APPLICATION OF INTEGRAL OPERATORS IN THE NUMERICAL SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT

Consider linear elliptic boundary value problems, with Dirichlet boundary conditions, for the following elliptic equations

$$-\Delta u + P(r^2)u = 0$$

$$-\Delta u + M(x_1)u = 0$$

$$-\Delta u + N(x_2)u = 0$$

$$-\Delta u + [M(x_1) + N(x_2)]u = 0,$$

where Δ = Laplace's operator, $r^2 = x_1^{\,2} + x_2^{\,2}$, and the functions P, M, and N are entire.

To solve these equations numerically, we construct general solutions using the operators of Bergman [6] and Vekua [53]. These general solutions are in terms of functions which satisfy Goursat problems, which have to be solved numerically. The boundary value problems are then solved using the method of particular solutions, and the boundary integral method.

For the equation $-\Delta u + P(r^2)u = 0$, we improve the numerical solution of the Goursat problem constructed by Gilbert and Linz [30]. The improved method is extended to the solution of the Goursat problem encountered for the other equations under consideration.

The method of particular solutions can then be applied in the normal way.

For the boundary integral method, we reformulate the integral equation for the density of the double-layer potential, and apply this new formulation to the boundary value problems under consideration.

We develop an error analysis for our method of solution of the Goursat problem.

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C O N T E N T S

CHAPTER I

INTRODUCTION

The solution of second order linear elliptic boundary value problems in three or more dimensions, by means of finite difference or finite element methods can lead to systems of linear algebraic equations, one for each internal point, with large matrices, which although sparse, nevertheless present problems of storage and processing time, even with the growing storage capacity of modern computers, the advent of parallel computers, and the innovation of rapid elliptic solver packages. Moreover, the user may find that he obtains approximate solutions at a large number of internal points where the solution is of no great interest and possibly not at just those points where the solution is of particular interest. Any increase in the required accuracy of the numerical solution can only compound the above problems.

This has led to renewed interest in methods which reduce the problem to solving equations constructed at points on the boundary only, with the approximate solution at any interior points expressed as a series or an integral. In these methods a general solution is adapted to suit the domain and boundary conditions of the problem. Boundary methods include the method of particular solutions and the boundary integral method.

The method of particular solutions is an old and trusted method. If we have an elliptic boundary value problem in two dimensions, we generate a system of functions, $\rho_k(x_1,x_2)$, $k = 0,1,\ldots$, which are formal solutions of the elliptic equation, these functions being independent of the domain of the boundary value problem, but being complete in the solution space of the problem. In our algorithm the integers, M,N, $N > M \geq 0$, are assigned and the coefficients, α_k, in the linear combination $\sum_{k=0}^{M} \alpha_k \rho_k(x_1,x_2)$ are determined so that the errors at N points on the boundary are minimized in the L_2 (least squares) norm.

In the integral equation method the solution of the boundary value problem is represented by a double layer potential. On the boundary the solution is known and the density of the double layer can be determined at a finite number of points, N, chosen to represent the boundary. The solution at any interior point is then given by the double-layer potential at these points.

Boundary methods lead to a significant reduction in the number of operations and the storage required, despite 'full' matrices, compared to the mesh-methods, such as finite differences and finite elements.

For Laplace's equation and Helmholtz' equation general solutions are known in closed forms, and consequently boundary problems for these equations have been extensively treated in two and three dimensions. (Burton [10], Cannon [11], Collatz [12], Davis and Rabinowitz [16], De Mey [18], [19], Fox, Henrici, and Moler [26].)

For Laplace's equation the general solution is in terms of harmonic functions. The fact that harmonic functions of two real variables can be expressed as the real (or imaginary) parts of analytic functions of one complex variable is extremely convenient since the corresponding translation of theorems on analytic functions into theorems on harmonic functions is almost immediate.

In all but a few cases the general solution of an elliptic equation is not known in closed form. However, by generalizing the operator <u>RE</u> ('take the real part of') it is possible to relate solutions of elliptic partial differential equations (p.d.e.s) in two, three and sometimes more variables, to complex analytic functions, thereby yielding a unified theory of an extensive class of linear p.d.e.s. The general solution is expressed in terms of an integral operator, operating on a complex analytic function. There are infinitely many of these integral operators.

S. Bergman [6] and I.N. Vekua [53], are independently responsible for a comprehensive theory for integral operators in two and three dimensions. The integral operator considered in this thesis is referred to by Bergman as the integral operator of the first kind; it has been shown by Henrici [32] to be completely equivalent to an operator developed by Vekua in terms of the Riemann function.

Every solution of the p.d.e.s considered can be represented by these operators. That part of the integrand which does not include an arbitrary analytic complex function of one variable will be called the generating function. The generating function is normally a function of three complex variables. This generating function has to be numerically approximated and consequently the integration has to be performed numerically, at each boundary node and at any internal point, when solving a boundary value problem.

In Bergmann's representation of the integral operator of the first kind the generating function is expressed as a power series in one of its arguments with the coefficients determined recursively in terms of the coefficients of the elliptic equation. The series can be truncated allowing a method of numerical approximation.

This approach has been used successfully to solve boundary value problems by the method of particular solutions for an equation of the type $-\Delta u + P(r^2)u = 0$, where Δ denotes Laplace's operator in two dimensions and P is a real polynomial in $r^2 = x_1{}^2 + x_2{}^2$ with $P(r^2) > 0$, by Bergman and Herriot [7], and for more general elliptic equations with polynomial coefficients by Schryer [47].

Series representations have also been used by Gilbert and Atkinson [29] in solving the equation $-\Delta u + P(r^2)u = 0$, where Δ denotes Laplace's operator in two dimensions and P is a real polynomial in $r^2 = x_1^2 + x_2^2$ with $P(r^2) > 0$, by means of Fredholm integral equations.

The generating function can also be expressed as the solution of a Goursat problem for a complex hyperbolic differential equation. In this case the generating function is obtained by solving the Goursat problem numerically and the integration is approximated using a numerical quadrature formula. This approach, by solution of a Goursat problem, was used by Gilbert and Linz [30] to solve boundary value problem with the equation $-\Delta u + P(r^2)u = 0$, where Δ denotes Laplace's operator in two dimensions and P is an analytic function of $r^2 = x^2 + y^2$, with $P(r^2) > 0$.

The approximation of the generating function separately has one major advantage. Gilbert [27] has shown that the generating function for the equation $-\Delta u + P(r^2)u = 0$ is independent of the dimension. That is, consider the equation $-\Delta_n u + P(r^2)u = 0$ where

$$\Delta_n u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} ,$$

and $r^2 = x_1^2 + x_2^2 + \ldots + x_n^2$, then the generating function $G(r,t)$, satisfies the same Goursat problem for all $n \geq 2$.

In this thesis we attempt to improve the solution of the Goursat problem which arises for the elliptic equation $-\Delta u + P(r^2)u = 0$, and use the improved solution in a particular solutions algorithm. We also improve on the integral equations algorithm of Atkinson, for this elliptic equation, and apply the Goursat solution in this algorithm. Finally we develop similar methods for the elliptic equations

$$-\Delta u + M(x_1)u = 0, \quad -\Delta u + N(x_2)u = 0, \quad -\Delta u + [M(x_1) + N(x_2)]u = 0,$$

where Δ is Laplace's operator and M and N are real analytic functions of x_1 and x_2 respectively. In Appendix I we use a heuristic argument to demonstrate that the generating function for these equations is the same in three dimensions. All the solved problems are two dimensional problems, we first have to understand and overcome the difficulties presented by these problems before we can solve three dimensional problems.

In Chapter 2, we explain the construction of Bergman's integral operator of the first kind and of Vekua's integral operator and look at some of the principal results of these authors.

In Chapter 3, we devise integral operators for the equations $-\Delta u + P(r^2)u = 0$, $-\Delta u + M(x_1)u = 0$, $-\Delta u + N(x_2)u = 0$, and $-\Delta u + [M(x_1) + N(x_2)]u = 0$, and consider methods of evaluating the integrals, including the formulation of the respective Goursat problems. These equations all afford one important simplification, the generating functions are functions of two real variables satisfying a Goursat problem for a real hyperbolic differential equation of two variables.

In Chapter 4, we look in detail at the method of solution of the Goursat problem and we investigate the error in our numerical approximation.

Chapters 5 and 6 are concerned with the construction of the particular solutions algorithm and the double-layer formulation of the boundary integral method, respectively, for each of the elliptic equations above. The numerical treatment of the double-layer formulation is given in Chapter 7.

Chapter 8, the final chapter, concerns the numerical results. The problems solved for the equation $-\Delta u + P(r^2)u = 0$, are those solved by Linz [30], Atkinson [29], [2], Herriot [7], and Schryer [48], which afford some measure of comparison. We also present problems for the equation $-\Delta u + M(x_1)u = 0$, and the equation $-\Delta u + [M(x_1) \dot{+} N(x_2)]u = 0$.

Throughout this thesis only Dirichlet boundary conditions have been considered. For the method of particular solutions Neumann and mixed boundary conditions can be treated in a similar way. In the boundary integral method for Neumann boundary conditions we would need to use single layer potentials to get a Fredholm equation of the second kind.

Also only homogeneous equations have been considered, non-homogeneous equations for particular non-homogeneous terms have been considered by Schryer [47] and Bergman [7].

Since the work has been seen as a step towards solving three dimensional problems, we have not sought practical applications in two dimensions, although they do occur. For instance, problems in hydro-dynamic lubrication such as those for fluid flow between a roller and absorbent compressible paper (A.B. Taylor [49]); and fluid flow in a complete journal bearing result in Navier-Stokes equations which under suitable assumptions reduce to Reynolds equation which can be transformed to equations of the type $-\Delta u + N(x_1)u = 0$ (C. Mason [39]).

Colton [13],[14] has given a complete generalization of the work of Bergman and Vekua in three dimensions, and there are many applications in three dimensions (Krzywublocki [36],[37]).

CHAPTER 2

AN INTRODUCTION TO THE WORK OF S. BERGMAN AND I.N. VEKUA ON THE GENERAL REPRESENTATION OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE IN TWO INDEPENDENT VARIABLES

1. Helmholtz' Equation

In their earliest papers, Bergman [5] and Vekua [51],[52] consider the Helmholtz equation in a domain D which is assumed to be simply-connected and containing the origin. We seek a general solution $u(x_1,x_2)$ of the Helmholtz equation

$$\Delta u + \lambda^2 u = 0 \quad \text{in } D,$$

Δ is the Laplace operator in two dimensions (2.01)

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \qquad \lambda = \text{constant}.$$

a) S. Bergman's approach.

Without loss suppose $\lambda = 1$ and rewrite the Helmholtz equation in polar coordinates (r,θ) given by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $r \geq 0$, $0 \leq \theta < 2\pi$. Then

$$\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial r^2}\right] + u = 0 \quad \text{in } D.$$

We seek separable solutions of this equation, of the form

$u = R(r)\Theta(\theta)$, where R is a function of r,

and Θ is a function of θ.

By separating the variables we find

$$\Theta(\theta) = \sin n\theta,\ \cos n\theta\ ; \quad n = 0,1,2,\ldots\ ;$$

$$R(r) = J_n(r),\ \ Y_n(r)\ ; \quad n = 0,1,2,\ldots\ .$$

n is an integer to insure continuity of u across $\theta = 0,\ 2\pi$.

J_n, Y_n are Bessel functions of the first and second kind. $Y_n(r)$ is discarded since $Y_n(r)$ is not finite at the origin $r = 0$. We are then led to consider the formal solution

$$u = \sum_{n=0}^{\infty} J_n(r)[A_n \cos n\theta + B_n \sin n\theta]\ ,$$

where A_n, B_n are constants.

Using the well known integral representation due to Poisson,

$$J_n(r) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\left(\frac{r}{2}\right)^n \int_0^1 (1-t^2)^{n-\frac{1}{2}} e^{irt}\, dt,$$

leads to

$$u = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\left(\frac{r}{2}\right)^n [A_n \cos n\theta + B_n \sin n\theta] \int_0^1 (1-t^2)^{n-\frac{1}{2}} e^{irt} dt.$$

Taking $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, $r = z\bar{z}$, and changing the order of summation and integration, we obtain

$$u = \int_{-1}^{1} e^{i\sqrt{z\bar{z}}\,t} \sum_{n=0}^{\infty} \left[\frac{C_n((z/2)(1-t^2))^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} + \frac{D_n((\bar{z}/2)(1-t^2))^n}{\Gamma\ (\frac{1}{2})\Gamma(n+\frac{1}{2})} \right] \frac{dt}{\sqrt{1-t^2}}\ ,$$

where $A_n \cos n\theta + B_n \sin n\theta = C_n e^{in\theta} + D_n e^{-in\theta}$. Assuming the series are convergent, let

$$f((z/2)(1-t^2)) = \sum_{n=0}^{\infty} \frac{C_n((z/2)(1-t^2))^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} ,$$

$$g((\bar{z}/2)(1-t^2)) = \sum_{n=0}^{\infty} \frac{D_n((\bar{z}/2)(1-t^2))^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} .$$

Then

$$u = \int_{-1}^{1} e^{i\sqrt{z\bar{z}}\,t}[f((z/2)(1-t^2)) + g((\bar{z}/2)(1-t^2))] \frac{dt}{\sqrt{1-t^2}} ,$$

and taking $g = \bar{f}$, the complex conjugate of f, gives a general representation of the real solutions of Helmholtz' equation, regular at the origin as follows

$$u(x,y) = \begin{matrix}\mathrm{RE}\\ \mathrm{IM}\end{matrix} \int_{-1}^{1} e^{i\sqrt{z\bar{z}}\,t} f((z/2)(1-t^2)) \frac{dt}{\sqrt{1-t^2}} ; \qquad (2.02)$$

where RE means 'take the real part of... '

IM means 'take the imaginary part of ....' .

b) I.N. Vekua's approach:

We suppose that Helmholtz' equation (2.01) is continued to complex values of x_1 and x_2; and introduce the two new independent variables

$$z = x_1 + ix_2, \qquad z^* = x_1 - ix_2 .$$

Notice that $z^* = \bar{z}$, the complex conjugate of z, only if x_1 and x_2 are real. Then

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) ,$$

and

$$\frac{\partial^2}{\partial z \partial z^*} = \frac{1}{4} \left(\frac{\partial^2}{\partial x_1{}^2} + i \frac{\partial^2}{\partial x_2{}^2}\right) = \frac{\Delta}{4} .$$

Let C be some simply connected domain such that $z \in C$, $z^* \in \overline{C}$. Then Helmholtz' equation becomes

$$\frac{\partial^2 U}{\partial z \partial z^*} + \frac{1}{4} \lambda^2 U = 0 \qquad \text{in } (C,\overline{C}), \quad z \in C, \quad z^* \in \overline{C} .$$

This is a hyperbolic equation in the complex domain. Integrating with respect to z^* gives

$$\frac{\partial U}{\partial z} + \frac{1}{4} \lambda^2 \int_0^{z^*} U(z,\eta) d\eta = f'(z) ,$$

where $f'(z)$ in an arbitrary function of z.

Integrating with respect to z gives

$$U(z,z^*) = f(z) + g(z^*) - \frac{1}{4} \lambda^2 \int_0^z d\xi \int_0^{z^*} U(\xi,\eta) d\eta,$$

where $g(z^*)$ is an arbitrary function of z^*.

This is a Volterra equation in the complex plane which Vekua solves by the method of successive approximations, setting

$$\phi_0 = \phi_0(z,z^*) = f(z) + g(z^*),$$

and

$$\phi_n(z,z^*) = \left(\frac{\lambda}{2}\right)^2 \int_0^z d\xi \int_0^{z^*} \phi_{n-1}(\xi,\eta) d\eta ; \qquad n = 1,2,\ldots ,$$

to obtain

$$\phi(z,z^*) = f(z) + g(z^*) - \int_0^z H(z,z^*,\xi,\tfrac{\lambda}{2})f(\xi)d\xi$$

$$- \int_0^{z^*} H(z,z^*,\xi,\tfrac{\lambda}{2})g(\xi)d\xi; \qquad (2.03)$$

where

$$H(z,z^*,\xi,\tfrac{\lambda}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^n(\frac{\lambda}{2})^{2n} z^{*n}(z-\xi)^{n-1}}{(n-1)!\ n!},$$

$$= -\frac{\lambda}{2}\left(\frac{z^*}{z-\xi}\right)^{\frac{1}{2}} J_1(\lambda\sqrt{z^*(z-\xi)}\ ),$$

$$= \frac{\partial}{\partial\xi} J_0(\lambda\sqrt{z^*(z-\xi)}\ ),$$

where J_0, J_1 are the Bessel functions of the first kind.

Taking $g = \overline{f}$, the complex conjugate of f, gives a general representation of real solutions of Helmholtz' equation as follows.

$$u(x_1,x_2) = \begin{matrix}\mathrm{RE}\\ \mathrm{IM}\end{matrix}\ f(z) - \int_0^z f(\xi)\frac{\partial}{\partial\xi} J_0(\lambda\sqrt{\overline{z}(z-\xi)}\ )\,d\xi \quad . \qquad (2.04)$$

2. The General Homogeneous Elliptic P.D.E.

We now seek the general solution of any homogeneous elliptic p.d.e. Let $u = u(x_1,x_2)$ satisfy

$$Lu = \Delta u + a(x_1,x_2)u_{x_1} + b(x_1,x_2)u_{x_2} + c(x_1,x_2)u = 0$$

$$\text{in } D, \qquad (2.05)$$

where Δ is Laplace's operator in two dimensions, $u_{x_1} = \frac{\partial u}{\partial x_1}$, $u_{x_2} = \frac{\partial u}{\partial x_2}$, a, b, and c are given functions of the variables x_1 and x_2. We assume that the coefficients a, b, and c are analytic functions in the closure of the domain D.

We suppose the coefficients a, b and c of the equation can be continued analytically into the domain of complex values of x_1 and x_2 taking new independent variables $z = x_1 + ix_2$, $z^* = x_1 - ix_2$ to get

$$U = U_{zz^*} + A(z,z^*)U_z + B(z,z^*)U_{z^*} + C(z,z^*)U = 0,$$

$$\text{in } (C,C^*) . \qquad (2.06)$$

where

$$U_{zz^*} = \frac{\partial^2 U}{\partial z \partial z^*} , \quad U_z = \frac{\partial U}{\partial z}, \quad U_z = \frac{\partial U}{\partial z^*} ,$$

and

$$A(z,z^*) = \frac{1}{4}(a(\frac{z+z^*}{2} , \frac{z-z^*}{2}) + ib(\frac{z+z^*}{2} , \frac{z-z^*}{2})) ,$$

$$B(z,z^*) = \frac{1}{4}(a(\frac{z+z^*}{2} , \frac{z-z^*}{2}) - ib(\frac{z+z^*}{2} , \frac{z-z^*}{2})) ,$$

and $C(z,z^*) = \frac{1}{4} c(\frac{z+z^*}{2} , \frac{z-z^*}{2})$.

Where $A(z,z^*)$, $B(z,z^*)$ and $C(z,z^*)$ are analytic functions of the two complex variables z, z^* in the cylindrical domain (C,C^*).

a) Bergman's integral operator

Bergman transforms (2.06) by taking

$$v(z,z^*) = \exp\left[\int_0^{z^*} A\,dz - n(z)\right] U(z,z^*),$$

where n(z) an arbitrary function of z and v(z,z*) satisfies

$$v_{zz^*} + D_1 v_{z^*} + F\,v = 0, \tag{2.07}$$

$$D_1 = n_z - \int_0^{z^*} A_z dz^* + B\,, \qquad F = -A_z - AB + C\,.$$

Bergman then seeks a solution of equation (2.07) as a generalization of equation (2.02) in the form

$$v(z,z^*) = \int_{-1}^{1} \tilde{E}(z,z^*,t)\, f\!\left(\tfrac{z}{2}(1-t^2)\right) \frac{dt}{\sqrt{1-t^2}}\,.$$

It is found that $\tilde{E}$ has to be a twice differentiable solution of the equation

$$(1-t^2)\tilde{E}_{z^*t} - \frac{1}{t}\tilde{E}_{z^*} + 2tz(\tilde{E}_{zz^*} + D\tilde{E}_{z^*} + F\tilde{E}) = 0 \tag{2.08}$$

for $|t| \leq 1$ with

$$\lim_{t=\pm 1} (1-t^2)^{\frac{1}{2}}\, \tilde{E}_{z^*}(z,z^*,t) = 0,$$

and $\frac{\tilde{E}_{z^*}}{zt}$ is continuous, for (z,z*) belonging to a 4 dimensional neighbourhood of the origin.

Thus Bergman develops an integral operator denoted by B which he writes

$$B(f(z)) = \int_{-1}^{1} \exp[-\int_{0}^{z^*} A dz^* + n(z)]\tilde{E}(z,z^*,t) f(\tfrac{1}{2}z(1-t^2)) \frac{dt}{\sqrt{1-t^2}} .$$

Putting

$$E(z,z^*,t) = \exp[-\int_{0}^{z^*} A dz^* + n(z)]\tilde{E}(z,z^*,t),$$

we obtain

$$B(f(z)) = \int_{-1}^{1} E(z,z^*,t) f(\tfrac{1}{2}z(1-t^2)) \frac{dt}{\sqrt{1-t^2}}$$

Bergman calls B his integral operator of the first kind.

b) Vekua's integral operator

Vekua integrates the complex hyperbolic equation (2.06) first with respect to z and then with respect to z* to obtain a Volterra equation equivalent to equation (2.06),

$$U(z,z^*) + \int_{0}^{z^*} A(z,\eta)U(z,\eta)d\eta + \int_{0}^{z} B(\xi,z^*)U(\xi,z^*)d\xi$$

$$+ \int_{0}^{z} d\xi \int_{0}^{z^*} D_1(\xi,\eta)U(\xi,\eta)d\eta = f(z) + g(z^*) ,$$

where f and g are arbitrary functions of z and z* respectively, and

$$D_1(z,z^*) = C(z,z^*) - A_z(z,z^*) - B_z(z,z^*).$$

Take $U_0(z,z^*) = U(z,z^*) + \int_{0}^{z^*} A(z,\eta)U(z,\eta)d\eta$, to obtain

$$U_0(z,z^*) + \int_0^z B(\xi,z^*)U_0(\xi,z^*)d\xi + \int_0^z d\xi \int_0^{z^*} D_2(z^*,\xi,\eta)U_0(\xi,\eta)d\eta$$

$$= f(z) + g(z^*),$$

where

$$D_2(z^*,\xi,\eta) = -B(\xi,z^*)A(\xi,\eta)\exp - \int_\eta^{z^*} A(\xi,\eta_1)d\eta_1$$

$$+ D_1(\xi,\eta) - A(\xi,\eta) \int_\eta^{z^*} D_1(\xi,\eta_1)\exp - \int_\eta^{\eta_1} A(\xi_1,\eta_2)d\eta_2 \, d\eta_1 .$$

Now take

$$V(z,z^*) = U_0(z,z^*) + \int_0^z B(\xi,z^*)U_0(\xi,z^*)d\xi,$$

to obtain an equation of the form

$$V(z,z^*) = \int_0^z d\xi \int_0^{z^*} K(z,z^*,\xi,\eta)V(\xi,\eta)d\eta + f(z) + g(z^*),$$

where

$$K(z,z^*,\xi,\eta) = -D_2(z^*,\xi,\eta) + B(\xi,\eta) \int_\xi^z D_2(z^*,\xi_1,\eta)\exp[- \int_\xi^z B(\xi_2,\eta)d\xi_2]d\xi_1 .$$

This is an ordinary Volterra equation in the complex plane, thus it has a solution of the form of equation (2.03), by solving in much the same way Vekua develops an integral operator denoted by V such that

$$V(f(z)) = {RE \atop IM} \{G(z,\eta_0,z,z^*)f(z) - \int_{z_0}^z f(t) \frac{\partial}{\partial t} G(t,\eta_0,z,z^*)dt\} \qquad (2.10)$$

where $G(z,\eta,t,\tau)$ is the Riemann function.

The Riemann function, G, satisfies the following equations

$$G(t,\zeta,t,\tau) = \exp \int_{\tau}^{\zeta} A(t,\eta)\,d\eta \,, \qquad t \in C; \quad \zeta,\tau \in \overline{C}.$$

$$G(z,\tau,t,\tau) = \exp \int_{t}^{z} B_1(\xi,\tau)\,d\xi, \qquad z,t \in C; \quad \tau \in \overline{C}\,.$$

3. Some Properties of the Operators of Bergman and Vekua

Consider the linear elliptic boundary value problem

$$
\begin{aligned}
Lu &= 0 && \text{on } D, \\
u &= g && \text{on } \partial D \text{ with } g \in C^{(0)}(\partial D), \\
& && u \in C^{(0)}(D \cup \partial D) \cap C^{(2)}D ,
\end{aligned}
\tag{2.11}
$$

where D is a bounded simply connected domain containing the origin. We consider the corresponding complex hyperbolic equation (2.06),

$$
\begin{aligned}
\mathcal{L}u &= 0 \qquad \text{on } (C, \overline{C}), \\
\mathcal{L}U &= U_{zz*} + AU_z + B_1 U_{z*} + C_1 U, \\
z &\in C, \quad z* \in \overline{C} .
\end{aligned}
$$

A, B_1 and C_1 are assumed to be analytic functions of z and z* on $(C,\overline{C})$.

THEOREM 1 Bergman [6], p.19

Let the coefficients A, B_1, C_1 of $\mathcal{L}U = 0$, be functions of two complex variables z,z*, which are regular in a sufficiently large domain. Then every real solution $U(z,\overline{z}) = u(x,y)$ which is regular in a domain β^2 of the real x,y-plane can be continued into the domain β^4 of the z,z* space. β^4 is the product domain $\beta_1{}^2 \times \beta_2{}^2$ where $\beta_1{}^2$ is the domain β^2 in the z plane, and $\beta_2{}^2$ is the same domain in the z* plane.

a) Bergman's operator

In (2.09) B is Bergman's integral operator of the first kind for the equation $Lu = 0$. There are a number of representations for Bergman's operator B; one alternative representation is constructed by Bergman by taking

$$g(z) = \int_{-1}^{1} f(\tfrac{1}{2}z(1-t^2)) \frac{dt}{\sqrt{1-t^2}} ,$$

$$n(z) = 0,$$

and

$$\tilde{E}(z,z^*,t) = 1 + \sum_{n=1}^{\infty} t^{2n} z^n \int_0^{z^*} P^{(2n)}(z,\eta)d\eta.$$

$\tilde{E}$ must satisfy equation (2.08) and by substituting for $\tilde{E}$ in this equation we obtain the following equations for $P^{(2n)}(z,z^*)$

$$P^{(2)}(z,z^*) = -2F(z,z^*),$$

$$(2n+1)P^{(2n+2)}(z,z^*) = -2[P_z^{(2n)}(z,z^*) + D_1(z,z^*)P^{2n}(z,z^*)$$

$$+ F(z,z^*) \int_0^{z^*} P^{(2n)}(z,\eta)d\eta , \qquad n = 1,2,\ldots , \qquad (2.12)$$

where D_1 and F are defined in (2.07).

Take $Q^{(n)}(z,z\ ) = \int_0^{z^*} P^{(2n)}(z,z^*)dz^*;$

then

$$B(g(z)) = \exp[-\int_0^{z^*} Adz^*] \ [g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,z^*)}{2^{2n}\beta(n,\ n+1)} \int_0^{z} (z-\zeta)^{n-1} g(\zeta)d\zeta . \qquad (2.13)$$

Thus Bergman's operator can be constructed directly from the coefficients, $a(x,y)$, $b(x,y)$, $c(x,y)$ of the equation $Lu = 0$.

Theorem 2 (Bergman [6], pp. 13)

Suppose that the coefficients A, B, C of the equation $LU = 0$ are analytic functions of two complex variables z, z^* regular in the bicylinder $[|z| \leq r, \ |z^*| \leq r]$, $r > 0$. Then $E(z,z^*,t)$ is regular in $[|z| < \frac{r}{3}, \ |z^*| < \frac{r}{3}, \ |t| \leq 1]$.

Thus for regular solutions Bergman requires the coefficients A, B, C to be analytic for z and z^* in the bicylinder $[|z| \leq 3r + \varepsilon, \ |z^*| \leq 3r + \varepsilon]$, $\varepsilon > 0$, $r = \max_{\partial D}[|z|, |z^*|]$.

Bergman [6], pp.22 , shows that the particular solutions of $Lu = 0$ generated by his integral operator are complete in the solution space of (2.11) over compact subsets of D. However, completeness has not been proved over the closure of D.

The operator B which transforms analytic functions $g(z)$ into complex solutions $U(z,z^*)$ of $LU = 0$, has an inverse. The formula expressing $g(z)$ in terms of U depends only on the coefficient $B(z,0)$. If $U = B(g(z))$, then:

$$(\mathrm{RE}\ B)^{-1}(U) = \mathrm{RE}(g(z)) = \mathrm{RE}\{2U(z,0) - U(0,0)\exp(-\int_0^z B(z',0)dz')\} \tag{2.14}$$

b) Vekua's Operator

Vekua assumes the coefficients A, B, C of the equation $LU = 0$ are analytic functions of z and z^* on $(C, \overline{C})$.

The operator, V, is a 1-1 map from the set of functions $g(z)$ which are analytic on C with real values at the origin, onto the space of solutions of $Lu = 0$ which are regular.

Definition. Hölder continuous

Let the function $f(x_1,x_2,\ldots,x_n)$ be given on the set M of points $(x_1,x_2,\ldots,x_n)$. Then f satisfies a Hölder condition on M if

$$|f(x_1',x_2', \ldots, x_n') - f(x_1,x_2,\ldots,x_n)| < K \sum_{k=1}^{n} |x_k' - x_k|^{\alpha} ,$$

for any two points $(x_1,x_2,\ldots,x_n)$, $(x_1',x_2',\ldots,x_n')$ of the set M, where K and α are positive constants, $0 < \alpha \leq 1$, which are independent of the choice of points $(x_1',x_2', \ldots, x_n')$, $(x_1,x_2,\ldots,x_n)$. f is said to be continuous in Hölder's sense on M.

We shall say that a simply-connected domain D is of class Ah, if the boundary, ∂D, is a simple closed smooth curve satisfying the condition that the angle $\theta(t)$ between the tangent to ∂D at the point t and the x-axis is continuous in Hölder's sense along ∂D.

Theorem 3 (Vekua [53], pp.128)

If D is a simply connected domain containing the origin, and of class Ah, and u is a regular solution of $Lu = 0$, which is continuous in Hölder's sense on the boundary ∂D, then the unique f(z), analytic on D with f(0) real, so that $u = V(f)$, is continuous in Hölder's sense on $D \cup \partial D$.

Theorem 4 (Vekua [53], pp.156)

Let D be a simply-connected domain of class Ah. Let $u(x_1,x_2)$ be a real regular solution in D of $Lu = 0$, which is continuous in $D \cup \partial D$, and is continuous in Hölder's sense on ∂D, then given any $\varepsilon > 0$ there are constants $c_1,\ldots,c_n$, such that

$$|u(x_1,x_2) - \sum_{k=1}^{n} c_k u_k(x_1,x_2)| < \varepsilon \qquad ,$$

in the closed domain $D \cup \partial D$.

That is the particular solutions of $Lu = 0$, generated by Vekua's operator are complete in the closure of D.

The operator V, which transforms analytic functions f(z) into solutions of $Lu = 0$, has an inverse, the formula expressiong f(z) in terms of U being

$$V^{-1}(U) = RE(f(z)) = RE\{2U(z,0) - U(0,0)G(0,0,z,0)\},$$

By the definition of the Riemann function G,

$$V^{-1}(U) = RE(f(z)) = RE\{2U(z,0) - U(0,0)\exp(-\int_0^z B(z',0)dz')\} . \qquad (2.15)$$

4. Summary

Bergman's operator has the advantage that it can be constructed directly from the coefficients, a, b, c, of the equation Lu = 0. However, Bergman does not prove the completeness of the particular solutions of Lu = 0 generated by his operator for the closure of the domain D.

Vekua has proved smoothness and completeness theorems over the domain D; however, his operator is expressed in terms of the Riemann function, for which there is no easy construction.

Equations (2.14) and (2.15) imply that for any regular solution of Lu = 0,

$$(\mathrm{RE}\ B)^{-1}(u) = V^{-1}(u)$$

the equivalence of the two operators was first proved by Henrici [32].

Since the two operators are identical they generate the same sets of particular solutions for a given set of regular functions. Thus it is possible to combine the constructive advantages of Bergman's operator, with the theoretical properties of Vekua's operator.

CHAPTER 3

INTEGRAL OPERATORS FOR THE EQUATIONS $-\Delta u + P(r^2)u = 0$, $-\Delta u + M(x_1)u = 0$

1. Introduction

In this chapter, integral operators are constructed for the equations $-\Delta u + P(r^2)u = 0$, and $-\Delta u + M(x_1)u = 0$. Subsequently, numerical methods of approximating the application of these integral operators are discussed, and finally the particular methods of calculation used in this thesis are presented in detail.

2. The equation $-\Delta u + P(r^2)u = 0$

Let $Lu \equiv -\Delta u + P(r^2)u = 0$, in D, (3.01)

Δ = Laplace operator, $P(r^2) > 0$, is an entire function of $r^2 = x_1{}^2 + x_2{}^2$ and D is a simply-connected domain containing the origin.

For the general homogeneous elliptic equation (2.05):

$$Lu = \Delta u + au_{x_1} + bu_{x_2} + cu = 0,$$

the solution U of the equivalent hyperbolic equation (2.06):

$$\mathcal{L}U = U_{zz^*} + AU_z + BU_{z^*} + CU = 0,$$

is given by Bergman (see (2.13)) as

$$U(z,z^*) = \exp[-\int_0^{z^*} A dz^*][g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,z^*)}{2^{2n}\beta(n,n+1)} \int_0^z (z-\zeta)^{n-1} g(\zeta)d\zeta,$$

where $u(x_1,x_2) = \frac{1}{2}(U(z,z^*) + \overline{U}(z^*,z))$,

(For real x_1 and x_2 $u(x_1,x_2)$ is found by taking the real and imaginary parts of $U(z,\overline{z})$, we will write $u(x_1,x_2) = {}^{RE}_{IM}\{U(z,\overline{z})\}$).

Let $\zeta = z\sigma^2$. Then:

$$U(z,z^*) = \exp[-\int_0^{z^*} A dz^*][g(z) + \sum_{n=1}^{\infty} \frac{z^n Q^{(n)}(z,z^*)}{2^{2n}\beta(n,\ n+1)} \int_0^1 2\sigma(1-\sigma^2)^{n-1} g(z\sigma^2)d\sigma. \tag{3.02}$$

Define $\underline{Q}^{(2n)}(z,z^*) \equiv z^n Q^{(n)}(z,z^*) = z^n \int_0^{z^*} P^{(2n)}(z,z^*)dz^*$, $n = 1,2,\ldots$, where $\{P^{(2n)}(z,z^*)\}$ are the functions introduced in (2.12).

Now

$$\underline{Q}^{(2n)}(z,z^*) = z^n \int_0^{z^*} P^{(2n)}(z,z^*)dz^*, \qquad n = 1,2,\ldots .$$

$$\therefore \quad \underline{Q}^{(2n)}_{z^*}(z,z^*) = z^n P^{(2n)}(z,z^*), \qquad n = 1,2,\ldots .$$

$$\therefore \quad \underline{Q}^{(2n)}_{zz*}(z,z*) = nz^{n-1}P^{(2n)}(z,z*) + z^nP^{(2n)}_z(z,z*),$$

$$\therefore \quad z^nP^{(2n)}_z(z,z*) = \underline{Q}^{(2n)}_{zz*}(z,z*) - \frac{n}{z}\,\underline{Q}^{(2n)}_{z*}(z,z*).$$

Take $z* = \bar{z}$ and $r^2 = z\bar{z}$; then for $-\Delta u + P(r^2)u = 0$, $A = B = D_1 = 0$ and $F = -P(r^2)/4$.

Thus (3.02) becomes

$$U(z,\bar{z}) = g(z) + \sum_{n=1}^{\infty} \frac{2\underline{Q}^{(2n)}(z,\bar{z})}{2^{2n}\beta(n,n+1)} \int_0^1 \sigma(1-\sigma^2)^{n-1}g(z\sigma^2)d\sigma, \qquad (3.03)$$

where $\underline{Q}^{(2)}_{\bar{z}} = -\dfrac{zP(r^2)}{2}$,

$$(2n+1)\underline{Q}^{(2n+2)}_{\bar{z}} = -2z[\underline{Q}^{(2n)}_{z\bar{z}} - \frac{P(r^2)}{4}\,\underline{Q}^{(2n)} - \frac{n}{z}\,\underline{Q}^{(2n)}_{\bar{z}}], \quad n = 1,2,\ldots, \qquad (3.04)$$

$$\underline{Q}^{(2)}_{\bar{z}} = \frac{\partial \underline{Q}^{(2)}}{\partial \bar{z}} = \frac{\partial \underline{Q}^{(2)}}{\partial (r^2)} \cdot \frac{\partial (r^2)}{\partial \bar{z}} = z\underline{Q}^{(2)}_{(r^2)} .$$

Thus $\underline{Q}^{(2)}_{(r^2)} = -\dfrac{P(r^2)}{2}$, and $\underline{Q}^{(2)}$ is a function of r^2 only. For $\underline{Q}^{(2n)}$, $n > 1$, proceed by induction. Rewrite (3.04) as follows

$$(2n+1)\underline{Q}^{(2n+2)}_{(r^2)} = -2[r^2\underline{Q}^{(2n)}_{(r^2)(r^2)} - \frac{P(r^2)}{4}\,\underline{Q}^{(2n)} - (n-1)\underline{Q}^{(2n)}_{(r^2)}],$$

where $\underline{Q}_{(r^2)(r^2)} = \dfrac{\partial^2 \underline{Q}}{\partial (r^2)^2}$. Then it is obvious that each $\underline{Q}^{(2n)}$, $n = 1,2,\ldots$, depends only on r^2, and we take $e_n(r^2) \equiv \underline{Q}^{(2n)}(z,\bar{z})$.

Rewriting equation (3.03), and noticing $u(x,y) = {}^{RE}_{IM}\{U(z,\bar{z})\}$

$$u(x,y) = {}^{RE}_{IM}\{g(z) + \sum_{n=1}^{\infty} \frac{2e_n(r^2)}{2^{2n}\beta(n,n+1)} \int_0^1 \sigma(1-\sigma^2)^{n-1}g(z\sigma^2)d\sigma\}.$$

Now ${}^{RE}_{IM}\{g(z)\}$ is a harmonic function $h(\underline{r})$ say where $\underline{r} = (x_1, x_2)$; $x_1 = r \cos\theta$, $x_2 = r \sin\theta$.

Hence

$$u(\underline{r}) = h(\underline{r}) + \sum_{n=1}^{\infty} \frac{e_n(r^2)}{2^{2n-1}\beta(n,n+1)} \int_0^1 \sigma(1-\sigma^2)^{n-1} h(\underline{r}\sigma^2)d\sigma.$$

Reversing the orders of integration and summation and putting

$$G(r,\ 1-\sigma^2) \equiv \sum_{n=1}^{\infty} \frac{e_n(r^2)(1-\sigma^2)^{n-1}}{2^{2n-1}\beta(n,n+1)} \quad ,$$

We have finally

$$u(\underline{r}) \quad = \quad h(\underline{r}) + \int_0^1 \sigma G(r,\ 1-\sigma^2)h(\underline{r}\sigma^2)d\sigma. \qquad (3.05)$$

We call $G(r,\ 1-\sigma^2)$ Gilbert's G-function.

Substituting for u in the p.d.e. $-\Delta u + P(r^2)u = 0$, we find that (Gilbert [28]) $G(r,t)$, $t = 1-\sigma^2$ satisfies

$$r[G_{rr} - PG] - G_r + 2(1-t)G_{rt} \quad = \quad 0, \qquad P = P(r^2) \quad ,$$

providing $\quad G_r(r,0) \quad = \quad rP(r^2),$

and $\quad G(0,t) \quad = \quad 0.$

Thus $\quad G(r,t)$ is the solution of a Goursat problem.

3. Numerical Approximation of the Integral Operator for $-\Delta u + P(r^2)u = 0$

a) By Truncated Series (Bergman & Herriot [7], Schryer [48].

We recall that

$$G(r,t) = \sum_{n=1}^{\infty} \frac{e_n(r^2)t^{n-1}}{2^{2n-1}\beta(n,n+1)} ,$$

Let

$$c_n(r^2) = \frac{e_n(r^2)}{2^{2n-1}\beta(n,n+1)} .$$

Then $$G(r,t) = \sum_{n=1}^{\infty} c_n(r^2)t^{n-1} ,$$

and from (3.04) we find that

$$\frac{\partial c_1(r^2)}{\partial r} = + r\, P(r^2),$$

$$2n \frac{\partial c_{n+1}(r^2)}{\partial r} = -r \frac{\partial^2}{\partial r^2} c_n(r^2) + (2n-1) \frac{\partial c_n(r^2)}{\partial r} + rP(r^2)c_n(r^2),$$

$$n \geq 1.$$

An approximation G_T of the G-function can then be found by truncating the series

$$G_T(r,t) = \sum_{n=1}^{T} c_n(r^2)t^{n-1},$$

b) By Solution of the Goursat Problem

$G(r,t)$ satisfies the following Goursat problem

$$r[G_{rr} - PG] - G_r + 2(1-t)G_{rt} = 0, \quad r > 0, \quad t > 0,$$

with
$$G(r,0) = \int_0^r \lambda P(\lambda^2)d\lambda, \quad r > 0, \tag{3.06}$$

$$G(0,t) = 0, \quad t > 0.$$

The hyperbolic p.d.e.in (3.06) satisfied by G can be simplified by reducing it to canonical form. This is accomplished by taking as the new independent variables the characteristic coordinates, or any function of the characteristic coordinates [see Appendix 2].

i) R.P. Gilbert and P. Linz [30].

Gilbert and Linz take as new variables

$$\rho = r\sqrt{1-t}, \quad t = t,$$

which correspond to the characteristics $\rho/r\sqrt{1-t}$ = const., t = const.

Also taking $\frac{W(\rho,t)}{1-t} = G(r,t)$, the Goursat problem for $W(\rho,t)$ becomes

$$W_{\rho t} = \rho \frac{P(\rho^2/(1-t))}{2(1-t)^2} W, \quad t > 0, \ \rho > 0, \ \rho \leq a\sqrt{1-t},$$

$$W(\rho,0) = \int_0^\rho \lambda P(\lambda^2)d\lambda, \quad \rho > 0, \tag{3.07}$$

$$W(0,t) = 0, \quad t > 0.$$

Integrating the hyperbolic equation (3.07) first with respect to ρ, and then with respect to t gives

$$W(\rho,t) = W(0,t) + W(\rho,0) + \int_0^\rho \int_0^t \rho' \frac{P(\rho'^2/(1-t')\ W(\rho',t')}{2(1-t')^2}\, dt'd\rho'.$$

This transformation has two major disadvantages. Firstly, the factor $\frac{1}{(1-t)^2}$ presents problems near $t = 1$. Secondly, the integration is over the parabolic region bounded by $t = 0$, $\rho = 0$, $\rho = a\sqrt{1-t}$, and the shape of this region makes it generally impractical to cover it with a regular grid, thus leading to low order methods of solution.

Gilbert and Linz use a product-integration technique to alleviate the problem at $t = 1$, approximating the integrand by a bi-linear function and integrating the result.

It is possible to find a canonical form of the hyperbolic equation without a singularity, by taking exponential functions of the solutions of the characteristic equations, however the resulting system is very inconvenient numerically.

ii) An alternative transformation

Take new independent variables

$t = t$, $\rho = \dfrac{r^2(1-t)}{r_0^2}$; r_0^2 a convenient constant whose rôle will become clear.

Let

$$\frac{2}{1-t} H(\rho,t) = G(r,t), \quad \text{where } H(\rho,t) \equiv H(\rho,t,\ r).$$

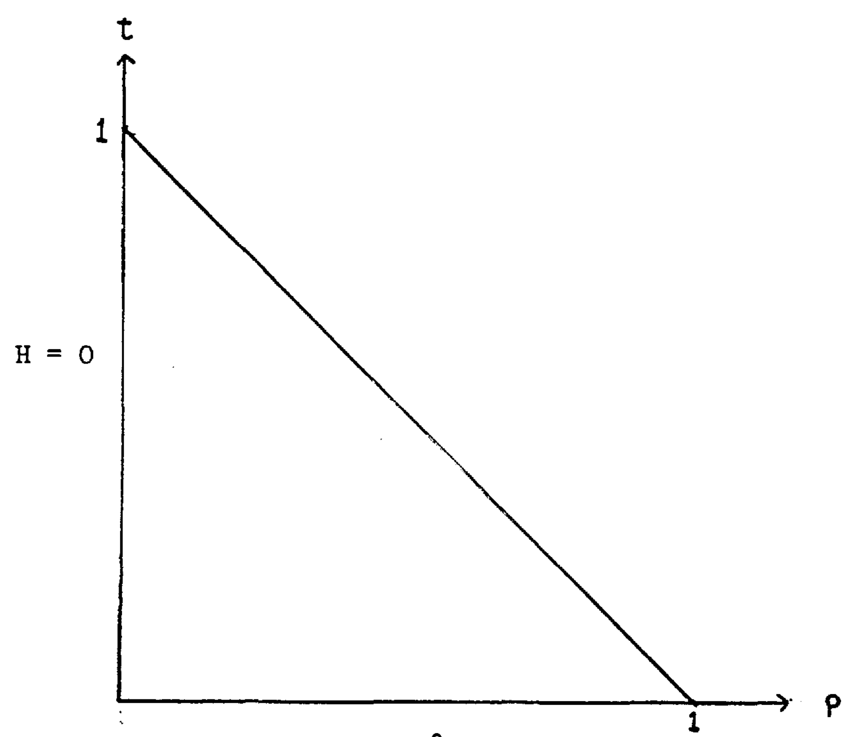
Then (see appendix 2) H satisfies

$$\begin{aligned} H_{\rho t} &= \frac{r_0^2}{4(1-t)^2}\ P\left(\frac{r_0^2\rho}{1-t}\right)H, && t > 0,\ \rho > 0 \\ & && \rho \le \left(\frac{r}{r_0}\right)^2(1-t) \\ H(0,t) &= 0, && t > 0\ , \qquad (3.08) \\ H(\rho,0) &= \frac{1}{4}\, r_0^2 \int_0^\rho P(r_0^2\lambda)d\lambda, && \rho > 0\ . \end{aligned}$$

Integrating the hyperbolic equation in H in equation (3.08) with respect to ρ and t.

$$H(\rho,t) = H(\rho,0) + H(0,t) + \frac{r_0^{\,2}}{4}\int_0^{\rho}\int_0^{t} \frac{P\left(\frac{r_0^{\,2}\rho'}{1-t'}\right)}{(1-t')^2}\, H(\rho',t')\,dt'\,d\rho' \,. \qquad (3.09)$$

The presence of the factor $\frac{1}{(1-t)^2}$ causes problems when t is close to 1. However, by a convenient choice of $r_0^{\,2}$, the solution $H(\rho,t)$ is required on the line $\rho + t = 1$, and the solution domain is as in figure 1.



$$H = \frac{1}{4}\, r_0^{\,2} \int_0^{\rho} P(r_0^{\,2}\lambda)\,d\lambda.$$

(Figure 1)

The most efficient mesh to construct on the domain is a square mesh.

The numerical method used to solve the Goursat problem for H, is described in detail in Chapter 4, "The Goursat problem".

We recall that

$$u(\underline{r}) = h(\underline{r}) + \int_0^1 \sigma G(r,1-\sigma^2)h(\underline{r}\sigma^2)\,d\sigma.$$

In terms of the function H, this becomes

$$u(\underline{r}) = h(\underline{r}) + \int_0^1 H\frac{(\tau,1-\tau)}{\tau}\, h(\underline{r}\tau)\, d\tau. \tag{3.10}$$

At $\tau = 0$, $H(0,1) = 0$, and thus there is no singularity. The limiting value of $H(\frac{\tau,1-\tau}{\tau})$ as $\tau \to 0$ is found by extrapolating on the line $\rho = 1-t$, this is described in Chapter 9.

Although H is written as a function of τ, it is clear from (3.08) that it is also a function of r.

4. <u>The equation $-\Delta u + M(x_1)u = 0$</u>

Let $L_2 u \equiv -\Delta u + M(x_1)u = 0$, in D, (3.11)

where Δ = Laplace operator in two dimensions and $M(x_1) > 0$ is an entire function of x_1. D is a simply connected domain containing the origin.

For (3.11) Eichler [22] proposes a solution of the type

$$u(x_1,x_2) = \genfrac{}{}{0pt}{}{RE}{IM}\{f(z) - \int_0^z S(x_1,x_2;\xi)f(\xi)d\xi\}. \tag{3.12}$$

This is a solution closely related to Vekua's solution (see equation 2.10, note $G(z,\eta_0,z,z^*) = 1$ for (3.11)). For $u(x_1,x_2)$ in (3.12) to satisfy $L_2 u = 0$ (3.11), S must satisfy the following conditions

$$\begin{aligned} &S_{x_1x_1} + S_{x_2x_2} + M(x_1)S = 0 \\ &\frac{\partial}{\partial x_1} S(x_1,x_2;z) + i\frac{\partial}{\partial x_2} S(x_1,x_2;z) = \tfrac{1}{2}M(x_1) . \end{aligned} \tag{3.13}$$

For numerical expediency we develop an operator of Bergman's type for equation (3.17).

Recall that from equation (2.13), noting that A = 0 for equation (3.11), every solution can be represented in the form

$$u(x,y) = \genfrac{}{}{0pt}{}{RE}{IM}\{g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,\bar{z})}{2^{2n}\beta(n,n+1)} \int_0^z (z-\xi)^{n-1}g(\xi)d\xi\} . \tag{3.14}$$

where $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$

$$Q^{(n)}(z,\bar{z}) = \int_0^{\bar{z}} P^{(2n)}(z,\bar{z})d\bar{z} ,$$

$$P^{(2)} = -2F,$$

and $(2n+1)P^{(2n+2)} = -2[P_z^{(2n)} + D_1P^{(2n)} + F\int_0^{\bar{z}} P^{(2n)}d\bar{z}]$, (see 2.12).

$n = 1,2,\ldots$.

where $D_1 = -\int_0^{\bar{z}} A_z d\bar{z} + B_1, \quad F = -A_z - AB_1 + C_1$.

For equation (3.11) $A = B = 0$, $D_1 = 0$, $F = C_1 = -\dfrac{M(x_1)}{4}$.

Thus we have

$$P^{(2)} = +\frac{M(x_1)}{2},$$

$$3P^{(4)} = -2[+ M(x_1)_{x_1} - \frac{M(x_1)}{4}\int_0^{x_1} \frac{M(x_1)}{2}\, dx]$$

Since $(2n+1)P^{2n+2} = -2[P_{x_1}^{(2n)} - \dfrac{M(x_1)}{4}\displaystyle\int_0^{x_1} P^{(2n)}\, dx]$

We see by induction that $P^{(2n)}$, $n = 1,2,\ldots$ depends only on x_1.

Thus $Q^{(n)}(z,\bar{z}) = Q^{(n)}(x_1)$, and we can rewrite (3.14) as

$$u(x_1,x_2) = {}^{RE}_{IM}\{g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(x_1)}{2^{2n}\beta(n,n+1)} \int_0^z (z-\xi)^{n-1} g(\xi)d\xi.$$

Reversing the orders of numeration and integration and taking

$$W(x_1, z-t) \equiv -\sum_{n=1}^{\infty} \frac{Q^{(n)}(x_1)}{2^{2n}\beta(n,n+1)} (z-t)^{n-1},$$

then

$$u(x_1,x_2) = {}^{RE}_{IM}\{g(z) - \int_0^z W(x_1,z-t)g(t)dt\}. \tag{3.15}$$

Taking $g(z) = f(z)$ in (3.12) we see that

$$S(x_1,x_2;t) = W(x_1,z-t) . \tag{3.16}$$

However, the properties (3.13) do not uniquely determine the generating function $W(x_1,z-t)$.

Definition. A generating function, W, satisfying equation (3.15) will be called a canonical generating function with respect to the origin, if $W(x_1,z+\bar{z}) = 0$ ($\bar{z}$ is the complex conjugate of z).

Theorem 5. (Eichler [22], p.261)

If $M(x_1)$ is regular in the neighbourhood of $x_1 = 0$, then there exists one and only one canonical generating function W, with respect to this point.

Canonical generating solutions allow us to represent solutions of $L_2u = 0$ in the following way

$$u(x_1,x_2) = {}^{\mathrm{RE}}_{\mathrm{IM}}\{g(z) - \int_{-\bar{z}}^{z} W(x_1,z-\xi)g(\xi)d\xi \quad ,$$

Let $t = \xi - iy$,

Then $$u(x_1,x_2) = {}^{\mathrm{RE}}_{\mathrm{IM}}\{g(z) - \int_{-x_1}^{x_1} W(x_1,x_1-t)g(t+ix_2)dt\}, \tag{3.17}$$

To verigy that this is a solution of $L_2u = 0$, substitute for u in $L_2u = 0$. This gives that (3.17) satisfies $-\Delta u + M(x_1)u = 0$, provided

$$W_{11} + W_{12} - WN(x_1) = 0,$$

$$W(x_1,0) = \frac{1}{2}\int_0^x N(\lambda)d\lambda \tag{3.18}$$

and $$W(x_1,2x_1) = 0$$

Thus W as defined in (3.1) is a canonical generating function when (3.17) is a solution of $-\Delta u + M(x_1)u = 0$.

[Note $W_1(x_1,x_2) = \frac{\partial}{\partial x_1} W(x_1,x_2)$, $W_2(x_1,x_2) = \frac{\partial}{\partial x_2} W(x_1,x_2)$, etc...]

5. Numerical Approximation of the Integral Operator for $-\Delta u + M(x_1)u = 0$

a) By truncated series (Eichler [22]).

We recall that $W(x_1,z-t) = -\sum_{n=1}^{\infty} \frac{Q^{(n)}(x_1)}{2^{2n}\beta(n,n+1)}(z-t)^{n-1}$.

Let

$$C_n(x_1) = -\frac{Q^{(n)}(x_1)}{2^{2n}\beta(n,n+1)} .$$

Then
$$W(x_1,z-t) = \sum_{n=1}^{\infty} C_n(x_1)(z-t)^{n-1}, \tag{3.19}$$

where $2C_1'(x_1) = M(x_1)$

and $2C_{n+1}'(x_1) = C_n''(x_1) - \frac{M(x_1)}{4} C_n(x_1), \quad n \geq 1.$

Since this requires infinitely many derivatives of $M(x_1)$ it makes sense only if $M(x_1)$ is an analytic function of x_1. The series (3.19) converges absolutely provided, $x_1 \neq 0$, $|x_1-t| < 2x_1$ [Eichler [22] , pp.271].

If $M(x_1)$ has singular points, Eichler [22] , pp.261 constructs a solution of Lu = 0, in terms of the derivatives of an analytic function of z

$$u(x_1,x_2) = q_0(x_1)f(z) + q_1(x_1)\frac{df(z)}{dz} + q_2(x_1)\frac{\partial^2 f(z)}{dz} + \cdots . \tag{3.20}$$

where the $q_n(x_1)$ satisfy the recurrence formula

$$q_0'' - Mq_0 = 0,$$
$$q_1'' - Mq_1 = -2q_0' ,$$
$$\cdots\cdots\cdots\cdots$$
$$q_{n+1}'' - Mq_{n+1} = -2q_n' ,$$
$$\cdots\cdots\cdots\cdots$$

The series (3.20) for the solution is convergent in a sufficiently small neighbourhood of $x_1 = 0$, for $0 < |x_1| < (z-z_0)/2$, where z_0 is the nearest singular point of f(z), Eichler [22] , pp.274 .

b) By solution of the Goursat problem.

From (3.18) we see that $w(x_1,x_1-t)$ satisfies the following Goursat problem,

$$W_{11} + 2W_{12} - M(x_1)W = 0, \qquad -x_1 < t < x_1 \qquad x_1 \in D$$

$$W(x_1,0) = \frac{1}{2}\int_0^{x_1} N(\lambda)d\lambda, \qquad x_1 \in D \tag{3.21}$$

$$W(x_1,2x_1) = 0 \quad . \qquad x_1 \in D \text{ .}$$

However, the hyperbolic p.d.e. satisfied by W in (3.21) can be simplified by reducing it to canonical form. This is achieved by taking as new independent variables ξ,η where ξ = constant, η = constant are the characteristics. As before any functions of ξ and η can also be used. [See Appendix 2].

In this case,

$$\xi = (2x_1-\tau)/2, \qquad \eta = \tau/2, \qquad \tau = x_1-t,$$

and we let $F(\xi,\eta) = W(x_1,\tau)$.

Then F satisfies

$$\frac{\partial^2 F}{\partial\xi\partial\eta} = M(\xi + \eta)F, \qquad \begin{array}{l} 0 < \xi < x_1 \\ 0 < \eta < x_1 \\ x_1 \in D \end{array} \tag{3.22}$$

$$F(\xi,0) = -\frac{1}{2}\int_0^{\xi} M(\lambda)d\lambda, \quad \xi > 0, \; F(0,\eta) = 0, \quad \eta > 0.$$

Note $\xi = x_1 - \eta$,

and $u(x_1,x_2) = {}^{RE}_{IM}\{g(z) - \int_{-x}^{x} F(\frac{x_1+t}{2}, \frac{x_1-t}{2})g(t+ix_2)dt\}$.

Take $x_1 t' = t$, then

$$u(x_1,x_2) = {}^{RE}_{IM}\{g(z) - x_1 \int_{-1}^{1} F(x_1(\frac{1+t'}{2}), x_1(\frac{1-t'}{2})g(x_1 t' + ix_2)dt'\} ,$$

Redefine $\xi = \frac{1+t}{2}$, $\eta = \frac{1-t}{2}$, $x_1\xi = x_1(1-\eta)$.

Then $F(x_1\xi, x_1\eta)$ satisfies

$$F_{12} = M(x_1(\xi+\eta))F \qquad \xi > 0, \eta > 0, \xi + \eta \leq 1,$$

$$F(x_1\xi,0) = -\frac{1}{2}\int_0^{x_1\xi} N(\lambda)d\lambda . \qquad \xi > 0,$$

$$F(0, x_1\eta) = 0 \qquad \eta > 0.$$

Thus the Goursat problem has to be solved in the triangular region bounded by the lines, $\xi = 0$, $\eta = 0$, $\xi + \eta = 1$.

This is essentially the same problem as had to be solved for the equation $-\Delta u + P(r^2)u = 0$, and it is solved in the same way.

6. The Equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$.

Consider $L_3 u = -\Delta u + N(x_2)u = 0$ in D, where $N(x_2) > 0$, Δu = Laplace's operator, and D is a simply connected domain containing the origin.

By analogy with the equation $L_2 u = 0$, the general solution to the equation $L_3 u = 0$ is given by

$$u(x_1,x_2) = \begin{matrix}\mathrm{RE}\\\mathrm{IM}\end{matrix}\{g(z) - x_2 \int_{-1}^{1} E(x_2(\frac{1+t}{2}), x_2(\frac{1-t}{2}))g(x_1+itx_2)dt\}, \qquad (3.23)$$

where $E(x_2\overline{\xi},x_2\overline{\eta})$ satisfies the following Goursat problem

$$E_{12} = N(x_2(\overline{\xi}+\overline{\eta}))E, 0 < \overline{\xi} < 1, \quad 0 < \overline{\eta} < 1$$

$$E(x_2\overline{\xi},0) = -\frac{1}{2} \int_0^{x_2\overline{\xi}} N(\lambda)d\lambda \qquad 0 < \overline{\xi} < 1 \qquad (3.24)$$

$$E(0, x_2\overline{\eta}) = 0 \qquad 0 < \overline{\eta} < 1$$

$x_2\overline{\xi} = x_2(1 - \overline{\eta})$.

Now consider the equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$

$L_4 u = -\Delta u + [M(x_1) + N(x_2)]u = 0$ in D.

$M(x_1) > 0$, $N(x_2) > 0$, $\Delta \equiv$ Laplace operator.

Let $F(\xi,\eta)$ be the generating function for $L_2 u = 0$, and let $E(\xi,\eta)$ be the generating function for the equation $L_3 u = 0$, then

$$u(x_1,x_2) = v(x_1,x_2) - x_1 \int_{-1}^{1} F(x_1(\frac{1+t}{2}), x_1(\frac{1-t}{2}))v(x_1 t,x_2)dt,$$

is the general solution of the equation $L_4 u = 0$, when

$$V(x_1,x_2) = \begin{matrix}RE\\IM\end{matrix}\{g(z) - x_2 \int_{-1}^{1} E(x_2(\tfrac{1+t}{2}),\ x_2(\tfrac{1-t}{2}))g(x_1 + itx_2)dt\},$$

where g is an arbitrary analytic function.

This can be verified by substitution.

The solution of $L_4 u = 0$ is thus given by

$$u(x_1,x_2) = \begin{matrix}RE\\IM\end{matrix}\{g(z) - x_2 \int_{-1}^{1} E(x_2(\tfrac{1+t}{2}),\ x_2(\tfrac{1-t}{2}))g(x_1+ix_2t)dt$$

$$-x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))g(x_1t + ix_2)dt$$

$$+ x_1x_2 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2})) \int_{-1}^{1} E(x_2(\tfrac{1+\tau}{2}),\ x_2(\tfrac{1-\tau}{2}))g(x_1t+ix_2\tau)d\tau dt. \tag{3.25}$$

7. Numerical Approximation of the Integral Operator for the Equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$

a) Truncation of Series.

Recall from Ch. 3, Section 5, that $W(x_1, z-t)$ is the canonical generating function for the equation $L_2 u = 0$, and that $W(x_1, z-t)$ can be expressed as an infinite series as follows

$$W(x_1, z-t) = \sum_{n=1}^{\infty} C_n(x_1)(z-t)^{n-1}, \tag{3.26}$$

where $C_n(x_1)$ satisfies a certain recurrence relationship. In an identical way a canonical generating function of the form $\tilde{W}(x_2, z-t)$ can be found for the equation $L_3 u = 0$, and there exists a function $D_n(x_2)$ such that

$$\tilde{W}(x_2, z-t) = \sum_{n=1}^{\infty} D_n(x_2)(z-t)^{n-1}, \tag{3.27}$$

where $D_n(x_2)$ satisfies a known recurrence relationship.

No new generating functions are introduced for the equation $L_4 u = 0$, and so the two generating functions to be evaluated can be evaluated with a knowledge of (3.26) and (3.27).

b) By solution of Goursat problems.

The generating functions to be evaluated by solution of Goursat problems are

$$E\left(x_2\left(\frac{1+t}{2}\right), x_2\left(\frac{1-t}{2}\right)\right) \quad \text{and} \quad F\left(x_1\left(\frac{1+t}{2}\right), x_1\left(\frac{1-t}{2}\right)\right).$$

We described how these were solved in section 5.

CHAPTER 4

THE NUMERICAL SOLUTION OF THE GOURSAT PROBLEM

1. Introduction

Consider the following Goursat problem

$$
\begin{aligned}
&H_{\rho t} = f(\rho,t,H) = F(\rho,t)H(\rho,t), \quad 0 < \rho < \alpha,\ 0 < t < \beta, \\
&H(\rho,0) = \phi(\rho), \qquad 0 < \rho < \alpha, \\
&H(0,t) = \psi(t), \qquad 0 < t < \beta, \text{ with } \phi(0) = \psi(0),
\end{aligned}
\tag{4.01}
$$

where $F(\rho,t)$ is defined everywhere in R_0, where R_0 encloses the rectangle $0 < \rho < \alpha$, $0 < t < \beta$.

Many familiar methods have been developed to find the numerical solution of this Goursat problem. We mention: the method of characteristics (Fox, [25], pp.211); a Gaussian-quadrature method (Day, [17]); an Euler-Cauchy polygon method (Diaz, [20]); a Runge-Kutta procedure (Moore, [42]); Tornig [50] generalizes the explicit and implicit Adams methods; Aziz and Hubbard [4] use a finite-difference method; and Duris [21] uses a Riemann-like method. All of these and many others could be used on this problem. However, in the interests of accuracy, we develop a difference scheme based on Simpson's 9-point quadrature rule.

2. The Goursat problem

If $f(\rho,t,H)$, $\phi(\rho)$, $\psi(t)$, satisfy the following conditions then it can be proved (Bernstein, [8] , p.109) that the Goursat problem (4.01), has a unique solution.

i) $f(\rho,t,H)$ is continuous in R_0, and all H.

ii) In any closed, bounded subrectangle of R_0, $R = P\times T$, where P is the interval $\rho_1 \leq \rho \leq \rho_2$, and T is the interval $t_1 \leq t \leq t_2$, f satisfies a Lipschitz condition.
That is there exists a constant $L \geq 0$, such that

$$|f(\rho,t,H) - f(\rho,t,H^*)| \leq L[|H-H^*|],$$

for $(\rho,t) \in R$ and all H and H^*.

iii) The function $\phi(\rho)$ possesses a continuous first derivative $\phi'(\rho)$, $0 < \rho < \alpha$.

(iv) The function $\psi(t)$ possesses a continuous first derivative $\psi'(t)$, $0 < t < \beta$.

(v) $\phi(0) = \psi(0)$.

In order to find a numerical solution for the Goursat problem ((4.01), taking $\alpha = \beta = 1$), on the line $\rho+t = 1$, we impose a square mesh over the domain bounded by $\rho = 0$, $t = 0$, $\rho+t = 1$, $\rho = 1 - \delta$, $t = 1 - \delta$ where $\delta > 0$. We can exclude the points (0,1), and (1,0) because the solution of the Goursat problem is known at these points. If we require N mesh points on $\rho+t = 1$, where N is an integer, then $h = \frac{1}{N+1}$ (in practice we take $\delta = h$). Let R_{ij} denote a square of the mesh such that $R_{ij} = P_i \times T_j$, where P_i is the interval $\rho_i \leq \rho \leq \rho_{i+1}$, and T_j is the interval $t_j \leq t \leq t_{j+1}$, and $\rho_{i+1} = \rho_i+h$, $t_{j+1} = t_j+h$, then the mesh consists of the squares R_{ij}, $i = 1,\dots,N+1-j$ $j = 1,\dots,N$. We assume that H, $\frac{\partial H}{\partial \rho}$, $\frac{\partial H}{\partial t}$ are known at the points (ρ_i, t_{j+1}), (ρ_i,t_j), $(\rho_{i+\frac{1}{2}},t_j)$, (ρ_{i+1},t_j), $(\rho_i, t_{j+\frac{1}{2}})$ and $\frac{\partial^2 H}{\partial \rho \partial t}$ is known at (ρ_i,t_j), where $\rho_{i+\frac{1}{2}} = \rho_i + \frac{h}{2}$, $t_{j+\frac{1}{2}} = t_j + \frac{h}{2}$. This is certainly true when $\rho_i = \rho_1 = 0$, $t_j = t_1 = 0$. We wish to calculate H, $\frac{\partial H}{\partial \rho}$, $\frac{\partial H}{\partial t}$ at the points $(\rho_{i+1},t_{j+\frac{1}{2}})$, $(\rho_{i+\frac{1}{2}},t_{j+1})$, (ρ_{i+1},t_{j+1}). (4.02)

Then by proceeding in a stepwise manner over the squares R_{ij}, $i = 1,\dots,N+1-j$, $j = 1,\dots,N$, we obtain N values of H on the line $\rho+t = 1$. In order to calculate H, $\frac{\partial H}{\partial \rho}$, $\frac{\partial H}{\partial t}$ at the points given in (4.02) we integrate the differential equation

$$H_{\rho t} = f(\rho,t,H)$$

over the square R_{ij} to obtain

$$H(\rho_{i+1},t_{j+1}) = H(\rho_{i+1},t_j) + H(\rho_i,t_{j+1}) - H(\rho_i,t_j) + \int_{\rho_i}^{\rho_{i+1}} \int_{t_j}^{t_{j+1}} f(\rho,t,H(\rho,t))\,d\rho\, dt \tag{4.03}$$

$$\left.\frac{\partial H}{\partial \rho}\right|_{\rho_{i+1} t_{j+1}} = \left.\frac{\partial H}{\partial \rho}\right|_{\rho_{i+1} t_j} + \int_{t_j}^{t_{j+1}} f(\rho_{i+1},t, H(\rho_{i+1},t))\,dt, \tag{4.04}$$

$$\left.\frac{\partial H}{\partial t}\right|_{\rho_{i+1} t_{j+1}} = \left.\frac{\partial H}{\partial t}\right|_{\rho_i, t_{j+1}} + \int_{\rho_j}^{\rho_{j+1}} f(\rho,t_{j+1},H(t_{j+1},\rho))\,d\rho. \tag{4.05}$$

Simpson's 9-point quadrature rule is used to evaluate the double integral in equation (4.03), taking $h_2 = \frac{h}{2}$,

$$\int_{\rho_i}^{\rho_{i+1}} \int_{t_j}^{t_{j+1}} f(\rho,t,H(\rho,t)d\rho dt = \frac{h_2^2}{9} \sum_{n=o}^{2} \sum_{m=o}^{2} W_{nm} f(\rho_i+nh_2, t_j+mh_2,$$

$$H(\rho_i+nh_2,t_j+mh_2)) - \frac{h_2^6}{45}(\Delta^2 f(\xi,\eta,H(\xi,\eta)) + O(h_2)^8, \qquad (4.06)$$

where $\Delta^2 = \frac{\partial^4}{\partial\xi^4} + \frac{\partial^4}{\partial\eta^4}$, $\quad \xi = \rho_i + h_2$, $\eta = t_j + h_2$,

and $W_{00} = W_{02} = W_{20} = W_{22} = 1$, $W_{01} = W_{10} = W_{21} = W_{12} = 4$, $W_{11} = 16$. (Bickley, [9]).

In order to calculate $H(\rho_{i+1}, t_{j+1})$ in (4.06), we need to evaluate $H(\rho_i + \frac{h}{2},\ t_j + \frac{h}{2})$, $H(\rho_i + h,\ t_j + \frac{h}{2})$, $H(\rho_i + \frac{h}{2},\ t_j + h)$, and $H(\rho_i + h,\ t_j + h)$.

The following estimates of order 4 using Taylor expansions have been used by Jain and Sharma [34].

For $0 \leq \sigma\ ,\ \tau\ \leq 1$,

$$\begin{aligned}
H(\rho_i+\sigma h,t_j+\tau h) = {} & [1+2(\sigma^3+\tau^5)-3(\sigma^2+\tau^2)]H(\rho_i,t_j) + [3\tau^2-2\tau^3]H(\rho_i+t_{j+1}) \\
& + [3\sigma^2-2\sigma^3]H(\rho_{i+1},t_j) + h[\sigma^3-2\sigma^2+\sigma-\sigma\tau^2] \left.\frac{\partial H}{\partial\rho}\right|_{\rho_i t_j} \\
& + h\sigma^2\tau^2 \left.\frac{\partial H}{\partial\rho}\right|_{\rho_i t_{j+1}} + h[\sigma^3-\sigma^2] \left.\frac{\partial H}{\partial\rho}\right|_{\rho_{i+1} t_j} \\
& + h[\tau^3-2\tau^2+\tau-\sigma^2\tau] \left.\frac{\partial H}{\partial\tau}\right|_{\rho_i t_j} + h\sigma^2\tau \left.\frac{\partial H}{\partial\tau}\right|_{\rho_{i+1} t_j} \\
& + h[\tau^3-\tau^2] \left.\frac{\partial H}{\partial\tau}\right|_{\rho_i t_{j+1}} + h^2\sigma\tau[1-\sigma-\tau]f(\rho_i,t_j,H(\rho_i,t_j)) \\
& + \frac{h^4}{4!}[\sigma^2(\sigma-1)^2]\left.\frac{\partial^4 H}{\partial\rho^4}\right|_{\rho_i t_j} + 4\tau\sigma^2[\sigma-1] \left.\frac{\partial^4 H}{\partial\rho^3\partial t}\right|_{\rho_i t_j} \\
& + 6\tau^2\sigma^2 \left.\frac{\partial^4 H}{\partial\rho^2\partial t^2}\right|_{\rho_i t_j} + 4\tau^2\sigma[\tau-1] \left.\frac{\partial^4 H}{\partial\rho\partial t^3}\right|_{\rho_i t_j} \\
& + \tau^2[\tau-1]^2 \left.\frac{\partial^4 H}{\partial t^4}\right|_{\rho_i t_j} + O(h^6)\ .
\end{aligned} \qquad (4.07)$$

We still require values of $\frac{\partial H}{\partial \rho}$, $\frac{\partial H}{\partial t}$ at $(\rho_{i+1} t_{j+1})$. For these the formulas (4.04), (4.05) are used, the integrals being approximated using Simpson's 3-point quadrature formula. This involves the values of $H(\rho_{i+1}, t_j + \frac{h}{2})$, $H(\rho_i + \frac{h}{2}, t_{j+1})$ and $H(\rho_{i+1}, t_{j+1})$.

The first two have already been approximated using equation (4.06), and the value of $H(\rho_{i+1}, t_{j+1})$ is calculated using equation (4.03).

The error in Simpson's rule over the interval ρ_i to ρ_{i+1}, with step-length $h_2 = h/2$ is

$$h \cdot \frac{h_2^{\;4}}{180} \left| \frac{\partial^4}{\partial \rho^4} f\,(\rho, t_{j+1}, H(\rho, t_{j+1})) \right|_{\rho=\xi_1} , \tag{4.08}$$

$$\rho_i < \xi_1 < \rho_{i+1} .$$

Similarly for equation (4.04) the error is given by

$$h \cdot \frac{h_2^{\;4}}{180} \left| \left[\frac{\partial^4}{\partial t^4} f\,(\rho_{i+1}, t, H(\rho_{i+1}, t)) \right] \right|_{t=\eta_1} , \tag{4.09}$$

$$t_j < \eta_1 < t_{j+1} .$$

The expressions (4.08) and (4.09) are of order h^5.

3. The equation $-\Delta u + P(r^2)u = 0$.

For the equation $-\Delta u + P(r^2)u = 0$, the following Goursat problem has to be solved,

$$\frac{\partial^2 H}{\partial\rho\partial t} = \frac{r_0^2}{4(1-t)^2} P\left(\frac{r_0^2\rho}{1-t}\right)H, \qquad \text{in } t > 0, \quad \rho > 0$$
$$\rho + t \leq 1,$$

$$H(0,t) = 0, \qquad t > 0, \tag{4.10}$$

$$H(\rho,0) = \tfrac{1}{2}r_0^2 \int_0^\rho P(r_0^2\lambda)d\lambda, \qquad \rho > 0,$$

Let R_0 be the domain $0 \leq \rho \leq 1-\delta$, $0 \leq t \leq 1-\delta$, $\delta > 0$. Then for the conditions i) and ii) in section 2 to hold, $\frac{r_0^2}{4(1-t)^2} P(\frac{r_0^2\rho}{1-t})$ must be continuous and bounded in R_0. This implies that t must not equal 1, and that $P(\frac{r_0^2\rho}{1-t})$ is a continuous function in R_0. Conditions iii) to v) follow from the initial values of $H(\rho,t)$, on the lines $t = 0$, and $\rho = 0$. Thus the Goursat problem has a unique solution.

We anticipate the presence of a singularity at $t = 1$, this will cause a deterioration in the numerical solution, particularly at points near $t = 1$.

4. The equations $-\Delta u + M(x_1)u = 0,$

$$-\Delta u + N(x_2)u = 0,$$

$$-\Delta u + [M(x_1)\ N(x_2)]u = 0.$$

For the equation $-\Delta u + M(x_1)u = 0$ the following Goursat problem has to be solved for $F(x_1\xi, x_2, \eta)$:

$$\begin{aligned} &F_{12} = M(x_1(\xi+\eta))F, && \xi > 0,\ \eta > 0,\ \xi + \eta \leq 1, \\ &F(x_1\xi, 0) = -\frac{1}{2}\int_0^{x_1\xi} M(\lambda)d\lambda, && \xi > 0 \\ &F(0, x_1\eta) = 0, && \eta > 0 \end{aligned} \tag{4.11}$$

The domain R_0 is $0 \leq x_1\xi \leq x_1 \quad 0 \leq x_1\eta \leq x_1$. Then for the conditions i) and ii) of section 2 to hold, $M(x_1(\xi+\eta))$ must be continuous and bounded in R_0. Conditions iii) to v) follow from the initial values of F on the lines $\xi = 0$, and $\eta = 0$.

Similarly for the equation $-\Delta u + N(x_2)u = 0$, the following Goursat problem has to be solved for $E(x_2\xi, x_2\eta)$:

$$\begin{aligned} &E_{12} = N(x_2(\xi+\eta))E, && \xi > 0,\ \eta > 0,\ \xi + \eta \leq 1, \\ &E(x_2\xi, 0) = -\frac{1}{2}\int_0^{x_2\xi} N(\lambda)d\lambda\ , && \xi > 0, \\ &E(0, x_2\eta) = 0, && \eta > 0. \end{aligned} \tag{4.12}$$

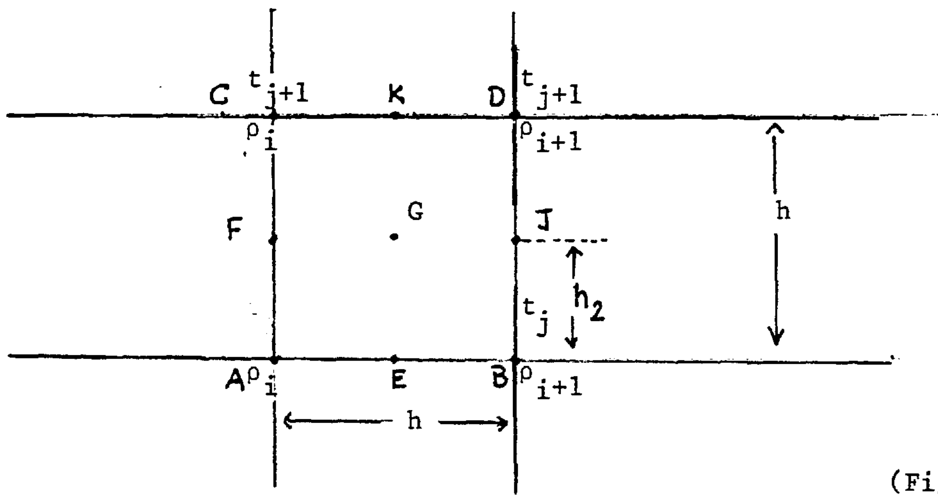
The domain R_0 is $0 \leq x_2\xi \leq x_2$, $0 \leq x_2\eta \leq x_2$. If $N(x_2(\xi+\eta))$ is continuous in R_0 then conditions i) to v) hold. Thus the Goursat problems (4.11) and (4.12) have unique solutions.

For the equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$, both the Goursat problems (4.11) and (4.12) have to be solved.

5. Error analysis for the Goursat problem

a) The local decretization error.

Let CABD, (see figure 2), be a typical square, R_{ij}, of the mesh over the triangular domain of the Goursat problem (section 2).



(Figure 2)

It is assumed that the values of H, $\frac{\partial H}{\partial \rho}$ and $\frac{\partial H}{\partial t}$ are known exactly on AB and AC.

The computed value of H at D, $\tilde{\tilde{H}}$, is then given by

$$\tilde{\tilde{H}}(\rho_{i+1},t_{j+1}) = H(\rho_{i+1},t_j) + H(\rho_i,t_{j+1}) - H(\rho_i,t_j)$$

$$+ \frac{h_2^2}{9} \sum_{n=0}^{2} \sum_{m=0}^{2} w_{nm}\, f(\rho_i + nh_2,\ t_j+mh_2, \tilde{H}(\rho_i + nh_2,\ t_j+mh_2)),$$

$$h_2 = \frac{h}{2}$$

where the terms $\frac{h_2^2}{9} \sum_{n=o}^{2} \sum_{m=0}^{2} w_{nm} f_{nm}$ are those of Simpson's 9-point rule, see equation (4.06), and $\tilde{H}$ denotes the approximate value of H at the points K, G, J and D respectively, found from Taylor expansions, see equation (4.07). It is clear that $\tilde{H}$ can be found at all points of the square from Taylor expansions.

Let $H(\rho_{i+1},t_{j+1})$ be the true solution of H at D, then

$$|H(\rho_{i+1},t_{j+1}) - \tilde{\tilde{H}}(\rho_{i+1},t_{j+1})| = |H(\rho_{i+1},t_{j+1}) - \Sigma H + \Sigma H - \tilde{\tilde{H}}(\rho_{i+1},t_{j+1})|,$$

where

$$\Sigma H \equiv H(\rho_{i+1},t_j) + H(\rho_i,t_{j+1}) - H(\rho_i,t_j) + \frac{h_2^2}{9}\sum_{n=0}^{2}\sum_{m=0}^{2} w_{nm} f(\rho_i+nh_2, t_j+mh_2, H).$$

Hence, from equation (4.06)

$$\begin{aligned}|H(\rho_{i+1},t_{j+1}) - \tilde{\tilde{H}}(\rho_{i+1},t_{j+1})| &\leq |H(\rho_{i+1},t_{j+1}) - \Sigma H|+|\Sigma H - \tilde{\tilde{H}}(\rho_{i+1},t_{j+1})| \\ &\leq \left|\frac{h_2^6}{45}\Delta^2 f(\xi,\eta,H(\xi,\eta)) + O(h_2{}^8)\right| \\ &\quad + \frac{h_2^2}{9}\sum_{n=1}^{2}\sum_{m=1}^{2}|w_{nm}||f(\rho_i+ nh_2, t_j+mh_2, H) - f(\rho_i+nh_2,t_j+mh_2,\tilde{H})|,\end{aligned} \tag{4.13}$$

where $\Delta^2 = \frac{\partial^4}{\partial\xi} + \frac{\partial^4}{\partial\eta}$ and $\Delta^2 f$ is evaluated at $\xi = \rho_i + h_2$, $\eta = t_j + h_2$.

Using equation (4.01)

$$\begin{aligned}&|f(\rho_i+nh_2, t_j+ mh_2, H) - f(\rho_i+nh_2, t_j+mh_2, \tilde{H})| \\ &= |F(\rho_i+nh_2, t_j+mh_2)[H(\rho_i+nh_2, t_j+mh_2) - \tilde{H}(\rho_i+nh_2, t_j+mh_2)]| \\ &\leq L[|H-\tilde{H}|] \quad \text{from the Lipschitz condition (ii) section 2).}\end{aligned}$$

For each n and m $|H-\tilde{H}|$ is the error arising from the Taylor expansions, equation (4.07). By calculating the error for each n and m, multiplying by the appropriate weight $|w_{nm}|$ and recalling that $h_2 = \frac{h}{2}$, we can rewrite (4.13) as

$$|H(\rho_{i+1},t_{j+1}) - \tilde{\tilde{H}}(\rho_{i+1},t_j)| \leq \frac{h^6}{45\times 2^6} \quad 2A \quad + \frac{h^6}{9\times 2^2} \cdot L \cdot \frac{2.5M+12N+24C}{4!} + O(h^8) \tag{4.14}$$

where

$$A \equiv \max\{\left|\frac{\partial^4 f}{\partial\xi^4}(\xi,\eta,H)\right|, \left|\frac{\partial^4 f}{\partial\eta^4}(\xi,\eta,H)\right| \;:\; \xi = \rho_i + \frac{h}{2},\ \eta = t_j + \frac{h}{2}\},$$

$$M \equiv \max\{\left|\frac{\partial^4 H}{\partial\xi^4}(\xi,\eta)\right|, \left|\frac{\partial^4 H}{\partial\eta^4}(\xi,\eta)\right| \;:\; \xi = \rho_i,\ \eta = t_j\},$$

$$N \equiv \max\{\left|\frac{\partial^2 f}{\partial\xi^2}(\xi,\eta,H)\right|, \left|\frac{\partial^2 f}{\partial\eta^2}(\xi,\eta,H)\right| \;:\; \xi = \rho_i,\ \eta = t_j\},$$

$$C \equiv \left|\frac{\partial^2 f}{\partial\xi\partial\eta}(\xi,\eta,H)\right|, \qquad \xi = \rho_i, \quad \eta = t_j,$$

$$L \equiv \frac{\partial f}{\partial H}(\xi,\eta,H), \qquad \xi = \rho_i,\ \eta = t_j.$$

b) An estimate of the global discretization error

In the numerical solution of ordinary differential equations, error bounds for the global discretization error, obtained using conventional error analysis, may be extremely pessimistic and unrepresentative. In partial differential equations, the increased complexity can only add to the inadequacy of the error bound. Error estimates for ordinary differential equations, have, however, been found to be extremely useful for monitoring the error.

Consider the Goursat problem (4.01):

$$\begin{aligned} H_{\rho t} - f(\rho,t,H) &= 0, && 0 < \rho < \alpha,\ 0 < t < \beta, \\ H(\rho,0) &= \phi(\rho), && 0 < \rho < \alpha, \\ H(0,t) &= \psi(t), && 0 < t < \beta, \end{aligned}$$

with $\phi(0) = \psi(0)$.

The difference scheme used to solve this problem can be written as follows:

$$\tilde{H}(\rho_{i+1},t_{j+1})-\tilde{H}(\rho_{i+1},t_j) - \tilde{H}(\rho_i,t_{j+1}) + \tilde{H}(\rho_i,t_j)$$

$$- \frac{h^2}{4\times 9} \sum_{n=0}^{2} \sum_{m=0}^{2} W_{nm} f(\rho_{i+n/2},t_{j+m/2}, \tilde{H}(\rho_{i+n/2}, t_{j+m/2})) = 0 ,$$

$$\text{with} \quad \tilde{H}(\rho_i,t_1) = \phi(\rho_i), \tag{4.15}$$

$$\tilde{H}(\rho_1,t_j) = \psi(t_j) ,$$

for $i = 1,2,\ldots,N-j$, $j = 1,2,\ldots,N-1$.

If this difference scheme is consistent of order q - that is the local discretization error, d_{i+1i+1} is of the form

$$d_{i+1\ i+1} = -h^q\xi(\rho_{i+1},t_{i+1}) - O(h^{q+1}) , \tag{4.16}$$

where ξ is some function of ρ and t independent of h - and if the initial values are accurate at least to order q+1, then we will show that

$$\tilde{H}(\rho_i,t_i) = H(\rho_i,t_i) + h^q e(\rho_i,t_i) + O(h^{q+1}), \tag{4.17}$$

where

$$\frac{\partial^2 e(\rho,t)}{\partial\rho\partial t} - \frac{\partial f}{\partial H} (\rho,t,H(\rho,t))e(\rho,t) = \xi(\rho,t), \tag{4.18}$$

and $e(\rho,0) = e(0,t) = 0$.

We will write

$$f_{ij} = f(\rho_i,t_j,H(\rho_i,t_j)),$$

$$\tilde{f}_{ij} = f(\rho_i,t_j, H_{ij}+h^q e(\rho_i,t_j))$$

$$H_{ij} = H(\rho_i,t_j),$$

$$e_{ij} = e(\rho_i,t_j) .$$

Substitute $H_{ii} + h^q e_{ii}$ into the left hand side of equation (4.15), then

$$H_{i+1\ i+1} - H_{i+1\ i} - H_{i\ i+1} + H_{ii} + h^q(e_{i+1\ i+1} - e_{i+1\ i} - e_{i\ i+1} + e_{ii}) - \frac{h^2}{4\times 9} \sum_{n=o}^{2} \sum_{m=o}^{2} w_{nm}\, \tilde{f}_{i+n/2,i+m/2} \tag{4.19}$$

From Taylor's series

$$f(\rho_{i+n/2}, t_{i+n/2}, H(\rho_{i+n/2}, t_{i+n/2}) + h^q e_{i+n/2\ i+n/2})$$

$$= f(\rho_{i+n/2}, t_{i+n/2}, H(\rho_{i+n/2}, t_{i+n/2})) + \frac{\partial f}{\partial H}(\rho_{i+n/2}, t_{i+n/2} H(\rho_{i+n/2}, t_{i+n/2})) \times \times\ h^q e_{i+n/2\ i+n/2} + O(h^{2q})\ . \tag{4.20}$$

Substituting (4.20) into (4.19) and using the definition of the local discretization error, (4.19) becomes

$$d_{i+1\ i+1} + h^q(e_{i+1\ i+1} - e_{i+1\ i} - e_{i\ i+1} + e_{ii}) - \frac{h^2}{4\times 9} \sum_{n=0}^{2} \sum_{m=0}^{2} \frac{\partial f}{\partial H}(\rho_{i+n/2}, t_{i+m/2}, H(\rho_{i+n/2}, t_{i+m/2})) \times \times\ h^q e_{i+n/2\ i+m/2} + O(h^{2q}) \tag{4.21}$$

using Taylor's series to expand all terms about the point (ρ_{i+1}, t_{i+1}) (4.21) becomes

$$h^q \frac{\partial^2}{\partial\rho\partial t} e_{i+1\ i+1} - h^q \frac{\partial f}{\partial H}\ i+1\ i+1 \times e_{i+1\ i+1} - h^q \xi_{i+1\ i+1} + O(h^{q+1})$$

$$= O(h^{q+1}), \qquad \text{from (4.18).}$$

Thus $\tilde{H}(\rho_i,t_i)$ and $H(\rho_i,t_i)$ satisfy sets of equations whose right hand sides differ by $O(h^{q+1})$, from the stability of the method this implies that $\tilde{H}(\rho_i,t_i)$ and $H(\rho_i,t_i)$ differ by $O(h^{q+1})$.

This result means that a method for which the principal part of the local discretization error is $h^q\xi_{i+1\ i+1}$ at ρ_{i+1}, t_{i+1}, has global discretization error with principal part $h^q e(\rho_{i+1},t_{i+1})$ and the deviation from the true solution is the same as that due to a perturbation $h^{(q)}\xi(\rho_{i+1},t_{i+1})$ in the analytic equation, provided that the initial conditions are correct to order q+1. Thus the solution of the equation

$$\frac{\partial^2 e}{\partial\rho\,\partial t}(\rho,t) - \frac{\partial f}{\partial H}(\rho,t,H(\rho,t))e(\rho,t) = \xi(\rho,t),$$

$$\text{with } e(\rho,0) = e(0,t) = 0, \qquad \rho > 0, \quad t > 0 \tag{4.22}$$

gives the leading term of the global discretization error.

Consider the equation

$$\frac{\partial^2 \varepsilon}{\partial\rho\,\partial t}(\rho,t) - L\,\varepsilon(\rho,t) = \bar{\xi},$$

$$\varepsilon(\rho,0) = \varepsilon(0,t) = 0\,, \qquad \rho > 0, \quad t > 0 \tag{4.23}$$

where $\quad L \geq |f_H(\rho,t,\, H(\rho,t))| \quad$ and $\bar{\xi} \geq \xi(\rho,t)$.

We solve (4.22) and (4.23) by the method of successive approximation, taking

$$e(\rho,t) = e_0(\rho,t) + e_1(\rho,t) + \ldots.. + e_n(\rho,t) + \ldots\;,$$

$$\varepsilon(\rho,t) = \varepsilon_0(\rho,t) + \varepsilon_1(\rho,t) + \ldots.. + \varepsilon_n(\rho,t) + \ldots\;.$$

where

$$|e_0(\rho,t)| = \int_0^\rho \int_0^t |\xi(\rho',t')| d\rho' dt' ,$$

$$\varepsilon_0(\rho,t) = \int_0^\rho \int_0^t \bar{\xi}\, d\rho'\, dt' = \bar{\xi}\rho t ,$$

Thus

$$|e_0(\rho,t)| \leqq \varepsilon_0(\rho,t) = \bar{\xi}\rho t.$$

$$|e_1(\rho,t)| = \int_0^\rho \int_0^t \left| \frac{\partial f}{\partial H} (\rho',t',H(\rho',t')e_0(\rho',t') \right| d\rho' dt',$$

$$\varepsilon_1(\rho,t) = \int_0^\rho \int_0^t L\varepsilon_0(\rho',t')d\rho' dt' = \int_0^\rho \int_0^t L\bar{\xi}\rho' t' d\rho' dt'$$

$$= \frac{L\rho^2 t^2}{2^2} \bar{\xi} ,$$

Thus $|e_1(\rho,t)| \leqq \varepsilon_1(\rho,t)$ since $L \geqq \left| \frac{\partial f}{\partial H} (\rho,t,H(\rho,t) \right|$.

Assume that for some n $|e_{n-1}(\rho,t)| \leqq \varepsilon_{n-1}(\rho,t)$,

then

$$|e_n(\rho,t)| = \int_0^\rho \int_0^t \left| \frac{\partial f}{\partial H} (\rho',t',H(\rho',t')e_{n-1}(\rho',t')) \right| d\rho'\, dt' ,$$

and $\varepsilon_n(\rho,y) = \int_0^\rho \int_0^t L\varepsilon_{n-1}(\rho',t')d\rho' dt'$,

Thus $|e_n(\rho,t)| \leqq \varepsilon_n(\rho,t)$, since

$$L \geqq \left| \frac{\partial f}{\partial H} \rho,t,H(\rho,t) \right| .$$

Hence by induction $|e_n(\rho,t)| \leqq \varepsilon_n(\rho,t)$ for all n.

So $\quad |e(\rho,t)| \leqq \varepsilon(\rho,t), \qquad \rho > 0, \quad t > 0$

where

$$\varepsilon(\rho,t) = \rho t\,\overline{\xi} + \frac{L\rho^2 t^2 \overline{\xi}}{(2!)^2} + \frac{L^2\rho^3 t^3 \overline{\xi}}{(3!)^2} + \ldots$$

$$\varepsilon(\rho,t) = \left[\frac{I_0(2\sqrt{4\rho t}) - 1}{L}\right] . \overline{\xi} \quad .$$

Taking

$$\overline{\xi} < \frac{h^4}{2^5 . 3^3}\,[0.6A + L(2.5M + 12N + 24C)], \qquad \text{see equation (4.14).}$$

then

$$|e(\rho,t)| < \frac{h^4}{2^5 . 3^3}\,[\frac{0.6A}{L} + 2.5M + 12N + 24C] . [I_0(2\sqrt{L\rho t}) - 1], \qquad (4.24)$$

$$\rho > 0, \ t > 0.$$

Where

$$A = \sup_{ij}\left\{\left|\frac{\partial^4 f}{\partial\rho^4}(\xi,\eta,H)\right|, \left|\frac{\partial^4 f}{\partial t^4}(\xi,\eta,H)\right| \ : \ \xi = \rho_i + \frac{h}{2},\ \eta = t_j + \frac{h}{2}\right\},$$

$$M = \sup_{ij}\left\{\left|\frac{\partial^4 H}{\partial\rho^4}(\xi,\eta)\right|, \left|\frac{\partial^4 H}{\partial t^4}(\xi,\eta)\right| \ : \ \xi = \rho_i,\ \eta = t_j\right\},$$

$$N = \sup_{ij}\left\{\left|\frac{\partial^2 f}{\partial\rho^2}(\xi,\eta,H)\right|, \left|\frac{\partial^2 f}{\partial t^2}(\xi,\eta,H)\right| \ : \ \xi = \rho_i,\ \eta = t_j\right\},$$

$$C = \sup_{ij}\left\{\left|\frac{\partial^2 f}{\partial\rho\,\partial t}(\xi,\eta,H)\right| \ : \ \xi = \rho_i,\ \eta = t_j\right\},$$

$$L = \sup_{ij}\left\{\left|\frac{\partial f}{\partial H}(\xi,\eta,H)\right| \ : \ \xi = \rho_i,\ \eta = t_j\right\},$$

$$i = 1,2,\ldots,N-j, \qquad j = 1,2,\ldots,N-1.$$

The error in the Goursat problem

$$|H(\rho,t) - \tilde{\tilde{H}}(\rho,t)| = e(\rho,t) + O(h^6),$$

where $|e(\rho,t)|$ is bounded by equation (4.24).

In Chapter 8, we consider the usefulness of the error estimate (4.24) in practical computation.

A similar result for a Gaussian quadrature method is given by Day [17], using the techniques discussed by Walter [55] and Henrici [33].

We will consider the following linear elliptic boundary value problems.

Let D be a simply-connected domain containing the origin, of class Ah (Chapter 2, Section 3(b)).

i) Problem A

Problem A is to find the solution $u(x_1,x_2)$ of the equation

$$Lu = \Delta u + a(x_1,x_2)u_{x_1} + b(x_1,x_2)u_{x_2} + c(x_1,x_2)u = 0, \qquad \text{(I)}$$

where $\Delta \equiv \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, Laplace's operator,

$u_{x_1} \equiv \dfrac{\partial u}{\partial x_1}$, $u_{x_2} \equiv \dfrac{\partial u}{\partial x_2}$, and a, b and c are analytic functions of x_1, x_2 for $(x_1,x_2) \in D$, and $c(x_1,x_2) < 0$.

$u(x_1,x_2)$ is regular in D, continuous in $D \cup \partial D$, and satisfies the boundary condition

$$u(x_1,x_2) = f(x_1,x_2) \qquad (x_1,x_2) \in \partial D,$$

where f is a given real function continuous on the boundary ∂D.

ii) Problem B

Problem B is to find the solution $u(x_1,x_2)$ of the equation

$$Lu \equiv -\Delta u + P(r^2)u = 0, \qquad \text{(II)}$$

where $r^2 = x_1^{\ 2} + x_2^{\ 2}$, Δ = Laplace's operator, and P is an analytic function of r^2, with $P(r^2) > 0$. $u(x_1,x_2)$ is regular for $(x_1,x_2) \in D$, continuous in $D \cup \partial D$, and satisfies the boundary condition

$$u(x_1,x_2) = f(x_1,x_2), \qquad (x_1,x_2) \in \partial D ,$$

where f is a given real function continuous on ∂D.

iii) Problem C

Problem C is to find the solution $u(x_1,x_2)$ of the equation

$$Lu = -\Delta u + M(x_1)u = 0 \qquad \text{(III)}$$

where Δ = Laplace's operator, and M is an analytic function of x, with $M(x_1) > 0$ $u(x_1,x_2)$ is regular when $(x_1,x_2) \in D$, continuous in $D \cup \partial D$ and satisfies the boundary condition

$$u(x_1,x_2) = f(x_1,x_2) \qquad \text{for } (x_1,x_2) \in \partial D ,$$

where f is a given real function continuous on ∂D.

We will also consider the equation

$$Lu = -\Delta u + N(x_2)u = 0, \qquad \text{on } D,$$

Δ = Laplace's operator, N an analytic function of x_2 with $N(x_2) \geq 0$, for problem C.

iv) Problem D

Problem D is to find the solution $u(x_1,x_2)$ of the equation

$$Lu = -\Delta u + [M(x_1) + N(x_2)]u = 0, \qquad \text{(IV)}$$

where Δ is Laplace's operator, and M and N are analytic functions of x_1 and x_2 respectively, with $M(x_1) > 0$, $N(x_2) > 0$.

$u(x_1,x_2)$ is regular for $(x_1,x_2) \in D$, continuous in $D \cup \partial D$, and satisfies the boundary condition

$$u(x_1,x_2) = f(x_1,x_2) , \qquad (x_1,x_2) \in \partial D,$$

where f is a given real function, continuous on the boundary ∂D.

CHAPTER 5

THE METHOD OF PARTICULAR SOLUTIONS

1. Introduction

Consider the linear elliptic boundary value problem

$$\begin{aligned} Lu &= 0, && \text{on } D, \\ u &= f && \text{on } \partial D, \quad f \in C^{0}(\partial D) \end{aligned} \tag{5.01}$$

where D is of class Ah, L is a linear elliptic operator and f is continuous on the boundary of D.

Definition (Colton and Gilbert [15])

For equation (5.01) the family of particular solutions $\{\rho_n\}_{n=0}^{\infty}$ is complete, if given any solution, u, of the equation (5.01), regular in D, any closed subset D_0 of D, and any positive number ε, there exists an integer M and a system of constants $c_1, c_2, \ldots, c_M$ such that

$$\left|u - \sum_{k=1}^{M} c_k \rho_k\right| < \varepsilon \quad , \qquad \text{at all points in } D_0.$$

Assuming there exists a complete family of particular solutions $-\rho_k(x_1,x_2)$, $k = 1,2,\ldots$, which are formal solutions of $Lu = 0$, we select a finite linear combination

$$\tilde{u}(x_1,x_2) \equiv \tilde{u}(\alpha,x_1,x_2) = \sum_{j=1}^{M} \alpha_j \rho_j(x_1,x_2) \ ,$$

where $L\tilde{u} = 0$, on D.

We want to choose the constants α_j, $j = 1,2,\ldots,M$, so that u approximates f on the boundary ∂D.

Choose points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ on ∂D. Define

$$\eta_N = \max_{\partial D} \min_{i} |\underline{x} - \underline{x}_i| , \qquad x \in \partial D.$$

We assume $\eta_N \to 0$ as $N \to \infty$.

There are a number of ways in which the constants α_j, $j = 1,2,\ldots,M$, can be chosen.

i) Collocation

This is a very old and simple method of using particular solutions. Take M = N, so that there are equal numbers of boundary points and particular solutions.

Now determine the constants α_j, $j = 1,2,\ldots,N$ so that

$$\tilde{u}(\underline{x}_i) = f(\underline{x}_i) = \text{given boundary data, } i = 1,2,\ldots,N,$$

i.e. we interpolate the boundary data by means of a linear combination of particular solutions. However, as with other interpolation processes convergence is not assured when $N \to \infty$. Collocation using particular solutions has been used by Fox, Henrici and Moler [26] in the eigenvalue problem for elliptic equations.

If N >> M, requiring the constants α_j, $j = 1,2,\ldots,M$ to satisfy

$$f(\underline{x}_i) = \sum_{j=1}^{M} \alpha_j \rho_j(\underline{x}_i), \quad i = 1,2,\ldots,N, \tag{5.03}$$

leads to N linear equations in M unknowns. This system is over-determined.

We define the following norms:

$$||f||_2 = \sqrt{\Sigma w_i f^2(x_i)} \;, \qquad \text{the two norm,}$$

where the weights, w_i, are chords or means of chords.

$$||f||_\infty = \sup_{1\le i\le N} |f(x_i)| \;, \qquad \text{the infinity norm.}$$

ii) Least Squares

Take N >> M, and for the over-determined system, (5.03), determine the constants α_j, $j = 1,2,\ldots,M$ so that

$$||f(\underline{x}_i) - \sum_{j=1}^{M} \alpha_j \rho_j(\underline{x}_i)||_2 = \text{minimum.}$$

We will show that for this method convergence as $M \to \infty$ holds under quite general conditions.

Applications of the least squares method to boundary value problems was popularized by Davis and Rabinowitz [16].

iii) Linear Programming

Take N >> M and determine the constants α_j, $j = 1,2,\ldots,M$, so that

$$||f(\underline{x}_i) - \sum_{j=1}^{M} \alpha_j \rho_j(\underline{x}_i)||_\infty = \text{minimum.}$$

We will show that for this method convergence as $M \to \infty$ holds under quite general conditions.

Linear programming methods with approximation in the infinity (Chebyshev) norm, have become increasingly popular, as linear programming algorithms become more efficient. Canon [11], Schryer [48] and Rabinwitz [45].

Although linear programming algorithms have some advantages over least squares algorithms, (for example, by definition, the maximum error achieved at the boundary points using a linear programming algorithm will be less than that using a least squares algorithm, for the same number of boundary points), we choose to use a least squares algorithm, since the calculation is almost twice as fast.

The theoretical success of the method of particular solutions depends upon finding a complete system of particular solutions of the elliptic equations. The numerical success depends upon finding a complete system that is readily computed. For the elliptic equations we consider, we develop a complete system of particular solutions, using the general solutions of Bergman and Vekua. That the system is complete is confirmed by Vekua (see Theorem 4, Ch. 2), and that the system is readily computable, is due to the construction of Bergman's operator (see Sect. 4, Ch. 2).

2. The Least Squares Approximation

THEOREM 6. The existence of a best linear approximation (Handscomb, [31]).

Let f be a member of some normed vector space, a subspace of which is spanned by m given linearly-independent vectors, ϕ_i, $i = 1,\ldots,m$. There exists a set of coefficients $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_M)$, such that

$$\| f - \sum_{i=1}^{m} \alpha_i\phi_i \| \leq \| f - \sum_{i=1}^{m} \mu_i\phi_i \| ,$$

for every set $\{\mu_i\}$, so that $\Sigma\alpha_i\phi_i$ is a best approximation to f.

THEOREM 7. The uniqueness of the best approximation (Handscomb, [31]).

The best linear approximation with respect to a strict norm is unique.

Definition. Strict Norm. If a norm satisfies the condition that $\| f + g\| < \| f\| + \| g\|$ unless $j = 0$, or $g = 0$, or $f = \theta g$, for some θ. Then it is a strict norm.

The uniform norm and the 2-norm are both strict norms. Handscomb [31]. Thus the uniqueness of the best linear approximation for the 2-norm is assured.

We now consider the convergence of the sequence of approximations in the uniform, and the 2-norm.

a) Convergence in the uniform norm.

We use the notations

$$\| f\| = \max_{t \in D \cup \partial D} |f(\underline{x})| ,$$

$$\| f\|_{\partial D} = \max_{t \in \partial D} |f(\underline{x})| ,$$

where D and ∂D are the domain and boundary in Problem A. The theorems

8, 9, 10 and proofs are adapted from Schryer [47].

Consider solutions of the elliptic p.d.e. (I) in problem A. Let $\phi_k(\underline{x})$, k = 1,2,... be a system of functions which are formal solutions of Lu = 0, and let these functions form a complete set, that is given $0 < \varepsilon < 1$, we can find M and $\alpha_i^{(M)}$ such that

$$\| f - \sum_{i=1}^{M} \alpha_i^{(M)} \phi_i \|_{\partial D} \leqq \varepsilon \quad . \tag{5.04}$$

Writing $u_M = \sum_{i=1}^{M} \alpha_i^{(M)} \phi_i$, then since u = f on ∂D, (5.04) is the same as

$$\| u - u_M \|_{\partial D} \leq \varepsilon .$$

Let $\rho_M > 0$ be such that

$$\| \phi_i \|_{\partial D} \geqq \rho_M \quad , \qquad i = 1,2,\ldots,M.$$

Define the compact set

$$S_M = \{ \alpha \,\big|\, |\alpha_i| \leqq (\| f \|_{\partial D} + 1)/\rho_M \}.$$

THEOREM 8. For $0 < \varepsilon < 1$, there exists an integer M and $\alpha^{(M)} \in S_M$ such that

$$\| u - \sum_{i=1}^{M} \alpha_i^{(M)} \phi_i \|_{\partial D} \leqq \varepsilon \; .$$

Proof. From the completeness of the ϕ_i, i = 1,2,... there exists an M and $\alpha^{(M)}$ such that

$$\| u - \sum_{i=1}^{M} \alpha_i^{(M)} \phi_i \| \leqq \varepsilon .$$

We need to show that $\alpha^{(M)} \in S_M$;

$$\| u_M \|_{\partial D} \leqq \| f \|_{\partial D} + 1, \qquad \text{see (5.04)}$$

$$\left\| \sum_{i=1}^{M} \alpha_i^{(M)} \phi_i \right\|_{\partial D} \geqq |\alpha_i^{(M)}| \rho_M ,$$

thus

$$|\alpha_i^{(M)}| \leqq (\| f \|_{\partial D} + 1)/\rho_M,$$

and $\qquad \alpha^{(M)} \in S_M.$

For any integer N we choose points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ on the boundary ∂D.

<u>Define</u> $\qquad \eta_N = \underset{\underline{x} \in \partial D}{\text{Max}} \; \underset{i}{\text{Min}} \; |\underline{x} - \underline{x}_i| \; .$

We impose the condition that $\eta_N \to 0$, as $N \to \infty$.

<u>Define</u> $\quad$ The continuous function E_{MN}, for $\alpha \in S_M$ by

$$E_{MN}(\alpha) = \max_{1 \leqq j \leq N} \left| f(\underline{x}_j) - \sum_{i=1}^{M} \alpha_i \phi_i(\underline{x}_j) \right| ,$$

and set $e_{MN} = \inf_{S_M} E_{MN}(\alpha)$.

Since E_{MN} is continuous on the compact set S_M, there is an $\alpha^{(MN)} \in S_M$ so that

$$e_{MN} = E_{MN}(\alpha^{(MN)}) \; .$$

<u>Define</u> $\qquad U_{MN}$ by

$$U_{MN}(\underline{x}_j) = \sum_{i=1}^{M} \alpha_i^{(MN)} \phi(\underline{x}_j) , \qquad j = 1,2,\ldots,N.$$

This is the best uniform approximation to u at the points $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$, using M linearly independent approximating functions.

<u>Define</u>
$$w(f,\eta) = \sup_{\substack{\underline{x}_1,\underline{x}_2 \in \partial D \\ |\underline{x}_1-\underline{x}_2| \leq \eta}} |f(\underline{x}_1) - f(\underline{x}_2)|,$$

the modulus of continuity of f on ∂D.

<u>THEOREM 9</u>

$$\| u - u_{MN} \| \leqq w(f,\eta_N) + e_{MN} + w(u_{MN},\eta_N).$$

<u>Proof</u>. By the maximum principle

$$\| u - u_{MN} \| \leqq \| u - u_{MN} \|_{\partial D} .$$

Let $\underline{x}$ be between $\underline{x}_j$ and $\underline{x}_{j+1}$ on ∂D. Then

$$|u(\underline{x}) - u_{MN}(\underline{x})| \leqq |u(\underline{x}) - u(\underline{x}_j)| + |u(\underline{x}_j) - u_{MN}(\underline{x}_j)| + |u_{MN}(\underline{x}_j) - U_{MN}(\underline{x})|$$
$$\leqq w(f,\eta_N) + e_{MN} + w(u_{MN},\eta_N).$$

For fixed M and varying N, α^{MN} are bounded, since they are in the compact set S_M. This means that the U_{MN} are equicontinuous on $D \cup \partial D$ for varying N and fixed M. Thus for fixed M as $\eta_N \to 0$ we have $w(u_{MN},\eta_N) \to 0$ and also $w(f,\eta_N) \to 0$.

<u>THEOREM 10</u>. For any $0 < \varepsilon < 1$ there exists $M(\varepsilon)$ and $N(\varepsilon)$ so that $N \geqq N(\varepsilon)$ implies

$$\| u - u_{M(\varepsilon)N} \| \leqq \varepsilon .$$

Proof

$$E_{MN}(\alpha) = \max_{1\leqq j\leq N} \left| f(\underline{x}_j) - \sum_{i=1}^{M} \alpha_i \phi_i(\underline{x}_j) \right|,$$

$$\leqq \left\| f(\underline{x}) - \sum_{i=1}^{M} \alpha_i \phi_i(\underline{x}) \right\|_{\partial D}$$

Thus

$e_{MN} \leqq \min_{S_N} \left\| f - \sum_{i=1}^{M} \alpha_i \phi_i \right\|$, and hence by theorem 7 there is an integer $M(\varepsilon)$ so that $e_{M(e)N} \leqq \varepsilon/3$ for all N.

Also since for fixed M as $\eta_N \to 0$, $w(u_{MN}, \eta_N) \to 0$, and $w(f, \eta_N) \to 0$, we may choose an integer $N(\varepsilon)$ so that $N \geqq N(\varepsilon)$ implies

$$w(f,\eta_N) \leqq \varepsilon/3 \quad \text{and} \quad w(u_{M(\varepsilon)N}, \eta_N) \leqq \varepsilon/3 .$$

Then by theorem 8 we see that for $N \geqq N(\varepsilon)$ we have

$$\| u - u_{M(\varepsilon)N} \| \leqq \varepsilon.$$

This shows that the sequence of approximations, u_{MN}, is uniformly convergent with respect to points on the boundary.

b) Convergence in the 2-norm:

If $\underline{x} \in \partial D$ then

$$\| f \|^2 = \int_{\partial D} (f(\underline{x}))^2 \, ds,$$

where s represents arc length.

Recall for $\underline{x}_i \in \partial D$, $i = 1,2,\ldots,N$

$$\| f \|_2^2 = \sum_{i=1}^{N} w_i \, f(\underline{x}_i)^2, \qquad \text{where } w_i > 0.$$

As $N \to \infty$ $\quad \| f \|_2^2 \to \| f \|^2$.

The weights w_i are chords or means of chords.

Consider solutions of the elliptic p.d.e. (I) in problem A. Let $\phi_k(\underline{x})$, $k = 1,2,\ldots$, be a system of functions which are formal solutions of $Lu = 0$, and let these functions form a complete set. That is given $0 < \varepsilon < 1$, we can find $M = M(\varepsilon)$, and $\alpha^M = (\alpha_1{}^M, \alpha_2{}^M, \ldots, \alpha_M{}^M)$, such that

$$\| f - \sum_{i=1}^{M} \alpha_i{}^M \phi_i \| \leqq \varepsilon.$$

THEOREM 11. The method of least squares converges. Mikhlin [40].

Proof. For any integer N we choose $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ on the boundary ∂D.

Define the continuous function

$$E_{MN}(\alpha) = \| f - \sum_{i=1}^{M} \alpha_i \phi_i \|_2 \quad , \qquad N >> M.$$

We could choose $N = 3M$, that is that $N = N(M)$ for example. ϕ_i, $i = 1,2,\ldots.$ is a complete system of linear independent functions which satisfy the elliptic equation (I).

Consider the continuous function

$$e_M(\alpha) = \| f - \sum_{i=1}^{M} \alpha_i \phi_i \| ,$$

as $N \to \infty$ $\quad E_{MN} \to e_M$.

Thus for any ε, $0 < \varepsilon < 1$, there exists $M(\varepsilon)$, and consequently an N, such that for $M > M(\varepsilon)$

$$| e_M(\alpha) - E_{MN}(\alpha) | \leqq \varepsilon/2.$$

Thus $E_{MN}(\alpha) \leqq \varepsilon/2 + e_M(\alpha)$, $\quad M > M(\varepsilon)$. $\qquad$ (5.06)

But from the completeness of the ϕ_i, $i = 1,2,\ldots$, for any $0 < \varepsilon/2 < 1$, there exists an M_1, and an α^{M_1} such that

$$e_{M_1}(\alpha_i^{M_1}) \leq \varepsilon/2.$$

Without loss of generality we can assume $M_1 > M(\varepsilon)$, since if $M_1 \leq M(\varepsilon)$, we can take for $i \geq M_1$, $\alpha_i^{M_1} = 0$.

Thus from equation (5.06)

$$E_{M_1 N}(\alpha_i^{M_1}) \leq \varepsilon . \tag{5.07}$$

However from Theorem 5 the best approximation in the 2-norm exists, thus there exists an $(a_i^{M_1})$ such that

$$E_{M_1 N}(a_i^{M_1}) = \text{Min},$$

thus from equation (5.07)

$$E_{M_1 N}(a_i^{M_1}) \leq \varepsilon .$$

It follows that the method of least squares converges in the mean on the boundary ∂D, that is that

$$\sum_{i=1}^{M} a_i^{(M)} \phi_i(t_j) \to f(t_j), \qquad j = 1,2,\ldots,N, \quad \text{as } M \to \infty.$$

However, consider

$$\left\| f - \sum_{i=1}^{M} \alpha_i \phi_i \right\|^2 = \int_{\partial D} [f(t) - \sum_{i=1}^{M} \alpha_i \phi_i(t)]^2 \,.ds.$$

The condition that

$$\left\| f - \sum_{i=1}^{M} \alpha_i \phi_i \right\| \leq \varepsilon,$$

does not ensure that

$$\left|f(t) - \sum_{i=1}^{M} \alpha_i \phi_i(t)\right| \le \varepsilon, \qquad \text{for all } t \in \partial D.$$

In practice a search of the boundary ∂D is performed to find

$$E_{max} = \max_{\partial D} \left|f(t) - \sum_{i=1}^{M} \alpha_i \phi_i(t)\right| ,$$

after the calculation of the α_i, $i = 1,2,\ldots,M$, by approximation in the 2-norm.

By the maximum principle the error on the interior can then be bounded by E_{max}.

With a given number N of boundary points and a given number M of particular solutions, we would expect that if the coefficients are calculated by minimizing the uniform norm, then from Chebyshev's theorem the maximum error will be attained at points all around the boundary. For approximation in the 2-norm, the maximum error on the boundary will necessarily be larger than the maximum error for uniform approximation. However, we would expect the maximum error of the least squares approximation to occur at only a few points around the boundary, while at the majority of points on the boundary, the error due to the least squares approximation to be smaller.

3. Construction of the Particular Solutions

a) For problem B, $(-\Delta u + P(r^2)u = 0)$.

In chapter 3 the following integral representation of the general form of the solutions of the equation $-\Delta u + P(r^2)u = 0$, was obtained:

$$u(x_1,x_2) = h(x_1,x_2) + \int_0^1 \frac{H(\tau,1-\tau)}{\tau} h(x_1\tau,x_2\tau)d\tau . \tag{5.08}$$

We can rewrite (5.08)

$$u = (I + H)h, \qquad \underline{x} = (x_1,x_2) \tag{5.09}$$

where I is the identity operator and

$$(Hh)(\underline{x}) = \int_0^1 \frac{H(\tau,1-\tau)}{\tau} h(\underline{x}\tau)\, d\tau .$$

$H(\tau,\eta)$ is the solution of a Goursat problem and $h(x_1,x_2) = h(\underline{x})$ is a harmonic function.

In order to construct particular solutions we take

$$h_k(x_1,x_2) = \left\{\begin{matrix} RE \\ IM \end{matrix}\right\} z^k \quad , \qquad z = x_1 + ix_2, \quad k = 0,1,2,...$$

and from equation (5.09)

$$u_i(x_1,x_2) = ((I+H)h_k)\,(x_1,x_2), \quad i = 1,2,...,$$
$$k = \text{integer part of } i/2.$$

b) For Problem C, $(-\Delta u + M(x_1)u = 0)$ and problem D, $(-\Delta u + [M(x_1)+N(x_2)]u=0)$.

In chapter 3 the following integral representation of the general form of the solution of equation $-\Delta u + M(x_1)u = 0$ was obtained:

$$u(x_1,x_2) = h(x_1,x_2) - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))h(x_1 t,x_2)dt, \quad (5.10)$$

where $F(\tau,\eta)$ is the solution of a Goursat problem and $h(x_1,x_2)$ is a harmonic function.

For the equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$, the corresponding representation was obtained:

$$\begin{aligned} u(x_1,x_2) = h(x_1,x_2) &- x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_2(\tfrac{1-t}{2}))h(x_1 t,x_2)dt \\ &- x_2 \int_{-1}^{1} E(x_2(\tfrac{1+t}{2}),\ x_2(\tfrac{1-t}{2}))h(x_1,x_2 t)dt \\ &+ x_1 x_2 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2})) \int_{-1}^{1} E(x_2(\tfrac{1+\tau}{2}),\ x_2(\tfrac{1-\tau}{2}))h(x_1 t,x_2\tau)d\tau dt, \end{aligned} \quad (5.11)$$

where $F(\tau,\eta)$, $E(\tau,\eta)$ are solutions of Goursat problems, and $h(x_1,x_2)$ is a harmonic function.

As in section a) take $h_k(x_1,x_2) = {}^{RE}_{IM}\{z^k\}$, $z = x_1+ix_2$. Then for equation $-\Delta u + M(x_1)u = 0$, the particular solutions are given by

$$u_i = (I - F)h_k \ , \qquad i = 1,2,\ldots \qquad (5.12)$$
$$k = \text{integer part of } i/2.$$

and for the equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$ the particular solutions are given by

$$u = (I - F)(I - E)h \ , \qquad i = 1,2,\ldots \ . \qquad (5.13)$$
$$k = \text{integer part of } i/2.$$

where I is the identity operator and the operators F and E are defined by

$$(Fh)(x_1,x_2) = x_1 \int_{-1}^{1} F\left(x_1\left(\frac{1+t}{2}\right),\ x_1\left(\frac{1-t}{2}\right)\right)h(x_1t,x_2)dt \ .$$

$$(Eh)(x_1,x_2) = x_2 \int_{-1}^{1} E\left(x_2\left(\frac{1+t}{2}\right),\ x_2\left(\frac{1-t}{2}\right)\right)h(x_1,x_2t)dt \ . \tag{5.14}$$

4. Completeness of the Sets of Particular Solutions

The particular solutions were all constructed using the integral operators of Bergman and Vekua. From theorem 4, Chapter 2, we see that the particular solutions constructed in this way are complete in the closure of the domain of the problems B, C and D respectively.

CHAPTER 6

INTEGRAL EQUATION FORMULATION VIA THE DOUBLE-LAYER POTENTIAL

1. Introduction

Consider the following equation

$$\lambda\phi(s) - \int_a^b k(s,t)\phi(t)dt = f(s), \qquad a \leq t \leq b, \quad a \leq s \leq b, \qquad (6.01)$$

where ϕ is an unknown function, while k and f are given functions and λ is constant. Such an equation is called an integral equation, since the unknown function appears under the integral sign. The function f is called "the right hand side", the function k is called the kernel, and the numerical coefficient λ is called the parameter of the equation. It will be assumed that a and b are finite constants. The parameter λ and the functions ϕ, k and f can be taken as real or complex quantities.

We consider the following types of integral equations. When $k(s,t)$ is continuous in the region $a \leq s \leq b$, $a \leq t \leq b$, or if the discontinuities of the kernel are such that the double integral

$$\int_a^b \int_a^b |k^2(s,t)|ds\, dt, \quad \text{has a finite value,}$$

the equation (6.01) is called an equation of Fredholm type.

When the kernel has the form

$$k(s,t) = \frac{h(s,t)}{|s-t|^{\alpha}} ,$$

where $h(s,t)$ is bounded and α is constant, satisfying the inequality $0 < \alpha < 1$ then the equation (6.01) is called an equation with a weak singularity.

Solutions of equations like (6.01) are not in general known in closed form, so they have to be approximated. However, approximate methods can only be applied with confidence when the solubility of the equation has been established beforehand.

For an equation (6.01) of Fredholm type or with a weak singularity, the solubility can be established by what is known as Fredholm's alternative:

Either the in-homogeneous equation ($f(s) \neq 0$) is soluble whatever the right hand side may be, or the corresponding homogeneous equation ($f(s) = 0$) has a non-trivial solution. (6.02)

Fredholm's alternative follows directly from Fredholm's 4 theorems on integral equations (Mikhlin [41]), and is very often used in the analysis of integral equations.

2. The solution of the Dirichlet problem for Laplace's equation

Let C be a simply-connected domain with contour ∂C, assumed to be smooth with continuous curvature. The Dirichlet problem is to find a function $u(x_1,x_2)$ harmonic in C, and satisfying the boundary condition $u = f(t)$ on ∂C, where t is the complex coordinate of a point on ∂C.

The function $u(x_1,x_2)$ can be regarded as the real part of a certain analytic function $\phi(z)$, $z = x_1+ix_2$, which is holomorphic in the domain C.

We will try to find $\phi(z)$ in the form of a Cauchy type integral

$$\phi(z) = \frac{1}{i} \int_{\partial C} \frac{\mu(t)}{t - z}\, dt, \qquad z \in C \tag{6.03}$$

where μ is an unknown real function satisfying a Hölder condition. The problem reduces to the determination of μ.

Let z in equation (6.03) tend from the inside to a point t_0 on the boundary. Using the Plemelj formula (Muskhelishvili [43])

$$\phi(t_0) = \pi\mu(t_0) + \frac{1}{i} \int_{\partial C} \frac{\mu(t)}{t-t_0}\, dt , \qquad t_0 \in \partial C, \tag{6.04}$$

provided that the integral is interpreted as a Cauchy principal value.

Taking the real part of equation (6.04)

$$\pi\mu(t_0) + \int_{\partial C} \mu(t)\, \mathrm{IM}\{\frac{dt}{t-t_0}\} = f(t_0) , \tag{6.05}$$

since $\mu(t)$ is real and $f(t_0) = \mathrm{RE}(\phi t_0))$.

Let $re^{i\theta} = t-t_0$

$$\mathrm{IM}\,\{\frac{dt}{t-t_0}\} = \mathrm{IM}\{d\,\ln(t-t_0)\} = d\theta = \frac{\partial\theta}{\partial s}\, ds,$$

where ds is the element of arc of the contour.

By the Cauchy-Riemann equations

$$\frac{\partial\theta}{\partial s} = -\frac{\partial \ln r}{\partial n} = -\frac{1}{r}\frac{\partial r}{\partial n},$$

$-\frac{\partial \ln r}{n}$ is known in two dimensional potential theory as the potential of a unit doublet where n is the inward normal to ∂C at t_0 and r is the radius vector from t to t_0.

Now $\frac{\partial r}{\partial n} = \cos(r,n)$, where r,n denotes the angle between the radius vector and the inward normal and.hence

$$\text{IM}\{\frac{dt}{t-t_0}\} = -\frac{\cos(r,n)}{r}\,ds.$$

Thus (6.04) becomes

$$\pi\mu(t_0) - \int_{\partial C} \mu(t)\,\frac{\cos(r,n)}{r}\,ds = f(t_0)\,. \tag{6.06}$$

This is a Fredholm integral equation of the second kind under the assumption of continuous curvature of the boundary ∂C, the kernal $r^{-1}\cos(r,n)$ is continuous.

In accordance with Fredholm's alternative (6.02), equation (6.06) is soluble, and has a unique solution, whatever the right hand side may be, provided the corresponding homogeneous equation has a trivial solution.

Let $f(t_0) \equiv 0$. Equation (6.06) then becomes homogeneous. Let $\mu_0(t_0)$ be any solution so that

$$\pi\mu_0(t_0) - \int_{\partial C} \mu_0(t)\,\frac{\cos(r,n)}{r}\,ds = 0 \tag{6.07}$$

Put

$$\phi_0(z) = \frac{1}{i}\int_{\partial C}\frac{\mu_0(t)}{t-z}\,dt, \tag{6.08}$$

where z is a point in C. The condition $f(t_0) \equiv 0$ shows that $RE\{\phi_0(t_0)\} = 0$, if t_0 is a point on ∂C. A harmonic function which is not constant attains its maximum and minimum values on the boundary of a domain and this implies that $RE\{\phi_0(z)\} = 0$ in the closure of the domain $C \cup \partial C$. It follows that $\phi_0(z) = iA$, where A is a real constant. Equation (6.08) can then be written

$$\frac{1}{i} \int_{\partial C} \frac{\mu_0(t) - iA}{t - z} dt = 0,$$

which is true for every point z in C. But then, by Cauchy's theorem, $\mu_0(t) - iA$ is the limiting value on ∂C of a certain function $\psi(z)$, regular outside ∂C and equal to zero at infinity. The imaginary part of this function is equal to A on ∂C, but this implies that $\psi(z)$ = constant. However, $\psi(z) = 0$ at infinity, thus $\psi(z) \equiv 0$. The function $\mu_0(t_0)$ is the value of $RE\{\psi(z)\}$ on ∂C, and so $\mu_0(t_0) \equiv 0$. Thus the only solution of the inhomogeneous equation (6.07) is the trivial solution.

Note: It can be shown by a different method (J. Radon, [46]) that the solution of the Dirichlet problem exists and is unique under much more general assumptions with regard to the boundary of the region. However, the solution can no longer be found in the form of a double-layer potential.

3. The Dirichlet problem for second order elliptic boundary value problems

Consider problem A and let C denote the simply connected domain, such that $z = x_1+ix_2 \in C$. Every solution can be written as (Ch. 2, and Vekua, pp.123, [5 3]),

$$u(x_1,x_2) = \mathrm{Re}[H_0(z)\phi(z) + \int_0^z H(z,t)\phi(t)dt] \ , \tag{6.09}$$

$$z = x_1+ix_2, \quad H_0(z) = G(z,0,z,\bar{z}), \quad H(z,t) = -\frac{\partial}{\partial t} G(t,0,z,\bar{z}) \ ,$$

where $G(t,\tau,z,\bar{z})$ is the complex Riemann function. $\phi(z)$ is an arbitrary holomorphic function in C, which may be assumed without loss of generality to be subject to the condition $\phi(0) = \overline{\phi(0)}$ (Vekua, pp.123, [53]).

$H_0(z)$ is an analytic function in C and by the definition of the Riemann function $G(t,\tau,z,\bar{z})$, $H_0(z) \neq 0$ anywhere in C. $H(z,t)$ is a holomorphic function of t for z on the boundary ∂C, and an analytic function of z in C.

By taking (Vekua, pp.130, [53])

$$\phi(z) = \frac{1}{i} \int_{\partial C} \frac{\mu(t)}{H_0(t)(t-z)} \, dt,$$

where $\mu(t)$ is a real function continuous in the Hölder sense on ∂C, and then substituting in equation (6.09) we obtain

$$u(x_1,x_2) = \int_{\partial C} M(z,t)\mu(t)t' \, ds, \qquad z \in C, \ t \in \partial C, \tag{6.10}$$

where $t' = \frac{dt}{ds}$, a continuous function, and

$$M(z,t) = \mathrm{RE}[\frac{H_0(z)}{iH_0(t)(t-z)} + \frac{1}{iH_0(t)} \int_0^z \frac{H(z,t_1)}{t-t_1} dt_1] \ .$$

It follows that

$$M(z,t) = \mathrm{RE}\left[\frac{H_0(z)}{iH_0(t)(t-z)} + \frac{H(z,t)}{iH_0(t)} \ln(1 - \frac{z}{t}) + \frac{1}{iH_0(t)} H^*(z,t)\right],$$

where

$$H^*(z,t) = \int_0^z \frac{H(z,t_1) - H(z,t)}{t-t_1} dt_1 .$$

Clearly $H^*(z,t)$ is an analytic function of z, and is also holomorphic as a function of t on ∂C; $\ln(1 - \frac{z}{t})$ is taken to mean its principal value

Taking the limit as z tends from the inside of C to t_0, where t_0 is the complex coordinate of a point on the boundary ∂C, and using the Plemelj formula,

$$\pi\mu(t_0) + \int_{\partial D} M(t_0,t)\mu(t)t'ds = f(t_0) , \tag{6.11}$$

where

$$M(t_0,t) = \mathrm{RE}\left[\frac{H_0(t_0)}{iH_0(t)(t-t_0)} + \frac{H(t_0,t)}{iH_0(t)} \ln(1 - \frac{t_0}{t}) + \frac{H^*(t_0,t)}{iH_0(t)}\right] .$$

Thus $M(t_0,t)$ has a logarithmic singularity, and equation (6.11) is a weakly singular integral equation. Thus in accordance with Fredholm's alternative, a solution to equation (6.11) exists and is unique if the corresponding homogeneous equation has only the unique solution. This can be proved in an analogous way to that used in the proof for the Dirichlet problem for Laplace's equation in the last section (Vekua, p.126, [53]).

4. Boundary value problems: B, $(-\Delta u + P(r^2)u = 0)$;

C, $(-\Delta u + M(x_1)u = 0)$; D, $(-\Delta u + (M(x_1) + N(x_2))u = 0)$.

Problems B, C and D are all special cases of problem A. For these special cases $H_0(z) = 1$, and so the holomorphic function $\phi(z)$ defined in section 3, reduces to

$$\phi(z) = \frac{1}{i} \int_{\partial D} \frac{\mu(t)}{t-z}\, dt,$$

where $\mu(t)$ is a real continuous function. $RE(\phi(z))$, with $\phi(z)$ defined as above is the potential

$$RE\ \phi(z) = \int_{\partial D} \frac{\partial}{\partial n} \ln r\ \mu\, ds, \qquad r = |t-z|, \qquad \text{(Muskelishvili, pp.23, [43])}, \tag{6.12}$$

where n is the inward normal at t.

Also for these special cases $H(z,t)$ is real and thus equation (6.09) becomes

$$u(x_1,x_2) = \int_{\partial D} \frac{\partial}{\partial n} \ln r\ \mu\, ds + \int_0^z H(z,t) \{ \int_{\partial D} \frac{\partial}{\partial n} \ln(r)\ \mu\, ds\} dt,$$

and equation (6.10) becomes

$$u(x_1,x_2) = \int_{\partial D} M(z,t)\mu(t)t'ds, \qquad z \in D,\ t \in \partial D,\ t' = \frac{dt}{ds}, \tag{6.13}$$

$$M(z,t) = \frac{\partial}{\partial n} \ln r + \int_0^z H(z,t_1) \frac{\partial}{\partial n} \ln|t_1-t|\, dt_1.$$

Let z tend from the inside of C to a point on the boundary ∂C. Then it follows directly from the general case that

$$\pi\mu(t_0) + \int_{\partial C} \mu(t)M(t_0,t)t'ds \;=\; f(t_0), \quad t_1,t_0 \in \partial C, \tag{6.14}$$

and that this equation has a unique solution.

By integrating by parts it is possible to show that $M(t_0,t)$ contains a logarithmic singularity, and so as in the general case will be weakly singular when $t_0 \in \partial C$.

Now consider the simply connected domain D with a smooth boundary ∂D, and let $\underline{x} = (x_1,x_2) \in D$. Then we rewrite equations (6.13)

$$u(\underline{x}) \;=\; \int_{\partial D} M(\underline{x},\underline{y})\mu(\underline{y})d\underline{y}, \quad \underline{x} \in D,\ \underline{y} \in \partial D. \tag{6.15}$$

where

$$M(\underline{x},\underline{y}) \;=\; \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| + \int_0^x H(\underline{x},t)\,\frac{\partial}{\partial n} \ln|t-\underline{y}|dt.$$

Let $\underline{x}$ tend to a point $\underline{x}_0 \in \partial D$. Then

$$\pi\mu(\underline{x}_0) + \int_{\partial D} \mu(\underline{y})M(\underline{x}_0,\underline{y})d\underline{y} \;=\; f(\underline{x}_0), \quad \underline{x}_0,\ \underline{y} \in \partial D. \tag{6.16}$$

We now find $M(\underline{x}_0,\underline{y})$ in the particular problems B, C and D.

a) For problem B $(-\Delta u + P(r^2)u = 0)$;

$$M(\underline{x}_0,\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}_0-\underline{y}| + \int_0^1 \frac{H(\tau,1-\tau)}{\tau}\,\frac{\partial}{\partial n} \ln|\underline{x}_0\tau-\underline{y}|d\tau.$$

b) For problem C $(-\Delta u + M(x_1)u = 0)$,

$$M(\underline{x}_0,\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}_0-\underline{y}| - \int_{-1}^{1} x_1F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))\frac{\partial}{\partial n} \ln|(x,t,x_2) - \underline{y}|dt,$$

$$\underline{x}_0 = (x_1,x_2).$$

For equation $-\Delta u + N(x_2)u = 0$,

$$M(\underline{x}_0,\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}_0-\underline{y}| - \int_{-1}^{1} x_2E(x_2(\frac{1+t}{2}),\ x_2(\frac{1-t}{2}))\ \frac{\partial}{\partial n} \ln|(x_1,x_2t) - \underline{y}|dt,$$

$$\underline{x}_0 = (x_1,x_2).$$

c) For problem D, $[-\Delta u + [M(x_1) + N(x_2)] = 0]$

$$M(\underline{x}_0,\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}_0-\underline{y}| - \int_{-1}^{1} x_1F(x_1(\frac{1+t}{2}),x_1(\frac{1-t}{2}))\frac{\partial}{\partial n} \ln|(x_1t,x_2) + \underline{y}|dt$$

$$- \int_{-1}^{1} x_2E(x_2(\frac{1+t}{2}),\ x_2(\frac{1-t}{2}))\frac{\partial}{\partial n} \ln|(x_1,x_2t) - \underline{y}|dt$$

$$+ \int_{-1}^{1} x_1F(x_1(\frac{1+t}{2}),\ x_1(\frac{1-t}{2})) \int_{-1}^{1} x_2E(x_2(\frac{1+\tau}{2}),\ x_2(\frac{1-\tau}{2}))\frac{\partial}{\partial n} \ln|(x_1t,x_2\tau)$$

$$- \underline{y}|d\tau\ dt, \quad \underline{x}_0 = (x_1,x_2).$$

CHAPTER 7

NUMERICAL APPROXIMATION OF THE INTEGRAL EQUATION FORMULATION

1. Introduction

Consider the boundary value problems, B, C and D. In order to find numerical solutions to these problems, the density of the double layer potential, μ, has to be approximated by solving numerically the following integral equation (see (6.16))

$$\pi\mu(\underline{x}) + \int_{\partial D} \mu(\underline{y})M(\underline{x},\underline{y})dy = f(\underline{x}), \qquad \underline{y},\underline{x} \in \partial D. \tag{7.01}$$

Having approximated μ, the solution $u(\underline{x})$, where $\underline{x}$ is a point of the domain, D, is found by substituting for μ, in the following equation

$$u(\underline{x}) = \int_{\partial D} \mu(\underline{y})M(\underline{x},\underline{y})dy, \qquad \underline{x} \in D,\ \underline{y} \in \partial D. \tag{7.02}$$

$M(\underline{x},\underline{y})$ represents the kernel associated with the problem, B, C or D respectively.

Since the procedure involves integrals around the boundary, ∂D, this method of solving boundary value problems is referred to as the boundary integral method (B.I.M.).

Let the curve ∂D have a sufficiently continuously differentiable parameterization, $\underline{r}(s) = (\xi(s), \eta(s))$, $0 \leq s \leq A$, where s need not refer to arc length. Then equations (7.01), (7.02) become

$$\pi\mu(s) + \int_0^A \mu(t)M(s,t)dt = f(s), \qquad 0 \leqq s \leqq A, \tag{7.03}$$

$$u(\underline{\rho}) = \int_0^A \mu(t)M(\underline{\rho},t)dt, \qquad \underline{\rho} = (x,y) \in D. \tag{7.04}$$

Observe that in (7.03), (7.04) the notation has changed in an obvious way.

There are many numerical methods for solving equation (7.03) (Atkinson [3]); when $M(s,t)$ is continuous perhaps the best known and most widely used is the Nyström method.

In the Nyström method the integral in (7.03) is approximated by a numerical quadrature scheme

$$\int_0^A \mu(t)M(s,t)dt = \sum_{j=1}^{n} W_j M(s,s_j)\mu(s_j) \ ,$$

the scheme must converge for all continuous functions on [0,A]. The weights of the quadrature and the nodes, s_j, will depend on n, the number of nodes chosen. Now by letting $s = s_1, s_2, \ldots, s_n$, the following linear system is defined for equation (7.03)

$$\pi\mu(s_i) + \sum_{j=1}^{n} W_j M(s_i,s_j)\mu(s_j) = f(s_i) \ ,$$

n equations for n unknowns. If these equations can be solved, the solution in the domain, D, is then approximated by

$$u(\underline{\rho}) = \sum_{j=1}^{n} W_j M(\underline{\rho}, s_j)\mu(s_j), \qquad \underline{\rho} = (x,y) \in D.$$

Complete error analyses of the Nyström method have been given (K. Atkinson [3]).

In the boundary integral element (B.I.E.) method, the boundary D is split into N arcs ∂D_j, $j = 1,\ldots,N$ and

$$\int_{\partial D} \mu(\underline{y})M(\underline{x},\underline{y})dy = \sum_{j=1}^{N} \int_{\partial D_j} \mu(\underline{y})M(\underline{x},\underline{y})dy.$$

$\mu(\underline{y})$ is chosen to be constant (classical B.I.E. method) quadratic or cubic etc. (improved B.I.E. method), on each of the boundary arcs ∂D_j. This leaves an integral independent of μ which is calculated analytically if possible or using some suitable numerical quadrature scheme. Equation (7.01) becomes

$$\pi\mu(\underline{x}) + \sum_{j=1}^{N} \int_{\partial D_j} \mu(\underline{y})M(\underline{x},\underline{y})dy = f(\underline{x}) \tag{7.05}$$

Let $\underline{x}_i$, $\underline{x}_{i+1}$, be the end points of the boundary arc ∂D_i, $\underline{x}_{N+1} = \underline{x}_1$. Then by taking $\underline{x} = \underline{x}_i$, $i = 1,2,\ldots,N$, in equation (7.05), the approximate density of the double-layer potential, μ, can be found at each of the N boundary nodes.

The B.I.E. method has been successfully used on a variety of problems, and is not confined to double-layer potential formulations, or equations of the second kind (Fairweather et al. [24]).

When the kernel $M(\underline{x},\underline{y})$ contains a singularity the convergence of the methods described above is affected adversely, unless special techniques are employed. A popular technique is to model the singularity with a function that can be integrated. Another technique is to deal with the singular part of the kernel directly by using product-integration techniques, (Atkinson [1]).

For certain kernels Atkinson is able to develop a complete error analysis for the numerical solution using product-integration techniques, of integral equations of the second kind with compact operators.

The B.I.E. method uses product-integration techniques and it should be possible to develop an error analysis for the numerical solution of integral equations of the second kind with compact operators, for particular kernels.

Kantorovich and Krylov [35] give a method of cancelling the singularity. Assume M(s,t), the kernel, is singular only if s = t. Then use

$$\int_0^A \mu(t)M(s,t)dt = \mu(s) \int_0^A M(s,t)dt + \int_0^A M(s,t)(\mu(t)-\mu(s))dt.$$

Kussmaul and Werner [38] give error estimates for the approximate solution of Fredholm equations using this technique, for weakly singular kernels of the form

$$M(s,t) = a(s,t) \ln|s-t| + b(s,t)$$

If a(s,t) and b(s,t) are 4 times differentiable the error is shown to be $O(h^3)$.

Atkinson [29], [2] in his numerical approximation of problem B, notes that the kernel, M(s,t), has a logarithmic singularity of the form

$$a(s,t)\ln|\underline{r}(s) - \underline{r}(t)| + b(s,t) \ . \tag{7.06}$$

where a and b are well behaved functions. However he is unable to find a and b explicitly.

If it were possible to express the kernel as in equation (7.06), where a and b are well behaved functions, then both the method of cancellation and Atkinson's generalized Simpson's rule (product integration formula) would be excellent methods of approximating the boundary integral. If a and b were sufficiently continuous functions a comprehensive error analysis for the numerical solution of the boundary value problem would be available.

The problems B, C and D all have kernels with logarithmic singularities; unfortunately an expression of the kind (7.06) with a and b well behaved functions, has not been found for any of these kernels.

2. Atkinson's Numerical Approximation of problem B (Atkinson, [29], [2]).

For $\underline{x} \in D$, the solution is represented by

$$u(\underline{x}) = \int_{\partial D} \mu(\underline{y}) M(\underline{x},\underline{y}) dy \, . \tag{7.07}$$

For $\underline{x} \in \partial D$ we have the integral equation

$$\pi\mu(\underline{x}) + \int_{\partial D} \mu(\underline{y}) M(\underline{x},\underline{y}) dy = f(\underline{x}) \, , \tag{7.08}$$

where

$$M(\underline{x},\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| + \int_0^1 G\left(\frac{\|\underline{x}\|}{2}, 1-\tau\right) \frac{\partial}{\partial n} \ln|\underline{x}\tau - \underline{y}| d\tau,$$

n being the inward normal at $\underline{y}$, $G(\|\underline{x}\|, 1-\tau)$ is Gilbert's G-function.

Let the curve ∂D have a twice continuously differentiable parameterization, $\underline{r}(s) = (\xi(s),\eta(s))$, $0 \leq s \leq A$, with $\|r'(s)\| \neq 0$; s need not refer to arc length.

Then (7.08) becomes

$$\pi\mu(s) + \int_0^A \mu(t) M(s,t) dt = f(t), \qquad 0 \leq s \leq A,$$

where

$$M(s,t) = \frac{\xi'(t)(\eta(s)-\eta(t))-\eta'(t)(\xi(s)-\xi(t))}{\|\underline{r}(s) - \underline{r}(t)\|}$$

$$+ \int_0^1 \frac{G(r(s),1-\tau)}{2} \left\{ \frac{\xi'(t)(\tau\eta(s)-\eta(t))-\eta'(t)(\tau\xi(s)-\xi(t))}{\|\tau\underline{r}(s) - \underline{r}(t)\|} \right\} d\tau \, .$$

Let $k(s,t) = \dfrac{\xi'(t)(\eta(s)-\eta(t))-\eta'(t)(\xi(s)-\xi(t))}{\|\underline{r}(s) - \underline{r}(t)\|}$,

$$\ell(s,t) = \int_0^1 \frac{G(r(s),1-\tau)}{2} \quad \frac{\xi'(t)(\tau\eta(s)-\ (t))-\eta'(t)(\tau\xi(s)-\xi(t))}{\|\tau\underline{r}(s) - \underline{r}(t)\|} \, d\tau.$$

$k(s,t)$ has a removable singularity at $t = s$, and

$$\lim_{t\to s} k(s,t) = \frac{1}{2} \; \frac{\xi'(s)\eta''(s)-\eta'(s)\xi''(s)}{\|\underline{r}'(s)\|^{3/2}} ,$$

which is one-half the curvature of ∂D at $\underline{r}(s)$. $\ell(s,t)$ has a logarithmic singularity as $t \to s$. Let $N > 0$, be an integer, $h = A/N$, and $t_j = jh$, $j = 0,1,\ldots,N$. Atkinson uses Simpson's rule as the numerical integration formula on $[0,A]$, and takes

$$\int_0^A k(s,t)\mu(t)dt \;=\; \sum_{j=0}^{N} W_j k(s_i,s_j)\mu(s_j) ,$$

where W_j are the usual weights associated with Simpson's rule.

For the second term divide the interval $[0,A]$ into $N/2$ intervals of length $2h$, and let $s = s_i$,

$$\int_0^A \ell(s_i,t)\mu(t)dt \;=\; \sum_{j=1}^{N/2} \int_{s_{2j-2}}^{s_{2j}} \ell(s_i,t)\mu(t)\, dt.$$

If $\min_{k=0,1,2} \{\| r(s_i) - r(s_{2j-k})\|\} > \varepsilon$, where $\varepsilon > 0$ is a preassigned number, then approximate the integral using Simpson's Rule:

$$\int_{s_{2j-2}}^{s_{2j}} \ell(s_i,t)\mu(t)dt \;=\; \frac{h}{3}[\ell(s_i,s_{2j-2})\mu(s_{2j-2}) + 4\ell(s_i,s_{2j+1})\mu(s_{2j-1}) + \ell(s_i,s_{2j})\mu(s_{2j})].$$

If $\min_{k=0,1,2} \{\| r(s_i) - r(s_{2j-k})\| \} < \varepsilon$,

take

$$\int_{s_{2j-2}}^{s_{2j}} \ell(s_i,t)\mu(t)dt = \alpha\mu(s_{2j-2}) + \beta\mu(s_{2j-1}) + \gamma\mu(s_{2j}) ,$$

where

$$\alpha = \frac{1}{2h^2} \cdot \int_{s_{2j-2}}^{s_{2j}} (s-s_{2j})(s-s_{2j-1})\ell(s_i,s)ds;$$

$$\beta = \frac{1}{2h^2} \int_{s_{2j-2}}^{s_{2j}} (s-s_{2j-1})(s-s_{2j-2})\ell(s_i,s)ds;$$

$$\gamma = \frac{1}{2h^2} \int_{s_{2j-2}}^{s_{2j}} (s-s_{2j-2})(s-s_{2j})\ell(s_i,s)ds.$$

In Gilbert and Atkinson [29], Atkinson notes that Simpson's rule is based on the approximation of the integrand by a piecewise quadratic interpolating function, and if the integrand is ill-behaved, Simpson's rule will not perform well. In the above only that part of the integrand which should behave reasonably well is approximated. Presumably that part of the integrand which does not behave reasonably well is ignored!

Solving the Nyström equations gives a numerical approximation of the density of the double-layer potential, μ, at N+1 points in the interval [0,A] and hence at N+1 points on the boundary ∂D.

To find a solution at a point $\underline{\rho}$ on the interior, D, equation (7.07) is used. Using the same parameterization of the boundary ∂D as for equation (7.08), and dividing the interval [0,A], into N equal intervals as before, the solution $u(\underline{\rho})$, at an interior point $\underline{\rho} = (x,y) \in D$, is given by

$$u(\underline{\rho}) = \sum_{j=0}^{N} W_j M(\underline{\rho}, s_j)\mu(s_j) ,$$

where W_j are the weights associated with Simpson's rule, and

$$M(\underline{\rho}, s) = \frac{\xi'(s)[y-\eta(s)]-\eta'(s)[x-\xi'(s)]}{\| \underline{\rho} - r(s)\|^2}$$

$$+ \int_0^1 G(\frac{\|\underline{\rho}\| ,1-\tau)}{2} \{\frac{\xi'(s)(\tau y-\eta(s))-\eta'(s)[\tau x-\xi(s))]}{\| \tau \underline{\rho} - \underline{r}(s)\|}\} d\tau .$$

Atkinson notes that when $\underline{\rho}$ is quite near the boundary, $M(\underline{\rho} ,s)$ becomes quite peaked on that part of ∂D which is nearest to $\underline{\rho}$. The integral should then be treated in the same way as when $\underline{x} \in \partial D$.

Although this method seems to exhibit convergence for a simple problem, the results are rather poor.

In Atkinson [2], those integrals for which $\min_{k=0,1,2} \{\| r(s_i)-r(s_{2j-k})\|\}>\varepsilon$, are evaluated by adaptive numerical methods for double integrals on rectangles.

Atkinson notes that this method works well in solving the equation

$$\pi\mu(s) + \int_0^A \mu(t)M(s,t)dt = f(t)$$

where $M(s,t)$ is singular at $s = t$, it is more difficult to judge its practicality. Certainly it would seem to be impracticable when the generating function G is not known explicitly. In order to evaluate the integral

$$\int_0^1 G(\frac{r(s),1-\tau}{2})\{\frac{\xi'(t)(\tau\eta(s)-\eta(t))-\eta'(t)(\tau\xi(s)-\xi(t))}{\| \tau\underline{r}(s) - \underline{r}(t)\|}\} d\tau,$$

when $\min_{k=0,1,2} \{\| r(s_i) - r(s_{2j-k})\|\} < \varepsilon$,

30, 60, and 120 evenly spaced sub-intervals of [0,1] were used followed by Aitken extrapolation for a final value.

For the adaptive numerical method 5 to 6 digits of accuracy were obtained with around 2000-6000 evaluations of the integrand.

Clearly the amount of computation involved in this method is on a different scale to that in the one we use (see following section), and it would be unfair to compare the two results. Moreover, Atkinson not only assumes a knowledge to 9 decimal places for the generating function G, but also only applies the method in smooth domains (bounded by circles and ellipses).

3. An Alternative Formulation, based on the smoothing of the singularity as given by Kantorovich and Krylov [35], p.101 .

In order to find the double-layer formulation for the problems B, C and D, we took the arbitrary harmonic function $\phi(\underline{x})$, $\underline{x} \in D$, defined by

$$\phi(\underline{x}) = \int_{\partial D} \frac{\partial}{\partial n} \ln |\underline{x}-\underline{y}| \mu(\underline{y}) d\underline{y},$$

where n is the inward normal, and the density function μ is real and Hölder continuous.

Let

$$\tilde{\phi}(\underline{x}) \equiv \int_{\partial D} [\mu(\underline{y}) - \mu(\tilde{\underline{x}})] \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| + \mu(\tilde{\underline{x}}) \int_{\partial D} \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| d\underline{y} .$$

Then $\quad \tilde{\phi}(\underline{x}) \equiv \phi(\underline{x}), \qquad \underline{x} \in D.$

Consider

$$\int_{\partial D} \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| d\underline{y} .$$

Putting $|\underline{x}-\underline{y}| = r$, and $d\underline{y} = ds$, then (Ch. 4. Sect. 2)

$$\int_{\partial D} \frac{\partial}{\partial n} \ln r \, ds = \int_{\partial D} \frac{d\theta}{ds} ds = 2\pi;$$

here θ is the angle that the radius r makes with some predetermined direction: Thus

$$\tilde{\phi}(\underline{x}) = \int_{\partial D} [\mu(\underline{y}) - \mu(\tilde{\underline{x}})] \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| d\underline{y} + 2\pi\mu(\tilde{\underline{x}}), \qquad (7.05)$$

$$\underline{x} \in D, \quad \tilde{\underline{x}} \in \partial D.$$

If we let $\underline{x} \in D$ tend to the point $\tilde{\underline{x}} \in \partial D$ then $\tilde{\phi}(\underline{x})$ tends uniformly to $\tilde{\phi}(\tilde{\underline{x}})$, (Muskelishvili [43], p.38), where

$$\tilde{\phi}(\tilde{\underline{x}}) = \int_{\partial D} [\mu(\underline{y}) - \mu(\tilde{\underline{x}})] \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| d\underline{y} + 2\pi\mu(\tilde{\underline{x}}) .$$

Let

$$\phi(\underline{x}) = \int_{\partial D} [\mu(\underline{y}) - \mu(\underline{x}_0)] \frac{\partial}{\partial n} \ln|\underline{x}_0-\underline{y}| \; d\underline{y} + 2\pi\mu(\underline{x}_0)$$

$$\underline{x}_0 \in \partial D.$$

Since μ is a Hölder continuous function, $\phi(\underline{x})$ is also a Hölder continuous function, $\underline{x} \in D \cup D$. This is because $\frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}|$ is continuous, $\underline{x} \in D \cup \partial D$, $\underline{y} \in \partial D$.

If $\phi(\underline{x})$ is a function on ∂D which satisfies a $H(\mu)$ condition, then

$$|\phi(\underline{x}) - \phi(\underline{y})| \leq A|\underline{x}-\underline{y}|^{\mu} \quad . \tag{7.10}$$

For the neighbourhoods of corner points other than cusps this definition of the $H(\mu)$ condition is equivalent to

$$|\phi(\underline{x}) - \phi(\underline{y})| \leqq A\sigma^{M}, \tag{7.11}$$

where σ is the length of that part of ∂D between $\underline{x}$ and $\underline{y}$, and $k_0 = \dfrac{|\underline{x}-\underline{y}|}{\sigma} \leq 1$.

However, in the neighbourhood of a cusp $\dfrac{|\underline{x}-\underline{y}|}{\sigma} < 1$ and may be as small as we like, and the conditions (7.10) and (7.11) are no longer equivalent. If (7.10) holds then $\phi(\underline{x})$ is said to satisfy the $H(\mu)$ condition in the strong form, if (7.11) holds then $\phi(\underline{x})$ is said to satisfy the $H(\mu)$ condition in the weak form. If $H(\mu)$ holds true in the strong form then $H(\gamma)$ holds true in the weak form, but not conversely.

If

$$\phi(\underline{x}) = \int_{\partial D} [\mu(\underline{y}) - \mu(\underline{x}_0)] \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| d\underline{y} + 2\pi\mu(\underline{x}_0), \quad \underline{x} \in D$$
$$\underline{x}_0, \underline{y} \in \partial D,$$

satisfies a $H(\gamma)$ condition in the weak form, then it can be shown, (Muskhelishvili, Appendix 2, [43]), that

$$\phi(\underline{x}_0) = \int_{\partial D} [\mu(\underline{y}) - \mu(\underline{x}_0] \frac{\partial}{\partial n} \ln|\underline{x}_0 - \underline{y}| d\underline{y} + 2\pi\mu(\underline{x}_0), \quad \underline{x}_0, \underline{y} \in \partial D$$

where $\underline{x}_0$ is a corner or a cusp point.

a) Problem B, $(-\Delta u + P(r^2)u = 0)$.

We define the operator (I+H), where I is the identity operator, so that

$$((I+H)h)(\underline{x}) \equiv h(\underline{x}) + \int_0^1 H\frac{(\tau,1-\tau)}{\tau} h(\tau\underline{x}) d\tau.$$

I+H is a linear operator, and a continuous operator, and so with $\phi(\underline{x})$ defined as in (7.09), the solution $u(\underline{x}) = ((I+H)\phi)(\underline{x})$, of problem B, will be continuous for $\underline{x} \in D \cup \partial D$.

$$u(\underline{x}) = ((I+H)\phi)(\underline{x}) = 2\pi\mu(\tilde{\underline{x}})[1 + \int_0^1 H\frac{(\tau,1-\tau)}{\tau} d\tau$$
$$+ \int_{\partial D} [\mu(\underline{y}) - \mu(\tilde{\underline{x}})] M(\underline{x},\underline{y}) d\underline{y}, \tag{7.12}$$

$\underline{x} \in D, \quad \tilde{\underline{x}} \in \partial D, \quad$ if $\underline{x} \in \partial D, \ \tilde{\underline{x}} = \underline{x}; \ u(\underline{x}) = f(\underline{x})$.

$$M(\underline{x},\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| + \int_0^1 H\frac{(\tau,1-\tau)}{\tau} \frac{\partial}{\partial n} \ln|\underline{x}\tau-\underline{y}| d\tau.$$

Note, $M(\underline{x},\underline{y})$ is the same as in equation (7.07), since

$$\frac{H(t,1-t)}{t} = \frac{G(\|\underline{x}\|,1-t)}{2} \quad . \tag{7.13}$$

[It should be stressed that H is a function of $\|\underline{x}\|$ as well as t. For convenience we suppress the dependence on $\|\underline{x}\|$ in our notation].

To find a numerical solution to problem B, take $\underline{x} \in D$ in equation (7.12)

$$2\pi\mu(\underline{x})[1 + \int_0^1 H\frac{(\tau,1-\tau)}{\tau}\,d\tau] + \int_{\partial D} [\mu(\underline{y}) - \mu(\underline{x})]M(\underline{x},\underline{y})d\underline{y} = f(\underline{x}) \tag{7.14}$$

Let the curve ∂D have a continuous parameterization, $r(s) = (\xi(s),\eta(s))$, $0 \leqq s \leqq A$ (s need not refer to arc length).

Define

$$h(r(s)) = 1 + \int_0^1 H\frac{(\tau,1-\tau)}{\tau}\,d\tau,$$

then equation (7.14) can be written

$$2\pi\mu(s)h(r(s)) + \int_0^A [\mu(t)-\mu(s)]M(s,t)dt = f(s), \quad 0 \leqq s \leqq A. \tag{7.15}$$

$$M(s,t) = \frac{\xi'(t)[\eta(s)-\eta(t)]-\eta'(t)[\xi(s)-\xi(t)]}{\|\underline{r}(s) - \underline{r}(t)\|}$$

$$+ \int_0^1 H\frac{(\tau,1-\tau)}{\tau} \; \frac{\xi'(t)[\tau\eta(s)-\eta(t)]-\eta'(t)[\tau\xi(s)-\xi(t)]}{\|\tau\underline{r}(s) - \underline{r}(t)\|}\,d\tau.$$

Proceeding as in the Nyström method, choose a numerical integration method which converges for all continuous functions on [0,A], equation (7.15) thus becomes

$$2\pi\mu(s)h(r(s)) + \sum_{j=1}^{n} w_j[\mu(s_j)-\mu(s)]M(s,s_j) = f(s), \quad 0 \leqq s \leqq A$$

The weights of the quadrature formula, w_j, and the nodes, s_j, depend on n, the number of nodes chosen.

By taking $s = s_1, s_2, \ldots, s_n$ the following linear system is defined for equation (7.15):

$$2\pi\mu(s_i)h(r(s_i)) + \sum_{j=1}^{n} w_j[\mu(s_j)-\mu(s_i)]M(s_i,s_j) = f(s_i), \quad i = 1,2,\ldots,n. \tag{7.16}$$

Having solved (7.16) for the unknown (approximate) values of the density, μ, at the n nodes, the approximate solution $u(\underline{\rho})$, $\underline{\rho} \in D \cup \partial D$, is found by substitution in the equation

$$u(\underline{\rho}) = 2\pi\mu(\tilde{s})h(r(\underline{\rho})) + \sum_{j=1}^{n} w_j[\mu(s_j)-\mu(\tilde{s})]M(\underline{\rho},s_j), \tag{7.17}$$

If $\underline{\rho} \in D$, $\tilde{s} \in \{s_i\}$, where s is chosen so that $|\underline{\rho}-\tilde{s}| = \min_{s_i}$.
If $\underline{\rho} \in \partial D$, a linear (quadratic or cubic) approximation is found to $\mu(\tilde{s}) = \mu(\underline{\rho})$, using the calculated values $\mu(s_i)$, $i = 1,2,\ldots,n$.

$$M(\underline{\rho},s) = \frac{\xi'(s)[x_2-\eta(s)]-\eta'(s)[x_1-\xi(s)]}{\|\underline{\rho} - \underline{r}(s)\|}$$

$$+ \int_0^1 H\frac{(\tau,1-\tau)}{\tau} \left\{\frac{\xi'(s)[x_2\tau-\eta(s)]-\eta'(s)[x_1\tau-\xi(s)]}{\|\underline{\rho}\tau - \underline{r}(s)\|}\right. d\tau.$$

$$\underline{\rho} = (x_1,x_2) .$$

An error analysis for the above method of numerical solution is not possible. The kernel, M, is a weakly-singular function, the integrand is Hölder continuous, so the first derivatives of the integrand exist, but they are not bounded, and so we are not able to bound the error in the quadrature formula.

Nevertheless this formulation offers many advantages; these will be discussed at the end of the chapter.

b) Problem C, $(-\Delta u + M(x_1)u = 0)$.

We define the operator (I-F), where I is the identity operator, and

$$((I-F)h)(\underline{x}) \equiv h(\underline{x}) - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}), x_1(\tfrac{1-t}{2}))h(x_1 t, x_2)dt,$$

$$\underline{x} = (x_1, x_2).$$

Since I-F, is a linear, continuous operator, the solution, $u(\underline{x})$, to problem C,

$$u(\underline{x}) \quad = \quad (I-F)\phi(\underline{x})$$

will be continuous for all $\underline{x} \in D \cup \partial D$, provided $\phi(\underline{x})$ is defined as in equation (7.09).

$$u(\underline{x}) = 2\pi\mu(\tilde{\underline{x}})[1-x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))dt] + \int_{\partial D}[\mu(\underline{y})-\mu(\underline{x})]M(\underline{x},\underline{y})d\underline{y}. \tag{7.18}$$

$$\underline{x} \in D \cup \partial D, \quad \underline{x} = (x_1, x_2), \quad \tilde{\underline{x}} \in \partial D.$$

If $\quad \underline{x} \in \partial D, \quad \tilde{\underline{x}} = \underline{x}, \quad u(\underline{x}) = f(\underline{x}).$

$$M(\underline{x},\underline{y}) = \frac{\partial}{\partial n} \ln|\underline{x}-\underline{y}| - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))\ \frac{\partial}{\partial n} \ln|(x_1 t, x_2) - \underline{y}|dt.$$

For an approximate numerical solution proceed as in a).

Take $\underline{x} \in \partial D$ in equation (7.18),

$$2\pi\mu(\underline{x})[1 - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}), x_1(\tfrac{1-t}{2}))dt] + \int_{\partial D}[\mu(\underline{y})-\mu(\underline{x})]M(\underline{x},\underline{y})d\underline{y} = f(\underline{x}), \tag{7.19}$$

Let the curve ∂D have a continuous parameterization, $\underline{r}(s) = (\xi(s),\eta(s))$, $0 \leqq s \leqq A$; again, s need not refer to arc length. Equation (7.19) can then be written as equation (7.15) with the appropriate definition of $h(r(s))$ and $M(s,t)$ for problem C. The procedure for solving (7.15) is the same as in a), and hence we find the solution $u(\underline{x})$, $\underline{x} \in D \cup \partial D$.

Due to the symmetry of the linear elliptic equations $-\Delta u + N(x_2)u = 0$, and $-\Delta u + M(x_1)u = 0$, the boundary value problem involving the former equation can be solved in an identical way to that of solving problem C.

c) Problem D, $(-\Delta u + [M(x_1) + N(x_2)]u = 0)$.

We define the integral operator $(I-E)(I-F)$, where I is the identity operator, and

$$((I-E)(I-F)h)(\underline{x}) = h(\underline{x}) - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+\tau}{2}),\ x_1(\tfrac{1-\tau}{2}))h(x_1\tau,x_2)d\tau$$

$$- x_2 \int_{-1}^{1} E(x_2(\tfrac{1+\tau}{2}),\ x_2(\tfrac{1-\tau}{2}))h(x_1,x_2\tau)d\tau$$

$$+ x_1x_2 \int_{-1}^{1} E(x_2(\tfrac{1+\tau}{2}),\ x_2(\tfrac{1-\tau}{2})) \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))h(x_1t,x_2\tau)\,dtd\tau.$$

Since $(I-E)(I-F)$ is a linear continuous operator the solution $u(\underline{x})$ to problem D

$$u(\underline{x}) = (I-E)(I-F)\phi(\underline{x}),$$

will be continuous for all $\underline{x} \in D \cup \partial D$, provided $(\underline{x})$ is defined as in equation (7.09). In order to approximate the solution to problem D, take $\underline{x} \in \partial D$, then

$$(I-E)(I-F)\phi(\underline{x}) \quad = \quad f(\underline{x}) \ . \tag{7.20}$$

Proceeding as in a) and b), we find a continuous parameterization of ∂D, equation (7.20) can then be written in the form of equation (7.15), with appropriate definitions of $h(r(s))$ and $M(s,t)$. The procedure for solving (7.15) is as before, as in the calculation of the solution, $u(\underline{x})$ $\underline{x} \in D \cup \partial D$.

4. The Advantages of the Alternate Formulation

The lack of an *a priori* bound on the error is not an uncommon problem for boundary integral methods. For problems, B, C and D, the maximum principle (see Results) holds, and so an *a posteriori* bound on the error is possible. In order to find this bound, it must be possible to find the numerical solution at any point on the boundary so that a boundary search can be made. This enables an approximation to be made of the largest error incurred on the boundary, since the true solution on the boundary is usually known, and this 'largest' error, bounds any error incurred on the interior.

The numerical integration of a continuous integrand would be expected to be more accurate than the numerical integration of a weakly singular integrand, particularly since the singularity occurs on the boundary. Thus we would expect the alternative formulation to give a more accurate error bound in this case. Moreover the alternative formulation is the same for a point on the boundary or in the interior thus simplifying the programming and improving the efficiency of the algorithm.

The presence of the singularity in the kernel, on the boundary, will usually affect the numerical solution at points 'near' the boundary; with the alternative formulation this is automatically alleviated.

In Atkinson's formulation a twice continuously differentiable parameterization of the boundary is required; in the alternative formulation, the curve need only have a continuous parameterization. This allows the use of the alternative formulation for boundaries containing corners or cusps. Moreover at a corner or a cusp the formulation is unchanged, simplifying the programming. The alternative formulation has been successfully applied to an L-shaped domain (see Results).

CHAPTER 8

NUMERICAL RESULTS

1. Introduction

The boundary value problems for the elliptic equation $-\Delta u + P(r^2)u = 0$, presented here, represent a cross-section of those that have been solved by other numerical analysts using integral-operator methods. We also present the results of boundary value problems for the equations $-\Delta u + M(x_1)u = 0$, and $-\Delta u + [M(x_1) + N(x_2)]u = 0$. These results give a true reflection of the performance of our algorithm.

Having obtained a numerical approximation to the solution it is essential that something is known of its accuracy. This will depend on the solution domain, the boundary conditions, and the method of solution. In section 2, we present the maximum principle and in section 3, we consider the estimation of the error in our numerical solutions.

Eisenstat [23] has proved that the degree of approximation of the solution, u, of the boundary value problem A, by generalized harmonic polynomials, depends on the smoothness of the boundary, or equivalently for domains with smooth boundaries on the smoothness of the boundary data. Since these factors directly affect the continuity of the density, μ, of the double layer potential, we would expect that our results for the integral equations algorithm will also be more accurate for smooth boundary conditions.

2. Maximum Principle (Protter and Weinberger [44])

THEOREM. Let $u(x_1,x_2,\ldots,x_n)$ satisfy the differential inequality

$$L[u] \equiv \sum_{i,j=1}^{n} a_{ij}(\underline{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\underline{x}) \frac{\partial u}{\partial x_i} \geq 0,$$

in a domain D where L is uniformly elliptic. Suppose the coefficients a_{ij} and b_{ij} are uniformly bounded. If u attains a maximum M at a point of D, then $u \equiv M$ on D.

THEOREM. Let u satisfy the differential inequality

$$(L+h)[u] \geq 0, \qquad \text{in a domain D}$$

with $h \leq 0$, and L and u given above, and the coefficients of L and h bounded. If u attains a nonnegative maximum M at an interior point of D, then $u \equiv M$.

Note: A maximum principle for functions satisfying $(L+h)[u] \leq 0$, $h \geq 0$, is obtained by applying the maximum principle to the function $-u$. Therefore a non-constant solution of the elliptic differential equation $(L+h)[u] = 0$ can attain neither a maximum nor a minimum at an interior point of D.

The maximum principle will apply to the function u in the equations $-\Delta u + P(r^2)u = 0$, $P(r^2) > 0$; $-\Delta u + M(x_1)u = 0$, $M(x_1) > 0$; and $-\Delta u + N(x_2)u = 0$, $N(x_2) > 0$. Thus the function $u(x_1,x_2)$ which satisfies the elliptic equation $Lu = 0$, in the boundary problems A, B, C, and D, will satisfy the maximum principle. In particular if $\tilde{u}$ is an approximation to u such that $L[u-\tilde{u}] = 0$, then the function $u-\tilde{u}$ will satisfy the maximum principle. That is

$$\max_{D \cup \partial D} |u-\tilde{u}| \leq \max_{\partial D} |u-\tilde{u}| .$$

3. The Error in the Numerical Solution

a) The particular solutions algorithm

Consider the boundary value problem, problem A. The solution is given by

$$u(x_1,x_2) = \sum_{j=1}^{\infty} \alpha_j \rho_j(x_1,x_2) .$$

where $\{\rho_j(x_1,x_2) \mid j = 1,2,\ldots\}$ is a complete set of particular solutions to $Lu = 0$, so that

$$L(\rho_j(x_1,x_2)) = 0, \qquad j = 1,2,\ldots .$$

To approximate u, take M, $M < N$, and determine the α_j, $j = 1,2,\ldots,M$, so that the error in the approximate solution $\sum_{j=1}^{M} \alpha_j \rho_j(x_1,x_2)$ at the N boundary points is a minimum in the least squares norm. Let $\tilde{u}_N(x_1,x_2)$ represent the approximate solution at the point (x_1,x_2) then

$$\tilde{u}_N(x_1,x_2) = \sum_{j=1}^{M} \alpha_{j,N} \rho_j(x_1,x_2);$$

where the $\alpha_{j,N}$ also depend on M.

Notice that $L\tilde{u}_N(x_1,x_2) = 0$ so that $L(u-\tilde{u}) = 0$ and hence $|u-\tilde{u}_N|_{D \cup \partial D} \leq |u-\tilde{u}_N|_{\partial D}$, by the maximum principle.

For the boundary value problems, B, C and D, the $\rho_j(x_1,x_2)$ are not in general known explicitly and have to be approximated. The $\rho_j(x_1,x_2)$ are known explicitly, when a closed form of the fundamental solution of the linear elliptic equation can be found.

Let the approximate particular solution be $\rho_j^{\,s}(x_1,x_2)$, and let

$$\tilde{u}_N^{\,s} = \sum_{j=1}^{M} \alpha_{j\,N}^{s} \rho_j^{\,s} .$$

Re-define

$$\tilde{u}_N = \sum_{j=1}^{M} \alpha^s_{j,N} \rho_j .$$

Then $L\tilde{u}_N = 0$, since L is a linear operator and

$$L(\rho_j) = 0, \qquad j = 1,2,\ldots,M.$$

However, $L\tilde{u}_N^s \neq 0$, since $L(\rho_j^{\,s}) \neq 0$.

Now consider $\max_{D \cup \partial D} |u-\tilde{u}_N^{\,s}|$. This is the maximum error over the closure of the domain of the approximate solution of the boundary value problem B, C or D, using the particular solutions algorithm.

$$\max_{D\cup\partial D} |u-\tilde{u}_N^{\,s}| \leqq \max_{D\cup\partial D} |u-\tilde{u}_N| + \max_{D\cup\partial D} |\tilde{u}_N-\tilde{u}_N^{\,s}|$$

$$\leqq \max_{\partial D} |u-\tilde{u}_N| + \max_{D\cup\partial D} |\tilde{u}_N-\tilde{u}_N^{\,s}| , \qquad (8.01)$$

by the maximum principle.

$$\max_{D\cup\partial D} |\tilde{u}_N-\tilde{u}_N^{\,s}| \leqq \max_{D\cup\partial D} \sum_{j=1}^{M} |\alpha^s_{j,N}|\,|\rho_j-\rho_j^{\,s}|, \qquad (8.02)$$

$$\max_{D\cup\partial D} |\tilde{u}_n-\tilde{u}_N^{\,s}| \leq \sum_{j=1}^{M} |\alpha^s_{jN}| \max_{D\cup\partial D} |\rho_j-\rho_j^{\,s}| .$$

The term $\max_{D\cup\partial D} |\rho_j-\rho_j^{\,s}|$ is the absolute error in the approximation of the particular solutions which we will consider for problems B, C and D in section 4.

$$\max_{\partial D}|u-\tilde{u}_N| \leq \max_{\partial D} |u-\tilde{u}_N^{\,s}| + \max_{\partial D}|\tilde{u}_N^{\,s}-\tilde{u}_N|$$

Where $\max_{\partial D}|u-\tilde{u}_N^{\,s}|$ is found by doing a search of the boundary.

Thus

$$\max_{D\cup\partial D} |u_n-\tilde{u}_N^{\,s}| \le \max_{\partial D}|u-\tilde{u}_N^{\,s}| + 2 \sum_{j=1}^{M} |\alpha_{jN}^{\,s}| \max_{D\cup\partial D} |\rho_j-\rho_j^{\,s}| .$$

Thus if we can estimate the error in our approximation of the particular solutions, we will have an estimate for the error in our numerical solutions, at any point in the closure of the domain.

b) The integral equation formulation (Chapter 7, Section 3).

Consider the boundary value problem A. The solution $u(\rho)$, at a point ρ in the domain D is given by

$$u(\rho) = 2\pi\mu(s)h(r(\rho)) + \int_0^A [\mu(t) - \mu(s)]M(s,t)dt, \qquad 0 \leqq s \leqq A. \tag{8.03}$$

where μ is found by solving the equation

$$2\pi\mu(s)h(r(s)) + \int_0^A [\mu(t) - \mu(s)]M(s,t)dt = f(s), \qquad 0 \leqq s \leqq A , \tag{8.04}$$

where f is the known boundary function.

This equation (8.04) for the density μ, is solved by choosing N points on [0,A], approximating the integral by a quadrature formula, and finding $\mu(s_i)$, i = 1,2,...,N, by collocating at the N points. We obtain

$$2\pi\tilde{\mu}(s_i)\tilde{h}(r(s_i)) + \sum_{j=1}^{N} w_j[\tilde{\mu}(s_j) - \tilde{\mu}(s_i)]\tilde{M}(s_i,s_j) = f(s_i),$$

$$i = 1,2,\ldots,N.$$

Where $\tilde{h}(r(s_i))$, $\tilde{M}(s_i,s_j)$, and w_j are the numerical approximations of $h(r(s_i))$, $M(s_i,s_j)$, and w_j are the weights of the quadrature formula.

This leads to the solution of a system of simultaneous equations for $\underline{\mu} = \{\tilde{\mu}(s_1), \tilde{\mu}(s_2), \ldots, \tilde{\mu}(s_n)\}$, which is solved using a N.A.G. library routine (Wilkinson and Reinach [55]).

The solution u, (8.03) is then calculated in a similar way. Taking the same N points on [0,A], and using the same quadrature formula for the integral, we obtain

$$\tilde{u}^{(N)}(\rho) = 2\pi\tilde{\mu}(\tilde{s})\tilde{h}(r(\rho)) + \sum_{j=1}^{N} w_j[\tilde{\mu}(s_j) - \tilde{\mu}(\tilde{s})]\tilde{M}(\rho, s_j),$$

where $\tilde{s} \in \{s_i, i = 1,2,\ldots,N\}$, and w_j are the same weights as before.

Clearly both the density μ and the solution u are subject to error. The main sources of error in μ are

i) the approximation of M by $\tilde{M}$ and h by $\tilde{h}$. This is due to the approximation of the kernel, when the kernel is not known in closed form.

ii) the quadrature formula.

iii) the application of the resulting discretized integral equation at only a finite number of points.

(iv) the approximation of the coefficients in the resulting simultaneous equations.

The main sources of error in u are

i) The error in M and h.

ii) The error in μ.

iii) The quadrature formula.

For the boundary value problems considered in this thesis the maximum principle holds. That is if the elliptic equation on the domain D is given by

$$Lu = 0,$$

then the function u satisfies the maximum principle. Let

$$u^{(N)} = 2\pi\tilde{\mu}(\tilde{s})h(r(\rho)) + \sum_{j=1}^{N} w_j|\tilde{\mu}(s_j) - \tilde{\mu}(\tilde{s})|M(\rho,s_j),$$

where $\tilde{s}$ is one of the collocating points s_j, $j = 1,2,\ldots,N$. Then $u^{(N)}$ satisfies the maximum principle since

$$Lu^{(N)} = 0.$$

However $L\tilde{u}^{(N)} \neq 0$ since $L\tilde{h} \neq 0$, $L\tilde{M} \neq 0$.

$$\max_{D\cup\partial D} |u-\tilde{u}^{(N)}| \leq \max_{D\cup\partial D} |u-u^{(N)}| + \max_{D\cup\partial D} |u^{(N)}-\tilde{u}^{(N)}|.$$

$$\leq \max_{\partial D}|u-u^{(N)}| + \max_{D\cup\partial D} |u^{(N)}-\tilde{u}^{(N)}|$$

by the maximum principle.

$$\max_{D\cup\partial D} |u^{(N)}-\tilde{u}^{(N)}| \leq \max_{D\cup\partial D} |2\pi\mu(\tilde{s})||h(r(\rho)) - \tilde{h}(r(\rho))|$$

$$+ \max_{D\cup\partial D} \sum_{j=1}^{N} |w_j||\tilde{\mu}(s_j) - \tilde{\mu}(\tilde{s})||M(\rho,s_j) - \tilde{M}(\rho,s_j)|.$$

$|h(r(\rho)) - \tilde{h}(r(\rho))|$ is the error in the first particular solution, at the point ρ.

$M(\rho,t)$ has a logarithmic singularity when $\rho \in \partial D$. Thus $[\mu(t) - \mu(s)]M(\rho,t)$ has discontinuities in its derivatives; when $\rho \in \partial D$. Since the same quadrature formula is used to approximate

$$\int_0^A [\mu(t) - \mu(s)]M(\rho,t)dt,$$

for ρ on the boundary or in the interior, we would expect the error in the quadrature to be greater when $\rho \in \partial D$.

4. The error in the particular solutions

Unfortunately the error estimate given in Chapter 4, for the solution of the Goursat problem is not very useful in practical computations. The estimate can only be calculated when the solution is known explicitly, which of course means that the error is known exactly.

However, it does tell us that the rate of convergence of the method is $O(h^4)$ provided that the solution function, H, and its fourth partial derivatives are smooth. We expect this to be true for problems C and D, but for problem B we expect the singularity at $t = 1$ in the hyperbolic equation to affect the convergence.

The error in the hyperbolic solver will directly affect the error in the particular solution.

a) Problem B, the equation $-\Delta u + P(r^2)u = 0$.

In Chapter 5 we showed that the particular solution for the equation $-\Delta u + P(r^2)u = 0$ are given by

$$\rho_j(x_1,x_2) = h_j(x_1,x_2) + \int_0^1 H(\tfrac{\tau,1-\tau}{\tau})\, h_j(x_1\tau,x_2\tau)d\tau, \qquad j = 1,2,\ldots. \tag{8.05}$$

where $h_j(x_1,x_2) = {}^{RE}_{IM}\{z^k\}$, k = integer part of $j/2$, and H is a function of r as well as τ.

In order to approximate the integral in (8.05) Simpson's composite quadrature rule is used with s intervals. $H(\tau,1-\tau)$ is found at the nodal points by solving a Goursat problem for $H(\rho,t)$. Thus

$$\rho_j^s(x_1,x_2) = {}^{RE}_{IM}\{z^k(1 + \sum_{i=0}^{s} w_i\tau_i^{k-1}\tilde{H}(\tau_i,1-\tau_i))\}, \tag{8.06}$$

where the w_i denote Simpson's rule weights, and k is the integer part of j/2.

When j = 1, k = 0, and

$$\rho_1^s(x_1,x_2) = 1 + \sum_{i=0}^{s} w_i \frac{\tilde{H}(\tau_i,1-\tau_i)}{\tau_i} . \qquad (8.07)$$

At i = 0, $\tau_i = 0$, and for each point (x_1,x_2) the value of $\frac{\tilde{H}(\tau_i,1-\tau_i)}{\tau_i}$ is indeterminate, and the limit has to be found. To find the limit we use a fourth order extrapolation process.

Let v_0 be the limiting value of $\frac{\tilde{H}(\tau_0,1-\tau_0)}{\tau_0}$, for the point (x_1,x_2) and v_i be the value of $\frac{\tilde{H}(\tau_i,1-\tau_i)}{\tau_i}$, i = 0,1,...,s, for the point (x_1,x_2). Let the error in v_i for the point (x_1,x_2) be e_i, i = 0,1,...,s.

Then v_0 is found by taking

$$v_0 = 4v_1 - 6v_2 + 4v_3 - v_4 .$$

The error e_0 in this extrapolation process satisfies

$$|e_0| \le 4|e_1| + 6|e_2| + 4|e_3| + |e_4| + h^4|v^{IV}|, \qquad (8.09)$$

where

$$|v^{IV}| = \max_{0<\tau<4h} \left| \frac{\partial^4}{\partial\tau^4} \frac{H(\tau,1-\tau)}{\tau} \right| ,$$

and $h = \frac{1}{s}$.

We recognise that $|e_0|$ calculated using (8.09) will be extremely pessimistic.

For the other particular solutions the extrapolation process is not required. The error in the j^{th} particular solution, for the point (x_1,x_2) is given by

$$|\rho_j-\rho_j^s| \le \frac{h^4}{180} \max_{0<\tau<1} \left|\frac{\partial^4}{\partial\tau^4} \tau^{k-1}H(\tau,1-\tau)\right| + \frac{h}{3} \sum_{i=1}^{s} |w_i\tau_i^k e_i|, \qquad (8.10)$$

This bound for the error in the particular solutions, for a point (x_1,x_2), if calculable, will be extremely pessimistic.

b) Problem C, the equation $-\Delta u + M(x_1)u = 0$.

In Chapter 5, we showed that the particular solutions for the equation, $-\Delta u + M(x_1)u = 0$, are given by

$$\rho_j(x_1,x_2) = h_j(x_1,x_2) - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+t}{2}),\ x_1(\tfrac{1-t}{2}))h_j(x_1 t,x_2)dt,$$

$$j = 1,2,\ldots \qquad (8.11)$$

where $h_j(x_1,x_2) = {}^{RE}_{IM}\{z^k\}$, and k is the integer part of j/2.

In order to approximate the integral in (8.11) Simpson's composite quadrature rule is used with s intervals. $F(x_1(\frac{1+t}{2}),\ x_1(\frac{1-t}{2}))$, is found at the abscissae by solving a Goursat problem for $F(x_1\xi,x_1\eta)$. Thus

$$\rho_j^{\,s}(x_1,x_2) = {}^{RE}_{IM}\{z^k - \sum_{j=1}^{s} w_j\, x_1\tilde{F}(x_1(\tfrac{1+t_j}{2}),\ x_1(\tfrac{1-t_j}{2}))(x_1 t_j + ix_2)^k\},$$

$$j = 1,2,\ldots\ . \qquad (8.12)$$

where w_j denote the weights of Simpson's rule, and k is the integer part of j/2.

Thus for the point (x_1,x_2) let

$$|e_i| = |x_1F(x_1(\tfrac{1+t_i}{2}),\ x_1(\tfrac{1-t_i}{2})) - x_1\tilde{F}(x_1(\tfrac{1+t_i}{2}),\ x_1(\tfrac{1-t_i}{2}))|\ ,\ i = 0,1,\ldots,s.$$

then

$$|\rho_j - \rho_j^s| \leq \frac{h^4}{180} x_1 \max_{-1\leq\tau\leq 1} \left| \frac{\partial^4}{\partial\tau^4} [F(x_1(\tfrac{1+\tau}{2}), x_1(\tfrac{1-\tau}{2}))(x_1\tau + ix_2)^k] \right|$$

$$+ \frac{h}{3} \left| \sum_{i=0}^{s} w_i (x_1\tau_i + ix_2)^k |e_i| \right| \quad , j = 1,2,\ldots \quad ,$$

where w_i are the weights of Simpson's rule.

Again we expect this bound for the error to be pessimistic, if indeed we can calculate it, and we will be looking for other ways of estimating the error in the particular solutions algorithm.

c) For Problem D, the equation $-\Delta u + [M(x_1) + N(x_2)]u = 0$.

The particular solution is given by

$$\rho_j(x_1,x_2) = {}^{RE}_{IM}\{z^k - x_1 \int_{-1}^{1} F(x_1(\tfrac{1+\tau}{2}), x_1(\tfrac{1-\tau}{2}))(x_1\tau + ix_2)^k d\tau$$

$$- x_2 \int_{-1}^{1} E(x_2(\tfrac{1+\tau}{2}), x_2(\tfrac{1-\tau}{2}))(x_1 + ix_2\tau)^k d\tau$$

$$+ x_1x_2 \int_{-1}^{1} F(x_1(\tfrac{1+\tau}{2}), x_2(\tfrac{1-\tau}{2})) \int_{-1}^{1} E(x_2(\tfrac{1+t}{2}), x_2(\tfrac{1-t}{2}))(x_1\tau + ix_2t)^k dt d\tau\}$$

(8.14)

For the double integration using Simpson's rule with s intervals for each integration the first term of the error at the point (t,τ), with $h = \frac{1}{s}$ is

$$\frac{h^4}{45}\left(\frac{\partial^4}{\partial t^4} + \frac{\partial^4}{\partial\tau^4}\right)[(F(x_1(\tfrac{1+\tau}{2}), x_1(\tfrac{1-\tau}{2}))E(x_2(\tfrac{1+t}{2}), x_2(\tfrac{1-t}{2}))(x_1\tau + ix_2t)^k],$$

(Bickley[9]), and a similar expression as in a), b) can be found for the error in the calculated particular solutions.

5. Tables

Table 1 gives a comparison of the actual error and the estimated error in the Goursat problem, for the equation $-\Delta u + P(r^2)u = 0$, where $P(r^2)$ is taken to be constant, $P(r^2) = \lambda^2$, for various values of λ.

The solution of the Goursat problem

$$\frac{\partial^2 H}{\partial\rho\partial t} = \frac{r_0^{\,2}\lambda^2}{4(1-t)^2} H = f(\rho,t,H), \qquad t > 0, \quad \rho > 0$$

$$H(0,t) = 0, \qquad t > 0$$

$$H(\rho,0) = \frac{1}{2} r_0^{\,2}\lambda^2\rho \qquad \rho > 0 ,$$

is $$H(\rho,t) = \frac{\lambda r_0}{2}\rho^{\frac{1}{2}} \frac{(1-t)^{\frac{1}{2}}}{t^{\frac{1}{2}}} I_1(r_0 \frac{\lambda\rho^{\frac{1}{2}}t^{\frac{1}{2}}}{(1-t)^{\frac{1}{2}}}) ;$$

The error of the numerical scheme for solving the Goursat problem is $e(\rho,t) + O(h^6)$, where

$$e(\rho,t) < \frac{h^4}{2^5.2^3}[\frac{0.6A}{L} + 2.5M + 12N + 24C].\ [I_0(2\sqrt{L\rho t}) - 1],$$

and

$$A = \sup_{i,j}\{\left|\frac{\partial^4 f}{\partial\rho^4}(\xi,\eta,H)\right|, \left|\frac{\partial^4 f}{\partial t^4}(\xi,\eta,H)\right|, \quad \xi = \rho_i + \frac{h}{2}, \eta = t_j + \frac{h}{2}\}$$

$$M = \sup_{i,j}\{\left|\frac{\partial^4 H}{\partial\rho^4}(\xi,\eta)\right|, \left|\frac{\partial^4 H}{\partial t^4}(\xi,\eta)\right|, \qquad \xi = \rho_i, \ \eta = t_j\}$$

$$N = \sup_{i,j}\{\left|\frac{\partial^2 f}{\partial\rho^2}(\xi,\eta,H)\right|, \left|\frac{\partial^2 f}{\partial t^2}(\xi,\eta,H)\right|, \xi = \rho_i, \ \eta = t_j\}$$

$$C = \sup_{i,j}\{\left|\frac{\partial^2 f}{\partial\rho\partial t}(\xi,\eta,H)\right|, \qquad \xi = \rho_i, \ \eta = t_j\}$$

$$L = \sup_{i,j}\{\left|\frac{\partial f}{\partial H}(\xi,\eta,H)\right|, \qquad \xi = \rho_i, \ \eta = t_j\},$$

$i = 1,2,\ldots,N+1-j, \quad j = 1,2,\ldots,N,$

where ρ_i, t_j are the coordinates of the bottom left hand corner of the square mesh R_{ij}, with width h.

TABLE 1 h = 0.1.

λ	ERROR IN H(0.1, 0.9)	ESTIMATED ERROR IN H(0.1,0.9)
.1	1.50×10^{-9}	1.22×10^{-9}
.5	7.41×10^{-7}	8.89×10^{-7}
1.0	1.53×10^{-6}	2.17×10^{-5}
2.0	7.53×10^{-4}	1.24×10^{-3}
10.0	3.84×10^{2}	9.71×10^{5}

For the equation $P(r^2) = \lambda^2$, Table 2 shows the ratio of the error when h is halved, at a point on the line $\rho+t = 1$ close to the singularity at t = 1, (0.1,0.9) and at a point away from the singularity (0.4,0.6), for $\lambda = 1$.

TABLE 2

$\lambda = 1$	h = 0.1	h = 0.05	Ratios	h = 0.05	h = 0.025	Ratios
Error in:						
H(0.1,0.9)	1.53×10^{-6}	3.11×10^{-7}	4.92	3.11×10^{-7}	2.25×10^{-8}	13.8
H(0.4,0.6)	9.44×10^{-8}	5.23×10^{-9}	18.0	5.23×10^{-9}	2.74×10^{-10}	19.1

This shows that the further away from the singularity, the more rapidly the approximate solution approaches the true solution.

We have employed many numerical techniques (Chapter 5) to tackle the relatively poor rate of convergence, in the numerical solution near the singularity. These have included product integration, surface fitting and iteration, however none of them significantly improved the accuracy or the rate of convergence. One technique that we are in the process of trying is a refinement of the mesh near the singularity; we shall discuss this in the conclusion.

It should be remembered that the solution values calculated on the line $\rho = 1-t$ are used in the numerical approximation of an integration (see 8.06). Thus as we reduce h, the mesh length, we increase the number of points, s-1, calculated on the line $\rho+t = 1$, and consequently the number of nodes, s+1, used in the approximation of the integral. However, we also calculate H at points closer to the line $t = 1$. When $h = 0.1$, we calculate the solution at 9 points on the line $\rho = 1-t$, the largest error occurring (predictably) at (.1,.9). When $h = 0.05$, we calculate the solution at 18 points on the line $\rho+t = 1$, the largest error occurring at (.05,.95), the error at this point being slightly greater in absolute size to the error at the point (.1,.9) for $h = 0.1$. The effect of this on the error in the particular solution can be seen from equation (8.10). For the second term $\frac{h}{3}\sum_{i=0}^{s}|w_i\tau_i^k e_i|$, the $\sum_{i=0}^{s}|w_i\tau_i^k e_i|$ will not have significantly decreased (if at all) for smaller h, thus the best we can expect for the rate of convergence of the particular solutions is O(h).

In table 3 we give the absolute error in the particular solutions $|\rho_j-\rho_j^s|$, for the equation $-\Delta u + P(r^2)u = 0$, for $P(r^2) = \lambda^2$, and for various values of λ. The particular solutions, ρ_1 and ρ_2, are found by analytically integrating (8.05), when

$$h_j(x_1,x_2) = 1, \quad \text{and} \quad h_j(x_1,x_2) = x_1,$$

and the value was found at $x_1 = 1$, $x_2 = 0$.

TABLE 3. ABSOLUTE ERROR IN

	λ	$\rho_1(1,0)$	$\rho_2(1,0)$
$h = 0.1$			
	.1	3.92×10^{-9}	2.53×10^{-10}
	.5	1.93×10^{-6}	1.78×10^{-7}
	1	3.89×10^{-6}	2.93×10^{-7}
	2	1.97×10^{-3}	1.18×10^{-4}
	10	9.70×10^{2}	8.07×10^{1}
$h = 0.05$			
	1	1.41×10^{-6}	7.49×10^{-8}
$h = 0.025$			
	1	6.26×10^{-7}	1.90×10^{-8}
RATIOS			
$\frac{\text{ERROR AT } h = 0.1}{\text{ERROR AT } h = 0.05}$		2.76	3.91
$\frac{\text{ERROR AT } h = 0.05}{\text{ERROR AT } h = 0.025}$		2.25	3.94

The lower ratios for the errors in $\rho_1(1,0)$ and the higher errors in the computed values of $\rho_1(1,0)$, reflect the added error due to the extrapolation process.

In Table 4 we consider the error in the numerical solution of the Goursat problem for the equation $-\Delta u + M(x_1)u = 0$, with $M(x_1) = \lambda^2$. We take $h = 0.1$, and various values of λ. The point (.7, .3), is where the error is greatest.

TABLE 4. ABSOLUTE ERROR IN NUMERICAL APPROXIMATION AT (.7, .3).

λ	
.1	9.41×10^{-15}
.5	4.74×10^{-10}
1	1.26×10^{-7}
2	3.74×10^{-5}
10	4.67×10^{2}

In Table 5 we look at the absolute error in our numerical solution of the Goursat problem at the point (.7, .3) for varying values of h. We take $\lambda = 1$.

TABLE 5. ABSOLUTE ERROR AT (.7, .3) FOR VARIOUS VALUES OF h.

$\lambda = 1$	h	0.1	0.05	Ratio	0.05	0.025	Ratio
(.7, .3)		1.26×10^{-7}	8.52×10^{-9}	14.8	8.52×10^{-9}	5.53×10^{-10}	15.4

In Table 6 we give the absolute error in the particular solutions $|\rho_j-\rho_j^s|$ for the equation $-\Delta u + M(x_1)u = 0$, $M(x_1) = \lambda^2$ for various values of λ. The particular solutions ρ_1 and ρ_2 are found by analytically integrating equation (8.11), when $h_j(x_1,x_2) = 1$, and $h_j(x_1,x_2) = x$, respectively, the values were found for $x_1 = 1$, $x_2 = 0$.

TABLE 6. ABSOLUTE ERROR IN

	λ	$P_1(1, 0)$	$P_2(1, 0)$
$h = 0.1$			
	.1	5.62×10^{-13}	1.33×10^{-9}
	.5	8.17×10^{-9}	8.09×10^{-7}
	1	4.19×10^{-7}	1.18×10^{-5}
	2	4.07×10^{-6}	1.26×10^{-4}
	10	5.49×10^{2}	6.12×10^{1}
$h = 0.05$			
	1	2.53×10^{-8}	7.34×10^{-7}
$h = 0.025$			
	1	1.55×10^{-9}	4.58×10^{-8}
RATIOS			
$\frac{\text{ERROR } h = 0.1}{\text{ERROR } h = 0.05}$		16.6	16.1
$\frac{\text{ERROR } h = 0.05}{\text{ERROR } h = 0.025}$		16.3	16.0

6. Numerical Results

In the following section we will display our results for some of the problems we have undertaken. The following symbols will be used.

N	represents the number of boundary points.
M	represents the number of particular solutions.
s	represents the number of intervals in Simpson's rule, in the approximation of the generating function.
$h = \frac{1}{s}$	h is the interval length in the above Simpson's rule, it is also the mesh length in the numerical solution of the Goursat problem.
E_B	represents the largest absolute error in the numerical solution on the boundary, as found by completing a boundary search.

For all the problems for which we have applied the integral equation algorithm, the trapezoidal quadrature rule was used to approximate the boundary integral, except for the L-shape problem for which Simpson's quadrature rule was applied. This is because we wanted to take advantage of the known improvement of the trapezoidal rule for periodic functions. For the L-shaped reason we do not expect the integrand to be periodic and so the trapezoidal rule offers no advantage.

PROBLEM I. $-\Delta u + u = 0$ on D,

$$u(x_1,x_2) = \frac{I_0\left(\sqrt{x_1^{\,2}+x_2^{\,2}}\right)}{I_0(1)}, \qquad \text{on } \partial D,$$

where ∂D is the ellipse, $x_1^{\,2} + 4x_2^{\,2} = 1$.

Here I_0 is the modified Bessel function of the first kind, and the solution is

$$u(x_1,x_2) = \frac{I_0\left(\sqrt{x_1^{\,2}+x_2^{\,2}}\right)}{I_0(1)}$$

This elementary problem was attempted by K. Atkinson [29], to illustrate the numerical approximation of R.P. Gilbert's double-layer potential formulation for problem B. Because Atkinson uses a truncated infinite series to approximate the kernel numerically, and we use the numerical solution of a Goursat problem, any discrepancy in the accuracy should be in Atkinson's favour.

Problem 1 gives us an opportunity of comparing Atkinson's algorithm with our integral equation algorithm for problem B, with $P(r^2) = 1$. Results for our particular solutions algorithm are also shown.

This problem has very smooth boundary conditions, in fact the density of the double-layer potential, μ, is a constant for the problem. So we would expect good numerical results from our algorithms.

To parameterize the problem for the integral equations algorithm, $x_1 = a \cos t$, $x_2 = b \sin t$, $0 \leqq t < 2\pi$, where t is the eccentric angle.

ALGORITHM		ATKINSON		INTEGRAL EQUATIONS		PARTICULAR SOLUTIONS	
Computer		CDC 3600		CDC 6400		CDC 6400	
Boundary nodes		N = 12	N = 24	N = 12		N = 16	
Other parameters				S = 10	S = 40	S = 10	S = 40
						M = 4	
INTERNAL PTS		RELATIVE ERROR					
X_1	X_2	$\times 10^{-1}$	$\times 10^{-2}$	$\times 10^{-5}$	$\times 10^{-7}$	$\times 10^{-5}$	$\times 10^{-6}$
0.0	0.0	0.274	0.054	0.0324	0.0246	0.325	0.819
0.0	0.125	0.425	0.22	0.314	0.237	0.316	0.795
0.0	0.25	1.16	1.18	0.271	0.203	0.276	0.696
0.25	0.0	0.068	0.13	0.364	0.271	0.343	0.864
0.5	0.0	0.186	0.011	0.245	0.234	0.250	0.635
E_B		NOT	GIVEN	1.4×10^{-6}	6.5×10^{-8}	5.18×10^{-6}	2.6×10^{-6}
Time (sec)		NOT	GIVEN	1.5	19.6	0.929	13.4

COMMENTS

In section 3, for the integral equations algorithm we concluded that we would expect the error on the boundary to exceed the error on the interior. For this problem that is not true. The reason for this is that the density, μ, is constant, thus the error in $h(r(s))$ becomes significant, and there is no reason why the error in $h(r(s))$ should be a maximum on the boundary. In fact $h(r(s))$ is the first particular solution, and the error in $h(r(0))$ is given in Table 3.

That Atkinson's algorithm works so poorly in comparison, for such a simple example, suggests that his formulation is inferior.

For the particular solutions algorithm we exploit the 4-fold symmetry of the problem. We see that the error on the interior is in this case bounded by E_B. This means that the error in solution due to the numerical approximation of the particular solutions is not greater than 5.18×10^{-6} for s = 11, and not greater than 2.6×10^{-6} for s = 41.

Problem 2. $-\Delta u + u = 0$ on D
$u + 1$ on ∂D

∂D is the ellipse $x_1^2 + 4x_2^2 = 1$.

This problem is more complicated than problem 1, where the boundary values were those of a particular solution of the elliptic equation. The problem was used by P. Linz [30], to illustrate his particular solution algorithm for problem B, which includes the numerical approximation of the kernel by the solution of a Goursat problem. The analytical solution to this problem is not known. For the integral equations algorithm the eccentric angle was taken as the parameter as in problem 1.

RESULTS

ALGORITHM		P. LINZ			PARTICULAR SOLNS		INTEGRAL EQUATIONS	
COMPUTER		Not given			CDC 6400		CDC 6400	
Parameters		N=31			M=4, N=40		N=12	N=24
		M=4	M=8	M=12	s=10	s=40	s=10	
Internal Points								
X_1	X_2							
0.0	0.0	.9062	.9064	.9063	.907163	.907165	.907541	.907189
0.1	0.0	.9068	.9079	.9079	.908053	.908055	.908318	.908080
0.0	0.1	.9105	.9107	.9107	.910812	.910814	.911299	.910839
0.5	0.0	.9297	.9295	.9295	.929645	.929647	.930308	.929691
0.0	0.25	.9298	.9301	.9300	.930056	.930058	.931112	.930115
E_B		7×10^{-4}	2×10^{-4}	1×10^{-4}	5.95×10^{-4}	5.95×10^{-4}	3.8×10^{-2}	1.9×10^{-2}
Time (sec)		NOT GIVEN			1.95	27.3	1.56	3.11

COMMENTS

Linz calculates E_B by finding the error at the N boundary points and taking the maximum as E_B. This is clearly not going to bound the error on the interior adequately.

For the particular solutions algorithm the coefficients of the four particular solutions are

$$\alpha_1 = 0.9072, \quad \alpha_2 = -0.1378, \quad \alpha_3 = 0.1079\times10^{-2}$$

$$\alpha_4 = -0.6650\times10^{-5}.$$

By combining this information with that of Table 3, suggests that the first 4 figures of our solutions, for the particular situation algorithm, are probably correct.

For the integral equations algorithm the density, μ, of the double layer potential is not constant.

PROBLEM 3 $\qquad -\Delta u + (4-2r^2)u = 0, \quad \text{on } D$

$$u = x_1^2 + x_2^2 \qquad \text{on } \partial D$$

Where ∂D is the 'squirkle'.

This problem has been solved numerically by Bergman and Herriot [7], and by Schryer [48]. The 'squirkle' is a square with sides $x_1 = \pm 1$, $x_2 = \pm 1$, but with its corners replaced by segments of circles. The northeast corner of the square is replaced by the northeast quadrant of the circle,

$$(x_1-x_0)^2 + (x_2-x_0)^2 = (1-x_0)^2,$$

where $x_0 = \tan(39^\circ)$.

The boundary conditions and the domain allow 8-fold symmetry which we have taken advantage of in the calculation, as with Bergman and Herriot; Schryer allows 4-fold symmetry.

Bergman and Herriot and Schryer approximate the kernel of the integral operator by truncation of an infinite series; both use particular solutions algorithms.

The number of terms taken before truncation is indicated by the letter T. It is possible to calculate the error in the solution on the interior due to the truncation of the infinite series, this error is indicated by E_A. Unfortunately for our own particular solutions algorithm we have been unable to obtain an estimate for the error in the solution on the interior due to the error in the particular solutions.

This problem was also solved using the integral equations algorithm, taking advantage of the 8-fold symmetry and taking the eccentric angle as parameter.

RESULTS FOR PROBLEM 3

ALGORITHM	SCHRYER		BERGMAN AND HERRIOT	PARTICULAR SOLNS.		INTEGRAL EQUATIONS
COMPUTER	IBM 360/67		IBM 7090	CDC 6400		CDC 6400
N	192	320	368	360		80
M	8	24	8	8		-
OTHER PARAMETERS	T=10	T=15,10*	T=4	S=10	S=40	S=10
INTERNAL POINTS X_1 X_2			SOLUTIONS			
0.0 0.0			0.5187	0.5190	0.5186	0.5189
0.2 0.0			0.5394	0.5398	0.5394	0.5397
0.6 0.0	NOT		0.7034	0.7039	0.7034	0.7038
0.8 0.0			0.8400	0.8408	0.8403	0.8407
0.2 0.2	GIVEN		0.5610	0.5614	0.5610	0.5613
0.6 0.2			0.7318	0.7322	0.7317	0.7321
0.8 0.2			0.8740	0.8746	0.8742	0.8745
0.6 0.6			0.9571	0.9572	0.9565	0.9569
0.6 0.8			1.1449	1.1449	1.1443	1.1449
0.8 0.8			1.3738	1.3734	1.3729	1.3735
E_B	3.1×10^{-4}	4.9×10^{-5}	1.4×10^{-3}	5.2×10^{-4}	5.2×10^{-4}	3.2×10^{-2}
E_A	1.6×10^{-5}	9.4×10^{-11}	less than 10^{-3}	NOT AVAILABLE		
Time (sec)	18	113	240	8.8	133	9.8

*Here the first 4 particular solutions used were truncated after 15 terms of the infinite series, and the rest after 10 terms of the infinite series.

COMMENTS

For the equation $-\Delta u + P(r^2)u = 0$, and $P(r^2) = \lambda^2$, we saw in Table 1 that as λ increased so did the error in our numerical solution of the Goursat problem, and in Table 3 we saw the corresponding effect in the error in the particular solutions.

In problem 3 $P(r^2) > 1$, and we expect the error in the numerical solutions due to the error in the particular solutions, to be greater than that obtained by Schryer.

However, our results are certainly comparable to those obtained by Bergman and Herriot. Notice in particular that the integral equations algorithm requires only 80 boundary nodes for comparable accuracy.

PROBLEM 4.

$$\Delta u - 4u = 0 \quad \text{on } D$$

$$u = 1 \quad \text{on } \partial D.$$

The domain (figure 3) is an L-shaped region.

This domain has a re-entrant corner with angle $3\pi/2$. When a domain has a corner in it, particularly a re-entrant corner, particular solutions algorithms are no longer practical. This is because the partial derivatives of the solution at the corner may become infinite, whereas the partial derivatives of the particular solutions are bounded over the closure of the domain. Schryer [48] overcomes this difficulty by creating additional particular solutions to represent the singularity and he uses these in combination with the usual particular solutions.

We use the integral equations algorithm, using Simpson's quadrature formula to approximate the boundary integral. We parameterize the boundary by taking the parameter $t = \pm x_2$ when x is constant (on AB, CD and EF, in Figure 3), and $t = \pm x_1$ when y is constant (on BC, DE and FA in figure 3).

The derivatives of t are not single valued at the corners. This difficulty is overcome by treating each corner as two points, for instance at the corner, B, (figure 3), one point is taken to be part of AB, $t = x_2$, and the derivative is 1, the other point is taken to be part of BC, $t = -x_1$, and the derivative is -1.

For this problem we also used a finite difference algorithm with an iterative conjugant gradient technique, and preconditioning, Wallcroft [54]. This algorithm is specifically designed for rectangular and L-shaped regions.

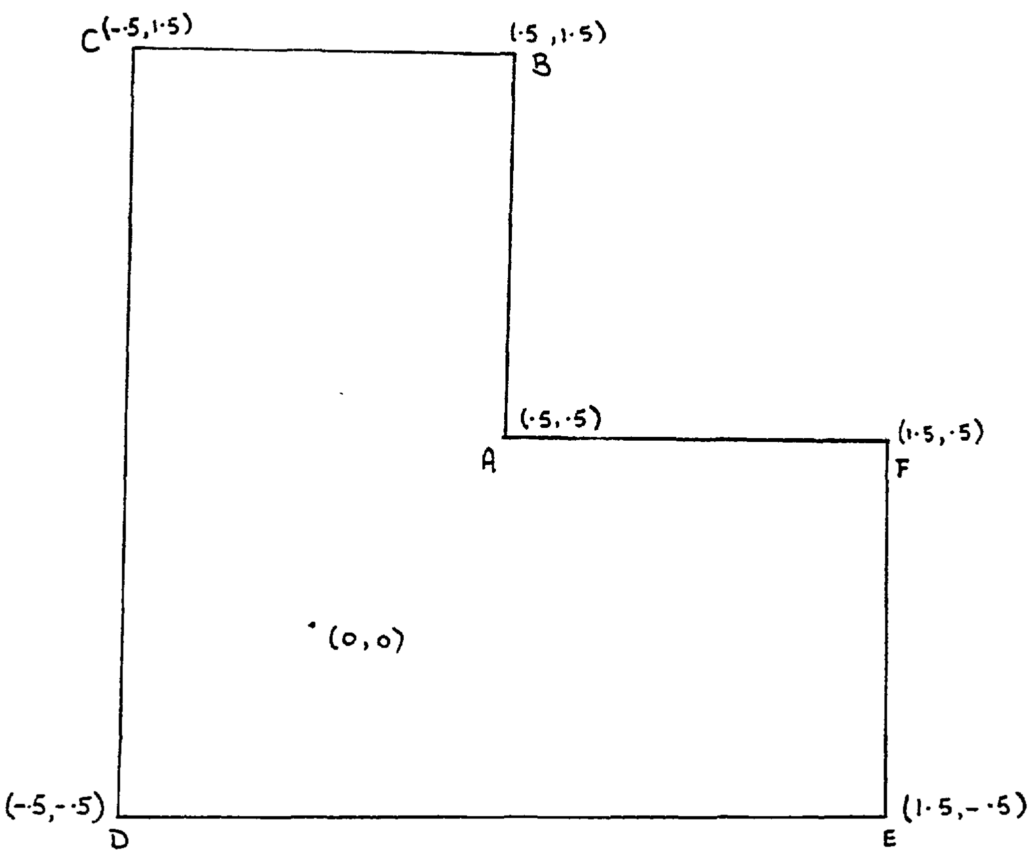


Figure 3

RESULTS FOR PROBLEM 4

ALGORITHM	SCHRYER		FINITE DIFFERENCES	INTEGRAL EQUATIONS	
COMPUTER	IBM 360/67		CDC 6400	CDC 6400	
N	385	385	124 mesh points	32	64
OTHER PARAMETERS	M=13+6* T=10	M=25+4* T=10	-	s=10	s=10
INTERNAL POINTS X_1 X_2					
0.0 0.0	NOT GIVEN		0.6388	0.6396	0.6384
0.25 0.25			0.6109	0.6166	0.6107
0.5 −0.25			0.7262	0.7276	0.7265
0.75 0.0			0.6560	0.6609	0.6571
1.0 0.25			0.7661	0.7730	0.7680
E_B	3.1×10^{-3}	5.0×10^{-4}	-	6.0×10^{-2}	2.1×10^{-2}
E_A**	3.6×10^{-5}	9.0×10^{-8}	-	-	-
Time (sec)	111	286	3.03	4.4	12.7

*The first number refers to the ordinary particular solutions and the second to the special particular solutions, the infinite series for both sets of particular solutions were truncated after 10 terms.

**E_A is defined in problem 3.

COMMENTS

From the results given by Schryer, it is impossible to give a thorough comparison of his algorithm and the integral equation algorithm. However, from E_B, we see that a reasonable degree of accuracy is achieved with very few boundary points.

The finite difference algorithm, although an extremely quick algorithm, is one that has been specifically designed to work best for domains with regular sides and with angles of multiples of 45°. Its disadvantage is its large storage requirements (the 31×31 mesh was the largest possible for running the algorithm at a time-sharing terminal). Moreover the solution is only known at the mesh points and for other points an interpolation process needs to be used.

PROBLEM 5

$$-\Delta u + x_1^2 u = 0 \qquad \text{on } D,$$
$$u = \cosh x_2 . e^{-\frac{1}{2}x_1^2} \qquad \text{on } \partial D.$$

The domain is a square with sides $x_1 = \pm 1$, $x_2 = \pm 1$. For this problem the boundary function is a solution of the elliptic equation, and so the solution over the closure of the domain $D \cup \partial D$ is $u = \cosh x_2 . e^{-\frac{1}{2}x_1^2}$.

For this example we use the particular solutions algorithm, and the finite difference algorithm. For the particular solutions algorithm full advantage of the 4-fold symmetry is taken.

Since the partial derivatives of the solution at the corners, and indeed over the closure of the domain are bounded, there is no necessity for the use of special singular solutions, as used by Schryer for the L-shaped domain.

RESULTS FOR PROBLEM 5

ALGORITHM		FINITE DIFFERENCE	PARTICULAR SOLUTIONS		
COMPUTER		CDC 6400	CDC 6400		
		Mesh points 124	N=60		
			s=10		
OTHER PARAMETERS			M=2	M=4	M=6
INTERNAL POINTS		RELATIVE ERROR			
X_1	X_2				
0.25	0.25	5.2×10^{-5}	1.7×10^{-2}	1.4×10^{-5}	4.1×10^{-6}
0.375	0.375	7.5×10^{-5}	1.4×10^{-2}	1.7×10^{-5}	4.1×10^{-6}
0.50	0.125	3.5×10^{-5}	2.0×10^{-2}	1.3×10^{-5}	5.4×10^{-6}
0.625	0.25	5.1×10^{-5}	2.0×10^{-2}	1.7×10^{-5}	5.5×10^{-6}
0.75	0.375	4.8×10^{-5}	4.0×10^{-2}	3.2×10^{-5}	5.0×10^{-6}
E_B		-	1.5×10^{-1}	1×10^{-4}	7×10^{-8}
Time (sec)		4.28	1	1.1	1.1

It was found that in the particular solutions algorithm when $M = 8$, $N = 60$, a reduction in h (where h is the mesh length in the hyperbolic solver and also the nodal length in Simpson's quadrature rule used to approximate the generating function) by $\frac{1}{2}$, led to a reduction in the error at the internal points of approximately $\frac{1}{16}$. This suggests a rate of convergence of h^4 in each particular solution.

COMMENTS

In the particular solutions algorithm, when $M = 2$ and 4, the error on the interior at the points shown is bounded by the error on the boundary. For $M = 6$ this is no longer true, the error in the solution due to the error in approximating the particular solution, E_n, is larger than the boundary error.

The smoothness of the solution of this problem is reflected by the rapidity of the convergence of the numerical solution to the solution as the number of particular solutions increases.

The speed of the particular solutions algorithm is due to the exploitation of the 4-fold symmetry in contrast to the inflexibility of the finite difference algorithm. We see that an increase in the number of particular solutions barely affects the speed of the algorithm.

PROBLEM 6

$$-\Delta u + (\sinh^2 x_1 + \sinh^2 x_2)u = 0 \qquad \text{on } D$$
$$u = \sinh(\sinh(x_1)\sin(x_2)), \qquad \text{on } \partial D.$$

Where the boundary, ∂D, is the ellipse $x_1^2 + 4x_2^2 = 1$.

For this problem we use the integral equations algorithm and the particular solutions algorithm.

The solution over the closure of the domain is $u = \sinh(\sinh(x_1)\sin(x_2))$. Because the boundary and boundary functions are smooth we expect our algorithms to give reasonable results, despite the complexity of the equation.

RESULTS FOR PROBLEM 6

ALGORITHM		INTEGRAL EQUATIONS (1)			INTEGRAL EQUATIONS (2)	PARTICULAR SOLUTIONS	
COMPUTER					CDC 6400		
N		16	32	64	32	44	
M		-	-	-	-	2	4
s		10	10	10	10	10	10
INTERNAL POINTS				RELATIVE	ERRORS		
x_1	x_2						
0.215	0.26	8.9×10^{-2}	1.7×10^{-3}	2.1×10^{-5}	1.4×10^{-3}	6.4×10^{-3}	9.1×10^{-6}
0.398	0.199	1.8×10^{-2}	3.0×10^{-4}	3.1×10^{-5}	5.2×10^{-4}	1.3×10^{-3}	1.2×10^{-7}
0.520	0.108	1.3×10^{-3}	3.0×10^{-5}	1.0×10^{-5}	2.6×10^{-5}	1.3×10^{-3}	5.1×10^{-6}
0.530	0.265	1.5×10^{-2}	5.1×10^{-4}	6.8×10^{-6}	7.3×10^{-3}	1.6×10^{-3}	1.0×10^{-6}
0.693	0.144	4.0×10^{-3}	1.2×10^{-4}	2.5×10^{-6}	6.1×10^{-4}	2.3×10^{-3}	1.0×10^{-6}
E_B		8.9×10^{-2}	2.2×10^{-2}	7.2×10^{-3}	1.04	7.5×10^{-2}	2.6×10^{-5}
Time (sec)		7	15.7	43	15	8.6	10.3

By varying h the mesh length in the hyperbolic solver and the nodal width in the Simpson's rule quadrature used to approximate the generating function, suggests a rate of convergence of just under $O(h^4)$.

COMMENTS

This problem illustrates the use of both particular solutions algorithm and the integral equations algorithm for an equation of the type $-\Delta u + (M(x_1)u + N(x_2))u = 0$.

The results headed 'Integral Equations (2)' were obtained from the integral equations algorithm but with the approximate solution at interior points calculated from

$$u(\underline{x}) = \int_{\partial D} \mu(\underline{y})M(\underline{x},\underline{y})d\underline{y}, \qquad \underline{x} \in D, \quad \underline{y} \in \partial D,$$

instead of from

$$u(\underline{x}) = 2\pi\mu(\underline{x}_0)h(r(\underline{x}_0)) + \int_{\partial D} [\mu(\underline{y}) - \mu(\underline{x}_0)]M(\underline{x},\underline{y})d\underline{y}, \quad \underline{x} \in D,\ \underline{x}_0,\ \underline{y} \in \partial D$$

('Integral equations (1)').

A comparison for 32 boundary points indicate that although there is some deterioration in the numerical solution at points away from the boundary, for points near and on the boundary 'Integral equations (1)' is far more accurate.

7. CONCLUSION

Problem 1 illustrates the improvement in the solution using the integral equations algorithms in place of Atkinson's for boundary value problems with the elliptic equation $-\Delta u + P(r^2)u = 0$. Atkinson only attempts problems with very smooth boundary conditions and so other comparisons are not possible. However, we show in problems 2, 3 and 4, that the integral equations algorithm does work well for a variety of domains and boundary conditions.

Problem 2 illustrates the improvement in the numerical solution of the Goursat problem. Problems 1-4 show that it is possible to achieve acceptable numerical solutions when the generating function is approximated by the numerical solution of a Goursat problem, in both the particular solutions algorithm and the integral equations algorithm.

For $P(r^2) = \lambda^2$, we have seen that the greater the value of λ, the greater the error in the hyperbolic solver, and this error directly affects the error in the numerical solution of the problem. To reduce this error we halve the steplength h, however this is expensive, approximately 4 times as long, and the improvement in the numerical solution is disappointing, we suggest an approximate rate of convergence of $O(h)$.

As λ increases more terms of the infinite series before truncation are needed to maintain the accuracy of the particular solutions. However, taking more terms in the infinite series appears less expensive to implement than the necessary reduction in the mesh length in the hyperbolic solver.

It is this aspect of the hyperbolic solver that we are currently trying to improve. The idea is to refine the mesh only for the point (or points) most affected by the singularity. For example, take

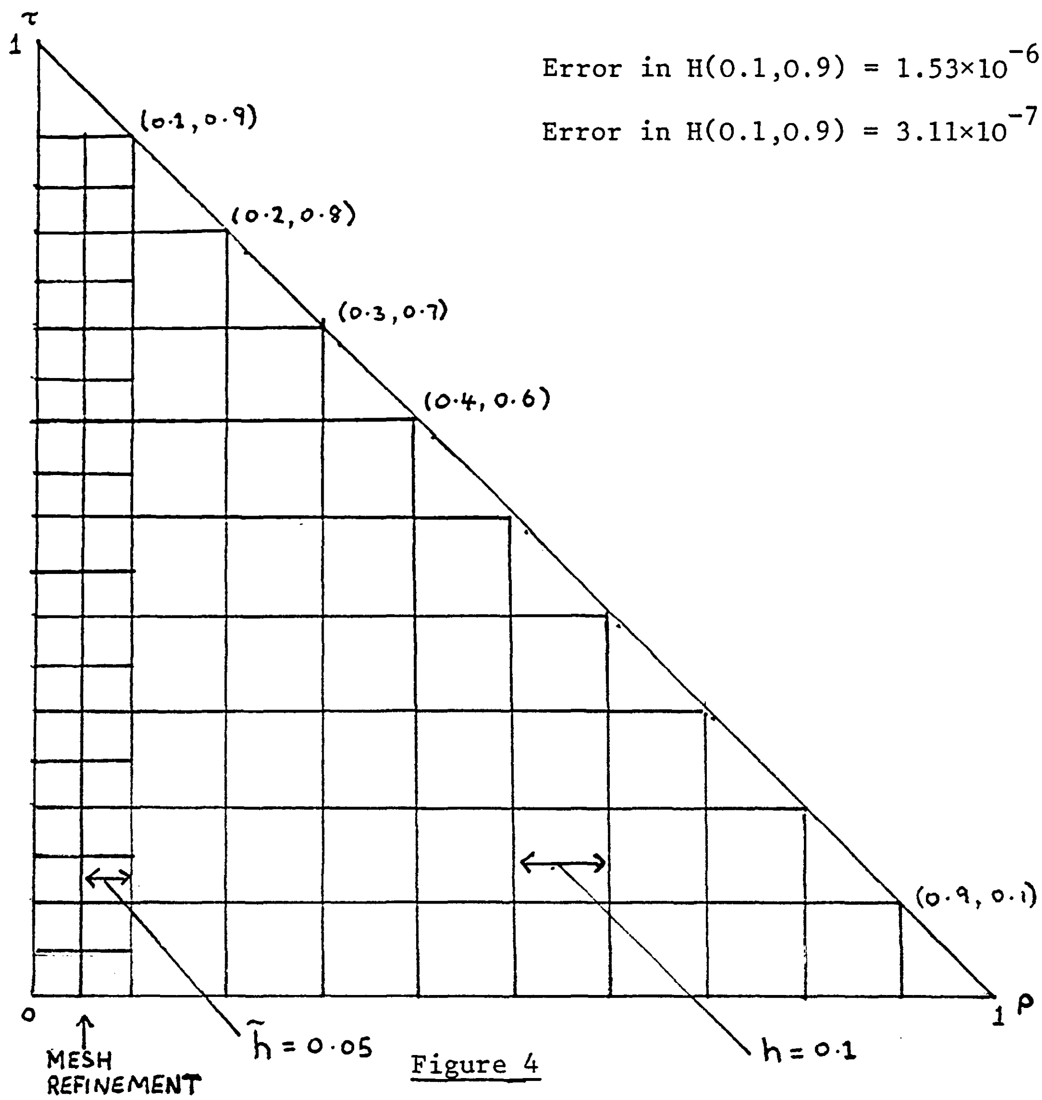


Figure 4

$P(r^2) = \lambda^2 = 1$. For $h = 0.1$, we obtain the numerical solution $\tilde{H}(\rho,t)$ at (0.1,0.9) with an error of 1.53×10^{-6}, at (0.2,0.8) with an error of 6.14×10^{-7}, by refining the mesh for the point (0.1,0.9) (see Figure 4), the value at (0.1,0.9) will become 3.11×10^{-7}. Moreover the values at the other nodal points on the line $\rho+t=1$, will improve due to the mesh refinement in the left hand column, these having the greatest improvement being those closest to the singularity. Also for the first particular solution, and in the integral equations algorithm the value of $\frac{H(\rho,t)}{(1-t)}$ at the point $t = 1$, which is indeterminate and has to be found using a fourth order extrapolation process, will also be more accurate, due to the increased accuracy in $\frac{H(\rho,t)}{(1-t)}$ at the points (0.1,0.9), (0.2,0.8), (0.3,0.7), (0.4,0.6). Moreover, the time taken by the algorithm is proportional to the number of squares in the mesh of the hyperbolic solver, thus all these improvements will be at a cost of approximately 1.8 times the cost for the standard mesh with $h = 0.1$.

Problems 5 and 6 illustrate the use of the solution of the Goursat problem method for particular solutions and integral equations algorithms, for boundary value problems for elliptic equations of the type $-\Delta u + [M(x_1) + N(x_2)]u = 0$. These have the advantages of approximately $O(h^4)$ rates of convergence, where h is the mesh length in the hyperbolic solver and the nodal width in the Simpson's quadrature rule used to approximate the integral in the particular solutions.

Methods for which the kernel function is approximated are particularly interesting because the kernel functions considered are unchanged in three dimensions. The numerical approximation of three dimensional boundary value problems for these elliptic equations present the same general difficulties as for boundary value problems with Laplace's equation and Helmholtz' equation. A particular problem is one of increased storage; the particular solutions algorithm and the integral equations algorithm reduce the problem to solving equations constructed on the boundary only, effectively reducing a three-dimensional problem to a two-dimensional problem.

APPENDIX I. THE EQUATIONS $-\Delta_3 u + M(x_1)u = 0$, $-\Delta_3 + N(x_2)u = 0$,

$-\Delta_3 u + [M(x_1) + N(x_2)]u = 0.$

Where

$$\Delta_3 u = \frac{\partial^2 u}{\partial x_1^{\,2}} + \frac{\partial^2 u}{\partial x_2^{\,2}} + \frac{\partial^2 u}{\partial x_3^{\,2}}\ , \qquad u = (x_1,x_2,x_3).$$

Let $h(x_1,x_2,x_3)$ be a harmonic function. That is $h_{11} + h_{22} + h_{33} = 0$. ($h_{11} = \frac{\partial^2 h}{\partial x_1^{\,2}}$ etc.)

For $-\Delta_3 u + M(x_1)u = 0$, we consider a solution of the form

$$u(x_1,x_2,x_3) = h(x_1,x_2,x_3) - \int_{-x_1}^{x_1} G(x_1,x_1-t)h(t,x_2,x_3)dt,$$

where G is twice differentiable with respect to each argument.

$$\begin{aligned}\frac{\partial^2 u}{\partial x_1^{\,2}} = h_{11} &- 2G_1(x_1,0)h(x_1,x_2,x_3) - G(x_1,0)h_1(x_1,x_2,x_3)\\ &- 2G_1(x_1,2x_1)h(-x_1,x_2,x_3) - 3G_2(x_1,2x_1)h(-x_1,x_2,x_3)\\ &- G_2(x_1,0)h(x_1,x_2,x_3) + G(x_1,2x_1)h_1(-x_1,x_2,x_3).\\ &- \int_{-x_1}^{x_1} [G_{11}(x_1,x_1-t) + 2G_{12}(x_1,x_1-t) + G_{22}(x_1,x_1-t)h(t,x_2,x_3)dt.\end{aligned}$$

Integrating by parts twice, we obtain

$$\begin{aligned}\int_{-x_1}^{x_1} G_{22}(x_1,x_1-t)h(t,x_2,x_3)dt = &-G_2(x_1,0)h(x_1,x_2,x_3) + G_2(x_1,2x_1)h(-x_1,x_2,x_3)\\ &-G(x_1,0)h_1(x_1,x_2,x_3) + G(x_1,2x_1)h_1(-x_1,x_2,x_3)\\ &+ \int_{-x_1}^{x_1} G(x_1,x_1-t)h_{11}(t,x_2,x_3)dt.\end{aligned}$$

Thus

$$\frac{\partial^2 u}{\partial x_1^2} = h_{11} - 2G_1(x_1,0)h(x_1,x_2,x_3) - 2G_1(x_1,2x_1)h(-x_1,x_2,x_3)$$

$$- 4G_2(x_1,2x_1)h(-x_1,x_2,x_3) - \int_{-x_1}^{x_1} [G_{11}+2G_{12}]h(t,x_2,x_3)dt$$

$$- \int_{-x_1}^{x_1} G(x_1,x_1-t)h_{11}(t,x_2,x_3)dt.$$

$$\frac{\partial^2 u}{\partial x_2^2} = h_{22} - \int_{-x_1}^{x_1} G(x_1,x_1-t)h_{22}(t,x_2,x_3)dt.$$

$$\frac{\partial^2 u}{\partial x_3^2} = h_{33} - \int_{-x_1}^{x_1} G(x_1,x_1-t)h_{33}(t,x_2,x_3)dt.$$

Thus

$$-\Delta_3 u + M(x_1)u$$

$$= 2G_1(x_1,0)h(x_1,x_2,x_3) + 2[G_1(x_1,2x_1) + 2G_2(x_1,2x_1)]h(-x_1,x_2,x_3)$$

$$+ \int_{-x_1}^{x_1} [G_{11} + 2G_{12}]h(t,x_2,x_3)dt$$

$$+ M(x_1)h(x_1,x_2,x_3) - \int_{-x_1}^{x_1} M(x_1)Gh(t,x_2,x_3)dt.$$

But $-\Delta_3 u + M(x_1)u = 0$

$\Rightarrow$ $\quad 2G_1(x_1,0) = M(x_1) \quad$ i.e. $G(x_1,0) = \frac{1}{2}\int_0^{x_1} M(\lambda)d\lambda$,

$G_1(x_1,2x_1) + 2G_2(x_1,2x_1) = 0 \quad$ i.e. $G(x_1,2x_1) = 0.$

$G_{11} + 2G_{12} - MG = 0.$

Let $\quad F(\xi,\eta) = G(x_1,x_1-t)$

where $\xi = \dfrac{x_1+t}{2}$, $\eta = \dfrac{x_1-t}{2}$

Then $$\frac{\partial^2 F}{\partial\xi\partial\eta} = MF \qquad \xi > 0, \quad \eta > 0$$

with $$F(\xi,0) = \frac{1}{2}\int_0^{\xi} M(\lambda)d\lambda \qquad \xi > 0, \tag{I.01}$$

$$F(0,\eta) = 0. \qquad \eta > 0.$$

$$u = h(x_1,x_2,x_3) - \int_{-x_1}^{x_1} F(\frac{x_1+t}{2}, \frac{x_1-t}{2})h(t,x_2,x_3)dt$$

$$\therefore \quad u = h(x_1,x_2,x_3) - x_1\int_{-1}^{1} F(x_1(\frac{1+t}{2}), x_1(\frac{1-t}{2}))h(tx_1,x_2,x_3)dt.$$

If we compare the above equations with the equations (3.22), we see that the kernel functions in 2 and 3 dimensions, and the Goursat problem they satisfy are the same.

APPENDIX II

REDUCTION TO CANONICAL FORM:

Consider the p.d.e.

$$R \frac{\partial^2 z}{\partial x_1{}^2} + S \frac{\partial^2 z}{\partial x_1, \partial x_2} + T \frac{\partial^2 z}{\partial x_2{}^2} + f(x_1, x_2, z, z_{x_1}, z_{x_2}) = 0 \qquad \text{(II.01)}$$

$$z = z(x_1, x_2) \ .$$

R, S, and T are functions of x_1 and x_2 possessing continuous partial derivatives of as high an order as necessary.

Take $\xi = \xi(x_1,x_2)$, $\eta = \eta(x_1,x_2)$ as new independent variables and write $z(x_1,x_2) = \zeta(\xi,\eta)$; then it is readily shown that (II.01) takes the form

$$A(\xi_{x_1}, \xi_{x_2}) \frac{\partial^2 \zeta}{\partial \xi^2} + 2B(\xi_{x_1}, \xi_{x_2}; \eta_{x_1}, \eta_{x_2}) \frac{\partial^2 \zeta}{\partial \xi \partial \eta}$$

$$+ A(\eta_{x_1}, \eta_{x_2}) \frac{\partial^2 \zeta}{\partial \eta^2} = F(\xi, \eta, \zeta, \zeta_{x_1}, \zeta_{x_2}), \qquad \text{(II.02)}$$

where $\quad A(v,u) = Ru^2 + Suv + Tv^2,$

and $\quad B(u_1,v_1;u_2,v_2) = Ru_1u_2 + \frac{1}{2} S(u_1v_2 + u_2v_1) + Tv_1v_2 \ .$

It is possible to choose ξ,η so that this equation takes the simplest possible form. We consider two cases:

Case a) $\qquad S^2 - 4RT > 0, \quad S^2 - 4RT < 0.$

If either of these are true then the roots λ_1, λ_2 of the equation $R\alpha^2 + S\alpha + T = 0$, are distinct, and the coefficients of $\frac{\partial^2 \zeta}{\partial \xi^2}$ and $\frac{\partial^2 \zeta}{\partial \eta^2}$

will vanish in (II.02) if ξ and η are chosen such that

$$\frac{\partial\xi}{\partial x_1} = \lambda_1 \frac{\partial\xi}{\partial x_2} , \qquad \frac{\partial\eta}{\partial x_1} = \lambda_2 \frac{\partial\eta}{\partial x_2} .$$

So a suitable choice for ξ and η is the general solutions of the characteristic equations

$$\frac{du}{dx_1} + \lambda_1(x_1,x_2) = 0, \qquad \text{(II.03)}$$

$$\frac{du}{dx_2} + \lambda_2(x_1,x_2) = 0. \qquad \text{(II.04)}$$

The solutions of these equations are $\xi(x_1,x_2)$ = constant, and $\eta(x_1,x_2)$ = constant.

Case b) $S^2 - 4RT = 0$

In this case the roots of the equation

$$R\alpha^2 + S\alpha + T = 0,$$

are equal. Define ξ as in case a) and take η as any function of x_1,x_2 which is independent of ξ.

i) Consider $r[G_r - PG] - G_r + 2(1-t)G_{rt} = 0.$

This is case a) where

$$S = 2(1-t), \quad R = r,$$

$$\therefore \quad r\alpha^2 + 2(1-t)\alpha = 0,$$

and $$\lambda_1 = 0, \quad \lambda_2 = \frac{-2(1-t)}{r} .$$

Equations of the characteristics are

$$\frac{dt}{dr} = 0 \quad => \quad t = \text{constant}.$$

$$\frac{dt}{dr} - \frac{2(1-t)}{r} = 0 \quad => \quad , \quad r^2(1-t) = \text{constant}$$

$$\text{or} \quad r\sqrt{1-t} = \text{constant}.$$

Thus possible new variables, τ and ρ are given by

$$\tau = t, \quad \rho = r\sqrt{1-t} \quad ; \qquad \text{Gilbert and Linz},$$

or $$\tau = t, \quad \rho = \frac{r^2(1-t)}{r_0{}^2} \quad ; \qquad r_0{}^2 = \text{constant}.$$

ii) Consider $W_{11} + 2W_{12} - M(x_1)W = 0$.

Then $S = 2$, $R = 1$, $T = 0$,

$$S^2 - 4RT = 4 > 0.$$

The roots of the equation $\alpha^2 + 2\alpha = 0$,

are $$\lambda_1 = 0, \qquad \lambda_2 = -2.$$

The equations of the characteristic are given by

$$\frac{d\eta'}{dx_1} = 2 \quad => \quad \eta' - 2x_1 = \text{constant}$$

and $$\frac{d\eta'}{dx_1} = 0 \quad => \quad \eta' = \text{constant}.$$

Thus one choice of new independent variables is

$$\eta = \eta'/2 \quad ,$$

$$\xi = (2x_1 - \eta')/2 \quad .$$

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