SOME INVESTIGATIONS INTO THE STRUCTURE OF LOCAL GAUGE-INVARIANT FIELD THEORIES

by

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ABSTRACT

This thesis is divided into three sections. The first deals with Ultra Violet behaviour in certain Lagrangian formulations of massive spin-one theories. A detailed analysis of the so called "Unitary Gauge" formulation is performed. The non analytic behaviour is explicitly calculated and used to derive a consistent renormalization scheme, from which finite scattering-matrix elements are obtained. Comparison is made with the "Renormalizable Gauge" which is formally related by a point transformation of the fields and a suitable limiting process. The conditions under which the two theories may be considered equivalent is carefully examined. The second section is concerned with on-shell behaviour of non-abelian gauge theories and the infra-red problems there encountered. The physical interpretation of these divergences is investigated and comparison is drawn between the similarities and differences in these theories and that of Quantum Electrodynamics. The third section is devoted to the dynamics of non-abelian gauge theories; in particular to the possibility of dynamical generation of mass. These investigations fall naturally into two parts: stability considerations and approximation schemes. Stability of the theory is considered using functional techniques. The "effective potential" for a composite field is constructed and used to determine the conditions under which dynamical breakdown may occur. The difficulties to be encountered in any approximation scheme are outlined and discussed. Calculations are performed with one such scheme.

PREFACE

The work presented in this thesis was carried out between October 1971 and August 1974 under the supervision of Professor T.W.B. Kibble. Except where stated, this work is original and has not been submitted for a degree of this or any other university.

The author wishes to thank Professor T.W.B. Kibble not only for his guidance and encouragement, but also for his readiness to discuss ideas of a more speculative nature. He would also like to thank Dr. D.A. Ross, all other members of the Theoretical Group at Imperial College, and Miss M. Chock for typing the thesis.

The financial support of the Science Research Council is gratefully acknowledged.

This thesis is dedicated to my parents,

EMILY and LEONARD

WOODHOUSE

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SECTION ONE

The Renormalizability of the U-Gauge

THE CANONICAL QUANTIZATION

The starting point for Canonical Quantization is the Lagrangian density,

$$\mathcal{L} = \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} B^{\mu\nu}(\partial_\nu B_\mu - \partial_\mu B_\nu) + \frac{1}{2} m^2 B_\mu B^\mu \left(1+\frac{e\phi}{m}\right)^2$$

$$+ \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{2}\mu^2 f \phi^3 - \frac{1}{8} f^2 \phi^4 \tag{1}$$

The Euler-Lagrange equations for this system are

$$B_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu \tag{2}$$

$$\partial^\nu B_{\nu\mu} = m^2 \left(1+\frac{e\phi}{m}\right)^2 B_\mu \tag{3}$$

$$(\Box + \mu^2)\phi = em\left(1+\frac{e\phi}{m}\right) B_\mu B^\mu - \frac{3}{2}\mu^2 f \phi^2 - \frac{1}{2} f^2 \phi^3 \tag{4}$$

Given a Lorentz frame one can perform the decomposition

$$\left.\begin{aligned} B_{ij} &= \partial_j B_i - \partial_i B_j \\ B_{oj} &= \partial_j B_o - \partial_o B_j \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} -\partial^j B_{oj} &= m^2\left(1+\frac{e\phi}{m}\right)^2 B_o \\ \partial^o B_{oj} + \partial^i B_{ij} &= m^2\left(1+\frac{e\phi}{m}\right) B_j \end{aligned}\right\} \tag{6}$$

$$(\Box+\mu^2)\phi = em\left(1+\frac{e\phi}{m}\right) B_o B^o + em\left(1+\frac{e\phi}{m}\right) B_j B^j - \frac{3}{2}\mu^2 f\phi^2 - \frac{1}{2} f^2\phi^3 \tag{7}$$

Since no time derivatives of B_o appear, it is evident that this component is not a true dynamical variable.

The conjugate momenta are defined by

$$\pi^B_j = \partial\mathcal{L}/\partial(\partial^0 B^j) = B_{0j} \tag{8}$$

$$\pi^\phi = \partial\mathcal{L}/\partial(\partial^0\phi) = \partial_0\phi \tag{9}$$

Quantization is implemented by the equal time commutation relations.

$$\begin{aligned}
&[B^i(\underline{x},t), \pi^B_j(\underline{y},t)]_- = i\delta^i_j\,\delta_3(\underline{x}-\underline{y})\\
&[B^i(\underline{x},t), B^j(\underline{y},t)]_- = 0\\
&[\pi^B_i(\underline{x},t), \pi^B_j(\underline{y},t)]_- = 0\\
&[\phi(\underline{x},t), \pi^\phi(\underline{y},t)]_- = i\delta_3(\underline{x}-\underline{y})\\
&[\phi(\underline{x},t), \phi(\underline{y},t)]_- = 0\\
&[\pi^\phi(\underline{x},t), \pi^\phi(\underline{y},t)]_- = 0
\end{aligned} \tag{10}$$

The field Equations (5), (6), and (7) may be written entirely in terms of the canonical variables

$$B_{ij} = \partial_j B_i - \partial_i B_j \tag{11}$$

$$\pi^B_j = \partial_j\left\{-\frac{1}{m^2}\left(1+\frac{e\phi}{m}\right)^{-2}\partial^k\pi^B_k\right\} - \partial_0 B_j \tag{12}$$

$$\partial^0\pi^B_j + \partial^i B_{ij} = m^2\left(1+\frac{e\phi}{m}\right)B_j \tag{13}$$

$$(\Box+\mu^2)\phi = \frac{e}{m^3}\left(1+\frac{e\phi}{m}\right)^{-3}(\partial^j\pi^B_j)(\partial^k\pi^B_k) + em\left(1+\frac{e\phi}{m}\right)B^jB_j$$
$$-\frac{3}{2}\mu^2 f\phi^2 - \frac{1}{2}f^2\phi^3 \qquad (14)$$

$$B_0 = -\frac{1}{m^2}\left(1+\frac{e\phi}{m}\right)^{-2}\partial^j\pi^B_j \qquad (15)$$

The Hamiltonian density is obtained from the expression

$$\mathcal{H} = \pi^B_j\partial^0 B^j + \pi^\phi\partial^0\phi - \mathcal{L} \qquad (16)$$

by rewriting the Lagrangian density as

$$\mathcal{L} = -\frac{1}{2}\pi^B_j\pi^j_B - \frac{1}{4}(\partial_j B_i - \partial_i B_j)(\partial^j B^i - \partial^i B^j)$$

$$+\frac{1}{2}m^2 B_j B^j\left(1+\frac{e\phi}{m}\right)^2 + \frac{1}{2m^2}\left(1+\frac{e\phi}{m}\right)^{-2}(\partial^j\pi^B_j)(\partial^k\pi^B_k)$$

$$+\frac{1}{2}\pi_\phi\pi_\phi + \frac{1}{2}(\partial_j\phi)(\partial^j\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{2}\mu^2 f\phi^3 - \frac{1}{8}f^2\phi^4 \qquad (17)$$

and using (9) and (12) (omitting a spatial divergence)

$$\mathcal{H} = -\frac{1}{2}\pi^j_B\pi^B_j + \frac{1}{4}(\partial_j B_i - \partial_i B_j)(\partial^j B^i - \partial^i B^j) + \frac{1}{2}\pi_\phi\pi_\phi$$

$$-\frac{1}{2}m^2 B_j B^j\left(1+\frac{e\phi}{m}\right)^2 + \frac{1}{2m^2}\left(1+\frac{e\phi}{m}\right)^{-2}(\partial^j\pi^B_j)(\partial^k\pi^B_k)$$

$$-\frac{1}{2}(\partial_j\phi)(\partial^j\phi) + \frac{1}{2}\mu^2\phi^2 + \frac{1}{2}\mu^2 f\phi^3 + \frac{1}{8}f^2\phi^4 \qquad (18)$$

That this Hamiltonian density is positive can be easily seen when written in usual 3-vector notation.

$$\mathbf{B} = (B^1, B^2, B^3), \qquad \pi^B = (\pi_B^1, \pi_B^2, \pi_B^3).$$

$$\mathcal{H} = \tfrac{1}{2}\pi_B^2 + \tfrac{1}{2}(\nabla\times B)^2 + \tfrac{1}{2}m^2 B^2\left(1+\frac{e\phi}{m}\right)^2 + \frac{1}{2m^2}\left(1+\frac{e\phi}{m}\right)^{-2}(\nabla\cdot\pi_B)^2$$

$$+\tfrac{1}{2}\pi_\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{2}\mu^2 f\phi^3 + \tfrac{1}{8}f^2\phi^4 \qquad (19)$$

aside from the ϕ^3 term, this is manifestly positive definite. The Hamiltonian density is readily decomposable into "free" and "interaction" pieces

$$\mathcal{H} = \mathcal{H}_B + \mathcal{H}_\phi + \mathcal{H}_I \qquad (20)$$

To facilitate transition to the interaction picture it is convenient to analyse the "free" pieces first.

FREE SCALAR FIELD

The Hamiltonian density,

$$\mathcal{H}_\phi = \tfrac{1}{2}\pi_\phi\pi_\phi - \tfrac{1}{2}(\partial_j\phi)(\partial^j\phi) + \tfrac{1}{2}\mu^2\phi^2 \qquad (21)$$

can be derived from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}\mu^2\phi^2 \qquad (22)$$

The theory of such a system is well known. Of particular relevence is the value of the time ordered two point function,

$$\langle 0|T[\phi(x)\phi(0)]|0\rangle = i\Delta_F(x;\mu)$$

$$\Delta_F(x;\mu) = \int d^4k\, e^{-ik\cdot x} \frac{1}{k^2-\mu^2+i\varepsilon} \tag{23}$$

FREE VECTOR FIELD

The theory of the free massive vector field is somewhat more involved, and will be treated here in a little more detail. A convenient starting point is Lagrangian density for the "Proca" field.

$$\mathcal{L} = \frac{1}{4} B_{\mu\nu}B^{\mu\nu} - \frac{1}{2} B^{\mu\nu}(\partial_\nu B_\mu - \partial_\mu B_\nu) + \frac{1}{2} m^2 B_\mu B^\mu \tag{24}$$

The equations of motion for this system are,

$$\begin{aligned} B_{\mu\nu} &= \partial_\nu B_\mu - \partial_\mu B_\nu \\ \partial^\nu B_{\nu\mu} &= m^2 B_\mu \end{aligned} \tag{25}$$

From which follows, for the case of non-vanishing mass,

$$\partial_\mu B^\mu = 0 \tag{26}$$

$$(\Box + m^2) B_\mu = 0 \tag{27}$$

Taking the canonical pair as,

$$B_j \quad ; \quad \pi_j = \partial\mathcal{L}/\partial(\partial^0 B^j) = B_{0j} \tag{28}$$

leads to the set of equations

$$\pi_j = \partial_j\left(-\frac{1}{m^2}\partial^k \pi_k\right) - \partial_0 B_j \tag{29}$$

$$B_{ij} = \partial_j B_i - \partial_i B_j \tag{30}$$

$$B_0 = -\frac{1}{m^2}\partial^j \pi_j \tag{31}$$

$$\partial^0 \pi_j + \partial^i B_{ij} = m^2 B_j \tag{32}$$

Rewriting (29) in the form

$$\left(\delta_j{}^k + \frac{\partial_j \partial^k}{m^2}\right)\pi_k = -\partial_0 B_j \tag{33}$$

allow the non-vanishing commutation relation,

$$[B^i(\underline{x},t), \pi_j(\underline{y},t)]_- = i\delta^i_j\, \delta_3(\underline{x}-\underline{y}) \tag{34}$$

to be written in the more convenient form,

$$[B^i(\underline{x},t), \partial_0 B_j(\underline{y},t)]_- = -i\left(\delta^i_j + \frac{\partial^i \partial_j}{m^2}\right)\delta_3(\underline{x}-\underline{y}) \tag{35}$$

Equation (27) is satisfied by

$$B_\mu(x) = \int \frac{d^3k}{2\omega_{\underline{k}}}\left[a_\mu(\underline{k})e^{-ik\cdot x} + a^*_\mu(\underline{k})\, e^{ik\cdot x}\right] \tag{36}$$

$$\omega_{\underline{k}}^2 = \underline{k}^2 + m^2$$

or inversely

$$a_\mu(\underline{k}) = i\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} B_\mu(\underline{x},t) \tag{37}$$

where $B_\mu(x)$ is hermitian by construction. Equations (33) and (37) together with,

$$\begin{aligned}[B_i(\underline{x},t) \, , \, B_j(\underline{y},t)]_- &= 0 \\ [\partial_0 B_i(\underline{x},t) \, , \, \partial_0 B_j(\underline{y},t)]_- &= 0\end{aligned} \tag{38}$$

imply the relations,

$$\begin{aligned}[a_i(\underline{k}) \, , \, a_j(\underline{k}')]_- &= 0 \\ [a_i(\underline{k}), a_j^*(\underline{k}')]_- &= -2\omega_{\underline{k}} \left(g_{ij} - \frac{k_i k_j}{m^2}\right) \delta_3(\underline{k}-\underline{k}')\end{aligned} \tag{39}$$

Using the orthogonality and completeness of the Fourier expansion (36), equation (26) takes the form,

$$a_0(\underline{k}) = -\frac{k^j a_j(\underline{k})}{\omega_{\underline{k}}} \tag{40}$$

so that (39) may be extended to

$$\begin{aligned}[a_\mu(\underline{k}), a_\nu(\underline{k}')]_- &= 0 \\ [a_\mu(\underline{k}), a_\nu^*(\underline{k}')]_- &= -2\omega_{\underline{k}} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right) \delta_3(\underline{k}-\underline{k}')\end{aligned} \tag{41}$$

Knowledge of the commutator enables evaluation of the vacuum expectation value

$$\langle 0|B_\mu(x)B_\nu(0)|0\rangle = \int \frac{d^3k}{2\omega_{\underline{k}}} \cdot \frac{d^3k'}{2\omega_{\underline{k}'}} \cdot e^{-ik\cdot x} \langle 0|[a_\mu(\underline{k}), a^*_\nu(\underline{k}')]|0\rangle$$

$$= -\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right) \int \frac{d^3k}{2\omega_{\underline{k}}} e^{-ik\cdot x}$$

$$= -i\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right) \Delta^{(+)}(x)$$

(42)

Conjugation gives,

$$\langle 0|B_\nu(0)B_\mu(x)|0\rangle = +i\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right)\Delta^{(-)}(x) \tag{43}$$

Hence the time ordered "propagator" follows at once from (42) and (43).

$$\langle 0|T[B_\mu(x)B_\nu(0)]|0\rangle = -i\theta(x^0)\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right)\Delta^{(+)}(x)$$
$$+ i\theta(-x^0)\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}\right)\Delta^{(-)}(x) \tag{44}$$

This expression may be simplified by the manipulation of contour integrals in momentum space. Corresponding results may also be obtained in coordinate space by first smearing (44) with a suitably smooth function and then using the properties of the "Schwinger" invariant function (appendix),

$$\Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) \tag{45}$$

With smearing understood, the result can be stated as

$$\langle 0|T[B_\mu(x)B_\nu(0)]|0\rangle = -i\left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{M^2}\right)\Delta_F(x) + \frac{i}{M^2} g_{0\mu} g_{0\nu}\delta_4(x)$$

$$\Delta_F(x) = \Theta(x^0)\Delta^{(+)}(x) - \Theta(-x^0)\Delta^{(-)}(x) \qquad (46)$$

INTERACTION PICTURE

Transition may now be made to an intermediate representation in which the field operators develop in time according to the free field Hamiltonians. The effective interaction Hamiltonian density is, in this picture

$$\mathcal{H}_I = -\frac{1}{2}m^2 B_j B^j\left[\left(1+\frac{e\phi}{m}\right)^2 - 1\right] + \frac{1}{2M^2}\left[\left(1+\frac{e\phi}{M}\right)^{-2} - 1\right](\partial^j\pi_j)(\partial^k\pi_k)$$

$$+ \frac{1}{2}\mu^2 f\phi^3 + \frac{1}{8}f^2\phi^4 \qquad (47)$$

Eliminating the conjugate momenta in favour of the time component of the vector field (31) gives a non-covariant expression

$$\mathcal{H}_I = -\frac{1}{2}m^2 B_j B^j\left[\left(1+\frac{e\phi}{m}\right)^2 - 1\right] + \frac{1}{2}m^2 B_0 B^0\left[\left(1+\frac{e\phi}{m}\right)^{-2} - 1\right]$$

$$+ \frac{1}{2}\mu^2 f\phi^3 + \frac{1}{8}f^2\phi^4 \qquad (48)$$

It is precisely these non-covariant terms in (46) and (48) which are responsible for the appearance of an additional interaction term, playing a rather important role in the matter of renormalizability. The following technique was first devised by T.D. Lee and C.N. Yang.[1] In the conversion of the time ordered products, occurring in the Scattering operator,

$$\mathcal{S} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \; T\left[\mathcal{H}_I(x_1) \ldots \mathcal{H}_I(x_n)\right] \tag{49}$$

into normal ordered products, the elementary pairing (46) occurs repeatedly. It is possible however to define a co-variant pairing,

$$\langle 0 | \left(B_\mu(x) B_\nu(0) \right)_+ | 0 \rangle = -i \left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2} \right) \Delta_F(x) \tag{50}$$

such that the additional non covariant term in (46) can be regarded as an extra contribution to the interaction Hamiltonian. Examination of (48) indicates that contractions over vector fields can only amount topologically, to chains without branches, or simple loops. Furthermore, the non covariant piece of (46) can only occur in contractions between vertices of the second type appearing in (48). In summary, one must add an infinite set of new interaction vertices to the Hamiltonian.

$$-i\delta H = \sum_{N=1}^{\infty} \left\{ h_N^{(1)} + h_N^{(2)} \right\} \tag{51}$$

$$h_N^{(1)} = \left(\frac{-im^2}{2}\right)^N \int dx_1 \ldots dx_N \, B_0(x_1) B_0(x_1) B_0(x_2) \ldots B_0(x_{N-1}) B_0(x_N) B_0(x_N) \, \Theta(x_1) \ldots \Theta(x_N)$$

$$h_N^{(2)} = \frac{(-im^2)^N}{2N} \int dx_1 \ldots dx_N \, B_0(x_1) B_0(x_1) B_0(x_2) \ldots B_0(x_{N-1}) B_0(x_N) B_0(x_N) \, \Theta(x_1) \ldots \Theta(x_N) \tag{52}$$

where

$$\Theta(x) = \left[\left(1 + \frac{e\phi(x)}{m}\right)^{-2} - 1 \right]$$

$$B_0(x) B_0(y) = -\frac{i}{m^2} \delta_4(x-y) \tag{53}$$

Substitution of (53) into (52) leads to a simplification.

$$h_N^{(1)} = \frac{(-1)^N}{2} (im^2) \int d^4x \, B_0(x) B_0(x) \, \Theta^N(x)$$

$$h_N^{(2)} = \frac{(-1)^N}{2N} \delta_4(0) \int d^4x \, \Theta^N(x) \tag{54}$$

Finally (54) and (51) lead after formal summation, to

$$-i\delta\mathcal{H} = -\frac{im^2}{2} \int d^4x \, B_0(x) B_0(x) \left[1 - \{1 + \Theta(x)\}^{-1} \right] - \frac{1}{2} \delta_4(0) \int d^4x \, \ln \left[1 + \Theta(x) \right] \tag{55}$$

$$= \frac{im^2}{2} \int d^4x \, B_0(x) B_0(x) \left[\left(1 + \frac{e\phi(x)}{m}\right)^2 - 1 \right] + \delta_4(0) \int d^4x \, \ln \left(1 + \frac{e\phi(x)}{m}\right) \tag{56}$$

so that the modification to the Hamiltonian density gives

$$\mathcal{H}_I^{EFF} = -\frac{1}{2} m^2 B_\mu B^\mu \left[\left(1 + \frac{e\phi}{m}\right)^2 - 1 \right] + \frac{1}{2} \mu^2 f \phi^3 + \frac{1}{8} f^2 \phi^4 + i \, \delta_4(0) \ln \left(1 + \frac{e\phi}{m}\right) \tag{57}$$

which aside from the last term is manifestly covariant.

In conclusion, the Canonical Quantization of the System (1) leads naturally to an interaction picture, in which the propagation of free scalar and vector particles are describable by the covariant Green's functions (23) and (50) respectively. In addition, their effective interactions are conveniently summarized by the Hamiltonian density (57). It is important to realise, however, that all parameters so far discussed are "bare" ones not physical ones. For this reason, this is not the most convenient interaction picture in which to do perturbation theory. As will be shown, however, the necessary modifications are minimal.

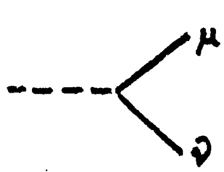
RADIATIVE CORRECTIONS

Having established the form of the effective Hamiltonian in the preceding section, we turn now to the perturbative evaluation of the scattering matrix. "Feynman Rules" for the system may be written down at once from (23), (50) and (57). To establish a scheme whereby the parameters occurring in the theory are physical observables, it is first necessary to perform a rescaling of (23), (50) and (57). The result of this is to reproduce equations (23), (50) and (57) where now the parameters and field operators are the renormalized ones. In addition there will be the well known counterterms. In this section we shall not be concerned with the explicit form of the counterterms; - it is sufficient to know only that they exist. Our attention will be centered upon the non renormalizable divergences, with which this theory is plagued. Postponing explicit form of the rescaling till later the Feynman Rules for system (1) may be expressed graphically as,

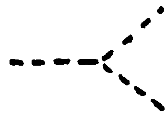
(58)

$$= -i\,\frac{(g_{\mu\nu} - k_\mu k_\nu/m^2)}{k^2 - m^2 + i\varepsilon}$$

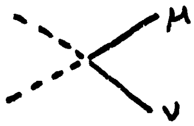
$$= \frac{i}{k^2 - \mu^2 + i\varepsilon}$$



$$= 2iem\,g_{\mu\nu}$$

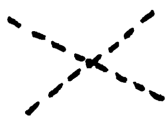


$$= -3i\mu f$$

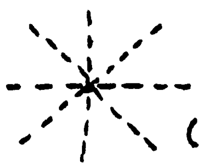


$$= 2ie^2 g_{\mu\nu}$$

(e.f. μ.m., Physical Parameters)



$$= -3if^2$$



(N-Legs)

$$= (-1)^{N+1}\,\delta_4(0)\,(N-1)!\,\left(\frac{e}{m}\right)^N$$

+ All Counterterm Graphs.

The unusual feature of this theory stems from the large momentum behaviour of the vector propagator (58). In conventional formulations, such a behaviour would almost certainly lead to a non-renormalizable theory. In the present case the contrived relations between the various interaction vertices result in a subtle cancellation between the more divergent pieces of the radiative corrections. Although at the present time there exists no conclusive proof, there are strong indications that the renormalized S-matrix elements are finite to every order of perturbation theory. In the following section the divergent pieces of all the vertex functions in the one loop approximation, will be explicitly evaluated. Armed with this information the conditions under which the cancellation of the non renormalizable divergences may occur, can readily be established.

This will in turn place constraints on the allowable forms of subtraction to be used in the full renormalization programme.

CALCULATION OF DIVERGENCES

The worst divergences come from loops of vector particles. It is therefore advantageous, in the one loop approximation, to consider the Green's functions corresponding to the propagation of scalar particles only. We commence by calculating the divergent pieces of the proper (one-particle irreducible) vertex functions. Since the system (58) possesses no derivative coupling interactions, attention may be focussed on the propagators alone. Consider a Feynman integral whose intermediate state is represented by a closed loop of N-vector propagators. The vertices, which may be of the vector-vector-scalar or vector-vector-scalar-scalar type, are labelled in the natural order (1, 2, .., N). The i^{th} Vertex is regarded as a source of momentum P_i. The propagator contribution is

$$\Phi(P_1,\ldots,P_N) = \int d^4k \; \mathrm{Tr}\left[\Delta(k_1)\cdot\Delta(k_2)\cdot\cdots\cdot\Delta(k_N)\right] \tag{59}$$

where the trace of two tensors is defined in the obvious way,

$$\mathrm{Tr}\left[A\cdot B\right] = g_{\mu\nu}\,(A\cdot B)^{\mu\nu} \tag{60}$$

It is clear from the form of the integrand (59) and the symmetry property of the trace, that Φ must be invariant under any cyclic interchange of its variables, corresponding to rotations and reflections of the original loop. The values of the 'source' momenta are arbitrary, subject only to momentum conservation.

It is convenient to decompose the vector propagator into transverse and londitudinal parts.

$$\Delta_{\mu\nu}(k) = \Delta^{T}_{\mu\nu}(k) + \Delta^{L}_{\mu\nu}(k) = -D(k,m)T_{\mu\nu}(k) + \frac{1}{m^2}L_{\mu\nu}(k) \tag{61}$$

Where the momentum space projection operators have been introduced

$$\begin{aligned} T_{\mu\nu}(k) &= g_{\mu\nu} - k_\mu k_\nu / k^2 \\ L_{\mu\nu}(k) &= k_\mu k_\nu / k^2 \\ D(k,m) &= \frac{1}{k^2 - m^2 + i\varepsilon} \end{aligned} \tag{62}$$

Substituting (61) into (59) and expanding gives,

$$\begin{aligned} \Phi = \int d^4k \, \mathrm{Tr} \Big\{ & \Delta^T(k_1)\Delta^T(k_2)\ldots\Delta^T(k_N) + \sum_{i\in(12..N)} \Delta^T(k_1)\cdot\cdot\Delta^L(k_i)\cdot\cdot\Delta^T(k_N) \\ & \cdots + \sum_{i<j\in(12..N)} \Delta^L(k_1)\ldots\Delta^T(k_i)\ldots\Delta^T(k_j)\ldots\Delta^L(k_N) \;+ \\ & \sum_{i\in(12..N)} \Delta^L(k_1)\ldots\Delta^T(k_i)\ldots\Delta^L(k_N) + \Delta^L(k_1)\Delta^L(k_2)\ldots\Delta^L(k_N) \Big\} \end{aligned} \tag{63}$$

As such this integral is undefined. A sensible regulator scheme is required. At the level of approximation to which we are working it is convenient to use the recently devised technique of dimensional regularization.[2] This entails an analytic continuation in the dimension of space time 'ω'. To this end we define a new quantity

$$\Phi(P_1,\ldots,P_N;\omega) = \int d\mu(k,\omega)\, \mathrm{Tr}\left\{\Delta^T(k_1)\Delta^T(k_2)\ldots\Delta^T(k_N) + \cdots \text{etc.}\right\} \tag{64}$$

The measure of integration is taken initially as the usual measure on $\mathbb{R}^\omega$ for positive integer ω. (i.e. after Wick rotation). The trace is likewise defined over this space. The assumption on which the method relies is that there exists a region of ω for which the integral is well defined, all other values being exhibited by analytic continuation.

In the limit of large momenta, the transverse and londitudinal parts of the vector propagator have the power law behaviour

$$\Delta^T(k) \sim \frac{1}{k^2} \qquad\qquad \Delta^L(k) \sim 1 \tag{65}$$

Simple power counting applied to (63) indicates that only the last three terms are potentially divergent. The dimensionally regulated expression contains an expression proportional to

$$\int d\mu(k,\omega) \tag{66}$$

It is not entirely transparent as to the meaning to be attributed to this expression since this expression is nowhere analytic. Some authors have claimed that one can consistently set this to zero. However, as will be shown, such a procedure is unnecessary and indeed the claim can be made that regarding (66) as zero is rather "dishonest" owing to the exceptional properties of the number. In the following (66) will be denoted by $\delta\omega(o)$ where for integer values of 'ω' this singular object is interpreted to be the Dirac delta function with vanishing argument. Separating out this piece, the last three terms of (64) give,

$$\begin{aligned}\Psi(P_1,\ldots,P_N;\omega) + \left(\tfrac{1}{M^2}\right)^N\delta_4(0) &= \sum_{i<j\in(12..N)} \int d\mu(k,\omega)\,\mathrm{Tr}\left[\Delta^L(k_1)..\Delta^T(k_i)..\Delta^T(k_j)\ldots\Delta^L(k_N)\right] + \sum_{i\in(12..N)}\int d\mu(k,\omega)\,\mathrm{Tr}\left[\Delta^L(k_1)\ldots\Delta^T(k_i)\ldots\Delta^L(k_N)\right] \\ &+ \int d\mu(k,\omega)\left\{\mathrm{Tr}\left[\Delta^L(k_1)\ldots\Delta^L(k_N)\right]-\left(\tfrac{1}{M^2}\right)^N\right\} + \left(\tfrac{1}{M^2}\right)^N\delta_\omega(0)\end{aligned} \tag{67}$$

As such $\psi(P_1 \ldots P_N;\ \omega)$ is a well defined quantity provided one combines the integrands in the third term of (67) before performing the integration. By expanding $\psi(P_1\ P_2 \ldots P_N;\ \omega)$ in a Laurent series about the point $\omega=4$ the divergences can be exhibited as a simple pole at $\omega=4$. Consequently, the net divergence structure of the original integral can be written as

$$\mathcal{D}i\omega\ \Phi(P_1,..,P_N) \cong \left(\tfrac{1}{M^2}\right)^N\delta(0) + \frac{\mathrm{Res}\ \Psi(P_1,..,P_N)}{(2-\omega/2)} \tag{68}$$

$$\mathrm{Res}\,\Psi(P_1,..,P_N) = \lim_{\omega\to 4} (2-\omega/2)\,\Psi(P_1,...,P_N;\omega)$$

The first of the three possible terms contributing to Resψ, although potentially logarithmically divergent, is in fact finite. This can be seen from the fact that a contracted pair

$$\Delta^L{}_{\mu\nu}(k_1)\,\Delta^{T\,\nu}{}_{\sigma}(k_2) \tag{69}$$

is one power lower in the loop momenta than the uncontracted pair; a simple consequence of the transverse and londitudinal natures of the propagators. By the same argument, the second term in (67) is only logarithmically divergent (for $\omega=4$).

To evaluate the pole term in this piece, the origin in momentum space is first shifted so that the transverse factor contains no dependence on source momenta. Expanding out the numerators and denominators, and keeping only the leading terms in powers of loop momenta gives

$$\left(\frac{1}{m^2}\right)^{N-1} \sum_{i\in(12..N)} \int d\mu(k,\omega)\; P_i^{\mu}\,\frac{(k^2 g_{\mu\nu} - k_\mu k_\nu)}{k^6}\,P_{i+1}^{\nu} \tag{70}$$

$$+ \text{ terms finite as } \omega\to 4$$

Performing the integration, and picking out the divergent piece in accordance with (68) gives the pole term,

$$\frac{3}{4}\,\frac{(i)}{(4\pi)^2}\left(\frac{1}{m^2}\right)^{N-1}\left\{\sum_{i\in(12..N)} P_i\cdot P_{i+1}\right\}\frac{1}{(2-\omega/2)} \quad : i\,(\mathrm{Mod}\ N) \tag{71}$$

The divergent piece of the third term in (67) requires a little more effort. Writing out the integrand in detail gives

$$\left(\frac{1}{m^2}\right)^N \left\{ \frac{k\cdot(k+P_1)}{k^2}\frac{(k+P_1)\cdot(k+P_1+P_2)}{(k+P_1)^2}\cdots\frac{(k+P_1+\cdots+P_{N-1})\cdot(k+P_1+\cdots+P_N)}{(k+P_1+\cdots+P_{N-1})^2} - 1 \right\}$$

$$= \left(\frac{1}{m^2}\right)^N \left\{ \prod_{\alpha=1}^{N}\left(1 - \frac{P_\alpha\cdot Q_\alpha}{Q_\alpha^2}\right) - 1 \right\} \qquad (72)$$

where $$Q_\alpha = k + P_1 + P_2 + \cdots + P_\alpha$$

Expanding the product (72) and applying simple power counting arguments shows that only the first four terms contain divergences when integrated.

$$\left(\frac{1}{m^2}\right)^N \left\{ -\sum_\alpha \frac{(P_\alpha\cdot Q_\alpha)}{Q_\alpha^2} + \sum_{\alpha<\beta} \frac{(P_\alpha\cdot Q_\alpha)}{Q_\alpha^2}\frac{(P_\beta\cdot Q_\beta)}{Q_\beta^2} \right.$$

$$- \sum_{\alpha<\beta<\gamma} \frac{(P_\alpha\cdot Q_\alpha)}{Q_\alpha^2}\frac{(P_\beta\cdot Q_\beta)}{Q_\beta^2}\frac{(P_\gamma\cdot Q_\gamma)}{Q_\gamma^2}$$

$$\left. + \sum_{\alpha<\beta<\gamma<\delta} \frac{(P_\alpha\cdot Q_\alpha)}{Q_\alpha^2}\frac{(P_\beta\cdot Q_\beta)}{Q_\beta^2}\frac{(P_\gamma\cdot Q_\gamma)}{Q_\gamma^2}\frac{(P_\delta\cdot Q_\delta)}{Q_\delta^2} \right\} \qquad (73)$$

Straight forward integration followed by Laurent expansion in ω, leads to the pole term.

$$\frac{(-i)(m^2)^{-N}}{24(4\pi)^2(2-\omega/2)} \Big\{ \sum_{\alpha<\beta}^{} \left[4(P_\alpha\cdot\Delta_{\alpha\beta})(\Delta_{\alpha\beta}\cdot P_\beta)+2(P_\alpha\cdot P_\beta)(\Delta^2_{\alpha\beta})\right]$$

$$-\sum_{\alpha<\beta<\gamma}\left[2(P_\alpha\cdot P_\beta)(P_\gamma\cdot\Delta_{\alpha\beta})-4(P_\alpha\cdot P_\gamma)(P_\beta\cdot\Delta_{\alpha\beta})+2(P_\beta\cdot P_\gamma)(P_\alpha\cdot\Delta_{\alpha\beta})\right. \tag{74}$$

$$\left.-4(P_\alpha\cdot P_\beta)(P_\gamma\cdot\Delta_{\alpha\gamma})+2(P_\alpha\cdot P_\gamma)(P_\beta\cdot\Delta_{\alpha\gamma})+2(P_\beta\cdot P_\gamma)(P_\alpha\cdot\Delta_{\alpha\gamma})\right]$$

$$-\sum_{\alpha<\beta<\gamma<\delta}\left[(P_\alpha\cdot P_\beta)(P_\gamma\cdot P_\delta)+(P_\alpha\cdot P_\gamma)(P_\beta\cdot P_\delta)+(P_\alpha\cdot P_\delta)(P_\beta\cdot P_\gamma)\right]$$

where
$$\Delta_{\alpha\beta} = P_{\alpha+1} + P_{\alpha+2} + \cdots + P_\beta$$

$$\alpha,\beta,\gamma,\delta \in \{1,2,\ldots,N\}$$

For future convenience (74) will be denoted by,

$$\left(\frac{1}{m^2}\right)^N \chi(P_1, P_2, \ldots, P_N) \tag{75}$$

THE VERTEX FUNCTIONS

Having evaluated the basic loop integral it remains only to sum over all possible diagrams corresponding to an N-leg vertex function. In the one loop approximation, none of the vertex functions will possess external legs corresponding to the vector particles, these occurring in the intermediate state only. It turns out to be convenient to add the contributions to the various pole terms which are of a given degree in the external momenta.

(a) Momentum Independent Divergences

The momentum independent divergences correspond to the quartic divergences in (59). Consider all graphs which possess a loop of vector particles. Consider, first, those graphs which have "a" vertices of the $\phi B_\mu{}^2$ type and "b" vertices of the $\phi^2 B_\mu{}^2$ type. Such graphs will possess a loop consisting of a+b vector propagators, and so each will contribute a term

$$\left(\frac{1}{m^2}\right)^{a+b} \cdot \delta_4(0) \tag{76}$$

The following factors must be included

(i) Propagators and Vertices: - $(i)^{a+b}(2iem)^a(2ie^2)^b$

(ii) Vertex arrangements: - $\frac{1}{2}\frac{(a+b-1)!}{a!\,b!}$

(iii) External legs: - $\left(\frac{1}{2}\right)^b N!$

Hence each graph will contribute an overall factor of

$$(-1)^{a+b}\,2^{a-1}\left(\frac{e}{m}\right)^N N!\,\frac{(a+b-1)!}{a!\,b!}\cdot\delta_4(0) \tag{77}$$

The net contribution from all possible graphs with 'N' external legs is obtained by summing over (77) for non-negative 'a' and 'b', subject to

$$a + 2b = N \tag{78}$$

namely

$$\frac{1}{2} N! \left(\frac{e}{m}\right)^N J_N \, \delta_4(0) \tag{79}$$

Where a convenient function has been introduced

$$J_N = \sum_{\substack{a+2b=N \\ a,b \geqslant 0}} (-1)^{a+b} 2^a \frac{(a+b-1)!}{a!\, b!}$$

There is one other graph which contributes to this class of divergences. This is the last graph in (58), and has the value

$$(-1)^{N+1} (N-1)! \left(\frac{e}{m}\right)^N \cdot \delta_4(0) \tag{80}$$

It is shown in the appendix that J_N may be evaluated explicitly,

$$J_N = \frac{2(-1)^N}{N} \tag{81}$$

The net coefficient of the $\delta_4(o)$ term in an N-point vertex function is, using (79), (80) and (81), precisely zero. This result was first obtained by K. Nishijima and T. Watanabe.[3)]

(b) Divergences, Quadratic in External Momenta

The typical contribution of this type is from (71)

$$\frac{3i\,(m^2)^{1-N}}{4(4\pi)^2(2-\omega/2)} \cdot \left[P_1 \cdot P_2 + P_2 \cdot P_3 + P_3 \cdot P_4 + \cdots P_{N-1} \cdot P_N + P_N \cdot P_1 \right] \tag{82}$$

The form of this expression allows a simple method of summation. Note that from the set of all possible scalar products of the form $(P_i.P_j)$, only those which correspond to adjacent vertices appear in (82). Although this is clearly exhibited by the choice of labelling being used it is in fact a characteristic of the loop itself. From this it follows that the only graphs which contribute a divergent piece proportional to $(P_i.P_j)$, are those which have the momenta P_i and P_j flowing through adjacent vertices. Consequently, to obtain the net coefficient of any $(P_i.P_j)$ term in the vertex function, one has only to sum over contributions from graphs which satisfy this condition. This is easily done. All the relevant graphs may be divided into four classes corresponding to the vertex structure of the lines carrying the momenta P_i and P_j.

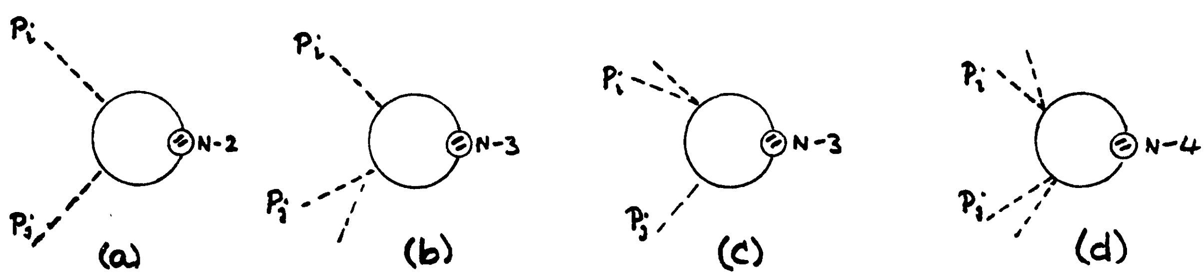


Figure 1

Initially 'N' may be taken as being greater or equal to five.

Type (a)

Consider all diagrams of the type (a,b) which have 'a' vertices of the $\phi B_\mu^{\ 2}$ type, and 'b' vertices of the $\phi^2 B_\mu^{\ 2}$ type, within the shaded circle. Summing over all graphs of this type generates the factors;

(i) Vertices:- $(2iem)^{a+2}(2ie^2)^b \quad : \; a+2b=N-2$

(ii) Vertex arrangements:- number of distinct ways of arranging the (a+b) vertices in a line = $\dfrac{(a+b)!}{a!\,b!}$

(iii) Propagators:- Each graph corresponds to a loop with (a+b+2) vector propagators. Contained in the quadratic monomials associated with each loop there is one term proportional to $(P_i . P_j)$. Only this piece is taken;

$$\frac{3(i)^{a+b+3}\;(P_i\cdot P_j)}{4(4\pi)^2(m^2)^{a+b+2}}\cdot\frac{1}{(2-\omega/2)}$$

(iv) External legs:- There are (N-2) legs unspecified. External momenta may be distributed among them in (N-2)! ways. This, however, amounts to an over counting due to the symmetry in the scalar legs of the $\phi^2 B_\mu{}^2$ vertices. The correct factor is;

$$\frac{(N-2)!}{2^b}$$

Combining these factors and summing over all allowed values of 'a' and 'b' gives

$$\frac{(3i)}{(4\pi)^2}\,(N-2)!\; I_{N-2}\left(\frac{e}{m}\right)^N \frac{(P_i\cdot P_j)}{(2-\omega/2)} \qquad (83)$$

where I_N is defined as

$$I_N = \sum_{\substack{a+2b=N \\ a,b\geqslant 0}} (-1)^{a+b}\, 2^a \, \frac{(a+b)!}{a!\, b!} \tag{84}$$

The remaining three classes are calculated in exactly the same way. The additional momenta flowing through the vertices i and j cause no problems, indeed for type (b) the momentum factor is

$$P_i \cdot (P_j + q_j) = (P_i \cdot P_j) + (P_i \cdot q_j) \tag{85}$$

Where q_j is any of the (N-2) other momenta. Clearly from (85) one still picks up a factor $(P_i . P_j)$ with the same weight as before.

The results, including (83) are simply,

Type (a): $$\frac{3i}{(4\pi)^2} \cdot (N-2)!\; I_{N-2} \left(\frac{e}{m}\right)^N \frac{(P_i \cdot P_j)}{(2-\omega/2)}$$

Type (b): $$\frac{3i}{(4\pi)^2} \cdot (N-2)!\; I_{N-3} \left(\frac{e}{m}\right)^N \frac{(P_i \cdot P_j)}{(2-\omega/2)}$$

Type (c): $$\frac{3i}{(4\pi)^2} \cdot (N-2)!\; I_{N-3} \left(\frac{e}{m}\right)^N \frac{(P_i \cdot P_j)}{(2-\omega/2)}$$

Type (d): $$\frac{3i}{(4\pi)^2} \cdot (N-2)!\; I_{N-4} \left(\frac{e}{m}\right)^N \frac{(P_i \cdot P_j)}{(2-\omega/2)}$$

Hence the net contribution is just the sum of these

$$\frac{3i}{(4\pi)^2}\cdot(N-2)!\left(\frac{e}{m}\right)^N\left[I_{N-2}+2I_{N-3}+I_{N-4}\right]\frac{(P_i\cdot P_j)}{(2-\omega/2)} \tag{87}$$

The value of I_M may be given explicitly. As shown in the appendix it is,

$$I_M = (-1)^M(1+M) \tag{88}$$

This implies that (87) vanishes identically. Consequently, since i and j have been chosen arbitrary throughout, the entire contribution to the vertex function from this type of divergent term must also vanish identically. Thus we arrive at the conclusion that (at the one loop level) the vertex functions with five or more external scalar legs possess no divergent terms which are quadratic monomials in the external momenta.

Turning to the vertex functions with less than five legs four new features appear.

(a) Loops with only two propagators give the $(P_i.P_j)$ factor a weight two. This being easily seen from (71) with N=2.

$$\frac{3i}{4(4\pi)^2}\left(\frac{1}{m^2}\right)\cdot\left\{P_1\cdot P_2+P_2\cdot P_1\right\} \tag{89}$$

(b) The four sets into which the diagrams have been divided, may not contain any elements when N is less than five.

(c) New divergences arise from graphs containing scalar particles in intermediate states.

(d) The counterterm vertices must be included at this level.

In practice effect (a) can be ignored since one must include an extra factor of $\frac{1}{2}$ for loops containing only two propagators. This result being a direct consequence of Wick's theorem. (c) and (d) will be taken as compensating for each other. It is not entirely obvious, however, that these extra divergences can be consistently absorbed by renormalization counterterms, since not all the counterterms can be chosen independently. This will be investigated in more detail later. Effect (b) is dealt with by treating each case separately.

N=4 : Each of the four classes of diagram contain at least one member, so result (87) is valid here also.

N=3 : There are no diagrams of the type (d) (see Fig. 1). However, substituting N=3 into (86) yields a factor I_{-1} which although meaningless in equation (84) is meaningful in expression (88), and fortunately has value zero. Hence the result once again follows here.

N=2 : In this case the results differ since there are only two graphs contributing with a vector loop and two with a scalar loop. Nevertheless, the divergences produced which are quadratic in the external momentum, are absorbed by wave function renormalization, so in this case also the result holds.

To summarise then:- None of the renormalized vertex functions (in the one loop approximation) possess divergent terms which are quadratic monomials in the external momenta. A result which holds both on or off mass-shell. It is interesting to note that in diagramatic language, this implies only loops consisting solely of "londitudinal propagators" can contribute to the overall divergence structure of vertex functions at this order. This peculiar result will be analysed further in a later section.

(c) Divergences, Quartic in External Momenta

The method used in the preceding section involved all but two of the external lines losing their identity; thus resulting in mere combinatorics. Examination of equation (74) indicates that this method will not work in such a straightforward way. A new approach is needed. This is furnished by a study of the most general form the divergent term may have while still satisfying the conditions of Bose statistics (invariance under particle interchange), and momentum conservation. Clearly the result must be a monomial of degree four. There are seven basic objects to consider.

$$A_N = \sum_N a^2 \cdot a^2 \qquad B_N = \sum_N a^2 \cdot ab$$

$$C_N = \sum_N a^2 \cdot b^2 \qquad D_N = \sum_N ab \cdot ab \tag{90}$$

$$E_N = \sum_N a^2 \cdot bc \qquad F_N = \sum_N ab \cdot ac$$

$$G_N = \sum_N ab \cdot cd$$

$$ab \cdot cd \equiv (P_a \cdot P_b)(P_c \cdot P_d)$$

In (90) distinct letters stand for distinct numbers in the summations. All letters take values running from (1,2..,N). The point to notice is that these seven basic forms are invariant under the full permutation group of order 'N'. Momentum conservation, however, implies that B,E,F and G can always be written in terms of A,C and D. For example,

$$\begin{aligned} B_N &= \sum_N a^2 \cdot ab \\ &= \sum_N a^2 \cdot a \Big(\sum_N b\Big) - \sum_N a^2 \cdot a^2 \\ &= -\sum_N a^2 \cdot a^2 = -A_N \end{aligned} \tag{91}$$

Where momentum conservation has been used in the form

$$\sum_N a = 0 \tag{92}$$

So the general form of the vertex function must be expressible in the form,

$$\mathcal{D} i \omega T^{(N)}(P_1, P_2, \ldots, P_N) = \mu A_N + \nu C_N + \lambda D_N \tag{93}$$

In practical calculations, however, it is convenient to re-express (93) in terms of the (N-1) independent momenta. This saves having to symmetrise all the results in order to recover the form (93). Substituting for P_N via (92) gives the reduction

$$A_N \to 2A_{N-1} + 4B_{N-1} + 2C_{N-1} + 4D_{N-1} + 4E_{N-1} + 8F_{N-1} + 8G_{N-1}$$

$$C_N \to A_{N-1} + 2B_{N-1} + 3C_{N-1} + 2E_{N-1} \tag{94}$$

$$D_N \to A_{N-1} + 2B_{N-1} + 3D_{N-1} + 2F_{N-1}$$

It is worth noting that some of the seven invariants (9)) can only be constructed for a sufficiently large 'N'. It can be checked that (94) remains true for all N provided one assigns the value zero to those with insufficient N-values. Substituting (94) into (93) gives the equivalent result,

$$\begin{aligned} \mathcal{D}iv\, T^{(N)}(p_1, p_2, \ldots, p_{N-1}, (-p_1 - \cdots - p_{N-1})) &= (2\mu+\nu+\lambda)A_{N-1} \\ &+ 2(2\mu+\nu+\lambda)B_{N-1} + (2\mu+3\nu)C_{N-1} + (4\mu+3\lambda)D_{N-1} \\ &+ (4\mu+2\nu)E_{N-1} + 2(4\mu+\lambda)F_{N-1} + 8\mu G_{N-1} \end{aligned} \tag{95}$$

So far we have achieved essentially nothing with respect to evaluation of the net divergent piece of the vertex function. A step in this direction comes from the following two points

(i) Once the coefficients (μ ν and λ) are known, the general result may be written down at once via (93) or (95).

(ii) It is no longer necessary to have all the N-external momenta non-zero in (95) in order to deduce the values of the coefficients. To make use of these two points, we must first note the result of setting one of the momenta to zero in the seven invariants. Let this momenta be P_N. The reduction simply amounts to lowering the index N by one; namely

$$\begin{aligned}
&A_N \rightarrow A_{N-1} \equiv A_N(P_1,\dots, P_{N-1}, P_N=0)\\
&B_N \rightarrow B_{N-1}\\
&C_N \rightarrow C_{N-1} \qquad E_N \rightarrow E_{N-1}\\
&D_N \rightarrow D_{N-1} \qquad F_N \rightarrow F_{N-1} \qquad G_N \rightarrow G_{N-1}
\end{aligned} \tag{96}$$

Putting all but three of the momenta in (95) to zero (which are conveniently taken as P_1 P_2 and P_3)

$$\begin{aligned}
\mathcal{D}iw\, T'^{(N)}(P_1,P_2,P_3,(-P_1-P_2-P_3),0,\dots,0) &= (2\mu+\nu+\lambda)A_3\\
&+2(2\mu+\nu+\lambda)B_3 + (2\mu+3\nu)C_3 + (4\mu+3\lambda)D_3\\
&+(4\mu+2\nu)E_3 + 2(4\mu+\lambda)F_3 \qquad [G_3=0]
\end{aligned} \tag{97}$$

Knowing the value of

$$\mathcal{D}iw\, T'^{(N)}(P_1 P_2, P_3, (-P_1-P_2-P_3),0,\dots,0) \tag{98}$$

the three coefficients μ,ν and λ can be calculated. One might think that perhaps one more momenta can be set to zero in (97) but this is not the case. The resulting expression does not lead to three independent equation and thus is insufficient for deduction of the parameters.

The procedure now is straightforward. (98) is calculated explicitly and inserted into (97), from which the values of μ, ν and λ follow at once. Computationally, the advantage of all this is that (98) is simple to evaluate since for any $N \geqslant 4$, $(N-4)$ of the external legs have completely lost their identity, as in the previous section. Hence as before, all diagrams can be grouped into classes according to the topology of the momentum carrying lines, and remaining sums falling into mere combinatorics. In the present case there are nine such classes.

Class (1): The three related diagrams are shown in figure (2)

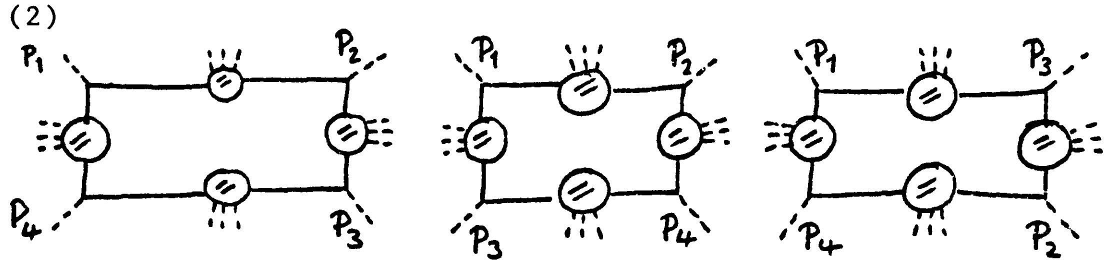
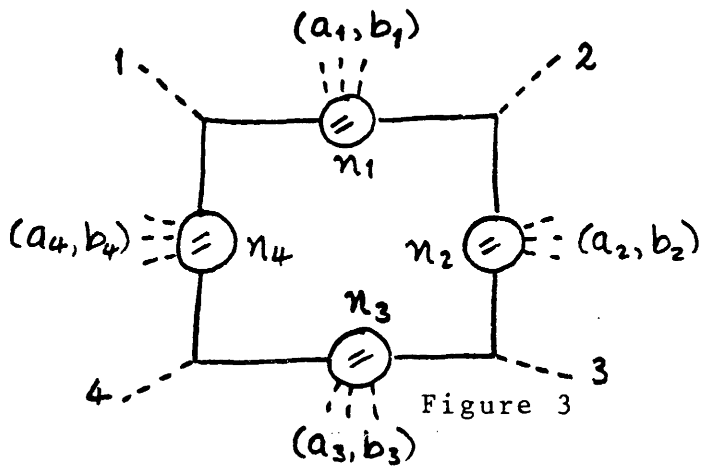


Figure 2

As before summation is first done over those graphs which have a given number of $\phi B_\mu{}^2$ and $\phi^2 B_\mu{}^2$ vertices in each shaded circle; figure (3).



$$
\begin{cases} a_\alpha + 2b_\alpha = n_\alpha \\ \sum_{\alpha=1}^{4} n_\alpha = N-4 \end{cases}
$$

Figure 3

(1) Vertex factors:- $(2iem)^4(2iem)^{\Sigma a_\alpha}(2ie^2)^{\Sigma b_\alpha}$

(2) Vertex arrangements:- $\prod_{\alpha=1}^{4}\dfrac{(a_\alpha+b_\alpha)!}{a_\alpha!\,b_\alpha!}$

(3) Propagators:- The relevent factor here is that corresponding to (74) and (75). A reduction, however, is possible due essentially to the projective nature of the londitudinal propagator,

$$\chi(P_1,0,\ldots,0,P_2,0,\ldots,0,P_3,0,\ldots,0,P_4,0,\ldots,0)=\chi(P_1,P_2,P_3,P_4) \qquad (99)$$

(Proof of this remark is given in the appendix.) Since the loop is equivalent, up to a factor, to one with only four propagators, the net contribution is

$$\frac{(i)^{4+\sum_\alpha^{\cdot}(a_\alpha+b_\alpha)}}{(m^2)^{4+\sum_\alpha^{\cdot}(a_\alpha+b_\alpha)}}\cdot\chi(P_1,P_2,P_3,P_4)$$

(4) External legs:- The (N-4) remaining legs may be labelled in (N-4)! ways. The symmetry of the $\phi^2B_\mu^{\ 2}$ vertex involves an overcounting by $2^{\sum_\alpha b_\alpha}$, giving an overall factor of,

$$\frac{(N-4)!}{2^{\sum_\alpha b_\alpha}}$$

Combining these factors gives

$$16\left(\frac{e}{m}\right)^N\cdot(N-4)!\,\chi(P_1,P_2,P_3,P_4)\left\{\prod_{\alpha=1}^{4}(-1)^{a_\alpha+b_\alpha}\cdot 2^{a_\alpha}\frac{(a_\alpha+b_\alpha)!}{a_\alpha!\,b_\alpha!}\right\} \qquad (100)$$

To complete the calculation it remains only to sum over all possible values of a_α and b_α,

$$16\left(\frac{e}{m}\right)^N\cdot(N-4)!\,\chi(P_1,P_2,P_3,P_4)\sum_{n_\alpha\geq 0}\left(I_{n_1}I_{n_2}I_{n_3}I_{n_4}\right) \tag{101}$$

subject to $n_1+n_2+n_3+n_4 = N-4$

Where definition (84) has been used. Substituting (88) into (101) gives

$$16\left(\frac{e}{m}\right)^N(-1)^N\cdot(N-4)!\,K_{N-4}\,\chi(P_1,P_2,P_3,P_4) \tag{102}$$

Where a new function has been introduced,

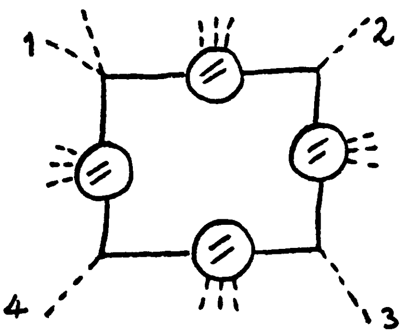
$$\begin{aligned} K_M &= \sum(a+1)(b+1)(c+1)(d+1) \\ & a,b,c,d \geq 0 \\ & a+b+c+d = M \quad (\text{integer}) \\ & K_M = 0 \quad \text{for } M<0 \end{aligned} \tag{103}$$

Adding the remaining contributions for the remaining two diagrams of FiG(2).

$$16(-1)^N\left(\frac{e}{m}\right)^N\cdot(N-4)!\,K_{N-4}\left[\chi(1,2,3,4)+\chi(1,2,4,3)+\chi(1,3,2,4)\right] \tag{104}$$

Class (2): There are twelve related diagrams in this class, a typical member of which is shown in Fig. (4). The other eleven are obtained by interchanging the (1234) labels in all distinct ways.

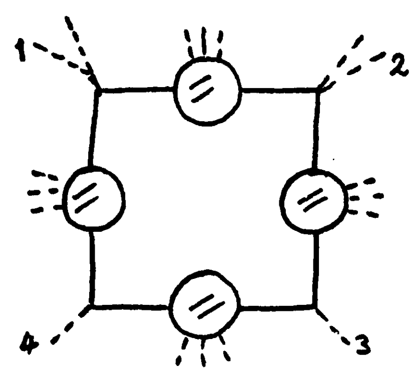
Figure 4



Result:- $-64(-1)^N\left(\frac{e}{m}\right)^N\cdot(N-4)!\,K_{N-5}\left[\chi(1,2,3,4)+\chi(1,2,4,3)+\chi(1,3,2,4)\right]$ (105)

Class (3): There are eighteen related diagrams, a typical member of which is shown in Fig. (5).

Figure 5

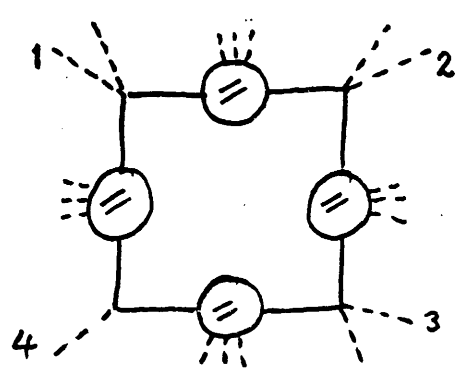


Result:-

$$96(-1)^N\left(\frac{e}{m}\right)^N\cdot(N-4)!\,K_{N-6}\left[\chi(1,2,3,4)+\chi(1,2,4,3)+\chi(1,3,2,4)\right] \quad (106)$$

Class (4): Twelve related diagrams, e.g. Fig. (6).

Figure 6

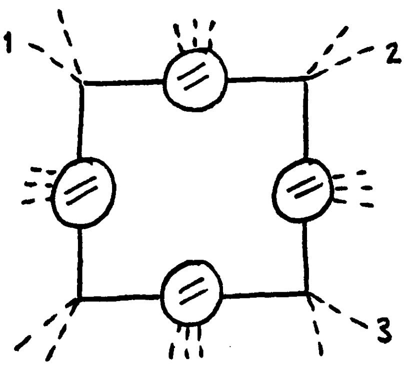


Result:-

$$-64(-1)^N\left(\frac{e}{m}\right)^N\cdot(N-4)!\,K_{N-7}\left[\chi(1,2,3,4)+\chi(1,2,4,3)+\chi(1,3,2,4)\right] \quad (107)$$

Class (5): Three related diagrams, e.g. Fig. (7)

Figure (7)



Result:

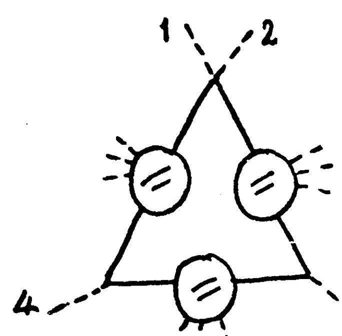
$$16(-1)^N\left(\tfrac{e}{m}\right)^N\cdot(N-4)!\;K_{N-8}\left[\chi(1,2,3,4)+\chi(1,2,4,3)+\chi(1,3,2,4)\right] \tag{108}$$

Class (6): The procedure here is the same as before except that onl, three shaded circles are needed. Correspondingly equation (101) has only three factors of I_n, which leads to a second kind of function instead of (103).

$$\begin{aligned} L_M &= \sum(a+1)(b+1)(c+1) \\ & a,b,c \geqslant 0 \\ & a+b+c = M \text{ (integer)} \\ & L_M = 0 \text{ for } M<0 \end{aligned} \tag{109}$$

The six related diagrams are characterised by Fig. (8).

Figure 8

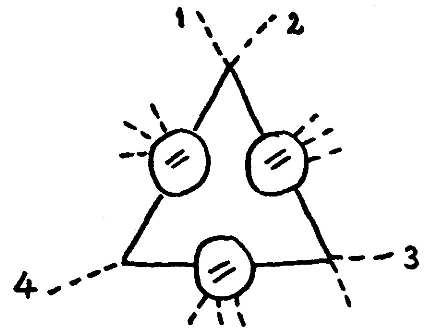


Result:-

$$-8(-1)^N\left(\tfrac{e}{m}\right)^N\cdot(N-4)!\;L_{N-4}\left[\chi(1+2,3,4)+\chi(1+3,2,4)\right.$$
$$\left.+\chi(1+4,2,3)+\chi(2+3,1,4)+\chi(2+4,1,3)+\chi(3+4,1,2)\right] \tag{110}$$

Class (7): Twelve related diagrams. See Fig. (9).

Figure 9

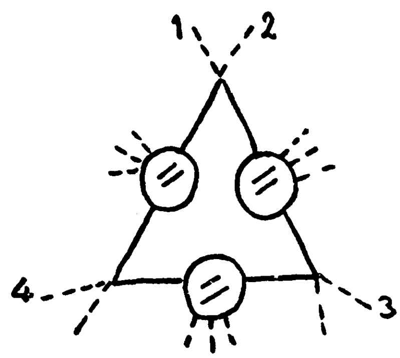


Result:

$$16(-1)^N\left(\tfrac{e}{m}\right)^N\cdot(N-4)!\,L_{N-5}\Big[\chi(1+2,3,4)+\chi(1+3,2,4)$$
$$+\chi(1+4,2,3)+\chi(2+3,1,4)+\chi(2+4,1,3)+\chi(3+4,1,2)\Big] \tag{111}$$

Class (8): Six related diagrams. See Fig. (10).

Figure 10



Result:

$$-8(-1)^N\left(\tfrac{e}{m}\right)^N\cdot(N-4)!\,L_{N-6}\Big[\chi(1+2,3,4)+\chi(1+3,2,4)$$
$$+\chi(1+4,2,3)+\chi(2+3,1,4)+\chi(2+4,1,3)+\chi(3+4,1,2)\Big] \tag{112}$$

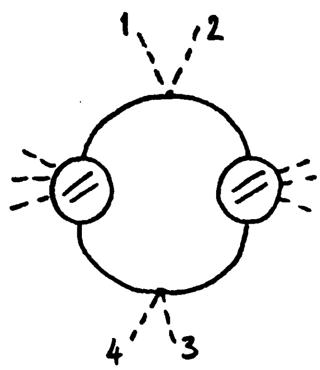
Class (9): This final class has three constituent types of diagram. Once again the procedure is straightforward; appearance of only two shaded circles results in the need for a third function,

$$M_N = \sum (a+1)(b+1)$$

$$a, b \geqslant 0$$
$$a+b = N \text{ (integer)}$$
$$M_N = 0 \quad \text{for} \quad N < 0 \tag{113}$$

In addition to before, however, it is necessary to include a factor of $\frac{1}{2}$, coming from the possibility of interchanging the two shaded circles

Figure 11



Result:

$$2(-1)^N\left(\frac{e}{m}\right)^N \cdot (N-4)!\, M_{N-4}\left[\chi(1+2,3+4)+\chi(1+3,2+4)+\chi(1+4,2+3)\right] \tag{114}$$

Finally, adding all these contributions gives

$$16(-1)^N(N-4)!\left(\frac{e}{m}\right)^N\left[K_{N-4}-4K_{N-5}+6K_{N-6}-4K_{N-7}+K_{N-8}\right]\times \tag{115}$$

$$\{\chi(1,2,3,4)+\chi(1,2,4,3)+\chi(1,3,2,4)\} - 8(-1)^N(N-4)!\left(\frac{e}{m}\right)^N\times$$

$$[L_{N-4}-2L_{N-5}+L_{N-6}]\cdot\{\chi(1+2,3,4)+\chi(1+3,2,4)+\chi(1+4,2,3)$$

$$+\chi(2+3,1,4)+\chi(2+4,1,3)+\chi(3+4,1,2)\} + 2(-1)^N(N-4)!$$

$$\left(\frac{e}{m}\right)^N M_{N-4}\{\chi(1+2,3+4)+\chi(1+3,2+4)+\chi(1+4,2+3)\}$$

It is shown in the appendix that

$$K_N = \binom{N+7}{7} \quad L_N = \binom{N+5}{5} \quad M_N = \binom{N+3}{3} \tag{116}$$

Substitution gives,

$$K_{N-4} - 4K_{N-5} + 6K_{N-6} - 4K_{N-7} + K_{N-8} = M_{N-4}$$

$$L_{N-4} - 2L_{N-5} + L_{N-6} = M_{N-4} \tag{117}$$

So that (115) reduces to

$$(-1)^N (N-1)! \left(\frac{e}{m}\right)^N \cdot \frac{1}{3} \Big[8 \sum_{3 \text{ Perms.}} \chi(a,b,c,d)$$

$$- 4 \sum_{6 \text{ Perms.}} \chi(a+b,c,d) + \sum_{3 \text{ perms.}} \chi(a+b,c+d) \Big] \tag{118}$$

<u>CALCULATION OF THE χ-FUNCTIONS</u>

Despite the enormous simplification so far achieved by setting all but three independent momenta to zero, the explicit calculation of the χ-functions appearing in (118) or (115) is by far the longest part of the calculation. The twelve χ's appearing in (115) must each be evaluated by substitution into the general formula of (74) and (75). Secondly, the dependent momenta P_4 is removed by using momentum conservation

$$P_4 = -(P_1 + P_2 + P_3) \tag{119}$$

Finally, after adding those sets which differ by permutation only (and checking the result is symmetric in the remaining three momenta P_1, P_2, P_3) they are separated into the seven invariants of (90). The result of this rather long calculation is,

$$\sum_{3\,\text{Perms.}} \chi(a,b,c,d) = \frac{i}{24(4\pi)^2(2-\omega/2)}\left[18A_3+36B_3+36C_3+18D_3+33E_3+30F_3\right] \tag{120}$$

$$\sum_{6\,\text{perms.}} \chi(a+b,c,d) = \frac{i}{24(4\pi)^2(2-\omega/2)}\left[30A_3+60B_3+48C_3+42D_3+48E_3+60F_3\right]$$

$$\sum_{3\,\text{perms.}} \chi(a+b,c+d) = \frac{i}{24(4\pi)^2(2-\omega/2)}\left[12A_3+24B_3+12C_3+24D_3\right]$$

So that (118) becomes,

$$\mathcal{D}i\omega\Gamma^{(N)}(P_1,P_2,P_3,(-P_1-P_2-P_3),0,..,0) = \frac{(-1)^N i(N-1)!\,(\frac{e}{m})^N}{2(4\pi)^2(2-\omega/2)} \cdot \left[A_3+2B_3+3C_3+2E_3\right] \tag{121}$$

This is precisely of the form (97) so that the values of the parameters μ, ν and λ are,

$$\mu = 0 \quad , \quad \nu = \frac{(-1)^N i(N-1)!\,(\frac{e}{m})^N}{2(4\pi)^2(2-\omega/2)} \quad , \quad \lambda = 0 \tag{122}$$

This immediately yields the result for the general case via equation (93) namely

$$\mathcal{D}i\omega\Gamma^{(N)}(p_1,p_2,\ldots,p_N)=\frac{(-1)^N i(N-1)!\,(\frac{e}{m})^N}{2(4\pi)^2(2-\omega/2)}\cdot C_N$$

$$C_N=\sum_{a<b}^{N}(p_a\cdot p_a)(p_b\cdot p_b) \qquad (123)$$

Vertex functions in two, three and four external legs require additional treatment. In these cases the contributions from intermediate scalar particles also give divergences. In each case however, these extra graphs contribute only to the renormalization constants, so that (123) remains valid provided the effects of subtraction are accounted for.

$$\mathcal{D}i\omega\Gamma^{(2)}(p,(-p))=\frac{i(\frac{e}{m})^2}{2(4\pi)^2(2-\omega/2)}\left[C_2(p,(-p))+\alpha\mu^2p^2+\beta\mu^4\right] \qquad (124)$$

$$\mathcal{D}i\omega\Gamma^{(3)}(p_1,p_2,(-p_1-p_2))=\frac{-i(\frac{e}{m})^3}{(4\pi)^2(2-\omega/2)}\left[C_3(p_1,p_2,(-p_1-p_2))+\gamma\mu^4\right]$$

$$\mathcal{D}i\omega\Gamma^{(4)}(p_1,p_2,p_3,(-p_1-p_2-p_3))=\frac{3i(\frac{e}{m})^4}{(4\pi)^2(2-\omega/2)}\left[C_4(p_1,p_2,p_3,(-p_1-p_2-p_3))+\delta\mu^4\right]$$

The numbers α, β, γ and δ remained undetermined. Only after specification of a subtraction procedure can they be given explicit values.

RENORMALIZATION AND THE S-MATRIX

In conventional field theories the renormalization charge, mass and "wavefunction" are usually expressed as conditions on the Vertex functions.

$$
\begin{aligned}
&\Gamma^{(2)}(p,(-p))\big|_{p^2=\mu^2} = 0 \\
&d\Gamma^{(2)}(p,(-p))/dp^2\big|_{p^2=\mu^2} = i \\
&\Gamma^{(3)}(p_1,p_2,p_3)\big|_{sp_3(\mu)} = -3i\mu f \\
&\Gamma^{(4)}(p_1,p_2,p_3,p_4)\big|_{sp_4(\mu)} = -3if^2
\end{aligned}
\tag{125}
$$

Where the symmetry point is defined by

$$
sp_N(\mu) \quad : (p_i \cdot p_j) = \mu^2 \left[\frac{N\delta_{ij}-1}{N-1}\right] \qquad i,j \in (1,2,\ldots,N)
\tag{126}
$$

Translating these conditions (125) over to the divergent pieces (124) gives simply

$$
\begin{aligned}
&\mathcal{D}iv\,\Gamma^{(2)}(p,(-p))\big|_{p^2=\mu^2} = 0 \\
&d\mathcal{D}iv\,\Gamma^{(2)}(p,(-p))/dp^2\big|_{p^2=\mu^2} = 0 \\
&\mathcal{D}iv\,\Gamma^{(3)}(p_1,p_2,p_3)\big|_{sp_3(\mu)} = 0 \\
&\mathcal{D}iv\,\Gamma^{(4)}(p_1,p_2,p_3,p_4)\big|_{sp_4(\mu)} = 0
\end{aligned}
\tag{127}
$$

A surprising feature of this theory is that the conditions (127) cannot lead to finite Scattering Matrix Elements.

This will appear in the following. Anticipating this result we pose the question of renormalizability in another way namely; is it possible to establish whether any tenable scheme exists. Our criteria for acceptability will be that the S-Matrix elements following from the theory must be finite. This requirement overdetermines the values of α, β, γ and δ; hence the consistency of the subtraction scheme is mirrored in the consistency of equations determining α, β, γ and δ.

PARTICLE PROCESSES

The propagation of a single particle under the action of self interactions alone is the lowest order Green's function to consider. The condition of finite S-Matrix translates to the first condition of (127). Substituting for C_2, i.e.,

$$C_2(p,(-p)) = (p^2)^2 \tag{128}$$

yields

$$1+\alpha+\beta = 0 \tag{129}$$

Substituting for α in this equation into (124) gives,

$$\mathscr{D}i\omega\,\Gamma^{(2)}(p,(-p)) = \frac{i\left(\frac{e}{m}\right)^2}{2(4\pi)^2(2-\omega/2)}\left[p^2-\mu^2\right]\left[p^2-\beta\mu^2\right] \tag{130}$$

Note that the second condition of (127) is not used. This is not a physical condition in our former sense. Arbitrariness in fixing the wave function normalisation is expressed succinctly by the renormalization Group. The corresponding statement of such a renormalization group for this theory deserves further investigation. Insight into the non-analytic behaviour may well be possible.

Proceeding with the programme, we turn next to the three particle interaction. The finiteness condition is expressed on the full Green's function. A word of caution is, however, necessary. The three particle interaction with external particles on mass shell is not a physically allowed process as can be seen from conservation of four momentum. It may however be realised (in principle) by extrapolation of a processes like figure (12)

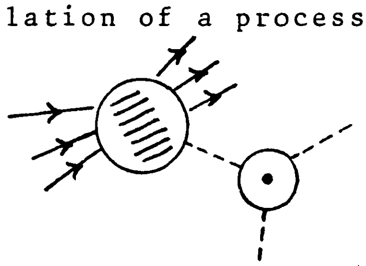


Figure 12

The situation is analogous to Quantum Electrodynamics where the renormalized charge is taken from the limit of zero momentum transfer of electron-electron scattering. The finiteness condition is

$$\mathcal{D}i\omega \Gamma^{(3)}(P_1,P_2,P_3) + \mathcal{D}i\omega \Gamma^{(2)}(P_1,(-P_1))\frac{i}{(P_1^2-\mu^2)}(-3i\mu f) \qquad (131)$$

$$+ \mathcal{D}i\omega \Gamma^{(2)}(P_2,(-P_2))\frac{i}{(P_2^2-\mu^2)}(-3i\mu f) + \mathcal{D}i\omega \Gamma^{(2)}(P_3,(-P_3))\frac{i}{(P_3^2-\mu^2)}(-3i\mu f)$$

$$= 0$$

Where the limits of P_1^2, P_2^2 and P_3^2 tending to the mass shell are taken using the alternative expression (130) for the two point vertex. The resulting constraint on the parameters is

$$9\beta + 2\gamma - 3 = 0 \tag{132}$$

Diagramatically equation (131) is given by Fig. (13).

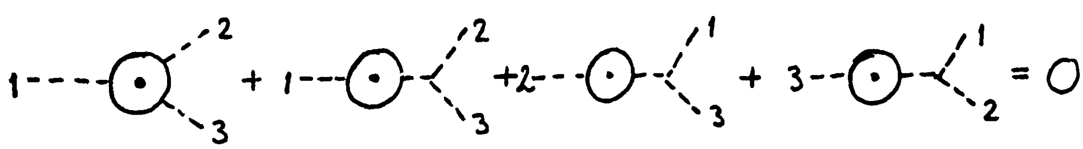
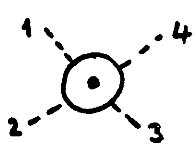
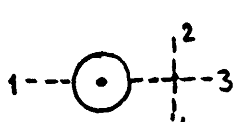


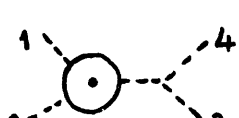
Figure 13

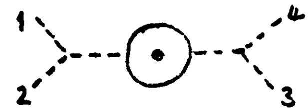
The dotted circles in this and the following stand for (123) or (124).

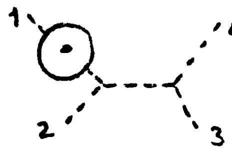
The four particle process may also be used. The contributions are listed below.

(a)  $= 3(6+\delta)\left(\frac{e}{m}\right)^4 \mu^4$

(b)  (4-diags.) $= 6(1-\beta)\left(\frac{e}{m}\right)^4 \mu^4$

(c)  (6-diags.) $= -36\left(\frac{e}{m}\right)^4 \mu^4 - 18(3+\gamma)\left(\frac{e}{m}\right)^4 \mu^4 \sum_{\alpha}^{6} \frac{1}{q_\alpha}$

(d)  $(3\text{diags}) = \frac{27}{2}\left(\frac{e}{m}\right)^4\mu^4 + \frac{9}{4}\left(\frac{e}{m}\right)^4(1-\beta)\mu^4 \sum_{\alpha}^{6} \frac{1}{q_\alpha}$

(e)  $(12\text{-diags}) = 9(1-\beta)\left(\frac{e}{m}\right)^4\mu^4 \sum_{\alpha}^{6} \frac{1}{q_\alpha}$

These results are all given with external legs on mass shell. The q_α are introduced for convenience,

$$q_\alpha \equiv q_{ij} = 1 + (p_i \cdot p_j)\mu^{-2} \tag{133}$$

$$\alpha = (ij) = 12,\ 13,\ 14,\ 23,\ 24,\ 34.$$

An inessential common factor has been omitted from the above. Adding these and equating to zero gives

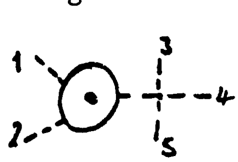
$$\frac{3}{2}(1-4\beta+2\delta)\left(\frac{e}{m}\right)^4\mu^4 - \frac{9}{4}(19+5\beta+8\gamma)\left(\frac{e}{m}\right)^4\mu^4\left\{\sum_{\alpha}^{6}\frac{1}{q_\alpha}\right\} \equiv 0 \tag{134}$$

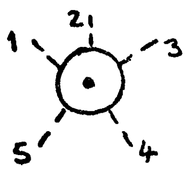
which can only be true provided

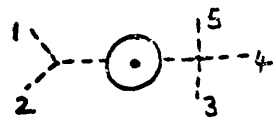
$$1-4\beta+2\delta = 0 \tag{135}$$

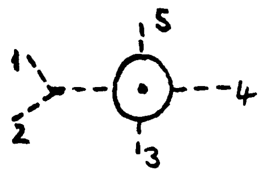
$$19+5\beta+8\gamma = 0 \tag{136}$$

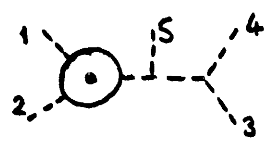
Lastly the five particle process is evaluated. The results only are listed. Calculations are as always straightforward but long.

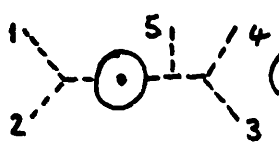
(a)  $(10\text{-diags.}) = -60\left(\frac{e}{m}\right)^5\mu^4 - 3(3+\gamma)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}$

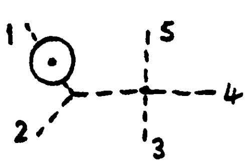
(b)  $= -120\left(\frac{e}{m}\right)^5\mu^4$

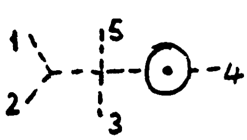
(c)  (10-diags.) $= 45\left(\frac{e}{m}\right)^5\mu^4 + \frac{9}{2}(1-\beta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}$

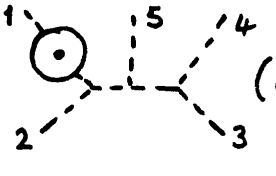
(d)  (10-diags.) $= 270\left(\frac{e}{m}\right)^5\mu^4 + 9(6+\delta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}$

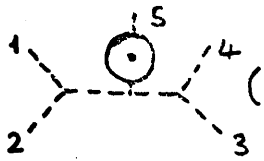
(e)  (30-diags.) $= -54\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\} - 18(3+\gamma)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha\beta}^{15}\frac{1}{q_\alpha q_\beta}\right\}$

(f)  (30-diags.) $= \frac{81}{2}\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\} + 27(1-\beta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha\beta}^{15}\frac{1}{q_\alpha q_\beta}\right\}$

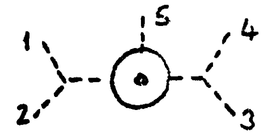
(g)  (20-diags.) $= 9(1-\beta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}$

(h)  (30diags.) $= \frac{27}{2}(1-\beta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}$

(i)  (60 diags.) $= 54(1-\beta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha\beta}^{15}\frac{1}{q_\alpha q_\beta}\right\}$

(j) 

$$\text{(15-diags.)} = \frac{27}{2}(1-\beta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha\beta}^{15}\frac{1}{q_\alpha q_\beta}\right\}$$

(k) 

$$\text{(15-diags.)} = -135\left(\frac{e}{m}\right)^5\mu^4 - 54\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}$$
$$-9(3+\gamma)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha\beta}^{15}\frac{1}{q_\alpha q_\beta}\right\}.$$

The q_α are the same as in (133) except α and β take the values.

$$\sum_{\alpha}^{10} : \alpha \equiv (i,j) = 12, 13, 14, 15, 23, 24, 25, 34, 35, 45.$$

$\sum_{\alpha\beta}^{15}$ $\alpha\beta \equiv (ij;kl) =$

23;45	24;35	25;34	13;45
14;35	15;34	12;45	14;25
15;24	12;35	13;25	15;23
12;34	13;24	14;23	

Adding all contributions for the five particle process,

$$\frac{3}{2}(3-18\beta-2\gamma+6\delta)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha}^{10}\frac{1}{q_\alpha}\right\}+\frac{27}{2}(1-7\beta-2\gamma)\left(\frac{e}{m}\right)^5\mu^4\left\{\sum_{\alpha\beta}^{15}\frac{1}{q_\alpha q_\beta}\right\}$$
$$\equiv 0 \tag{137}$$

which can only be satisfied if,

$$3-18\beta-2\gamma+6\delta=0 \tag{138}$$

$$1-7\beta-2\gamma=0 \tag{139}$$

Examination of (129), (132), (135), (136), (138) and (139) shows that they are all mutually consistent and have a unique solution

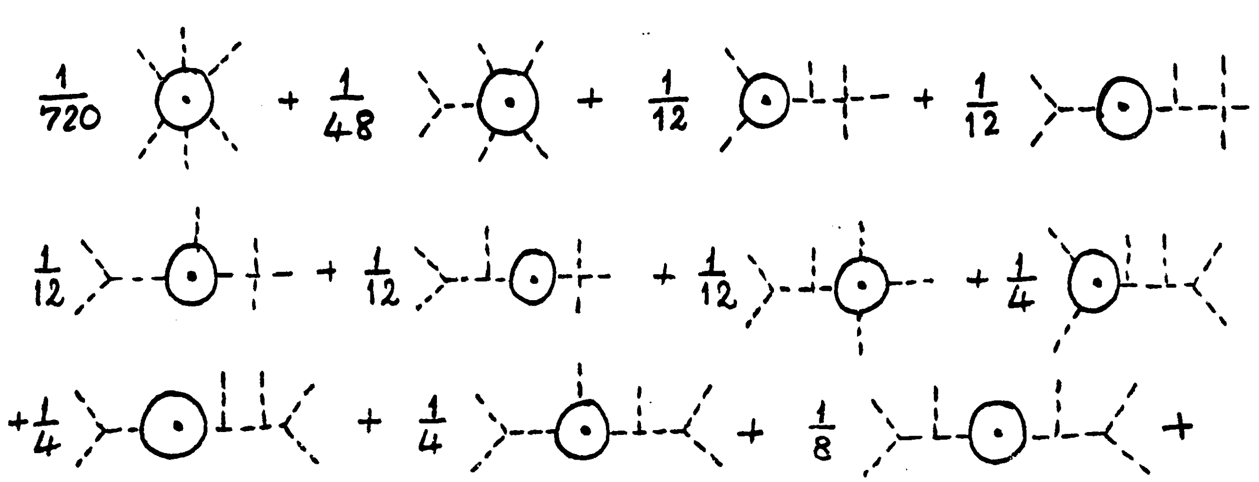
$$\alpha=-2 \qquad \beta=1 \qquad \gamma=-3 \qquad \delta=3/2 \tag{140}$$

Of particular interest is the value of δ. Had conditions (125) or equivalently (127) been used one would have arrived at

$$\alpha=-2 \qquad \beta=1 \qquad \gamma=-3 \qquad \delta=-6 \tag{141}$$

This discrepancy substantiates the claim made at the beginning of this section, that the conventional conditions do not lead to finite S-Matrix elements. The reason why only the four point function differs from usual is not difficult to see. In normal theories, the renormalized vertex functions are finite both on and away from the point of subtraction. This, however, is no longer true in the present case; the renormalized vertex functions (and indeed the full Green's functions) are finite only at the subtraction point. The decomposition of the full (connected) Green's function into one-particle irreducible pieces, shows that the four particle scattering process is the lowest order process which involves an off-shell (i.e. off subtraction point) vertex function in an intermediate state. This residual divergence must be cancelled by a corresponding one in the four-point vertex function if the scattering process is to be finite.

As a further check for consistency, the six-particle and seven-particle processes may also be evaluated. In order to reduce the large number of distinct graphs, the calculations are restricted to the corresponding six and seven particle symmetry points (126). The price to be paid here, is one of lost generality. (It is not inconceivable that finiteness in a given region of physical momenta may imply finiteness for all such physical momenta; - neither proof nor disproof of this possibility has been found.) The calculations are straightforward but rather long. It suffices to say that the cancellation does occur thus rendering the S-Matrix elements finite (at least to this order of approximation). The results are recorded as two diagramatic equations Fig. (14) and Fig. (15).



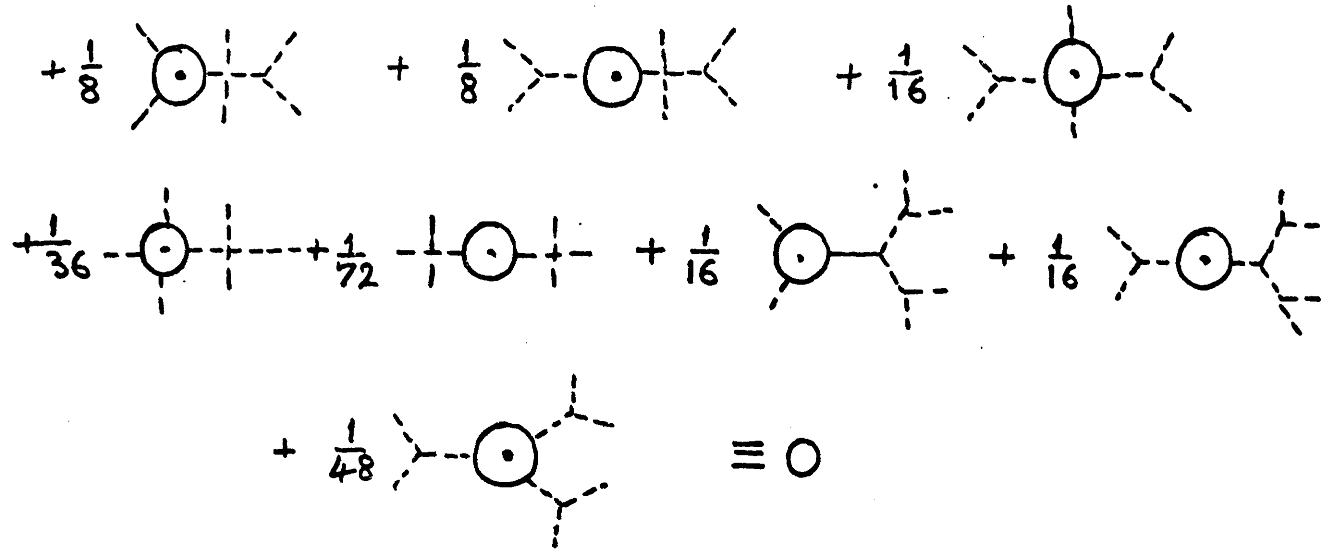
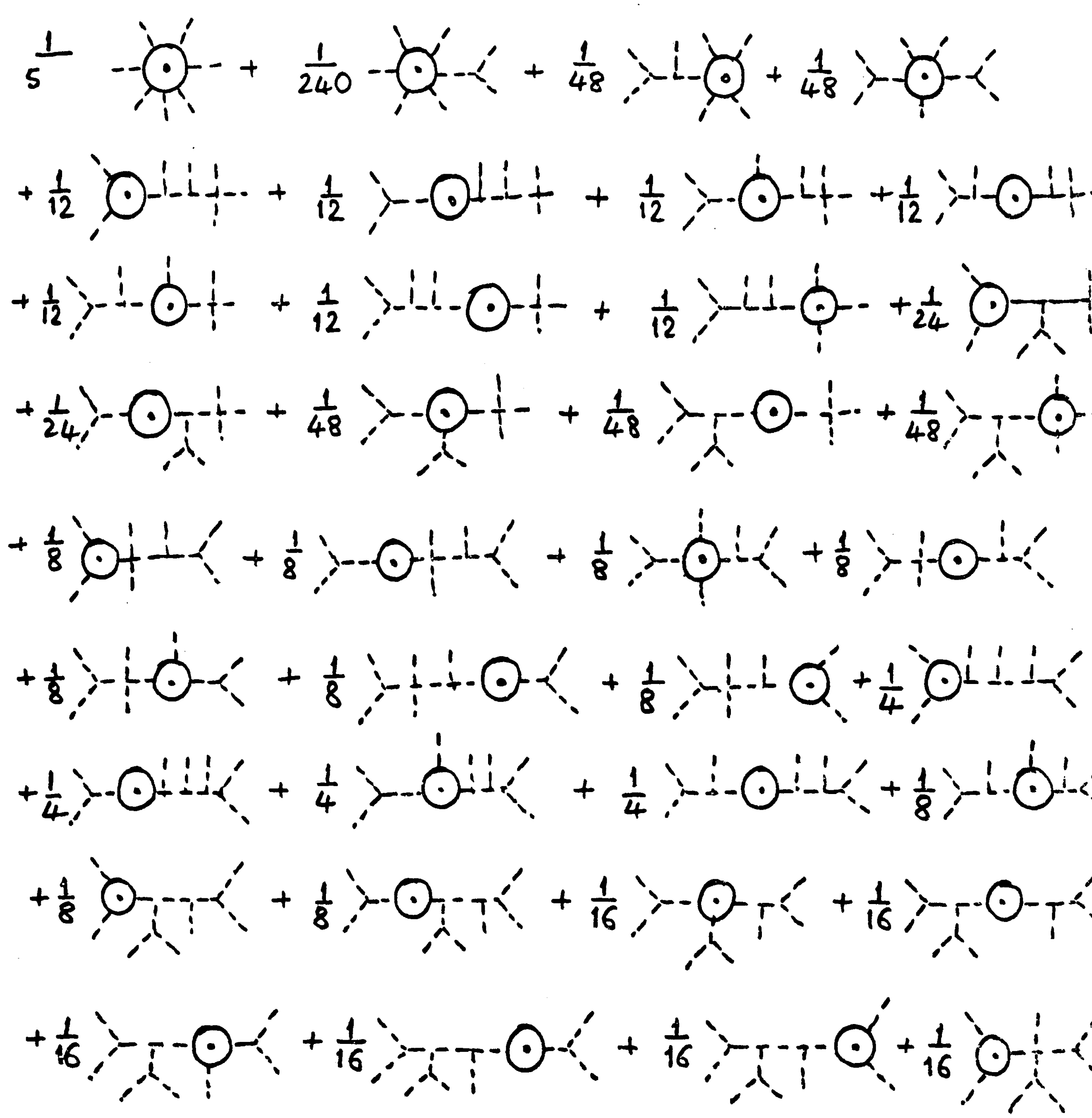

+ 1/8 + 1/8 + 1/16
+ 1/36 + 1/72 + 1/16 + 1/16
+ 1/48 ≡ 0

Figure 14


1/5 + 1/240 + 1/48 + 1/48
+ 1/12 + 1/12 + 1/12 + 1/12
+ 1/12 + 1/12 + 1/12 + 1/24
+ 1/24 + 1/48 + 1/48 + 1/48
+ 1/8 + 1/8 + 1/8 + 1/8
+ 1/8 + 1/8 + 1/8 + 1/4
+ 1/4 + 1/4 + 1/4 + 1/8
+ 1/8 + 1/8 + 1/16 + 1/16
+ 1/16 + 1/16 + 1/16 + 1/16

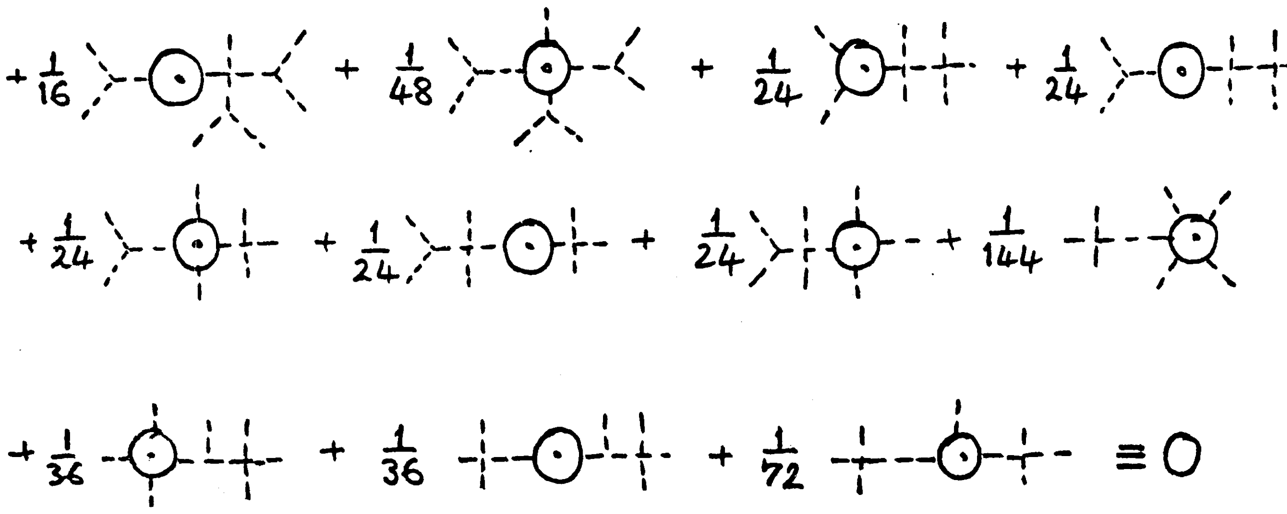


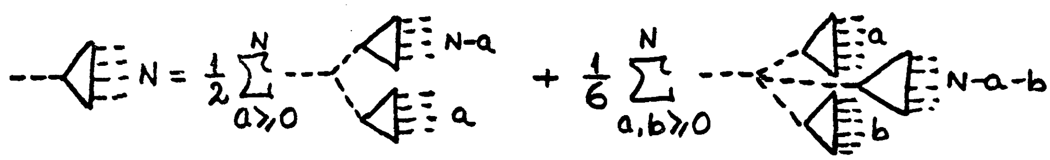
Figure 15

Since the divergent piece of the general N-leg vertex function is known (123) and (124), it would be somewhat more satisfactory to show explicitly the cancellation to all orders of particle processes. To date, all attempts in this direction have failed. Perhaps the most promising method along these lines is that of an inductive proof. This however runs into severe difficulties owing essentially to the non linear nature of system. A second method would be a proof of cancellation at the symmetry point; to be supplemented by a theorem of the type referred to above. Even with such simplifying restrictions, the non linearity still appears insurmountable. A brief indication of the difficulties involved is as follows.

Of primary interest to establishing a proof is knowledge of the value of the tree-like structures which are attached to the vertex 'loop'. We attempt to evaluate such a sum of trees by taking all the branches but one to have the same symmetrical nature as would arise if inserted into a larger diagram which is being restricted to its symmetry point, i.e.

$$q^2 = \mu^2 \qquad\qquad q_i \cdot q_j = -\alpha\mu^2 \;:i \neq j \tag{142}$$

Diagramatically the "N-branched tree" satisfies the equation



Denoting the left hand side by R_N, where the N legs satisfy (142) and the incoming leg carries the momentum required by the conservation low. Using the rules (58) gives,

$$\begin{aligned} R_N = {} & \frac{1}{2}\sum_{a\geqslant 0}^{N}\binom{N}{a}(-3i\mu f)\, iR_{N-a}\, iR_a\left[(q_1+\cdots+q_a)^2-\mu^2\right]^{-1} \times \\ & \left[(q_{a+1}+\cdots+q_N)^2-\mu^2\right]^{-1} + \frac{1}{6}\sum_{a,b\geqslant 0}^{N}\frac{N!}{a!\,b!\,(N-a-b)!}(-3if^2)\, iR_{N-a-b} \times \\ & iR_a\, iR_b\left[(q_1+\cdots+q_a)^2-\mu^2\right]^{-1}\left[(q_{a+1}+\cdots+q_{a+b})^2-\mu^2\right]^{-1}\left[(q_{a+b+1}+\cdots+q_N)^2-\mu^2\right]^{-1} \end{aligned} \tag{143}$$

which reduces owing to (142). Introducing a second quantity

$$S_a = \frac{i R_a f}{a!\,\mu}\left[(q_1+\cdots+q_a)^2-\mu^2\right]^{-1}\Big|_{sp} \tag{144}$$

yields on substitution into (143)

$$\begin{aligned} \left[(q_1+\cdots+q_N)^2-\mu^2\right]_{sp} S_N &= \frac{3}{2}\mu^2\sum_{a\geqslant 0}^{N} S_a S_{N-a} + \frac{1}{2}\mu^2\sum_{a,b\geqslant 0}^{N} S_a S_b S_{N-a-b} \\ &= \left[p^2-\mu^2\right] S_N \end{aligned} \tag{145}$$

where p is the ingoing momenta. The form of (145) suggests immediately a Fourier transform. Taking

$$\mathcal{S}(q) = \sum_{N=-\infty}^{+\infty} \mathcal{S}_N \, e^{iNq}$$

$$\mathcal{S}_N = \frac{1}{2\pi}\int_{-\pi}^{+\pi} dq \, \mathcal{S}(q) \, e^{-iNq} \tag{146}$$

and hoping that the factorial included in the definition of $\mathcal{S}_N$ (144) is sufficient to make $\mathcal{S}_N$ vanish for negative N gives, after summing over N,

$$\frac{3}{2}\mu^2 \mathcal{S}(q)^2 + \frac{1}{2}\mu^2 \mathcal{S}(q)^3 = \sum_{N=-\infty}^{+\infty} [p^2 - \mu^2] \, \mathcal{S}_N \, e^{iqN} \tag{147}$$

To evaluate the right hand side of (147) one must first find the N-dependence of p^2 from (142).

$$p^2 = (q_1 + \cdots + q_N)^2 = N \cdot q^2 + 2\binom{N}{2}(q_1 \cdot q_2)$$

$$= N\mu^2 - N(N-1)\alpha\mu^2 \tag{148}$$

Therefore,

$$p^2 - \mu^2 = (N-1)(1-\alpha N)\mu^2 \tag{149}$$

So that the right hand side of (147) can be written as,

$$\mu^2\left(-i\frac{\partial}{\partial q} - 1\right)\left(1 + i\alpha\frac{\partial}{\partial q}\right) \sum_{N=-\infty}^{+\infty} \mathcal{S}_N \, e^{iqN} \tag{150}$$

$$= \mu^2 \left\{ \alpha \frac{\partial^2 S(q)}{\partial q^2} - i(1+\alpha) \frac{\partial S(q)}{\partial q} - S(q) \right\}$$

Finally therefore, our required Equation is,

$$\alpha \frac{\partial^2 S(q)}{\partial q^2} - i(1+\alpha) \frac{\partial S(q)}{\partial q} = S(q) + \frac{3}{2} S^2(q) + \frac{1}{2} S^3(q) \tag{151}$$

In principle, one could solve (151), Fourier invert using (146) and find S_N and subsequently R_N via (144). The non-linearity of (151) does not however permit a simple method of solution. This method has been taken no further. With regard to an inductive proof it is also worth mentioning that given an N-particle process is finite on shell, it is not easy to relate this information to an (N+1) particle process, since although the former may be considered "nested" inside the latter it no longer has the corresponding legs on-mass shell.

There are many paths which can be followed from this point. Perhaps the most obvious one is the peculiar feature noted just before the section on 'Divergences, quartic in external momenta'. This was the observation that only the loops of londitudinal propagators produce the troublesome divergences. All other contributions, from mixtures of transverse and londitudinal propagators, cancel completely and independently of momentum flowing in their external legs.

GENERALIZATIONS

Consider any Feynman diagram derived from the set of rules (58). This may possess any number of loops and external legs. If attention is focussed on the lines representing the vector particles, then from the topological structure of the rules (58), one finds that these lines have no branches; i.e. it is possible to follow the path of a vector particle from one external point through the network and back out to a second external point, or alternatively, follow the vector until it closes back on itself in the form of a simple loop. In pictures, Figures (16) and (17) represent the general manner in which a vector line can appear in a diagram.

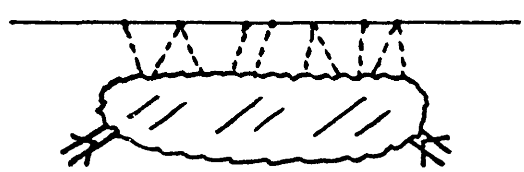


Figure 16

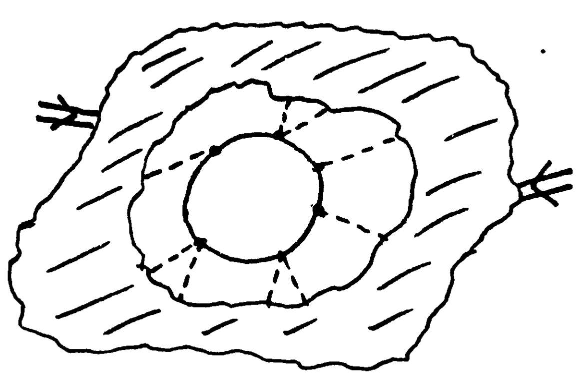


Figure 17

The arrowed doubled line indicates all external lines, which may include vector and scalar particles. The shaded portions represent the remaining part of the diagram whose structure is not explicitly exposed. The interesting feature which is immediately apparent from Fig. (17), is that even in the most complicated diagrams, the simple vector loop already studied plays a predominant role. Let us take a step further. Notice first that there are a definite number of scalar lines connecting the vector loop to the remainder of the diagram. (In the case of Fig. (17) there are nine such lines). Let us now add together all graphs which (1) have the same number of scalar lines connecting the vector loop and (2) have exactly the same network structure except for the coupling to the vector loop. In other words, in Fig. (17), the shaded piece is kept the same while the way in which the vector loop couples is varied. This allows the vector loop to be replaced by the full one loop approximation to the corresponding vertex function. So that in the graphical notation previously used whereby the dotted circle represents the fully symmetrized (1-loop) vertex, one has Fig. (18).

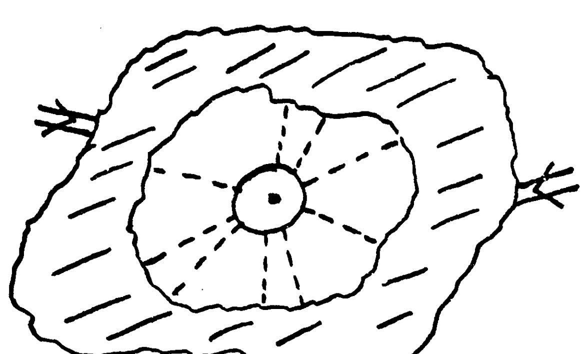


Figure 18

The point is that the non cancelling divergences hidden in the vertex function are known, from previous analysis, to be due to those graphs which are obtained by replacing the full vector propagator by its londitudinal part. This leads us naturally to the question with which the remainder of this section will be devoted, namely; can one associate the origin of all non-cancelling divergences to graphs of the type shown in Fig. (17) where now a continuous line is taken to represent only the londitudinal piece of the vector propagator?

OVERLAPPING DIVERGENCES AND SUBGRAPHS

It is convenient at this point to return to Fig. (17) and examine the subintegrations necessarily involved. First a graphical notation is introduced. The decomposition of the propagator (61) is denoted by

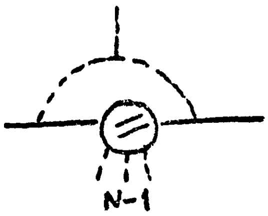
$$\begin{aligned} \Delta_{\mu\nu}(k) &= \Delta^{T}_{\mu\nu}(k) + \Delta^{L}_{\mu\nu}(k) \\ &= \text{---}\!\!\times\!\!\text{---} + \text{------} \end{aligned} \tag{152}$$

(Note that a continuous line now represent only the londitudinal piece of the propagator - in contrast to before.) Introducing this decomposition into an arbitrary vector loop Fig. (17) results in 2^N new diagrams - where N is the number of scalar lines connected to the loop. Consider any diagram which has a given number (non zero) of transverse propagators. The question is whether the net divergence arising from this graph is cancelled by another graph's contribution or not. The new feature here, is that the divergences arise not only from the subintegration over the loop momenta, but from all diverging subintegrations which include some subsection of the loop. It is known already that the graphs which cancel with the vector loop subintegration are those which possess the same number of transverse propagators and correspond to the same vertex function. Hence it is natural to examine whether the subintegrations of these same subsets of cancelling graphs, also yield no net divergent contribution.

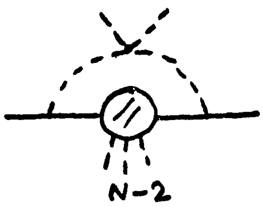
It is not difficult to convince oneself that there are essentially only two subintegrations of immediate relevance. Those which involve two scalar propagators and londitudinal vector propagators (logarithmically divergent), and those involving one scalar propagator and londitudinal propagators (quadratically divergent). At first sight it might appear that subintegrations involving one scalar propagator and one transverse propagator would yield a logarithmically divergent integration. This however is only true provided there are no contractions with londitudinal pieces, since in this case the transversality reduces the integration momenta by at least one power rendering it convergent.

<u>LOGARITHMICALLY DIVERGENT SUBINTEGRATIONS</u>

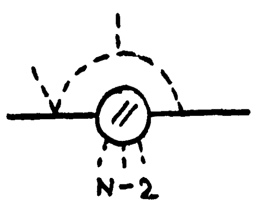
For a logarithmically diverging subintegration involving two scalar propagators, the sum of all graphs may be established using the same technique as before. The general case of a graph with 'N' external scalar legs is considered. Owing to the logarithmic nature of the divergence the values of the momenta flowing in and out of the diagram are irrelevent to the combinators. As before results are merely quoted.

(a) 

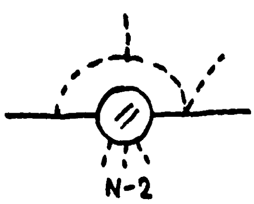
$$= 12\left(\frac{e}{m}\right)^{N+2}(m\mu)^2 I_{N-1}$$

(b) 

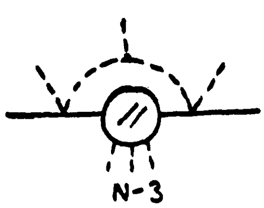
$$= 6\left(\frac{e}{m}\right)^{N+2}(m\mu)^2 I_{N-2}$$

(c) 

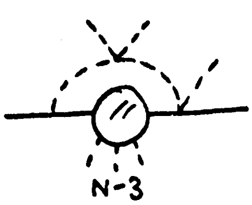
$$= 12\left(\frac{e}{m}\right)^{N+2}(m\mu)^2 I_{N-2}$$

(d) 

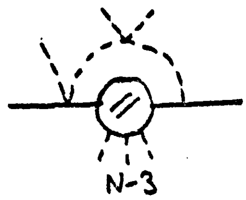
$$= 12\left(\frac{e}{m}\right)^{N+2}(m\mu)^2 I_{N-2}$$

(e) 

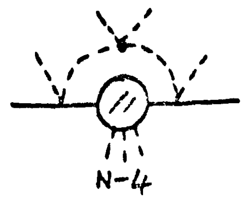
$$= 12\left(\frac{e}{m}\right)^{N+2}(m\mu)^2 I_{N-3}$$

(f) 

$$= 6\left(\frac{e}{m}\right)^{N+2}(m\mu)^2 I_{N-3}$$

(g) 

$$= 6\left(\frac{e}{m}\right)^{N+2}(m\mu)^2\, I_{N-3}$$

(h) 

$$= 6\left(\frac{e}{m}\right)^{N+2}(m\mu)^2\, I_{N-4}$$

In all these graphs the common divergent factor corresponding to the loop integration has been omitted. Furthermore the relationship,

$$\mu/m = f/e \tag{153}$$

has been used. The origin of (153) will be discussed in greater detail in a following section. Adding the contributions from (a) to (h) gives,

$$6\left(\frac{e}{m}\right)^{N+2}(m\mu)^2\left[2I_{N-1} + 5I_{N-2} + 4I_{N-3} + I_{N-5}\right] \tag{154}$$

Substituting for the value of I_M as given in (88) give zero value for (154). So that by summing over the same subset of graphs, one finds that logarithmically divergent subintegrations cancel. For future convenience, we denote any divergence which is cancelled by summing over a suitable subset of graphs; "non-permanent" whereas those which do not have this property are denoted "permanent". It is the occurrence of such 'permanent' divergences within this theory which gives the (on-shell) S-matrix a privileged role. From the above result and those previous, one sees that the divergences associated with the vector loop insertion of Fig. (19) are of the non-permanent type.

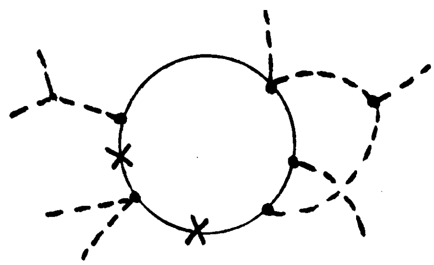
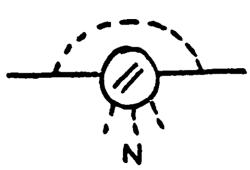


Figure 19

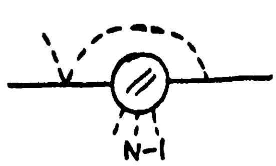
With regard to the quadratically divergent subintegrations the same procedure can be carried out. From general considerations, it is expected that the divergent term contains a piece depending on the momenta and piece independent of the momenta. It is instructive to examine this latter type first. This is easy, since all external momenta in the graphs are set to zero. Results are,

(a)



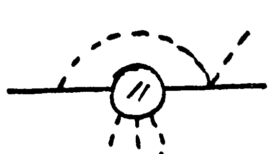
$$= 4\left(\frac{e}{m}\right)^{N+2} m^2 I_N$$

(b)

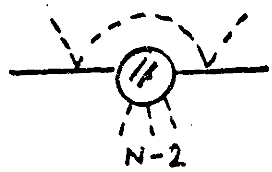


$$= 4\left(\frac{e}{m}\right)^{N+2} m^2 I_{N-1}$$

(c)



$$= 4\left(\frac{e}{m}\right)^{N+2} m^2 I_{N-1}$$

(d) 

$$= 4\left(\frac{e}{m}\right)^{N+2} m^2 I_{N-2}$$

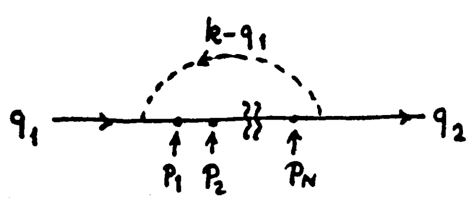
As before the common diverging factor has been omitted. Addition gives

$$4\left(\frac{e}{m}\right)^{N+2} m^2 \left[I_N + 2I_{N-1} + I_{N-2} \right] \tag{155}$$

which in virtue of (88) gives a vanishing result. This result is certainly encouraging, and demands an investigation of the momentum dependent divergence also hidden in this class of graphs. Unfortunately, this is rather more involved calculation.

THE MOMENTUM DEPENDENT DIVERGENCE

The basic object is shown in Fig. (20).



$$q_2 = q_1 + \sum_{\alpha=1}^{N} P_\alpha$$

Fig (20)

The relevant integral is,

$$\int d\mu(k,\omega)\, L(k) \cdot L(k+P_1) \cdot \ldots\ldots L\left(k + \sum_{\alpha=1}^{N} P_\alpha\right) \frac{1}{(k-q_1)^2 - \mu^2} \tag{156}$$

Where definition (152) has been used. Writing out the numerator of the integrand,

$$\frac{k_\mu (k+\sum_{\alpha=1}^{N} P_\alpha)_\nu}{k^2} \left\{ \frac{k\cdot(k+P_1)}{(k+P_1)^2} \frac{(k+P_1)\cdot(k+P_1+P_2)}{(k+P_1+P_2)^2} \cdots \frac{(k+P_1+\cdots+P_{N-1})\cdot(k+\cdots+P_N)}{(k+P_1+\cdots+P_N)^2} \right\} \tag{157}$$

The α^{th} term may be written,

$$\frac{(k+P_1+\cdots+P_{\alpha-1})\cdot(k+P_1+\cdots+P_\alpha)}{(k+P_1+\cdots+P_\alpha)^2} = \left(1 - \frac{P_\alpha\cdot Q_\alpha}{Q_\alpha^2}\right) \tag{158}$$

Hence the integrand is simply,

$$\frac{k_\mu (k+\sum_{\alpha=1}^{N} P_\alpha)_\nu}{k^2([k-q_1]^2-\mu^2)} \cdot \prod_{\alpha=1}^{N} \left(1 - \frac{P_\alpha\cdot Q_\alpha}{Q_\alpha^2}\right) \tag{159}$$

The form (159) is convenient for picking out the divergent pieces. Only the first three terms of the expanded product contribute.

$$\int d\mu(k,\omega) \frac{k_\mu (k+q_2-q_1)_\nu}{k^2([k-q_1]^2-\mu^2)} \left\{ 1 - \sum_{\alpha=1}^{N} \frac{P_\alpha\cdot Q_\alpha}{Q_\alpha^2} + \sum_{\alpha<\beta}^{N} \frac{(P_\alpha\cdot Q_\alpha)(P_\beta\cdot Q_\beta)}{Q_\alpha^2 \; Q_\beta^2} \right\} \tag{160}$$

The integration of (160) is straightforward using standard techniques. Ignoring finer details, the result is

$$\frac{i}{24(4\pi)^2(2-\omega/2)} \left\{ (12\mu^2 g_{\mu\nu} - 2q_1\cdot q_2\, g_{\mu\nu} - 4\sum_{\alpha} P_\alpha\cdot\Delta_\alpha\, g_{\mu\nu}) \right. \tag{161}$$

$$-6\left[q_1^\mu q_1^\nu + q_2^\mu q_2^\nu\right] + 16 q_1^\mu q_2^\nu + 4 q_1^\nu q_2^\mu + 2\sum_\alpha \left(P_\mu^\alpha \Delta_\nu^\alpha + \Delta_\mu^\alpha P_\nu^\alpha\right)$$

$$g_{\mu\nu}\sum_{\alpha<\beta}(P_\alpha \cdot P_\beta) + \sum_{\alpha<\beta}\left(P_\mu^\alpha P_\nu^\beta + P_\nu^\alpha P_\mu^\beta\right)\Big\}$$

For future convenience of notation, (161) will be denoted by

$$\frac{(+i)\,\Psi(q_1; P_1, \ldots, P_N)}{24(4\pi)^2(2-\omega/2)} \tag{162}$$

In (161) the Δ_α is defined as,

$$\Delta_\alpha = P_1 + P_2 + \cdots + P_\alpha \tag{163}$$

As before the implications of Bose symmetry are used to find the net contribution. Consider the one-particle irreducible vertex function corresponding to the Greens function, (Heisenberg Picture)

$$\langle 0 | \left(B_\mu(q_1) B_\nu(-q_2)\,\phi(P_1)\ldots\phi(P_N)\right)_+ | 0 \rangle^{CON} \tag{164}$$

In particular the one loop approximation to it can be written as the sum of two types of diagrams, namely those involving two scalar propagators, and those involving only one. The other possibilities do not give rise to divergent terms. It has already been shown that the divergences of the first type are "non-permanent", from this, it follows that the net contribution from the graphs of the second type is Bose symmetric in its arguments. Suppose q_2 is eliminated using momentum conservation, then,

$$\mathcal{D}i\omega T^{(2,N)}_{\mu\nu}(q_1;P_1,..,P_N) = \frac{i\,F_{\mu\nu}(q_1;P_1,..,P_N)}{24(4\pi)^2(2-\omega/2)} \tag{165}$$

Where F_μ is symmetric in the 'N'-scalar momenta. The most general structure that F_μ can take follows from (161) as

$$F_{\mu\nu}(q;P_1,..,P_N) = g_{\mu\nu}\mu^2 A + G_{\mu\nu}(q;P_1,...,P_N) \tag{166}$$

Where G_μ is a sum over monomials of degree two in the (N+1) momenta, i.e.,

$$G_{\mu\nu}(q;P_1,...,P_N) = \alpha_1 q^2 g_{\mu\nu} + \alpha_2 q_\mu H_\nu(P_1,..,P_N)$$
$$+\alpha_3 q_\nu J_\mu(P_1,..,P_N) + \alpha_4 K_{\mu\nu}(P_1,..,P_N) + \alpha_5 q^\mu q^\nu \tag{167}$$

Here, H and J are symmetric functions of their arguments of degree one, and K is a similar symmetric function but of degree two. Clearly,

$$H(P_1,...,P_N) = J(P_1,...,P_N) = P_1+P_2+\cdots+P_N \tag{168}$$

and

$$K_{\mu\nu}(P_1,..,P_N) = \beta_1 g_{\mu\nu}\sum_{\alpha<\beta}(P_\alpha\cdot P_\beta) + \beta_2\sum_\alpha(P_\alpha^2)g_{\mu\nu}$$
$$+\beta_3\sum_\alpha P^\alpha_\mu P^\alpha_\nu + \beta_4\sum_{\alpha\neq\beta}P^\alpha_\mu P^\beta_\nu \tag{169}$$

Finally,

$$G_{\mu\nu}(q;P_1,..,P_N) = \alpha_1 q^2 g_{\mu\nu} + \alpha_2 (P_1+\cdots+P_N)_\nu q_\mu + \tag{170}$$

$$\alpha_3 (P_1+\cdots+P_N)_\mu q_\nu + \alpha_4 q_\mu q_\nu + \alpha_5 g_{\mu\nu} \sum_{\alpha<\beta} (P_\alpha \cdot P_\beta) + \alpha_6 g_{\mu\nu} \sum_{\alpha} P_\alpha^2$$

$$\alpha_7 \sum_{\alpha} P_\mu^\alpha P_\nu^\alpha + \alpha_8 \sum_{\alpha\neq\beta} P_\mu^\alpha P_\nu^\beta + \alpha_9 g_{\mu\nu} q\cdot(P_1+\cdots+P_N)$$

From (170) it follows that everything can be recovered from a calculation which has only two of the scalar legs carrying non zero momenta. It remains only to route the two momenta through the diagrams immediately preceding equation (155) in all possible ways. There are fourteen basic configurations to consider, which are conveniently regrouped as follows.

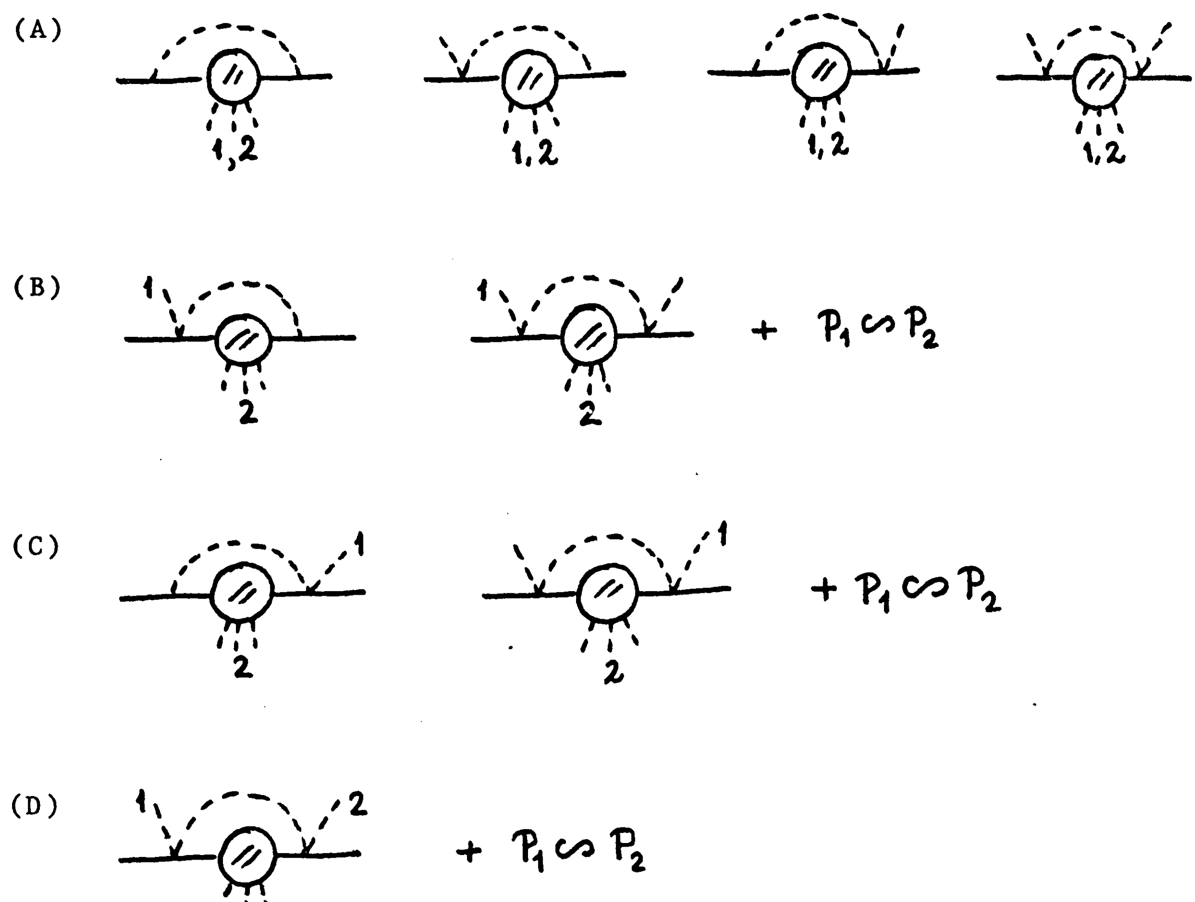
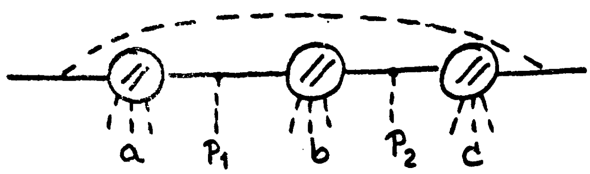


Figure 21

Consider the diagrams of the type shown in (A). The manner in which the two non zero momenta flow into to shaded circle, may be outlined as,

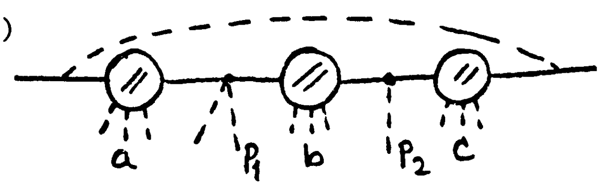
(a)



$+\ P_1 \leftrightarrow P_2$

$$a+b+c = N-2$$

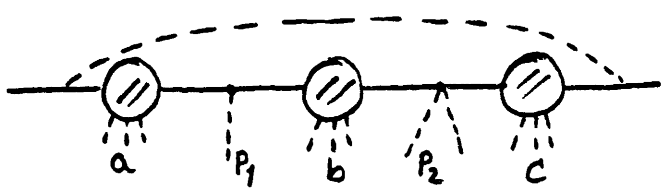
(b)



$+\ P_1 \leftrightarrow P_2$

$$a+b+c = N-3$$

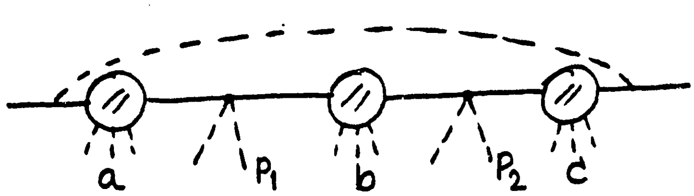
(c)



$+\ P_1 \leftrightarrow P_2$

$$a+b+c = N-3$$

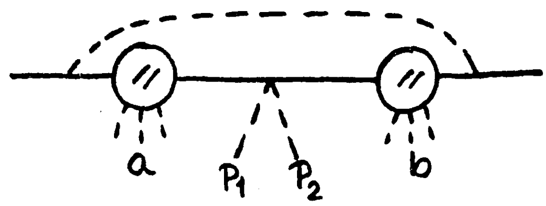
(d)



$+\ P_1 \leftrightarrow P_2$

$$a+b+c = N-4$$

(e)



$a + b = N-2$

The shaded circle with 'a' external scalar legs, is just the sum over all possible diagrams,

$$= \sum_{\alpha+2\beta=a} \frac{(2iem)^{\alpha}(2ie^2)^{\beta}(i)^{\alpha+\beta-1}}{(m^2)^{\alpha+\beta-1}} \frac{(\alpha+\beta)!}{2^{\beta}\,\alpha!\,\beta!} \tag{171}$$

$$= -im^2\left(\frac{e}{m}\right)^a I_a$$

Where the auxillary function I_M has been reintroduced; previously defined in (84). The contribution of (a) to (170) is,

$$\sum_{a+b+c=N-2} (-im^2)^3\left(\frac{e}{m}\right)^{N-2} I_a I_b I_c (2iem)^4 \frac{i^7}{m^{12}} \times \left\{\psi(q;P_1,P_2) + \psi(q;P_2P_1)\right\} \tag{172}$$

Using (109) gives,

$$16(-1)^N\left(\frac{e}{m}\right)^{N+2} m^2 L_{N-2}\left[\psi(q;P_1,P_2) + \psi(q;P_2,P_1)\right] \tag{173}$$

Likewise for (b) + (c) we have

$$(-1)^{N+1}\, 32\left(\frac{e}{m}\right)^{N+2} m^2 L_{N-3}\left[\psi(q;P_1,P_2) + \psi(q;P_2,P_1)\right] \tag{174}$$

For (d)

$$16(-1)^N\left(\frac{e}{m}\right)^{N+2} m^2 L_{N-4}\left[\psi(q;P_1,P_2) + \psi(q;P_2,P_1)\right] \tag{175}$$

Finally for (e),

$$8(-1)^{N+1}\left(\frac{e}{m}\right)^{N+2} m^2 M_{N-2}\,\psi(q;\,P_1+P_2) \tag{176}$$

Where the definition (113) has been used. Hence the net contribution from the first diagram of (A) in Fig. (21) is given by the sum of (173), (174), (175) and (185)

$$16(-1)^{N}\left(\frac{e}{m}\right)^{N+2} m^2 \left(L_{N-2}-2L_{N-3}+L_{N-4}\right)\left\{\psi(q;P_1,P_2)\right.$$
$$\left.+\,\psi(q;P_2,P_1)\right\} + 8(-1)^{N+1}\left(\frac{e}{m}\right)^{N+2} m^2 M_{N-2}\,\psi(q;\,P_1+P_2) \tag{177}$$

Which reduces according to (117),

$$8(-1)^{N}\left(\frac{e}{m}\right)^{N+2} m^2 M_{N-2}\left\{2\psi(q;P_1P_2)+2\psi(q;P_2P_1)-\psi(q;P_1+P_2)\right\} \tag{178}$$

Including the remaining three diagrams of (A) in Fig. (21) gives,

$$8(-1)^{N}\left(\frac{e}{m}\right)^{N+2} m^2 \left\{M_{N-2}-2M_{N-3}+M_{N-4}\right\}\times$$
$$\left\{2\psi(q;P_1,P_2)+2\psi(q;P_2,P_1)-\psi(q;P_1+P_2)\right\} \tag{179}$$

A third relation derivable from (116), namely,

$$M_{N-2}-2M_{N-3}+M_{N-4} = N-1 \tag{180}$$

gives for the net contribution of class (A)

$$8(-1)^{N}\left(\frac{e}{m}\right)^{N+2} m^2 (N-1)\left\{2\psi(q;P_1,P_2)+2\psi(q;P_2,P_1)\right.$$
$$\left.-\,\psi(q;P_1+P_2)\right\} \tag{181}$$

In precisely the same manner, all contributions from the diagrams of Fig. (21) may be evaluated to give a net total.

$$4(-1)^N\left(\frac{e}{m}\right)^{N+2} m^2 (N-1)\Big\{4\psi(q;P_1P_2)+4\psi(q;P_2P_1)$$
$$-2\psi(q;P_1+P_2)-2\psi(q+P_1;P_2)-2\psi(q+P_2;P_1)-2\psi(q;P_1)$$
$$-2\psi(q;P_2)+\psi(q+P_1;0)+\psi(q+P_2;0)\Big\} \tag{182}$$

The final part of the calculation is the explicit form of the ψ-functions. These are found by direct substitution into the general formulae (161) and (162). The result of a long but straightforward calculation yields finally for (182)

$$24(-1)^N\left(\frac{e}{m}\right)^{N+2} m^2 (N-1)\Big\{2(P_1+P_2)_\nu q_\mu - 2(P_1+P_2)_\mu q_\nu \tag{183}$$
$$-g_{\mu\nu}(P_1^2+P_2^2)-2(P_1^\mu P_1^\nu+P_2^\mu P_2^\nu)-6(P_1^\mu P_2^\nu+P_2^\mu P_1^\nu)\Big\}$$

The coefficients in (170) can be read off directly from (183), from which the general result follows. Note also, that the divergent term vanishes when the momenta are set to zero - in agreement with that found previously. The conclusion to be drawn from (183), however, is that one cannot isolate the graphs Fig. (18) containing only londitudinal vector propagators as containing the only "permanent" divergences. This is because from (192) we see that there exist "permanent" divergences in the subintegrations of graphs containing both londitudinal and transverse vector propagators.

The conclusion of the latter part of this section must thus be that it does not appear possible to identify "permanent" and "non permanent" divergences by simple graphical terms. An interesting feature which arose from this investigation is that, just as before, all divergences except those containing the highest order monomials in momenta, cancel after summing over suitable subsets of graphs. Clearly, it is of some interest to find a general underlying reason for this peculiar behaviour.

FUNCTION'L INTEGRAL APPROACH TO THE U-GAUGE

Although the Feynman rules for the system (1) have been derived from the Canonical Quantization scheme, there exists a second method, which although of a much more formal method, still leads to the same results. This method is to be presented here in order to establish in a simple way the phenomenon of spontaneous symmetry breakdown, and the Higgs' mechanism.[4)] The starting point is a simple two component (or complex) scalar field with quartic interactions, which has been coupled to the electromagnetic field via the "minimal substitution" prescription.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (\nabla_\mu\bar{\Phi})^\dagger(\nabla^\mu\bar{\Phi}) + \bar{\mu}^2\bar{\Phi}^\dagger\bar{\Phi} - \frac{1}{2}f^2(\bar{\Phi}^\dagger\bar{\Phi})^2 \tag{184}$$

Where

$$\begin{aligned} F_{\mu\nu} &= \partial_\nu A_\mu - \partial_\mu A_\nu \\ \nabla_\mu &= \partial_\mu + ieA_\mu \\ \bar{\Phi} &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \end{aligned} \tag{185}$$

(Note that we do not write the Langrangian density in first order form only for ease of notation). Inspection of (184) shows that the 'mass' term has the wrong sign. This results in an instability in the system described by (184), producing a non vanishing vacuum expectation value for the scalar field. In physical terms, it becomes energetically more favourable for the ground state of the system to be characterized by a "sea" of scalar particles. The effect of this coupled with the long range electromagnetic phenomenon, leads to vector excitations with a massive nature. This is known as the Higgs phenomenon. By a change of parameterization of the fields

$$\Phi(x) = \frac{1}{\sqrt{2}}(u+\phi(x))\, e^{i\zeta(x)/u}\,; \quad A_\mu(x) = B_\mu(x) - \frac{1}{eu}\partial_\mu \zeta(x) \tag{186}$$

one derives a second form for the Lagrangian density (184). Namely,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(\partial_\nu B_\mu - \partial_\mu B_\nu)^2 + \frac{1}{2}m^2 B_\mu^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 \\ &+ (em)\phi B_\mu^2 + \frac{1}{2}e^2\phi^2 B_\mu^2 - \frac{1}{2}\mu f \phi^3 - \frac{1}{8}f^2\phi^4\end{aligned} \tag{187}$$

where

$$m = (ue) \qquad\qquad \mu = (uf) \tag{188}$$

and the parameter 'u' has been fixed at the minimum of the classical potential.

$$u = \sqrt{2}\cdot \bar{\mu}/f \tag{189}$$

The ratio of the two masses in (197) is independent of the value of 'u'. This relation was cited before in (153).

The functional integral approach allows one to write down the vacuum to vacuum transition amplitude in presence of an external source, generating T^* products.

$$\langle 0^+|0^-\rangle = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \mathcal{D}A_\mu \exp i\int d^4x\, \mathcal{L}[\phi_1\phi_2 A_\mu; x) \tag{190}$$

Where in (190) the sources have been ignored since they are not relevent for our purposes. The measure of integration is formally represented by,

$$\mathcal{D}\phi = \prod_x (d\phi)(x) \tag{191}$$

The change of field variables (186) results, as in usual integration theory to a Jacobian of transformation. The partial derivatives give

$$\frac{\delta\phi_1(x)}{\delta\phi(y)} = \cos\{\zeta(x)/u\}\,\delta(x-y) \tag{192}$$

$$\frac{\delta\phi_2(x)}{\delta\phi(y)} = \sin\{\zeta(x)/u\}\,\delta(x-y)$$

$$\frac{\delta A_\mu(x)}{\delta\phi(y)} = 0$$

$$\frac{\delta\phi_1(x)}{\delta\zeta(y)} = -\left(1+\frac{\phi(x)}{u}\right)\sin\{\zeta(x)/u\}\,\delta(x-y)$$

$$\frac{\delta\phi_2(x)}{\delta\zeta(y)} = \left(1+\frac{\phi(x)}{u}\right)\cos\{\zeta(x)/u\}\,\delta(x-y)$$

$$\frac{\delta A_\mu(x)}{\delta\zeta(y)} = -\frac{1}{eu}\,\partial_\mu^x\,\delta(x-y)$$

$$\frac{\delta\phi_1(x)}{\delta B_\mu(y)} = 0$$

$$\frac{\delta\phi_2(x)}{\delta B_\mu(y)} = 0$$

$$\frac{\delta A_\mu(x)}{\delta B_\nu(y)} = g_{\mu\nu}\,\delta(x-y)$$

The Jacobian of transformation is the determinant of these derivatives, i.e.

$$\frac{\partial(\phi_1\phi_2 A_\mu;x)}{\partial(\phi\zeta\ B_\nu;\,y)} = \mathcal{D}et\left\{\left(1+\frac{e\phi(x)}{m}\right)\delta(x-y)\right\} \tag{193}$$

Generalizing the formula,

$$\mathrm{Det}\{\exp A\} = \exp\{\mathrm{Tr}\,A\}. \tag{194}$$

Where A is a square matrix, (193) can be written as

$$\begin{aligned}&\exp\,\mathrm{Tr}\left\{\mathcal{L}n\left(1+\frac{e\phi(x)}{m}\right)\delta(x-y)\right\}\\ &\qquad = \exp\,\delta_4(0)\int d^4x\,\ln\left(1+\frac{e\phi(x)}{m}\right)\end{aligned} \tag{195}$$

Hence the change of variables in (190) must be accompanied by an addition to the lagrangian of the term,

$$\mathcal{L} \rightarrow \mathcal{L} - i\delta_4(0) \ln\left(1 + \frac{c\phi}{m}\right) \tag{196}$$

This is precisely the additional term found in changing from the time ordered products to the covariant $\{T^*\}$ products, in the canonical quantization approach. Any formulation of this theory which contains no spurious (unphysical) poles is known as a Unitary formulation. This particular field parameterization is known as the U-gauge form. The functional formalism is rather more formal in nature than the canonical approach but if care is taken, it almost always gives the same results. In some respects one could argue that in a theory involving such severe divergences, it amounts only to self deception, to use the "more rigorous" approach. However, although there is a great deal of truth in such an argument, it is clearly better to be as precise as possible in an investigation of the type undertaken. Nevertheless, the functional integral is a very useful tool for rapid testing of ideas and will be used in the following on more than one occasion.

THE U' FORMULATION

In the previous sections it was shown that many of the divergences occurring in individual Feynman graphs are "non-permanent"; i.e. they are cancelled by divergences in other diagrams. Such a behaviour lends support to the idea that there may exist another change of variables besides

(186), which automatically performs some of the cancellation. This indeed appears to be so. Our starting point is the Lagrangian density (187) and (196). Previous work suggests that a decomposition of the vector propagator into transverse and londitudinal parts should be of some assistance. Writing

$$B_\mu = A_\mu + \frac{1}{m}\partial_\mu \rho \tag{197}$$

Where A_μ is transverse

$$\partial_\mu A^\mu = 0 \tag{198}$$

induces the following decomposition of the Lagrangian density (187),

$$\begin{aligned} \mathcal{L} \rightarrow & -\tfrac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) + \tfrac{1}{2}m^2 A_\mu^2 + \tfrac{1}{2}(\partial_\mu \rho)^2 + \tfrac{1}{2}(\partial_\mu \phi)^2 \\ & -\tfrac{1}{2}\mu^2\phi^2 + em\phi A_\mu^2 + (\tfrac{e}{m})\phi(\partial_\mu\rho)^2 + 2e\phi A_\mu(\partial^\mu\rho) \\ & +\tfrac{1}{2}e^2\phi^2 A_\mu^2 + \tfrac{1}{2}(\tfrac{e}{m})^2\phi^2(\partial_\mu\rho)^2 + \tfrac{e^2}{m}\phi^2 A_\mu(\partial^\mu\rho) \\ & -\tfrac{1}{2}\mu f\phi^3 - \tfrac{1}{8}f^2\phi^4 \end{aligned} \tag{199}$$

Where in deriving (199), condition (198) has been used and total divergences been discarded. (199) can be written;

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\phi + \mathcal{L}_\rho + \mathcal{L}_I \tag{200}$$

$$\mathcal{L}_A = \tfrac{1}{2}A_\mu \Box A^\mu + \tfrac{1}{2}m^2 A_\mu A^\mu$$

$$\mathcal{L}_\phi = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu^2\phi^2$$

$$\mathcal{L}_\rho = \tfrac{1}{2}(\partial_\mu\rho)^2(1 + \tfrac{e}{m}\phi)^2$$

$$\mathcal{L}_I = em\phi A_\mu^2 + 2e\phi A_\mu \partial^\mu \rho + \frac{1}{2}e^2\phi^2 A_\mu^2$$
$$+ \frac{e^2}{m}\phi^2 A_\mu \partial^\mu \rho - \frac{1}{2}\mu f \phi^3 - \frac{1}{8} f^2 \phi^4$$

The clue for the next substitution comes from the form of $\mathcal{L}_\rho$. Suppose instead of 'ρ' one chose to use the field

$$\psi = \left(1 + \frac{e\phi}{m}\right)\rho \tag{201}$$

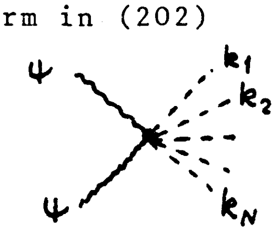
Then rewriting $\mathcal{L}_\rho$ gives

$$\mathcal{L}_\rho \to \mathcal{L}_\psi = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{1}{2}\left(\frac{e}{m}\right)\psi^2\left(1 + \frac{e\phi}{m}\right)^{-1}\Box\phi \tag{202}$$

Now although (202) appears more complicated, the transformation (201) has a distinct advantage; the Jacobian factor associated with this change of variables precisely cancels the divergent term in (196). Furthermore, substituting for ρ in $\mathcal{L}_I$ leads also to a simple expression involving only polynomial interactions

$$\mathcal{L}_I = em\phi A_\mu^2 + \frac{1}{2}e^2\phi^2 A_\mu^2 - \frac{1}{2}\mu f \phi^3 - \frac{1}{8} f^2\phi^4 - 2e\psi A_\mu(\partial^\mu \phi) \tag{203}$$

As a result of these substitutions, we arrive at a new parameterization of (184), which has no $\delta_4(o)$ terms, and yet the same number of field degrees of freedom. This will be called the U'-gauge parameterization. Consider one of the infinite set of Feynman graphs given by the non polynomial term in (202)



$$: \; -i\left(\frac{e}{m}\right)^{N+1}(-1)^N\left(k_1^2 + k_2^2 + \cdots + k_N^2\right)$$

Figure 22

Examination of the form of this coupling gives a rather nice way of looking at the peculiar cancellations noted at the end of the previous section. For example, suppose one wished to repeat the calculation to find the one-loop divergences of the N-point scalar vertex functions. In this formulation only <u>one</u> type of graph contributes.

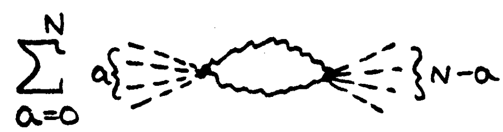


Figure 23

A short calculation, involving a summation over number of legs in each vertex yields the result obtained first in (123) but with a drastic reduction in the effort required. Note also that the form of Fig. (23) indicates even before doing the calculation, that the answer must consist of fourth order monomials in the scalar momenta. Likewise, an inspection of the remaining Feynman rules indicates the form of the scalar-vector vertex found in (183). It is interesting to speculate that the divergences of the U' formulation may all be of the "permanent" type. Before closing this section some comments are in order. The derivation of this U' formulation could hardly be described as conclusive. It is not clear that any choice of field parameterization will in fact lead to the same S-Matrix elements, owing to the highly singular nature of the theory in general. Another possible difficulty is that the field 'ψ' is massless, and may introduce infrared difficulties, not present in the original U-gauge formulation. Nevertheless, it is felt that it does give a rather interesting way of seeing cancellation clearly exhibited, even if it is in a rather formal way.

RENORMALIZATION

It has already been shown that a consistent cancellation of the residual divergences appearing in the U-gauge calculations (in the one loop approximation) could only be achieved by renormalizing the S-matrix elements themselves. This result being a direct consequence of the divergent nature of off shell Green's functions, even after renormalization has been carried out. In those sections, however, we were concerned primarily in establishing the existence of finite S-matrix elements, rather than showing that the subtraction constants necessarily introduced, satisfied the additional constraints imposed by the spontaneously broken nature of the theory. Such constraints derive from the fact that the unsymmetric theory possesses a larger number of interaction vertices and masses than the corresponding symmetric theory, and hence implies that the independent subtraction constants available are fewer in number than the primitively divergent S-matrix processes. In the following we shall only concern ourselves with the infinite pieces of these subtraction constants. The aim is to see whether the afore mentioned constraints can be satisfied in an acceptable manner.

It is convenient to start from the Lagrangian density describing the unstable form of scalar electrodynamics. See equations (184) and (185). The field transformation analogous to (186) is introduced except that the value of the parameter 'u' is not fixed at the value of the classical minimum. The result is,

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\nu B_\mu - \partial_\mu B_\nu)^2 + \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}(eu)^2 B_\mu^2 + e^2 u B_\mu^2 \phi \\ &-\tfrac{1}{2}(fu)^2\phi^2 + \tfrac{1}{2}(\bar{\mu}^2 - \tfrac{1}{2}f^2u^2)\phi^2 - \tfrac{1}{2}f^2 u\,\phi^3 - \tfrac{1}{8}f^2\phi^4 \\ &+ u(\bar{\mu}^2 - \tfrac{1}{2}f^2u^2)\phi + \tfrac{1}{2}e^2\phi^2 B_\mu^2 \;.\end{aligned} \tag{204}$$

Note that this is the same as (187) and (188) except for two extra terms;

$$\tfrac{1}{2}(\bar{\mu}^2 - \tfrac{1}{2}f^2u^2)\phi^2 + u(\bar{\mu}^2 - \tfrac{1}{2}f^2u^2)\phi \tag{205}$$

It is not difficult to convince oneself that the canonical quantization previously performed, will be unchanged by the addition of the terms (205). Essentially, this is because they do not involve any vector fields. With the understanding that covariant T^* products are used, the theory may be described by the Lagrangian density (204) plus the extra term coming from the T^* convention.

$$-i\delta_4(0)\ln\left(1 + \frac{\phi(x)}{u}\right) \tag{206}$$

The parameters and fields appearing in (204) and (206) are "bare" or "unrenormalized" quantities. To convert to "renormalized" parameters, a rescaling must be done. First note however, that an extra parameter has been smuggled in through the transformation (186). It follows therefore, that in principle only five independent rescalings are permissible in the new theory. Hence one relationship must exist among the six parameters occurring in (204). This relationship is in fact that the new fields must correspond asymptotically to operators which create (annihilate) positive energy, one particle states from the ground state of the asymmetric vacuum. That a value for 'u' can be found which satisfies this property, must at this stage be regarded as a working hypothesis. If then, 'u' is to be taken as a meaningful quantity, it should in principle be expressible in terms of the three bare parameters appearing in the symmetric theory (184)

$$u = u\,(e, f, \bar{\mu}) \tag{207}$$

Hence a transition to rescaled quantities induces the correct rescaling of 'u'.

In practice, however, one would have to choose 'u' to satisfy the above prescriptions in the renormalized theory and so having deduced (207) for renormalized parameters, could then proceed to the corresponding relationships between bare quantities. The essential point is, however, that the rescaling of 'u' is a passive one.

To determine the correct rescaling of (204) it is customary to adjust 'u' so that the vacuum expectation value of the scalar field $\phi(x)$ vanishes. It is far from clear that such a procedure is meaningful in the present case. This requirement involves an off-shell (Tadpole) Greens function, which as we have seen in previous sections, do not exist in any reasonable mathematical sense. Ideally a condition on the S-matrix elements would be preferable, although at the present date, no such method appears to be available.

The renormalization parameters are defined by

$$\begin{aligned} B_\mu &\rightarrow Z_B^{1/2} B_\mu & f^2 &\rightarrow (Z_1 f^2)/Z_\phi^2 \\ \phi &\rightarrow Z_\phi^{1/2} \phi & e^2 &\rightarrow (Z_2 e^2)/Z_B Z_\phi \end{aligned} \tag{208}$$
$$\bar{\mu}^2 \rightarrow \tfrac{1}{2} f^2 u^2 - \delta\mu^2/Z_\phi$$

Owing to the symmetrical way in which 'u' and 'ϕ'(x) appear in (186), we can take as a first attempt,

$$u \rightarrow Z_\phi^{1/2} u \tag{209}$$

The result of these substitutions gives,

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_\phi + \mathcal{L}_I + \mathcal{L}_{CT} \tag{210}$$

where

$$\mathcal{L}_B = -\tfrac{1}{4}(\partial_\nu B_\mu - \partial_\mu B_\nu)^2 + \tfrac{1}{2}(eu)^2 B_\mu^2$$

$$\mathcal{L}_\phi = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}(fu)^2 \phi^2$$

$$\begin{aligned} \mathcal{L}_I = {} & e^2 u B_\mu^2 \phi + \tfrac{1}{2} e^2 B_\mu^2 \phi^2 - \tfrac{1}{2}(f^2 u)\phi^3 \\ & - \tfrac{1}{8} f^2 \phi^4 - i\delta_4(0) \ln\left(1 + \frac{\phi}{u}\right) \end{aligned}$$

$$\mathcal{L}_{CT} = -\tfrac{1}{4}(Z_B-1)(\partial_\nu B_\mu - \partial_\mu B_\nu)^2 + \tfrac{1}{2}(Z_2-1)(ue)^2 B_\mu^2$$

$$+\tfrac{1}{2}(Z_\phi-1)(\partial_\mu\phi)^2 - \tfrac{1}{2}(Z_1-1)(fu)^2\phi^2 + (Z_2-1)(e^2u)B_\mu^2\phi$$

$$\tfrac{1}{2}(Z_2-1)e^2 B_\mu^2\phi^2 - \tfrac{1}{2}(Z_1-1)(f^2u)\phi^3 \quad -\tfrac{1}{8}(Z_1-1)f^2\phi^4$$

$$-\tfrac{1}{2}\,\delta\mu^2\phi^2 - u\,\delta\mu^2\phi$$

The mass ratio relationship can also be reintroduced between renormalized quantities

$$\left.\begin{aligned} m &= (eu) \\ \mu &= (fu) \end{aligned}\right\} \quad (mf) = (\mu e) \tag{211}$$

It is not possible to fix the values of m, μ, e and f, to all orders of perturbation independently. Anticipating renormalizability of the theory, however, modifications should only involve finite corrections. In the following, any finite corrections are ignored and the four parameters treated effectively as independent.

COMPUTATION OF COUNTERTERMS

By straightforward application of the graphical rules derivable from (210), the divergent pieces of the four lowest order scalar vertex functions may be evaluated.

(1)

$$d\omega\Gamma^{(1)}(0) = \frac{i}{(4\pi)^2}\,\frac{u^3}{(2-\omega/2)}\cdot\left\{\tfrac{3}{2}f^4 + 3e^4\right\} - iu\,\delta\mu^2 \tag{212}$$

(2)
$$\mathrm{div}\,\Gamma^{(2)}(p,(-p)) = \frac{i}{(4\pi)^2}\frac{u^2}{(2-\omega/2)}\left\{6f^4+9e^4-3e^2f^2\left(\frac{p^2}{\mu^2}\right)\right.$$

$$\left.+\frac{1}{2}f^4\left(\frac{p^2}{\mu^2}\right)^2\right\} + i(Z_\phi-1)p^2 - i(Z_1-1)(fu)^2 - i\delta\mu^2 .$$

(3)
$$\mathrm{div}\,\Gamma^{(3)}(P_1,P_2(-P_1-P_2)) = \frac{i}{(4\pi)^2}\frac{u}{(2-\omega/2)}\left\{\frac{27}{2}f^4+18e^4\right.$$

$$\left.-\frac{f^4}{\mu^4}C_3(P_1,P_2,(-P_1-P_2))\right\} - 3i(Z_1-1)(f^2u) .$$

(4)
$$\mathrm{div}\,\Gamma^{(4)}(P_1,P_2,P_3(-P_1-P_2-P_3)) = \frac{i}{(4\pi)^2}\cdot\frac{1}{(2-\omega/2)}\left\{\frac{27}{2}f^4\right.$$

$$\left.+18e^4+\frac{3f^4}{\mu^4}C_4(P_1,P_2,P_3,(-P_1-P_2-P_3))\right\} -3i(Z_1-1)f^2 .$$

Note that the notation div $\Gamma^{(N)}$ has been used to represent contributions from both vector and scalar intermediate states. This is to be distinguished from Div $\Gamma^{(N)}$ which includes only the vector intermediate states. The function $C_N(P_1,P_2\ ..,P_N)$ was first introduced in (1·23). If one tries to fix the counterterm $\delta\mu^2$ in (1) so that the tadpole vanishes, serious difficulties are encountered. Fixing the pole and residue of the propagator, via the self energy (2), and combining the results with the tadpole condition, fixes the renormalization constants, $\delta\mu^2$, Z_1 and $Z\phi$. These values are in complete disagreement with the requirement that the three point and four point S-matrix processes (involving 3 & 4) should be finite. This is clear from substitution. To remedy the situation, the tadpole condition is dropped, and the original scheme whereby finite S-matrix elements are demanded, is readopted. This method is still overdetermined, so that the question of consistency is still an important one. Firstly the two-point and three point vertex functions must have tadpole contributions included Fig. (24)

Figure 24

$$d_\omega \Gamma^{(2)}(p,(-p)) = \frac{iu}{(4\pi)^2(2-\omega/2)}\left\{6f^4+9e^4-3e^2f^2\left(\frac{p^2}{\mu^2}\right)+\frac{1}{2}f^4\left(\frac{p^2}{\mu^2}\right)^2\right\} \tag{213}$$

$$\frac{-3iu^2}{(4\pi)^2(2-\omega/2)}\left\{\frac{3}{2}f^4+3e^4\right\}+i(z_\phi-1)p^2-i(z_1-1)(fu)^2+2i\delta\mu^2.$$

$$d_\omega \Gamma^{(3)}(p_1,p_2,(-p_1-p_2)) = \frac{iu}{(4\pi)^2(2-\omega/2)}\left\{\frac{27}{2}f^4+18e^4-\frac{f^4}{\mu^4}C_3[p]\right\} \tag{214}$$

$$\frac{-3iu}{(4\pi)^2(2-\omega/2)}\left\{\frac{3}{2}f^4+3e^4\right\}-3i(z_1-1)(f^2u)+\frac{3i\delta\mu^2}{u}$$

Fixing the pole and residue of the propagator, via the self-energy (213) gives the conditions.

$$Z_\phi = 1 + \frac{[3e^2 - f^2]}{(4\pi)^2(2-\omega/2)} \tag{215}$$

$$(Z_1 - 1)(fu)^2 - 2\delta\mu^2 = \frac{u^2 f^4}{(4\pi)^2(2-\omega/2)}$$

So that the renormalized self-energy takes on the form,

$$d\omega\Gamma^{(2)}(p,(-p)) = \frac{iu^2 f^4}{2(4\pi)^2(2-\omega/2)} \cdot \frac{[p^2-\mu^2]^2}{\mu^4} \tag{216}$$

The three point vertex is also fixed on mass shell giving,

$$(Z_1 - 1)(f^2 u) - \frac{\delta\mu^2}{u} = \frac{u[2f^4 + 3e^4]}{(4\pi)^2(2-\omega/2)} \tag{217}$$

This together with (215) determines all the renormalization parameters,

$$Z_\phi = 1 + \frac{[3e^2 - f^2]}{(4\pi)^2(2-\omega/2)}$$

$$Z_1 = 1 + \frac{[3f^2 + 6e^4/f^2]}{(4\pi)^2(2-\omega/2)}$$

$$\delta\mu^2 = \frac{u^2[f^4 + 3e^4]}{(4\pi)^2(2-\omega/2)} \tag{218}$$

The crucial test comes when (218) is substituted into (212):(4) for the four point function. The value is,

$$d i \omega T^{(4)}(P_1,P_2,P_3,P_4) = \frac{3 i f^4 \left[C_4(P_1,P_2,P_3,P_4) + 3/2\, \mu^4 \right]}{(4\pi)^2 (2-\omega/2) \cdot \mu^4} \tag{219}$$

This is precisely the result found earlier in (124) and (140) based on the requirement of finiteness of the S-matrix elements. Furthermore, substituting (218) into the tadpole (212):(1) gives

$$d i \omega T^{(1)}(0) = \frac{i u^3 f^4}{2(4\pi)^2(2-\omega/2)} \tag{220}$$

which is in agreement with earlier conclusion with regards to the non existence of this object (in the mathematical sense). It seems likely that for higher order calculations, the re-scaling (209) is too restrictive. To the one loop order, however, this is acceptable.

VECTOR PROCESSES

Straightforward evaluation of the one loop contributions to the vector self energy, leads via (210) to

$$i d i \omega \Pi_{\mu\nu}(p) = \frac{i g_{\mu\nu} e^2 [\mu^2 - 3m^2 + \frac{1}{3}p^2]}{(4\pi)^2(2-\omega/2)} + \frac{i g_{\mu\nu} e^2 \mu^2}{(4\pi)^2(2-\omega/2)} \tag{221}$$

$$+ \frac{4}{3}\frac{i e^2 p_\mu p_\nu}{(4\pi)^2(2-\omega/2)} - \frac{i e^2 \mu^2 g_{\mu\nu}}{(4\pi)^2(2-\omega/2)} + i(z_B - 1)(p_\mu p_\nu - p^2 g_{\mu\nu}) + i(z_2 - 1) m^2 g_{\mu\nu}$$

Introducing the spin projectors, the spin-one and spin-zero contributions to the self energy can be separated

$$\Pi_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi_1(p^2) + \frac{p_\mu p_\nu}{p^2}\Pi_0(p^2) \tag{222}$$

One must ensure that extra singularities are not introduced. The condition necessary for this is,

$$\Pi_1(0) = \Pi_0(0) \tag{223}$$

Inverting (222) gives

$$\begin{aligned} \Pi_0(p^2) &= \frac{1}{p^2}\cdot p^\mu \Pi_{\mu\nu}(p)\, p^\nu \\ \Pi_1(p^2) &= \frac{1}{3}\left(\Pi_\mu{}^\mu(p) - \Pi_0(p^2)\right) \end{aligned} \tag{224}$$

Applying (224) to (221) gives,

$$d\omega\Pi_0(p^2) = \frac{e^2[\mu^2 - 3m^2 + \frac{5}{3}p^2]}{(4\pi)^2(2-\omega/2)} + (Z_2 - 1)m^2 \tag{225}$$

$$d\omega\Pi_1(p^2) = \frac{e^2[\mu^2 - 3m^2 + \frac{5}{3}p^2]}{(4\pi)^2(2-\omega/2)} - (Z_B - 1)p^2 + (Z_2 - 1)m^2$$

Note how the condition (223) is automatically satisfied independently of the value of Z_2 and Z_B. In fact

$$d\omega\Pi_1(p^2) = d\omega\Pi_0(p^2) - (Z_B - 1)p^2 \tag{226}$$

The inverse propagator is given by,

$$\Delta'^{-1}_{\mu\nu}(p) = p_\mu p_\nu - p^2 g_{\mu\nu} + m^2 g_{\mu\nu} + (g_{\mu\nu} - p_\mu p_\nu/p^2)\Pi_1(p^2) + p_\mu p_\nu/p^2\,\Pi_0(p^2) \tag{227}$$

So after inversion and use of (226), one has

$$\Delta'_{\mu\nu}(p) = \frac{(-g_{\mu\nu} + P_\mu P_\nu/p^2)}{p^2 - m^2 - \Pi_0(p^2) + (Z_B - 1)p^2} + \frac{P_\mu P_\nu/p^2}{m^2 + \Pi_0(p^2)} \tag{228}$$

The renormalization condition chosen, is to fix the transverse piece of the propagator to behave like a simple pole with unit residue at $p^2 \approx m^2$. This fixes the counterterms,

$$Z_2 = 1 + \frac{[3e^2 - f^2]}{(4\pi)^2(2-\omega/2)}$$

$$Z_B = 1 - \frac{5e^2/3}{(4\pi)^2(2-\omega/2)} \tag{229}$$

The londitudinal piece of the propagator remains divergent, even after renormalization. Nevertheless, the S-matrix element is well defined, owing to the transverse nature of the polarization vectors.

$$\partial_\mu B^\mu(x) = 0 \tag{230}$$

The lowest radiative corrections to the $B_\mu{}^2\phi$ vertex, may also be calculated in the standard way.

$$\operatorname{div} T^{(3)}_{\mu\nu}(q,p:(-q-p)) = \frac{3ie^2f^2u g_{\mu\nu}}{(4\pi)^2(2-\omega/2)} + \frac{ie^2[g_{\mu\nu}(3\mu^2 - 9m^2 - p^2) + 4P_\mu P_\nu]}{3u(4\pi)^2(2-\omega/2)} \tag{231}$$

$$+ \frac{ie^2[g_{\mu\nu}\{\frac{4}{3}(p^2+q^2) + 2(pq) - 2\mu^2\} + \frac{1}{3}(P_\mu P_\nu + q_\mu q_\nu) - \frac{1}{2}(P_\mu q_\nu - 3q_\mu P_\nu)]}{u(4\pi)^2(2-\omega/2)}$$

$$\frac{ie^2[g_{\mu\nu}(3\mu^2-9m^2-q^2)+4q_\mu q_\nu]}{3u(4\pi)^2(2-\omega/2)} - \frac{3ie^2f^2ug_{\mu\nu}}{(4\pi)^2(2-\omega/2)} .$$

Picking out the terms proportional to $g_{\mu\nu}$ from (231), and adjusting them so that they vanish when the external legs of the vertex are put on shell, fixes the counterterm Z_2,

$$Z_2 = 1 + \frac{[3e^2-f^2]}{(4\pi)^2(2-\omega/2)} \tag{232}$$

Fortunately this is in complete agreement with (229). Despite the large body of consistency obtained so far, we now encounter a serious difficulty. Examination of (231) reveals that the polarization vectors $\varepsilon_\mu(q)$ and $\varepsilon_\nu(p)$, which are included in the relevant scattering process, are insufficient to remove the remaining terms via condition (230). In particular, the term,

$$\frac{ie^2\,\frac{1}{2}p_\mu q_\nu}{(4\pi)^2u(2-\omega/2)} \tag{232}$$

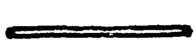
remains. This implies that even the renormalized S-matrix fails to exist as the regulator is removed ($\omega \to 4$). In an attempt to understand this result we digress to a brief account of the R_α - "Renormalizable Gauge".

RENORMALIZABLE GAUGE

Alternative to the U-gauge used so far there exists a class of gauges, known collectively as the 'R_α' or 'renormalizable' gauge.[5] The advantage of using such a gauge stems from the improved high energy behaviour of the vector propagator which renders the theory manifestly re-normalizable by simple power-counting arguments. In such a gauge one encounters non-physical "ghost" particles which although a necessary consequence of working in the R_α gauge, obscure the unitary nature the S-matrix.

The S-matrix in this formulation has been shown to be independent of any choice of gauge taken from the R_α class, where α may take any finite value. The Unitary gauge corresponds to the limiting case $\alpha \to \infty$; a limit which is somewhat formal owing to the highly singular nature of the limiting theory. Without going into details, the Feynman Rules for this formulation are

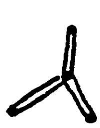
$$= i\Delta^A_{\mu\nu}(k) = \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu (1-\alpha)/(k^2-\alpha\mu_0^2)\right)}{k^2-\mu_0^2+i\varepsilon}$$



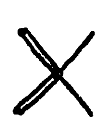
$$= i\Delta_\psi(k) = \frac{i}{k^2-m_\psi^2+i\varepsilon}$$

$$= i\Delta_\chi(k) = \frac{i}{k^2-\alpha\mu_0^2+i\varepsilon}$$

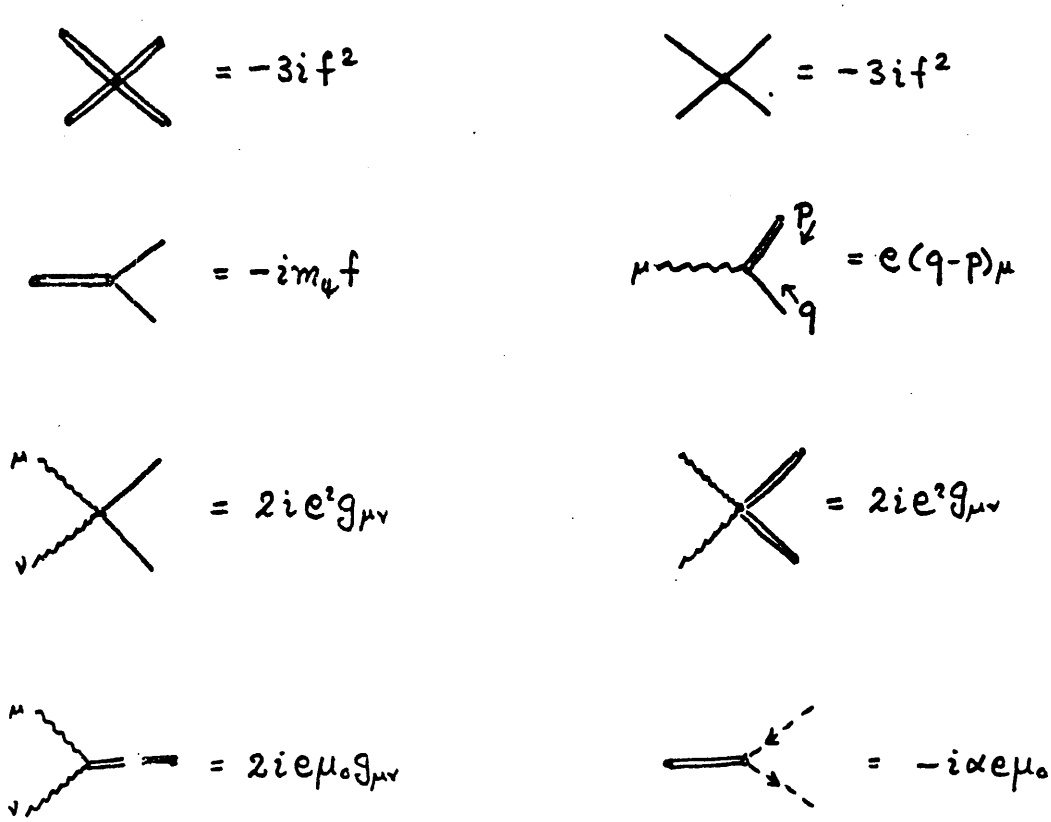
$$= i\Delta_\omega(k) = \frac{i}{k^2-\alpha\mu_0^2+i\varepsilon}$$



$$= -3im_\psi f$$



$$= -if^2$$



+ Counterterms

Figure 25

Note that in the limit $\alpha \to \infty$, the rules themselves go over into the U-gauge rules, (58)

$$
\begin{array}{ll}
m_\psi \to \mu \qquad\qquad & \psi \to \phi \\
\mu_0 \to m & \chi \to 0 \\
 & A_\mu \to B_\mu \;.
\end{array}
\tag{233}
$$

While the ghost loops go over into the 'log' term. The conditions under which the U and R_α gauges may be taken as completely equivalent, (and not just at the tree-graph level), can be given as follows. Consider all Feynman graphs G contributing to a process M at the N-loop order. Then the contribution can be written as an integral over the loop momenta

$$M(P_1, P_2, \ldots, P_M) = \int (dk) \sum_G I_G^{(R)}(k; P_1, \ldots, P_M : \alpha) \tag{234}$$

Where the external momenta $P_1 \ldots P_M$ are restricted to mass shell values, the index R indicates renormalized integrands, and the absence of the parameter 'α' on the left hand side is a statement of gauge independence of the S-matrix. The condition for equivalence is that the limit

$$M(P_1, P_2, \ldots, P_M) = \underset{\alpha \to \infty}{Lt} \int (dk) \sum_G I_G^{(R)}(k; P_1, \ldots, P_M : \alpha) \tag{235}$$

can be taken through the integration and summation sign; i.e. there exists a uniform convergence with respect to these two operations. The result (232) indicates that this is not satisfied, owing to the singular nature of the U-gauge. Our conclusion must be, therefore, that the U-gauge and $R_{\alpha \to \infty}$ gauge formulations, as they stand, are indeed inequivalent. An important question to be faced here is whether the Unitary Gauge is inconsistent (with the implication that the R_α-gauge is not Unitary) or whether the manipulation of such highly singular operators of the U-gauge parameterization, is inadequate. Although no definite conclusion can be drawn as yet, it seems to me that the latter hypothesis is more likely. Nevertheless it may be possible to add an extra piece to the Lagrangian density resulting in a well defined theory.

SECTION TWO

The Infra-Red Problem in Non-Abelian Gauge Theories

THE INFRA-RED PROBLEM

The origin of the infra-red problem in Quantum Electrodynamics is known to stem from an invalid assumption, (implicit in the conventional perturbation approach), that both initial and final asymptotic states contain a finite number of photons. The correct asymptotic states are in fact ones which in general contain an infinite number of photons. Perhaps one of the nicest ways of viewing the problem is in the decomposition of the Hamiltonian into transient and persistent effects. The essential point is that particles at asymptotic times are not characterized by free field equations, but by equations which include all kinds of persistent interactions. In massive field theories, one usually needs to account only for the clouds of virtual quanta which surround each asymptotic particle. In contrast theories involving massless fields, have interactions which fall off according to the inverse square law. Unfortunately, this power law decrease is so slow that two charged particles still produce an effect on each other, even for asymptotic times. In addition to this any deviation from free particle motion is sufficient to induce photon radiation. In summary, therefore, the asymptotic behaviour of the charged matter and radiation fields are described by including self interactions and long range coulomb forces together with "coherent" showers of photons. It is indeed fortunate that in the present instance these latter effects are essentailly classical in nature and may be accounted for via non perturbative solutions to the relevant equations of motion. Explicit knowledge of these solutions proves to be the crucial factor in obtaining a full understanding of the problem. In the case of the Yang-Mills field there are at the present time no such relevant non-perturbative solutions. It is this fact which prohibits a direct test of the ideas presented below and for this reason, can only be presented as speculation or hypothesis. The essential difference between the Yang-Mills field and Quantum Electrodynamics is, in the present context, that the former gauge field undergoes self interactions whereas the latter does not. Owing to the masslessness of the field, the kinematics do not prohibit a single particle from decaying into two other particles; the three trajectories being colinear. The essential point is that this transition mode is a persistent one and consequently must be included in the description of the asymptotic motion of the initial and final states.

Asymptotic States

It is instructive to examine the motion of a single quantum of radiation. Let it be monochromatic at t = o, and thus describable at this instant by a state from a Fock space.

$$|\psi(0);E\rangle = |\omega\rangle_{Fock.} \qquad (1)$$
$$E = \hbar\omega$$

After time 't' the state can again be expanded on the Fock basis,

$$|\psi(t);E\rangle = \sum_{i} |\omega_i\rangle a_i(t) + \sum_{ij} |\omega_i\omega_j\rangle a_{ij}(t) + \cdots \tag{2}$$

In other words at time 't' the state $|\psi\rangle$ has a finite probability for any number of quanta being present provided that the sum of the energies of the constituent quanta, is equal to the original energy 'E'. In fact since this effect is always present one expects in the limit of large times that the state has a large probability of describing an infinite number of quanta with net energy 'E'. Hence we reach the conclusion that the final state does not describe a fixed number of quanta, but rather a co-linear "cascade" of quanta of indefinite number but finite total energy. In mathematical terms, any attempted description in terms of a conventional Fock state is utterly hopeless. It is clearly meaningful to describe scattering only between such cascade states. To do this however requires an exact solution of the asymptotes. The unavailability of such a solution has essentially stopped further progress in this direction.

The Persistent Effects

The claim in the previous section that the cascade effect is a persistent phenomenon, can be shown in a rather transparent way by adapting a method introduced by Faddeev and Kulish[6)] to deal with the Infra-Red problem of Quantum Electrodynamics. Consider for simplicity the massless scalar field with ϕ^3 interactions,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{3!} g \phi^3 \tag{3}$$

In the usual interaction picture one has for the "perturbing potential"

$$\begin{aligned} V(t) = \frac{g}{3!}\int d\mu(\underline{k}_1)d\mu(\underline{k}_2)d\mu(\underline{k}_3)\{ & a_1 a_2 a_3 e^{i(-k_1^0-k_2^0-k_3^0)t}\delta_3(\underline{k}_1+\underline{k}_2+\underline{k}_3) \\ & + a_1 a_2 a_3^* e^{i(-k_1^0-k_2^0+k_3^0)t}\delta_3(\underline{k}_1+\underline{k}_2-\underline{k}_3) + a_1 a_2^* a_3 e^{i(-k_1^0+k_2^0-k_3^0)t} \\ & \times \delta_3(\underline{k}_1-\underline{k}_2+\underline{k}_3) + a_1^* a_2 a_3 e^{i(k_1^0-k_2^0-k_3^0)t}\delta_3(\underline{k}_1-\underline{k}_2-\underline{k}_3) + a_1 a_2^* a_3^* e^{i(-k_1^0+k_2^0+k_3^0)t} \\ & \times \delta_3(\underline{k}_1-\underline{k}_2-\underline{k}_3) + a_1^* a_2 a_3^* e^{i(k_1^0-k_2^0+k_3^0)t}\delta_3(\underline{k}_1-\underline{k}_2+\underline{k}_3) + a_1^* a_2^* a_3 e^{i(k_1^0+k_2^0-k_3^0)t} \\ & \times \delta_3(\underline{k}_1+\underline{k}_2-\underline{k}_3) + a_1^* a_2^* a_3^* e^{i(k_1^0+k_2^0+k_3^0)t}\delta_3(\underline{k}_1+\underline{k}_2+\underline{k}_3)\}. \end{aligned} \tag{4}$$

To pick out the persistent part of this potential one has to see which parts contribute for large times ($t \to \pm\infty$). Owing to the exponential factor, it will be those regions of k-space for which the phase factors vanish, since any other value will lead to an infinitely oscillating phase which will average to zero. As an example consider the term

$$\int d\mu(\underline{k}_1)\, d\mu(\underline{k}_2)\, d\mu(\underline{k}_3)\; a_1 a_2 a_3^* \; e^{i(-k_1^0 - k_2^0 + k_3^0)t}\, \delta(\underline{k}_1 + \underline{k}_2 - \underline{k}_3) \tag{5}$$

At asymptotic times one needs

$$k_1^0 + k_2^0 - k_3^0 = k_1^0 + k_2^0 - \sqrt{(\underline{k}_1 + \underline{k}_2)^2} \approx 0 \tag{6}$$

This will hold provided either (a) k_1^o or k_2^o is very small, which is the infrared region, or (b) $\underline{k}_1 // \underline{k}_2$ which is the cascade region. This procedure can be applied to the entire potential (4) and the persistent contribution seperated. The result is not presented here owing to its unattackability. In principle one could then use this to determine the precise nature of the asymptotic states. Many people have suggested that in the case of Yang-Mills this cascade decay is forbidden by gauge invariance. It is felt, however, that owing to the highly singular nature of the asymptotic form of the propagator this argument could well be false. Finally, there exists a very general theorem proved by T.D. Lee and M. Nauenberg[7)] that by averaging over sufficiently large ensembles of initial and final states, finite transition probabilities may be obtained. In the language of this section these ensembles correspond to Fock space projections of the cascade states.

SECTION THREE

The Dynamical Breakdown of Non-Abelian Gauge Fields

CONVENTIONS

The final section of this thesis is concerned with the dynamics of the Yang-Mills Field,[8] expressing a local invariance under the Ispin group SU(2). The following remarks should be sufficient to fix notational conventions.

The local symmetry amounts to the invariance of the theory under the replacements

$$\begin{aligned} \Psi(x) &\rightarrow \Psi'(x) = \Omega(x)\,\Psi(x) \\ \Omega(x) &= \exp[\,ie\phi(x)] \end{aligned} \tag{1}$$

where any two fields belonging to the same representation obey the properties

$$\begin{aligned} [A, B] &= i\,(A\times B) \\ (A\times B)_a &= \varepsilon_{abc}\,A_b A_c \end{aligned} \tag{2}$$

Transition from a globally invariant theory to a local one, is generally achieved by replacing ordinary derivatives (appearing in the Lagrangian density) by their "covariant" counterparts ∇_μ

$$\partial_\mu f \rightarrow \nabla_\mu f = \partial_\mu f - e\,B_\mu \times f \tag{3}$$

where the "gauge field" B_μ transforms according to the rule,

$$B_\mu \rightarrow B'_\mu = \Omega B_\mu \Omega^{-1} - \frac{i}{e}\,\Omega\,\partial_\mu \Omega^{-1} \tag{4}$$

Although (4) appears to depend on the representation matrices Ω, it is, in fact, only dependent on the structure constants. The inhomogeneous nature of this transformation is so contrived that ∇_μ transforms homogeneously. The tensor

$$\mathcal{B}_{\mu\nu} = \partial_\nu \mathcal{B}_\mu - \partial_\mu \mathcal{B}_\nu + e\,\mathcal{B}_\mu \times \mathcal{B}_\nu \tag{5}$$

also transforms homogeneously and serves for the construction of an invariant Lagrangian density from which the field equations are to be derived

$$\mathcal{L}_{YM} = -\frac{1}{4}\mathcal{B}_{\mu\nu}{}^2 \tag{6}$$

In equation (6) the 'square' indicates summation over group, as well as Lorentz, indices. To work with (6) it is also necessary to choose a gauge condition. Addition of the term

$$\mathcal{L}_\alpha = -\frac{1}{2\alpha}(\partial_\mu \mathcal{B}^\mu)^2 \tag{7}$$

leads to a family known collectively as the linear α-gauge, provided however that additional fields, obeying Fermi statistics, are included

$$\mathcal{L}_{FP} = (\partial_\mu \bar{\omega})\cdot(\partial^\mu \omega) + e\,\mathcal{B}_\mu \cdot (\partial^\mu \bar{\omega} \times \omega) \tag{8}$$

Finally a coupling to the classical source $J_\mu(x)$ for later manipulations, leads to the theory

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial_\mu B^\mu)^2 + (\partial_\mu \bar{\omega})\cdot(\partial^\mu \omega) + e B_\mu \cdot (\partial^\mu \bar{\omega} \times \omega) + J_\mu B^\mu \quad (9)$$

Application of the Euler-Lagrange equations leads to three coupled field equations.

(a) $$\nabla^\nu B_{\mu\nu} + \frac{1}{\alpha} \partial_\mu \partial^\nu B_\nu + e\, \partial_\mu \bar{\omega} \times \omega + J_\mu = 0$$

(b) $$\partial^\mu (\nabla_\mu \cdot \omega) = 0 \quad (10)$$

(c) $$\Box \bar{\omega} - e B_\mu \times (\partial^\mu \bar{\omega}) = 0$$

It is the system (10) which we take to describe the dynamics of our theory.

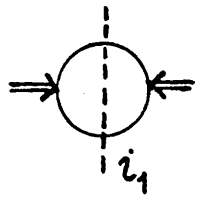
SCHWINGER MECHANISM

The conditions under which the "massless" field equations (10) can result in a system with massive characteristics was first given by Schwinger. The complete inverse vector propagator is of the form,

$$G^{-1}_{\mu\nu}(k) = -k^2 g_{\mu\nu} + k_\mu k_\nu - \frac{1}{\alpha} k_\mu k_\nu + \Pi_{\mu\nu}(k)$$

$$i\, G_{\mu\nu}(x) = \left\langle \left(B_\mu(x) B_\nu(0) \right)_+ \right\rangle^{CON}_{J=0} \quad (11)$$

differs from its free field couterpart by the addition of a "Self Energy" term describing all the radiative corrections which are one-particle irreducible with respect to a cut between the external legs

$i\,\Pi_{\mu\nu} =$  (12)

Inversion of (11) is achieved by first splitting the self energy contribution into transverse and londitudinal pieces, i.e.

$$\Pi_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu / k^2)\,\Pi(k^2) \tag{13}$$

where we have used the fact that the londitudinal piece of the self energy vanishes. This is a non-perturbative result which follows from a Ward identity. The dressed propagator is thus,

$$G_{\mu\nu}(k) = -(g_{\mu\nu} - k_\mu k_\nu / k^2)\left[k^2 - \Pi(k^2)\right]^{-1} - \alpha k_\mu k_\nu / k^4 \tag{14}$$

As before the Ispin labels have been suppressed. Schwinger's observation was simply, that if $\pi(o)$ is positive, non-zero, the self energy (13) has a pole at zero momentum, and the vector propagator (14) acquires a massive nature. Furthermore, due to the irreducibility condition imposed on the self energy (12), the pole is clearly of a dynamical origin and expresses the formation of a composite zero-mass bound state. This additional degree of freedom is absorbed by the vector field, producing a massive state in accordance with the phenomenon first investigated in the context of field theory by Higgs. Manipulation of bound state modes introduces, however, certain technical difficulties related to the necessity of infinite resummation of perturbation graphs. In order to circumvent this problem, one generally resorts to the weaker criteria of "self-consistency" and in so doing reduces ones conclusions from relative certainty to mere possibility. (It is in this sense that a verdict of impossibility may be regarded as carrying a slightly greater force.)

FORM OF THE SPONTANEOUSLY BROKEN SOLUTION

Although all possible forms of spontaneously broken solutions are of interest in themsleves, one form stands out as being particularly "natural" for SU(2). To indicate this it is convenient to express the gauge field in the so called "charged basis" rather than the "cartesian basis". The former basis rendering the third component of Ispin diagonal. This permits one to associate a charge quantum number with the field quanta. Perhaps the most natural breaking which can be envisaged is that whereby the positively and negatively charged particles acquire an equal mass 'm' while the neutral particle remains massless (photon). Hence the hypothesised breaking is that by which the full SU(2) symmetry is reduced to its U(1) subgroup.

In the charged basis,

$$|+\rangle \qquad |0\rangle \qquad |-\rangle \tag{15}$$

the generators take the form,

$$I_1^Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad I_2^Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{16}$$

$$I_3^Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

while the mass matrix is

$$M_Q^2 = m^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Transition back to the Cartesian basis is implemented by the transformation matrix <C|Q>

$$|c\rangle = \sum_{Q\in\{+,0,-\}} |Q\rangle\langle Q|C\rangle \tag{18}$$

$$\langle Q|C\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{bmatrix}$$

This gives for the mass matrix in the Cartesian basis,

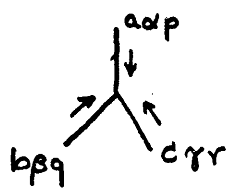
$$M_C^2 = m^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = m^2\left(I_3^C\right)^2 \tag{19}$$

As the final expression indicates, this is clearly invariant under the U(1) subgroup.

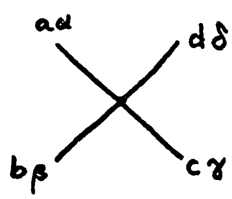
THE DYSON EQUATION FOR THE SELF ENERGY

It is convenient to write down an equation for the self energy in the Dyson form. First consider the interaction picture diagrams corresponding to the Lagrangian density (9)

$$: \; i\Delta^{ab}_{\mu\nu}(k) = -i\,\frac{(g_{\mu\nu}-k_\mu k_\nu/k^2)\delta_{ab}}{k^2} - i\alpha\,\frac{k_\mu k_\nu\,\delta_{ab}}{k^4}$$

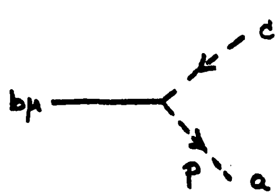


$$: \; e\,\varepsilon_{abc}\left[g_{\alpha\beta}(q-p)_\gamma + g_{\beta\gamma}(r-q)_\alpha + g_{\gamma\alpha}(p-r)_\beta\right]$$



$$\begin{aligned} : \; & -ie^2\varepsilon_{gac}\,\varepsilon_{gbd}\left[g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\gamma\beta}\right] \\ & -ie^2\varepsilon_{gad}\,\varepsilon_{gbc}\left[g_{\alpha\beta}g_{\delta\gamma} - g_{\alpha\gamma}g_{\delta\beta}\right] \\ & -ie^2\varepsilon_{gab}\,\varepsilon_{gcd}\left[g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right] \end{aligned}$$

$$: \; i\Delta^{ab}(k) = \frac{i}{k^2}$$



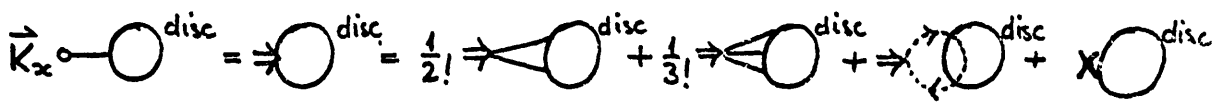
$$: \; e\,\varepsilon_{abc}\,p_\mu \qquad\qquad a\mu \xrightarrow{q} \times \; : \; i\tilde{J}_{a\mu}(q)$$

Figure 1

By taking the vacuum expectation value of equation (10a) between states describing the vacuum at very early and very late times and replacing the vector fields by functional derivatives, one arrives at an expression for the generating functional for disconnected Greens functions,

$$Z^{disc}[J] = {}_J^{out}\langle 0|0\rangle_J^{in} = \bigcirc^{disc} \tag{20}$$

in terms of the same with two ghost field insertions; although straightforward to write equations in algebraic form, it is a considerable simplification to represent them by the interaction picture diagrams. This avoids the long equations which tend to hide the underlying simplicity of the idea involved



(21)

Where K_x is the operator defining the free vector propagator via,

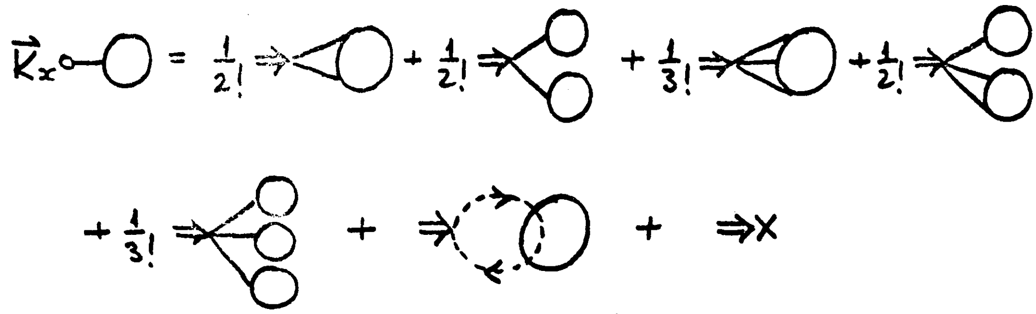
$$\vec{K}_x\cdot\Delta(x) = (\Box g_{\mu\nu} - \partial_\mu\partial_\nu + \tfrac{1}{\alpha}\partial_\mu\partial_\nu)\Delta_{\nu\lambda}(x) = g_{\mu\lambda}\delta(x) \tag{22}$$

The double arrow indicates the amputation of a vector propagator, whilst all other symbols are given in Fig. (1) and (20). Restriction to the connected parts of the Green's functions is achieved via the substitution,

$$\ln Z^{disc}[J] = iW[J] = \bigcirc \tag{23}$$

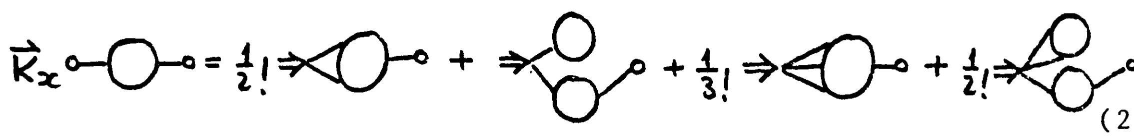
$$iW[J] = \sum_{N=1}^{\infty} \frac{i^N}{N!}\int dx_1\dots dx_N\, J_{\mu_1 a_1}(x_1)\dots J_{\mu_N a_N}(x_N)\langle 0|(B_{\mu_1 a_1}(x_1)\dots B_{\mu_N a_N}(x_N))_+|0\rangle^{CON}$$

which converts (21) into,

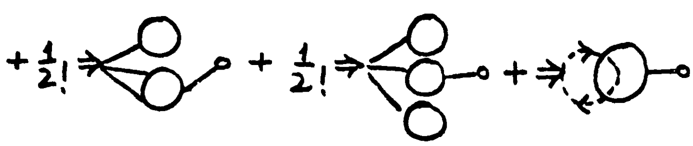


(24)

after division by the common factor of $Z^{disc}[J]$. We now differentiate functionally once to add a second leg to the bubble.

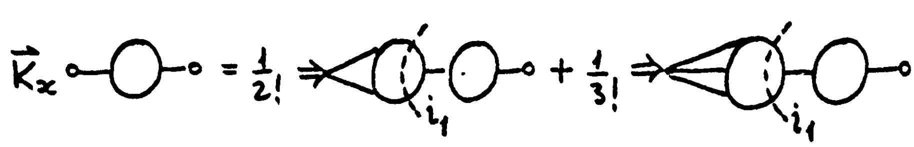


(25)

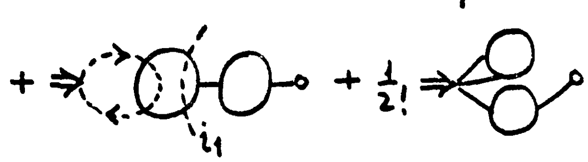


+ contact term.

Separating out the full two point connected Green's function on this second leg so that the remainder is one particle irreducible, and turning off the external source gives,



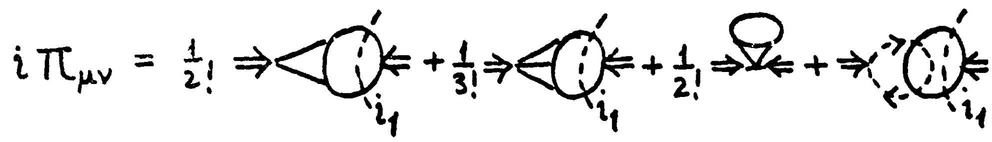
(26)



+ contact term.

Where the assumption has been made that the "tadpole" graph vanishes with the external source whereas the "seagull" graph does not. Note that by this assumption the Lagrangian density (9) cannot be taken Wick-ordered. By regrouping

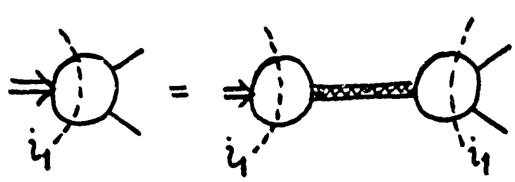
(26) and using the definition (11) for the inverse 'dressed' propagator gives



(27)

This is the Dyson equation for the self energy in terms of the 'dressed' vertices of Fig. (1).

The appearance of a pole in the self energy can now be attributed to a bound state in a simple way. Consider the first diagram on the righthand side of (27). The vertex function appearing here is one particle irreducible with respect to the external leg, and thus contains no poles corresponding to the elementary field $B_{\mu a}(x)$. The assumption is, however, that there exists a bound state pole here, corresponding to a composite object, diagramatically,



+ Regular (28)

Where the 'ladder' is meant to represent the bound state pole. As it stands, this diagram alone is not gauge invariant. It must be combined with the remainder of (27), for which the corresponding statements can be made.

SELF CONSISTENT APPROXIMATION

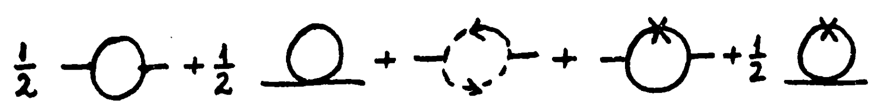
One method of trying to establish an approximation scheme is to add a mass term directly to the Lagrangian density (9) and attempt to do calculations in a self consistent manner. The main obstacle to this program is that the loss of gauge invariance results in a theory which has every indication of being non-renormalizable. This calculation is performed here to the lowest order of approximation not only for the insight to be gained, but also for future use in a more realistic version of approximation. Adding a mass term to the Lagrangian density (9) and at the same time assuming that the bare mass vanishes, is equivalent to fixing the physical mass and the mass-counterterm to be equal and opposite. Let this term be,

$$\delta\mathcal{L} = \tfrac{1}{2} M^2_{ab} B_{\mu a} B^{\mu}{}_{b} \tag{29}$$

and denote its graph by,

$$= iM^2_{ab}\, g_{\mu\nu} \tag{30}$$

Only the divergent terms are considered since it is these which determine whether consistency can be achieved. Finite corrections can be dealt with later. The one loop contributions to (27) are



+ counter terms.

The first three diagrams are those of the usual massless Yang-Mills field and consequently are gauge invariant. They give a contribution

$$\frac{ie^2\delta_{ab}}{(4\pi)^2}\,\frac{(10/3)}{(2-\omega/2)}\,(g_{\mu\nu}k^2 - k_\mu k_\nu) \tag{31}$$

The last two graphs give the net contribution,

$$\frac{3}{4}\cdot\frac{i}{(4\pi)^2}\cdot\frac{e^2 g_{\mu\nu}}{(2-\omega/2)}\left[\delta_{ab}\,\mathrm{Tr}\,M^2 - M^2_{ab}\right] \tag{32}$$

Note however, that (32) is not gauge invariant. This is a serious difficulty. Let us bypass this difficulty momentarily by adding to (32) an extra term to give a net gauge invariant result,

$$\frac{3}{4}\,\frac{i}{(4\pi)^2}\,e^2\,\frac{\left[\delta_{ab}\,\mathrm{Tr}\,M^2 - M^2_{ab}\right]}{(2-\omega/2)}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{33}$$

The net result (31) and (33) gives

$$\mathscr{D}i\omega\; i\Pi^{ab}_{\mu\nu}(k) = \frac{i e^2\left[\frac{10}{3}k^2\delta_{ab} + \frac{3}{4}\delta_{ab}\,\mathrm{Tr}\,M^2 - \frac{3}{4}M^2_{ab}\right]\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{(4\pi)^2(2-\omega/2)} \tag{34}$$

+ (counter terms).

This result looks promising but there still remains the problems of gauge invariance and non-renormalizability. In order to try to overcome these problems, we turn to a very new approach suggested by J. Cornwall.[9]

MASSIVE GAUGE-INVARIANT YANG-MILLS FIELDS

Cornwall's idea was to add to the massless Yang-Mills Lagrangian density a mass-like term which is itself gauge invariant. The price one has to pay for such a method is that resulting term is both non-local and non-polynomial.

It is very easy to construct a gauge invariant vector by using the group property of the gauge transformations as follows. Consider the action of the transformation (4) which we now write as,

$$B_\mu \rightarrow B_\mu^\Omega = f_\mu(B,\Omega) = \Omega B_\mu \Omega^{-1} - \frac{i}{e}\Omega \partial_\mu \Omega^{-1} \tag{35}$$

The action of two transformations may be represented by the diagram

$$\begin{tikzcd} & B_\mu \arrow[dl, "\Omega"'] \arrow[dr, "\theta"] & \\ B_\mu^\Omega \arrow[rr, "\theta\Omega^{-1}"'] & & B_\mu^\theta \end{tikzcd}$$

where

$$\begin{aligned} B_\mu^\Omega &= f_\mu(B,\Omega) \\ B_\mu^\theta &= f_\mu(B,\theta) \end{aligned} \tag{36}$$

using the group property of the transformation (35) one also has,

$$B_\mu^\theta = f_\mu(B^\Omega, \theta\Omega^{-1})$$

hence from (36),

$$B_\mu^\theta = f_\mu(B,\theta) = f_\mu(B^\Omega, \theta\Omega^{-1}) \tag{37}$$

This is the required relation; $f_\mu(B,\theta)$ is an invariant provided one imposes the transformation law,

$$f_\mu(B,\Theta): \quad \begin{aligned} B_\mu &\to \Omega B_\mu \Omega^{-1} - \frac{i}{e}\Omega\partial_\mu\Omega^{-1} \\ \Theta &\to \Theta\Omega^{-1} \end{aligned} \tag{38}$$

As (38) stands however, it involves one extra field '$\theta(x)$. The trick is to choose $\theta(x)$ as a function of $B_\mu(x)$ in such a way that the transformation (35) acting on B_μ automatically induces the correct transformation of $\theta(x)$

$$\Theta(B;x)\,\Omega^{-1}(x) = \Theta(B^{\Omega};x) \tag{39}$$

The invariant $f_\mu(B,\theta)$ is in fact representation independent, as noted previously. In component form (35) becomes,

$$\begin{aligned} f_{\mu a} &= \alpha_{ab}(\phi)B_\mu^b - \beta_{ab}(\phi)\partial_\mu\phi^b \\ \Theta(x) &= \exp[ie\phi_a(x)T_a] \end{aligned} \tag{40}$$

Consider a second quantity $R_{\mu a}$, which transforms like a group vector

$$\begin{aligned} R_{\mu a} &= B_{\mu a} - \gamma_{ab}\,\partial_\mu\phi_b \\ \gamma_{ab}(\phi) &= \alpha^{-1}_{ac}(\phi)\beta_{cb}(\phi) \end{aligned} \tag{41}$$

The advantage of this form is that it is possible to construct an invariant equation relating $B_{\mu a}$ and θ (or equivalently ϕ) which is found to also satisfy the condition (39). Using the homogeneous transformation law for the covariant derivative one finds that,

$$\nabla_\mu^{ab} R^\mu_b = 0 \tag{42}$$

is an invariant equation. Using (3) and (41) this can be cast into a more convenient form.

$$\begin{aligned} \Box \phi^a + \Theta_\mu^{ab} \partial_\mu \phi^b &= \psi^a \\ \Theta_\mu^{ab} &= (\gamma^{-1} \cdot \nabla_\mu \cdot \gamma)^{ab} \\ \psi^a &= (\gamma^{-1} \cdot \partial^\mu B_\mu)^a \end{aligned} \tag{43}$$

These equations form the basis of the gauge invariant method - one adds the term,

$$\delta \mathcal{L} = \frac{1}{2} f_\mu^a M_{ab}^2 f^{b\mu} \tag{44}$$

to the Lagrange density (9). By construction this term is gauge invariant (38), and can be computed from (40) and (43) to any desired order of approximation. Explicit knowledge of 'f_μ' allows one to perform gauge invariant calculations in a straightforward manner. Owing to the non-linear nature of (43), an exact solution does not seem possible. It is therefore necessary to work within a perturbative scheme. Since calculations will be performed only to the one loop level it is sufficient to know 'f_μ' to the second power of the coupling constant. Two further comments are necessary. Firstly, the additional term (44) should be adjusted (with respect to the corresponding counterterm) so that the bare mass vanishes. More precisely stated, the bare mass must vanish as the regulator is removed. This can be checked by using a "Callan-Symanzik" type equation. Secondly, the vector 'f_μ' is invariant under a restricted class of gauge transformations (1) and (4). Those obeying

$$\Box\psi^a(x) = 0$$
$$\Omega(x) = \exp[ie\,\psi^a(x)T_a] \tag{45}$$

must be excluded. This condition removes terms like $\Box^{-1}0$ from the gauge transformed f^a_μ.

CALCULATION OF f^a_μ

We have the definitions

$$f_\mu(B,\Omega) = \Omega B_\mu \Omega^{-1} - \frac{i}{e}\Omega\partial_\mu\Omega^{-1}$$
$$[A,B] = i\,(A\times B) \tag{46}$$
$$\Omega(x) = \exp[ie\phi] \quad : \phi = \phi^a T_a$$

Firstly one must convert (46) to the form

$$f_\mu(B,\Omega) = T_a\,\alpha_{ab}(\phi)\,B_{b\mu} - T_a\,\beta_{ab}(\phi)\,\partial_\mu\phi_b \tag{47}$$

Using the identity,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots \tag{48}$$

gives

$$\Omega B_\mu \Omega^{-1} = B_\mu + ie\,[\phi, B_\mu] + \frac{1}{2}(ie)^2[\phi,[\phi,B_\mu]] + \cdots$$
$$= B_\mu - e(\phi\times B_\mu) + \frac{1}{2}e^2\phi\times(\phi\times B_\mu) + O(e^3) \tag{49}$$

Secondly, using the identity,

$$e^A\partial_\mu e^{-A} = -\left\{\partial_\mu A + \frac{1}{2!}[A,\partial_\mu A] + \frac{1}{3!}[A,[A,\partial_\mu A]] + \cdots\right\}$$

gives,

$$\Omega\partial_\mu\Omega^{-1} = -\{ie\,\partial_\mu\phi + \tfrac{1}{2}(ie)^2[\phi,\partial_\mu\phi] + \tfrac{1}{6}(ie)^3[\phi,[\phi,\partial_\mu\phi]] + \cdots\}$$

$$= -ie\partial_\mu\phi + \tfrac{1}{2}ie^2(\phi\times\partial_\mu\phi) - \tfrac{1}{6}ie^3\phi\times(\phi\times\partial_\mu\phi) + \cdots$$

Hence,

$$-\frac{i}{e}\Omega\partial_\mu\Omega^{-1} = -\partial_\mu\phi + \tfrac{1}{2}e(\phi\times\partial_\mu\phi) - \tfrac{1}{6}e^2\phi\times(\phi\times\partial_\mu\phi) + O(e^3) \qquad (50)$$

From this follows the value for $f_\mu(B,\Omega)$;

$$f_\mu = B_\mu - e(\phi\times B_\mu) + \tfrac{1}{2}e^2\phi\times(\phi\times B_\mu) - \partial_\mu\phi + \tfrac{1}{2}e(\phi\times\partial_\mu\phi) - \tfrac{1}{6}e^2\phi\times(\phi\times\partial_\mu\phi) + O(e^3) \qquad (51)$$

'ϕ' must be determined from (43) to order e^2,

$$\phi = A + eE + e^2F \qquad (52)$$

substituting into (51) gives,

$$f_\mu = B_\mu - e(A\times B_\mu) - e^2(E\times B_\mu) + \tfrac{1}{2}e^2A\times(A\times B_\mu) - \partial_\mu A - e\partial_\mu E - e^2\partial_\mu F + \tfrac{1}{2}e(A\times\partial_\mu A) + \tfrac{1}{2}e^2(A\times\partial_\mu E) + \tfrac{1}{2}e^2(E\times\partial_\mu A) - \tfrac{1}{6}e^2A\times(A\times\partial_\mu A) \qquad (53)$$

The matrix 'α' appearing in (47) follows from (49),

$$\alpha\cdot g = g - e(\phi\times g) + \tfrac{1}{2}e^2\phi\times(\phi\times g) + \cdots \qquad (54)$$

where g is any group vector. The inverse of (54) is thus,

$$\alpha^{-1}\cdot g = g + e(\phi\times g) + \tfrac{1}{2}e^2\phi\times(\phi\times g) + \cdots \qquad (55)$$

which on application to (51) gives

$$\mathcal{R}_\mu = \alpha^{-1}\cdot f_\mu = B_\mu - \partial_\mu\phi - \tfrac{1}{2}e\phi\times\partial_\mu\phi - \tfrac{1}{6}e^2\phi\times(\phi\times\partial_\mu\phi) + \cdots \tag{56}$$

So that from (56) and (41) we have,

$$\begin{aligned} \gamma\cdot g &= g + \tfrac{1}{2}e(\phi\times g) - \tfrac{1}{6}e^2\phi\times(\phi\times g) + O(e^3) \\ \gamma^{-1}\cdot g &= g - \tfrac{1}{2}e(\phi\times g) + \tfrac{5}{12}e^2\phi\times(\phi\times g) + O(e^3) \end{aligned} \tag{57}$$

Applying the result (57) to equation (43) yields after some simplification

$$\begin{aligned} \Box\phi &= \partial^\mu B_\mu + e\{B_\mu\times\partial^\mu\phi - \tfrac{1}{2}\phi\times\partial^\mu B_\mu\} + e^2\{\tfrac{5}{12}\phi\times(\phi\times\partial^\mu B_\mu) \\ &\quad + \tfrac{1}{2}(\partial_\mu\phi)\times(\phi\times B^\mu) - \tfrac{1}{6}(\partial_\mu\phi)\times(\phi\times\partial^\mu\phi)\} + O(e^3) \end{aligned} \tag{58}$$

This equation is easily solved by interation. Substituting (52) into (58) gives

$$\begin{aligned} A &= \Box^{-1}\partial_\mu B^\mu \\ E &= \Box^{-1}\{B_\mu\times\partial^\mu A - \tfrac{1}{2}A\times\partial^\mu B_\mu\} \\ F &= \Box^{-1}\{B_\mu\times\partial^\mu E - \tfrac{1}{2}E\times\partial^\mu B_\mu + \tfrac{5}{12}A\times(A\times\partial^\mu B_\mu) \\ &\qquad + \tfrac{1}{2}(\partial_\mu A)\times(A\times B^\mu) - \tfrac{1}{6}(\partial_\mu A)\times(A\times\partial^\mu A)\} \end{aligned} \tag{59}$$

Finally from (44), (51) and (59) one arrives at the mass term order e^3;

$$\begin{aligned} \delta\mathcal{L} &= \tfrac{1}{2}(B_\mu - \partial_\mu A)\cdot M^2\cdot(B^\mu - \partial^\mu A) + e(B_\mu - \partial_\mu A)\cdot M^2\cdot(B_\mu\times A \\ &\quad + \tfrac{1}{2}A\times\partial_\mu A) + e^2(B_\mu - \partial_\mu A)\cdot M^2\cdot\{B_\mu\times E + \tfrac{1}{2}A\times(A\times B_\mu) + \end{aligned} \tag{60}$$

$$+\tfrac{1}{2}(A\times\partial_\mu E)+\tfrac{1}{2}(E\times\partial_\mu A)-\tfrac{1}{6}A\times(A\times\partial_\mu A)\big\}+\tfrac{1}{2}e^2\,(B_\mu\times A$$
$$-\partial_\mu E+\tfrac{1}{2}A\times\partial_\mu A)\cdot M^2\cdot(B_\mu\times A-\partial_\mu E+\tfrac{1}{2}A\times\partial_\mu A)$$

Enormous simplification may be achieved by working in the Landau gauge ($\alpha = o$). Only two types of graph topology occur in applying (60) to (27).

(61)

where the "square" three and four point vertices are those corresponding to (60). Rather than use symmetrized vertices it is computationally more convenient to start directly from the Gell-Mann Low formula and pick out those pairings which do not vanish under a transverse projection. The first graph of (61) has a vanishing contribution. This was first observed by Cornwall. He completely missed the graphs of the second type in (61) and wrongly concluded that the break-down does not appear at the one-loop level. Inclusion of these latter graphs does in fact produce the net term,

$$-\frac{3}{4}\,\frac{e^2 i}{(4\pi)^2}\;\frac{[\mathrm{Tr}\,M^2\delta_{ab}-M^2_{ab}]}{(2-\omega/2)}\;\frac{p_\mu p_\nu}{p^2} \qquad (62)$$

which is precisely the addition that was made to (32) in order to reach a gauge invariant answer. Occurring among the contributions to (62) are terms of the form $1/p^4$ and $1/0$. Fortunately these all cancel. If this method is to be acceptable, such cancellations must occur to all orders of perturbation. Whether this is so remains at present an open question.

Although this method looks very promising there are two difficulties to be faced. Firstly a brief examination of (34) indicates that all three vector particles will acquire a mass. This is in conflict with our requirement of naturalness (17) and (19). The trouble is that the restricted gauge transformations are not sufficient keep the "photon" massless in finite order perturbation theory and even if one begins with the mass matrix (19), the third component field acquires a mass already at the one loop level. The second difficulty is that this approach gives no clue as to the stability of the broken solution. In view of these facts we turn to another approach which appears to offer a way round these difficulties.

THE FUNCTIONAL APPROACH

This method involves computing an effective potential for the composite field operator,

$$B_{\mu a} B^{\mu}{}_{b} \tag{63}$$

The following is incomplete in as much as renormalization has not been fully accounted for, and may even be an insurmountable problem. The method was devised by Gross and Neveu[10] for dealing with ϕ^4 and $(\bar{\psi}\psi)^2$ interactions.

The first stage is to modify the four point vertex appearing in the Yang-Mills Lagrangian density by introducing a Lagrangian multiplier field.

Normal Vertex: $$-\frac{1}{4}e^2 B^a_\mu B^b_\nu \{ B^a_\mu B^b_\nu - B^b_\mu B^a_\nu \} \tag{64}$$

Modified Vertex: $$-\frac{1}{4} S_{ab} S_{ab} + \frac{1}{4} S_{aa} S_{bb}$$

$$-\frac{1}{2} e S_{aa} B^b_\mu B^b_\mu + \frac{1}{2} e S_{ab} B^a_\mu B^b_\mu$$

The equation of motion for the S-field is simply

$$S_{ab} = e B_{\mu a} B^{\mu}{}_{b} \tag{65}$$

so that the two forms in (64) are indeed equivalent. One is interested in the object.

$$\exp iW_2[J] = \int \mathcal{D}B_\mu \mathcal{D}\bar{\omega} \mathcal{D}\omega \exp i \int d^4x \{ \mathcal{L}_{YM} + e J_{ab} B^a_\mu B^b_\mu \} \tag{66}$$

Or using the multiplier formulation,

$$\exp iW_2[J] = \int \mathcal{D}B_\mu \mathcal{D}S \mathcal{D}\bar{\omega} \mathcal{D}\omega \exp i \int d^4x \{ \mathcal{L}' + e J_{ab} B^a_\mu B^b_\mu \} \tag{67}$$

where $\mathcal{L}'$ is the same Lagrangian density as before but with the modification (64). The trick is to perform a change of variables in (67)

$$S_{ab} \rightarrow S'_{ab} = S_{ab} - 2J_{ab} + \delta_{ab} J_{cc} \tag{67}$$

giving the form,

$$\exp iW_2[J] = \int \mathcal{D}B_\mu \mathcal{D}S \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp i \int d^4x \{ \mathcal{L}' - \frac{1}{2} J_{aa}^2 + J_{ab} J_{ab} + S_{ab} J_{ab} \} \tag{68}$$

or simply

$$W_2[J] = W_1[J] + \int d^4x \left(J_{ab} J^{ab} - \frac{1}{2} J_{aa}^2 \right) \tag{69}$$

where,

$$\exp iW_1[J] = \int \mathcal{D}B_\mu \mathcal{D}S \mathcal{D}\omega \mathcal{D}\bar{\omega} \exp i\int d^4x \{ \mathcal{L}'(x) + S_{ab}J_{ab} \} \tag{70}$$

The point of this procedure is that the graphs contributing to the effective potential corresponding to (70), are easily classified by their topological properties, whereas those contributing to (67) are not. Defining the conjugate variables

$$\frac{\delta W_1[J]}{\delta J_{ab}(x)} = \langle S^J_{ab}(x) \rangle = \sigma^J_{ab}(x)$$

$$\frac{\delta W_2[J]}{\delta J_{ab}(x)} = e\langle (B^a_\mu B^b_\mu)^J_+ \rangle = \beta^{ab}_J(x) \tag{71}$$

gives for the Legendre transforms,

$$\Gamma_1[\sigma^J] = W_1[J] - \int d^4x\, \sigma^J_{ab}(x) J_{ab}(x)$$

$$\Gamma_2[\beta^J] = W_2[J] - \int d^4x\, \beta^J_{ab}(x) J_{ab}(x) \tag{72}$$

Taking σ and β to translationally invariant and introducing the effective potentials via

$$\Gamma_1 = -\int d^4x\, V_1(x)$$

$$\Gamma_2 = -\int d^4x\, U(x) \tag{73}$$

gives the relations

$$U_2(\beta) = V_1(\sigma) + J_{ab}J^{ab} - \tfrac{1}{2} J_{aa}J^{bb} \tag{74}$$

$$J_{ab} = \partial V_1(\sigma)/\partial \sigma_{ab}$$

$$\beta_{ab} = \sigma_{ab} + 2J_{ab} - \delta_{ab} J_{cc}$$

Hence calculating $V_1(\sigma)$ gives (in principle) the value of $U_2(\beta)$. The one loop calculation has been performed in the usual way, by adding all one-particle irreducible contributions. The result is

$$V_1(\sigma) = -\tfrac{1}{2}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \tfrac{3}{4}\left(\tfrac{1}{4\pi}\right)^2 \Big\{ -2e\Lambda^2[\sigma_1+\sigma_2+\sigma_3] \tag{75}$$

$$-\tfrac{1}{2}e^2[\sigma_1^2+\sigma_2^2+\sigma_3^2] - \tfrac{1}{2}e^2[\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1]$$

$$+\tfrac{1}{4}e^2[\sigma_2+\sigma_3]^2 \ln \frac{e^2(\sigma_2+\sigma_3)^2}{\Lambda^4} + \tfrac{1}{4}e^2[\sigma_1+\sigma_3]^2 \ln \frac{e^2(\sigma_1+\sigma_3)^2}{\Lambda^4}$$

$$+\tfrac{1}{4}e^2[\sigma_1+\sigma_2]^2 \ln \frac{e^2(\sigma_1+\sigma_2)^2}{\Lambda^4} \Big\}$$

Where

$$\sigma_{ab} = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{pmatrix}$$

and Λ is a large momentum cutoff. Renormalization is performed by demanding that the symmetric point $(\sigma_1, \sigma_2, \sigma_3) = \underline{0}$ be a stationary point and that the potential has its classical value at $\sigma_1 = \sigma_2 = \sigma_3 = \Sigma$:

$$V_1(\sigma) = -\tfrac{1}{2}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \tfrac{3}{2}\left(\tfrac{e}{4\pi}\right)^2 \Big[-\tfrac{3}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) \tag{76}$$

$$-\tfrac{3}{2}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \tfrac{1}{2}[\sigma_2+\sigma_3]^2 \ln \left(\frac{\sigma_2+\sigma_3}{2\Sigma}\right)$$

$$+\tfrac{1}{2}[\sigma_1+\sigma_3]^2 \ln\left(\frac{\sigma_1+\sigma_3}{2\Sigma}\right) + \tfrac{1}{2}[\sigma_1+\sigma_2]^2 \ln\left(\frac{\sigma_1+\sigma_2}{2\Sigma}\right) \Big]$$

In principle this is the required result. One can obtain U_2 and β parametrical in terms of σ's and thus investigate the asymmetric solutions. This is not done here however since it is not clear that (76) is meaningfull. The difficulty is that the external sources have been coupled to unrenormalized operators, whereas one should really couple them to renormalized fields to perform these sorts of arguments. Consequently some of the above equations may cease to exist as the cutoff 'Λ' becomes infinite. Whether this difficulty can be overcome is at present an open question.

CONCLUSION AND OUTLOOK

The first section of this thesis dealt with the renormalizability in the U-gauge parameterization of the Higgs' model. The divergent pieces of scalar and scalar-vector vertex functions have been calculated in the one-loop approximation. The existence of residual divergences which are not removed by the renormalization procedure prevent one from continuing scattering amplitudes off shell. A direct consequence of this is that one must introduce the "physical" masses and charges via S-matrix processes in contrast to the usual methods involving proper vertex functions. The resulting rescaled S-matrix elements are almost all well defined (finite) an exception being the lowest radiative correction to the $B_\mu{}^2\phi$ vertex which owing to the quadratic nature of its divergence, cannot be rendered finite by charge renormalisation, even when restricted by mass-shell and transversality conditions. Whether this difficulty can be overcome is an open question. Nevertheless, theories of this type deserve further study since they mark the borderline between renormalizable and non-renormalizable theories and may provide valuable insight into latter. Many people have speculated that non-renormalizability may be connected with non-analyticity in the coupling constant. Owing to computational difficulties this conjecture remains unproven. In the present instance, however, the situation is far more encouraging since the apparent non-renormalizable nature of the U-gauge can be directly related to the non-analytic limit of the $R_{\alpha\to\infty}$ parameterization. It is felt that investigation along these lines may yield interesting results. A possible tool for this approach is a suitable generalization of the "pole identities" recently noted by t'Hooft. These give a rather interesting characterization of renormalizability and appear to hold more detailed information than the usual power counting approach.

The second section was concerned with the infra-red problem of non-abelian gauge theories. It is felt that further progress along the directions indicated here, can only be achieved by a computational advance. I feel that the physical basis underlying this phenomenon has been given and it remains only to find a realistic model allowing a direct computation of the asymptotic states, to give a direct test of these ideas. In the particular case of the non-abelian gauge fields this may be a pseudoproblem, since there are strong reason to suggest that this class of theory acquires mass dynamically and in doing so removes the infra-red difficulty. Nevertheless the discussion of this section is applicable to any massless, self interacting field and for this reason remains a potent question.

The third and last section of this thesis was concerned with the dynamical breakdown of the Yang-Mills field. This line of investigation seems to me to be the most pressing one of all. This is for two reasons. Firstly, the dynamics of this system are deduced from the very reasonable postulate of local SU(2) invariance and unlike many other field theories, does not have to justify its existence on the grounds of renormalizability. Secondly, this theory possesses the curious behaviour of asymptotic freedom, which aside from the computational advantages this may offer in strong interaction physics, is well worth investigation on account of its non intuitive nature and the improved flexibility of outlook that may thus be acquired. The main obstacle to be faced here is that of finding a reliable approximation scheme which retains gauge invariance and renormalizability. This is by no means a trivial task.

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APPENDIX

INVARIANT FUNCTIONS

The notation used in the text can be summarized as,

$$\Delta^{(+)}(x) = -i\int d^4k\, e^{-ik\cdot x}\,\Theta(k^0)\,\delta(k^2-m^2)$$

$$\Delta^{(-)}(x) = +i\int d^4k\, e^{-ik\cdot x}\,\Theta(-k^0)\,\delta(k^2-m^2) \tag{1}$$

From which the invariant "Schwinger" function,

$$\Delta = \Delta^{(+)} + \Delta^{(-)}$$

$$\Delta = -i\int d^4k\, e^{-ik\cdot x}\,\varepsilon(k^0)\,\delta(k^2-m^2) \tag{2}$$

Performing the k^0 integration in (2) gives

$$\Delta(x) = -\int d^3k\, e^{+i\underline{k}\cdot\underline{x}}\,\frac{\sin(\omega_{\underline{k}}x^0)}{\omega_{\underline{k}}} \tag{3}$$

$$\omega_{\underline{k}}^2 = \underline{k}^2 + m^2$$

The alternative form (3) gives the two properties used in the text, namely

$$\partial_0\Delta(\underline{x}, x^0)\big|_{x^0=0} = -\delta_3(\underline{x}) \tag{4}$$

$$\partial_j\Delta(\underline{x}, x^0)\big|_{x^0=0} = 0 \tag{5}$$

THE COMBINATORIAL FUNCTIONS

$$I_M = \sum_{a+2b=M} (-1)^{a+b} 2^a \frac{(a+b)!}{a!\,b!} \tag{1}$$

$a,b \geq 0$ $\qquad$ M : integer

Using the simple identity,

$$\frac{(a+b)!}{a!\,b!} = \frac{(a+b-1)!}{(a-1)!\,b!} + \frac{(a+b-1)!}{a!\,(b-1)!}$$

(1) can be written

$$I_M = \sum_{\substack{a+2b=M\\ a,b\geq 0}} (-1)^{a+b} 2^a \frac{(a+b-1)!}{(a-1)!\,b!} + \sum_{\substack{a+2b=M\\ a,b\geq 0}} (-1)^{a+b} 2^a \frac{(a+b-1)!}{a!\,(b-1)!} \tag{2}$$

In the first summation of (2) we change variables a → a+1 and in the second summation b → b+1

$$I_M = \sum_{\substack{a+2b=M-1\\ a\geq -1,\, b\geq 0}} (-1)^{a+b+1} 2^{a+1} \frac{(a+b)!}{a!\,b!} + \sum_{\substack{a+2b=M-2\\ a\geq 0,\, b\geq -1}} (-1)^{a+b+1} 2^a \frac{(a+b)!}{a!\,b!} \tag{3}$$

The ranges of summation can be reduced due to the vanishing of the summand.

$$I_M = -2\sum_{\substack{a+2b=M-1\\ a,b\geq 0}} (-1)^{a+b} 2^a \frac{(a+b)!}{a!\,b!} - \sum_{\substack{a+2b=M-2\\ a,b\geq 0}} (-1)^{a+b} 2^a \frac{(a+b)!}{a!\,b!} \tag{4}$$

so that by definition (1),

$$I_M + 2I_{M-1} + I_{M-2} = 0 \tag{5}$$

Relation (5) can be used to evaluate I_M. Defining the generating function

$$G(t) = \sum_{M=0}^{\infty} I_M t^M \tag{6}$$

We can sum over equation (5) after the replacement M → M+2.

$$\sum_{M=0}^{\infty} I_M t^M + 2\sum_{M=0}^{\infty} I_{M+1} t^M + \sum_{M=0}^{\infty} I_{M+2} t^M = 0$$

hence,

$$t^2 G(t) + 2t\{G(t) - I_0\} + \{G(t) - I_0 - I_1 t\} = 0$$

From equation (1) we have $I_o = 1$, $I_1 = -2$ giving,

$$t^2 G(t) + 2t\{G(t) - 1\} + \{G(t) - 1 + 2t\} = 0$$

i.e.

$$(1+t)^2 G(t) = 1 \tag{7}$$

Expanding (7) in a power series in 't' gives

$$G(t) = (1+t)^{-2} = \sum_{M=0}^{\infty} (-1)^M (1+M) t^M \tag{8}$$

equating (6) and (8) gives the required result.

$$I_M = (-1)^M (1+M) \tag{9}$$

To calculate J_M we use (1) and (9).

$$J_M = \sum_{\substack{a+2b=M \\ a,b\geqslant 0}} (-1)^{a+b} 2^a \frac{(a+b-1)!}{a!\,b!} \tag{10}$$

Multiplying (10) by M, and using the fact, that inside the summation M = a + 2b gives,

$$M\cdot J_M = \sum_{\substack{a+2b=M\\ a,b\geqslant 0}} (-1)^{a+b} 2^a \frac{(a+b-1)!}{(a-1)!\, b!} + 2\sum_{\substack{a+2b=M\\ a,b\geqslant 0}} (-1)^{a+b} 2^a \frac{(a+b-1)!}{a!\,(b-1)!} \tag{11}$$

In the first summation, change a → a+1 while in the second b → b+1. Reducing the range of summation as in (4) gives,

$$M\cdot J_M = -2\sum_{\substack{a+2b=M-1\\ a,b\geqslant 0}} (-1)^{a+b} 2^a \frac{(a+b)!}{a!\, b!} - 2\sum_{\substack{a+2b=M-2\\ a,b\geqslant 0}} (-1)^{a+b} 2^a \frac{(a+b)!}{a!\, b!} \tag{12}$$

Applying definition (1) to (12)

$$M\cdot J_M = -2\,I_{M-1} - 2\,I_{M-2}$$

which on substitution of (9) gives

$$M\cdot J_M = 2(-1)^M M - 2(-1)^M (M-1)$$

or (13)

$$J_M = \frac{2(-1)^M}{M}$$

To evaluate K, L and M it is convenient to define a more general function.

$$\Theta_{M,N} = \sum_{\substack{a_1+a_2+\cdots+a_M=N\\ a_i\geqslant 0}} \left\{ \prod_{i=1}^{M} (a_i+1) \right\} \quad : \quad \begin{array}{l} M\geqslant 1 \\ N\geqslant 0 \end{array} \tag{14}$$

For M ⩾ 2, N ⩾ 1 we can write

$$\Theta_{M,N-1} = \sum_{\substack{a_1+a_2+\cdots+a_M=N-1 \\ a_i \geq 0}} \left\{ \prod_{i=1}^{M} (a_i+1) \right\} = \sum_{\substack{a_1+\cdots+a_M=N \\ a_i \geq 0}} a_M \left\{ \prod_{i=1}^{M-1} (a_i+1) \right\}$$

so that,

$$\Theta_{M,N-1} = \sum_{\substack{a_1+\cdots+a_M=N \\ a_i \geq 0}} \left\{ \prod_{i=1}^{M} (a_i+1) \right\} - \sum_{\substack{a_1+\cdots+a_M=N \\ a_i \geq 0}} \left\{ \prod_{i=1}^{M-1} (a_i+1) \right\} \tag{15}$$

Hence from (14) and (15) we have

$$\Theta_{M,N-1} = \Theta_{M,N} - \sum_{a=0}^{N} \Theta_{M-1,a} \tag{16}$$

Iterating (16) gives

$$\Theta_{M,N} = \Theta_{M,0} + \sum_{b=1}^{N} \sum_{a=0}^{b} \Theta_{M-1,a} \tag{17}$$

From (14) one always has,

$$\Theta_{M,0} = 1 \tag{18}$$

so that (17) and (18) give

$$\Theta_{M,N} = \sum_{b=0}^{N} \sum_{a=0}^{b} \Theta_{M-1,a} \tag{19}$$

We proceed by induction; suppose that for all $b \geq o$ for a given $a = M-1$,

$$\Theta_{a,b} = \binom{a+2b-1}{2b-1} \tag{20}$$

Substituting this into (19) gives,

$$\Theta_{M,N} = \sum_{b=0}^{N} \sum_{a=0}^{b} \binom{a+2M-3}{2M-3} = \sum_{b=0}^{N} \binom{b+2M-2}{2M-2} = \binom{N+2M-1}{2M-1} \qquad (21)$$

where we have used the tabulated result;

$$\sum_{a=0}^{N} \binom{a+b}{b} = \binom{N+b+1}{b+1}$$

so that from (21) if the result (20) is true for, a = M-1 then it is also true for, a=M. To complete the calculation we have only to show (20) holds in the simplest case. Writing out (14) for the case, a = 1, b = N, we have only one term.

$$\Theta_{1,N} = \sum_{a_1=N} (a_1+1) = N+1 = \binom{N+1}{1}$$

From which follows the general result;

$$\Theta_{M,N} = \binom{N+2M-1}{2M-1} \qquad M \geqslant 1 \;,\; N \geqslant 0 \qquad (22)$$

So from (14) and (22)

$$M_N = \Theta_{2,N} = \binom{N+3}{3} \;;\; L_N = \Theta_{3,N} = \binom{N+5}{5} \;;\; K_N = \Theta_{4,N} = \binom{N+7}{7} \qquad (23)$$

SYMMETRY PROPERTIES OF χ-FUNCTIONS AND ψ-FUNCTIONS

$$\delta_\omega(0) + \chi(P_1,..,P_N) = \int d\mu(k,\omega)\, Tr\,\{L(k_1)\cdot L(k_2)\ldots L(k_N)\}$$

It is clear from properties of a trace that the χ-functions are unchanged under cyclic and anticyclic permutations of the variables $\{P_1,..,P_n\}$. If we set the first two momenta variables equal, then from the way the integrand has been defined,

$$\delta_\omega(0) + \chi(P_1,P_1,P_3,..,P_N) = \int d\mu(k,\omega)\, Tr\,\{L(k_1)\cdot L(k_1)\ldots L(k_N)\}$$

but from

$$L(k) = \frac{k_\mu k_\nu}{k^2}$$

it follows that

$$L(k_1)\cdot L(k_1) = L(k_1)$$

so

$$\delta_\omega(0) + \chi(P_1,P_1,\ldots P_N) = \int d\mu(k,\omega)\, Tr\,\{L(k_1)\cdot L(k_3)\ldots L(k_N)\}$$

or $$\chi(P_1,P_1,P_3,..,P_N) = \chi(P_1,P_3,..,P_N)$$

By cyclic symmetry, this is true for any pair of adjacent variables.

$$\chi(P_1,P_2,\ldots,P_i,P_{i+1}=P_i,P_{i+2},..,P_N) = \chi(P_1,..,P_i,P_{i+2},..,P_N)$$

In particular we have the property,

$$\chi(0,0,..,P_i,0,0,..,P_j,0,0,..,P_k,Q..) = \chi(P_i,P_j,P_k,..)$$

This is the contraction property referred to in the text. By an identical procedure one also has

$$\psi(q;0,..,0,p_i,0,..,0,p_j,\ldots) = \psi(q;p_i,p_j,\ldots)$$