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PROBABILISTIC ASPECTS OF BIVARIATE COUNTING SYSTEMS

BY

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Cold hearted orb that rules the night,
Removes the colours from our sight.
Red is grey and yellow white,
But we decide which is right.
And which is illusion?

The Moody Blues, "Days of Future Passed."

ABSTRACT

This thesis is concerned largely with a bivariate counting system arising in radio-activity measurement. To estimate the disintegration rate of a radio-active source use is made of three independent Poisson processes. These Poisson processes cannot be observed individually. The first and third processes are recorded on one counter, the second and third processes on a second counter. Both counters are subject to dead-time effects; following a recorded event on a counter there is a constant period during which no other events can be recorded. Thus events from all three processes are lost to observation. The estimation of the rates of the three Poisson processes usually involves the use of either the covariance between the numbers of recorded events on the two counters in a given time interval, or the coincidence rate; that is the rate of pairs of events occurring close together.

This system is generalised in three ways in order to model the true process of disintegrations more realistically:

(i) Events from the second and third processes are of two types, each type invoking a different property in the second counter.

(ii) An event from the third process occurs on the second counter an exponentially distributed period after its occurrence on the first counter.

(iii) Each event on either counter is displaced by a random amount.

The coincidence rate is calculated for generalisations (i), (ii) and (iii), the covariance function for generalisations (ii) and (iii).

Two univariate counting systems are also considered; they are :

(iv) When all three processes in (ii) are recorded on a single counter.

(v) When the rate of events into a counter is decaying exponentially.

The expected number of recorded events in a given interval is calculated for (iv) and the expectation and variance for (v).

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THE PRINCE AND THE MAGICIAN

Once upon a time there was a young prince, who believed in all things but three. He did not believe in princesses, he did not believe in islands, he did not believe in God. His father, the King, told him that such things did not exist. As there were no princesses or islands in his father's domaines, and no sign of God, the young prince believed his father.

But then, one day, the prince ran away from his palace. He came to the next land. There, to his astonishment, from every coast he saw islands, and on these islands, strange and troubling creatures whom he dared not name. As he was searching for a boat, a man in full evening dress approached him along the shore.

'Are those real islands?' asked the young prince.

'Of course they are real islands,' said the man in evening dress.

'And those strange and troubling creatures?'

'They are all genuine and authentic princesses.'

'Then God also must exist!' cried the prince.

'I am God,' replied the man in full evening dress, with a bow.

The young prince returned home as quickly as he could.

'So you are back,' said his father, the king.

'I have seen islands, I have seen princesses, I have seen God,' said the prince reproachfully.

The king was unmoved.

'Neither real islands, nor real princesses, nor a real God, exist.'

'I saw them!'

'Tell me how God was dressed.'

'God was in full evening dress.'

'Were the sleeves of his coat rolled back?'

The prince remembered that they had been. The king smiled.

'That is the uniform of a magician. You have been deceived.'

At this, the prince returned to the next land, and went to the same shore, where once again he came upon the man in full evening dress.

'My father the king has told me who you are,' said the young prince indignantly. 'You deceived me last time, but not again. Now I know that those are not real islands and real princesses, because you are a magician.'

The man on the shore smiled.

'It is you who are deceived, my boy. In your father's kingdom there are many islands and many princesses. But you are under your father's spell, so you cannot see them.'

The prince returned pensively home. When he saw his father he looked him in the eyes.

'Father, is it true that you are not a real king, but only a magician?'

The king smiled and rolled back his sleeves.

'Yes, my son, I am only a magician.'

'Then the man on the shore was God.'

'The man on the shore was another magician.'

'I must know the real truth, the truth beyond magic.'

'There is no truth beyond magic,' said the king.

The prince was full of sadness.

He said, 'I will kill myself.'

The king by magic caused death to appear. Death stood in the door and beckoned to the prince. The prince shuddered. He remembered the beautiful but unreal islands and the unreal but beautiful princesses.

'Very well,' he said. 'I can bear it.'

'You see, my son,' said the king, 'You too now begin to be a magician.'

John Fowles, 'The Magus.'

CHAPTER 1. A GENERAL INTRODUCTION

1.1 Simplest Physical Situation

Consider a source that emits a stream of particles which form a Poisson process. An example of such a source is a radio-active isotope, the emissions being due to the disintegration process that the isotope is continually undergoing. If the disintegration rate, the rate of the Poisson process, is unknown, then it is of importance to estimate it. The procedures for this estimation fall into two broad categories, which are distinguished by the way in which we observe the process. Either we look at the time intervals between successive emissions or we may count the number of emissions in a given time interval. The former is impracticable, in the context of the above example with which we shall be concerned, due to the high rates involved. The reason is that the intervals between the emissions are then too small to be measured accurately and too many to be stored feasibly.

Given that we are counting numbers of emissions we examine the properties of the counting mechanism, namely an electronic counter. Ideally we would require the counter to record whenever an emitted particle reaches and passes through it. Unfortunately real counters are not perfect, and for the purpose of illustrating these imperfections the counting mechanism considered here consists of two distinct components in series: a detector followed by the counter itself. The effect of the detector is such that whenever an emission occurs it is only detected and so allowed to pass through to the counter with a certain probability, which is less than unity and known as the efficiency of the counter. This efficiency is unknown. Furthermore the detection of each emission is independent of the detection of all other emissions. We now turn our attention to the second of the two components, the counter itself. This suffers from dead-time effects. Following a recorded emission, the counter is dead for a time during which no further events can be recorded. Basically, two types of dead-time are usually considered:

(i) The non-extended type where emissions occurring within a dead-time have no effect on the dead-time.

(ii) The extended type where emissions occurring within a dead-time induce more dead-times and so prolong the existing dead-time.

Both types are approximations to the true dead-time behaviour of a counter, but type (i) is usually considered the closer approximation, particularly if the dead-time is of constant length, τ say. For this reason constant dead-times of type (i) will be considered throughout this thesis except in the final chapter, which has no immediate practical consequence but is of great theoretical importance.

1.2 One Counter

Much statistical/probabilistic work has been done on the single counter system, particularly in the 1950's. Albert and Nelson (1953) noted that true dead-time behaviour is a compromise between the type (i) and type (ii) counters of section 1.1. They modelled such a compromise by assuming that the dead-time behaviour of their counter was type (ii) with each induced dead-time being of constant length, except that an event during a dead-time only induces another dead-time with a certain fixed probability. They then proceeded to determine the distribution of the number of recorded events and a confidence interval for the rate of events into the counter, assuming the process of events to be Poisson. Takács (1958) also determined the distribution of the number of recorded events for this problem but with random dead-times. Hammersley (1953) discussed the problems of counting blood cells electronically. The counting system Hammersley considered is equivalent to one in which a Poisson process of events is transformed in sequence first by a type (ii) counter and then by a type (i) counter; the dead-times of both counters being of random length. Hammersley investigated the univariate distributions of events before and after type (i) deletion; his approach was unusual and complex in that he substituted the circumference of a circle for the investigated time interval. The two distributions of recorded events, before and after type (i) deletion, was found to be asymptotically normal as the circumference of the circle tends to infinity; the asymptotic means and variances were also calculated. Takâcs (1956) used much simpler methods to treat the same problems, and obtained a slightly different result from Hammersley for the limiting variance of the number of recorded events after type (i) deletion, in a given interval, as the length of the interval tends to infinity. This section of the literature is neatly tied up by Smith (1957) and Pyke (1958). Smith solved Hammersley's (1953) problem by renewal type arguments and also noted that the asymptotic variance mentioned above was erroneous. Pyke considered type (i) and type (ii) counters in a general renewal framework. Many references to the early work on counters may be found in Smith (1958).

However, although the above work is of great theoretical importance and many elegant methods have been devised to tackle one counter problems, from our practical viewpoint they are of little use. This is because the detector in the counting system ensures that the rate of events into the counter is not λ, the parameter of interest, but $\lambda\varepsilon$, where ε is the efficiency of the detector and is unknown. Clearly the single counter system is inadequate for the purpose of estimating the rate of disintegrations λ.

1.3 Two Counters

1.3.1 A More Practicable Counting System

Each emission described in section 1.1 in fact consists of two particles, a beta particle and a gamma particle. We can use this property of the disintegration process by introducing a second counter. One counter is made insensitive to beta particles while the other is made insensitive to gamma particles. Because of the differences in response of the detectors to the beta and gamma particles and because of other possible differences between the two counters, we denote the efficiency of the beta counter by ε_β and the constant dead-time by τ_β, with ε_γ and τ_γ denoting corresponding parameters for the gamma counter. Since the detectors on the two counters work independently there are three types of event which may occur at the detectors following an emission,

(i) a beta particle only is detected with rate

$$\lambda_\beta = \lambda\varepsilon_\beta(1 - \varepsilon_\gamma) ,$$

(ii) a gamma particle only is detected with rate

$$\lambda_\gamma = \lambda\varepsilon_\gamma(1 - \varepsilon_\beta) ,$$

(iii) a beta-gamma pair of particles is detected with rate

$$\lambda_{\beta\gamma} = \lambda\varepsilon_\beta\varepsilon_\gamma .$$

Thus, so far as the counters are concerned, the original Poisson process of rate λ is equivalent to the superposition of three independent Poisson processes of rates λ_β, λ_γ and $\lambda_{\beta\gamma}$. By looking solely at the beta counter it is possible to estimate the rate of particles entering that counter, $\lambda_\beta + \lambda_{\beta\gamma}$, and similarly for the gamma counter. However, to estimate all three parameters, that is λ_β, λ_γ and $\lambda_{\beta\gamma}$, or equivalently λ, ε_β and ε_γ, a measurement which is affected by dependence between the two counters has to be taken. This dependence is caused by the simultaneous events of rate $\lambda_{\beta\gamma}$. The most commonly used methods of incorporating such a measurement into the estimation procedure are those of coincidence counting and covariance calculation.

1.3.2 Coincidence Counting

A third counter, known as a coincidence counter, may be connected to the beta and gamma counters so that the sequence of events recorded by the coincidence counter is obtained by superimposing that sequence recorded on the beta counter, upon that sequence recorded on the gamma counter. A coincident event then occurs whenever there is an event simultaneously on each counter, i.e. a double event on the coincidence counter. However, the coincidence counter cannot resolve between two events that are separated by some small time interval and therefore a count is made of the number of pairs of events that are less than some suitable time h apart. If h is chosen so that it is less than the minimum dead-time of the two counters, i.e. $h < \min(\tau_\beta, \tau_\gamma)$ then the two events must be recorded on different counters, which is more like a coincidence rate. Consequently, the coincidence method will capture all the genuine simultaneous events plus some "accidental" coincidences. These accidental coincidences are hopefully small in number and this number is dependent on the size of the resolving time h.

1.3.3 Covariance Calculation

By measuring the number of recorded events on each of the two counters over a given interval of length t and repeating this many times the covariance between the two counts may be observed. The main advantages of this over coincidence counting are,

(i) no restrictions are placed on the sizes of dead-times other than those imposed by the individual counters,

(ii) the extra coincidence measurement is no longer needed, all necessary information being contained in a sequence of pairs of counts obtained from the beta and gamma counters.

1.4 Problems to be Considered

The counting system of section 1.3 was considered by Cox and Isham (1977) who calculated both the covariance function and the coincidence rate for constant dead-times, where the larger of the two dead-times is an integer multiple of the smaller, and for exponentially distributed dead-times. In this thesis the basic disintegration process and counting system of section 1.3 are generalised in three separate ways to achieve a slightly more realistic system:

(i) Two different types of gamma particles are emitted from the source and they invoke different properties in the gamma counter; see Chapter 2.

(ii) The gamma particle of each beta-gamma pair is delayed by an exponentially distributed period; see Chapter 3.

(iii) Jitter is allowed to enter the counting mechanism so that simultaneous emissions are never recorded simultaneously; see Chapter 3.

In addition to (i), (ii) and (iii) two univariate systems are considered which are relevant to the true physical processes when,

(iv) the beta and gamma particles of (ii) are indistinguishable and have to be recorded on the same counter, see Chapter 4,

(v) the disintegration rate decays exponentially, see Chapter 5.

The objective of this thesis is not to provide estimates for the disintegration rate λ in cases (i)-(v) but to provide properties of the numbers of recorded events that may be used in an estimation procedure; although a possible estimate for the system of section 1.3 is mentioned in section 1.5. The functions of the numbers of recorded events are, the expectation for (i)-(v), the variance for (iv), the covariance for (ii) and (iii), the coincidence rate for (i)-(iii). The methods used successfully by Cox and Isham (1977) will be used to a large extent in this work and provides an invaluable basis for the calculations of Chapters 2-4.

Cox and Isham (1977) noted that the leading term in their expression for the covariance was the same for constant and exponentially distributed dead-times; for the case of constant dead-times, the larger dead-time was constrained to be an integer multiple of the smaller dead-times. They also conjectured that this leading term could be derived by a simple probabilistic argument for arbitrary dead-times. Kingman (1977) partially answered this conjecture by proving that the leading term in the covariance was the same for any dead-time distribution. However Kingman's analysis is elegant but by no means a simple probabilistic argument. Similar leading terms appear in the covariances for (ii) and (iii), but no simple probabilistic argument was found here either.

A wealth of semi-empirical formulae appears in the physical literature for a variety of counting systems that include (i)-(v), and a short survey may be found in Müller (1973). Perhaps the most relevant papers for this thesis are Lewis, Smith and Williams (1973) for (ii), (iv) and (v), Williams and Campion (1965) for (iii) and Axton and Ryves (1963) for (v).

1.5 A Possible Estimation Procedure

....,λ,..... but this still figure, benign, all-powerful, yet unable to intervene or speak, able simply to be and constitute.

John Fowles, "The Magus."

If we assume that the counting system of section 1.3 is the most relevant to a particular practical situation, then assuming the results of Cox and Isham (1977),

(i) the number of recorded events on the beta counter in a given time interval can be used to estimate $\rho_\beta = \lambda_\beta + \lambda_{\beta\gamma} = \lambda\varepsilon_\beta$,

(ii) the number of recorded events on the gamma counter in a given time interval can be used to estimate $\rho_\gamma = \lambda_\gamma + \lambda_{\beta\gamma} = \lambda\varepsilon_\gamma$,

(iii) either the covariance between or the coincidence rate of, the number of recorded events on each counter in a given time interval, in conjunction with (i) and (ii), can be used to estimate $\lambda_{\beta\gamma} = \lambda\varepsilon_\beta\varepsilon_\gamma$.

If the three estimates given by (i), (ii) and (iii) are denoted by $\hat{\rho}_\beta$, $\hat{\rho}_\gamma$ and $\hat{\lambda}_{\beta\gamma}$ then we can estimate λ by $\hat{\lambda}$, where

$$\hat{\lambda} = \hat{\rho}_\beta\hat{\rho}_\gamma/\hat{\lambda}_{\beta\gamma} ,$$

since

$$\lambda = \rho_\beta \rho_\gamma / \lambda_{\beta\gamma} .$$

The efficiency on each counter is unknown and yet controllable, so that the experiment that gave (i), (ii) and (iii) could be repeated at different efficiencies and a number of times at each efficiency. The point estimates obtained from a series of experiments could then be averaged to produce an overall estimate of the original disintegration rate λ.

Unfortunately, as Campion (1959) pointed out the beta counter can be slightly sensitive to gamma particles, that is gamma particles are occasionally recorded on the beta counter provided the beta particle is not recorded. In this case the effective efficiency, or rather the effective detection rate on the beta detector is

$$\varepsilon'_\beta = \varepsilon_\beta + (1 - \varepsilon_\beta)\varepsilon_{\beta\gamma},$$

where ε_β is still the probability that a beta is detected and $(1 - \varepsilon_\beta)\varepsilon_{\beta\gamma}$ is the probability that the gamma is detected by the beta counter, given that the beta is not. If we assume that the beta counter gets the chance of detecting the gamma particle first, then the gamma particle cannot be detected on both counters. Hence the efficiency of the gamma counter is now slightly less than it is in the absence of gamma sensitivity of the beta counter.

If we note that the rate into the beta counter is now

$$\lambda\varepsilon'_\beta = \lambda\{1 - [1 - \varepsilon_{\beta\gamma}][1 - \varepsilon_\beta]\} ,$$

and $\hat{\lambda}_{\beta\gamma}/\hat{\rho}_\gamma$ estimates ε_β, then by regressing ε_β on the rate of events into the beta counter, an estimate of the original disintegration rate λ may be found.

The above estimation procedure is not discussed further and the reader is referred to Lewis, Smith and Williams (1973) for a particular case of practical application. We now move to the first of the cases (i)-(v) of section 1.4, namely case (i).

CHAPTER 2. IN AND OUT OF CHANNEL GAMMA EVENTS

2.1 Physical Reality and Theoretical Representation

We now generalise the basic model, as described in the introduction, by relaxing an unrealistic theoretical constraint.

The process of events arriving at the counter mechanism is the same as that described in the introduction and our generalisation concerns the mechanisms' response to the gamma stream of particles. Each gamma particle emitted from the source has an associated energy level. In the gamma counter particles with different energy levels invoke differing efficiencies and differing dead-times. To model this situation fully, with the purpose of calculating the covariance between the beta and gamma counters and/or some sort of coincidence rate, is complex. The true situation may be approximated in several ways to arrive at a theoretical model; the possibilities include the following:

(i) We may totally ignore the effects of different energy levels by averaging the true efficiencies and dead-times, and using these averages in the model of section 1.3 of the General Introduction. Thus

$$\varepsilon_\gamma = \sum_{\text{levels}} \left\{ \begin{array}{l} \text{efficiency} \\ \text{at each} \\ \text{different level} \end{array} \right\} \text{pr} \left\{ \begin{array}{l} \text{a particle having} \\ \text{a corresponding} \\ \text{energy level} \end{array} \right\},$$

$$\tau_\gamma = \sum_{\text{levels}} \left\{ \begin{array}{l} \text{dead-time} \\ \text{at each} \\ \text{different level} \end{array} \right\} \text{pr} \left\{ \begin{array}{l} \text{a particle having} \\ \text{a corresponding} \\ \text{energy level} \end{array} \right\}.$$

But, to acknowledge the different energy levels by ignoring their effects, is no step forward.

(ii) We may partially ignore the effects of different energy levels by accepting differing efficiencies, but assuming identical dead-time reaction to all gamma particles, regardless of their energy level. This may be achieved either by the averaging process of (i), applied only to the dead-time, and implementation of this average as the effective dead-time in the model, or by physically setting all the dead-times equal to the largest dead-time.

(iii) We may compromise between modelling the true physical situation and that proposed in (i) by grouping the different energy levels into two bands, and treating each band as in (i). This is particularly appropriate when we are interested in those gamma particles whose energy levels fall inside a certain band of energies, so-called in-channel gamma particles. Those particles whose energy levels fall outside this particular band are referred to as out-of-channel gamma particles. The counting mechanism is then assumed to react to all in-channel particles in the same way and to all out-of-channel particles in a different way.

Cox and Isham (1977) calculated the covariance between the beta and gamma counts and the coincidence rate between beta's and in-channel gamma's when the in-channel and out-of-channel dead-times are equal. In the rest of this chapter we shall relax the assumption of equal dead-times to obtain a slightly more realistic model for the problem of different energy levels.

Superficially it would appear advantageous to set the gamma counter mechanism to react only to in-channel gamma particles. However, if this were the case, then whenever a gamma particle entered the mechanism, the mechanism would first have to decide whether or not the particle's energy level is such that it is classified as in-channel or out-of-channel.

During the time that this decision process is in operation the mechanism would be unable to count any other gamma particles that happen to arrive. In effect the mechanism would be dead. Therefore the out-of-channel particles cannot be ignored, for otherwise the rate of in-channel particles emitted by the detector would no longer be equal to the rate of in-channel particles arriving at the counter, assuming that the classification process takes place between the detection and counting stages. (In (iii), the decision process time is added to the true dead-time, resulting in the effective dead-time.)

We now develop the model proposed in (iii). In addition to the total count on the gamma counter we are able to record the number of in-channel gamma particles, and hence the number of out-of-channel gamma particles that occur in any given time interval. Due to the action of the detectors the original Poisson process can be considered as consisting of five independent Poisson processes. If we define p_{in} to be the probability that a particular gamma particle is an in-channel gamma particle, and $\varepsilon_{\gamma in}$ to be the efficiency of the gamma detector to an in-channel gamma particle, then assuming corresponding results for out-of-channel particles, the five different types of event corresponding to the five independent processes which may occur at the detectors are the arrival of

(i) a beta particle only, detected with rate .

$$\lambda_\beta = \lambda\varepsilon_\beta(1 - \varepsilon_{\gamma in})p_{in} + \lambda\varepsilon_\beta(1 - \varepsilon_{\gamma out})p_{out},$$

(ii) an in-channel gamma particle only, detected with rate

$$\lambda_{\gamma in} = \lambda\varepsilon_{\gamma in}p_{in}(1 - \varepsilon_\beta) ,$$

(iii) an out-of-channel gamma particle only, detected with rate

$$\lambda_{\gamma out} = \lambda\varepsilon_{\gamma out}P_{out}(1 - \varepsilon_\beta) ,$$

(iv) a beta particle with an in-channel gamma particle, i.e. a true in-channel coincidence, detected with rate

$$\lambda_{\beta\gamma in} = \lambda\varepsilon_\beta\varepsilon_{\gamma in}P_{in} ,$$

(v) a beta particle with an out-of-channel gamma particle, i.e. a true out-of-channel coincidence, detected with rate

$$\lambda_{\beta\gamma out} = \lambda\varepsilon_\beta\varepsilon_{\gamma out}P_{out} .$$

In the above formulation there are four independent parameters λ, ε_β, $\varepsilon_{\gamma in}P_{in}$ and $\varepsilon_{\gamma out}P_{out}$. Therefore to estimate the original disintegration rate λ, four independent measurements on the counting system are required. These may be chosen from the following (see section 2.5),

(i) the expected number of recorded beta particles in a given time interval,

(ii) the expected number of recorded (a) in-channel, (b) out-of-channel and (c) all, gamma particles in a given time interval,

(iii) the coincidence rate between beta particles and (a) in-channel, (b) out-of-channel and (c) all, gamma particles.

It should be noted that, in (ii) and (iii), any two functions readily give the third by simple addition or subtraction.

For notational convenience, we relabel $\beta \equiv 1$, $\gamma_{in} \equiv 2$ and $\gamma_{out} \equiv 3$, the beta counter as counter 1 and the gamma counter as counter 2. However, the physical interpretation of all events will be kept, i.e. λ_{12} will still be referred to as a true in-channel coincidence.

2.2 The Expectation on Each Counter

If $N_i(t)$ is defined to be the total number of recorded events in time t, on counter i for i = 1,2, then $N_i(t)$ may be represented as

$$N_i(t) = \int_0^t dN_i(u), \qquad \text{for } i = 1,2 \tag{2.1}$$

Therefore, if the processes on each counter are in statistical equilibrium at the start of the interval (0,t], and if p_i denotes the equilibrium probability that counter i is open for i = 1,2, then taking expectations through (2.1) we have that for i = 1,2,

$$E\{N_i(t)\} = \int_0^t pr\{dN_i(u) = 1\}$$

$$= \int_0^t p_i(\text{total rate on counter } i)du \ . \tag{2.2}$$

The total rate on counter 1 is $\rho_1 = \lambda_1 + \lambda_{12} + \lambda_{13} = \rho_{12} + \lambda_{13} = \rho_{13} + \lambda_{12}$ and the total rate on counter 2 is $\rho_{23} = \lambda_2 + \lambda_3 + \lambda_{12} + \lambda_{13} = \rho_2 + \lambda_3 + \lambda_{13}$ $= \lambda_2 + \lambda_{12} + \rho_3$. Thus calculating the expectations reduces to calculating the equilibrium probabilities p_i for i = 1 and 2.

Now the sequence of states on counter 1 alternates between exponentially distributed open periods of mean ρ_1^{-1} and dead-times of constant length τ_1. On counter 2 the sequence of states alternates between exponentially distributed open periods of mean ρ_{23}^{-1} and dead-times of variable length T, where

$$T = \tau_i \text{ with probability } \rho_i\rho_{23}^{-1} \quad \text{for } i = 2,3.$$

Hence the equilibrium probabilities that the counters are open are

(i) for counter 1,

$$p_1 = \rho_1^{-1}\{(\rho_1^{-1} + \tau_1)\}^{-1} = (1 + \rho_1\tau_1)^{-1} ; \qquad (2.3)$$

(ii) for counter 2,

$$p_2 = \rho_{23}^{-1}\{\rho_{23}^{-1} + E(T)\}^{-1} = (1 + \rho_2\tau_2 + \rho_3\tau_3)^{-1}. \qquad (2.4)$$

The expectations may now be calculated using (2.2)-(2.4),

$$E\{N_1(t)\} = \int_0^t p_1\rho_1 du = p_1\rho_1 t = \rho_1 t(1 + \rho_1\tau_1)^{-1} , \qquad (2.5)$$

and

$$E\{N_2(t)\} = \int_0^t p_2\rho_{23} du = p_2\rho_{23} t = \rho_{23} t(1 + \rho_2\tau_2 + \rho_3\tau_3)^{-1} . \qquad (2.6)$$

It may be seen that $E\{N_2(t)\}$ is the sum of two components, $E\{N_{22}(t)\}$ the expected number of in-channel gamma particles that are recorded in (0,t], and $E\{N_{23}(t)\}$ the expected number of out-of-channel gamma particles that are recorded in (0,t]. We have that

$$E\{N_{2i}(t)\} = \rho_i t(1 + \rho_2\tau_2 + \rho_3\tau_3)^{-1} , \quad \text{for } i = 2,3. \qquad (2.6a)$$

Some measure of the dependence between the two counters is now calculated.

2.3 The Possible States of the Counting System

To calculate the coincidence rates we need to obtain the probability that both counters are open simultaneously and, if a counter is closed, when the counter reopens. So, if either counter is closed we need to know how long it has been closed; and for counter 2 we also need to know which type of event closed it, an in-channel or an out-of-channel event, due to the differing dead-times associated with these events. Thus, the following state probabilities and probability densities are defined:

(i) p_{12}, the joint probability that both counters are open;

(ii) $q_1(u)$, the joint probability density that counter 1 has been closed for a period u, $0 \leq u \leq \tau_1$, and counter 2 is open;

(iii) $q_{2i}(v)$, the joint probability density that counter 1 is open and that counter 2 has been closed for a period v; $0 \leq v \leq \tau_i$, where the event that caused the closure is (a) in-channel (i = 2) or (b) out-of-channel (i = 3),

(iv) $q_{12i}(u,v)$, the joint probability density that counter 1 has been closed for a period u, $0 \leq u \leq \tau_1$, and that counter 2 has been closed for a period v, $0 \leq v \leq \tau_i$, where the event that caused the closure is (a) in-channel (i = 2) or (b) out-of-channel (i = 3).

Therefore, there are six possible states for the counting system, and to calculate the corresponding state probabilities defined above, the equilibrium equations representing the possible transitions from one state to another, may be set up as follows. First

$$\rho p_{12} = q_1(\tau_1) + q_{22}(\tau_2) + q_{23}(\tau_3) \,, \qquad (2.7)$$

where the period that both counters are open simultaneously is exponentially distributed with mean ρ^{-1}, $\rho = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_{12} + \lambda_{13}$. Also, we have that

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + q_{122}(u,\tau_2) + q_{123}(u,\tau_3), \tag{2.8}$$

$$\frac{dq_{2i}(u)}{du} = -\rho_1 q_{2i}(u) + q_{12i}(\tau_1,u), \qquad i = 2,3, \tag{2.9}$$

$$\frac{\partial q_{12i}(u,v)}{\partial u} + \frac{\partial q_{12i}(u,v)}{\partial v} = 0 , \qquad i = 2,3 . \tag{2.10}$$

A set of boundary conditions corresponding to the instant when one of the counters becomes blocked, may also be formulated,

$$q_1(0) = p_{12}\lambda_1 , \qquad q_{2i}(0) = p_{12}\lambda_i , \quad i = 2,3, \tag{2.11}$$

$$q_{12i}(u,0) = \rho_i q_1(u), \quad q_{12i}(0,v) = \rho_1 q_{2i}(v), \; i = 2,3. \tag{2.12}$$

Because both counters may be closed simultaneously with non-zero probability, then $q_{12i}(u,v)$ for i = 2,3 may be separated into

$$q_{12i}(u,v) = q_{12is}(u)\delta(u-v) + q_{12ic}(u,v) , \tag{2.13}$$

where $\delta(\cdot)$ is the Dirac delta function, and both $q_{12is}(u)$ and $q_{12ic}(u,v)$ are absolutely continuous. The simultaneous closing of both counters may now be expressed as

$$q_{12is}(u) = \lambda_{1i}p_{12} , \qquad i = 2,3 . \tag{2.14}$$

There may be up to two more boundary conditions, depending on the relative sizes of the three dead-times, and caused by possible simultaneous closure of both counters. Thus, for the counter with the largest dead-time, the probability density that it is closed and the other counter is open, has a discontinuity when the period of closure is equal to the smallest of the three dead-times, and possibly the median. For example, if $\tau_1 > \tau_2 > \tau_3$, then

$$q_1(\tau_3^+) = q_1(\tau_3^-) + p_{12}\lambda_{13},$$

$$q_1(\tau_2^+) = q_1(\tau_2^-) + p_{12}\lambda_{12} .$$

To calculate the coincidence rate we need only find $p_{12}, q_1(u)$ and $q_{2i}(v)$ for i = 2,3 (see section 2.4). With this objective we eliminate the joint probability densities $q_{12i}(u,v)$, for i = 2,3, from equations (2.8) and (2.9).

The solution of (2.10) subject to (2.12) is for i = 2,3

$$q_{12ic}(u,v) = \begin{cases} \rho_i q_1(u-v) & \text{for } u > v \\ \rho_1 q_{2i}(v-u) & \text{for } v > u. \end{cases}$$

Upon substitution into (2.8) and (2.9) we find that for i = 2,3,

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + \begin{Bmatrix} \rho_3 q_1(u-\tau_3) & \text{for } u > \tau_3 \\ \rho_1 q_{23}(\tau_3-u) & \text{for } u < \tau_3 \end{Bmatrix} + \begin{Bmatrix} \rho_2 q_1(u-\tau_2) & \text{for } u > \tau_2 \\ \rho_1 q_{22}(\tau_2-u) & \text{for } u < \tau_2 \end{Bmatrix} \tag{2.15}$$

and

$$\frac{dq_{2i}(v)}{dv} = -\rho_1 q_{2i}(v) + \begin{cases} \rho_1 q_{2i}(v-\tau_1) & \text{for } v > \tau_1 \\ \rho_i q_1(\tau_1 - v) & \text{for } \tau_1 > v. \end{cases} \qquad (2.16)$$

The solution of (2.15) and (2.16) greatly simplifies if the two larger dead-times are integer multiples of the smallest dead-time. Therefore, we assume that $\tau_1 = m\tau$, $\tau_2 = n\tau$ and $\tau_3 = \ell\tau$ where ℓ,m and n are positive integers such that $\min(\ell,m,n) = 1$. The ranges of ℓ, m and n that will be considered are $m \geq n \geq \ell$ and $n \geq m \geq \ell$ with the exception of $\ell = m = n$ which has been dealt with by Cox and Isham (1977). Note that some ranges are omitted, these are (a) the minimum of the two gamma channel dead-times is greater than the beta dead-time, i.e. $\min(\tau_2,\tau_3) > \tau_1$, which is physically improbable, (b) the out-of-channel dead-time being at least as large as the beta dead-time which in turn is at least as large as the in-channel dead-time, i.e. $\tau_3 \geq \tau_1 \geq \tau_2$, and (c) the in-channel dead-time being less than the out-of-channel dead-time which in turn is less than the beta dead-time, i.e. $\tau_1 > \tau_3 > \tau_2$. Note that ranges (b) and (c) may be obtained by interchanging the physical interpretation of 3 ≡ out-of-channel and 2 ≡ in-channel.

To calculate the coincidence rates, $q_1(u)$ and $q_{2i}(v)$ for i = 2,3 are required, see section 2.4. However, exact closed form solutions are not attempted for these probability densities. Instead, the range of each density is split into integer multiples of the smallest dead-time of the two counters, and separate first order Taylor expansions of each density are made within each section of the range. It is assumed that $(\rho\tau)$ is small, so that terms of order $(\rho\tau)^2$ may be neglected; such terms will be omitted throughout the rest of this section. Thus approximate forms for the probability densities $q_1(u)$ and $q_{2i}(v)$, $i = 2,3$, are calculated for combinations of the relative sizes of the three dead-times τ_1, τ_2 and τ_3. We have that for $\tau_1 > \tau_2 > \tau_3$, i.e. $m > n$ and $n > 1$, (2.15) and (2.16) are

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + \begin{cases} \rho_1 q_{22}(\tau_2-u)+\rho_1 q_{23}(\tau_3-u) & \text{for } 0<u<\tau_3 \\ \rho_1 q_{22}(\tau_2-u)+\rho_3 q_1(u-\tau_3) & \text{for } \tau_3<u<\tau_2 \\ \rho_2 q_1(u-\tau_2) + \rho_3 q_1(u-\tau_3) & \text{for } \tau_2<u<\tau_1 \end{cases} , \tag{2.17}$$

and for $i = 2,3$

$$\frac{dq_{2i}(v)}{dv} = -\rho_1 q_{2i}(v) + \rho_i q_1(\tau_1-v), \quad \text{for } 0 < v < \tau_i . \tag{2.18}$$

In addition to the boundary conditions (2.11), we have discontinuity conditions

$$q_1(\tau_3^{+}) = q_1(\tau_3^{-}) + p_{12}\lambda_{13} ,$$

and

$$q_1(\tau_2^{+}) = q_1(\tau_2^{-}) + p_{12}\lambda_{12} . \tag{2.19}$$

Therefore, the solution of (2.17) and (2.18), subject to (2.11) and (2.19), yields approximate forms for the probability densities $q_1(u)$ and $q_{2i}(v)$, $i = 2,3$, when $\tau_1 > \tau_2 > \tau_3$, as follows:

$$q_1(u)=p_{12}\begin{cases}\lambda_1+a_1u & 0\le u\le\tau_3\\ (\lambda_1+\lambda_{13})+a_1\tau_3+a_2(u-\tau_3) & \tau_3<u\le 2\tau_3\\ (\lambda_1+\lambda_{13})+\{a_1+a_2+(r-3)a_3\}\tau_3+a_3\{u-(r-1)\tau_3\} & (r-1)\tau_3<u\le r\tau_3\\ & \text{for } r=3,\ldots,n\\ \rho_1+\{a_1+a_2+(n-2)a_3\}\tau_3+a_4(u-n\tau_3) & \tau_2<u\le(n+1)\tau_3\\ \rho_1+\{a_1+a_2+(n-2)a_3+a_4+(s-n-2)a_5\}\tau_3+a_5\{u-(s-1)\tau_3\} & (s-1)\tau_3<u\le s\tau_3\\ & \text{for } s=n+2,\ldots,2n\\ \rho_1+\{a_1+a_2+(n-2)a_3+a_4+(n-1)a_5\}\tau_3 & 2n\tau_3<u\le\tau_1;\end{cases} \tag{2.20}$$

where

$$a_1=\rho_1(\lambda_2+\lambda_3)-\lambda_1(\rho_2+\rho_3),\qquad a_2=\lambda_{12}(\lambda_2-\lambda_1)-\lambda_{13}(\lambda_{12}+\rho_3)\ ,$$

$$a_3=\lambda_{12}(\rho_2-\rho_1),\qquad a_4=-\lambda_{12}(\lambda_{12}\rho_{23}+\lambda_{13}\rho_2)\ ,$$

$$a_5=-\lambda_{12}\rho_2\ .$$

If $n = 2$, then the third line in (2.20) is omitted, and if $m \le 2n$, then the sixth line in (2.20) is omitted and the range in the fifth line is replaced by $s = n+2,\ldots,m$. The in-channel probability density is

$$q_{22}(u)=p_{12}\begin{cases}\lambda_2+\lambda_{12}\rho_1 u & 0\le u\le\tau_1-\tau_2\\ \lambda_2+\lambda_{12}\rho_2(\tau_1-\tau_2)+\lambda_{12}(\rho_1-\rho_2)u & \tau_1-\tau_2<u\le\tau_2\ .\end{cases} \tag{2.21}$$

where the second line in (2.21) is omitted if $\tau_1 > 2\tau_2$. The out-of-channel probability density is

$$q_{23}(v) \;=\; p_{12}(\lambda_3+\lambda_{13}\rho_1 v), \qquad 0 \le v \le \tau_3 \;. \qquad (2.22)$$

The next range of dead-times to be considered is when the two gamma channel dead-times are equal, but are smaller than the beta dead-time, i.e. $\tau_1 > \tau_2 = \tau_3$. The equilibrium equation for the beta probability density, (2.15), becomes

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + \begin{cases} \rho_1 q_{22}(\tau_3-u) + \rho_1 q_{23}(\tau_3-u) & 0 \le u \le \tau_3 \\ \rho_2 q_1(u-\tau_3) + \rho_3 q_1(u-\tau_3) & \tau_3 < u \le \tau_1 \;, \end{cases} \qquad (2.23)$$

and the equilibrium equations for the two gamma probability densities (2.16) are given by (2.18) as before. The sole discontinuity condition is on the beta probability density, and is

$$q_1(\tau_3^{+}) \;=\; q_1(\tau_3^{-}) + p_{12}(\lambda_{12} + \lambda_{13}) \;. \qquad (2.23a)$$

Therefore the solution of (2.18) and (2.23) subject to (2.11) and (2.23a) yields approximate forms for the probability densities $q_1(u)$ and $q_{2i}(v)$ $i = 2,3$, when $\tau_1 > \tau_2 = \tau_3$, as follows:

$$q_1(u) = p_{12} \begin{cases} \lambda_1 + \lambda_1.(\rho_{23}-\rho_1)u & 0 \le u \le \tau_3 \\ \rho_1 + \lambda_1.(2\rho_{23}-\rho_1)\tau_3-\rho_{23}\lambda_1.u & \tau_3 < u \le 2\tau_3 \\ \rho_1 - \rho_1\lambda_1.\tau_3 & 2\tau_3 < u \le \tau_1 \\ & \text{provided } \tau_1 > 2\tau_3 \;, \end{cases} \qquad (2.24)$$

where $\lambda_{1.} = \lambda_{12} + \lambda_{13}$, and

$$q_{22}(v) = p_{12}\lambda_2 + p_{12}\{\rho_1\lambda_{12} + \lambda_2(\rho_1-\rho_{23})\}v \qquad 0 \leq u \leq \tau_3 = \tau_2, \tag{2.25}$$

$$q_{23}(v) = p_{12}\lambda_3 + p_{12}\{\lambda_{13}\rho_1 + \lambda_3(\rho_1-\rho_{23})\}v \qquad 0 \leq u \leq \tau_3 . \tag{2.26}$$

When the beta and in-channel gamma dead-times are equal and greater than the out-of-channel gamma dead-time, i.e. $\tau_1 = \tau_2 > \tau_3$, the equilibrium equations for the two gamma probability densities are again given by (2.18), but that for the beta probability density is now

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + \begin{cases} \rho_1 q_{22}(\tau_1-u) + \rho_1 q_{23}(\tau_3-u) & 0 \leq u \leq \tau_3 \\ \rho_1 q_{22}(\tau_1-u) + \rho_3 q_1(u-\tau_3) & \tau_3 < u \leq \tau_1 . \end{cases} \tag{2.27}$$

With the boundary conditions (2.11) and the discontinuity condition

$$q_1(\tau_3^{+}) = q_1(\tau_3^{-}) + p_{12}\lambda_{13} ,$$

the three equations (2.18) and (2.25) give, for $\tau_1 = \tau_2 > \tau_3$, the beta probability density $q_1(u)$ as

$$q_1(u) = p_{12}\begin{cases} \lambda_1 + \lambda_{1.}(\rho_{23}-\rho_1)u & 0\leq u\leq \tau_3 \\ \rho_{13} + \{\lambda_{13}(\rho_1-\lambda_1) + \lambda_3\lambda_{1.}\}\tau_3 + \{\lambda_{12}(\rho_2-\rho_1)-\lambda_{13}\rho_3\}u & \tau_3<u\leq 2\tau_3 \\ \rho_{13} + \{\lambda_{13}(\rho_2-\rho_{13})+\lambda_3\lambda_{12}\}\tau_3+\lambda_{12}(\rho_2-\rho_1)u & 2\tau_3<u\leq\tau_1 \end{cases}$$

provided $\tau_1>2\tau_3$,

(2.28)

the in-channel gamma probability density $q_{22}(v)$ as

$$q_{22}(v) = p_{12}\begin{cases} \lambda_2+\lambda_{12}(\rho_1-\rho_2)v & 0\leq v\leq \tau_1-\tau_3 \\ \lambda_2+\lambda_{13}\rho_2(\tau_1-\tau_3)+(\lambda_1\lambda_{12}-\lambda_2\lambda_{1\cdot})v & \tau_1-\tau_3<v\leq \tau_1-\tau_2, \end{cases} \tag{2.29}$$

and, finally, the out-of-channel probability density $q_{23}(v)$ as

$$q_{23}(v) = p_{12}\lambda_3+p_{12}(\lambda_{13}\rho_{13}-\lambda_3\lambda_{12})v \qquad 0\leq v\leq \tau_3 \ . \tag{2.30}$$

The penultimate range of dead-times that will be considered occurs when the in-channel gamma dead-time is larger than the beta dead-time, which in turn is larger than the out-of-channel gamma dead-time, i.e. $\tau_2 > \tau_1 > \tau_3$. Within this range the equilibrium equations (2.15) and (2.16) for $q_1(u)$ and $q_{22}(v)$ respectively, become

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + \begin{cases} \rho_1 q_{22}(\tau_2-u)+\rho_1 q_{23}(\tau_3-u) & 0\leq u\leq \tau_3 \\ \rho_1 q_{22}(\tau_2-u)+\rho_3 q_1(u-\tau_3) & \tau_3<u\leq \tau_1, \end{cases} \tag{2.31}$$

and

$$\frac{dq_{22}(v)}{dv} = -\rho_1 q_{22}(v) + \begin{cases} \rho_2 q_1(\tau_1-v) & 0\leq v\leq \tau_1 \\ \rho_1 q_{22}(v-\tau_1) & \tau_1<v\leq \tau_2 \ . \end{cases} \tag{2.32}$$

The equilibrium equation for $q_{23}(v)$ remains as (2.18) for i = 3. The discontinuity conditions reflect the behaviour on both counters, and are

$$\begin{aligned} q_1(\tau_3^{+}) &= q_1(\tau_3^{-}) + p_{12}\lambda_{13}, \\ q_{22}(\tau_1^{+}) &= q_{22}(\tau_1^{-}) + p_{12}\lambda_{12} \; . \end{aligned} \qquad (2.33)$$

The solution of (2.31) and (2.32) subject to (2.11) and (2.33) is in two parts depending on the relative sizes of τ_1 and τ_2 subject to $\tau_2 > \tau_1$. We have that for $\tau_2 = \tau_1 + \tau_3$, the beta probability density $q_1(u)$ is

$$q_1(u) = p_{12} \begin{cases} \lambda_1 + (\rho_{23}\lambda_{1.} - \lambda_{13}\rho_1)u & 0 \leq u \leq \tau_3 \\ \rho_{13} + \{\lambda_{13}\rho_2 + \lambda_{12}(\rho_{13}+\rho_3)\}\tau_3 + \{\lambda_{12}(\rho_2-\rho_1) - \lambda_{13}\rho_{13}\}u & \tau_3 < u \leq 2\tau_3 \\ \rho_{13} + \{\lambda_{12}(\rho_{12}+\rho_3) + \lambda_{13}(\rho_{23}-2\rho_{13})\}\tau_3 + \lambda_{12}(\rho_2-\rho_1)u & 2\tau_3 < u \leq \tau_1 \end{cases}$$

provided $\tau_1 > 2\tau_3$,

(2.34)

and when $\tau_2 > \tau_1 + \tau_3$

$$q_1(u) = p_{12} \begin{cases} \lambda_1 + (\rho_{23}\lambda_{1.} - \rho_1\lambda_{13})u & 0 \leq u \leq \tau_3 \\ \rho_{13} + \{\rho_3\lambda_{1.} + \lambda_{13}(\rho_{23}-\rho_1)\}\tau_3 + (\lambda_{12}\rho_2 - \lambda_{13}\rho_3)u & \tau_3 < u \leq 2\tau_3 \\ \rho_{13} + \{\lambda_{12}\rho_3 + 2\lambda_{12}\rho_1 + \lambda_{13}(\rho_2-\rho_1)\}\tau_3 + \lambda_{12}(\rho_2-\rho_1)u & 2\tau_3 < u \leq \tau_1 \end{cases}$$

provided $\tau_1 > 2\tau_3$.

(2.35)

However, both the gamma probability densities have single forms when $\tau_2 > \tau_1 > \tau_3$, for

$$q_{22}(v) = p_{12}\begin{cases} \lambda_2+\lambda_{12}(\rho_1-\rho_2)v & 0\leq v\leq\tau_1-\tau_3 \\ \lambda_2+\lambda_{13}\rho_2(\tau_1-\tau_3) + (\lambda_1\rho_2-\lambda_2\rho_1)v & \tau_1-\tau_3<v\leq\tau_1 \\ \rho_2 + (2\rho_{13}-\lambda_2+\lambda_{12})\tau_1-\lambda_{13}\rho_2\tau_3-\lambda_{12}\rho_1 v & \tau_1<v\leq 2\tau_1 \\ \rho_2-\lambda_{12}\rho_2\tau_1-(\lambda_{13}\rho_2+\lambda_{12}\rho_1)\tau_3 & 2\tau_1<v\leq\tau_2 \end{cases}$$

provided $\tau_2>2\tau_1$,

(2.36)

and

$$q_{23}(v) = p_{12}\lambda_3 + p_{12}(\lambda_{13}\rho_{13} - \lambda_3\lambda_{12})v \qquad 0\leq v\leq\tau_3 \ .$$

(2.37)

Finally, we consider the case when the beta dead-time is equal to the out-of-channel dead-time but smaller than the in-channel dead-time, i.e. $\tau_2 > \tau_1 = \tau_3$. The equilibrium equations for the counter probability densities $q_1(u)$ and $q_{22}(v)$, (2.15) and (2.16), when $\tau_2 > \tau_1 = \tau_3$ are

$$\frac{dq_1(u)}{du} = -\rho_{23}q_1(u) + \rho_1 q_{22}(\tau_2-u) + \rho_1 q_{23}(\tau_3-u) \tag{2.38}$$

and

$$\frac{dq_{22}(v)}{du} = -\rho_1 q_{22}(v) + \begin{cases} \rho_2 q_1(\tau_3-v) & 0\leq v\leq\tau_3 \\ \rho_1 q_{22}(v-\tau_3) & \tau_3<v\leq\tau_2 \ . \end{cases} \tag{2.39}$$

The single discontinuity condition corresponding to this range of dead-times refers to the in-channel probability density

$$q_{22}(\tau_3^{+}) = q_{22}(\tau_3^{-}) + p_{12}\lambda_{12} \; . \qquad (2.40)$$

To calculate the three counter state probability densities $q_1(u)$, $q_{2i}(v)$, $i = 2,3$, for $\tau_2 > \tau_1 = \tau_3$, we solve (2.18) for $i = 3$ and (2.38), (2.39) subject to (2.11) and (2.40). Therefore

$$q_1(u) = p_{12}\lambda_1 + p_{12}\{\lambda_{1.}(\rho_{23}-\lambda_{13})-\lambda_1\lambda_{13}\}u \qquad 0 \le u \le \tau_1 = \tau_3 \; , \qquad (2.41)$$

$$q_{22}(v) = p_{12}\begin{cases} \lambda_2 + (\lambda_1\lambda_{12}-\lambda_2\lambda_{1.})v & 0 \le v \le \tau_3 \\ \rho_2 + \{\lambda_{12}(\lambda_1+\rho_1)-\lambda_2\lambda_{1.}\}\tau_3 - \lambda_{12}\rho_1 v & \tau_3 < v \le 2\tau_3 \\ \rho_2 - \rho_2\lambda_{1.}\tau_3 & 2\tau_3 < v \le \tau_2 \text{ provided } \tau_2 > 2\tau_3 \end{cases} \qquad (2.42)$$

and

$$q_{23}(v) = p_{12}\lambda_3 + p_{12}(\lambda_1\lambda_{13}-\lambda_3\lambda_{1.})v \qquad 0 \le v \le \tau_3 = \tau_1 \qquad (2.43)$$

Having calculated the three counter state probability densities $q_1(u)$, $q_{2i}(v)$ for $i = 2,3$ for a variety of cases, dependent on the relative sizes of the three dead-times τ_1, τ_2 and τ_3, we can now calculate three coincidence rates. These three coincidence rates are between the beta and in-channel gamma, the beta and out-of-channel gamma, and consequently the beta and total gamma, series of recorded events.

2.4 The Coincidence Rate

As in the Introduction, a coincidence is defined to be the occurrence in the combined output of the two counters of two recorded events within a time span h of each other. The coincidence rate is calculated for $h < \min(\tau_1, \tau_2, \tau_3)$, so that the coincident events are from different counters. Formally, the coincidence rate is defined to be

$$\lim_{\delta h \to 0} \frac{\mathrm{pr}\left\{\begin{array}{l}\text{a recorded event on one counter in } [0,\delta h) \\ \text{and a recorded event on the other counter in } [0,h)\end{array}\right\}}{\delta h} .$$

Given that we require a recorded event in $[0,\delta h)$ there must be at least one counter open at 0. Then the possible states for the two counters at 0 are:

(i) Both are open, with probability p_{12}.

(ii) Counter 1 has been closed for u and counter 2 is open, with probability density $q_1(u)$.

(iii) Counter 1 is open and counter 2 has been closed for v the closing event being of type i, with probability density $q_{2i}(v)$ $i = 2,3$.

Now there are three ways in which an in-channel coincidence may occur:

(i)' A beta in $[0,\delta h)$ and an in-channel gamma in $[0,\delta h)$.

(ii)' A beta in $[0,\delta h)$ and an in-channel gamma in $[\delta h,h)$.

(iii)' A beta in $[\delta h,h)$ and an in-channel gamma in $[0,\delta h)$.

Corresponding to state (i) the sum of the rates corresponding to the three ways (i)', (ii)' and (iii)' is

$$p_{12}\lambda_{12} + p_{12}\lambda_1 \frac{\rho_2}{\rho_{23}} (1 - e^{-\rho_{23}h}) + p_{12}\lambda_2(1 - e^{-\rho_1 h}) , \tag{2.44}$$

while from state (ii) the only contribution is from way (iii)', and is

$$\int_{\tau_1-h}^{\tau_1} \rho_2 q_1(u)[1 - \exp\{-\rho_1(h-\tau_1+u)\}]du , \tag{2.45}$$

since the counter reopens at τ_1-u.

Corresponding to state (iii) the only contribution is from way (ii)', so that we have

$$\int_{\tau_2-h}^{\tau_2} \rho_1 q_{22}(v)\rho_2\rho_{23}^{-1}[1 - \exp\{-\rho_{23}(h-\tau_2+v)\}]dv \quad \text{for } i = 2 \tag{2.46}$$

and

$$\int_{\tau_3-h}^{\tau_3} \rho_1 q_{23}(v)\rho_2\rho_{23}^{-1}[1 - \exp\{-\rho_{23}(h-\tau_3+v)\}]dv \quad \text{for } i = 3. \tag{2.47}$$

In total, the in-channel coincidence rate is therefore the sum of (2.44)-(2.47), and is

$$\begin{aligned} & p_{12}\lambda_{12} + p_{12}\lambda_1\rho_2\rho_{23}^{-1}(1 - e^{-\rho_{23}h}) + p_{12}\lambda_2(1 - e^{-\rho_1 h}) \\ & + \int_{\tau_1-h}^{\tau_1} \rho_2 q_1(u)[1-\exp\{-\rho_1(h-\tau_1+u)\}]du + \int_{\tau_2-h}^{\tau_2} \rho_1\rho_2\rho_{23}^{-1}q_{22}(v)[1-\exp\{-\rho_{23}(h-\tau_2+v)\}]dv \\ & + \int_{\tau_3-h}^{\tau_3} \rho_1\rho_2\rho_{23}^{-1}q_{23}(v)[1-\exp\{-\rho_{23}(h-\tau_3+v)\}]dv. \end{aligned} \tag{2.48}$$

Neglecting terms of order $(\rho h)^3$ the six terms of (2.48) are reduced to

$$p_{12}\lambda_{12}+p_{12}(\lambda_1\rho_2+\lambda_2\rho_1)h - p_{12}(\lambda_1\rho_2\rho_{23}+\lambda_2\rho_1{}^2)\frac{h^2}{2} + p_{12}\rho_1\rho_2\{\rho-\sum_{i=2}^{3}\lambda_{1i}\delta(\tau_1,\tau_i)\}\frac{h^2}{2}, \tag{2.50}$$

for τ_1 and τ_2 integer multiples of τ_3 such that $\min(\tau_1,\tau_2) \geq \tau_3$, except $\tau_1 = \tau_2 = \tau_3$ and where

$$\delta(\tau_1,\tau_i) = \begin{cases} 1 & \text{for } \tau_1 = \tau_i, \quad i = 2 \text{ or } 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2.51}$$

The out-of-channel coincidence rate may be obtained by substituting λ_{13} for λ_{12}, λ_3 for λ_2 and therefore ρ_3 for ρ_2. The total coincidence rate between a beta particle and any gamma particle is then obtained upon summation of the in-channel and out-of-channel coincidence rates, i.e.

$$p_{12}(\lambda_{12}+\lambda_{13})+p_{12}\lambda_1(1-e^{-\rho_{23}h}) + p_{12}(\lambda_2+\lambda_3)(1-e^{-\rho_1 h})$$
$$+ (\rho_2+\rho_3)\int_{\tau_1-h}^{\tau_1} q_1(u)[1-\exp\{-\rho_1(h-\tau_1+u)\}]du + \rho_1\int_{\tau_2-h}^{\tau_2} q_{22}(v)[1-\exp\{-\rho_{23}(h-\tau_2+v)\}]dv$$
$$+ \rho_1 \int_{\tau_3-h}^{\tau_3} q_{23}(v)[1-\exp\{-\rho_{23}(h-\tau_2+v)\}]dv\ . \tag{2.52}$$

Upon neglecting terms of order $(\rho h)^3$ in (2.52) the total coincidence rate is approximated by

$$p_{12}(\lambda_{12}+\lambda_{13}) + p_{12}\{\lambda_1\rho_{23} + (\lambda_2+\lambda_3)\rho_1\}h - p_{12}\{\lambda_1\rho_{23}^2+(\lambda_2+\lambda_3)\rho_1^2\}\frac{h^2}{2}$$

$$+ p_{12}\rho_1\rho_{23}\{\rho - \sum_{i=2}^{3} \lambda_{1i}\delta(\tau_1,\tau_i)\}\frac{h^2}{2} , \qquad (2.53)$$

for τ_1 and τ_2 integer multiples of τ_3 such that $\min(\tau_1,\tau_2) \geq \tau_3$ with the exception of $\tau_1 = \tau_2 = \tau_3$ and where $\delta(\cdot)$ is given by (2.51).

2.5 Conclusion

In the Introduction, it was noted that, under the assumed model, to estimate the rate λ of disintegrations, four independent functions of the bivariate process of beta and gamma events are needed because of the differing effects of in-channel and out-of-channel particles on the gamma counter. The four functions may be chosen from the following,

(i) the expected number of recorded events on the beta counter in time t,

$$E\{N_1(t)\} = \rho_1 t(1 + \rho_1\tau_1)^{-1} ,$$

(ii) the expected number of recorded in-channel gamma events in time t,

$$E\{N_{22}(t)\} = \rho_2 t(1 + \rho_2\tau_2 + \rho_3\tau_3)^{-1} ,$$

(iii) the expected number of recorded out-of-channel gamma events in time t,

$$E\{N_{23}(t)\} = \rho_3 t(1 + \rho_2\tau_2 + \rho_3\tau_3)^{-1} ,$$

(iv) the expected total number of recorded events on the gamma counter in time t,

$$E\{N_2(t)\} = \rho_{23} t(1 + \rho_2\tau_2 + \rho_3\tau_3)^{-1} ,$$

(v) the in-channel coincidence rate,

$$p_{12}\lambda_{12}+p_{12}(\lambda_1\rho_2+\lambda_2\rho_1)h-p_{12}(\lambda_1\rho_2\rho_{23}+\lambda_2\rho_1{}^2)\frac{h^2}{2} + p_{12}\rho_1\rho_2\{\rho - \sum_{i=2}^{3}\lambda_{1i}\delta(\tau_1,\tau_i)\}\frac{h^2}{2},$$

(vi) the out-of-channel coincidence rate,

$$p_{12}\lambda_{13}+p_{12}(\lambda_1\rho_3+\lambda_3\rho_1)h-p_{12}(\lambda_1\rho_3\rho_{23}+\lambda_3\rho_1{}^2)\frac{h^2}{2} +p_{12}\rho_1\rho_3\{\rho - \sum_{i=2}^{3}\lambda_{1i}\delta(\tau_1,\tau_i)\}\frac{h^2}{2},$$

(vii) the total coincidence rate,

$$p_{12}\lambda_{1.} + p_{12}\{\lambda_1\rho_{23} + (\lambda_2+\lambda_3)\rho_1\}h - p_{12}\{\lambda_1\rho_{23}^2 + (\lambda_2+\lambda_3)\rho_1{}^2\}\frac{h^2}{2}$$
$$+ p_{12}\rho_1\rho_{23}\{\rho - \sum_{i=2}^{3} \lambda_{1i}\delta(\tau_1,\tau_i)\} \frac{h^2}{2},$$

for τ_1 and τ_2 integer multiples of τ_3 such that $\min(\tau_1,\tau_2) \geq \tau_3$, except $\tau_1 = \tau_2 = \tau_3$ and where $\delta(\cdot)$ is given by (2.51). To apply any of the above three coincidence rates, the equilibrium probability that both counters are open simultaneously, p_{12}, is needed. To obtain p_{12}, (2.7) may be used together with the approximations for $q_1(u)$ etc. Perhaps more simply we may note that

$$p_1 = p_{12} + \int_0^{\tau_2} q_{22}(v)dv + \int_0^{\tau_3} q_{23}(v)dv,$$

and then use the approximation for p_1,

$$p_1 = 1 - \rho_1\tau_1 ,$$

together with those for $q_1(u)$ etc. This approach leads to

$$p_{12} = p_1 p_2 \begin{cases} 1 + \lambda_{12}\tau_2 + \lambda_{13}\tau_3 & \tau_1 \geq \tau_2 \geq \tau_3 \\ 1 + \lambda_{12}\tau_1 + \lambda_{13}\tau_3 & \tau_2 \geq \tau_1 \geq \tau_3 \end{cases} .$$

Here again terms of order $(\rho\tau_3)^2$ are neglected.

Smith (1978), considered in and out-of-channel gamma events and derived an estimate for the disintegration rate λ based upon the following relationship between the three parameters ρ_1, ρ_2 and λ_{12};

$$\lambda = \frac{\rho_1 \rho_2}{\lambda_{12}} .$$

Since Smith's analysis used the in-channel coincidence rate as calculated by Cox and Isham (1977), his results are restricted by equal dead-times for in-channel and out-of-channel gamma events. The properties of the counts listed above as (i)-(vii) obey no such restriction and the estimation of the disintegration rate λ is now possible for all ranges of dead-time subject to the two largest dead-times being integer multiples of the smallest dead-time. The estimation of λ is not discussed further apart from noting that the three rates ρ_1, ρ_2 and ρ_3 may be estimated by $\hat{\rho}_1$, $\hat{\rho}_2$ and $\hat{\rho}_3$, where

$$\hat{\rho}_1 = \frac{n_1(t)}{t - n_1(t)\tau_1} ,$$

$$\hat{\rho}_2 = \frac{\{t+n_{23}(t)\hat{\rho}_2\tau_2\tau_3\}n_{22}(t)}{\{t-n_{22}(t)\tau_2\}\{t-n_{23}(t)\tau_3\}} \quad \text{and} \quad \hat{\rho}_3 = \frac{\{t+n_{22}(t)\hat{\rho}_3\tau_2\tau_3\}n_{23}(t)}{\{t-n_{22}(t)\tau_2\}\{t-n_{23}(t)\tau_3\}} ;$$

where in the above $n_1(t)$, $n_{22}(t)$ and $n_{23}(t)$ are the observed values of the variable counts $N_1(t)$, $N_{22}(t)$ and $N_{23}(t)$ respectively.

CHAPTER 3. DELAYED STATE AND JITTER

3.1 Introduction

Two situations are now considered that are physically quite different but which are modelled, mathematically, so as to facilitate the use of the same technique of investigation. The two situations will be referred to as Delayed State and Jitter, the prime objective being as before, to estimate the disintegration rate of the source.

3.2 Delayed State

3.2.1 Physical Description of Delayed State

We now consider an important sub-class of the class of radioactive isotopes that disintegrate by emitting pairs consisting of a beta particle and a gamma particle. The emissions from isotopes in this sub-class are no longer simultaneous, unlike those of section 1.3. But our aim is the same, that is to obtain properties of the disintegration process essential for estimating the rate of disintegration.

The gamma particle, known as the daughter, in each pair is delayed relative to the beta particle, known as the parent, by a period which is exponentially distributed with mean η^{-1}. The parameter η is referred to as the half-life of the intermediate state and the parameter λ is again used to denote the disintegration rate of the source. The beta train of particles is passed through one counting mechanism, the gamma through another and counts are recorded for the purpose of estimating the disintegration rate λ. The half-life η is assumed to be known.

Because each counting mechanism is in two parts, with the actual recording of events taking place in the second part, it is perhaps conceptually simpler to regard the beta-gamma emissions as simultaneous and to build the delay, in the model of the process, into the detector mechanism, rather than modelling the true process with the delay at source. In such a model we therefore assume that the detector delays the gamma half of a beta-gamma pair by an exponentially distributed period only after it has been detected. This is purely a theoretical assumption and is not a direct representation of the true physical process, but the two situations are entirely equivalent so far as the properties of interest are concerned.

The three types of event which may now occur at the detectors are:

(i) a beta particle only, detected with rate

$$\lambda_\beta = \lambda\varepsilon_\beta(1 - \varepsilon_\gamma) ,$$

(ii) a gamma particle only, detected with rate

$$\lambda_\gamma = \lambda\varepsilon_\gamma(1 - \varepsilon_\beta) ,$$

the gamma particle then being held for a time t_1 with probability $\eta e^{-\eta t_1}$. However, we shall see in section 3.3.4 that the delaying of gamma only particles is effectively ignored,

(iii) a beta-gamma pair of particles, detected with rate,

$$\lambda_{\beta\gamma} = \lambda\varepsilon_\beta\varepsilon_\gamma ,$$

this is a "simultaneous emission" the gamma particle then being delayed for a time t_2 with probability density $\eta e^{-\eta t_2}$.

In this manner the original disintegration process of our model, being a Poisson process, may be considered to be the sum of three independent Poisson processes which are then fed into the relevant counters to be recorded. As before three measurements are required to estimate the three unknown parameters λ_β, λ_γ and $\lambda_{\beta\gamma}$, and subsequently the disintegration rate λ. These are the individual counts on the two counters and the covariance between them. The covariance measurement is preferred to the coincidence calculation for reasons outlined in section 1.3.3, and also because of one unavoidable effect due to the very nature of the Delayed State problem and the way in which coincidences are recorded. This will be described in the following section (however, the coincidence rate will also be calculated for the sake of completeness).

3.2.2 The Main Disadvantage of Coincidence Counting in Delayed State

When a gamma particle is delayed relative to its "parent" beta the choice of a suitable resolving time h for the coincidence counter becomes dependent on the delay parameter η. For, in order to have a fair chance of capturing both the parent and its corresponding daughter, which would constitute a true or genuine coincidence, the resolving time h must be at least as big as the mean delay η^{-1}. For some decay schemes η^{-1} can be many times a typical normal dead-time and therefore the dead-times on both counters must be increased to attain the inequality $h < \min(\tau_\beta, \tau_\gamma)$. (This inequality ensures that the two events comprising a coincidence, arise from different counters: see section 1.3.2.) This results in the number of events missed due to counter blocking being increased, which is clearly undesirable.

3.3 The Presence of Jitter in Electronic Counting Systems

3.3.1 Physical Description of Jitter

The disintegration process and the counter mechanisms are now assumed to be the same as described in section 1.3.1, but, because of imperfections in the counter mechanisms that until now have been ignored, there are no simultaneous events on the counter.

The gamma of a beta-gamma pair of particles appears to occur a time T after the beta, where T has probability density function f (t), which is typically of the form shown in Fig. 3.1.

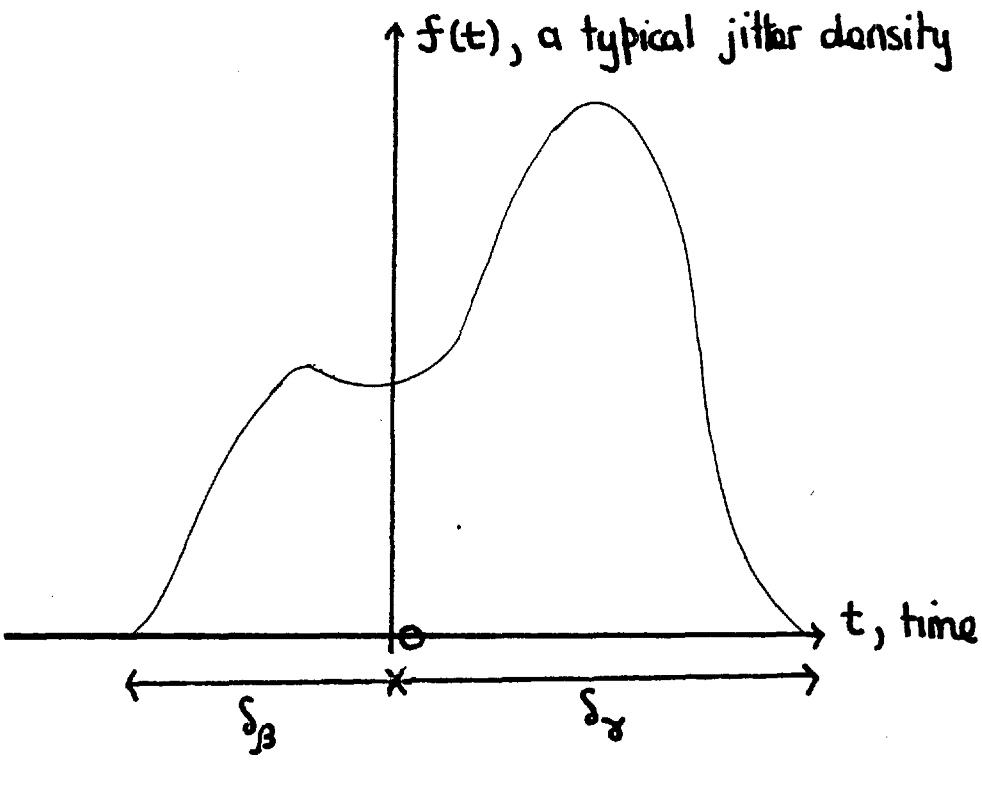


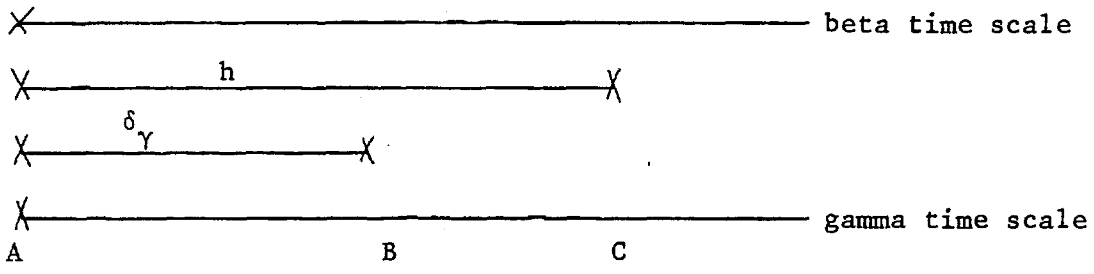
Fig. 3.1: An Example of a Jitter Distribution

In Fig. 3.1 the point O is the point at which the beta of a beta-gamma pair of particles occurs, so that a negative t implies the gamma is observed before the beta. The two values δ_β and δ_γ are the maximum times by which a gamma may occur before or after a beta from the same beta-gamma pair.

The lack of simultaneous events in Delayed State and Jitter leads to a similarity between the two situations, and this similarity will be exploited at a later stage; see section 3.3.4.

3.3.2 The Effect of Jitter

The effect of jitter is perhaps most easily seen on the coincidence counter described in section 1.3.2. In the absence of jitter there are two types of coincidence: true and accidental. The main effect of jitter is that there are no longer any true coincidences although ignoring dead-time effects they will still be recorded as such if $\max(\delta_\beta, \delta_\gamma) < h$. For example, in Fig. 3.2 suppose there is a simultaneous emission at time point A and the beta particle occurs at A, the gamma particle occurring before or at B, which is δ_γ after A. Then if point B is before point C, point C being h later than point A, in the absence of dead-time effects there will be a recorded coincidence.



A: simultaneous emission, the beta occurring instantaneously.

B: furthest point to which the gamma can be delayed.

C: the end point of the resolving period.

Fig. 3.2: Possible Arrangement of Events on the Coincidence Counter for $\max(\delta_\beta, \delta_\gamma) < h$.

The delay of one particle of a pair relative to another is so much smaller in Jitter than in Delayed State, that it is feasible in Jitter physically to set the resolving time h to satisfy $\max(\delta_\beta, \delta_\gamma) < h$, whereas for Delayed State a similar inequality was not feasible; see section 3.2.2. Now it is physically possible to set the resolving time h to be smaller than the maximum delay δ_β or δ_γ, although the effects of Jitter then increase. The extra effects are due mainly to pairs of particles that constitute coincidences for $\max(\delta_\beta, \delta_\gamma) < h$ possibly being jittered more than h apart.

3.3.3 De-centring the Jitter Distribution

The maximum time by which a beta of a beta-gamma pair of particles, may precede the gamma of the same pair, was defined to be δ_γ in section 3.3.1. If the detector of the beta counting mechanism is allowed to delay every beta that is detected by δ_γ, then the beta will never precede the gamma of a beta-gamma pair of particles. Thus f(t), the probability density of the original Jitter distribution, will be transformed to g(t), the probability density of the de-centred Jitter distribution; see Fig. 3.3 and refer to Fig. 3.1.

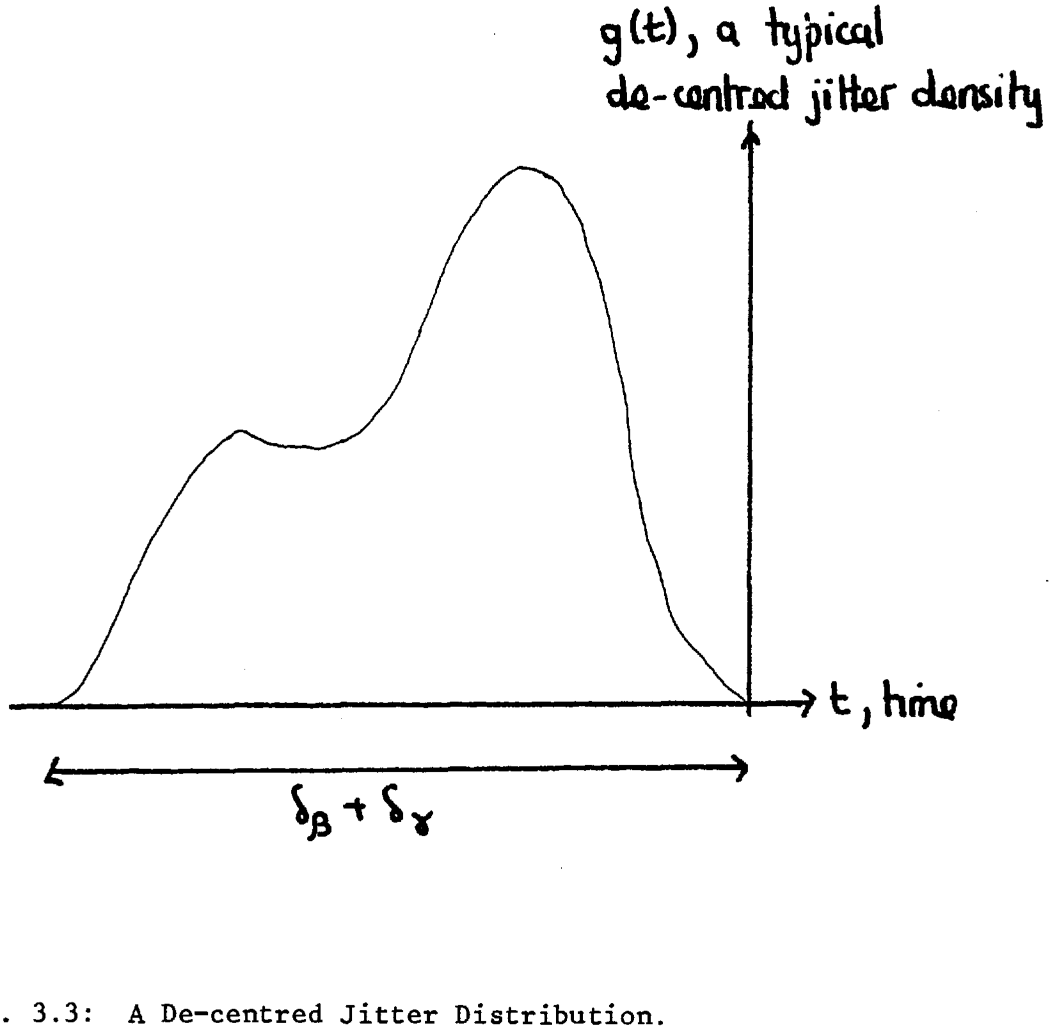


Fig. 3.3: A De-centred Jitter Distribution.

Traditionally the de-centred distribution g(t) has been used more than the distribution f(t). The main advantage of de-centring is that it is considerably easier conceptually to consider one particle delayed relative to another in a single direction only, rather than two directions. For a semi-empirical treatment of de-centred Jitter see Williams and Campion (1965). In this form, the Jitter problem becomes very similar to that of Delayed State, and would be the same if the roles of beta and gamma were interchanged and g(t) were to be an exponential distribution. In both problems one particle of a pair of particles is delayed relative to another particle of the pair. Also, in both problems there is no restriction on the number of particles that may be delayed, and so waiting to occur at any particular instant, although the average delay is of an order smaller in Jitter than it is in Delayed State. However, we have that delay is bounded in Jitter and unbounded in Delayed State.

In any study of Jitter, certain properties of the Jitter distribution will have to be estimated. These estimates are then incorporated into normal Jitter calculations and used to produce an estimate of the disintegration rate. When de-centring, the maximum time by which a beta may precede a gamma, δ_γ, is estimated and then a displacement of this size inserted in the beta train of events to delay each beta that is detected, and so de-centre the Jitter distribution. Estimates of this de-centred distribution are then used, for example, in the calculation of the coincidence rate, to produce an estimate of the disintegration rate. It is possible that when de-centring by this means, the error associated with estimating δ_γ, may be compounded with the error associated with the de-centred estimate of the coincidence rate, to use the same example as before, to produce a more inaccurate estimate than would be obtained if the Jitter distribution was not de-centred; especially if δ_γ were to be underestimated.

It is the aim of this chapter to produce the theory necessary to calculate the expected number of recorded events on each counter, the covariance between the two counters and the rate of coincidences, for both the normal, and de-centred jitter distributions. By modelling the jitter as described in the following section the above functions will also be obtained for the Delayed State problem.

3.3.4 Modelling the Jitter Distribution

As stated in section 3.3.1 a typical jitter distribution is of the form shown in Fig. 3.1. Two points to note are,

(i) the support of the distribution is finite, i.e. δ_β and δ_γ are both finite, and

(ii) there may be multiple turning points.

However, we wish to choose the model for the jitter distribution to make the theory tractable enough to treat the normal and de-centred distributions in the same way, and yet keep the essential properties of the effect of jitter. For this reason the finite support restriction is ignored and a combination of exponential distributions used. Graphically if we take a mixture of an exponential random variable and a gamma random variable then we obtain the distribution shown in Fig. 3.3.

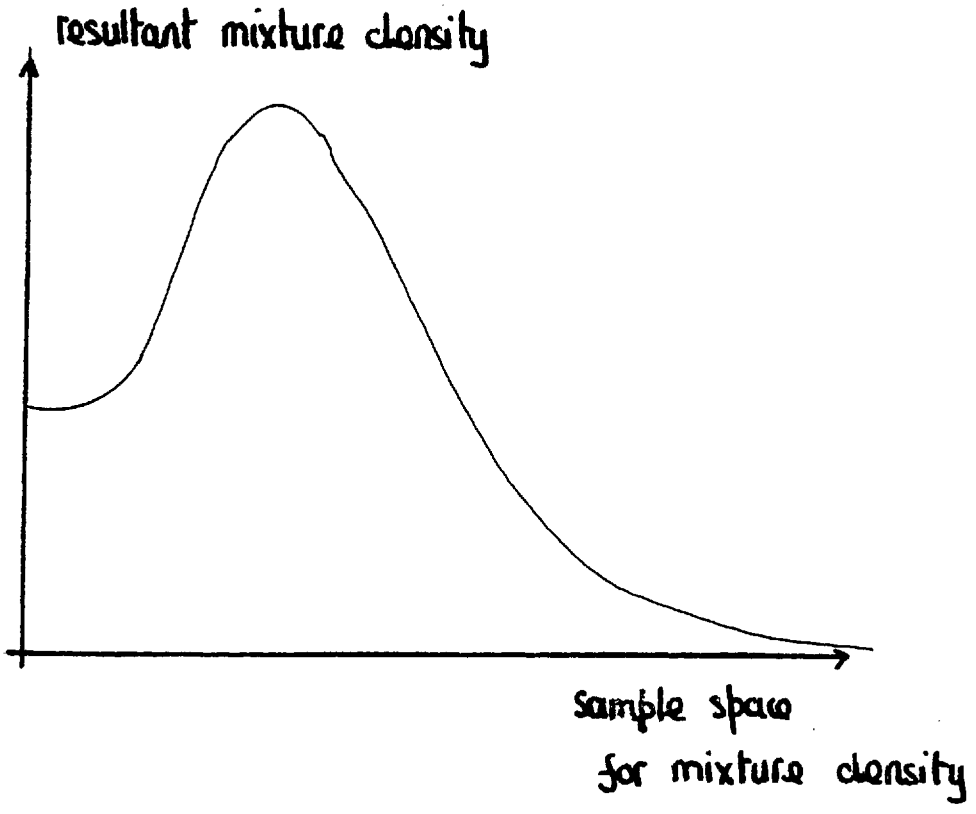


Fig. 3.4: Mixing an Exponential Random Variable with a Gamma Random Variable

By following a similar procedure on the negative side and mixing this with that obtained on the positive side, it is possible to obtain a density as pictured in Fig. 3.5.

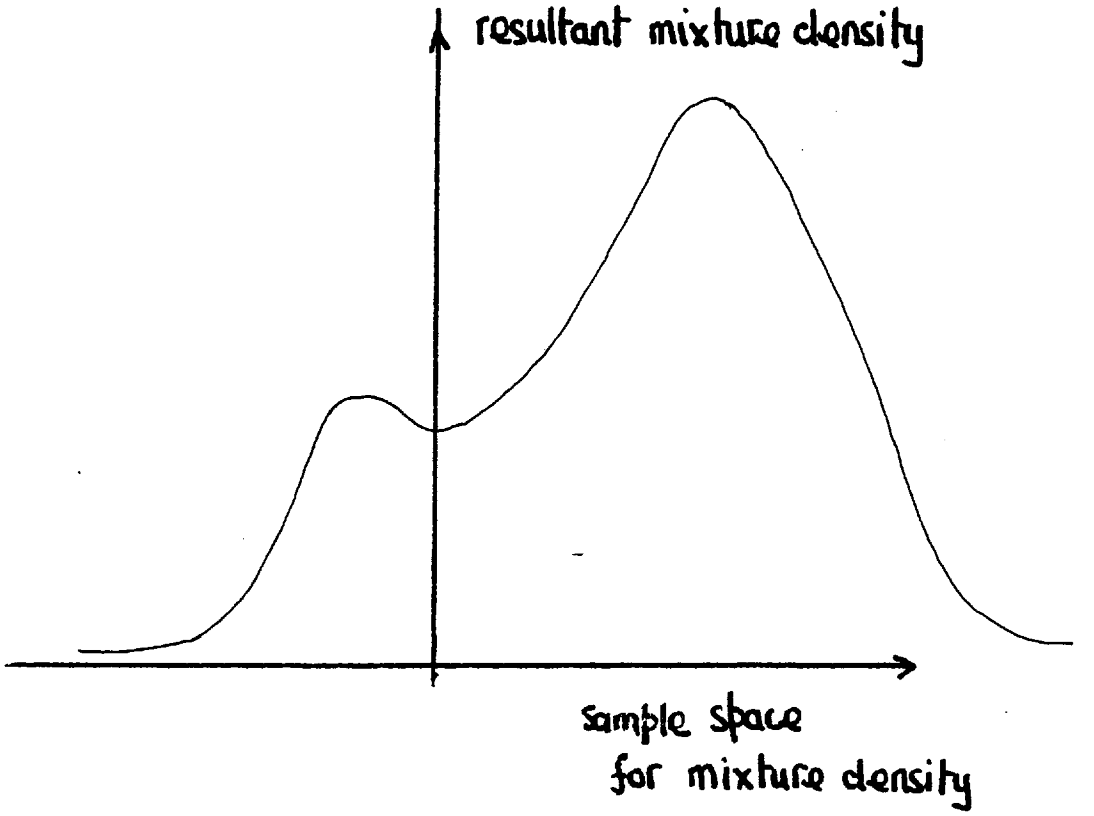


Fig. 3.5: A Combination of a Negative and Positive Mixture of an Exponential Random Variable with a Gamma Random Variable

A density of the form seen in Fig. 3.5 would appear to model the true jitter distribution quite well, see Fig. 3.1, apart from the finite support restriction. If f(t) denotes the jitter density of the model, then the approximating density described so far is of the form

$$f(t)\alpha \begin{cases} f_\beta(-t) & \text{for } t < 0 \\ f_\gamma(t) & \text{for } t > 0, \end{cases}$$

where

$$f_i(t) = \begin{cases} \phi_{i1}\mu_{i1}e^{-\mu_{i1}t} + (1-\phi_{i1})\mu_{i2}\dfrac{(\mu_{i2}t)^{K_{i2}-1}}{(K_{i2}-1)!}e^{-\mu_{i2}t} & \text{for } t > 0 \\ 0 & \text{for } t < 0, \end{cases}$$

for some $\phi_{i1},\mu_{i1},\mu_{i2}$ and K_{i2}, $i = \beta,\gamma$. A more general distribution of this form is

$$f_i(t) = \begin{cases} \sum_{j=1}^{B(i)} \phi_{ij}\mu_{ij}\dfrac{(\mu_{ij}t)^{K_{ij}-1}}{(K_{ij}-1)!}e^{-\mu_{ij}t} & \text{for } t > 0 \\ 0 & \text{for } t < 0, \end{cases}$$

where $\sum_{j=1}^{B(i)} \phi_{ij} = 1$ for some $B(i)$ and ϕ_{ij}, some μ_{ij} and K_{ij} for $j = 1,\dots,B(i)$ and $i = \beta,\gamma$. By setting $B(\gamma) = 0$ the de-centred distribution of section 3.3.3 may be obtained. By setting $B(\beta) = 0$, $B(\gamma) = 1$, $K_{\gamma 1} = 1$ and $\mu_{\gamma 1} = \eta$, we obtain the distribution $f_\gamma(t) = \eta e^{-\eta t}$ for the Delayed State problem. Thus, in its final form, we assume that the jitter distribution of the model, $f(t)$, satisfies

$$f(t) = \theta_\beta \sum_{j=1}^{B(\beta)} \phi_{\beta j} g_{\beta j}(-t)H(-t) + \theta_\gamma \sum_{j=1}^{B(\gamma)} \phi_{\gamma j} g_{\gamma j}(t)H(t), \tag{3.1}$$

where $\theta_\beta + \theta_\gamma = 1$, $\sum_{j=1}^{B(i)} \phi_{ij} = 1$, $g_{ij}(t)$ is the probability density of a gamma random variable with mean K_{ij}/μ_{ij} and index K_{ij}, for $j = 1,\ldots,B(i)$; $i = \beta,\gamma$, and $H(t)$ is the unit Heaviside function. The above equation, (3.1), may be interpreted as follows. Whenever a pair of particles is emitted and therefore arrives at the detectors in the counting mechanism, with probability θ_β the beta particle is delayed and the gamma occurs immediately. Furthermore, the beta particle is delayed for a time t and with probability $\phi_{\beta j}$ the distribution of delay time is gamma with mean $K_{\beta j}/\mu_{\beta j}$ and index $K_{\beta j}$, $j = 1,\ldots,B(\beta)$. With probability θ_γ the situation is reversed and the gamma particle follows the beta particle. For single events on the detectors the effect of jitter is non-existent. This is because the single events on either counter form a Poisson process, and if the points of a Poisson process are subjected to independent and identically distributed displacements, then the resultant process is a Poisson process with the original rate. Thus, there are four types of event which may occur at the detectors,

(i) a beta particle only, detected with rate

$$\lambda_\beta = \lambda\varepsilon_\beta(1 - \varepsilon_\gamma) ,$$

(ii) a gamma particle only, detected with rate

$$\lambda_\gamma = \lambda\varepsilon_\gamma(1 - \varepsilon_\beta),$$

(iii) a beta-gamma pair of particles in which the beta occurs immediately and the gamma is delayed for a time T where $T \sim \Gamma(K_{\gamma j}, /\mu_{\gamma j})$ with probability $\phi_{\gamma j}$, $j = 1,\ldots,B(\gamma)$, is detected with rate

$$\lambda_{\beta\gamma}\theta_\gamma = \lambda\varepsilon_\beta\varepsilon_\gamma\theta_\gamma,$$

(iv) a beta-gamma pair of particles in which the gamma occurs immediately and the beta is delayed for a time T where $T \sim \Gamma(K_{\beta j}, \mu_{\beta j})$ with probability $\phi_{\beta j}$, $j = 1, \ldots, B(\beta)$, is detected with rate

$$\lambda_{\beta\gamma}\theta_{\beta} = \lambda\varepsilon_{\beta}\varepsilon_{\gamma}\theta_{\beta} .$$

Therefore the original Poisson process of disintegrations of rate λ, can be thought of as made up of four independent Poisson processes which then reach the counters.

We now calculate the following four functions for the Jitter and consequently the Delayed State problems. These are, the two average counts, the covariance between the counts, and the coincidence rate. Before these calculations are tackled a much simpler problem is studied and a method formulated, which will then be developed at a later stage for use in Jitter. This problem is a simplification of Delayed State. The problem and its solution were devised as a first attempt to obtain the solution of Jitter. Subsequent solution of Jitter made the solution of the simpler problem redundant, however it is now included for the sake of completeness.

3.4 A Maximum of One Delayed Gamma Particle

3.4.1 Description of the Process

One of the simplest models that still retains the parent-daughter relationship of section 3.2, is one in which at most one gamma particle may be delayed at any particular instant. That is, following the first beta-gamma coincident event on the detectors and during the period for which its gamma particle is delayed, the gamma particle in any other arriving beta-gamma pair is irretrievably lost, i.e. the beta-gamma pair converts to a beta only. This conversion process continues until the original gamma (daughter) is released, so that the number of gamma's delayed reverts to zero. At this point the system is then capable of delaying a gamma particle again. The gamma only events remain unaffected.

In general this is a rather unrealistic model but if certain restrictions on the parameters are satisfied then it should be a good approximation to the true process. The average time between successive beta-gamma pairs is $\lambda_{\beta\gamma}^{-1}$ and, on average, the gamma then occurs η^{-1} later. Therefore if the average time between pairs is substantially greater than the average delay time, i.e. $\lambda_{\beta\gamma}^{-1} >> \eta^{-1}$ or equivalently $\eta >> \lambda_{\beta\gamma}$, then the true probability of more than one gamma being delayed at any particular instant, will be negligible. This is the model which is studied first. Again for notational convenience we take $\beta \equiv 1$ and $\gamma \equiv 2$ but keep the physical interpretation of the events.

3.4.2 The Covariance Between the Two Counts in the Absence of Dead-Time Effects

The total count, $N_1(t)$, on counter 1 in (0,t] may be split into two parts,

(i) $N_{11}(t)$, the number of beta's that arrive in (0,t] without a gamma ie, beta only events,

(ii) $N_{12}(t)$, the number of beta-gamma pairs that arrive in (0,t], in which the gamma may occur subsequently to the beta.

The total count $N_2(t)$ on counter 2 in (0,t], may be split similarly into two components,

(i) $N_{21}(t)$, the number of gamma's that arrive in (0,t] without a beta, i.e. gamma only events,

(ii) $N_{22}(t)$, the number of beta-gamma pairs from which the delayed gamma occurs in (0,t].

Thus

$$N_i(t) = N_{i1}(t) + N_{i2}(t) \qquad \text{for } i = 1,2 .$$

The component counts are not all Poisson variables, despite the disrupting influence on the distribution of the counts, due to dead-time effects, being absent. Note that all the pairs of $N_{ij}(t)$ for i,j = 1,2, are independent except for $N_{12}(t)$ and $N_{22}(t)$, so that the covariance between $N_1(t)$ and $N_2(t)$ reduces to the covariance between $N_{12}(t)$ and $N_{22}(t)$. This situation is now analogous to a single server queue with at most one person in the system at any one time, the arrival process being Poisson with rate λ_{12}, and the service distribution being exponential with mean η^{-1}. Then $N_{12}(t)$ may be identified with $N_a(t)$, the number of arrivals in (0,t], and $N_{22}(t)$ identified with $N_s(t)$, the number served in (0,t]. Thus, in the absence of dead-time effects we have that

$$\begin{aligned} \operatorname{cov}(N_{12}(t),N_{22}(t)) &= \operatorname{cov}(N_a(t),N_s(t)) \\ &= \frac{\lambda_{12}\eta^2}{(\lambda_{12}+\eta)^2}\{t - \frac{1 - e^{-(\lambda_{12}+\eta)t}}{\lambda_{12} + \eta}\}, \end{aligned} \qquad (3.2)$$

which is an adaptation of a result in Conolly (1975, page 21).

3.4.3 The Covariance Between the Two Counts Subject to Dead-Time Effects

Denote the covariance between the two counts by c(t), i.e.

$$\begin{aligned} c(t) &= \operatorname{cov}\{N_1(t),N_2(t)\} \\ &= E\{N_1(t)N_2(t)\} - E\{N_1(t)\}E\{N_2(t)\} . \end{aligned}$$

Let p_{ij} denote the equilibrium probability that counter i is open and j gamma's are delayed for i = 1,2; j = 0,1. Also let

$$p_i = p_{i0} + p_{i1} \qquad \text{for } i = 1,2.$$

The instantaneous rate of events on counter 1 is independent of the number of gamma's delayed, and is equal to $\rho_1 = \lambda_1 + \lambda_{12}$. Therefore, if the process of events on the two counters starts the interval (0,t] from statistical equilibrium, and if we represent each process by

$$N_i(t) = \int_0^t dN_i(u) \qquad \text{for } i = 1,2 ,$$

then

$$E\{N_1(t)\} = \int_0^t pr\{dN_1(u) = 1\}$$

$$= \int_0^t p_1\rho_1 du = p_1\rho_1 t \ . \qquad (3.3)$$

The instantaneous rate of events on counter 2 is directly dependent on the number of gamma's delayed, and furthermore the process of such events is no longer Poisson. So

$$E\{N_2(t)\} = \int_0^t pr\{dN_2(u) = 1\}$$

$$= \int_0^t \{\lambda_2 p_{20} + (\lambda_2 + \eta)p_{21}\}du$$

$$E\{N_2(t)\} = \{\lambda_2 p_{20} + (\lambda_2 + \eta)p_{21}\}t \ . \qquad (3.4)$$

The equilibrium probabilities p_{ij} for $i = 1,2$; $j = 0,1$, will be determined in section 3.4.4.

The cross-product term in the covariance may be written as

$$E\{N_1(t)N_2(t)\} = E\{\int_0^t \int_0^t dN_1(u)dN_2(v)\}$$

$$= \int_0^t \int_0^t pr\{dN_1(u) = dN_2(v) = 1\} \ . \qquad (3.5)$$

Because the probability of simultaneous events on the two counters is negligible, there is no contribution from $v = u$ and (3.5) may be split into two ranges $v < u$ and $v > u$, hence

$$E\{N_1(t)N_2(t)\} = \int_0^t \int_u^t \mathrm{pr}\{dN_1(u) = dN_2(v) = 1\} + \int_0^t \int_v^t \mathrm{pr}\{dN_1(u) = dN_2(v) = 1\}. \tag{3.6}$$

If we define the joint probability densities

$$h_{ij}(x) = \lim_{\substack{\delta x \to 0^+ \\ \delta y \to 0^+}} \frac{\mathrm{pr}\left\{\begin{array}{l}\text{a recorded event on counter } j \text{ in } (x, x+\delta x) \text{ and} \\ \text{a recorded event on counter } i \text{ in } (0, \delta y)\end{array}\right\}}{\delta x . \delta y}$$

for $x > 0$ and $i,j = 1,2;\ i \neq j$, then (3.6) becomes

$$E\{N_1(t)N_2(t)\} = \int_0^t \int_u^t h_{12}(v-u)dvdu + \int_0^t \int_v^t h_{21}(v-u)dudv . \tag{3.7}$$

Upon substitution of (3.7), (3.4) and (3.3) into (3.2), we have that

$$c(t) = \int_0^t \int_u^t h_{12}(v-u)dvdu + \int_0^t \int_v^t h_{21}(u-v)dudv - p_1\rho_1\{\lambda_2 p_{20} + (\lambda_2+\eta)p_{21}\}t^2. \tag{3.8}$$

The Laplace Transform of c(t), denoted by $c^*(s)$, satisfies

$$c^*(s) = \frac{h^*_{12}(s)}{s^2} + \frac{h^*_{21}(s)}{s^2} - p_1\rho_1\{\lambda_2 p_{20} + (\lambda_2+\eta)p_{21}\}\frac{2}{s^3} , \tag{3.9}$$

where

$$c^*(s) = \int_0^t e^{-st}c(t)dt \qquad \text{etc.}$$

To calculate the two joint probability densities $h_{ij}(x)$, $i,j = 1,2$, $i \neq j$, and the equilibrium probabilities p_{ij}, $i = 1,2;\ j = 0,1$, the states of the counters are represented as a Markov process. Thus, the following equilibrium probabilities and probability densities are defined,

(i) p_{12i}, the probability that both counters are open with i gamma's delayed and waiting to occur,

(ii) $q_{ji}(u)$, the probability density that counter j has been closed for a period u, with i gamma's delayed, $0 \leq u \leq \tau_j$, j = 1,2; i = 0,1,

(iii) $q_{12i}(u,v)$, the probability density that counter 1 has been closed for a period u, counter 2 for a period v and i gamma's delayed, $0 \leq u \leq \tau_1$, $o \leq v \leq \tau_2$ and i = 0,1.

If any of the above probability (densities) appear without the subscript i, then this probability (density) is obtained by summing the relevant probability (densities) that are defined above, over i = 0 and i = 1.

3.4.4 The Possible States of the Counting System

The equations representing the probabilities of change from one state to another, when the system is in equilibrium, can be written down in terms of the probability (densities) defined above. As the purpose of solving these equations is purely exploratory, developed merely to suggest a method of solution to the full model as opposed to the restricted one we are considering at the moment, equal dead-times are imposed. So, for $\tau_1 = \tau_2 = \tau$, the equilibrium equations are

$$(\rho+i\eta)p_{12i} = q_{1i}(\tau) + q_{2i}(\tau) \qquad \text{for } i = 0,1 . \qquad (3.10)$$

Here $\rho = \rho_1+\lambda_2 = \lambda_1+\rho_2 = \lambda_1+\lambda_2+\lambda_{12}$, and

$$\frac{dq_{10}(u)}{du} = -\rho_2 q_{10}(u) + q_{120}(u,\tau)$$

$$\frac{dq_{11}(u)}{du} = -(\lambda_2+\eta)q_{11}(u) + \lambda_{12}q_{10}(u) + q_{121}(u,\tau)$$

$$\frac{dq_{20}(u)}{du} = -\rho_1 q_{20}(u) + \eta q_{21}(u) + q_{120}(\tau,u)$$

$$\frac{dq_{21}(u)}{du} = -(\rho_1+\eta)q_{21}(u) + q_{121}(\tau,u) \qquad (3.11)$$

$$\frac{\partial q_{120}(u,v)}{\partial u} + \frac{\partial q_{120}(u,v)}{\partial v} = -\lambda_{12}q_{120}(u,v) + \eta q_{121}(u,v)$$

$$\frac{\partial q_{121}(u,v)}{\partial u} + \frac{\partial q_{121}(u,v)}{\partial v} = -\eta q_{121}(u,v) + \lambda_{12}q_{120}(u,v)$$

The corresponding boundary conditions for this set of equations are:

$$q_{10}(0) = \lambda_1 p_{120}, \qquad q_{11}(0) = \rho_1 p_{121}+\lambda_{12}p_{120},$$

$$q_{20}(0) = \lambda_2 p_{120}+\eta p_{121}, \qquad q_{21}(0) = \lambda_2 p_{121},$$

$$q_{120}(u,0) = \lambda_2 q_{10}(u)+\eta q_{11}(u), \quad q_{121}(u,0) = \lambda_2 q_{11}(u),$$

$$q_{120}(0,v) = \lambda_1 q_{20}(v), \quad q_{121}(0,v) = \rho_1 q_{21}(v)+\lambda_{12}q_{20}(v). \qquad (3.12)$$

An exact closed form solution of (3.10) and (3.11) subject to (3.12) is not attempted. Instead, separate Taylor expansions are made for each function and $\rho\tau$ is considered small enough for terms of order

$(\rho\tau)^2$ to be neglected; furthermore terms of order $(\rho\tau)^2$ will be neglected throughout section 3.4. The solution can be shown to be

$$\begin{aligned}
p_{121} &= \frac{\lambda_{12}}{\eta} p_{120}(1 - \eta\tau) , \\
q_{10}(u) &= \lambda_1 p_{120} , \\
q_{11}(u) &= \frac{\lambda_{12}}{\eta} p_{120}(\rho_1+\eta - \rho_1\eta\tau - \eta^2 u) , \\
q_{20}(u) &= p_{120}(\rho_2 - \lambda_{12}\eta\tau+\lambda_{12}\eta u) , \\
q_{21}(u) &= \frac{\lambda_2\lambda_{12}}{\eta} p_{120}(1 - \eta\tau) ,
\end{aligned} \qquad (3.13)$$

$$q_{120}(u,v) = p_{120}\begin{cases} \lambda_1\lambda_2+\rho_1\lambda_{12}+\lambda_{12}\eta(1-\rho_1\tau)-\lambda_{12}\eta^2 u + \{\lambda_{12}(\lambda_2-\rho_1-\eta) + \eta^2\}\lambda_{12}v & \text{for } u \geq v \\ \lambda_1\{\rho_2-\lambda_{12}\eta\tau\} + \lambda_{12}\{\lambda_{12}(\lambda_2-\lambda_1)+\eta(\rho_2-\lambda_1)\}u + \lambda_1\lambda_{12}\eta v & \text{for } u \leq v \end{cases}$$

(3.13)

and finally,

$$q_{121}(u,v) = \frac{\lambda_{12}}{\eta}p_{120}\begin{cases} \lambda_2(\rho_1+\eta-\rho_1\eta\tau) - \lambda_2\eta^2 u +\lambda_{12}\eta(\rho_1-\lambda_2+\eta)v & \text{for } u \geq v \\ \rho_1\lambda_2+\eta\rho_2-\eta\tau(\rho_1\lambda_2+\eta\lambda_{12}) + \{\lambda_1\lambda_{12}-\lambda_{12}\eta- \lambda_2\lambda_{12}-\eta\rho_2\}u\eta + \lambda_{12}\eta^2 v & \text{for } u \leq v. \end{cases}$$

(3.13)

In this solution note that the introduction of the probability density $q_{12}(u,v)$ was needed solely to obtain the probability (densities) p_{12i}, $q_{1i}(u)$ and $q_{2i}(v)$ for i = 0,1. These are the functions that are of direct use in the calculation of $h_{12}(x)$ and $h_{21}(x)$. Further, note that p_{120} remains unknown. Approximate formulae for p_{120} and the equilibrium probabilities p_{1i} and p_{2i} for i = 0,1 are now found. Two of the ways in which these unknowns may be calculated are as follows:

(i) For each counter define the equilibrium probability density that the counter is closed with i gammas delayed for i = 0,1. Then set up equilibrium equations similar to (3.10) and (3.11) which may be solved. (Note that these are univariate whilst (3.10) and (3.11) are bivariate.) The solutions of each set will be in terms of an unknown and these two unknowns may then be determined using the normalizing condition that, for each counter, the probability that the counter is closed plus the probability that the counter is open is equal to unity.

(ii) Counter 1 alternates between open periods exponentially distributed with mean ρ_1^{-1} and dead-times of constant length τ. Therefore the equilibrium probability that counter 1 is open, p_1 satisfies,

$$p_1 = \frac{\rho_1^{-1}}{\rho_1^{-1} + \tau} = \frac{1}{1 + \rho_1\tau} .$$

Neglecting terms of order $(\rho\tau)^2$,

$$p_1 = 1 - \rho_1\tau \quad . \qquad (3.14)$$

From (3.13) we see that,

$$p_{12} = p_{120} + p_{121} = p_{120}(1 + \frac{\lambda_{12}}{\eta} - \lambda_{12}\tau) ,$$

and

$$q_2(u) = q_{20}(u)+q_{21}(u) = p_{120}(\rho_2 + \frac{\lambda_2\lambda_{12}}{\eta} - \lambda_{12}\eta\tau - \lambda_2\lambda_{12}\tau + \lambda_{12}\eta u) .$$

Using the normalizing condition that

$$p_1 = p_{12} + \int_0^\tau q_2(u)du , \qquad (3.15)$$

we have

$$1 - \rho_1\tau = p_{120}(1 + \frac{\lambda_{12}}{\eta} - \lambda_{12}\tau) + p_{120}\tau(\rho_2 + \frac{\lambda_2\lambda_{12}}{\eta})$$

$$p_{120} = (1 + \frac{\lambda_{12}}{\eta})^{-1}(1 - \rho\tau) . \qquad (3.16)$$

The remaining unknowns on counter 1, p_{1i} for i = 0,1, may be determined by adding the subscript i to (3.15), thus

$$p_{10} = \frac{\eta}{\lambda_{12}+\eta}(1 - \lambda_1\tau),$$

$$p_{11} = \frac{\lambda_{12}}{\lambda_{12}+\eta}\{1 - (\rho_1+\eta)\tau\} . \qquad (3.17)$$

By interchanging the roles of counter 1 and counter 2 in (3.15),

$$p_{20} = \frac{\eta}{\lambda_{12}+\eta}(1 - \rho_2\tau) ,$$

$$p_{21} = \frac{\lambda_{12}}{\lambda_{12}+\eta}(1 - \lambda_2\tau) . \qquad (3.18)$$

Therefore

$$p_2 = \frac{1}{\lambda_{12}+\eta}\{(\lambda_{12}+\eta)(1-\lambda_2\tau) - \lambda_{12}\eta\tau\} . \qquad (3.19)$$

An approximate form for p_{12}, the equilibrium probability that both counters are open simultaneously, may now be written down:

$$p_{12} = (\lambda_{12}+\eta)(\lambda_{12}+\eta - \lambda_{12}\eta\tau)p_{10}p_{20}/\eta^2$$

$$= (\lambda_{12}+\eta)(\lambda_{12}+\eta+\eta^2\tau)p_{11}p_{21}/\lambda_{12}^2 \quad , \qquad (3.20)$$

and so

$$p_{12} = p_1 p_2 \quad . \qquad (3.21)$$

The corresponding formula for the equilibrium probability p_{12} in Cox and Isham (1977) is

$$p_{12} = p_1 p_2 (1 + \lambda_{12}\tau) \quad . \qquad (3.21a)$$

Upon comparison of (3.21) with (3.21a) it is plausible that (3.21) is "less dependent" than (3.21a), due to lack of "direct dependence" through simultaneous events that do not appear in the present problem but do appear in the problem of Cox and Isham. However, (3.21) implies that the two counters act independently; clearly this is not exactly correct and it should be noted that (3.21) is only a first order approximation in $\rho\tau$ to p_{12}.

Before progressing to the calculation of the two joint probability densities $h_{12}(x)$ and $h_{21}(x)$, which we are now in a position to do, the expected number of recorded events on each counter is calculated. From (3.3) and (3.4) we have that

$$E\{N_1(t)\} = p_1 \rho_1 t \quad , \qquad (3.22)$$

and

$$E\{N_2(t)\} \quad = \quad \{\lambda_2 p_{20} + (\lambda_2+\eta)p_{21}\}\ t\ . \tag{3.23}$$

Using (3.14) and (3.18) the expectations may be approximated as

$$E\{N_1(t)\} \quad = \quad \rho_1 t(1 - \rho_1\tau)\ , \tag{3.24}$$

and

$$E\{N_2(t)\} \quad = \quad \{(\lambda_2 + \frac{\lambda_{12}\eta}{\lambda_{12}+\eta}) - \lambda_2(\lambda_2 + \frac{2\lambda_{12}\eta}{\lambda_{12}+\eta})\}\ t\ . \tag{3.25}$$

We now consider the sequences of events on each of the two counters.

3.4.5 The Sequence of Events on Counter 1

The probability density

$$h_{21}(x)\delta x\delta y = \mathrm{pr}\left\{\begin{array}{l}\text{a recorded event on counter 1 in } (x,x+\delta x) \\ \text{and a recorded event on counter 2 in } (0,\delta y)\end{array}\right\}\ ,$$

for small δx and δy, is now calculated. For a recorded event on counter 2 in $(0,\delta y)$, counter 2 must be open at 0. Therefore summing over the possible states of counter 1 at 0 we have that

$$h_{21}(x)\delta x\delta y \quad =$$

$$\sum_{i=0}^{1} \mathrm{pr}\left\{\begin{array}{l}\text{a recorded event on counter 1 in } (x,x+\delta x)\text{, and} \\ \text{a recorded event on counter 2 in } (0,\delta y)\text{, and} \\ \text{both counters are open at 0 with } i \text{ gammas delayed}\end{array}\right\} \quad +$$

$$\sum_{i=0}^{1} \mathrm{pr}\left\{\begin{array}{l}\text{a recorded event on counter 1 in } (x,x+\delta x)\text{, and} \\ \text{a recorded event on counter 2 in } (0,\delta y)\text{, and counter 1} \\ \text{closed at 0, counter 2 open at 0, } i \text{ gammas delayed}\end{array}\right\}\ .$$

The states of the two counters at 0 and the type of recorded event on counter 2 in $(0,\delta y)$, which is either a gamma only, or a delayed gamma from a beta-gamma pair, give the state of the two counters at δy. However the sequence of events on counter 1 is independent of the number of gammas delayed in the system at any specific point. In particular the sequence is independent of the number of gammas delayed at δy. Therefore the recorded event on counter 2 in $(0,\delta y)$ does not affect the subsequent sequence of recorded events on counter 1, and so

$$h_{21}(x)\delta x\delta y = \sum_{i=0}^{1} \mathrm{pr}\left(\begin{matrix}\text{a recorded event on}\\ \text{counter 1 in } (x,x+\delta x)\end{matrix}\middle|\begin{matrix}\text{counter 1}\\ \text{open at } 0+\delta y\end{matrix}\right)(\lambda_2+i\eta)p_{12i}\delta y$$

$$+ \sum_{i=0}^{1} \int_0^{\tau} \mathrm{pr}\left(\begin{matrix}\text{a recorded event}\\ \text{on counter 1 in}\\ (x,x+\delta x)\end{matrix}\middle|\begin{matrix}\text{counter 1 for } u\\ \text{closed at } 0+\delta y\end{matrix}\right)(\lambda_2+i\eta)q_{1i}(u)\delta y du \;. \tag{3.26}$$

The sequence of events on counter 1 forms a renewal process, the interval between successive recorded events having density $\rho_1 e^{-\rho_1(x-\tau)}$ for $x \geq \tau$. If $g(x)$ denotes the renewal density of this process, then taking the limit of (3.26) as δx and δy tend to zero from the right,

$$h_{21}(x) = \sum_{i=0}^{1} \{\rho_1 e^{-\rho_1 x} + \int_0^x \rho_1 e^{-\rho_1 y} g(x-y)dy\}(\lambda_2+i\eta)p_{12i} +$$

$$+ \sum_{i=0}^{1} \int_0^{\tau} g(x+u)(\lambda_2+i\eta)q_{1i}(u)du \;. \tag{3.27}$$

Taking Laplace Transforms of (3.27) with respect to x, we have

$$h_{21}^*(s) = \sum_{i=0}^{1} (\lambda_2+i\eta)[p_{12i}\{\frac{\rho_1}{\rho_1+s} + \frac{\rho_1}{\rho_1+s} g^*(s)\} + g^*(s) \int_0^{\tau} e^{su}q_{1i}(u)du \,] \,, \tag{3.28}$$

where

$$g^*(s) = \rho_1\{(\rho_1+s)e^{s\tau} - \rho_1\}^{-1} . \tag{3.29}$$

For small $s\tau$, (3.29) may be approximated by

$$g^*(s) = \frac{p_1\rho_1}{s}(1 - p_1 s\tau) , \tag{3.30}$$

where terms of order $(\rho\tau)^2$ are neglected and $s < K\rho$, for some positive constant K. Upon substitution of the approximations for the probability densities $p_{12i}, q_{1i}(u)$ and $q_{2i}(u)$, for $i = 0,1$, found in (3.13), and the approximation for $g^*(s)$ given by (3.30), into (3.28), the joint probability density $h^*_{21}(s)$ approximates to

$$h^*_{21}(s) = p_1\rho_1\{\lambda_2 p_{20} + (\lambda_2+\eta)p_{21}\}/s . \tag{3.31}$$

3.4.6 <u>The Sequence of Events on Counter 2</u>

The probability density

$$h_{12}(x)\delta x\delta y = \mathrm{pr}\left\{\begin{array}{l}\text{a recorded event on counter 2 in } (x,x+\delta x)\text{, and}\\ \text{a recorded event on counter 1 in } (0,\delta y)\end{array}\right\} ,$$

for small δx and δy is now calculated. For a recorded event on counter 1 in $(0,\delta y)$, counter 1 must be open at 0. Therefore summing over the possible states of the two counters at 0 we have that

$$h_{12}(x)\delta x\delta y =$$

$$\sum_{i=0}^{1} \text{pr}\left\{\begin{array}{l}\text{a recorded event on counter 2 in } (x,x+\delta x)\text{, and}\\ \text{a recorded event on counter 1 in } (0,\delta y)\text{, and}\\ \text{both counters are open at 0 with i gammas delayed}\end{array}\right\}$$

$$+ \sum_{i=0}^{1} \text{pr}\left\{\begin{array}{l}\text{a recorded event on counter 2 in } (x,x+\delta x)\text{, and}\\ \text{a recorded event on counter 1 in } (0,\delta y)\text{, and}\\ \text{counter 1 open, counter 2 closed at 0 with i gammas delayed}\end{array}\right\} .$$

The recorded event on counter 1 in $(0,\delta y)$ is either a beta only (λ_1) or a beta-gamma pair (λ_{12}). Therefore if there is no gamma delayed at 0 and if the recorded event is due to the beta from a beta-gamma pair (λ_{12}) then the gamma is delayed. Otherwise there is no change in the state of counter 2. So

$$h_{12}(x)\delta x\delta y = \text{pr}\left\{\begin{array}{l}\text{a recorded event}\\ \text{on counter 2 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{at } \delta y \text{ counter 2}\\ \text{is open with}\\ \text{0 gamma's delayed}\end{array}\right\} \lambda_1 p_{120}\delta y$$

$$+ \text{pr}\left\{\begin{array}{l}\text{a recorded event}\\ \text{on counter 2 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{at } \delta y \text{ counter 2}\\ \text{is open with}\\ \text{1 gamma delayed}\end{array}\right\} (\lambda_{12}p_{120}+\rho_1 p_{121})\delta y$$

$$+ \int_0^{\tau} \text{pr}\left\{\begin{array}{l}\text{a recorded event}\\ \text{on counter 2 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{at } \delta y \text{ counter 2}\\ \text{is closed for u}\\ \text{with 0 gamma's delayed}\end{array}\right\} \lambda_1 q_{20}(u)\delta y du \quad . \quad (3.32)$$

$$+ \int_0^{\tau} \text{pr}\left\{\begin{array}{l}\text{a recorded event}\\ \text{on counter 2 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{at } \delta y \text{ counter 2 is}\\ \text{closed for u with}\\ \text{1 gamma delayed}\end{array}\right\} \{\lambda_{12}q_{20}(u)+\rho_1 q_{21}(u)\}\delta y du .$$

The subsequent sequence of events on counter 2 is not independent of whether or not there is a gamma delayed at δy, so that the probabilities in (3.32) cannot be written down simply in terms of a single renewal density. Instead define two sets of functions, for $i = 0$ and 1 we have

$$m_i(x)\delta x = \text{pr}\left\{\begin{array}{l|l}\text{recorded event} & \text{counter 2 is} \\ \text{on counter 2 in} & \text{open at } 0^+ \\ (x,x+\delta x) & \text{with i gamma's}\end{array}\right\} ,$$

$$n_i(x,u)\delta x = \text{pr}\left\{\begin{array}{l|l}\text{recorded event} & \text{counter 2} \\ \text{on counter 2 in} & \text{is closed for u} \\ (x,x+\delta x) & \text{at } 0^+ \text{ with i gamma's}\end{array}\right\} ,$$

for small δx and with $m_i(x) = 0$ for all $x \leq 0$, $n_i(x,u) = 0$ for all $x \leq \tau-u$. The probability density $h_{12}(x)$ may now be expressed in terms of the sets of functions $\{m_i(x)\}$ and $\{n_i(x,u)\}$,

$$h_{12}(x) = \lambda_1 p_{120} m_0(x) + (\lambda_{12}p_{120}+\rho_1 p_{121})m_1(x)$$

$$+ \int_0^\tau \lambda_1 q_{20}(u) n_0(x,u)du + \int_0^\tau \{\lambda_{12}q_{20}(u) + \rho_1 q_{21}(u)\}n_1(x,u)du.$$

Alternatively in terms of the Laplace transforms,

$$h_{12}^*(s) = \lambda_1 p_{120} m_0^*(s) + (\lambda_{12}p_{120}+\rho_1 p_{121})m_1^*(s) + \int_0^\tau \lambda_1 q_{20}(u) n_0^*(s,u)du +$$

$$+ \int_0^\tau \{\lambda_{12}q_{20}(u)+\rho_1 q_{21}(u)\}n_1^*(s,u)du . \tag{3.33}$$

If the densities $\{m_i(x)\}$ and $\{n_i(x,u)\}$ or equivalently $\{m_i^*(s)\}$ and $\{n_i^*(s,u)\}$ can be expressed in terms of known functions then clearly $h_{12}(x)$ or $h_{12}^*(s)$ may be found. For this purpose the relationship between $\{m_i(x)\}$ and $\{n_i(x,u)\}$ is investigated. Now

$$m_0(x) = \text{pr}\left\{\begin{array}{l|l}\text{recorded event} & \text{counter 2 is} \\ \text{on counter 2} & \text{open at } 0^+ \text{ with} \\ \text{at 'x'} & \text{no gamma delayed}\end{array}\right\} .$$

This may be split into two parts depending upon whether a beta-gamma pair and hence a delayed gamma occurs before x or not. Thus

$$m_0(x) = \text{pr}\left\{\begin{array}{l}\text{recorded event}\\ \text{on counter 2 in 'x'}\\ \text{and first } \lambda_{12} \text{ after x}\end{array}\middle|\begin{array}{l}\text{counter 2 is}\\ \text{open at } 0^+ \text{ with}\\ \text{no gamma delayed}\end{array}\right\} +$$

$$+ \text{ pr}\left\{\begin{array}{l}\text{recorded event on}\\ \text{counter 2 in 'x'}\\ \text{and first } \lambda_{12} \text{ before 'x'}\end{array}\middle|\begin{array}{l}\text{counter 2 is}\\ \text{open at } 0^+\\ \text{with no gamma}\end{array}\right\}$$

$$= e^{-\lambda_{12}x} r_2(x) + \int_0^x \lambda_{12}e^{-\lambda_{12}y}\{m_1(x-y)p_2(y) + \int_0^\tau n_1(x-y,v)pc_2(y,v)dv\}dy \ . \tag{3.34}$$

In (3.34) $r_2(x)$, $p_2(y)$ and $pc_2(y,v)$ refer to the process consisting solely of gamma only events in (0,y] and are defined as follows,

(i) $r_2(x)$ is the renewal density for the process starting with an open interval,

(ii) $p_2(y)$ is the probability density that counter 2 is open at y conditional on the counter being open at 0^+,

(iii) $pc_2(y,v)$ is the probability density that counter 2 is closed for v at y conditional on the counter being open at 0^+.

If $m_1(x)$ is split into two parts depending upon whether or not the gamma delayed at 0^+ remains delayed until after x, then,

$$m_1(x) = e^{-\eta x}r_2(x) + \eta e^{-\eta x}p_2(x) +$$

$$+ \int_0^x \eta e^{-\eta y}\{n_0(x-y,0)p_2(y) + \int_0^\tau n_0(x-y,v)pc_2(y,v)dv\}dy \ . \tag{3.35}$$

Note that the second term represents the probability density of the delayed gamma occurring on counter 2 at x, with the counter being open at that point.

To express $\{n_i(x,u)\}$ in terms of $\{m_i(x)\}$, condition on the possible states of counter 2 when the counter reopens at $t = \tau-u$. Then

$$n_i(x,u) = \sum_{j=0}^{1} m_i(x+u-\tau) r_{ij}(\tau-u), \tag{3.36}$$

where

$$r_{ij}(y) = \text{pr}\left\{ \begin{matrix} j \text{ gamma's} \\ \text{delayed at} \\ y \end{matrix} \middle| \begin{matrix} i \text{ gamma's} \\ \text{delayed at} \\ 0^+ \end{matrix} \right\}, \quad i,j = 0,1.$$

Here, $r_{ij}(y)$ represent the transition probabilities for a single server queue, total system size one, with the input process Poisson of rate λ_{12} the service discipline being Poisson of rate η. It may be shown easily that

$$r_{00}(y) = \frac{\eta}{\lambda_{12}+\eta} + \frac{\lambda_{12}}{\lambda_{12}+\eta} \exp\{-(\lambda_{12}+\eta)y\} ,$$

and

$$r_{10}(y) = \frac{\eta}{\lambda_{12}+\eta} [1 - \exp\{-(\lambda_{12}+\eta)y\}] .$$

Substituting these values of the transition probabilities in (3.36) we find that

$$n_0(x,u) = m_0(x+u-\tau) \ [\eta + \lambda_{12}\exp\{-(\lambda_{12}+\eta)(\tau-u)\}](\lambda_{12}+\eta)^{-1}$$

$$+ \lambda_{12} m_1(x+u-\tau)[1 - \exp\{-(\lambda_{12}+\eta)(\tau-u)\}](\lambda_{12}+\eta)^{-1} , \tag{3.37}$$

and

$$n_1(x,u) = \eta m_0(x+u-\tau)[1 - \exp\{-(\lambda_{12}+\eta)(\tau-u)\}](\lambda_{12}+\eta)^{-1}$$

$$+ m_1(x+u-\tau)[\lambda_{12} + \eta \exp\{-(\lambda_{12}+\eta)(\tau-u)\}](\lambda_{12}+\eta)^{-1} . \quad (3.38)$$

Taking Laplace Transforms of (3.34), (3.35), (3.37) and (3.38) we find that

$$m_0^*(s) = r_2^*(s+\lambda_{12}) + \lambda_{12}\{p_2^*(s+\lambda_{12})m_1^*(s) + \int_0^\tau pc_2^*(s+\lambda_{12},v)n_1^*(s,v)dv\} , \quad (3.39)$$

$$m_1^*(s) = r_2^*(s+\eta) + \eta[p_2^*(s+\eta)\{1+n_0^*(s,0)\} + \int_0^\tau pc_2^*(s+\eta,v)n_0^*(s,v)dv] , \quad (3.40)$$

$$(\lambda_{12}+\eta)e^{s(\tau-u)}n_0^*(s,u) = m_0^*(s)[\eta + \lambda_{12}\exp\{-(\lambda_{12}+\eta)(\tau-u)\}]$$

$$+ \lambda_{12}m_1^*(s) [1 - \exp\{-(\lambda_{12}+\eta)(\tau-u)\}] , \quad (3.41)$$

$$(\lambda_{12}+\eta)e^{s(\tau-u)}n_1^*(s,u) = \eta m_0(s)[1 - \exp\{-(\lambda_{12}+\eta)(\tau-u)\}]$$

$$+ m_1^*(s)[\lambda_{12} + \eta \exp\{-(\lambda_{12}+\eta)(\tau-u)\}] . \quad (3.42)$$

Upon the application of elementary renewal theory arguments it may be shown that

$$p_2^*(s) = \lambda_2^{-1}e^{sv}pc_2^*(s,v) = \lambda_2^{-1}r_2^*(s) = A(s) ,$$

where

$$A(s) = e^{s\tau}\{(s+\lambda_2)e^{s\tau} - \lambda_2\}^{-1} .$$

Now we are in a position to solve for $\{m_i^*(s)\}$ and $\{n_i^*(s,v)\}$ exactly in terms of known functions. However, these densities were used merely to calculate the probability density $h_{12}^*(s)$ which is central to the problem of finding the covariance between the two counters. In the transformed density $h_{12}^*(s)$ the counter state probabilities $\{p_{12i}\}$ and $\{q_{2i}(u)\}$ appear not in their exact form but approximately, i.e. terms of order $(\rho\tau)^2$ are neglected. So in order to keep the level of approximation consistent within $h_{12}^*(s)$ a zeroth order approximation to $n_i^*(s,u)$ is made, that is for i = 0 and 1,

$$n_i^*(s,u) = m_i^*(s) ,$$

neglecting terms of order $(\rho\tau)$. This gives a first order approximation to $h_{12}^*(s)$ upon substitution in (3.33), and thus (3.33) becomes

$$h_{12}^*(s) = \lambda_1 p_{120} m_0^*(s) + (\lambda_{12} p_{120} + \rho_1 p_{121}) m_1^*(s)$$

$$+ \int_0^\tau \lambda_1 q_{20}(u) m_0^*(s) du + \int_0^\tau \{\lambda_{12} q_{20}(u) + \rho_1 q_{21}(u)\} m_1^*(s) du. \qquad (3.43)$$

If it is noted that for i = 0 and 1

$$p_{1i} = p_{12i} + \int_0^\tau q_{2i}(u) du ,$$

then (3.43) simplifies further to

$$h_{12}^*(s) = \lambda_1 p_{10} m_0^*(s) + \lambda_{12} p_{10} m_1^*(s) + \rho_1 p_{11} m_1^*(s) . \qquad (3.44)$$

Therefore in order to complete the calculation of the probability density $h^*_{12}(s)$ all that remains is to obtain a first order approximation to $\{m^*_i(s)\}$. Now solving (3.39) and (3.40) exactly, we have that

$$m^*_0(s) = \frac{\lambda_2+m^*_1(s)G^*_0(s)}{s+\lambda_2(1-e^{-s\tau}) + G^*_0(s)} ,$$

and

$$m^*_1(s) = \frac{\lambda_2+\eta+m^*_0(s)G^*_1(s)}{s+(\lambda_2+\eta)(1-e^{-s\tau}) + G^*_1(s)} ,$$

where

$$G^*_0(s) = \lambda_{12} + \frac{\lambda_2\lambda_{12}e^{-s\tau}}{\lambda_{12}+\eta} \{1 - \exp\{-(\lambda_{12}+\eta)\tau\} ,$$

and

$$G^*_1(s) = \frac{\eta e^{-s\tau}}{\lambda_{12}+\eta} [\lambda_2+\eta+(\lambda_{12}-\lambda_2)\exp\{-(\lambda_{12}+\eta)\tau\}] .$$

These results lead to

$$m^*_0(s) = \frac{\lambda_2\{(s+(\lambda_2+\eta)(1-e^{-s\tau}) + G^*_1(s)\} + (\lambda_2+\eta)G^*_0(s)}{\{s+\lambda_2(1-e^{-s\tau})\}[\{s+\lambda_2(1-e^{-s\tau})\} + G^*_0(s)+G_1(s)+\eta(1-e^{-s\tau})] + \eta G^*_0(s)(1-e^{-s\tau})} ,$$

and

$$m^*_1(s) = \frac{(\lambda_2+\eta)\{s+\lambda_2(1-e^{-s\tau}) + G^*_0(s)\} + \lambda_2 G^*_1(s)}{\{s+\lambda_2(1-e^{-s\tau})\}[\{s+(\lambda_2+\eta)(1-e^{-s\tau})\}+G^*_0(s)+G^*_1(s)] + \eta G^*_0(s)(1-e^{-s\tau})} .$$

Clearly both $m_0^*(s)$ and $m_1^*(s)$ have a simple pole at $s = 0$, furthermore it may be shown that $m_0^*(s)$ and $m_1^*(s)$ also have the same residue at $s = 0$. It is assumed that all other poles of $m_0^*(s)$ and $m_1^*(s)$ have negative real parts. Then $\{m_i^*(s)\}$ may be written as

$$m_i^*(s) = \frac{a_i}{s} + b_i + c_i s + d_i^*(s) \quad , \quad i = 0,1,$$

where $d_i^*(s)$ is analytic in some half-plane $Re(s) > -\gamma_i$ with $\gamma_i > 0$ for $i = 0,1$.

The constants $\{a_i, b_i, c_i\}$, $i = 0, 1$ are approximated by

$$a_0 = \lambda_2 p_{20} + (\lambda_2+\eta)p_{21} \; , \qquad b_0 = -\frac{\lambda_{12}\eta}{(\lambda_{12}+\eta)^2}(1-2\lambda_2\tau) \; ,$$

$$c_0 = \frac{\lambda_{12}\eta}{(\lambda_{12}+\eta)^3}(1-2\lambda_2\tau), \qquad a_1 = a_0 \; ,$$

$$b_1 = \frac{\eta^2}{(\lambda_{12}+\eta)^2}(1-2\lambda_2\tau) \; , \qquad c_1 = -\frac{\eta^2}{(\lambda_{12}+\eta)^3}(1-2\lambda_2\tau) \; .$$

Upon substitution in (3.44) the probability density $h_{12}^*(s)$ is approximated by

$$h_{12}^*(s) = \frac{\{\lambda_2 p_{20}+(\lambda_2+\eta)p_{21}\}p_1\rho_1}{s} + \frac{\lambda_{12}\eta 2}{(\lambda_{12}+\eta)^2} p_1^2 p_2^2 (1+\lambda_{12}\tau + \frac{2\lambda_{12}\eta\tau}{\lambda_{12}+\eta})$$

$$- \frac{\lambda_{12}\eta^2}{(\lambda_{12}+\eta)^2} p_1^2 p_2^2 (1 + \lambda_{12}\tau + \frac{2\lambda_{12}\eta\tau}{\lambda_{12}+\eta})s + d_{01}^*(s) \quad . \qquad (3.45)$$

Here $d_{01}^*(s)$ is a linear combination of $d_0^*(s)$ and $d_1^*(s)$. This, together with the first order approximation for $h_{21}^*(s)$, see (3.31), gives a first order approximation to $c^*(s)$, the Laplace Transform of the covariance between the two counters. Thus,

$$c^*(s) = \frac{\lambda_{12}\eta^2}{(\lambda_{12}+\eta)^2} p_1{}^2 p_2{}^2 \{1 + \lambda_{12}\tau + \frac{2\lambda_{12}\eta}{(\lambda_{12}+\eta)}\tau\}\{\frac{1}{s^2} - \frac{(\lambda_{12}+\eta)^{-1}}{s}\} + d^*(s),$$

where $d^*(s)$ is analytic in some half-plane $Re(s) > -\gamma$ with $\gamma > 0$. It now follows that the behaviour of $c(t)$ for large t is such that

$$c(t) = \frac{\lambda_{12}\eta^2}{(\lambda_{12}+\eta)^2} p_1{}^2 p_2{}^2 \left\{1 + \lambda_{12}\tau + \frac{2\lambda_{12}\eta}{(\lambda_{12}+\eta)}\tau\right\}\left\{t - (\lambda_{12}+\eta)^{-1}\right\} + O(e^{-\gamma t}) \quad . \tag{3.46}$$

3.4.7 Simulations and Conclusions

The formula for the covariance between the recorded numbers of events on the two counters given by (3.46), which is approximate in dead-time, τ, and measuring interval, t, is now examined in three ways.

The first and perhaps the most obvious way is to allow the common dead-time of the two counters, τ, to tend to zero. Thus (3.46) becomes

$$c_0(t) = \lim_{\tau\to 0} c(t) = \frac{\lambda_{12}\eta^2}{(\lambda_{12}+\eta)^2}\{t - (\lambda_{12}+\eta)^{-1}\} \quad . \tag{3.47}$$

Comparing this with the exact zero dead-time covariance given by (3.2), which is

$$\frac{\lambda_{12}\eta^2}{(\lambda_{12}+\eta)^2}\{t - \frac{1 - e^{-(\lambda_{12}+\eta)t}}{\lambda_{12} + \eta}\}, \tag{3.48}$$

we see that (3.47) and (3.48) are in agreement for large t.

Secondly we allow the half-life of the intermediate state to tend to zero. This corresponds to η tending to infinity. We may then compare the resulting covariance with the covariance given by (26) in Cox and

Isham (1977), which is a first order approximation in dead-time behaviour to the exact covariance for zero half-life. Upon rewriting (3.46) we may obtain

$$c(t) \;=\; \lambda_{12}\left(\frac{\lambda_{12}+\eta}{\eta}\right)^2 (p_{10}p_{20})^2(1 + \lambda_{12}\tau)t \;, \qquad (3.49)$$

where terms of order $(\rho t)^0$, in addition to terms of order $(\rho\tau)^2$, have been omitted in obtaining (3.49), and where p_{10}, p_{20} are the equilibrium probabilities that counter 1 and counter 2 respectively, are open with no gamma's waiting to occur. If the half-life now tends to zero, then the time that any gamma is delayed also tends to zero, and so becomes a simultaneous emission. Therefore as the half-life tends to 0 we would expect p_{i0} to tend to p_i of Cox and Isham (1977) for i = 1 and 2, where p_i, i = 1 and 2, are the equilibrium probabilities that each counter is open in the case of simultaneous events.

If this limiting process were valid then from (3.49) we see that

$$\lim_{\eta\to\infty} c(t) \;=\; \lambda_{12}p_1^{\,2}p_2^{\,2}(1 + \lambda_{12}\tau)t, \qquad (3.50)$$

which is in agreement with the leading term of (26) of Cox and Isham (1977). However, the equilibrium probabilities p_{10} and p_{20} in (3.49) are approximations to the true p_{10} and p_{20}, see (3.17) and (3.18), and in making these approximations it was assumed that terms of order $(\eta\tau)^2$ could be neglected. Therefore, taking the limit as the half-life tends to zero of (3.49) contravenes this assumption, and hence the limiting procedure of (3.50) is not valid. But, if p_{10} and p_{20} are obtained by the method outlined in (i) of (3.4.4) then we have the exact results that

$$p_{10} = \frac{\eta(1+\rho_1\tau)^{-1}(\rho_1+\eta+\lambda_{12}-\rho_1 e^{-\Delta\tau})}{(\lambda_{12}+\eta)(\rho_1+\eta - \lambda_1 e^{-\Delta\tau})}$$

and

$$p_{20} = \frac{\eta\{\lambda_2+\eta+(\lambda_{12}-\lambda_2)e^{-\Delta\tau}}{(1+\lambda_2\tau)[\lambda_{12}(\rho_2+\eta)+\eta(\lambda_2+\eta)+\{\eta(\lambda_{12}-\lambda_2)-\lambda_2\lambda_{12}\}e^{-\Delta\tau}]+\lambda_{12}\eta\tau(\rho_2+\eta-\lambda_2 e^{-\Delta\tau})} , \tag{3.51}$$

where $\Delta = \lambda_{12}+\eta$. If we now take the limit as $\eta \to \infty$ of (3.51) then for $i = 1$ and 2,

$$\lim_{\eta\to\infty} p_{i0} = (1 + \rho_i\tau)^{-1} \quad .$$

The complement of this result is that for $i = 1$ and 2,

$$\lim_{\eta\to\infty} p_{i1} = 0 .$$

Therefore if, in the formula for the covariance given by (3.46), the approximate forms for p_{10} and p_{20} given by (3.17) and (3.18) are replaced by the exact forms given by (3.51), then the covariance would remain unchanged to first order in dead-time but the limiting process of (3.50) would now be valid.

The covariance between the number of recorded events on counter 1, and the number of recorded events on counter 2, was simulated for the true model of Delayed State, i.e. that with no restriction on the number of gamma's that may be delayed. These simulations were devised and computed by Dr. D. Smith of the National Physical Laboratory and although the six simulations have a small but physically typical range of parameter values it is apparent from Table 1 that the values predicted by the covariance c(t) of (3.46), are greatly different from the simulated values. The possible reasons for this discrepancy include,

(i) the value of t, the period over which the covariance is calculated, is not large enough compared with other parameter values, and in particular the common dead-time τ;

(ii) the model is inadequate because the number of gamma's delayed often exceeds one.

If simulation 2 is compared with 5, and simulation 4 is compared with 6, then this would seem to exclude possibility (i). For, if small t were to be the reason for the large discrepancies between the simulated results and those given by (3.46), then the increase in t in both these pairs would reduce the differences by a considerable amount, clearly it does not.

Now the expected queue size for each of the six simulations is given and the smallness of this quantity would appear to exclude possibility (ii). However, another measure of the inadequacy of the model is investigated; this is the ratio of the covariance in the unrestricted model to the covariance for the model with at most one gamma delayed, both for zero dead-time. Therefore we need to calculate the zero dead-time covariance of the unrestricted model of section 3.2; using the model of section 3.3.4, this may be obtained as a special case of the zero dead-time covariance of Jitter, which we shall now calculate.

In a manner reminiscent of that employed in section 3.4.2, the number of recorded events on each counter is split into its component parts. For counter 1, the number of recorded events in (0,t], $N_1(t)$, consists of

(i) $N_{10}(t)$: the number of beta's that arrive in (0,t] without a corresponding gamma, i.e. beta only events,

Table 1: Simulated covariance for Delayed State and estimates from the Restricted Model

Simulation no.	1	2	3	4	5	6
t	2	2	2	2	10	10
ρ_1	0.9	0.9	1	0.9	0.9	0.9
ρ_2	0.4	0.4	1	0.4	0.4	0.4
λ_{12}	0.36	0.36	1	0.36	0.36	0.36
η	20	10	5	10	10	10
Expected queue	0.018	0.036	0.2	0.036	0.036	0.036
τ	0.002	0.005	0.02	0.03	0.005	0.03
Simulated cov.	0.698	0.675	1.671	0.6341	3.511	3.299
Standard error	0.001	0.001	0.006	0.0005	0.006	0.001
$c(t)$	0.676	0.632	1.247	0.610	3.296	3.175
Ratio	1.036	1.073	1.44	1.073	1.073	1.073
$c(t) \times$ Ratio	0.700	0.678	1.795	0.655	3.538	3.408

The standard error is that of the simulated covariance.

(ii) $N_{11}(t)$: the number of beta's that arrive in (0,t] and have been delayed from a beta-gamma pair,

(iii) $N_{12}(t)$: the number of beta-gamma pairs that arrive in (0,t] in which the gamma is delayed.

Similarly on counter 2 the number of recorded events in (0,t], $N_2(t)$, splits into

(i) $N_{20}(t)$: the number of gamma's that arrive in (0,t] without a corresponding beta, i.e. gamma only events,

(ii) $N_{21}(t)$: the number of gamma's that arrive in 0,t] that have been delayed from a beta-gamma pair,

(iii) $N_{22}(t)$: the number of beta-gamma pairs that arrive in (0,t] in which the beta is delayed.

Therefore, for i = 1 and 2,

$$N_i(t) = N_{i0}(t) + N_{i1}(t) + N_{i2}(t) .$$

Due to the absence of dead-time effects the only dependencies are between $N_{11}(t)$ and $N_{22}(t)$, $N_{12}(t)$ and $N_{21}(t)$. (In the one-sided Jitter situation $N_{12}(t)$ and $N_{21}(t)$ are both identically zero; and in the Delayed State situation $N_{11}(t)$ and $N_{22}(t)$ are both identically zero.) The covariance between $N_1(t)$ and $N_2(t)$ now simplifies as follows,

$$\begin{aligned}\mathrm{cov}\{N_1(t),N_2(t)\} &= \mathrm{cov}\{N_{10}(t)+N_{11}(t)+N_{12}(t),\ N_{20}(t)+N_{21}(t)+N_{22}(t)\}\\ &= \mathrm{cov}\{N_{11}(t)+N_{12}(t),\ N_{21}(t)+N_{22}(t)\}\\ &= \mathrm{cov}\{N_{11}(t),\ N_{22}(t)\} + \mathrm{cov}\{N_{12}(t),\ N_{21}(t)\}.\end{aligned}$$

The two covariances may be written down by appealing to an infinite server queue analogy. Referring to section 3.4.2.1, in the first covariance the situation is equivalent to an infinite server queue with Poisson arrival rate $\lambda_{12}\theta_1$ and service distribution $\Gamma(K_{1i},\mu_{1i})$, with probability ϕ_{1i} for i = 1,...,B(1), in which,

(i) $N_{11}(t)$ is identified with the number of arrivals in (0,t], and

(ii) $N_{22}(t)$ is identified with the number served in (0,t].

Then

$$\mathrm{cov}\{N_{11}(t),N_{22}(t)\} = \lambda_{12}\theta_1[t - \mu_1 + \int_t^\infty \{1 - F_1(w)\}dw] \, , \qquad (3.54)$$

where μ_1 is the expected service time and $F_1(w)$ is the cumulative distribution function of service time, see Conolly (1975, page 121) for example; a similar expression for the covariance between $N_{12}(t)$ and $N_{21}(t)$ may also be found.

If $\theta_1\phi_{1i}$, $\theta_2\phi_{2i}$, K_{1i}, K_{2i}, μ_{1i} and μ_{2i} are replaced by θ_i, θ_{B+i}, K_i, K_{B+i}, μ_i and μ_{B+i} where B = B(1), then the covariance between $N_1(t)$ and $N_2(t)$ may be obtained upon summation of (3.54) and its counterpart for $N_{12}(t)$ and $N_{21}(t)$, so that

$$\mathrm{cov}\{N_1(t),\ N_2(t)\} =$$

$$\lambda_{12}\{t - \sum_{i=1}^{A} \theta_i \frac{K_i}{\mu_i} + \sum_{i=1}^{A} \frac{\theta_i}{\mu_i} e^{-\mu_i t} \sum_{j=0}^{K_i-1} (K_i-j) \frac{(\mu_i t)^j}{j!} \}, \qquad (3.55)$$

where A = B+B(2).

Therefore the ratio of the covariance for the unrestricted model for Delayed State, to the covariance when there is a maximum of one delayed gamma, for zero dead-time, is approximately

$$(1 + \lambda_{12}\eta^{-1})^2 , \tag{3.56}$$

for large t, where in (3.55) $A = 1$, $K_1 = 1$ and $\mu_1 = \eta$. (The zero dead-time covariance for the restricted model is given by (3.2).)

If the dead-time influenced covariance given by (3.46) is now multiplied by the ratio (3.56) then much of the discrepancy between the first, second and fifth simulations and the results given by (3.46) disappears. For these simulations the terms neglected, those of order $(\rho\tau)^2$, are of an order less than the error in the simulated covariance. Therefore comparisons between the simulated covariance and the covariance given by the model are valid. However, for simulations three, four and six the modified covariance, $c(t) \times$ Ratio, is as much above the simulated results as $c(t)$ was below. The reason for this is two-fold,

(i) the terms neglected in calculating the covariance as given by (3.46) are of the same order as the error in the simulated covariance and so make valid comparisons impossible,

(ii) since the effect of dead-time is to deflate both covariances, deflating the covariance for the unrestricted model by the greater amount, the ratio (3.56) obtained by comparing the zero dead-time covariances is therefore an over-estimate of the difference between the dead-time influenced covariances. Furthermore, as the dead-time increases so does the amount by which the ratio over-estimates the difference between the two models.

We therefore conclude that the restricted model with a maximum of one delayed gamma is inadequate even for systems with expected queue size quite small (0.018). But, as the restricted model was formulated to provide a basis for the study of the unrestricted model and not as an alternative, and is invaluable in this context, the above comments are of theoretical and not practical interest.

3.5 Delayed State and Jitter, the Full Solution

3.5.1 The General Form for the Covariance

The method developed in section 3.4 for the calculation of the covariance function for the restricted model, is now extended to cope with the capacity of the detectors to delay, not one, but any number of particles at a particular instant. So, denote the covariance between the two counts by $c(t)$, then

$$\begin{aligned} c(t) &= \operatorname{cov}\{N_1(t), N_2(t)\} \\ &= E\{N_1(t)N_2(t)\} - E\{N_1(t)\}E\{N_2(t)\} \end{aligned} \tag{3.57}$$

Following the same procedure as that in section 3.4.3, we may write (3.57) as

$$c(t) = \int_0^t \int_u^t h_{12}(v-u)\,dv\,du + \int_0^t \int_v^t h_{21}(u-v)\,du\,dv - E\{N_1(t)\}E\{N_2(t)\} \tag{3.58}$$

where the joint probability densities $h_{12}(x)$ and $h_{21}(x)$ are defined to be,

$$h_{ij}(x) = \lim_{\substack{\delta x \to 0^+ \\ \delta y \to 0^+}} \frac{\operatorname{pr}\left\{\begin{array}{l}\text{a recorded event on counter } j \text{ in } (x, x+\delta x) \text{ and} \\ \text{a recorded event on counter } i \text{ in } (0, \delta y)\end{array}\right\}}{\delta x . \delta y},$$

for $x > 0$ and $i,j = 1,2,\ i \neq j$.

The univariate series of events on the ith counter is Poisson with rate $\rho_i = \lambda_i + \lambda_{12} \sum_{i=1}^{A} \theta_i = \lambda_i + \lambda_{12}$, for $i = 1,2$, see sections 3.3.4 and 3.4.6. So, if p_i denotes the equilibrium probability that counter i is open and the process on each counter starts from equilibrium at 0, then for i = 1 and 2,

$$E\{N_i(t)\} = \int_0^t p_i\rho_i du$$

$$= p_i\rho_i t \ . \qquad (3.59)$$

If the Laplace Transform of (3.58) is taken after substitution of (3.59) into (3.58), then,

$$c^*(s) = \frac{h_{12}^*(s)}{s^2} + \frac{h_{21}^*(s)}{s^2} - \frac{2p_1p_2\rho_1\rho_2}{s^3} , \qquad (3.60)$$

where s is the transform variable and an asterisk denotes a transformed function.

As in section 3.4 the problem of calculating the covariance is equivalent to that of calculating the probability densities $h_{ij}(x)$, or their transforms $h_{ij}^*(s)$, for i,j = 1,2; i ≠ j, and the equilibrium probabilities p_i for i = 1,2. For this purpose the states of the counters are represented as a Markov Process.

3.5.2 The Possible States of the Counting System

Using the notation of section 3.4.7 together with the physical interpretation of section 3.3.4, whenever there is a simultaneous event on the detectors, with probability θ_i, i = 1,...,B, the gamma occurs immediately and the beta is delayed for a period which has a gamma distribution with mean K_i/μ_i and index K_i. For i = B+1,...,A, the situation is reversed and the beta occurs immediately. Now, we may proceed as if the delay period consists of K_i stages, the lengths of which are independently exponentially distributed with mean μ_i^{-1}. The collection of K_i stages will be referred to as the ith branch; for i = 1,...,B we assume that whenever a beta particle is chosen for the ith branch it first enters the K_i th stage and works backwards until it reaches the first stage and subsequently occurs on the gamma counter. The above comments apply when the beta is delayed; if the gamma particle is delayed then interchange beta for gamma and consider i = B+1,...,A.

To represent the state of the counters as a Markov process, the states must be defined in such a way that the instantaneous transitions from one state to another are influenced only by the current state. Therefore if a counter is closed, we need to know for how long it has been closed. Furthermore, for each particle that is delayed, we need to know not only which branch the particle is in but also at which stage within a branch it is. All the necessary information needed to satisfy this latter condition is contained in $(N,\underset{\sim}{n})$, where

$$\underset{\sim}{n} = (\underset{\sim}{n}_1,\ldots,\underset{\sim}{n}_A) \;, \quad \underset{\sim}{n}_i = (N_i, n_{i1},\ldots,n_{iK_i}), \quad i = 1,\ldots,A,$$

$$N_i = \sum_{j=1}^{K_i} n_{ij}, \quad i = 1,\ldots,A, \quad N = \sum_{i=1}^{A} N_i \,.$$

Here $\underset{\sim}{n}_i$ refers to the ith branch and n_{ij} is the total number of particles delayed in the jth stage of the ith branch. Therefore,

(i) $\sum_{i=1}^{B} N_i$ is the total number of beta's delayed, and,

(ii) $\sum_{i=B+1}^{A} N_i$ is the total number of gamma's delayed, and,

(iii) N is the total number of particles delayed.

We are now in a position to define equilibrium probabilities and probability densities for the states of the counters,

(i) $p_{12N}(\underset{\sim}{n})$, the probability that both counters are open with N particles delayed and waiting to occur on the counters, their state being $\underset{\sim}{n}$,

(ii) $q_{jN}(u,\underset{\sim}{n})$, the probability density that counter j has been closed for a period u, where $0 \leq u \leq \tau_j$, the other counter being open, and with N particles delayed their state being $\underset{\sim}{n}$, for j = 1,2.

(iii) $q_{12N}(u,v,\underset{\sim}{n})$, the joint probability density that counter 1 has been closed for a period u, where $0 \leq u \leq \tau_1$, counter 2 has been closed for a period v, where $0 \leq v \leq \tau_2$, and with N particles delayed their state being $\underset{\sim}{n}$.

Note that each n_{ij} may take any non-negative integer value. The equations representing the possible changes from one state to another, in equilibrium, can now be written down in terms of the probabilities defined above. But first a little notation is introduced, let

$$\underset{\sim}{n}_i^{-1} = (N_i-1, n_{i1}, \ldots, n_{iK_i-1}, n_{iK_i}-1), \qquad \underset{\sim}{n}_i^{+1} = (N_i+1, n_{i1}+1, n_{i2}, \ldots, n_{iK_i}),$$

$$\underset{\sim}{n}_{ij} = (N_i, n_{i1}, \ldots, n_{ij-2}, n_{ij-1}-1, n_{ij}+1, n_{ij+1}, \ldots, n_{iK_i}) ,$$

with all other branches unchanged for i = 1,...,A and j = 1,...,K_i. Further define

$$N.\mu. = \sum_{i=1}^{A} N_i \mu_i .$$

Therefore the equilibrium equations are

$$(\rho+N_{.}\mu_{.})p_{12N}(\underset{\sim}{n}) = \sum_{i=1}^{A}\sum_{j=2}^{K_i}(n_{ij}+1)\mu_i p_{12N}(\underset{\sim}{n}_{ij})+q_{1N}(\tau_1,\underset{\sim}{n})+q_{2N}(\tau_2,\underset{\sim}{n}), \tag{3.61}$$

$$\frac{dq_{1N}(u,\underset{\sim}{n})}{du} = -(\rho_2+N_{.}\mu_{.})q_{1N}(u,\underset{\sim}{n}) + \sum_{i=B+1}^{A}\lambda_{12}\theta_i q_{1N-1}(u,\underset{\sim}{n}_i^{-1})$$
$$+\sum_{i=1}^{B}(n_{i1}+1)\mu_i q_{1N+1}(u,\underset{\sim}{n}_i^{+1}) + \sum_{i=1}^{A}\sum_{j=2}^{K_i}(n_{ij}+1)\mu_i q_{1N}(u,\underset{\sim}{n}_{ij})$$
$$+ q_{12N}(u,\tau_2,\underset{\sim}{n}), \tag{3.62}$$

$$\frac{dq_{2N}(u,\underset{\sim}{n})}{du} = -(\rho_1+N_{.}\mu_{.})q_{2N}(u,\underset{\sim}{n}) + \sum_{i=1}^{B}\lambda_{12}\theta_i q_{2N-1}(u,\underset{\sim}{n}_i^{-1})$$
$$+\sum_{i=B+1}^{A}(n_{i1}+1)\mu_i q_{2N+1}(u,\underset{\sim}{n}_i^{+1}) + \sum_{i=1}^{A}\sum_{j=2}^{K_i}(n_{ij}+1)\mu_i q_{2N}(u,\underset{\sim}{n}_{ij})$$
$$+ q_{12N}(\tau_1,u,\underset{\sim}{n}), \tag{3.63}$$

$$\frac{\partial q_{12N}(u,v,\underset{\sim}{n})}{\partial u}+\frac{\partial q_{12N}(u,v,\underset{\sim}{n})}{\partial v} = -(\lambda_{12}+N_{.}\mu_{.})q_{12N}(u,v,\underset{\sim}{n})$$
$$+\sum_{i=1}^{A}\lambda_{12}\theta_i q_{12N-1}(u,v,\underset{\sim}{n}_i^{-1}) + \sum_{i=1}^{A}(n_{i1}+1)\mu_i q_{12N+1}(u,v,\underset{\sim}{n}_i^{+1})$$
$$+\sum_{i=1}^{A}\sum_{j=2}^{K_i}(n_{ij}+1)\mu_i q_{12N}(u,v,\underset{\sim}{n}_{ij}), \tag{3.64}$$

for $n_{ij} = 0,1,\ldots,\infty$ and $j = 1,\ldots,K_i$, $i = 1,\ldots,A$. In the above $\rho = \lambda_1+\rho_2 = \rho_1+\lambda_2 = \lambda_1+\lambda_2+\lambda_{12}$. The equations (3.61)-(3.64) are subject to certain boundary conditions, which are

$$q_{1N}(0,\underset{\sim}{n}) = \lambda_1 p_{12N}(\underset{\sim}{n}) + \lambda_{12}\sum_{i=B+1}^{A}\theta_i p_{12N-1}(\underset{\sim}{n}_i^{-1}) + \sum_{i=1}^{B}(n_{i1}+1)\mu_i p_{12N+1}(\underset{\sim}{n}_i^{+1}), \tag{3.65}$$

$$q_{2N}(0,\underset{\sim}{n}) = \lambda_2 p_{12N}(\underset{\sim}{n}) + \lambda_{12} \sum_{i=1}^{B} \theta_i p_{12N-1}(\underset{\sim}{n}_i^{-1}) + \sum_{i=B+1}^{A} (n_{i1}+1)\mu_i p_{12N+1}(\underset{\sim}{n}_i^{+1}), \tag{3.66}$$

$$q_{12N}(u,0,\underset{\sim}{n}) = \lambda_2 q_{1N}(u,\underset{\sim}{n}) + \lambda_{12} \sum_{i=1}^{B} \theta_i q_{1N-1}(u,\underset{\sim}{n}_i^{+1})$$

$$+ \sum_{i=B+1}^{A} (n_{i1}+1)\mu_i q_{1N+1}(u,\underset{\sim}{n}_i^{+1}) , \tag{3.67}$$

$$q_{12N}(0,v,\underset{\sim}{n}) = \lambda_1 q_{2N}(v,\underset{\sim}{n}) + \sum_{i=B+1}^{A} \theta_i q_{2N-1}(v,\underset{\sim}{n}_i^{-1})$$

$$+ \sum_{i=1}^{B} (n_{i1}+1)\mu_i q_{2N+1}(v,\underset{\sim}{n}_i^{+1}) , \tag{3.68}$$

for $n_{ij} = 0,1,\ldots,\infty$; $j = 1,\ldots,K_i$; $i = 1,\ldots,A$. The solution of (3.61)-(3.64) subject to (3.65)-(3.68) is considered for equal dead-times, that is $\tau_1 = \tau_2 = \tau$. The results obtained will be shown to extend trivially to unequal dead-times for Jitter, and for Delayed State, they are extended in the appendix. As in previous problems, see sections 2.3 and 3.4.4, the counter state probabilities are determined approximately, i.e. terms of order $(\rho\tau)^2$ are neglected. The solution depends upon the K_i; for $K_i > 2$, $i = 1,\ldots,A$ we have that

$$p_{12N}(n) = \{1 - (\rho + \sum_{i=1}^{A} n_{iK_i}\mu_i)\tau\} \prod_{i=1}^{A} \left(\frac{\lambda_{12}\theta_i}{\mu_i}\right)^{N_i} \frac{\exp(-K_i\lambda_{12}\theta_i/\mu_i)}{n_{i1}!\cdots\cdots n_{iK_i}!}$$

$$= \{1 - (\rho + \sum_{i=1}^{A} n_{iK_i}\mu_i)\tau\} p_N(\underset{\sim}{n}) , \tag{3.69}$$

$$q_{1N}(u,\underset{\sim}{n}) = p_{12N}(\underset{\sim}{n})\{\lambda_1 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i(1+\mu_i\tau)$$

$$+ \sum_{i=B+1}^{A} (n_{iK_i-1} - n_{iK_i})\mu_i^2 u \} , \tag{3.70}$$

$$q_{2N}(u,\underset{\sim}{n}) = p_{12N}(\underset{\sim}{n})\{\lambda_2 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{iK_i}\mu_i(1+\mu_i\tau)$$

$$+ \sum_{i=1}^{B} (n_{iK_i-1} - n_{iK_i})\mu_i^2 u \} , \qquad (3.71)$$

$$q_{12N}(u,v,\underset{\sim}{n}) =$$

$$p_{12N}(\underset{\sim}{n})[\{\lambda_1 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i(1+\mu_i\tau)\}\{\lambda_2 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{iK_i}\mu_i(1+\mu_i\tau)\}$$

$$+ (\lambda_2 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{iK_i}\mu_i) \sum_{i=B+1}^{A} (n_{iK_i-1} - n_{iK_i})\mu_i^2 u$$

$$+ (\lambda_1 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i) \sum_{i=1}^{B} (n_{iK_i-1} - n_{iK_i})\mu_i^2 v]. \quad (3.72)$$

By using the probability law that

$$p_{1N}(\underset{\sim}{n}) = p_{12N}(\underset{\sim}{n}) + \int_0^{\tau} q_{2N}(u,\underset{\sim}{n})du , \qquad (3.73)$$

and a similar version for $p_{2N}(\underset{\sim}{n})$, we obtain

$$p_{1N}(\underset{\sim}{n}) = p_N(\underset{\sim}{n})\{1 - (\lambda_1 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i)\tau\} , \qquad (3.74)$$

and

$$p_{2N}(\underset{\sim}{n}) = p_N(\underset{\sim}{n})\{1 - (\lambda_2 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{iK_i}\mu_i)\tau\} . \qquad (3.75)$$

This solution is not valid for Delayed State because of the constraint that $K_i > 2$ for $i = 1,.., A$, and so a separate solution has to be calculated. This solution is

$$p_{12N} = (\frac{\lambda_{12}}{\mu})^N \frac{e^{\frac{\lambda_{12}}{\mu}}}{N!} \{1 - (\rho + N\mu)\tau\} , \qquad (3.76)$$

$$q_{1N}(u) = p_{12N}\{\lambda_1 + N\mu(1 + \mu\tau) + N\mu^2 u\} , \qquad (3.77)$$

$$q_{2N}(u) = p_{12N}\{\rho_2 + \lambda_{12}\mu(u-\tau)\} , \qquad (3.78)$$

$$q_{12N}(u,v) = p_{12N} \begin{cases} \rho_2(\lambda_1+N\mu)+\lambda_{12}\mu-\mu\tau(\lambda_1\lambda_{12}-\lambda_2 N\mu) - (N\rho_2+\lambda_{12})\mu^2 u \\ \quad + \lambda_{12}\mu(\rho_2+\mu)v \qquad \text{for } u > v \\ \rho_2(\lambda_1+N\mu)-\mu\tau(\lambda_1\lambda_{12}-\lambda_2 N\mu) + \{\rho_2\mu(\lambda_{12}-N\mu)-\lambda_{12}\mu(\lambda_1+N\mu)\}u \\ \quad + \lambda_{12}\mu(\lambda_1+N\mu)v \qquad \text{for } u < v. \end{cases}$$

$$(3.79)$$

Using the Delayed State equivalent of (3.73) we have that

$$p_{1N} = (\frac{\lambda_{12}}{\mu})^N e^{-\frac{\lambda_{12}}{\mu}} \{1 - (\lambda_1+N\mu)\tau\} ,$$

$$(3.80)$$

$$p_{2N} = (\frac{\lambda_{12}}{\mu})^N \frac{e^{-\frac{\lambda_{12}}{\mu}}}{N!} (1 - \rho_2\tau) .$$

Thus the solution of (3.61)-(3.64) subject to (3.65)-(3.68) has been obtained for all cases except when $K_i = 1$ or 2 for at least one i in $i = 1,\ldots,A$; $A > 1$. In section 3.5.4 it will be demonstrated that it is not necessary to calculate the solution for these cases for the purpose of calculating a first order approximation to the covariance between the two counts.

The equilibrium probabilities p_1 and p_2 that each counter is open may be calculated by summing (3.74) and (3.75) over the number of particles delayed. Alternatively, by noting that the series of events on counter 1 alternates between open periods exponentially distributed with mean ρ_1^{-1} and constant dead-times of length τ, the probability that counter 1 is open in equilibrium is given by

$$p_1 = (1 + \rho_1\tau)^{-1} \quad . \tag{3.81}$$

Similarly

$$p_2 = (1 + \rho_2\tau)^{-1} \; . \tag{3.82}$$

Therefore the expectations (3.59) for i = 1,2, are

$$E\{N_i(t)\} = \frac{\rho_i t}{1 + \rho_i\tau} \qquad i = 1,2,$$

which can be approximated by

$$E\{N_i(t)\} = \rho_i t(1 - \rho_i\tau) \qquad i = 1,2 \; . \tag{3.83}$$

The cross-product term in the covariance between the two counts is now calculated via the probability densities $h_{ij}(x)$ $i,j = 1,2$; $i \neq j$ defined in section 3.5.1. This is achieved by studying the series of events on the single counters from 0^+ to x. Because the Jitter problem is symmetric it is sufficient to consider one counter only.

3.5.3 The Process of Events on a Single Counter

Without loss of generality we consider the process of events on the beta counter, counter 1, and calculate the probability density $h_{21}(x)$ where

$$h_{21}(x)\delta x\delta y = \text{pr}\begin{Bmatrix}\text{a recorded event on counter 1 in } (x,x+\delta x)\text{, and,}\\ \text{a recorded event on counter 2 in } (0,\delta y)\end{Bmatrix}$$

for small δx and δy. For a recorded event on counter 2 in $(0,\delta y)$, counter 2 must be open at 0. Therefore summing over the possible states of the two counters at 0 we have

$$h_{21}(x)\delta x\delta y = \sum_{(N,\underset{\sim}{n})} \text{pr}\begin{Bmatrix}\text{both counters open at 0 with N particles delayed,}\\ \text{their state of delay being } \underset{\sim}{n}\text{, a recorded event on}\\ \text{counter 2 in } (0,\delta y)\text{, a recorded event on counter 1}\\ \text{on } (x,x+\delta x)\end{Bmatrix}$$

$$+ \sum_{(N,\underset{\sim}{n})} \text{pr}\begin{Bmatrix}\text{counter 2 open and counter 1 closed at 0 with N particles delayed}\\ \text{their state of delay being } \underset{\sim}{n}\text{, a recorded event on counter 2}\\ \text{on } (0,\delta y) \text{ and a recorded event on counter 1 in } (x,x+\delta x)\end{Bmatrix} \tag{3.84}$$

The summation over the delayed particles is

$$\sum_{(N,\underset{\sim}{n})} = \sum_{N=0}^{\infty} \Sigma\{N_i : \sum_{i=1}^{A} N_i = N\}\ \Sigma\{n_{ij} : \sum_{j=1}^{K_i} n_{ij} = N_i\}. \tag{3.85}$$

The middle summation in (3.85) is over all N_i's such that $\sum_{i=1}^{A} N_i = N$ and the right-hand summation is over all n_{ij}'s such that $\sum_{j=1}^{K_i} n_{ij} = N_i$. The recorded event on counter 2 is one of three types and may change the state of the delayed particles. This is the only way in which the state of counter 1 may be changed by the event because there are no simultaneous events. The three possible events and the changes they cause on the state of the delayed particles are

(i) a gamma only event; the state of delayed particles is then unchanged,

(ii) a beta-gamma pair in which the beta is delayed, the number of delayed beta's then increasing by one,

(iii) a delayed gamma from a beta-gamma pair, the number of delayed gamma's then decreasing by one.

So

$$h_{21}(x)\delta x\delta y = \sum_{(N,\underset{\sim}{n})} \{\lambda_2 \delta y \; \mathrm{pr}\left\{\begin{matrix}\text{a recorded}\\ \text{event on}\\ \text{counter 1}\\ \text{in } (x,x+\delta x)\end{matrix} \;\middle|\; \begin{matrix}\text{counter 1}\\ \text{open at } \delta y\\ \text{with state}\\ \text{of delays } (N,\underset{\sim}{n})\end{matrix}\right\} +$$

$$+ \sum_{i=1}^{B} \lambda_{12}\theta_i \; \delta y \; \mathrm{pr}\left\{\begin{matrix}\text{a recorded event on}\\ \text{counter 1 in}\\ (x,x+\delta x)\end{matrix} \;\middle|\; \begin{matrix}\text{counter 1 open}\\ \text{at } \delta y \text{ with state}\\ \text{of delays } (N+1,\underset{\sim}{n}_i^{+1})\end{matrix}\right\} +$$

$$+ \sum_{i=B+1}^{A} n_{i1}u_i \delta y \; \mathrm{pr}\left\{\begin{matrix}\text{a recorded event}\\ \text{on counter 1}\\ \text{in } (x,x+\delta x)\end{matrix} \;\middle|\; \begin{matrix}\text{counter 1}\\ \text{open at 0 with}\\ \text{state of delays } (N-1,\underset{\sim}{n}_i^{-1})\end{matrix}\right\} p_{12N}(\underset{\sim}{n}) +$$

$$+ \left(\begin{matrix}\text{an equivalent set}\\ \text{of terms with counter 1 open at 0 replaced by counter 1 closed at 0}\end{matrix}\right). \qquad (3.86)$$

In (3.86) the notational device $(N,\underset{\sim}{n})$ of section 3.5.2 is somewhat changed, we now have

$$(N+1,\underset{\sim}{n}_i^{+1}) = (N+1,\underset{\sim}{n}_1,\ldots,\underset{\sim}{n}_{i-1},\underset{\sim}{n}_i^{+1},\underset{\sim}{n}_{i+1},\ldots), \quad \underset{\sim}{n}_i^{+1} = (N_i+1,n_{i1},\ldots,n_{iK_i}+1).$$

It is not possible to express the above probability densities in terms of some single renewal density and therefore we define

$$I_N(x,\underset{\sim}{n})\delta x = \text{pr}\left\{\begin{array}{l}\text{a recorded event}\\ \text{on counter 1 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{counter 1 is open}\\ \text{at 0, with the}\\ \text{state of delays }(N,\underset{\sim}{n})\end{array}\right\}, \tag{3.87}$$

and

$$J_N(x,u,\underset{\sim}{n})\delta x = \text{pr}\left\{\begin{array}{l}\text{a recorded event}\\ \text{on counter 1 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{counter 1 is closed}\\ \text{for u with the state}\\ \text{of delays }(N,\underset{\sim}{n})\end{array}\right\}, \tag{3.88}$$

where $I_N(x,\underset{\sim}{n}) = 0$ for $x \leq 0$ and $J_N(x,u,\underset{\sim}{n}) = 0$ for $x \leq \tau - u$, small δx and $n_{ij} = 0,1,\ldots,\infty$ for $j = 1,\ldots,K_i$, $i = 1,\ldots,A$.

The probability density $h_{21}(x)$ as defined in section 3.5.1 and expressed in (3.86), may now be written in terms of $I_N(x,\underset{\sim}{n})$ and $J_N(x,u,\underset{\sim}{n})$ defined in (3.87) and (3.88). Before doing so, the possible changes in $(N,\underset{\sim}{n})$ which we started to describe above, are now completed.

$$\underset{\sim}{n}_i^{-1} = (N_i-1,n_{i1}-1,n_{i2},\ldots,n_{iK_i}),\ \underset{\sim}{n}_{ij} = (N_i,n_{i1},\ldots,n_{ij-1}+1,n_{ij}-1,n_{ij+1},\ldots,),$$

for $j = 1,\ldots,K_i$ and $i = 1,\ldots,A$; all other branches remaining unchanged. Therefore, taking the limit of (3.86) as δx and δy both tend to zero from the right,

$$h_{21}(x) = \sum_{(N,\underset{\sim}{n})}\{\lambda_2 I_N(x,\underset{\sim}{n}) + \sum_{i=1}^{B}\lambda_{12}\theta_i I_{N+1}(x,\underset{\sim}{n}_i^{+1}) + \sum_{i=B+1}^{A} n_{i1}\mu_i I_{N-1}(x,\underset{\sim}{n}_i^{-1})\}p_{12N}(\underset{\sim}{n})$$
$$+ \sum_{(N,\underset{\sim}{n})}\int_0^{\tau}\{\lambda_2 J_N(x,u,\underset{\sim}{n}) + \sum_{i=1}^{B}\lambda_{12}\theta_i J_{N+1}(x,u,\underset{\sim}{n}_i^{+1}) + \sum_{i=B+1}^{A} n_{i1}\mu_i J_{N-1}(x,u,\underset{\sim}{n}_i^{-1}\}q_{1N}(u,\underset{\sim}{n})du. \tag{3.89}$$

The intrinsic relationship between $I_N(x,\underset{\sim}{n})$ and $J_N(x,u,\underset{\sim}{n})$ is now developed with the intention of solving for $h_{21}(x)$ in terms of known functions. Hence the probability density $h_{21}(x)$ may be found; this is central to the calculation of the covariance function. Consider $I_N(x,\underset{\sim}{n})$ as defined in (3.87). Now $I_N(x,\underset{\sim}{n})$ may be split into several parts depending upon when the first change in the state of the delayed particles takes place and the type of change it is. Thus,

$$I_N(x,\underset{\sim}{n}) = \text{pr}\left\{\begin{array}{l}\text{recorded event on}\\ \text{counter 1 in}\\ (x,x+\delta x)\end{array}\middle|\begin{array}{l}\text{counter 1 is open}\\ \text{at } 0^+ \text{ with the}\\ \text{state of delays } (N,\underset{\sim}{n})\end{array}\right\}$$

$$= \text{pr}\left\{\begin{array}{l}\text{recorded event on counter 1}\\ \text{in } (x,x+\delta x) \text{ and the first}\\ \text{change in } \underset{\sim}{n} \text{ at or before } x\end{array}\middle|\begin{array}{l}\text{counter 1 is open at } 0^+\\ \text{with the state of delays}\\ (N,\underset{\sim}{n})\end{array}\right\}$$

$$+ \text{pr}\left\{\begin{array}{l}\text{recorded event on counter 1 in}\\ (x,x+\delta x) \text{ and the first change}\\ \text{in } \underset{\sim}{n} \text{ occurring after } x\end{array}\middle|\begin{array}{l}\text{counter 1 is open at } 0^+\\ \text{with the state of delays}\\ (N,\underset{\sim}{n})\end{array}\right\}$$

$$= \{r_1(x) + \sum_{i=1}^{B} n_{i1}\mu_i + \sum_{i=B+1}^{A} \lambda_{12}\theta_i\}\exp\{-(\lambda_{12}+N.\mu.)x\} +$$

$$+ \int_0^x \exp(-(\lambda_{12}+N.\mu.)y)\{\lambda_{12}\sum_{i=B+1}^{A}\theta_i[J_{N+1}(x-y,0,\underset{\sim}{n}_i^{+1})p_1(y)+ \int_0^\tau J_{N+1}(x-y,v,\underset{\sim}{n}_i^{+1})pc_1(y,v)]$$

$$+ \lambda_{12}\sum_{i=1}^{B}\theta_i[I_{N+1}(x-y,\underset{\sim}{n}_i^{+1})p_1(y) + \int_0^\tau J_{N+1}(x-y,v,\underset{\sim}{n}_i^{+1})pc_1(y,v)dv] +$$

$$+ \sum_{i=1}^{B} n_{i1}\mu_i[J_{N-1}(x-y,0,\underset{\sim}{n}_i^{-1})p_1(y) + \int_0^\tau J_{N-1}(x-y,v,\underset{\sim}{n}_i^{-1})pc_1(y,v)dv] +$$

$$+ \sum_{i=B+1}^{A} n_{i1}\mu_i[I_{N-1}(x-y),\underset{\sim}{n}_i^{-1})p_1(y) + \int_0^\tau J_{N-1}(x-y,v,\underset{\sim}{n}_i^{-1})pc_1(y,v)dv]$$

$$+ \sum_{i=1}^{A}\sum_{j=2}^{K_i} n_{ij}\mu_i[I_N(x-y,\underset{\sim}{n}_{ij})p_1(y) + \int_0^\tau J_N(x-y,v,\underset{\sim}{n}_{ij})pc_1(y,v)dv]\}dy,$$

(3.90)

where $r_1(x)$, $p_1(y)$ and $pc_1(y,v)$ refer to the process with inter-event arrivals consisting of open periods of average length λ_1^{-1} and dead-times of constant length τ, and are such that

(i) $r_1(x)$ is the renewal density for the process starting with an open interval,

(ii) $p_1(y)$ is the probability that counter 1 is open at y, conditional on the counter being open at 0^+, and,

(iii) $pc_1(y,v)$ is the probability that counter 1 is closed for v at y, conditional on the counter being open at 0^+.

The Laplace Transforms of $r_1(x)$, $p_1(y)$ and $pc_1(y,v)$ are related in a fairly simple way,

$$p_1^*(s) \quad = \quad \lambda_1^{-1} pc_1^*(s,v) \quad = \quad \lambda_1^{-1} r_1^*(s) \quad = \quad A(s) \ ,$$

where

$$A(s) \quad = \quad e^{s\tau}\{(s + \lambda_1)e^{s\tau} - \lambda_1\}^{-1} \ . \tag{3.90a}$$

To obtain $J_N(x,v,\underset{\sim}{n})$ in terms of $I_N(x,\underset{\sim}{n})$ we condition on the possible state of delays at $t = \tau - v$, the instant at which the blocked counter reopens. Then

$$J_N(x,v,\underset{\sim}{n}) \quad = \sum_{(M,\underset{\sim}{m})} r_{NM}(\tau - v;\underset{\sim}{n},\underset{\sim}{m}) I_M(x+v-\tau,\underset{\sim}{m}), \tag{3.91}$$

where

$$r_{NM}(y;\underset{\sim}{n},\underset{\sim}{m}) = \text{pr}\left\{ \begin{matrix}\text{state of delays}\\ \text{at } y \text{ is } (M,\underset{\sim}{m})\end{matrix} \;\middle|\; \begin{matrix}\text{state of delays}\\ \text{at } 0^+ \text{ is } (N,\underset{\sim}{n})\end{matrix} \right\} . \tag{3.92}$$

Note that $\{r_{NM}(y;\underset{\sim}{n},\underset{\sim}{m})\}$ are the transition probabilities of the number of customers in an infinite server queue, with arrivals forming a Poisson process of rate λ_{12} and where the service distribution has density f(x) say, with

$$f(x) = \sum_{i=1}^{A} \theta_i \frac{\mu_i(\mu_i x)^{K_i - 1}}{(K_i - 1)!} .$$

Considerable simplification of (3.90) ensues upon substitution of (3.91) into (3.90). Subsequent Laplace Transformation of (3.90) yields

$$A^{-1}(s+\lambda_{12}+N_{\cdot}\mu_{\cdot})I_N^*(s,\underset{\sim}{n}) = \lambda_1 + \sum_{i=1}^{B} n_{i1}\mu_i + \sum_{i=B+1}^{A} \lambda_{12}\theta_i$$

$$+ \sum_{i=1}^{B} \lambda_{12}\theta_i I_{N+1}^*(s,\underset{\sim}{n}_i^{+1}) + \sum_{i=B+1}^{A} n_{i1}\mu_i I_{N-1}(s,\underset{\sim}{n}_i^{-1}) + \sum_{i=1}^{A}\sum_{j=2}^{K_i} n_{ij}\mu_i I_N^*(s,\underset{\sim}{n}_{ij})$$

$$+ \sum_{(M,\underset{\sim}{m})} I_M^*(s,\underset{\sim}{m})R_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m})e^{-s\tau} , \qquad (3.93)$$

where

$$R_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) = \sum_{i=B+1}^{A} \lambda_{12}\theta_i r_{N+1M}(\tau;\underset{\sim}{n}_i^{+1},\tilde{m}) + \sum_{i=1}^{B} n_{i1}\mu_i r_{N-1M}(\tau;\underset{\sim}{n}_i^{-1},\underset{\sim}{m})$$

$$+ \lambda_1 \int_0^{\tau} \exp(-(\lambda_{12}+N_{\cdot}\mu_{\cdot})v)\{\sum_{i=1}^{A} \lambda_{12}\theta_i r_{N+1M}(\tau-v;\underset{\sim}{n}_i^{+1},\underset{\sim}{m}) + \sum_{i=1}^{A} n_{i1}\mu_i r_{N-1M}(\tau-v;\underset{\sim}{n}_i^{-1},\underset{\sim}{m})$$

$$+ \sum_{i=1}^{A}\sum_{j=2}^{K_i} n_{ij}\mu_i r_{NM}(\tau-v;\underset{\sim}{n}_{ij},\underset{\sim}{m})\}dv. \qquad (3.94)$$

The set of transition probabilities as defined in (3.92), has generating function $F_N(x,\underset{\sim}{z},\underset{\sim}{n})$, where

$$F_N(\underset{\sim}{x},\underset{\sim}{z},\underset{\sim}{n}) = \sum_{(M,m)} r_{NM}(x;\underset{\sim}{n},\underset{\sim}{m}) \prod_{i=1}^{A} \prod_{j=1}^{K_i} (Z_{ij})^{m_{ij}} .$$

$$= \prod_{i=1}^{A} \prod_{j=1}^{K_i} F_{ij}^{n_{ij}}(x) \exp\{ \sum_{i=1}^{A} \frac{\lambda_{12}\theta_i}{\mu_i} \sum_{j=1}^{K_i} (Z_{ij}-F_{ij}(x)\}, \qquad (3.95)$$

and

$$F_{ij}(x) = 1 - e^{-\mu_i x} \sum_{\ell=1}^{j} \frac{(1-Z_{i\ell})(\mu_i x)^{j-\ell}}{(j-\ell)!} . \qquad (3.96)$$

This is a generalisation of a result which may be found in Gross and Harris (1974, page 117) which is valid for exponential service distributions. A first order approximation to $r_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m})$, that is, neglecting terms of order $(\rho\tau)$ may be obtained from (3.95) and (3.96) or more directly by simple probabilistic arguments. We have

$$r_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) = \begin{cases} n_{i1}\mu_i\tau & \text{for } M = N-1, \ \underset{\sim}{m} = (\underset{\sim}{n}_i^{-1}), \ i = 1,\dots,A \\ n_{ij}\mu_i\tau & \text{for } M = N, \ \underset{\sim}{m} = (\underset{\sim}{n}_{ij}), \ i = 1,\dots,A, \ j = 2,\dots,K_i \\ 1-(\lambda_{12}+N.\mu.)\tau & \text{for } M = N, \ \underset{\sim}{m} = \tilde{n}, \\ \lambda_{12}\theta_i\tau & \text{for } M=N+1, \ \underset{\sim}{m} = (\underset{\sim}{n}_i^{+1}), \ i = 1,\dots,A . \end{cases} \qquad (3.97)$$

For any $K_i = 1$, the second line in (3.97) is absent. Upon substitution in (3.94) we obtain a first order approximation to $R_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m})$. Thus

$$
R_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) = \begin{cases}
\lambda_{12}^{2}\theta_a^{2}\tau & M = N+2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+2}) \\
\lambda_{12}^{2}\theta_a\theta_b\tau & M = N+2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1},\underset{\sim}{n}_b^{+1}) \\
\lambda_{12}\theta_a \; 1-\{\lambda_{12}+(N_a+1)\mu_a+N_{\odot}\mu_{\odot}\}\tau & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1}) \\
\lambda_{12}\theta_a n_{bc}\mu_b\tau & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1},\underset{\sim}{n}_{bc}) \\
\lambda_{12}\theta_a n_{b1}\mu_b\tau & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1}, \underset{\sim}{n}_b^{-1}) \\
n_{a1}\mu_a(n_{a1}-1)\mu_a\tau & M = N-2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-2}) \\
n_{a1}\mu_a n_{b1}\mu_b\tau & M = N-2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1},n_b^{-1}) \\
n_{a1}\mu_a n_{bc}\mu_c\tau & M = N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1},\underset{\sim}{n}_{bc}) \\
n_{a1}\mu_a \; 1-\{\lambda_{12}+(N_a-1)\mu_a+N_{\odot}\mu_{\odot}\}\tau & M=N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1}) \\
n_{a1}\mu_a\lambda_{12}\theta_b\tau & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1},\underset{\sim}{n}_b^{+1}) \\
\lambda_2\lambda_{12}\theta_b\tau & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_b^{+1}) \\
\lambda_2 n_{bc}\mu_b\tau & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_{bc}) \\
\lambda_2 n_{b1}\mu_b\tau & M = N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_b^{-1})
\end{cases} \qquad (3.98)
$$

The ranges for a, b and c in (3.98) are: $a = 1,\ldots,B$ for the first five entries, $a = B+1,\ldots,A$ for the next five entries, $b = 1,\ldots,A$ and $c = 2,\ldots,K_b$ for all relevant entries. Note that the fourth entry in (3.98) becomes $\lambda_{12}\theta_a(n_{aK_a}+1)\mu_a$ when $a = b$ and $c = K_b$. Also in (3.98) we use the notation

$$\underset{\sim}{n}_a^{+2} \equiv (N_a+2,\ n_{a1},\ldots,\ n_{aK_a-1}, n_{aK_a}+2),$$

$$\underset{\sim}{n}_a^{-2} \equiv (N_a-2,\ n_{a1}-2, n_{a2},\ldots,n_{aK_a}) ,$$

and

$$N_{\odot}\mu_{\odot} = \sum_{\substack{i=1\\ i\neq a}}^{A} N_i\mu_i .$$

Also $A^{-1}(s)$ may be approximated by

$$A^{-1}(s) = s\ (1+\lambda_1\tau).$$

Therefore

$$A^{-1}(s+\lambda_{12}+N_{.}\mu_{.}) = (s+\lambda_{12}+N_{.}\mu_{.})(1+\lambda_1\tau) . \tag{3.99}$$

A first order approximation to equation (3.93) may now be produced, this is

$$\sum_{\underset{\sim}{m}} a_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m})I_M^*(s,\underset{\sim}{m}) = \lambda_1 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{i1}\mu_i, \tag{3.100}$$

where

$$a_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) = R^{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) - e^{-s\tau}R_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}), \tag{3.101}$$

and

$$R^{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) = \begin{cases} -\lambda_{12}\theta_a & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1}) \quad a = B+1,\ldots,A \\ A^{-1}(s+\lambda_{12}+N_{.}\mu_{.}) & M = N \quad \underset{\sim}{m} = \underset{\sim}{n} \\ -n_{ab}\mu_a & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_{ab}) \quad a = 1,\ldots,A;\ b = 2,\ldots,K_a \\ -n_{a1}\mu_a & M = N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1}) \quad a = 1,\ldots,B. \end{cases} \tag{3.102}$$

Using the approximation for $A^{-1}(s+\lambda_{12}+N_{.}\mu_{.})$ in (3.99), and that for $R_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m})$ in (3.98), it is possible to express $a_{NM}(\tau;\tilde{n},\tilde{m})$ in the form

$$a_{NM}(\tau;\underset{\sim}{n},\underset{\sim}{m}) \;=\; (1+\lambda_1\tau)b_{NM}(\underset{\sim}{n},\underset{\sim}{m}) + \tau C_{NM}(\underset{\sim}{n},\underset{\sim}{m}), \tag{3.103}$$

where

$$b_{NM}(\underset{\sim}{n},\, \underset{\sim}{m}) = \begin{cases} -\lambda_{12}\theta_a & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1}) \quad a = 1,\dots,A \\ s+\lambda_{12}+N_{.}\mu_{.} & M = N \quad \underset{\sim}{m} = \underset{\sim}{n} \\ -n_{ab}\mu_a & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_{ab}) \quad a = 1,\dots,A;\ b = 2,\dots,K_a \\ -n_{a1}\mu_a & M = N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1}) \quad a = 1,\dots,A, \end{cases} \tag{3.104}$$

and

$$c_{NM}(\underset{\sim}{n},\underset{\sim}{m}) = \begin{cases} -\lambda_{12}\theta_a\theta_b & M = N+2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1},\underset{\sim}{n}_b^{+1}) \\ \lambda_{12}\theta_a\{s+\lambda_{12}+(N_a+1)\mu_a+N_{\odot}\mu_{\odot}\} & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1}) \\ -\lambda_{12}\theta_a n_{bc}\mu_b & M = N+1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1},\underset{\sim}{n}_{bc}) \\ -\lambda_{12}\theta_a n_{b1}\mu_b & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1},\underset{\sim}{n}_b^{-1}) \\ -\lambda_{12}\theta_a n_{b1}\mu_b & M = N \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{+1},\underset{\sim}{n}_b^{-1}) \\ n_{a1}\mu_a\{s+\lambda_{12}+(N_a-1)\mu_a+N_{\odot}\mu_{\odot}\} & M = N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1}) \\ -n_{a1}\mu_a n_{bc}\mu_b & M = N-1 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1},\underset{\sim}{n}_{bc}) \\ -n_{a1}\mu_a n_{b1}\mu_b & M = N-2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-1},\underset{\sim}{n}_b^{-1}) \\ -n_{a1}\mu_a(n_{a1}-1)\mu_a & M = N-2 \quad \underset{\sim}{m} = (\underset{\sim}{n}_a^{-2}) \;. \end{cases} \tag{3.105}$$

Note that in (3.105) the third entry becomes $\lambda_{12}\theta_a(n_{aK_a}+1)\mu_a$ when $a = b$ and $c = K_a$; the ranges of a,b and c in (3.105) are: $a = 1,\ldots,B$ for the first four entries, $a = B+1,\ldots,A$ for the last five entries, $b = 1,\ldots,A$ and $c = 2,\ldots,K_b$ for all the entries. The equivalent form of (3.103) in matrix notation is

$$a = (1 + \lambda_1\tau)b + c\tau, \tag{3.106}$$

where $a = (a_{NM})$, $b = (b_{NM})$ and $c = (c_{NM})$ are all square matrices of infinite dimensionality. Therefore, if we define the infinite dimensional vectors $\underset{\sim}{I}^*$ and $\underset{\sim}{\Lambda}$ by

$$\underset{\sim}{I}^* = \{I_N^*(s,\underset{\sim}{n})\} \qquad \text{and} \quad \underset{\sim}{\Lambda} = (\lambda_1 + \sum_{i=1}^{B} n_{i1}\mu_i + \sum_{i=B+1}^{A} \lambda_{12}\theta_i) \quad ,$$

then (3.100) can be written as

$$a\underset{\sim}{I}^* = \underset{\sim}{\Lambda}. \tag{3.107}$$

Assuming that the inverses of the matrices a and b exist and are unique, then

$$\underset{\sim}{I}^* = a^{-1}\underset{\sim}{\Lambda}$$

$$= \{(1 + \lambda_1\tau)b + \tau c\}^{-1}\underset{\sim}{\Lambda} \quad .$$

an approximation to which, is

$$\underset{\sim}{I}^* = \{(1 - \lambda_1\tau)b^{-1} - \tau b^{-1}cb^{-1}\}\, \underset{\sim}{\Lambda} \quad . \tag{3.108}$$

If the dead-time τ is zero then (3.108) becomes

$$\underset{\sim}{I}^*(\tau = 0) = b^{-1}\underset{\sim}{\Lambda} \quad . \tag{3.109}$$

Therefore, denoting $\underset{\sim}{I}^{*}$ for $\tau = 0$ by $\underset{\sim}{I}^{0*}$, we have

$$\underset{\sim}{I}^{*} = (1 - \lambda_1\tau)\underset{\sim}{I}^{0*} - \tau b^{-1}c\underset{\sim}{I}^{0*} \quad . \tag{3.110}$$

Thus if we can find an inverse of the tridiagonal matrix b then an approximate form for $\underset{\sim}{I}^{*}$ will be available. This will then be used to calculate the probability density $h_{21}(x)$ which is the remaining unknown function in the formulae for the covariance function. Since the matrix b is associated with the zero dead-time situation we again consider this. We have that if $I_N^{0}(x,\underset{\sim}{n})$ is the $\underset{\sim}{n}^{th}$ element of $\underset{\sim}{I}^{0}$ then

$$I_N^{0}(x,\underset{\sim}{n}) = \sum_{(M,\underset{\sim}{m})} (\lambda_1 + \sum_{i=1}^{B} m_{i1}\mu_i + \sum_{i=B+1}^{A} \lambda_{12}\theta_i) r_{NM}(x;\underset{\sim}{n},\underset{\sim}{m}) \quad , \tag{3.111}$$

since an event can happen in any one of three ways;

(i) a beta only event,

(ii) a delayed beta from a beta-gamma pair, and

(iii) a beta-gamma pair in which the gamma is delayed.

Now (3.111) may be calculated as follows,

$$\begin{aligned} I_N^{0}(x,\underset{\sim}{n}) &= \lambda_1 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{(M,\underset{\sim}{m})} \sum_{i=1}^{B} m_{i1}\mu_i r_{NM}(x;\underset{\sim}{n},\underset{\sim}{m}) \\ &= \lambda_1 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} \mu_i \, E\left\{ \begin{matrix}\text{number delayed} \\ \text{in branch } i \\ \text{at } x\end{matrix} \,\middle|\, \begin{matrix}\text{state of} \\ \text{delays at } 0\end{matrix} \right\} \\ &= \lambda_1 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} \mu_i \left.\frac{\partial F_N}{\partial z_{i1}}\right|_{\underset{\sim}{z}=\underset{\sim}{1}} \quad , \end{aligned} \tag{3.112}$$

where $F_N = F_N(x;\underset{\sim}{z},\underset{\sim}{n})$ is defined in (3.95) to be the generating function of the transition probabilities $\{r_{NM}(x;\underset{\sim}{n},\underset{\sim}{m})\}$ and $\underset{\sim}{1}$ is the unit vector. Since

$$\left.\frac{\partial F_N}{\partial z_{i1}}\right|_{\underset{\sim}{z}=\underset{\sim}{1}} = \frac{\lambda_{12}\theta_i}{\mu_i}\ \{1 - \sum_{a=1}^{K_i} \frac{(\mu_i x)^{a-1} e^{-\mu_i x}}{(a-1)!}\} + \sum_{a=1}^{K_i} \frac{n_{ia}(\mu_i x)^{a-1} e^{-\mu_i x}}{(a-1)!} ,$$

then (3.112) becomes

$$I_N^O(x,\underset{\sim}{n}) = \rho_1 + \sum_{i=1}^{B} \sum_{j=1}^{K_i} (n_{ij}\mu_i - \lambda_{12}\theta_i) \frac{(\mu_i x)^{j-1} e^{-\mu_i x}}{(j-1)!} .$$

Alternatively, after Laplace Transformation

$$I_N^{O*}(s,\underset{\sim}{n}) = \frac{\rho_1}{s} + \sum_{i=1}^{B} \sum_{j=1}^{K_i} \frac{(n_{ij}\mu_i - \lambda_{12}\theta_i)\mu_i^{j-1}}{(s+\mu_i)^j} . \qquad (3.113)$$

If we represent (3.111) in matrix form, then

$$\underset{\sim}{I}^O = R\underset{\sim}{\Lambda} ,$$

or

$$\underset{\sim}{I}^{O*} = R^*\underset{\sim}{\Lambda} , \qquad (3.114)$$

where

$$R^* = \{r_{NM}^*(s;\underset{\sim}{n},\underset{\sim}{m})\} . \qquad (3.115)$$

Comparing (3.114) with (3.109) we see that

$$b^{-1} = R^*.$$

Therefore, we are now in a position to calculate $b^{-1}c\underset{\sim}{I}^{O*}$. If $G_N(\underset{\sim}{n})$ is defined to be the $\underset{\sim}{n}$th element of $c\underset{\sim}{I}^{O*}$, then

$$G_N(\underset{\sim}{n}) = (\lambda_1 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i)(\sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{i1}\mu_i) + (\sum_{i=B+1}^{A} \lambda_{12}\theta_i)(\sum_{i=1}^{B} n_{i1}\mu_i) +$$

$$+ \sum_{i=1}^{B} n_{i1}(n_{i1}-1)\mu_i^{\,2} + \sum_{\substack{i=1 \\ 1\neq j}}^{B} \sum_{j=1}^{B} n_{i1}n_{j1}\mu_i\mu_j \ . \qquad (3.116)$$

So that the $\underset{\sim}{n}$th of $b^{-1}c\underset{\sim}{I}^{0*}$ is

$$\sum_{(M,\underset{\sim}{m})} G_M(\underset{\sim}{m})r_{NM}^{*}(s;\underset{\sim}{n},\underset{\sim}{m}) \qquad (3.117)$$

This involves terms like

$$\sum_{(M,\underset{\sim}{m})} m_{i1}\mu_i r_{NM}^{*}(s;\underset{\sim}{n},\underset{\sim}{m}), \quad \text{and,} \quad \sum_{(M,\underset{\sim}{m})} m_{i1}m_{j1}\mu_i\mu_j r_{NM}^{*}(s;\underset{\sim}{n},\underset{\sim}{m}), \qquad (3.118)$$

for $i,j = 1,\ldots,B$. When $i = j$ in the second summation of (3.118), m_{j1} is replaced by $m_{i1}-1$. Now (3.117) can be calculated using $F_N(x;\underset{\sim}{z},\underset{\sim}{n})$ the generating function of $\{r_{NM}(x;\underset{\sim}{n},\underset{\sim}{m})\}$, for example

$$\sum_{(M,\underset{\sim}{m})} m_{i1}\mu_i r_{NM}^{*}(s;\underset{\sim}{n},\underset{\sim}{m}) = \mathcal{L}\left\{ \mu_i \left.\frac{\partial F_N}{\partial z_{i1}}\right|_{\underset{\sim}{z}=1} \right\} , \qquad (3.119)$$

and

$$\sum_{(M,\underset{\sim}{m})} m_{i1}m_{j1}\mu_i\mu_j r_{NM}^{*}(s;\underset{\sim}{n},\underset{\sim}{m}) = \mathcal{L}\left\{ \mu_i\mu_j \left.\frac{\partial^2 F_N}{\partial z_{i1}\partial z_{j1}}\right|_{\underset{\sim}{z}=\underset{\sim}{1}} \right\} \qquad (3.120)$$

where $\mathcal{L}$ denotes the Laplace Transform operator. Hence, using the form for $F_N(x;\underset{\sim}{z},\underset{\sim}{n})$ given by (3.95) and (3.96), calculation of (3.119), (3.120) and subsequent substitution into the relevant parts of (3.117), an approximate form for $I_N^{*}(s,\underset{\sim}{n})$ may be found, to wit

$$I_N^{*}(s,\underset{\sim}{n}) = \frac{p_1\rho_1}{s} + p_1^{\,2} \sum_{i=1}^{B} \sum_{j=1}^{K_i} \frac{(n_{ij}\mu_i - \lambda_{12}\theta_i)\mu_i^{\,j-1}}{(s+\mu_i)^j} + L_N(\underset{\sim}{n})\tau \ , \qquad (3.121)$$

where $L_N(n)$ is such that

$$\sum_{(N,\underset{\sim}{n})} p_N(\underset{\sim}{n}) L_N(\underset{\sim}{n}) = 0 ,$$

$p_N(\underset{\sim}{n})$ given by (3.67).

Upon substitution of (3.121) into the Laplace Transformed version of (3.91) an approximation to $J_N^*(s,v,\underset{\sim}{n})$ is found. However, to obtain a first order approximation to the probability density $h_{21}^*(s)$, we need only a zeroth order approximation to $J_N^*(s,v,\underset{\sim}{n})$, and this is $I_N^*(s,\underset{\sim}{n})$. This zeroth order approximation becomes first order upon substitution into (3.89), indeed we have that

$$h_{21}^*(s) = \sum_{(N,\underset{\sim}{n})} \{\lambda_2 I_N^*(s,\underset{\sim}{n}) + \sum_{i=1}^{B} \lambda_{12}\theta_i I_{N+1}^*(s,\underset{\sim}{n}_i^{+1})$$

$$+ \sum_{i=B+1}^{A} n_{i1}\mu_i I_{N-1}^*(s,\underset{\sim}{n}_i^{-1})\} p_{2N}(\underset{\sim}{n}) , \qquad (3.122)$$

where all terms of order $(\rho\tau)^2$ have been neglected in (3.122). Finally we work (3.121) into (3.122), which yields

$$h_{21}^*(s) = \frac{p_1 p_2 \rho_1 \rho_2}{s} + \sum_{i=1}^{B} p_1^{\,2} p_2^{\,2} \lambda_{12}\theta_i \left(\frac{\mu_i}{\mu_i + s}\right)^{K_i} . \qquad (3.123)$$

3.5.4 The Covariance Function and Comments

Having obtained the joint probability density $h_{21}^*(s)$, we may write down the corresponding density for the gamma counter by interchanging beta for gamma throughout section 3.5.3. This may be achieved by replacing the summation $\sum_{i=1}^{B}$ by the summation $\sum_{i=B+1}^{A}$ in (3.123), thus

$$h_{12}^{*}(s) = \frac{p_1 p_2 \rho_1 \rho_2}{s} + \sum_{i=B+1}^{A} p_1^{\,2} p_2^{\,2} \lambda_{12} \theta_i \left(\frac{\mu_i}{\mu_i+s}\right)^{K_i} . \qquad (3.124)$$

An approximate form for the transformed covariance function may now be obtained upon substitution of (3.123) and (3.124) into (3.60), whence,

$$c^{*}(s) = \frac{\lambda_{12} p_1^{\,2} p_2^{\,2}}{s^2} \sum_{i=1}^{A} \theta_i \left(\frac{\mu_i}{\mu_i+s}\right)^{K_i} . \qquad (3.125)$$

If (3.125) is inverted with respect to the transform variable s, then

$$\begin{aligned} \mathrm{cov}\{N_1(t),N_2(t)\} &= c(t) \\ &= \lambda_{12} p_1^{\,2} p_2^{\,2} \{ t - \sum_{i=1}^{A} \theta_i \frac{K_i}{\mu_i} \\ &\quad + \sum_{i=1}^{A} \frac{\theta_i e^{-\mu_i t}}{\mu_i} \sum_{j=0}^{K_i-1} (K_i - j) \frac{(\mu_i t)^j}{j!} \}. \end{aligned} \qquad (3.126)$$

Now the common dead-time τ of the two counters enters the covariance function given by (3.126) only through $p_1^{\,2}$ and $p_2^{\,2}$, and a first order approximation in dead-time to the covariance function with unequal dead-times, is obtained simply by setting $\tau = \tau_i$ in each of p_i, i.e. $p_i = (1+\rho_i\tau_i)^{-1}$ for i = 1 and 2. This may be seen by considering $h_{21}^{*}(s)$: the gamma dead-time τ_2 enters $h_{21}(s)$ only through equation (3.89) and consequently through $p_{2N}(\underset{\sim}{n})$ in (3.122), and the text between equations (3.90) and (3.122) is valid for the beta dead-time equal to τ_1. Thus, $h_{21}^{*}(s)$ is valid for unequal dead-times, therefore so are $h_{12}^{*}(s)$ and $c^{*}(s)$.

We now examine the validity of the covariance function given by (3.126) for various ranges of the K_i. The whole of section 3.5.3 is seen to hold for any values that the K_i might take, provided the approximate forms (3.74) and (3.75) for $p_{1N}(\underset{\sim}{n})$ and $p_{2N}(\underset{\sim}{n})$ apply. The two joint probabilities $p_{1N}(\underset{\sim}{n})$ and $p_{2N}(\underset{\sim}{n})$ were obtained using the normalising condition (3.73) and an equivalent condition for counter 2, (3.73) did not depend on the coefficient of u in $q_{2N}(u,\underset{\sim}{n})$ and it is only this coefficient that takes different forms for different ranges of the K_i. Thus, the covariance function given by (3.126) is valid for any or all of the K_i equal to unity or two, $i = 1,\ldots,A$. In particular, by setting $B = 0$, $A = 1$, $\theta_1 = 1$, $K_1 = 1$ and $\mu_1 = \eta$ we obtain the Delayed State covariance,

$$\mathrm{cov}\{N_1(t),N_2(t)\} = \lambda_{12}p_1^{\,2}p_2^{\,2}(t - \frac{1-e^{-\eta t}}{\eta}) \ . \qquad (3.127)$$

If the dead-time influenced covariance given by (3.126) is compared with the zero dead-time covariance given by (3.55), then it is seen that the former is a multiple, $p_1^{\,2}p_2^{\,2}$, of the latter. Therefore, it appears that to first order in the dead-time the covariance is equivalent to that which would be obtained in the zero dead-time situation if the efficiencies of each counter were $p_i^{\,2}\varepsilon_i$ instead of ε_i, $i = 1,2$; i.e. the probability of a random deletion of a particle on counter i is raised from $1-\varepsilon_i$ to $1-p_i^{\,2}\varepsilon_i$, $i = 1,2$.

To complete the study of Jitter we now calculate an approximation to the recorded coincidence rate; all terms of order $(\rho\tau)^3$ are neglected in section 3.5.5, where the counters will be assumed to have common dead-time τ.

3.5.5 The Coincidence Rate

As in section 1.3.2 a coincidence is defined to be the occurrence, in the combined output of the two counters, of two recorded events within a period h of one another, with $h < \min(\tau_1, \tau_2)$, so that the events are from different counters; thus the coincidence rate is

$$\lim_{\delta h \to 0} \frac{\mathrm{pr}\left\{ \begin{array}{l} \text{a recorded event on one counter in } [0,\delta h], \text{ and} \\ \text{a recorded event on the other counter in } [0,h) \end{array} \right\}}{\delta h} \quad . \tag{3.128}$$

Since the probability of a simultaneous event is negligible, the coincidence must be due to either,

(i) a recorded event on counter 1 in $[0,\delta h)$, and a recorded event on counter 2 in $[\delta h, h)$, or,

(ii) a recorded event on counter 2 in $[0,\delta h)$, and a recorded event on counter 1 in $[\delta h, h)$.

Now each recorded event is one of three types;

(i)' a single event,

(ii)' an event that causes a delayed event on the other counter, or,

(iii)' a delayed event.

Types (ii)' and (iii)' change the state of those particles queueing where (i)' leaves the state unchanged.

For equal dead-times, i.e. $\tau_1 = \tau_2 = \tau$, the rate of coincidences due to type (ii) events is

$$\sum_{(N,\underset{\sim}{n})} \int_0^h dx\{\lambda_2 I_N(x,\underset{\sim}{n}) + \sum_{i=1}^{B} \lambda_{12}\theta_i I_{N+1}(x,\underset{\sim}{n}_i^{+1}) + \sum_{i=B+1}^{A} n_{i1}\mu_i I_{N-1}(x,\underset{\sim}{n}_i^{-1})\}p_{12N}(\underset{\sim}{n})$$

$$+ \sum_{(N,\underset{\sim}{n})} \int_{\tau-h}^{\tau} du \int_0^h dx\{\lambda_2 J_N(x,u,\underset{\sim}{n}) + \sum_{i=1}^{B} \lambda_{12}\theta_i J_{N+1}(x,u,\underset{\sim}{n}_i^{+1}) +$$

$$+ \sum_{i=B+1}^{A} n_{i1}\mu_i J_{N-1}(x,u,\underset{\sim}{n}_i^{-1})\}q_{1N}(u,n). \quad (3.129)$$

In (3.129), $I_N(x,\underset{\sim}{n})$ and $J_N(x,u,\underset{\sim}{n})$ are defined in (3.87) and (3.88) respectively; the terms involving $I_N(x,\underset{\sim}{n})$ represent the probability of both counters being open at 0, an event on counter 2 at 0^+ and an event on counter 1 before h; the terms involving $J_N(x,u,\underset{\sim}{n})$ represent a similar probability but with counter 1 closed at 0. Now, using the zeroth order approximation to $J_N(x,u,\underset{\sim}{n})$, namely $I_N(x-\tau+u,\underset{\sim}{n})$, and defining

$$g_N(x;\underset{\sim}{n}) = \lambda_2 I_N(x;\underset{\sim}{n}) + \sum_{i=1}^{B} \lambda_{12}\theta_i I_{N+1}(x,\underset{\sim}{n}_i^{+1}) + \sum_{i=B+1}^{A} n_{i1}\mu_i I_{N-1}(x,\underset{\sim}{n}_i^{-1}) , \quad (3.130)$$

then neglecting terms of order $(\rho\tau)^3$, (3.129) may be approximated by

$$\sum_{(N,n)} \{p_{12N}(\underset{\sim}{n}) \int_0^h g_N(x,\underset{\sim}{n})dx + \int_{\tau-h}^{\tau} \int_0^h q_{1N}(u,\underset{\sim}{n})g_N(x-\tau+u,\underset{\sim}{n})dx\, du \}$$

$$= \sum_{(N,\underset{\sim}{n})} \{p_{12N}(\underset{\sim}{n}) \int_0^h g_N(x,\underset{\sim}{n})dx + \int_{\tau-h}^{\tau} \int_0^{h-\tau+u} q_{1N}(u,\underset{\sim}{n})g_N(x,\underset{\sim}{n})dx\, du\}. \quad (3.131)$$

In obtaining (3.131) remember $I_N(x,\underset{\sim}{n}) = 0$ for $x \le 0$. If the order of integration in the second term of (3.131) is reversed, and $q_{1N}(u,\underset{\sim}{n})$ is approximated by $(\lambda_2 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i)p_{12N}(\underset{\sim}{n})$ then (3.131) is

$$\sum_{(\underset{\sim}{N},n)} p_{12N}(\underset{\sim}{n}) \int_0^h g_N(x,\underset{\sim}{n}) \{1 + (\lambda_2 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i)(h-x)\}dx$$

$$= \int_0^h \sum_{(\underset{\sim}{N},n)} p_{1N}(\underset{\sim}{n})g_N(x,\underset{\sim}{n}) \{1 + (\lambda_2 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i)(h-x-\tau)\}dx \ , \tag{3.132}$$

since, from (3.69), (3.74) and (3.75)

$$p_{12N}(\underset{\sim}{n}) = p_{2N}(\underset{\sim}{n})\{1 - (\lambda_1 + \sum_{i=1}^{B} \lambda_{12}\theta_i + \sum_{i=B+1}^{A} n_{iK_i}\mu_i)\tau\} + O(\tau^2)$$

$$= p_{1N}(\underset{\sim}{n})(1 - (\lambda_2 + \sum_{i=B+1}^{A} \lambda_{12}\theta_i + \sum_{i=1}^{B} n_{iK_i}\mu_i)\tau) + O(\tau^2) \ .$$

In progressing through the next two stages, (3.122) and (3.123) are used, thus the rate of coincidences due to type (ii) events is

$$\int_0^h h_{21}(x)\{1 + \rho_1(h-x-\tau)\}dx$$

$$= p_{12}p_1\{(1+\rho_1 h)[\rho_1\rho_2 h + p_{12} \sum_{i=1}^{B} \lambda_{12}\theta_i\{1 - \sum_{j=0}^{K_i-1} \frac{(\mu_i h)^j e^{-\mu_i h}}{j!}\}\}\ ]$$

$$- \rho_1[\rho_1\rho_2 \frac{h^2}{2} + p_{12} \sum_{i=1}^{B} \lambda_{12}\theta_i \frac{K_i}{\mu_i} \{1 - \sum_{j=0}^{K_i} \frac{(\mu_i h)^j e^{-\mu_i h}}{j!}\}\ ]\ \} \ . \tag{3.133}$$

Equation (3.133) takes different forms depending on the value of the K_i, and specifically, on whether any of the K_i are equal to unity. We consider only two cases, the first when all the K_i are greater than unity and the second when B = 1 and $K_1 = 1$.

For $K_i > 1$, $i = 1,\ldots,B$, (3.133) is

$$p_{12}p_1 h\{\rho_1\rho_2 + \rho_1^2\rho_2 \frac{h}{2} + p_{12} \sum_{i=1}^{B} \lambda_{12}\theta_i\mu_i \frac{h}{2}(\mu_i - \rho_1 K_i)\} \quad . \tag{3.134}$$

By interchanging the roles of the subscripts 1 and 2 in (3.134) we may obtain the rate of coincidences due to type (i) events for $K_i > 1$, $i = B+1,\ldots,A$. Therefore the total coincidence rate for $K_i > 1$, $i = 1,\ldots,A$ is

$$p_{12}h\{\rho_1\rho_2(p_1+p_2) + \rho_1\rho_2 \frac{h}{2}(p_1\rho_1+p_2\rho_2) + p_{12}\frac{h}{2}\sum_{i=1}^{A}\lambda_{12}\theta_i\mu_i(\mu_i-\rho_i K_i)\} \ , \tag{3.135}$$

where

$$\rho_i = \begin{cases} \rho_1 & \text{for } 1 = 1,\ldots,B \\ \rho_2 & \text{for } i = B+1,\ldots,A. \end{cases}$$

Before we calculate the coincidence rate due to type (ii) events for the second case, i.e. $B = 1$ and $K_1 = 1$, the coincidence rate due to type (i) events is calculated when there are no delayed events on counter 2. This is necessary for the calculation of the coincidence rate for De-centred Jitter and Delayed State. The rate of coincidences due to type (i) events is now

$$\sum_{(N,\underset{\sim}{n})} [p_{12N}(\underset{\sim}{n})(1-e^{-\rho_2 h}) + \int_{\tau-h}^{\tau} q_{2N}(u,\underset{\sim}{n})\{1-e^{-\rho_2(h-\tau+u)}\}du](\lambda_1+\sum_{i=1}^{B} n_{i1}\mu_i). \tag{3.136}$$

The two components of (3.136) correspond to counter 2 being open or closed at 0, and if counter 2 is closed for u at 0 then it is noted that the counter reopens at $\tau-u$. The approximations to $p_{12N}(\underset{\sim}{n})$ and $q_{2N}(u,\underset{\sim}{n})$

of Section 3.5.2 are now used to calculate an approximation to (3.136); it is fairly easy to show that for $K_i > 1$, $i = 1,\ldots,B$, (3.136) becomes

$$p_{12}\rho_1\rho_2 h, \tag{3.137}$$

and for $B = 1$, $K_1 = 1$, (3.136) becomes

$$p_{12}h(\rho_1\rho_2 - \rho_2\lambda_{12}\mu_1\tau + \rho_2\lambda_{12}\mu_1 \frac{h}{2}) . \tag{3.138}$$

Finally we calculate the coincidence rate due to type (ii) events for $B = 1$ and $K_1 = 1$, for this case (3.133) becomes

$$p_{12}p_1 h\{\rho_1\rho_2 + p_{12}\lambda_{12}\mu_1(1- \frac{\mu_1 h}{2}) + \frac{\rho_1 h}{2}(\rho_1\rho_2 + p_{12}\lambda_{12}\mu_1)\}. \tag{3.139}$$

Therefore the total rate of coincidences for De-centred Jitter with $K_i > 1$ for $i = 1,\ldots,B$, is the addition of (3.134) and (3.137), i.e.

$$p_{12}h\{\rho_1\rho_2(1+p_1) + \rho_1^2\rho_2 \frac{h}{2} + p_{12}\sum_{i=1}^{B} \lambda_{12}\theta_i\mu_i \frac{h}{2}(\mu_i-\rho_1 K_i)\} . \tag{3.140}$$

The total rate of coincidences for delayed state is obtained when (3.138) is added to (3.139), the roles of subscripts 1 and 2 interchanged and $\mu_1 = \eta$, i.e.

$$p_{12}h\{\rho_1\rho_2(1+p_2) - \rho_1\lambda_{12}\eta\tau + \rho_1\lambda_{12}\eta \frac{h}{2} + p_{12}\lambda_{12}\eta(1-\eta\frac{h}{2}) + \frac{\rho_2 h}{2}(\rho_1\rho_2+p_{12}\lambda_{12}\eta)\}. \tag{3.141}$$

3.5.6 Summary

To estimate the original disintegration rate λ for the problem of Delayed State, the three functions necessary to any estimation procedure may be chosen from the following,

(i) the expected number of recorded events on each counter,

$$E\{N_i(t)\} = p_i \rho_i t ,$$

where

$$p_i = (1 + \rho_i \tau_i)^{-1} \qquad \text{for} \quad i = 1 \text{ and } 2,$$

(ii) the covariance between the recorded events on each counter,

$$\text{cov}\{N_1(t), N_2(t)\} = \lambda_{12} p_1{}^2 p_2{}^2 \{1 + O(\rho\tau)^2\}(t - \frac{1-e^{-\eta t}}{\eta}) ,$$

(iii) the coincidence rate between the recorded events on each counter,

$$p_{12} h\{\rho_1\rho_2(1+p_2) - \rho_1\lambda_{12}\eta\tau + \rho_1\lambda_{12}\frac{\eta h}{2} + p_{12}\lambda_{12}\eta(1 - \frac{\eta h}{2}) + \frac{\rho_2 h}{2}(\rho_1\rho_2 + p_{12}\lambda_{12}\eta)$$

$$+ O(\rho\tau)^3 ,$$

where $p_{12} = p_1 p_2$. For reasons outlined in section 3.2.2 (iii) is not usually used in practice.

When the Jitter model is applicable then assuming that the parameters in model (3.1) have been estimated, the three functions required for the estimation of the original disintegration rate may be chosen from (i),

(iv) the covariance between the recorded events on each counter, which for large t and small τ may be approximated by

$$\text{cov}\{N_1(t), N_2(t)\} = \lambda_{12} p_1{}^2 p_2{}^2 (t - m_1 - m_2) ,$$

where m_1 is the average delays of a beta particle given that the beta particle is delayed; similarly for m_2 and a gamma particle,

(v) the coincidence rate between the recorded events on each counter,

$$p_{12}h\{\rho_1\rho_2(p_1+p_2)+\rho_1\rho_2 \frac{h}{2} (p_1\rho_1+p_2\rho_2) + p_{12} \frac{h}{2} \sum_{i=1}^{A} \lambda_{12}\theta_i\mu_i(\mu_i-\rho_iK_i)\} + O(\rho\tau)^3,$$

provided $K_i > 1$ for $i = 1,\ldots,A$. This applies to the normal Jitter distribution, while for the De-centred Jitter distribution we have a coincidence rate of

$$p_{12}h\{\rho_1\rho_2(1+p_1) + \rho_1^2\rho_2 \frac{h}{2} + p_{12} \sum_{i=1}^{B} \lambda_{12}\theta_i\mu_i \frac{h}{2} (\mu_i-\rho_1K_i)\} + O(\rho\tau)^3 ,$$

provided $K_i > 1$ for $i = 1,\ldots,B$. It should be noted that the last two formulae are not of great practical importance because of the $K_i > 1$ restriction for various i, although it would not be too difficult to obtain rates for some $K_i = 1$ from (3.133).

For the problem of Delayed State, one possible estimate of the disintegration rate λ, based upon the relationship

$$\lambda = \rho_1\rho_2\lambda_{12}^{-1} ,$$

is

$$\frac{n_{1t}n_{2t}(t - n_{1t}\tau_1)(t - n_{2t}\tau_2)}{t^4c(t)(t - \eta^{-1} + \eta^{-1}e^{-\eta t})} ,$$

where n_{it} is the observed mean number of recorded events on counter i for i = 1,2, and c(t) is the observed covariance between the two numbers of recorded events, in time t.

For the Jitter problem a very similar estimate of λ may be found.

Appendix 1. A Closer Approximation to the Covariance for Delayed State

We now give an outline of the calculations that produce a second order approximation to the covariance between the number of recorded events on each counter, for the Delayed State situation. We will not restrict the calculations to equal dead-times. From equation (3.65) we have that the Laplace Transform of the covariance between $N_1(t)$ and $N_2(t)$ is given by

$$c^*(s) = \frac{h^*_{12}(s)}{s^2} + \frac{h^*_{21}(s)}{s^2} - \frac{2p_1p_2\rho_1\rho_2}{s^3}, \tag{3.142}$$

where $h_{21}(x)$ is given by (3.89) and $h_{12}(x)$ is obtained by interchanging beta for gamma throughout (3.89). In Delayed State there are no delayed beta's and each gamma is delayed by an exponentially distributed period of mean η^{-1}. Therefore we set $A = 1$, $B = 0$, $\theta_1 = 1$, $K_1 = 1$, $\mu_1 = \eta$ and $(N,\underset{\sim}{n}) = N$ so that (3.89) reduces to

$$h^*_{21}(s) = \sum_{N=0}^{\infty} \{\lambda_2 I^*_N(s) + N\eta\, I^*_{N-1}(s)\}\, p_{12N}$$
$$+ \sum_{N=0}^{\infty} \int_0^{\tau_1} \{\lambda_2 J^*_N(s,u) + N\eta J^*_{N-1}(s,u)\} q_{1N}(u)du \; . \tag{3.143}$$

The sequence of events on counter 1 forms a renewal process, the intervals between successive events having density $\rho_1\exp\{-\rho_1(x-\tau)\}$ for $x \geq \tau$. Furthermore the sequence of events is independent of the number of gamma's delayed at any specific point. In particular the sequence is independent of the number at zero. Thus, if $K(x)$ denotes the renewal density of the renewal process on counter 1 then

$$I_N(x) = \rho_1 e^{-\rho_1 x} + \int_0^x \rho_1 e^{-\rho_1 y} K(x-y)dy, \tag{3.144}$$

and

$$J_N(x,u) = K(x+u) . \tag{3.145}$$

As a consequence of (3.144) and (3.145), (3.143) becomes

$$h_{21}^*(s) = p_{12} \frac{\rho_1\rho_2}{\rho_1+s} \{1 + K^*(s)\} + \int_0^{\tau_1} e^{su} \rho_2 q_1(u) K^*(s) du, \tag{3.146}$$

where

$$K^*(s) = \rho_1\{(\rho_1 + s)e^{s\tau_1} - \rho_1\}^{-1} . \tag{3.147}$$

To obtain a second order approximation to $h_{21}^*(s)$ by neglecting terms of order $(\rho\tau_1)^3$, we merely note that $p_{12} = p_1 p_2$ to first order. This may be seen from the leading terms in the expansions of $q_{1N}(u)$ and $q_{2N}(u)$ as given by (3.77) and (3.78) respectively, which are unchanged when the counter dead-times are unequal and by using the two normalizing conditions

$$p_{iN} = p_{12N} + \int_0^{\tau_i} q_{iN}(u)du \qquad \text{for } i = 1,2,$$

and summing over N. It then follows that

$$h_{21}^*(s) = \frac{p_1 p_2 \rho_1 \rho_2}{s} . \tag{3.148}$$

To obtain $h_{12}^*(s)$ we first interchange the subscripts 1 and 2 throughout (3.89), then we set the various parameters to be compatible with the Delayed State problem, i.e.

$$h_{12}^*(s) = \sum_{N=0}^{\infty} \{\lambda_1 I_N^*(s) + \lambda_{12} I_{N-1}^*(s)\} p_{12N}$$

$$+ \sum_{N=0}^{\infty} \int_0^{\tau_2} \{\lambda_1 J_N^*(s,u) + \lambda_{12} J_{N-1}^*(s,u)\} q_{2N}(u)du \ , \qquad (3.149)$$

where now

$$I_N(x)\delta x = \mathrm{pr}\left\{ \begin{array}{l} \text{a recorded event} \\ \text{on counter 2 in} \\ (x, x+\delta x) \end{array} \middle| \begin{array}{l} \text{counter 2 is open} \\ \text{at } 0^+ \text{ with N} \\ \text{gamma's delayed} \end{array} \right\} ,$$

$$J_N(x,u)\delta x = \mathrm{pr}\left\{ \begin{array}{l} \text{a recorded event} \\ \text{on counter 2 in} \\ (x, x+\delta x) \end{array} \middle| \begin{array}{l} \text{counter 2 is closed} \\ \text{for u at } 0^+ \text{ with N} \\ \text{gamma's delayed} \end{array} \right\} ,$$

for small δx, $I_N(x) = 0$ for $x \leq 0$ and $J_N(x,u) = 0$ for $x \leq \tau_2 - u$, $N = 0,1,\ldots,\infty$. In section 3.5.3 the intrinsic relationship between $I_N(x)$ and $J_N(x,u)$ was developed with the intention of solving for $I_N(x)$, and consequently $J_N(x,u)$, in terms of known functions. That development is now extended to include terms of order $(\rho\tau)^2$, where τ is the smaller of τ_1 and τ_2.

From (3.93) we have that

$$A^{-1}(s+\lambda_{12}+N\eta)I_N^*(s) = \lambda_2+N\eta+\lambda_{12}I_{N+1}^*(s) + \sum_{M=0}^{\infty} I_M^*(s)R_{NM}(\tau_2)e^{-s\tau_2} \ , \qquad (3.150)$$

where

$$R_{NM}(\tau_2) = N\eta r_{N-1M}(\tau_2) + \lambda_2 \int_0^{\tau_2} \exp\{-(\lambda_{12}+N\eta)v\}\{\lambda_{12} r_{N+1M}(\tau_2-v)$$

$$+ N\eta r_{N-1M}(\tau_2-v)\}dv, \qquad (3.151)$$

and A(s) is given by (3.90a) but with λ_1, τ replaced by λ_2, τ_2 respectively. Furthermore, $\{r_{NM}(x)\}$ are the transition probabilities for an infinite server queue with arrivals forming a Poisson process of rate λ_{12} and the service distribution being exponential with mean η^{-1}. The set $\{r_{NM}(x)\}$ have generating function $F_N(x,z)$, where

$$F_N(x,z) = \sum_{N=0}^{\infty} z^M r_{NM}(x)$$

$$= \{1 + (z-1)e^{-\eta x}\}^N \exp\{-\frac{\lambda_{12}}{\eta}(1 - e^{-\eta x})(1 - z)\}, \qquad (3.152)$$

see Gross and Harris (1974, page 117). We may rewrite (3.150) as

$$\sum_{M=0}^{\infty} a_{NM}(\tau_2) I_M^*(s) = \lambda_2 + N\eta, \qquad (3.153)$$

where $\{a_{NM}(\tau_2)\}$ is a function of $\{r_{NM}(\tau_2)\}$. If a second order approximation to $\{r_{NM}(\tau_2)\}$ is obtained by neglecting terms of order $(\lambda_{12}\tau_2)^3$, for small η, in (3.152) when $x = \tau_2$, then $\{a_{NM}(\tau_2)\}$ may be seen to satisfy

$$a_{NM}(\tau_2) = \{1 + \lambda_2\tau_2 - \lambda_2(s+\lambda_{12}+N\eta)\frac{\tau_2^2}{2} + N\lambda_{12}\eta\frac{\tau_2^2}{2}\} b_{NM}$$

$$+ \lambda_2\lambda_{12}\frac{\tau_2^2}{2} b_{N+1M} + \{1+\lambda_2\tau_2 - (s+\lambda_{12}+(N-1)\eta)\frac{\tau_2}{2}\}N\eta\tau_2 b_{N-1M}$$

$$+ N(N-1)\eta^2\frac{\tau_2^2}{2} b_{N-2M}, \qquad (3.154)$$

where

$$b_{NM} = \begin{cases} -N\eta & M = N-1 \\ s + \lambda_{12} + N\eta & M = N \\ -\lambda_{12} & M = N+1 \, . \end{cases} \qquad (3.155)$$

If we represent (3.153) in matrix form and assume that the inverse of the matrix with elements b_{NM} exists and is unique, and if the elements of the inverse are denoted by b_{NM}^{-1} then in the same way as section 3.5.3 $b_{NM}^{-1} = r_{NM}^*(s)$, where the untransformed elements $r_{NM}(x)$ were defined by (3.152). If terms of order $(\rho\tau_2)^3$ are neglected in the inversion of (3.153) then it can be shown that

$$I_N^*(s) = \frac{\rho_2}{s}\left(p_2 + s\rho_2\frac{\tau_2^{\,2}}{2}\right) + \left(\frac{N\eta-\lambda_{12}}{s+\eta}\right)\{p_2^{\,2}+\rho_2(s+\tfrac{\eta}{2})\tau_2^{\,2}\}$$
$$+\left\{\frac{\lambda_{12}^{\,2}-2N\lambda_{12}\eta+N(N-1)\eta^2}{s+2\eta}\right\}\tau_2\left[-1+\{3\rho_2+\frac{(s+\eta)}{2}\}\tau_2\right]$$
$$+\frac{-\lambda_{12}^{\,3}+3N\lambda_{12}^{\,2}\eta-3\lambda_{12}\eta^2N(N-1)+N(N-1)(N-2)\eta^3}{s+3\eta}\tau_2^{\,2}\ . \qquad (3.156)$$

From (3.91) we have that

$$J_N^*(s,u) = e^{-s(\tau_2-u)}\sum_{M=0}^{\infty} I_N^*(s)r_{NM}(\tau_2-u)\quad .$$

Therefore using a first order approximation to $r_{NM}(\tau_2-u)$ we may produce the following first order approximation to $J_N^*(s,u)$,

$$J_N^*(x,u) = N\eta(\tau_2-u)I_{N-1}^*(s) + \{1-(s+\lambda_{12}+N\eta)(\tau_2-u)\}I_N^*(s)$$
$$+\lambda_{12}(\tau_2-u)I_{N+1}^*(s)\ . \qquad (3.157)$$

If the first order approximation to $J_N^*(s,u)$, (3.157), is substituted into (3.149) then

$$h_{12}^*(s) = \sum_{N=0}^{\infty}\lambda_1\{p_{1N}I_N^*(s)+p_{12N}[N\eta I_{N-1}^*(s)-(s+\lambda_{12}+N\eta)I_N^*(s)+\lambda_{12}I_{N+1}^*(s)]\rho_2\frac{\tau_2^{\,2}}{2}\}$$
$$+\sum_{N=0}^{\infty}\lambda_{12}\{p_{1N}I_N^*(s)+p_{12N}[(N+1)\eta I_N^*(s)-\{s+\lambda_{12}+(N+1)\eta\}I_{N+1}^*(s)+\lambda_{12}I_{N+2}(s)]$$
$$\times\rho_2\frac{\tau_2^{\,2}}{2}\}\ . \qquad (3.158)$$

Hence, to produce a second order approximation to $h^*_{12}(s)$ we need p_{1N} to second order. By considering the sequence of events on counter 1 it can be shown that

$$p_{1N} = \left(\frac{\lambda_{12}}{\eta}\right)^N \frac{e^{-\lambda_{12}/\eta}}{N!} \{1 - (\lambda_1+N\eta)\tau_1 + [\lambda_1(\lambda_1+N\eta)+\{\lambda_1+(N-1)\eta\}N\eta-(\lambda_{12}-N\eta)\tfrac{\eta}{2}]\tau_1^2 + O(\rho\tau_1)^3 \}. \tag{3.159}$$

By substituting (3.156) and (3.159) into (3.158) and using

$$p_{12N} = \left(\frac{\lambda_{12}}{\eta}\right)^N \frac{e^{-(\lambda_{12}/\eta)}}{N!} + O(\rho\tau),$$

it may be shown that

$$h^*_{12}(s) = \frac{p_1 p_2 \rho_1 \rho_2}{s} + \frac{p_1^2 p_2^2 \lambda_{12}\eta}{s+\eta}\{1 + \frac{(\rho_1\tau_1^2-\rho_2\tau_2^2)\eta}{2}\} + \frac{2\lambda_{12}^2\eta^2\tau_1\tau_2}{s+2\eta} . \tag{3.160}$$

Finally, by combining (3.148) and (3.160) in (3.142) we have

$$\begin{aligned}\mathrm{cov}\{N_1(t),N_2(t)\} &= c(t)\\ &= \lambda_{12}p_1^2 p_2^2\{1 + \frac{(\rho_1\tau_1^2-\rho_2\tau_2^2)\eta}{2}\}\left(t - \frac{1-e^{-\eta t}}{\eta}\right)\\ &\quad + \lambda_{12}^2\eta\tau_1\tau_2\left(t - \frac{1-e^{-2\eta t}}{2\eta}\right) .\end{aligned} \tag{3.161}$$

Appendix 2. Simulations for Delayed State

We now refer to tables 2-5 which give simulated covariances for the model of Delayed State and were devised and computed by Dr. D. Smith of the National Physical Laboratory. Tables 2-5 also give the covariances predicted by (3.127) and (3.161), the difference between (3.127) and (3.161) being that second order dead-time effects are omitted in (3.127) but included in (3.161). Before discussing the simulations we adopt the following convention, whenever terms neglected in the calculation of the covariance (3.127) or (3.161), are of at least the same order as the standard error in the simulated covariance, we will speak of a type i error with i = 2 for (3.127) and i = 3 for (3.161).

Before the results that appear in Appendix 1 were calculated it was thought that the first order approximation to the covariance given by (3.127), i.e.

$$c_1(t) = \lambda_{12} p_1^{\,2} p_2^{\,2}\left(t - \frac{1-e^{-\eta t}}{\eta}\right) , \qquad (3.127)$$

might in fact be exact since the two dead-times τ_1 and τ_2 do not appear explicitly. So, the covariance was simulated for a variety of dead-times including some that did not satisfy $\rho_i \tau_i < 1$ for i = 1 and 2, which was an essential assumption of section 3.5.

Of all the simulations only the second and third do not have type 2 errors. Although it must be noted that dead-time effects are small for these two simulations, the results predicted by the first order approximation to the covariance given by $c_1(t)$ are in remarkable agreement with the simulated results. Simulations 1, 4, 5, 7-12 and 14-17 have type 2 errors but all satisfy the inequality $\rho_i \tau_i < 1$ for i = 1 and 2. Despite the presence of type 2 errors in these simulations the results predicted by $c_1(t)$ are still within two standard errors of the simulated covariance. This is perhaps the most relevant point to state that in calculating the predicted value of the covariance given by $c_1(t)$ the exact form of the equilibrium probability p_i is used, i.e. $p_i = (1+\rho_i\tau_i)^{-1}$, rather than the first order approximation $1 - \rho_i\tau_i$, for i = 1 and 2. It will become apparent that the majority of the dead-time effects on the covariance are accounted for by the factor $p_1{}^2 p_2{}^2$. This may be seen in the aforementioned simulations numbered 1, 4, 5, 7-12 and 14-18, where the effect on the covariance of terms of order $(\rho\tau_i)^2$, over and above that accounted for by $p_1{}^2 p_2{}^2$, is negligible, and most particularly in simulations 6, 13 and 18. Simulations 6, 13 and 18 have both a type 2 error and $\rho_i\tau_i > 1$ for i = 1 and/or 2. In these cases the covariance predicted by $c_1(t)$ differs from the simulated result but not by a substantial amount. Therefore, by comparing $c_1(t)$ with the simulated covariance it is apparent that $c_1(t)$ is not exact. However, we may conclude that at least 99% of the dead-time behaviour is accounted for by $p_1{}^2 p_2{}^2$ when $\rho_i\tau_i \leq 0.1$ for i = 1 and 2, and at least 80% when $\rho_i\tau_i \leq 1.8$ for i = 1 and 2.

Table 2: Simulated Covariance for Delayed State

Sim. no.	1	2	3	4	5
t	2	2	2	2	2
λ_{12}	0.36	0.36	0.36	1	0.4
η	10	10	20	5	3
ρ_1	0.9	0.9	0.9	1	0.8
ρ_2	0.4	0.4	0.4	1	0.5
τ_1	0.03	0.005	0.002	0.02	0.04
τ_2	0.03	0.005	0.002	0.02	0.02
Sim. cov.	0.6341	0.675	0.698	1.671	0.6129
St. err.	0.0005	0.001	0.001	0.006	0.0003
$c_1(t)$	0.6332	0.675	0.698	1.663	0.6139
$c_2(t)$	0.6369	0.675	0.698	1.667	0.6156

St. err. is the standard error on sim. cov., the simulated covariance.

Table 3: Simulated Covariance for Delayed State

Sim. no.	6	7	8	9	10
t	5	5	5	5	5
λ_{12}	0.36	1	1	1	0.9
η	0.5	0.25	0.25	0.75	4
ρ_1	0.9	1	1	1	1
ρ_2	0.4	1	1	1	1.8
τ_1	2	0.1	0.2	0.05	0.04
τ_2	2	0.2	0.1	0.1	0.02
Sim. cov.	0.0503	1.2413	1.2315	2.782	3.7014
St. err.	0.0004	0.002	0.003	0.001	0.0006
$c_1(t)$	0.0448	1.232	1.232	2.772	3.6826
$c_2(t)$	1.1058	1.243	1.252	2.781	3.7017

St. err. is the standard error on sim. cov., the simulated covariance.

Table 4: Simulated Covariance for Delayed State

Sim. no.	11	12	13	14
t	5	5	10	10
λ_{12}	0.9	1	0.36	0.36
η	8	0.75	0.5	10
ρ_1	1	1	0.9	0.9
ρ_2	1.8	1	0.4	0.4
τ_1	0.04	0.1	1.5	0.03
τ_2	0.02	0.05	3	0.045
Sim. cov.	3.812	2.786	0.1297	3.261
St. err.	0.004	0.002	0.0009	0.007
$c_1(t)$	3.779	2.772	0.1079	3.261
$c_2(t)$	3.818	2.796	2.6898	3.278

St. err. is the standard error on sim. cov., the simulated covariance.

Table 5: Simulated Covariance for Delayed State

Sim. no.	15	16	17	18
t	10	10	10	10
λ_{12}	0.36	0.36	0.48	0.36
η	10	10	10	0.5
ρ_1	0.9	0.9	0.6	0.9
ρ_2	0.4	0.4	0.8	0.4
τ_1	0.03	0.005	0.03	2
τ_2	0.03	0.005	0.045	2
Sim. cov.	3.299	3.511	4.305	1.287
St. err.	0.001	0.006	0.008	0.0015
$c_1(t)$	3.299	3.518	4.272	0.1136
$c_2(t)$	3.318	3.519	4.280	2.5032

St. err. is the standard error on sim. cov., the simulated covariance.

We shall now discuss the second order approximation to the covariance for Delayed State given by (3.161), namely

$$c_2(t) = \lambda_{12}p_1{}^2p_2{}^2\{1 + (\rho_1\tau_1{}^2-\rho_2\tau_2{}^2)\ \frac{\eta}{2}\ \}(t - \frac{1-e^{-\eta t}}{\eta})$$

$$+ \lambda_{12}\eta\tau_1\tau_2(t - \frac{1-e^{-2\eta t}}{2\eta}) \quad . \qquad (3.161)$$

Simulations 1-4, 11 and 14-17 do not have type 3 errors and $c_2(t)$ is within two standard errors for all these simulations except numbers 1, 14, 15 and 17. Of the four simulations for which $c_2(t)$ is not within two standard errors of the simulated covariance, $c_2(t)$ is within three standard errors for all but number 14. It is only for this simulation that $c_2(t)$ seems incompatible with the simulated covariance. It should be noted that the covariance given by $c_2(t)$ gives wildly inaccurate results for simulations 6, 13 and 18; but this was to be expected since for each of these three simulations $\rho_i\tau_i \geq 1$ for $i = 1$ and/or 2.

To compare the covariance predicted by $c_1(t)$ and $c_2(t)$ we look at those simulations which do not have type 3 errors, to wit simulations 1-4, 11, 14-17. Of these nine simulations $c_1(t)$ is incompatible only with number 11 and $c_2(t)$ incompatible only with number 1; although $c_2(t)$ is only within approximately three standard errors of the simulated covariance for simulations 14 and 15. Of those simulations that do not have type 3 errors and satisfy $\rho_i\tau_i < 1$ for $i = 1$ and 2, numbers 7, 9 and 10 favour $c_2(t)$ rather than $c_1(t)$.

We conclude this discussion by stating that $c_1(t)$ predicts the covariance very accurately for $\rho_i\tau_i \leq 1$, $i = 1,2$, and furthermore, $c_1(t)$ should be used in preference to $c_2(t)$ whenever $\rho_i\tau_i \geq 1$ for $i = 1$ and/or 2.

Unfortunately this simulation study does not allow any positive conclusions to be drawn about the conditions under which $c_2(t)$ should be used in preference to $c_1(t)$. It is hoped that this will be rectified at a later date.

CHAPTER 4. DELAYED STATE ON ONE COUNTER

4.1 Introduction

If, in the Delayed State problem as described in section 3.2, gamma particles are converted to beta particles within the source and so become indistinguishable from beta particles, then both are recorded on the same counter. Therefore we must study the disintegration process on a single counter and for this we calculate two functions, the expected number of recorded events in (0,t] and the variance of the number of recorded events in (0,t]. The analysis of this chapter is very similar to that of chapter 3 and in particular, complements to some extent the appendices to chapter 3.

We now examine the different types of event that may occur on the detector of the counting mechanism. As in section 3.2 we assume for simplicity of calculation that the beta and gamma emissions are simultaneous at source, and that only if both particles are detected is the gamma delayed, the delay period being exponentially distributed with mean η^{-1}. Therefore there are three different types of event that may occur on the detector, namely,

(i) a beta particle only, detected with rate

$$\lambda\varepsilon_\beta(1 - \varepsilon_\gamma) ,$$

(ii) a gamma particle only, detected with rate

$$\lambda\varepsilon_\gamma(1 - \varepsilon_\beta) ,$$

(iii) a beta-gamma pair of particles, detected with rate

$$\lambda\varepsilon_\beta\varepsilon_\gamma ,$$

the gamma particle then being delayed for a period which is exponentially distributed with mean η^{-1}.

We define two types of process, one consisting solely of single particles, the other consisting of pairs of particles, thus

$$\lambda_1 = \lambda\varepsilon_\beta(1 - \varepsilon_\gamma) + \lambda\varepsilon_\gamma(1 - \varepsilon_\beta) ,$$

$$\lambda_2 = \lambda\varepsilon_\beta\varepsilon_\gamma .$$

Then processes 1 and 2 are allowed to reach the counter to be recorded with process 2 causing a gamma particle to be delayed.

If p_N denotes the equilibrium probability that the counter is open with N gamma's delayed and if N(t) is the number of recorded events on the counter in (0,t], then as in previous chapters,

$$\begin{aligned} E\{N(t)\} &= \int_0^t \mathrm{pr}\{dN(u) = 1\} \\ &= \{\sum_{N=0}^{\infty} (\lambda_1+\lambda_2+N\eta)p_N\}t . \end{aligned} \tag{4.1}$$

The variance of N(t) may be expressed as

$$\mathrm{var}\{N(t)\} = \int_0^t \int_0^t \mathrm{pr}\{dN(u) = dN(v) = 1\} - E^2\{N(t)\}. \tag{4.2}$$

If the range of integration in (4.2) is split into the three components u>v, u<v and u = v, and a joint probability density h(x) defined to be,

$$h(x) = \lim_{\substack{\delta x \to 0^+ \\ \delta y \to 0^+}} \frac{\mathrm{pr}\left\{\begin{array}{l}\text{a recorded event in } (x,x+\delta x), \text{ and} \\ \text{a recorded event in } (0,\delta y)\end{array}\right\}}{\delta x \delta y} , \tag{4.3}$$

for $x > 0$, then the variance expressed in (4.2) may be written in terms of $h(x)$ and the expectation $E\{N(t)\}$, i.e.

$$\operatorname{var}\{N(t)\} = 2 \int_0^t \int_{u^+}^t h(v-u)dvdu + E\{N(t)\} - E^2\{N(t)\} . \tag{4.4}$$

To proceed further we need to study in depth the sequence of events on the counter and for this the dead-time behaviour of the counter has to be defined. The first type of behaviour that was considered is the physically unrealistic one where each dead-time is exponentially distributed with mean μ^{-1} independently of any other dead-time. This was originally meant to be a prelude to a gamma distributed dead-time and hence include the constant dead-time behaviour in which we were really interested. This would hopefully result in a less approximate analysis than that of previous chapters. However, the expectation and the variance could not be found for the gamma distributed dead-time and we are left with the exponentially distributed dead-time results. For this reason constant dead-times are considered in their own right and the exponential dead-time analysis is given for completeness.

4.2 Exponential Dead-time Behaviour

To calculate the expectation as given by (4.1) we need to determine the equilibrium probability p_N that the counter is open with N gamma particles delayed. To do so we define the complementary function q_N to be the equilibrium probability that the counter is closed with N gamma's delayed. The forward equation representing the change from the closed state to the open state is

$$(\lambda_1 + \lambda_2 + N\eta)p_N = \mu q_N , \tag{4.5}$$

and the equation representing the change from the open state to the closed state is,

$$(\mu+\lambda_2+N\eta)q_N = \lambda_2(p_{N-1}+q_{N-1}) + \lambda_1 p_N + (N+1)\eta(p_{N+1}+q_{N+1}) \ , \qquad (4.6)$$

for N = 0,1,...,∞. The solution of (4.5) and (4.6) for p_N is made considerably easier by noting that,

$$p_N + q_N = \text{pr}\binom{N \text{ gamma's}}{\text{queueing}} = \left(\frac{\lambda_2}{\eta}\right)^N \frac{e^{-\frac{\lambda_2}{\eta}}}{N!} = r_N \ , \qquad (4.7)$$

using a well known result from infinite server queue theory where the input process is Poisson of rate λ_2 and the service distribution is exponential of mean η^{-1}; see Gross and Harris (1974, page 272) for example. Thus,

$$p_N = \frac{\mu r_N}{\lambda_1+\lambda_2+N\eta+\mu} \ , \qquad q_N = \frac{\lambda_1+\lambda_2 N\eta}{\lambda_1+\lambda_2+N\eta+\mu} \cdot r_N, \qquad (4.8)$$

for N = 0,1,...,∞. On substituting for p_N into the expectation as given by (4.1) results in

$$\frac{E\{N(t)\}}{t} = \sum_{N=0}^{\infty} \frac{(\lambda_1+\lambda_2+N\eta)\mu . r_N}{\lambda_1+\lambda_2+N\eta+\mu} = \mu(1-p) \ , \qquad (4.9)$$

where p is the equilibrium probability that the counter is open and may be obtained by summing the p_N over N = 0,1,...,∞. An approximation to the expectation per unit time given by (4.9) may be found by considering $E(AX+a)^{-1}$, where X is a Poisson random variable with mean θ,

$$(AX+a)^{-1} = \frac{1}{a} + X\left(-\frac{A}{a^2}\right) + \frac{X^2}{2!}\frac{2A^2}{a^3} + O\left(\frac{1}{a^4}\right) \ , \qquad (4.10)$$

therefore

$$E(AX+a)^{-1} = \frac{1}{a} - \frac{\theta A}{a} + \frac{(\theta^2+\theta)A^2}{a^3} + O(\frac{1}{a^4}) \quad . \tag{4.11}$$

If we identify $A = \eta$, $a = \lambda_1+\lambda_2+\mu$ and $\theta = \lambda_2\eta^{-1}$ then

$$\frac{E\{N(t)\}}{t} = (\lambda_1+2\lambda_2) - \frac{\{(\lambda_1+2\lambda_2)^2+\lambda_2\eta\}}{\mu} + O(\lambda\mu^{-1})^2$$

$$= (\lambda_1+2\lambda_2)p\ \{1 - \frac{\lambda_2\eta}{(\lambda_1+2\lambda_2)\mu}\} + O(\lambda\mu^{-1})^2 \ , \tag{4.12}$$

where $p = 1 - (\lambda_1+2\lambda_2)\mu^{-1} + O(\lambda\mu^{-1})^2$ and $\lambda = \lambda_1+2\lambda_2$.

The problem of calculating the variance function is equivalent to that of calculating the joint probability density function h(x), as defined by (4.3). Because the sequence of events on the counter is not independent of the number of gamma's queueing at any instant, then we cannot express h(x) in terms of a single renewal density. Instead define two sets of functions similar to those of section 3.5.3, define

$$I_N(x)\delta x \simeq \text{pr}\left\{\begin{matrix}\text{a recorded} \\ \text{event in} \\ (x,x+\delta x)\end{matrix} \;\middle|\; \begin{matrix}\text{the counter} \\ \text{is open at } 0^+ \\ \text{with N gammas delayed}\end{matrix}\right\},$$

$$J_N(x)\delta x \simeq \text{pr}\left\{\begin{matrix}\text{a recorded} \\ \text{event in} \\ (x,x+\delta x)\end{matrix} \;\middle|\; \begin{matrix}\text{the counter is closed} \\ \text{at } 0^+ \text{ with N} \\ \text{gamma's delayed}\end{matrix}\right\},$$

for small δx and with $I_N(x) = J_N(x) = 0$ for $x \leq 0$; $N = 0,1,\ldots,\infty$.

We may now express h(x) in terms of $\{I_N(x)\}$ and $\{J_N(x)\}$, i.e.

$$h(x) = \sum_{N=0}^{\infty} \{\lambda_1 J_N(x) + \lambda_2 J_{N+1}(x) + N\eta J_{N-1}(x)\}p_N . \qquad (4.13)$$

Again as in section 3.5.3 we highlight the relationship between $\{I_N(x)\}$ and $\{J_N(x)\}$ in order to solve for $\{J_N(x)\}$ in terms of known functions. We have

$$I_N(x) = (\lambda_1+\lambda_2+N\eta)\exp\{-(\lambda_1+\lambda_2+N\eta)x\}$$

$$+ \int_0^x \exp\{-(\lambda_1+\lambda_2+N\eta)y\}\{\lambda_1 J_N(x-y)+\lambda_2 J_{N+1}(x+y)+N\eta J_{N-1}(x-y)\}dy , \qquad (4.14)$$

where in (4.14) the first term on the right hand side represents the probability that the first event to occur at or before x actually occurs at x, the second term represents the probability that the first event occurs before x. Also we have

$$J_N(x) =$$

$$\int_0^x \exp\{-(\lambda_2+N\eta+\mu)y\}\{\lambda_2 J_{N+1}(x-y)+N\eta J_{N-1}(x-y)+\mu I_N(x-y)\}dy . \qquad (4.15)$$

In (4.15) the first two terms in the brackets of the integrand represent the probability that the first event to occur is at y, and this event results in a change in the number of gamma's queueing, the last term in the brackets represents the probability that the first event to occur is the opening of the counter. If we now take Laplace Transforms of (4.14) and (4.15) it will be seen that,

$$I_N^*(s) = (s+\lambda_1+\lambda_2+N\eta)^{-1}\{\lambda_1+\lambda_2+N\eta+\lambda_1 J_N^*(s)+\lambda_2 J_{N+1}^*(s)+N\eta J_{N-1}^*(s)\} \quad (4.16)$$

$$J_N^*(s) = (s+\lambda_2+N\eta+\mu)^{-1}\{\lambda_2 J_{N+1}^*(s)+N\eta J_{N-1}^*(s)+\mu I_N^*(s)\}. \quad (4.17)$$

It then follows directly from (4.16) and (4.17) that,

$$I_N^*(s) = \frac{\lambda_1+\lambda_2+N\eta}{s+\lambda_1+\lambda_2+N\eta+\mu} + J_N^*(s) , \quad (4.18)$$

whence

$$(s+\lambda_2+N\eta)J_N^*(s)-\lambda_2 J_{N+1}^*(s) - N\eta J_{N-1}^*(s) = \frac{\mu(\lambda_1+\lambda_2+N\eta)}{s+\lambda_1+\lambda_2+N\eta+\mu} . \quad (4.19)$$

To invert (4.19) we use a result of section 3.5.3 that was an integral part of that analysis. The inverse of the tri-diagonal matrix that gives rise to the left-hand-side of (4.19) consists of elements $r_{NM}^*(s)$, where $r_{NM}^*(s)$ are the Laplace Transforms of the transition probabilities $r_{NM}(t)$ of an infinite server queue with Poisson arrival rate λ_2 and Poisson service rate η. These transition probabilities have generating function defined in (3.152) if λ_2 is identified with λ_{12}. Therefore,

$$J_N^*(s) = \sum_{M=0}^{\infty} r_{NM}^*(s)\mu . \frac{\lambda_1+\lambda_2+M\eta}{s+\lambda_1+\lambda_2+M\eta+\mu} . \quad (4.20)$$

Expanding (4.20) for small s,

$$J_N^*(s) = \sum_{M=0}^{\infty} r_{NM}^*(s)\mu . \frac{\lambda_1+\lambda_2+M\eta}{\lambda_1+\lambda_2+M\eta+\mu} \{1 - \frac{s}{\lambda_1+\lambda_2+M\eta+\mu} + \frac{s^2}{(\lambda_1+\lambda_2+M\eta+\mu)^2} + \dots \} . \quad (4.21)$$

We now use the same approximation technique as in (4.10) and (4.11) to produce an approximation to $J_N(s)$ for small s, where now

$$E(X) = \mathcal{L}\left\{ \frac{\lambda_2}{\eta}(1 - e^{-\eta t}) + Ne^{-\eta t} \right\},$$

$$E\{X(X-1)\} = \mathcal{L}\left\{N(N-1)e^{-2\eta t} + 2N\frac{\lambda_2}{\eta}e^{-\eta t}(1-e^{-\eta t}) + \left(\frac{\lambda_2}{\eta}\right)^2(1-e^{-\eta t})^2\right\},$$

and $\mathcal{L}$ denotes the Laplace Transform operator with respect to t. This approach leads to

$$J_N^*(s) = \frac{K_N}{s} + \ell_N + m_N s + n_N^*(s), \tag{4.22}$$

where $n_N^*(s)$ is analytic in some half-plane $\mathrm{Re}(s) > -\gamma_N$ with $\gamma_N > 0$, and

$$K_N = (\lambda_1+2\lambda_2) - \frac{((\lambda_1+2\lambda_2)^2+\lambda_2\eta)}{\mu} + O(\lambda\mu^{-1})^2,$$

$$\ell_N = \frac{N\eta-\lambda_2}{\eta} - \left[\frac{(N\eta-\lambda_2)\{2(\lambda_1+2\lambda_2)+\eta\}}{\eta\mu} + \frac{N(N-1)\eta^2-2N\eta\lambda_2+\lambda_2^2}{2\eta\mu}\right] + O(\lambda\mu^{-1})^2,$$

$$m_N = -\frac{(N\eta-\lambda_2)}{\eta^2} + \left\{\frac{2(N\eta-\lambda_2)(\lambda_1+2\lambda_2)}{\eta\mu} + \frac{N(N-1)\eta^2-2N\eta\lambda_2+\lambda_2^2}{2\eta\mu}\right\} + O(\lambda\mu^{-1})^2.$$

(4.23)

The Laplace Transform of the joint probability density function h(x) can be calculated in two ways:

(i) Using (4.18) and (4.22) calculate a similar expression to (4.22) for $I_N^*(s)$ and then note that,

$$h^*(s) = \sum_{N=0}^{\infty} \{(s+\lambda_1+\lambda_2+N\eta)I_N^*(s)-(\lambda_1+\lambda_2+N,\eta)\}p_N,$$

and calculate the summation using (4.8) for p_N, or

(ii) calculate each of the terms in (4.13) by using (4.22) and then use p_N as given by (4.8).

Both approaches give the same result for $h^*(s)$ which

$$h^*(s) = \frac{a}{s} + b + cs + d^*(s) \tag{4.24}$$

where $d^*(s)$ is analytic in some half-plane $Re(s) > -d$, $d > 0$ and

$$a = \left[\frac{E\{N(t)\}}{t}\right]^2 , \quad b = \lambda_2 p^3(1 - \frac{\eta}{\mu}) - \frac{(\lambda_1+2\lambda_2)(\lambda_1+3\lambda_2)}{\mu} , \quad c = -\frac{\lambda_2}{\mu} p^4, \tag{4.25}$$

where terms of order $(\lambda\mu^{-1})^2$ have been neglected in calculating a, b and c.

An approximate form for the variance function may now be found using (4.12), (4.24), (4.25) and then inverting the Laplace Transform, thus for large t

$$\text{var}\,\{N(t)\} = p^4\{(\lambda_1+4\lambda_2)t - \frac{2\lambda_2}{\eta}\} + \frac{t}{\mu}\{(\lambda_1+2\lambda_2)^2 - 3\lambda_2\eta\} . \tag{4.26}$$

4.3 Constant Dead-Time Behaviour

We now calculate the expectation (4.1) and the variance (4.4) for dead-times of constant length τ. The calculations of this section are very similar to those of section 3.5.2 and certain results from that section will be used to shorten the analysis here. We start by determining p_N, the equilibrium probability that the counter is open with N gamma's delayed. To do so we again define the complementary function $q_N(u)$ for $u \leq \tau$, the probability density that the counter has been closed for a period u and N gamma's are delayed. The equation representing the change from the closed state to the open state is now

$$(\lambda_1+\lambda_2+N\eta)p_N = q_N(\tau) , \qquad (4.27)$$

and the equation representing the change from the open state to the closed state is

$$q_N(0) = \lambda_1 p_N+\lambda_2 p_{N-1} + (N+1)\eta p_{N+1} , \quad N = 0,1,\ldots,\infty. \qquad (4.28)$$

Two additional equations are

$$\frac{dq_N(u)}{du} = -(\lambda_2+N\eta)q_N(u)+\lambda_2 q_{N-1}(u) + (N+1)\eta q_{N+1}(u) , \qquad (4.29)$$

which represents change within the closed state, and

$$p_N + \int_0^\tau q_N(u)du = (\frac{\lambda_2}{\eta})^N \frac{e^{-(\frac{\lambda_2}{\eta})}}{N!} = r_N , \qquad (4.30)$$

which represents the probability that N gamma's are delayed, $N = 0,1,\ldots,\infty$. As in previous sections we solve (4.27)-(4.30) by expanding p_N and $q_N(u)$ in Taylor Series and here we ignore terms of order $(\lambda\tau)^3$. Therefore,

$$p_N = r_N[1 - (\lambda_1+\lambda_2+N\eta)\tau + \{(\lambda_1+\lambda_2+N\eta)^2 - \frac{(N\eta-\lambda_2)}{2}\eta\} \tau^2],$$

$$q_N(u) = r_N(\lambda_1+\lambda_2+N\eta)\{1-(\lambda_1+\lambda_2+N\eta)\tau\} + r_N(N\eta-\lambda_2)\eta(\tau-u) . \qquad (4.31)$$

Substituting p_N from (4.31) into the expression for the expectation it then follows that

$$\frac{E\{N(t)\}}{t} = (\lambda_1+2\lambda_2)-\{(\lambda_1+2\lambda_2)^2+\lambda_2\eta\}\tau + \{(\lambda_1+2\lambda_2)^3+3\lambda_2\eta(\lambda_1+2\lambda_2) + \frac{\lambda_2\eta^2}{2}\} \tau^2 + O(\lambda\tau)^3 . \qquad (4.32)$$

We may find the equilibrium probability that the counter is open by summing p_N for $N = 0,1,\ldots,\infty$. We have

$$p = \sum_{N=0}^{\infty} p_N$$
$$= 1 - (\lambda_1+2\lambda_2)\tau + \{(\lambda_1+2\lambda_2)^2+\lambda_2\eta\}\tau^2 + O(\lambda\tau)^3 . \qquad (4.33)$$

If τ is identified with μ^{-1} of section 4.2 then (4.32) and (4.33) are equivalent to their counter parts (4.12), at least to order $(\lambda\tau)$.

To calculate the variance function (4.4) we now follow section 3.5.3; using $\{I_N(s)\}$ and $\{J_N(x,u)\}$ as defined in (3.87) and (3.88), the joint probability density $h(x)$ may be expressed as

$$h(x) = \sum_{N=0}^{\infty} \{\lambda_1 J_N(x) + \lambda_2 J_{N+1}(x) + N\eta J_{N-1}(x)\}p_N , \qquad (4.34)$$

where $J_N(x,0)$ is abbreviated to $J_N(x)$, $N = 0,1,\ldots,\infty$. Two equations that link $I_N(x)$ and $J_N(x)$ are

$$I_N(x) = (\lambda_1+\lambda_2+N\eta)\exp\{-(\lambda_1+\lambda_2+N\eta)x\}$$

$$+ \int_0^x \exp\{-(\lambda_1+\lambda_2+N\eta)y\}\{\lambda_1 J_N(x-y)+\lambda_2 J_{N+1}(x-y)+N\eta J_{N-1}(x-y)\}dy , \qquad (4.35)$$

and

$$J_N(x) = \sum_{M=0}^{\infty} I_M(x-\tau)r_{NM}(\tau) , \qquad (4.36)$$

for $N = 0,1,\ldots,\infty$. Taking Laplace Transforms of, and eliminating $J_N^*(s)$ between, (4.35) and (4.36) leads to

$$(s+\lambda_1+\lambda_2+N\eta)I_N^*(s) = \lambda_1+\lambda_2+N\eta + e^{-s\tau} \sum_{M=0}^{\infty} I_M^*(s)R_{NM}(\tau)$$

i.e.

$$\sum_{M=0}^{\infty} a_{NM}I_M^*(s) = \lambda_1+\lambda_2+N\mu \ , \qquad (4.37)$$

where

$$a_{NM} = (s+\lambda_1+\lambda_2+N\eta)\delta_{NM} - \{\lambda_1 r_{NM}(\tau) + \lambda_2 r_{N+1M}(\tau) + N\eta r_{N-1M}(\tau)\}e^{-s\tau} \ , \qquad (4.38)$$

and δ_{NM} is the Kronecker delta, $N = 0,1,\ldots,\infty$ and $M = 0,1,\ldots,\infty$. A second order approximation to $r_{NM}(\tau)$ may be obtained from (3.152) and if this is then substituted into (4.38) it is found that each a_{NM} may be expressed in terms of b_{N-1M}, b_{NM} and b_{N+1M} where $\{b_{NM}\}$ are given by (3.155), and terms of order $(\lambda\tau)^3$ have been neglected, $N = 0,1,\ldots,\infty$ and $M = 0,.,\ldots,\infty$. We now apply a method that appeared in the first appendix to chapter 3 to invert equation (4.37). If we represent (4.37) in matrix form and assume that the inverse of the matrix with elements b_{NM} exists and is unique, and if the elements of the inverse are denoted by b_{NM}^{-1}, then as in section 3.5.3 $b_{NM}^{-1} = r_{NM}^*(s)$, where the untransformed elements $r_{NM}(x)$ were defined by (3.152). If terms of order $(\lambda\tau)^3$ are neglected in (4.37) then it can be shown that,

$$I_N^*(s) = \left[\frac{E\{N(t)\}}{t}\right]\frac{1}{s} + \{(\lambda_1+2\lambda_2)^2+\lambda_2\eta\}\frac{\tau^2}{2} + \{\frac{N\eta-\lambda_2}{s+\eta}\}\{p^2+(s+\eta)(\lambda_1+2\lambda_2)\tau^2 - \frac{\lambda_1\eta}{2}\tau^2\}$$

$$- \{\frac{\lambda_2^2-2N\lambda_2\eta + N(N-1)\eta^2}{s+2\eta}\}\{p^3 - (s+2\eta)\frac{\tau}{2} + \frac{\eta\tau}{2}\}\tau$$

$$+ \{\frac{-\lambda_2^3+3N\eta\lambda_2^2 - 3\lambda_2\eta^2N(N-1) + N(N-1)(N-2)\eta^3}{s + 3\eta}\} \quad \tau^2 . \qquad (4.39)$$

for $N = 0,1,\ldots,\infty$ and where $E\{N(t)\}/t$ is given by (4.32), p is given by (4.33).

If the Laplace Transform of the joint probability density h(x) as given by (4.34) is taken, then upon comparison with (4.35) it is apparent that

$$h^*(s) = \sum_{N=0}^{\infty} (s+\lambda_1+\lambda_2+N\eta)I_N^*(s)p_N - \sum_{N=0}^{\infty} (\lambda_1+\lambda_2+N\eta)p_N$$

$$= \sum_{N=0}^{\infty} (s+\lambda_1+\lambda_2+N\eta)I_N^*(s)p_N - [\frac{E\{N(t)\}}{t}] . \qquad (4.40)$$

If the Laplace Transform of (4.4) is taken then

$$\mathcal{L}\,[var\{N(t)\}] = \frac{2h^*(s)}{s^2} + \mathcal{L}\,[E\{N(t)\} - E^2\{N(t)\}] \quad . \qquad (4.41)$$

From (4.32), (4.39)-(4.41) and consequent inversion of the Laplace Transform of the variance function, it follows that

$$var\{N(t)\} = 2\lambda_2p^4(1+2\lambda_2\eta\tau^2)\{ t - \frac{1-e^{-\eta t}}{\eta}\} + p^2(1+\lambda_2\eta\tau^2)E\{N(t)\}$$

$$+ 2\lambda_2^2\eta\tau^2 \{t - \frac{1-e^{-2\eta t}}{2\eta} \} + \{(\lambda_1+2\lambda_2)^2+\lambda_2\eta\}\tau^2 . \qquad (4.42)$$

Where $E\{N(t)\}$ and p are given by (4.32) and (4.33) respectively, furthermore terms of order $(\lambda\tau)^3$ have been neglected in (4.32), (4.33) and (4.42).

4.4 Comments

We first investigate the differences, if any, between the exponentially distributed dead-time results and the constant dead-time results. To do so all terms of order $(\lambda\tau)^2$ in section 4.3 are ignored.

It was noted in section 4.3 that the expected number of recorded events in $(0,t]$ and the equilibrium probability that the counter is open, are the same for both dead-time distributions if the mean of the exponential distribution is equal to the constant dead-time of section 4.3. To compare variances we ignore terms of order $(\lambda\tau)^2$ in (4.42), thus for constant dead-times,

$$\text{var}\{N(t)\} = (\lambda_1+4\lambda_2)p^4t + \frac{2\lambda_2}{\eta}(e^{-\eta t}-1)p^4 + (\lambda_1+2\lambda_2)^2 t\tau - \lambda_2\eta t\tau. \tag{4.43}$$

When (4.43) is compared with (4.26) its equivalent for exponentially distributed dead-time, it is seen that (4.43) exceeds (4.26) by $2\lambda_2\eta\tau$, if $\tau = \mu^{-1}$ and t is large. In other words, the variance for constant dead-times is greater than that for exponentially distributed dead-times; apart from noting the presence of the delayed state variable η, no qualitative explanation is provided for this unexpected result.

Since the internal conversion process of section 4.1 is present to some extent in those isotopes considered in Chapter 3, then the results of this chapter may be used in conjunction with those of Chapter 3 relevant to Delayed State, to obtain a more practicable estimate of the disintegration rate λ; see Lewis, Smith and Williams (1973).

CHAPTER 5. EXPONENTIAL DECAY OF SOURCE

5.1 Introduction

One overriding assumption made when considering the problems of the previous chapters has been that the disintegration rate of the source is constant. Usually this assumption is reasonable physically because the intervals over which the process of disintegrations is studied are short when compared with the half-life of the source. However, if this is not the case, then the theory developed in earlier chapters is clearly inadequate.

In this chapter a preliminary study of the process subject to decay of source is made via the expected number of recorded events on a single counter in time t. That is, we have a Poisson process with a time varying rate $\lambda_0 e^{-\lambda t}$ which is fed into a single counter. Because we cannot estimate the disintegration rate with only one counter, see section 1.2, then the problem of calculating the expected number of recorded events in time t is unchanged if the efficiency of the detector is assumed to be unity.

So, if N(t) is the number of recorded events or counts in time t, then as in earlier chapters,

$$E\{N(t)\} = \int_0^t pr\{dN(u) = 1\}. \qquad (5.1)$$

If $\pi(t)$ is the probability that the counter is open at time t, given the initial state of the counter, which must be open when the counter is switched on, then

$$E\{N(t)\} = \int_0^t \lambda(u)\pi(u)du. \qquad (5.2)$$

Therefore, calculating the expected number of recorded events in $(0,t]$, is equivalent to finding the probability that the counter is open at any specific instant. Two methods of tackling this problem are,

(i) to treat the problem in its own right by modifying the counter state equations of previous chapters for a transient analysis with a time-varying disintegration rate, or,

(ii) to develop a method that will allow the use of constant rate analysis.

Both methods will be considered, the latter first.

5.2 Step Function Approximation to an Exponential Function

Suppose that we approximate the exponentially decaying process, represented by the rate $\lambda(t) = \lambda_0 e^{-\lambda t}$, by the rate $\Lambda(t)$ where

$$\Lambda(t) = \lambda_i \qquad \text{for } t \in (t_{i-1}, t_i], \qquad i = 1,\ldots,K,$$

and $0 = t_0 < t_1 < \ldots. < t_K = t$ for some K. If we then redefine $\pi(t)$ to be $\pi_i(t)$, the probability that the counter is open at time t when $t \in (t_{i-1}, t_i]$ for $i = 1,\ldots,K$, it follows that (5.2) becomes

$$E\{N(t)\} = \sum_{i=1}^{K} \int_{t_{i-1}}^{t_i} \lambda_i \pi_i(u)du \; . \tag{5.3}$$

Therefore, if we can find a sequence of λ_i's and a partition set $(t_0,\ldots,t_K)$ of $(0,t]$ that give a good approximation to $\lambda(t)$, and consequently an approximation to $\pi_i(t)$, then the expectation (5.3) may be found.

To calculate $\pi_i(t)$ we form a recursive scheme by conditioning on the possible state of the counter at t_{i-1} given the state of the counter at $t_0 = 0$, i.e.

$$\pi_i(t) = \mathrm{pr}\left\{\begin{matrix}\text{counter is}\\ \text{open at}\\ \text{time } t\end{matrix}\;\middle|\;\begin{matrix}\text{open}\\ \text{at}\\ t_{i-1}\end{matrix}\right\} \mathrm{pr}\left\{\begin{matrix}\text{open}\\ \text{at}\\ t_{i-1}\end{matrix}\right\} + \mathrm{pr}\left\{\begin{matrix}\text{counter is}\\ \text{open at}\\ \text{time } t\end{matrix}\;\middle|\;\begin{matrix}\text{closed}\\ \text{at}\\ t_{i-1}\end{matrix}\right\} \mathrm{pr}\left\{\begin{matrix}\text{closed}\\ \text{at}\\ t_{i-1}\end{matrix}\right\} . \tag{5.4}$$

Now if $\Lambda(t)$ is to be a "good approximation" to $\lambda(t)$, then the partitioning of $(0,t]$ must be fine, that is $\max_{i=1,\ldots,K} (t_i - t_{i-1})$ must be small relative to the inverse of the decay rate λ. Consequently, the interval over which the sequence of events on the counter is studied to calculate the conditional probabilities in (5.4), must also be small relative to the inverse of the decay rate λ. In previous chapters the dead-times of the counters were taken to be constant, for this is a reasonable approximation to the physical situation. However, if the dead-time of the counter was taken to be constant in (5.4), then the conditional probabilities appearing there would be extremely difficult to calculate. For simplicity, we therefore assume that in all future calculations the dead-time is exponentially distributed with mean μ^{-1}; hence most dead-times have length less than μ^{-1} while occasionally there is a dead-time far greater than μ^{-1}. With this exponential dead-time assumption,

$$\pi_i(t) = \left[\frac{\mu}{\lambda_i+\mu} + \frac{\lambda_i}{\lambda_i+\mu}\exp\{-(\lambda_i+\mu)(t-t_{i-1})\}\right] p_i^{\,0}$$

$$+ \left[\frac{\mu}{\lambda_i+\mu} - \frac{\mu}{\lambda_i+\mu}\exp\{-(\lambda_i+\mu)(t-t_{i-1})\}\right] p_i^{\,c} , \tag{5.5}$$

where

$$p_i^{\,0} = \text{pr}\left\{\begin{matrix}\text{counter}\\ \text{is open}\\ \text{at } t_{i-1}\end{matrix}\right\} = \pi_{i-1}(t_{i-1}) = 1 - p_i^{\,c} ,$$

for $i = 1,\ldots,K$ and with $\pi_0(0) = 1$, i.e. the counter is open initially. So

$$\pi_i(t) = \exp\{-(\lambda_i+\mu)(t-t_{i-1})\}\pi_{i-1}(t_{i-1})$$

$$+ \frac{\mu}{\lambda_i+\mu}[1 - \exp\{-(\lambda_i+\mu)(t-t_{i-1})\}] \quad . \qquad (5.6)$$

Upon substitution of (5.6) into (5.3) the expectation becomes

$$E\{N(t)\} = \sum_{i=1}^{K} \lambda_i \left[\frac{1 - \exp\{-(\lambda_i+\mu)(t_i-t_{i-1})\}}{\lambda_i+\mu}\right]\pi_{i-1}(t_{i-1})$$

$$+ \sum_{i=1}^{K} \lambda_i\left[t_i - t_{i-1} - \frac{1 - \exp\{-(\lambda_i+\mu)(t_i-t_{i-1})\}}{\lambda_i+\mu}\right] \frac{\mu}{\lambda_i+\mu} \;. \qquad (5.7)$$

The remaining unknown quantity in (5.7), apart from the choice of λ_i and t_i, is the probability that the counter is open at t_{i-1}, i.e. $\pi_{i-1}(t_{i-1})$, this may be found by solving (5.6) recursively to obtain,

$$\pi_i(t_i) = \exp\{-\sum_{j=1}^{i} (\lambda_i+\mu)(t_i-t_{i-1})\} \; [1 +$$

$$+ \sum_{h=1}^{i} \frac{\mu}{\lambda_h+\mu} \cdot \frac{1 - \exp\{-(\lambda_h+\mu)(t_h-t_{h-1})\}}{\exp\{-\sum_{g=1}^{h} (\lambda_g+\mu)(t_g-t_{g-1})\}} \;] \; , \qquad (5.8)$$

for $i = 1,\ldots,K$. The expectation given by (5.7) may now be written as

$$E\{N(t)\} = \sum_{i=1}^{K} \left[\frac{\lambda_i \mu}{\lambda_i + \mu}\{t_i - t_{i-1}\} + \frac{\lambda_i}{\lambda_i + \mu}\{\pi_{i-1}(t_{i-1}) - \pi_i(t_i)\}\right], \qquad (5.9)$$

where $\pi_i(t_i)$ for $i = 1,\ldots,K$ is given by (5.8) and $\pi_0(0) = 1$.

To proceed further, a partition $(t_1,\ldots,t_K)$ of $(0,t]$ and a set $(\lambda_1,\ldots,\lambda_K)$ of steps have to be defined. Since the intention is to let K tend to infinity, the choice of partition and steps is dependent solely on the ease of calculation of the expectation as defined in (5.9). For this reason choose

$$t_j = j\,\frac{t}{K} \quad \text{and} \quad \lambda_j = \lambda_0\left\{\frac{e^{-\lambda t_j} + e^{-\lambda t_{j-1}}}{2}\right\}, \qquad (5.10)$$

for $j = 1,\ldots,K$. The first term in the expression for the expectation (5.9) is

$$\sum_{i=1}^{K} \frac{\lambda_i \mu}{\lambda_i + \mu}(t_i - t_{i-1}) = \sum_{i=1}^{K} \frac{\lambda_i}{1+\lambda_i \mu^{-1}} \cdot \frac{t}{K}. \qquad (5.11)$$

For those sources for which $\lambda_i < \mu$ for $i = 1,\ldots,K$ we may use the Binomial Theorem to expand (5.11); the result of substituting (5.10) into this expansion and then summing over K, is to produce the following expression for (5.11),

$$\sum_{r=0}^{\infty} \frac{(1 + e^{-\lambda t K^{-1}})^{r+1}}{(-\mu)^r\, 2^{r+1}} \; \frac{1 - e^{-\lambda t(r+1)}}{1 - e^{-\lambda t(r+1)K^{-1}}} \; \frac{t}{K}\, \lambda_0^{\,r+1}. \qquad (5.12)$$

Since the summation of (5.12) is uniformly convergent for each K and each partial sum is continuous, then we may take the limit as $K \to \infty$ inside the summation so that (5.12) becomes

$$\frac{1}{\lambda\mu^{-1}} \log\left(\frac{1 + \lambda_0 \mu^{-1}}{1 + \lambda_0 \mu^{-1} e^{-\lambda t}}\right). \qquad (5.13)$$

To calculate the second term in the expansion for the expectation given by (5.9) is difficult. However, for large μ, i.e. small dead-time, the contribution from this term is negligible since

$$\frac{\lambda_i}{\lambda_i+\mu} \{\pi_{i-1}(t_{i-1}) - \pi_i(t_i)\} ,$$

is of order μ^{-2}.

Therefore, for small mean dead-time the expected number of recorded events in (0,t] is approximately

$$\frac{\mu}{\lambda} \log \left(\frac{1 + \lambda_0\mu^{-1}}{1 + \lambda_0\mu^{-1}e^{-\lambda t}}\right) , \tag{5.14}$$

at least to first order in dead-time effects.

We shall now briefly consider the effect of large dead-times on the expectation defined in (5.2). When the mean dead-time μ^{-1} is large, then the expectation as given by (5.9) is increasingly dominated by $\pi_i(t_i)$. For the case of infinite dead-times the form of the expectation given by (5.9) reduces to

$$E\{N(t)\} = \sum_{i=1}^{K} \{\pi_{i-1}(t_{i-1}) - \pi_i(t_i)\} , \tag{5.15}$$

where $\pi_i(t_i)$ is now a monotonically decreasing function of t_i. Upon application of similar methods to those employed to obtain (5.14), (5.15) becomes

$$E\{N(t)\} = 1 - \exp\{-\frac{\lambda_0}{\lambda}(1 - e^{-\lambda t})\}. \tag{5.16}$$

However, (5.16) may be obtained by use of direct probabilistic arguments, so that the step function analysis is redundant for infinite dead-times. But for large rather than infinite dead-times, although it is not difficult to obtain an approximation to the expectation using the step function approach, it would appear to be virtually impossible using direct probabilistic arguments.

We shall now determine an approximation to the expectation (5.2) by a transient type analysis in which the level of approximation may be readily extended to any order; this is not true of the step function approach. However, there may be circumstances under which the step function analysis is a viable alternative to the transient analysis; this justifies its appearance.

5.3 Transient Type Analysis

To find our objective, the expected number of recorded events in (0,t] as given by (5.2), i.e.

$$E\{N(t)\} = \int_0^t \lambda(u)\pi(u)\, du,$$

we calculate $\pi(u)$ directly, where $\pi(u)$ is the probability that the counter is open at time u, given that the counter is open at zero. We still assume an exponentially distributed dead-time behaviour so that we can compare our results with those obtained using the step function method. The forward equation for $\pi(t)$ expressing the change from the closed state to the open state is

$$\frac{d\pi(t)}{dt} = -\lambda(t)\pi(t) + \mu\{1 - \pi(t)\} . \qquad (5.17)$$

The initial condition that the counter is open at 0 may be represented as $\pi(0) = 1$. The solution of (5.17) subject to the initial condition $\pi(0) = 1$ is

$$\pi(t)\exp(\mu t - \frac{\lambda_0}{\lambda} e^{-\lambda t}) - e^{-\frac{\lambda_0}{\lambda}} = \int_{e^{-\lambda t}}^{1} \frac{\mu}{\lambda} x^{-(\frac{\mu}{\lambda}+1)} e^{-\frac{\lambda_0}{\lambda}x} dx, \tag{5.18}$$

where $\lambda(t) = \lambda_0 e^{-\lambda t}$. The integral on the right-hand side of (5.18) is expanded for large μ using Laplace's Method as described by Olver (1974, page 80). This results in

$$\int_{e^{-\lambda t}}^{1} \frac{\mu}{\lambda} x^{-(\frac{\mu}{\lambda}+1)} e^{-\frac{\lambda_0}{\lambda}x} dx = \frac{\mu}{\lambda}(1 + \frac{\mu}{\lambda})^{-1}\exp(\mu t - \frac{\lambda_0}{\lambda} e^{-\lambda t}) \{1 + \frac{\lambda-\lambda_0 e^{-\lambda t}}{\lambda + \mu} + \frac{\lambda^2-3\lambda_0\lambda e^{-\lambda t} + \lambda_0^2 e^{-2\lambda t}}{(\lambda+\mu)^2} + O(\frac{\lambda}{\mu})^3\}, \tag{5.19}$$

Note that in obtaining (5.19) via Laplace's Method it was assumed that 1 and $e^{-\lambda t}$ are distinct, i.e. $t \neq 0$. Therefore any approximation to $\pi(t)$ obtained using the method of Laplace will not satisfy the initial condition $\pi(0) = 1$; this is commented upon in section 5.4. Substituting (5.19) into (5.18) we have that

$$\pi(t) = 1 - \frac{\lambda_0}{\mu} e^{-\lambda t} + (\frac{\lambda_0}{\mu})^2 e^{-2\lambda t} - \frac{\lambda_0\lambda}{\mu^2} e^{-\lambda t} + O(\frac{\lambda}{\mu})^3 . \tag{5.20}$$

We may now calculate the expectation as given by (5.2), thus

$$E\{N(t)\} = \frac{\lambda_0}{\lambda}(1 - e^{-\lambda t}) - \frac{\lambda_0^2}{2\mu^2}(1 - e^{-2\lambda t}) + \frac{\lambda_0^3}{3\lambda\mu^2}(1 - e^{-3\lambda t}) - \frac{\lambda_0^2}{2\lambda\mu}(1 - e^{-2\lambda t}) + O(\frac{\lambda}{\mu})^3 . \tag{5.21}$$

5.4 Comments

We first note that the expression for the expectation given by the step-function method, (5.14), is identical with that given by the transient type analysis, (5.21), if terms of order $(\lambda/\mu)^2$ are ignored and $\lambda_0 = R\lambda$ for some positive R. It is only when these second order terms are included that (5.15) and (5.21) differ. It is thought that this difference is accounted for by the second term in the expression for the expectation given by (5.9) which was only bounded for large μ. Also, in principle the analysis of section 5.3 can be extended to any order, whereas that of section 5.2 would be extremely difficult to extend.

In the context of the single counter system considered above, it should be noted that the transient analysis produced an approximation to $\pi(t)$ that failed to satisfy $\pi(0) = 1$; the implication here is that t would have to be fairly large in some sense for the method to be applicable in a practical situation. Therefore if the object of an experiment, in which the mean number of recorded events is measured, is to determine the original rate λ_0, then the above calculations would be valid. But if the object is to determine the current rate by taking a quick measurement, then some expansion other than (5.19) is required for small t.

Therefore, it is proposed that if the original disintegration rate of a source is to be estimated, then the transient type analysis of section 5.3 should be used in a bivariate counter situation similar to that of section 1.3.

Cold hearted orb that rules the night,

Removes the colours from our sight.

Red is grey and yellow white,

But we decide which is right.

And which is illusion???

The Moody Blues, "Days of Future Passed."

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