FILTERING, REGULATION AND AN INTERNAL MODEL PRINCIPLE FOR LINEAR DELAY SYSTEMS

by

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ABSTRACT

This work is concerned with two problems involving time delay systems.

The first part of this thesis deals with the classical regulator problem of control theory. In particular, we investigate the necessary structural features of a controller which yields output regulation and internal stability despite uncertainty in some of the system's and controller parameters. Our approach consists in transforming the original delay differential system into an evolution equation in an infinite dimensional Hilbert space. In this abstract setting, under the assumption of internal stability, a useful characterization of the regulation condition is obtained by means of a linear operator equation. Then, it is shown that stabilizability and detectability of both, the system and controller, are necessary conditions for internal stability to hold. The concepts of readability and internal model are extended for the class of evolution systems of our concern. Next, it is shown that a structurally stable controller incorporates feedback of the regulated variables, together with an internal model of the dynamic structure of the external signals which the controller is required to process. Necessity of these structural features constitutes the Internal Model Principle for delay systems. The sufficiency of the Internal Model Principle is investigated. Necessary and sufficient conditions are derived, in terms of the system's parameters, to assure the existence of a structurally stable controller. Also, a design procedure to construct such controller is obtained. We point out that these results are known for finite dimensional systems with no delays. However, the appropriate manner in which the Internal Model Principle should be formulated for delay equations is by no means obvious, and the technical

problems in obtaining the main analogues of the known delay-free results are quite considerable.

The second part of this thesis is concerned with the optimal filtering problem for linear systems involving time delays in the state, observations and noise process. To our knowledge, this is the first rigorous treatment of linear systems containing point delays in the noise process. Our approach is based on projection methods in the Hilbert space of square integrable random vectors. It is shown that the filtered estimate satisfies a stochastic functional differential equation which is coupled with the integral equation for the smoothed estimates. The optimal filter is characterized by two gains. One of the these gains is the usual error covariance matrix function. The second gain is expressed in terms of the error covariance and the fundamental matrix associated with the homogeneous part of the delay differential system. The error covariance function satisfies a set of three coupled Riccati-type partial differential equations. Two of these equations involve the fundamental matrix previously mentioned. When no delays occur in the state and observations, the second gain may be expressed in terms of the fundamental matrix associated with the error functional differential equation. In this case, the gains involved in the optimal filter are shown to be unique solution of two coupled Riccati-type differential equations. Next, a dual optimal control problem is obtained. The dual system contains delays in the state, control and observations. The optimization problem consists in minimizing a quadratic functional of the observations and controls. In the case of no delays in the state and controls, a feedback realization for the optimal control is obtained by exploiting our results on the filtering problem.

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CONTENTS

PART I

REGULATION AND AN INTERNAL MODEL PRINCIPLE FOR LINEAR TIME DELAY DYSTEMS

CHAPTER I

INTRODUCTION

A problem of major interest in control theory is that of synthesizing controllers which regulate a given linear system and provide internal stability. A more practical problem is the design of controllers which preserve regulation and internal stability despite uncertainty in some of the system's and controller parameters. Such class of controllers are referred to as being structurally stable.

The above problems have been widely studied for linear systems modelled by ordinary differential equations [W1], [W2], [F1] - [F5], [S1]. The main result of these investigations may be summarized as the Internal Model Principle (IMP), that is the necessary structural features of a controller which is structurally stable. The sufficiency of the IMP has also been investigated [F2]. Necessary and sufficient conditions to assure the existence of a structurally stable controller and procedures to design such controllers have been established in [F4], [S1].

Recently, the regulation and internal stability problem has been investigated by Bhat [B1] for a larger class of linear systems, namely those described by abstract evolution equations. In this setting an Internal Model Principle was derived and applications to time delay systems were investigated. However, Bhat's version of the IMP is incomplete as compared with available results for ordinary systems. More precisely, in [B1] it is assumed that the controller is 'driven' by the regulated variables while this feedback structure constitutes an essential part of the IMP for ordinary systems. Also, Bhat's treatment of time delays systems contains a significant mistake which restricts the validity of his results. To be precise, Bhat claims [B1, Chapter 6, 6.5.1] that variations in the elements of the matrices in the delay equation correspond to 'bounded' perturbations of the parameters in the associated evolution system. Contrary to this claim it will be shown later in Chapter 3, that some of such variations of matrix parameters yield 'unbounded' perturbations of the parameters in the corresponding evolution equation. Since Bhat's developments are confined to deal with bounded parameter perturbations, it turns out that his results are not completely satisfactory when applications to time delay systems are considered.

In this thesis the problem of main concern is that of obtaining a full version of the IMP for time delay systems. From this point of view our work is a generalization of Bhat's results. Our approach consists in transforming the original delay system into an equivalent evolution equation. By introducing this abstract representation we are able to study a larger class of delay differential systems, e.g. systems with multiple and distributed delays. In constrast with Bhat's work, we will restrict our treatment to those evolution systems arising from delay equations, but certain class of unbounded parameter perturbations will be considered. It will turn out however, that our results will be valid for a larger class of evolution systems provided that the parameters of these systems satisfy certain conditions which will be determined by properties of the parameters of time delay systems. (We point out that at the present it is very difficult, if not impossible, to obtain significant results when we allow unbounded parameter perturbations without making strong assumptions on the evolution system).

In the following we briefly describe the development of this work. In Chapter 2 we formulate our problem in an abstract setting, that is we write our original delay system as an evolution on an infinite dimensional Hilbert space. We will then obtain a useful characterization of the regulation condition. Finally, from the requirement of internal stability we will derive some necessary features of our system and controller. Most of the results of this chapter are extensions of Bhat's work [B1, Chapter 5] .

In Chapter 3 we will obtain an IMP for time delay systems. Also, necessary conditions for the existence of a structurally stable controller will be derived and some concepts and results used in establishing the IMP for the delay-free case [F1] will be extended to the class of systems of our concern. Our developments will be based on the 'decomposition of a linear operator equation' (as in the delay-free case [F1]). We mention that an alternative approach is possible. In fact, we could analyze directly this 'linear operator equation', as in [B1, Chapter 5,6] (also see [W1, Chapter 8] for the delay-free situation). However, this approach would increase the technical difficulties considerably and the understanding of our problem would be obscured. We finally point out that the special properties of time delay systems will play a fundamental role throughout this chapter.

In Chapter 4 we will derive necessary and sufficient conditions to assure the existence of a structurally stable controller. These conditions will be given in terms of the system parameters. The sufficiency of the IMP will also be investigated. A procedure for constructing a structurally stable synthesis will be obtained. The devlopment of this chapter will require some of the results obtained by Bhat, in particular, the observer theory for evolution systems in [B1, Chapter 4](see also [B9] where applications to delay systems are considered).

CHAPTER 2

REGULATION AND INTERNAL STABILITY

This chapter deals with the regulation and internal stability problem for linear delay systems. We shall first state our problem and then we will give an abstract formulation in an infinite dimensional vector space. Under the assumption of internal stability we will obtain necessary and sufficient conditions for regulation to hold. These conditions will constitute our point of departure for further developments in Chapters 3 and 4. We will then show that it is possible to obtain an equivalent 'reduced' problem in which part of our original system is modelled by an ordinary differential equation. Finally, the necessity of certain stabilizability and detectability conditions, for both the system and controller, will be established.

2.1 Problem Formulation

Consider the time delay differential system

$$\dot{\hat{x}}_1(t) = \hat{A}_0\hat{x}_1(t) + \hat{A}_1\hat{x}_1(t-h) + \hat{A}_4\hat{x}_2(t) + \hat{A}_5\hat{x}_2(t-h) + \hat{B}_1u(t) \tag{2.1}$$

$$\dot{\hat{x}}_2(t) = \hat{A}_2\hat{x}_2(t) + \hat{A}_3\hat{x}_2(t-h) \tag{2.2}$$

$$y(t) = \hat{C}_1\hat{x}_1(t) + \hat{C}_2\hat{x}_2(t) \tag{2.3}$$

$$z(t) = \hat{D}_1\hat{x}_1(t) + \hat{D}_2\hat{x}_2(t) \tag{2.4}$$

where $h > 0$, $\hat{x}_1 \in \mathbb{R}^{n_1}$, $\hat{x}_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ the initial segments $\hat{\phi}_1(\theta)$, $\hat{\phi}_2(\theta)$, $\theta \in [-h,0]$ are elements of the function spaces $L_2[-h,0];\mathbb{R}^{n_1})$ and $L_2([-h,0];\mathbb{R}^{n_2})$ respectively.

(2.1) represents the system's dynamics

(2.2) is a model for disturbance and/or reference signals

(2.3) corresponds to the observation process

(2.4) are the variables to be regulated

The regulation and internal stability problem consists in determining a controller for the system (2.1) - (2.4) such that

i) $z(t) \to 0$ as $t \to \infty$, i.e. $z(t)$ is regulated

ii) the plant (2.1) together with the controller are asymptotically stable, i.e. the closed system is internally stable

In order to provide an adequate setting for our problem, we will write (2.1) - (2.4) as evolution equations in the infinite dimensional Hilbert spaces $X_1 = M_2^{n_1} = \mathbb{R}^{n_1} \times L_2([-h,0]; \mathbb{R}^{n_1})$ and $X_2 = M_2^{n_2} = \mathbb{R}^{n_2} \times L_2([-h,0]; \mathbb{R}^{n_2})$. It can be shown [B2] - [B4], [D1], [D2] that (2.1) - (2.4) can be equivalently represented by

$$\frac{dx_1(t)}{dt} = A_1x_1(t) + A_3x_2(t) + B_1u(t) \qquad (2.5)$$

$$\frac{dx_2(t)}{dt} = A_2x_2(t) \qquad (2.6)$$

$$y(t) = C_1x_1(t) + C_2x_2(t) \qquad (2.7)$$

$$z(t) = D_1x_1(t) + D_2x_2(t) \qquad (2.8)$$

where $x_1 = (\hat{x}_1^0, \hat{x}_1^1) \in X_1 = M_2^{n_1}$, $x_2 = (\hat{x}_2^0, \hat{x}_2^1) \in X_2 = M^{n_2}$, $u \in \mathbb{R}^m = U$, $y \in \mathbb{R}^p = Y$, $z \in \mathbb{R}^q = Z$, $x_1(0) = (\hat{\emptyset}_1(0), \hat{\emptyset}_1)$, $x_2(0) = (\hat{\emptyset}_2(0), \hat{\emptyset}_2)$ and all the operators are bounded†, except A_1, A_2 and A_3 which are unbounded. A_1 and A_2 are closed with dense domains $D(A_1)$ and $D(A_2)$ respectively.

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† B_1, C_1, C_2, D_1 and D_2 are in fact compact since either their domain or range are finite dimensional. This will be of crucial importance in further developments.

$$B_1u = (\hat{B}_1u,0)$$

$$C_1x_1 = \hat{C}_1\hat{x}_1^0 \quad , \quad D_1x_1 = \hat{D}_1\hat{x}_1^0$$

$$C_2x_2 = \hat{C}_2\hat{x}_2^0 \quad , \quad D_2x_2 = \hat{D}_2\hat{x}_2^0$$

$$A_1x_1 = (\hat{A}_0\hat{x}_1^0 + \hat{A}_1\hat{x}_1^1(-h), \frac{d\hat{x}_1^1}{d\theta}) \quad , \quad x_1 \in D(A_1)$$

$$A_2x_2 = (\hat{A}_2\hat{x}_2^0 + \hat{A}_3\hat{x}_2^1(-h), \frac{d\hat{x}_2^1}{d\theta}) \quad , \quad x_2 \in D(A_2)$$

$$A_3x_2 = (\hat{A}_4\hat{x}_2^0 + \hat{A}_5\hat{x}_2^1(-h), \ 0 \ )$$

$$D(A_1) = \left\{x_1 \in M_2^{n_1} \,|\, \hat{x}_1^1 \text{ is absolutely continuous, } \hat{x}_1^1(0) = \hat{x}_1^0 \text{ and } \frac{d\hat{x}_1^1}{d\theta}(\theta) \in L_2([-h,0]; \mathbb{R}^{n_1})\right\}$$

$$D(A_2) = \left\{x_2 \in M^{n_2} \,|\, \hat{x}_2^1 \text{ is absolutely continuous, } \hat{x}_2^1(0) = \hat{x}_2^0 \text{ and } \frac{d\hat{x}_2^1}{d\theta}(\theta) \in L_2([-h,0]; \mathbb{R}^{n_2})\right\}$$

The inner product in $X_1 = M_2^{n_1}$ is defined by

$$\langle x_1, z_1\rangle_{X_1} = \langle \hat{x}_0^0 , \hat{z}_1^0\rangle_{\mathbb{R}^{n_1}} + \langle \hat{x}_1^1, \hat{z}_1^1\rangle_{L_2([-h,0];\mathbb{R}^{n_1})}$$

and X_1 is endowed with the norm induced by this inner product. (Similarly for X_2).

Some features associated with the operators A_1, A_2 and A_3 will be useful in later developments. We first consider the operator A_1

P1) A_1 is the infinitesimal generator of a strongly continuous semigroup of bounded operators $S_1(t)$, $t \geq 0$. $S_1(t)$ is differentiable and compact for $t \geq h$. [S2].

P2) For $\lambda \in \rho(A_1)$, i.e. λ belongs to the resolvent set of A_1, the resolvent operator $(A_1-\lambda)^{-1}$ is compact [S2].

P3) The spectrum of A_1 consists of eigenvalues (i.e. $\sigma(A_1)$=point spectrum) with finite multiplicites, and the number of eigenvalues with real part greater than a given (arbitrary) constant is finite, that is, the set $\{\lambda \in \sigma(A_1) | \mathrm{Re}\,\lambda > w\}$ is finite for any number w, [S2][V1].

P4) The exponential growth (stability) of the semigroup $S_1(t)$ is determined by the spectrum of A_1 [T1] [S2], i.e. for each $w > w_0$ there is a constant $M_w < \infty$ such that $\| S_1(t) \| \leq M_w e^{wt}$ $t \geq 0$ where†

$$w_0 \underline{\underline{\mathrm{def}}} \lim_{t\to\infty} \ln \| S_1(t) \| / t = \sup \mathrm{Re}\, \sigma(A_1)$$

Clearly the operator A_2 also satisfies $P_1 - P_4$ above. We point out that these properties are interconnected. Indeed, P4 follows from the compactness of $S_1(t)$ for $t \geq h$ (P4 is also satisfied in a number of other situations, see [T1, Section 2]). The first assertion in P3 is a consequence of the compactness of the resolvent operator $(A_1-\lambda)^{-1}$ (see, e.g. [K1. p.187, th. 6.29]). The second part in P3 can be deduced from the compactness of the semigroup $S_1(t)$, $t \geq h$ as in [V1].

Concerning the unbounded operator A_3, we further note that it is not even closable, i.e. does not have a closed extension. However, A_3 is an A_2-compact operator. Indeed, let $x_2 \in D(A_2)$ then we may write

$$\hat{x}_2^1(-h) = \hat{x}_2^0 - \int_{-h}^{0} \dot{\hat{x}}_2^1(\theta)d\theta$$

also define

$$|||x_2|||^2 = \| (x_2, A_2x_2) \|^2_{X_2\times X_2} = \| x_2 \|^2_{X_2} + \| A_2x_2 \|^2_{X_2}$$

Since A_2 is closed, it follows that $D(A_2)$ becomes a Banach space with the norm $|||\cdot|||$. We now show that A_3 is bounded on $D(A_2)$ under this norm.

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† Triggiani [T1] refers to this identity as the 'spectrum determined growth assumption'.

$$\| A_3x_2 \|_{X_1} \leq \| \hat{A}_4 + \hat{A}_5 \| \quad \| \hat{x}_2^0 \|_{\mathbb{R}^{n_2}} + \| \hat{A}_5 \| \sqrt{h} \; \| \dot{\hat{x}}_2^1 \|_{L_2([-h,0];\mathbb{R}^{n_1})}$$

thus, defining $M = (\| \hat{A}_4 + \hat{A}_5 \|^2 + \| \hat{A}_5 \|^2 h)^{\frac{1}{2}}$ we obtain†

$$\| A_3x_2 \|_{X_1} \leq M \, (\| \hat{x}_2^0 \|^2_{\mathbb{R}^{n_2}} + \| \dot{\hat{x}}_2^1 \|^2_{L_2})^{\frac{1}{2}}$$

but

$$\| \dot{\hat{x}}_2^1 \|^2_{L_2} = \| (A_2x_2)^1 \|^2_{L_2}$$

and since

$$\| x_2 \|^2_{X_2} = \| \hat{x}_2^0 \|^2_{\mathbb{R}^{n_2}} + \| \dot{\hat{x}}^1_2 \|^2_{L_2}$$

$$\| A_2x_2 \|^2_{X_2} = \| (A_2x_2)^0 \|^2_{\mathbb{R}^{n_2}} + \| (A_2x_2)^1 \|^2_{L_2}$$

we have

$$\| A_3x_2 \|_{X_1} \leq M \; |||x_2||| \qquad , \quad x_2 \in D(A_2)$$

Hence A_3 is A_2-bounded (see Appendix B), and since the range of A_3 is finite dimensional we conclude that A_2-compact.

By considering the evolution system (2.5) - (2.8) it is now clear that we are able to study the regulation and internal stability problem for a larger class of system than those modelled by (2.1) - (2.4), e.g. systems with multiple and distributed delays, or even those evolution systems with parameters having the properties mentioned in the preceding paragraphs (in particular P1-P4).

In this abstract setting, the controller equation may be written as follows

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† in the sequel we will write L_2 in place of $L_2([-h,0];\mathbb{R}^n)$.

$$u(t) = F_c x_c(t) + G_c y(t) \tag{2.9}$$

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + B_c y(t) \tag{2.10}$$

where $x_c \in \mathcal{X}_c = M_2^{n_c}$, F_c, G_c and B_c are bounded operators.

We assume that A_c is an unbounded closed operator with dense domain $D(A_c)$ and it is convenient, but not unreasonable, to suppose that A_c satisfies P1) - P4), i.e. A_c shares the properties of A_1 (and A_2).

As mentioned previosuly, the purpose of the controller is two-fold, that is, to regulate z(t) (given by (2.8)) and to yield internal stability, i.e. the closed loop operator (without the exosystem) must be asymptotically stable. We will show later that under our assumptions on A_c, internal stability will be determined by the spectrum of the closed loop operator.

Finally, it is important to note that the controller (2.9) - (2.10) has only access to the measured variables y(t) and we do not assume any a priori relation between y(t) and z(t). These conditions constitute the main difference between Bhat's formulation and ours.

2.2 Characterizations of Internal Stability and Regulation

In this section we will show that internal stability of the closed loop system is determined by the $\sigma(A_L)$ where A_L denotes the closed loop operator (without the disturbance signals). Then we will obtain a useful characterization of regulation.

Consider the loop operator

$$A_L = \begin{pmatrix} A_1+B_1G_cC_1 & B_1F_c \\ B_cC_1 & A_c \end{pmatrix} : D(A_1) \times D(A_c) \to \mathcal{X}_1 \times \mathcal{X}_c$$

Clearly A_L may be decomposed as $A_L = \tilde{A} + \tilde{B}$

where

$$\tilde{B} = \begin{pmatrix} B_1 G_c C_1 & B_1 F_c \\ B_c C_1 & 0 \end{pmatrix} \text{ is a bounded operator on } X_L = X_1 \times X_c$$

and

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_c \end{pmatrix} : D(A_L) = D(A_1) \times D(A_c) \to X_L \text{ is an unbounded operator.}$$

We can now study the properties of A_L via the 'simpler' operator $\tilde{A}$.

<u>Lemma 2.1</u>: a) A_L is a closed unbounded operator with dense domain $D(A_L) \subset X_L$.

b) A_L is the infinitesimal generator of a strongly continuous semigroup $S_L(t)$, $t \geq 0$.

c) $S_L(t)$ is compact for $t \geq h$.

<u>Proof</u>: a) Since $\tilde{A}$ is closed (A_1 and A_c are closed) and $\tilde{B}$ is bounded the result follows from the fact that closedness is a stable property under bounded perturbations [K1, p.203, th.2.14]. Clearly $\tilde{A}$ is densely defined since both $D(A_1)$ and $D(A_c)$ are dense in X_1 and X_c respectively.

b) It is easy to see that $\tilde{A}$ is the infinitesimal generator of the strongly continuous semigroup

$$S_{\tilde{A}}(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_c(t) \end{pmatrix}, \quad t \geq 0$$

where $S_1(t)$ and $S_c(t)$ are the semigroups generated by A_1 and A_c respectively. The property of being a generator is stable under bounded perturbations [K1, p. 497, th. 2.1]. This proves b)

c) $S_L(t)$, $t \geq 0$ satisfies the perturbation formula [K1, p.497]

$$S_L(t) = S_{\tilde{A}}(t) + \int_0^t S_{\tilde{A}}(t-s)\tilde{B}\, S_L(s)ds$$

Clearly $S_{\tilde{A}}(t)$ is compact for $t \geq h$, and since Im $\tilde{B}$ is finite dimensional the second term in the expression above is compact (see the arguments in [T3, Lemma 2.1] or [S3]). Thus, $S_L(t)$ is also compact for $t \geq h$.

Our next result is concerned with the stability of A_L. We first need the following

Definition. We say that the infinitesimal generator $A : D(A) \to X$ of a strongly continuous semigroup $S(t)$, $t \geq 0$ is (asymptotically) stable, if for all $x \in X$ there are constants $M < \infty$ and $\omega < 0$ such that

$$\| S(t)x \| \leq Me^{\omega t} \| x \| , \qquad t \geq 0 \qquad \text{all } x \in X.$$

A fundamental difficulty regarding the stability of an unbounded operator is that the inclusion of its spectrum in the open left half plane is not sufficient to guarantee its stability. However, in our case we have the following result

Lemma 2.2: The semigroup $S_L(t)$, $t \geq 0$ is asymptotically stable if and only if $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(A_L)$.

Proof: We first note that the infinitesimal generator A_L is asymptotically stable if and only if there exist two constants $M < \infty$ and $\omega < 0$ such that

$$\| S_L(t) \| \leq Me^{\omega t}, \qquad t \geq 0$$

Now, from semigroup theory, it is known [D3, part I, Chapter VIII], [H1, pp. 306 and 457] that for any $\varepsilon > 0$ there is a constant $M_\varepsilon < \infty$ such that $\| S_L(t) \| \leq M_\varepsilon e^{(\omega_0+\varepsilon)t}$, where

$$\omega_0 \underset{=}{\text{def}} \lim_{t\to\infty} \ln \| S_L(t) \| /t \geq \sup \mathrm{Re}\, \sigma(A_L)$$

therefore if $S_L(t)$ is asymptotically stable we must have $\sup \mathrm{Re}\, \sigma(A_L) < 0$. Now suppose that $\sup \mathrm{Re}\, \sigma(A_L) < 0$ then, since $S_L(t)$ is compact for $t \geq h$, it follows [Z1, Lemma 1] ([T1, Section 2]) that

$$\omega_0 = \sup \mathrm{Re}\, \sigma(A_L)$$

hence $S_L(t)$ is asymptotically stable.

The following result is concerned with the resolvent operator and spectrum of A_L.

Lemma 2.3: a) The resolvent operator $(A_L-\lambda)^{-1}$ is compact for λ belonging to the resolvent set $\rho(A_L)$.

b) The spectrum of A_L consists of isolated eigenvalues of finite multiplicities, i.e. A_L has point spectrum only.

Proof: a) clearly $\tilde{A}$ has compact resolvent for some λ, since A_1 and A_c have compact resolvents for some λ_1 and λ_c. Since $\tilde{B}$ is bounded the result follows from [P1, Theorem 4.3]

b) is a consequence of a) [K1, p.187, th. 6.29].

In further developments we will need the following spectral decomposition results.

As mentioned previously in Section 2.1, the spectrum of A_2 consists of eigenvalues of finite multiplicities and the number of eigenvalues with real part greater than a given (arbitrary) constant is finite [V1]. It follows that A_2 satisfies the spectrum decomposition in [K. 1, p.178, th. 6.17] , that is

i) $X_2 = X_2^- \oplus X_2^+$ with X_2^- and X_2^+ both invariant under A_2

X_2^+ denotes the unstable subspace associated with A_2, and X_2^- corresponds to the stable subspace, and

$X_2^+ = P_2X_2$, $X_2^- = (I-P_2)X_2$ where $P_2 : X_2 \to X_2^+$ is the projection on X_2^+ along X_2, and $P_2D(A_2) \subset D(A_2)$

ii) $\sigma(A_2^\pm) = \sigma^\pm(A_2)$ where $A_2^\pm = A_2|X_2^\pm$, $\sigma^+(A_2)$ (resp. $\bar{\sigma}(A_2)$) is the spectrum of A_2 contained in C^+ (resp. C^-). A_2^+ is a bounded operator on X_2^+ .

iii) P_2 commutes with A_2, i.e. for each $x_2 \in D(A_2)$, $x_2^+ = P_2x_2 \in D(A_2)$ and $P_2A_2x_2 = A_2P_2x_2 = A_2^+x_2^+$. Similarly $(I-P_2)$ commutes with A_2.

iv) A_2^+ and A_2^- are closed operators. Furthermore, A_2^+ is bounded with $D(A_2^+) = X_2^+$ and $D(A_2^-) = X_2^- \cap D(A_2)$.
In addition we have

v) P_2 and $(I-P_2)$ commute with $S_2(t)$, $t \geq 0$. This is a consequence of iii) (see Appendix 2 in [T1]). Furthermore, $S_2^+(t)$ is a uniformly continuous and analytic group. $S_2^-(t)$ is compact for $t \geq h$, and therefore its growth is determined by $\sigma(A_2^-)$.

vi) X_2^+ is finite dimensional. In fact we have $\dim[X_2^+] = \sum$ algebraic multiplicities of eigenvalues of A_2 with $\text{Re}\,\lambda \geq 0$. (This result follows from the compactness of $(A_2-\lambda)^{-1}$ see [K1, th.6.29, p.187 and p. 181])

A_1 and A_c may be decomposed similarly.

According to the above decomposition we may write (2.6) in more detail

$$\begin{pmatrix} \dfrac{dx_2^-(t)}{dt} \\ \\ \dfrac{dx_2^+(t)}{dt} \end{pmatrix} = \begin{pmatrix} A_2^- & 0 \\ \\ 0 & A_2^+ \end{pmatrix} \begin{pmatrix} x_2^-(t) \\ \\ x_2^+(t) \end{pmatrix} \tag{2.11}$$

Using this representation we have that the closed loop system, together with the exosystem is given by

$$\begin{pmatrix} \dot{x}_L(t) \\ \dot{x}_2^-(t) \\ \dot{x}_2^+(t) \end{pmatrix} = \begin{pmatrix} A_L & B_L^- & B_L^+ \\ 0 & A_2^- & 0 \\ 0 & 0 & A_2^+ \end{pmatrix} \begin{pmatrix} x_L(t) \\ x_2^-(t) \\ x_2^+(t) \end{pmatrix}, \quad t \geq 0 \tag{2.12}$$

$$z(t) = [D_L \quad D_2^- \quad D_2^+] \begin{bmatrix} x_L(t) \\ x_2^-(t) \\ x_2^+(t) \end{bmatrix} \tag{2.13}$$

where

$$A_L = \begin{bmatrix} A_1+B_1G_cC_1 & B_1F_c \\ B_cC_1 & A_c \end{bmatrix}, \qquad D_L = [D_1 \quad 0],\ x_L(t) = \begin{bmatrix} x_1(t) \\ x_c(t) \end{bmatrix}$$

$$B_L^- = \begin{bmatrix} A_3^-+B_1G_cC_2^- \\ B_cC_2^- \end{bmatrix}, \qquad B_L^+ = \begin{bmatrix} A_3^++B_1G_cC_2^+ \\ B_cC_2^+ \end{bmatrix}$$

and

$$A_3^\pm = A_3|X_2^\pm \ , \quad C_2^\pm = C_2|X_2^\pm \ , \qquad D_2^\pm = D_2|X_2^\pm$$

We point out that B_L^+ is a bounded operator. Indeed, since A_3 is bounded on $D(A_2)$ (with the norm $|||\cdot|||$) and $X_2^+ = P_2D(A_2) \subset D(A_2)$ we have

$$\| A_3^+x_2^+\|^2 = \| A_3P_2x_2^+\|^2 \leq M^2(\| P_2x_2^+\|^2 + \| A_2x_2^+\|^2)$$

where $$M^2 = \| \hat{A}_4+\hat{A}_5\|^2 + \| \hat{A}_5\|^2h$$

but

$$P_2x_2^+ = x_2^+$$

$$A_2P_2x_2^+ = P_2A_2x_2^+ = A_2^+x_2^+$$

and since A_2^+ is bounded we obtain

$$\| A_3^+x_2^+\|^2 \leq M^2(1 + \| A_2^+\|^2)\| x_2^+\|^2 \quad , \quad x_2^+ \in X_2^+$$

So A_3^+ is bounded and therefore B_L^+ is also bounded.

The next two lemmas provide a characterization of the regulation condition, they are minor extensions of Bhat's results [B1, Chapter 5].

Lemma 2.4: Suppose A_L is stable, then regulation is attained if and only if

$$X_s^+(A_s) \subset \text{Ker } D_s$$

where

$$A_s = \begin{pmatrix} A_L & B_L^- & B_L^+ \\ 0 & A_2^- & 0 \\ 0 & 0 & A_2^+ \end{pmatrix} : D(A_s) = D(A_L) \times D(A_2^-) \times X_2^+ \to X_s = X_L \times X_2^- \times X_2^+$$

$$D_s = [D_L \quad D_2^- \quad D_2^+] : X_s \to Z$$

Proof: The closed-loop system is given by (2.12) - (2.13). Now since A_L is stable then $\sigma(A_L) \subset C^-$ (Lemma 2.2) and $\sigma(A_2^-) \subset C^-$, $\sigma(A_2^+) \subset C^+$. Moreover, $\sigma(A_s) = \sigma(A_L) \cup \sigma(A_2^-) \cup \sigma(A_2^+)$ therefore

$\sigma^+(A_s) = \sigma(A_2^+)$ and $X_s^+(A_s)$ is given by

$$X_s^+(A_s) = P_s X_s$$

where

$$P_s = -\frac{1}{2\pi i}\int_\Gamma (A_s-\lambda)^{-1} d\lambda$$

and Γ encloses $\sigma(A_2^+)$. It is easily seen that A_s satisfies the decomposition described previously, therefore (2.12) is decomposed as

$$\begin{pmatrix} \dot{x}_s^-(t) \\ \dot{x}_s^+(t) \end{pmatrix} = \begin{pmatrix} A_s^- & 0 \\ 0 & A_s^+ \end{pmatrix} \begin{pmatrix} x_s^-(t) \\ x_s^+(t) \end{pmatrix}$$

For an arbitrary initial condition $x_s(0) = (x_s^-(0), x_s^+(0))$, the solution of (2.12) decomposes as $x_s(t) = x_s^+(t) + x_s^-(t)$ where

$$x_s^+(t) = e^{A_s^+ t} x_s^+(0)$$

By stability of A_s^-, $x_s^-(t) \to 0$ as $t \to \infty$, and $z(t)$ is given by

$$z(t) = D_s x_s(t) = D_s [e^{A_s^+ t} x_s^+(0) + x_s^-(t)]$$

regulation requires that

$$D_s e^{A_s^+ t} x_s^+(0) \to 0 \qquad \text{as } t \to \infty$$

Since $x_s^+(0)$ is arbitrary in X_s^+ and X_s^+ is invariant under A_s^+, then regulation holds if and only if

$$X_s^+(A_s) \subset \text{Ker } D_s$$

<u>Lemma 2.5</u>: $X_s^+(A_s) = \text{Im} \begin{pmatrix} -X_L \\ 0 \\ I \end{pmatrix}$

where $X_L : X_2^+ \to X_L$ is a bounded operator, which is the unique solution of

$$A_L X_L - X_L A_2^+ = B_L^+ \tag{2.14}$$

and I is the identity operator on X_2^+ and 0 is the zero operator on X_2^-.

<u>Proof</u>: Since the spectra of A_L and A_2^+ in the extended complex plane do not intersect, then the operator equation (2.14) has a unique solution, see [K2, p. 316] or [B1, Chapter 5, Lemma 5.1.2.].

Now

$$X_s^+(A_s) = P_s X_s = \text{Im } P_s$$

where P_s is defined in the proof of Lemma 2.4.

Thus,

$$P_s = -\frac{1}{2\pi i}\int_\Gamma \begin{pmatrix} (\tilde{A}_L-\lambda)^{-1} & (\tilde{A}_L-\lambda)^{-1}\tilde{B}_L^+(A_2^+-\lambda)^{-1} \\ 0 & (A_2^+-\lambda)^{-1} \end{pmatrix} d\lambda$$

where

$$\tilde{A}_L = \begin{pmatrix} A_L & B_L^- \\ 0 & A_2^- \end{pmatrix} \quad , \qquad \tilde{B}_L^+ = \begin{pmatrix} B_L^+ \\ 0 \end{pmatrix}$$

For $x_s = (\tilde{x}_L, x_2^+) \in X_s$, $\tilde{x}_L = (x_L, x_2^-)$, we have

$$P_s x_s = -\frac{1}{2\pi i}\int_\Gamma \begin{pmatrix} (\tilde{A}_L-\lambda)^{-1}\tilde{x}_L + (\tilde{A}_L-\lambda)^{-1}\tilde{B}_L^+(A_2^+-\lambda)^{-1}x_2^+ \\ (A_2^+-\lambda)^{-1}x_2^+ \end{pmatrix} d\lambda \tag{2.15}$$

Since A_2^+ is bounded and Γ encloses $\sigma(A_2^+)$,

$$-\frac{1}{2\pi i}\int_\Gamma (A_2^+-\lambda)^{-1}d\lambda = I$$

On the other hand, since $\sigma(\tilde{A}_L) = \sigma(A_L) \cup \sigma(A_2^-)$ and $\sigma(A_2^+)$ do not intersect in the extended complex plane†, then there is a unique solution $\tilde{X}_L$ to the following operator equation,

$$\tilde{A}_L\tilde{X}_L - \tilde{X}_L A_2^+ = \tilde{B}_L^+ \tag{2.16}$$

that is, there are λ s.t. $(\tilde{A}_L-\lambda)^{-1}$ and $(A_2^+-\lambda)^{-1}$ exists and are (both) bounded operators, therefore for these λ (2.16) can be written as

$$(\tilde{A}_L-\lambda)^{-1}\tilde{X}_L-\tilde{X}_L(A_2^+-\lambda)^{-1} = -(\tilde{A}_L-\lambda)^{-1}\tilde{B}_L^+(A_2^+-\lambda)^{-1} \tag{2.17}$$

---

† Extended complex plane is the one-point compactification of the ordinary complex plane by adjunction of the point ∞.

hence, (2.17) yields

$$-\frac{1}{2\pi i}\int_{\Gamma}(\tilde{A}_L-\lambda)^{-1}\tilde{B}_L^{+}(A_2^{+}-\lambda)^{-1}x_2^{+}d\lambda$$

$$=\frac{1}{2\pi i}\int_{\Gamma}((\tilde{A}_L-\lambda)^{-1}\tilde{X}_L-\tilde{X}_L(A_2^{+}-\lambda)^{-1})x_2^{+}\,d\lambda$$

and since $\sigma(\tilde{A}_L)$ lies extirely outside of Γ the integral of the terms involving $(\tilde{A}_L-\lambda)^{-1}$ is zero. Thus, (2.15) reduces to

$$P_s x_s = \begin{pmatrix} -\tilde{X}_L x_2^{+} \\ x_2^{+} \end{pmatrix} \tag{2.18}$$

However writing (2.16) in more detail we obtain

$$\begin{pmatrix} A_L & B_L^{-} \\ 0 & A_L^{-} \end{pmatrix}\begin{pmatrix} X_L \\ X_2^{-} \end{pmatrix} - \begin{pmatrix} X_L \\ X_2^{-} \end{pmatrix} A_2^{+} = \begin{pmatrix} B_L^{+} \\ 0 \end{pmatrix}$$

and, again, since $\sigma(A_2^{-})$ and $\sigma(A_2^{+})$ do not intersect, we conclude that $X_2^{-} = 0$, and (2.18) gives

$$P_s x_s = \begin{pmatrix} -X_L x_2^{+} \\ 0 \\ x_2^{+} \end{pmatrix}$$

therefore

$$X_s^{+}(A_s) = \text{Im } P_s = \text{Im}\begin{pmatrix} -X_L \\ 0 \\ I \end{pmatrix}$$

As a consequence of these two lemmas, we have the following

Proposition 2.6: If A_L is stable, then regulation is equivalent to the existence of a bounded operator $X_L : X_2^+ \to X_L$ such that

$$A_L X_L - X_L A_2^+ = B_L^+ \tag{2.19}$$

$$D_L X_L = D_2^+ \tag{2.20}$$

The expressions (2.19) - (2.20) will play an important role in determining an IMP since they contain 'information' about the structure of the controller. We further note that, while (2.19) always has a (unique) solution (2.20) might not be satisfied, i.e. internal stability and regulation are not compatible requirements necessarily. Also note that the above expressions do not involve the operators restricted to the stable subspace X_2^- and therefore all the terms involving such operators may be discarded. Finally, we mention that the results of this section hold in the case that X_2^+ is an infinite dimensional vector space, and A_2^+ is an arbitrary bounded operator with $\sigma(A_2^+) \subset C^+$.

2.3 An Equivalent Reduced Problem

From the spectral decomposition results for time delay systems in Appendix A, it can be shown that the projection operator $P_2 : X_2 \to X_2^+$ is characterized by

$$x_2^+(t) = P_2 x_2(t) = \Phi_2^+ \langle\langle \Psi_2^+, x_2(t) \rangle\rangle$$

where Φ_2^+, Ψ_2^+ and $\langle\langle \cdot, \cdot \rangle\rangle$ are defined in Appendix A.

Now, define

$$w(t) = \langle\langle \Psi_2^+, x_2(t) \rangle\rangle \in \mathbb{R}^N , \quad N = \dim X_2^+$$

then

$$x_2^+(t) = \Phi_2^+ w(t)$$

and $w(t)$ satisfies the ordinary differential equation

$$\dot{w}(t) = A_w w(t), \quad t \geq 0 \tag{2.21}$$

where A_w is an N×N real matrix, $w(0) = \langle\langle \Psi_2^+, x_2(0) \rangle\rangle$

and $\quad \sigma(A_w) = \sigma(A_2^+)$,

and A_w satisfies

$$A_2\Phi_2^+ = \Phi_2^+ A_w, \qquad w \in \mathbb{R}^N \tag{2.22}$$

Furthermore, $\Phi_2^+ : \mathbb{R}^N \to X_2^+$ is an isomorphism, i.e. Φ_2^+ is bijective and bounded with $(\Phi_2^+)^{-1}$ being bounded.

Now consider the closed-loop system (2.12) - (2.13) with $x_2^+(t)$ replaced by $\Phi_2^+ w(t)$ and $w(t)$ satisfying (2.21). Clearly we may use Lemmas 2.4 and 2.5 to conclude that when A_L is stable regulation is attained if and only if there is a bounded operator $\hat{X}_L : \mathbb{R}^N \to X_L$ such that

$$A_L\hat{X}_L - \hat{X}_L A_w = (A_3^+ + B_1 G_c C_2^+)\Phi_2^+ \tag{2.23}$$

$$D_L\hat{X}_L = D_2^+\Phi_2^+ \tag{2.24}$$

The following result establishes the equivalence of the expressions (2.23) - (2.24) and proposition 2.6.

Lemma 2.7: Suppose A_L is stable, then there exists a bounded operator $X_L : X_2^+ \to X_L$ satisfying (2.19) - (2.20) if and only if there is a bounded operator $\hat{X}_L : \mathbb{R}^N \to X_L$ satisfying (2.23) - (2.24).

Proof: From (2.22) is easy to see that Im Φ_2^+ is an A_2 invariant subspace. Since Φ_2^+ is injective it follows that $A_2|\text{Im}\,\Phi_2^+$ is isomorphic to A_w. But Im $\Phi_2^+ = X_2^+$, thus

$$A_2^+ \approx A_w$$

In fact the isomorphism is give by

$$A_2^+ = \Phi_2^+ A_w (\Phi_2^+)^{-1}$$

The remainder of the proof is easily obtained.

The above result means that our original problem is equivalent to a problem in which the disturbances and/or reference signals are modelled by an ordinary differential equation.

2.4 Stabilizability and Detectability of the System and Controller

In this section we will show that stabilizability of the pairs (A_1,B_1) and (A_c,B_c) and detectability of (C_1,A_1) and (F_c,A_c) are necessary conditions for the solvability of the regulation and internal stability problem. In fact these conditions are a consequence of the requirement of internal stability. Before obtaining these results we need some preliminary definitions and technical lemmas.

We say that pair (A_1, B_1) is *stabilizable* if there exists a bounded linear operator $F_1 : X_1 \to U$ such that $A_1 + B_1F_1$ is stable. Similarly, the pair (C_1,A_1) is *detectable* if there is a bounded linear operator $K_1 : Y \to X_1$ such that $A_1 + K_1C_1$ is stable.

The following lemmas provide convenient characterizations of stabilizability and detectability.

Lemma 2.8: The pair (A_1,B_1) is stabilizable if and only if

$$\mathrm{Im}(A_1-\lambda) + \mathrm{Im}B_1 = X_1, \quad \lambda \in C^+$$

Lemma 2.9: The pair (C_1, A_1) is detectable if and only if

$$\mathrm{Ker}(A_1-\lambda) \cap \mathrm{Ker}\, C_1 = 0, \ \lambda \in C^+.$$

A proof of these lemmas is given in [B1] (also see [B5]). We point out that these results are consequence of the properties of the operator A_1, namely that the unstable subspace X_1^+ associated with A_1 is finite dimensional and $A_1^- = A_1|X_1^-$ is stable. Also, by our assumptions on A_c, the above lemmas hold for (A_c,B_c) and (F_c,A_c).

Now we can prove the following

Lemma 2.10: The stabilizability of the pairs (A_1,B_1) and (A_c,B_c) and detectability of (C_1,A_1) and (F_c,A_c) are necessary conditions for the stability of the closed loop system.

Proof: Stability of A_L implies that $\sigma(A_L) \cap C^+ = \phi$ so that $C^+ \subset \rho(A_L)$, and since A_L is closed we have ([S4, p. 179]),

$$\mathrm{Im}(A_L-\lambda) = X_L = X_1 \times X_c \ , \ \lambda \in C^+$$

$$\mathrm{Ker}(A_L-\lambda) = 0 \qquad \lambda \in C^+$$

that is

$$\mathrm{Im}\begin{pmatrix} A_1+B_1G_cC_1-\lambda & B_1F_c \\ B_cC_1 & A_c-\lambda \end{pmatrix} = X_1 \times X_c \ , \ \lambda \in C^+$$

and

$$\mathrm{Ker}\begin{pmatrix} A_1+B_1G_cC_1-\lambda & B_1F_c \\ B_cC_1 & A_c-\lambda \end{pmatrix} = 0 \qquad , \ \lambda \in C^+$$

In particular we have, for $\lambda \in C^+$

$$\mathrm{Im}(A_1+B_1G_cC_1-\lambda) + \mathrm{Im}\, B_1 F_c = X_1 \tag{2.25a}$$

$$\mathrm{Im}\, B_c C_1 + \mathrm{Im}(A_c-\lambda) = X_c \tag{2.25b}$$

$$\mathrm{Ker}(A_1+B_1G_cC_1-\lambda) \cap \mathrm{Ker}\, B_c C_1 = 0 \tag{2.25c}$$

$$\mathrm{Ker}\, B_1 F_c \cap \mathrm{Ker}(A_c-\lambda) = 0 \tag{2.25d}$$

hence, for $\lambda \in C^+$

$$\mathrm{Im}(A_1-\lambda) + \mathrm{Im}\, B_1 = X_1 \tag{2.26a}$$

$$\mathrm{Im}\, B_c + \mathrm{Im}(A_c-\lambda) = X_c \tag{2.26b}$$

$$\mathrm{Ker}(A_1-\lambda) \cap \mathrm{Ker}\, C_1 = 0 \tag{2.26c}$$

$$\mathrm{Ker}(A_c-\lambda) \cap \mathrm{Ker}\, F_c = 0 \tag{2.26d}$$

(2.26a) - (2.26d) together with lemmas 2.8 and 2.9 give the desired result.

We note that if the closed-loop system is stable, i.e. A_L is stable, then the stronger conditions (2.25a) - (2.25d) must be satisfied. These expressions may be interpreted as providing us with a 'geometric picture' of how certain subspaces associated with the parameters of the controller have to be "placed" with respect to the subspaces associated with the system's parameters. This idea will be useful for the developments of Chapter 3. We finally point out that the expressions (2.26a) - (2.26d) may be reduced to controllability and observability conditions in the finite dimensional subspaces X_1^+ and X_c^+, see [B1] or [B5].

2.5 Conclusions and Remarks

The development of this chapter follows closely Bhat's work [B1, Chapter 5]. Our results are modifications of those in [B1] to accomodate the fact that the regulated variables are not directly available to the controller. The assumptions on the operator A_c are motivated by the dynamic structure of time delay systems. The results of this chapter, in particular proposition 2.6 (and Lemma 2.7) provide the basis for obtaining an IMP for delay systems.

CHAPTER 3

STRUCTURAL STABILITY OF A CONTROLLER :
AN INTERNAL MODEL PRINCIPLE FOR TIME DELAY SYSTEMS

This chapter deals with the problem of determining an Internal Model Principle for time delay systems, that is, determine the necessary structural features of a controller which yields output regulation and internal stability under small perturbations of certain parameters.

Before solving our main problem we need some preliminary results. We shall first specify the class of perturbation operators. Also we will make precise the meaning of smallness of the perturbations. Then, as in delay-free case, we will introduce the concepts of readability and internal model (the latter must not be confused with the IMP). Convenient characterizations of these concepts will be obtained. As a general outline of the results that will be derived, we will briefly summarize the IMP for ordinary linear systems.

The approach for solving our problem will consist of several steps. We shall first allow 'variations' in one parameter while the remaining parameters will be fixed. This will allow us to show the necessity of some feature, either of the controller, or the system. We will then assume that this particular feature holds and perturbations in another parameter will be introduced to establish the necessity of another feature. We will proceed in this manner until an IMP is obtained.

Our first result will establish the necessity of readability, which is a condition on the system's parameters. Next, we will establish the necessity of the internal model, that is that the controller dynamics must incorporate a 'suitable' reduplication of the dynamics of the disturbance and/or reference signals. Finally, the feedback structure will be justified that is that the internal model must be driven by the regulated variables.

In fact, as in the delay-free situation, we will show that the internal model is controllable by the regulated variables, and observable by the control(controllability and observability will be defined on an 'adequate' finite dimensional subspace of X_c).

For reference we write the abstract equation associated with our delay system

$$\frac{dx_1(t)}{dt} = A_1x_1(t) + A_3x_2(t) + B_1u(t) \tag{3.1}$$

$$\frac{dx_2(t)}{dt} = A_2x_2(t) \tag{3.2}$$

$$y(t) = C_1x_1(t) + C_2x_2(t) \tag{3.3}$$

$$z(t) = D_1x_1(t) + D_2x_2(t) \tag{3.4}$$

where $x_1 \in X_1 = M_2^{n_1}$, $x_2 \in X_2 = \mathbb{R}^{n_2}$, $u \in U = \mathbb{R}^m$

$y \in Y = \mathbb{R}^p$, $z \in Z = \mathbb{R}^q$

As discussed previously all the operators are bounded except A_1. We assume that A_2 is a bounded (linear) operator defined on the finite dimensional space $X_2 = \mathbb{R}^{n_2}$ and $\sigma(A_2) \subset C^+$. There is no loss of generality in this assumption since, by the results of sections 2.2 and 2.3, we can always reduce our problem to this case. We mention that throughout this chapter we consider the operator A_2 to be represented by an $n_2 \times n_2$ matrix, where $n_2 = \dim [X_2]$.

In addition we may assume that

$$[C_1 \quad C_2]: \; X_1 \times X_2 \to Y = \mathbb{R}^p \text{ is surjective} \tag{3.5}$$

otherwise we may replace Y by $\text{Im}\, C_1 + \text{Im}\, C_2$. Also we assume that

$$D_1 : X_1 \to Z = \mathbb{R}^q \text{ is surjective} \tag{3.6}$$

since a necessary condition for output regulation is $\text{Im } D_2 \subset \text{Im } D_1$ (see (2.20) and (2.24)), and hence we may set $\mathcal{Z} = \text{Im } D_2 + \text{Im } D_1 = \text{Im } D_1$.

Also we suppose that (A_1, B_1) is stabilizable and (C_1, A_1) is detectable, since by Lemma 2.10 both conditions are necessary for internal stability.

Finally we mention that the finite dimensionality of the spaces $\mathcal{X}_2$, $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{Z}$ will play an important role in our developments.

3.1 Class of Admissible Perturbations

In general, it is difficult to relate arbitrary perturbations of the infinitesimal generator A_1 in (3.1) to the original delay system. Even for certain finite dimensional perturbations of A_1, the corresponding semigroup cannot be described by a delay differential equation alone† [S5]. Furthermore, some of the properties of the operator A_1 may be destroyed by an arbitrary perturbation, e.g. closedness, the property of being an infinitesimal generator, etc. .

On the other hand, it is of physical intrest to consider perturbations of the operators associated with the abstract evolution equation (3.1) which correspond to variations in the elements of the matrices of the original delay system. It is readily verified that variations $\delta\hat{A}_0$ in the elements of the matrix $\hat{A}_0$ in (2.1) correspond to certain bounded finite dimensional (compact) perturbations of the infinitesimal generator A_1 in (3.1). Moreover, when the delay system contains terms of the type $\int_{-h}^{0} \hat{A}(s)\, \hat{x}_1(t+s)ds$, i.e. distributed delays, we find that

† This is the case when state feedback is used for systems with delays in the controls and in the state. Also, this situation arises when output feedback is used, and the output mapping contains delays.

changes $\delta\hat{A}(s)$ in the matrix function $\hat{A}(s)$ also correspond to certain compact perturbations of the associated infinitesimal generator. We mention that terms of this type also arise naturally from state feedback and in the theory of observers for delay systems. When the elements of the matrix $\hat{A}_1$ in (2.1) are allowed to vary, the corresponding perturbations of A_1 turn out to be unbounded operators which are not even closable. However, we will show below that such perturbations correspond to certain class of A_1- compact operators† and their ranges are finite dimensional. Furthermore, since the adjoint operator A_1^* is densely defined, it can be shown that the above perturbation operators have A_1-bound-zero [K1, p. 196].

Let $\delta\hat{A}_1$ denote the variations in the elements of the matrix $\hat{A}_1$ and let δA_1 be the corresponding perturbation of A_1, then for $x_1 \varepsilon D(A_1)$

$$(A_1+\delta A_1)x_1 = (\hat{A}_0 x_1^0 + (\hat{A}_1 + \delta\hat{A}_1)\hat{x}_1^1(-h), \quad \frac{d\hat{x}_1^1}{d\theta} \quad)$$

so that δA_1 is given by

$$\delta A_1 x_1 = (\delta\hat{A}_1 \hat{x}_1^1(-h), 0)$$

Now δA_1 is unbounded, for $\| \hat{x}_1^1(-h) \|_{\mathbb{R}^{n_1}}$ can be arbitrarily large for $\| x_1 \|_{X_1} = 1$. However, δA_1 is bounded on $D(A_1)$ with the graph norm, i.e. $|||x_1|||^2 = \| x_1 \|^2_{X_1} + \| A_1 x_1 \|^2_{X_1}$, $x_1 \varepsilon D(A_1)$. Indeed, for $x_1 \varepsilon D(A_1)$ we may write

$$\hat{x}_1^1(-h) = \hat{x}_1^0 - \int_{-h}^{0} \dot{\hat{x}}_1^1(\theta)\, d\theta$$

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† see Appendix B for the definition of relative compactness and relative boundedness.

thus

$$\| \delta A_1 x_1 \|_{X_1} \le \| \delta \hat{A}_1 \| (\| \hat{x}_1^0 \| + \int_{-h}^{0} \| \dot{\hat{x}}_1^1(\theta) \| d\theta)$$

$$\le \| \delta \hat{A}_1 \| (1+h)^{\frac{1}{2}} (\| \hat{x}_1^0 \|^2 + \| \dot{\hat{x}}_1^1 \|^2_{L_2})^{\frac{1}{2}}$$

but

$$\| \dot{\hat{x}}_1^1 \|^2_{L_2} = \| (A_1 x_1)^1 \|^2_{L_2}$$

and since

$$\| A_1 x_1 \|^2_{X_1} = \| (A_1 x_1)^0 \|^2 + \| (A_1 x_1)^1 \|^2_{L_2}$$

$$\| x_1 \|^2_{X_1} = \| \hat{x}_1^0 \|^2 + \| \hat{x}_1^1 \|^2_{L_2}$$

we obtain

$$\| \delta A_1 x_1 \|_{X_1} \le \| \delta \hat{A}_1 \| (1+h)^{\frac{1}{2}} ||| x_1 |||$$

Hence, δA_1 is A_1-bounded and since Im δA_1 is a finite dimensional subspace of X_1, we conclude that δA_1 is A_1-compact.

Thus, in general, variations in the elements of the matrices $\hat{A}_0$ and $\hat{A}_1$ in (2.1) correspond to certain perturbations δA_1 of A_1 which are A_1-bounded operators with finite dimensional ranges. For such perturbations, we find that the operator $A_1+\delta A_1$ has the following properties†

1) $A_1+\delta A_1$ is closed with domain $D(A_1+\delta A_1) = D(A_1)$

2) $A_1+\delta A_1$ is the infinitesimal generator of a strongly continuous semigroup $S_{A_1+\delta A_1}(t)$, $t \ge 0$.

3) The semigroup $S_{A_1+\delta A_1}(t)$ is compact for $t \ge h$.

4) The resolvent operator $(A_1+\delta A_1-\lambda)^{-1}$ is compact for all $\lambda \in \rho(A_1+\delta A_1)$.

---

† Replace $\hat{A}_0$ and $\hat{A}_1$ in (2.1) by $\hat{A}_0+\delta\hat{A}_0$ and $\hat{A}_1+\delta\hat{A}_1$ respectively.

We point out that for an arbitrary A_1-bounded perturbation with finite dimensional range some properties of the operator A_1 are stable, e.g. closedness, compactness of the resolvent (see Appendix B or [K1, Chapter IV], [G1, Chapter V]). However, in general, it is not known whether the property of being an infinitesimal generator of a strongly continuous semigroup is stable under an arbitrary A_1-bounded perturbation with finite dimensional range.

We shall therefore limit the class of admissible perturbations δA_1 of A_1 to those satisfying conditions 1) - 4) and so in particular to those corresponding to variations in the elements of the matrices $\hat{A}_0$, $\hat{A}_1$ in (2.1). Such class of perturbations will be denoted by $\mathcal{F}(A_1)$. For all other operators which are not only bounded but compact (since either, they are defined on a finite dimensional space or their ranges are finite dimensional), the perturbation class consists of arbitrary bounded operators between the appropriate spaces. Finally, we mention that the restriction on the perturbation class for A_1 will not affect our results on the Internal Model Principle, since the conditions will be principally determined by the perturbation class of A_3.

Having specified the class of perturbation operators we now make precise what is meant by a small perturbation. For this, we need to introduce the concept of gap between two operators. The following definitions are given in [K1, pp. 197-205].

<u>Definition 3.1</u>: Let X be a Banach space, and S be a closed subspace of X. Then for $x \in X$, the distance from x to the subspace S is given by

$$\operatorname{dist}(x,S) = \inf_{y \in S} \| x-y \| \quad . \qquad (3.7)$$

Definition 3.2: For a pair of closed subspaces R and S of a Banach space X, define

$$\delta(R,S) = \sup_{\substack{x\in R \\ \|x\|=1}} \operatorname{dist}(x,S) \tag{3.8}$$

and

$$\hat{\delta}(R,S) = \max[\delta(R,S),\ \delta(S,R)] \tag{3.9}$$

$\hat{\delta}(R,S)$ is called the gap between R and S. We note that (3.8) has no meaning if $R = 0$; in this case we define $\delta(0,S) = 0$ for any S. Also, for $R \neq 0$, $\delta(R,0) = 1$. The following relations follow directly from the above definition

$$\delta(R,S) = 0 \text{ if and only if } R \subset S$$

$$\hat{\delta}(R,S) = 0 \text{ if and only if } R = S$$

$$0 \leq \delta(R,S) \leq 1 \quad , \quad 0 \leq \hat{\delta}(R,S) \leq 1$$

We now define the gap between two closed operators. Recall that an operator $A : X \to Y$ is closed if and only if its graph G(A) is a closed subspace of the product space $X \times Y$[†]. Thus, we have the following definition

Definition 3.3: For $A, B \in C(X,Y)$ the set of all closed (linear) operators from X to Y, define

$$\delta(A,B) = \delta(G(A),\ G(B)) \tag{3.10}$$

and

$$\hat{\delta}(A,B) = \hat{\delta}(G(A),G(B)) \tag{3.11}$$

$\hat{\delta}(A,B)$ is called the gap between A and B.

This definition of gap leads to the following concept of convergence of closed operators

Definition 3.4: $\{A_n\} \in C(X,Y)$ is said to *converge in the generalized sense* to $A \in C(X,Y)$ if $\hat{\delta}(A_n,A) \to 0$.

[†]Throughout this work we consider that the norm for the space $X\times Y$ is given by

$$\|(x,y)\|_{X\times Y} = (\|x\|_X^2 + \|y\|_Y^2)^{\frac{1}{2}}$$

This notion of generalized convergence enables us to make precise the smallness of perturbation. Thus, we say that for a closed operator $A : X \to Y$ a perturbation δA is small if the gap $\hat{\delta}(A+\delta A, A)$ is in a neighbourhood of zero. In case δA belongs to the set $B(X,Y)$ of all bounded (linear) operators, this is equivalent to $\|\delta A\|$ being small, [K1, p.203, th. 2.14]. Also we note that on the subset $B(X,Y)$ of $C(X,Y)$, the topology induced by $\hat{\delta}(A,B)$ coincides with the topology induced by the metric $\|A-B\|$.

Before concluding this section, we mention that the gap function is not, in general, a proper distance function since it does not satisfy the triangle inequality (unless the underlying space is Hilbert). It can be shown that the above definition of gap can be adequately modified to provide a distance function for the set of all closed subspaces. However, when we consider the topology of the set of closed subspaces the two functions give the same results and usually the gap function is more convenient to use for applications. For details see [K1, pp.197-205].

3.2 Readability

Recall from section 2.1 that the controller (2.9) - (2.10) is restricted to process the measurable output y. We will show later that a synthesis may be structrually stable only if the controller has access to the regulated variable z. This motivates the following definition, which is given in [F1].

Definition 3.5: We say that z is *readable* from y if there is a bounded (linear) operator $Q : Y \to Z$ such that

$$z = Qy \tag{3.12}$$

The following lemma gives a convenient characterization of readability.

Lemma 3.6: z is readable from y if and only if

$$\mathrm{Ker}[C_1 \ C_2] \subset \mathrm{Ker}[D_1 \ D_2] \tag{3.13}$$

Proof: The proof of this result is the same as in the finite dimensional case [F1], however we give it here for completeness.

(Necessity). Suppose that z is readable from y.

Then, by definition, there is a bounded operator $Q : \mathcal{Y} \to \mathcal{Z}$ such that

$$z = [D_1 \ D_2]x = Q[C_1 \ C_2]x \qquad x \in \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$$

Thus

$$\mathrm{Ker}(Q[C_1 \ C_2]) = \mathrm{Ker}[D_1 \ D_2]$$

and (3.13) follows from the above expression

(Sufficiency). We first note that

$$\mathrm{Im}[C_1 \ C_2] \approx \frac{\mathcal{X}}{\mathrm{Ker}[C_1 \ C_2]} \ , \quad \mathrm{Im}[D_1 \ D_2] \approx \frac{\mathcal{X}}{\mathrm{Ker}[D_1 \ D_2]}$$

Now suppose that (3.13) holds, from (3.5) and (3.6), we have

$$\dim[\mathcal{Y}] = \dim[\mathrm{Im}[C_1 \ C_2]] \geq \dim[\mathrm{Im}[D_1 \ D_2]] = \dim[\mathcal{Z}]$$

Thus we can define $\mathcal{Y}$ according to

$$\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z} \tag{3.14}$$

where $\mathcal{W}$ is a suitable complement of $\mathcal{Z}$. Then

$$[C_1 \ C_2] = \begin{pmatrix} E_1 & E_2 \\ D_1 & D_2 \end{pmatrix} \tag{3.15}$$

for some bounded operators $E_1 : \mathcal{X}_1 \to \mathcal{W}$, $E_2 : \mathcal{X}_2 \to \mathcal{W}$. Defining $w = E_1x_1 + E_2x_2$ we have

$$y = \begin{pmatrix} w \\ z \end{pmatrix} \in \mathcal{W} \oplus \mathcal{Z}$$

Now consider the natural projection $Q : \mathcal{W} \oplus \mathcal{Z} \to \mathcal{Z}$. Clearly this Q yields the desired result, i.e. that z is readable from y.

3.3 Internal Model

The concept of internal model has been introduced in [F1], [F2] for linear operators acting on finite dimensional spaces. In this case the internal model is defined as follows

Let $A : X \to X$ and $A_2 : X_2 \to X_2$ denote two linear operators and suppose that X and X_2 are finite dimensional.

Definition 3.7: We say that $A : X \to X$ incorporates an *internal model* of $A_2 : X_2 \to X_2$ if the minimal polynomial of A_2 divides at *least* $q = \dim [Z]$ invariant factors† of A.

The above definition of internal model is not adequate without the assumptions on the dimensions of X and X_2. To see this, we first note that for arbitrary linear operators defined on infinite dimensional spaces the concepts of cyclic subspaces and minimal polynomial are rather 'difficult' to define. In fact, the idea of minimal polynomial is restricted to very special operators, e.g. bounded operators with rational resolvent (see [T2 pp. 336-337]). However, if we only assume that X_2 is finite dimensional, then definition 3.7 is still of some use. Indeed, definition 3.7 may be paraphrased by saying that the internal model is at least a q-fold reduplication in A of the maximal cyclic component of A_2. This interpretation motivates an alternative definition of internal model, which of course is equivalent to definition 3.7 when X is finite dimensional. Before generalizing the concept of internal model we need the following preliminaries.

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†recall that the invariant factors of $A:X \to X$ are the minimal polynomials of the cyclic components in a rational canonical decomposition of X relative to A. Of course X is a finite dimensional space [W1, pp.16-17]

Rational Canonical Decomposition. (X_2 is finite dimensional).

Let

$$X_2 = \bigoplus_{i=1}^{k} X_{2i}$$

be a rational canonical decomposition of X_2 relative to A_2[W1, pp.16-17]. Then the following holds

i) X_{2i} is A_2-invariant for $i = 1, 2,\ldots,k$

ii) $A_{2i} = A_2|X_{2i}$ is cyclic for $i = 1, 2,\ldots,k$

iii) the minimal polynomial (m.p) of A_{2i} divides the m.p. of A_{2i+1} for $i = 1, 2,\ldots,k-1$

iv) the m.p. of A_{2k} is the same as that of A_2

v) the integer k is called the cyclic index of A_2 and $k = \max\{\dim[\mathrm{Ker}(A_2-\lambda)] | \lambda \in \sigma(A_2)\}$

Now define

$$\begin{aligned}\tilde{X}_2 &= X_{2k} \oplus X_{2k} \oplus \ldots \oplus X_{2k} \\ &= [X_{2k}]^{\ell} \ (\ell\text{-fold direct sum})\end{aligned} \tag{3.16}$$

and

$$\tilde{A}_2 : \tilde{X}_2 \to \tilde{X}_2 \ , \ \tilde{A}_2|X_{2k} = A_{2k} \tag{3.17}$$

that is, $\tilde{A}_2$ is an ℓ-fold direct sum of the largest cyclic component of A_2.

We now give the following definition .

Let X_2 be finite dimensional and suppose that $A : X \to X$ is a closed operator with dense domain $D(A)$ in the Banach space X.

Definition 3.8: We say that $A : X \to X$ contains an *internal model* $A_2 : X_2 \to X_2$ if there is a bounded injective operator $R : \tilde{X}_2 \to X$ such that on the domain of A the following diagram commutes, i.e. for all $\tilde{x}_2 \in \tilde{X}_2$, $R\,\tilde{x}_2 \in D(A)$ and

$$AR\,\tilde{x}_2 = R\tilde{A}_2\,\tilde{x}_2 \tag{3.18}$$

$$\begin{array}{ccc} D(A) & \xrightarrow{A} & D(A) \\ R \uparrow & & \uparrow R \\ \tilde{X}_2 & \xrightarrow{\tilde{A}_2} & \tilde{X}_2 \end{array} \tag{3.19}$$

where $\tilde{X}_2$ and $\tilde{A}_2$ are given by (3.16) and (3.17) and $\ell \geq q$ with $q = \dim[Z]$.

The interpretation of definition 3.8 is that Im R is an A-invariant subspace of X and that $A|\text{Im } R$ is isomorphic to $\tilde{A}_2$. Thus, we have extended the concept of internal model for closed operators with dense domains.

A different definition of internal model is given by Bhat[B1, Chapter 5] via the commutative diagram (3.19) with $\tilde{A}_2$ and $\tilde{X}_2$ replaced by A_2 and X_2. In this case we have that $A|\text{Im } R$ is isomorphic to A_2. Thus, Bhat's definition does not involve the idea of a q-fold reduplication in A of certain features of the dynamic structure of the exosystem. Since such reduplication plays an important role in establishing an IMP, we prefer to define the internal model in terms of this reduplication (as in the finite dimensional case).

To conclude this section we give a useful characterization of the internal model.

<u>Lemma 3.9</u>: $A : X \to X$ incorporates an internal model of A_2 if and only if for each $\lambda \in \sigma(A_2)$

$$\dim[\text{Ker}(A-\lambda) \cap \text{Im}(A-\lambda)^{k_\lambda - 1}] \geq q \tag{3.20}$$

where k_λ is the degree of the factor $(s-\lambda)$ in the minimal polynomial of A_2.

Proof: (a simple proof of this result is given in [F1, Lemma 3] for X finite dimensional).

By definition 3.8 A contains an internal model of A_2 if there is a bounded injective operator $R : \tilde{X}_2 \to X$ such that

$$AR = R\tilde{A}_2 \tag{3.20a}$$

where

$$\tilde{A}_2 = \begin{pmatrix} A_{2k} & & & \\ & A_{2k} & & \\ & & \ddots & \\ & & & A_{2k} \end{pmatrix}$$

is an ℓ-fold direct sum of the largest cyclic component of A_2, for some $\ell \geq q$. Let $R = [R_1 \;\; R_2 \; \ldots \; R_\ell]$ where $R_i : X_{2k} \to X$ for $1 \leq i \leq \ell$, and since R is injective, it is easily verified that each R_i is injective and $\text{Im } R_i \cap \text{Im } R_j = 0$, $i \neq j$. Restricting (3.20a) to X_{2k} we obtain

$$A\, R_i = R_i\, A_{2k} \quad , \qquad i = 1, 2, \ldots , \ell \geq q \tag{3.20b}$$

Let

$$X_{2k} = \bigoplus_{j=1}^{m} X_{2\lambda j}$$

where m is the number of distinct eigenvalues of A_2, and since A_{2k} is cyclic it follows that $A_{2k}|X_{2\lambda j}$ is also cyclic. Select a basis for X_{2k} such that each $A_{2k}|X_{2\lambda j}$ is in Jordan canonical form, that is $A_{2k}|X_{2\lambda j} = J(\lambda_j)$ is an $k_{\lambda j} \times k_{\lambda j}$ matrix where $k_{\lambda j}$ is the degree of the factor $(s-\lambda j)$ in the minimal polynomial of A_2. Further, since R_i is injective, it is easy to see that

$$R_i|X_{2\lambda j} = [\gamma_{i1}^{\lambda j} \ \gamma_{i2}^{\lambda j} \ \dots \ \gamma_{ik_{\lambda j}}^{\lambda j}]$$

is injective for i = 1, 2,...ℓ, j = 1, 2,...m and that

$$\mathrm{Im}(R_i|X_{2\lambda j}) \cap \mathrm{Im}(R_{p_1}|X_{2\lambda_{p_2}}) = 0 \quad i \neq p_1, \quad j \neq p_2$$

for i, p_1 = 1, 2, ...ℓ j , p_2 = 1, 2,..m that is, the vectors

$$\{\gamma_{i1}^{\lambda j} \ \gamma_{i2}^{\lambda j} \ \dots \ \gamma_{ik_{\lambda j}}^{\lambda j} \ ; \ i = 1,2\dots\ell, \ j = 1,2\dots m\}$$

are linearly independent. Furthermore, restricting (3.20b) to $X_{2\lambda j}$ we obtain, for each i and j

$$\begin{aligned}
(A-\lambda j)\ \gamma_{i1}^{\lambda j} &= 0 \\
(A-\lambda j)\ \gamma_{i2}^{\lambda j} &= \gamma_{i1}^{\lambda j} \\
&\vdots \\
(A-\lambda j)\ \gamma_{ik_{\lambda j}}^{\lambda j} &= \gamma_{ik_{\lambda j}-1}^{\lambda j}
\end{aligned} \tag{3.20c}$$

which in turn imply that

$$\gamma_{ip}^{\lambda j} \in N_p(\lambda j) = \mathrm{Ker}(A-\lambda j)^p \cap \mathrm{Im}(A-\lambda j)^{k_{\lambda j}-p} \tag{3.20d}$$

for p = 1,2,...$k_{\lambda j}$, j = 1, 2,...m, i = 1,2... ℓ

Now, for $\lambda j \in \sigma(A_2)$, it is easy to see from (3.20d), that

$$N_1(\lambda j) \subset N_2(\lambda j) \subset \dots N_{k_{\lambda j}}(\lambda j)$$

thus we may conclude that

$$\dim[N_1(\lambda j)] \geq \ell \geq q$$

since $\gamma_{11}^{\lambda j}, \gamma_{21}^{\lambda j} \dots \gamma_{\ell 1}^{\lambda j} \varepsilon N_1(\lambda j)$ are linearly independent. This establishes the necessity of (3.20).

To prove the sufficiency of (3.20) consider λj fix. Clearly (3.20) implies that there are at least q linearly independent vectors.

$$\gamma_{11}^{\lambda j} , \gamma_{21}^{\lambda j} \dots \gamma_{q1}^{\lambda j} \varepsilon N_1(\lambda j) \tag{3.20e}$$

We will now show that there are at least q-linearly independent vectors

$$\gamma_{1p}^{\lambda j} , \gamma_{2p}^{\lambda j} \dots \gamma_{qp}^{\lambda j} \varepsilon N_p(\lambda j), \quad p = 2, 3, \dots k_{\lambda j} \tag{3.20f}$$

such that the vectors

$$S(\lambda j) = \{\gamma_{i1}^{\lambda j} , \gamma_{i2}^{\lambda j} \dots \gamma_{ik_{\lambda j}}^{\lambda j} ; \quad i = 1,2, \dots q\} \tag{3.20g}$$

are linearly independent.

It can be shown [T2, Theorem 6.3, p. 291]that

$$N_1(\lambda j) \approx \frac{\mathrm{Ker}(A-\lambda j)^{k_{\lambda j}}}{\mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1}}$$

In fact, $(A-\lambda j)^{k_{\lambda j}-1}$ maps $\mathrm{Ker}(A-\lambda j)^{k_{\lambda j}}$ onto $N_1(\lambda j)$. Hence, (3.20e) implies that there are q independent vectors $\gamma_{ik_{\lambda j}}^{\lambda j}$ such that

$$(A-\lambda j)^{k_{\lambda j}-1} \gamma_{ik_{\lambda j}}^{\lambda j} = \gamma_{i1}^{\lambda j} , \quad i = 1, 2 \dots q \tag{3.20h}$$

i.e. for $1 \leq i \leq q$

$$\gamma_{ik_{\lambda j}}^{\lambda j} \varepsilon \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}}, \gamma_{ik_{\lambda j}}^{\lambda j} \notin \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1} \tag{3.20i}$$

and since $\gamma_{i1}^{\lambda j} \varepsilon \mathrm{Ker}(A-\lambda j) \subset \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ it follows that the vectors

$$\{\gamma_{i1}^{\lambda j} , \gamma_{ik_{\lambda j}}^{\lambda j} ; \quad i = 1, 2, \dots q\}$$

are linearly independent.

Now set

$$(A-\lambda j)^{k_{\lambda j}-2} \gamma_{ik_{\lambda j}}^{\lambda j} = \gamma_{i2}^{\lambda j} \quad , \quad i = 1,2 \ldots q \qquad (3.20j)$$

and note that $(A-\lambda j)^{k_{\lambda j}-2}$ maps $\mathrm{Ker}(A-\lambda j)^{k_{\lambda j}}$ onto $N_2(\lambda j)$. Clearly the vectors $\gamma_{i2}^{\lambda j}$ are independent. We next show that

$$\{\gamma_{i1}^{\lambda j}, \gamma_{i2}^{\lambda j}, \gamma_{ik_{\lambda j}}^{\lambda j} \quad ; \quad i = 1, 2 \ldots q\} \qquad (3.20k)$$

are linearly independent. It is easy to see that $\{\gamma_{i2}^{\lambda j}, \gamma_{ik_{\lambda j}}^{\lambda j}\}$ are independent since $\gamma_{i2}^{\lambda j} \in \mathrm{Ker}(A-\lambda j)^2 \subset \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ and $\gamma_{ik_{\lambda}}^{\lambda j} \notin \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ (see (3.20i). Also, $\{\gamma_{i1}^{\lambda j}, \gamma_{i2}^{\lambda j}\}$ are linearly independent since

$$(A-\lambda j)\gamma_{i2}^{\lambda j} = \gamma_{i1}^{\lambda j}$$

otherwise

$$(A-\lambda j)\gamma_{i2}^{\lambda j} = 0$$

which in turn implies

$$(A-\lambda j)^{k_{\lambda j}-1} \gamma_{ik_{\lambda j}}^{\lambda j} = 0$$

i.e. $\gamma_{ik_{\lambda j}}^{\lambda j} \in \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ which is not possible (see (3.20i))

thus,

$$\gamma_{i2}^{\lambda j} \in \mathrm{Ker}(A-\lambda j)^2, \ \gamma_{i2}^{\lambda j} \notin \mathrm{Ker}(A-\lambda j) \qquad (3.20\ell)$$

Hence the vectors in (3.20k) are linearly independent.

We now set

$$(A-\lambda j)^{k_{\lambda j}-3} \gamma_{ik_{\lambda j}}^{\lambda j} = \gamma_{i3}^{\lambda j} \quad , \quad i = 1, 2 \ldots q \qquad (3.20m)$$

and note that $(A-\lambda j)^{k_{\lambda j}-3}$ maps $\mathrm{Ker}(A-\lambda j)^{k_{\lambda j}}$ onto $N_3(\lambda j)$. It is readily verified that $\gamma_{i3}^{\lambda j}$ are independent. We next show that

$$\{\gamma_{i1}^{\lambda j}, \gamma_{i2}^{\lambda j}, \gamma_{i3}^{\lambda j}, \gamma_{ik_{\lambda j}}^{\lambda j} ; \quad i = 1,2\ldots q\} \qquad (3.20n)$$

are independent. First observe that $\{\gamma_{i3}^{\lambda j}, \gamma_{ik_{\lambda j}}^{\lambda j}\}$ are independent since $\gamma_{i3}^{\lambda j} \in \text{Ker}(A-\lambda j)^3 \subset \text{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ and $\gamma_{ik_{\lambda j}}^{\lambda j} \notin \text{Ker}(A-\lambda j)^{k_{\lambda j}-1}$.
On the other hand we have the vectors $\{\gamma_{i1}^{\lambda j}, \gamma_{i2}^{\lambda j}, \gamma_{i3}^{\lambda j}\}$ are independent since

$$(A-\lambda j)^2\gamma_{i3}^{\lambda j} = \gamma_{i1}^{\lambda j}$$

otherwise

$$(A-\lambda j)^2\gamma_{i3}^{\lambda j} = 0$$

which implies

$$(A-\lambda j)^{k_{\lambda j}-1}\gamma_{ik_{\lambda j}}^{\lambda j} = 0$$

i.e. $\gamma_{ik_{\lambda j}}^{\lambda j} \in \text{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ which is not possible (see (3.20i))
thus

$$\gamma_{i3}^{\lambda j} \in \text{Ker}(A-\lambda j)^3 , \gamma_{i3}^{\lambda j} \notin \text{Ker}(A-\lambda j)^2 \supset \text{Ker}(A-\lambda j)$$

hence the vectors in (3.20n) are linearly independent.

Now consider any $3 \leq t \leq k_{\lambda j} -1$. Let

$$(A-\lambda j)^{k_{\lambda j}-t}\gamma_{ik_{\lambda j}}^{\lambda j} = \gamma_{it}^{\lambda j} , \quad i = 1,2\ldots q \qquad (3.20p)$$

Observe that $(A-\lambda j)^{k_{\lambda j}-t}$ maps $\text{Ker}(A-\lambda j)^{k_{\lambda j}}$ onto $N_t(\lambda j)$. Clearly the vectors $\gamma_{it}^{\lambda j}$ are independent. We next show that

$$\{\gamma_{i1}^{\lambda j}, \gamma_{i2}^{\lambda j}, \gamma_{i3}^{\lambda j}, \ldots\ldots\ldots, \gamma_{it}^{\lambda j}, \gamma_{ik_{\lambda j}}^{\lambda j} ; \quad i = 1, 2\ldots q\} \qquad (3.20q)$$

are also independent. It is easy to see that $\{\gamma_{it}^{\lambda j}, \gamma_{ik_{\lambda j}}^{\lambda j}\}$ are independent since $\gamma_{it}^{\lambda j} \in \text{Ker}(A-\lambda j)^t \subset \text{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ and $\gamma_{ik_{\lambda j}}^{\lambda j} \notin \text{Ker}(A-\lambda j)^{k_{\lambda j}-1}$

On the other hand we have that $\{\gamma_{ip}^{\lambda j}\ ;\ \ i = 1, 2\ldots q,\ p = 1,2\ldots t\}$ are independent since

$$(A-\lambda j)^{t-1}\ \gamma_{it}^{\lambda j} = \gamma_{i1}^{\lambda j}$$

otherwise

$$(A-\lambda j)^{t-1}\gamma_{it}^{\lambda j} = 0$$

and this implies

$$(A-\lambda j)^{k_{\lambda j}-1}\ \gamma_{ik_{\lambda j}}^{\lambda j} = 0$$

which is not possible, since $\gamma_{ik_{\lambda j}}^{\lambda j} \notin \mathrm{Ker}(A-\lambda j)^{k_{\lambda j}-1}$ (see (3.20i)) thus,

$$\gamma_{it}^{\lambda j} \in \mathrm{Ker}(A-\lambda j)^{t},\ \gamma_{it}^{\lambda j} \notin \mathrm{Ker}(A-\lambda j)^{t-1} \supset \mathrm{Ker}(A-\lambda j)^{t-2} \ldots \supset \mathrm{Ker}(A-\lambda j)$$

Hence, the vectors in (3.20q) are independent.

We may now conclude that the vectors $S(\lambda j)$ in (3.20g) are linearly independent. Furthermore, the $\{\gamma_{i1}^{\lambda j}, \gamma_{i2}^{\lambda j} \ldots \gamma_{ik_{\lambda j}}^{\lambda j}\}$ satisfy the expressions (3.20c) for each $1 \leq i \leq q$, and for each $\lambda j \in \sigma(A_2)$. Also it is easy to see that

$$S(\lambda i) \cap S(\lambda j) = 0, \qquad i \neq j \qquad i,j = 1,2\ldots m$$

therefore, there exist a bounded injective operator $R : \tilde{X}_2 \to X$ satisfying (3.20a), and by definition this implies that A contains an internal model of A_2. This completes the proof of Lemma 3.9.

3.4 The Internal Model Principle for Ordinary Linear Systems

In this section we first state the IMP for finite dimensional linear systems. Then a brief description of the steps involved in establishing this result is given. For details see [F1] [F2] [W1].

The Internal Model Principle: A regulator synthesis, that is a controller which yields output regulation and internal stability of the closed loop system, is structurally stable only if it utilizes feedback of the regulated variables, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process.

We now outline the steps required in proving the IMP. The main technical results are given by various lemmas.

Step 1: This step consists in proving that readability of z from y is a necessary condition for structural stability. A preliminary result is given by the following

Lemma 3.10: A synthesis is structurally stable at A_3 only if

$$\text{Ker } C_1 \subset \ker D_1$$

From the above result we can write $\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z}$, then

$$C_1 = \begin{pmatrix} E_1 \\ D_1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} E_2 \\ \tilde{D}_2 \end{pmatrix}, \quad B_c = [B_{cw} \;\; B_{cz}], \; F = [F_w \;\; F_z] \qquad .. \; (3.21)$$

for some E_1, E_2 and $\tilde{D}_2$ where $B_{cw} = B_c|\mathcal{W}$, $B_{cz} = B_c|\mathcal{Z}$, $F_w = F|\mathcal{W}$ and $F_z = F|\mathcal{W}$.

Considering perturbations in some other parameter it can be shown that structural stability of the controller requires $\tilde{D}_2 = D_2$. This result may be expressed as follows

Lemma 3.11: (Necessity of Readability) . A synthesis is structurally stable at $(A_3, B_c|\mathcal{Z} = B_{cz})$ only if

$$\text{Ker}[C_1 \;\; C_2] \subset \text{Ker}[D_1 \;\; D_2]$$

Step 2: Having proved the necessity of readability we now consider a synthesis in which this condition is satisfied. Also, we adopt the representation (3.21) with $\tilde{D}_2 = D_2$.

This step consists in establishing the existence of an A_c-invariant subspace R_c such that the operator $\bar{A}_c : \bar{X}_c \to \bar{X}_c$ induced by A_c in $\bar{X}_c = X_c / R_c$ incorporates an internal model of A_2

Define

$$R_c = \langle A_c | B_{cw} E_1 \text{ Ker } D_1 \rangle^{\dagger} \qquad (3.22)$$

then $X_c = R_c \oplus X_{c2}$ and

$$A_c = \begin{pmatrix} A_{c1} & A_{c3} \\ 0 & A_{c2} \end{pmatrix}, \quad B_c = \begin{pmatrix} B_{cw1} & B_{cz1} \\ B_{cw2} & B_{cz2} \end{pmatrix}, \quad F_c = [F_{c1} \quad F_{c2}] \qquad .. \quad (3.23)$$

A_{c2} must contain an internal model of A_2. This result is expressed as follows

<u>Lemma 3.12</u>: (Necessity of the internal model). A synthesis is structurally stable at A_3 only if the controller incorporates an internal model of A_2. Moreover, the internal model is observable by u, that is A_2-modes of A_c are observable by F_c, i.e. $\text{Ker } F_c \cap \text{Ker}(A_c - \lambda) = 0, \quad \lambda \in \sigma(A_2)$.

<u>Step 3</u>: This step consists in proving the necessity of the feedback structure. This result will follow once we show that

$$\text{Im } B_{cw} \subset R_c \qquad (3.24)$$

or equivalently

$$\langle A_c | \text{Im } B_{cw} \rangle = R_c$$

where R_c is given by (3.22). The necessity of this condition is expressed as follows

<u>Lemma 3.13</u>: There is no synthesis in which (3.24) fails and is structurally

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† Let $A : X \to X$ and consider a subspace R of X. Then

$\langle A|R \rangle = R + AR + \ldots + A^{n-1}R$ is an A-invariant subspace of X[W1]. (Of course we assume that X is finite dimensional).

stable at $(A_3, P_c B_{cw} = \bar{B}_{cw})$, where $P_c : X_c \to \bar{X}_c = \frac{X_c}{R_c}$ is the canonical projection.

The interpretation of (3.24) is that $B_{cw_2} = 0$ in the representation (3.23), i.e. $P_c B_{cw} = 0$. In this case we have the following

Lemma 3.14: If (3.24) holds, then the internal model is controllable by z, that is A_2-modes of A_c are controllable by $P_c B_{cz} = \bar{B}_{cz}$, i.e.

$$\bar{X}_c = \text{Im}(\bar{A}_c - \lambda) + \text{Im Im } \bar{B}_{cz}, \quad \lambda \in \sigma(A_2).$$

This establishes the necessity of the feedback structure.

To close this section we give the following result concerning the sufficiency of the IMP [F2].

Lemma 3.15: Suppose z is readable from y, the closed-loop system is internally stable and the controller incorporates an internal model of A_2 which is controllable by z and observable by u. Then the synthesis is structurally stable with respect to the parameters $(A_1, A_3, B_1, F_{c1}, F_{c2}, F_w, F_2, A_{c1}, A_{c3}, B_{cw1}, B_{cz1}, B_{cz2})$.

The only part of the controller which we do not allow to vary is A_{c2}, i.e. the part containing the internal model of A_2. Also we mention that while A_3 is allowed to vary arbitrarily, the size of the perturbations of the remainding parameters is restricted to preserve internal stability, i.e. $(A_L + \delta A_L)$ must be stable.

3.5 Structural Stability of Stabilizability and Detectability

In this section we will establish that stabilizability of (A_1, B_1) and detectability of (C_1, A_1) are stable properties with respect to certain small perturbations of A_1, B_1 and C_1 (see Section 3.1).

Proposition 3.16. Stabilizability of the pair (A_1, B_1) is a stable property with respect to small bounded perturbations of B_1 and small perturbations of A_1 of class $F(A_1)$.

Proof: The proof of this result consists of several steps.

1. We know that A_1 satisfies the spectral decomposition described in Section 2.2. In particular, the finite dimensionality of X_1^+ and the compactness of the semigroup $S_{A_1}(t)$, $t \geq h$ imply, by Lemma 2.8, that (A_1, B_1) is stabilizable if and only if

$$\mathrm{Im}(A_1-\lambda) + \mathrm{Im}\, B_1 = X_1 \quad , \quad \lambda \in C^+ \tag{3.25}$$

2. Now assume B_1 is fixed and only A_1 is allowed to perturb. Let $\{\lambda_i\}$, $i = 1, 2,\ldots,k$ be the distinct eigenvalues of A_1 in C^+, and let m_i denote the algebraic multiplicity of λ_i. The total multiplicity of the eigenvalues of A_1 in C^+ is $N = \sum_{i=1}^{k} m_i$. Now enclose each λ_i by a closed curve Γ_i so that Γ_i contains λ_i only. Then, by Theorem B.III.2 in Appendix B and the fact that $\{\lambda_i\}$ is a finite system of eigenvalues, we may conclude that there is a $\delta > 0$, depending on A_1 and Γ_i's, such that for any δA_1 of class $F(A_1)$ with $\hat{\delta}(A_1+\delta A_1, A_1) < \delta^{\dagger}$, the spectrum of $(A_1+\delta A_1)$ is likewise seperated by Γ_i's, and the total multiplicity of the eigenvalues of $(A_1+\delta A_1)$ in Γ_i is m_i for each $i = 1,2\ldots k$. Furthermore, the change of each λ_i is small if $\hat{\delta}(A_1+\delta A_1, A_1)$ is small. In addition, the upper semicontinuity of the spectrum of A_1, assures that no eigenvalues of A_1 in C^- move to C^+. Therefore the total multiplicity of the eigenvalues of $(A_1+\delta A_1)$ in C^+ is equal to N.

---

† a more explicit condition is given by Theorem B.III.4 in Appendix B

3. Consider any perturbation δA_1 as described in Step 2. It follows that $(A_1+\delta A_1)$ is the infinitesimal generator of a strongly continuous semigroup $S_{(A+\delta A_1)}(t)$, $t \geq 0$. Moreover, $S_{(A_1+\delta A_1)}(t)$ is compact for $t \geq h$ (see Section 3.1). Thus, by Lemma 2.8, we have that $(A_1+\delta A_1,B_1)$ is stabilizable if and only of

$$\mathrm{Im}(A_1+\delta A_1-\lambda') + \mathrm{Im}\, B_1 = X_1 \quad , \lambda' \varepsilon\ C^+ \qquad (3.26)$$

Thus the proposition will be proved if the above condition is verified at each $\lambda' \varepsilon\ \sigma(A_1+\delta A_1) \cap C^+$.

4. It can be shown that $\mathrm{Im}(A_1-\lambda)$ and $\mathrm{Im}(A_1+\delta A_1-\lambda)$ are closed subspaces† for every $\lambda \varepsilon\ C$. Clearly $\mathrm{Im}\, B_1$ is colsed since it is finite dimensional. Therefore we may use the results of Appendix B concerning pairs of closed subspaces.

Since $\mathrm{Im}(A_1-\lambda) + \mathrm{Im}\, B_1 = X_1$, $\lambda \varepsilon\ C^+$, Theorem B.I.1 implies that there is an $\varepsilon > 0$ such that

$$1 \geq \gamma(\mathrm{Im}(A_1-\lambda), \mathrm{Im}\, B_1) \geq \varepsilon \quad , \quad \lambda \varepsilon\ \sigma(A_1) \cap C^+$$

Now, since $\mathrm{Ker}(A_1-\lambda)$ is finite dimensional and δA_1 is A_1 - compact, Theorem B.IV.12 gives

$$\delta(\mathrm{Im}(A_1-\lambda), \mathrm{Im}(A_1+\delta A_1-\lambda')) \leq \frac{a}{\gamma(A_1-\lambda)} + b + |\lambda-\lambda'|$$

where a, b†† are non-negative constants so that $\| \delta A_1 x_1 \| \leq a \| x_1 \| + b \| A_1 x_1 \|$, $x_1 \varepsilon\ D(A_1)$, and $\gamma(A_1-\lambda) > 0$ by Theoem B.IV.1. Thus, for a sufficiently small perturbation δA_1 (in the sense of $\hat{\delta}(A_1+\delta A_1,A_1)$ being small)

$$\delta(\mathrm{Im}(A_1-\lambda), \mathrm{Im}(A_1+\delta A_1-\lambda')) < \varepsilon \leq \gamma(\mathrm{Im}(A_1-\lambda), \mathrm{Im}\, B_1)$$

---

† this follows from the fact that $(A_1-\lambda)$ and $(A_1+\delta A_1-\lambda)$ are Fredholm operators from all λ (see Lemma 3.21 in Section 3.7).

†† b may be chosen arbitrarily small, since since the range of $\delta A_1 \varepsilon F(A_1)$ is finite dimensional and A_1 is densely defined (Section 3.1).

Theorem B.I.2 now gives

$$\text{def}(\text{Im}(A_1+\delta A_1-\lambda'),\text{Im } B_1) \leq \text{def}(\text{Im}(A_1-\lambda), \text{ Im } B_1) = 0$$

hence

$$\text{Im}(A_1+\delta A_1-\lambda') + \text{Im } B_1 = X \qquad \lambda' \in C^+.$$

The case when B_1 is perturbed can be proved in a similar way. Furthermore, if A_1 and B_1 are perturbed simultaneously Theorem B.I.8 may be used to obtain

$$\text{Im}(A_1+\delta A_1-\lambda') + \text{Im}(B_1+\delta B_1) = X_1 \quad , \quad \lambda' \in C^+ \tag{3.27}$$

We next prove the structural stability of detectability.

<u>Proposition 3.17</u>. Detectability of the pair (C_1, A_1) is a stable property with respect to small bounded perturbations of C_1, and small perturbations of A_1 of class $F(A_1)$.

<u>Proof</u>: The proof is similar to that of proposition 3.16.

1. We have from Lemma 2.9 that (C_1, A_1) is detectable if and only if

$$\text{Ker}(A_1-\lambda) \cap \text{Ker } C_1 = 0, \qquad \lambda \in C^+ \tag{3.28}$$

2. The discussion in step 2 of the previous proof is valid in this case also.

3. Again, from Lemma 2.9, we may conclude that $(C_1, A_1+\delta A_1)$ is detectable if and only if

$$\text{Ker}(A_1+\delta A_1-\lambda') \cap \text{Ker } C_1 = 0, \qquad \lambda' \in C^+ \tag{3.29}$$

4. Clearly Ker C_1 is a closed subspace of X_1, further since Im C_1 is finite dimensional, it is easy to see that

$$\text{codim}[\text{Ker } C_1] < \infty$$

thus $[\text{Ker}(A_1-\lambda) + \text{Ker } C_1]$ and $[\text{Ker}(A_1+\delta A_1-\lambda') + \text{Ker } C_1]$ are both closed subspaces [T2, p.73].

It follows, by Theorem B.I.1, that there is an $\varepsilon > 0$ such that

$$1 \geq \gamma(\mathrm{Ker}\ C_1, \mathrm{Ker}(A_1-\lambda)) \geq \varepsilon \quad , \quad \lambda \in \sigma(A_1) \cap C^+$$

Now, since $(A_1-\lambda)$ and $(A_1+\delta A_1-\lambda')$ are Fredholm operators we have that $\gamma(A_1-\lambda) > 0$, and Theorem B.IV.10 yields

$$\delta(\mathrm{Ker}(A_1+\delta A_1-\lambda'), \mathrm{Ker}(A_1-\lambda)) \leq [2 + \gamma^{-2}(A_1-\lambda)]^{\frac{1}{2}}\delta(A_1+\delta A_1-\lambda', A_1-\lambda)$$

but, since δA_1 is A_1-bounded with A_1-bound zero, Theorem B.II.5 gives

$$\delta(A_1+\delta A_1-\lambda', A_1-\lambda) \leq (1-b)^{-1}[(a+\lambda-\lambda')^2 + b^2]^{\frac{1}{2}}$$

thus for a sufficiently small perturbation δA_1 we have

$$\delta(\mathrm{Ker}(A_1+\delta A_1-\lambda'), \mathrm{Ker}(A_1-\lambda)) < \varepsilon \leq \gamma(\mathrm{Ker}\ C_1, \mathrm{Ker}(A_1-\lambda))$$

and Theorem B.I.2 implies

$$\mathrm{nul}(\mathrm{Ker}(A_1+\delta A_1-\lambda'), \mathrm{Ker}\ C_1) \leq \mathrm{nul}(\mathrm{Ker}(A_1-\lambda), \mathrm{Ker}\ C_1) = 0$$

hence

$$\mathrm{Ker}(A_1+\delta A_1-\lambda') \cap \mathrm{Ker}\ C_1 = 0 \quad , \quad \lambda' \in C^+$$

The case when C_1 is perturbed can be proved in a similar way. Furthermore, if A_1 and C_1 are perturbed simultaneously, Theorem B.I.7 may be used to obtain

$$\mathrm{Ker}(A_1+\delta A_1-\lambda') \cap \mathrm{Ker}(C_1+\delta C_1) = 0 \ , \ \lambda \ \in C^+ \qquad (3.30)$$

We point out that (3.27) and (3.30) also hold for small perturbations δA_1 of A_1 which are A_1-bounded, with A_1-bounded less than 1. However, for such perturbations, (3.27) and (3.30) cannot be interpreted as conditions for stabilizability and detectability, simply because the perturbed operator $A_1+\delta A_1$ is not, in general, an infinitesimal generator of a strongly continuous semigroup. Even, for small bounded perturbations of A_1, (3.27) and (3.30) may no longer be sufficient conditions to assure stabilizability and detectability (although $A_1+\delta A_1$ does generate a strongly continuous semigroup when δA_1 is bounded operator).

We show below how such situation may arise.

Let

$$A_1 = \begin{pmatrix} A_1^- & 0 \\ 0 & A_1^+ \end{pmatrix} \quad , \quad B_1 = \begin{pmatrix} 0 \\ B_1^+ \end{pmatrix}$$

with $\sigma(A_1^-) \subset C^-$, $\sigma(A_1^+) \subset C^+(\sigma(A_1^+)$ consists of a finite number of eigenvalues). We further know that the semigroup $S_{A_1}(t)$ is compact for $t \geq h$. Therefore $S_{A_1^-}(t)$ is also compact for $t \geq h$ and since $\sigma(A_1^-) \subset C^-$ we may conclude that there is an $M = M_{(\alpha-\varepsilon)} < \infty$ such that

$$\| S_{A_1^-}(t) \| \leq M\, e^{-(\alpha-\varepsilon)t} \quad , \text{ arbitrary } \varepsilon > 0,\ t \geq 0$$

where

$$-\alpha = \sup \operatorname{Re} \sigma(A_1^-)$$

Suppose that (A_1, B_1) is stabilizable, then by Lemma 2.8 this is equivalent to (3.25). Now consider the bounded perturbation

$$\delta A_1 = \begin{pmatrix} \delta A_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

such that $\sigma(A_1^- + \delta A_{11}) \subset C^-$

Clearly (3.27) is satisfied with $\delta B_1 \equiv 0$, however the semigroup $S_{(A_1^-+\delta A_{11})}(t)$ is not necessarily compact for $t \geq h$, and the inclusion of $\sigma(A_1^-+\delta A_{11})$ in the open left half plane is not sufficient to guarantee the stability of the semigroup. In this case (3.27) alone is not a sufficient condition to determine the stabilizability of the pair $(A_1+\delta A_1, B_1)$.

On the other hand we have a rough estimate for the growth of $S_{(A_1^-+\delta A_{11})}(t)$ [K1, Theorem 2.1, p.497]

$$\| S_{(A_1^-+\delta A_{11})}(t) \| \leq M\, e^{(M\| \delta A_{11} \| - (\alpha-\varepsilon))t} \quad , \quad t \geq 0$$

So if

$$M\| \delta A_{11} \| < (\alpha-\varepsilon)$$

and (3.27) holds then the pair $(A_1+\delta A_1, B_1)$ is stabilizable.

3.6 Necessity of Readability

In this section we will establish the counterpart of Lemmas 3.10 and 3.11 of Section 3.4. As in the finite dimensional case, we assume that A_2, C_1, C_2, D_1 and D_2 are fixed. We denote by $S_c = (\mathcal{X}_c, A_c, B_c, F_c, G_c)$ any synthesis which yields regulation of z and internal stability of the closed loop system. For such S_c we have, by proposition 2.6, that there is a bounded operator $X_L : \mathcal{X}_2 = R^{n_2} \to \mathcal{X}_L = \mathcal{X}_1 \times \mathcal{X}_c$ such that

$$A_L X_L - X_L A_2 = B_L \tag{3.31}$$

$$D_L X_L = D_2 \tag{3.32}$$

where

$$A_L = \begin{pmatrix} A_1 + B_1 G_c C_1 & B_1 F_c \\ B_c C_1 & A_c \end{pmatrix} : D(A_1) \times D(A_c) \to \mathcal{X}_1 \times \mathcal{X}_c$$

$$B_L = \begin{pmatrix} A_3 + B_1 G_c C_2 \\ B_c C_2 \end{pmatrix} : \mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_c$$

$$D_L = [D_1 \quad 0] : \mathcal{X}_1 \times \mathcal{X}_c \to \mathcal{Z}$$

We can now prove the following partial result

Proposition 3.18. A synthesis S_c is structurally stable at A_3 only if

$$\operatorname{Ker} C_1 \subset \operatorname{Ker} D_1 \tag{3.33}$$

Proof: Let $\delta A_3 : \mathcal{X}_2 \to \mathcal{X}_1$ be a bounded operator (clearly it is compact since $\mathcal{X}_2$ is finite dimensional) and suppose that S_c is a synthesis which is structurally stable. Then the perturbed versions of (3.31) - (3.32) must have a solution $\hat{X}_L = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_c \end{pmatrix}$, i.e.,

$$\begin{pmatrix} A_1+B_1G_cC_1 & B_1F_c \\ B_cC_1 & A_c \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_c \end{pmatrix} - \begin{pmatrix} \hat{X}_1 \\ \hat{X}_c \end{pmatrix} A_2 = \begin{pmatrix} A_3+\delta A_3+B_1G_cC_c \\ B_cC_2 \end{pmatrix}$$

$$[D_1 \quad 0] \begin{pmatrix} \hat{X}_1 \\ \hat{X}_c \end{pmatrix} = D_2$$

Let $\hat{X}_L - X_L = \delta X_L = \begin{pmatrix} \delta X_1 \\ \delta X_c \end{pmatrix}$, then from the above expressions

and (3.31) - (3.32) we obtain

$$\begin{pmatrix} A_1+B_1G_cC_1 & B_1F_c \\ B_cC_1 & A_c \end{pmatrix} \begin{pmatrix} \delta X_1 \\ \delta X_c \end{pmatrix} - \begin{pmatrix} \delta X_1 \\ \delta X_c \end{pmatrix} A_2 = \begin{pmatrix} \delta A3 \\ 0 \end{pmatrix} \tag{3.34}$$

$$[D_1 \quad 0] \begin{pmatrix} \delta X_1 \\ \delta X_c \end{pmatrix} = 0$$

Now consider a decomposition of $\mathcal{X}_2$ into into prime subspaces [W1], that is

$$\mathcal{X}_2 = \bigoplus_{\lambda} \bigoplus_{j=1}^{t(\lambda)} \mathcal{X}_{2\lambda}^j$$

where $\lambda \in \sigma(A_2)$ and $t(\lambda) = \dim [\mathrm{Ker}(A_2-\lambda)]$ then, it can be shown that $A_2|\mathcal{X}_{2\lambda}^j$ is cyclic with minimal polynomial $(S-\lambda)^{k(\lambda,j)}$, $k(\lambda,j) = \dim[\mathcal{X}_{2\lambda}^j]$.

Let k_λ denote the degree of the factor $(s-\lambda)$ in the minimal polynomial of A_2, then k_λ is given by

$$k_\lambda = \max[k(\lambda,j);\; j = 1,2,\dots t(\lambda)]$$

We now fix $\lambda \in \sigma(A_2)$ and choose any prime subspace $X^j_{2\lambda}$ corresponding to λ. $\dim[X^j_{2\lambda}] = k(\lambda,j)$. Select a basis for $X^j_{2\lambda}$ such that $A^j_{2\lambda} = A_{2\lambda}|X^j_{2\lambda}$ is represented by its Jordan form. In this basis we can write

$$D_2|X^j_{2\lambda} = [d_{21}\ d_{22}\ \cdots\ d_{2k}]$$

$$C_2|X^j_{2\lambda} = [c_{21}\ c_{22}\ \cdots\ c_{2k}]$$

$$\delta X_1|X^j_{2\lambda} = [x_{11}\ x_{12}\ \cdots\ x_{1k}]$$

$$\delta X_c|X^j_{2\lambda} = [x_{c1}\ x_{c2}\ \cdots\ x_{ck}]$$

$$\delta A_3|X^j_{2\lambda} = [a_{31}\ a_{32}\ \cdots\ a_{3k}]$$

where, for simplicity in notation we have written k in place of $k(\lambda,j)$.

Now, from (3.34) we obtain (for each $\lambda \in \sigma(A_2)$ and each prime subspace $X^i_{2\lambda}$ associated with λ) the following equations

$$\begin{pmatrix} A_1+B_1G_cC_1 & B_1F_c \\ B_cC_1 & A_c \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{c1} & x_{c2} & \cdots & x_{ck} \end{pmatrix}$$

$$- \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{c1} & x_{c2} & \cdots & x_{ck} \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & & \lambda \end{pmatrix}_{k\times k} = \begin{pmatrix} a_{31} & a_{32} & \cdots & a_{3k} \\ 0 & 0 & & 0 \end{pmatrix}$$

$$D_1[x_{11}\ x_{12}\ \cdots\ x_{1k}] = [0\quad 0\ \cdots\ 0]_{1\times k}$$

or equivalently

$$\begin{bmatrix} A_1+A_1G_cC_1-\lambda & B_1F_c \\ B_cC_1 & \lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{c1} \end{bmatrix} = \begin{bmatrix} a_{31} \\ 0 \end{bmatrix} \qquad (3.35)$$

$$\begin{bmatrix} A_1+B_1G_cC_1-\lambda & B_1F_c \\ B_cC_1 & A_c-\lambda \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{ci} \end{bmatrix} = \begin{bmatrix} a_{3i} \\ 0 \end{bmatrix} + \begin{bmatrix} x_{1i-1} \\ x_{ci-1} \end{bmatrix}, \; i = 2, 3 \ldots k$$

(3.36)

$$D_1x_{1i} = 0 \quad , \quad i = 1, 2, 3 \ldots k \qquad (3.37)$$

An inspection of (3.35) - (3.37) reveals that these equations will have a solution only if there is a solution to (3.35) with

$$\begin{bmatrix} x_{11} \\ x_{c1} \end{bmatrix} \varepsilon \text{ Ker } D_L = \text{Ker}[D_1 \quad 0]$$

and since a_{31} is arbitrary in X_1, we obtain[†]

$$\begin{bmatrix} X_1 \\ 0 \end{bmatrix} \subset (A_L-\lambda)[\text{Ker } D_L \cap D(A_L)] \;, \; \lambda \; \varepsilon \; \sigma(A_2) \qquad (3.38)$$

Now define

$$T = [A_1+B_1G_cC_1-\lambda \quad B_1F_c] : \; D(T) = D(A_1) \times X_c \to X_1 \quad (3.39)$$

$$V = [B_cC_1 \quad A_c-\lambda] : D(V) = X_1 \times D(A_c) \to X_c \qquad (3.40)$$

T and V are closed densely defined operators and, since $\sigma(A_L) \subset C^-$, we have[††]

$$\text{Ker}(A_L-\lambda) = \text{Ker } T \cap \text{Ker } V = 0, \quad \lambda \; \varepsilon \; \sigma(A_2) \subset C^+$$

moreover

$$\text{Im } T = X_1 \text{ and Im } V = X_c$$

---

† for simplicity in notation we will sometimes write AM in place of A[M∩D(A)]

†† it can also be shown that Ker T + Ker V is closed and equals X_L, i.e. $\text{Ker } T \oplus \text{Ker } V = X_L$

So from (3.38)

$$(A_L-\lambda)[\text{Ker } V \cap D(A_L)] \subset (A_L-\lambda)[\text{Ker } D_L \cap D(A_L)]$$

and since $(A_L-\lambda)$ has a bounded inverse for all $\lambda \in \sigma(A_2) \subset C^+$ we obtain (the bar denotes closure)

$$\overline{\text{Ker } V \cap D(A_L)} \subset \overline{\text{Ker } D_L \cap D(A_L)}$$

Now, in a Hilbert space we can write (for any subspace L) $\bar{L} = L^{\perp\perp}$, where $\perp$ denotes orthogonal complement, thus

$$\overline{\text{Ker } V \cap D(A_L)} = [\text{Ker } V \cap D(A_L)]^{\perp\perp}$$

and since

$$[D(A_L)]^{\perp} = [\overline{D(A_L)}]^{\perp} = X_L^{\perp} = \{0\}$$

we obtain

$$[\text{Ker} V \cap D(A_L)]^{\perp} = [\text{Ker } V]^{\perp} + [D(A_L)]^{\perp}$$

$$= [\text{Ker } V]^{\perp}$$

thus

$$\overline{\text{Ker } V \cap D(A_L)} = [\text{Ker } V]^{\perp\perp} = \overline{\text{Ker } V} = \text{Ker } V$$

since the null space of a closed operator is closed. Similarly we obtain

$$\overline{\text{Ker } D_L \cap D(A_L)} = \text{Ker } D_L$$

hence

$$\text{Ker } V \subset \text{Ker } D_L \tag{3.41}$$

But

$$\text{Ker } B_c C_1 \times \text{Ker}(A_c-\lambda) \subset \text{Ker } V$$

and

$$\text{Ker } D_L = \text{Ker } D_1 \times X_c$$

(3.41) now gives

$$\text{Ker } B_c C_1 \subset \text{Ker } D_1 \tag{3.42}$$

which in turn gives

$$\text{Ker } C_1 \subset \text{Ker } D_1$$

this completes the proof.

Remarks: In the proof of Proposition 3.18 we have obtained the stronger conditions (3.41) and (3.42). These expressions will be useful in further developments.

We now consider a synthesis S_c which is structurally stable at A_3. Thus by proposition 3.18 Ker $C_1 \subset$ Ker D_1 and from (3.5) - (3.6) we conclude that

$$\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z}$$

where $\mathcal{W}$ is a complement of $\mathcal{Z}$ in $\mathcal{Y}$. According to this decomposition we may write

$$C_1 = \begin{pmatrix} E_1 \\ D_1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} E_2 \\ \tilde{D}_2 \end{pmatrix}, \quad B_c = [B_{cw} \quad B_{cz}], \; G_c = [G_{cw} \quad G_{cz}] \quad .. \quad (3.43)$$

where E_1 and E_2 are bounded (linear) operators from X_1 and X_2 respectively, into $\mathcal{W}$, and $B_{cw} = B_c|\mathcal{W}$, $B_{cz} = B_c|\mathcal{Z}$, $G_{cw} = G_c|\mathcal{W}$, $G_{cz} = G_c|\mathcal{Z}$.

The necessity of readability will be established once we show that $\tilde{D}_2 = D_2$.

Theorem 3.19: (Necessity of readability). A synthesis S_c is structurally stable at (A_3, B_{cz}) only if

$$\text{Ker}[C_1 \quad C_2] \subset \text{Ker}[D_1 \quad D_2] \qquad (3.44)$$

Proof: Suppose that $\tilde{D}_2 \neq D_2$. Substitution of (3.43) in (3.31) gives

$$(A_1 + B_1 G_c C_1) X_1 + B_1 F_c X_c - X_1 A_2 = A_3 + B_1 G_c C_2 \qquad (3.45a)$$

$$B_{cw} E_1 X_1 + B_{cz} D_1 X_1 + A_c X_c - X_c A_2 = B_{cw} E_2 + B_{cz} \tilde{D}_2 \qquad (3.45b)$$

$$D_1 X_1 = D_2 \qquad (3.45c)$$

Now consider perturbations δA_3 and δB_{cz} of A_3 and B_{cz} respectively. We mention that while the size of δB_{cz} is restricted to preserve internal stability, the size of δA_3 is arbitrary. Structural stability of S_c implies that there must exist $\hat{X}_1$ and $\hat{X}_c$ such that (3.45a) - (3.45c) are satisfied when A_3, B_{cz}, X_1 and X_c are replaced by $A_3 + \delta A_3$, $B_{cz} + \delta B_{cz}$, $\hat{X}_1$ and $\hat{X}_c$.

Define $\delta X_1 = \hat{X}_1 - X_1$ and $\delta X_c = \hat{X}_c - X_c$. Then

$$\begin{bmatrix} A_1+B_1G_cC_1 & B_1F_c \\ B_cC_1 & \end{bmatrix} \begin{bmatrix} \delta X_1 \\ \delta X_c \end{bmatrix} - \begin{bmatrix} \delta X_1 \\ \delta X_c \end{bmatrix} A_2 = \begin{bmatrix} \delta A_3 \\ \delta B_{cz}(\tilde{D}_2 - D_2) \end{bmatrix} \quad \text{.. (3.46a)}$$

$$[D_1 \quad 0] \begin{bmatrix} \delta X_1 \\ \delta X_c \end{bmatrix} = 0 \quad \text{(3.46b)}$$

As in the proof of proposition 3.18, let $X_2 = \underset{\lambda}{\oplus} \overset{t(\lambda)}{\underset{j=1}{\oplus}} X^j_{2\lambda}$, for $\lambda \in \sigma(A_2)$. From our initial assumption there is some $\lambda \in \sigma(A_2)$ and some prime subspace $X^j_{2\lambda}$ corresponding to λ such that

$$\tilde{D}_2 | X^j_{2\lambda} \neq D_2 | X^j_{2\lambda}$$

that is

$$(\tilde{d}_{2i} - d_{2i}) \neq 0, \text{ at least for one } 1 \leq i \leq k$$

writing (3.46a) - (3.46b) in more detail we obtain, for $\lambda \in \sigma(A_2)$

$$\begin{bmatrix} A_1+B_1G_cC_1-\lambda & B_1F_c \\ B_cC_1 & A_c-\lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{c1} \end{bmatrix} = \begin{bmatrix} a_{31} \\ \delta B_{cz}(\tilde{d}_{21} - d_{21}) \end{bmatrix}$$

$$\begin{bmatrix} A_1+B_1G_cC_1-\lambda & B_1F_c \\ B_cC_1 & A_c-\lambda \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{ci} \end{bmatrix} = \begin{bmatrix} a_{3i} \\ \delta B_{cz}(\tilde{d}_{2i} - d_{2i}) \end{bmatrix} + \begin{bmatrix} x_{1i-1} \\ x_{ci-1} \end{bmatrix}$$

$$i = 2,3 \ldots k$$

$$D_1 x_{1i} = 0 \quad , \quad i = 1,2 \ldots k$$

By inspection of the above expressions we find that for any $r_1 \in X_1$ and $r_c \in X_c$, there must exist $x_1 \in X_1$ and $x_c \in X_c$ such that, for $\lambda \in \sigma(A_2)$

$$(A_L-\lambda)\begin{pmatrix} x_1 \\ x_c \end{pmatrix} = \begin{pmatrix} r_1 \\ r_c \end{pmatrix} \quad , \quad \begin{pmatrix} x_1 \\ x_c \end{pmatrix} \in \text{Ker } D_L$$

but this implies that

$$\overline{(A_L-\lambda)^{-1}[X_1 \times X_c]} = \overline{D(A_L)} \subset \text{Ker } D_L$$

and since $D(A_L)$ is dense in $X_L = X_1 = X_c$ this can only happen for $D_L \equiv 0$ i.e. there are no variables to be regulated. Therefore $\tilde{D}_2 = D_2$ after all.

Having proved the necessity of readability we need only consider synthesis in which z is readable from y.

3.7 Necessity of Internal Model

In this section we will establish that a structurally stable synthesis S_c necessarily incorporates a reduplication, in the sense of Section 3.3, of the dynamic structure of the exogenous signals.

First, we develope some preliminary results

Definition 3.20 [G1, p. 103]. A closed operator A from X to Y, i.e. $A \in C(X,Y)$ is said to be a Fredholm operator if

1) $\dim[\text{Ker } A] < \infty$
2) $\text{codim}[\text{Im } A] = [Y/\text{Im } A] < \infty$

in this case the index of A is defined by

$$\text{ind}[A] = \dim[\text{Ker } A] - \text{codim } [\text{Im } A] < \infty$$

if in addition, X and Y are Banach spaces then (2) implies that Im A is closed in Y.

Lemma 3.21: [Gl. p.119][†]. Let X be a Banach space and A a closed operator in X. Suppose that there exist a λ_0 such that $(A-\lambda_0 I)^{-1}$ is compact. Then for all $\lambda \in C$, $(A-\lambda I)$ is a Fredholm operator with zero index, i.e.

$$\dim[\mathrm{Ker}(A-\lambda I)] = \mathrm{codim}[\mathrm{Im}(A-\lambda I)] < \infty, \quad \forall\lambda \in C$$

Lemma 3.22: [Gl, p.103]. Let M be a closed subspace of a Banach space X, with codim $[M] < \infty$, then

i) For any subspace L of X there is a finite dimensional subspace $N \subset L$ such that

$$\bar{L} = (\bar{L} \cap M) \oplus N$$

ii) If L is dense in X, then $L \cap M$ is dense in M.

Lemma 3.23: Let A be a closed operator in the Banach space X. Suppose that A is a Fredholm operator and let S be a subspace of X with finite codimension. Then

$$\mathrm{codim}[AS] + \mathrm{ind}[A] = \mathrm{codim}[S] + \dim[S\cap \mathrm{Ker}\, A] \quad (<\infty)$$

In particular if $\mathrm{ind}[A] = 0$, then

$$\mathrm{codim}[AS] \geq \mathrm{codim}[S]$$

where the equality holds if $S \cap \mathrm{Ker}\, A = 0$

Proof: We give a proof of this result since it is apparently not available in the literature. We proceed by a number of steps

1. Let $S \cap \mathrm{Ker}\, A = N_1$, then $N_1 \subset D(A)$ and $\dim[N_1] = n_I < \infty$

2. Let N_2 be a subspace of Ker A such that $\mathrm{Ker}\, A = N_1 \oplus N_2$

then

$$\dim[N_2] = n_2 < \infty \quad , N_2 \subset D(A) \quad , \quad S \cap N_2 = 0$$

and

$$\dim\left(\frac{\mathrm{Ker}\, A}{S\cap \mathrm{Ker}\, A}\right) = \dim[N_2]$$

† this result is valid for $(A-\lambda_0 I)^{-1}$ being strictly singular, see [Gl, Chapters III-IV] for details

3. From Lemma 3.22, there is a finite dimensional subspace $N_3 \subset D(A)$ with $\dim[N_3] = n_3 < \infty$ and such that

$$X = S \oplus N_2 \oplus N_3$$

then

$$\dim[\tfrac{X}{S}] = \dim[N_2] + \dim[N_3]$$

4. Since $\mathrm{Ker}\ A \subset S \oplus N_2$ we have

$$\mathrm{Im}\ A = AX = AS \oplus AN_3$$

and since A is 1-1 on all N_3

$$\dim\left(\frac{\mathrm{Im}A}{AS}\right) = \dim[AN_3] = \dim N_3$$

5. On the other hand

$$\dim\left(\frac{X}{\mathrm{Im}A}\right) = \left(\frac{X/AS}{\mathrm{Im}A/AS}\right) = \dim\left(\frac{X}{AS}\right) - \dim\left(\frac{\mathrm{Im}A}{AS}\right)$$

6. Combining 2 - 5 we obtain

$$\dim\left(\frac{X}{\mathrm{Im}A}\right) = \dim\left(\frac{X}{AS}\right) + \dim\left(\frac{X}{S}\right) - \dim\left(\frac{\mathrm{Ker}\ A}{S \cap \mathrm{Ker}\ A}\right)$$

hence

$$\dim\left(\frac{X}{AS}\right) + \mathrm{ind}[A] = \dim\left(\frac{X}{S}\right) + \dim[S \cap \mathrm{Ker}\ A]$$

The last part of the lemma follows easily.

We can now prove the following result concerning structural stability of the synthesis S_c.

Prosposition 3.24 . A synthesis S_c is structurally stable at A_3 only if for every $\lambda \in \sigma(A_2)$,

$$(A_1-\lambda)[\mathrm{Ker}\ D_1 \cap D(A_1)] + \mathrm{Im}B_1 = X_1$$

and this implies

$$\dim[\mathrm{Im}\ B_1] \geq \dim[Z] = q.$$

Proof: From the proof of proposition 3.18 we have, for $\lambda \varepsilon \sigma(A_2)$

$$T \text{ Ker } D_L = [A_1+B_1G_cC_1-\lambda \quad B_1F_c]\text{Ker } D_L = X_1$$

But

$$\text{Ker } D_L = \text{Ker } D_1 \times X_c$$

thus

$$(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1 + \text{Im } B_1 F_c = X_1 \qquad (3.47)$$

but

$$\text{Im } B_1 F_c \subset \text{Im } B_1$$

hence

$$(A_1-\lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im } B_1 = X_1$$

for the last part of the proposition we have from (3.47)

$$\text{codim}[(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1] = \dim\left(\frac{(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1 + \text{Im } B_1F_c}{(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1}\right) \quad .. \quad (3.48)$$

but the left hand side of the above expression, by lemma 3.23, equals

$$\text{codim}[(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1] = \text{codim}[\text{Ker } D_1] + \dim[\text{Ker}(A_1+B_1G_cC_1-\lambda) \cap \text{Ker } D_1] \quad .. \quad (3.49)$$

and the right hand side equals

$$\dim\left(\frac{(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1+\text{Im}B_1F_c}{(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1}\right) = \dim\left(\frac{\text{Im } B_1 F_c}{\text{Im } B_1 F_c \cap (A_1+B_1G_cC_1-\lambda)\text{ker } D_1}\right) \quad .. \quad (3.50)$$

Combining (3.48) - (3.50), and noting that $\text{codim}[\text{Ker } D_1] = \dim[Z] = q$ we obtain

$$\dim[\text{Im } B_1 F_c] = q + \dim[\text{Ker}(A_1+B_1G_cC_1-\lambda)\cap \text{Ker } D_1]$$
$$+ \dim[\text{Im } B_1 F_c\cap(A_1+B_1G_cC_1-\lambda)\text{Ker } D_1]$$

so $$\dim[\text{Im } B_1 F_c] \geq q \qquad (3.51)$$

since $\text{Im } B_1 F_c \subset \text{Im } B_1$ the result follows.

The above result is well known for ordinary linear systems [W1,Chapter 8]. A similar result was obtained by Bhat, for evolution systems, under the assumption $[C_1\ C_2] = [D_1\ D_2]$ [B1, Chapter 6]. However, the condition $\dim[\text{Im } B_1] \geq q$ is not derived in his work. We point out that proposition 3.24 is a 'nice' result since it provides necessary conditions in terms of the systems paramters. The sufficiency of this result will be investigated in Chapter 4.

Proposition 3.25. A synthesis S_c is structurally stable at A_3 only if each $\lambda \in \sigma(A_2)$ also belongs to $\sigma(A_c)$.

Proof: Recall that A_c only has point spectrum (by assumption), therefore, for a proof by contradiction, suppose that such $\lambda \notin \sigma(A_c)$ exists. This implies that $\lambda \in \rho(A_c)$ and $(A_c-\lambda)^{-1}$ exists and is a bounded operator (in fact, it is compact). Now, structural stability of S_c with respect to A_3 implies (see (3.41) - (3.42) in the proof of proposition 3.18)

$$\text{Ker } V \subset \text{Ker } D_L \text{ and } \text{Ker } B_c C_1 \subset \text{Ker } D_1$$

by Lemma 3.22 we can write

$$X_1 = \text{Ker } B_c C_1 \oplus P, \quad P \subset D(A_1) \text{ and } \dim[P] < \infty$$

thus

$$\text{Ker } D_1 = \text{Ker } B_c C_1 \oplus Q, \; Q = P \cap \text{Ker } D_1$$

Now Ker V may be expressed as

$$\text{Ker } V = \text{Ker}[B_c C_1 \quad A_c - \lambda] = (\text{Ker } B_c C_1 \times \{0\}) \oplus \mathcal{V}$$

where

$$\mathcal{V} = \{(x_1, x_c) \mid x_1 \varepsilon Q \subset D(A_1),\ x_c = -(A_c - \lambda)^{-1} B_c C_1 x_1\}$$

and $\dim[\mathcal{V}] = \dim[Q]$, $\mathcal{V} \subset D(T) = D(A_1) \times \mathcal{X}_c$. Let

$$\mathcal{W} = T \text{ Ker } V = [A_1 + B_1 G_c C_1 - \lambda \quad B_1 F_c] \text{Ker } V$$

since $\text{Ker } T \cap \text{Ker } V = 0$ we obtain

$$\mathcal{W} = (A_1 + B_1 G_c C_1 - \lambda) \text{Ker } B_c C_1 \oplus T \mathcal{V} \tag{3.52}$$

For structural stability we must have $\mathcal{W} = \mathcal{X}_1$. We will now show that this is not the case for $\lambda \in \rho(A_c)$. From the proof of Lemma 2.10 we have

$$\text{Ker}(A_1 + B_1 G_c C_1 - \lambda) \cap \text{Ker } B_c C_1 = 0$$

and by Lemmas 3.21 and 3.23 we obtain

$$\text{codim}[(A_1 + B_1 G_c C_1 - \lambda) \text{Ker } B_c C_1] = \text{codim Ker}[B_c C_1] \tag{3.53}$$

Also, since T is 1 -1 on all of $\mathcal{V}$, we have

$$\dim[T\mathcal{V}] = \dim[\mathcal{V}] = \dim[Q] \tag{3.54}$$

On the other hand

$$\text{codim}[\mathcal{W}] = \dim \left(\frac{\mathcal{X}_1 / (A_1 + B_1 G_c C_1 - \lambda) \text{Ker } B_c C_1}{\mathcal{W} / (A_1 + B_1 G_c C_1 - \lambda) \text{Ker } B_c C_1} \right)$$

$$= \text{codim}[(A_1 + B_1 G_c C_1 - \lambda) \text{Ker } B_c C_1] - \dim[T\mathcal{V}]$$

hence, from (3.53) and (3.54)

$$\text{codim}[\mathcal{W}] + \dim[Q] = \text{codim}[\text{Ker } B_c C_1] \tag{3.55}$$

but

$$q = \text{codim}[\text{Ker } D_1] = \text{codim Ker}[B_c C_1] - \dim[Q]$$

thus (3.55) gives

$$\text{codim}[\mathcal{W}] = q$$

this implies that $\mathcal{W}$ is a proper subspace of X_1 and therefore $\lambda \varepsilon \sigma(A_2)$ must belong to $\sigma(A_c)$ after all.

As a consequence of proposition 3.25 we may write, for $\lambda \varepsilon \sigma(A_2)$

$$\text{Ker } V = \{\text{Ker } B_c\, C_1 \times \text{Ker}(A_c-\lambda)\} \oplus S \qquad (3.56)$$

where

$$S = \{(x_1,x_c)\,|\,x_1 \varepsilon Q \subset D(A_1),\ x_c \varepsilon\ X_{c0} \cap D(A_c),\ B_c C_1 x_1 + (A_c-\lambda)x_c = 0\} \qquad (3.57)$$

and $\dim[S] \leq \dim[Q]$, $S \subset D(T)$, X_{c0} is a complement of $\text{Ker}(A_c-\lambda)$ in X_c , i.e. $X_c = X_{c0} \oplus \text{Ker}(A_c-\lambda)$.

The following proposition shows that we can always choose $\text{Ker}(A_c-\lambda)$ of a suitable high dimension to assure $X_1 = T\ \text{Ker } V$ and $\text{Ker } V \subset \text{Ker } D_L$, for all $\lambda \varepsilon \sigma(A_2)$.

<u>Proposition 3.26</u>. A synthesis S_c is structurally stable at A_3 only if for each $\lambda \varepsilon \sigma(A_2)$

$$q = \dim[Z] \leq \dim[\text{Ker}(A_c-\lambda)] \qquad (<\infty)$$

<u>Proof</u>: Let $L = T\ \text{Ker } V$, where Ker V is given by (3.56) - (3.57). Now, since $\text{Ker } V \cap \text{Ker } T = 0$ we have

$$\{\text{Ker } B_c\, C_1 \times \text{Ker}(A_c-\lambda)\} \cap \text{Ker } T = 0$$

thus

$$(A_1+B_1G_cC_1-\lambda)\ \text{Ker } B_c\, C_1 \cap B_1\, F_c\, \text{Ker}(A_c-\lambda) = 0$$

hence

$$L = (A_1+B_1G_cC_1-\lambda)\text{Ker } B_c\, C_1 \oplus B_1\, F_c\, \text{Ker}(A_c-\lambda) \oplus T\, S$$

As in the proof of proposition 3.25 we obtain

$$\text{codim}[L] = \text{codim}[\text{Ker } B_c\, C_1] - \dim[B_1\, F_c\, \text{Ker}(A_c-\lambda)] - \dim[S]$$

Structural stability at A_3 now requires that $L = X_1$, thus

$$\dim[B_1 F_c \text{Ker}(A_c-\lambda)] = \text{codim}[\text{Ker } B_c C_1] - \dim[S]$$
$$= q + \dim[Q] - \dim[S]$$

but from the proof of Lemma 2.10 we have

$$\text{Ker } B_1 F_c \cap \text{Ker}(A_c-\lambda) = 0$$

and since $\dim[Q] \geq \dim[S]$ we conclude that

$$\dim[\text{Ker}(A_c-\lambda)] \geq \dim[Z] = q \ , \quad \forall \lambda \in \sigma(A_2)$$

<u>Remark</u>: We point out that if B_c and A_c are 'adequately' chosen, equality may be achieved in proposition 3.26. For example if

$$B_c C_1 Q \subset \text{Im}(A_c-\lambda) \ , \qquad \forall \lambda \in \sigma(A_2)$$

then $\dim[S] = \dim[Q]$ (see (3.57)), and in this case we obtain

$$\dim[\text{Ker } (A_c-\lambda)] = q \quad , \qquad \forall \lambda \in \sigma(A_2)$$

The proofs of proposition 3.25 and 3.26 are based in a decomposition of the subspace Ker V. We next give an alternative proof of these results by exploiting the stability of A_L.

<u>Proposition 3.26a</u>: A Synthesis S_c is structurally stable at A_3 only if each $\lambda \in \sigma(A_2)$ belongs to $\sigma(A_c)$ and

$$q = \dim[Z] \leq \dim[\text{Ker}(A_c-\lambda)] \qquad (<\infty)$$

<u>Proof</u>: Recall that for $\lambda \in \sigma(A_2) \subset C^+$, $\lambda \in \rho(A_L)$ therefore from Lemmas 3.21 and 3.23 we have

$$\text{codim}[(A_L-\lambda)\text{Ker } D_L] = \text{codim}[\text{Ker } D_L] = q \qquad (3.52a)$$

and from (3.39) - (3.40)

$$(A_L-\lambda) = \begin{pmatrix} T \\ V \end{pmatrix}$$

with

$$\operatorname{Im}\begin{pmatrix}T\\V\end{pmatrix} = \mathcal{X}_L \quad , \quad \operatorname{Ker} T \cap \operatorname{Ker} V = 0 \tag{3.52b}$$

We next show that Ker T + Ker V is closed and equals $\mathcal{X}_L$, i.e.

$$\operatorname{Ker} T \oplus \operatorname{Ker} V = \mathcal{X}_L \tag{3.52c}$$

CLearly we may write

$$(A_L-\lambda) = (\tilde{A}-\lambda) + \tilde{B}$$

where $\tilde{A}$ and $\tilde{B}$ are defined in Chapter 2, Section 2.2. It now follows from Theorem B.IV.9 in Appendix B

$$(A_L-\lambda)^* = (A-\lambda)^* + B^*$$

$$= \begin{pmatrix}(A_1-\lambda)^* & 0\\ 0 & (A_c-\lambda)^*\end{pmatrix} + \begin{pmatrix}C_1^* G_c^* B_1^* & C_1^* B_c^*\\ F_c^* B_1^* & 0\end{pmatrix}$$

$$= [T^* \quad V^*] \tag{3.52d}$$

and since $\lambda \varepsilon \rho(A_L)$ we have†

$$\operatorname{Im}(A_L-\lambda)^* = \operatorname{Im}T^* + \operatorname{Im} V^* = \mathcal{X}_L{}^{\dagger\dagger} \tag{3.52e}$$

$$\operatorname{Ker}(A_L-\lambda)^* = 0 \tag{3.52f}$$

but, from Theorem B.IV.7, (3.52e) yields

$$\operatorname{Im} T^* + \operatorname{Im} V^* = (\operatorname{Ker} T)^{\perp} + (\operatorname{Ker} V)^{\perp} = \mathcal{X}_L$$

hence, by Theorem B.I.4, Ker T + Ker V is closed. To obtain (3.52c) we note that (3.52f) implies

$$\operatorname{Im} T^* \cap \operatorname{Im} V^* = 0$$

Again, by Theorems B. IV.7 and B.I.4 we obtain

$$\operatorname{codim}[\operatorname{Ker} T + \operatorname{Ker} V] = \dim[(\operatorname{Ker} T)^{\perp} \cap (\operatorname{Ker} V)^{\perp}]$$

$$= \dim[\operatorname{Im} T^* \cap \operatorname{Im} V^*] = 0$$

---

† see [G1, p.66] or [T2, pp. 237-245]

†† Since $\mathcal{X}_L$ is a Hilbert space we may write $\mathcal{X}_L^* = \mathcal{X}_L$ see [T2, th. 5.1, p.142]

which in turn, together with (3.52b), give (3.52c).

Now, for structural stability with respect to A_3 we must have (see (3.38) and (3.41) in the proof of proposition 3.18).

$$\begin{pmatrix} \mathcal{X}_1 \\ 0 \end{pmatrix} \subset (A_L-\lambda)\mathrm{Ker}\ D_L \tag{3.52g}$$

$$\mathrm{Ker}\ V \subset \mathrm{Ker}\ D_L \tag{3.52h}$$

therefore

$$\mathrm{Ker}\ D_L = \mathrm{Ker}\ V \oplus (\mathrm{Ker}\ T \cap \mathrm{Ker}\ D_L)$$

and

$$T\ \mathrm{Ker}\ D_L = T\ \mathrm{Ker}\ V = \mathcal{X}_1$$

$$V\ \mathrm{Ker}\ D_L = V(\mathrm{Ker}\ T \cap \mathrm{Ker}\ D_L) = \mathcal{W}$$

from (3.52a) we may now conclude that

$$\mathrm{codim}[\mathcal{W}] = q$$

that is

$$\mathrm{codim}[B_c\ C_1\ \mathrm{Ker}\ D_1 + \mathrm{Im}(A_c-\lambda)] = q \tag{3.52i}$$

here we have used the identity $\mathrm{Ker}\ D_L = \mathrm{Ker}\ D_1 \times \mathcal{X}_c$. But

$$\mathrm{codim}[B_c C_1 \mathrm{Ker} D_1 + \mathrm{Im}(A_c-\lambda)] = \dim\left(\frac{\mathcal{X}_c/\mathrm{Im}(A_c-\lambda)}{\dfrac{B_c\ C_1\ \mathrm{Ker}\ D_1 + \mathrm{Im}(A_c-\lambda)}{\mathrm{Im}(A_c-\lambda)}} \right)$$

$$= \mathrm{codim}[\mathrm{Im}(A_c-\lambda)] - \dim\left(\frac{B_c\ C_1\ \mathrm{Ker}\ D_1}{B_c\ C_1\ \mathrm{Ker}\ D_1 \cap \mathrm{Im}(A_c-\lambda)} \right)$$

and since $\mathrm{codim}[\mathrm{Im}(A_c-\lambda)] = \dim[\mathrm{Ker}(A_c-\lambda)]$ (see Lemma 3.21)we obtain

$$\dim[\mathrm{Ker}(A_c-\lambda)] + \dim[B_c\ C_1\ \mathrm{Ker}\ D_1 \cap \mathrm{Im}(A_c-\lambda)] - \dim[B_c\ C_1\ \mathrm{Ker}\ D_1] = q \tag{3.52j}$$

Now, suppose that $\lambda \notin \sigma(A_c)$, then $\lambda \in \rho(A_c)$ i.e.

$$\mathrm{Ker}(A_c-\lambda) = 0 \quad , \quad \mathrm{Im}(A_c-\lambda) = X_c$$

and the left hand side of (3.52j) becomes zero, therefore $\lambda \in \sigma(A_2)$ must belong to $\sigma(A_c)$ and since

$$\dim[B_c\ C_1\ \mathrm{Ker}\ D_1] \geq \dim[B_c\ C_1\ \mathrm{Ker}\ D_1 \cap \mathrm{Im}(A_c-\lambda)]$$

we conclude that

$$\dim[\mathrm{Ker}(A_c-\lambda)] \geq q \qquad , \lambda \in \sigma(A_2)$$

Remarks: The interpretation of Propositions 3.25 - 3.26a is that for each $\lambda \in \sigma(A_2)$ the subspace Ker V must be "large" enough to yield T Ker V = X_1 and at the same time the condition Ker V $\subset$ Ker D_L must be satisfied. Note that if $\lambda \in \rho(A_c)$ then V Ker $D_L = X_c$ and by structural stability with respect to A_3 T Ker $D_L = X_1$, i.e. $(A_L-\lambda)$ Ker $D_L = X_L$, and this can only happen if Ker D_L is the whole space X_L, i.e. $D_L \equiv 0$.

As a result of Proposition 3.25 and 3.26, and since A_c has compact resolvent (by assumption) we my write

$$X_c = X_{c1} \oplus X_{c2}$$

where

$$X_{c1} = \bigcap_{i=1}^{\ell} \mathrm{Im}(A_c-\lambda_i)^{p_i}$$

$$X_{c2} = \bigoplus_{i=1}^{\ell} \mathrm{Ker}(A_c-\lambda_i)^{p_i}$$

$\{\lambda_1,\lambda_2\ldots\lambda_\ell\}$ are all distinct eigenvalues of A_2

$\ell q \leq \dim[X_{c2}]$ = total algebraic multiplicity of $\{\lambda_i\}$'s as eigenvalues of A_c

According to this decomposition A_c, B_c and F_c are represented by

$$A_c = \begin{pmatrix} A_{c1} & 0 \\ 0 & A_{c2} \end{pmatrix}, \quad B_c = \begin{pmatrix} B_{c1} \\ B_{c2} \end{pmatrix}, \quad F_c = [F_{c1} \;\; F_{c2}] \quad .. \quad (3.58)$$

where A_{c2} is a bounded operator and $\sigma(A_{c2})$ coincides with $\sigma(A_2)$ except in multiplicities. Also $\sigma(A_2) \subset \rho(A_{c1})$.

We can now write the operator equations (3.31) - (3.32) in more detail

$$\begin{pmatrix} A_1+B_1G_cC_1 & B_1F_{c1} & B_1F_{c2} \\ B_{c1}C_1 & A_{c1} & 0 \\ B_{c2}C_1 & 0 & A_{c2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_{c1} \\ x_{c2} \end{pmatrix} - \begin{pmatrix} x_1 \\ x_{c1} \\ x_{c2} \end{pmatrix} A_2 = \begin{pmatrix} A_3+B_1G_cC_2 \\ B_{c1}C_2 \\ B_{c2}C_2 \end{pmatrix} \quad .. \quad (3.59)$$

$$D_L \begin{pmatrix} x_1 \\ x_{c1} \\ x_{c2} \end{pmatrix} = D_2 \quad (3.60)$$

and when the operator A_3 is perturbed by δA_3 we obtain, as in the proof of Proposition 3.18, the following set of equations, for $\lambda \varepsilon \sigma(A_2)$

$$\begin{pmatrix} A_1+B_1G_cC_1-\lambda & B_1F_{c1} & B_1F_{c2} \\ B_{c1}C_1 & A_{c1}-\lambda & 0 \\ B_{c2}C_1 & 0 & A_{c2}-\lambda \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{c1i} \\ x_{c2i} \end{pmatrix} = \begin{pmatrix} a_{3i} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_{1i-1} \\ x_{c1i-1} \\ x_{c2i-1} \end{pmatrix} \quad .. \quad (3.61)$$

$$D_1 x_{1i} = 0 \quad (3.62)$$

where $1 \leq i \leq k$ and $\begin{pmatrix} x_{10} \\ x_{c10} \\ x_{c20} \end{pmatrix} \equiv 0$

So far, we have obtained several results by analyzing some of the equations in (3.61) - (3.62), namely for i = 1. In general, it is difficult, even in the finite dimensional case, to extract the information contained in these expressions when considered simultaneously. However, for our next result on structural stability it is necessary to examine a few more of them.

Writing out the last equation in (3.61) we obtain, for $\lambda \varepsilon \sigma(A_2)$

$$\begin{aligned} B_{c2} C_1 x_{11} + (A_{c2}-\lambda)x_{c21} &= 0 \\ B_{c2} C_1 x_{12} + (A_{c2}-\lambda)x_{c22} &= x_{c21} \\ &\vdots \\ B_{c2} C_1 x_{1k} + (A_{c2}-\lambda)x_{c2k} &= x_{c2k-1} \end{aligned} \tag{3.63}$$

and $$D_1 x_{1i} = 0 \ , \quad i = 1,2,\ldots k \tag{3.64}$$

Then, since A_{c2} is bounded and X_{c2} finite dimensional, is easy to see that[†]

$$x_{c21} \varepsilon R = \langle A_{c2} | B_{c2} C_1 \text{ Ker } D_1 \rangle + \text{Im}(A_{c2}-\lambda)^{k-1} \tag{3.65}$$

where

$$\langle A_{c2} | B_{c2} C_1 \text{ Ker } D_1 \rangle = B_{c2} C_1 \text{ Ker } D_1 + A_{c2} B_{c2} C_1 \text{ Ker } D_1 + \ldots$$
$$\ldots + A_{c2}^{n_{c_2}-1} B_{c2} C_1 \text{ Ker } D_1 \tag{3.66}$$

and $n_{c2} = \dim[X_{c2}]$

now, we define the subspace M as follows

$$M = \{(x_{11}, x_{c1}, x_{c2}); \begin{pmatrix} B_{c1} C_1 & A_{c1}-\lambda & 0 \\ B_{c2} C_1 & 0 & A_{c2}-\lambda \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{c1} \\ x_{c2} \end{pmatrix} = 0, \tag{3.67}$$
$$x_{11} \varepsilon \text{ Ker } D_1, \ x_{c2} \varepsilon R\}$$

[†]Note that $\langle A_{c2}-\lambda | B_{c2} C_1 \text{ Ker } D_1 \rangle = \langle A_{c2} | B_{c2} C_1 \text{ Ker } D_1 \rangle$

If we consider a_{31} to be arbitrary in X_1, then structural stability requires that

$$[A_1+B_1\ G_c\ C_1-\lambda \quad B_1\ F_{c1} \quad B_1\ F_{c2}]\ M = X_1 \tag{3.68}$$

The following result gives a necessary condition for (3.68) to hold.

Theorem 3.27: (Necessity of the Internal Model).

A Synthesis S_c is structurally stable with respect to A_3 only if the controller incorporates an internal model A_2.

Before proving this result we need a technical lemma.

Lemma 3.28:[†] Let U and X be finite dimensional spaces. Consider the linear bounded operators $B : U \to X$ and $A : X \to X$ and let

$$N = \{(u,x);\quad Bu + Ax = 0,\ x \in \langle A|\mathrm{Im}\ B\rangle + \mathrm{Im}\ A^{k-1}\}$$

then

$$\dim[N] \leq \dim[U] + \dim[\mathrm{Ker}\ \bar{A} \cap \mathrm{Im}\ \bar{A}^{k-1}]$$

where $\bar{A}$ is the operator induced by A on the space $\dfrac{X}{\langle A|\mathrm{Im}\ B\rangle}$

Proof of Theorem 3.27

We first decompose M given by (3.67) according to

$$\mathrm{Ker}\ D_1 = \mathrm{Ker}\ B_c\ C_1 \oplus Q$$

where $\dim[Q] < \infty$, $Q \subset D(A_1)$ (see proof of proposition 3.25)

Define, for $\lambda \in \sigma(A_2)$

$$M_1 = \{(x_{11},\ 0,\ 0);\quad x_{11} \in \mathrm{Ker}\ B_c\ C_1\} \tag{3.69}$$

$$M_2 = \{(x_{11},\ x_{c1},\ x_{c2});\quad x_{11} \in Q \subset D(A),\ x_{c2} \in R,\ x_{c1} = (A_{c1}-\lambda)^{-1} B_{c1}\ C_1\ x_{11},\ B_{c2}\ C_1\ x_{11} + (A_{c2}-\lambda)\,x_{c2} = 0\} \quad .. \tag{3.70}$$

† A proof of this result is given in [F1]

then $M = M_1 \oplus M_2$

Let $T = [A_1 + B_1 G_c C_1 - \lambda \quad B_1 F_{c1} \quad B_1 F_{c2}]$

so, by structural stability and since $\text{Ker } T \cap M = 0$ we obtain

$$X_1 = T M_1 \oplus T M_2$$

or equivalently

$$\text{codim}[T M_1 \oplus T M_2] = 0$$

which implies

$$\text{codim}[T M_1] = \dim[T M_2] \tag{3.71}$$

thus, since $\text{Ker } B_c C_1 \cap \text{Ker } (A_1 + B_1 G_c C_1 - \lambda) = 0$ and T is $1-1$ on all of M_2, (3.69) and (3.71) give

$$\text{codim}[\text{Ker } B_c C_1] = \dim[M_2] \tag{3.72}$$

Next, we show that Lemma 3.28 may be applied to M_2.

Define

$$\tilde{Q} = \{(x_{11}, (A_{c1}-\lambda)^{-1} B_{c1} C_1 x_{11}) \; ; \quad x_{11} \in Q\} \tag{3.73}$$

clearly $\dim[\tilde{Q}] = \dim[Q]$

Observe that, (Since $\text{Ker } B_c C_1 \subset \text{Ker } B_{c2} C_1$)

$$\langle A_{c2} | B_{c2} C_1 \text{ Ker } D_1 \rangle = \langle A_{c2} | B_{c2} C_1 Q \rangle \tag{3.74}$$

and let

$$\tilde{B} = [B_{c2} C_1 \quad 0] : \; \tilde{X} = X_1 \times X_{c1} \to X_{c2} \tag{3.75}$$

then from (3.74) - (3.75) we have

$$\langle A_{c2} | B_{c2} C_1 \text{ Ker } D_1 \rangle = \langle A_{c2} | \tilde{B} \tilde{Q} \rangle \tag{3.76}$$

therefore M_2 may be written as follows

$$M_2 = \{(\tilde{x}, x_{c2}) ; \tilde{x} \in \tilde{Q}, \; \tilde{B} \tilde{x} + (A_{c2}-\lambda) x_{c2} = 0$$
$$x_{c2} \in \langle A_{c2} | \tilde{B} \tilde{Q} \rangle + \text{Im}(A_{c2}-\lambda)^{k-1}\} \tag{3.77}$$

hence, by Lemma 3.28 we obtain

$$\dim[M_2] \leq \dim[Q] + \dim[\mathrm{Ker}(\bar{A}_{c2}-\lambda) \cap \mathrm{Im}(\bar{A}_{c2}-\lambda)^{k-1}] \qquad (3.78)$$

where $\bar{A}_{c2}$ is the operator induced by A_{c2} on $\dfrac{X_{c2}}{\langle A_{c2}|B_{c2}\ C_1\ \mathrm{Ker}\ D_1\rangle}$

combining (3.78) and (3.72), and since

$$\mathrm{codim}[\mathrm{Ker}\ B_c\ C_1] = \mathrm{codim}[\mathrm{Ker}\ D_1] + \dim[Q]$$

we obtain

$$\dim[\mathrm{Ker}(\bar{A}_{c2}-\lambda) \cap \mathrm{Im}(\bar{A}_{c2}-\lambda)^{k-1}] \geq q \qquad (3.79)$$

and since (3.79) must hold for every prime subspace $X_{2\lambda}^{j}$ corresponding to λ we conclude by Lemma 3.9 that $\bar{A}_{c2}$ contains an internal model of A_2. This completes the proof.

Comments: We will show in Chapter 4, by constructing a structurally stable synthesis, that equality may be achieved in (3.79), that is, if there exists a structurally stable synthesis it can be chosen such that the order of the internal model is minimal, i.e. A_c contains a g-fold reduplication of the maximal cyclic component of A_2.

3.8 Necessity of Feedback

To complete our work on the Internal Model Principle, we have to show that a structurally stable controller requires feedback of the regulated variables. First, we derive a preliminary result.

It is easy to see that B_c may be written as follows (see (3.58) and (3.43)).

$$\begin{pmatrix} B_{c1} \\ B_{c2} \end{pmatrix} = \begin{pmatrix} B_{cw1} & B_{cz1} \\ B_{cw2} & B_{cz2} \end{pmatrix} \qquad (3.80)$$

then, from (3.80) and (3.43)

$$\langle A_{c2}|B_{c2}\ C_1\ \mathrm{Ker}\ D_1\rangle = \langle A_{c2}|B_{cw2}\ E_1\ \mathrm{Ker}\ D_1\rangle \qquad (3.81)$$

Let $P_{c2} : X_{c2} \to \overline{X}_{c2} = X_{c2}/\langle A_{c2}|B_{cw2} E_1 \text{ Ker } D_1\rangle$ be the canonical projection, and define

$$\bar{B}_{cw2} = P_{c2} B_{cw2} \tag{3.82}$$

$$\bar{B}_{cz2} = P_{c2} B_{cz2} \tag{3.83}$$

The following proposition shows that for structural stability we must have $\bar{B}_{cw2} \equiv 0$, this in turn is equivalent to

$$\text{Im } B_{cw2} \subset \langle A_{c2}|B_{cw2} E_1 \text{ Ker} D_1\rangle^{\dagger} \tag{3.84}$$

Proposition 3.29: There is no synthesis in which (3.84) fails and which is structurally stable at $(A_3, \bar{B}_{cw2})$.

We mention that the proof of proposition 3.29 is exactly the same as in the finite dimensional case. This fact is not unexpected, after all we have isolated the finite dimensional part of the controlller containing the internal model.

Proof of Proposition 3.29. This proof can be found [F1], however we give it here for completeness.

Suppose, in contradiction, that there is such synthesis. From (3.59) - (3.60) there are $X_1 : X_2 \to X_1$ and $X_{c2} : X_2 \to X_{c2}$ such that

$$B_{c2} C_1 X_1 + A_{c2} X_{c2} - X_{c2} A_2 = B_{c2} C_2 \tag{3.85}$$

$$D_1 X_1 = D_2 \tag{3.86}$$

Now, let X_{11} be an arbitrary completement of Ker D_1 in X_1, and since D_1 is surjective it has a right inverse $\check{D}_1$ with Im $\check{D}_1 = X_{11}$. So any X_1 satisfying (3.86) may be written as

$$X_1 = \hat{X}_1 + \check{D}_1 D_2 \text{ with Im } \hat{X}_1 \subset \text{Ker } D_1 \tag{3.87}$$

---

†(3.84) is also equivalent to $\langle A_{c2}|\text{Im } B_{cw2}\rangle = \langle A_{c2}|B_{cw2} E_1 \text{ Ker } D_1\rangle$

Then, since

$$C_1 = \begin{pmatrix} E_1 \\ D_1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} E_2 \\ D_2 \end{pmatrix} \text{ and } B_{c2} = [B_{cw2} \quad B_{cz2}]$$

(3.85) becomes

$$B_{cw2}\, E_1\, \hat{X}_1 + A_{c2}\, X_{c2} - X_{c2}\, A_2 = B_{cw2}(E_2 - E_1\check{D}_1 D_2)$$

Applying P_{c2} to both sides of the above expression, and since $P_{c2}\, B_{cw2}$ E_1 Ker $D_1 = 0$, we obtain

$$\bar{A}_{c2}\, \bar{X}_{c2} - \bar{X}_{c2}\, A_2 = \bar{B}_{cw2}(E_2 - E_1\check{D}_1 D_2) \qquad (3.88)$$

where $\quad \bar{X}_{c2} = P_{c2}\, X_{c2}$

Next we show that if (3.84) does not hold then

$$(E_2 - E_1\, \check{D}_1\, D_2) \neq 0 \qquad (3.89)$$

Since $[C_1 \; C_2]$ is surjective, we have that for each $w \in \mathcal{W}$ there are $\hat{x}_1 \in$ Ker D_1 and $x_2 \in X_2$ such thar

$$w = E_1\, \hat{x}_1 + (E_2 - E_1\check{D}_1 D_2)x_2$$

thus

$$\mathcal{W} = E_1 \text{ Ker } D_1 + \text{Im}(E_2 - E_1\check{D}_1 D_2)$$

hence

$$\text{Im } B_{cw2} = B_{cw2}\, E_1 \text{ Ker } D_1 + B_{cw2} \text{ Im}(E_2 - E_1\check{D}_1 D_2)$$

and since (3.84) is assumed to fail we conclude that (3.89) must hold.

Now consider an arbitrary (small) bounded perturbations $\overline{\delta B}_{cw2}$ of $\bar{B}_{cw2}$. Then by structural stability, there is $\overline{\delta X}_{c2} : X_2 \rightarrow X_{c2}$ such that

$$\bar{A}_{c2}\, \overline{\delta}\, X_{c2} - \overline{\delta X}_{c2} A_2 = \overline{\delta B}_{cw2}(E_2 - E_1\, \check{D}_1\, D_2) \qquad (3.90)$$

Choose $\lambda \in \sigma(A_2)$ and a prime subspace $X^i_{2\lambda}$ corresponding to λ

such that $(E_2 - E_1 \check{D}_1 D_2)|X^i_{2\lambda} \neq 0$. Fix a basis for $X^j_{2\lambda}$ such that A_2 is represented by its Jordan form.

Let

$$\overline{\delta X}_{c2}|X^j_{2\lambda} = [\bar{r}_1, \bar{r}_2 \dots \bar{r}_k]$$

$$(E_2 - E_1 \check{D}_1 D_2)|X^j_{2\lambda} = [e_1, e_2 \dots e_k]$$

where $e_i \neq o$, at least for one $1 \leq i \leq k = \dim[X^j_{2\lambda}] \leq k_\lambda$

Then, restricting (3.90) to $X^j_{2\lambda}$ and assuming that the first $e_i \neq 0$ is the ℓ^{th}, we obtain

$$\begin{aligned} (\bar{A}_{c2} - \lambda)\bar{r}_1 &= 0 \\ (\bar{A}_{c2} - \lambda)\bar{r}_2 &= \bar{r}_1 \\ &\vdots \\ (\bar{A}_{c2} - \lambda)\bar{r}_{\ell-1} &= \bar{r}_{\ell-2} \\ (\bar{A}_{c2} - \lambda)\bar{r}_\ell &= \bar{r}_{\ell-1} + \overline{\delta B}_{cw2}\, e_\ell \end{aligned} \tag{3.91}$$

but $\overline{\delta B}_{cw2}\, e_\ell$ is arbitrary in $\bar{X}_{c2}$, so (3.91) implies that for any $\bar{x}_{c2} \in X_{c2}$ there is an $\bar{r}_\ell \in X_{c2}$ such that

$$(\bar{A}_{c2} - \lambda)^\ell \bar{r}_\ell = (\bar{A}_{c2} - \lambda)^{\ell-1} \bar{x}_{c2}$$

hence

$$\mathrm{Im}(\bar{A}_{c2} - \lambda)^{\ell-1} \subset \mathrm{Im}(\bar{A}_{c2} - \lambda)^\ell \tag{3.92}$$

On the other hand

$$\bar{X}_{c2} = \mathrm{Im}(\bar{A}_{c2} - \lambda)^0 \supset \mathrm{Im}(\bar{A}_{c2} - \lambda) \supset \dots \supset \mathrm{Im}(\bar{A}_{c2} - \lambda)^{\ell-1} \supset \mathrm{Im}(\bar{A}_{c2} - \lambda)^\ell \;.. \tag{3.93}$$

So (3.92) and (3.93) give

$$\mathrm{Im}(\bar{A}_{c2} - \lambda)^{\ell-1} = \mathrm{Im}(\bar{A}_{c2} - \lambda)^\ell$$

that is the descent of $(\bar{A}_{c2} - \lambda)$ is less than or equal to $\ell-1$, and since $(\bar{A}_{c2} - \lambda)$ is bounded we may conclude that the ascent of $(\bar{A}_{c2} - \lambda)$ is also less than or equal to $\ell-1$ [T2, th. 6.2 p.290].

Therefore

$$\mathrm{Ker}(\bar{A}_{c2}-\lambda)^{\ell-1} \cap \mathrm{Im}(\bar{A}_{c2}-\lambda)^{\ell-1} = 0 \qquad (3.94)$$

but

$$\mathrm{Ker}(\bar{A}_{c2}-\lambda) \subset \mathrm{Ker}(\bar{A}_{c2}-\lambda)^{\ell-1}$$

and, since $k_\lambda \geq k \geq \ell$

$$\mathrm{Im}(\bar{A}_{c2}-\lambda)^{k_\lambda-1} = \mathrm{Im}(\bar{A}_{c2}-\lambda)^{\ell-1}$$

hence (3.94) implies

$$\mathrm{Ker}(\bar{A}_{c2}-\lambda) \cap \mathrm{Im}(\bar{A}_{c2}-\lambda)^{k_\lambda-1} = 0$$

so $\bar{A}_{c2}$ does not contain an internal model of A_2 which contradicts Theorem 3.27. This completes the proof.

We can now prove the following result

Theorem 3.30: (Necessity of Feedback). Let S_c be a synthesis in which (3.84) holds. Then S_c is structurally stable with respect to A_3 only if the controller incorporates an internal model of A_2 which is controllable by z and observable by u.

Proof: from Theorem 3.27 we know that $\bar{A}_{c2}$ contains an internal model of A_2. We now show that the internal model is controllable by z and observable by u. Of course controllability and observability are defined as in the finite dimensional case.

Since A_L is stable this implies that for $\lambda \in \sigma(A_2) \subset C^+$, $(A_L-\lambda)$ has a bounded inverse. Thus

$$\mathrm{Ker}(A_{c2}-\lambda) \cap \mathrm{Ker}\, B_1 F_{c2} = 0 \qquad (3.95)$$

$$\mathrm{Im}\, B_{c2}C_1 + \mathrm{Im}(A_{c2}-\lambda) = X_{c2} \qquad (3.96)$$

From (3.95) is easy to see that

$$\mathrm{Ker}(A_{c2}-\lambda) \cap \mathrm{Ker}\, F_{c2} = 0 \quad , \quad \lambda \in \sigma(A_2)$$

hence the internal model is observable by u.

From (3.96) we obtain

$$\mathrm{Im}\, B_{c2} + \mathrm{Im}(A_{c2}-\lambda) = X_{c2} \quad , \quad \lambda \in \sigma(A_2)$$

using (3.80), the above expression yields

$$\mathrm{Im}\, B_{cw2} + \mathrm{Im}\, B_{cz2} + \mathrm{Im}(A_{c2}-\lambda) = X_{c2} \tag{3.97}$$

Applying P_{c2} to both sides of (3.97)

$$\mathrm{Im}\, \bar{B}_{cz2} + \mathrm{Im}(\bar{A}_{c2}-\lambda) = \bar{X}_{c2} \ , \quad \lambda \in \sigma(A_2)$$

since $\bar{B}_{cw2} \equiv 0$ by(3.84). Hence the internal model is controllable by z. This completes the proof.

3.9 Conclusions and Remarks

The main result of this chapter may now be summarized as follows

Theorem 3.31. The Internal Model Principle

A synthesis $S_c = (X_c, A_c, B_c, F_c, G_c)$ which is structurally stable with respect to the parameters (A_3, B_{cz}). necessarily utilizes feedback of the regulated variables and contains, at least, a q- fold reduplication of the dynamic structure of the disturbance and reference signals which the controller is required to process.

Thus, we have obtained a complete version of the Internal Model Principle for time delay systems. Also, necessary conditions for the existence of structurally stable controller have been derived. These conditions are given in terms of the system's parameters and are easy to verify.

Finally we mention that in the special case y = z, it is easy to establish the equivalence between our results and those obtained in [B1,Chapter 6].

CHAPTER 4

ON THE SUFFICIENCY OF THE INTERNAL MODEL PRINCIPLE

The major concern of this chapter is to establish the sufficiency of the Internal Model Principle, as well as obtain necessary and sufficient conditions to assure the existence of a structurally stable controller. Our developments will yield a procedure to construct such a controller. This procedure is based on the observer theory for evolution systems developed by Bhat [B1, Chapter 4].

For reference we write the system's equations

$$\dot{x}_1(t) = A_1\, x_1(t) + A_3\, x_2(t) + B_1\, u(t) \tag{4.1a}$$

$$\dot{x}_2(t) = A_2\, x_2(t) \tag{4.1b}$$

$$y(t) = C_1\, x_1(t) + C_2\, x_2(t) \tag{4.1c}$$

$$z(t) = D_1\, x_1(t) + D_2\, x_2(t) \tag{4.1d}$$

The controller to be synthesized is given by

$$\dot{x}_c(t) = A_c\, x_c(t) + B_c\, y(t) \tag{4.2a}$$

$$u(t) = F_c\, x_c(t) + G_c y(t) \tag{4.2b}$$

Throughout this chapter we make the following assumptions

1) the pair (A_1, B_1) is stabilizable
2) the pair (C_1, A_1) is detectable
3) A_2 is a bounded operator with $\sigma(A_2) \subset C^+$
4) the spaces X_2, U, Y and Z are finite dimensional
5) $Y = \text{Im}\, C_1 + \text{Im}\, C_2$ and $\text{Im}\, D_2 \subset \text{Im}\, D_1 = Z$
6) z is readable from y, i.e. $\text{Ker}[C_1\ C_2] \subset \text{Ker}[D_1\ D_2]$
7) $\dim[\text{Im}\, B_1] \geq \dim[Z] = q$

4.1 Preliminaries

Before solving our main problem we need several preliminary results.

The following propositions will be needed in the actual construction of a synthesis.

Proposition 4.1: Let X_3 be a finite dimensional space. Then, for any bounded operators $M : X_3 \to X_3$ and $T : X_3 \to X_1$ with $\sigma(M) \subset \sigma(A_2)$, there exist bounded operators $X_1 : X_3 \to X_1$ and $U : X_3 \to U$ such that

$$A_1 X_1 - X_1 M + B_1 U = T \qquad (4.3a)$$

$$D_1 X_1 = 0 \qquad (4.3b)$$

if and only if

$$(A_1 - \lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im } B_1 = X_1, \quad \forall \lambda \in \sigma(A_2) \qquad .. \quad (4.4)$$

Furthermore (4.4) is a stable property under small bounded perturbations of B_1 and small perturbations of A_1 of class $F(A_1)$.

Proof: We first prove the proposition when $X_3 = X_2$, $M = A_2$ and $T = X_2 \to X_1$. As in the proof of proposition 3.18, consider a decomposition of X_2 into prime subspaces. Then it is easy to see that (4.3a) - (4.3b) have a solution if and only if they have a solution when restricted to any prime subspace corresponding to each $\lambda \in \sigma(A_2)$. Now, fix $\lambda \in \sigma(A_2)$ and choose any prime subspace $X_{2\lambda}^j$ associated with λ. Select a basis for $X_{2\lambda}^j$ such that $A_2|X_{2\lambda}^j$ is in Jordan form and let

$$X_1|X_{2\lambda}^j = [x_1, x_2 \ldots x_k]$$

$$U|X_{2\lambda}^j = [u_1, u_2 \ldots u_k]$$

$$T|X_{2\lambda}^j = [t_1, t_2 \ldots t_k]$$

where $k = k(\lambda,j) = \dim[X^j_{2\lambda}]$. Then restricting (4.3a) - (4.3b) to $X^j_{2\lambda}$ we obtain

$$(A_1-\lambda)x_i + B_1u_i = t_i + x_{i-1} \tag{4.5a}$$

$$D_1 x_i = 0 \tag{4.5b}$$

for $i = 1, 2 \ldots k$, where $x_0 \triangleq 0$.

Since $t_i \in X_1$ is arbitrary, we conclude that (4.5a) - (4.5b) have a solution if and only if

$$(A_1-\lambda)[\mathrm{Ker}\ D_1 \cap D(A_1)] + \mathrm{Im}\ B_1 = X_1$$

When $X_3 \neq X_2$, $M \neq A_2$, we consider a decomposition of X_3 into prime subspaces, and since $\sigma(M) \subset \sigma(A_2)$, (4.4) follows as above.

We next show that (4.4) is a stable property when A_1 and B_1 are subjected to small perturbations. Consider a bounded perturbation δB_1 of B_1. Then Im B_1 and $\mathrm{Im}(B_1+\delta B_1)$ are both finite dimensional and therefore closed subspaces of X_1. Furthermore, we have that both the Ker B_1 and $\mathrm{Ker}(B_1+\delta B_1)$ are also finite dimensional. So B_1 and $B_1+\delta B_1$ are semi-Fredholm operators, and Theorem B.IV.12 yields

$$\delta(\mathrm{Im}\ B_1, \mathrm{Im}(B_1+\delta B_1)) \leq \frac{\| \delta B_1 \|}{\gamma(B_1)} \tag{4.6a}$$

where $\gamma(B_1) > 0$.

Consider a perturbation δA_1 of A_1 of class $F(A_1)$. It is readily verified that $D(A_1+\delta A_1) = D(A_1)$, since δA_1 is A_1-compact. Also, since $(A_1-\lambda)$ is a Fredholm operator, Theorem B.IV.5 implies that $(A_1+\delta A_1-\lambda)$ is also Fredholm. Therefore $\mathrm{Im}(A_1-\lambda)$ and $\mathrm{Im}(A_1+ A_1-\lambda)$ are closed subspaces of X_1. Moreover, by Theorem B.IV.2 we have that $(A_1-\lambda)$ Ker D_1 and $(A_1+\delta A_1-\lambda)$ Ker D_1 are closed subspaces.

It now follows that the subspaces

$$(A_1+\delta A_1-\lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im } B_1,\ (A_1-\lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im}(B_1+\delta B_1)$$

$$(A_1+\delta A_1-\lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im}(B_1+\delta B_1)$$

are closed.

On the other hand we may decompose X_1 as follows (see Lemma 3.22 in Section 3.7, Chapter 3)

$$X_1 = \text{Ker } D_1 \oplus N$$

where $N \subset D(A_1)$ is finite-dimensional. Now. let $\hat{A}_1$ be the restriction of A_1 to Ker $D_1 \cap D(A_1)$. It is readily verified that $\hat{A}_1$ is closed (since Ker D_1 is a closed subspace and A_1 is closed operator) with

$D(\hat{A}_1) = \text{Ker } D_1 \cap D(A_1)$ and $\text{Im}(\hat{A}_1-\lambda) = (A_1-\lambda)\text{Ker } D_1$. Also we have that $D(\hat{A}_1+\delta A_1) = D(\hat{A}_1)$ and $\text{Im}(\hat{A}_1+\delta A_1-\lambda) = (A_1+\delta A_1-\lambda)\text{Ker } D_1$. Therefore, Theorem B.IV.12 gives

$$\delta((A_1-\lambda)\text{Ker } D_1,\ (A_1+\delta A_1-\lambda)\text{Ker } D_1) = \delta(\text{Im}(\hat{A}_1-\lambda),\ \text{Im}(\hat{A}_1+\delta A_1-\lambda))$$

$$\leq \frac{a+b\ \gamma(\hat{A}_1-\lambda)}{\gamma(\hat{A}_1-\lambda)} \qquad (4.6b)$$

where $\gamma(\hat{A}_1-\lambda) > 0$, since $\text{Im}(\hat{A}_1-\lambda)$ is closed, and a, b are non-negative constants such that, for all $x_1 \in D(\hat{A}_1)$

$$\| \delta A_1\ x_1 \| \leq a \| x_1 \| + b \| \hat{A}_1\ x_1 \|$$

Now, since (4.4) holds there is an $\varepsilon > 0$ such that

$$\gamma((A_1-\lambda)\text{Ker } D_1,\ \text{Im } B_1) \geq \varepsilon$$

therefore for sufficiently small perturbations δB_1 and δA_1 we conclude, from Theorem B.I.8, that

$$(A_1+\delta A_1-\lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im}(B_1+\delta B_1) = X_1, \quad \lambda \in \sigma(A_2)$$

this completes the proof.

Proposition 4.2: Let $A_{2e} : X_{2e} \to X_{2e}$ be a q-fold direct sum of the largest cyclic component of A_2, i.e. $X_{2e} = [X_{2k}]^q$, $A_{2e} = \mathrm{diag}[A_{2k}, A_{2k} \dots A_{2k}]$(see Section 3.3). Define

$$\hat{X} = X_1 \oplus X_{2e}$$

$$\hat{A} = \begin{pmatrix} A_1 & A_{3e} \\ 0 & A_{2e} \end{pmatrix} \tag{4.7a}$$

$$\hat{D} = [D_1 \quad 0] \tag{4.7b}$$

If the pair (D_1, A_1) is detectable, then there is a bounded operator $A_{3e} : X_{2e} \to X_1$ such that $(\hat{D},\hat{A})$ is detectable.

Proof: Since (D_1, A_1) is detectable, it is easy to see that detectability of $(\hat{D},\hat{A})$ is equivalent to the conditions

a) $\mathrm{Ker}(A_{2e}-\lambda) \cap \mathrm{Ker}\, A_{3e} = 0 \quad , \qquad \lambda \in \sigma(A_{2e})$

b) $(A_1-\lambda)[\mathrm{Ker}\, D_1 \cap D(A_1)] \cap A_{3e}\, \mathrm{Ker}(A_{2e}-\lambda) = 0 \;, \; \lambda \in \sigma(A_{2e})$

We now will show that there is an A_{3e} satisfying a) and b) above. By Lemmas 3.21 and 3.22 we have

$$\mathrm{codim}[(A_1-\lambda)[\mathrm{Ker}\, D_1]] = \mathrm{codim}[\mathrm{Ker}\, D_1] = q \tag{4.8}$$

Thus, there are q-linearly independent vectors in $X_1/(A_1-\lambda)[\mathrm{Ker}\, D_1]$ with $\bar{x}_i = x_i + (A_1-\lambda)\mathrm{Ker}\, D_1$, $i = 1,2 \dots q$ where $x_i \in X_1$.

On the other hand,

$$\dim[\mathrm{Ker}\,(A_{2e}-\lambda)] = q \tag{4.9}$$

So, there are q independent elements $\{y_1, y_2, \dots y_q\}$ in $\mathrm{Ker}(A_{2e}-\lambda)$. Define $A_{3e}(\lambda)$ by

$$x_i = A_{3e}(\lambda) y_i \quad , \quad i = 1, 2, \dots q$$

and let

$$A_{3e} = \bigoplus_{\lambda \in \sigma(A_{2e})} A_{3e}(\lambda) \tag{4.10}$$

Then it is easy to verify that A_{3e} given by (4.10) satisfies a) and b).

The next result provides a necessary and sufficient condition to assure the existence of a synthesis in a very special case.

Proposition 4.3: Suppose that $[C_1\ C_2] = [D_1\ D_2]$ and that the pair (D,A) is detectable where

$$D = [D_1\ D_2] \qquad A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$$

Then, there is a synthesis for the system (4.1a) - (4.1d) if and only if there are bounded operators $X_1 : \mathcal{X}_2 \to \mathcal{X}_1$ and $U : \mathcal{X}_2 \to \mathcal{U}$ such that

$$A_1 X_1 - X_1 A_2 + B_1 U = A_3 \tag{4.11a}$$

$$D_1 X_1 = D_2 \tag{4.11b}$$

Proof: A Proof of this result is given in [B1, Chapter 5, pp.73-77].† However we give it here since it will be useful in the construction of a synthesis.

Necessity: This part follows from proposition 2.6

Sufficiency : This part consists in obtaining a synthesis by means of an observer for the system (4.1a) - (4.1d). The detectability condition is required in this part of the proof.

Since (A_1,B_1) is stabilizable, there is an $F_1 : \mathcal{X}_1 \to \mathcal{U}$ such that $(A_1+B_1 F_1)$ is stable. Now, by (4.11a) - (4.11b), we may choose an $F_2 : \mathcal{X}_2 \to \mathcal{U}$ such that there is an $\hat{X}_1 : \mathcal{X}_2 \to \mathcal{X}_1$ satisfying

$$(A_1+B_1F_1)\hat{X}_1 - \hat{X}_1 A_2 = A_3 + B_1 F_2 \tag{4.12a}$$

$$D_1 \hat{X}_1 = D_2 \tag{4.12b}$$

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† also see [F5] for the finite dimensional case

this gives a feedback

$$u(t) = F_1 \, x_1(t) + F_2 \, x_2(t)$$

We now show that this feedback may be implemented via an observer which has z(t) as input. Since, by assumption, (D,A) is detectable, there is a bounded operator $K : Z \to X_1 \times X_2$ such that (A+KD) is stable.

Define $X_c = X_{c1} \times X_{c2} = X_1 \times X_2$. Then the equation

$$\dot{x}_c(t) = (A+KD)x_c(t) - K\, z(t) + B_1 u(t) \qquad (4.13)$$

is an observer for the system (4.1a) - (4.1b). Now writing

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

and

$$u(t) = F_1 \, x_{c1}(t) + F_2 \, x_{c2}(t)$$

(4.5) may be written as

$$\begin{pmatrix} \dot{x}_{c1}(t) \\ \dot{x}_{c2}(t) \end{pmatrix} = \begin{pmatrix} A_1 + K_1 \, D_1 + B_1 \, F_1 & A_3 + K_1 \, D_2 + B_1 \, F_2 \\ K_2 \, D_1 & A_2 + K_2 \, D_2 \end{pmatrix} \begin{pmatrix} x_{c1}(t) \\ x_{c2}(t) \end{pmatrix}$$

$$- \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} z(t)$$

Our synthesis is now complete with $G_c \equiv 0$, $B_c = -K$, $F_c = F = [F_1 \;\; F_2]$ and $A_c = (A+KD + BF)$ where $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$.

To complete the proof we need to show that the closed-loop system is internally stable and that z(t) is regulated. To prove the Stability of the closed-loop system it is necessary and sufficient to show that $\sigma(A_L) \subset C^-$. The closed-loop system is given by

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_{c1}(t) \\ \dot{x}_{c2}(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_1 & B_1F_1 & B_1F_2 & A_3 \\ -K_1D_1 & A_1+K_1D_1+B_1F_1 & A_3+K_1D_2+B_1F_2 & -K_1D_2 \\ -K_2D_1 & K_2D_1 & A_2+K_2D_2 & -K_2D_2 \\ 0 & 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_{c1}(t) \\ x_{c2}(t) \\ x_2(t) \end{pmatrix} \quad \ldots \quad (4.14)$$

define $e_1(t) = x_{c1}(t) - x_1(t)$ and $e_2(t) = x_{c2}(t) - x_2(t)$, then it is easy to see that

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_1+B_1F_1 & B_1F_1 & B_1F_2 & A_3+B_1F_2 \\ 0 & A_1+K_1D_1 & A_3+K_1D_2 & 0 \\ 0 & K_2D_1 & A_2+K_2D_2 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ e_1(t) \\ e_2(t) \\ x_2(t) \end{pmatrix}$$

Clearly $\sigma(A_L) = \sigma(A_1+B_1F_1) \cup \sigma(A+KD) \subset C^-$ and we conclude that (4.14) is internally stable.

To prove regulation define

$$\hat{X}_c = \begin{pmatrix} -\hat{X}_1 \\ I \end{pmatrix}$$

where $\hat{X}_1$ satisfies (4.12a) - (4.12b) and I is the identity operator on X_2.

Let $$\hat{X} = \begin{pmatrix} \hat{X}_1 \\ -\hat{X}_c \end{pmatrix}$$

then , with $F_c = F=[F_1 \; F_2]$, we have from (4.12a)

$$A_1 \hat{X}_1 - \hat{X}_1 A_2 = A_3 + B_1 F_c \hat{X}_c$$

and since

$$D_1 \hat{X}_1 = D_2$$

it is easy to see that

$$A_c \hat{X}_c - \hat{X}_c A_2 = KD_2 - KD_1 \hat{X}_1$$
$$= 0$$

then, for the closed-loop system we have

$$A_L \hat{X} - \hat{X}A_2 = B_L$$

$$D_L \hat{X} = D_2$$

where

$$A_L = \begin{pmatrix} A_1 & B_1F_c \\ -KD_1 & A_c \end{pmatrix}, \quad B_L = \begin{pmatrix} A_3 \\ -KD_2 \end{pmatrix}, \quad D_L = [\, D_1 \;\; 0 \,]$$

Now Proposition 2.6 establishes that z is regulated.

The following proposition will be needed in further developments

<u>Proposition 4.4</u>: Suppose that $[C_1 \; C_2] = [D_1 \; D_2]$, i.e. $y = z$, and that $S_c = (X_c, A_c, B_c, F_c, G_c)$ is synthesis which provides internal stability and A_c contains an internal model of A_2. Then, the internal model is precisely a q-fold reduplication in A_c of the largest cyclic component of A_2 and

$$\text{Ker } B_c = 0 \tag{4.15}$$

$$\text{Im } B_c \cap \text{Im}(A_c-\lambda) = 0 \quad , \qquad \lambda \in \sigma(A_2) \tag{4.16}$$

$$\text{Ker}(A_c-\lambda)^{k-1} \subset \text{Im}(A_c-\lambda), \qquad \lambda \in \sigma(A_2), 1 \le k \le k_\lambda \tag{4.17}$$

where k_λ is the degree of the factor $(s-\lambda)$ in the minimal polynomial of A_2.

<u>Proof</u>: The loop operator is given by

$$A_L = \begin{pmatrix} A_1+B_1G_cD_1 & B_1F_c \\ B_cD_1 & A_c \end{pmatrix}$$

Now, stability of A_L implies that, for $\lambda \in \sigma(A_2)$

$$\text{Im } B_c\ D_1 + \text{Im}(A_c-\lambda) = X_c$$

but $\text{Im } B_c\ D_1 = B_c\ \text{Im } D_1 = \text{Im } B_c$ since D_1 is surjective, thus

$$\text{Im } B_c + \text{Im}(A_c-\lambda) = X_c \tag{4.18}$$

On the other hand, since A_c contains an internal model of A_2, there is an injective operator $X_c : X_{2e} \to X_c$ such that

$$A_c\ X_c = X_c\ A_{2e}$$

where $X_{2e} = [X_{2k}]^{\ell}$ and A_{2e} is an ℓ-fold direct sum of A_{2k} for some $\ell \geq q$. From this it follows that $\text{Im } X_c$ is an A_c-invariant subspace of X_c, therefore we may write

$$X_c = \text{Im } X_c \oplus X_c^0$$

and accordingly

$$A_c = \begin{pmatrix} A_{2e} & A_{c2} \\ 0 & A_{c1} \end{pmatrix} \tag{4.19}$$

Now, using the representation (4.19), it is easy to see that

$$\text{Ker}(A_{2e}-\lambda) \times \{0\} \subset \text{Ker}(A_c-\lambda)$$

and since $\dim[\text{Ker}(A_{2e}-\lambda)] = \ell \geq q$ we obtain

$$\dim[\text{Ker}(A_c-\lambda)] \geq \ell \geq q \tag{4.20}$$

Also, since $B_c : Z \to X_c$

$$\dim[\text{Im } B_c] \leq \dim[Z] = q \tag{4.21}$$

Now, from our assumptions on A_c, $\mathrm{Im}(A_c-\lambda)$ is a closed subspace of X_c and

$$\mathrm{Im}(A_c-\lambda) \approx \frac{X_c}{\mathrm{Ker}(A_c-\lambda)} \qquad , \lambda \in \sigma(A_2)$$

thus

$$X_c = \mathrm{Ker}(A_c-\lambda) \oplus X_c^1 \tag{4.22}$$

where $\quad X_c^1 = \dfrac{X_c}{\mathrm{Ker}(A_c-\lambda)} \approx \mathrm{Im}(A_c-\lambda)$

hence, from (4.22), (4.18) and (4.21)

$$\begin{aligned}\dim[\mathrm{Ker}(A_c-\lambda)] &= \dim[\mathrm{Im}\ B_c] - \dim[\mathrm{Im}\ B_c \cap \mathrm{Im}(A_c-\lambda)] \\ &\leq q\end{aligned}$$

this, together with (4.20), give

$$\dim[\mathrm{Ker}(A_c-\lambda)] = q$$

(4.15) and (4.16) now follow

Also, the above expression yields

$$\dim[\mathrm{Ker}(A_{2e}-\lambda)] = q$$

which in turn implies that A_{2e} is a q-fold reduplication of A_{2k} i.e., $\ell = q$.

To prove (4.17), first observe that, for $\lambda \in \sigma(A_2)$

$$\mathrm{Ker}(A_{2e}-\lambda) \subset \mathrm{Ker}(A_{2e}-\lambda)^2 \subset \ldots \subset \mathrm{Ker}(A_{2e}-\lambda)^{k_\lambda - 1^2} = \mathrm{Im}(A_{2e}-\lambda) \tag{4.23}$$

Now, since $\mathrm{Ker}(A_{2e}-\lambda) \times \{0\} \subset \mathrm{Ker}(A_c-\lambda)$ with

$\dim[\mathrm{Ker}(A_{2e}-\lambda) \times \{0\}] = \dim[\mathrm{Ker}(A_c-\lambda)] = q$ we conclude

$$\mathrm{Ker}(A_{2e}-\lambda) \times \{0\} = \mathrm{Ker}(A_c-\lambda) \qquad , \lambda \in \sigma(A_2) \tag{4.24}$$

and using the representation (4.19) we obtain

$$\mathrm{Im}(A_c-\lambda) = \mathrm{Im}\begin{pmatrix} A_{2e}-\lambda \\ 0 \end{pmatrix} + \mathrm{Im}\begin{pmatrix} A_{c2} \\ A_{c1}-\lambda \end{pmatrix} \tag{4.25}$$

this together with (4.23), and (4.24) yield

$$\mathrm{Ker}(A_c-\lambda) \subset \mathrm{Im}(A_c-\lambda) \qquad , \lambda \in \sigma(A_2). \tag{4.26}$$

To complete the proof we use the fact [T2, p. 291] that

$$\frac{\mathrm{Ker}(A_c-\lambda)^{i+j}}{\mathrm{Ker}(A_c-\lambda)^{i}} \simeq \mathrm{Im}(A_c-\lambda)^{i} \cap \mathrm{Ker}(A_c-\lambda)^{i} \tag{4.27}$$

for i, j = 0, 1, 2 ....

Thus, for i = j = 1, (4.27) and (4.26) give

$$\dim\left(\frac{\mathrm{Ker}(A_c-\lambda)^2}{\mathrm{Ker}(A_c-\lambda)}\right) = \dim[\mathrm{Ker}(A_c-\lambda)] = q$$

hence

$$\dim[\mathrm{Ker}(A_c-\lambda)^2] = 2q$$

this, together with

$$\mathrm{Ker}(A_{2e}-\lambda)^2 \times \{0\} \subset \mathrm{Ker}(A_c-\lambda)^2$$

implies that

$$\mathrm{Ker}(A_{2e}-\lambda)^2 \times \{0\} = \mathrm{Ker}(A_c-\lambda)^2$$

therefore, from (4.23) and (4.25) we conclude that

$$\mathrm{Ker}(A_c-\lambda)^2 \subset \mathrm{Im}(A_c-\lambda) \qquad , \quad \lambda \in \sigma(A_2) \tag{4.28}$$

Similarly we obtain from (4.27), with i = 1, j = 2, and (4.28)

$$\dim\left(\frac{\mathrm{Ker}(A_c-\lambda)^3}{\mathrm{Ker}(A_c-\lambda)}\right) = \dim[\mathrm{Ker}(A_c-\lambda)^2] = 2q$$

which in turn gives

$$\mathrm{Ker}(A_c-\lambda) \subset \mathrm{Im}(A_c-\lambda) \qquad , \quad \lambda \in \sigma(A_2)$$

Proceeding in this manner (4.17) is obtained. This completes the proof.

We point out that the assumption $[C_1 \ C_2] = [D_1 \ D_2]$ in Proposition 4.4 is merely a convenient one. Indeed, the proof of the proposition depends on the special structure of the loop operator A_L, namely that the controller contains an internal model of A_2 and utilizes feedback of the regulated variables z. Hence the proposition will remain valid if we assume, in place of $y = z$, that the controller is driven by the regulated variables. In particular, this feedback assumption is justified if $\mathrm{Ker}[C_1 \ C_2] \subset \mathrm{Ker}[D_1 \ D_2]$, i.e. z is readable from y, and the pair (D_1, A_1) is detectable, the latter being a requirement for stability of A_L. The readability condition constitutes a basic assumption throughout this chapter (see 1) - 7)). However, detectability of the pair (D_1, A_1) is not guaranteed by our basic assumptions 1) - 7). Nevertheless, in case this condition is not satisfied, it is always possible to construct a dynamic controller to achieve detectability of a related pair $(\tilde{D}_1, \tilde{A}_1)$, as we will show in Section 4.3.

4.2 Sufficiency of the Internal Model Principle

The sufficiency of the Internal Model Principle is essentially established by the following

Theorem 4.5: Suppose that $[C_1 \ C_2] = [D_1 \ D_2]$, i.e. $y = z$ and that $S_c = (\mathcal{X}_c, A_c, B_c, F_c, G_c)$ is a synthesis which provides internal stability and A_c contains an internal model of A_2. Then S_c is a structurally stable synthesis with respect to the parameters

$$P = (A_1, B_1, A_3, B_c, F_c, G_c)$$

Proof: We first show that internal stability is preserved under small perturbations of the parameters A_1, B_1, B_c, F_c, G_c.

It is easy to see that for perturbations of A_1 of class $F(A_1)$, and bounded perturbations of the remaining operators, the perturbed operator $(A_L+\delta A_L)$ generates a strongly continuous semigroup $S_{(A_L+\delta A_L)}(t)$ which is compact for $t \geq h$. Therefore, the stability of $(A_L+\delta A_L)$ will follow once we show that $\sigma(A_L+\delta A_L) \subset C^-$ for a sufficiently small pertutbation. Clearly $\sigma(A_L) \subset C^-$ consists of eigenvalues of finite multiplicities, and for any real number $\alpha > 0$, the set $\{\lambda \varepsilon \sigma(A_L); -\alpha \leq R_e \lambda \leq 0\}$ is finite. Let $C^{\alpha-} = \{\lambda \varepsilon C; R_e \lambda < -\alpha\}$ and $C^{\alpha+} = \{\lambda \varepsilon C; R_e \lambda \geq -\alpha\}$. Now enclose each $\lambda_i \varepsilon \sigma(A_L) \cap C^{\alpha+}$ by a circle Γ_i of small radius so that $\Gamma_i \subset C^-$. It now follows from Theorem B.III.2 in Appendix B, that there is a $\delta > 0$ (depending on A_L and Γ_i's) such that for any δA_L with $\hat{\delta}(A_L+\delta A_L, A_L) < \delta$, the spectrum of $(A_L+\delta A_L)$ is likewise separated by the Γ_i's and the total multiplicity of the eigenvalues of $(A_L+\delta A_L)$ in Γ_i equals the multiplicity of the eigenvalue of A_L in Γ_i. Further, the upper semicontinuity of $\sigma(A_L)$ asures that no eigenvalues of A_L in $C^{\alpha-}$ move to $C^{\alpha+}$. Hence, $\sigma(A_L+\delta A_L) \subset C^-$ for a sufficiently small perturbation δA_L.

Next we show that regulation is preserved under small perturbations of A_L and arbitrary bounded perturbations of B_L.

Choose δA_L, such that $(A_L+\delta A_L)$ is stable. Note that A_c, D_1 and D_2 are not allowed to vary. Let δB_L be a bounded perturbation of B_L. Then, there is a unique bounded operator $X_L : \mathcal{X}_2 \to \mathcal{X}_L$ such that

$$(A_L+\delta A_L)X_L - X_L A_2 = (B_L+\delta B_L) \tag{4.29}$$

Let $X_L = \begin{pmatrix} X_1 \\ X_c \end{pmatrix}$, writing (4.29) in detail we obtain

$$[A_1+\delta A_1+[B_1 G_c + \delta(B_1 G_c)]D_1]X_1 - X_1 A_2$$
$$+[B_1 F_c+\delta(B_1 F_c)]X_c = A_3+\delta A_3+[B_1 G_c+\delta(B_1 G_c)]D_2 \tag{4.30a}$$

$$[B_c+\delta B_c]D_1X_1 + A_cX_c - X_cA_c = (B_c+\delta B_c)D_2 \qquad (4.30b)$$

defining $\hat{B}_c = B_c + \delta B_c$, (4.30b) gives

$$\hat{B}_c(D_1 X_1 - D_2) + A_c X_c - X_c A_2 = 0 \qquad (4.31)$$

thus output regulation will be guaranteed once we show that

$$D_1 X_1 - D_2 = 0 \qquad (4.32)$$

Now, consider a decomposition of X_2 into prime subspaces. Fix $\lambda \in \sigma(A_2)$ and choose any prime subspace $X_{2\lambda}^j$ corresponding to λ. Select a basis for $X_{2\lambda}^j$ such that $A_2|X_{2\lambda}^j$ is in Jordan form, and let

$$(D_1 X_1 - D_2)|X_{2\lambda}^j = [r_1,r_2\ldots r_k]$$
$$X_c|X_{2\lambda}^j = [p_1,p_2\ldots p_k]$$

where $k = k(\lambda;j) = \dim[X_{2\lambda}^j]$.

Restricting (4.31) to $X_{2\lambda}^j$ we obtain

$$\hat{B}_c r_i + (A_c-\lambda)p_i = p_{i-1} , \; i = 1,2\ldots k \qquad (4.33)$$

where $p_0 \triangleq 0$.

Since $(A_L+\delta A_L)$ is stable and A_c contains an internal model of A_2, proposition 4.4 yields

$$\text{Ker } \hat{B}_c = 0 \qquad (4.34)$$

$$\text{Im } \hat{B}_c \cap \text{Im}(A_c-\lambda) = 0 \qquad (4.35)$$

$$\text{Ker}(A_c-\lambda)^{k_\lambda-1} \subset \text{Im}(A_c-\lambda) \qquad (4.36)$$

Now, for i = 1, (4.33), (4.34) and (4.35) give $r_1 = 0$. This implies that $p_1 \in \text{Ker}(A_c-\lambda)$ which in turn gives, by (4.36), $p_1 \in \text{Im}(A_c-\lambda)$. For i = 2, (4.33), (4.34) and (4.35) yield, $r_2 = 0$. Hence $p_2 \in \text{Ker}(A_c-\lambda)^2$ which implies, by (4.36), that $p_2 \in \text{Im}(A_c-\lambda)$. Proceeding in this manner we obtain $r_i = 0$, i = 1, 2...k. Therefore

$$(D_1 X_1 - D_2)|X_{2\lambda}^j = 0$$

Since $\lambda \in \sigma(A_2)$ and $X^j_{2\lambda}$ were chosen arbitrarily we conclude that (4.32) holds. This completes the proof.

We mention that the discussion following proposition 4.4 also applies to Theorem 4.5. That is, the crucial factor in the proof of the Theorem is not the assumption y = z, but the fact that the controller utilizes feedback of the regulated variables, and A_c contains an internal model of A_2.

Having established that the synthesis S_c in Theorem 4.5 is structurally stable at $P = (A_1, B_1, A_3, B_c, F_c, G_c)$, we now show that this property is maintained under certain small perturbations δA_c of the operator A_c. In fact the class of perturbation operator δA_c of A_c, consists of those operators for which $(A_c+\delta A_c)$ contains an internal model of A_2 and the closed-loop system is internally stable.

Consider the decomposition

$$X_c = X_{c1} \oplus X_{c2} \tag{4.37}$$

where

$$X_{c1} = \bigcap_{i=1}^{\ell} \operatorname{Im}(A_c-\lambda_i)^{p_i}$$

$$X_{c2} = \bigoplus_{i=1}^{\ell} \operatorname{Ker}(A_c-\lambda_i)^{p_i}$$

$\{\lambda_1, \lambda_2 \ldots \lambda_\ell\}$ are the distinct eigenvalues of A_2

$p_1, p_2 \ldots p_\ell$ are finite integers and X_{c2} is finite dimensional.

According to (4.37) we may write

$$F_c = [F_{c1} \quad F_{c2}] \tag{4.38a}$$

$$B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \tag{4.38b}$$

$$A_c = \begin{pmatrix} A_{c1} & 0 \\ 0 & A_{c2} \end{pmatrix} \qquad (4.38c)$$

where A_{c2} is a bounded operator and contains an internal model of A_2, in fact $\sigma(A_{c2})$ coincides with $\sigma(A_2)$ except in multiplicities and $\sigma(A_{c2}) \cap \sigma(A_{c1}) = \emptyset$.

Using the representation (4.38a) - (4.38c), A_L and B_L may be written as

$$A_L = \begin{pmatrix} \bar{A}_1 & \bar{B}_1 F_{c2} \\ B_{c2}\bar{D}_1 & A_{c2} \end{pmatrix} \quad , \quad B_L = \begin{pmatrix} \bar{A}_3 \\ B_{c2} D_2 \end{pmatrix} \qquad (4.39a)$$

where

$$\bar{A}_1 = \begin{pmatrix} A_1 + B_1 G_c D_1 & B_1 F_{c1} \\ B_{c1} D_1 & A_{c1} \end{pmatrix} \quad : \mathcal{X}_1 \oplus \mathcal{X}_{c1} \rightarrow \mathcal{X}_1 \oplus \mathcal{X}_{c1}$$

$$\bar{A}_3 = \begin{pmatrix} A_3 + B_1 G_c D_2 \\ B_{c1} D_2 \end{pmatrix} \quad : \quad \mathcal{X}_2 \rightarrow \mathcal{X}_1 \oplus \mathcal{X}_{c1} \qquad (4.39b)$$

$$\bar{B}_1 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} : \mathcal{U} \rightarrow \mathcal{X}_1 \oplus \mathcal{X}_{c1}$$

$$\bar{D}_1 = [D_1 \quad 0] \quad : \quad \mathcal{X}_1 \oplus \mathcal{X}_{c1} \rightarrow \mathcal{Z}$$

It is now clear, from proposition 4.4 and Theorem 4.5 (with appropriate modifications), that $S_c = (\mathcal{X}_c, A_c, B_c, F_c, G_c)$ is structurally stable with respect to the parameters $\mathcal{P}_1 = (A_1, B_1, A_3, B_{c1}, B_{c2}, F_{c1}, F_{c2}, G_c, A_{c1})$. Hence, according to the representation (4.38c), the class of admissible perturbations δA_c of the operator A_c, correspond to small perturbations δA_{c1} of A_{c1}, and since A_{c2} is fixed it is readily verified that

$$A_c+\delta A_c = \begin{bmatrix} A_{c1}+\delta A_{c1} & 0 \\ 0 & A_{c2} \end{bmatrix}$$

contains an internal model of A_2. Thus, we may conclude that for any representation of A_c, the admissible perturbations of A_c, consist of those (small) operators δA_c, such that $(A_c+\delta A_c)$ contains an internal model of A_2; however, we mention that in this case it may be difficult to determine explicitly which perturbations preserve the internal model.

In the remaining part of this section we relax the assumption $y = z$, and we assume that z is, readable from y, i.e. $\mathrm{Ker}[C_1 \; C_2] \subset \mathrm{Ker}[D_1 \; D_2]$. Also, it is assumed that $S_c = (\mathcal{X}_c, A_c, B_c, F_c, G_c)$ is a synthesis for the system (4.1a) - (4.1d) such that the closed-loop system is internally stable, A_c contains an internal model of A_2 and the internal model is controllable by the regulated variables z, that is, the controller incorporates a feedback structure. We will show below that S_c is structurally stable. First, a convenient representation for the loop operator A_L, will be derived.

From the readability assumption we may write

$$\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z} \tag{4.40}$$

where $\mathcal{W}$ is a complement of $\mathcal{Z}$ in $\mathcal{Y}$. According to this decomposition we have

$$[C_1 \;\; C_2] = \begin{bmatrix} E_1 & E_2 \\ D_1 & D_2 \end{bmatrix} \tag{4.41}$$

Clearly, since A_c contains an internal model of A_2, we may adopt the representation (4.38a) - (4.38c). Further, according to (4.40), (4.38b) can be written in more detail as follows

$$B_c = \begin{bmatrix} B_{cw1} & B_{cz1} \\ B_{cw2} & B_{cz2} \end{bmatrix} \tag{4.42}$$

Also, we have

$$G_c = [G_{cw} \quad G_{cz}] \tag{4.43}$$

Now, since A_L is stable, it is easy to see that the pair $(A_{c2}, [B_{cw2} \ \ B_{cz2}])$ is controllable†, i.e.

$$\text{Im } B_{cw2} + \text{Im } B_{cz2} + \text{Im}(A_{c2}-\lambda) = X_{c2} \quad , \lambda \in \mathcal{C} \tag{4.44}$$

But A_{c2} is the part of A_c containing the internal model, and since the internal model is controllable by z (by assumption) we conclude that either

a) $B_{cw2} \equiv 0$ and (A_{c2}, B_{cz2}) is controllable

or

b) X_{c2} may be decomposed as follows

$$X_{c2} = X^1_{c2} \oplus X^2_{c2} \tag{4.45a}$$

where $X^1_{c2} = \langle A_{c2} | \text{Im } B_{cw2} \rangle$ and X^2_{c2} is a complement of X^1_{c2} in X_{c2}.
According to (4.45a) we can write

$$F_{c2} = [F^1_{c2} \quad F^2_{c2}] \tag{4.45b}$$

$$[B_{cw2} \quad B_{cz2}] = \begin{pmatrix} B^1_{cw2} & B^1_{cz2} \\ 0 & B^2_{cz2} \end{pmatrix} \tag{4.45c}$$

$$A_{c2} = \begin{bmatrix} A^1_{c2} & A^3_{c2} \\ 0 & A^2_{c2} \end{bmatrix} \tag{4.45d}$$

A^2_{c2} contains the internal model, and the pair (A^2_{c2}, B^2_{cz2}) is controllable.

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† recall that A_{c2} is a bounded operator and X_{c2} is finite dimensional

In fact a) is a special case of b), therefore we may assume that b) holds. So, combining (4.38a), (4.38c), (4.41), (4.42), (4.43), (4.45b) - (4.45d) we obtain

$$A_L = \begin{pmatrix} \bar{A}_1 & \bar{B}_1 \\ B^2_{cz2}\bar{D}_1 & A^2_{c2} \end{pmatrix} \quad , \quad B_L = \begin{pmatrix} \bar{B}_L \\ B^2_{cz2}\ D_2 \end{pmatrix} \tag{4.46}$$

where

$$\bar{A}_1 = \begin{pmatrix} A_1+B_1G_{cw}E_1+B_1G_{cz}D_1 & B_1F_{c1} & B_1F^1_{c2} \\ B_{cw1}E_1+B_{cz1}D_1 & A_{c1} & 0 \\ B^1_{cw2}E_1+B^1_{cz2}D_1 & 0 & A^1_{c2} \end{pmatrix} : \ X_1 \oplus X_{c1} \oplus X^1_{c2} \to X_1 \oplus X_{c1} \oplus X^1_{c2}$$

$$\bar{B}_1 = \begin{pmatrix} B_1\ F^2_{c2} \\ 0 \\ A^3_{c2} \end{pmatrix} \quad : X^2_{c2} \to X_1 \oplus X_{c1} \oplus X^1_{c2}$$

$$\bar{D}_1 = [D_1 \quad 0 \quad 0] \quad : X_1 \oplus X_{c1} \oplus X^1_{c2} \to Z$$

$$\bar{B}_L = \begin{pmatrix} A_3+B_1G_{cw}E_2+B_1G_{cz}D_2 \\ B_{cw1}E_2+B_{cz1}D_2 \\ B^1_{cw2}E_2+B^1_{cz2}D_2 \end{pmatrix} \quad : \quad X_2 \to X_1 \oplus X_{c1} \oplus X^1_{c2}$$

Now using the representation (4.46) it is easy to see, from the proof of proposition 4.4 and Theorem 4.5, that the synthesis S_c is structurally stable with respect to the parameters

$$P_2 = (A_1, B_1, A_3, F_{c1}, F^1_{c2}, F^2_{c2}, G_{cw}, G_{cz}, B_{cw1}, B_{cz1}, B^1_{cw2}, B^1_{cz2}, B^2_{cz2}, A_{c1}, A^1_{c2}, A^3_{c2})$$

The main result of this section can be summarized in the following

Theorem 4.6: Suppose that S_c is a synthesis which provides internal stability, utilizes feedback of the regulated variables and incorporates, in the feedback path, an internal model of the dynamic structure of the exogenous signals which the controller is required to process. Then output regulation is maintained when the system's and controller parameters undergo small perturbations which preserve internal stability and the internal model.

4.3 Construction of a Structurally Stable Synthesis

In this section we will establish a sufficient condition, in terms of the systems parameters, to guarantee the existence of a structurally stable controller. Furthermore, a procedure to construct such controller will be obtained.

The main result of this section is given by the following

Theorem 4.7: Suppose that 1) - 7) are satisfied. If in addition, the system (4.1a) -(4.1d) satisfies the condition

$$(A_1-\lambda)[\text{Ker } D_1 \cap D(A_1)] + \text{Im } B_1 = \mathcal{X}, \quad \lambda \in \sigma(A_2), \tag{4.47}$$

then, there is a synthesis $S_c = (\mathcal{X}_c, A_c\ B_c, F_c, G_c)$ which is structurally stable.

Proof: The proof of Theorem 4.7 consists of a procedure for the construction of a structurally stable synthesis and will be given in several steps.

Step 1: This step consists in augmenting our system by means of a dynamic controller, to achieve certain detectability condition. That is, if $\tilde{A}_1$ represents the dynamics of the augmented system, then we require the pair $(\tilde{D}_1, \tilde{A}_1)$ to be detectable, where $\tilde{D}_1$ is of the form $[D_1\ 0]$. If the pair

(D_1, A_1) is detectable then proceed to Step 2.

Consider the system

$$\dot{x}_{c1}(t) = (A_1+K_1C_1)x_{c1}(t) - K_1y(t) + B_1u(t) \qquad (4.48a)$$

where $x_{c1} \varepsilon \mathcal{X}_{c1}$, A_1 C_1 and B_1 denote copies† of the operators defined before, but now with $\mathcal{X}_1$ replaced by $\mathcal{X}_{c1}$, and $K_1 : \mathcal{Y} \to \mathcal{X}_{c1}$ is chosen such that $(A_1+K_1C_1)$ is stable. Since (C_1, A_1) is detectable (by assumption) such K_1 clearly exists.

Now let

$$u(t) = F_{c1}x_{c1}(t) + v(t) \qquad (4.48b)$$

where v(t) is an external input and $F_{c1} : \mathcal{X}_{c1} \to \mathcal{U}$ is a bounded operator such that $(A_1+B_1F_{c1})$ is stable. Clearly the existence of F_{c1} is guaranteed by the stabilizability of the pair (A_1, B_1).

Combining (4.1a) - (4.1d) with (4.48a) - (4.48b) we obtain

$$\dot{x}_1(t) = \tilde{A}_1 \tilde{x}_1(t) + \tilde{A}_3 \tilde{x}_2(t) + \tilde{B}_1 v(t) \qquad (4.49a)$$

$$\dot{x}_2(t) = A_2 x_2(t) \qquad (4.49b)$$

$$y(t) = \tilde{C}_1 \tilde{x}_1(t) + C_2 x_2(t) \qquad (4.49c)$$

$$z(t) = \tilde{D}_1 \tilde{x}_1(t) + D_2 x_2(t) \qquad (4.49d)$$

where

$$\tilde{\mathcal{X}}_1 = \mathcal{X}_1 \oplus \mathcal{X}_{c1}$$

$$\tilde{A}_1 = \begin{bmatrix} A_1 & B_1F_{c1} \\ -K_1C_1 & A_1+B_1F_{c1}+K_1C_1 \end{bmatrix} \qquad \tilde{B}_1 = \begin{bmatrix} B_1 \\ B_1 \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} A_3 \\ -K_1C_2 \end{bmatrix}, \quad \tilde{C}_1 = [C_1 \quad 0], \quad \tilde{D}_1 = [D_1 \quad 0]$$

We now show that the pair $(\tilde{D}_1, \tilde{A}_1)$ is detectable. First note that,

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† Throughout this section we will always use the same symbols to denote the corresponding copies of the operators previously defined.

since $K_1\ C_1$ and $B_1\ F_{c1}$ are compact, the semigroup $S_{\tilde{A}_1}(t)$ is compact fot $t \geq h$, therefore detectability of $(\tilde{D}_1, \tilde{A}_1)$ is equivalent to the condition

$$\text{Ker}\ \tilde{D}_1 \cap \text{Ker}\ (\tilde{A}_1 - \lambda) = 0, \qquad \lambda \in C^+ \tag{4.50}$$

Next, define $e(t) = x_{c1}(t) - x_1(t)$, then it is easy to see that

$$\begin{pmatrix} \dot{x}_1(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} A_1 + B_1 F_{c1} & B_1 F_{c1} \\ 0 & A_1 + K_1 C_1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ e(t) \end{pmatrix} + \begin{pmatrix} A_3 \\ -(A_3 + K_1 C_2) \end{pmatrix} x_2(t) + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} v(t)$$

thus

$$\sigma(\tilde{A}_1) = \sigma(A_1 + B_1 F_{c1}) \dot{\cup}\ \sigma(A_1 + K_1 C_1) \subset C^-$$

hence (4.50) is satisfied for $\lambda \in C^+$ and $(\tilde{D}_1, \tilde{A}_1)$ is detectable.

Before proceeding to the next step we prove the following result

<u>Proposition 4.8</u>: Let $\tilde{X}_1$, $\tilde{A}_1$, $\tilde{B}_1$ and $\tilde{D}_1$ be as in Step 1. Then

$$(\tilde{A}_1 - \lambda)[\text{Ker}\ \tilde{D}_1 \cap D(\tilde{A}_1)] + \text{Im}\ \tilde{B}_1 = \tilde{X}_1, \qquad \lambda \in \sigma(A_2) \tag{4.51}$$

if and only if

$$(A_1 - \lambda)[\text{Ker}\ D_1 \cap D(A_1)] + \text{Im}\ B_1 = X_1, \qquad \lambda \in \sigma(A_2) \tag{4.52}$$

<u>Proof</u>: We first note the following

$$(\tilde{A}_1 - \lambda)\text{Ker}\ \tilde{D}_1 + \text{Im}\ \tilde{B}_1 = \left\{ \begin{pmatrix} A_1 - \lambda & 0 \\ -K_1 C_1 & A_1 + K_1 C_1 - \lambda \end{pmatrix} + \begin{pmatrix} 0 & B_1 F_{c1} \\ 0 & B_1 F_{c1} \end{pmatrix} \right\} \text{Ker}\ \tilde{D}_1 + \text{Im} \begin{pmatrix} B_1 \\ B_1 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 - \lambda & 0 \\ -K_1 C_1 & A_1 + K_1 C_1 - \lambda \end{pmatrix} \text{Ker}\ \tilde{D}_1 + \text{Im} \begin{pmatrix} B_1 \\ B_1 \end{pmatrix} \qquad .. \qquad (4.53)$$

Necessity: Suppose (4.51) holds, then (4.53) gives

$$\begin{bmatrix} A_1-\lambda & \\ -K_1C_1 & A_1+K_1C_1-\lambda \end{bmatrix}\begin{bmatrix} \text{Ker } D_1 \\ X_{c1} \end{bmatrix} + \text{Im}\begin{bmatrix} B_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_{c1} \end{bmatrix} \quad , \quad \lambda \in \sigma(A_2)$$

and (4.52) follows.

Sufficiency: Suppose (4.52) holds, then for any $x_1 \in X_1$, there are $r_1 \in \text{Ker } D_1 \cap D(A_1)$ and $u_1 \in U$ such that

$$(A_1-\lambda)r_1 + B_1u_1 = x_1 \quad , \quad \lambda \in \sigma(A_2)$$

for (4.51) to hold we need to show that for any $x_{c1} \in X_{c1}$ there is an $r_{c1} \in D(A_1) \subset X_{c1}$ such that

$$-K_1C_1r_1 + B_1u_1 + (A_1+K_1C_1-\lambda)r_{c1} = x_{c1}, \qquad \lambda \in \sigma(A_2)$$

Since $(A_1+K_1C_1)$ is stable clearly such r_{c1} exists, thus

$$\begin{bmatrix} A_1-\lambda & 0 \\ -K_1C_1 & A_1+K_1C_1-\lambda \end{bmatrix} \text{Ker } \tilde{D}_1 + \text{Im}\begin{bmatrix} B_1 \\ B_1 \end{bmatrix} = \tilde{X}_1, \qquad \lambda \in \sigma(A_2)$$

and from (4.53) we obtain (4.51). This completes the proof.

We now continue with the proof of Theorem 4.6.

<u>Step 2</u>: Let $A_{2e} : X_{2e} \to X_{2e}$ be a q-fold direct sum of the largest cyclic component of A_2. That is, $A_{2e} = \text{diag}[A_{2k}\ A_{2k} \cdots A_{2k}]$, and $X_{2e} = [X_{2k}]^q$. Define

$$X_e = \tilde{X}_1 \oplus X_{2e}$$

and let

$$\dot{x}_e(t) = A_ex_e(t) + B_e\, v(t) \tag{4.54a}$$

$$z_e(t) = D_ex_e(t) \tag{4.54b}$$

where

$$A_e = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_{3e} \\ 0 & A_{2e} \end{bmatrix} : X_e \to X_e \qquad B_e = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} : U \to X_e$$

$$D_e = [\tilde{D}_1 \quad 0] : X_e \to Z$$

and the bounded operator $\tilde{A}_{3e} : X_{2e} \to \tilde{X}_1$ is chosen such that the pair (D_e, A_e) is detectable. The existence of such $\tilde{A}_{3e}$ is guaranteed by detectability of the pair $(\tilde{D}_1, \tilde{A}_1)$ and proposition 4.2 gives a way of obtaining $\tilde{A}_{3e}$.

We now show that there is a synthesis for the system (4.54a) - (4.54b). This result will follow from proposition 4.3 once we show that there are bounded operators $X_{1e} : X_{2e} \to \tilde{X}_1$ and $U_e : X_{2e} \to U$ such that

$$\tilde{A}_1 X_{1e} - X_{1e} A_{2e} + \tilde{B}_1 U_e = \tilde{A}_{3e} \tag{4.55a}$$

$$\tilde{D}_1 X_{1e} = 0 \tag{4.55b}$$

By assumption, (4.47) holds, this proposition 4.8 implies that (4.51) holds and by proposition 4.1 we conclude that there are X_{1e} and U_e satisfying (4.55a) - (4.55b). Hence there is a synthesis for the system (4.54a) - (4.54b).

When (D_1, A_1) is detectable, replace $\tilde{A}_1$, $\tilde{B}_1$ $\tilde{A}_{3e}$ and $\tilde{D}_1$ in (4,54a) - (4.54b) by A_1, B_1, A_{3e} and D_1, where $A_{3e} : X_{2e} \to X_1$ is chosen such that (D_e, A_e) is detectable. As before, it is easy to see that there is a solution of (4.55a) - (4.55b) with $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{A}_{3e}$ and $\tilde{D}_1$ replaced by A_1, B_1, A_{3e} and D_1, therefore we conclude that there is a synthesis for the system (4.54a) - (4.54b).

Step 3: In this step we construct a synthesis for the system (4.54a) - (4.54b), via an observer, as in the proof of Proposition 4.3.

Choose an $F_{2e}: \mathcal{X}_{2e} \to \mathcal{U}$ such that there is an $\tilde{R}_{1e} : \mathcal{X}_{2e} \to \tilde{\mathcal{X}}_1$ satisfying

$$\tilde{A}_1 \tilde{R}_{1e} - \tilde{R}_{1e} A_{2e} = \tilde{A}_{3e} + \tilde{B}_1 F_{2e} \tag{4.56a}$$

$$\tilde{D}_1 \tilde{R}_{1e} = 0 \tag{4.56b}$$

Clearly such F_{2e} exists, since (4.55a) - (4.55b) have a solution as was shown in Step 2. Furthermore $\tilde{R}_{1e}$ is unique since $\sigma(\tilde{A}_1) \cap \sigma(A_{2e}) = \emptyset$. Choose Q_e such that $(A_e + Q_e D_e)$ is stable. The existence of Q_e is guaranteed by detectability of the pair (D_e, A_e). The observer for the system (4.54a) - (4.54b) is given by

$$\dot{x}_{c2}(t) = (A_e + Q_e D_e) x_{c2}(t) - Q_e z_e(t) + B_e v(t) \tag{4.57a}$$

$$v(t) = F_e x_{c2}(t) \tag{4.57b}$$

where $x_{c2} \in \mathcal{X}_{c2} = \tilde{\mathcal{X}}_{c2} \oplus \mathcal{X}_{2e}$ and $F_e = [0 \quad F_{2e}]$

The synthesis for the system (4.54a) - (4.54b) is now complete. As in the proof of Proposition 4.3 it is easy to see that the closed-loop system (4.54a), (4.57a) and (4.57b) is internally stable and that z_e is regulated.

When (D_1, A_1) is detectable, the expressions (4.56a) - (4.56b) are modified as follows. First, select an $F_{1e} : \mathcal{X}_1 \to \mathcal{U}$ such that $(A_1 + B_1 F_{1e})$ is stable. The existence of such F_{1e} is guaranteed by stabilizability of the pair (A_1, B_1). Also, note, that in the previous case we have set $F_{1e} = 0$ since $\tilde{A}_1$ is stable. Now choose $F_{2e} : \mathcal{X}_{2e} \to \mathcal{U}$ such that there is an $R_{1e} : \mathcal{X}_{2e} \to \mathcal{X}_1$, satisfying

$$(A_1+B_1F_{1e})R_{1e} - R_{1e}A_{2e} = A_{3e} + B_1F_{2e}$$

$$D_1R_{1e} = 0$$

Clearly such F_{2e} exists, since (4.55a) - (4.55b) have a solution with $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{A}_{3e}$ and $\tilde{D}_1$ replaced by A_1, B_1, A_{3e} and D_1. The remaining part of the construction follows as described above, but now F_e in (4.57b) is given by

$$F_e = [F_{1e} \quad F_{2e}]$$

Step 4: Observe that (4.57a) has as input z_e, and by assumption only y is directly accessible, hence (4.57a) cannot be implemented as it stands. However, since z is readable from y we may assume that (4.57a) is driven by z in place of z_e. Indeed, readability implies that there is a bounded $P : \mathcal{Y} \rightarrow \mathcal{Z}$ such that

$$z = Py$$

Furthermore, using the representation (4.41) we obtain

$$P = [0 \quad I_Z]$$

thus

$$Q_e z = Q_e P_y = [0 \quad Q_e]y$$

hence (4.57a), together with (4.57b) gives

$$\dot{x}_{c2}(t) = (A_e+Q_eD_e+B_eF_e)x_{c2}(t) - [0 \quad Q_e]y(t) \qquad (4.58)$$

The remaining part of the proof consists in showing that (4.48a), (4.48b), (4.57b) and (4.58) constitute a structurally stable synthesis for the system (4.1a) - (4.1b). This will follow from Theorem 4.6 once we show that synthesis utilizes feedback of the regulated variables, incorporates in the feedback path an internal model of A_2, and provides internal stability. From (4.58) it is clear that our synthesis utilizes feedback of z, by virtue of the decomposition $\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z}$, i.e. $y = \begin{pmatrix} w \\ z \end{pmatrix}$. Also, from (4.56a) - (4.56b) it is easy to see that $(A_e+Q_eD_e+B_eF_e)$ contains an internal model of A_2. Indeed, let

$$R = \begin{pmatrix} -\tilde{R}_{1e} \\ I_{2e} \end{pmatrix} \tag{4.59}$$

where $\tilde{R}_{1e}$ satisfies (4.56a) - (4.56b) and I_{2e} is the identity operator on $\mathcal{X}_{2e}$. Clearly R is injective.

Let

$$Q_e = \begin{pmatrix} Q_{e1} \\ Q_{e2} \end{pmatrix} : \mathcal{Z} \to \tilde{\mathcal{X}}_{c2} \oplus \mathcal{X}_{2e} = \mathcal{X}_{c2}$$

then we may write $(A_c+Q_e\ D_e + B_e\ F_e)$ in detail

$$A_e + Q_e\ D_e + B_e\ F_e = \begin{pmatrix} \tilde{A}_1 + Q_{e1}\ \tilde{D}_1 & \tilde{A}_{3e} + \tilde{B}_1\ F_{2e} \\ Q_{e2}\ \tilde{D}_1 & A_{2e} \end{pmatrix} \tag{4.60}$$

Now it is readily verified, using (4.56a), (4.56b), (4.59) and (4.60), that

$$(A_e + Q_e\ D_e + B_e\ F_e)R = R\ A_{2e} \tag{4.61}$$

therefore, definition 3.8 implies that $(A_e + Q_e\ D_e + B_e\ F_e)$ contains an internal model of A_2. (The case (D_1, A_1) detectable follows from the above discussion after appropriate modifications).

Thus, the structural stability of our synthesis will be established if we show that the closed-loop system is internally stable.

Step 5: In this step we will show that the loop operator A_L is stable. The closed loop system is given by

$$\begin{pmatrix} \dot{x}_L(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_L & B_L \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_L(t) \\ x_2(t) \end{pmatrix} \tag{4.62a}$$

$$z(t) = [D_L \quad D_2] \begin{pmatrix} x_L(t) \\ x_2(t) \end{pmatrix} \tag{4.62b}$$

where $X_L = \tilde{X}_1 \oplus \tilde{X}_{c2} \oplus X_{2e}$

$$A_L = \begin{pmatrix} \tilde{A}_1 & 0 & \tilde{B}_1 F_{2e} \\ -Q_{e1}\tilde{D}_1 & \tilde{A}_1 + Q_{e1}\tilde{D}_1 & \tilde{A}_{3e} + \tilde{B}_1 F_{2e} \\ -Q_{e2}\tilde{D}_1 & Q_{e2}\tilde{D}_1 & A_{2e} \end{pmatrix} \tag{4.62c}$$

$$B_L = \begin{pmatrix} \tilde{A}_3 \\ -Q_{e1}D_2 \\ -Q_{e2}D_2 \end{pmatrix} \tag{4.62d}$$

$$D_L = [\tilde{D}_1 \quad 0 \quad 0] \tag{4.62e}$$

Now, since $\tilde{B}_1 F_{2e}$, $Q_{e1}\tilde{D}_1$, $Q_{e2}\tilde{D}_1$, $\tilde{A}_{3e}$ are compact operators, it is easy to see that $S_{A_L}(t)$ is compact for $t \geq h$. Therefore internal stability will be established once we show that $\sigma(A_L) \subset C^-$. Define

$$e(t) = \begin{pmatrix} \tilde{e}(t) \\ x_{2e}(t) \end{pmatrix} = \begin{pmatrix} \tilde{x}_{c2}(t) - \tilde{x}_1(t) \\ x_{2e}(t) \end{pmatrix}$$

then, it is readily verified that

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{e}}(t) \\ \dot{x}_{2e}(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 & 0 & \tilde{B}_1 F_{2e} & \tilde{A}_3 \\ 0 & \tilde{A}_1 + Q_{e1}\tilde{D}_1 & \tilde{A}_{3e} & -(\tilde{A}_3 + Q_{e1}D_2) \\ 0 & Q_{e2}\tilde{D}_1 & A_{2e} & -Q_{e2}D_2 \\ 0 & 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{e}(t) \\ x_{2e}(t) \\ x_2(t) \end{pmatrix}$$

but

$$A_e+Q_eD_e = \begin{bmatrix} \tilde{A}_1+Q_{e1}\tilde{D}_1 & \tilde{A}_{3e} \\ Q_{e2}\tilde{D}_1 & A_{2e} \end{bmatrix}$$

thus

$$\sigma(A_L) = \sigma(\tilde{A}_1) \cup \sigma(A_e+Q_eD_e)$$

Since $\tilde{A}_1$ and $(A_e+Q_eD_e)$ are stable we obtain $\sigma(A_L) \subset C^-$, i.e. A_L is stable. (When (D_1, A_1) is detectable, similar arguments are used to establish the stability of A_L).

It now follows from Theorem 4.5, that our synthesis is structurally stable with respect to $\tilde{P} = (\tilde{A}_1, \tilde{A}_3, \tilde{B}_1, Q_e, F_{2e})$.† Moreover, from the discussion following Theorem 4.5 we conclude that we may allow small perturbations of $(A_e+Q_eD_e+B_eF_e)$ whenever the internal model and the stability of A_L are preserved. It is easy to see (from (4.59) - (4.61)) that small perturbations of $(A_c+Q_eD_e+B_eF_e)$ arising from small perturbations of Q_e preserve the internal model. Unfortunately, it is difficult to determine explicitly other perturbations of the operator $(A_e+Q_eD_e+B_eF_e)$ which preserve the internal model. (The case when (D_1,A_1) is detectable follows similarly with $\tilde{P}$ replaced by $P = (A_1,A_3,B_1,Q_e,F_{1e},F_{2e})$). This completes the proof of Theorem 4.7.

The condition (4.47) in Theorem 4.7 is given in terms of the operators of the abstract evolution system (4.1a) - (4.1d). Our next result gives a necessary and sufficient condition for the existence of a structurally stable synthesis in terms of the matrices $\hat{A}_0$, $\hat{A}_1$, $\hat{B}_1$ and $\hat{D}_1$ of the corresponding delay system.

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† Here we do not consider perturbations of the corresponding "copies" of $\tilde{A}_1, \tilde{B}_1, Q_e$ and F_{2e} in $(A_e+Q_eD_e+B_eF_e)$.

<u>Theorem 4.9</u>: Suppose that 1) - 7) hold. Then, there exists a structurally stable controller for the delay system

$$\dot{\hat{x}}_1(t) = \hat{A}_0\hat{x}_1(t) + \hat{A}_1\hat{x}_1(t-h) + \hat{A}_3\hat{x}_2(t) + \hat{B}_1u(t)$$

$$\dot{\hat{x}}_2(t) = \hat{A}_2\hat{x}_2(t)$$

$$y(t) = \hat{C}_1\hat{x}_1(t) + \hat{C}_2\hat{x}_2(t)$$

$$z(t) = \hat{D}_1\hat{x}_1(t) + \hat{D}_2\hat{x}_2(t)$$

if and only if

$$\text{Rank}\begin{pmatrix} \hat{A}_0 + \hat{A}_1e^{-\lambda h} - & \hat{B} \\ \hat{D}_1 & 0 \end{pmatrix} = n_1+q \quad , \quad \forall \ \lambda \in \sigma(\hat{A}_2) \qquad (4.63)$$

<u>Proof</u>: The necessity of (4.47) was established in Chapter 3 (see Proposition 3.24). Therefore it is only required to show that (4.47) is equivalent to (4.63).

From (4.47), for any $x_1 = (\hat{x}_1^0, \hat{x}_1^1) \in X_1 = M_2^{n_1}$, there exist $\phi_1 = (\hat{\phi}_1^0, \hat{\phi}_1^1) \in D(A_1)$ and $u \in U = \mathbb{R}^m$ such that

$$\hat{A}_0 \hat{\phi}_1^0 + \hat{A}_1\hat{\phi}_1^1(-h) - \lambda \hat{\phi}_1^0 + \hat{B}_1u = x_1^0 \qquad (4.64a)$$

$$\frac{d\hat{\phi}_1^1(\theta)}{d\theta} - \lambda\hat{\phi}_1^1(\theta) = \hat{x}_1^1(\theta) \qquad (4.64b)$$

$$\hat{D}_1 \hat{\phi}_1^0 = 0 \qquad (4.64c)$$

Now, since $\hat{\phi}_1^1(0) = \hat{\phi}_1^0$, (4.64b) gives

$$\hat{\phi}_1^1(\theta) = e^{\lambda\theta} \hat{\phi}_1^0 - \int_{\theta}^{0} e^{\lambda(\theta-\sigma)} \hat{x}_1^1(\sigma)d\sigma \ , \ -h \le \theta \le 0$$

thus, form (4.64a) - (4.64c) we obtain

$$(\hat{A}_0+\hat{A}_1e^{-\lambda h}-\lambda)\hat{\phi}_1^0 + \hat{B}_1u = \hat{x}_1^0 + \int_{-h}^{0}e^{-\lambda(h+\sigma)} \hat{A}_1\hat{x}_1^1(\sigma)d\sigma$$

$$\hat{D}_1 \hat{\phi}_1^0 = 0$$

and since $x_1 \in X_1$ is arbitrary we obtain (4.63)

Now suppose that (4.63) holds, then for any $f^0 \in \mathbb{R}^{n_1}$, there are $g^0 \in \mathbb{R}^{n_1}$ and $u \in \mathbb{R}^m$ such that

$$(\hat{A}_0 + \hat{A}_1 e^{-\lambda h} - \lambda) g^0 + \hat{B}_1 u = f^0 \tag{4.65a}$$

$$\hat{D}_1 g^0 = 0 \tag{4.65b}$$

For any $f^1(\theta) \in L_2([-h,0]; \mathbb{R}^{n_1})$ define $g^1(\theta)$ as the solution of the differential equation

$$\frac{dg^1(\theta)}{d\theta} = \lambda g^1(\theta) + f^1(\theta) \qquad -h \leq \theta \leq 0 \tag{4.66}$$

with $g^1(0) = g^0$.

It follows that $g = (g^1(0), g^1) \in D(A_1)$ and (4.65a) gives

$$\hat{A}_0 g^1(0) + \hat{A}_1 g^1(-h) - \lambda g^1(0) + \hat{B}_1 u = f^0 - \int_{-h}^{0} e^{-\lambda(h+\sigma)} \hat{A}_1 f^1(\sigma) d\sigma \tag{4.67}$$

Since $f^0 \in \mathbb{R}^{n_1}$ and $f^1(\theta) \in L_2([-h,0]; \mathbb{R}^{n_1})$ are arbitrary, (4.47) follows from (4.67), (4.66) and (4.65b). This completes the proof.

We point out that the condition (4.63) is easy to verify. Also, the conditions 3) - 7) are easy to check in terms of the matrices of the delay system. Verification of the conditions 1) and 2), that is stabilizability and detectability of the pairs (A_1, B_1) and (C_1, A_1) respectively, is slightly more difficult since we need to compute the eigenvalues of A_1 in C^+. We conclude this section with the following lemmas.

<u>Lemma 4.10</u>: The pair (A_1, B_1) is stabilizable if and only if

$$\text{Rank}[\hat{A}_1 + \hat{A}_1 e^{-\lambda h} - \lambda,\ \hat{B}_1] = n_1, \qquad \forall \lambda \in C^+.^{\dagger}$$

---

† It is necessary and sufficient to verify this condition for all $\lambda \in C^+$ such that $\det(\hat{A}_0 + \hat{A}_1 e^{-\lambda h} - \lambda) = 0$, i.e. $\lambda \in \sigma(A_1) \cap C^+$.

Lemma 4.11: The pair (C_1, A_1) is detectable if and only if

$$\text{Rank} \begin{pmatrix} A_0 + A_1 e^{-\lambda h} - \lambda \\ C_1 \end{pmatrix} = n_1, \qquad \forall \lambda \in C^+ \quad {}^{\dagger}$$

the proof of these results follows easily from Lemmas 2.8 and 2.9 in Chapter 2.

4.4 Conclusions and Remarks

The sufficiency of the Internal Model Principle is a major result of this chapter. Precisely, we have shown that a controller which provides internal stability utilizes feedback of the regulated variables and incorporates, in the feedback path, an internal model of the dynamics of the exogenous signals, preserves output regulation when the parameters of the system and controller undergo small perturbations, provided that internal stability and the internal model are maintained. Thus we have attained a greater degree of structural stability than was initially required.

We have also derived simple conditions, in terms of the matrices of the delay systems, to assure the existence of a structurally stable controller. A design procedure to construct such controller has been obtained. It is important to note, that the dynamics of such controller become unpleasantly "large". This is the price to be paid for insisting on regulation in the presence of arbitrary perturbations of the operator A_3.

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† It is necessary and sufficient to verify this condition for all $\lambda \in C^+$ such that $\det(A_0 + A_1 e^{-\lambda h} - \lambda) = 0$, i.e. $\lambda \in \sigma(A_1) \cap C^+$

CHAPTER 5

CONCLUSIONS

The central subject of the preceding chapters has been the validation of the Internal Model Principle for linear systems involving time delays in the state.

First, the regulation and internal stability problem for delay systems is formulated in an abstract setting. In this formulation the controller equations are written in concise manner and necessary features of both, the system and the controller are obtained. Under the additional requirement of structural stability, the necessity of the Internal Model Principle is established. Next, the sufficiency of these features is investigated and conditions under which a structurally stable controller exists are derived. Such conditions are then expressed in terms of the matrices of the original delay system. A method to synthesize a structurally stable controller is also obtained.

Thus, we have widen the class of linear systems for which the Internal Model Principle is valid. In fact, our results are derived in an abstract framework and they are applicable to certain class of evolution systems, provided that the system's operators have similar properties to those of the operators arising from the class of delay systems considered in this thesis. Further research is needed in this area to determine specific classes of systems to which our results are applicable.

Other directions in which this research may be pursued further involve systems with delays in the controls and observations. Also, perturbations of the delay interval must be studied and efficient methods for constructing structurally stable controllers need to be developed.

PART II

FILTERING FOR LINEAR DELAY SYSTEMS

CHAPTER 6

INTRODUCTION

Recently, the study of linear delay differential systems has received considerable attention. Both filtering and optimal control problems have been investigated. In particular, the filtering theory of Kalman and Bucy has been extended to systems with delays in the state and observations [B7], [K4], [K6]. Duality relations between estimation and control have also been obtained [L1] and versions of the separation theorem have been proved for the linear quadratic gaussian problem [L2] , [K5]. However, it seems that none of the available literature has considered the case of delays in the noise process. The occurence of delays in the noise may arise in several ways. In general, taking into account that time delays are inherent in the transport of materials and information, in our actions and in the measurement of variables, we anticipate that the dynamic behaviour of a great number of physical systems may be modeled more adequately by functional differential equations in which the 'forcing terms' themselves are functionals. Such 'forcing terms' may consist of a 'control action' and/or a deterministic stochastic perturbation. For example the 'control action' maybe corrupted by an additive 'white' noise. Systems described by delay differential equations are found in several fields of applied science such as biology, economics, industrial processes, ect. .

The main subject of this work is the optimal filtering problem for linear systems involving delays in the state, observations and in the noise process. We assume that the observations are contaminated by an additive 'white' noise (measurement noise) which is independent of the noise process and without delays. The approach that will be used for solving this problem is based on projection methods in a Hilbert space. We will first show that the filtered estimate satisfies a stochastic functional differential equation which is coupled with the integral equation for the smoothed estimates. Then, a set of partial differential equations satisfied by the error covariance function will be obtained. In the case of no delays in the state and observations we will derive a set of alternative differential equations satisfied by the gains involved in the optimal filter and the uniqueness of solutions to these equations will be established (this results are reported in [M6]). To conclude this work it will be shown that our filtering problem is 'equivalent' to an optimal control problem. The 'dual' system will involve delays in the state and in the controls. The 'dual' problem will consist in minimizing a quadratic functional with delays.

CHAPTER 7

OPTIMAL FILTERING FOR LINEAR SYSTEMS WITH DELAYS IN THE NOISE

This chapter deals with the optimal filtering problem for linear systems with delays in the states, observations and noise process. Our main interest is to obtain a characterization of the optimal filter and derive a set of partial differential equations satisfied by the 'gains' involved in the filter. These questions will be solved in Section 7.3. Next, we consider the class of linear systems involving delays in the noise process only. For such systems we will obtain an alternative characterization of the optimal filter. This will enable us to establish uniqueness of solutions to the set of differential equations satisfied by the 'gains' in the optimal filter. Finally, a dual optimal control problem will be formulated.

7.1 Problem Formulation

Consider the system

$$dx(t) = A_1x(t)dt + A_2x(t-h)dt + B_1dw(t) + B_2dw(t-h), \; t \in [0,T] \tag{7.1}$$

$$dy(t) = C_1x(t)dt + C_2x(t-h)dt + D\,dv(t), \; t \in [0,T] \tag{7.2}$$

$$x(\theta) = x_0(\theta), \; \theta \in [-h,0]$$

$$w(\theta) = w_0(\theta), \; \theta \in [-h,0], \; w(-h) = 0$$

$$y(0) = 0$$

where the vector $x(t)$ takes values in $\mathbb{E}^n$, $y(t)$ in $\mathbb{E}^p$. The noise processes $\{w(s), -h \leq s \leq T\}$ and $\{v(s), 0 \leq s \leq T\}$ are independent standard vector Wiener processes in $\mathbb{E}^m$ and $\mathbb{E}^p$ respevtively. A_1, A_2, B_1, B_2, C_1, C_2 and D are constant matrices of appropriate dimensions.

D is assumed to be nonsingular so that $D D' = R > 0$, i.e. R^{-1} exists. h is a positive constant. $x_0(\theta)$ is a gaussian vector process on $[-h,0]$ completely independent of $\{w(s)\}$ and $\{v(s)\}$, with zero mean and $E[|x_0(\theta)|^2] < \infty \quad \theta \in [-h,0]$ ($|\cdot|$ denotes Euclidean norm). All stochastic processes are defined relative to a given probability space.

We point out that (7.1) - (7.2) must be interpreted as integral equations, since w(t) and v(t) are not differentiable at any point (with probability 1). The integrals throughout this work are defined in the Lebesgue or quadratic mean (stochastic) sense. Moreover, to assure that the Lebesgue integrals of a given stochastic process are well defined, it will be considered in the sequel that a measurable version is used. This is justified as the processes involved are quadratic mean continuous (which is a sufficient condition to assure the existence of a measurable version).

It can be shown, following the arguments in[L1], that (7.1) - (7.2) have a unique sample continuous solution almost surely. The filtering problem for the system (7.1) - (7.2) consists in determining the best estimate of x(t) in the least squares sense, i.e. determine $\hat{x}(t|t) = E[x(t)|\mathcal{Y}^t]$ where $\mathcal{Y}^t$ denotes the σ-algebra generated by the observations $\{y(s), 0 \leq s \leq t\}$. Since all the processes involved are gaussian $\hat{x}(t|t)$ must be a linear functional of past observations. Thus linear estimation methods may be used to obtain $\hat{x}(t|t)$. We mention that if w(t) and v(t), in (7.1) - (7.2), are replaced by any stationary orthogonal increments processes, then it is no longer necessarily true that $E[x(t)|\mathcal{Y}^t]$ is a linear functional of $\{y(s), 0 \leq s \leq t\}$. However, the results presented in this work are still valid if we are only interested in determining the best linear least squares estimate of x(t).

7.2 Notations

H will denote the Hilbert space of square integrable random vectors x, i.e. $E[|x|^2] < \infty$, $|\cdot|$ denotes Euclidean norm. H_t^z will denote the Hilbert space spanned by the process $\{z(s), -h \le s \le t\}$ for $z(s) \in H$ for each s.

P_t^z is defined as a linear map which takes any element in H into its projection onto H_t^z.

$L_2[a,b]$ denotes the space of square integrable functions on [a,b]. A vector function $h(\cdot)$ is said to be an L_2- vector function on [a,b] if $\int_a^b |h(t)|^2 dt < \infty$, $|\cdot|$ is Euclidean norm. Similarly a matrix function $K(\cdot,\cdot)$ is said to be an L_2-kernel on the square $[a,b] \times [a,b]$ if $\int_a^b \int_a^b |K(t,s)|^2 ds\, dt < \infty$, where the norm of a matrix is defined by $|K(t,s)|^2 = \sum_{i,j} k_{ij}(t,s)^2 = \text{trace } K(t,s)K'(t,s)$ (prime stands for transposition).

7.3 The Optimal Filter

As in the case of no time delays in the noise, the equation for the filtered estimate $\hat{x}(t|t)$ will involve some smoothed estimates $\hat{x}(t-\theta|t)$, $\theta > 0$. It will become apparent in later developments that this is also true for systems without delays in the states and observations (section 7.4). Therefore it is convenient to consider the general smoothing problem for the system (7.1) - (7.2).

We define the innovations process corresponding to the observation equation (7.2) to be

$$\nu(t) = y(t) - \int_0^t C_1 \hat{x}(u|u)\, du - \int_0^t C_2 \hat{x}(u-h|u)\, du \tag{7.3}$$

Now, it is readily shown [B7](D[5]) that $\{\nu(s), 0 \leq s \leq t\}$ spans the same family of subspaces as the observations $\{y(s), 0 \leq s \leq t\}$ for each $t \in [0,T]$. Furthermore,

$$E[\nu(t)\nu'(s)] = R\, E[v(t)v'(s)] = R \min(t,s) \tag{7.3a}$$

thus we can write

$$\hat{x}(s|t) = \int_0^t N(s,u)R^{-1}d\nu(u) \tag{7.4}$$

where $N(s,u) = \frac{\partial}{\partial u} E[x(s)\nu'(u)]$ a.e.† is an L_2-kernel measurable in (s,u)

Thus, our problem is reduced to characterizing N(s,u) subject to the dynamics (7.1). Define

$$\tilde{x}(s|t) = x(s) - \hat{x}(s|t)$$

then since x(s) is orthogonal to H_t^ν, it is easy to see from (7.3) that

$$N(s,u) = E[x(s)\tilde{x}'(u|u)]C_1' + E[x(s)\tilde{x}'(u-h|u)]C_2' \quad \text{a.e.} \tag{7.5}$$

Now, let

$$P(t,s,u) \triangleq E[\tilde{x}(t|u)\tilde{x}'(s|u)] \tag{7.6}$$

By the projection theorem, which we recall states that $\tilde{x}(s|u)$ is orthogonal to H_u^ν, we obtain from (7.4) - (7.6)

$$\hat{x}(s|t) = \int_0^t [P(s,u,u)C_1' + P(s,u-h,u)C_2']R^{-1}d\nu(u) \tag{7.7}$$

which clearly may be written as

$$\hat{x}(s|t) = \hat{x}(s|s) + \int_s^t [P(s,u,u)C_1' + P(s,u-h,u)C_2']R^{-1}d\nu(u) \tag{7.8}$$

Note that (7.7) - (7.8) are exactly the same equations that one obtains in the case of no delays in the noise. This fact will be of crucial importance in further developments.

Next we derive a differential equation for the filtered estimate $\hat{x}(t|t)$. Let s = t in (7.5). Integrating (7.1) on [u,t] it is easy to see, using the projection theorem and (7.6), that

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† $u \to E[x(s)\nu'(u)]$ is absolutely continuous [D5], therefore differentiable a.e.

$$N(t,u) = P(u,u,u)C_1' + P(u,u-h,u)C_2' + \int_u^t A_1P(r,u,u)\,drC_1' \quad \text{a.e.}$$
$$+ \int_u^t A_1P(r,u-h,u)\,drC_2' + \int_u^t A_2P(r-h,u,u)\,drC_1' \quad \cdots \quad (7.9)$$
$$+ \int_u^t A_2\,P(r-h,u-h,u)\,drC_2' + E[\int_{u-h}^{t-h} B_2dw(r)\cdot\tilde{x}'(u|u)]C_1'$$
$$+ E[\int_{u-h}^{t-h} B_2dw(r)\cdot\tilde{x}'(u-h)|u)]C_2'$$

here we have used Fubini's theorem to interchange the order of integration. Now consider the last two terms in the right hand side of (7.9). Since, by the projection theorem, $\tilde{x}(r|u) - P_t^w\,\tilde{x}(r|u)$ is orthogonal to H_t^w, we may replace $\tilde{x}(u|u)$ and $\tilde{x}(u-h|u)$, in (7.9), by $P_t^w\,\tilde{x}(u|u)$ and $P_t^w\,\tilde{x}(u-h|u)$ respectively. It can be shown [D5] that

$$P_t^w\,\tilde{x}(r|u) = \int_{-h}^t K(u,r,\sigma)\,dw(\sigma) \tag{7.10}$$

where $K(u,r,\sigma) = \frac{\partial E}{\partial\sigma}[\tilde{x}(r|u)w'(\sigma)]$ a.e. is an L_2-kernel, measurable in (r,u,σ) (since $P_t^w\,\tilde{x}(r|u)$ is a second order quadratic mean continuous process and $E[|x(r)|^2]$ is bounded).

Hence, from (7.4) (with s = t) (7.9) - (7.10) and the properties of Wiener integrals we have

$$\hat{x}(t|t) = \int_0^t [P(u,u,u)C_1' + P(u,u-h,u)C_2']R^{-1}d\nu(u)$$
$$= \int_0^t \int_u^t A_1[P(r,u,u)C_1' + P(r,u-h,u)C_2']R^{-1}dr\;d\nu(u)$$
$$+ \int_0^t \int_u^t A_2[P(r-h,u,u)C_1' + P(r-h,u-h,u)C_2']R^{-1}dr\;d\nu(u) \tag{7.11}$$
$$+ \int_0^t \int_u^t B_2[K'(u,u,r-h)C_1' + K'(u,u-h,r-h)C_2']R^{-1}dr\;d\nu(u)$$

by a Fubini type theorem [D6, p. 431], we may interchange Lebesgue and stochastic integration in (7.11) and from (7.7) (with t = s = r and t = r , s = r = h), (7.11) yields

$$d\hat{x}(t|t) = A_1\hat{x}(t|t)dt + A_2\hat{x}(t-h|t)dt$$
$$+ [P(t,t,t)C_1' + P(t,t-h,t)C_2']R^{-1}d\nu(t) \tag{7.12}$$
$$+ \int_{t-h}^{t} B_2[K'(u,u,t-h)C_1' + K'(u,u-h,t-h)C_2']R^{-1}d\nu(u)dt$$
$$\hat{x}(\theta|\theta) = 0, \qquad \theta \in [-h,0]$$

in the last term of (7.12) we have used the fact†

$$K(u,r,\sigma) = 0 \qquad r \le u < \sigma \quad (\text{or } u \le r < \sigma)$$

Thus the filtered estimate satisfies the stochastic differential equation (7.12). As previously stated the filter equation involves some smoothed estimates of $x(t)$ (note that the innovations process (7.3) itself depends on smoothed estimates).

It remains to characterize $K(t,s,u)$ and $P(t,s,u)$. For this purpose, we shall first derive on alternative representation for $\hat{x}(s|t)$ based on the Projection theorem.

It can be shown [B7], [D5], [D6] that

$$\hat{x}(s|t) = \int_0^t Q(s,r,t)dy(r) \tag{7.13}$$

for some L_2-kernel $Q(s,r,t)$. Now, by the projection theorem, $\tilde{x}(s|t)$ is orthogonal to H_t^y so that for any $0 \le \sigma \le t$

$$E[\tilde{x}(s|t)y'(\sigma)] = 0$$

which in turn gives, using (7.2) and Fubini's theorem

$$\int_0^{\sigma}\{E[\tilde{x}(s|t)x'(r)]C_1' + E[\tilde{x}(s|t)x'(r-h)]C_2'\}dr + E[\tilde{x}(s|t)v'(\sigma)]D' = 0 \qquad .. \tag{7.14}$$

but, by the Projection theorem, $E[\tilde{x}(s|t)x'(\alpha)] = P(s,\alpha,t)$

---

† note that $P_t^w\,\tilde{x}(r|u) = \begin{cases} P_u^w\,\tilde{x}(r|u), & r \le u \le t \\ P_r^w\,\tilde{x}(r|u), & u \le r \le t \end{cases}$

since $\tilde{x}(r|u)$ is independent of $\begin{cases} w(t)-w(u), & r \le u < t \\ w(t)-w(r), & u \le r < t \end{cases}$

Also, it is easy to see from (7.13) (note that $x(s)$ is independent of $v(\sigma)$) that

$$E[\tilde{x}(s|t)v'(\sigma)]D' = -E[\hat{x}(s|t)v'(\sigma)]D'$$

$$= -\int_0^{\sigma} Q(s,r,t)R\,dr \qquad (7.15)$$

thus, (7.14) - (7.15) yield

$$Q(s,r,t) = [P(s,r,t)C_1' + P(s,r-h,t)C_2']R^{-1} \qquad \text{a.e.}$$

hence

$$\hat{x}(s|t) = \int_0^t [P(s,r,t)C_1' + P(s,r-h,t)C_2']R^{-1}dy(r) \qquad (7.16)$$

<u>Characterization of $K(t,s,u)$</u>

We shall first obtain a representation for the covariance

$$E[\tilde{x}(s|t)w'(u)] = E[x(s)w'(u)] - E[\hat{x}(s|t)w'(u)] \qquad (7.17)$$

Let $\Phi(t,s)$ be the fundamental matrix solution associated with the homogeneous part of (7.1). It can be shown [H2] [L1] [D2] [D8], that $\Phi(t,s)$ is bounded on $[0,T] \times [0,T]$, $t \to \Phi(t,s)$ is absolutely continuous for $t > s$, $s \to \Phi(t,s)$ is absolutely continuous for $s < t$ and $\Phi(t,s)$ satisfies

$$\frac{\partial\Phi(t,s)}{\partial t} = A_1\Phi(t,s) + A_2\Phi(t-h,s) \qquad \text{a.e.} \qquad t > s$$

$$\Phi(s,s) = I \qquad (7.18)$$

$$\Phi(t,s) = 0 \qquad , t < s$$

Now, the solution of (7.1) may be written as ($s \geq 0$)

$$x(s) = \Phi(s,0)x_0(0) + \int_{-h}^{0} \Phi(s,u+h)A_2x_0(u)du \qquad (7.19)$$

$$+ \int_0^s \Phi(s,u)B_1dw(u) + \int_{-h}^{s-h} \Phi(s,u+h)B_2dw(u)$$

thus, defining $R(s,u) = E[x(s)w'(u)]$, we obtain $(u \geq -h)$

$$R(s,u) = \begin{cases} \int_{-h}^{\min(s,u)} \Phi(s,\sigma)B_1 X(\sigma)\,d\sigma + \int_{-h}^{\min(s-h,u)} \Phi(s,\sigma+h)B_2\,d\sigma \ , & s \geq 0 \\ 0 \ , & s \leq 0 \end{cases} \tag{7.20}$$

where

$$X(\sigma) = \begin{cases} 1 \ , & \sigma > 0 \\ 0 & \sigma \leq 0 \end{cases}$$

Combining (7.2), (7.16), (7.17), (7.20) and using the fact that $\{w(s)\}$ and $\{v(s)\}$ are independent we have

$$E[\tilde{x}(s|t)w'(u)] = R(s,u) - \int_0^t [P(s,\sigma,t)C_1' + P(s,\sigma-h,t)C_2']R^{-1} \cdot \\ \cdot [C_1 R(\sigma,u) + C_2 R(\sigma-h,u)]\,d\sigma \tag{7.21}$$

By the properties of $\Phi(s,\sigma)$ the kernel $R(s,u)$ is piecewise continuously differentiable with respect to its arguments, therefore for $s \in [-h,T]$

$$\frac{\partial R(s,u)}{\partial u} = \Phi(s,u)B_1 X(u) + \Phi(s,u+h)B_2 \quad \text{a.e. in } u \in [-h,T] \quad \ldots \tag{7.22}$$

here and in the sequel we define $\Phi(s,\cdot) \equiv 0$, for $s < 0$. It now follows that $E[\tilde{x}(s|t)w'(u)]$ is piecewise continuously differentiable with respect to u (in fact, it can be shown that $u \to E[\tilde{x}(s|t)w'(u)]$ is absolutely continuous see the arguments in [D5])

Thus from (7.21) and (7.22) we obtain (using the fact that $\Phi(s,\sigma) = 0$, $s < \sigma$)

$$K(t,s,u) = \Phi(s,u)B_1X(u) + \Phi(s,u+h)B_2$$

$$- \int_u^t [P(s,\sigma,t)C_1' + P(s,\sigma-h,t)C_2']R^{-1}C_1 \, \Phi(\sigma,u)B_1 d\sigma \, X(u)$$

$$- \int_{u+h}^t [P(s,\sigma,t)C_1' + P(s,\sigma-h,t)C_2']R^{-1}C_1 \, \Phi(\sigma,u+h)B_2 d\sigma \qquad \text{a.e.} \quad (7.23)$$

$$- \int_{u+h}^t [P(s,\sigma,t)C_1' + P(s,\sigma-h,t)C_2']R^{-1}C_2\Phi(\sigma-h,u)B_1 d\sigma \, X(u)$$

$$- \int_{u+2h}^t [P(s,\sigma,t)C_1' + P(s,\sigma-h,t)C_2']R^{-1}C_2\Phi(\sigma-h,u+h)B_2 d\sigma$$

We may now write the filter equation (7.12) in more detail. Using (7.23) and since $\Phi(s,t) = 0 \; s < t$, (7.12) yields (note that $P'(t,s,u) = P(s,t,u)$)

$$\begin{aligned} d\hat{x}(t|t) &= A_1\hat{x}(t|t)dt + A_2 \, \hat{x}(t-h|t)dt \\ &+ [P(t,t,t)C_1' + P(t,t-h,t)C_2']R^{-1}d\nu(t) \\ &+ B_2B_1'X(t-h)\int_{t-h}^t \Phi'(u,t-h)C_1'R^{-1}d\nu(u)dt \qquad (7.24) \\ &- B_2B_1'X(t-h)\int_{t-h}^t \int_{t-h}^u \Phi'(\sigma,t-h)C_1'R^{-1}C_1[P(\sigma,u,u)C_1' + \\ &\qquad + P(\sigma,u-h,u)C_2']\cdot R^{-1}d\sigma d\nu(u)dt \\ &- B_2B_1'X(t-h)\int_{t-h}^t \int_{t-h}^u \Phi'(\sigma,t-h)C_1' R^{-1}C_2[P(\sigma-h,u,u)C_1' + \\ &\qquad + P(\sigma-h,u-h,u)C_2']\cdot R^{-1}d\sigma d\nu(u)dt \end{aligned}$$

An inspection of (7.24) and (7.8) (with $s = t-\theta$, $\theta \leq 0$) reveals that the optimal filter is completely characterized by $\hat{x}(t-\theta|t)$ $\theta \in [0,h]$, the error covariance $P(t-\theta_1, t-\theta_2, t)$ $\theta_1, \theta_2 \in [0,2h]$ and the fundamental matrix $\Phi(t-\theta,t-h)$ $\theta \in [0,h]$. Also observe that, on the interval $t \in [0,h]$, (7.24) suggests that the optimal filter behaves as if no delay was present in the noise process. However, we will show later that this is not the case, except in a very particular situation, i.e. $w(s) \equiv 0$ $s \in [-h,0]$ †.

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† if $w(s) \equiv 0$ $s \in [-h,0]$ the second term in (7.22) should be replaced by $\Phi(s,u+h)B_2X(u)$. (7.23) is then modified in an obvious manner. The filter equation (7.24) remains unchanged however.

The last two terms in (7.24) may also be written in terms of smoothed estimates. Indeed, by a Fubini type theorem [D6, p.431], we may interchane Lebesgue and stochastic integration in (7.24). Then using (7.8) we find that

$$\hat{x}(\sigma|t) - \hat{x}(\sigma|\sigma) = \int_{\sigma}^{t} [P(\sigma,u,u)C_1' + P(\sigma,u-h,u)C_2']R^{-1}d\nu(u)$$

and

$$\hat{x}(\sigma-h|t) - \hat{x}(\sigma-h|\sigma) = \int_{\sigma}^{t} [P(\sigma-h,u,u)C_1' + P(\sigma-h,u-h,u)C_2']R^{-1}d\nu(u)$$

inserting these expressions in (7.24) (after changing the order of integration in the last two terms) and using (7.3) we finally obtain

$$\begin{aligned} d\hat{x}(t|t) &= \{A_1-[P(t,t,t)C_1' + P(t,t-h,t)C_2']R^{-1}C_1\}\hat{x}(t|t)dt \qquad (7.25) \\ &+ \{A_2-[P(t,t,t)C_1' + P(t,t-h,t)C_2']R^{-1}C_2\}\hat{x}(t-h|t)dt \\ &- B_2B_1'X(t-h) \int_{t-h}^{t} \Phi'(\sigma,t-h)C_1'R^{-1}[C_1\hat{x}(\sigma|t)+ C_2\hat{x}(\sigma-h|t)]d\sigma dt \\ &+ [P(t,t,t)C_1' + P(t,t-h,t)C_2']R^{-1}dy(t) \\ &+ B_2B_1'X(t-h) \int_{t-h}^{t} \Phi'(\sigma,t-h)C_1'R^{-1}dy(\sigma)dt \end{aligned}$$

and from (7.16)

$$\hat{x}(t+\sigma|t) = \hat{x}(t+\sigma|t+\sigma) + \int_{t+\sigma}^{t} [P(t+\sigma,r,t)C_1' + P(t+\sigma,r-h,t)C_2']R^{-1}dy(r) \quad .. \quad (7.26)$$

In contrast with equation (7.24), the above representation of the optimal filter requires the smoothed estimates $\hat{x}(t-\theta|t)$ $\theta \in [0,2h]$. However, if no delays occur in the observations, i.e $C_2 \equiv 0$, then (in both representations) we need only to compute $\hat{x}(t-\theta|t)$, $\Phi(t-\theta,t-h)$ $\theta \in [0,h]$ and $P(t-\theta_1,t-\theta_2,t)$ $\theta_1, \theta_2 \in [0,h]$. In this case (7.25) and (7.24) may be written as delay differential equations. Furthermore, when no delays are present in the state, i.e. $A_2 \equiv 0$, then (7.25) shows that the

the optimal filter still involves some smoothed estimates of x(t). This special case will be treated in more detail in Section 7.4.

Characterization of P(t,s,u)

We will first derive three integral equations for P(t,s,u). By definition

$$P(t,s,u) = E[\tilde{x}(t|u)\tilde{x}'(s|u)] \tag{7.27}$$

by the projection theorem (7.27) gives

$$P(t,s,u) = E[x(t)x'(s)] - E[\hat{x}(t|u)\hat{x}'(s|u)] \tag{7.28}$$

defining $M(t,s) = E[x(t)x'(s)]$ and using (7.3a), (7.7), we obtain from (7,28) (note that $P'(t,s,u) = P(s,t,u)$)

$$P(t,s,u) = M(t,s) - \int_0^u [P(t,\sigma,\sigma)C_1' + P(t,\sigma-h,\sigma)C_2']R^{-1}\cdot [C_1P(\sigma,s,\sigma) + C_2P(\sigma-h,s,\sigma)]d\sigma \tag{7.29}$$

To obtain the second equation for P(t,s,u) we now use the representation (7.16)for $\hat{x}(t|u)$ in place of (7.7). First notice that, by the projection theorem, (7.27) may be written as

$$P(t,s,u) = E[x(t)x'(s)] - E[\hat{x}(t|u)x'(s)] \tag{7.30}$$

which in turn gives, since x(u) is independent of v(u)

$$P(t,s,u) = M(t,s) - \int_0^u [P(t,\sigma,u)C_1' + P(t,\sigma-h,u)C_2']R^{-1}\cdot [C_1M(\sigma,s) + C_2M(\sigma-h,s)]d\sigma \tag{7.31}$$

The third integral equation is obtained in a similar manner (also note that $M'(s,t) = M(t,s)$).

$$P(t,s,u) = M(t,s) - \int_0^u [M(t,\sigma)C_1' + M(t,\sigma-h)C_2']R^{-1}\cdot [C_1P(\sigma,s,u) + C_2P(\sigma-h,s,u)]d\sigma \tag{7.32}$$

We point out that (7.31) - (7.32) have been previously derived by Kwong [K5] for linear system without delays in the noise. The fact that these equations are also valid in the case of delays in the noise process is a direct consequence of the smoothing equations (7.7) - (7.16) (which also hold in the case of no delays in the noise process). We also mention that (7.31) - (7.32) may be written as Fredholm integral equations. Indeed, along the lines of [K5], (7.31) and (7.32) yield

$$P(t,s,u) = M(t,s) - \int_{-h}^{u} P(t,\sigma,u)W(\sigma,s)d\sigma \tag{7.31a}$$

$$P(t,s,u) = M(t,s) - \int_{-h}^{u} W'(\sigma,t)P(\sigma,s,u)d\sigma \tag{7.32a}$$

where

$$W(\sigma,s) = [C_1'R^{-1}C_1M(\sigma,s) + C_1'R^{-1}C_2M(\sigma-h,s)]X_{[0,u]}(\sigma)$$

$$+ [C_2'R^{-1}C_1M(\sigma+h,s) + C_2'R^{-1}C_2M(\sigma,s)]X_{[-h,u-h]}(\sigma)$$

$$X_{[s,t]}(\sigma) = \begin{cases} 1 & s \leq \sigma \leq \\ 0 & \text{otherwise} \end{cases}$$

Thus, for fixed s and u, (7.32a) is a Fredholm integral equation for P(t,s,u) in t. We may apply standard Fredholm theory to conclude that (7.32a) has a unique L_2-solution P(t,s,u). Furthermore, it can be shown [K6] that P(t,s,u) is continuous in its arguments.

Having established the integral equations (7.30) - (7.32) we now show that P(t,s,u) is piecewise continuously differentiable with respect to its arguments.

From (7.29) it is easy to see that

$$\frac{\partial P(t,s,u)}{\partial u} = - [P(t,u,u)C_1' + P(t,u-h,u)C_2']R^{-1} \cdot [C_1P(u,s,u) + C_2P(u-h,s,u)] \quad \text{a.e.} \tag{7.33}$$

To calculate the partial derivates of P(t,s,u) with respect to t and s we first summarize some properties of M(t,s). It is readily verified, using the variation of constants formula that for $t \geq 0$, $s \geq 0$

$$\begin{aligned} M(t,s) = {} & \Phi(t,0)E[x_0(0)x_0'(0)]\Phi'(s,0) \\ & + \int_{-h}^{0} \Phi(t,0)E[x_0(0)x_0'(u)]A_2'\Phi'(s,u+h)\,du \\ & + \int_{-h}^{0} \Phi(t,u+h)A_2E[x_0(u)x_0'(0)]\Phi'(s,0)\,du \\ & + \int_{-h}^{0}\int_{-h}^{0} \Phi(t,u+h)A_2E[x_0(u)x_0'(\sigma)]A_2'\Phi'(s,\sigma+h)\,d\sigma du \\ & + \int_{0}^{\min(t,s)} \Phi(t,u)[B_1B_1' + B_2B_2']\Phi'(s,u)\,du \\ & + \int_{0}^{\min(t,s-h)} \Phi(t,u)B_1B_2'\Phi'(s,u+h)\chi(u)\,du \\ & + \int_{0}^{\min(t-h,s)} \Phi(t,u+h)B_2B_1'\Phi'(s,u)\chi(u)\,du \end{aligned} \qquad (7.34)$$

and

$$M(t,s) = E[x_0(t)x_0'(s)] \qquad t,s \in [-h,0] \qquad (7.34a)$$

It now follows that M(t,s) is continuous in t and s. Moreover, M(t,s) is piecewise continuously differentiable with respect to t and s (except at a finite number of lines). By the porperties of $\Phi(t,s)$ we obtain from (7.34)

$$\begin{aligned} \frac{\partial M(t,s)}{\partial t} = {} & A_1M(t,s) + A_2M(t-h,s) + [B_1B_1'+B_2B_2']\Phi'(s,t) \\ & + B_1B_2'\Phi'(s,t+h)\chi(t) + B_2B_1'\Phi'(s,t-h)\chi(t-h) \end{aligned} \qquad \text{a.e.} \quad (7.35)$$

$$\begin{aligned} \frac{\partial M(t,s)}{\partial s} = {} & M(t,s)A_1' + M(t,s-h)A_2' + \Phi(t,s)[B_1B_1' + B_2B_2'] \\ & + \Phi(t,s-h)B_1B_2'\chi(s-h) + \Phi(t,s+h)B_2B_1'\chi(s) \end{aligned} \qquad \text{a.e} \quad (7.36)$$

$$\frac{dM(t,t)}{dt} = A_1 M(t,t) + M(t,t)A_1' + A_2 M(t-h,t) + M(t,t-h)A_2'$$
$$+ B_1 B_1' + B_2 B_2' + \Phi(t,t-h)B_1 B_2' X(t-h) + B_2 B_1' \Phi'(t,t-h)X(t-h)$$
$$\text{.. a.e.} \quad (7.37)$$

Now, from (7.32) - (7.35) we finally obtain

$$\frac{\partial P(t,s,u)}{\partial t} = A_1 P(t,s,u) + A_2 P(t-h,s,u)$$
$$\text{a.e.} \quad (7.38)$$
$$+ [B_1 B_1' + B_2 B_2']S(t,s,u)$$
$$+ B_1 B_2' \; S(t+h,s,u)X(t) + B_2 B_1' \; S(t-h,s,u)X(t-h)$$

where

$$S(t,s,u) = \Phi'(s,t) - \int_0^u [\Phi'(\sigma,t)C_1' + \Phi'(\sigma-h,t)C_2']R^{-1} \cdot$$
$$(7.39)$$
$$\cdot [C_1 P(\sigma,s,u) + C_2 P(\sigma-h,s,u)]d\sigma$$

note that

$$S(t,s,u) = \begin{cases} I , & s = u = t \\ 0 , & s < u \leq t \end{cases} \quad (7.39a)$$

Similarly, from (7.31) and (7.36) we obtain

$$\frac{\partial P(t,s,u)}{\partial s} = P(t,s,u)A_1' + P(t,s-h,u)A_2'$$
$$\text{a.e.} \quad (7.40)$$
$$+ S'(s,t,u)[B_1 B_1' + B_2 B_2']$$
$$+ S'(s-h,t,u)B_1 B_2' X(s-h) + S'(s+h,t,u)B_2 B_1' X(s)$$

As previously discussed, we need to specify the error covariance function $P(t-\theta_1,t-\theta_2,t)$ θ_1, θ_2 $\varepsilon[0,2h]$ for each t $\varepsilon[h,T]$ and θ_1, θ_2 $\varepsilon[0,h]$ for t $\varepsilon[0,h]$. Therefore it is convenient to characterize $P(t-\theta_1,t-\theta_2,t)$ by its derivatives with respect to t, θ_1, and θ_2.

From (7.29) we have

$$P(t-\theta_1,t-\theta_2,t) = M(t-\theta_1,t-\theta_2) - \int_0^t [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1} \cdot$$

$$\cdot [C_1(P(\sigma,t-\theta_2,\sigma) + C_2P(\sigma-h,t-\theta_2,\sigma)]d\sigma \qquad .. \quad (7.41)$$

Now, it is easy to verify that

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right\}M(t-\theta_1-t-\theta_2) = 0 \qquad (7.41a)$$

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_1}\right\}P(t-\theta_1,u,v) = 0 \qquad (7.41b)$$

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_2}\right\}P(u,t-\theta_2,v) = 0 \qquad (7.41c)$$

thus (7.41) gives

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right\}P(t-\theta_1,t-\theta_2,t) = -[P(t-\theta_1,t,t)C_1' + P(t-\theta_1,t-h,t)C_2']R^{-1} \cdot$$

$$\cdot [C_1P(t,t-\theta_2,t) + C_2P(t-h,t-\theta_2,t)] \qquad \text{a.e.} \quad (7.42)$$

We point out that (7.42) is also valid in the case of no delays in the noise process.

Now, set $\theta_2 = 0$ in (7.41), then it is easy to see that

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_1}\right\}M(t-\theta_1,t) = M(t-\theta_1,t)A_1' + M(t-\theta_1,t-h)A_2' + \Phi(t-\theta_1,t-h)B_1B_2'X(t-h) \qquad \text{a.e.} \quad (7.43)$$

Combining (7.40), (7.41b), (7.43), (7.29) and (7,39a) we obtain

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_1}\right\}P(t-\theta_1,t,t) = P(t-\theta_1,t,t)A_1' + P(t-\theta_1,t-h,t)A_2'$$

$$- [P(t-\theta_1,t,t)C_1' + P(t-\theta_1,t-h,t)C_2']R^{-1} \cdot$$

$$\cdot [C_1P(t,t,t) + C_2P(t-h\ ,t,t)] +$$

$$+ \Phi(t-\theta_1, t-h)B_1B_2'X(t-h) \qquad \text{a.e.} (7.44)$$
$$- L(t-\theta_1, t-h, t)B_1B_2'X(t-h)$$

where

$$L(t-\theta_1, t-h, t) = \int_0^t [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1} \cdot$$
$$\cdot [C_1S'(t-h,\sigma,\sigma) + C_2S'(t-h,\sigma-h,\sigma)]d\sigma \qquad (7.45)$$

using (7.39) and the fact that $\Phi(t,s) = 0$ $s > t$, this expression yields

$$L(t-\theta_1, t-h, t) = \int_{t-h}^t [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1}C_1\Phi(\sigma,t-h)d\sigma$$
$$- \int_{t-h}^t \int_{t-h}^\sigma [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1} \cdot$$
$$\cdot [C_1P(\sigma,\zeta,\sigma) + C_2P(\sigma-h,\zeta,\sigma)]C_1'R^{-1}C_1\Phi(\zeta,t-h)d\zeta d\sigma$$
$$- \int_{t-h}^t \int_{t-h}^\sigma [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1} \cdot$$
$$\cdot [C_1P(\sigma,\zeta-h,\sigma) + C_2P(\sigma-h,\zeta-h,\sigma)]C_2'R^{-1}C_1\Phi(\zeta,t-h)d\zeta d\sigma$$

changing the order of integration in the second and third terms of this expression and using the identities (which easily obtained from (7.29))

$$P(t-\theta_1,\zeta,\zeta) - P(t-\theta_1,\zeta,t) = \int_\zeta^t [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1} \cdot$$
$$\cdot [C_1P(\sigma,\zeta,\sigma) + C_2P(\sigma-h,\zeta,\sigma)]d\sigma$$

and

$$P(t-\theta_1,\zeta-h,\zeta) - P(t-\theta_1,\zeta-h,\zeta) = \int_\zeta^t [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1} \cdot$$
$$\cdot [C_1P(\sigma,\zeta-h,\sigma) + C_2\ P(\sigma-h,\zeta-h,\sigma)]d\sigma$$

we obtain

$$L(t-\theta_1,t-h,t) = \int_{t-h}^{t} [P(t-\theta_1,\sigma,\sigma)C_1' + P(t-\theta_1,\sigma-h,\sigma)C_2']R^{-1}C_1\Phi(\sigma,t-h)d\sigma$$

$$- \int_{t-h}^{t} [P(t-\theta_1,\zeta,\zeta) - P(t-\theta_1,\zeta, t)]C_1'R^{-1}C_1\Phi(\zeta,t-h)d\zeta$$

$$- \int_{t-h}^{t} [P(t-\theta_1,\zeta-h,\zeta) - P(t-\theta_1,\zeta-h,t)]C_2'R^{-1}C_1\Phi(\zeta,t-h)d\zeta$$

which in turn gives

$$L(t-\theta_1,t-h,t) = \int_{t-h}^{t} [P(t-\theta_1,\zeta,t)C_1' + P(t-\theta_1,\zeta-h,t)C_2']R^{-1}C_1\Phi(\zeta,t-h)d\zeta$$

hence

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial\theta_1}\right)P(t-\theta_1,t,t) = P(t-\theta_1,t,t)\{A_1' - C_1'R^{-1}[C_1P(t,t,t) + C_2P(t-h,t,t)]\}$$

$$+ P(t-\theta_1,t-h,t)\{A_2'-C_2'R^{-1}[C_1P(t,t,t)+C_2P(t-h,t,)]\}$$

$$+ \Phi(t-\theta_1,t-h)B_1B_2'X(t-h) \qquad \text{a.e.} \qquad (7.46)$$

$$- \int_{t-h}^{t} [P(t-\theta_1,\sigma,t)C_1' + P(t-\theta_1,\sigma-h,t)C_2']R^{-1}\cdot$$

$$\cdot\ C_1\Phi(\sigma,t-h)d\sigma \cdot B_1B_2'X(t-h)$$

We mention that on the interval $t \in [0,h]$, (7.46) is also satisfied if no delays occur in the noise.

Finally we derive a differential equation for $P(t,t,t)$. Setting $\theta_1 = \theta_2 = 0$ in (7.41) and combining (7.29), (7.37) - (7.40) we find

$$\frac{dP(t,t,t)}{dt} = A_1P(t,t,t) + P(t,t,t)A_1' + A_2\,P(t-h,t,t) + P(t,t-h,t)A_2'$$

$$- [P(t,t,t)C_1' + P(t,t-h,t)C_2']R^{-1}[C_1P(t,t,t) + C_2P(t-h,t,t)]$$

$$+ B_1B_1' + B_2\,B_2' + V(t,t-h)B_1\,B_2'\,X(t-h) + B_2\,B_1'\,V'(t,t-h)X(t-h)$$

$$\text{a.e.} \qquad (7.47)$$

where

$$V'(t,t-h) = \Phi'(t,t-h) - \int_{t-h}^{t} \Phi'(\sigma,t-h)C_1'R^{-1}[C_1P(\sigma,t,\sigma) + C_2P(\sigma-h,t,\sigma)]d\sigma$$

$$+ \int_{t-h}^{t}\int_{t-h}^{\sigma} \Phi'(\theta,t-h)C_1'R^{-1}C_1[P(\theta,\sigma,\sigma)C_1' +$$

$$+ P(\theta,\sigma-h,\sigma)C_2']R^{-1} \cdot$$

$$\cdot [C_1P(\sigma,t,\sigma) + C_2P(\sigma-h,t,\sigma)]d\theta d\sigma$$

$$+ \int_{t-h}^{t}\int_{t-h}^{\sigma} \Phi'(\theta,t-h)C_1'R^{-1}C_2[P(\theta-h,\sigma,\sigma)C_1' +$$

$$+ P(\theta-h,\sigma-h,\sigma)C_2']R^{-1} \cdot$$

$$\cdot [C_1P(\sigma,t,\sigma) + C_2P(\sigma-h,t,\sigma)]d\theta d\sigma$$

.. (7.48)

changing the order of integration in the last two terms of (7.48) and using the identities (which are obtained form (7.29))

$$P(\theta,t,\theta) = P(\theta,t,t) = \int_{\theta}^{t}[P(\theta,\sigma,\sigma)C_1' + P(\theta,\sigma-h,\sigma)C_2']R^{-1} \cdot$$

$$\cdot [C_1P(\sigma,t,\sigma) + C_2P(\sigma-h,t,\sigma)]\, d\sigma$$

and

$$P(\theta-h,t,\theta) - P(\theta-h,t,t) = \int_{\theta}^{t} [P(\theta-h,\sigma,\sigma)C_1' + P(\theta-h,\sigma-h,\sigma)C_2']R^{-1} \cdot$$

$$\cdot [C_1P(\sigma,t,\sigma) + C_2P(\sigma-h,t,\sigma)]d\sigma$$

(7.48) gives

$$V'(t,t-h) = \Phi'(t,t-h) - \int_{t-h}^{t} \Phi^{1}(\sigma,t-h)C_1'R^{-1}[C_1P(\sigma,t,\sigma) + C_2P(\sigma-h,t,\sigma)]d\sigma$$

$$+ \int_{t-h}^{t} \Phi'(\theta,t-h) C_1' R^{-1} C_1 [P(\theta,t,\theta) - P(\theta,t,t)] d\theta$$

$$+ \int_{t-h}^{t} \Phi'(\theta,t-h) C_1' R^{-1} C_2 [P(\theta-h,t,\theta) - P(\theta-h,t,t)] d\theta$$

this in turn yields

$$V'(t,t-h) = \Phi'(t,t-h) - \int_{t-h}^{t} \Phi'(\theta,t-h) C_1' R^{-1} [C_1 P(\theta,t,t) + C_2 P(\theta-h,t,t)] d\theta$$

therefore

$$\begin{aligned} \frac{dP(t,t,t)}{dt} &= A_1 P(t,t,t) + P(t,t,t) A_1' + A_2 P(t-h,t,t) + P(t,t-h,t) A_2' \\ &- [P(t,t,t) C_1' + P(t,t-h,t) C_2'] R^{-1} [C_1 P(t,t,t) + C_2 P(t-h,t,t)] \\ &+ B_1 B_1' + B_2 B_2' + \Phi(t,t-h) B_1 B_2' X(t-h) + B_2 B_1' \phi'(t,t-h) X(t-h) \\ &- \int_{t-h}^{t} [P(t,\theta,t) C_1' + P(t,\theta-h,t) C_2'] R^{-1} C_1 \Phi(\theta,t-h) d\theta \cdot B_1 B_2' X(t-h) \\ &- B_2 B_1' \int_{t-h}^{t} \Phi'(\theta,t-h) C_1' R^{-1} [C_1 P(\theta,t,t) + C_2 P(\theta-h,t,t)] d\theta \cdot X(t-h) \end{aligned}$$

$$\text{a.e.} \qquad (7.49)$$

We point out that if $w(s) = s \; \varepsilon[-h,0]$, then the term $B_2 B_2'$ in (7.49) should be replaced by $B_2 B_2' X(t-h)$. This implies that on the interval $t \; \varepsilon[0,h]$, (7.49) coincides with the corresponding equation in the case of the no delays in the noise.

We may now summarize the main result of this section

Theorem 7.1: The filtered estimate $\hat{x}(t|t)$ for the system (7.1) - (7.2) satisfies the following equations $t \; \varepsilon[0,T]$:

$$d\hat{x}(t|t) = A_1\hat{x}(t|t)dt + A_2\hat{x}(t-h|t)dt + [P(t,t,t)C_1' +$$

$$+ P(t,t-h,t)C_2']R^{-1}d\nu(t)$$

$$+ B_2\ B_1'X(t-h)\int_{t-h}^{t} \Phi'(u,t-h)C_1'\ R^{-1}d\nu(u)dt$$

$$- B_2\ B_1'X(t-h)\int_{t-h}^{t}\int_{t-h}^{u} \Phi'(\sigma,t-h)C_1'R^{-1}\{C_1[P(\sigma,u,u)C_1' +$$

$$+ P(\sigma,u-h,u)C_2'] + C_2[P(\sigma-h,u,u)C_1'+P(\sigma-h,u-h,u)C_2']\}R^{-1}d\sigma d\nu(u)dt$$

$$\hat{x}(t-\theta|t) = \hat{x}(t-\theta|t-\theta) + \int_{t-\theta}^{t} [P(t-\theta,\sigma,\sigma)C_1'+P(t-\theta,\sigma-h,\sigma)C_2']R^{-1}d\nu(\sigma)$$

$$\hat{x}(\theta|0) = 0 \quad , \theta \ \varepsilon[-h,0]$$

where
$$X(\sigma) = \begin{cases} 1, & \sigma > 0 \\ 0, & \sigma \leq 0 \end{cases}$$

$$\frac{\partial\Phi(t,s)}{\partial t} = A_1\Phi(t,s) + A_2\Phi(t-h,s) \qquad \text{a.e.} \qquad t > s$$

$$\Phi(t,s) = \begin{cases} I, & t = s \\ 0, & t < s \end{cases}$$

and the error covariance matrix function $P(t-\theta_1, t-\theta_2, t)$ satisfies equations (7.42), (7.46) and (7.49) almost everywhere with $P(\theta_1,\theta_2,0) = E[x_0(\theta_1)x_0'(\theta_2)]$, θ_1, θ_2 $\varepsilon[-h,0]$.

7.4 Systems with delays in the noise process only

In this section we will specialize the results of Section 7.3 to linear systems involving delays in the noise process only. In particular, we will establish the uniqueness of solutions to the differential equations satisfied by the 'gains' involved in the optimal filter. We first prove the following

Theorem 7.2: Let $A_2 = 0$, $C_2 = 0$, $A_1 = A$ and $C_1 = C$. Then the filtered estimate $x(t\ t)$ for the system (7.1) - (7.2) satisfies the stochastic functional differential equation

$$d\hat{x}(t|t) = A\hat{x}(t|t)dt + P(t)C'R^{-1}d\nu(t) + B_2\ B_1' X(t-h)\int_{t-h}^{t} \Psi'(s,t-h)C'R^{-1}d\nu(s)dt \qquad t\ \varepsilon[0,T] \tag{7.50}$$

$$\hat{x}(0|0) = 0$$

where $P(t) = P(t,t,t)$ is the error covariance matrix, $\Psi(t,s)$ is the fundemental matrix solution associated with the homogeneous part of the error differential equation and $X(s)$ is the step function previously defined. Furthermore $P(t)$ and $\Psi(t,s)$ satisfy two coupled Riccati-type differential equations

$$\frac{dP(t)}{dt} = A\ P(t) + P(t)A' - P(t)C'R^{-1}CP(t) + B_1B_1' + B_2B_2' + X(t-h)[\Psi(t,t-h)B_1\ B_2' + B_2\ B_1'\Psi'(t,t-h)] \quad t \geq 0 \ \text{a.e.} \quad .. \tag{7.51}$$

$$\frac{\partial\Psi(t,s)}{\partial t} = [A-P(t)C'R^{-1}C]\Psi(t,s) - B_2B_1'\ X(t-h)\int_{t-h}^{t} \Psi'(u,t-h)C'R^{-1}C\ \Psi(u,s)du \quad t > s \geq 0 \ \text{a.e} \quad .. \tag{7.52}$$

$$P(0) = E[x_0(0)x_0'(0)]\ , \quad \Psi(t,s) = \begin{cases} I, & t = s \\ 0, & t < s \end{cases}$$

Proof: Setting $A_2 = 0$ and $C_2 = 0$, (7.12) shows that we only need to characterize $P(t) = P(t,t,t)$ and $K(u,t-h) = K(u,u,t-h)$ $u\ \varepsilon[t-h,t]$. It now follows that

$$K(u,t-h) = \begin{cases} \Phi(u,t-h)B_1X(t-h) & , t-h \le u < t \\ \Phi(u,t-h)B_1X(t-h) + \Phi(u,t)B_2 & , t \le u \end{cases} \qquad \text{a.e.} \quad (7.53)$$

$$+$$

$$\begin{cases} \int_{t-h}^{u} P(u,\sigma,u)C'R^{-1}C\ \Phi(\sigma,t-h)B_1 d\sigma\ X(t-h) & , t-h \le u < t \\ \int_{t-h}^{u} P(u,\sigma,u)C'R^{-1}C\ \Phi(\sigma,t-h)B_1 d\sigma\ X(t-h) \\ \qquad + \int_{t}^{u} P(u,\sigma,u)C'\ R^{-1}C\ \Phi(\sigma,t)B_2 d\sigma & , \quad t \le u \end{cases}$$

(note that $\Phi(t,s) \neq 0$, $t < s$; this fact cpmplicates some calculations compare (7.53) with (7.23) for $t = s = u$ and $u = t-h$) .

Now, from (7.53) (with $u = t$) (7.49) may be written as ($A_2 = 0$, $C_2 = 0$)

$$\frac{dP(t)}{dt} = AP(t) + P(t)A' - P(t)C'R^{-1}C\ P(t) + B_1B_1' + K(t,t-h)B_2' + B_2\ K'(t,t-h) - B_2\ B_2' \qquad \text{a.e.} \quad (7.54)$$

Thus the optimal filter is completely characterized by (7.12) and (7.54) in terms of $P(t)$ and $K(u,t-h)$ $u \in [t-h,t]$. We next show that $K(u,t-h)$ may be represented in terms of the fundamental solution associated with the homogeneous part of the error differential equation

$$\begin{aligned} d\tilde{x}(t|t) = {} & [A-P(t)C'R^{-1}C]\tilde{x}(t|t)dt - B_2\int_{t-h}^{t} K'(s,t-h)C'R^{-1}C\tilde{x}(s|s)ds\ dt \\ & + B_1 dw(t) + B_2 dw(t-h) - P(t)C'R^{-1}dv(t) \\ & - B_2\int_{t-h}^{t} K'(s,t-h)C'R^{-1}dv(s)dt \end{aligned} \qquad (7.55)$$

$$\tilde{x}(0|0) = x_0(0) \quad , \quad \tilde{x}(s|s) = 0 \quad s < 0$$

Let $\Psi(t,s)$ be the fundamental matrix associated with (7.55). It can be shown [L1], [H2] that $\Psi(t,s)$ is bounded on $[0,T] \times [0,T]$, $t \to \Psi(t,s)$ is absolutely continuous, $s \to \Psi(t,s)$ is of bounded variation and $\Psi(t,s)$ satisfies

$$\frac{\partial \Psi(t,s)}{\partial t} = [A-P(t)C'R^{-1}C]\Psi(t,s) - B_2 \int_s^t K'(\sigma,t-h)C'R^{-1}C\,\Psi(\sigma,s)d\sigma$$

$$\text{a.e.} \quad t > s \geq 0 \qquad (7.56)$$

$$\Psi(t,s) = \begin{cases} I, & t = s \\ 0, & t < s \end{cases}$$

the solution of (7.55) is now given by (for $s \geq 0$)

$$\tilde{x}(s|s) = \Psi(s,0)x_0(0) + \int_0^s \Psi(s,\sigma)B_1 dw(\sigma) + \int_{-h}^{s-h} \Psi(s,\sigma+h)B_2 dw(\sigma)$$

$$- \int_0^s \Psi(s,\sigma)P(\sigma)C'R^{-1}dv(\sigma) - \int_0^s\int_0^\sigma \Psi(s,\sigma)B_2K'(r,\sigma-h)C'R^{-1}dv(r)d\sigma$$

$$.. \qquad (7.57)$$

here we have used the fact that $K(r,\sigma-h) = 0$ $r < \sigma-h$. It now follows, since $w(u) = \int_{-h}^u dw(r)$, $\Psi(t,s) = 0$ $t < s$ and by the properties of Wiener integrals, that

$$E[\tilde{x}(s|s)w'(u)] = \int_{-h}^{u} [\Psi(s,\sigma)B_1X(\sigma) + \Psi(s,\sigma+h)B_2]d\sigma$$

hence

$$K(s,u) = \frac{\partial}{\partial u}[\tilde{x}(s|s)w'(u)]$$

$$= \Psi(s,u)B_1X(u) + \Psi(s,u+h)B_2 \qquad \text{a.e.} \qquad (7.58)$$

Combining (7.12) and (7.58) and noting that $K(u,t-h)$ and $\Psi(u,t-h)B_1X(t-h)$ are equivalent L_2-kernels on the region of integration, i.e.

$$\int_{t-h}^t \Psi(u,t)B_2\, B_2'\Psi'(u,t)du = 0$$

we obtain (7.50). Similarly, (7.52) is obtained from (7.56) and (7.58). Finally, (7.51) follows easily from (7.54) and (7.58) with $s = t$ and $u = t-h$. This completes the proof.

The existence and uniqueness of solutions to (7.50) can be established using standard arguments (see [L1]). In the remainder if this section we will establish the uniqueness of solutions to (7.51) - (7.52).

Theorem 7.3: The system of equations (7.51) - (7.52) have a unique solution $P(t)$, $\Psi(t,s)$ in the class of symmetric matrix functions $P(\cdot)$ which are absolutely continuous, and matrix functions $\Psi(t,s)$ which are locally absolutely continuous in $t \in [s,\infty)$ for each $s \geq 0$ and of bounded variation in $s \in [0,t]$ for each t.

Proof: Clearly $\Phi(t,s)$ and $\Psi(t,s)$ are bounded on $[0,T] \times [0,T]$ and $P(t)$ is bounded, i.e. $|P(t)| \leq E[|x(t)|^2] < \infty$. It then follows that

$$P(t) = E[x(t)x'(t)] - E[\hat{x}(t|t)\hat{x}'(t|t)]$$

is Lipschitz continuous in t. Hence, $P(t)$ is absolutely continuous and we may integrate (7.51) to calculate the error covariance function.

Now, consider (7.51) - (7.52) on the interval $t \in [0,h]$. Since these expressions are decoupled on this interval, we may conclude (using standard arguments) that $P(t)$ and $\Psi(t,s)$ are unique for $t \in [0,h]$ and $s \in [0,t]$. We only need to show that the solution of (7.51) - (7.52) is unique for $t > h$.

Let $\{P(t), \Psi(t,s)\}$ be a solution to (7.51) - (7.52) and suppose that $\{P^*(t), \Psi^*(t,s)\}$ is another solution. Define

$$Q(t) = P(t) - P^*(t)$$

and

$$\Delta(t,s) = \Psi(t,s) - \Psi^*(t,s)$$

then $Q(t)$ and $\Delta(t,s)$ satisfy

$$\frac{dQ(t)}{dt} = [A-P(t)C'R^{-1}C]Q(t) + [A-P^*(t)C'R^{-1}C]'$$

$$+ X(t-h)[B_2B_1'\Delta'(t,t-h) + \Delta(t,t-h)B_1B_2'] \qquad \text{a.e} \qquad (7.59)$$

$$\frac{\partial\Delta(t,s)}{\partial t} = [A-P(t)C'R^{-1}C]\Delta(t,s) - Q(t)C'R^{-1}C\Psi^*(t,s)$$

$$- B_2B_1'X(t-h)\int_{t-h}^{t} \Psi'(\sigma,t-h)C'R^{-1}C\Delta(\sigma,s)\,d\sigma$$

$$- B_2B_1'X(t-h)\int_{t-h}^{t} \Delta'(\sigma,t-h)C'R^{-1}C\Psi^*(\sigma,s)\,d\sigma \qquad \text{a.e.} \qquad (7.60)$$

$$Q(0) = 0, \quad \Delta(t,s) = 0 \qquad t \le s$$

from the above expressions we obtain $Q(t) \equiv 0$, $t \in [0,h]$ and $\Delta(t,s) \equiv 0$ on the square $[0,h] \times [0,h]$. Now, for $t \ge h$ (7.59) and (7.60) are equivalent to

$$Q(t) = \int_h^t \Phi_1(t,\sigma)[B_2B_1'\,\Delta(\sigma,\sigma-h)B_1B_2']\Phi_2'(t,\sigma)\,d\sigma \qquad (7.61)$$

$$\Delta(t,s) = -\int_s^t \Psi(t,\sigma)Q(\sigma)C'R^{-1}C\Psi^*(\sigma,s)\,d\sigma$$

$$-\int_s^t\int_\sigma^t \Psi(t,u)B_2B_1'\Delta'(\sigma,u-h)C'R^{-1}C\Psi^*(\sigma,s)X(u-h)\,du\,d\sigma \qquad (7.62)$$

where $\Phi_1(t,s)$ and $\Phi_2(t,s)$ are the transition matrices associated with $A-P(t)C'R^{-1}C$ and $A-P^*(t)C'R^{-1}C$ respectively. Now, consider $h \le t \le t_1$ and $0 \le s \le t_1$ and let

$$\|Q\| = \sup_{h\le\sigma\le t_1} |Q(\sigma)| = \sup_{0\le\sigma\le t_1} |Q(\sigma)|$$

$$\|\Delta\| = \sup_{h\le\sigma\le t_1}\ \sup_{0\le u\le t_1} |\Delta(\sigma,u)| = \sup_{0\le\sigma\le t_1}\ \sup_{0\le u\le t_1} |\Delta(\sigma,u)|$$

From (7.61) we obtain, for some $K < \infty$

$$|Q(t)| \leq K \int_h^t |\Delta(u,u-h)|du \leq K|t-h| \, \|\Delta\|$$

this yields

$$\|Q\| \leq K|t_1-h| \, \|\Delta\| \tag{7.63}$$

Similarly from (7.62) we have, for some $K_1 < \infty$ and $K_2 < \infty$

$$\begin{aligned}|\Delta(t,s)| &\leq K_1 \int_s^t |Q(\sigma)|)\sigma + K_2 \int_s^t \int_\sigma^t |\Delta(\sigma,u-h)|du \, d\sigma \\ &\leq K_1 \int_s^t |Q(\sigma)|d\sigma + K_2 \int_h^t (t-\sigma) \sup_{0 \leq u \leq t} |\Delta(\sigma,u)|d\sigma \\ &\leq K_1(t-h)\|Q\| + \frac{K_2|t-h|^2}{2} \|\Delta\|\end{aligned}$$

thus

$$\|\Delta\| \leq K_1|t_1-h| \, \|Q\| + \frac{K_2|t_1-h|}{2} \|\Delta\| \tag{7.64}$$

substitute (7.63) in (7.64) to obtain

$$\|\Delta\| \leq M(t_1-h)^2 \|\Delta\| \quad , \quad M < \infty \tag{7.65}$$

Now choose t_1 such that $M(t_1-h)^2 < 1$, this implies $\|\Delta\| \equiv 0$ and from (7.63) we obtain $\|Q\| \equiv 0$. Therefore we may conclude that there is a unique solution to (7.51) - (7.52) on the intervals $0 \leq t \leq t_1$ and $0 \leq s \leq t_1$. Clearly the same arguments hold for $t_1 \leq t \leq t_2$ and $0 \leq s \leq t_2$. Therefore $P(t)$ and $\Psi(t,s)$ are the unique solutions to (7.51) - (7.52).

7.5 A Dual Optimal Control Problem

In this section we will show that the filtering problem posed in

Section 7.1 is in certain sense equivalent to a problem of control.

Theorem 7.4: Consider the optimal filtering problem over the interval [0,T] for the system (7.1) - (7.2) with $x_0(\theta) = 0$, $\theta \in [-h,0)$. Define the dual control system by

$$\frac{dz(t)}{dt} = -A_1' z(t) - A_2' z(t+h) - C_1' u(t) - C_2' u(t+h) \tag{7.66}$$

$$q(t) = B_1' z(t) + B_2' z(t+h) \tag{7.67}$$

with $z(T) = b$, $z(s) = 0 \quad s > T$, $u(s) = 0 \quad s > T$

The dual control problem is defined as follows

Determine an L_2-vector function $u: [0,T] \to \mathbb{R}^p$ to minimize the cost index

$$J_T(b,u) = z'(0)E[x_0(0)x_0'(0)]z(0) + \int_0^h z'(s)B_2B_2'z(s)\,ds + \int_0^T [q'(s)q(s) + u'(s)Ru(s)]ds \tag{7.68}$$

Then the optimal solution u_T to this problem of control is related to $\hat{x}(T|T)$ by

$$b'\hat{x}(T\ T) = -\int_0^T u_T'(s)dy(s) \tag{7.69}$$

Proof: The proof of this result follows easily from the work of Lindquist [L1]. We give it here for completeness.

First we note that the following integration by parts formula is valid

$$z'(T)x(T) - z'(0)x_0(0) = \int_0^T z'(s)dx(s) + \int_0^T x'(s)dz(s) \tag{7.70}$$

where the first integral exists in the Riemann-Stieltjes sense since $z(t)$ is absolutely continuous † on [0,T]. Now, using (7.1) and (7.66) and since $x_0(\theta) = 0\ \theta < 0$, $z(\theta) = 0$ and $u(\theta) = 0\ \theta > T$, (7.70) gives

$$z'(T)x(T) - z'(0)x_0(0) = \int_0^T z'(s)B_1 dw(s) + \int_0^T z'(s)B_2 dw(s-h)$$

$$- \int_0^T u'(s)[C_1 x(s) + C_2 x(s-h)]ds$$

since $z(T) = b$, this expression gives, from (7.2)††

$$b'x(T) + \int_0^T u'(s)dy(s) = z'(0)x_0 + \int_0^T u'(s)Ddv(s) \tag{7.71}$$

$$+ \int_0^T z'(s)B_1 dw(s) + \int_0^T z'(s)B_2 dw(s-h)$$

Now, as $x_0(0)$, $\{v(s),\ 0 \le s \le T\}$ and $\{w(s),\ -h \le s \le T\}$ are independent, it is easy to see that

$$E[b'x(T) + \int_0^T u'(s)dy(s)]^2 = z'(0)E[x_0(0)x_0'(0)]z(0)$$

$$+ \int_0^T u'(s)Ru(s)ds \tag{7.72}$$

$$+ \int_0^T z'(s)[B_1B_1' + B_2B_2']z(s)ds$$

---

† It is sufficient that $z(t)$ is of bounded variation on [0,T]. In this case the second integral in (7.70) is well defined in the Lebesgue-Stieltjes sense

†† Since $u \in L_2$ and y is of unbounded variation (almost surely) the integral in the left hand side of (7,71) should be understood as

$$\int_0^T u'(s)dy(s) = \int_0^T u'(s)[C_1 x(s) + C_2\ x(s-h)]ds + \int_0^T u'(s)Ddv(s)$$

where the first integral in the right hand side is defined in the Lebesgue sense and the second in quadratic mean.

$$+ \int_0^{T-h} z'(s)B_1B_2'z(s+h)\,ds$$

$$+ \int_0^{T-h} z'(s+h)B_2B_1'z(s)\,ds$$

and since $z(s) = 0 \; s > T$ (7.72) yields

$$E[b'x(T) + \int_0^T u'(s)dy(s)]^2 = J_T(b,u) \tag{7.73}$$

Thus minimizing (7.43) is equivalent to minimizing $J_T(b,u)$. Hence the least squares estimate of $b'x(T)$, which is $b'\hat{x}(T|T)$, is given $-\int_0^T u_T'(s)\,dy(s)$. This completes the proof.

Thus the dual control problem consists in minimizing a quadratic cost which contains delayed terms (note that(7.66) runs backwards in time)). We mention that certain problems of optimal control with delays in the cost functional have been studied in [D7][L3]. However, the results for such systems are rather incomplete as compared with known results for the case of no delays in the cost. In particular, Lee [L3] considers a linear system with delays in the states and the controls and quadratic cost of the form

$$x'(T)Qx(T) + \int_0^T [\sum_{i=1}^{k} \sum_{j=1}^{k} x'(s-h_i)P_i'P_j x(s-h_j) + u'(s)Ru(s)]ds \qquad .. \; (7.74)$$

$$0 = h_1 < h_2 \ldots < h_k = h$$

For this problem, Lee establishes the existence and uniqueness of the optimal control. Also, he obtains a representation for the optimal control in terms of the solution to an 'adjoint equation' (see [L3] for details). We point out that the cost (7.68) is slightly different from Lee's cost†.

---

† to compare the cost (7.68) with (7.74) let $\bar{z}(t) = z(T-t)$ and $\bar{u}(t) = u(T-t)$ and write (7.68) in terms of $\bar{z}(t)$ and $\bar{u}(t)$

However, under the assumption $w(s) = 0$ $s \in [-h,0]$, the cost given in Theorem 7.4 coincides with Lee's cost (the second term in (7.68) disappears, see also (7.71) - (7.72))). Thus, the results of the previous sections may be used in an attempt to complete Lee's work, e.g. obtain a feedback realization of the optimal control. Our final result shows how this can be done in the special case $A_2 = 0$, $C_2 = 0$, i.e. no delays in the state nor in the controls.

Proposition 7.5: Let $A_2 = 0$, $C_2 = 0$, $A_1 = A$ and $C_1 = C$. Then the optimal solution u_T to the control problem (7.66) - (7.68) is given by

$$u_T(t) = -R^{-1}C[P(t)z_T(t) + \int_t^{t+h} \Psi(t,s-h)B_1B_2'z_T(s)X(s-h)\,ds], t \in [0,T] \quad \text{.. a.e.} \qquad (7.75)$$

where z_T is the solution to (7.66) with $u = u_T$

Proof: from (7.50) it is easy to see

$$\hat{x}(T|T) = \int_0^T [\Psi(T,s)P(s) + \int_s^T \Psi(T,\sigma)B_2B_1'\Psi'(s,\sigma-h)X(\sigma-h)\,d\sigma]C'R^{-1}dy(s)$$

and from (7.69) we obtain

$$u_T(s) = -R^{-1}C[P(s)\Psi'(T,s) + \int_s^T \Psi(s,\sigma-h)B_1B_2'\ \Psi'(T,\sigma)X(\sigma-h)\,d\sigma]b \quad \text{.. a.e.} \qquad (7.76)$$

Next we show that $z_T(t) = \Psi'(T,t)b$, $t \in [0,T]$. From (7.66) we have, for $t \in [0,T]$

$$z(t) = b + \int_t^T A'z(s)\,ds + \int_t^T C'u(s)\,ds$$

this expression, together with (7.76) give

$$z_T(t) = \int_t^T A' z_T(s)ds + \{I - \int_t^T C'R^{-1}CP(s)\Psi'(T,s)ds \tag{7.77}$$

$$- \int_t^T \int_s^T C'R^{-1}C\Psi(s,\sigma-h)B_1B_2' \; \Psi'(T,\sigma)X(\sigma-h)d\sigma ds\} \; b$$

Now, it can be shown [L1] [H2], that $\Psi(t,s)$ satisfies

$$\Psi(t,s) = I - \int_s^t \Psi(t,\sigma)[A-P(\sigma)C'R^{-1}C]d\sigma$$

$$- \int_s^t \int_s^\sigma \Psi(t,\sigma)B_2B_1'\Psi'(\theta,\sigma-h)d\theta d\sigma \; , \qquad s \leq t \tag{7.78}$$

so (7,77) and (7.78) yield

$$[z_T(t) - \Psi'(T,t)b] = \int_t^T A'[z_T(s) - \Psi'(T,s)b]ds$$

which in turn gives

$$z_t(t) = \Psi'(T,t)b \quad , \qquad t \in [0,T] \tag{7.79}$$

thus, from (7.76) and (7.79) we obtain

$$u_T(s) = -R^{-1}C[P(s)z_T(s) + \int_s^T \Psi(s,\sigma-h)B_1B_2'z_T(\sigma)X(\sigma-h)d\sigma$$

using the fact that $\Psi(s,\sigma-h) = 0 \; s < \sigma-h$ and $z_T(\sigma) = z(\sigma) = 0 \; \sigma > T$

(7.75) follows easily from the expression above.

CHAPTER 8

CONCLUSIONS

In the preceding chapter we have extended known results for systems with delays in the state and observations, to systems containing delays in the noise process. In particular, it has been shown that the filtered estimate of a linear system with a delay in the state, observations and in the noise process satisfies a stochastic differential equation. This equation involves some smoothed estimates even when there is no delay in the state and observations. The 'gains' involved in the optimal filter are characterized in terms of the error covariance matrix function and the fundamental matrix associated with the homogeneous part of the system's dynamics. A set of partial differential equations for the error covariance have also been obtained. These equations resemble the corresponding expressions obtained for systems without a delay in the noise, plus a number of 'correction' terms due to the delay in the noise. Such 'correction' terms have no effect on the interval $t \in [0,h]$ and, under the assumption that the initial noise segment $w(s)$ $s \in [-h,0]$ is zero, the optimal filter behaves as if no delay was present in the noise (of course on $t \in [0,h]$). Unlike the case of no delays in the noise, we need to specify the error covariance function $P(t-\theta_1, t-\theta_2,t)$ on the intervals $\theta_1, \theta_2 \in [0,2h]$ for each $t \in [h,T]$ and $\theta_1, \theta_2 \in [0,h]$ for $t \in [0,h]$ (also we need to determine the fundamental matrix $\Phi(t-\theta,t-h)$ on $\theta \in [0,h]$ for $t \in [h,T]$). If no delay occurs in the observations, then it is sufficient to compute the error covariance on the intervals $\theta_1, \theta_2 \in [0,h]$ for each $t \in [0,T]$.

This is also the case for systems with delays in the noise process only; however, for this class of systems we have obtained an alternative characterization of the optimal filter. This characterization is given in terms of the covariance P(t,t,t) and the fundamental matrix associated with the homogeneous part of the corresponding error differential equation. We point out that this alternative representation allows us to reduce the number of differential equations satisfied by the 'gains' involved in the optimal filter, and to establish uniqueness of solutions to these equations.

We have also shown that the filtering problem posed in Section 7.1 is equivalent to a problem of optimal control. The dual system contains delays in the state, controls and observations. The dual optimization problem consists in minimizing a quadratic functional with delays. This problem has been previously studied by Lee [L3] , but his results are rather incomplete from the point of view that a feedback realization of the optimal control has not been obtained. In the special case of no delays in the state and controls we have obtained a feedback representation for the optimal control by exploiting our results on the filtering problem.

Finally we mention that our results are easily extended to systems with muliple point delays. The case of distributed delays in the noise, state and observations needs further research. We mention that for this class of systems Briggs [B11] has obtained some results by considering the time delay differential system as a stochastic evolution equation; however, the developments in [B11] are not applicable to systems containing point delays in the noise process. (The case of distributed delays in the state and observations has been studied in [K6]). More work is also needed to establish uniqueness of solutions to the set of partial differential equations satisfied by the error covariance function. The existence and stability of the stationary filter remains an open question. We believe that the study of the dual optimal control problem might be useful in solving the infinite time filtering problem. Also, the filtering problem for nonlinear stochastic delay systems should be studied (some results have been obtained by Kwong and Willsky [K6] when no delays occur in the noise process).

APPENDIX A

SPECTRAL DECOMPOSITION FOR TIME DELAY SYSTEMS

In this appendix we briefly describe the state space decomposition for time delay systems. The proofs of the results presented here are to be found in [S2] or [H2]†.

Consider the time delay system given by

$$\dot{x}(t) = A_1 x(t) + A_2 x(t-h) + Bu(t) , \quad t \geq 0 \tag{A.1}$$

$$x(\theta) = x_0(\theta) \quad , \quad \theta \in [-h,0]$$

$$y(t) = Cx(t) \tag{A.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and A_1, A_2 B, C are real constant matrices between the appropriate spaces.

Transformed into an evolution equation in $X = M_2$, (A.1) - (A.2) become

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \quad , \quad t \geq 0 \tag{A.3}$$

$$\tilde{x}(0) = x_0$$

$$y(t) = \tilde{C}\tilde{x}(t) \tag{A.4}$$

where $\tilde{x} \in M_2$, $\tilde{B}$ and $\tilde{C}$ are bounded operators and $\tilde{A}$ is a closed, unbounded operator with dense domian $D(\tilde{A})$ given by

$$D(\tilde{A}) = \{(x^0,x^1) \in M_2 | x^1 \in L_2([-h,0]; \mathbb{R}^n) \text{ is absolutely continuous,}$$
$$x^1(0) = x^0 \text{ and } \frac{dx^1(\theta)}{d\theta} \in L_2([-h,0];\mathbb{R}^n)\}$$

---

† Hale's results are developed in the space of continuous functions $C([-h,0];\mathbb{R}^n)$; however, they are essentially the same as those developed in the Hilbert space $M_2 = \mathbb{R}^n \times L_2([-h,0];\mathbb{R}^n)$.

Also, $\tilde{A}$ is the infinitesimal generator of a strongly continuous semigroup $S_{\tilde{A}}(t)$, $t \geq 0$ and this semigroup is compact for $t \geq h$. Furthermore, the spectrum of $\tilde{A}$ consists of eigenvalues of finite multiplicities, and the number of eigenvalues with real part greater than or equal to a given arbitrary constant is finite.

Now, ket Λ be a finite symmetric subset of $\sigma(\tilde{A})$ and define the subspaces

$$X_\Lambda = \bigoplus_{i=1}^{N} \mathrm{Ker}(\tilde{A}-\lambda_i)^{k_i} \quad , \; X^\Lambda = \bigcap_{i=1}^{N} \mathrm{Im}(\tilde{A}-\lambda_i)^{k_i} \tag{A.5}$$

where $\lambda_i \in \Lambda$ and N is the number of distinct eigenvalues of $\tilde{A}$ in Λ. Let m_i denote the algebraic multiplicity of $\lambda_i \in \Lambda$. It then follows, since $\dim[\mathrm{Ker}(\tilde{A}-\lambda)^{k_i}] = m_i$, that

$$k_i \leq m_i \qquad , \qquad i = 1, 2 \ldots N$$

and

$$\dim[X_\Lambda] = \sum_{i=1}^{N} m_i = M(\text{total multiplicity of the eigenvalues of } \tilde{A} \text{ in } \Lambda)$$

Furthermore, from the spectral theory of operators with compact resolvent, we have that

$$X = X_\Lambda \oplus X^\Lambda$$

and that the projection P_Λ of X onto X_Λ along X^Λ is given by

$$P_\Lambda = -\frac{1}{2\pi_i} \int_\Gamma (\tilde{A}-\lambda)^{-1} d\lambda \tag{A.6}$$

where $\Gamma = \bigcup_{i=1}^{N} \Gamma_{\lambda_i}$ is a circle around $\lambda_i \in \Lambda$ such that no other eigenvalue of

$\tilde{A}$ lies in the interior of Γ_{λ_i}.

In the remainder of this appendix, we will obtain another representation of the projection operator P_Λ. First, we summarize some preliminary results.

For $\psi, \phi \in M_2$ define the bilinear form †

$$<<\psi, \phi>> = \psi^{0^T} \phi^0 + \int_{-h}^{0} \psi^{1^T}(-\theta-h) A_2 \phi^1(\theta) d\theta \tag{A.7}$$

The operator $\tilde{A}^T$ adjoint to $\tilde{A}$ with respect to the bilinear form (A.7) is defined to satisfy, for $\psi \in D(\tilde{A}^T)$, $\phi \in D(\tilde{A})$

$$<<\tilde{A}^T\psi, \phi>> = <<\psi, A\phi>>$$

Simple computations now give $D(\tilde{A}^T) = D(\tilde{A})$ and

$$[\tilde{A}^T\psi]^0 = A_1^T \psi^0 + A_2^T \psi^1(-h)$$

$$[\tilde{A}^T\psi]^1 = \frac{d\psi^1(\theta)}{d\theta} \quad , \quad \theta \in [-h, 0]$$

$\tilde{A}^T$ is then a closed operator with dense domain. We also have that $\tilde{A}^T$ has point spectrum only, $\sigma(\tilde{A}^T) = \sigma(\tilde{A})$ and for $\lambda \in \sigma(\tilde{A}^T)$ the generalized eigenspaces are finite dimensional. We point out that $\tilde{A}^T$ must not be confused with the topological adjoint $\tilde{A}^*$; however, there is an interesting and useful relationship between them (see [D4] for details).

The spectrum of the infinitesimal generator $\tilde{A}$ is characterized by the n×n matrix function

$$\Delta(\lambda) = A_1 + A_2 e^{-\lambda h} - \lambda I \tag{A.8}$$

---

† when multiple and distributed delays are present in (A.1) we define the bilinear form by

$$<<\psi, \phi>> = \psi^{0^T} \phi^0 + \sum_{i=2}^{r} \int_{-h_i}^{0} \psi^{1^T}(-\theta-h_i) A_i \phi^1(\theta) d\theta + \int_{-h_r}^{0} \int_{-h_r}^{\theta} \psi^{1^T}(s-\theta) A(s) \phi^1(\theta) ds d\theta$$

where $0 < h_2 < h_3 \ldots < h_r$

in fact, we have

$$\sigma(\tilde{A}) = \{\lambda \in C \,|\, \det\Delta(\lambda) = 0\} \tag{A.9}$$

Now, let M_2^c denote the complex extension of M_2, i.e., $M_2^c = C^n \times L_2\ ([-h,0];\ C^n)$. Similarly $D(\tilde{A})^c$ denotes the complex function space corresponding to $D(\tilde{A})$. Define the complex valued functions $\alpha_\lambda^k : [-h,0] \to C$ for $\lambda \in C$ and $k = 0, 1\ldots$ by

$$\alpha_\lambda^k(\theta) = \frac{\theta^k}{k!}\ e^{\lambda\theta} \quad , \qquad -h \le \theta \le 0 \tag{A.10}$$

and let $*$ denote the convolution between two functions on $[-h,0]$

$$\alpha * \beta(\theta) = \int_\theta^0 \alpha(\theta-\sigma)\beta(\sigma)\,d\sigma \quad , \qquad -h \le \theta \le 0$$

We now give the following results

<u>Theorem A.1</u>: Let $\lambda \in \sigma(\tilde{A})$, $\phi\ ,\psi \in M_2^c$. Then, for $k = 1,2\ldots$ the equation $(\tilde{A}-\lambda I)^k\phi = \psi$ holds if and only of there exist vectors $\phi_0, \phi_1 \ldots \phi_{k-1} \in \mathbb{R}^n$ such that

$$\phi^0 = \phi_0\ ,\phi^1 = \sum_{j=0}^{k-1}\ \phi_j\ \alpha_\lambda^j - \alpha_\lambda^{k-1} * \psi^1 \tag{A.11}$$

and for $\nu = 0,\ldots k-1$

$$\sum_{j=0}^{\nu} \frac{1}{j!}\ \frac{d^j\Delta(\lambda)}{d\lambda^j}\ \phi_{k-1-\nu+j} = \langle\langle \alpha_\lambda^\nu\ ,\psi\rangle\rangle \tag{A.12}$$

(A.12) in Theorem A.1 can be written in matrix notation as

$$\begin{pmatrix} \Delta(\lambda) & \frac{d\Delta(\lambda)}{d\lambda} & \cdots & \frac{1}{(k-1)!} & \frac{d^{k-1}\Delta(\lambda)}{d\lambda^{k-1}} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \frac{d\Delta(\lambda)}{d\lambda} \\ & & & \ddots & \Delta(\lambda) \end{pmatrix} \begin{pmatrix} \phi_0 \\ \cdot \\ \cdot \\ \cdot \\ \phi_{k-2} \\ \phi_{k-1} \end{pmatrix} = \begin{pmatrix} \langle\langle \alpha_\lambda^{k-1}, \psi\rangle\rangle \\ \\ \\ \langle\langle \alpha_\lambda^{1}, \psi\rangle\rangle \\ \langle\langle \alpha_\lambda^{0}, \psi\rangle\rangle \end{pmatrix} \tag{A.13}$$

In the following the $nk \times nk$ matrix in this equation will be denoted by $\Delta_k(\lambda)$.

<u>Theorem A.2</u>: Let $k = 1,2 \ldots$ and $\phi \in M_2^c$. Then $\phi \in \mathrm{Im}(\tilde{A}-\lambda I)^k$ if and only if

$$\langle\langle \psi, \phi \rangle\rangle = 0$$

for all $\psi \in \mathrm{Ker}(\tilde{A}^T-\lambda I)^k$.

As a consequence of Theorem A.1 we have that the subspaces $\mathrm{Ker}(\tilde{A}-\lambda I)^k$ and $\mathrm{Ker}(\tilde{A}^T-\lambda I)^k$ of M_2^c are of the same dimension nk - rank $\Delta_k(\lambda)$ for every $k = 1,2\ldots$. Moreover, these subspaces are spanned by functions of the form

$$\phi(\theta) = \sum_{j=0}^{k-1} \phi_j \frac{\theta^j}{j!} e^{\lambda\theta} \quad , \quad -h \leq \theta \leq 0 \tag{A.14}$$

and

$$\psi(\theta) = \sum_{j=0}^{k-1} \psi_j \frac{\theta^j}{j!} e^{\lambda\theta} \quad , \quad -h \leq \theta \leq 0 \tag{A.15}$$

respectively, where

$$\gamma = \begin{pmatrix} \phi_0 \\ \cdot \\ \cdot \\ \phi_{k-2} \\ \phi_{k-1} \end{pmatrix} \text{ satisfies } \Delta_k(\lambda)\gamma = 0$$

and

$$\beta = \begin{pmatrix} \psi_{k-1} \\ \psi_{k-2} \\ \\ \psi_0 \end{pmatrix} \quad \text{satisfies } \beta^T \; \Delta_k(\lambda) = 0$$

It now follows that the generalized eigenspaces

$$Z_\lambda = \bigcup_k \operatorname{Ker}(\tilde{A}-\lambda I)^k \quad , \quad Z_\lambda^1 = \bigcup_k \operatorname{Ker}(\tilde{A}^T-\lambda I)^k$$

have the same dimension which equals the algebraic multiplicity of $\lambda \in \sigma(\tilde{A}) = \sigma(\tilde{A}^T)$. Furthermore, since the resolvent operators of $\tilde{A}$ and $\tilde{A}^T$ are compact, there is a minimal integer k_λ such thay

$$Z_\lambda = \operatorname{Ker}(\tilde{A}-\lambda I)^{k_\lambda} \quad , \quad Z_\lambda^1 = \operatorname{Ker}(\tilde{A}^T-\lambda I)^{k_\lambda}$$

Theorem A.2 now shows that, for the same k_λ, we have

$$Z^\lambda = \bigcap_k \operatorname{Im}(\tilde{A}-\lambda I)^k = \operatorname{Im}(\tilde{A}-\lambda I)^{k_\lambda}$$

and that $\phi \in Z^\lambda$ if and only if $\langle\langle\psi,\phi\rangle\rangle = 0$ for all $\psi \in Z_\lambda^1$.

Now define

$$Z_\Lambda = \bigoplus_{\lambda\varepsilon\Lambda} Z_\lambda \quad , \quad Z_\Lambda^1 = \bigoplus_{\lambda\varepsilon\Lambda} Z_\lambda^1 \quad , \quad Z^\Lambda = \bigcap_{\lambda\varepsilon\Lambda} Z^\lambda$$

then $M_2^c = Z_\Lambda \oplus Z^\Lambda$

Let $\{\phi_1,\ldots,\phi_M\}$ and $\{\psi_1,\ldots, \psi_M\}$ be a real† bases of Z_Λ and Z_Λ^1 respectively. Moreover define Φ $\Psi \varepsilon \mathbb{R}^{n\times M} \times L_2([-h,0]; \mathbb{R}^{n\times M})$ by

$$\Phi = (\Phi^0, \Phi^1) = [\phi_1 \ldots \phi_M] \tag{A.16}$$

$$\Psi = (\Psi^0, \Psi^1) \quad [\psi_1 \ldots \psi_M] \tag{A.17}$$

and let $<<\Psi,\Phi>>$ denote the real M×M matrix with entries $<<\psi_j,\phi_i>>$, $i,j = 1,2\ldots M$, i.e.

$$<<\Psi,\Phi>> = \Psi^{0^T} \Phi^0 + \int_{-h}^0 \Psi^{1^T}(-\sigma-h)A_2\Phi^1(\sigma)d\sigma$$

For $\phi \varepsilon M_2^c$ let $<<\Psi,\phi>> \varepsilon C^M$ be the vector with components $<<\psi_j,\phi>>$, $j = 1,2\ldots M$.

It can be shown that the matrix $<<\Psi,\Phi>>$ is non-singular. Thus the bases $\{\phi_1,\ldots,\phi_M\}$ and $\{\psi_1,\ldots,\psi_M\}$ can be chosen (without loss of generality) such that

$$<<\Psi,\Phi>> = I \tag{A.18}$$

We now give the following result

<u>Theorem A.3</u>: If (A.18) holds, then P_Λ, defined by

$$P_\Lambda \phi = \Phi<<\Psi,\phi>> \quad \phi \varepsilon M_2^c \tag{A.19}$$

projects M_2^c onto Z_Λ along Z^Λ.

Now, define the real subspaces

$$X_\Lambda = Z_\Lambda \cap M_2 \quad , \quad X^\Lambda = Z^\Lambda \cap M_2$$

then we obtain $X_\Lambda = \{\Phi x_\Lambda | x_\Lambda \varepsilon \mathbb{R}^M\}$ and $\phi \varepsilon X^\Lambda$ if and only if $\phi \varepsilon M_2$

---

† such bases clearly exist since $\underset{\sim}{\Lambda}$ is symmetric (with respect to the real axis). Note also that $\sigma(\tilde{A})$ is symmetric, since the coefficients of $\det\Delta(\lambda)$ are real.

and $\langle\langle\Psi,\phi\rangle\rangle = 0$. Hence P_Λ also projects M_2 onto X_Λ along X^Λ, and $M_2 = X_\Lambda \oplus X^\Lambda$.

Finally let us analyze the decomposition of the inhomogeneous system (A.1) with respect to the above decomposition of M_2. Since X_Λ is invariant under $\tilde{A}$, it can be shown that there is a (unique) real M×M matrix A_Λ satisfying.

$$\tilde{A}\,\Phi = \Phi\,A_\Lambda \tag{A.20}$$

<u>Theorem A.4</u>: Let (A.18) be satisfied and A_Λ defined by (A.20). Then

(i) $\tilde{A}^T\Psi = \Psi A_\Lambda^T$

(ii) $\Phi^1(\theta) = \Phi^0\, e^{A_\Lambda\theta}$, $\Psi^1(\theta) = \Psi^0\, e^{A_\Lambda^T\theta}$, $-h \leq \theta \leq 0$

(iii) $\sigma(A_\Lambda) = \Lambda$

(iv) $S(t)\Phi = \Phi e^{A_\Lambda t}$, $S^T(t)\Psi = \Psi e^{A_\Lambda^T t}$, $t \geq 0$

where $S(t)$ and $S^T(t)$ are the semigroups generated by $\tilde{A}$ and $\tilde{A}^T$ respectively .

(v) $P_\Lambda S(t) = S(t)P_\Lambda$

The above theroem, in particular (v), shows that both X^Λ and X_Λ are invariant under $S(t)$ for all $t \geq 0$. Hence the operator family

$$S^\Lambda(t) \triangleq S(t)|X^\Lambda$$

defines a strongly continuous semigroup of bounded linear operators on X^Λ with infinitesimal generator $\tilde{A}^\Lambda$ defined by

$$\tilde{A}^\Lambda\,\phi = \tilde{A}\,\phi\ ,\ \phi \in D(\tilde{A}^\Lambda) = D(\tilde{A}) \cap X^\Lambda$$

Clearly $S^\Lambda(t)$ is still compact for $t \geq h$ and the resolvent operator $(\tilde{A}^\Lambda-\lambda I)^{-1} = (\tilde{A}-\lambda I)^{-1}|X^\Lambda$ of $\tilde{A}^\Lambda$ is compact for $\lambda \in \rho(\tilde{A})$. In particular $S^\Lambda(t)$ is stable if and only if $R_e\,\lambda < \omega$ for all $\lambda \in \sigma(A^\Lambda) = \sigma(\tilde{A})|\Lambda$, for some $w \leq 0$.

We close this appendix with the following result

<u>Theorem A.5</u>: Let $\phi \in M_2$ and suppose that $u(\cdot)$ is locally integrable. Let $x(t) \in \mathbb{R}^n$ $t \geq -h$ be the unique solution of (A.1). Then the projection of $(x(t),x_t)$ into X_Λ is given by

$$P_\Lambda(x(t),x_t) = \Phi\, x_\Lambda(t) \tag{A.21}$$

where $x_\Lambda(t) = \langle\langle \Phi,(x(t),x_t)\rangle\rangle \in \mathbb{R}^M$, $t \geq 0$ is the unique solution of the ordinary differential equation

$$\dot{x}_\Lambda(t) = A_\Lambda x_\Lambda(t) + \Psi^{0^T} Bu(t), \; t \geq 0 \tag{A.22}$$

with initial value

$$x_\Lambda(0) = \langle\langle \Psi, x_0 \rangle\rangle \quad . \tag{A.23}$$

APPENDIX B

Most of the definition and results of this appendix are found in [K1].

I. Some Results On Pairs Of Closed Subspaces

Let M and N be two closed subspaces of a Banach space X. The nullity of the pair M,N is defined by

$$\text{nul}(M,N) = \dim[M \cap N]$$

The difficiency of the pair M, N is defined by

$$\text{def}(M,N) = \text{codim}[M+N] = \dim\left(\frac{X}{M+N}\right)$$

We also define

$$\gamma(M,N) = \inf_{\substack{x \in M \\ x \notin N}} \frac{\text{dist}(x,N)}{\text{dist}(x,M\cap N)} (\leq 1)$$

when $M \subset N$ we set $\gamma(M,N) = 1$.

Also we mention that $\gamma(M,N)$ is not in general equal to $\gamma(N,M)$ †, but they satisfy

$$\gamma(N,M) \geq \frac{\gamma(M,N)}{1+\gamma(M,N)}$$

$\hat{\gamma}(M,N) = \min[\gamma(M,N),\gamma(N,M)]$ is the minimum gap between M and N

Theorem B.I.1: [K1, Th. 4.2, p. 219]

The subspace M+N is closed if and only if $\gamma(M,N) > 0$

---

† if X is a Hilbert space then $\gamma(M,N) = \gamma(N,M)$

Theorem B.I.2: [K1, Th, 4.18, p. 226, Th, 4.24, p. 227]

Let M, N and M' be closed subspaces of a Banach space X. Suppose that M+N and M'+N are closed in X. Then

$$\delta(M',M) < \gamma(N,M) \text{ implies } \operatorname{nul}(M',N) \leq \operatorname{nul}(M,N)$$

and

$$\delta(M,M') < \gamma(M,N) \text{ implies } \operatorname{def}(M',N) \leq \operatorname{def}(M,N)$$

Theorem B.I.3: [K1, p.200]

If M and N are closed subspaces of a Banach space X. Then

$$\delta(M,N) < 1 \text{ implies } \dim M \leq \dim N$$

$$\hat{\delta}(M,N) < 1 \text{ implies } \dim M = \dim N$$

For any subspace M of a Banach space X, the annihilator of M in the adjoint space X^* is denoted by $M^\perp$ † and is a closed subspace Also, for any subspaces M, N of X we have

$$(M+N)^\perp = M^\perp \cap N^\perp$$

The dual relation $M^\perp + N^\perp = (M \cap N)^\perp$ does not always hold because $(M \cap N)^\perp$ is closed but $M^\perp + N^\perp$ need not be closed.

Theorem B.I.4: [K1, Th. 4.8, p. 221]

Let M and N be closed subspaces of a Banach space X. Then M+N is closed if and only if $M^\perp + N^\perp$ is closed in X^*. In this case

$$M^\perp + N^\perp = (M \cap N)^\perp$$

and

$$\operatorname{nul}(M^\perp,N^\perp) = \operatorname{def}(M,N), \quad \operatorname{def}(M^\perp,N^\perp) = \operatorname{nul}(M,N)$$

† if X is a Hilbert space, then $M^\perp$ denotes the orthogonal complement of M in $X = X^*$.

$$\gamma(M^\perp,N^\perp) = \gamma(N,M)^\dagger \quad , \quad \hat{\gamma}(M^\perp,N^\perp) = \hat{\gamma}(M,N)^\dagger$$

Theorem B.I.5: [K1, Th.8.9, p. 201]

Let M and N be closed subspaces of a Banach space X. Then

$$\delta(M,N) = \delta(N^\perp,M^\perp), \quad \hat{\delta}(M,N) = \hat{\delta}(M^\perp,N^\perp)$$

Theorem B.I.6: [K1, Lemma 2.2, p. 199]

For any closed subspaces M,N of X and any $u \in X$, we have

$$[1+\delta(M,N)]\ \mathrm{dist}(u,M) \geq \mathrm{dist}(u,N) - \delta(M,N)\|u\|$$

We conclude this section with two results concerning simultaneous perturbations of two closed subspaces.

Theorem B.I.7:

Let M, M', N and N' be closed subspaces of a Banach space X. Assume that $M \cap N = 0$ and M+N, M'+N, M+N', M'+N' are closed. If

$$\max[\delta(M',M),\delta(N',N)] < \min\left(\frac{\gamma(N,M)}{2+\gamma(N,M)}, \frac{\gamma(M,N)}{2+\gamma(M,N)}\right)^{\dagger\dagger} \qquad \text{(b.I.1)}$$

then $M' \cap N = 0$, $M \cap N' = 0$ and $M' \cap N' = 0$.

Proof: Suppose that $\max[\delta(M',M),\delta(N',N)] = \delta(M',M)$ then (b.I.1) implies

$$\delta(M',M) < \gamma(N,M)$$

and theorem B.I.2 gives

$$\dim[M' \cap N] \leq \dim[M \cap N] = 0$$

---

† These expressions are valid even if M+N is not closed.

†† If X is a Hilbert space then $\gamma(N,M) = \gamma(M,N)$ and this condition becomes $\max[\delta(M',M),\delta(N',N)] < \gamma(M,N)/2+\gamma(M,N)$

Hence $M' \cap N = 0$.

We next show that $\gamma(M',N) > \delta(M',M) \geq \delta(N',N)$. Let $u \in N$, then since $M \cap N = 0$

$$\text{dist}(u,M) \geq \gamma(N,M)\,\text{dist}(u,M\cap N) = \gamma(N,M)\,\|u\|$$

It follows, from the above inequality and Theorem B.I.6. (with the substitution $M \to M'$, $N \to M$) that

$$\text{dist}(u,M') \geq [1+\delta(M',M)]^{-1}[\gamma(N,M) - \delta(M'.M)]\,\|u\|$$

Since this is true for any $u \in N$, we obtain

$$\gamma(N,M') \geq \frac{\gamma(N,M) - \delta(M',M)}{1+\delta(M',M)}$$

which in turn implies

$$\gamma(M',N) \geq \frac{\gamma(N,M')}{1+\gamma(N,M')} \geq \frac{\gamma(N,M) - \delta(M',M)}{1+\gamma(N,M)} \tag{b.I.2}$$

On the other hand we have

$$\delta(M',M) < \frac{\gamma(N,M)}{2+\gamma(N,M)}$$

thus

$$\delta(M',M) < \frac{\gamma(N,M) - \delta(M',M)}{1+\gamma(N,M)} \tag{b.I.3}$$

Combining (b.I.3) and (b.I.2) we obtain

$$\gamma(M',N) > \delta(M',M) \geq \delta(N',N)$$

It is now clear, from Theorem B.I.2 (with the substitution $M' \to N'$ $M \to N$, $N \to M'$) that

$$\dim[M' \cap N'] \leq \dim[M' \cap N] = 0$$

therefore $M' \cap N' = 0$.

Now suppose that $\max[\delta(M',M),\delta(N',N)] = \delta(N',N)$ then, we obtain (as above) that $\gamma(M',N) > \delta(M',M)$ but we cannot conclude that $\gamma(M',N) > \delta(N',N)$.

However in this case

$$\delta(N',N) < \gamma(M,N)$$

and therefore B.I.2 gives (with the substitution $M' \to N'$, $M \to N, N \to M$) $\dim[M \cap N'] = 0$, hence $M \cap N' = 0$.

Next we show that $\gamma(N',M) > \delta(N',N) \geq \delta(M',M)$. Let $u \in M$, then since $M \cap N = 0$ we obtain

$$\text{dist}(u,N) \geq \gamma(M,N)\,\text{dist}(u,M \cap N) = \gamma(M,N)\,\|u\|$$

It follows from the above inequality and theorem B.I.6 (with the substitution $M \to N'$) that

$$\text{dist}(u,N') \geq [1 + \delta(N',N)]^{-1}[\gamma(M,N) - \delta(N',N)]\,\|u\|$$

Since this is true for any $u \in M$ we obtain

$$\gamma(M,N') \geq \frac{\gamma(M,N) - \delta(N',N)}{1+\delta(N',N)}$$

thus, from the above inequality and since $\delta(N',N) < \dfrac{\gamma(M,N)}{2+\gamma(M,N)}$

it is easy to see that

$$\gamma(N',M) \geq \frac{\gamma(M,N')}{1 + \gamma(M,N')} \geq \frac{\gamma(M,N) - \delta(N',N)}{1 + \gamma(M,N)} > \delta(N',N)$$

hence

$$\gamma(N'M) > \delta(N',N) \geq \delta(M',M)$$

and theorem B.I.2 (with the substitution $N \to N'$) gives

$$\dim[M' \cap N'] \leq \dim[M \cap N'] = 0$$

thus $\qquad M' \cap N' = 0$

This completes the proof.

Theorem B.I.8:

Let M, M', N and N' be closed subspaces of a Banach Space X.

Assume that M + N = X and that M' + N, M+N' and M'+N' are closed. If

$$\max[\delta(M,M'),\ \delta(N,N')] < \min\left\{\frac{\gamma(M,N)}{2+\gamma(M,N)},\ \frac{\gamma(N,M)}{2+\gamma(N,M)}\right\}$$

then

$$M' + N = X \quad , \quad M + N' = X \text{ and } M' + N' = X.$$

Proof: If sufices to apply Theorem B.I.7 to M,M',N and N' replaced by their annihilators. (Note Theorems B.I.5 and B.I.4)

II. Relative Boundedness and Relative Compactness

Let T and A be two operators with domains in a Banach space X, but not necessarily with the same range space. Let D(T) and D(A) be the domains of T and A respectively.

Definition B.II.1: A is said to be relatively bounded with respect to T, or simply T-bounded if $D(T) \subset D(A)$ and there are non-negative constants a, b such that

$$\| Ax\| \leq a\| x\| + b\| Tx\| \quad , \quad x \in D(T) \tag{b.1}$$

The greatest lower bound b_0 of all possible constants b in (b.1) will be called the relative bound of A with respect to T, or simply the T-bound of A. Clearly a bounded operator is T-bounded with T-bound zero.

Definition B.II.2: A is said to be relatively compact with respect to T, or simply T-compact, if $D(T) \subset D(A)$ and for any sequence $\{x_n\} \in D(T)$ with both $\{x_n\}$ and $\{Tx_n\}$ bounded, $\{Ax_n\}$ contains a convergent subsequence. Clearly if A is T-compact it is also T-bounded.

When T is closed, i.e. $T \in C(X,Y)$ we may introduce the graph norm on $D(T)$, that is

$$|||x|||^2 = \|x\|^2 + \|Tx\|^2 \quad , \quad x \in D(T)$$

Under the norm $|||\cdot|||$, $D(T)$ becomes a Banach space D_1 (since T is closed and this implies that D_1 is complete).
Let A_1 be the restriction of A to $D(T)$. Then it is easy to see that A is T-bounded if and only if A_1 is bounded, i.e.

$$\|Ax\| = \|A_1 x\| \leq M(\|x\|^2 + \|Tx\|^2)^{\frac{1}{2}} = M|||x||| \quad , \quad x \in D(T)$$

Similarly, it can be shown, that A is T-compact of and only if A_1 is compact.

We conclude this section with the following results

Theorem B.II.3: [Kl, Th. 1.1, p. 190]

Let T and A be operators from X to Y, and let A be T-bounded with T-bound smaller than 1. Then $S = T+A$ is closable if and only if T is closable and in this case the closures of T and S have the same domain. In particular S is closed if and only if T is closed.

We further note that for $b < 1$ in (b.1) and $S = T+A$, the operator A is S-bounded with S-bound $\leq b(1-b)^{-1}$. In fact for any operator that is T-bounded with T-bound b_1 is also S-bounded with S-bound $\leq b_1(1-b)^{-1}$.

Theorem B.II.4: [Kl, Th. 1.11, p. 194]

Let T, A be operators from X to Y and let A be T-compact. If T is closable, $S = T + A$ is also closable and the closures of T and S have the same domain and A is S-compact. In particular S is closed if T is closed. (Note that no assumption is made on the "size" of A).

Theorem B.II.5: [Kl, Th. 2.14, p. 203]

Let $T \in C(X,Y)$ and let A be T-bounded with T-bound less than 1. Then $S = T+A \in C(X,Y)$ and

$$\delta(S,T) \leq (1-b)^{-1}(a^2+b^2)^{\frac{1}{2}} \quad , \ \delta(T,S) \leq (a^2+b^2)^{\frac{1}{2}}$$

$$\hat{\delta}(S,T) = \max[\delta(S,T),\ \delta(T,S)] \leq (1-b)^{-1}(a^2+b^2)^{\frac{1}{2}}$$

In particular if A is bounded then

$$\hat{\delta}(S,T) \leq \| A\|$$

III. Perturbation Of The Spectrum Of Closed Operators

Theorem B.III.1: [Kl, Th. 3.1, p. 208]

Let $T \in C(X)$ and let Γ be a compact subset of the resolvent set $\rho(T)$. Then there is a $\delta > 0$[†] such that $\Gamma \subset \rho(S)$ for any $S \in C(X)$ with $\hat{\delta}(S,T) < \delta$.

The above result may be interpreted to imply that $\sigma(T)$ is upper semicontinuous. The following result establishes that each separated part of the $\sigma(T)$ is upper semicontinuous.

Theorem B.III.2: [Kl, Theorem 3.16, 212]

Let $T \in C(X)$ and let $\sigma(T)$ be separated into two parts $\sigma_1(T)$, $\sigma_2(T)$ by a closed curve Γ containing $\sigma_1(T)$ in its interior and $\sigma_2(T)$ in its exterior. Let $X = X_1(T) \oplus X_2(T)$ be the associated decomposition of X. Then there is a $\delta > 0$[††] depending on T and Γ with the

---

† $\delta = \min_{\lambda\in\Gamma} \frac{1}{2}(1+|\lambda|^2)^{-1}\ (1+\|\ (T-\lambda)^{-1}\|^2)^{-\frac{1}{2}}$

†† δ may be chosen as in Theorem B.III.1.

following properties. Any $S \in C(X)$ with $\hat{\delta}(S,T) < \delta$ has spectrum $\sigma(S)$ likewise separated by Γ into two parts $\sigma_1(S), \sigma_2(S)$ (Γ itself running in $\rho(S)$). In the associated decomposition $X = X_1(S) \oplus X_2(S)$, $X_1(S)$ and $X_2(S)$ are respectively isomorphic with $X_1(T)$ and $X_2(T)$. In particular $\dim X_1(S) = \dim X_1(T)$, and $\dim X_2(S) = \dim X_2(T)$ and both $\sigma_1(S)$ and $\sigma_2(S)$ are nonempty if this is true for T. The decompsotion $X = X_1(S) \oplus X_2(S)$ is continuous in S in the sense that the projection $P(S)$ of X onto $X_1(S)$ along $X_2(S)$ tends to $P(T)$ in norm as $\hat{\delta}(S,T) \to 0$.

When $\sigma_1(T)$ in the theorem above is finite system of eigenvalues then $\dim X_1(T) = m < \infty$, where m is the total multiplicity of the eigenvalues under consideration. In this case we may choose a closed curve Γ enclosing $\sigma_1(T)$, in such a way that for any $S \in C(X)$ with $\hat{\delta}(S,T) < \delta$, Γ also separates $\sigma(S)$ into two parts $\sigma_1(S)$, $\sigma_2(S)$ with $\sigma_1(S)$ (contained in Γ) being a finite system of eigenvalues with total multiplicity m, i.e. $\dim X_1(S) = m$. The same result holds when $\sigma_1(T)$ is replaced by anyone of the eigenvalues in $\sigma_1(T)$. Thus we conclude that the change of a finite system of eigenvalues of a closed operator T is small, when T is subjected to a small perturbation in the sense of $\hat{\delta}(S,T)$ being small, where $S \in C(X)$ denotes the perturbed operator .

The above results are rather general but not very convenient for applications. Next we give two results which are more directly useful.

<u>Theorem B.III.3</u>: [Kl. Th.3.17, p.214]

Let $T \in C(X)$ and A an operator in X which is T-bounded. If there is a point $\lambda \in \rho(T)$ such that

$$a\| (T-\lambda)^{-1} \| + b\| T(T-\lambda)^{-1} \| < 1$$

then $S = T + A$ is closed and $\lambda \in \rho(S)$ with

$$\| (S-\lambda)^{-1} \| \leq \| (T-\lambda)^{-1} \| (1-a \| (T-\lambda)^{-1} \| -b \| T(T-\lambda)^{-1} \|)^{-1}$$

If in particular T has compact resolvent, S has compact resolvent.

Theorem B.III.4: [K1, Th.3.18, p. 214]

Let T, A and S be as in the preceding theorem. Let $\sigma(T)$ be separated into two parts by a closed curve Γ as in Theorem B.III.2.

$$\sup_{\lambda\varepsilon\Gamma}(a \| (T-\lambda)^{-1} \| + b \| T(T-\lambda)^{-1} \|) < 1$$

thus $\sigma(S)$ is likewise separated by Γ and the results of Theorem B.III.2 hold.

In Theorem B.III.4, $\| P(S) - P(T) \|$ can be made arbitrarily small if $\| A(T-\lambda)^{-1} \|$ is sufficiently small for all $\lambda \varepsilon \Gamma$, which is the case if a, b are sufficiently small. (Actually a, b need not be too small but the condition in Theorem B.III.4 suffices).

IV. Some Results On Closed Operators In Banach Space

Let T be a closed operator from X to Y. The reduced minimum modulus of T, denoted by $\gamma(T)$ is defined by

$$\gamma(T) = \inf_{x\varepsilon D(T)} \frac{\| Tx \|}{\text{dist}(x,\text{Ker } T)}$$

where $\frac{0}{0}$ is defined to be ∞.

The reduced minimum modulus may also be defined as [K1, p. 231]

$$\gamma(T) = \| \tilde{T}^{-1} \|^{-1}$$

where $\tilde{T}$ is the 1-1 operator induced by T on X/Ker T. If $\tilde{T}^{-1}$ is unbounded $\gamma(T) = 0$.

Theorem B.IV.1: [K1, Th. 5.2, p.231]

$T \varepsilon C(X,Y)$ has closed range if and only if $\gamma(T) > 0$.

Theorem B.IV.2: [G1, Lemma IV. 2.9, p. 104]

Let $T \varepsilon C(X,Y)$ have closed range. If M is a subspace (not necessarily closed) of X, such that M + Ker T is closed then TM is closed. In particular, if M is closed and dim[Ker T] $< \infty$, then TM closed.

We now give a general theorem on perturbation of Fredholm operators.

Theorem B.IV.3: [K1, Th. 5.17, p. 235]

Let T, S $\varepsilon C(X,Y)$ and let T be Fredhom [semi-Fredholm].[†] If $\hat{\delta}(S,T) < \gamma(T)(1+\gamma^2(T))^{-\frac{1}{2}}$, then S is Fredholm [semi-Fredholm] and

dim[Ker S] $\leq$ dim[Ker T], codim[Im S] $\leq$ codim[Im T]

Furthermore there is a $\delta > 0$[††] such that $\hat{\delta}(S,T) < \delta$ implies[†††]

ind[S] = ind[T]

---

† An operator $T \varepsilon C(X,Y)$ is said to be semi Fredholm if [Im T] is closed and at least one of dim[Ker T] or codim[Im T] is finite. When both are finite T is said to be Fredholm operator. Observe that if codim[Im T] $< \infty$ then [Im T] is closed since X and Y are Banach spaces.

†† We may choose $\delta = \gamma(T)\ (1 + \gamma^2(T))^{-\frac{1}{2}}$ if X and Y are Hilbert spaces. In general it is difficult to give a simple estimate of δ.

††† ind[T] = dim[Ker T] - codim[Im T].

The next two results are more directly applicable.

Theorem B.IV.4: [Kl, Th. 5.22, p. 236]

Let $T \in C(X,Y)$ be semi-Fredholm (so that $\gamma(T) > 0$). Let A be a T-bounded operator from X to Y, so that (b.1) holds for some $a > 0$, $b > 0$. If

$$a < (1-b)\gamma(T) \qquad \text{(this implies } b < 1\text{)}$$

then $S = T + A \in C(X,Y)$, S is semi-Fredholm and

$$\dim[\text{Ker } S] \leq \dim[\text{Ker } T], \ \text{codim}[\text{Im } S] \leq \text{codim}[\text{Im } T]$$

$$\text{ind}[S] = \text{ind}[T]$$

Theorem B.IV.5: [Kl, Th. 5.26, p. 238]

Let $T \in C(X,Y)$ be semi-Fredholm. If A is a T-compact operator from X to Y, then $S = T + A \in C(X,Y)$ is also semi-Fredholm with

$$\text{ind}[S] = \text{ind}[T]$$

Theorem B.IV.6: [Kl, Th. 5.29, Th. 5.30, pp. 168-169]

Let X and Y be reflexive Banach spaces, and let $T \in C(X,Y)$ be densely defined. Then the adjoint of T, denoted by $T^* \in C(Y^*,X^*)$ and is densely defined. Furthermore $T^{**} = T$. If in addition T^{-1} exists and belongs to $B(Y,X)$, then $(T^*)^{-1}$ exists $\in B(X^*, Y^*)$ and $(T^*)^{-1} = (T^{-1})^*$.

Theorem B.IV.7: [Kl, Th. 5.13, p. 234]

Assume T^* exists. Then $\text{Im}\,T$ is closed if and only if $\text{Im}\,T^*$ is closed. In this case we have

$$(\text{Im } T)^\perp = \text{Ker } T^*, \qquad (\text{Ker } T)^\perp = \text{Im } T^*$$

$$\dim[\text{Ker } T^*] = \text{codim}[\text{Im } T], \ \text{codim}[\text{Im } T^*] = \dim[\text{Ker } T]$$

$$\gamma(T^*) = \gamma(T) \qquad \text{(this holds even if Im } T \text{ is not closed)}$$

In adition T is a Fredhomn operator (semi Fredholm) if and only if T^* is. In the case we have

$$\text{ind}[T^*] = -\text{ind}[T]$$

Theorem B.IV.8: [K1, Th. 2.18, p. 204]

Let T, S ε $C(X,Y)$ be desnsely defined. Then

$$\delta(T,S) = \delta(S^*,T^*) \text{ and } \hat{\delta}(T,S) = \hat{\delta}(T^*,S^*)$$

Theorem B.IV.9:

Let T ε $C(X,Y)$ be densely defined, B ε $B(X,Y)$ and C ε $B(Y,Z)$. Then

$$(T+B)^* = T^* + B^* \quad \text{and } (CB)^* = B^*C^*$$

Theorem B.IV.10: [C1]

Let T, S ε $C(X)$ be semi Fredholm. Then

$$\delta(\text{Ker } S, \text{ Ker } T) \leq [2 + \gamma^{-2}(T)]^{\frac{1}{2}} \delta(S,T)$$

Theorem B.IV 11:

Let T, S be as in the previous theorem. If in addition T and S are densely defined, then

$$\delta(\text{Im } T, \text{ Im } S) \leq [2 + \gamma^{-2}(T)]^{\frac{1}{2}} \delta(T,S)$$

Proof: from Theorems B.I.5 and B.IV.7 we obtain

$$\delta(\text{Im } T, \text{ Im } S) = \delta(\text{Im } S^{\perp}, \text{ Im } T^{\perp}) = \delta(\text{Ker } S^*, \text{ Ker } T^*)$$

and since S^* and T^* are ε $C(X^*)$ and semi Fredholm with $\gamma(T^*) = \gamma(T)$ Theorem B.IV.10 yields

$$\delta(\text{Im } T, \text{ Im } S) \leq [2 + \gamma^{-2}(T)]^{\frac{1}{2}} \delta(S^*,T^*)$$

the desired result follows from Theorem B.IV.8.

Theorems B.IV. 10 and B.IV.11 are rather general and provide crude estimates of δ(Ker S, Ker T) and δ(Im Y, Im S). The following result gives a better estimate of δ(Im T, Im S) under certain assumptions.

Theorem B.IV.12:

Let T ε $C(X)$ be semi-Fredholm with dim[Ker T] $< \infty$.

If A is a T-compact operator and S = T + A then

$$\delta(\text{Im } T, \text{Im } S) \quad \frac{a+b\,\gamma(T)}{\gamma(T)}$$

where a, b are positive constants such that

$$\| Au \| \leq a \| u \| + b \| Tu \| \qquad , u \in D(T)$$

<u>Proof</u>: Theorem B.IV.5 implies that $S = T+A \in C(X)$ is semi Fredholm and therefore Im S a closed subspace of X. Now suppose that Ker T = 0, so that T^{-1} exists and is bounded. Let $x \in \text{Im } T$ with $\| x \| = 1$ and let y be such that x = Ty (note that y is unique). Let z = (T+A)y, then

$$\| x-z \| = \| Ay \| \leq a \| y \| + b \| Ty \|$$

but

$$y = T^{-1}x$$

so that

$$\| x-z \| \leq (a \| T^{-1} \| + b) \| x \| = \frac{a}{\gamma(T)} + b$$

Hence

$$\delta(\text{Im } T, \text{Im } S) \leq \| x-z \| \leq \frac{a+b\,\gamma(T)}{\gamma(T)}$$

Now let Ker T $\neq$ 0, since it is finite dimensional we may write

$$X = X_0 \oplus \text{Ker } T$$

Let $\tilde{T}$ be the restriction of T to X_0, then Ker $\tilde{T}$ = 0, Im $\tilde{T}$ = Im T and $\text{Im}(\tilde{T}+A) \subset \text{Im}(T+A)$. Thus

$$\delta(\text{Im } T, \text{Im } S) \leq \delta(\text{Im } \tilde{T}, \text{Im}(\tilde{T}+A))$$

$$\leq a\|\tilde{T}^{-1}\| + b = \frac{a}{\gamma(T)} + b$$

This completes the proof.

We mention that the above result holds if A is T-bounded with $a + b\,\gamma(T) < \gamma(T)$ since in this case $S = T+A \in C(X)$ is semi Fredholm (see Theorem B.IV.4). Furthermore, we obtain

$$\delta(\text{Im } T, \text{Im } S) \leq \frac{a+b\gamma(T)}{\gamma(T)} < 1$$

and since (Theorem B.I.5)

$$\delta(\text{Im } T, \text{Im } S) = \delta(\text{Im } S^{\perp}, \text{Im } T^{\perp})$$

we have (Theorem B.I.3),

$$\text{codim}[\text{Im } S] = \dim[\text{Im } S^{\perp}] \leq \dim[\text{Im } T^{\perp}] = \text{codim}[\text{Im } T]^{\dagger}$$

Also, it is now clear that the estimate given by Theorem B.IV.11 is unnecessarily large. Indedd, from Theorems B.IV.11 and B.II.5 we would have

$$\delta(\text{Im } T, \text{Im } S) \leq [2 + \gamma^{-2}(T)]^{\frac{1}{2}}(a^2+b^2)^{\frac{1}{2}}$$

But

$$\frac{a}{\gamma(T)} + b \leq [1+\gamma^{-2}(T)]^{\frac{1}{2}} (a^2+b^2)^{\frac{1}{2}} < [2+\gamma^{-2}(T)]^{\frac{1}{2}}(a^2+b^2)^{\frac{1}{2}}$$

Thus, Theorem B.IV.12 provides a better estimate (from a quantitative of view) of $\delta(\text{Im } T, \text{Im } S)$.

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† for any closed subspace M of a Banach space X we have $\text{codim}[M] = \dim[M^{\perp}]$

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