number of columns or rows (γ=M or γ=N). From equation (4.122)

$$C(z_1,z_2) = \prod_{j=0}^{N} \left[\sum_{n=0}^{\alpha} \sum_{m=0}^{\beta} c_j(m,n) z_1^m z_2^n \right] = \prod_{j=0}^{N} C_j'(z_1,z_2)$$

where α and β are truncation limits. The spatial (time) domain coefficients $\{c_j(m,n)\}$ can be found by using equations (4.77), (4.83), and (4.84).

Indeed, from equation (4.77),

$$c_j(0,0) = \begin{cases} \exp[\hat{c}(0,0)] & j = 0 \\ 1 & j \neq 0 \end{cases} \tag{4.127}$$

and all the other coefficients can be found from equations (4.83), (4.84). Then, it can be shown that:

$$C_j'(z_1,z_2) = 1 + z_2 f_{j1}(z_1) + z_2 f_{j2}(z_1) + z_2 f_{j3}(z) + \ldots\ldots$$

$$j = 1,2,3,\ldots,N \tag{4.128}$$

By considering only the first two terms,

$$\tilde{C}_j'(z_1,z_2) = 1 + z_2 f_{j1}(z_1) \tag{4.129}$$

where $\tilde{C}_j'(z_1,z_2)$ is the approximation to $C_j'(z_1,z_2)$. And finally $\tilde{C}(z_1,z_2)$, the approximation to $C(z_1,z_2)$ becomes:

$$\tilde{C}(z_1,z_2) = \tilde{C}'(z_1)\tilde{C}'(z_1,z_2)\tilde{C}_2'(z_1,z_2)\ldots. \tag{4.130}$$

or

$$\tilde{C}(z_1,z_2) = f_0(z_1) | \prod_{j=1}^{N} \left[1 + z_2 f_{j1}(z_1) \right] \tag{4.131}$$

where $f_0(z_1) = \exp\left[\hat{c}(0,0) \right]$.

4.6.3 Testing the Stability of Stabilized Filter Function

The first term in equation (4.131) is a function of z_1. Hence a one-dimensional stability test can be used. The second terms in the multiplication can be tested by using the Jury-Anderson Method [22] . The stability testing problem of these terms can be reduced into finding the number of roots in the unit circle of a nonreciprocal polynomial (Cohn's theorem).

It is only necessary to show that

$$|f_{j1}(0)| < 1 \tag{4.132}$$

and $f_{j1}(z_1)f_{j1}(z_1^{-1}) > 1$ all $|z_1| = 1$ $j= 1,2,\ldots,N$ (4.133)

Equations (4.132) and (4.133) satisfy the first and the second conditions of the Huang theorem [19].

3.6.4 Example:

We consider Pistor's second example in reference [31]

$$b = \begin{array}{c} \begin{bmatrix} 0.68850 & 2.7639 & 4.15082 & 2.7639 & 0.68850 \\ 2.76390 & 11.0930 & 16.65830 & 11.0930 & 2.76390 \\ 4.15082 & 16.6583 & 25.29030 & 2.7639 & 4.15082 \\ 2.76390 & 11.0930 & 16.65830 & 11.0930 & 2.76390 \\ 0.68850 & 2.7639 & 4.15082 & 2.7639 & 0.68850 \end{bmatrix} \begin{matrix} \\ \\ \longleftarrow \\ \\ \\ \end{matrix} \\ \uparrow \end{array} \qquad (4.134)$$

where the vertical and the horizontal arrows indicate m=0 and n=0, respectively. The filter function in (4.134) is zero-phase and non-imaginary for all values of z_1 and z_2.

Fig.13(a) shows the magnitude response of the original filter. Fig.14(b) are the magnitude response of the stabilized filter. The stabilized filter function in the first quadrant is:

$$\tilde{C}(z_1,z_2)=\Big[1.40365 + 0.709042z_2\Big]\Big[1 + z_1(0.50514 + 0.30878z_2)\Big] \qquad (4.135)$$

From a comparison of Fig.13 and Fig.14, we can conclude that the approximation is reasonably sufficient. This can in particular be seen from the contours of the frequency magnitude of the original and stabilized filters.

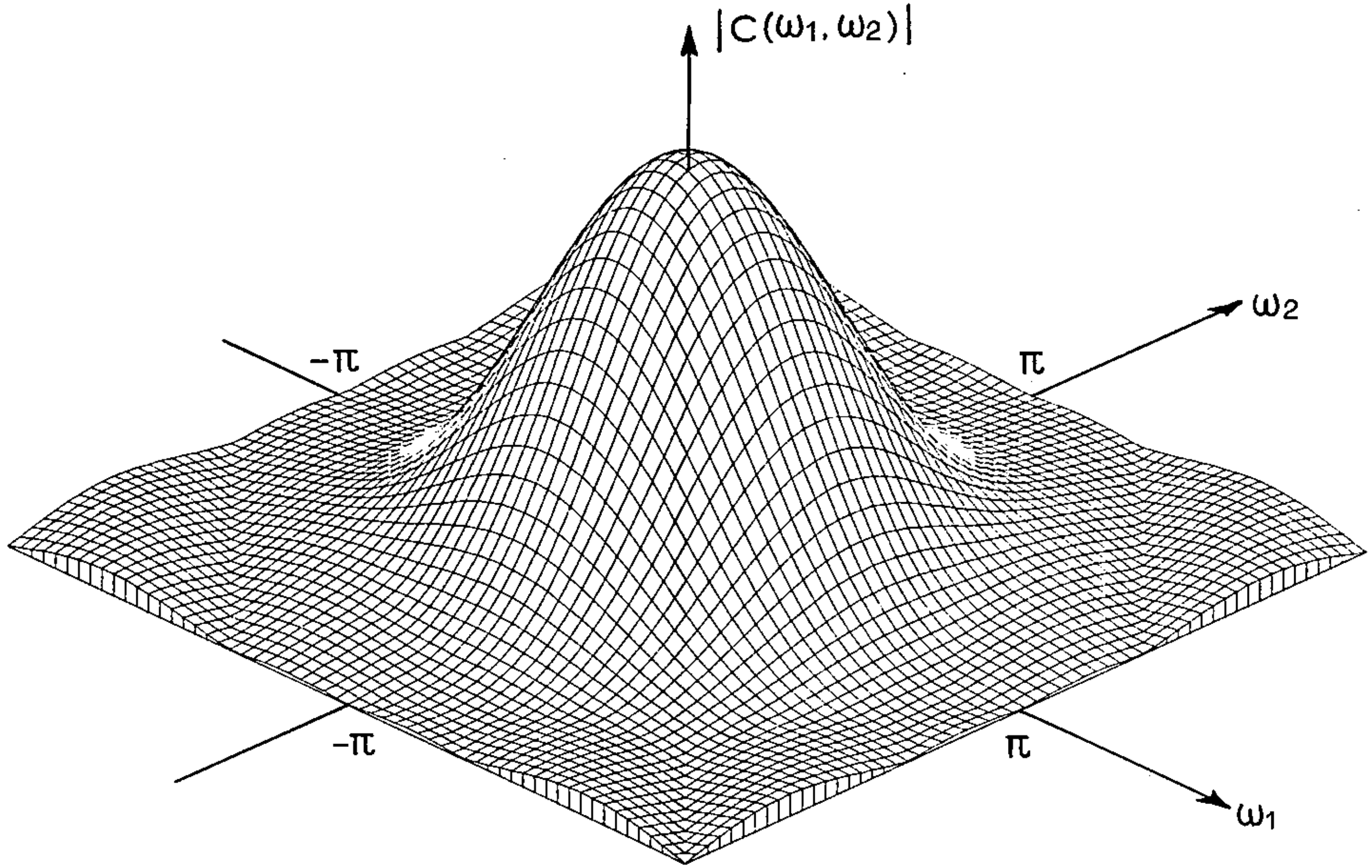


Fig. 4·13(a) Magnitude plot of the original filter

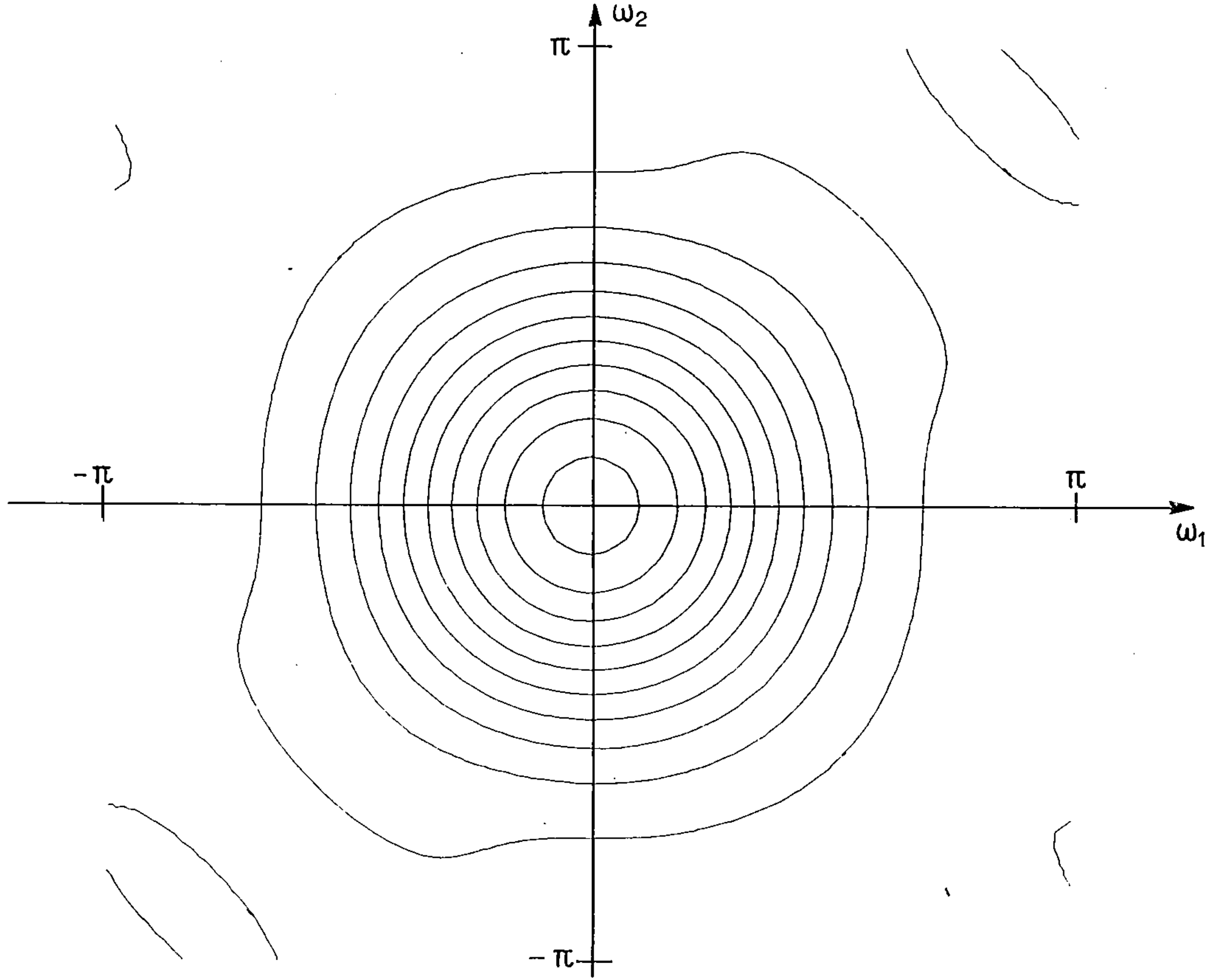


Fig. 4·13 (b) Contour plot of original filter

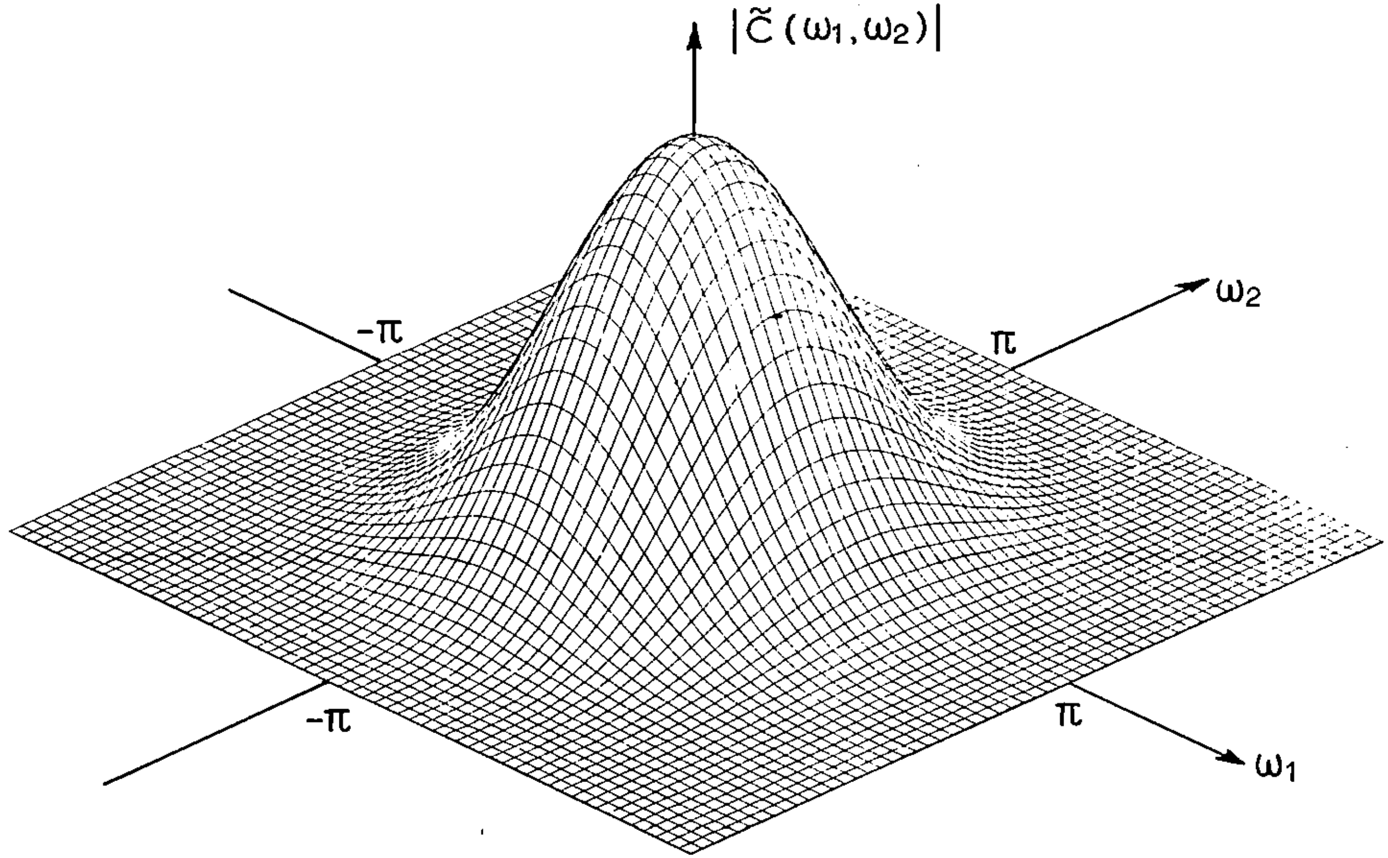


Fig. 4·14(a) Magnitude plot of stabilized filter.

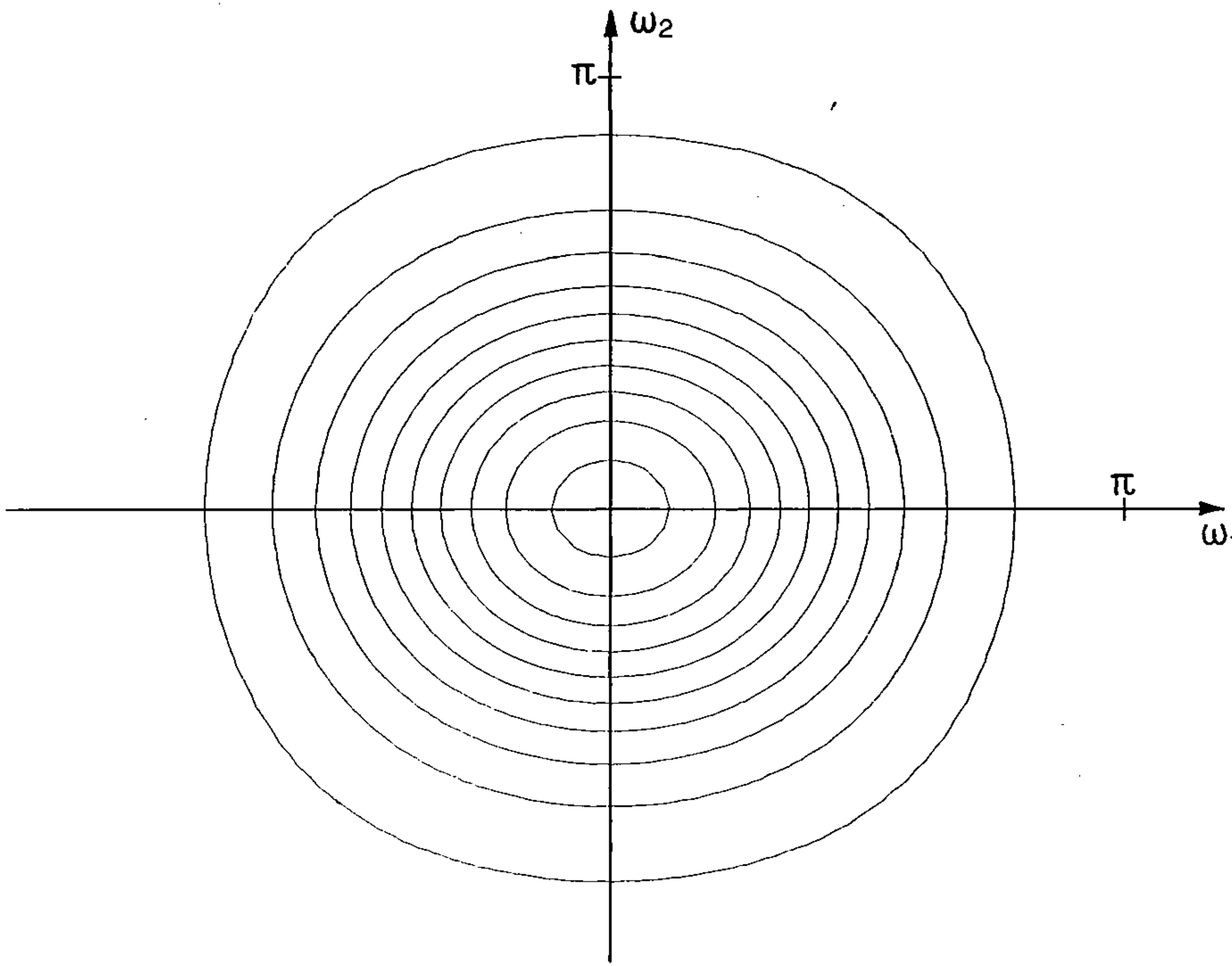


Fig. 4·14 (b) Contour plot of stabilized filter.

CHAPTER 5

DESIGN OF TWO-DIMENSIONAL RECURSIVE DIGITAL FILTERS

5.1 INTRODUCTION

Design of IIR or recursive two-dimensional filters can be divided into two categories; frequency domain approaches, and spatial domain techniques.

Because of the inherent difficulties of the multidimensional problem, namely the fact that the "fundamental theorem of algebra" does not apply, most basic questions of stability remain to be answered. At the present stage, it is not possible to use stability as a constraint in design procedures.

In the spatial domain approach, a filter transfer function is chosen to approximate a finite extent two-dimensional impulse response. The main drawback to this method is that there areno means of ensuring the stability of the output array when the true error is used as the criterion and that techniques for stabilization of a filter in the spatial domain are not available. However, it is still desirable to look briefly at spatial domain design techniques in order to appreciate the advantages of the frequency domain techniques.

The frequency domain design problem is based on an approximation to a frequency response characteristic at a discrete number of frequencies and use of an optimization technique to minimize the error between the desired multidimensional frequency response and the specified response. This is a problem of rational approximation. Most of these design methods can be grouped into two categories; those involving spectral transformations and those involving computer-aided optimization.

After discussing all existing design methods both in time and frequency domain, a new two-variable reactance function is proposed for designing two-dimensional recursive filters having circular symmetry and zero-phase. Next, a group of linear transformations is presented. It is shown that it is possible to obtain stable fan filters via the suggested transformations. Finally, a design technique for the fan and quadrant-fan filters with complex transformations is introduced.

5.2 SPATIAL DOMAIN DESIGN TECHNIQUES

The earliest work presented on this problem is due to Shanks et al [7]. They proposed a solution to the design problem by minimizing a false error function in order to obtain a recursive filter. Unfortunately, their recursive filter design technique, in general, does not lead to stable filters. To overcome this problem, the authors suggest using planar least square inverse (PLSI) approach to arrive at a stable approximation. In Section 4.4, we showed that Shanks' conjecture [7] is not valid under the new conditions introduced by Jury [112]. Furthermore, such an approach may lead to a filter whose impulse response does not adequately approximate the prescribed specifications.

Parker and Souchon [18] have used the Taylor series expansion to express the impulse response of a multidimensional filter in terms of the transfer function coefficients. Based upon these rules, by inverting the process, a synthesis procedure for obtaining a low-order recursive transfer function to approximate a given impulse response was presented. The stability of the filter after each successive approximation must be checked. However, this can not be done easily except for low-order two-dimensional sections where stability conditions are known.

Another notable contribution in spatial time domain designs is due to Nowrouzian et al [139]. They have reviewed and generalized four well-known design techniques proposed by Kalman [140], Shanks [141], Bertran [142], and Bodner [143]. All these methods are primarily least-square methods, with the first three techniques using an approximate measure of the error. It is also shown that the Bordner technique appears most desirable in that it is guaranteed to be stable whereas the other techniques require a concluding stability test.

Recently, Shaw and Mersereau have presented several modifications to the techniques suggested by Nowrouzian [144]. These modifications permit inclusion of an arbitrary error weighting function, design of filters with separable denominators, and improved convergence of the algortithm.

Lal [145] proposed a technique to remove the difficulty of Shanks' method [7]. Instead of attempting to obtain a transfer function which approximates to the complete specified impulse response and hence to optimize to a very high order transfer function, Lal partioned the desired impulse response into a number of smaller arrays. Each of these groups may be approximated to a relatively close degree such that simple constraints on the denominator will ensure stability.

Aly and Fahmy [146] proposed an l_p-optimization technique to design first quadrant and half-plane filters with specified impulse response. The filter is represented by its local state space model [96], rather than by its transfer function. The matrix derivative linear operator is used to calculate the performance measure gradient-vector.

5.3 FREQUENCY DOMAIN DESIGN TECHNIQUES

The problem of designing filters in the frequency domain involves determining a stable IIR transfer function which satisfactorily approximates a frequency domain specification. The required specification is usually given in the form of a two-dimensional magnitude spectrum. The approximating design is determined by using either any of spectral transformation approaches or computer-aided optimization techniques.

5.3.1 Spectral Transformations

In the spectral transformation approach, a two-dimensional digital filter is designed by using a transformation of complex variables from a one-dimensional or a two-dimensional prototype filter with known frequency response characteristics. In general, spectral transformations are complex maps which carry a stable transfer function into another stable transfer function and can be used to obtain two-dimensional filters with desirable characteristics.

The first technique for designing such filters was first advanced by Shanks et al [7] . The method consists of mapping one-dimensional filters into two-dimensional ones with arbitrary directivity in a two-dimensional frequency response. These filters are called rotated filters because they are obtained by rotating one-dimensional filters. There are two basic problems associated with this technique. First, the rotated two-dimensional filter may not be stable even though its one-dimensional prototype is stable. Second, the frequency response of the rotated version is not simply related to that of its parent filter because of the warping effects of the bilinear transformation on the frequency response. Therefore, it is difficult to design two-dimensional filter with prescribed frequency response characteristics.

A modification to the technique of Shanks has been introduced by Costa and Ventsanopoulos [8]. They attempted to design a near-circular symmetric filter by cascading a number of "Shanks' filters" having different angles of rotation. These angles of rotation are uniformly distributed over 180°. A stability criterion was developed which showed that angles of rotation of the designed filter from 0° to -90° resulted in stable filters. Hence the design technique could not achieve the required total span and the cut-off boundary was inevitably far from circular.

An alternative design procedure also involving the rotation of frequency responses has been presented by Chang and Aggarwal [147],[148]. The resulting filters are separable to the rotated frequency axes whereas Shanks' rotated filters are not. The transfer function is essentially a product of one-dimensional functions. The only difficulty with this technique is that filters are implemented by employing interpolated filter systems.

Ahmadi, Constantinides, and King [9] suggested a second order two-variable reactance function [159] for the design of circularly symmetric low-pass filters. The transformation is applied to a one-dimensional low-pass filter. The cut-off boundary of the filter depends on the choice of the cut-off frequency of the one-dimensional filter and the coefficients of the two-variable reactance function. Ahmadi's design approach has been used by several investigators [10] ,[32], [149] for the design of two -dimensional recursive filters.

The spectral transformation approach has also been used by Bernabo et al [30], [128]. This technique is based on the use of the transformation method of McClellan [150] followed by the decomposion method of Pistor [31] in order to obtain stable single quadrant filters which recurse in different directions.

5.3.2 Computer-Aided Optimization Methods

In the computer-aided optimization approaches, a nonlinear optimization procedure is used to adjust iteractively the filter coefficients to minimize a specified error criterion. A major cumbersome problem is ensuring the stability of the resultant two-dimensional transfer function.

Maria and Fahmy [11] developed an optimization algorithm to minimize the p-error criterion. In order to avoid the stability problem they designed the filters having transfer functions which are products of simple first and/or second-order terms. This facilitates testing stability of the approximate function at each step of the approximation as these low order terms can be tested using a simple set of inequalities associates with filter coefficients.

A differential correction optimization algorithm is used by Badner [13] for the approximation of a given frequency response. Stability is checked after each iteration of the optimization of a given frequency response using a stability testing algorithm similar to that outlined in Section 2.4.1. Because of the computational complexity of the stability test, this technique appears to be most appropriate for the design of low order filters. A similar approach was also used by Dudgeon [151]. Dudgeon used the discrete Hilbert transform to calculate the analytic phase function from the magnitude squared frequency response.

Alternative optimization approaches involve the design of a two-variable analog transfer function and the use of bilinear transformations to obtain the digital transfer function. Dubois and Blostein [152] have presented such an approach. They first design a two-variable passive analoge filter via computer-aided optimization technique, then obtain a two-dimensional recursive transfer function by applying a double bilinear transformation on the transfer function of the analog filter. The passivity (stability) is ensured by constraining the passive components always to have nonnegative values. A somewhat similar approach has been taken by Ramamoorthy and Bruton [14], [153],[154]. They express the denominator polynomial of a two-variable, analog transfer function in a suitable algebraic form which is always guaranteed to be realizable by a passive network thus ensuring stability. The design approach of Ramamoorthy and Bruton was later modified by Prasad and Reddy [155].

Recently, Ahmadi and Ramachandran [156] have presented a new method for the design of two-dimensional analog filters and recursive digital filters with guaranteed stability. Their method is based on the properties of positive definite matrices and their applications in generating very strictly Hurwitz polynomials. An unconstrained optimization technique is utilized to minimize some error function of the filter's magnitude response.

The two-dimensional stability testing problem can also be avoided by designing a separable two-dimensional recursive filter approximating the frequency response characteristic. For a separable filter, the two-dimensional transfer function can be expressed as the product of two one-dimensional transfer functions:

$$H(z_1,z_2) = H_1(z_1)H_2(z_2) \tag{5.1}$$

In this case, the stability testing reduces to that of checking the stability of one-dimensional filters, which is considerably simpler. Moreover, a separable filter is also more economical to implement. Twogood and Mitra [15], Abramatic et al [17],[157], and Antoniou and others [158] describe methods of designing separable filters. The approach of Twogood and Mitra makes use of the singular value decomposition of a matrix obtained by sampling the two-dimensional magnitude response matrix. The second method [17] is based on minimization of the mean-square error between the synthesized filter and a given prototype. The last technique develops the design technique of one-dimensional recursive filters due to Antoniou [175] to the design of separable two-dimensional filters.

5.4 A NEW TRANSFORMATION TECHNIQUE FOR THE DESIGN OF TWO-DIMENSIONAL STABLE RECURSIVE DIGITAL FILTERS

5.4.1 Introduction

A two-variable reactance function (2-RF) is a complex map that carries a one dimensional stable transfer function into another transfer function preserving some desirable characteristics in the two-dimensional frequency domain; if the 2-RF is also a stable function, the transformed two-dimensional function is also assured of stability. The design of two-dimensional recursive digital filters via 2-RF has been investigated in references [7],[8]-[10], [32], and [160].

We present a novel 2-RF transformation for the design of two-dimensional stable digital filters having zero phase. By optimising the parameters of the proposed 2-RF, it is possible to obtain circular symmetry over a wide range of cut-off frequencies. The 2-RF will also maintain some of the characteristics of the one-dimensional analogue prototype.

5.4.2 The two-variable reactance function

The most general form of two-variable reactance function is of the form

$$T(s_1, s_2) = \frac{\sum_{k=0}^{N} \sum_{j=0}^{2k+1} a_{jk} s_1^{j} s_2^{2k-j+1}}{\sum_{k=0}^{M} \sum_{j=0}^{2k} b_{jk} s_1^{j} s_2^{2k-j}} \qquad (5.2)$$

where M=N <u>or</u> N + 1

It may be noted that the simplest member of the transformation is the McClellan transformation, whose stability is simply determined. The Ahmadi transformation is a degenerate form of the second member of the general transformation (5.2) in which the stability is again assured if the coefficients are all positive.

The next higher member is of a second order namely:

$$T_2(s_1,s_2) = \frac{a_1 s_1 + a_2 s_2}{1 + b_1(s_1^2 + s_2^2) + b_2 s_1 s_2} \tag{5.3}$$

The constraints on the parameters a_1, a_2, b_1, b_2 to ensure stability of T_2 are (see Appendix C).

$$b_2 > 0 \tag{5.4}$$

$$b_1 > \frac{b_2^2}{4} - b_1^2 > 0 \tag{5.5}$$

5.4.3 Design Procedure

The design procedure involves choosing a one dimensional prototype, Butterworth, Chebyshev, etc. and applying the transformation $s = T_2(s_1,s_2)$ to obtain a two-dimensional recursive filter. Each filter must now be cascaded with a guard filter as the transformation function (5.3) does not preserve the filter response in all radial directions. The resulting one quadrant filter is finally cascaded with four similar single quadrant filter recursing in the remaining three cardinal directions.

The final transfer function $H(z_1,z_2)$ is thus given by:

$$H(z_1,z_2) = F(z_1,z_2)F(z_1,z_2^{-1})F(z_1^{-1},z_2)F(z_1^{-1},z_2^{-1}) \qquad (5.6)$$

where

$$F(z_1,z_2) = F_0 B(z_1,z_2)G(z_1,z_2) \qquad (5.7)$$

$B(z_1,z_2)$ is a two-dimensional function obtained from the one-dimensional prototype $B(s)$ via the transformation $s = T_2(s_1,s_2)$ and converted to the digital domain by means of the bilinear transformation:

$$s_i = k_i \frac{1-z_i}{1+z_i} \ ; \quad z_i = \exp(-s_i T_i) \ ; \quad i = 1,2 \qquad (5.8)$$

$G(z_1,z_2)$ is the transfer function of a guard filter which is used to eliminate the high pass regions along all radii except the coordinate axes. In its simplest, and adequate form it may be written

$$G(z_1,z_2) = \frac{(1+z_1)(1+z_2)}{(d_1+z_1)(d_2+z_2)} \qquad (5.9)$$

with real coefficients d_1 and d_2 so chosen as to attenuate the amplitude characteristics at high frequencies [160].

F_0 is a design parameter to correct the amplitude at the origin.

Optimisation

Although the optimisation technique is constrained by conditions (5.4) it may be rendered an unconstrained problem by modification of the variables. The unconstrained optimization algorithm due to Gill and Murray [161] was used to optimize the parameters a_1, a_2, b_1, b_2, k_1, k_2, d_1, d_2, F_0.

5.4.4 Example

As an example we design a circularly symmetric filter based on a third order Butterworth filter

$$B(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{5.10}$$

The parameters obtained after optimization are:

$a_1 = 0.96572$

$a_2 = 1.00363$

$b_1 = 0.1089$

$b_2 = 0.29020$

$k_1 = 0.80112$

$k_2 = 0.77385$

$d_1 = 11.27362$

$d_2 = 11.25402$

$F_0 = 35.2600$

The frequency response and contour plot are shown in Figs. 5.1 and 5.2. The cut-off, 3dB, frequency is at f_{-3} = 0.95 radians and the stop frequency of 60dB is at f_{-60} = 1.885 radians.

The error at the end of the optimization is of the order 10^{-5}. The resulting two-dimensional recursive function is inherently quadrantally stable and has approximately the same amplitude performance as the one dimensional prototype.

5.5 A SIMPLE DESIGN TECHNIQUE FOR STABLE FAN FILTERS

In this section, a group of linear transformations is presented. It is shown that it is possible to obtain stable fan filters via suggested transformations. The Ahmadi filter [133], is found to be useful in the design of fan filters due to the diamond shaped profile.

5.5.1 Single Variable Transformations

We considered the lowpass to highpass transformations developed by Constantinides [173],

$$Z_1 = -\frac{a_1 + z_1}{1 + a_1 z_1} \quad , \qquad Z_2 = -\frac{a_2 + z_2}{1 + a_2 z_2} \tag{5.11}$$

where a_1 and a_2 are real parameters.
When a_1 and a_2 are both zero, Z_1 and Z_2 becomes:

$$Z_1 = -z_1 \; , \qquad Z_2 = -z_2 \tag{5.12}$$

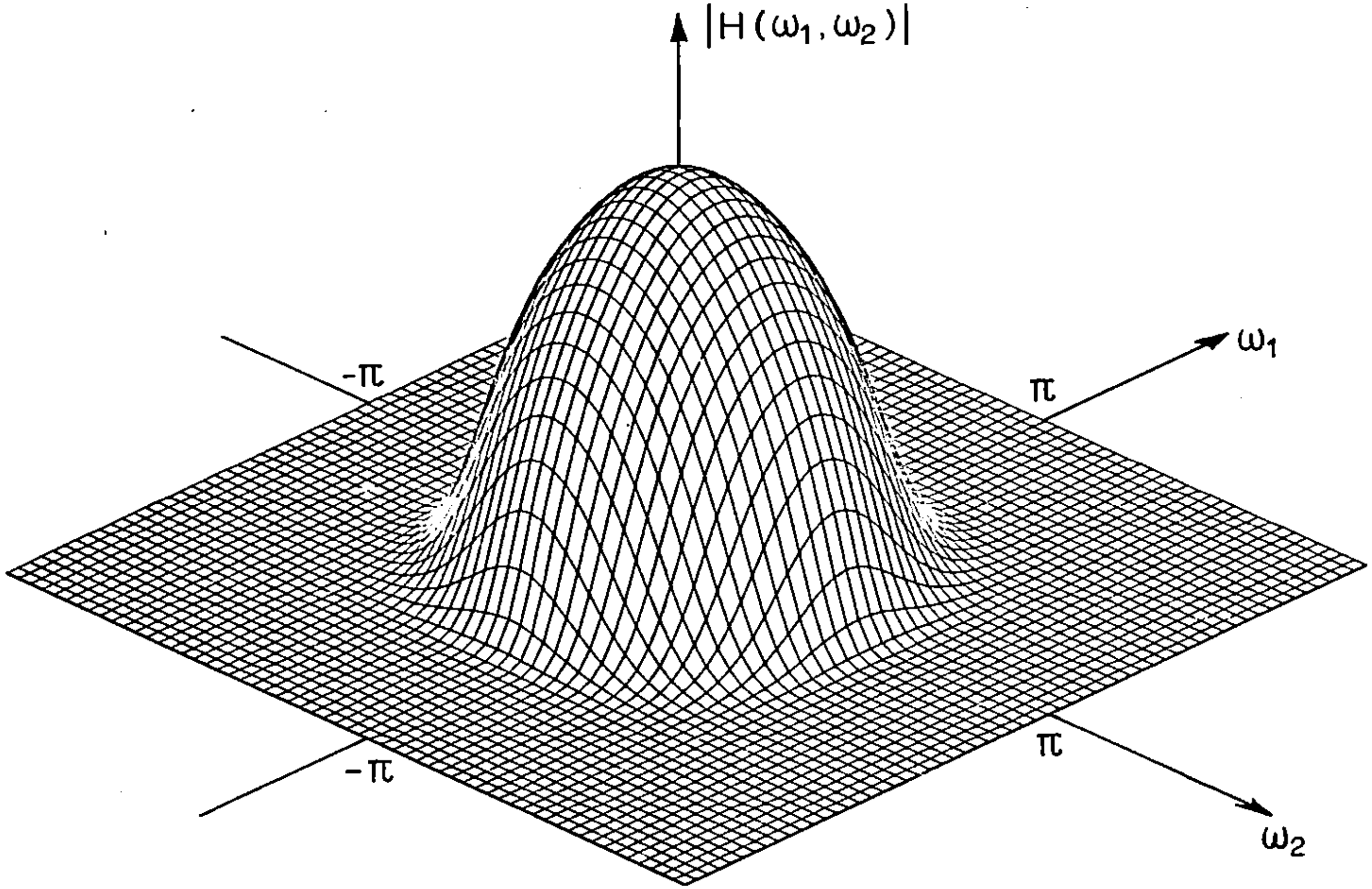


Fig. 5·1 Amplitude response of 2-dimensional recursive filter, using transformation of $T_2(s_1, s_2)$ in eqn. 5·3

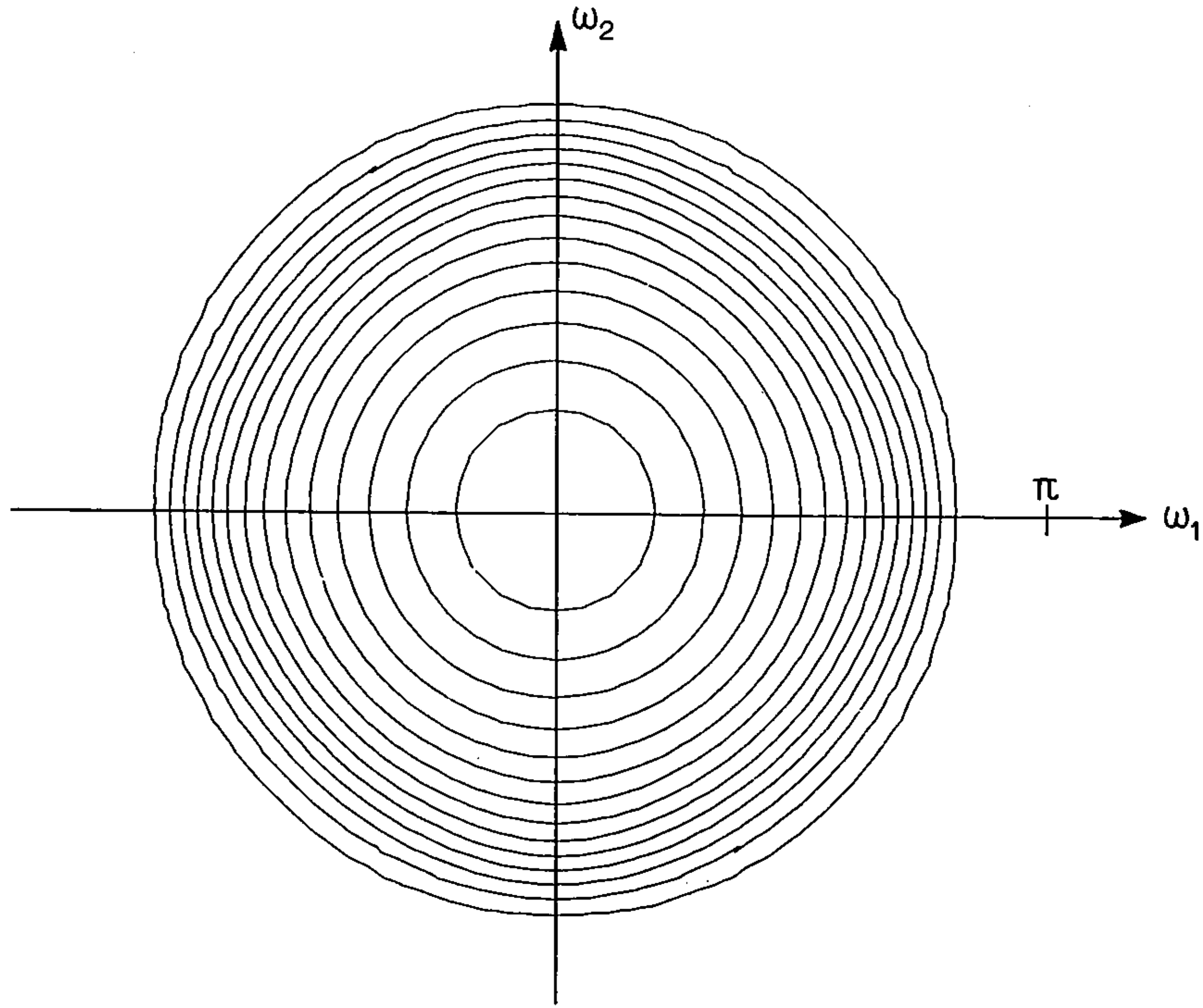


Fig. 5·2 Contour plot of response in Fig. 5·1

However, both Z_1 and Z_2 may not be equal to their negatives simultaneously. Hence three different types of single variable transformations can be defined as:

Transformation Type-I : $Z_1 = - z_1$, $Z_2 = + z_2$

Transformation Type-II : $Z_1 = + z_1$, $Z_2 = - z_2$

Transformation Type-III : $Z_1 = - z_1$, $Z_2 = - z_2$

The effect of these transformations can be seen in Fig.5.3.

a) $H(\omega_1, \omega_2)$ Identity

b) $H(\omega_1 - \pi, \omega_2)$ Type-I

c) $H(\omega_1, \omega_2 - \pi)$ Type-II

d) $H(\omega_1 - \pi, \omega_2 - \pi)$ Type-III

One may say that the type-I transformation and Type-II transformation replace the right half-plane with the left half-plane and the upper half-plane with the lower half-plane, respectively. The transformation type-III replaces the first quadrant and the second quadrant with the third and fourth ones, respectively.

5.5.2 Design of Fan Filters via Ahmadi Transformation [133]

In 1976, Ahmadi et al [9] proposed a simple two-variable reactance function,

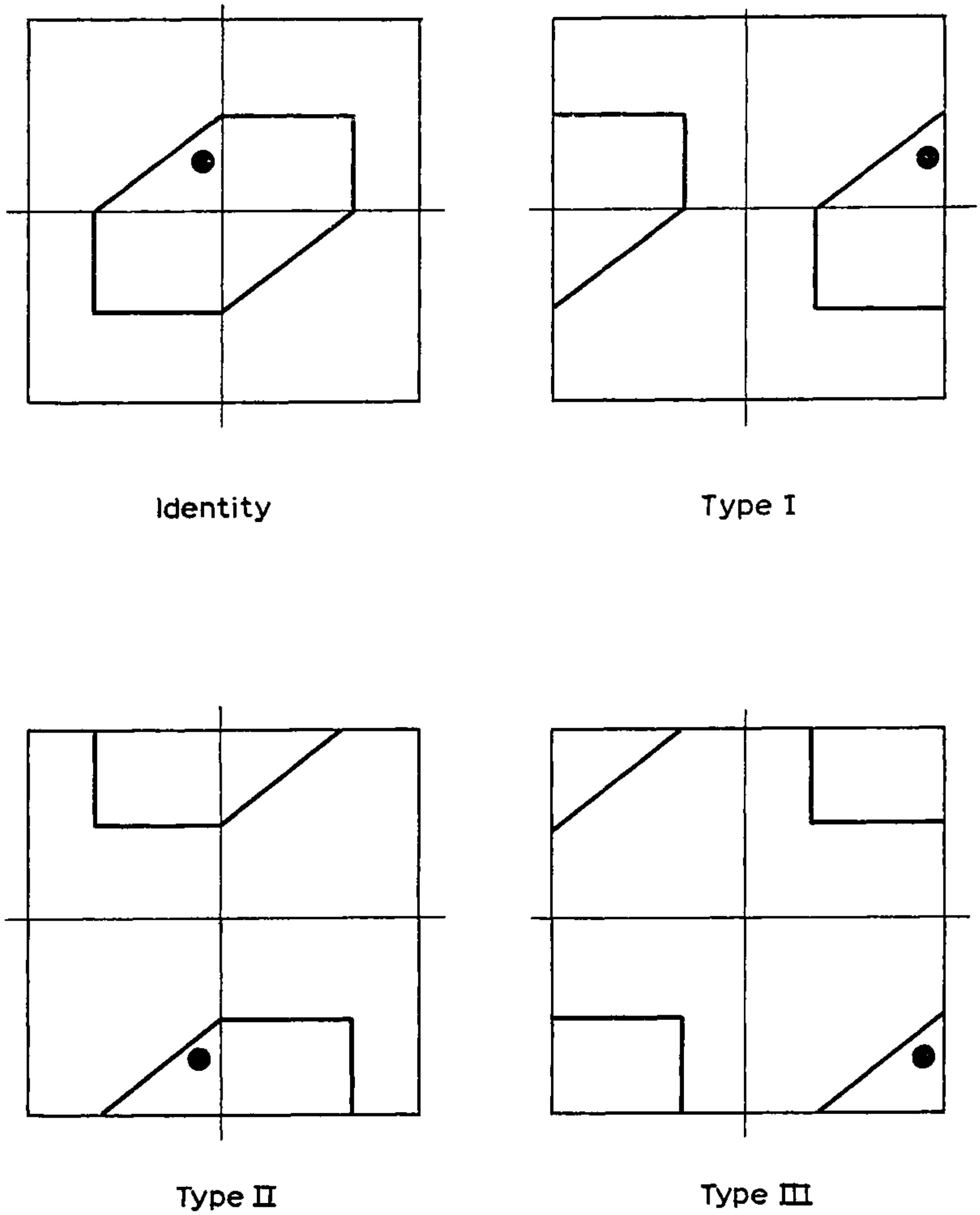


Fig. 5·3 Single variable transformations

$$s = \frac{a_1 s_1 + a_2 s_2}{1 + b s_1 s_2} \qquad (5.13)$$

This was used to transform a given one-dimensional lowpass filter function to a two-dimensional continuous lowpass function. The parameters a_1 and a_2 are frequency scaling factors, and the parameter b controls the shape of the cutoff boundary.

Since the Ahmadi transformation is not globally type preserving [10], a guard filter was used to remove highpass regions [160]. The parameters a_1 and a_2 will provide flexibility to obtaining a cutoff profile which is diamond shaped. After obtaining this diamond shaped filter, we may apply the transformation type-I or type-II to obtain fan filters. Since these transformations do not change the location of singularities, the resulting fan filters are stable.

5.5.3 Example

The fifth order Butterworth analog filter was used with transformation parameters $a_1 = 0.1$, $a_2 = 2.0$, and $b = 0.6$. The magnitude response and its contour plots can be seen in Fig. 5.4 and 5.5, respectively.

After designing the above filter, one may apply the transformation type-I to obtain the fan filter, Fig. 5.6 and 5.7 show the magnitude response and contour plots of the obtained fan filter.

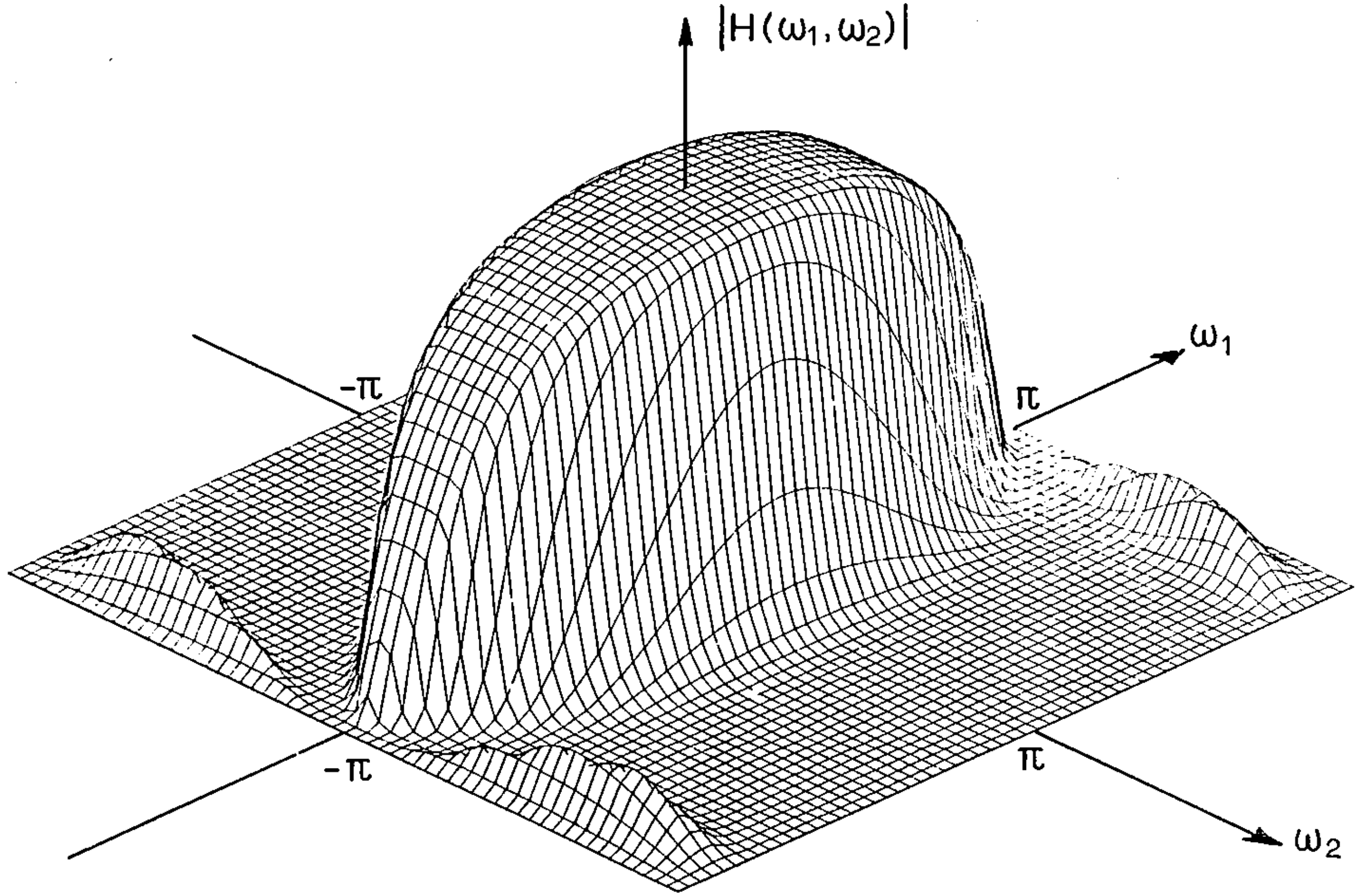


Fig. 5·4 Amplitude response of the filter obtained from a third order Butterworth filter.

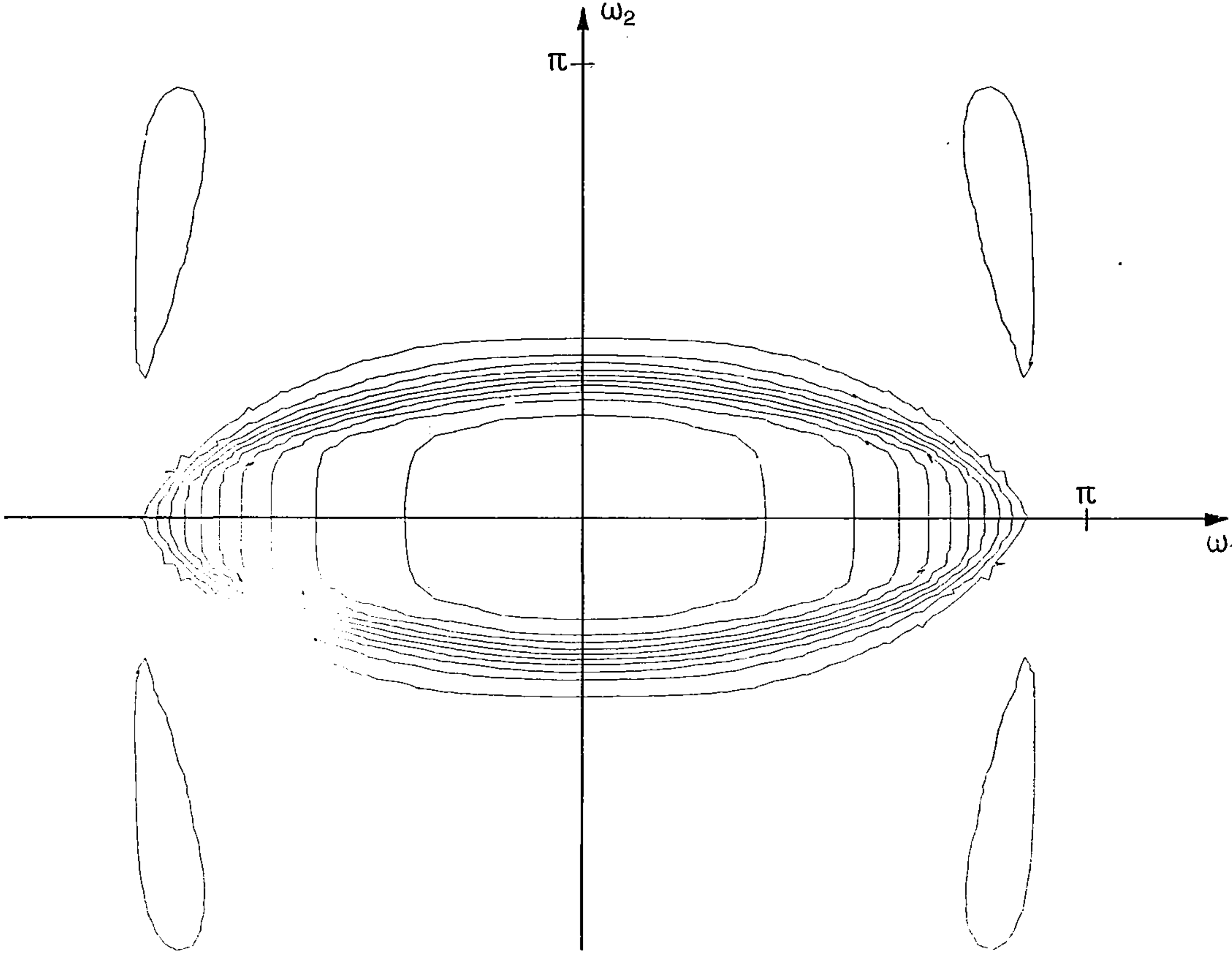


Fig. 5·5 Contour plot of response in Fig. 5·4

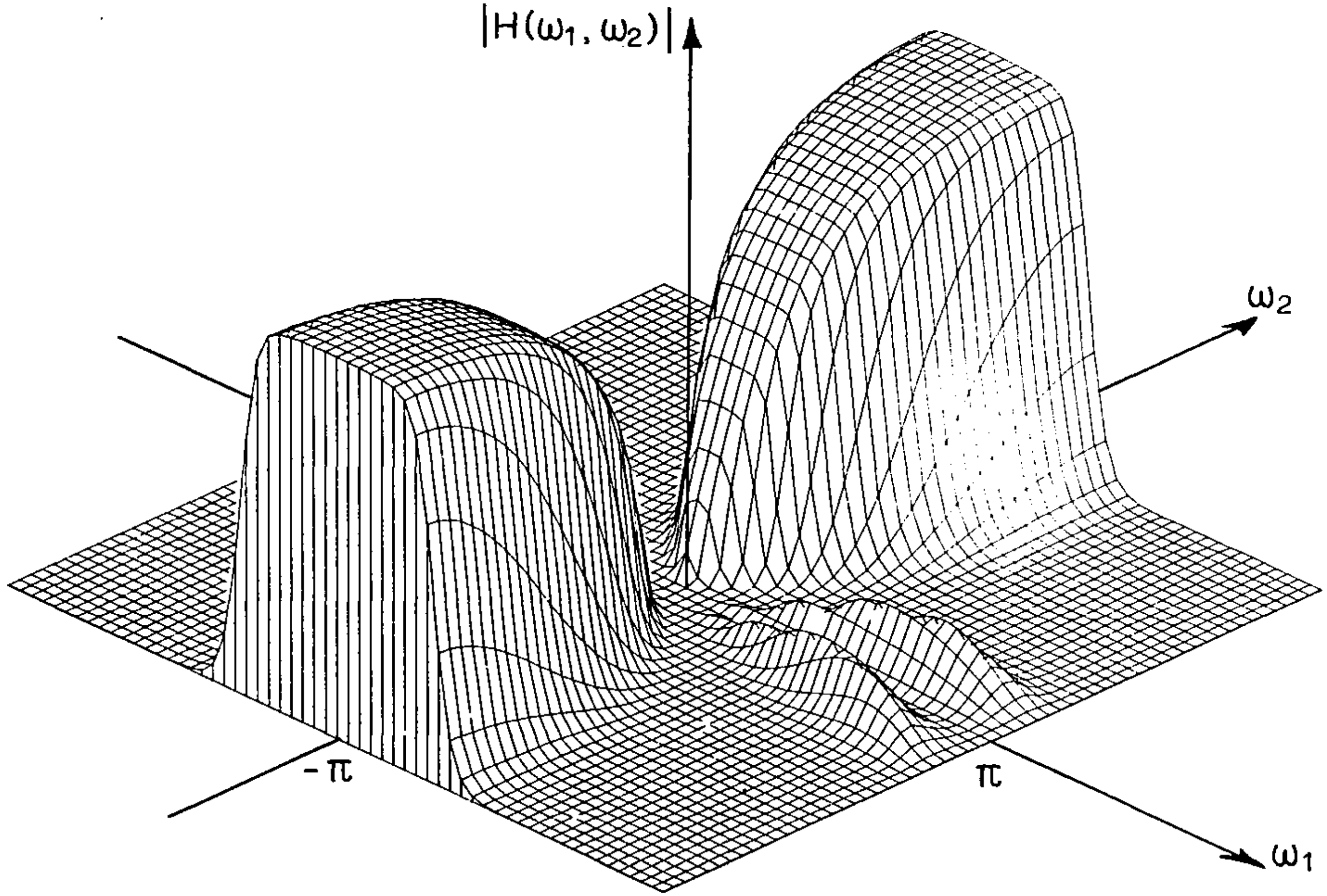


Fig. 5·6 Amplitude response of the filter after the transformation type I.

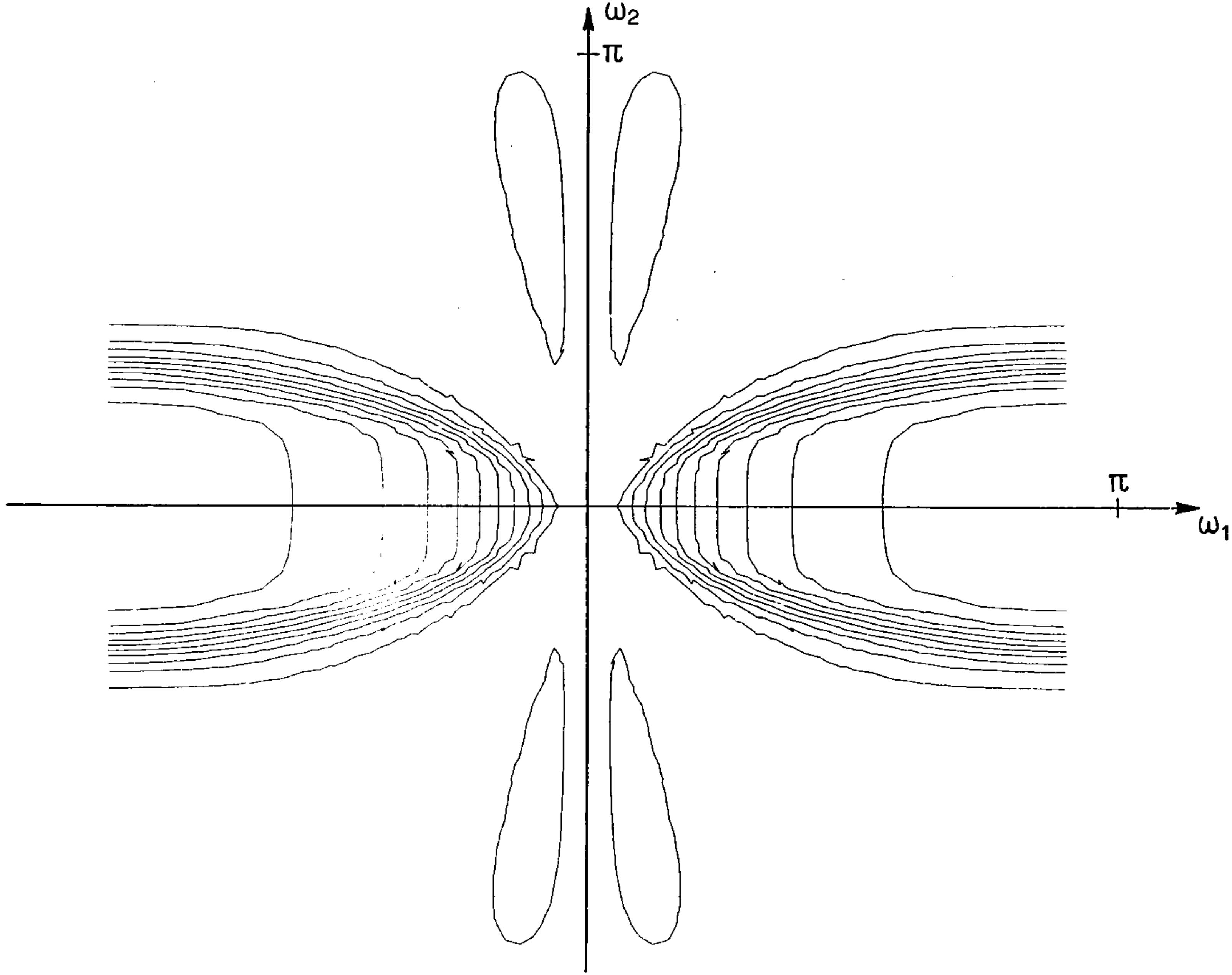


Fig. 5·7 Contour plot of the filter in Fig. 5·6

5.6 DESIGN OF RECURSIVE AND NON-RECURSIVE FILTERS WITH COMPLEX TRANSFORMATIONS

5.6.1 Introduction

A simple design technique for fan filters is presented using complex transformations. The only requirement of the present technique is to select a one-dimensional digital filter with a suitable cut-off frequency. The unique advantage of this method is that it is applicable for both FIR and IIR filters. As the resulting IIR fan filters are inherently stable, the proposed method requires neither any stabilization procedure nor a stability test. On the other hand, since one-dimensional prototype characteristics are preserved, an optimization is not needed for optimality. The design complexity is less than all existing ones [117], [150], and [162]-[169].

5.6.2 Complex Transformations

The frequency transformation techniques have been extended to two-dimensional filters by Pendergrass et al [172]. They consider only those that generate real two-dimensional network functions from real one-dimensional functions. The problem is to find the transformation

$$G:\ C^2 \rightarrow C^2$$

with

$$G: \begin{cases} z_1 \longrightarrow G_1(z_1, z_2) \\ \\ z_2 \longrightarrow G_2(z_1, z_2) \end{cases} \tag{5.14}$$

$$H(z_1,z_2) \rightarrow H(G_1(z_1,z_2),\ G_2(z_1,z_2))$$

$$= \frac{A(z_1,z_2)}{B(z_1,z_2)} \tag{5.15}$$

The frequency transformation should satisfy the following conditions. The transformation (5.14) must

(1) Produce stable first quadrant (causal) transfer functions from stable first quadrant functions (the original has to be stable).

(2) Transform real rational functions; G_1 and G_2 are real rational functions.

(3) Preserve some important basic characteristics of the amplitude response (such as ripple magnitude in the pass and stop regions) while altering other characteristics (such as cut-off frequencies or the number and shape of stop and pass regions).

Now we will consider a transformation which represents a complex function of the two-dimensional variables with rational powers and show that the limitations imposed by Pendergrass et al [172] on useful transformation functions are by no means mandatory.

For a causal and stable two-dimensional filter, described as in (2.1) by:

$$H_2(z_1,z_2) \rightarrow H_1(z_1) \tag{5.16}$$

consider the following transformation

$$z_1 \longrightarrow e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2} \tag{5.17}$$

The corresponding frequency transformation is:

$$\exp[j\omega_1] \longrightarrow \exp[j\phi].\exp\left[j\ \frac{\alpha_1}{\beta_1}\ \omega_1\right].\exp\left[j\ \frac{\alpha_2}{\beta_2}\ \omega_2\right]$$

$$= \exp\left[j(\phi + \frac{\alpha_1}{\beta_1}\ \omega_1 + \frac{\alpha_2}{\beta_2}\ \omega_2)\right]$$

or

$$\omega_1 \longrightarrow \phi + \frac{\alpha_1}{\beta_1}\ \omega_1 + \frac{\alpha_2}{\beta_2}\ \omega_2 \tag{5.18}$$

The amplitude and contour plots of the frequency response after transformations $\alpha_1/\beta_1 = \frac{1}{2}$, $\alpha_2/\beta_2 = \frac{1}{2}$, and $\phi = 90^{\circ}$ are shown in Fig. 5.9(a) and 5.9(b), respectively. The original filter (5.16) represents a low-pass filter with $\omega_c = \frac{1}{2}\pi$ as shown in Fig. 5.8.

There are three effects of transformation (5.17) on the resulting filter

$$H(e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2}) = H_1(z_1)\Big|_{z_1 = e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2}} \tag{5.19}$$

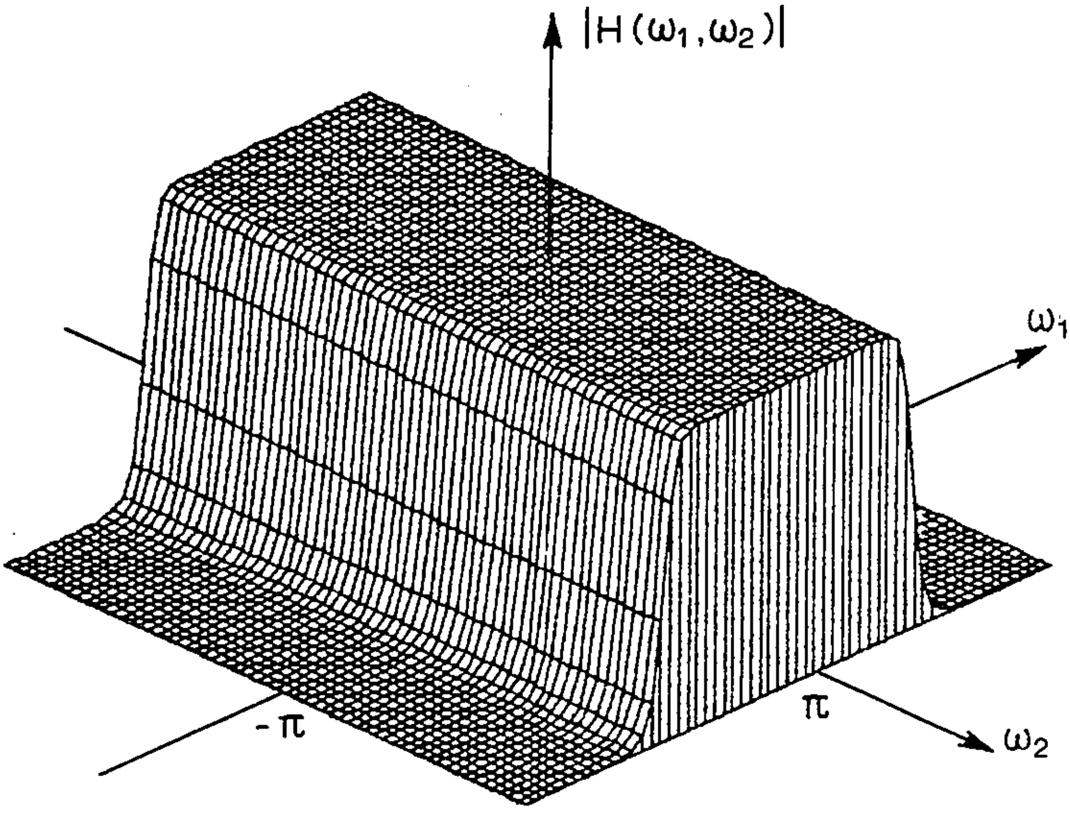


Fig. 5·8 Original low-pass amplitude response (tenth order Butterworth)

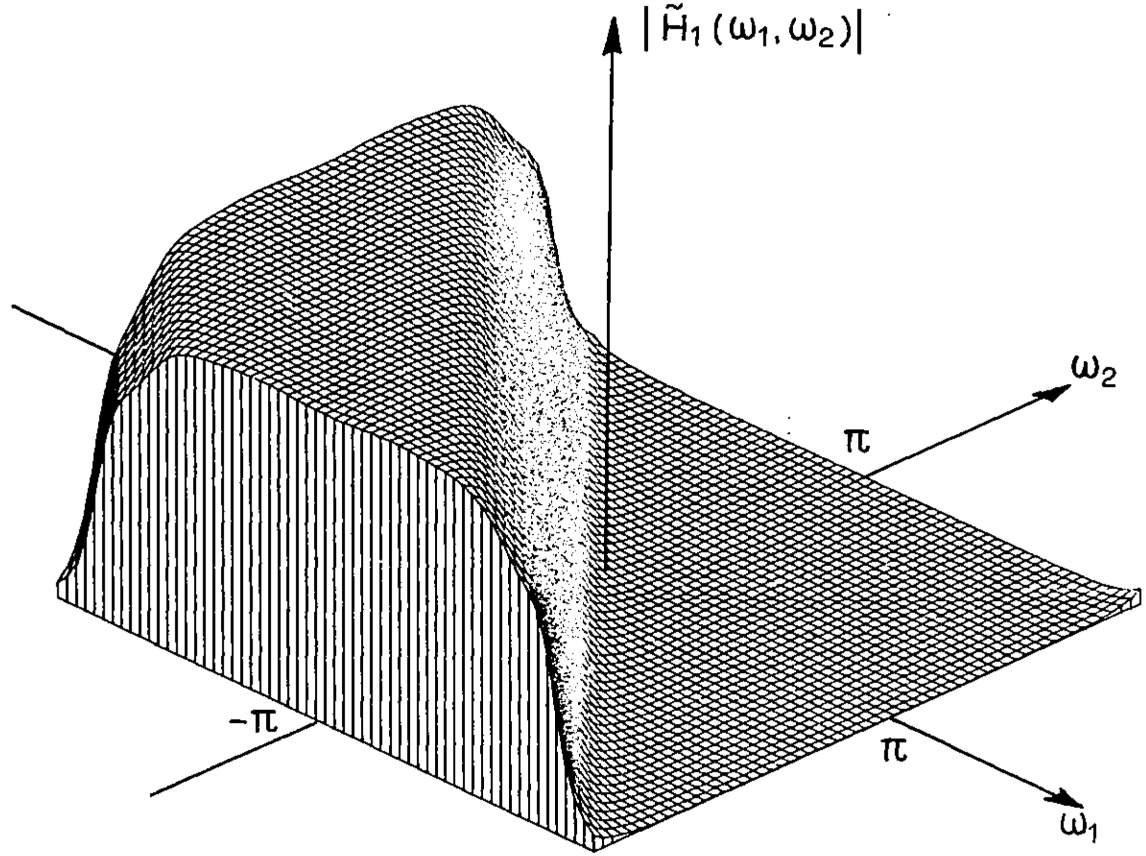


Fig. 5·9 (a) Frequency response of the filter after the complex transformation

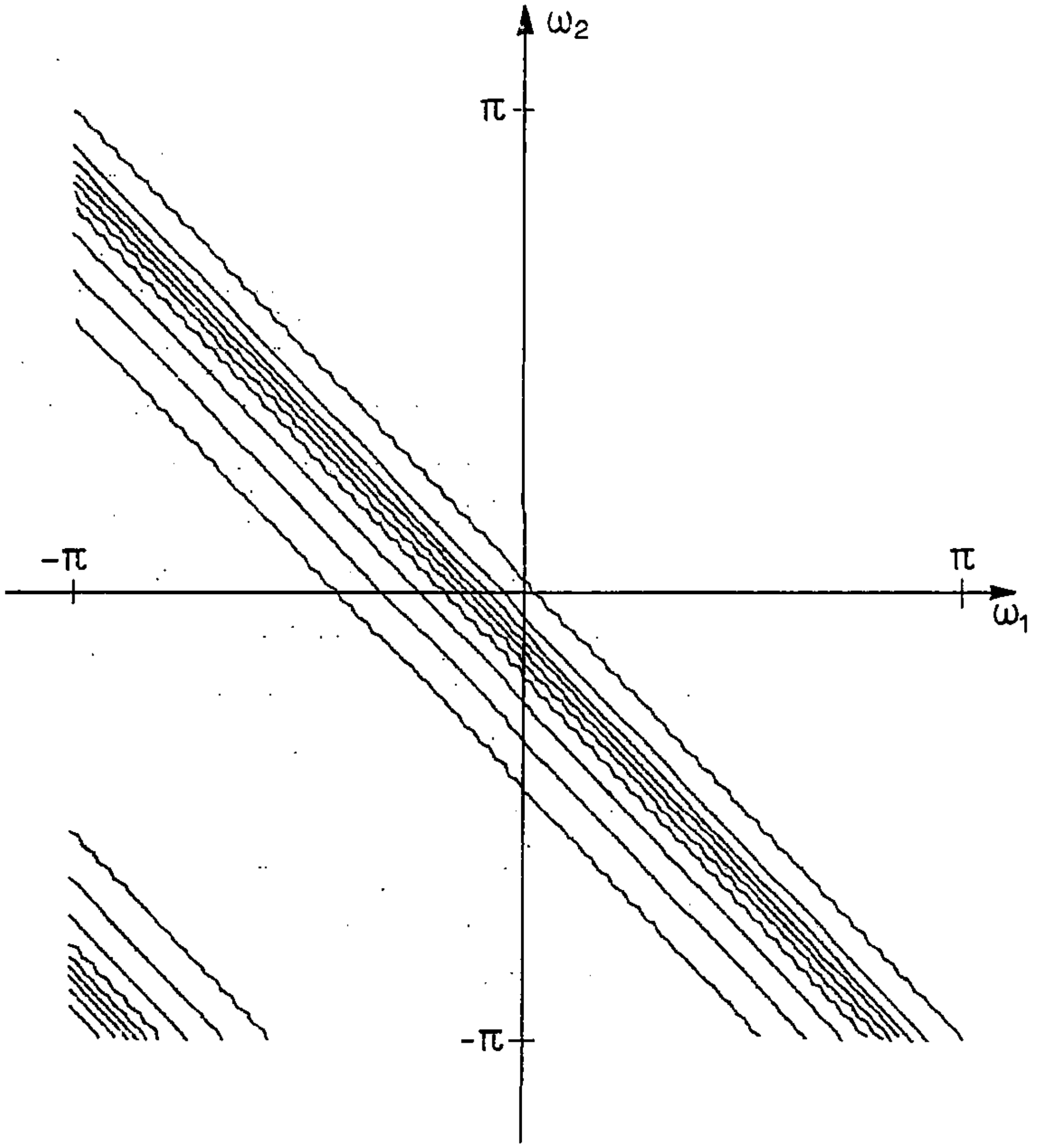


Fig. 5·9 (b) Contours of the frequency response of the filter shown in Fig. 5·9 (a)

(a) Frequency shifting along ω_1-axis: the frequency response of the resulting filter will be shifted by ϕ along ω_1-axis. The choice of $\phi = \pm \frac{\pi}{2}$ will lead to a symmetric fan filter.

(b) Rotation of the frequency response: the angle of rotation is:

$$\theta = \text{arc tan}\left(\frac{\alpha_2}{\beta_2}\right) \tag{5.20}$$

Since the original filter (5.16) is one-dimensional and function of z_1, the angle of rotation will be defined by the rational power of z_2. More detailed discussion of the rotation in (z_1,z_2) -plane can be found in [148].

(c) Scaling the frequency response along ω_1-axis: the rational power of z_1 will scale the frequency response by a factor β_1/α_1. However, the periodicity of the frequency response will be $\frac{\alpha_1}{\beta_1} 2\pi$ instead of 2π. The choice of $\frac{\alpha_1}{\beta_1} = \frac{1}{2}$ will lead to a fan symmetry. This will be discussed in the following section.

The other effects of the transformation on the resulting filter may be specified as follows:

(1) When $\frac{\alpha_1}{\beta_1} > 0$ and $\frac{\alpha_2}{\beta_2} > 0$, the transformation is causal, otherwise it is non-causal. However, the transformation $\frac{\alpha_1}{\beta_1} < 0$ or (and) $\frac{\alpha_2}{\beta_2} < 0$ may be implemented in the spatial time domain. In Section II(c), it is shown that for a finite array, one can choose a causal recursion direction by reorientating the input signal array.

(2) When $\phi \neq 0$, the resulting filter cannot be implemented in the spatial time domain. On the other hand, it will be shown that transformed filters may be combined in appropriate ways so that complex values do not exist in the final transfer function.

(3) Both rotation and frequency scaling are equivalent to the rotation of recursion direction with a new sampling interval. However, for the design of fan symmetric filters, it is found that their transfer functions have complex variables with integer powers. Hence the sampling interval will not change. In the following section this fact is proved.

(4) The stability of the resulting filter is not affected with $\frac{\alpha_1}{\beta_1} > 0$ and $\frac{\alpha_2}{\beta_2} > 0$. For $\frac{\alpha_1}{\beta_1} < 0$ or $\frac{\alpha_2}{\beta_2} < 0$ the transformed filter will be unstable for the causal recursion direction. However, if the orientation of the finite area array (input) is changed, there is always a non-causal recursion direction where the filter function is stable. During the implementation step, this fact will be explained in greater detail.

<u>Remark 5.1:</u>

The terms rotation and frequency scaling for the two-dimensional case can be used interchangeably. If one considers $\theta = \arc\tan\left(\frac{\alpha_1}{\beta_1}\right)$ as the rotation of the frequency response, then $\frac{\alpha_2}{\beta_2}$ will determine the scaling of frequency on the ω_2-axis. In this work, the causal, stable prototype function (5.16) is $H(z_1,z_2) = H_1(z_1)$.

5.6.3 Design of Fan Filters with Complex Transformations

A. Symmetric Fan Filters

In this section, a design procedure for fan filters is outlined by using the proposed complex transformation (5.17). For this design, the ideal fan filter specification is shown in Fig. 5.10. The specification is:

$$H_f(e^{j\omega_1},e^{j\omega_2}) = \begin{cases} 1 & |\omega_1| \geq |\omega_2| \\ 0 & \text{otherwise} \end{cases} \tag{5.21}$$

Consider the ideal prototype filter shown in Fig. 5.8. It is a low-pass filter with a cut-off frequency at $\omega_c = \pi/2$.

$$H(z_1,z_2) = H_1(z_1)$$

on replacing

$$z_1 = e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2}$$

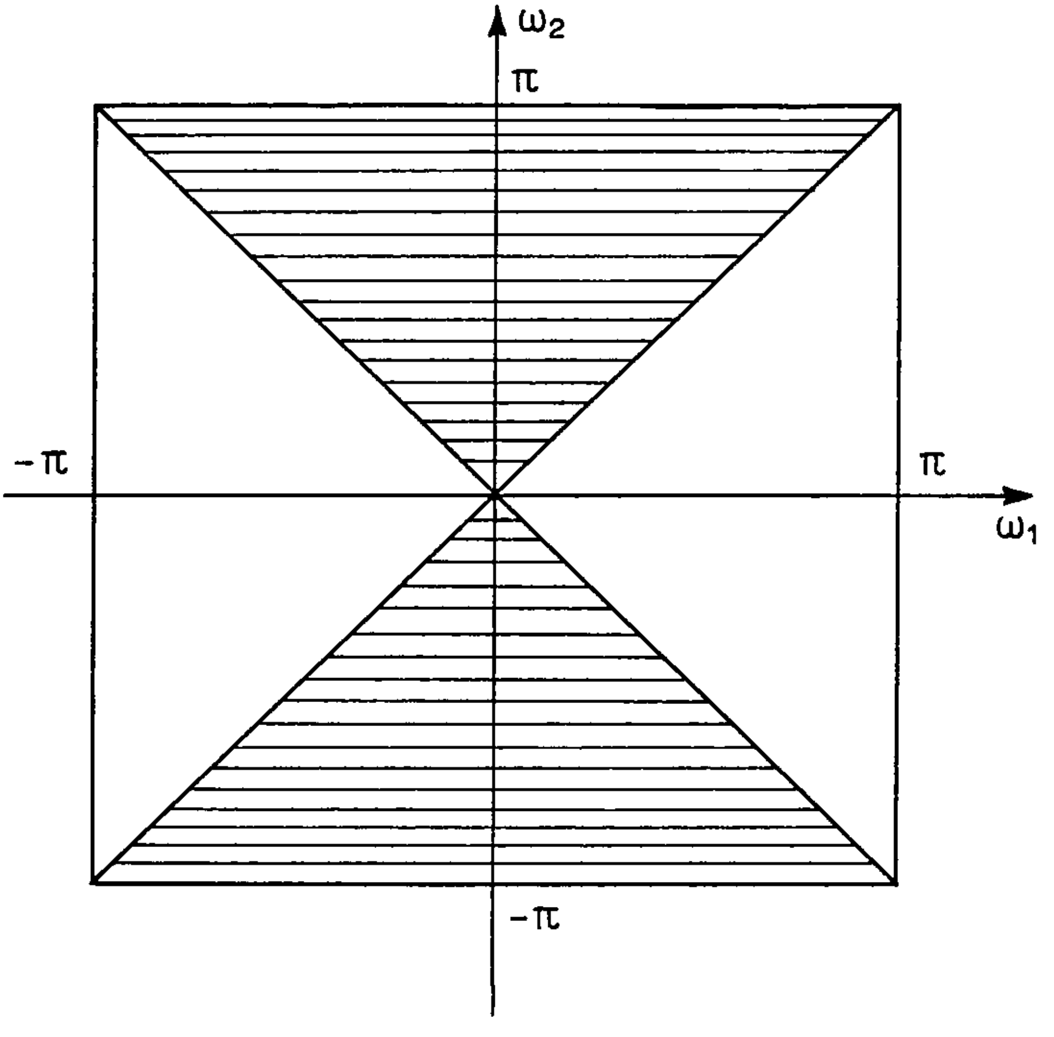


Fig. 5·10 Pass and stop region of an ideal symmetrical fan filter in the (ω_1, ω_2)-plane.

in $H(z_1,z_2)$, one obtains the shifted, scaled and rotated characteristics in the frequency domain. Let us denote the transformed filter by

$\tilde{H}(z_1,z_2, \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \phi)$, where

$$\tilde{H}(z_I,z_2, \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \phi) = H_1(z_1)\Bigg|_{z_1= e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2}} \tag{5.22}$$

In general, the filter coefficients in H will be complex and the variables z_1 and z_2 have rational non-integral powers. However, appropriate combinations of transformed filters will remove both these difficulties.

Let

$$\tilde{H}_1(z_1,z_2) = \tilde{H}(z_1,z_2, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{\pi}{2})$$

$$\tilde{H}_2(z_1,z_2) = \tilde{H}(z_1,z_2,-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{\pi}{2})$$

$$\tilde{H}_3(z_1,z_2) = \tilde{H}(z_1,z_2, \tfrac{1}{2},-\tfrac{1}{2}, \tfrac{\pi}{2})$$

$$\tilde{H}_4(z_1,z_2) = \tilde{H}(z_1,z_2,-\tfrac{1}{2},-\tfrac{1}{2}, \tfrac{\pi}{2}) \tag{5.23}$$

The frequency responses of transformed filters, $\tilde{H}_i(z_1,z_2)$, $i = 1,2,3,4$, can be found in Table 5.1. The original prototype filter is an ideal low-pass filter as shown in Fig. 5.8.

In this design procedure, filters in Table I will be used as main building blocks for fan filter with a specification in (5.21). One can construct the following filter characteristics:

$$H_{11}(z_1,z_2) = \tilde{H}_1(z_1,z_2)\tilde{H}_2(z_1,z_2)\tilde{H}_3^*(z_1,z_2)\tilde{H}_4^*(z_1,z_2) + \tilde{H}_1^*(z_1,z_2)\tilde{H}_2^*(z_1,z_2)\tilde{H}_3(z_1,z_2)\tilde{H}_4(z_1,z_2) \tag{5.24a}$$

and

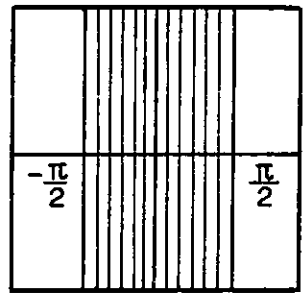
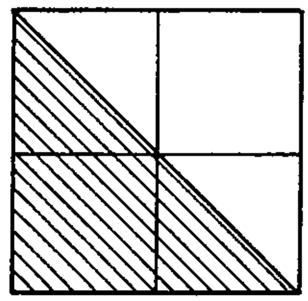
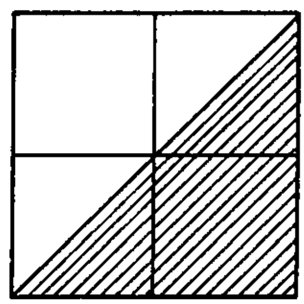
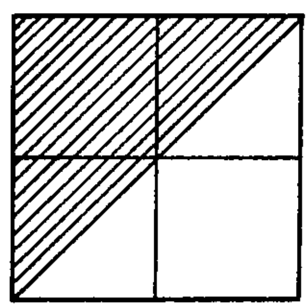
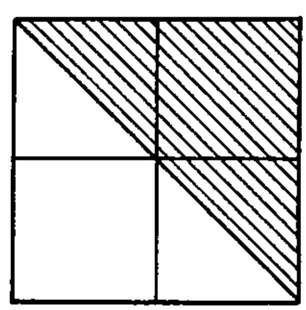
$$H_{22}(z_1,z_2) = \tilde{H}_1(z_1,z_2)\tilde{H}_3(z_1,z_2)\tilde{H}_2^*(z_1,z_2)\tilde{H}_4^*(z_1,z_2) + \tilde{H}_1^*(z_1,z_2)\tilde{H}_3^*(z_1,z_2)\tilde{H}_2(z_1,z_2)\tilde{H}_4(z_1,z_2) \tag{5.24b}$$

where $\tilde{H}_1^*$ denotes the filter with coefficients which are the complex conjugate of those of $\tilde{H}_1$, and so on.

The frequency characteristics of the obtained filter, H_{11} corresponds to the fan filter shown in Fig.5.10. H_{22} has a frequency characteristic which is a clockwise 90° rotated version of H_{11}.

The first and second terms in the right hand side of (5.24a) and (5.24b) individually represent zero-phase filters with complex coefficients. However, the coefficients

Table 5-I Basic building blocks for fan filter design.

	Prototype Filter $H(z_1, z_2)$	
1	$\tilde{H}(z_1, z_2, \frac{1}{2}, \frac{1}{2}, \frac{\pi}{2})$	
2	$\tilde{H}(z_1, z_2, -\frac{1}{2}, \frac{1}{2}, \frac{\pi}{2})$	
3	$\tilde{H}(z_1, z_2, \frac{1}{2}, -\frac{1}{2}, \frac{\pi}{2})$	
4	$\tilde{H}(z_1, z_2, -\frac{1}{2}, -\frac{1}{2}, \frac{\pi}{2})$	

of resulting filters $H_{11}(z_1,z_2)$ and $H_{22}(z_1,z_2)$ are real because each term is the complex conjugate of the other. During the implementation step, it will be shown that filter functions H_{11} and H_{22} are functions of complex variables z_1 and z_2 with integer powers (terms with rational powers of z_1 and z_2 will be cancelled).

Remark 5.2:

The equations (5.24a) and (5.24b) create an overlap in the frequency domain. In the ideal case, this overlap will exist at one point, at the origin. In order to remove the overlap in (5.24a) and (5.24b) one may consider the following technique [170].

$$H'_{11}(z_1,z_2) = H_{11}(z_1,z_2) - \hat{H}_{11}(z_1,z_2)\hat{H}^*_{11}(z_1,z_2) \tag{5.25a}$$

and

$$H'_{22}(z_1,z_2) = H_{22}(z_1,z_2) - \hat{H}_{22}(z_1,z_2)\hat{H}^*_{22}(z_1,z_2) \tag{5.25b}$$

where

$$\hat{H}_{11}(z_1,z_2) = \tilde{H}_1(z_1,z_2)\tilde{H}_2(z_1,z_2)\tilde{H}^*_3(z_1,z_2)\tilde{H}^*_4(z_1,z_2)$$

and

$$\hat{H}_{22}(z_1,z_2) = \tilde{H}_1(z_1,z_2)\tilde{H}_3(z_1,z_2)\tilde{H}^*_2(z_1,z_2)\tilde{H}^*_4(z_1,z_2)$$

However, in practice, pass and stop regions do not have sharp cut-off boundaries. There always exists a transition region between pass and stop bands. Therefore, this overlap problem at the origin may be neglected in most practical cases. In Section 5.65, it will be shown that this overlap depends on the prototype specifications of the original filter.

B. Quadrant Fan Filters

The frequency characteristic of a quadrant fan filter is shown in Fig.5.11. The specification is:

$$H_q(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 1 & \omega_1\omega_2 \geq 0 \\ 0 & \omega_1\omega_2 < 0 \end{cases} \tag{5.26}$$

Consider the same ideal prototype filter in Fig.5.8. Then, the following transformed filters are obtained,

$$\tilde{H}_{12}(z_1,z_2) = \tilde{H}(z_1,z_2,1,0,\tfrac{1}{2}\Pi)$$

$$\tilde{H}_{14}(z_1,z_2) = \tilde{H}(z_1,z_2,0,1,\tfrac{1}{2}\Pi)$$

$$\tilde{H}_{34}(z_1,z_2) = \tilde{H}(z_1,z_2,-1,0,\tfrac{1}{2}\Pi)$$

$$\tilde{H}_{23}(z_1,z_2) = \tilde{H}(z_1,z_2,0,-1,\tfrac{1}{2}\Pi) \tag{5.27}$$

where the subscripts on $\tilde{H}$ refers to quadrants to which the low-pass characteristics has been shifted. Table II shows the amplitude response of transformed filters in (5.27).

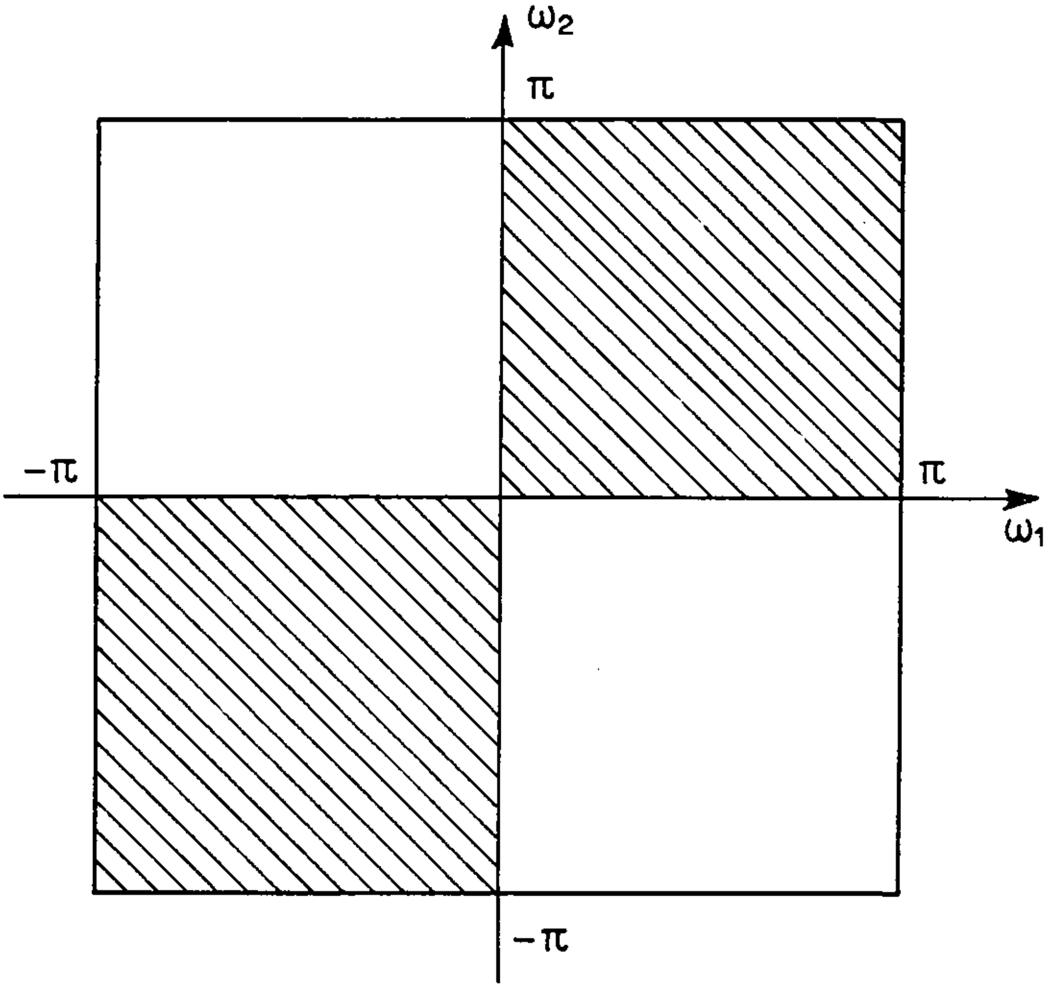
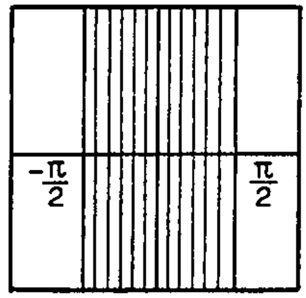
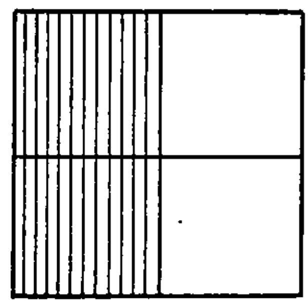
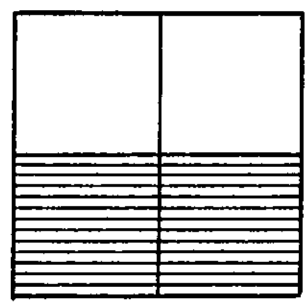
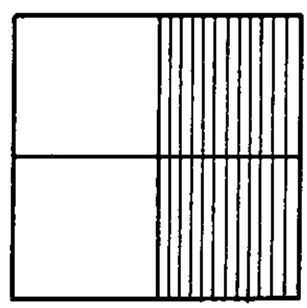
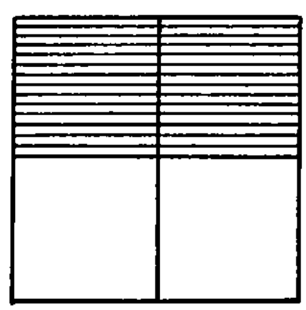


Fig. 5·11 Pass and stop regions of an ideal quadrant fan filter in the (ω_1, ω_2)-plane.

Table 5·II Basic building blocks for quadrant fan filter design.

	Prototype Filter $H(z_1, z_2)$	
1	$\tilde{H}(z_1, z_2, 1, 0, \frac{\pi}{2})$	
2	$\tilde{H}(z_1, z_2, 0, 1, \frac{\pi}{2})$	
3	$\tilde{H}(z_1, z_2, -1, 0, \frac{\pi}{2})$	
4	$\tilde{H}(z_1, z_2, 0, -1, \frac{\pi}{2})$	

Next, one can construct the following filter characteristics,

$$H_{13}(z_1,z_2) = \tilde{H}_{12}(z_1,z_2)\tilde{H}_{14}(z_1,z_2)\tilde{H}^*_{34}(z_1,z_2)\tilde{H}^*_{23}(z_1,z_2)$$

$$+ \tilde{H}^*_{12}(z_1,z_2)\tilde{H}^*_{14}(z_1,z_2)\tilde{H}_{34}(z_1,z_2)\tilde{H}_{23}(z_1,z_2)$$

(5.28a)

$$H_{24}(z_1,z_2) = \tilde{H}_{12}(z_1,z_2)\tilde{H}_{34}(z_1,z_2)\tilde{H}^*_{14}(z_1,z_2)\tilde{H}^*_{23}(z_1,z_2)$$

$$+ \tilde{H}^*_{12}(z_1,z_2)\tilde{H}^*_{34}(z_1,z_2)\tilde{H}_{14}(z_1,z_2)\tilde{H}_{23}(z_1,z_2)$$

(5.28b)

where $\tilde{H}^*_{12}$ denotes the filter with coefficients which are the complex conjugate of those of $\tilde{H}_{12}$, and so on.

The frequency characteristics of the resulting filters H_{13} correspond to quadrantal fan filter as shown in Fig. 5.11. H_{24} has a frequency characteristic which is a 90° clockwise rotated version of H_{13}.

Both H_{13} and H_{24} are zero phase. Comments in Remarks 5.2 are also valid for this case.

5.6.4 Implementation of Fan Filters

In Section 5.6.3, a design technique has been described for fan filters using the complex transformation of (5.17). Since the transformation is complex, each transformed filter cannot be implemented in a real time process. On the other hand, the transformation introduces complex variables z_1 and z_2 with rational powers. Hence the resulting filter functions of (5.24) and (5.28) are not applicable to rectangular arrays. Interpolated filter systems must be used to evaluate the undefined new grid points (assuming $|\frac{\alpha_1}{\beta_1}| < 1$, $|\frac{\alpha_2}{\beta_2}| < 1$). However, it can be shown that the obtained fan filter functions (5.24) and (5.28) do not have complex coefficients and the complex variables z_1 and z_2 have integral exponents. This is the result of the structure used in the design equations.

The prototype low-pass filter can be either recursive or non-recursive with a cut-off frequency at $\omega_c = \frac{1}{2}\pi$. In this section, both cases will be studied.

A. Non-Recursive (FIR)

Consider one-dimensional non-recursive digital filter function as:

$$H_1(z_1) = \sum_{i=0}^{N} a(i) z_1^i \tag{5.29}$$

Apply the complex transformation (5.17) to give

$$\tilde{H}(z_1,z_2) = H_1(e^{j\phi} z_1^{\alpha_1/\beta_1} z_2^{\alpha_2/\beta_2}) \tag{5.30}$$

From (5.28a) one can construct the fan filter function as:

$$H_{11}(z_1,z_2) = \hat{H}_{11}(z_1,z_2) + \hat{H}^*_{11}(z_1,z_2) \tag{5.31}$$

where:

$$\hat{H}_{11}(z_1,z_2) = H_1(jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})H_2(jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})H_3(-jz_1^{+\frac{1}{2}}z_2^{-\frac{1}{2}})H_4(-jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}}) \tag{5.32}$$

for $\phi = \frac{\pi}{2}$, $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{1}{2}$,

and $\hat{H}^*_{11}$ denotes the filter with coefficients which are complex conjugate of those of $\hat{H}_{11}$. Next, $\hat{H}_{11}(z_1,z_2)$ can be written as:

$$\hat{H}_{11}(z_1,z_2) = \hat{H}^r_{11}(z_1,z_2) + j\hat{H}^i_{11}(z_1,z_2) \tag{5.33}$$

then the filter function of (21) is:

$$H_{11}(z_1,z_2) = 2\hat{H}^r_{11}(z_1,z_2) \tag{5.34}$$

$\hat{H}^r_{11}$ has only real coefficients. Now, it can be readily shown that $\hat{H}^r_{11}$ is a function of integral powers of the complex variables z_1 and z_2.

The prototype transfer function can be factorized as:

$$H_1(z_1) = \prod_{i=1}^{N} (p_i + z_1) \tag{5.35}$$

where p_i, $i = 1,2,\ldots,N$ are zeroes of H_1.

From (5.35) H_{11} becomes:

$$H_{11}(z_1,z_2) = \prod_{i=1}^{N}\left[(p_i + jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})(p_i - jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})(p_i + jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})(p_i - jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}})\right]$$

$$= \prod_{i=1}^{N}\left[(p_i+1)^2 - p_i^2(z_2 + z_2^{-1} - z_1 - z_1^{-1})\right]$$

$$+ j\left[(p_i^2 + 1)p_i(z_1^{\frac{1}{2}}z_2^{\frac{1}{2}} + z_1^{-\frac{1}{2}}z_2^{\frac{1}{2}} - z_1^{-\frac{1}{2}}z_2^{\frac{1}{2}} - z_1^{\frac{1}{2}}z_2^{-\frac{1}{2}})\right]$$

$$\tag{5.36}$$

From the above equation (5.36), it can be observed that the terms with rational powers of z_1 and z_2 will exist only in the imaginary part in each factorized section. If the multiplication is carried on, the form will not be changed (i.e., complex variables with rational powers exist only in the imaginary part). Finally, the resulting transfer function of $\hat{H}_{11}$ may be rewritten as:

$$\hat{H}_{11}(z_1,z_2) = \sum_{i=-N}^{N}\sum_{j=-N}^{N} h(i,j)z_1^i z_2^j + j\left[\text{terms with rational powers}\right] \tag{5.37}$$

From equation (5.37), the filter function will be:

$$H_{11}(z_1,z_2) = 2 \sum_{i=-N}^{N} \sum_{j=-N}^{N} h(i,j) z_1^i z_2^j \qquad (5.38)$$

where $\{h(i,j)\}_{i,j=-N}^{N}$ are real numbers and obtained from the coefficients of the prototype filter as explained above.

Since $H_{11}(z_1,z_2)$ is zero-phase filter, it can be written as [7]

$$H_{11}(z_1,z_2) = \tilde{H}_1(z_1,z_2) + \tilde{H}_1(z_1^{-1},z_2^{-1}) + \tilde{H}_2(z_1^{-1},z_2) + \tilde{H}_2(z_1,z_2^{-1})$$

where (5.39)

$$\tilde{H}_1(z_1,z_2) = h(0,0) + \sum_{i=0}^{N} \sum_{j=0}^{N} 2h(i,j) z_1^i z_2^j \qquad (5.40)$$

$$(i,j) \neq (0,0)$$

and

$$\tilde{H}_2(z_1,z_2) = \sum_{i=1}^{N} \sum_{j=1}^{N} 2h(-i,j) z_1^i z_2^j \qquad (5.41)$$

The direct implementation of the final transfer function is in Fig.5.12. However, the complexity can be reduced by half if one considers the input as a finite area array. It can be shown (Appendix D) that all recursion

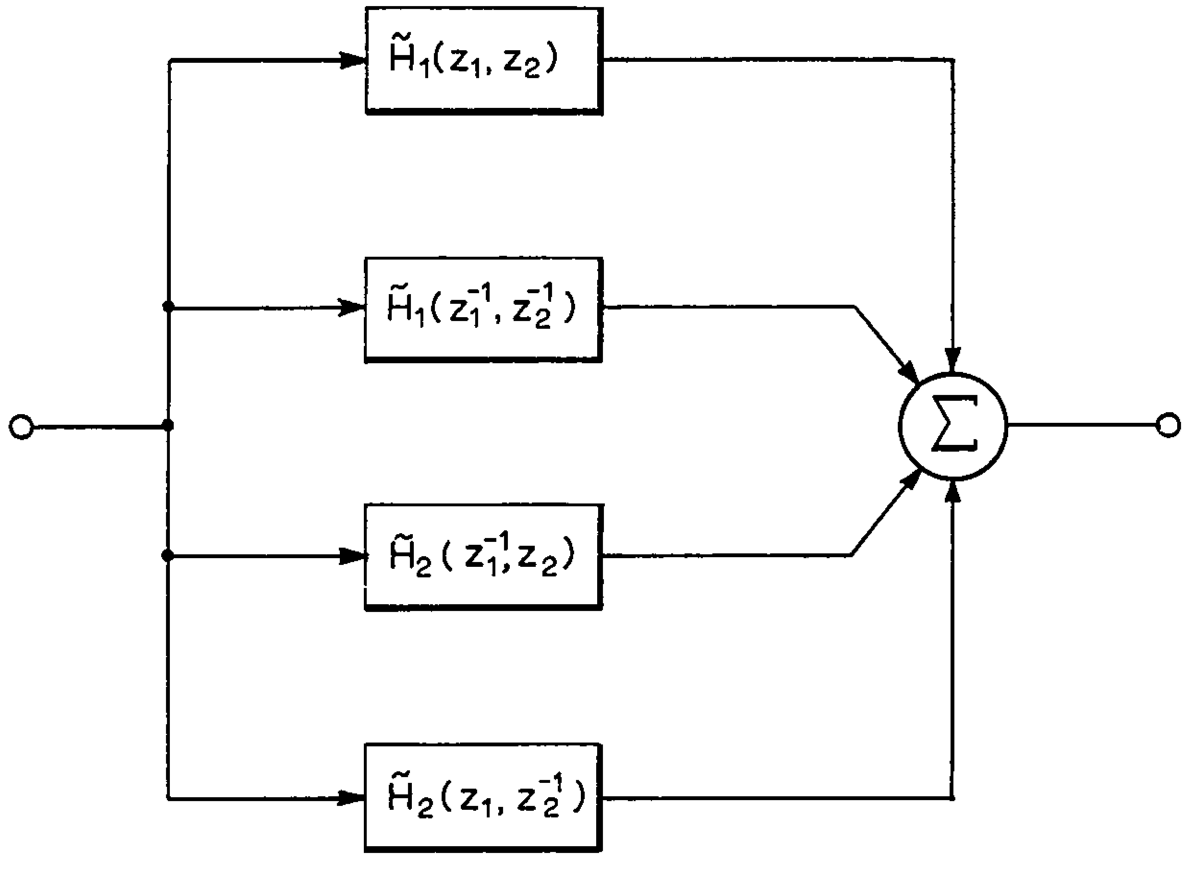


Fig. 5·12 Direct implementation of $H_{11}(z_1, z_2)$ in (5·39) directions may be obtained with the difference equation (2.1) by reorientating the input. Fig.5.13 shows the implementation with the finite area array (input) reorientation.

B. Recursive (IIR) Prototype

Consider the one-dimensional recursive filter function as:

$$H(z_1,z_2) = H_1(z_1) = \frac{\sum_{i=0}^{N_1} a(i) z_1^i}{\sum_{j=0}^{N_2} b(j) z_1^j} = \frac{A(z_1)}{B(z_1)} \tag{5.42}$$

After applying the complex transformation (5.17), one can construct a recursive fan filter having transfer function

$$H_R(z_1,z_2) = \frac{A_R(z_1,z_2)}{B_R(z_1,z_2)}$$

$$= \frac{A_{R_1}(z_1,z_2) B^*_{R_1}(z_1,z_2) + A^*_{R_1}(z_1,z_2) B_{R_1}(z_1,z_2)}{B_{R_1}(z_1,z_2) B^*_{R_1}(z_1,z_2)} \tag{5.43}$$

where

$$A_{R_1}(z_1,z_2) = A(jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})A(jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})A(-jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})A(-jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}}) \tag{5.44}$$

$$B_{R_1}(z_1,z_2) = B(jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})B(jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})B(-jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})B(-jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}}) \tag{5.45}$$

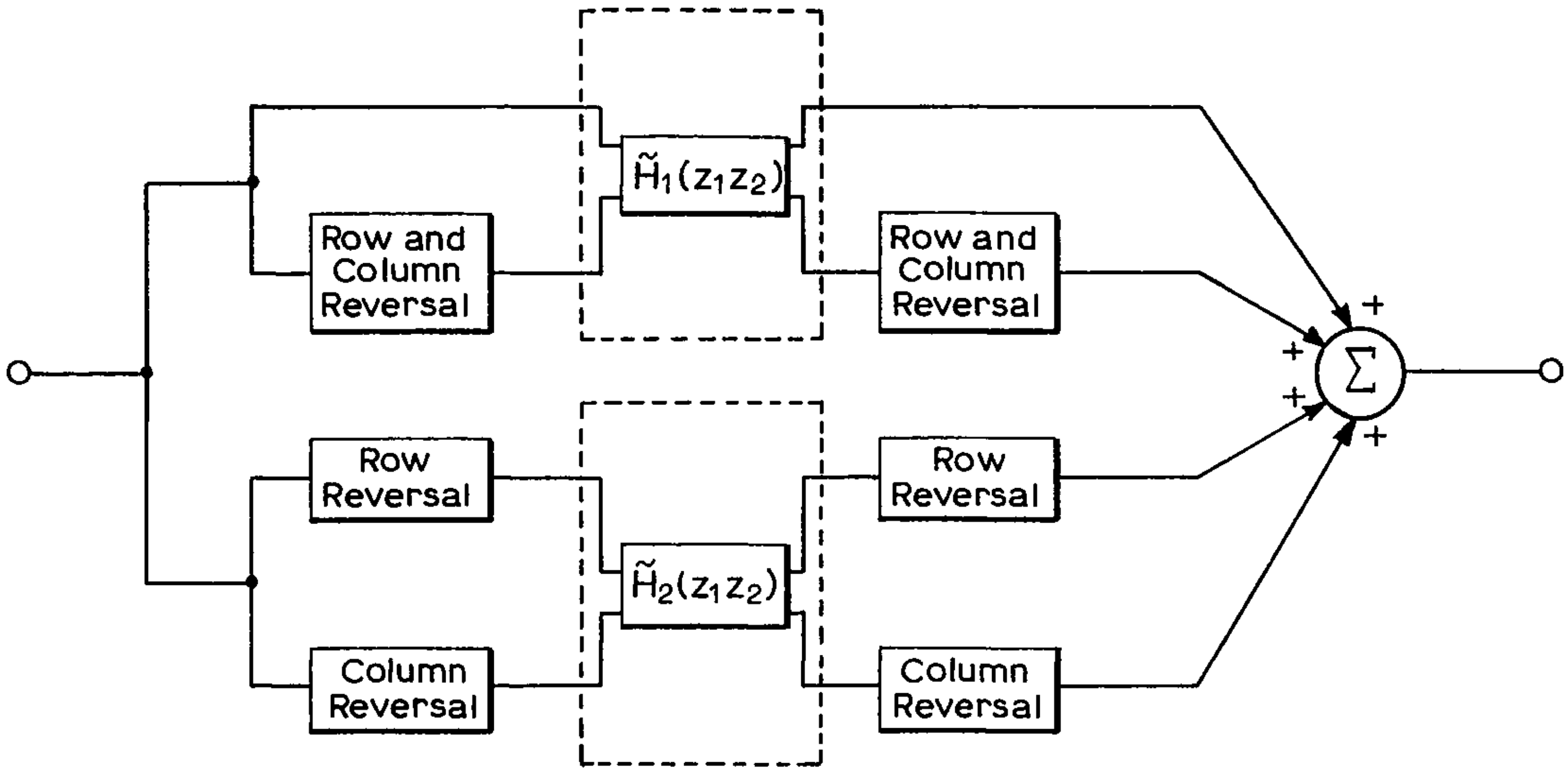


Fig. 5·13 Implementation of a zero-phase filter involving only causal recursion.

where $A_{R_1}^*$ and $B_{R_1}^*$ denotes the function with coefficients which are complex conjugates of theose of A_{R_1} and B_{R_1}, respectively.

It has been shown that the resulting transfer function of fan filter has only real coefficients as well as only integer powers of the complex variables of z_1 and z_2. The proof of this fact is omitted because it is similar to the non-recursive one. The final form of the transfer function (5.43) will be:

$$H_R(z_1,z_2) = \frac{\sum_{m=-M}^{M} \sum_{n=-M}^{M} a_r(m,n) z_1^m z_2^n}{\tilde{B}_R(z_1 z_2)\tilde{B}_R(z_1^{-1} z_2)\tilde{B}_R(z_1 z_2^{-1})\tilde{B}_R(z_1^{-1} z_2^{-1})} \tag{5.46}$$

where $M = N_1 + N_2$ and

$$B_R(z_1 z_2) = B(jz_1^{\frac{1}{2}} z_2^{\frac{1}{2}})B(-jz_1^{\frac{1}{2}} z_2^{\frac{1}{2}}) \tag{5.47}$$

By factorizing the denominator of (5.42) as:

$$B(z_1) = \prod_{i=1}^{N_2} (p_i + z_1) \tag{5.48}$$

we may write

$$\tilde{B}_R(z_1 z_2) = \prod_{i=1}^{N_2} \left[(p_i + j z_1^{\frac{1}{2}} z_2^{\frac{1}{2}})(p_i - j z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}) \right]$$

$$= \prod_{i=1}^{N_2} \left[p_i^2 + z_1 z_2 \right] \qquad (5.49)$$

From (5.49),we conclude that B_R can simply be obtained by replacing the roots of $B(z_1)$, p_i with p_i^2. This property will increase the speed of the design procedure. Clearly, $\tilde{B}_R(z_1 z_2)$ is stable, since poles of z_1 within the unit circle map directly into two dimensional poles within the unit bidisc.

It can readily be seen that $\tilde{B}_R(z_1^{-1} z_2)$, $\tilde{B}_R(z_1 z_2^{-1})$ and $\tilde{B}_R(z_1^{-1} z_2^{-1})$ are also stable in second, fourth and third quadrants, respectively.

Due to its zero phase, $A_R(z_1, z_2)$ can be written as:

$$A_R(z_1,z_2) = \tilde{A}_{R_1}(z_1,z_2) + \tilde{A}_{R_1}(z_1^{-1},z_2) + \tilde{A}_{R_2}(z_1^{-1},z_2^{-1}) + \tilde{A}_{R_2}(z_1,z_2^{-1}) \qquad (5.50)$$

where

$$\tilde{A}_{R_1}(z_1,z_2) = a_r(0,0) + \sum_{m=0}^{M} \sum_{n=0}^{M} a_r(m,n) z_1^m z_2^n$$

$$(m,n) \neq (0,0)$$

and

$$\tilde{A}_{R_2}(z_1, z_2) = \sum_{m=1}^{M} \sum_{n=1}^{M} a_r(m,n) z_1^m z_2^n \tag{5.52}$$

The direct implementation of H_R is in Fig.5.14; the final configuration of $H_R(z_1, z_2)$ involving only causal recursion in Fig. 5.15.

<u>An Example for Implementation</u>

Consider the recursive prototype filter to be of the form

$$H(z_1) = \frac{A(z_1)}{B(z_1)}$$

$$= \frac{z_1^2 + a_1 z_1 + a_2}{z_1^2 + b_1 z_1 + b_2} \tag{5.53}$$

For fan filter design, from (5.43)

$$H(z_1, z_2) = \frac{A_R(z_1, z_2)}{B_R(z_1, z_2)} \tag{5.54}$$

where

$$A_R(z_1, z_2) = \tilde{A}_R(z_1, z_2) + \tilde{A}_R^*(z_1, z_2) \tag{5.55}$$

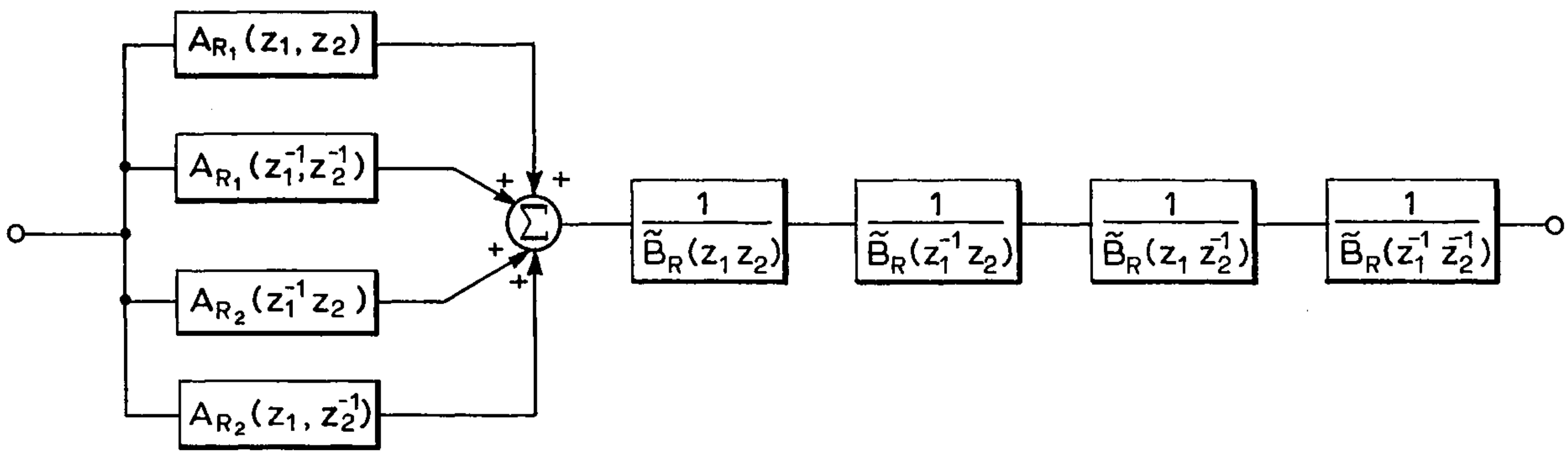


Fig. 5·14 Direct implementation of the fan filter function $H_R(z_1, z_2)$ involving non-causal recusion

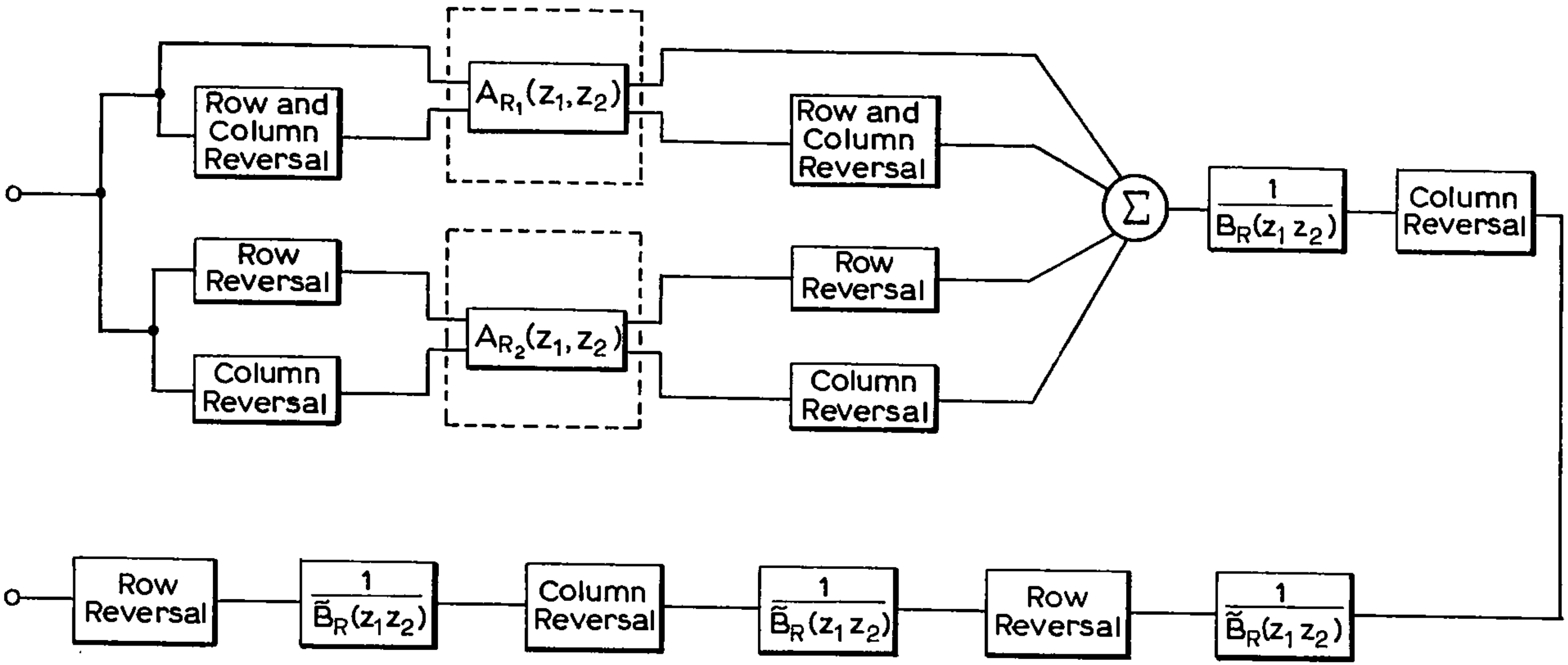


Fig. 5·15 Final configuration of $H_R(z_1, z_2)$ involving only causal recursion.

For the transfer function (5.53) $\tilde{A}_R$ can be written as:

$$\tilde{A}_R(z_1,z_2)=\left[A(jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})B(-jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})\right]\left[A(jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})B(-jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})\right]\left[A(-jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}})B(jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}})\right]$$

$$\bullet\left[A(-jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})B(jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})\right] \qquad (5.56)$$

The terms on the right hand side are obtained as:

$$A(jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})B(-jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}}) = \left[(a_2-z_1z_2) - ja_1z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\right]\left[(b_2-z_1z_2) + jb_1z_1^{\frac{1}{2}}z_2^{\frac{1}{2}}\right]$$

$$= \left[z_1^2z_2^2 - (a_2 + b_2 - a_1b_1)z_1z_2 + a_2b_2\right]$$

$$+ jz^{\frac{1}{2}}z^{\frac{1}{2}}\left[(b_1 - a_1)z_1z_2 + (a_1b_2 - a_2b_1)\right] \qquad (5.57)$$

$$= H_A(z_1,z_2) + jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}}H_B(z_1,z_2) \qquad (5.60a)$$

where

$$H_A(z_1,z_2) \triangleq z_1^2z_2^2 - (a_2 + b_2 - a_1b_1)z_1z_2 + a_2b_2 \qquad (5.58)$$

$$H_B(z_1,z_2) \triangleq (b_1 - a_1)\ z_1z_2 + a_1b_2 - a_2b_1 \qquad (5.59)$$

and similarly,

$$A(jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}})B(-jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}}) = H_A(z_1^{-1},z_2) + jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}}H_B(z_1^{-1},z_2) \tag{5.60b}$$

$$A(-jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}})B(jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}}) = H_A(z_1,z_2^{-1}) - jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}}H_B(z_1,z_2^{-1}) \tag{5.60c}$$

$$A(-j_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})B(jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}}) = H_A(z_1^{-1},z_2^{-1}) - jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}}H_B(z_1^{-1},z_2^{-1}) \tag{5.60d}$$

From (5.55) it is immediately seen that

$$A_R(z_1,z_2) = 2\mathrm{Re}\left\{\widetilde{A}_R(z_1,z_2)\right\} \tag{5.61}$$

where Re denotes the real part. Now from (5.56) and (5.60) we have

$$\widetilde{A}_R(z_1,z_2) = \left[H_A(z_1z_2) + jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}}H_B(z_1z_2)\right]\left[H_A(z_1^{-1}z_2^{-1}) - jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}}H_B(z_1^{-1}z_2^{-1})\right]$$

$$\bullet\left[H_A(z_1^{-1}z_2) + jz_1^{-\frac{1}{2}}z_2^{\frac{1}{2}}H_B(z_1^{-1}z_2)\right]\left[H_Z(z_1z_2^{-1}) - jz_1^{\frac{1}{2}}z_2^{-\frac{1}{2}}H_B(z_1z_2^{-1})\right] \tag{5.62}$$

and hence from (5.61)

$$A_R(z_1,z_2) = 2\mathrm{Re}\left\{\tilde{A}_R(z_1,z_2)\right\} = \left[H_A(z_1z_2)H_A(z_1^{-1}z_2^{-1}) + H_B(z_1z_2)H_B(z_1^{-1}z_2^{-1})\right]$$

$$\bullet\left[H_A(z_1^{-1}z_2)H_A(z_1z_2^{-1}) - H_B(z_1^{-1}z_2)H_B(z_1z_2^{-1})\right]$$

$$-\left[H_A(z_1^{-1}z_2^{-1})H_B(z_1z_2)z_2 - H_A(z_1z_2)H_B(z_1^{-1}z_2^{-1})z_1^{-1}\right]$$

$$\bullet\left[H_Z(z_1,z_2^{-1})H_B(z_1^{-1},z_2) - H_A(z_1^{-1}z_2)H_B(z_1z_2^{-1})z_2\right] \qquad (5.63)$$

The denominator function $B_R(z_1,z_2)$ can be found as:

$$B_R(z_1,z_2) = \tilde{B}_R(z_1z_2)\tilde{B}_R(z_1^{-1}z_2^{-1})\tilde{B}_R(z_1^{-1}z_2)\tilde{B}_R(z_1,z_2^{-1}) \qquad (5.64)$$

where

$$\tilde{B}_R(z_1z_2) = B(jz_1^{\frac{1}{2}}z_2^{\frac{1}{2}})B(-jz_1^{-\frac{1}{2}}z_2^{-\frac{1}{2}})$$

$$= z_1z_2 + (b_1 - 2b_2)z_1z_2 + b_2 \qquad (5.65)$$

The final configuration of the fan filter is given in Fig.5.16.

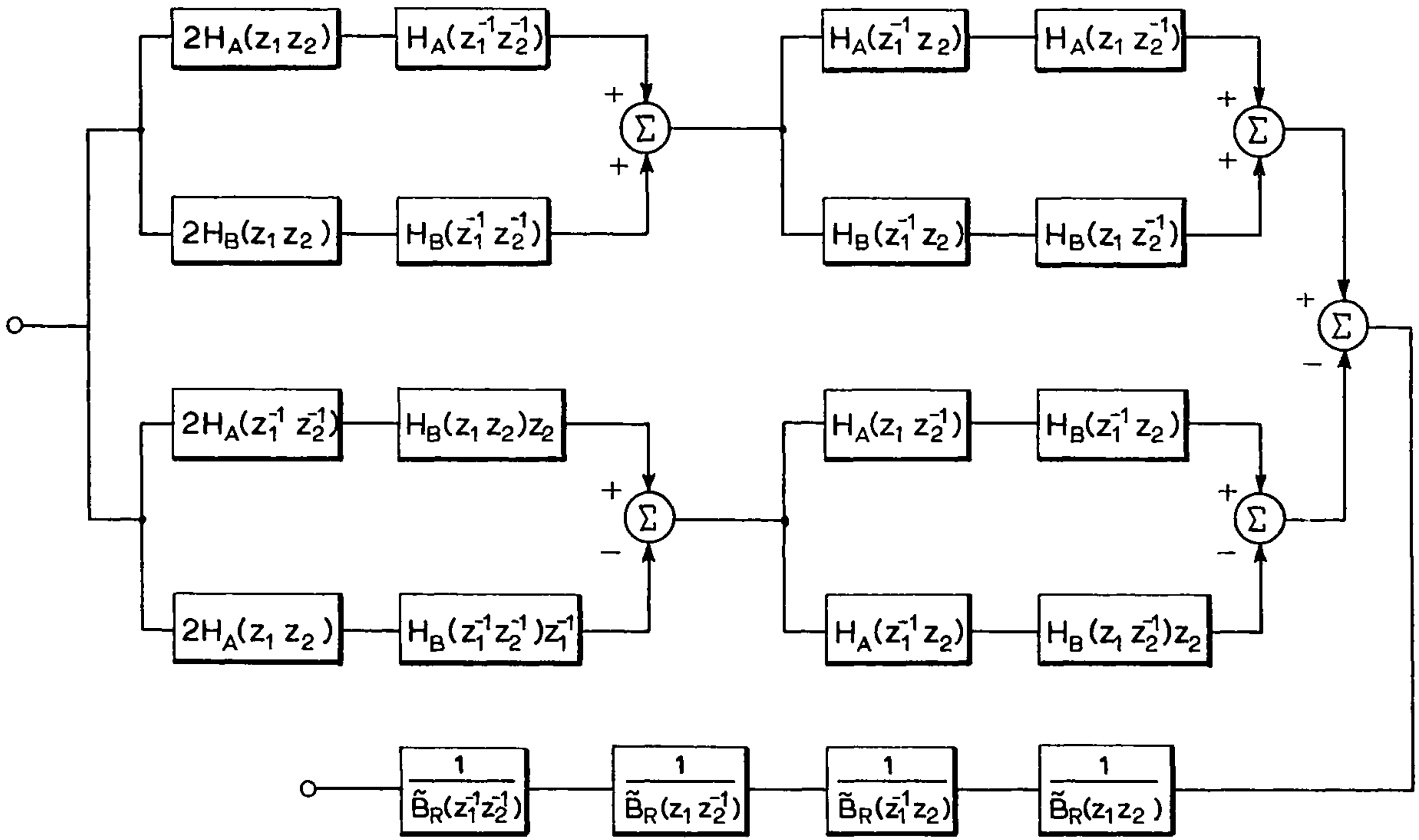


Fig. 5·16 Final configuration of the fan filter function in (5·59)

Remark 5.3:

It is interesting to note that the building blocks H_A and H_B in the final configuration (Fig.5.16) can be implemented as a one-dimensional filtering process. Fig.5.17 shows the recursion directions of $H_A(z_1 z_2)$ and $H_B(z_1 z_2)$. This consideration will reduce both storage and computational time.

5.6.5 Numerical Examples:

Example 1

Consider the low-pass one-dimensional filter function obtained by Charalambous [27].

$$H(z) = (0.03112) \prod_{k=1}^{3} \left[\frac{z^2 + a_k z + b_k}{z^2 + c_k z + d_k} \right] \tag{5.66}$$

The coefficients of the filter function (5.66) are given in Table 5.III. This filter function has the following specifications for the magnitude characteristics;

$$H(e^{j\omega}) = \begin{cases} 1 \pm \delta_p & \text{for} \quad \omega \in \psi_p \\ \delta_s & \text{for} \quad \omega \in \psi_s \end{cases} \tag{5.67}$$

where

$\psi_p = [0, 0.3333333\pi]$ is the pass band.

$\psi_s = [0.376111\pi, \pi]$ is the stop band.

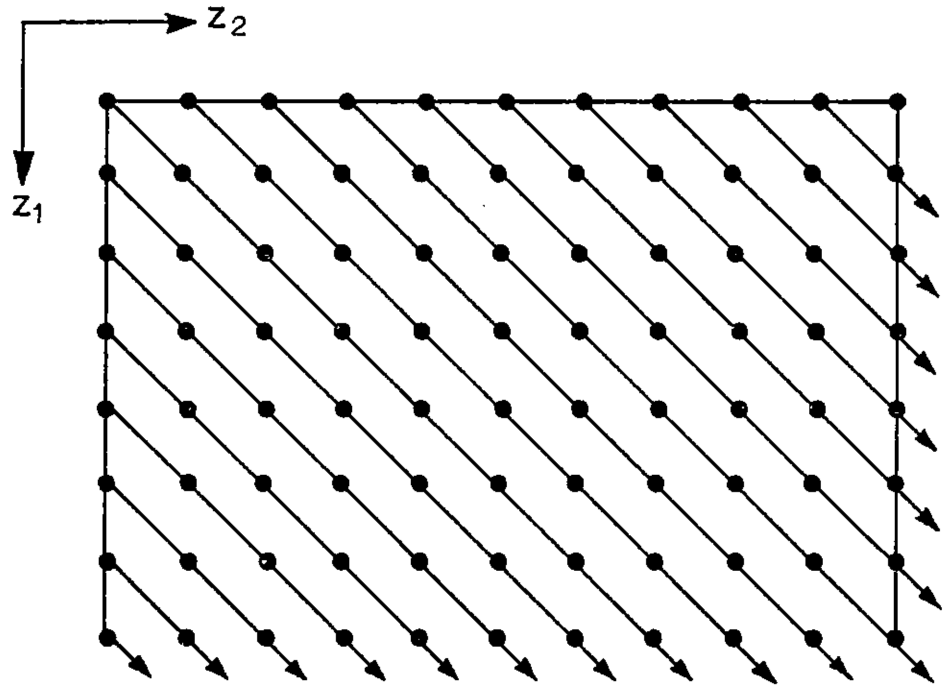


Fig. 5·17 Recursion direction of the filter blocks in Fig. 5·16

TABLE 5.III

Coefficient Values of the Filter in Example 2.

i	a_i	b_i	c_i	d_i
1	-0.72968	1.0	-0.94645	0.93267
2	1.17818	1.0	-1.09725	0.38569
3	-0.38510	1.0	-0.99086	0.71690

and

$$\delta_p = 0.01 \ , \qquad \delta_s = 0.001$$

Since the filter function does not have a cut-off frequency at $\omega_c = \frac{1}{2}\pi$, a low-pass to low-pass frequency transformation [173] is used to obtain a filter with a cut-off frequency at $\omega_c = \frac{1}{2}\pi$. Then, it is readily applied to equation (5.24a) to get a symmetric fan filter. Fig.5.18(a) and 5.18(b) show the magnitude and contour plots of the resulting fan filter.

Similarly, a quadrant filter can be obtained from (5.28a). The magnitude and contour plots of the designed quadrant fan filter are shown in Fig.19(a)-(b). From Figs.18-19, we conclude that the one dimensional prototype characteristics are clearly preserved in both the symmetric and quadrantal fan filters. This can be observed from the ripples in the pass band and stop band as well as the transition region.

Remark 5.4:

For this particular example, since the pass and stop band regions are very close to each other, the cut-off frequency was considered as ω_c= 0.343333π instead of ω_c = 0.3333333π. This modification enabled us to remove the overlap. Otherwise, there will be overlap at one point at the origin due to the sharpness of the transition region.

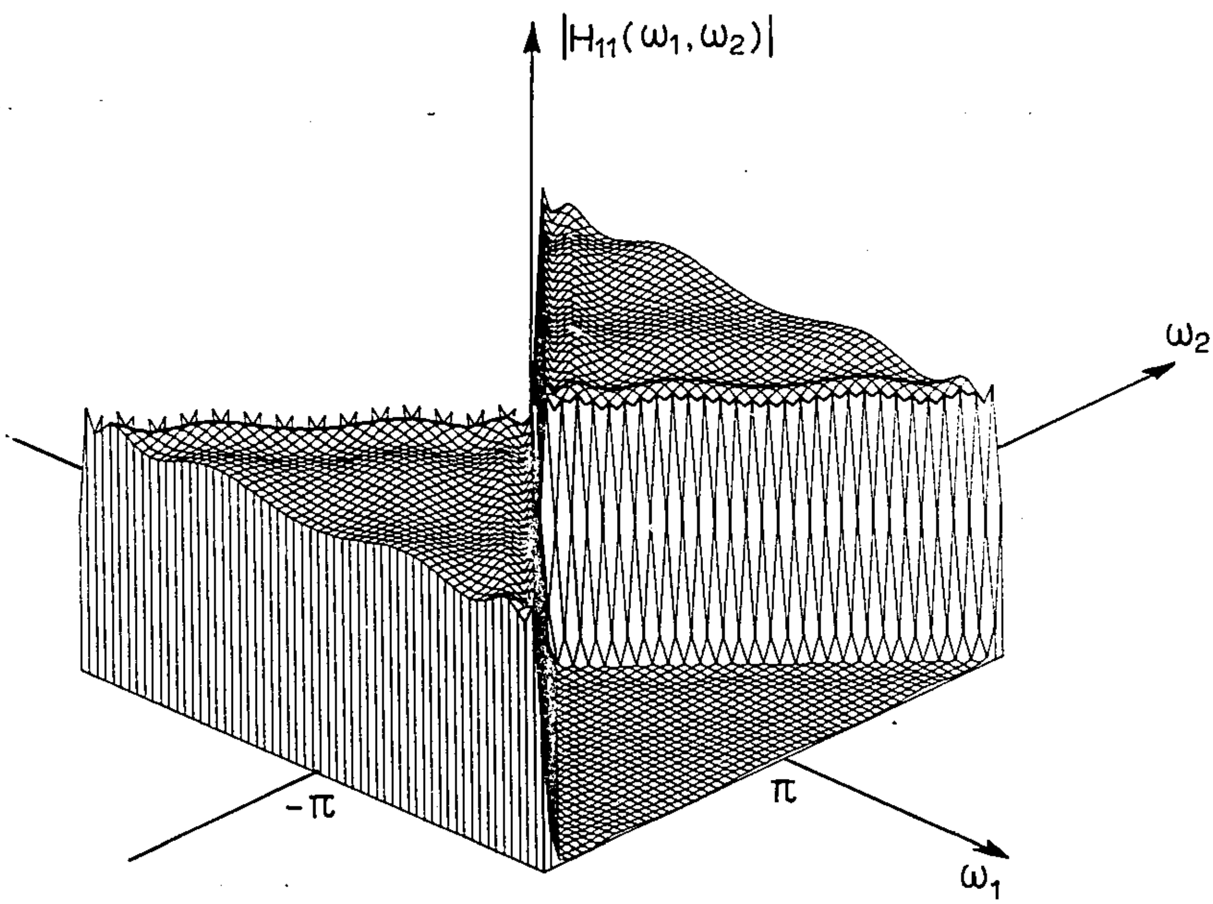


Fig. 5·18(a) Magnitude characteristics of the fan filter in Example 1

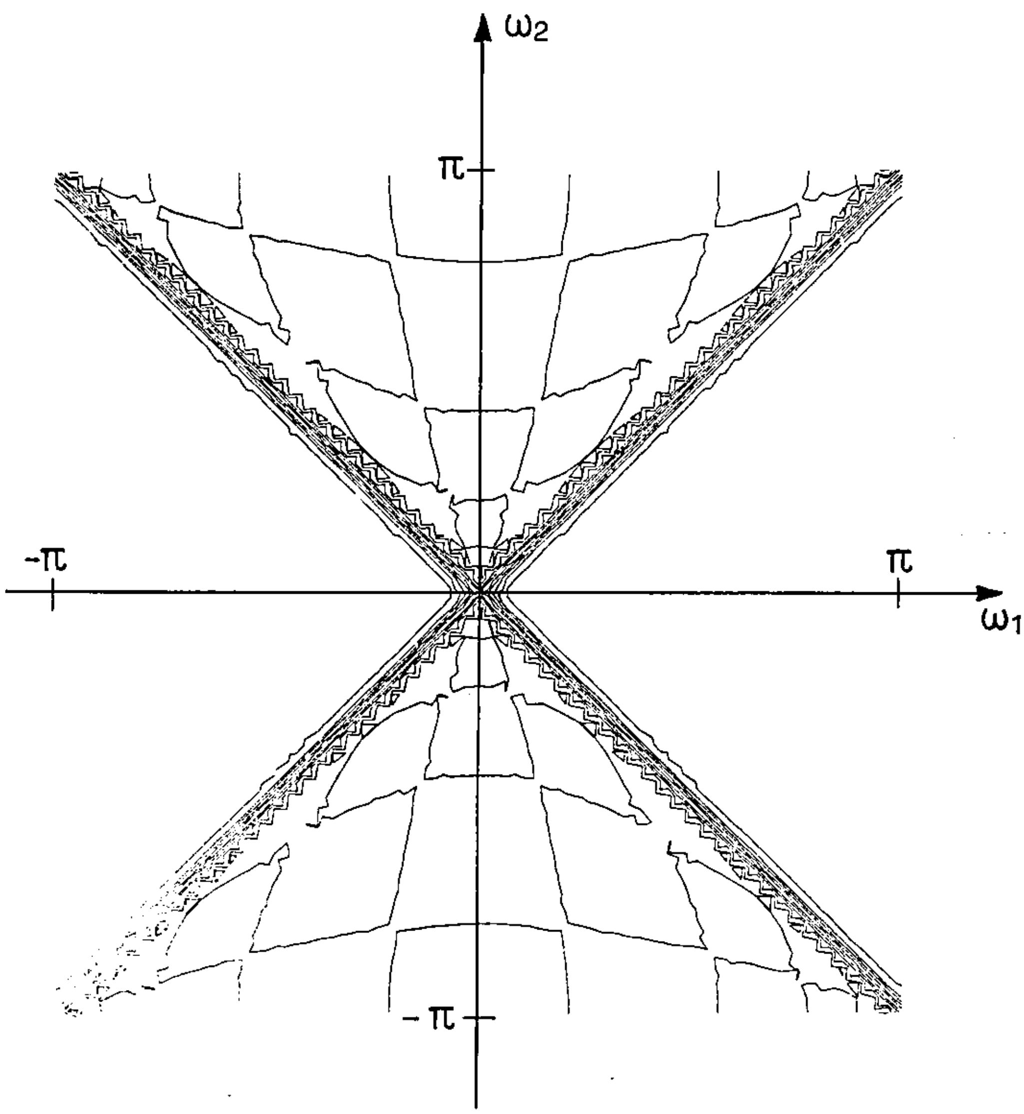


Fig. 5·18(b) Contour plot of the fan filter in Example 1

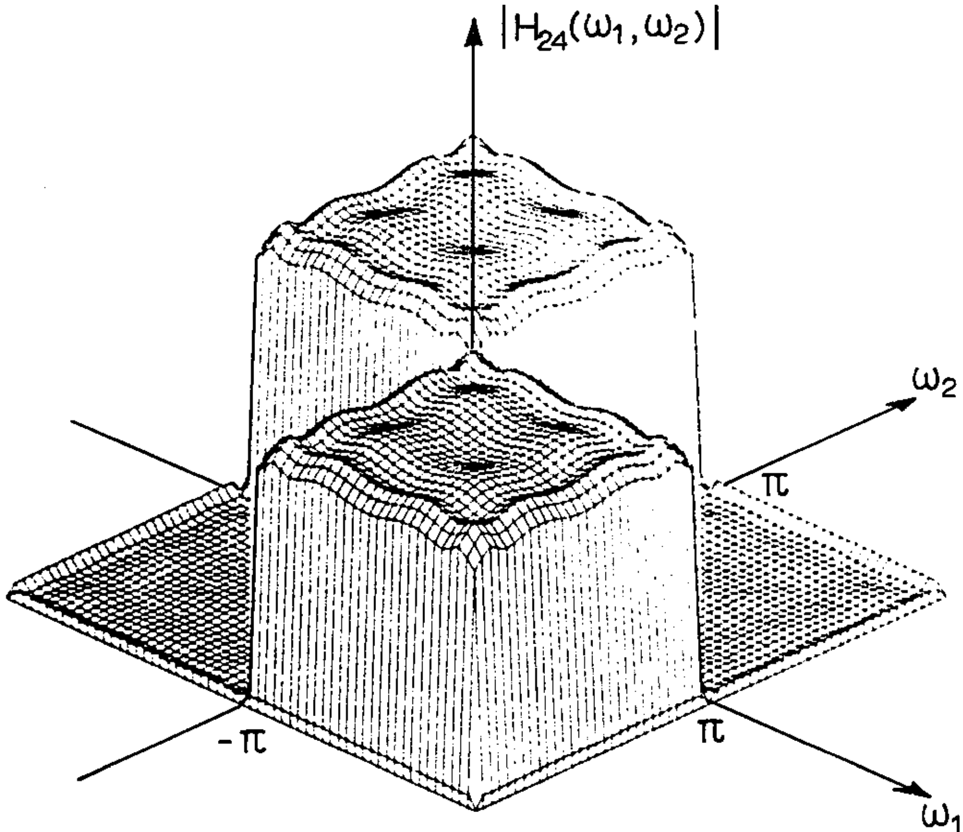


Fig. 5·19 (a) Magnitude characteristics of the quadrant fan filter in Example 1.

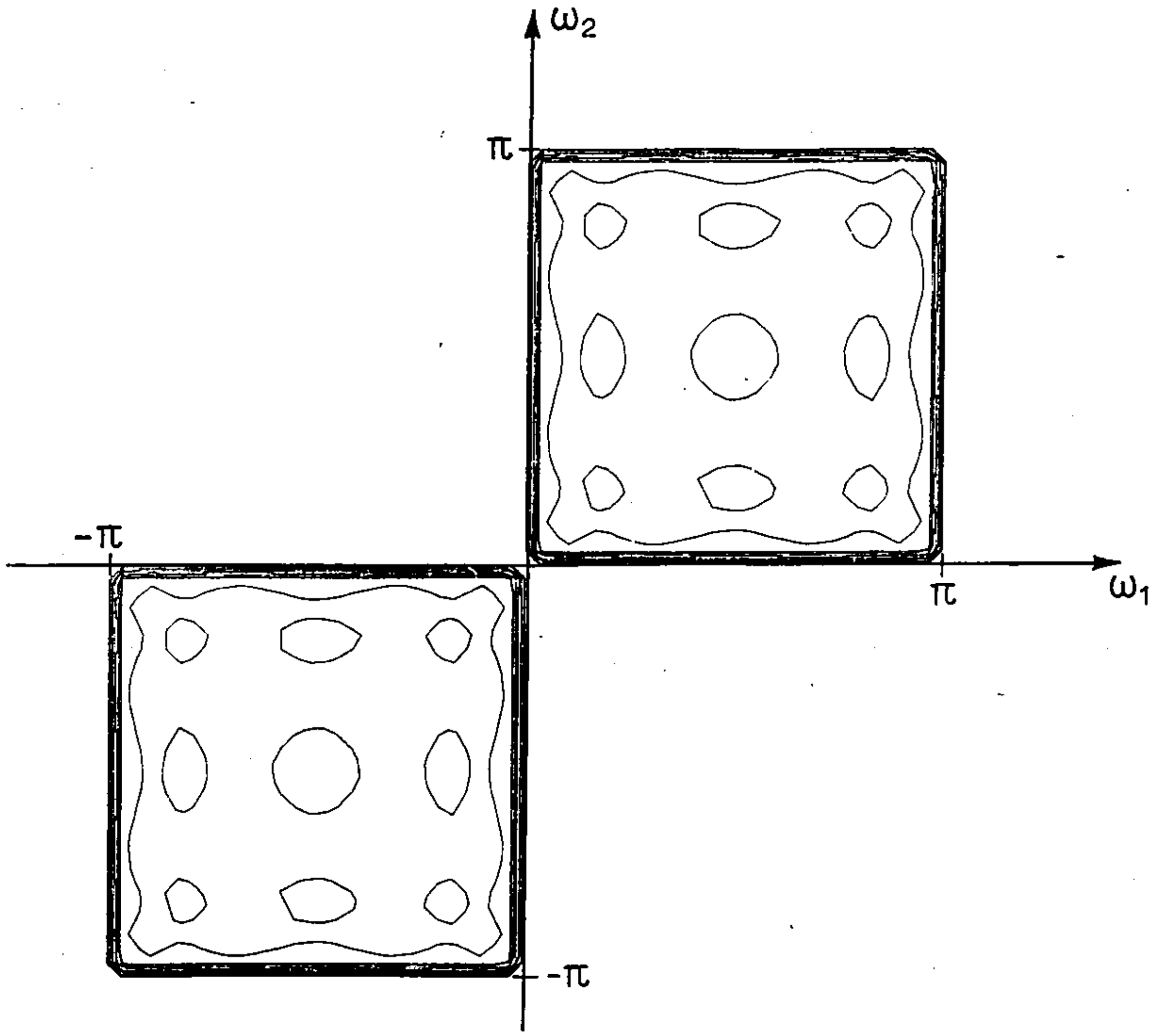


Fig. 5·19(b) *Contour plot of the quadrant fan filter in Example 1.*

Example 2

As a second example, we select as prototype a tenth order Butterworth filter [175]. The transfer function is of the form

$$F(s) = \frac{1}{\sum_{i=0}^{10} b_i s^i} \tag{5.68}$$

To enter the digital domain, substitute the well-known bilinear transformation

$$s = \frac{1-z}{1+z} \tag{5.69}$$

giving as the digital transfer function H(z)

$$H(z) = F(s)\Big|_{s=\frac{1-z}{1+z}} = \frac{(1+z)^{10}}{\sum_{i=0}^{10} b_i' z^i} \tag{5.70}$$

where $\{b_i'\}_{i=0}^{10}$ are real coefficients and can be obtained from $\{b_i\}_{i=0}^{10}$

The cut-off frequency of H(z), ω_c, can be found from (5.69).

$$\begin{aligned}\omega_c &= 2\arctan(\theta_c)\\ &= 2\arctan(1)\\ &= 2\frac{\pi}{4} = \frac{1}{2}\pi \qquad (5.71)\end{aligned}$$

where $\theta_c = 1$ rad. is the cut-off frequency of the prototype filter function (5.68).

Finally, equation (5.24a) is used to obtain symmetric fan filter. Fig.20(a) and Fig.20(b) show the magnitude and contour plots of the resulting filter. For the quadrant fan filter design, the cut-off frequency ω_c is changed to $\omega_c = 1.15$ rad. This alteration will change the cut-off frequency of the H(z) to $\omega_c = 0.545\pi$. The magnitude and contour plots of the designed quadrant fan filter can be seen in Fig.21(a) and Fig.21(b), respectively.

Example 3

To illustrate the FIR fan filter design with complex transformations, we consider the following linear phase filter,

$$H(z) = h(27) + \sum_{i=1}^{26}\left[h(i)z^{i-1} + z^{i+26})\right] \qquad (5.72)$$

The coefficients of H(z) are obtained using Remez Method [24]. The order of the filter is N = 53. The specifications of this low-pass filter are $\delta_p = 0.01175$, $\delta_s = 0.00588$,

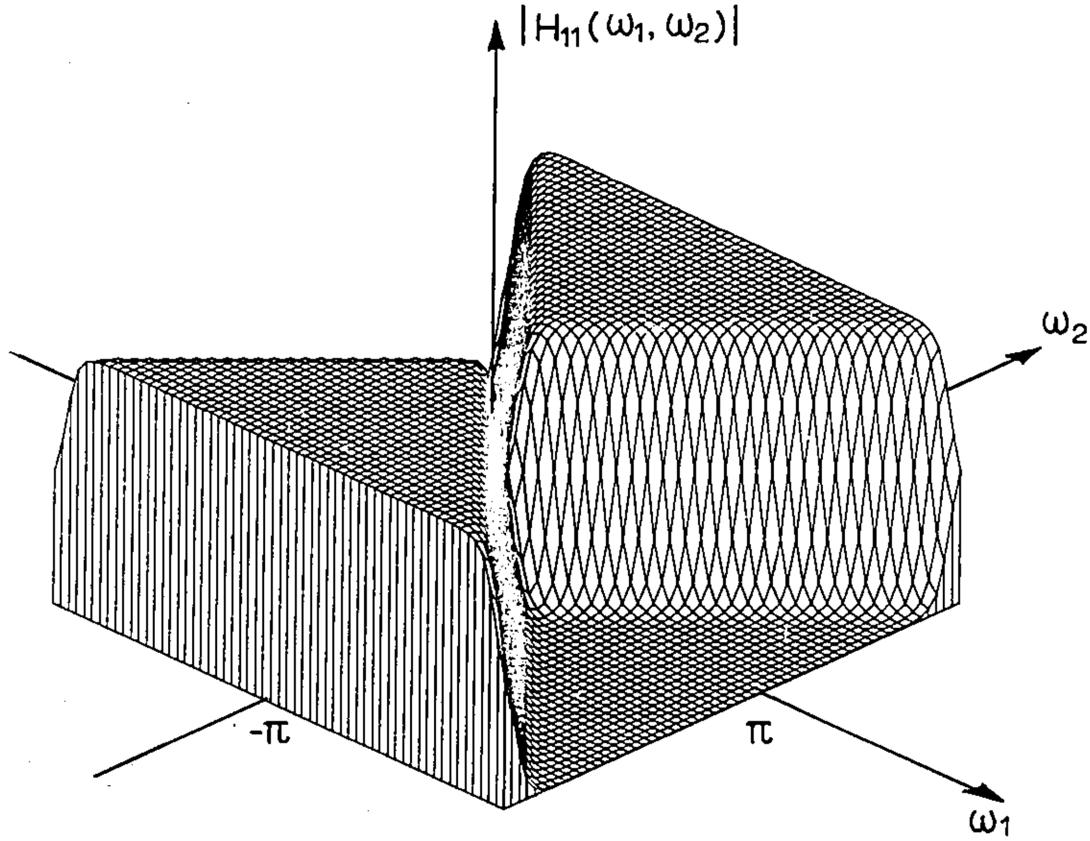


Fig. 20(a) Magnitude characteristics of the fan filter in Example 2.

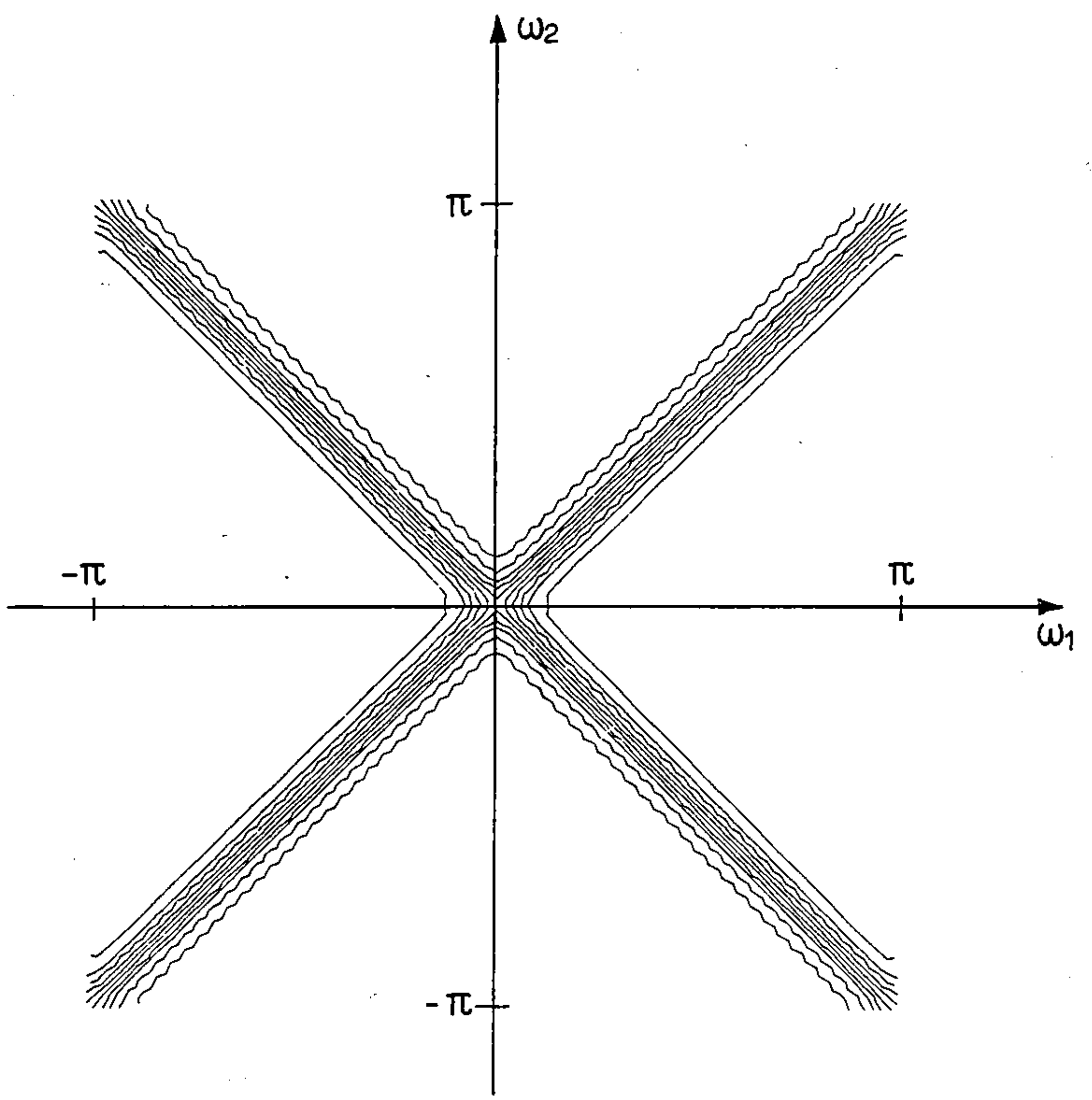


Fig. 5·20(b) Contour plot of the fan filter in Example 2.

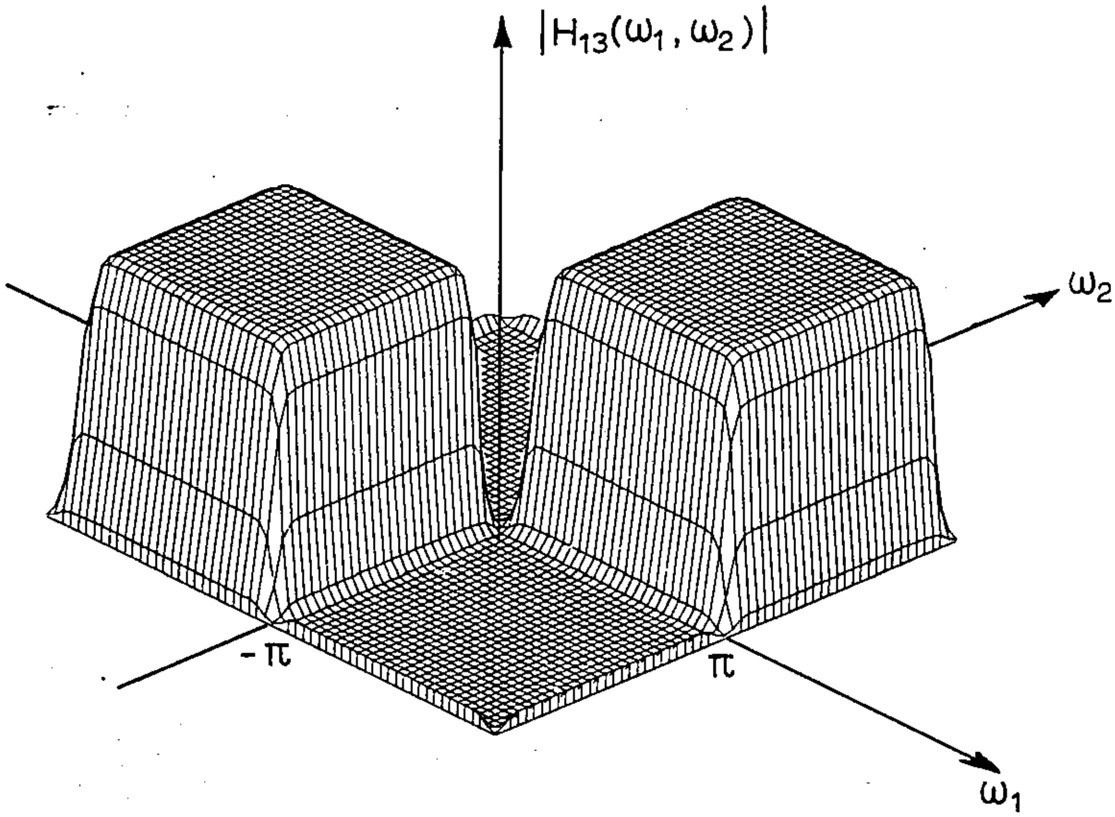


Fig. 5·21(a) Magnitude characteristics of the quadrant fan filter in Example 2.

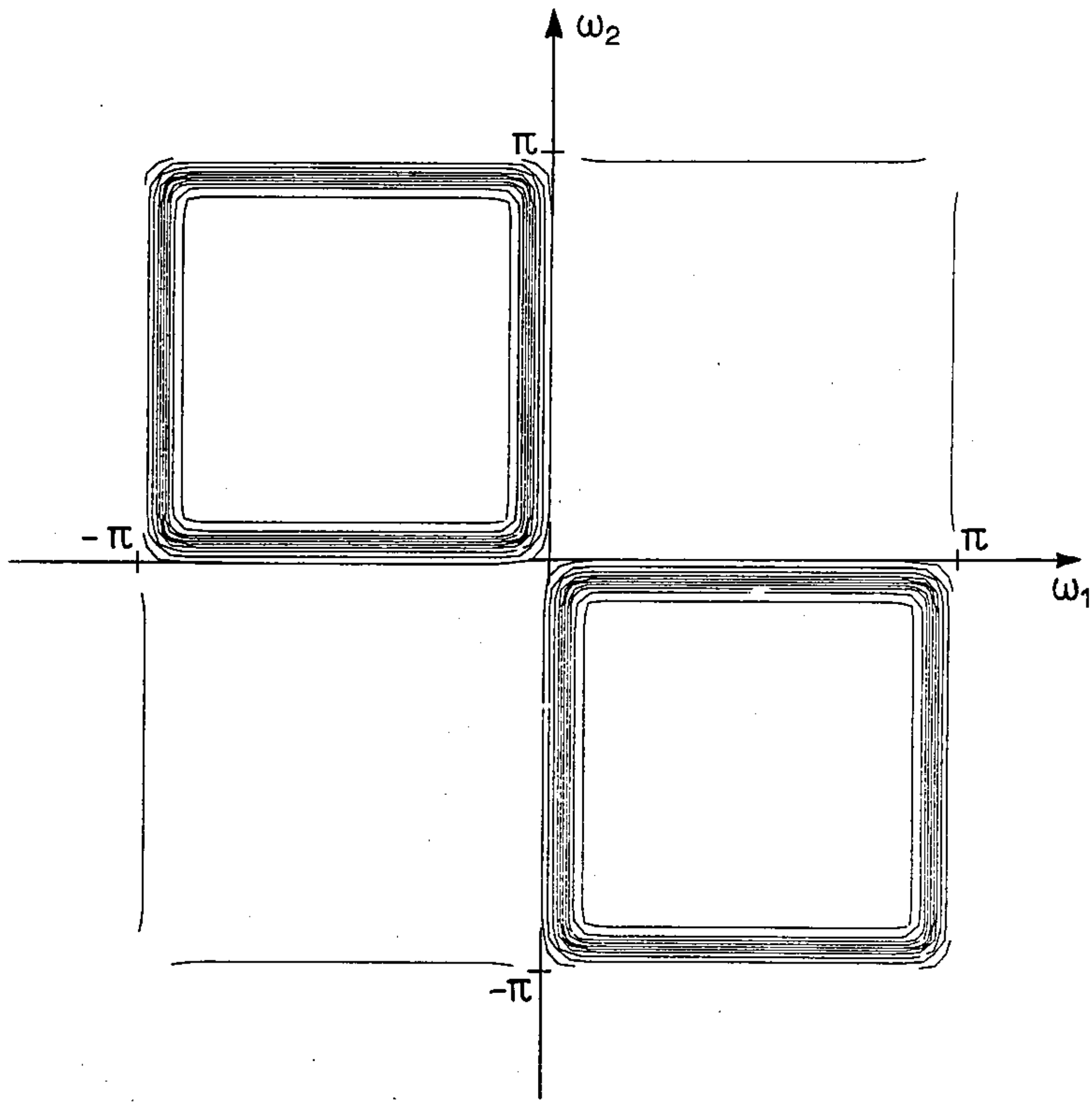


Fig. 5·21(b) Contour plot of the quadrant fan filter in Example 2.

and $\omega_s = 0.54$. The coefficients can be found in Table 2.IV.

Fig.22(a) and Fig.22(b) show the magnitude and contour plots of the frequency characteristics of the designed fan filter, respectively. From the ripples in the pass band and the shape of the transition region, it can be seen that one-dimensional characteristics of the prototype are completely preserved. If one compares this FIR fan filter with filters designed using McClellen's transformation [150], the superiority of the complex transformation method can be seen. It is interesting to note that the sharpness of the transition band of the fan filter in Fig.5.21(a) is also comparable with the other FIR fan filter techniques [164], [167].

The obtained quadrant fan filter is given in Fig.23(a) and Fig.23(b). It is possible to compare this filter with the one designed by using the generalised McClellen's transformation given by Mersereau [162]. Both filters have the same length (53 x 53). However, it can be shown that the approximation of the ideal response (5.26) of the designed filter is much better than the one suggested in [162]. The comparison can be done using both amplitude and contour plots of the filters.

Similarly, designed IIR fan and quadrant fan filters in Examples 1 and 2 can be compared with the filters obtained using previous IIR methods [150], [164], [167]. It is found that the suggested design technique

TABLE 5.IV

Coefficient Values of the Filter in Example 3.

h(1)	=	0.22112579E-02
h(2)	=	0.61455960E-02
h(3)	=	-0.10175120E-02
h(4)	=	-0.43941110E-02
h(5)	=	0.78730746E-03
h(6)	=	0.61650464E-02
h(7)	=	-0.10972550E-02
h(8)	=	-0.81230640E-02
h(9)	=	0.12600113E-02
h(10)	=	0.10617225E-01
h(11)	=	-0.14219410E-02
h(12)	=	-0.13755620E-01
h(13)	=	0.15621832E-02
h(14)	=	0.17762213E-01
h(15)	=	-0.16950150E-02
h(16)	=	-0.23061480E-01
h(17)	=	0.18116964E-02
h(18)	=	0.30431992E-01
h(19)	=	-0.19070420E-02
h(20)	=	-0.41552210E-01
h(21)	=	0.19880677E-02
h(22)	=	0.60814066E-01
h(23)	=	-0.20455840E-02
h(24)	=	-0.10437360E+00
h(25)	=	0.20808409E-02
h(26)	=	0.31773067E+00
h(27)	=	0.49790402E+00

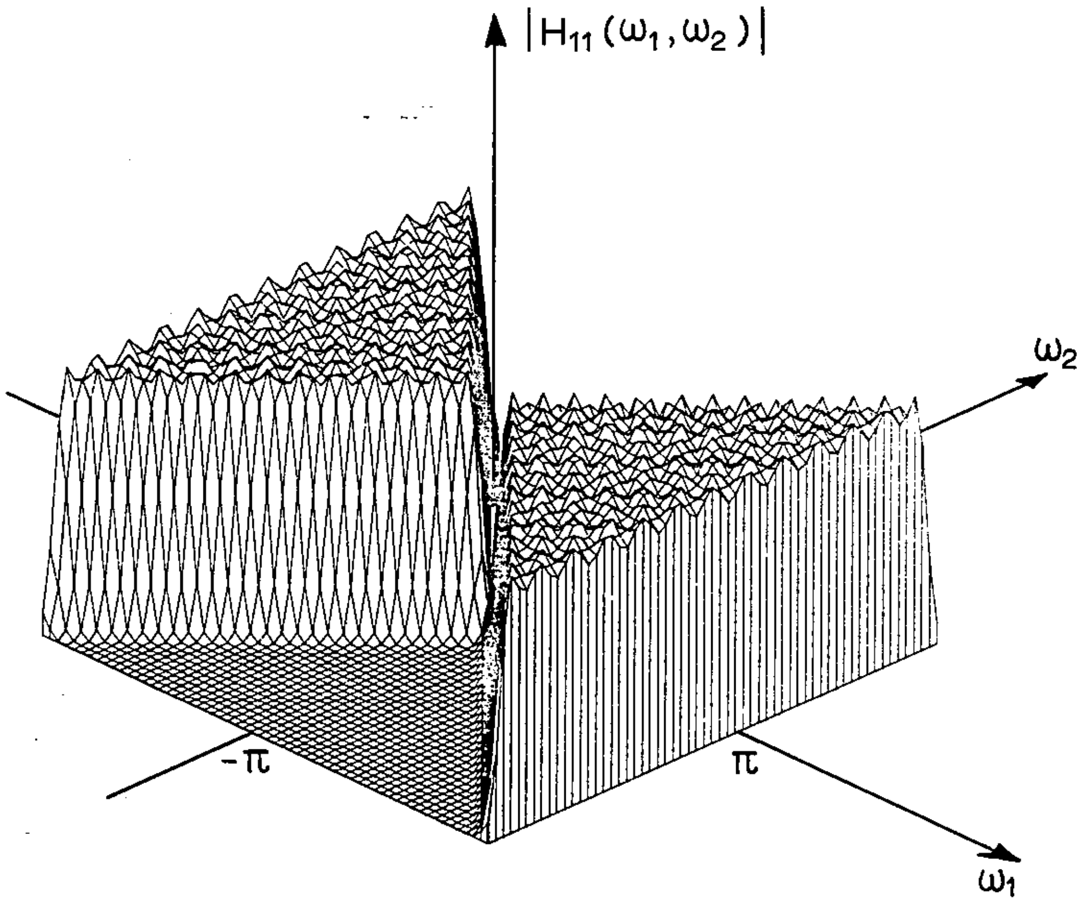


Fig. 5·22(a) Magnitude characteristics of the fan filter in Example 3.

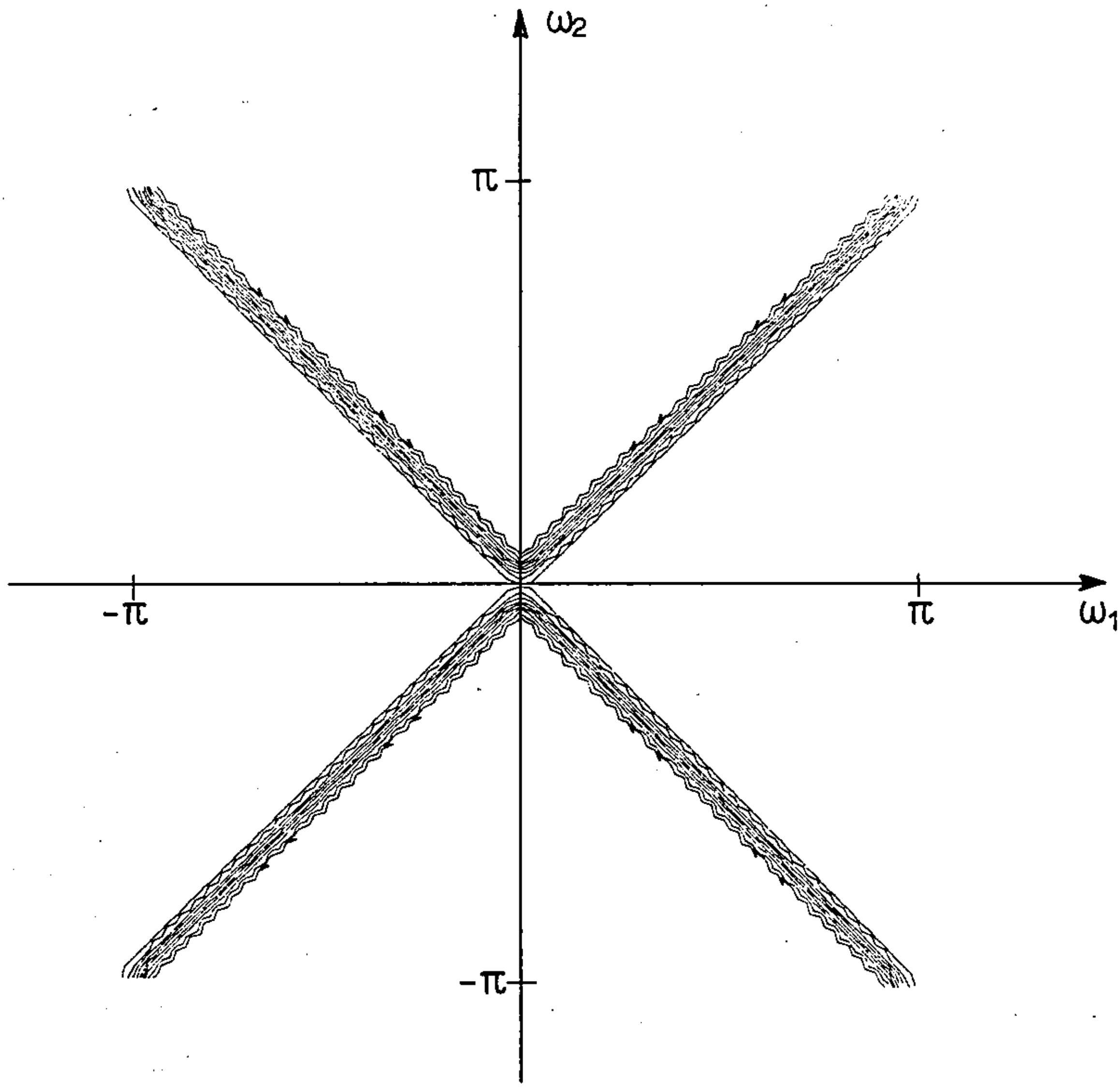


Fig. 5·22(b) Contour plot of the fan filter in Example 3.

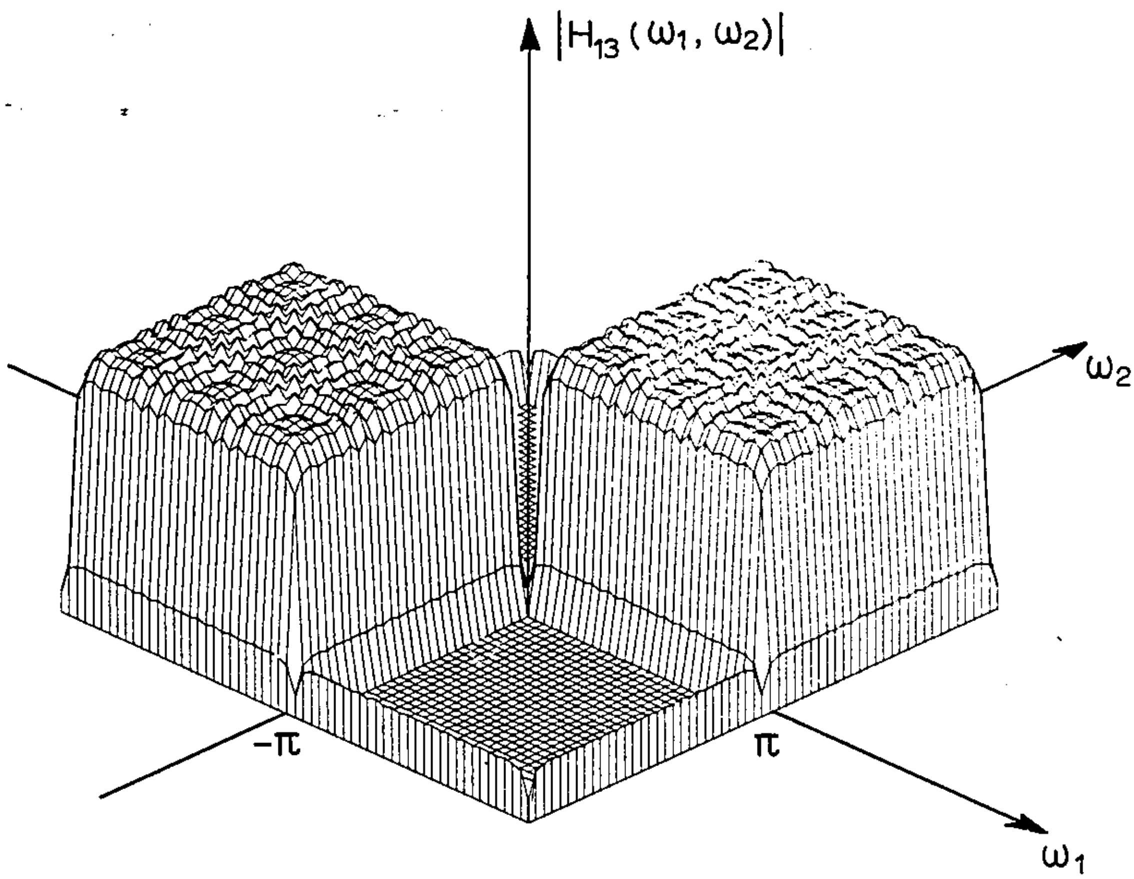


Fig. 5·23 (a) Magnitude characteristics of the quadrant fan filter in Example 3.

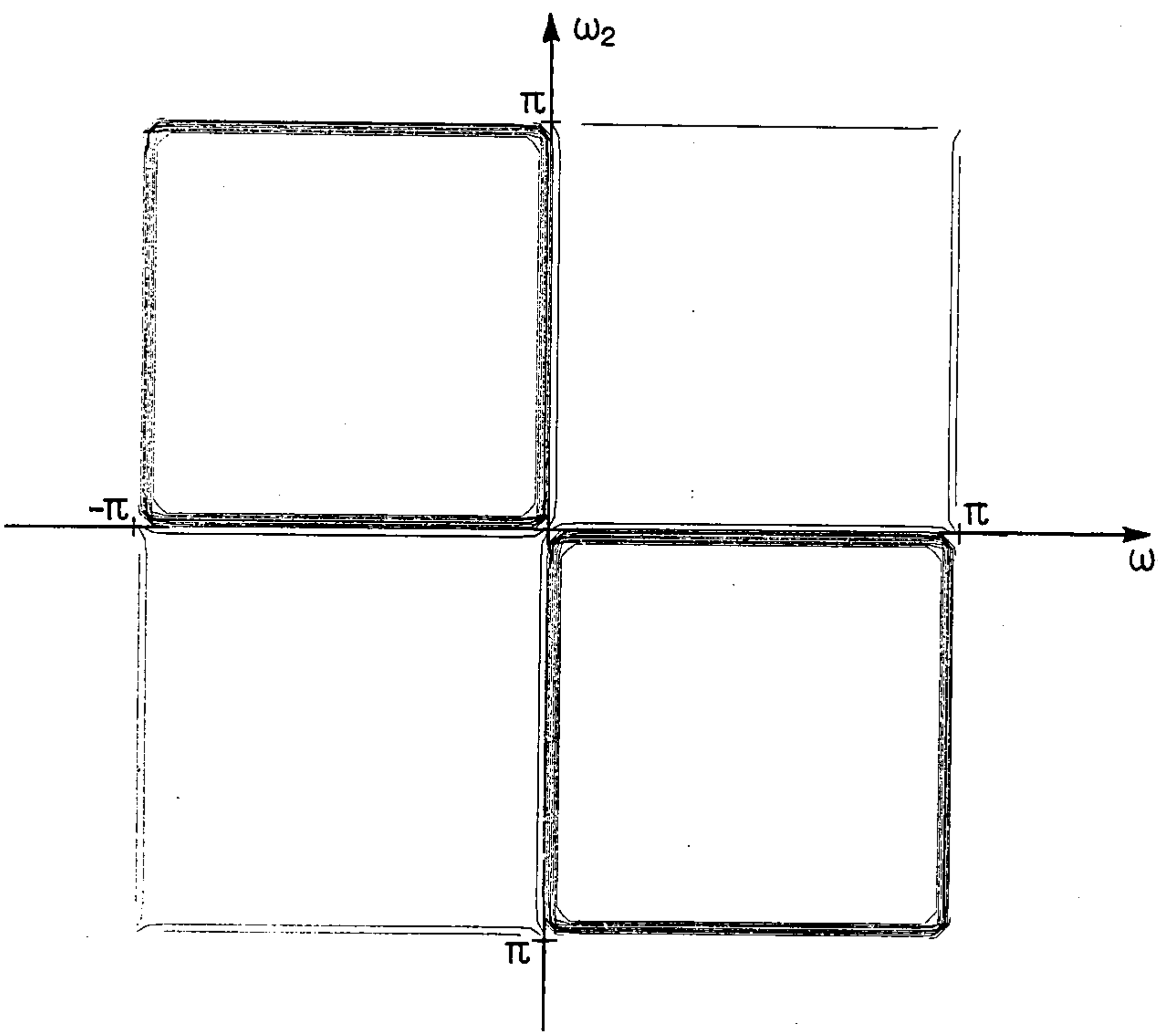


Fig. 5·23(b) Contour plot of the quadrant fan filter in Example 3.

is not only simpler than the existing ones but also its performance is superior.

Example 4

As a final example a specification was based on that proposed by McClellan [177]. Here the cut-off boundary for the fan filter extends from $\omega_1 = \frac{\pi}{5}$, $\omega_2 = 0$ at an angle of 45°. The design was based on the transformation of equation in which the cut-off frequency is at $\omega_c = \frac{\pi}{2} - \frac{\pi}{10}$ and the value of ϕ set at $\frac{\pi}{2} + \frac{\pi}{10}$. The resulting response and contour plots are shown in Figs. 24(a) and 24(b). The design is based on a 3rd order Chebyshev prototype [175].

5.6.6 Conclusions:

In this chapter we have introduced a more general class of two-dimensional spectral transformation, called complex transformations and we have developed a design procedure for designing zero-phase fan filters and quadrant fan filters.

The method offers a complete solution to the design problem of fan filters and is based on the use of a set of transformed filters obtained from the one-dimensional prototype with complex transformations. The superiority of the technique can be summarized as follows:

(1) the designed filters and the prototype both either FIR or IIR.

(2) The procedure does not introduce any stability problem. The resulting IIR filters are inherently stable.

(3) Due to the fact that the original one-dimensional filter characteristics are preserved, solutions are optimal. Hence no optimization is needed.

(4) The performance of the designed filters in the frequency domain is better than all the existing designs*.

(5) The designed zero-phase two-dimensional filter functions can be implemented as a one-dimensional filtering process.

*This comparison has been made with the technique of McClellan, Mersereau, and Marzetta, based on the error criteria defined in example 4.5.2, the simplicity of computation and complexity of design. This comparison has been made with the published results in references [150], [162] - [166]. This technique also has the advantage of achieving any given specification.

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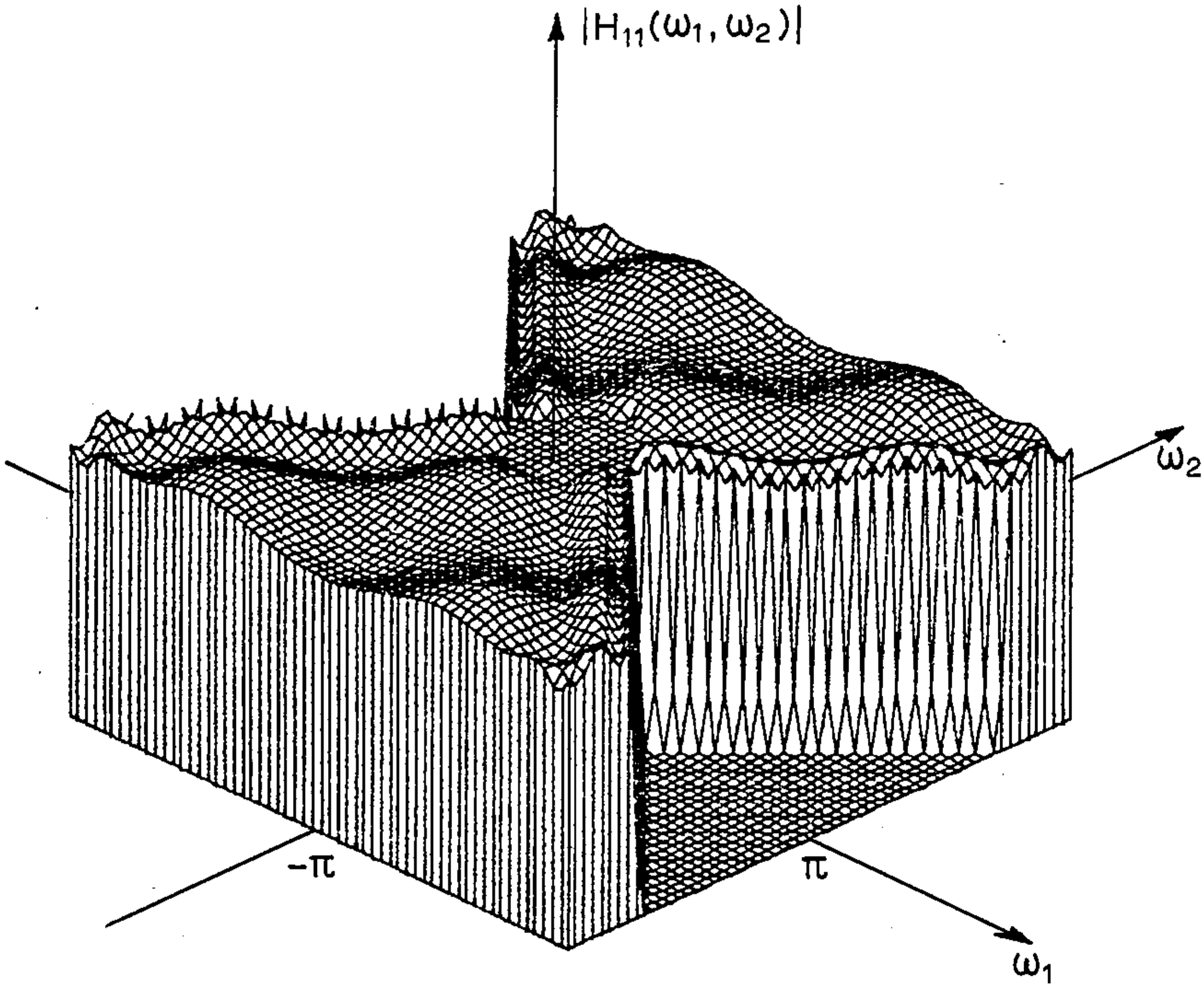

Fig. 5·24 (a) Magnitude plot of filter of Example 4

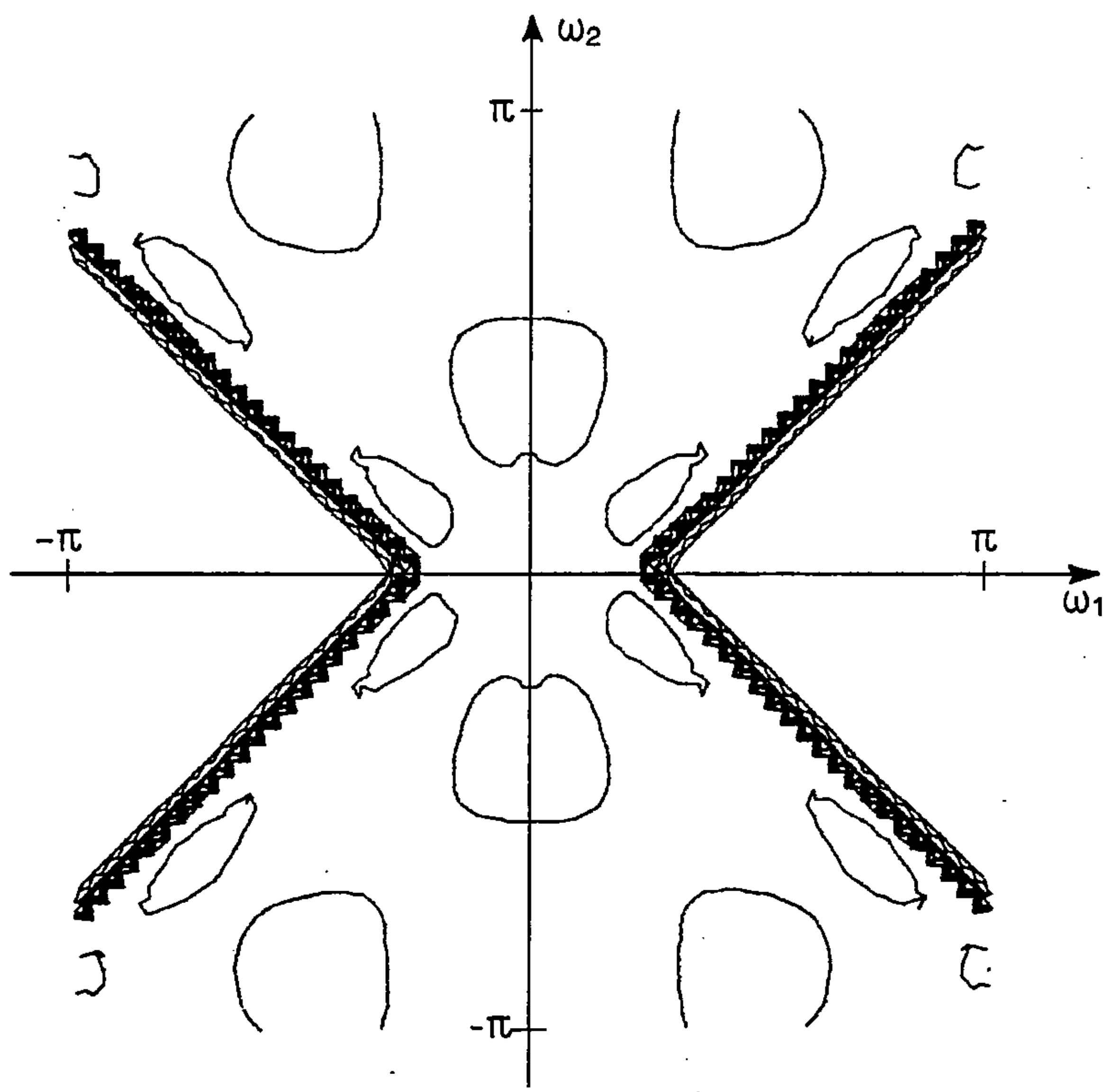

Fig. 5.24(b) Contour plot of filter of Example 4

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

6.1 CONCLUSIONS

The aim of this thesis is the investigation of the stability, stabilization and design of multi-dimensional recursive digital filters. The following problems have been considered in detail:

- Development of test for stability of two-dimensional systems.
- Investigation of the problem of the extension of Lyapunov's test into multidimensional case and, an extension of this test for the stability of Roesser's model.

A simple test has been derived for two-dimensional filters using the properties of inner determinants. The proposed test takes the form of a local positivity test applied to a two-variable polynomial with real coefficients. Furthermore, the extension of Lyapunov's stability test to Roesser's model has been developed. Some difficulties with this problem have been also shown.

- The existence of N-dimensional complex cepstrum.
- Development of a cepstral test for N-dimensional digital filters.

It has been shown that an N-dimensional complex cepstrum exists for N-dimensional rational polynomials. A method of testing the stability of N-dimensional recursive filters has been presented.

- Investigation of the two-dimensional planar least squares inverse stabilization technique and a counter-example for Jury's conjecture.
- Methods to stabilize multidimensional filters and their relation with stability.
- Development of an algorithm for Pistor's stabilization method.

Two stabilization methods, based on the Pistor decomposition have been developed. It has been shown that the Pistor technique is applicable to a more general class of recursive filters. An algorithm has been presented for computation of the spectral factors of unstable digital filter functions.

- Design of circularly symmetric filters with spectral transformations.
- Design of fan filters with Ahmadi transformation and complex transformations.

A design technique for circularly symmetric filters has been developed. A novel two-dimensional reactance function has been used to transform a one-dimensional continuous low-pass filter function to a two-dimensional continous low-pass function. Two design techniques for design of stable fan filters have been suggested. The first of them has been obtained by using the Ahmadi transformation and the second one with complex transformations.

6.2 SUGGESTIONS FOR FUTURE WORK

The following topics seem to be of interest for further research:

(1) In view of the complexity of multidimensional systems, the need for obtaining sufficiency conditions for structural stability (avoiding non-essential singularities of the second kind on the unit bidisc) is warranted.

Simple sufficiency conditions are needed for design of two-dimensional recursive digital filters.

(2) Most of the known stability tests developed so far apply to linear time invariant systems. In practice, the nonlinear effects of quantization round-off error, finite arithmetic, etc., should be taken into account for stable design. Hence, the extension of the known methods for stability analysis to nonlinear and time-varying systems is a major task.

(3) The validity of the use of the double bilinear transformation is needed for establishing the stability of two-dimensional digital filters. If we let

$$G(s_1,s_2) = \frac{1}{1+s_1+s_2} = \frac{1}{B(s_1,s_2)}$$, this is a stable function

Using $$s_1 = \frac{1-z_1}{1+z_1} \quad s_2 = \frac{1-z_2}{1+z_2}$$

we get

$$H(z_1,z_2) = \frac{(1 + z_1)(1 + z_2)}{3 + z_1 + z_2 + z_1 z_2}$$

Nonessential singularities of the second kind ($z_1 = -1$, $z_2 = -1$) now exist. $H(z_1,z_2)$ is shown to be unstable. Under what conditions does isomorphism hold? Fig. 6.1 shows the isomorphism.

(4) The extension of the Nyquist-like stability test for testing - the sign of the multidimensional polynomial on the distinguished boundary of the unit polydisc.

(5) If nonessential singularities exist on the unit bidisc how does one determine the stability of the two-dimensional filter function. In Section 2.2.1, it was shown that:

$$H_1(z_1,z_2) = \frac{(1 - z_1)^8 (1 - z_2)^8}{2 - z_1 - z_2} \quad \text{is stable}$$

but $$H_2(z_1,z_2) = \frac{(1 - z_1)(1 - z_2)}{2 - z_1 - z_2} \quad \text{is unstable}$$

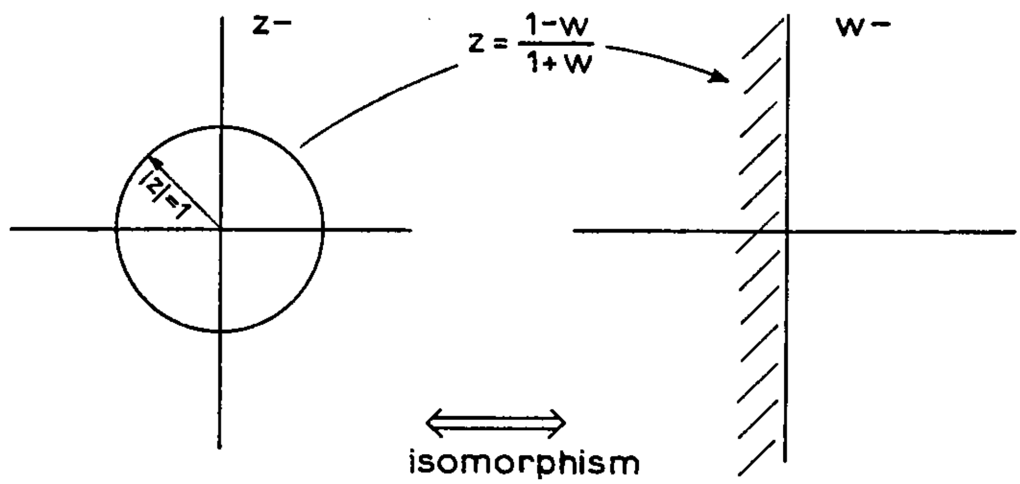

Fig. 6·1 Bilinear transformation

Also, a method is needed for testing nonessential singularities of the second kind.

(6) How does the distance from the boundary of the unit bidisc (termed as marginal stability) influence the system response and quantization error. Fig.6.2 shows the marginal stability of a stable system.

(7) Shanks' conjecture (1972) is false in general. Genin and Kamp (1975) gave a counterexample to disprove the conjecture. Later, in 1976 Jury imposed an additional constraint that if the least squares inverse are of the same degree then Shank's conjecture might be valid. However, recently, Kayran and King (1980) came up with another counterexample for Jury's conjecture. Therefore, it is of interest for effective design either to verify or refute this conjecture and in the same vein to obtain whatever additional constraints are needed to be imposed to verify the conjecture.

(8) It is well known that the one-dimensional discrete Hilbert transform can be used for the stabilisation of recursive filters. Furthermore, the magnitude function of the filter is unimpaired. In extending this method to the two-dimensional case one encounters many difficulties and, indeed, stabilisation cannot always be achieved and in addition the frequency magnitude is changed. Wood had shown in a counterexample that stability can not be achieved by a finite order filter.

Hence it is conjectured that stability can be obtained by using infinite order recursive two-dimensional filters. The verification of this conjecture is needed.

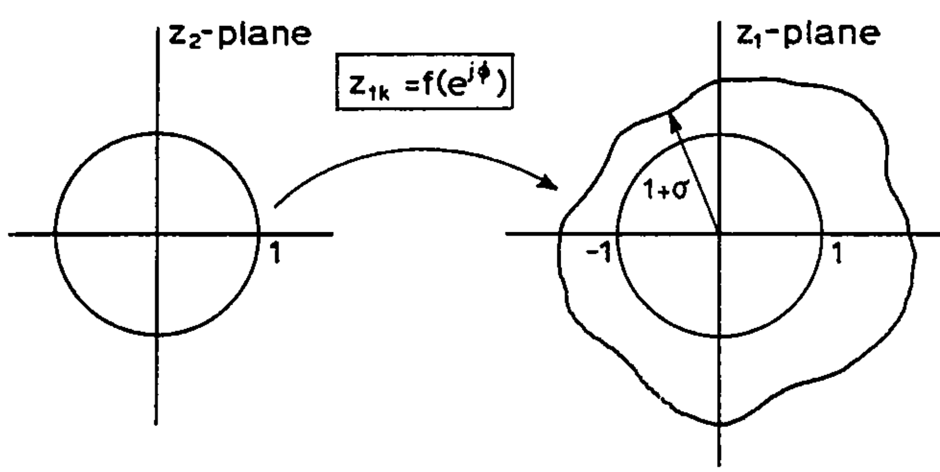

Fig. 6·2 Marginal stability

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APPENDIX - A

Let A(z) and B(z) be the following polynomials:

$$A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_o \qquad \text{(A-1)}$$

$$B(z) = b_m z^m + b_{m-1}z^{m-1} + \dots + b_o \qquad \text{(A-2)}$$

Where z is a complex variable and a_i and b_i are real or complex. We assume that $m \leq n$. A basic classical result is that the determinant of the (m+n) - order Sylvester matrix [22].

$$\left[\Delta_{m+n}\right] = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_o & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_1 & a_o & \cdots & 0 \\ \vdots & \vdots & & & & & & \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_o \\ \vdots & & \cdots & b_m & b_{m-1} & b_{m-2} & \cdots & b_o \\ 0 & b_m & b_{m-1} & \cdots & \cdots & b_1 & \cdots & 0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & \cdots & b_o & \cdots & 0 \end{bmatrix}$$

is nonzero if and only if A(z) and B(z) are relatively prime (that is, no common zeroes exist between A(z) and B(z)). This determinant is called the resultant R[A,B] of A and B.

APPENDIX - B

'If' part. Suppose that $|z_1| = |z_2| = 1$ is a root of $B(z_1, z_2)$. Recall

$$G(x,y) = D(x,z_2)\ D(x,z_2^{-1}) \tag{B-1}$$

where

$$D(x,z_2) = B(z_1,z_2)\ B(z_1^{-1},z_2) \tag{B-2}$$

The complex variables z_1 and z_2 are replaced by the corresponding real variables x and y defined by:

$$x = \frac{z_1 + z_1^{-1}}{2} \quad \text{on } |z_1| = 1 \tag{B-3}$$

$$y = \frac{z_2 + z_2^{-1}}{2} \quad \text{on } |z_2| = 1 \tag{B-4}$$

From equations (3) and (4), it can easily be seen that $x \in [-1,1]$ and $y \in [-1,1]$ is also a root of $G(x,y)$.

'Only if' part. Necessity can be shown by contradiction. Consider $G(x,y)$ has a real root (x,y) such that $x,y \in [-1,1]$, and suppose that $B(z_1,z_2) \neq 0$ for $|z_1| = |z_2| = 1$. Since $B(z_1,z_2) \neq 0$ on the distinguished boundary, from equation (1) and (2):

$$D(x,z_2) \neq 0 \qquad x \in [-1,1] \quad , \quad |z_2| = 1 \tag{B-5}$$

$$D(x,z_2) \neq 0 \qquad y \in [-1,1] \quad , \quad |z_2| = 1 \tag{B-6}$$

From equations (5) and (6) ;

$$G(x,y) \neq 0 \quad \text{for} \quad x,y \in [-1,1] \tag{B-7}$$

However, this contradicts our assumption. Hence when $G(x,y) \neq 0$ for $x,y \in [-1, 1]$, there exists some z_1 and z_2 such that $B(z_1,z_2) = 0$ for $|z_1|=|z_2|=1$.

APPENDIX-C

A two dimensional analogue function

$$T(s_1,s_2) = \frac{A(s_1,s_2)}{B(s_1,s_2)}$$

is stable if $B(j\omega_1,s_2) \neq 0 \quad Re(s_2) \geqslant 0$

$$B(s_1,1) \neq 0 \quad Re(s_1) \geqslant 0$$

Considering $$T(s_1,s_2) = \frac{a_1s_1 + a_2s_2}{1 + b_1(s_1^2 + s_2^2) + b_2s_1s_2} \qquad (C1)$$

First condition

$$B(j\omega_1,s_2) = b_1s_2 + j\omega_1b_2s_2 + (1-\omega_1^2b_1) \qquad (C2)$$

Equating $B(j\omega_1,s_2) = 0$ gives roots:

$$s_2 = \frac{-j\omega_1b_2 \pm \sqrt{\omega_1^2(4b_1^2 - b_2^2) - 4b_1}}{2b_1} \qquad (C3)$$

To satisfy $Re(s_2) < 0$

$$\omega_1^2(4b_1^2 - b_2^2) - 4b_1 < 0 \quad \text{for all} \quad \omega_1$$

$$\rightarrow b_1 > 0 \quad , \quad 4b_1^2 - b_2^2 < 0 \tag{C4}$$

<u>Second condition</u>

$$B(s_1,1) = 1 + b_1(s_1^2 + 1) + b_2 s_1 \tag{C5}$$

Equating $B(s_1,1)$ to zero gives:

$$s_1 = \frac{-b_2 \pm \sqrt{b_2^2 - 4b_1(b_1 + 1)}}{2b_1} \tag{C6}$$

To satisfy $Re(s_1) < 0$

$$b_2 > 0 \quad \text{and} \quad b_2^2 - 4b_1(b_1 + 1) < 0$$

$$\text{or} \quad b_2 > 0 \quad \text{and} \quad - 4b_1(b_1 + 1) < 0 \tag{C7}$$

Conditions (C4) and (C7) may be combined with

$$b_2 > 0$$

$$b_1 > \frac{b_2^2}{4} - b_1^2 > 0 \tag{C8}$$

APPENDIX -D

Recurse Directions of Finite Area Array:

A causal filter recurses in the (+m,+n) direction. The causal recursion starts at the NW corner of an input array as indicated in Fig.4.3.

Finite Area Array: A two-dimensional array that is nonzero for only a finite area in the spatial domain is referred to as a finite area array [147].

For a finite area array I(MxN), the following array operations are used for different recurse directions:

1) 180° rotation M/2 -axis (row reversal)

$$I_1(m,n) = I(M-m+1,n) \tag{D1}$$

2) 180° rotation about N/2 -axis (column reversal)

$$I_2(m,n) = I(m,N-n+1) \tag{D2}$$

3) Clockwise 180° rotation (row-column reversal)

$$I_3(m,n) = I(M-m+1,N-n+1) \tag{D3}$$

One can obtain from equation (2.1) recursive filters recursing in the (-m,n), (m,-n), and (-m,-n) directions with $I_1(m,n)$, $I_2(m,n)$, $I_3(m,n)$, respectively.