GRAVITATIONAL AND GAUGE INTERACTIONS

by

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ABSTRACT

Some aspects of the gravitational and gauge interactions are studied with a particular view to their generalisation. After a brief review of the conventional descriptions of these interactions attention is first focused on the Lagrangian for torsion-containing extensions of the general theory of relativity. It is shown that the general structure of metric-torsion theories allows a parity-violating contribution to the complete action which is linear in the curvature and vanishes identically in the absence of torsion. The resulting action involves, apart from the Newtonian constant, an extra coupling which governs the strength of the predicted parity non-conserving 'interactions' mediated by torsion. This theory is then studied in the presence of a Proca field and shown to lead to a parity-violating term in the field equations in contrast to the Einstein-Cartan-Sciama-Kibble theory.

The problem of coupling torsion to gauge fields in such a manner as to retain gauge invariance is considered next. It is shown that by modifying the Yang-Mills-Shaw field strength and using a generalisation of the minimal coupling procedure allows a simple but non-trivial type of dynamic torsion to couple to all gauge fields in a consistent manner. This allows, for the first time, a framework in which no spinning particle is required to be exempted from both generating and reacting to torsion. Apart from the introduction of a new scalar field, one may view the two modifications as being the replacement of all gauge couplings everywhere by space-time dependent gauge couplings.

PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, London, between October 1977 and June 1980, under the supervision of Professor Abdus Salam, to whom I am deeply indebted for invaluable advice and assistance throughout this period. Except where otherwise stated, this work is original and has not been submitted for a degree of this or any other university.

The work here described was originally reported in refs. 16 and 20. Some other published work carried out by the author and somewhat related to the subject matter of the thesis has also been attatched at the end.

I would like to thank my colleagues, Roberto Hojman, Chandrasekher Mukku, Mirza Mohammad Ali Namazie, Asghar Qadir and Anais Smailagic for providing opportunities for a free, though at times heated, exchange of ideas, so necessary for carrying out fruitful research.

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بِسْمِ اللّٰهِ الرَّحْمٰنِ الرَّحِيْمِ

قُلْ لَّوْ كَانَ الْبَحْرُ مِدَادًا لِّكَلِمٰتِ رَبِّيْ لَنَفِدَ الْبَحْرُ
قَبْلَ اَنْ تَنْفَدَ كَلِمٰتُ رَبِّيْ وَلَوْ جِئْنَا بِمِثْلِهٖ مَدَدًا ۝

In the name of Allah, the Most Gracious, Ever Merciful.

Tell them: If the ocean became ink for transcribing the words of my Lord, surely the ocean would be exhausted before the words of my Lord came to an end, even though We augmented it with the like thereof.

(The Holy Quran, Ch. 18, verse 110)

کیا عجب تُو نے ہر اِک ذرہ میں رکھے ہیں خواص
کون پڑھ سکتا ہے سارا دفتر اِن اَسرار کا
تیری قدرت کا کوئی بھی انتہا پاتا نہیں
کس سے کُھل سکتا ہے پیچ اِس عقدۂ دشوار کا

(Hazrat Mirza Ghulam Ahmad Qadiani, Durethamin)

This thesis is dedicated

to

my parents and teachers

CONTENTS

CHAPTER ONE

INTRODUCTION

He who attempts to deal with questions of natural science without the help of geometry is attempting the infeasible.

{Galileo, Dialogues Concerning Two New Sciences, Ch.VII}

1.1 *MOTIVATION*

At the macroscopic level the general theory of relativity appears to describe gravitational phenomena very well, while on the microscopic scale the theories of quantum electrodynamics(QED), the $SU(2)\times U(1)$ electroweak unification scheme, and the $SU(3)$ quantum chromodynamics scheme for strong interactions have attained great successes. Although each of these theories is built upon several important and distinct physical assumptions, the central idea in each case is the assumption that the laws being formulated to describe the particular interactions under study are invariant under some given set of transformations which form a group.

In the former case of general relativity the assumed invariance is the largest and requires that the theory be invariant under general co-ordinate transformations of the assumed four-dimensional Riemannian structure of space-time, while in the latter case of various gauge theories, the laws governing QED and electronuclear phenomena are taken to be invariant under a set of local space-time dependent gauge transformations acting in appropriately chosen internal spaces.

It is hardly necessary to even outline the experimental successes of these theories, suffice it to say that the ideas of general coordinate and gauge invariance have to date attained such a large measure of experimental support that their detailed study becomes imperative. This thesis is, therefore, devoted to the study of some aspects of these theories which we now describe briefly.

1.2 SUMMARY OF CONTENTS

In particular, we shall be concerned in this thesis with problems related to torsion-containing generalisations of Einstein's general theory of relativity and to problems which arise when we try to couple such theories to gauge fields. The next two brief chapters are, therefore, concerned with the presentation of those essential elements of these theories which we shall need later. Of course, it is not the purpose of these chapters to give sufficient material to enable the reader to master the subjects of gauge theories and extensions of the general theory of re~~l~~lativity: a basic knowledge of tensor analysis and group theory is assumed.

In chapter four we consider the choice of the Lagrangian which is generally used for the Einstein-Cartan-Sciama-Kibble(ECSK) theory, viz. the curvature scalar constructed out of the asymmetric connection on which this theory is based. This connection contains, apart from the symmetric Christoffel part, an additional term called the contorsion which is constructed from the antisymmetric part of the full connection. The work carried out in this chapter shows that the traditional requirements for determining the action do not in fact single out the conventional choice and that an extra contribution, involving the pseudo-tensor density $\varepsilon_{\mu\nu\alpha\beta}$, still linear in the curvature is allowed. The analogue of the additional term that we motivate has been considered in the past for the pure Einstein theory but is known there to vanish identically — leaving the standard choice of the Lagrangian for Einstein's theory as the unique candidate upto the addition of a cosmical term. The generalised action we propose does, therefore, involve, apart from the Newtonian constant, an additional coupling parameter which governs the strength of the new 'interactions'.

It is worth pointing out, however, that the new term cannot on its own, be used to provide an adequate description of the gravitational interactions since this term is not capable of providing the dynamics for the metric and, being a pseudoscalar, is parity violating, whereas the classically observed gravitational phenomena are parity conserving. The new Lagrangian still only involves torsion in an algebraic form since it does not contain any terms involving derivatives of the torsion (once some divergences have been removed) as indeed must be the case for all theories based on Lagrangians linear in the curvature tensor. One consequence of this is that torsion again vanishes by virtue of the field equations in the absence of matter as is the case for the ECSK theory.

However, if we accept the view that torsion is the geometrical analogue of spin just as curvature 'represents' mass and if we accept that gravitation is due to a spin-two particle, then we may reasonably demand that some form of dynamic torsion be present even in the absence of matter. This 'vacuum torsion' would then, in some sense, represent the torsional effects due to the spin-two nature of gravitation.

We consider possible ways of achieving this within the confines of a linear curvature theory and are led to examine a very restricted but dynamic form of torsion generated by a scalar field which makes an appearence also in the next chapter where it is argued to arise from a completely different point of view.

In the concluding section of this chapter we go on to give an example of a situation where the Lagrangian we propose can give rise to new effects not present in the ECSK theory. The example we consider shows that when our theory is analysed in the presence of the Proca field new parity violating terms arise in the field equations which would be absent for the similar situation in the ECSK theory.

The new feature of these generalised theories which distinguishes them from Einstein's theory is that they are, as already stated, based on an asymmetric torsion containing connection. The physical interpretation of the new contribution is tied up with the spin angular momentum of matter and it is argued that just as mass, or more correctly the energy-momentum tensor of matter, gives rise to the gravitational field in Einstein's theory based on the Riemannian structure of space-time, the spin angular momentum of matter should be the source of the non-Riemannian aspects of the theory based on the torsion containing connection.

A natural consequence of adopting this interpretation is that all spinning matter should couple to the torsion field of this theory. It turns out, however, as will be explicitly demonstrated in chapter five, that no gauge fields can be coupled to theories containing torsion in the usual manner of coupling matter to gravity theories without at the same time losing the gauge invariance of the original theory.

The problem that we tackle in chapter five is then to search for a way in which this deficiency can be overcome, thereby allowing us to maintain the standard interpretation of torsion and at the same time keeping gauge invariance. The method which allows us to do this does, however, require the generalisation of the usual concept of minimally coupling gauge fields to 'charged' matter fields and a slight modification of the Yang-Mills-Shaw field strength.

Two interesting consequences of the theory which emerges are:

(i) That all gauge couplings of nature become space-time dependent;

(ii) That torsion enters the theory in the very restricted but dynamic form that we motivate in chapter four.

A more detailed treatment of all this will be given in chapter five.

The recurrence of this particular type of dynamic torsion leads us to wonder whether it plays any fundamental role in nature and whether there are any experimental consequences of such torsion. This question has been examined for the case of the work of chapter five for the abelian case in the literature. We discuss these results and several other possible lines of further research suggested by the work of chapters four and five in the penultimate chapter.

The thesis ends on a rather optimistic note in the final very brief chapter where we record some speculative remarks provoked by the work detailed in the body of the thesis and some other, not altogether unrelated work attached as subsidiary material at the end of the thesis.

CHAPTER TWO

THE GRAVITATIONAL INTERACTION

From the absolute to nothing absolutely

2.1 *EINSTEIN'S GENERAL THEORY OF RELATIVITY* /F1/

Einstein's general theory of relativity/1-9/ is a theory of gravitation as fundamentally different from the Newtonian theory as it is possible to conceive. It abolishes the central idea of forces on which the extremely successful structure of Newton's theory rests. Instead it explains the observed phenomena of gravitation to an even greater degree of accuracy than the theory of Newton through geometrical means by providing a dynamical understanding of the structure of space-time.

The fundamental object with which Einstein's theory operates is the metric $g_{\mu\nu}$ (the components of which we shall refer to as the gravitational fields) of the assumed four dimensional Riemannian structure of space-time. Any point of this space-time model is labelled by real co-ordinates x^μ, with $\mu = 0,1,2,3$, where 0 refers to the time co-ordinate and 1,2,3, refer to the space co-ordinates. The theory further assumes:

(i) The equivalence of all four-dimensional systems of co-ordinates obtained from any one of them by an arbitrary general co-ordinate transformation, and

(ii) That the four-dimensional continuum has a metrical connection impressed upon it.

The meaning of the latter requirement is that at every point a certain quadratic form of the co-ordinate differntials,

$$g_{\mu\nu}dx^\mu dx^\nu,$$

called the square of the interval between the two points in question, has a fundamental meaning invariant under the aforesaid transformations.

This last requirement determines the connection of this theory to be the so-called Christoffel connection which depends on the metric $g_{\mu\nu}$ as follows:

$$\left\{ {\lambda \atop \mu\nu} \right\} = \tfrac{1}{2}g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \qquad (2.1.1)$$

where a comma denotes partial differentiation, thus

$$g_{\sigma\mu,\nu} = \partial g_{\sigma\mu}/\partial x^{\nu}.$$

In order to obtain the dynamics for the gravitational fields from a least action principle which obeys the requirements of general co-ordinate invariance and is such as to yield the Newtonian results in some limit it is necessary first to construct a tensor out of the metric tensor and its first and second derivatives which is also linear in the second derivatives of $g_{\mu\nu}$. Remembering that the Christoffel connection has the following transformation law

$$\{{}^{\lambda}_{\mu\nu}\} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \{{}^{\tau}_{\rho\sigma}\}' + \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\, \partial x^{\nu}} , \qquad (2.1.2)$$

it is easy to show that the only such tensor is the Riemann Christoffel tensor

$$R_{\mu\nu\lambda}{}^{\sigma}(\{\}) = \{{}^{\sigma}_{\nu\lambda}\}_{,\mu} - \{{}^{\sigma}_{\mu\lambda}\}_{,\nu} + \{{}^{\sigma}_{\mu\tau}\}\{{}^{\tau}_{\nu\lambda}\} - \{{}^{\sigma}_{\nu\tau}\}\{{}^{\tau}_{\mu\lambda}\}. \qquad (2.1.3)$$

Having obtained this tensor it is straightforward to see that the following action for gravitation yields the desired field equations for $g_{\mu\nu}$ by requiring stationarity under infinitesimal variations in the metric tensor:

$$I_G = \frac{1}{16\pi G_N} \int \sqrt{-g}\, R(\{\})\, d^4x . \qquad (2.1.4)$$

Here g denotes det $g_{\mu\nu}$, G_N is the Newtonian coupling constant and $R(\{\}) = g^{\nu\lambda} g^{\mu\sigma} R_{\mu\nu\lambda}{}^{\sigma}(\{\})$, is the Ricci curvature scalar. It will be useful to record here some of the formulae needed in the derivation of the field equations for $g_{\mu\nu}$.

From the definition of the curvature scalar $R(\{\})$ it follows that

$$\delta(\sqrt{-g}\, R(\{\})) = \delta(\sqrt{-g})\, R(\{\}) + \sqrt{-g}\, \delta g^{\nu\lambda} R_{\nu\lambda}(\{\}) + \sqrt{-g}\, g^{\nu\lambda}\, \delta R_{\nu\lambda}(\{\}) , \qquad (2.1.5)$$

where $R_{\nu\lambda}(\{\})$ is the Ricci tensor. It is an old established result that the last term in (2.1.5) can be expressed as a pure divergence and therefore drops out when we integrate over all space. Using now the results:

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu}\,\delta g_{\mu\nu}\ , \qquad (2.1.6)$$

and

$$\delta g^{\mu\nu} = -\, g^{\mu\rho}\, g^{\nu\sigma}\,\delta g_{\rho\sigma} \qquad (2.1.7)$$

we finally obtain

$$\delta I_G = \frac{-1}{16\pi G_N}\int \sqrt{-g}\ (\, R^{\mu\nu}(\{\}) - \tfrac{1}{2}g^{\mu\nu}\, R(\{\})\,)\ \delta g_{\mu\nu}\, d^4x\ . \qquad (2.1.8)$$

If the total action I_T is written as

$$I_T = I_M + I_G\ , \qquad (2.1.9)$$

and if we define the matter energy momentum tensor through

$$I_M = \tfrac{1}{2}\int d^4x\ \sqrt{-g}\ T^{\mu\nu}\ \delta g_{\mu\nu} \qquad (2.1.10)$$

then we may obtain the complete set of Einstein's field equations:

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}R(\{\}) - 8\pi G_N\, T_{\mu\nu} = 0\ . \qquad (2.1.11)$$

So much for the material we shall later need to be familiar with from Einstein's theory. The next section of this chapter contains an equally brief account of the ECSK theory.

2.2 THE ECSK THEORY

When we venture forth into the microphysical realm of matter, we find that spin angular momentum also comes into play ... The hypothesis is near at hand that spin angular momentum is the source of a field too, in fact the source of a gravitational field /14/.

The basic reason behind all attempts to generalise the theory of Einstein has been summed up beautifully by Schrodinger /3/ which we now paraphrase slightly: Of the two principles on which Einstein's theory is based, the second - the adoption of a metrical connection straight away - does not seem to be the simplest way of obtaining it. The reason for this is that the concepts on which this theory hinges such as invariant differentiation, Riemann-Christoffel tensor, curvature, variational principle etc. are not at all peculiar to the metrical connection. Indeed, they come in in a much simpler way when one only introduces as much of a connection as the idea of differentiation calls out for in view of the general co-ordinate invariance one has admitted. This is the so-called affine connection and leads to theories based on this more general connection inaugurated by Weyl as early as 1918 /10/ .

As a digression it is amusing to note that it was in this 1918 work of Weyl that the first local gauge invariance principle was suggested and amazingly enough it was introduced as an addition to Einstein's theory in an attempt to obtain geometrical unification of electromagnetism and gravitation. It should be mentioned, of course, that Weyl's attempted unification failed as it led to deductions in contradiction with experiment /F2/. The idea of gauge invariance here introduced, and from where it derives its name, was revived a decade or so later /11/ by Weyl himself, after the advent of quantum mechanics in the form of a U(1) gauge - a locally space-time dependent phase factor for charged fields -

instead of scale invariance. This abstract 'internal' gauge invariance was Weyl's second and much more successful definition. However, it was the generalisation to the non-abelian case, exactly a quarter of a century later, by Yang, Mills, and Shaw /12-13/ which led to gauge unification schemes so popular today. A more complete discussion of gauge theories will be relevant in the next chapter.

We return to the discussion of the ECSK theory. This theory /14/ differs from that of Einstein in that it employs, apart from the metric tensor $g_{\mu\nu}$, an extra set of 24 independent fields which arise in the theory on giving up the purely symmetric Christoffel connection. The asymmetric connection $\Gamma_{\mu\nu}{}^{\sigma}$ contains an antisymmetric part called the torsion,

$$S_{\mu\nu}{}^{\lambda} = \tfrac{1}{2} (\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda}) \tag{2.2.1}$$

$$= - S_{\nu\mu}{}^{\lambda} , \tag{2.2.2}$$

which is antisymmetric in its first two indices.

The full connection may be decomposed /E3/ upon imposition of the requirement of metricity, i.e. the requirement that space-time be locally Minkowskian, into,

$$\Gamma_{\mu\nu}{}^{\lambda} = \{ {}^{\lambda}_{\mu\nu} \} - K_{\mu\nu}{}^{\lambda} \tag{2.2.3}$$

where $K_{\mu\nu}{}^{\lambda}$, the contorsion tensor depends on the metric and the torsion in the following manner:

$$K_{\mu\nu}{}^{\lambda} = -S_{\mu\nu}{}^{\lambda} + S_{\nu}{}^{\lambda}{}_{\mu} - S^{\lambda}{}_{\mu\nu} \tag{2.2.4}$$

$$= - K_{\mu}{}^{\lambda}{}_{\nu} , \tag{2.2.5}$$

and is antisymmetric in its last two indices. The extra 24 independent fields of this theory may be identified with either of the two 24-component tensors $S_{\mu\nu}{}^{\lambda}$ or $K_{\mu\nu}{}^{\lambda}$.

A simple proof of (2.2.3) follows. Metricity implies that

$$g_{\mu\nu;\rho} = 0 \tag{2.2.6}$$

i.e.

$$g_{\mu\nu,\rho} - \Gamma_{\rho\mu}{}^{\alpha}\, g_{\alpha\nu} - \Gamma_{\rho\nu}{}^{\alpha}\, g_{\mu\alpha} = 0 \,. \tag{2.2.7}$$

We can obtain the following two equations from this by permutation:

$$g_{\nu\rho,\mu} - \Gamma_{\mu\nu}{}^{\alpha}\, g_{\alpha\rho} - \Gamma_{\mu\rho}{}^{\alpha}\, g_{\nu\alpha} = 0 \,, \tag{2.2.8}$$

and

$$g_{\rho\mu,\nu} - \Gamma_{\nu\rho}{}^{\alpha}\, g_{\alpha\mu} - \Gamma_{\nu\mu}{}^{\alpha}\, g_{\rho\alpha} = 0 \,. \tag{2.2.9}$$

Adding (2.2.8) and (2.2.9) and subtracting (2.2.7) from the result yields,

$$g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho} - \Gamma_{\mu\nu}{}^{\alpha}\, g_{\alpha\rho} - \Gamma_{\nu\mu}{}^{\alpha}\, g_{\alpha\rho} - \Gamma_{\mu\rho}{}^{\alpha}\, g_{\nu\alpha} -$$

$$- \Gamma_{\nu\rho}{}^{\alpha}\, g_{\alpha\mu} + \Gamma_{\rho\nu}{}^{\alpha}\, g_{\alpha\mu} + \Gamma_{\rho\mu}{}^{\alpha}\, g_{\nu\alpha} = 0 \tag{2.2.10}$$

The last six terms can be written in terms of torsion and the symmetric part of Γ as follows:

$$- 2\, g_{\alpha\rho} \overset{s}{\Gamma}_{\mu\nu}{}^{\alpha} - 2\, g_{\nu\alpha}\, S_{\mu\rho}{}^{\alpha} - 2\, g_{\alpha\mu}\, S_{\nu\rho}{}^{\alpha} \,, \tag{2.2.11}$$

where

$$\overset{s}{\Gamma}_{\mu\nu}{}^{\alpha} = \tfrac{1}{2}\, (\Gamma_{\mu\nu}{}^{\alpha} + \Gamma_{\nu\mu}{}^{\alpha}) \,. \tag{2.2.12}$$

Multiplying (2.2.10) by $\frac{1}{2}g^{\beta\rho}$ we obtain

$$\left\{{}^{\,\beta}_{\mu\nu}\right\} - \overset{s}{\Gamma}_{\mu\nu}{}^{\beta} - S_{\mu}{}^{\beta}{}_{\nu} - S_{\nu}{}^{\beta}{}_{\mu} = 0 \,. \tag{2.2.13}$$

Adding and subtracting an antisymmetric part of the connection to this equation gives,

$$\left\{{}^{\,\beta}_{\mu\nu}\right\} - \overset{s}{\Gamma}_{\mu\nu}{}^{\beta} - S_{\mu\nu}{}^{\beta} + S_{\mu\nu}{}^{\beta} - S_{\mu}{}^{\beta}{}_{\nu} - S_{\nu}{}^{\beta}{}_{\mu} = 0 \,. \tag{2.2.14}$$

However, the second and the third terms together are just $-\Gamma_{\mu\nu}{}^{\beta}$, so that we finally obtain the desired result that

$$\Gamma_{\mu\nu}{}^{\beta} = \{{}^{\beta}_{\mu\nu}\} + S_{\mu\nu}{}^{\beta} - S_{\nu}{}^{\beta}{}_{\mu} + S^{\beta}{}_{\mu\nu} \ . \qquad (2.2.15)$$

As before, we must now construct an action which can provide the dynamics for gravity. For this purpose the usual choice is to work with the curvature tensor $R_{\mu\nu\lambda}{}^{\sigma}(\Gamma)$ which has the same form in terms of Γ as the Riemann-Christoffel tensor has in terms of the Christoffel symbols $\{\}$:

$$R_{\mu\nu\lambda}{}^{\sigma}(\Gamma) = \Gamma_{\nu\lambda}{}^{\sigma}{}_{,\mu} - \Gamma_{\mu\lambda}{}^{\sigma}{}_{,\nu} + \Gamma_{\mu\tau}{}^{\sigma}\Gamma_{\nu\lambda}{}^{\tau} - \Gamma_{\nu\tau}{}^{\sigma}\Gamma_{\mu\lambda}{}^{\tau} \ . \qquad (2.2.16)$$

The action used conventionally for obtaining the field equations of this theory is then taken to be,

$$I_{ECSK} = \frac{1}{16\pi G_N} \int \sqrt{-g}\; R(\Gamma)\; d^4x \ , \qquad (2.2.17)$$

where now the curvature scalar is the one obtained by contracting $R_{\mu\nu\lambda}{}^{\sigma}(\Gamma)$.

It is worth pointing out here that this action does not involve any extra couplings apart from the Newtonian constant. It is a simple enough matter once again to derive the field equations for this theory. The only extra work needed over and above that needed to overcome the complications of the Einstein case is due to the variation of either the torsion or the contortion fields to obtain their field equations, though the techniques remain the same.

In the next chapter we shall go on to give a brief introduction to gauge theories.

CHAPTER THREE

THE GAUGE INTERACTIONS

Replacing everywhere always by here now

3.1 ABELIAN GAUGE INVARIANCE /F4/

In the last chapter we have outlined Einstein's description of the phenomena of gravitation. We saw there that a successful description of the effects of gravity was obtained by ascribing them to the geometrical structure of space-time. The dynamics of this structure then determined the laws of gravitation in a manner precisely dictated by the requirements of general co-ordinate invariance.

In the present chapter we shall be concerned with invariances not altogether dissimilar to those of Einstein, the only difference being that now we shall ascribe the new interactions, called gauge interactions, to the geometrical structure of appropriately defined internal spaces. At present it is believed that there are four fundamental interactions in nature. In order of increasing strength these are the gravitational, the weak, the electromagnetic, and the strong interactions. With the recent successes of the electroweak unification scheme of Salam and Weinberg on the one hand and the very encouraging though as yet only qualitative successes of the quantum chromodynamics model of the strong interactions on the other it is now widely accepted that the last three of these interactions can be successfully comprehended within the framework of gauge theories, while the theory of general relativity is, at present, the best candidate for the gravitational interaction. In view of the fact that all these interactions can be described in a 'geometrical' framework it becomes clear how very prophetic were the words of Galileo quoted earlier that an adequate description of the workings of nature is impossible without the aid of geometry.

In the present section we shall show how the laws of electrodynamics arise as a consequence of assuming a local abelian U(1) gauge invariance. This procedure shall then be generalised to non-abelian gauge groups in the second section of this chapter /15/ .

For the purposes of illustrating the basic concepts of gauge theories let us start by considering a set of fields Φ, the dynamics of which are determined by a Lagrange density which depends on Φ and $\partial_\mu\Phi$: $L(\Phi\ ,\ \partial_\mu\Phi)$.Suppose that each field Φ_i has charge q_i (in units of e the electron charge). Then define a group of transformations on the fields by

$$\Phi_i(x) \rightarrow \exp\ (\ -i\ q_i\ \Lambda\ )\ \Phi_i(x)\ , \tag{3.1.1}$$

where Λ is a constant. This group is the group of unitary transformations in one dimension U(1).

It is not hard to see that L must be invariant under these transformations. Every term in L is a product of the fields $\Phi_1 \ldots . \Phi_n$. Under the above transformation this term goes to

$$\exp\ (\ -i(q_1 + q_2 + \ldots + q_n)\Lambda)\ \Phi_1\ \Phi_2\ \ldots\ \Phi_n\ , \tag{3.1.2}$$

but charge conservation requires that L be neutral; therefore the sum $q_1 + q_2 + \ldots + q_n$ must vanish so that all such terms are invariant. However, some terms in L contain derivatives of the fields as well as the fields themselves. Nevertheless, since Λ is independent of x,

$$\partial_\mu\Phi_i \rightarrow\ \exp\ (\ -i\ q_i\ \Lambda\ )\partial_\mu\Phi_i \tag{3.1.3}$$

as well so that these terms are also invariant. The infinitesimal form of (3.1.1) is

$$\delta\Phi_i = -i\Lambda\ q_i\ \Phi_i\ , \tag{3.1.4}$$

where Λ is an arbitrary infinitesimal parameter.

It is well known, however, that electrodynamics possesses a symmetry larger than global(Λ not function of x) transformations of the above type. Indeed invariance is maintained under the much larger set of transformations obtained by allowing Λ in (3.1.1) to be space-time dependent. The invariance is much larger, not because we have enlarged the rank of the group, but because we have assumed that there is a U(1) invariance at each point of space-time whereas before we had a single global U(1). The finite and the infinitesimal form of these new local gauge transformations are, of course, just the expressions (3.1.1) and (3.1.4) where Λ is allowed to be a space-time dependent function.

Now we note that although the terms in the Lagrangian which depend only on the fields are once again invariant, terms involving the derivatives of the fields, such as the kinetic energy term, need to be cosidered a little more carefully since $\partial_\mu \Phi_i$ no longer transforms as Φ_i. Indeed $\partial_\mu \Phi_i$ transforms to

$$\partial_\mu \Phi_i \rightarrow \exp(-iq_i \Lambda(x)) \partial_\mu \Phi_i - iq_i(\partial_\mu \Lambda(x))\exp(-iq_i \Lambda(x)) \Phi_i . \qquad (3.1.5)$$

The second term in this expression is the difference between the way the derivative of Φ_i and Φ_i transform. Note, however, that the Lagrangian will be invariant only if it is a product of terms all of which transform like (3.1.1) with the sum of q_i vanishing.

This is achieved in electrodynamics by introducing the photon field according to the rule of minimal coupling which is an operator form of the classical $p_\mu \rightarrow p_\mu - eA_\mu$ transformation which takes us from classical mechanics to classical electrodynamics of charged particles. This rule requires that a derivative of the charged field appear in the Lagrangian only in conjunction with the photon field A_μ, in the combination $(\partial_\mu - ie\, q_i A_\mu) \Phi_i$. A_μ is a spin-one field - the photon - which is our first example of a gauge field. We require it to transform in a special way, so that the combination $(\partial_\mu - ie\, q_i A_\mu) \Phi_i$ transforms like Φ_i. That is

$$(\partial_\mu - ieq_i A'_\mu) \Phi'_i(x) = \exp(-iq_i\Lambda(x)) (\partial_\mu - ieq_i A_\mu)\Phi_i . \qquad (3.1.6)$$

If this can be arranged then $L(\Phi_i, (\partial_\mu - ieq_i A_\mu)\Phi_i)$ will be invariant under local gauge transformations also. Substituting into (3.1.6) the transformation law for the fields Φ_i we obtain:

$$(\partial_\mu - ieq_i A'_\mu)\exp(-iq_i\Lambda)\Phi_i = \exp(-iq_i\Lambda)(\partial_\mu - ieq_i A_\mu)\Phi_i \qquad (3.1.7)$$

or,

$$\exp(-iq_i\Lambda)(\partial_\mu - ieq_i A_\mu)\Phi_i - iq_i\partial_\mu\Lambda\exp(-iq_i\Lambda)\Phi_i - ieq_i\exp(iq_i\Lambda)\Phi_i\delta A_\mu$$

$$= \exp(-iq_i\Lambda)(\partial_\mu - ieq_i A_\mu)\Phi_i \ , \qquad (3.1.8)$$

which solves to yield the transformation law for A_μ

$$\delta A_\mu = A'_\mu - A_\mu = -\frac{1}{e}\,\partial_\mu\Lambda(x) \ . \qquad (3.1.9)$$

In addition to the terms coupling the photon field to the charged fields, we need also a quadratic kinetic energy term for the photon in order to provide the dynamics for this field. This term is constructed from the gauge invariant field strength,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (3.1.10)$$

and is taken conventionally to be

$$L_{em} = -\tfrac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \ . \qquad (3.1.11)$$

An action principle can then easily be shown to lead to the familiar field equations of electrodynamics in a manner completely analogous to the case of the gravitational field equations we obtained in the last chapter.

The generalisation of local gauge invariance to non-Abelian groups was considered first by Yang and Mills, and Shaw /12-13/ and is the subject of the next section. There it will be seen that the fundamental new ingredient that emerges is the possibility of self-coupling gauge fields for non-Abelian groups - a feature which is absent from electrodynamics because of its abelian nature which requires that the photon be neutral. The presence of this self-interaction term for non-Abelian gauge fields means that the theory is intrinsically non-linear and therefore resembles the theory of Einstein much more than the Abelian case of electrodynamics.

3.2 THE NON ABELIAN THEORY

For the purposes of outlining the non-Abelian theory let us consider an internal symmetry group G with generators T_i which satisfy the commutation relations,

$$[T_i \, , \, T_j] = i \, C_{ijk} \, T_k \, , \tag{3.2.1}$$

where the C_{ijk} are the structure constants of the group G. A collection of fields Φ_i which we shall denote simply by Φ transforms according to

$$\Phi(x) \rightarrow \Phi'(x) = \exp(-iL^j \Lambda^j) \, \Phi(x)$$

$$\equiv U(\Lambda) \, \Phi(x) \, , \tag{3.2.2}$$

where $\Phi(x)$ is a column vector and L^j is a matrix representation of the generators of the group under which Φ transforms as above. The Lagrangian L is assumed to be invariant under transformations with constant Λ^j. The problem is then to construct a theory which is invariant under the larger set of transformations obtained by replacing Λ^j in (3.2.2) by $\Lambda^j(x)$ exactly as for the Abelian case. We shall do this, in analogy with the case of electrodynamics by introducing a set of vector fields A^j_μ . Under local gauge transformations,

$$\Phi(x) \rightarrow U(\Lambda) \, \Phi(x) \tag{3.2.3}$$

and therefore,

$$\partial_\mu \Phi(x) \rightarrow U(\Lambda) \, \partial_\mu \Phi(x) + (\partial_\mu U(\Lambda)) \, \Phi(x) \, . \tag{3.2.4}$$

The idea is to introduce a gauge covariant derivative $D_\mu \Phi(x)$ which transforms like $\Phi(x)$. Thus

$$D_\mu \Phi(x) \rightarrow U(\Lambda) \, D_\mu \Phi(x) \tag{3.2.5}$$

Then if $\partial_\mu\Phi(x)$ appears in L only as a part of $D_\mu\Phi(x)$, L will retain its invariance under transformations of the type (3.2.2) even when Λ is allowed to be a space-time dependent function.

We define the covariant derivative as follows:

$$D_\mu\Phi(x) = (\partial_\mu - ig\, L\cdot A_\mu)\,\Phi(x) \tag{3.2.6}$$

where we have introduced one gauge field for each generator and where the coupling constant g is the analogue of the electromagnetic e. The transformation property of A^j_μ is determined by the requirement that

$$\begin{aligned} D'_\mu\Phi'(x) &= \partial_\mu\Phi'(x) - ig\, A'^j_\mu L^j\Phi'(x) \\ &= (\partial_\mu U(\Lambda))\Phi + U(\Lambda)\partial_\mu\Phi - igA'_\mu\cdot L\, U(\Lambda)\Phi \end{aligned} \tag{3.2.7}$$

is equal to

$$U(\Lambda)\, D_\mu\Phi = U(\Lambda)\, (\partial_\mu - ig\, A_\mu\cdot L)\Phi\ . \tag{3.2.8}$$

This solves straightforwardly to yield

$$-igA'_\mu\cdot L\, U(\Lambda)\,\Phi = -igU(\Lambda)\, A_\mu\cdot L\Phi - (\partial_\mu U(\Lambda))\Phi \tag{3.2.9}$$

or, since this must hold for all Φ

$$\begin{aligned} A'_\mu\cdot L &= U(\Lambda)\, A_\mu\cdot L\, U^{-1}(\Lambda) - \frac{i}{g}\,(\partial_\mu U(\Lambda))\, U^{-1}(\Lambda) \\ &= U(\Lambda)\, \{A_\mu\cdot L - \frac{i}{g}\, U^{-1}(\Lambda)\, \partial_\mu U(\Lambda)\}\, U^{-1}(\Lambda)\ . \end{aligned} \tag{3.2.10}$$

The appearance of L in this equation might seem to suggest that the transformation law for the gauge fields depends on the representation under which the charged fields transform, whereas infact it depends only on the commutators $[L^i\ , L^j]$ whose form is representation independent. This fact becomes apparent from the infinitesimal transformation law as follows. From (3.2.10) we obtain for the infinitesimal case,

$$L^j\ \delta A^j_\mu = -\frac{1}{g}\, L^j\partial_\mu\Lambda^j + iL^iA^i_\mu\Lambda^jL^j - i\Lambda^jL^jA^i_\mu L^i$$

$$= -\frac{1}{g} L^j \partial_\mu \Lambda^j + i\Lambda^j A^i_\mu [L^i , L^j]$$

$$= -\frac{1}{g} L^j \partial_\mu \Lambda^j - \Lambda^j A^i_\mu c_{ijk} L^k \quad (3.2.11)$$

and since the L^j are linearly independent one obtains for δA^i_μ the formula,

$$\delta A^i_\mu = -\frac{1}{g} \partial_\mu \Lambda^i + c_{ijk} \Lambda^j A^k_\mu , \quad (3.2.12)$$

which shows that the transformation properties do, in fact, not depend on the representation matrices L^i .

Before going on to discuss the construction of the kinetic energy term it is worth showing that these transformations satisfy the group property. If we perform two gauge transformations successively on A_μ then we obtain,

$$A'_\mu \cdot L = U(\Lambda_1)\{A_\mu \cdot L - \frac{i}{g} U^{-1}(\Lambda_1)\partial_\mu U(\Lambda_1)\} U^{-1}(\Lambda_1) \quad (3.2.13)$$

and

$$A''_\mu \cdot L = U(\Lambda_2)\{A'_\mu \cdot L - \frac{i}{g} U^{-1}(\Lambda_2)\partial_\mu U(\Lambda_2)\} U^{-1}(\Lambda_2) . \quad (3.2.14)$$

Upon substitution for A'_μ from (3.2.13) into (3.2.14) we obtain,

$$A''_\mu \cdot L = U(\Lambda_2) \ [U(\Lambda_1) \ \{A_\mu \cdot L - \frac{i}{g} U^{-1}(\Lambda_1)\partial_\mu U(\Lambda_1)\} U^{-1}(\Lambda_1) -$$

$$- \frac{i}{g} U^{-1}(\Lambda_2)\partial_\mu U(\Lambda_2)\} U^{-1}(\Lambda_2) . \quad (3.2.15)$$

However, the last two terms reduce to,

$$- \frac{i}{g} (\partial_\mu U(\Lambda_3)) \ U^{-1}(\Lambda_3) \quad (3.2.16)$$

where

$$U(\Lambda_3) = U(\Lambda_2) \ U(\Lambda_1) . \quad (3.2.17)$$

So we obtain finally the result that

$$A'''_\mu \cdot L = U(\Lambda_3) \ \{A_\mu \cdot L - \frac{i}{g} U^{-1}(\Lambda_3)\partial_\mu U(\Lambda_3)\}U^{-1}(\Lambda_3) \quad (3.2.18)$$

which has the same form as (3.2.10) and, therefore, shows that the group property is indeed satisfied.

Next we consider the problem of constructing the kinetic energy term for these non-Abelian gauge fields. Now we recall that unlike the case of electromagnetism, not all members of the gauge field multiplet A_μ^i are neutral under all the generators T_i of the group. This means that the 'free' kinetic energy part of the gauge Lagrangian can no longer be of the same simple form encountered in electromagnetism. Indeed, it follows from (3.2.12) that

$$\delta[\partial_\mu A_\nu^i - \partial_\nu A_\mu^i] = C_{ijk}\Lambda_j(\partial_\mu A_\nu^k - \partial_\nu A_\mu^k) + C_{ijk}\{(\partial_\mu \Lambda^j)A_\nu^k - (\partial_\nu \Lambda^j)A_\mu^k\}. \quad (3.2.19)$$

It is easy to see, however, that a kinetic energy term of the F^2 type for non-Abelian gauge fields would be gauge invariant if it was constructed from a tensor $F_{\mu\nu}^i$ which is gauge covariant. It is necessary, therefore, to add something to $\partial_\mu A_\nu^i - \partial_\nu A_\mu^i$ which cancels the unwanted terms in (3.2.19). Furthermore, this term must involve a self-coupling of the gauge fields for the reasons discussed already.

Now, we know from (3.2.12) that

$$g\, C_{ijk}\, \delta[A_\mu^j, A_\nu^k] = - C_{ijk}\{(\partial_\mu \Lambda^j)A_\nu^k - (\partial_\nu \Lambda^j A_\mu^k\} +$$
$$+ \frac{1}{g} C_{ijk}(C_{jlm}\Lambda^l A_\mu^m A_\nu^k + C_{klm}\Lambda^l A_\nu^m A_\mu^j) \quad (3.2.20)$$

The first terms cancel the unwanted terms in (3.2.19) while the last two can be written, using the antisymmetry of the structure constants as

$$[C_{imk}C_{kjl} - C_{ijk}C_{kml}]\, \Lambda^l A_\mu^j A_\nu^m , \quad (3.2.21)$$

which can be further simplified to read

$$C_{ilk}C_{kjm}\, \Lambda^l\, A_{\mu j}\, A_{\nu m} \quad (3.2.22)$$

upon use of the Jacobi identity for the structure constants

$$C_{imk}C_{kjl} - C_{ijk}C_{kml} - C_{ilk}C_{kjm} = 0 . \quad (3.2.23)$$

So that if we define

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\, C_{ijk} A^j_\mu A^k_\nu \,, \qquad (3.2.24)$$

then

$$\delta F^i_{\mu\nu} = C_{ijk}\, \Lambda^j\, F^k_{\mu\nu} \quad , \qquad (3.2.25)$$

which shows that $F^i_{\mu\nu}$ transforms gauge covariantly. This allows us to take

$$-\frac{1}{4} F^i_{\mu\nu}\, F^i_{\mu\nu} \;,$$

as the gauge kinetic energy term.

Once again, a mass term of the form $-\frac{m^2}{2} A^i_\mu\, A^{i\mu}$ violates gauge invariance and is, therefore not permissible. Finally, then, we can retain gauge invariance of the local type if we take as our Lagrangian

$$L_{\mathrm{Total}} = -\frac{1}{4} F^i_{\mu\nu}\, F^i_{\mu\nu} + L(\Phi,(\partial_\mu - igA_\mu \cdot L)\Phi) \;. \qquad (3.2.26)$$

Of course, massless gauge fields which do not correspond to the photon have little to do with observed phenomena. This does not mean, however, that they have no role to play in a successful description of nature. What it does mean is that several other ingredients are needed before they can be used with any success. These are the well-known phenomena of spontaneous symmetry breaking and the Higgs-Kibble mechanism, which we shall not in fact be using in the work carried out in this thesis.

CHAPTER FOUR

LAGRANGIANS FOR METRIC-TORSION THEORIES OF GRAVITATION

4.1 *MOTIVATION*

As the title of this chapter suggests we shall here be concerned with the choice of a suitable Lagrangian for metric-torsion theories of gravitation. The point being that once we give up the Einsteinian choice of the Christoffel connection there is no reason to assume that the simple choice $R(\Gamma)$ for the metric-torsion Lagrangian will be the only possibility permitted by the requirement of general co-ordinate invariance as is the case for the choice $R(\{\})$ for the Einstein theory.

We shall consider this question in the following section of this chapter where it will be argued that the Lagrangian for such theories ought to be constructed solely out of linear combinations of the curvature tensor thereby removing the possibility of admitting other tensors, such as the torsion tensor or its covariant derivatives, from appearing in the Lagrangian. It will be shown, however, that even this restriction does not limit the choice of the Lagrangian to the conventional one, viz. $R(\Gamma)$.

The new Lagrangian /16/ that we propose is motivated in the next section and involves, apart from the usual choice, an extra term constructed from the pseudo-tensor density $\varepsilon^{\mu\nu\alpha\beta}$ and the curvature tensor $R_{\mu\nu\alpha\beta}(\Gamma)$. The complete action is, of course, still linear in the curvature but leads to new parity violating effects in the presence of torsion which are not present in the ordinary ECSK theory. The analogue of the additional term our action involves has been considered before /17/ for the pure Einstein theory but is known there to vanish identically because of the cyclicity property of the Riemann-Christoffel tensor $R_{\mu\nu\alpha\beta}(\{\})$.

The Lagrangian density we propose can be written symbolically as

$$L = L_{ECSK} + L_A ,$$

where L_{ECSK} is the usual expression for the ECSK theory given in eqn. (2.2.15) and involves the Newtonian coupling constant.

$L_A(\sim \varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}(\Gamma))$ is the additional contribution which will be shown to be non-zero for torsion containing theories. The standard procedure for incorporating torsion into Einstein's theory involves working only with L_{ECSK} and does not, therefore, require the introduction of extra couplings. For L, however, an additional coupling is seen to be necessary and governs the strength of the parity-violating effects mediated by torsion.

4.2 THE NEW LAGRANGIAN

Let us begin by outlining the standard considerations which lead, for the pure Einstein case, to the unique (up to a cosmical term) Lagrangian density

$$L_E \sim \sqrt{-g}\, R(\{\}) \ . \tag{4.2.1}$$

The proof of this begins by noting that the Riemann-Christoffel tensor

$$R_{\mu\nu\lambda}{}^{\kappa}(\{\}) = \{^{\kappa}_{\nu\lambda}\}_{,\mu} - \{^{\kappa}_{\mu\lambda}\}_{,\nu} + \{^{\kappa}_{\mu\sigma}\}\{^{\sigma}_{\nu\lambda}\} - \{^{\kappa}_{\nu\sigma}\}\{^{\sigma}_{\mu\lambda}\} \tag{4.2.2}$$

is the only tensor that can be constructed from the metric tensor and its first and second derivatives and which is linear in the second derivatives. This tensor is, therefore, the simplest object at our disposal when we come to write down an action for gravity. We must now begin to contract indices in such a way as to construct all possible scalars linear in the curvature from $R_{\mu\nu\lambda\sigma}(\{\})$. The most general Lagrangian would then just be a sum of all these scalars with appropriate couplings in front.

It is an elementary exercise to show that only two such scalars can be constructed. However, one of them /F5/$\sim\eta^{\mu\nu\lambda\sigma}\ R_{\mu\nu\lambda\sigma}(\{\})$ vanishes identically due to following cyclicity identity

$$R_{\mu\nu\lambda\sigma}(\{\}) + R_{\mu\lambda\sigma\nu}(\{\}) + R_{\mu\sigma\nu\lambda}(\{\}) = 0 \ . \tag{4.2.3}$$

This implies that one can only use $R(\{\})$ to construct an action for Einstein's theory, thus the choice (2.1.4) is unique.

The generalisation to the case when torsion is present begins with the curvature tensor formed out of the non-symmetric connection $\Gamma_{\mu\nu}^{\ \ \alpha}$. The expression for this has been given already in (2.2.16). It is immediately clear from this definition that this curvature tensor is antisymmetric in its first two indices. In the general case (i.e. without any assumptions of metricity, etc.) this is the only /18/ symmetry property of $R_{\mu\nu\alpha}{}^{\beta}(\Gamma)$. If we demand metricity, we gain, in addition, antisymmetry in the last two indices. A simple proof follows.

Metricity implies that $\Gamma_{\mu\nu}^{\alpha}$ can be written (symbolically) as

$$\Gamma = \{\} - K , \qquad (4.2.4)$$

where K represents the contorsion tensor. Because of the non-tensorial nature of $\{\}$, one can choose a co-ordinate system where it vanishes (but not simultaneously $\partial\{\}$) and in such a system, the curvature tensor $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ can be split as

$$R_{\mu\nu\lambda}{}^{\kappa}(\Gamma) = R_{\mu\nu\lambda}{}^{\kappa}(\{\}) - (\partial_{\mu}K_{\nu\lambda}{}^{\kappa} - \partial_{\nu}K_{\mu\lambda}{}^{\kappa})$$
$$+ (K_{\mu\rho}{}^{\kappa}K_{\nu\lambda}{}^{\rho} - K_{\nu\rho}{}^{\kappa}K_{\mu\lambda}{}^{\rho}). \qquad (4.2.5)$$

Using the well known symmetry properties of $R_{\mu\nu\lambda}{}^{\kappa}(\{\})$ and $K_{\mu\nu}{}^{\lambda}$, it is now easy to see from this equation that $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ is indeed antisymmetric in its last two indices also if metricity is demanded. These two antisymmetry properties of the curvature tensor are sufficient to ensure that the Ricci tensor ($R_{\nu\lambda}(\Gamma) = R_{\mu\nu\lambda}{}^{\mu}(\Gamma)$) and the Ricci scalar ($R(\Gamma) = R_{\nu}{}^{\nu}(\Gamma)$) are the only essential contractions of $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$.

Now, we come to the important question of whether we can form a non-zero scalar using the pseudo-tensor density $\varepsilon^{\mu\nu\lambda\sigma}$. Recall that the scalar so constructed in the Einstein case vanished identically by virtue of the cyclicity property of the Riemann Christoffel tensor $R_{\mu\nu\lambda\sigma}(\{\})$. When torsion is present, no such relation holds and so the non-zero scalar density $\varepsilon^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}(\Gamma)$ is a perfectly good quantity which can contribute to the total action of an ECSK-type theory. Indeed, if the requirements which determine the choice of our action for a metric-torsion theory are just the requirements of general co-ordinate invariance and linearity in the curvature tensor, then the general structure of this theory requires the presence of this term and, therefore, predicts parity violating interactions mediated by torsion which might be expected to show up as deviations from the predictions of the general theory of relativity at the microscopic level.

The new Lagrangian density may now be written as /F6/

$$L_G = \frac{1}{16\pi G_N} \sqrt{-g}\ R(\Gamma) + \frac{1}{16\pi G_P} \varepsilon^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}(\Gamma) \ , \qquad (4.2.6)$$

where G_N is the Newtonian coupling constant and G_P is the analagous quantity which governs the strength of the parity non-conserving interactions present in L_G .

In the next section we simplify the form of this expression and compare and contrast this action with the one used in the ECSK theory. There, we will find that when we have removed some total divergences, the Lagrangian contains, apart from the simple Einstein expression, terms quadratic in the contorsion tensor $K_{\mu\nu}{}^{\lambda}$. Since $K_{\mu\nu}{}^{\lambda}$ is a tensor one might consider the most general quadratic expression in the contorsion fields as forming the Lagrangian for torsion. This is also discussed in some detail in the next section.

4.3 OTHER LAGRANGIANS

Consider the following Lagrangian densities

$$L_{ECSK} \sim \sqrt{-g}\; R(\Gamma) \tag{4.3.1}$$

and

$$L_A \sim \varepsilon^{\mu\nu\lambda\sigma}\; R_{\mu\nu\lambda\sigma}(\Gamma). \tag{4.3.2}$$

Recall that the components of the non-symmetric connection $\Gamma_{\mu\nu}{}^{\lambda}$ can, upon imposition of metricity be written as

$$\Gamma_{\mu\nu}{}^{\lambda} = \{{}^{\lambda}_{\mu\nu}\} - K_{\mu\nu}{}^{\lambda} \tag{4.3.3}$$

where

$$\{{}^{\lambda}_{\mu\nu}\} = \tfrac{1}{2}\, g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \;, \tag{4.3.4}$$

$$K_{\mu\nu}{}^{\lambda} = -S_{\mu\nu}{}^{\lambda} + S_{\nu}{}^{\lambda}{}_{\mu} - S^{\lambda}{}_{\mu\nu}$$

$$= -K_{\mu}{}^{\lambda}{}_{\nu}\;, \tag{4.3.5}$$

and

$$S_{\mu\nu}{}^{\lambda} = \tfrac{1}{2}\,(\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda})\;, \tag{4.3.6}$$

is the torsion. Note that for metric-torsion theories the position of the indices is important and we work with the usual convention that in all covariant derivatives the first of the lower indices on the connections is the differentiating index.

In order to obtain the field equations we must choose an appropriate set of independent fields for variational purposes. Because of metricity,

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma_{\lambda\mu}{}^{\tau} g_{\tau\nu} - \Gamma_{\lambda\nu}{}^{\tau} g_{\mu\tau} = 0 \tag{4.3.7}$$

we have, as has been shown in section two of chapter two, apart from the ten $g_{\mu\nu}$'s another 24 independent components in $\Gamma_{\mu\nu}{}^{\lambda}$. For our present purposes, we shall take these to be the 24 components of the contorsion tensor $K_{\mu\nu}{}^{\lambda}$. Nevertheless, the field equations one obtains by varying with the contorsion can easily be related to those one would obtain by varying with the torsion using the relationship between $K_{\mu\nu}^{\cdot\cdot\lambda}$ and $S_{\mu\nu}^{\cdot\cdot\lambda}$ given in (4.3.5) above.

Before plunging ourselves into variation of the Lagrangians written above, it is advisable to first obtain their simplest form by discarding total divergences and by using the symmetry properties of $g_{\mu\nu}$ $K_{\mu\nu}^{\ \ \lambda}$, etc. This procedure yields the following simple expression for L_{ECSK} :

$$L_{ECSK} \sim \sqrt{-g}\,\{R(\{\}) + g^{\mu\lambda}(K_{\nu\sigma}^{\ \ \nu} K_{\mu\lambda}^{\ \ \sigma} - K_{\mu\sigma}^{\ \ \nu} K_{\nu\lambda}^{\ \ \sigma})\}$$

$$+ \text{(Total divergence)}; \tag{4.3.8}$$

which may be written as

$$L_{ECSK} \sim L_E + L_C + \text{(Total divergence)} ; \tag{4.3.9}$$

where L_E denotes the usual Einstein-Hilbert expression and L_C denotes the terms quadratic in $K_{\mu\nu}^{\cdot\cdot\lambda}$.

Similarly, we obtain for L_A the result:

$$L_A \sim \varepsilon^{\mu\nu\lambda\beta}\, g_{\beta\kappa}\, K_{\mu\sigma}^{\ \ \kappa}\, K_{\nu\lambda}^{\ \ \sigma} + \text{(Total divergence)}. \tag{4.3.10}$$

We see that the contorsion terms enter both Lagrangians quadratically and that no derivatives of the contorsion fields appear once some total divergences are removed. This is a general consequence of restriction to theories linear in the curvature and is, therefore, unchanged even with the addition of L_A. Stated more dramatically, this implies that if we use a linear combination of L_A and L_{ECSK} as the Lagrangian density of our system, then we will not be able to obtain propagating torsion.

The interesting thing to note is, however, that the effective contribution of contorsion to L_{ECSK} is a particular linear combination of two of the three possible scalars quadratic in the contorsion tensor (contracting with $g_{\mu\nu}$), the third being $K_{\nu\sigma}{}^{\lambda} K^{\nu\sigma}{}_{\lambda}$. One may, at this point, argue that an equally valid approach to determine an action for the torsion would be to consider all possible linear combinations of quadratics in the contorsion fields and simply add these to L_E. Such an approach would, however, necessitate the introduction of at least three other arbitrary parameters into the theory apart from the Newtonian coupling constant.

As regards L_A one can also think of three other scalars quadratic in contorsion (contracting with $\varepsilon^{\mu\nu\lambda\sigma}$ and $g_{\mu\nu}$) apart from the one selected by L_A, namely $\eta^{\mu\nu\lambda\sigma} K_{\alpha}{}^{\alpha}{}_{\mu} K_{\nu\lambda\sigma}$, $\eta^{\mu\nu\lambda\sigma} K_{\alpha\mu\nu} K_{\lambda\sigma}{}^{\alpha}$, and $\eta^{\mu\nu\lambda\sigma} K_{\alpha\mu\nu} K^{\alpha}{}_{\lambda\sigma}$. Thus, the most general such Lagrangian density for torsion would contain seven contractions all with different and arbitrary coefficients. In view of this it seems much simpler, and indeed more natural, to restrict oneself to Lagrangians obtained directly by contracting $R_{\mu\nu\lambda\sigma}(\Gamma)$ in all possible ways to form a scalar.

Having simplified the form of the expressions (4.3.1) and (4.3.2) we now go on to consider the matter-free theory.

4.4 NO MATTER, NO TORSION

Let us first consider the contorsion field equations for both the ordinary and the generalised ECSK action in the absence of matter. We have already shown that L_{ECSK} can be decomposed as in (4.3.8).

Taking $g_{\mu\nu}$ and $K_{\lambda\sigma}{}^{\tau}$ as our independent variables, the field equations obtained by the $K_{\lambda\sigma}{}^{\tau}$ variation are,

$$K^{\lambda\mu\nu} + K^{\nu\lambda\mu} - K_{\alpha}{}^{\alpha\nu}\, g^{\mu\lambda} - K_{\alpha}{}^{\lambda\alpha}\, g^{\mu\nu} = 0 \qquad (4.4.1)$$

These are 24 equations because of the antisymmetry property of $K_{\mu\nu}{}^{\lambda}$. Contracting μ and λ (or ν and λ) gives

$$K_{\nu}{}^{\nu}{}_{\beta} = 0 \ . \qquad (4.4.2)$$

Using this in (4.4.1) we obtain

$$K^{\lambda\mu\nu} + K^{\nu\lambda\mu} = 0 \ . \qquad (4.4.3)$$

By cyclically permuting this equation we get the two equations,

$$K^{\mu\nu\lambda} + K^{\lambda\mu\nu} = 0 \qquad (4.4.4)$$

and

$$K^{\nu\lambda\mu} + K^{\mu\nu\lambda} = 0 \ . \qquad (4.4.5)$$

Adding (4.4.3), (4.4.4) and (4.4.5) and using the last equation to simplify the sum, one can easily verify that

$$K^{\lambda\mu\nu} = 0 \ .$$

The same calculation can be repeated for the theory based on L_G. The analogue of equation (4.4.1) now reads,

$$K^{\lambda\mu\nu} + K^{\nu\lambda\mu} - K_{\alpha}{}^{\alpha\nu}\, g^{\mu\lambda} - K_{\alpha}{}^{\lambda\alpha}\, g^{\mu\nu} -$$
$$-2a(\eta^{\mu\nu\sigma\rho}\, K_{\sigma}{}^{\lambda}{}_{\rho} + \eta^{\mu\lambda\sigma\rho}\, K_{\sigma\rho}{}^{\nu}) = 0 \qquad (4.4.6)$$

After a certain amount of tedious algebra and index manipulation, one can again explicitly verify the result that torsion vanishes in the absence of matter. These results follow in fact from quite general considerations as outlined below.

If one has a Lagrangian which involves the contorsion fields in a non-dynamic manner (no second derivatives of $K_{\mu\nu}{}^{\lambda}$, or equivalently, terms quadratic in the derivatives of $K_{\mu\nu}{}^{\lambda}$), then stationarity under variations in the contorsion fields will give rise to an algebraic equation for these fields which can, in principle, be solved to yield an expression for $K_{\mu\nu}{}^{\lambda}$. The solution of this equation must, therefore, be expressible in terms of the other quantities that are present in the theory. In our case we have at our disposal only the objects,

$$g_{\mu\nu}\ ,\ \varepsilon_{\mu\nu\alpha\beta}\ ,\ g_{\mu\nu,\alpha}\ ,\ \text{and}\ \ g_{\mu\nu,\alpha\beta}$$

out of which we must be able to construct a three index tensor if torsion is not to vanish identically.

Now, it is immediately clear that since the process of contraction always removes two indices every time a contraction is made, that no such object can be formed from the quantities

$$g_{\mu\nu}\ ,\ g_{\mu\nu,\alpha\beta},\ \text{and}\ \varepsilon_{\mu\nu\alpha\beta}$$

only. Thus the first derivatives of the metric, $g_{\mu\nu,\alpha}$ must enter each term of the expression for the contorsion field. However, we can always choose a co-ordinate system in which $g_{\mu\nu,\alpha}$ vanishes since the partial derivative of the metric is not a tensor. Thus $K_{\mu\nu}{}^{\lambda}$ will vanish in this co-ordinate system and, by virtue of its tensorial character, in all co-ordinate systems. It should be noted, of course, that we are not at liberty to use the covariant derivative of the metric, $g_{\mu\nu;\alpha}$, since this vanishes because of metricity.

It follows, therefore, that both the (matter-free) theories are identical to Einstein's general theory of relativity. So long as torsion is algebraic, this identity between the two matter-free theories and the theory of Einstein will remain.

However, as remarked in the introduction, it is reasonable to expect torsion to be non-zero even in the absence of matter to represent the spin effects of torsion. In the next section we go on to consider possible ways of implementing these ideas.

4.5 DYNAMIC TORSION AND FIELD EQUATIONS

We now wish to consider possible ways of incorporating dynamic torsion into the matter-free theory. One approach is to work with Lagrangians quadratic in the curvature tensor. However these lead to rather cumbersome higher than second order differential field equations and in any case such Lagrangians give rise to non-positive-definite Hamiltonians even at the classical level. Another approach consists essentially in adding to L_{ECSK} a scalar density quadratic in the covariant derivatives of the contorsion fields. However, the most general such scalar density, L_K , would contain an enormous /F7/ number of independent terms (see Appendix A for its explicit form) involving an equally large number of arbitrary parameters and would be quite useless unless one is able to eliminate most of these terms on some physical grounds - and this seems unlikely. So how else can one modify the theory in order to obtain dynamic torsion?

Recall that torsion vanished in the absence of matter by virtue of the field equations essentially because of the non-existence in the theory of an odd-index object using which we could construct a three index tensor. Since torsion itself is represented by a three index tensor, the simplest possibility for having non-zero torsion is to allow for a new one index quantity in the theory in terms of which $K_{\mu\nu}{}^{\lambda}$ can be expressed. Coupled with the requirement that this new field be dynamical we are led to examine the following form /F8/ for the contorsion:

$$K_{\alpha\beta\gamma} = \Phi_\beta\, g_{\alpha\gamma} - \Phi_\gamma\, g_{\alpha\beta} \qquad (4.5.1)$$

where $\Phi_\alpha = \Phi_{,\alpha}$.

In the rest of this section we shall restrict ourselves to such a form of contorsion. We may now proceed in two different ways. One is to simply substitute the motivated form of contorsion into L_G, eliminate

$K_{\mu\nu}{}^{\lambda}$, and obtain the field equations for $g_{\mu\nu}$ and Φ by variation. We prefer to avoid this approach and consider it more appropriate to treat (4.5.1) as a constraint which will be implemented by introducing an appropriate set of Lagrange multiplier fields into L_G.

Let us, therefore, consider the following Lagrangian density,

$$L = \sqrt{-g}\,\{\tilde{R} + g^{\mu\lambda}\,(K_{\nu\sigma}{}^{\nu}\,K_{\mu\lambda}{}^{\sigma} - K_{\mu\sigma}{}^{\nu}\,K_{\nu\lambda}{}^{\sigma}) + $$
$$+ 2a\,\varepsilon^{\mu\nu\lambda\beta}\,g_{\beta\kappa}\,K_{\mu\sigma}{}^{\kappa}\,K_{\nu\lambda}{}^{\sigma}\} +$$
$$+ \Lambda^{\alpha\beta}{}_{\gamma}\,(K_{\alpha\beta}{}^{\gamma} - \Phi_{\beta}\,\delta^{\gamma}_{\alpha} + \Phi_{\lambda}\,g^{\lambda\gamma}\,g_{\alpha\beta}) \; ; \qquad (4.5.2)$$

where the $\Lambda^{\alpha\beta}{}_{\gamma}$ are the Lagrange multipliers introduced to ensure satisfaction of (4.5.1) and where we introduce the notation that a tilde on any quantity indicates that it is to be constructed from the Christoffel connection, thus, for example,

$$\tilde{R} = R(\{\}) \quad , \quad R = R(\Gamma), \text{ etc.}$$

Variations with respect to $g_{\mu\nu}$, $K_{\alpha\beta}{}^{\gamma}$ and Φ yield the following equations:

$$\delta g \quad : \quad \sqrt{-g}\,(\tilde{R}^{ab} - \tfrac{1}{2}\tilde{R}\,g^{ab}) +$$
$$+\tfrac{1}{2}\sqrt{-g}\,(g^{ab}\,g^{\mu\lambda} - g^{\mu a}\,g^{\lambda b} - g^{\mu b}\,g^{\lambda a})(K_{\mu\sigma}{}^{\nu}\,K_{\nu\lambda}{}^{\sigma} - K_{\nu\sigma}{}^{\nu}\,K_{\mu\lambda}{}^{\sigma}) -$$
$$-\tfrac{1}{2}\,(\Lambda^{\alpha}{}_{\alpha}{}^{a}\,\Phi^{b} + \Lambda^{\alpha}{}_{\alpha}{}^{b}\,\Phi^{a}) + \tfrac{1}{2}\,(\Lambda^{ab\mu} + \Lambda^{ba\mu})\,\Phi_{\mu} +$$
$$+a\,(\varepsilon^{\mu\nu\lambda a}\,K_{\mu\sigma}{}^{b}\,K_{\nu\lambda}{}^{\sigma} + \varepsilon^{\mu\nu\lambda b}\,K_{\mu\sigma}{}^{a}\,K_{\nu\lambda}{}^{\sigma}) = 0 \; ; \qquad (4.5.3)$$

$$\delta K \quad : \quad \Lambda^{abc} - \sqrt{-g}\,\{(K^{cab} + K^{bca} - K_{\lambda}{}^{\lambda b}\,g^{ac} - K_{\lambda}{}^{c\lambda}\,g^{ab}) +$$
$$+ 2a\,(\eta^{ab\nu\lambda}\,K_{\nu}{}^{c}{}_{\lambda} + \eta^{ac\nu\lambda}\,K_{\nu\lambda}{}^{b})\} = 0; \qquad (4.5.4)$$

and

$$\delta\Phi \quad : \quad (\Lambda^{\alpha\beta}{}_{\alpha} - \Lambda^{\alpha}{}_{\alpha}{}^{\beta})_{,\beta} = 0 \; ; \qquad (4.5.5)$$

while the Λ variation yields the desired constraint.

Eliminating K and Λ from the above equations one obtains,

$$(\sqrt{-g} \ \Phi^{\beta})_{,\beta} = 0 \tag{4.5.6}$$

and

$$\tilde{R}^{ab} - \tfrac{1}{2}g^{ab} \tilde{R} = -6 \ (\Phi^{a}\Phi^{b} - \tfrac{1}{2}g^{ab}\Phi_{\lambda}\Phi^{\lambda}) \ . \tag{4.5.7}$$

Note that no parity violating term now remains. This is due to the special form we have taken for the contorsion and implies that the vacuum theory is parity conserving.

4.6 COUPLING TO MATTER FIELDS

In this section we wish to give an example where our Lagrangian predicts parity violating effects but where the ECSK Lagrangian does not. It is a little unfortunate that there are not many matter fields one can study at the Lagrangian level. Indeed when studying matter fields on a Riemann-Cartan space-time one is further restricted, for example, by the fact that one cannot couple gauge fields to torsion /F9/ in a gauge invariant manner, so that the study of gauge fields on a Riemann-Cartan space-time does not lead to any new physics than on a Riemannian space-time. We cannot use the Dirac field for our present purposes as the ECSK theory already predicts a parity violating effect for this field and the distinctive effects of our theory would be only blurred by this.

So we are left with the Proca (massive vector) field, which due to its non-zero mass does not present problems of gauge (non-) invariance when coupled to torsion. We take the usual Lagrangian for the Proca field:

$$L_m = \sqrt{-g}\ (-\frac{1}{4}\ G_{\mu\nu}G^{\mu\nu} - \tfrac{1}{2}m^2\ A_\mu A^\mu) \qquad (4.6.1)$$

with the field strength tensor $G_{\mu\nu}$ given by (∇_μ denotes the full space-time covariant derivative),

$$\begin{aligned} G_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - 2\ A_\sigma\ S_{\mu\nu}{}^\sigma\ , \end{aligned}$$

where it should be noted that only the antisymmetric part of the full connection enters $G_{\mu\nu}$. This can be written in terms of the Christoffel covariant derivative $\tilde{\nabla}$ as follows:

$$G_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu - 2\ A_\sigma\ S_{\mu\nu}{}^\sigma\ , \qquad (4.6.2)$$

where the antisymmetry of $G_{\mu\nu}$ removes the Christoffel contribution to $G_{\mu\nu}$.

Let us now define

$$B_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu \tag{4.6.3}$$

then

$$G_{\mu\nu} = B_{\mu\nu} - 2 A_\sigma S_{\mu\nu}{}^\sigma , \tag{4.6.4}$$

and L_m can, therefore, be written as,

$$L_m = \sqrt{-g} \, (- \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} + B^{\alpha\beta} S_{\alpha\beta}{}^\sigma A_\sigma -$$

$$- g^{\mu\alpha} g^{\nu\beta} S_{\alpha\beta}{}^\sigma S_{\mu\nu}{}^\rho A_\sigma A_\rho - \tfrac{1}{2} m^2 A_\mu A^\mu) \; . \tag{4.6.5}$$

Now the spin-angular momentum tensor of matter is defined by

$$\sqrt{-g} \, \tau_k{}^{ji} = \frac{\delta L_m}{\delta K_{ij}{}^k} \tag{4.6.6}$$

where $\frac{\delta}{\delta K}$ denotes the variational derivative.

For the Lagrangian in (4.6.5) it is easy to show that

$$\tau_{kji} = G_{i[j} A_{k]} \; , \tag{4.6.7}$$

or

$$\tau_{kji} = A_k \tilde{\nabla}_{[j} A_{i]} - A_j \tilde{\nabla}_{[k} A_{i]} - 2 \, S_{i[j}{}^\sigma A_{k]} A_\sigma \; . \tag{4.6.8}$$

Let us now write for the total Lagrangian,

$$L = L_{ECSK} + L_A + L_m \; . \tag{4.6.9}$$

As L does not contain any derivatives of the torsion, the Euler-Lagrange equations obtained by variation of $K_{ij}{}^k$ are simply,

$$\frac{\partial L}{\partial K_{ij}{}^k} = 0 \; . \tag{4.6.10}$$

And since,

$$\sqrt{-g} \, \tau_k{}^{ji} = \frac{\partial L_m}{\partial K_{ij}{}^k} \; , \tag{4.6.11}$$

(4.6.10) gives,

$$\frac{\partial L_{ECSK}}{\partial K_{ij}{}^{k}} + \frac{\partial L_A}{\partial K_{ij}{}^{k}} = -\sqrt{-g}\,\tau_k{}^{ji} \ . \qquad (4.6.12)$$

Note also that

$$\frac{\partial L_{ECSK}}{\partial K_{ij}{}^{k}} = \sqrt{-g}\,T^{j}{}_{k}{}^{i} \quad , \qquad (4.6.13)$$

where T_{ijk} is the so-called modified torsion tensor /14/ , and is defined as follows,

$$T_{ijk} = S_{ijk} + g_{ik}\,S_{jl}{}^{l} - g_{jk}\,S_{il}{}^{l} \ . \qquad (4.6.14)$$

Writing

$$L_A = 2a\sqrt{-g}\,\eta^{\mu\nu\lambda\beta} g_{\beta\rho} K_{\mu\sigma}{}^{\rho}\,K_{\nu\lambda}{}^{\sigma} \quad , \qquad (4.6.15)$$

it is not difficult to show that

$$\frac{\partial L_A}{\partial K_{ij}{}^{k}} = 2a\sqrt{-g}\,(\eta^{i\nu\lambda}{}_{k}\,K_{\nu\lambda}{}^{j} + \eta^{i\nu\lambda j}\,K_{\nu k\lambda}) \ . \qquad (4.6.16)$$

Therefore equation (4.6.12) finally gives,

$$-\sqrt{-g}(T_k{}^{ji} - \tau_k{}^{ji} - 2a(\eta^{i\nu\lambda}{}_{k}\,K_{\nu\lambda}{}^{j} + \eta^{i\nu\lambda j}\,K_{\nu k\lambda})) = 0 \qquad (4.6.17)$$

or,

$$T_{kji} = \tau_{kji} + 2a\,\eta^{\nu\lambda}{}_{i\sigma}K_{\nu\lambda\rho}(\delta^{\sigma}_{k}\delta^{\rho}_{j} - \delta^{\sigma}_{j}\delta^{\rho}_{k}) \ . \qquad (4.6.18)$$

As the Proca field is just a massive Maxwell field, the field equations for this field can be written down immediately as,

$$\nabla_\rho G_\mu{}^{\rho} - m^2 A_\rho = 0 \quad , \qquad (4.6.19)$$

or,

$$\nabla_\rho B_\mu{}^{\rho} - 2\,\nabla_\rho\,(A_\sigma\,S_\mu{}^{\rho\sigma}) - m^2 A_\mu = 0 \ . \qquad (4.6.20)$$

In order to eliminate the non-Riemannian part of equation (4.6.20), we must first invert equation (4.6.18) for the torsion. Remembering the definitions of T_{ijk} and $K_{\nu\lambda\rho}$ as given in equations (4.6.14) and (4.3.5) respectively, we can write equation (4.6.18) as,

$$S_{kji} + g_{ki}\, S_{jl}{}^{l} - g_{ji}\, S_{kl}{}^{l} =$$

$$= \tau_{kji} + 2a\eta^{\nu\lambda}{}_{i\sigma}\delta^{\sigma\rho}_{kj}(S_{\lambda\rho\nu} - S_{\nu\lambda\rho} - S_{\rho\nu\lambda}) \; . \tag{4.6.21}$$

Now, because of the antisymmetry of $\eta^{\nu\lambda}{}_{i\sigma}$ in $\nu\lambda$,

$$\eta^{\nu\lambda}{}_{i\sigma}\, S_{\lambda\rho\nu} = -\,\eta^{\nu\lambda}{}_{i\sigma}\, S_{\nu\rho\lambda} \quad , \tag{4.6.22}$$

and antisymmetry of $S_{\nu\rho\lambda}$ in its first two indices further implies that,

$$\eta^{\nu\lambda}{}_{i\sigma}\, S_{\lambda\rho\nu} = \eta^{\nu\lambda}{}_{i\sigma}\, S_{\rho\nu\lambda} \quad . \tag{4.6.23}$$

Substituting this into equation (4.6.18) gives,

$$S_{kji} + g_{ki}\, S_{jl}{}^{l} - g_{ji}\, S_{kl}{}^{l} =$$

$$= \tau_{kji} - 2a\, \eta^{\nu\lambda}{}_{i\sigma}\, \delta^{\sigma\rho}_{kj}\, S_{\nu\lambda\rho} . \tag{4.6.24}$$

At this stage we note that by putting a equal to zero we can recover the result that the ECSK theory does not predict any parity violating effects when it is coupled to a massive vector field.

For the purposes of solving equation (4.6.18) for the torsion, we simlify equation (4.6.24) with the help of equation (4.6.14) to obtain

$$T_{kji} = \tau_{kji} + 2a\, \eta^{\nu\lambda}{}_{\sigma i}\, \delta^{\sigma\rho}_{kj}\, S_{\nu\lambda\rho} \quad . \tag{4.6.25}$$

Multiplying this by $\eta^{kj\alpha\beta}$ gives,

$$\eta^{kj\alpha\beta}\, T_{kji} = \tau_{kji}\, \eta^{kj\alpha\beta} + 2a\, \eta^{\nu\lambda\sigma}{}_{i}\, \eta^{kj\alpha\beta}\, \delta^{\sigma\rho}_{kj}\, S_{\nu\lambda\rho} \; . \tag{4.6.26}$$

Now,

$$2\,\eta^{kj\alpha\beta}\,\delta^{\sigma\rho}_{kj} = \eta^{\sigma\rho\alpha\beta} - \eta^{\rho\sigma\alpha\beta}$$

$$= 2\,\eta^{\sigma\rho\alpha\beta} \qquad (4.6.27)$$

and

$$2\,\eta^{\nu\lambda}{}_{\sigma i}\,\eta^{\sigma\rho\alpha\beta} = 2\,g_{i\mu}\,\eta_{\sigma}{}^{\nu\lambda\mu}\,\eta^{\sigma\rho\alpha\beta}$$

$$= -2\,g_{i\mu}\,\delta^{\nu\lambda\mu,\rho\alpha\beta} \qquad (4.6.28)$$

which can be written interms of the metric alone upon further use of the identities given in footnote five as follows,

$$= -2\,g_{i\mu}\{g^{\nu\rho}g^{\lambda\alpha}g^{\mu\beta} - g^{\nu\rho}g^{\lambda\beta}g^{\mu\alpha} +$$

$$+ g^{\nu\alpha}g^{\lambda\beta}g^{\mu\rho} - g^{\nu\alpha}g^{\lambda\rho}g^{\mu\beta} +$$

$$+ g^{\nu\beta}g^{\lambda\rho}g^{\mu\alpha} - g^{\nu\beta}g^{\lambda\alpha}g^{\mu\rho}\} \qquad (4.6.29)$$

Therefore, we finally find that

$$2\,\eta^{\nu\lambda}{}_{\sigma i}\,\eta^{kj\alpha\beta}\,\delta^{\sigma\rho}_{kj}\,S_{\nu\lambda\rho} = 2\,\eta^{\nu\lambda}{}_{\sigma i}\,\eta^{\sigma\rho\alpha\beta}\,S_{\nu\lambda\rho}$$

$$= -2\,g_{i\mu}\{S^{\rho\alpha}{}_{\rho}g^{\mu\beta} - S^{\rho\beta}{}_{\rho}g^{\mu\alpha} + S^{\alpha\beta\mu} -$$

$$- S^{\alpha\rho}{}_{\rho}g^{\mu\beta} + S^{\beta\rho}{}_{\rho}g^{\mu\alpha} - S^{\beta\alpha\mu}\}$$

$$= -2g_{i\mu}\{2(S^{\alpha\beta\mu} + S^{\beta}g^{\mu\alpha} - S^{\alpha}g^{\mu\beta})$$

$$= -4\,T^{\alpha\beta}{}_{i}\ . \qquad (4.6.30)$$

Substituting this result back into equation (4.6.26) gives

$$\eta^{kj\alpha\beta}\,T_{kji} + 4ag^{k\alpha}g^{j\beta}T_{kji} = \tau_{kji}\,\eta^{kj\alpha\beta}\ . \qquad (4.6.31)$$

Multiplying this now by $\eta_{\alpha\beta\rho\sigma}$ yields

$$\eta_{\alpha\beta\rho\sigma}\,\eta^{kj\alpha\beta}\,T_{kji} + 4a\,g^{k\alpha}\,g^{j\beta}\,\eta_{\alpha\beta\rho\sigma}\,T_{kji} =$$

$$= \tau_{kji}\,\eta^{kj\alpha\beta}\,\eta_{\alpha\beta\rho\sigma}\ , \qquad (4.6.32)$$

or,

$$2\,\tau_{kji}\,\delta^{kj}_{\rho\sigma} = 2T_{kji}\,\delta^{kj}_{\rho\sigma} - 4a\,\eta^{kj}{}_{\rho\sigma}\,T_{kji}\ , \tag{4.6.33}$$

which can be further simplified to,

$$\tau_{\rho\sigma i} = T_{\rho\sigma i} - 2a\,\eta^{kj}{}_{\rho\sigma}\,T_{kji}\ , \tag{4.6.34}$$

or,

$$T^{\alpha\beta}{}_{i} - 2a\,\eta^{kj\alpha\beta}\,T_{kji} = \tau^{\alpha\beta}{}_{i}\ . \tag{4.6.35}$$

From equation (4.6.31) we see that

$$2a\,\eta^{kj\alpha\beta}\,T_{kji} + 8\,a^2\,g^{k\alpha}\,g^{j\beta}\,T_{kji} =$$

$$= 2a\,\tau_{kji}\,\eta^{kj\alpha\beta}\ . \tag{4.6.36}$$

Substituting this into equation (4.6.35) we obtain

$$T^{\alpha\beta}{}_{i} + 8a^2\,T^{\alpha\beta}{}_{i} - 2a\,\tau_{kji}\,\eta^{kj\alpha\beta} = \tau^{\alpha\beta}{}_{i}\ , \tag{4.6.37}$$

or,

$$(1 + 8a^2)\,T_{kji} = \tau^{\alpha\beta}{}_{i}\,(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2a\,\eta^{kj\alpha\beta})\ . \tag{4.6.38}$$

We have therefore,

$$S_{kji} + g_{ki}\,S_{jl}{}^{l} - g_{ji}\,S_{kl}{}^{l} =$$

$$= \frac{\tau_{\alpha\beta i}}{(1+8a^2)}\,(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2a\,\eta^{kj\alpha\beta})\ . \tag{4.6.39}$$

Tracing over the i and j indices gives,

$$S_k = -\tfrac{1}{2}(1+8a^2)^{-1}\{\tau_k + 2a\,\tau_{\alpha\beta i}\,\eta_k{}^{i\alpha\beta}\} \tag{4.6.40}$$

Substitution of S_k into equation (4.6.39) finally gives the solution for torsion as

$$S_{kji} = (1+8a^2)^{-1}\{\tau_{\alpha\beta i}(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2a\,\eta^{\alpha\beta}{}_{kj}) -$$

$$-\tfrac{1}{2}g_{ij}(\tau_k + 2a\eta_k{}^{\gamma\alpha\beta}\tau_{\alpha\beta\gamma}) + \tfrac{1}{2}g_{ik}(\tau_j + 2a\eta_j{}^{\gamma\alpha\beta}\tau_{\alpha\beta\gamma})\}\ ,$$

$$\tag{4.6.41}$$

where we have used the abbreviations,

$$A_j = A_{jl}{}^{l} \qquad \text{and} \qquad A^j = g^{jl} A_l ,$$

where A is some arbitrary tensor, for the traces of the torsion and spin-angular momentum tensors .

It is clear that upon substitution of the expression for torsion given in equation (4.6.41) into equation (4.6.20) we shall indeed have parity-violating interaction terms, which would not be present in the usual ECSK theory — thus demonstrating that the new Lagrangian proposed in the present work predicts new parity violating effects.

In the next chapter we go on to consider the problem of coupling torsion to gauge fields and leave a discussion of the results of the work of this and the following chapter to chapter six.

CHAPTER FIVE

COUPLING TORSION TO GAUGE FIELDS

5.1 *MOTIVATION*

The motivation for introducing torsion into Einstein's theory of general relativity is well known and has been discussed briefly in the Introduction and in the last chapter. In a sentence, torsion is argued to arise naturally if we are to incorporate the spin-angular momentum of matter into a theory of gravitation, or in other words if we are to understand spin in a geometrical fashion analogous to the understanding which Einstein's theory provides for mass.

An immediate consequence of adopting the above interpretation is that all spinning matter is required to both generate and react to torsion. However, and this fact is not so strongly emphasised in the literature as it deserves, it is not possible to consistently couple torsion to gauge fields in the conventional manner using minimal coupling (as applied to the coupling of gravity to matter) in such a way as to retain gauge invariance (what is meant precisely by this will become clear in the next section).

The general attitude to this problem /14/ has been to simply abandon the notion of a coupling between torsion and all gauge fields by using only the Christoffel part of the full asymmetric torsion containing connection in the space-time covariant derivatives which are needed when coupling gravity to gauge fields. For coupling gravity to other spinning matter, though, the full connection may be employed. This prescription leads, of course, to several amusing situations. For example, when coupling to a photon one must use only the Christoffel connection, but if coupling to a massive vector boson is desired one should use the full connection.

The argument generally put forward for adopting this procedure being that when faced by the choice between either gauge invariance or torsion-gauge field coupling, then the natural thing to do is to opt for the more fundamentally justifiable of the two — namely gauge invariance — and forget about coupling torsion to any gauge fields. The fact that there is no obvious way of getting round this problem has indeed led to the adoption of the view that gauge invariance infact forbids the coupling of torsion to gauge fields just as gauge invariance forbids the appearance of a mass term for gauge fields.

Such a solution is, however, unappealing on two counts. Firstly, because of the different way in which it treats the coupling of metric-torsion theories to matter of the gauge variety and other non-gauge matter. Secondly, and more importantly, because such a procedure runs counter to the basic reasoning which goes into incorporating torsion into a gravitational framework in the first place.

The first attempt at resolving this problem in a satisfactory manner was made by Hojman, Rosenbaum, Ryan and Shepley /19/. These authors showed that by using a slight generalisation of the standard minimal coupling procedure for the coupling of electrically charged fields to the electromagnetic potential, it is infact possible to couple a simple, though non-trivial type of dynamic torsion to the photon. An extremely interesting feature of this approach is that not only does it resolve the problem of coupling torsion to gauge fields in a gauge invariant manner but it also provides restrictions on the type of torsion that can infact couple to gauge fields.

With the quantitative experimental successes of the Salam-Weinberg theory of electroweak interactions on the one hand and the very encouraging, though as yet only qualitative successes of quantum chromodynamics on the other, it is now widely believed that non-Abelian gauge fields have as central a role to play in any successful and complete description of nature as the one experimentally verified abelian gauge field — namely the photon.

It is natural, therefore, to enquire whether it is possible to extend the work of Hojman, Rosenbaum, Ryan and Shepley to encompass a self-consistent coupling between all (Abelian and non-Abelian) gauge fields and some form of torsion /20/ . Clearly if the methods of Hojman et. al. do not allow any successful generalisation which allows us to achieve this, we shall have to face the same problem as before, only now for non-Abelian gauge fields. It is this question to which we shall address ourselves in the present chapter.

It will be shown that it is possible to answer this question in the affirmative provided that we use, apart from the generalisation of the usual minimal coupling procedure introduced by Hojman et. al., a modified form of the Yang-Mills-Shaw field strength for non-Abelian gauge fields.

Apart from the introduction of a new scalar field, named the tlaplon in ref. 19, which acts as a potential for the torsion, the two modifications lead to a particularly interesting consequence in that they require the replacement of all gauge coupling constants of nature everywhere by an effective space-time dependent coupling:

$$g \rightarrow g/f(x) \;;$$

where the space-time dependence is given by the function $f(x)$ which is determined by the same scalar field the derivatives of which also determine the type of torsion that this procedure allows to couple to all gauge fields.

Interestingly enough the form of torsion that is found for both the Abelian and the non-Abelian theories is the same and has the form that we motivated in the last chapter as the vacuum torsion that ought to exist even in the absence of matter to represent the torsional effects expected to be present because of the spin-2 nature of gravitation.

In conclusion, therefore, the work here carried out allows us to present a completely consistent way of coupling all gauge fields to a metric-torsion theory of gravitation containing a specified type of dynamic torsion provided that we are willing to accept that all gauge couplings must appear as being space-time dependent in a manner dictated by torsion.

5.2 *THE PROBLEM*

To begin, let us recall some essentials of gauge theories and the usual procedure for coupling gravity to matter. For this purpose let us consider a Lagrangian density of the form /F10/

$$L_\psi = L(\psi , \partial_\mu \psi) \qquad (5.2.1)$$

where the ψ_i are a set of complex scalar fields which transform according to some representation Θ of an internal space symmetry group, G . Now suppose that this Lagrangian density is invariant under a set of global gauge transformations,

$$\psi \rightarrow \psi' = e^{-i\Theta\cdot\Lambda} \psi \qquad (5.2.2)$$

$$\Theta\cdot\Lambda = \Theta_i \Lambda^i , \qquad (5.2.3)$$

where the Λ's are a set of arbitrary, but space-time independent, real parameters. Consistent with the physical requirement that all dynamical laws must be local in nature, we now demand invariance under the larger group of transformations given in (5.2.2) but where the Λ's are now arbitrary, but well behaved, functions of space-time.

In order to retain invariance under these transformations, L_ψ has to be modified, as already shown in chapter three, in such a way as to account for the misbehaviour of $\partial_\mu \psi$ under these new local gauge transformations. The standard procedure, known as minimal coupling, is to replace all partial derivatives ∂_μ which appear in the Lagrangian by gauge covariant derivatives D_μ which is defined in such a way as to ensure that the covariant derivative of ψ transform in the same manner as the fields ψ themselves, i.e.

$$D'_\mu \psi' = e^{-i\Theta\cdot\Lambda} D_\mu \psi . \qquad (5.2.4)$$

This is achieved by introducing in D_μ a field (or a set of fields) —

the gauge field(s) —— which compensates, or corrects, for the ill behaviour of the partial derivatives under the local transformations. Carrying out the standard algebra of chapter three, one finds that if the covariant derivative is required to be linear in the gauge connection A_μ and has the following form,

$$D_\mu = \partial_\mu - ig\, A_\mu \cdot \Theta \tag{5.2.5}$$

then the compensating set of gauge potentials must transform in the following manner in order to satisfy (5.2.4) :

$$A^i_\mu \rightarrow A'^i_\mu = A^i_\mu - \frac{1}{g}\Lambda^i{}_{,\mu} - C^i{}_{jk}\, A^j_\mu\, \Lambda^k \, . \tag{5.2.6}$$

This procedure then ensures that the Lagrangian density

$$L_\psi(\psi \, , \, D_\mu\psi)$$

obtained from $L_\psi(\psi \, , \, \partial_\mu\psi)$ by replacing ∂_μ everywhere by D_μ, is now invariant under the local gauge transformations. Of course, one must supplement L_ψ with a part that describes the dynamics of these gauge fields. This is constructed using the Yang-Mills-Shaw field strengths $F^i_{\mu\nu}$, where

$$F^i_{\mu\nu} = A^i_{\nu,\mu} - A^i_{\mu,\nu} + g\, C^i{}_{jk}\, A^j_\mu\, A^k_\nu \quad , \tag{5.2.7}$$

and where the $C^i{}_{jk}$ are the structure constants of the group G, and the $F^i_{\mu\nu}$ transform gauge covariantly under the gauge transformations (see chapter three for the details). The Lagrangian density for the complete system is then,

$$L_{\psi,A} = L_\psi(\psi \, , \, D_\mu\psi) - \frac{1}{4}\, F_{\mu\nu}\cdot F^{\mu\nu} \quad . \tag{5.2.8}$$

All this is, of course, in flat space. We consider next the usual formalism for coupling gravity (without torsion) to this system. This proceeds in two steps. First, one replaces all /F11/ $\eta_{\mu\nu}$'s in the flat space Lagrangian density by a general space-time metric $g_{\mu\nu}(x)$. Second, one replaces all partial derivatives by the space-time covariant derivatives defined in an analogous manner to the gauge covariant derivative /F12/ :

$$(\text{S.R.}) \xrightarrow[\partial_\mu \;\to\; \tilde{\nabla}_\mu]{\eta_{\mu\nu} \;\to\; g_{\mu\nu}} (\text{G.R.}) \; . \qquad (5.2.9)$$

For a scalar

$$\tilde{\nabla}_\mu \Phi \;=\; \partial_\mu \Phi \quad , \qquad (5.2.10)$$

while for a vector field A_ν we have that

$$\tilde{\nabla}_\mu A_\nu \;=\; \partial_\mu A_\nu - \{{}^{\sigma}_{\mu\nu}\} A_\sigma \quad , \qquad (5.2.11)$$

where $\{{}^{\sigma}_{\mu\nu}\}$ are the components of the Christoffel connection defined already in (4.3.4).

It should be stressed again that the above procedure entails the replacement of all partial derivatives by the space-time covariant derivatives and that this procedure is guaranteed (or, rather so constructed as) to take full care of the space-time invariance properties which we wish to maintain. The gauge invariance properties of the flat space Lagrangian are quite separate and unconnected to this process.

The question then is: Does this procedure for constructing a general relativistic Lagrangian from a given special relativistic one leave any gauge invariance of the original flat space Lagrangian undisturbed? It will be shown presently that for the Einstein theory which employs the Christoffel connection in all space-time covariant derivatives the answer is in the affirmative. It should not be concluded though that this renders the earlier observations trivial, since, as we shall see later, when torsion is present in the space-time connection and the full torsion containing connection is used in the space-time covariant derivatives, the gauge invariance of the flat space Lagrangian is in fact lost and one has to search for ways of retaining it if the coupling of torsion to gauge fields is still desired.

Let us now make these points clearer by considering the coupling of gravity to the Lagrangian system (5.2.8). First, it should be noted that for that part of the complete Lagrangian which only involves the scalar fields nothing is changed as far as the gauge invariance properties of this part are cocerned by carrying out the procedure of equations (5.2.9) - (5.2.11) since the space-time covariant derivative for a scalar field is just the ordinary derivative and gravity couples to the scalar fields only inasmuch as the space-time dependent metric must be used to contract all indices instead of the flat space metric.

As far as the gauge part is concerned, such a procedure respects the local gauge invariance properties so long as no torsion is present, and the form of $F_{\mu\nu}$ given in (5.2.7) remains gauge covariant when the partial derivatives are replaced by the covariant derivative given in (5.2.11). It is easy to see how this comes about.

Under (5.2.11),

$$\begin{aligned} F^i_{\mu\nu} \rightarrow F'^i_{\mu\nu} &= \tilde{\nabla}_\mu A^i_\nu - \tilde{\nabla}_\nu A^i_\mu + g\, C^i{}_{jk}\, A^j_\mu\, A^k_\nu \\ &= F^i_{\mu\nu} - \{{}^{\sigma}_{\mu\nu}\}\, A^i_\sigma + \{{}^{\sigma}_{\nu\mu}\}\, A^i_\sigma \\ &= F^i_{\mu\nu} \quad , \end{aligned} \tag{5.2.12}$$

due to the symmetry in the lower two indices of the Christoffel symbols $\{{}^{\sigma}_{\mu\nu}\}$. So that if $F^i_{\mu\nu}$ is gauge covariant, then so is $F'^i_{\mu\nu}$.

In the presence of torsion, however, the covariant derivative for a vector field becomes,

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}{}^{\sigma} A_\sigma \quad , \tag{5.2.13}$$

where the $\Gamma_{\mu\nu}{}^{\sigma}$'s are the components of the full non-symmetric connection,

$$\Gamma_{\mu\alpha}{}^{\sigma} = \{{}^{\sigma}_{\mu\alpha}\} - (S_{\alpha}{}^{\sigma}{}_{\mu} - S_{\mu\alpha}{}^{\sigma} - S^{\sigma}{}_{\mu\alpha}) \qquad (5.2.14)$$

where $S_{\mu\nu}{}^{\lambda}$, the torsion, is the antisymmetric part of the connection.

It is worth pointing out, however, that in theories containing an asymmetric contribution to the connection one has the possibility of defining two types of space-time covariant derivatives - each giving rise to a generally co-ordinate invariant theory. This is due to the fact that one can use either only the Christoffel part of the full connection — since this part on its own possesses all the properties required of a connection — or one can employ the full connection containing contributions from the torsion. The two choices lead, of course, to different physics, for example, one leads to problems when attempting a gauge invariant coupling between torsion and gauge fields while the other does not. No problems arise if one uses the Christoffel connection. The interesting case, which we shall now examine, is, therefore, the case when we use the full connection. For this we have that

$$\begin{aligned} F'^{i}_{\mu\nu} &= \nabla_{\mu}A^{i}_{\nu} - \nabla_{\nu}A^{i}_{\mu} + g\, C^{i}{}_{jk}\, A^{j}_{\mu}\, A^{k}_{\nu} \\ &= F^{i}_{\mu\nu} - \Gamma_{\mu\nu}{}^{\sigma}\, A^{i}_{\sigma} + \Gamma_{\nu\mu}{}^{\sigma}\, A^{i}_{\sigma} \qquad (5.2.15) \end{aligned}$$

Once again the contribution from the Christoffel part of $\Gamma_{\mu\nu}{}^{\sigma}$ will cancel out to yield

$$F'^{i}_{\mu\nu} = F^{i}_{\mu\nu} + K_{\mu\nu}{}^{\sigma}\, A^{i}_{\sigma} - K_{\nu\mu}{}^{\sigma}\, A^{i}_{\sigma}\,. \qquad (5.2.16)$$

Substituting for $K_{\mu\nu}{}^{\sigma}$, the contorsion tensor, we obtain,

$$\begin{aligned} F'^{i}_{\mu\nu} = F^{i}_{\mu\nu} &+ S_{\nu}{}^{\sigma}{}_{\mu}A^{i}_{\sigma} - S_{\mu\nu}{}^{\sigma}A^{i}_{\sigma} - S^{\sigma}{}_{\mu\nu}A^{i}_{\sigma} \\ &- S_{\mu}{}^{\sigma}{}_{\nu}A^{i}_{\sigma} + S_{\nu\mu}{}^{\sigma}A^{i}_{\sigma} + S^{\sigma}{}_{\nu\mu}A^{i}_{\sigma}\,, \qquad (5.2.17) \end{aligned}$$

which, using the antisymmetry of $S_{\mu\nu}{}^{\sigma}$ in its first two indices finally gives

$$F'^{i}_{\mu\nu} = F^{i}_{\mu\nu} - 2\, A^{i}_{\sigma}\, S_{\mu\nu}{}^{\sigma} . \qquad (5.2.18)$$

For the purposes of not having to write down factors of 2 everywhere, let us define another tensor for torsion by

$$T^{\sigma}{}_{\mu\nu} = -2\, S_{\mu\nu}{}^{\sigma} . \qquad (5.2.19)$$

We shall now use this $T^{\sigma}{}_{\mu\nu}$ throughout instead of $S_{\mu\nu}{}^{\sigma}$. In terms of this the expression for $F^{i}_{\mu\nu}$ becomes,

$$F'^{i}_{\mu\nu} = F^{i}_{\mu\nu} + A^{i}_{\sigma}\, T^{\sigma}{}_{\mu\nu} . \qquad (5.2.20)$$

For non-zero torsion, the presence of the last term in this expression ruins the gauge covariance of $F'^{i}_{\mu\nu}$, which is necessary if a gauge invariant Lagrangian for the gauge fields is to be constructed from it. The fact that $F'^{i}_{\mu\nu}$ does not now transform gauge covariantly is easily seen, since, on the r.h.s. of (5.2.20) the first term transforms gauge covariantly, while the last term transforms like the gauge fields.

Here, then, is the dilemma. If one wants torsion to couple to gauge fields through the usual mechanisms, one must give up the very fundamental notion of gauge invariance.

There are two ways out of this impasse. One is the rather unsatisfactory approach of abandoning the notion of a coupling between torsion and gauge fields, to which we have referred already, by constructing all field strengths etc. using only the symmetric Christoffel part of the full connection. However this is tantamount to rejecting the fundamental reason for which one would like to incorporate torsion into Einstein's theory. We are, therefore, more inclined towards the second approach which consists of trying somehow to modify the usual definition(s) of the gauge covariant derivative and/or $F_{\mu\nu}$, so as to retain both gauge invariance and the coupling of torsion to gauge fields. The rest of this chapter is devoted to showing that such a procedure can in fact be set up and leads to several interesting results which we promised in the introduction.

5.3 *MODIFIED MINIMAL COUPLING*

Let us consider first the work of Hojman, Rosenbaum, Ryan and Shepley (HR^2S) who modified the minimal coupling procedure for electromagnetism in order to solve the above problem for the photon. Interpreting the principle of minimal coupling as applied to the coupling of electromagnetism and the charged matter fields in the Lagrangian density to mean that the new derivative should depend linearly on the gauge connection A_μ but not on its derivatives, they proposed that the gauge covariant derivative be defined as

$$D_\mu = \partial_\mu - iq\, b_\mu^\alpha A_\alpha \qquad (5.3.1)$$

where it is to follow the transformation law for the charged fields as before, viz.

$$\psi \to \psi' = e^{iq\,\Lambda(x)}\,\psi\ . \qquad (5.3.2)$$

In (5.3.1) the $b_\mu^\alpha(x)$ are sixteen functions of space-time (but not of A_μ). With this definition of D_μ, we must now repeat the calculations of chapter three to determine the new transformation law for the compensating gauge field A_μ. The requirement which determines this is

$$D_\mu' \psi' = e^{iq\,\Lambda(x)}\, D_\mu\psi\ , \qquad (5.3.3)$$

which implies that

$$(\partial_\mu - iq\, b_\mu^\alpha A_\alpha)e^{iq\Lambda}\psi = e^{iq\Lambda}(\partial_\mu - iq\, b_\mu^\alpha A_\alpha)\ , \qquad (5.3.4)$$

or

$$e^{iq\Lambda}\psi(iq\partial_\mu\Lambda - iqb_\mu^\alpha\delta A_\alpha)+e^{iq\Lambda}D_\mu\psi = e^{iq\Lambda}D_\mu\psi\ , \qquad (5.3.5)$$

where $\delta A_\alpha = A_\alpha' - A_\alpha$ and where we assume that the fields b_μ^α are invariant under the gauge transformations.

This expression simplifies to give

$$b_{\mu}^{\alpha}\,\delta A_{\alpha} = \partial_{\mu}\Lambda \quad , \tag{5.3.6}$$

which upon multiplication by C_{ν}^{μ} , the inverse of b_{μ}^{ν} , yields

$$\delta A_{\nu} = C_{\nu}^{\mu}\,(\partial_{\mu}\Lambda) \; . \tag{5.3.7}$$

Note, however, that C_{ν}^{μ} is still not fixed and it is this extra degree of freedom introduced by generalising the usual procedure of minimal coupling which allows a restricted form of dynamical torsion to interact with gauge fields without destroying gauge invariance. We shall now use this freedom of choice of C_{ν}^{μ} to choose it in such a way as to ensure that

$$F'_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} + A_{\sigma}\,T^{\sigma}{}_{\mu\nu}, \tag{5.3.8}$$

be gauge invariant (since for the Abelian case the Maxwell field strength is gauge invariant as well as gauge covariant) under this modified transformation property of A_{μ} .

This requires that the variation in $F'_{\mu\nu}$ be zero under (5.3.7) for arbitrary Λ , i.e.

$$\begin{aligned}\delta F'_{\mu\nu} &= \partial_{\mu}(\,C_{\nu}^{\alpha}\;\Lambda_{,\alpha}\,) - \partial_{\nu}\,(C_{\mu}^{\alpha}\,\Lambda_{,\alpha}) + C_{\sigma}^{\alpha}\,T^{\sigma}{}_{\mu\nu}\,\Lambda_{,\alpha} \\ &= (C^{\alpha}_{\nu,\mu} - C^{\alpha}_{\mu,\nu} + C_{\sigma}^{\alpha}T^{\sigma}{}_{\mu\nu})\Lambda_{,\alpha} + C_{\nu}^{\alpha}\Lambda_{,\alpha\mu} - C_{\mu}^{\alpha}\Lambda_{,\alpha\nu} \\ &= 0 \quad , \end{aligned} \tag{5.3.9}$$

where $\Lambda_{,\alpha\beta}$ denotes $\partial_{\alpha}\partial_{\beta}\Lambda$.

This is possible only if the coefficients of $\Lambda_{,\alpha}$ and $\Lambda_{,\alpha\beta}$ vanish separately. This implies that we must solve the following two equations,

$$C^{\alpha}_{\nu,\mu} - C^{\alpha}_{\mu,\nu} + C_{\sigma}^{\alpha}\,T^{\sigma}{}_{\mu\nu} = 0 \quad , \tag{5.3.10}$$

and

$$C_{\nu}^{(\alpha}\,\delta_{\mu}^{\beta)} - C_{\mu}^{(\alpha}\,\delta_{\nu}^{\beta)} = 0 \quad , \tag{5.3.11}$$

where round brackets denote symmetrization.

Multiplying (5.3.11) by δ^{ν}_{β} we obtain

$$\begin{aligned} C^{\alpha}_{\mu} &= \frac{1}{4} C^{\beta}_{\beta}\, \delta^{\alpha}_{\mu} \\ &= f(x)\,\delta^{\alpha}_{\mu} \end{aligned} \tag{5.3.12}$$

where f(x) is some function of space-time. Substituting this into (5.3.10) we get

$$\delta^{\alpha}_{\nu} f_{,\mu} - \delta^{\alpha}_{\mu} f_{,\nu} + f\, T^{\alpha}{}_{\mu\nu} = 0 \ , \tag{5.3.13}$$

which may be written as

$$T^{\sigma}{}_{\mu\nu} = \delta^{\sigma}_{\mu} (\ln f)_{,\nu} - \delta^{\sigma}_{\nu} (\ln f)_{,\mu} \ . \tag{5.3.14}$$

Discarding the singular solution (when $f = 0$) , we find that the requirement that

$$b^{\alpha}_{\mu} \to \quad \text{when} \quad T^{\sigma}{}_{\mu\nu} \to 0 \ , \tag{5.3.15}$$

(which ensures that we recover the ordinary torsion free theory as one limit) allows us to write

$$f = e^{\Phi} \ ,$$

where Φ — the tlaplon field — is a scalar field which acts as a potential for the torsion.

So for the Abelian case we can retain gauge invariance and the coupling of torsion to the photon if we follow the prescription

$$\psi \to \psi' = e^{iq\Lambda}\, \psi \tag{5.3.16}$$

$$\begin{aligned} \partial_{\mu} \to D_{\mu} &= \partial_{\mu} - iq\, e^{-\Phi} A_{\mu} \\ &= \partial_{\mu} - i \frac{q}{f(x)} A_{\mu} \end{aligned} \tag{5.3.17}$$

where

$$A_{\mu} \to A'_{\mu} = A_{\mu} + e^{\Phi} \Lambda_{,\mu} \tag{5.3.18}$$

and where the usual procedure is employed for coupling gravity to the system with torsion being given (in terms of Φ) by

$$T^{\sigma}{}_{\mu\nu} = \delta^{\sigma}_{\mu} (\partial_{\nu}\Phi) - \delta^{\sigma}_{\nu} (\partial_{\mu}\Phi) \quad . \tag{5.3.19}$$

We consider now the non-Abelian case. In order to see how to proceed, we try to see if the non-Abelian form of the modified minimal coupling law of HR^2S given in (5.3.1);

$$D_{\mu} = \partial_{\mu} - ig\, b^{\alpha}_{\mu} A_{\alpha} \tag{5.3.20}$$

is sufficient to allow $F'^{i}_{\mu\nu}$ to transform gauge covariantly for a suitable choice of torsion and b^{α}_{μ} . Such a procedure turns out not to be sufficient but this exercise does give us a hint as to how to solve the problem.

The above definition for D_{μ} means that if $D_{\mu}\psi$ is to transform like ψ , then following the same sort of procedure as for the Abelian case, it is easy to show that the gauge fields A^{i}_{μ} must transform as

$$A^{i}_{\mu} \rightarrow A'^{i}_{\mu} = A^{i}_{\mu} - \frac{1}{g} C^{\alpha}_{\mu}\Lambda^{i}{}_{,\alpha} - C^{i}{}_{jk}\, A^{j}_{\mu}\, \Lambda^{k} \ , \tag{5.3.21}$$

where the Λ^{i}'s are arbitrary infinitesimal functions of space-time. Using this , we obtain the following expression for $\delta F'^{i}_{\mu\nu}$ (some details of the calculations needed to arrive at this result have been collected in Appendix B):

$$\begin{aligned} \delta F'^{i}_{\mu\nu} = {} & \frac{1}{g} (C^{\alpha}_{\mu}\, \delta^{\beta}_{\nu} - C^{\alpha}_{\nu}\, \delta^{\beta}_{\mu})\, \Lambda^{i}{}_{,\alpha\beta} + \\ & + \{ \frac{1}{g}\, \delta^{i}_{k}\, (C^{\alpha}_{\mu,\nu} - C^{\alpha}_{\nu,\mu} - C^{\alpha}_{\sigma}\, T^{\sigma}{}_{\mu\nu}) + \\ & \quad + C^{i}{}_{jk} A^{j}_{\mu} (\delta^{\alpha}_{\nu} - C^{\alpha}_{\nu}) + C^{i}{}_{jk} A^{j}_{\nu} (C^{\alpha}_{\mu} - \delta^{\alpha}_{\mu}) \} \Lambda^{k}{}_{,\alpha} + \\ & + \{ C^{i}{}_{jk} (A^{j}_{\mu,\nu} - A^{j}_{\nu,\mu} - A^{j}_{\sigma}\, T^{\sigma}{}_{\mu\nu}) - \end{aligned}$$

$$-g(C^{i}{}_{jm}C^{m}{}_{lk}A^{j}_{\mu}A^{l}_{\nu} + C^{i}{}_{ml}C^{m}{}_{jk}A^{j}_{\mu}\,A^{l}_{\nu})\}\Lambda^{k} \quad . \qquad (5.3.22)$$

Now, however, we must ensure that $F'^{i}_{\mu\nu}$ transforms gauge co-variantly, thus

$$\delta F'^{i}_{\mu\nu} = C^{i}{}_{jk}\,\Lambda^{j}\,F'^{k}_{\mu\nu} \quad . \qquad (5.3.23)$$

This leads then to the following equations for C^{ν}_{μ} and $T^{\sigma}{}_{\mu\nu}$ which we obtain by requiring the coefficients of $\Lambda^{i}{}_{,\alpha\beta}$, $\Lambda^{k}{}_{,\alpha}$, and Λ^{k} to vanish separately.

(i) Coefficient of $\Lambda^{i}{}_{,\alpha\beta}$:

$$C^{(\alpha}_{\mu}\,\delta^{\beta)}_{\nu} - C^{(\alpha}_{\nu}\,\delta^{\beta)}_{\mu} = 0 \quad , \qquad (5.3.24)$$

where round brackets () again denote symmetrisation over the indices enclosed.

(ii) Coefficient of $\Lambda^{k}{}_{,\alpha}$:

$$\frac{1}{g}(C^{\alpha}_{\mu,\nu} - C^{\alpha}_{\nu,\mu} - C^{\alpha}_{\sigma}\,T^{\sigma}{}_{\mu\nu})\,\delta^{i}_{k} \; +$$

$$+ C^{i}{}_{jk}A^{j}_{\mu}(\delta^{\alpha}_{\nu} - C^{\alpha}_{\nu}) + C^{i}{}_{jk}A^{j}_{\nu}(C^{\alpha}_{\mu} - \delta^{\alpha}_{\mu}) = 0. \qquad (5.3.25)$$

(iii) Finally, the coefficient of Λ^{k} , using equation (5.3.24) in equation (5.3.23) can be reduced, after some tedious index manipulations, to the following,

$$C_{imk}C_{mjl} - C_{ijm}C_{mlk} - C_{iml}C_{mjk} = 0 \quad , \qquad (5.3.26)$$

which is satisfied identically, due to the Jacobi identity (3.2.23) satisfied by the structure constants. Indeed (5.3.26) is the Jacobi identity.

Equation (5.3.24) is identical to equation (5.3.11) encountered for the Abelian case, and which has been shown already to solve to yield,

$$C_\mu^\nu = f(x)\, \delta_\mu^\nu \quad . \tag{5.3.27}$$

Substituting this into (5.3.25) we obtain,

$$\frac{1}{g}\,(f_{,\nu}\delta^\alpha_\mu - f_{,\mu}\delta^\alpha_\nu - f\,T^\alpha{}_{\mu\nu})\,\delta^i_k \; + $$
$$+ \; C^i{}_{jk}(f-1)(A^j_\nu\delta^\alpha_\mu - A^j_\mu\delta^\alpha_\nu) = 0 \quad . \tag{5.3.28}$$

However, the only solution to these equations is the trivial one, namely,

$$T^\alpha{}_{\mu\nu} = 0 \quad ,$$

which gives the ordinary torsionless gravity theory coupled to the usual non-Abelian gauge fields.

Clearly, something more than just the HR^2S modification of minimal coupling is called for.

5.4 *MODIFIED FIELD STRENGTH*

We saw in the last section that on its own the generalisation of the usual gauge covariant derivative as proposed by Hojman et. al. was not sufficient to allow us to couple torsion to non-Abelian gauge fields. The work of the last section does suggest, however, that if we modify the Yang-Mills-Shaw field strengths, $F^i_{\mu\nu}$, in the following way, we may obtain a satisfactory solution of the problem.

Let us define,

$$\ddot{F}_{\mu\nu} = A^i_{\nu,\mu} - A^i_{\mu,\nu} + g\, C^i_{\;jk} B^\alpha_\mu B^\beta_\nu A^j_\alpha A^k_\beta \qquad (5.4.1)$$

Replacing the partial derivatives by covariant derivatives and carrying out all the above calculational procedures for these modified field strengths and the modified minimal coupling of equation (5.3.20), we find that instead of equations (5.3.24) - (5.3.26) we must solve the following set of equations:

(5.3.24) and (5.3.26) remain the same as before while (5.3.25) is modified to:

$$\begin{aligned} &\frac{1}{g} (C^\alpha_{\mu,\nu} - C^\alpha_{\nu,\mu} - C^\alpha_\sigma T^\sigma_{\mu\nu}) \delta^i_k \; + \\ &+ C^i_{\;jk} (A^j_\mu \delta^\alpha_\nu - B^\gamma_\mu B^\beta_\nu A^j_\gamma C^\alpha_\beta) \; + \\ &+ C^i_{\;jk} (B^\gamma_\mu B^\beta_\nu A^j_\beta C^\alpha_\gamma - A^j_\nu \delta^\alpha_\mu) = 0 \; . \end{aligned} \qquad (5.4.2)$$

It is a straightforward matter to discover that the following is a solution to these equations;

$$C^\nu_\mu = f(x) \delta^\nu_\mu \quad , \qquad (5.4.3)$$

$$B^\nu_\mu = f^{-\frac{1}{2}}(x) \delta^\nu_\mu \; , \qquad (5.4.4)$$

$$f\, T^\sigma_{\mu\nu} = f_{,\mu} \delta^\alpha_\nu - f_{,\nu} \delta^\alpha_\mu . \qquad (5.4.5)$$

Equation (5.4.3) implies that since,

$$b^{\alpha}_{\mu} C^{\nu}_{\alpha} = \delta^{\nu}_{\mu} , \qquad (5.4.6)$$

$$b^{\nu}_{\mu} = f^{-1}(x) \delta^{\nu}_{\mu} . \qquad (5.4.7)$$

As (5.4.7) is singular for f(x) = 0, we shall require f(x) to be non-zero everywhere in space-time. Further, requiring (5.3.20) to reduce to its usual definition ($b^{\nu}_{\mu} \to \delta^{\nu}_{\mu}$ for zero torsion) fixes the sign of f(x) to be everywhere positive. Hence we write

$$f(x) = e^{\Phi(x)} . \qquad (5.4.8)$$

The expression for torsion then takes the form

$$T^{\sigma}{}_{\mu\nu} = \delta^{\sigma}_{\mu} \Phi_{,\nu} + \delta^{\sigma}_{\nu} \Phi_{,\mu} \qquad (5.4.9)$$

and the connection is, therefore, given by

$$\Gamma_{\mu\nu}{}^{\sigma} = \{{}^{\sigma}_{\mu\nu}\} - \delta^{\sigma}_{\mu} \Phi_{,\nu} + g_{\mu\nu} \Phi^{,\sigma} \qquad (5.4.10)$$

We have found, therefore, that it is necessary to modify the Yang-Mills-Shaw field strength to

$$\ddot{\acute{F}}{}^{i}_{\mu\nu} = A^{i}_{\nu,\mu} - A^{i}_{\mu,\nu} + g\, C^{i}_{jk} B^{\alpha}_{\mu} B^{\beta}_{\nu} A^{j}_{\alpha} A^{k}_{\beta} - A^{i}_{\sigma} T^{\sigma}{}_{\mu\nu} \qquad \underline{(5.4.11)}$$

where the B^{α}_{μ} are functions of space-time but not of the gauge fields, in order to solve the problem of coupling gauge fields to torsion for the non-Abelian case. Making this modification and using the modified minimal coupling procedure for gauge fields (5.3.20) we see that it is possible to retain gauge invariance while coupling torsion to all gauge fields provided that

$$B^{\alpha}_{\mu} = f^{-\frac{1}{2}}(x) \delta^{\alpha}_{\mu} \qquad (5.4.12)$$

where f, b^{α}_{μ} (or C^{α}_{μ}), and $T^{\sigma}{}_{\mu\nu}$ are as for the Abelian case. Substituting our solution back into the expressions for D_{μ} and $\ddot{F}^{\,i}_{\mu\nu}$ gives us

$$D_{\mu} = \partial_{\mu} - i\,\frac{g}{f}\,A_{\mu}\cdot\Theta \tag{5.4.13}$$

and

$$\ddot{F}^{\,i}_{\mu\nu} = A^{i}_{\nu,\mu} - A^{i}_{\mu,\nu} + \frac{g}{f}\,C^{i}{}_{jk}A^{j}_{\mu}A^{k}_{\nu} - A^{i}_{\sigma}T^{\sigma}{}_{\mu\nu}, \tag{5.4.14}$$

where $T^{\sigma}{}_{\mu\nu}$ is given by (5.3.19).

We have written D_{μ} and $\ddot{F}^{\,i}_{\mu\nu}$ in the above form to make the point that the effect of coupling our particular form of torsion to gauge fields by the method described above is essentially equivalent to defining an effective coupling constant

$$g(x) = g/f(x)$$

which is a function of the space-time point at which the interaction takes place. A more complete discussion of the implications of this will be given in the next chapter.

The total Lagrangian density for a system of complex scalar fields, metric, and gauge fields considered above can, then, be easily shown to take the form,

$$L = \frac{\sqrt{-g}}{16\pi}\,(\tilde{R} - 6\Phi^{,\mu}\Phi_{,\mu} - \ddot{F}_{\mu\nu}\cdot\ddot{F}^{\mu\nu} - 4\psi^{*}_{.\mu}\psi^{.\mu}) \quad ,$$

where the dot before an index (thus $.\mu$) implies use of the modified covariant derivative (5.4.13).

CHAPTER SIX

DISCUSSION

Of all the fundamental interactions observed in nature, the gravitational interaction is by far the weakest and though all present day evidence from macrophysics attests to the validity of Einstein's general relativistic description of it, at the microscopic level it is the least well understood. Almost all generalisations of the general theory of relativity, therefore, try to modify the theory in such a way as to allow deviations from it in the small — at the elementary particle scale.

At this scale we find, however, that gauge theories describe the other observed fundamental interactions, namely the elecronuclear interactions. These theories assume for the space-time symmetries the Poincaré group and classify the particles by means of the irreducible unitary representations of this group. These representations are labelled by the mass m, and spin s of the elementary particles.

Now, mass, which is connected with the translational part of the Poincaré group, finds a beautiful interpretation in terms of the geometrical notion of the curvature of space-time in the general theory of relativity. However, spin, the other parameter necessary for a complete classification of these elementary particles and one which is associated with the rotational part of the Poincaré group, is not afforded a similar geometric interpretation in Einstein's theory of gravitation.

This notion of spin, s, of the elementary particles is, therefore, the quantity which the Einstein-Cartan-Sciama-Kibble theory tries to incorporate into a geometrical framework more general than that of Riemann which Einstein employed. The ECSK theory achieves this by introducing an asymmetric contribution, called the contorsion, in addition to the Christoffel connection into the space-time connection which it employs. The non-Riemannian aspects introduced in this way are then attributed to the spin-angular momentum of matter.

In chapter four we have considered the choice of a suitable action for this theory which could determine the dynamics of both the metric $g_{\mu\nu}$, and the contorsion $K_{\mu\nu}{}^{\lambda}$ fields. There we have shown that the Lagrangian density which is conventionally employed for this purpose is not the most general allowed, even if it is required to be linear in the curvature tensor formed out of the complete asymmetric connection. We have argued that another parity violating contribution constructed from the pseudo-tensor density $\varepsilon^{\mu\nu\alpha\beta}$ and the curvature tensor $R_{\mu\nu\alpha\beta}(\Gamma)$ ought to contribute to the complete action for such a theory.

Pseudoscalar actions for the ECSK theory have been considered before by Purcell /21/ who generalised this theory by the addition of the most general action which is bilinear in the antisymmetric part of the connection and linear in the Levi-Civita density. For his theory the net effect of the new additions to the action was to reduce the spin-spin coupling constant of the ECSK theory. There are two points worth mentioning with regard to the connection between the work of Purcell and the work here reported. Firstly, it is worth noting that whereas Purcell has allowed the contorsion tensor a role analogous to that of the curvature tensor inasmuch as they are both allowed to appear in the action, we have argued that the action ought, for the sake of a reduction in the number of arbitrary parameters that would be needed, to be constructed solely out of the linear combinations of the curvature tensor. The second point we wish to make is that the introduction of the Levi-Civita density into the theory does not only imply a reduction in the strength of the spin-spin interaction. Indeed, we have shown in section six of chapter four that new effects not at all predicted by the ECSK theory may be expected.

We have also argued in chapter four that torsion should be present in a dynamic form even in the absence of matter to represent the torsional effects due to the spin-2 nature of gravitation. We arrived at a particularly simple form of this vacuum torsion which is generated by a scalar field and for which the parity-violating effects due to the additional term in the action we motivated vanish.

In order to illustrate that the generalised theory we have put forward does lead to effects not present in the ordinary ECSK framework, we went on to show that parity-violating effects not required by the ECSK theory for the Proca field are expected to be present in a theory based on our action. This example is not altogether an academic exercise and devoid of any physical content, since massive spin-one particles, such as the ρ, ω, and ϕ are known to exist for which, at least in principle, our discussion in section six of chapter four might have some relevance. Some further work in this direction is, therefore, conceivable and might center on trying to explain some of the observed features of these massive spin-one objects.

The presence of the parity-violating contribution appears to us to be the most distinctive feature of torsion-containing theories which could serve to distinguish them from Einstein's theory and which could provide the basis for the experimental verification or rejection of such theories.

However, in the present work we have not developed the theory to a stage where it can be confronted with experiment. Further work in this direction is also possible. The first task that one might carry out in order to bring the theory closer to making experimentally testable predictions is to incorporate fermions into the theory and then discuss a particular laboratory situation in which the distinguishing features of this theory would be illustrated. The extreme weakness of the effects expected does, however, mean that there is, at present, little hope for a direct confrontation between experiment and the predictions of this theory.

The work that we carry out in chapter five is, however, much closer to experimental testing since it makes definite statements about the coupling of photons to a particularly simple form of dynamic torsion that we motivated also in chapter four.

The work of chapter five concentrated on trying to resolve the problem of coupling torsion to all spinning matter - in particular matter of the gauge field variety. The problem arises, as was illustrated in the second section of this chapter, because a straightforward attempt to achieve such a coupling leads to the loss of gauge invariance. We found that a coupling between gauge fields and a simple form of torsion could be achieved provided that one generalised the usual concept of minimal coupling of gauge fields to charged matter and provided that a modified form of the gauge field strengths was employed.

This approach led to one particularly interesting consequence in that the whole procedure could be viewed as the replacement of all space-time independent gauge coupling by ones which depended on space-time in a manner determined by the strength of the torsion. Of course, this means for the Abelian case that test bodies with different electromagnetic energy contents would behave differently and therefore have implications for the null experimental results of the Eotvos-Dicke-Braginsky experiments.

This question has been considered in detail by Wei-Tou Ni /22/ who has claimed that for the Sun, the scalar field ϕ , which generates the torsion, would be around 0.67×10^{-4} U where U is the Newtonian potential at the surface of the earth. This, he further claims would lead to the prediction from this theory that the gravitational accelerations of aluminium and gold would differ by

$$2 \times 10^{-7} \vec{\nabla} U ,$$

which disagrees with the null experiments of precision

$$10^{-11} \vec{\nabla} U \quad \text{and} \quad 10^{-12} \vec{\nabla} U$$

performed respectively by Roll, Krotkov, and Dicke and by Braginsky and Panov /23-24/.

One may draw two conclusions from this - both somewhat discouraging.

The most straightforward, and perhaps honest, is to conclude that the modified form of minimal coupling proposed by Hojman, Rosenbaum, Ryan, and Shepley in reference 19 is not made use of by nature and that one ought after all to leave gauge fields uncoupled to torsion, or at least one should look for other ways of achieving it.

Of course, one might also conclude that perhaps the generalisation of Einstein's theory to include torsion is not a useful one in the first place and that, therefore, torsion has no role to play in a description of the gravitational interactions. Both these conclusions are, therefore, negative and somewhat discouraging though, of course, one can think of more exotic situations which would explain away these negative results.

It is conceivable, for example, that the region of space-time that we happen to be in is a zero torsion region and that perhaps non-zero torsion exists in other regions of the universe, where the effects studied in chapter five might in fact occur. This rather optimistic situation might appear less unrealistic if, for example, the ideas of Hanson and Regge /25/ turn out to be right and the absence of torsion in conventional gravity could be explained in a dynamical manner. These authors have suggested that a gravitational Meissner effect might be responsible for producing instanton-like vortices of non-zero torsion concentrated at four-dimensional points. Such torsion vortices would be the analogues of magnetic flux vortices in a type II superconductor, while ordinary torsion-free space-time would correspond to the field-free superconducting region of a superconductor.

However, no convincing demonstration of the occurence of such effects in a metric-torsion theory have been reported as yet.

We shall end this discussion here since we are already bordering on the very speculative which is the subject of the next very brief chapter.

CHAPTER SEVEN

SPECULATIONS

In this very brief chapter we discard all pretences of rigour and make some speculative remarks and suggest possible lines along which further work might proceed.

Torsion containing theories are not only useful for describing gravitational interactions. It has been known for some time that torsion containing geometries have a very useful role to play in continuum physics. Based mainly on the work of Kondo, Bilby, Bullough and Smith, and Kroner /26/ it has become clear that torsion plays a central role in the continuum theory of crystal dislocations where the torsion is identified with the physical notion of a dislocation density. It is amusing to speculate whether the generalised Lagrangian density we have proposed in chapter four and the concepts outlined in chapter five may not in fact find some more useful applications in such theories of crystal dislocations.

Another possible link between the work of chapter four and some physical situations might arise if we compare this work with the speculative and rather vague remarks of Stueckelberg /27/ who tried to explain the experimental results of parity violation observed in the weak interactions in 1957 by ascribing them to a cosmological distinction between left and right. He claimed that a cosmological asymmetry was perfectly compatible with Riemannian space-time of ordinary general relativity and tried to explain this by proposing the existence of a 'field' $\varepsilon_{\alpha\beta\gamma\delta}(x)$ whose covariant derivative vanished everywhere. However the major problem faced by us while trying to work along these lines has been to actually understand what exactly Stueckelberg had in mind - this not being clear from his extremely brief work of reference 27 .

It is also interesting to consider the possibility of including an εR type term in the Lagrangian for the theory of supergravity which is based simply on the Einstein Lagrangian and the use of the torsionless

Christoffel connection when coupling to matter. It is also worth asking whether it would still be possible to write down a theory of supergravity in which the full torsion containing connection was employed for the purposes of coupling to matter rather than the 'minimal' approach traditionally adopted. Still with supergravity, it may be useful to examine in a superspace formulation of the theory the field content of an εR type term, where the ε field is now the superspace analogue of the simple Levi-Civita density of ordinary general relativity.

It would also be very interesting to see if the concepts of modified minimal coupling etc. employed in the work of chapter five can be made use of in supergravity or even in an ordinary global super-symmetric framework.

Finally, we wish to close by speculating about the possible role torsion might play in strong interaction physics. It has been suggested by Isham, Salam and Strathdee /28/ that the spin-2 aspects of strong interaction phenomena may be understood from a geometrical point of view by a two-tensor f-g theory.

This theory is the gravitational analogue of the vector-meson dominance hypothesis for hadron electrodynamics and attempts to describe the gravitational interactions of hadrons and leptons through an f-g mixing which resembles the ρ^0-ω mixing of lepton hadron interactions. The action which this theory employs uses, apart from the Einstein-Hilbert expressions for each of the two spin-2 fields, a generally covariant f-g mixing term and may be written symbolically as

$$L_{fg} = L_{EH}(g) + L_{EH}(f) + L_{PF} ,$$

where L_{PF} is the mixing term and is just a generally covariant form of the well known Pauli-Fierz expression for a massive spin-2 field.

The point to note is that we are, here, concerned with phenomena at the microscopic level where, as we have repeatedly emphasised, the spin-angular momentum of matter might be expected to play a significant role. In view of this the idea comes immediately to mind that one should attempt to incorporate torsion into the strong metric (f) part of this theory and study any consequences to which this modification might give rise.

APPENDICES

APPENDIX A

In this appendix we give the most general form for L_K mentioned in section five of chapter four. The G_i are arbitrary parameters and we have only given the terms for the ordinary ECSK theory. Allowing the additional term L_A the situation can only get more complicated.

$$L_K = \sum_{i=1}^{16} \frac{1}{16\pi G_i} Q_i \quad , \text{ with}$$

$$Q_1 = K^{\alpha\beta\lambda}{}_{;\sigma} K_{\lambda\alpha\beta;}{}^{\sigma}$$

$$Q_2 = K_{\alpha}{}^{\alpha\lambda}{}_{;\sigma} K_{\beta\lambda}{}^{\beta}{}_{;}{}^{\sigma}$$

$$Q_3 = K^{\alpha\beta\lambda}{}_{;\sigma} K_{\alpha\beta\lambda;}{}^{\sigma}$$

$$Q_4 = K^{\alpha\beta\sigma}{}_{;\alpha} K_{\lambda\sigma}{}^{\lambda}{}_{;\beta}$$

$$Q_5 = K^{\beta\alpha\sigma}{}_{;\alpha} K_{\lambda\sigma}{}^{\lambda}{}_{;\beta}$$

$$Q_6 = K^{\sigma\alpha\beta}{}_{;\alpha} K_{\lambda\sigma}{}^{\lambda}{}_{;\beta}$$

$$Q_7 = K^{\alpha\sigma\lambda}{}_{;\alpha} K^{\beta}{}_{\sigma\lambda;\beta}$$

$$Q_8 = K^{\alpha\sigma\lambda}{}_{;\alpha} K_{\sigma}{}^{\beta}{}_{\lambda;\beta}$$

$$Q_9 = K^{\beta\sigma\lambda}{}_{;\alpha} K^{\alpha}{}_{\sigma\lambda;\beta}$$

$$Q_{10} = K^{\beta\sigma\lambda}{}_{;\alpha} K_{\sigma}{}^{\alpha}{}_{\lambda;\beta}$$

$$Q_{11} = K^{\sigma\alpha\lambda}{}_{;\alpha} K_{\sigma}{}^{\beta}{}_{\lambda;\beta}$$

$$Q_{12} = K^{\sigma\beta\lambda}{}_{;\alpha} K_{\sigma}{}^{\alpha}{}_{\lambda;\beta}$$

$$Q_{13} = K^{\sigma\alpha\lambda}{}_{;\alpha} K_{\lambda}{}^{\beta}{}_{\sigma;\beta}$$

$$Q_{14} = K^{\sigma\beta\lambda}{}_{;\alpha} K_{\lambda}{}^{\alpha}{}_{\sigma;\beta}$$

$$Q_{15} = K^{\sigma\alpha}{}_{\sigma;\alpha} K^{\lambda\beta}{}_{\lambda;\beta}$$

$$Q_{16} = K^{\sigma\beta}{}_{\sigma;\alpha} K^{\lambda\alpha}{}_{\lambda;\beta}$$

APPENDIX B

In this appendix we derive the expression for $\delta F'^{i}_{\mu\nu}$ given in (5.3.22) using the expression for δA^{i}_{μ} given in (5.3.21), viz.

$$\delta A^{i}_{\mu} = -\frac{1}{g} C^{\alpha}_{\mu} \Lambda^{i}{}_{,\alpha} - C^{i}{}_{jk} A^{i}_{\mu} \Lambda^{k} \tag{B.1}$$

and go on to show that use of (5.3.23) yields the equations (5.3.24) - (5.3.26).

Now,

$$F'^{i}_{\mu\nu} = F^{i}_{\mu\nu} + A^{i}_{\sigma} T^{\sigma}{}_{\mu\nu}$$

$$= \partial_{\mu} A^{i}_{\nu} - \partial_{\nu} A^{i}_{\mu} + g\, C^{i}{}_{jk} A^{j}_{\mu} A^{k}_{\nu} + A^{i}_{\sigma} T^{\sigma}{}_{\mu\nu}\,, \tag{B.2}$$

so that

$$\delta F'^{i}_{\mu\nu} = \partial_{\mu} \delta A^{i}_{\nu} - \partial_{\nu} \delta A^{i}_{\mu} + g\, C^{i}{}_{jk} (\delta A^{j}_{\mu} A^{k}_{\nu} + A^{j}_{\mu} \delta A^{k}_{\nu}) + \delta A^{i}_{\sigma} T^{\sigma}{}_{\mu\nu}. \tag{B.3}$$

However,

$$\partial_{\mu} \delta A^{i}_{\nu} = -\frac{1}{g}(C^{\alpha}_{\nu,\mu} \Lambda^{i}{}_{,\alpha} + C^{\alpha}_{\nu} \Lambda^{i}{}_{,\alpha\mu}) - C^{i}{}_{jk}(A^{j}_{\nu,\mu} \Lambda^{k} + A^{j}_{\nu} \Lambda^{k}{}_{,\mu}) \tag{B.4}$$

Inserting this into (B.3) yields,

$$\begin{aligned} \delta F'^{i}_{\mu\nu} = & -\frac{1}{g}\{(C^{\alpha}_{\nu,\mu} \Lambda^{i}{}_{,\alpha} + C^{\alpha}_{\nu} \Lambda^{i}{}_{,\alpha\mu}) - (C^{\alpha}_{\mu,\nu} \Lambda^{i}{}_{,\alpha} + C^{\alpha}_{\mu} \Lambda^{i}{}_{,\alpha\nu})\} + \\ & + g\, C^{i}{}_{jk}\{A^{k}_{\nu}(-\frac{1}{g} C^{\alpha}_{\mu} \Lambda^{j}{}_{,\alpha} - C^{j}{}_{lm} A^{l}_{\mu} \Lambda^{m}) + \\ & + A^{j}_{\mu}(-\frac{1}{g} C^{\alpha}_{\nu} \Lambda^{k}{}_{,\alpha} - C^{k}{}_{lm} A^{l}_{\nu} \Lambda^{m})\} + \\ & + T^{\sigma}{}_{\mu\nu} (-\frac{1}{g} C^{\alpha}_{\sigma} \Lambda^{i}{}_{,\alpha} - C^{i}{}_{jk} A^{j}_{\sigma} \Lambda^{k}) \,. \end{aligned} \tag{B.5}$$

Finally, collecting terms linear in $\Lambda^{i}{}_{,\alpha\beta}$, $\Lambda^{k}{}_{,\alpha}$, and Λ^{k} gives us the result that

$$\delta F'^{i}_{\mu\nu} = \frac{1}{g} (C^{\alpha}_{\mu} \delta^{\beta}_{\nu} - C^{\alpha}_{\nu} \delta^{\beta}_{\mu}) \Lambda^{i}{}_{,\alpha\beta} +$$

$$+ \frac{1}{g}\{\delta^{i}_{k}(C^{\alpha}_{\mu,\nu} - C^{\alpha}_{\nu,\mu} - C^{\alpha}_{\sigma}\, T^{\sigma}{}_{\mu\nu}) +$$

$$+ C^{i}{}_{jk}A^{j}_{\mu}(\delta^{\alpha}_{\nu} - C^{\alpha}_{\nu}) + C^{i}{}_{jk}A^{j}_{\nu}(C^{\alpha}_{\mu} - \delta^{\alpha}_{\mu})\ \Lambda^{k}{}_{,\alpha}\} +$$

$$+ \{C^{i}{}_{jk}\ (A^{j}_{\mu,\nu} - A^{j}_{\nu,\mu} - A^{j}_{\sigma}\, T^{\sigma}{}_{\mu\nu}) -$$

$$- g(C^{i}{}_{jm}C^{m}{}_{lk}A^{j}_{\mu}A^{l}_{\nu} + C^{i}{}_{ml}C^{m}{}_{jk}A^{j}_{\mu}A^{l}_{\nu})\}\Lambda^{k} \quad . \qquad \text{(B.6)}$$

Now requiring the gauge covariance of $F'^{i}_{\mu\nu}$ under (B.1) will lead to equation (5.3.23) :

$$\delta F'^{i}_{\mu\nu} = C^{i}{}_{kj}\ \Lambda^{k}\ F'^{j}_{\mu\nu} \quad . \qquad \text{(B.7)}$$

The r.h.s. of this equation does not involve any derivatives of Λ , so that when we come to obtain equation (5.3.24) - (5.3.26) by setting the coefficients of $\Lambda^{i}{}_{,\alpha\beta}$, $\Lambda^{k}{}_{,\alpha}$, and Λ^{k} in (B.7) equal to zero, the first two of these equations can be read off simply from (B.6). The last equation does, however, require that we subtract from the coefficient of Λ^{k} in (B.6) the quantity on the r.h.s. of (B.7). This then easily leads to equation (5.3.26).

FOOTNOTES

F1. The material for this section is taken essentially from references 3, 6 and 9.

F2. A very useful and concise account of this attempted unification of electromagnetism and gravity is given by W. Pauli /4/ in section 65 of his book on relativity, where he also discusses in detail why the theory led to contradictions with experimental results.

F3. This is proved on the next page.

F4. The material for this section is essentially taken from the excellent review on gauge theories by Abers and Lee in reference 15.

F5. For convenience we define the pseudo-tensors $\eta^{\mu\nu\lambda\sigma}$ and $\eta_{\mu\nu\lambda\sigma}$ from the usual Levi-Civita (pseudo-) tensor densities $\varepsilon^{\mu\nu\lambda\sigma}$ and $\varepsilon_{\mu\nu\lambda\sigma}$ as

$$\eta^{\mu\nu\lambda\sigma} = (-g)^{-\frac{1}{2}} \varepsilon^{\mu\nu\lambda\sigma} ,$$

$$\eta_{\mu\nu\lambda\sigma} = (-g)^{\frac{1}{2}} \varepsilon_{\mu\nu\lambda\sigma} .$$

They satisfy the following properties:

$$\eta^{\mu\nu\lambda\sigma} \eta_{\mu\alpha\beta\gamma} = -\delta^{\nu\lambda\sigma}_{\alpha\beta\gamma} ,$$

$$\eta^{\mu\nu\lambda\sigma} \eta_{\mu\nu\beta\gamma} = -2 \delta^{\lambda\sigma}_{\beta\gamma} ,$$

$$\eta^{\mu\nu\lambda\sigma} \eta_{\mu\nu\lambda\gamma} = -6 \delta^{\sigma}_{\gamma} ,$$

$$\eta^{\mu\nu\lambda\sigma} \eta_{\mu\nu\lambda\sigma} = -24 ,$$

where the tensor $\delta^{\mu\nu\lambda\cdots}_{\alpha\beta\gamma\cdots}$ is a generalised Kronecker symbol obeying the following rules: If $\mu,\nu,\lambda,\cdots$ are all different and $\alpha,\beta,\gamma,\cdots$ are obtained from them by a certain permutation, then it is equal to +1 or −1 depending on whether the permutation $\begin{smallmatrix}\mu\nu\lambda\cdots\\ \alpha\beta\gamma\cdots\end{smallmatrix}$ is even or odd, in the remaining cases it is equal to zero.

F6. For later purposes we define

$$a = (16\pi G_P)^{-1}.$$

For the most part we shall work with $(16\pi G_N)$ set to unity.

F7. This is illustrated beautifully in the work of Michael Hovak and Pemeter Krupka (Int. J. Theor. Phys. 17(1978) 543) in which these authors consider the problem of finding all first order invariant Einstein-Cartan structures. They consider Lagrangians containing terms linear or quadratic in the following objects: g_{ij} , $g_{ij;k}$, T^i_{jk} , S^i_{jk} (the latter two are in their notation the antisymmetric and symmetric parts of the connection) $R^i{}_{jkl}$, and $T^i{}_{jk;l}$. Assuming compatibility they deduce that for a four-dimensional theory there exist at most 194 such functionally independent generally invariant Lagrangians. If we allow for the use of the tensor density ε_{ijkl}, not considered by them, the number of such independent Lagrangian structures would increase further. In view of this the generalised action we propose (by restricting ourselves to linear curvature theories) seems to be a very reasonable choice.

F8. We may here point out that (4.5.1) is not the most general form for the contorsion that we can write if we allow the use of ε_{ijkl}. In fact it is possible (while still only introducing one index fields) to consider the following choice for K_{ijk}:

$$K_{ijk} = \Phi_j\, g_{ik} - \Phi_k\, g_{ij} + \varepsilon_{ijkl}\, \psi^{l}$$

where we have introduced a pseudoscalar field ψ (ψ^{δ} bein ψ'^{δ}) which, like Φ would be a dynamical field once we incorporate this type of contorsion into our Lagrangian. For simplicity, however, we do not consider this choice in the present work.

F9. We shall in fact be considering this problem in the next chapter where modifications of the usual minimal coupling procedure and the use of a modified Yang-Mills-Shaw field strength does enable us to couple a simple form of torsion to gauge fields. For the present, however, we shall not complicate the discussion by considering such a possibility.

F10. Wherever possible we shall suppress all internal indices.

F11. $\eta_{\mu\nu}$ is the Minkowskian metric, diagonal (+,-,-,-).

F12. Here S.R. stands for special relativitivistic and G.R. for general relativistic.

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SUBSIDIARY MATERIAL

Monopole solutions for strong gravity coupled to SO(3) gauge fields

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We find a class of static spherically symmetric monopole solutions to the coupled Einstein-SO(3) gauge field equations for f-g theory in the limit that weak gravity is neglected. These solutions reduce, in the appropriate limits, to the Salam-Strathdee class of solutions and the Wang solutions for the pure Einstein-SO(3) theory. We comment on possible extensions and the relevance these solutions may have for hadronic physics.

Exact spherically symmetric solutions for the pure f-g theory of Isham, Salam, and Strathdee[1] have recently[2,3] been obtained. In this note we incorporate SO(3) gauge fields in the theory and obtain static spherically symmetric monopole solutions for the gauge fields and the f metric. The full Lagrangian density for such a system is given by

$$\mathfrak{L} = -\frac{1}{\kappa_g^2}\sqrt{-g}\,R(g) - \frac{1}{\kappa_f^2}\sqrt{-f}\,R(f) + \mathfrak{L}_{fg} + \mathfrak{L}_{YM}\,, \tag{1}$$

that is, the Einstein expressions for the g and f fields, a generally covariant mixing term which, at the linearized level, is responsible for the f field mass, and a Yang-Mills part for the SO(3) gauge fields. This latter contribution to $\mathfrak{L}$ is constructed using $f_{\mu\nu}$ as a metric tensor. Such a procedure is consistent with the prescription employed in f-g theory. Hadronic matter parts of the Lagrangian are to be formed using $f_{\mu\nu}$ as a metric tensor, while for the leptonic parts one must use $g_{\mu\nu}$. The underlying physics is that while leptons interact directly with gravitation, hadrons do so only through an f-g mixing, analogous to the ρ^0-γ mixing in hadron electrodynamics. f-g theory without leptons may thus be described as hadron geometrodynamics. In detail,

$$\mathfrak{L}_{fg} = -\frac{M^2}{4\kappa_f^2}\sqrt{-f}\left(\frac{\sqrt{-g}}{\sqrt{-f}}\right)^{\alpha}(f^{\mu\nu}-g^{\mu\nu})(f^{\rho\sigma}-g^{\rho\sigma}) \times (g_{\mu\rho}g_{\nu\sigma}-g_{\mu\nu}g_{\rho\sigma})\,, \tag{2}$$

$$\mathfrak{L}_{YM} = -\tfrac{1}{4}\sqrt{-f}\,f^{\mu\nu}f^{\rho\sigma}F^a_{\mu\rho}F^a_{\nu\sigma}\,, \tag{3}$$

where

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\epsilon_{abc}W^b_\mu W^c_\nu\,. \tag{4}$$

We use the following notation. Greek indices run from 0 to 3, while Latin indices denote 1, 2, and 3. ϵ_{abc} is the usual ϵ symbol, with $\epsilon_{123}=1$. $\kappa_f^2 = 8\pi G_S$, $\kappa_g^2 = 8\pi G_N$, where $G_S \sim 1\ \mathrm{GeV}^{-2}$, $G_N \sim 10^{-39}\ \mathrm{GeV}^{-2}$. $g = \det(|g_{\mu\nu}|)$, $f = \det(|f_{\mu\nu}|)$ and $c=1$ is assumed throughout.

We are interested in solutions of (1) in the limit that weak gravity is neglected. In this case (1) reduces to

$$\mathfrak{L} = -\frac{1}{\kappa^2}\sqrt{-f}\,R(f) + \mathfrak{L}_{mass} + \mathfrak{L}_{YM}\,, \tag{5}$$

where $\mathfrak{L}_{mass}$ is the expression given in (2) but with $g_{\mu\nu}$ everywhere replaced by $\eta_{\mu\nu}$, the flat space-time metric. For convenience we choose to work in spherical polar coordinates, where our signs are such that $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -r^2\sin^2\theta)$. We have dropped the f label from κ_f^2.

It should be noted that $\mathfrak{L}_{mass}$ is not generally covariant; it is the flat-space approximation to (2). Restricting ourselves to the static spherically symmetric case and considering monopole-type solutions, we write[1]

$$W^a_0 = 0,\quad W^a_{i,0} = 0,\quad W^a_i = \epsilon_{iab}\frac{x^b U(r)}{r^2} \tag{6}$$

and

$$f_{\mu\nu}dx^\mu dx^\nu = C dt^2 - 2D\,dt\,dr - A\,dr^2 - B(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{7}$$

where A, B, C, D, and U are functions of r only, and where $W^a_{i,0}$ denotes the time derivative of W^a_i.

Using (6) and (7) and performing the θ, ϕ integrations we obtain (the prime denoting differentiation with respect to r)

$$\begin{aligned} E &= -\int \mathfrak{L}\,dr\,d\theta\,d\phi \\ &= -\frac{4\pi}{\kappa^2}\int\left(2\sqrt{\Delta} - \frac{B'C'}{\sqrt{\Delta}} + \frac{B'^2C}{2B\sqrt{\Delta}}\right)dr + \frac{M^2\pi}{\kappa^2}\int\left(\frac{r^2}{B\sqrt{\Delta}}\right)^{\alpha}\left[[-2B + 6B(A+C) - 4r^2(A+C)]\frac{1}{\sqrt{\Delta}}\right. \\ &\quad \left. + \left(12r^2 - \frac{2r'}{B} - 12B\right)\sqrt{\Delta}\right]dr + 2\pi\int\left[\frac{2CU'^2}{\sqrt{\Delta}} + (cu^2 + 2u)^2\frac{\sqrt{\Delta}}{B}\right]dr\,, \end{aligned} \tag{8}$$

where $\Delta = AC + D^2$ (>0). We note that in (8) D always occurs in the combination $\Delta = AC + D^2$. Making use of this, we shall exchange D for Δ as the variable of choice. It is clear that stationarity of E under small variation of A, C, and Δ will yield independent equations only when $D \neq 0$. Krive and Sitenko[5] have considered the $D = 0$ solutions of the linearized equations obtained from E. In this note we confine our attention to solving the full equations, but with $D \neq 0$. Variations with A, B, C, Δ, and U yield Eqs. (9)–(13):

$$D \neq 0, \quad 6B - 4r^2 = 0 \ , \tag{9}$$

$$\left(\frac{C'}{\sqrt{\Delta}} + \frac{2C}{r\sqrt{\Delta}}\right)' + \frac{2C}{r^2\sqrt{\Delta}} - \frac{9\kappa^2}{8r^4}\sqrt{\Delta}\,(eu^2 + 2u)^2 + \frac{M^2}{4}\left(\frac{3}{2\sqrt{\Delta}}\right)^\alpha \left[\frac{2}{\sqrt{\Delta}}(3A + 3C - 1 + \alpha) - \frac{3\sqrt{\Delta}}{2}(\alpha + 5)\right] = 0 \ , \tag{10}$$

$$\Delta'/\Delta = \tfrac{3}{2}\kappa^2 U'^2/r \ , \tag{11}$$

$$1 - \frac{2(rC)'}{3\Delta} - \frac{M^2}{4}\left(\frac{3}{2\sqrt{\Delta}}\right)^\alpha \left[\frac{2}{3\Delta}(1+\alpha) + \tfrac{1}{2}(1-\alpha)\right] r^2 - \frac{\kappa^2}{2}\left[\frac{3}{4r^2}(eu^2 + 2u)^2 - \frac{Cu'^2}{\Delta}\right] = 0 \ , \tag{12}$$

$$\left(\frac{Cu'}{\sqrt{\Delta}}\right)' - \frac{3\sqrt{\Delta}}{2r^2}(eu^2 + 2u)(eu + 1) = 0 \ . \tag{13}$$

We have used the solution to (9) to simplify Eqs. (10)–(13).

The following forms for B, U, A, C, and Δ can easily be shown to satisfy these equations:

$$B = \tfrac{2}{3} r^2,$$

$$U = -\beta/e, \quad \beta = 0, 1, 2 \ ,$$

$$\Delta = \Delta_0, \quad \text{a constant of integration} \ ,$$

$$C = \frac{3\Delta_0}{2} + \frac{C_0}{r} + \left[\frac{3\kappa\beta(\beta - 2)}{4e}\right]^2 \frac{\Delta_0}{r^2} - \frac{M^2}{12}\left(\frac{3}{2\sqrt{\Delta_0}}\right)^\alpha \left[1 + \alpha + \frac{3\Delta_0}{4}(1+\alpha)\right] r^2 \ ,$$

and $A(r) = \frac{2}{3} + \frac{3}{2}\Delta_0 - C(r)$ determines $A(r)$. C_0 is another constant of integration.

We note that the function $D (= (\Delta - AC)^{1/2})$ may become imaginary for some values of r unless we restrict the parameters in our solution in such a way as to make D real everywhere. Requiring $\Delta - AC > 0$ implies

$$C^2 - (\tfrac{2}{3} + \tfrac{3}{2}\Delta_0)C + \Delta_0 > 0 \ . \tag{14}$$

One particularly simple way of satisfying (14) is to choose $\Delta_0 = \frac{4}{9}$ and (so as to exclude $D = 0$) $C \neq \frac{2}{3}$. This latter condition may be satisfied by suitably restricting the other parameters which appear in C. This is only one way of ensuring that D is real everywhere—other more complicated possibilities exist and a particular choice may be relevant for a discussion of confinement in hadron physics.[6]

We note also that in the limit that the gauge fields vanish ($\beta = 0$ or 2)[7] our solutions reduce to those obtained by Salam and Strathdee.[2] Recently, Wang[8] considered the analogous problem in the Einstein-SO(3) theory (essentially the $M \to 0$ limit of our theory). In this limit, our solutions reduce to those found by him. Note that $\beta = 1$ must be chosen to obtain the explicit solutions presented by Wang. Further, a transformation of the time coordinate (which diagonalizes $f_{\mu\nu}$) is necessary for a formally identical result.

We end with a few comments on possible extensions of the work presented here. As has been pointed out,[3] although $g_{\mu\nu} = \eta_{\mu\nu}$ may be a physically reasonable approximation, many important questions cannot be answered within this framework—the role played by coordinate singularities being one. Extension to the $g_{\mu\nu} \neq \eta_{\mu\nu}$ case would, therefore, be worthwhile. We do not anticipate any difficulty in this extension. Another possible extension, and one perhaps more relevant to hadronic physics, is to the more reasonable group SU(3). Yet another possibility is to consider nonsingular 't Hooft[9] type solutions for the gauge fields—whether such solutions are possible in the context of f-g theory is an interesting question to examine. We hope to be able to consider these problems in the near future.

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HAWKING RADIATION AND STRONG GRAVITY BLACK HOLES

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We show that the strong gravity theory of Salam et al. places severe restrictions on black hole evaporation. Two major implications are that: mini black holes (down to masses $\sim 10^{-16}$ kg) would be stable in the present epoch; and that some suggested mini black hole mechanisms to explain certain astrophysical phenomena would not work. The first result implies that f-gravity appears to make black holes much safer by removing the possibility of extremely violent black hole explosions suggested by Hawking.

1. Introduction. Within the framework of the two-tensor f–g theory of Isham et al. [1], gravity couples to hadrons via an f–g mixing analogous to the γ–ρ^0 coupling of the vector meson dominance model of hadron electrodynamics. At the simplest level the theory incorporates two spin-2 particles (the f and the g) which are governed by a modified Einstein-type lagrangian containing a mixing term, which provides the f–g coupling, and the usual Einstein lagrangians for the f and g fields. For our purposes the essential difference between the g and the f is that the coupling for g is $G_g = 6.67 \times 10^{-11}$ (mks), while for f it is the hadronic coupling $G_f \sim 10^{28}$ (mks).

It has been suggested that if hadrons are pictured as f-black holes [2], Hawking radiation [3,4] type ideas may provide an interesting explanation of the concept of hadronic temperature [5] in particle physics. In the present work we wish to pursue these ideas with reference to neutron stars and big-bang cosmologies, where hadronic environments exist and hence f–g theory may be expected to play an important role. The suggestion that black holes may be formed in the cores of neutron stars has been put forward [6] as a possible explanation of various astrophysical phenomena. We analyse these suggestions and consider the implications of f-gravity in such a discussion.

Strong gravity may also play a vital role in black hole evaporation processes to determine whether a mini black hole of a given mass would evaporate away [7,8]. We find that, if f-gravity is accepted, much smaller mini black holes from the initial big bang could be expected to have survived up to now than is otherwise supposed [7,9].

Our basic argument is that at sufficiently small distances the f and the g can be regarded as equivalent (for our purposes) except for the difference of couplings $-G_f/G_g \sim 10^{39}$. Thus, for distances within the range of the f meson, f-gravity would be expected to dominate. It follows that the surface temperature of an f-black hole would be some thirty-nine orders of magnitude lower than that of an equally massive g-black hole. Thus much smaller mini black holes could be expected to survive to the present epoch. This sharp reduction of the masses of mini black holes also makes it much less unlikely that such holes may be forming now.

By comparing f-gravity predictions for neutron stars with those obtained by using g-gravity only [6], it may be possible to test the validity of f-gravity theory (assuming black hole evaporation to be valid) using the ideas presented in this note.

After discussing the concept of hadrons as f-black holes, in the next section we go on to investigate the

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implications of f-gravity in some astrophysical situations. The last section consists of a summary of our results and a brief discussion.

2. Hadrons as f-black holes? Following Salam and Strathdee [5], we assume that we can obtain results for f-black holes from the formulae for g-black holes by replacing G_g by G_f, while leaving everything else unaltered. However, we must bear in mind that the range of distances over which f-gravity may be assumed to be applicable is (approximately) given by

$$\text{Range} \sim \hbar/m_f c \sim 10^{-16}\ \text{m}\,, \tag{1}$$

$\hbar$ being Planck's constant, c being the speed of light, and m_f – the mass of the f meson – is a typical spin-2 mass from the particle data booklet [10]. For greater distances, f-gravity effects will obviously be negligible.

Taking over the g-gravity formulae, we find that the surface temperature of an f-black hole is given by

$$T = K\hbar/2\pi kc\,, \tag{2}$$

where k is Boltzmann's constant and K – the surface gravity – is given by

$$K = 4\pi(Rc^2 - G_f M)A^{-1}\,. \tag{3}$$

The radius R and area A of the trapped surface being given by

$$Rc^2 = G_f M + (G_f^2 M^2 - J^2c^2/M^2 - G_f Q^2)^{1/2}\,, \tag{4}$$

$$Ac^4 = 4\pi G_f\,[2G_f M^2 - Q^2 + 2(G_f^2 M^4 - J^2c^2 - G_f M^2 Q^2)^{1/2}]\,, \tag{5}$$

where M, J and Q are the mass, angular momentum and charge of the Kerr–Newman black hole (in mks units).

Assuming that hadrons may be regarded as f-black holes and that Hawking radiation ideas apply, the question arises whether the proton – treated as an f-black hole – is stable, or will it evaporate away? A simple calculation using the above formulae shows that an f-black hole having proton mass, charge and angular momentum radiates at a temperature of $\sim 2 \times 10^{11}$ K. The radius of its event horizon is $\sim 3 \times 10^{-16}$ m.

Of course, there is nothing sacred about the value of G_f taken here. It could even be an order of magnitude different from the value taken. Thus, it could be argued that if $G_f \sim 5 \times 10^{28}$ (mks) hadrons of ~ 5 GeV/c^2 mass could be treated as f-black holes, having the correct radii. The problem, however, is that the surface temperature of these "hadrons" is of order 10^{11} K. If this picture of the hadronic world is to survive it is clear that we must assume the hadrons are in a heat bath of $T \sim 10^{11}$ K. It turns out that the concept of temperature already exists in hadron physics [11]. One of the latest manifestations of the use of this concept is in the work of Bartke et al. [12], who show that hadronic spectra, when expressed in terms of the transverse energy, can be fitted with a universal type of thermodynamical distribution with one common temperature which is approximately $kT \approx 120$ MeV or $T \approx 10^{12}$ K. If one now assumes that this is the temperature of the hadronic world, then hadrons may be thought of as stable f-black holes.

One may, however, question the stability of single hadrons. For these it may, at first sight, be argued that in a bootstrap type model each hadron, being composed of many others, is automatically in a hadronic environment. On closer examination, however, this explanation breaks down, as the black hole temperature is viewed from *outside*, where only the mass, charge and angular momentum of the black hole are apparent but no other internal structure. If the black hole surface temperature is much above the ambient temperature, the black hole would evaporate away violating baryon-number conservation. Thus, unless some mechanism can be found, whereby decaying hadrons preserve baryonic number as they evaporate away, it does not seem possible to construct a black-hole picture of hadrons. Even if we could construct such a mechanism, it is not entirely clear how the stability of the proton could be accounted for, in view of the fact that the temperature of radiation *increases* as the mass of the black hole *decreases*.

3. f-black holes, neutron stars and the big bang. So far, we have been considering hadrons in free space, but it is inside a neutron star that the concepts of f-black holes really come into their own – in a hadronic environment. Thus, when considering mini black holes forming inside neutron stars we can expect f-gravity to have a decisive influence. We shall consider this situation next.

Let us define the *minimum mass, m,* of a black hole as that mass at which a black hole will be stable (i.e. in thermodynamic equilibrium with its environment) at some given temperature. Using eqs. (2)–(5) we may write this minimum mass for a Schwarzschild black hole as

$$m_g = \hbar c^3/8\pi k\, G_g T\,. \tag{6}$$

A less massive black hole would have a higher temperature than its environment and would radiate away all its energy more and more rapidly. The minimum mass for an f-black hole would be

$$m_f = \hbar c^3/8\pi k\, G_f T\,, \tag{7}$$

so that

$$m_f/m_g = G_g/G_f \sim 7 \times 10^{-39}\,. \tag{8}$$

This is the basis of our claim that the minimum mass of an f-black hole is much less than that of a g-black hole at the same temperature.

The radius of a g-black hole (Schwarzschild) is

$$R_g = 2G_g M/c^2\,. \tag{9}$$

Thus, for just-stable f-black holes compared with just-stable g-black holes we see that

$$\frac{R_f}{R_g} = \frac{2G_f m_f/c^2}{2G_g m_g/c^2} = \frac{G_f m_f}{G_g m_g} = 1! \tag{10}$$

Also, the density of the black hole is given by

$$\rho = 3c^6/32\pi G^3 M^2\,. \tag{11}$$

Thus for the minimum mass black holes

$$\frac{\rho_f}{\rho_g} = \frac{G_g^3 m_g^2}{G_f^3 m_f^2} = \frac{G_g}{G_f} \sim 7 \times 10^{-39}\,. \tag{12}$$

So mini f-black holes are much less dense than mini g-black holes.

It is interesting to consider whether f-black holes might reproduce the Jacobs–Seitzer [6] mechanisms with lower densities. It should be noted that these authors require "density spikes" of $\sim 7 \times 10^{55}$ kg/m^3 for their largest black holes (10^{12} kg) in an average density of $\sim 10^{18}$ kg/m^3, their "spike density" is thirty-eight orders of magnitude larger than the surrounding density! For their smaller black holes $\sim 10^{-8}$ kg the "spike" density would be $\sim 7 \times 10^{95}$ kg/m^3. It could be expected that f-black holes would form at much lower densities.

For f-black holes the usual ideas of collapse, based on a long-range gravitational force cannot be used. Instead, here we require that the matter to be collapsed should be inside a volume of radius $\sim 10^{-16}$ m. For a single nucleon this would mean that it should be compressed to a density of $\sim 10^3$ times the nuclear density, i.e. 10^{21} kg/m^3. For more nucleons, a proportionately higher density would be required. It is obvious that before the density could be reached where many nucleons would be compressed, individual nucleons would collapse to f-black holes. Certainly, long before the Jacobs–Seitzer "spike" could be reached, nucleons would form f-black holes, at densities $\sim 10^{53}$–10^{93} lower than theirs.

What could be expected to happen when the f-black holes start forming? If the core temperature were much below the black hole's temperature, $\sim 5 \times 10^{11}$ K, the black hole would radiate its energy away, thereby causing a reduction in the density. Thus, even if 10^3 nuclear densities were to occur in the core of the neutron star, they would disappear. If the core temperature did not allow the black hole to evaporate, the stable f-black hole would have an event horizon encompassing $\sim 10^3$ nucleons, which would presumably also collapse forming a much larger (and hence more stable) black hole, which would take in many more nucleons and so collapse the whole star. We must conclude that this process does not occur, as neutron stars are seen to exist. Thus, on the basis of Hawking radiation and f–g theory we must conclude that either densities $\sim 10^3$ nuclear densities do not occur, or the temperature of the core of a neutron star is much less than 5×10^{11} K.

Let us now briefly consider the effects of f-gravity on black holes produced in a big-bang cosmology. We have already shown that f-black holes of a given mass radiate at a much lower temperature than g-black holes. Thus, whereas a g-black hole of 10^{-16} kg would radiate at a temperature of 10^{22} K and thus disappear instantly, f-black holes would be stable, at a background temperature of 2.7 K. Thus very much smaller black holes produced in the big bang could be expected to survive to this day!

4. Conclusion. We have seen that mini black holes, which would be expected to "evaporate" by the Hawking process according to ordinary gravity theory, might be "held together" as it were by f-gravity. This could be regarded as being due to a "higher potential" through which the radiation must tunnel for the black hole to evaporate. It would not matter whether the hole was formed by f-gravity processes or not – it would be held together by f-gravity. Thus, if f-gravity is valid then very small black holes could exist.

The very attractive mechanisms of Jacobs and Seitzer would have to be discarded if f–g theory holds, since the energy released by a few nucleons collapsing into an f-black hole and radiating would be ~10–100 J, which would not be adequate for any of their mechanisms, except for avoiding neutron star collapse. The extremely high "spike" required for their mechanisms might anyhow make their theory unlikely. If, however, their theory *can* be validated, it would disprove f–g theory.

It should be remarked that f-gravity appears to make mini black holes much safer in that they are not likely to evaporate with such violence as they would if f-gravity did not exist. This implies that we cannot hope to see an explosion of a mini black hole as it would pass utterly unnoticed.

We finally point out that whereas the radius of a hadron is approximately its f-gravity Schwarzschild radius and its f-black-hole temperature is the hadronic temperature, it does not, at present, appear to be feasible to regard a hadron as an f-black hole.

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PHYSICAL REVIEW LETTERS

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Finite-Temperature Gauge-Theory Effects on Calculations of the Cosmological Baryon Excess

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The relevance of finite-temperature gauge-theory effects on computations of the cosmological baryon-to-entropy ratio is discussed.

Recently, several authors[1-3] have invoked an interesting synthesis of big-bang cosmology on the one hand and unified gauge theories on the other to achieve a dynamical understanding of the observed baryon-to-entropy ratio of the present day universe. It would appear that there are three essential ingredients to the problem of baryon-excess generation: (a) The existence of an epoch in the evolution of the universe during which certain of the respective number densities of the various species of particles present (photons, leptons, intermediate vector bosons, quarks, and Higgs bosons) were out of thermal equilibrium; (b) that this epoch coincided with a CP- and C-nonconserving phase.[4] (c) The existence of baryon-nonconserving interactions. The first of these (a), is largely a question of cosmology and statistical mechanics, while the other requirements are met within the framework of current grand unified models of strong and electroweak interactions which predict such exotic interactions. Evidently, the relevant epoch is thought to have occurred at approximately 10^{-35} sec after the initial singularity when the ambient temperature was of the order of 10^{15} GeV or 10^{28} K. What we wish to remark is that in considering requirement (c) it may reasonably be expected, in view of the extremely high temperatures involved, that finite-temperature gauge-theory effects play a significant role. The point of this note is to make a rough estimate of such corrections.

Before proceeding, it is important to first distinguish the purely thermodynamic aspects of the baryon-excess computation from the unified-gauge-theory input. The former has been dealt with at considerable length[1-3] and need not concern us here. Following the procedure and notation of Weinberg[2] one obtains for the baryon-number-to-entropy ratio:

$$kn_B/s \approx 0.13(N_X/N)\Delta B\,, \tag{1}$$

where N is the total number of particle states with masses less than m_X and N_X is the total number of X- and $\bar{X}$-boson states of mass m_X, while ΔB is the mean net baryon number generated in a single X or $\bar{X}$ decay.

It should be noted that the above analysis has thus far had to do with thermodynamics and cosmology only; it is in the computation of the quanity ΔB that recourse has to be made to a particular unified gauge theory. Thus it is here that finite-temperature gauge-theory effects may be expected to be relevant.

The main contribution to ΔB is thought[3] to come from X-boson decay. Weinberg and Nanopoulos have computed the various baryon-number-nonconserving decay amplitudes (in the zero-temperature gauge-theory limit) which contribute to ΔB. To illustrate our point that a finite-temperature gauge theory may change the estimate of ΔB, it will be sufficient to consider (see Ref. 3), for example, X_s decay only. Symbolically, one has

$$\Delta B \sim (\mathrm{Tr}\,\Gamma_i^\dagger\Gamma_i)^{-1}\{\mathrm{Im}f(\Gamma_i, g_j)\,\mathrm{Im}I_{sv}[(m_{X_v})_j/(m_{X_s})_i] + \mathrm{Im}f(\Gamma_i, g_j)\,\mathrm{Im}\,I_{sv'}[(m_{X_{v'}})_j/(m_{X_s})_i] + \mathrm{Im}f(\Gamma_i, \Gamma_j)\,\mathrm{Im}I_{ss}[(m_{X_s})_j/(m_{X_s})_i]\} + \dots \qquad (2)$$

with species i not the same as species j. Here, the f's are complex functions of the various couplings involved in the respective processes that contribute; i, j denote various species of bosons $X_{v,v'}$ (gauge bosons) and X_s (Higgs bosons), and also summation over fermion and internal symmetry indices. The important point to note is that the only place finite-temperature corrections would be anticipated is in the Feynman integrals,

$$I_{sv,v'}[(m_{X_{v,v'}})_j/(m_{X_s})_i] \text{ and } I_{ss}[(m_{X_s})_j/(m_{X_s})_i].$$

This is because, as is known,[5] the leading contribution of nonzero temperature is to the mass terms in the effective Lagrangian [stated differently, the effect of finite temperature can be accounted for, to leading order, by replacing the zero-temperature mass m by an effective temperature-dependent mass $m(T)$].

It may be argued that since only ratios of masses are involved, finite-temperature effects would tend to cancel. However, to have a nonzero ΔB in the first place, the gauge vector and Higgs structure of the theory must be sufficiently complicated so as to include at least two species (see Ref. 3 for a discussion of this point) with their respective *different* couplings, etc. Thus, at finite temperatures the numerator and denominator in $[m_{X_{v,v'}}(T)]_j/[m_{X_s}(T)]_i$ and $[m_{X_s}(T)]_j/[m_{X_s}(T)]_i$ might vary appreciably differently under temperature changes for their ratio to depart from its zero-temperature value.

To make this quantitative let us consider a local O(n) gauge theory with one n vector of Higgs fields. The potential ($\mu_0^2 > 0$ and is the bare mass) is given by

$$V(\varphi) = -\tfrac{1}{2}\mu_0^2\varphi_i\varphi_i + \tfrac{1}{4}\lambda(\varphi_i\varphi_i)^2. \qquad (3)$$

One-loop temperature corrections can be taken into account by working with the finite-temperature effective potential

$$V_{\mathrm{eff}}(\varphi) = \tfrac{1}{2}\mu^2(T)\varphi_i\varphi_i + \tfrac{1}{4}\lambda(\varphi_i\varphi_i)^2, \qquad (4)$$

where

$$\mu^2(T) = -\mu^2(0) + \tfrac{1}{12}\lambda T^2(2+n) + \tfrac{1}{16}g^2T^2(n-1) \qquad (5)$$

for $T < T_c$. Here $\mu^2(0)$ is the renormalized mass and g is the gauge coupling.[6] Extrapolating to the more complicated grand unified theory responsible for baryon-nonconserving processes, one will obtain something like the following expression for the ratio of the temperature-dependent masses discussed earlier:

$$\frac{[m_{X_s}(T)]_j}{[m_{X_s}(T)]_i} \sim \frac{-[m_{X_s}^2(0)]_j + T^2f(\lambda_j, g_j)}{-[m_{X_s}^2(0)]_i + T^2f(\lambda_i, g_i)}, \qquad (6)$$

where f and h are some functions of the couplings and group theoretic parameters. Since the couplings[7] λ_j, g_j are different from λ_i, g_i, the finite-temperature ratio could, in principle, be different from the corresponding zero-temperature ratio.

In order to see if this is likely to happen, let us examine the ratio of finite-temperature to the zero-temperature Higgs mass in the O(n) example. This is given by

$$M^2(T)/M^2(0) = -2\mu^2(T)/-2\mu^2(0) = 1 - T^2/T_c^2; \quad T < T_c \qquad (7)$$

and the values of this ratio just below the symmetry restoration temperature are exhibited in Table I.

It can be seen that one has to be extremely

TABLE I. $M^2(T)/M^2(0)$ as a function of T for $T_c = 10^{15}$ GeV.

T (GeV)	$M^2(T)/M^2(0)$
10^{15}	0
$10^{14.8}$	0.6
$10^{14.5}$	0.9
10^{14}	0.99

close to the critical temperature[8] to obtain a significant departure from unity. By the time the temperature has fallen below T_c by about one order of magnitude the scalar (and gauge) field has acquired its full zero-temperature field-theory mass.

Assuming the same sort of qualitative behavior to pertain for a more realistic grand unified model, one would be tempted to conclude that finite temperatures would not cause any substantial modification; i.e., the temperature-dependent corrections in Eq. (6) are negligible. However, one is prevented from doing so because, lacking a more specific unified gauge model than is available at present, one does not know the precise masses of the X bosons and hence how close in fact these masses are to the symmetry restoration temperature T_c.

In this note we have contented ourselves with pointing out the possible relevance of finite-temperature gauge-theory effects on the baryon-excess calculation. A less superficial treatment than the one above does not seem to be merited at the present juncture, in the main because the zero-temperature gauge-theory value for kn_B/s is at best determined to be in the fairly wide range of 10^{-12} to 10^{-7}.[9] One really has to await a specific grand unified model before deciding the issue. The burden of our remarks here has been to suggest that, given such a unified gauge model,[10] finite-temperature effects should be taken into account before deciding whether the mechanisms proposed can really explain the baryon-to-entropy ratio.

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Note added.—One thing to note from the above considerations is that at temperatures greater than T_c, the Higgs scalars become physical massive particles with masses[11] of order kT while all other species of particles (vector bosons, fermions, and photons) are massless.[12] It is reasonable then to conclude that these Higgs scalars would be as abundant, at these early stages of the evolution of the universe,[13] as photons and other massless particles. Such a picture of the universe close to $t=0$ is drastically different from the conventional assumption that at these early times the universe consisted only of massless radiation.

It is intriguing to pose the question whether such a scenario would have any cosmological consequences which might have left some present-day residual effects. If such were the case, one would have independent cosmological evidence for the existence or otherwise of the Higgs scalar. We shall return to this point elsewhere.

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[7]Strictly one should be working with the running, or effective, values of these couplings at the mass scale relevant for the energies at which the baryon excess is produced.

[8]A word of warning; it should be noted that in the perturbative framework (see Ref. 5) with which we have treated the $O(n)$ example, perturbation theory is not expected to remain valid within an order of magnitude or so of T_c. However, this inadequacy can pre-

sumably be transcended by the use of a renormalization-group-improved formalism.

[9]This value has been obtained by Weinberg and Nanopoulos (see Ref. 3) in the limit in which vector boson contributions to scalar decay are neglected. If the former are, in fact, included, the baryon-to-entropy ratio is shifted to the range 10^{-9} to 10^{-4}. This is to be compared with the present experimental value of $\simeq 10^{-9}$.

[10]The strategy then would be to determine precisely which (if any) of the X bosons are close (say within an order of magnitude) to the symmetry restoration temperature. If there are none such, finite-temperature effects can presumably safely be ignored. However, if this is not the case, finite-temperature effects would certainly seem to be crucial. Furthermore, in the latter event, a renormalization-group formalism would have to be used because of the breakdown of validity of the naive perturbative approach near T_c.

[11]The finite-temperature mass of the Higgs scalar is proportional to the product of the temperature and the (running) coupling constants of the theory. If asymptotic-freedom arguments persist beyond T_c, then one may conclude that $\mu(T) < kT$ unless the non-asymptotically-free self-couplings become extremely large. With $\mu(T) < kT$ the Higgs scalars would certainly be copiously produced thermally.

[12]The masslessness of the vector bosons above T_c has been called into question by certain authors [M. B. Kislinger and P. D. Morley, Phys. Rev. D 13, 2765 (1976)]. However, the difficulties in handling the infrared divergences of a non-Abelian theory renders their conclusions suspect [see A. D. Linde, Rep. Prog. Phys. 42, 389 (1975) for a discussion of this point].

[13]The lack of a sensible field-theoretic description at such high energies ($10^{16} < T < 10^{19}$ GeV) and in particular the absence of a satisfactory quantum theory of gravity confine one to making only very general speculative remarks. Having said this, it is rather remarkable all the same the Hawking (see the Proceedings of the Marcel Grossmann Meeting, Trieste, 1979), from completely different (topological) considerations, has envisaged a primordial scenario (at a length scale of $M_{\rm Planck}^{-1}$) in which scalar fields have "induced" masses of $O(M_{\rm Planck})$ whereas fermions and vector bosons are effectively massless. There is thus a highly suggestive similarity between this and the emergent scenario from grand unified gauge theories above T_c (in which the Higgs scalars are not only the *only* massive particles, but also have masses tending towards $M_{\rm Planck}$ as the temperature rises).

Spontaneous *CP* Nonconservation in Theories with More Than Four Quarks

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It is shown that the requirements of spontaneous CP breaking and natural flavor conservation lead to a class of theories where CP nonconservation is due solely to Higgs exchange, for an arbitrary number of fermion generations.

Although CP nonconservation can be easily incorporated[1] in unified gauge theories, one is still faced with the challenge of understanding the smallness[2] of the violation. It has been pointed out by Lee[3] and Weinberg,[4] that within unified gauge theories of weak and electromagnetic interactions, the Higgs bosons can provide a mechanism for a naturally small CP-invariance violation. In unified gauge theories, the fermion Yukawa interactions are such that Higgs-boson exchange leads to an effective Fermi interaction of strength $G_F\, m_F^2/m_H^2$ (where G_F is the Fermi coupling constant, and m_F and m_H are the fermion and Higgs-boson masses, respectively). Thus, in theories where Higgs-particle exchanges are solely responsible for CP nonconservation, the smallness of the violation is naturally understood; it merely reflects the fact that Higgs bosons are much heavier[5] than the light fermions. It is clear that in order for this explanation of the size of CP nonconservation to hold, it is necessary that CP nonconservation arises only through Higgs-boson exchange and from no other sector of the theory. An example of this class of theories has been given by Weinberg,[3] in a model with four quarks and three Higgs doublets. It is well known that if there are only four quarks and no right-handed currents,[6] the gauge interactions of the vector mesons automatically conserve CP. However, for three or more Higgs doublets,[4] CP invariance will be violated through Higgs-particle exchange. In a theory with three quark doublets (as it seems to be required by experimental evidence), the situation is more complicated, since in general the Cabibbo-like mixing matrix contains a CP-nonconserving phase δ. In this case, one loses control over the strength of CP nonconservation, since the phase δ is is general arbi-

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RADIATIVE CORRECTIONS AND GAUGE HIERARCHIES

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ABSTRACT

We consider the question of hierarchies in a simple O(3) model completely broken to O(1). It is shown that the one loop upper bound to the hierarchy is not independent of the scalar potential parameters. We discuss higher loop effects but point out that, as at the tree level, order of magnitude estimates can be misleading. Our analysis, which proceeds along the lines of Gildener, is of considerable help in determining whether, if at all, recent arguments put forward by some authors are in contradiction with the results of Gildener.

A large hierarchy, or the much stronger breaking of some gauge symmetries than others, is an essential ingredient of all grand unified gauge models incorporating spontaneous symmetry breaking. The latter is usually studied at the semiclassical level where one assumes that the minima of the tree-level effective potential give the true minima of the theory. It turns out however, as was amply demonstrated in the paper of Coleman and E.Weinberg [1], that radiative corrections to the tree-level potential can drastically change the vacuum structure of a theory. Indeed, as these authors showed, in certain cases, radiative corrections can even be the dominant force causing symmetry breaking.

Several other interesting results have been obtained using the ideas developed in ref. 1. One of the more recent is contained in a paper of Gildener [2] who has conjectured that a superstrong hierarchial breakdown cannot be obtained in the usual way. It is claimed that one cannot artificially establish a gauge hierarchy of any desired magnitude by adjusting the scalar-field parameters in the Lagrangian and using the tree-level approximation to the potential. Radiative corrections set an upper bound on such a hierarchy which is independent of the scalar field tachyonic masses and their self couplings.

The problem of gauge hierarchies has, more recently, been discussed by several authors [3,4] who evidently obtain different results. Unfortunately, none of these authors follow the work of ref. 2. closely enough to allow one to ascertain precisely where, if at all, their arguments are in contradiction with those of ref. 2. We present in this letter the results of an explicit investigation of this problem to the one-vector loop level which was carried out along the lines of Gildener's [2] work for the case of an O(3) model, which is the simplest O(N) model allowing the study of hierarchies. Such an analysis provides interesting insight into the problem.

Our treatment shows clearly that although one-vector-loop corrections provide an upper bound to the allowed hierarchy, the bound is not independent of the scalar potential parameters. It is in fact possible to choose a set of parameters which allows an arbitrarily large hierarchy both at the tree and at the one-loop level.

We also point out that order of magnitude estimates of higher loop effects used by some authors [4] can, as at the tree level, give misleading results.

Consider, for simplicity, an O(3) model which contains, in addition to the gauge fields $\underline{W}_\mu$, two scalar fields $\underline{A}$ and $\underline{B}$ which transform as vectors under O(3). Assuming the theory possesses the discrete symmetry $\underline{A} \to -\underline{A}$, the tree level approximation to the effective potential is

$$V_o = -\frac{1}{2}m_1^2\underline{A}^2 - \frac{1}{2}m_2^2\underline{B}^2 + \frac{1}{4}f_1(\underline{A}^2)^2 + \frac{1}{4}f_2(\underline{B}^2)^2 + \frac{1}{2}f_3\,\underline{A}^2\underline{B}^2 + \frac{1}{2}f_4(\underline{A}\cdot\underline{B})^2 \qquad (1)$$

We take m_1^2, $m_2^2 > o$ to ensure that the origin is unstable and spontaneous symmetry breaking occurs. The requirement of a hierarchy is that

$$R = M_H^2 / M_L^2 \gg 1 \qquad (2)$$

where M_H and M_L are the masses of the heavy and light gauge bosons. At the tree level it can be shown [2] that R is given by

$$R = a^2/b^2 = \frac{f_2}{\Delta f_3}\left(1 - \frac{f_3\, m_2^2}{f_2\, m_1^2}\right) \qquad (3)$$

where $a^2 = <\underline{A}>^2$, $b^2 = <\underline{B}>^2$ in the symmetric non-trivial vacuum $\Delta f_3 = f_1\frac{m_2^2}{m_1^2} - f_3$, and where $<\underline{A}>$ is associated with the initial stage of the breaking. The various constraints to which the minimization is subject then show that $\Delta f_3 > o$ and that $R < \frac{2f_2}{\Delta f_3}$, where f_2 has been assumed to be greater than or equal to f_1 so as to maximise the tree level hierarchy.

Gildener [2] now argues that one-loop contributions to the effective potential will set a lower bound on Δf_3 (which we denote by δf_3) which is independent of the parameters appearing in (1) so that,

$$R < 2f_2 / \delta f_3 \qquad (4)$$

Arguing on the basis of order of magnitude estimates of one-loop effects he further claims that

$$R < \frac{\partial f_3}{\delta f_3} \ll O(\alpha^{-1}) \tag{5}$$

if the gauge coupling is taken to be the electromagnetic coupling. This, of course, implies that the upper bound on the hierarchy cannot be transcended by adjusting the scalar potential parameters. The essential result of this letter is to show by an explicit evaluation of δf_3 that this is not so.

To do this we repeat the above analysis for the one-loop corrected potential but take into account only the vector loops which, for weakly coupled scalars, are the most important.

The same problem has been treated by the authors of ref. 4. who arrive at the final conclusion that there are no limits on gauge hierarchies due to radiative corrections contrary to the assertions of Gildener. Our work seems to indicate disagreement with this last conclusion, though we too find that the one-loop bound can be transcended.

Following Coleman and E.Weinberg [1], we write down the one-loop corrections to the effective potential induced by vector particles. In the Landau gauge these are

$$V_1 = \frac{3}{64\pi^2} \mathrm{Tr}\left[M_v^4 \ln \frac{M_v^2}{\Lambda^2} \right] \tag{6}$$

where M_v^2 is the vector boson mass matrix and Λ^2 is an arbitrary renormalisation subtraction point. Explicitly,

$$V_1 = \frac{3g^4}{32\pi^2} \left[\lambda_+^2 \ln \frac{\lambda_+}{\Lambda^2} + \lambda_-^2 \ln \frac{\lambda_-}{\Lambda^2} + (\underline{A}^2 + \underline{B}^2)^2 \ln \frac{(\underline{A}^2 + \underline{B}^2)}{\Lambda^2} \right] \tag{7}$$

where $$\lambda_\pm = \frac{1}{2}\left[\underline{A}^2 + \underline{B}^2 \pm \sqrt{(\underline{A}^2 - \underline{B}^2)^2 + 4(\underline{A}\cdot\underline{B})^2} \right] \tag{8}$$

The new extrema are now determined.* Following some tedious algebra one arrives at the result that in the true vacuum of the theory, R is given by (we work with $\Lambda^2 = \langle A^2 \rangle$ in the true vacuum).

$$R = \frac{f_2'}{\Delta f_3'}\left[1 - \frac{f_3' m_2^2}{f_2' m_1^2}\right] , \tag{9}$$

where

$$f_2' = f_2 + \frac{3g^4}{8\pi^2} - \frac{3g^4}{16\pi^2}\beta , \tag{10}$$

$$f_3' = f_3 + \frac{9g^4}{32\pi^2} , \tag{11}$$

$$\Delta f_3' = \Delta f_3 + \frac{3g^4}{32\pi^2}\left(\frac{2m_2^2}{m_1^2} - 1\right) \tag{12}$$

and $e^{\beta} = R \gg 1$.

If $g = e$ then $\frac{3g^4}{32\pi^2} \simeq \frac{3\alpha^2}{2}$, and we obtain our main result that

$$\delta f_3 = \frac{3\alpha^2}{2}\left(\frac{2m_2^2}{m_1^2} - 1\right) \tag{13}$$

---

* We have restricted ourselves to the case $\langle \underline{A} \rangle \cdot \langle \underline{B} \rangle = 0$ but it should be noted that such a condition is more stringent than is required for the existence of a hierarchy. Consider

$V(1,2,3) \equiv V(\underline{A}^2, \underline{B}^2, (\underline{A}\cdot\underline{B})^2)$, then at the stationary points,

$$A_i V_{,1} + (\underline{A}\cdot\underline{B}) B_i V_{,3} = 0$$

and

$$B_i V_{,2} + (\underline{A}\cdot\underline{B}) A_i V_{,3} = 0 .$$

The requirement of a hierarchy demands that $\langle \underline{A} \rangle$ is not parallel to $\langle \underline{B} \rangle$. This is satisfied if

$$V_{,1} = 0 , \quad V_{,2} = 0 \quad \text{and} \quad V_{,3} = 0 \quad \text{or} \quad \langle \underline{A} \rangle \cdot \langle \underline{B} \rangle = 0 .$$

For the tree-level potential the last two constraints are, of course, identical, but this need not be the case for higher loops.

We see that the one-vector-loop contributions to the effective potential provide a lower bound on $\Delta f_3'$ given by (8) which is not independent of the scalar parameters. Indeed, it is possible to consistently obtain an arbitrarily large gauge hierarchy at both the tree and the one-vector-loop levels.*)

What of higher loops? Some authors [4] have argued that higher vector loops will be negligible if

$$g^2\beta << 1 \tag{14}$$

and that this, using $g \sim e$, "leads" to $\beta << 100$. And since $R = \exp\{\beta\}$, the higher loops do not really limit the hierarchy. We wish to point out that such an order of magnitude estimate of the power of e is extremely dangerous. For illustrative purposes let us return to eqn.(14) from which it is concluded $\beta << 100$. Since the authors of ref. 4. (as indeed we) have been using the results of ref. 1., if we use their definition of α (deducible from eqn. 6.14 of ref. 1) we find that the bound on the hierarchy provided by (14) is $\sim O(10^4)$ rather than $(10)^{40}$!

What we wish to state simply is that without an accurate analysis of higher loop effects a definite statement of the bound on the hierarchy cannot be made. Thus the question of whether there is an inherent bound on the gauge hierarchy for any particular model is still [2] completely open.

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*) Assuming the f's to be $O(\alpha)$, we see that $R \sim \frac{O(\alpha)}{\delta f_3} \sim$ arbitrary if Δf_3 is set to zero and choosing $2m_2^2 \simeq m_1^2$. Of course, another possibility is to choose $\Delta f_3' = 0$ which implies the less stringent constraint $(f_3 + \frac{3\alpha^2}{2}) = \frac{m_2^2}{m_1^2}(f_1 + 3\alpha^2)$. However, the point of Gildener's analysis would seem to be to determine the bound imposed on the hierarchy attainable at the tree level by contributions to the effective potential of all higher loops. In this approach one fixes the parameters so as to give the maximum possible hierarchy at the tree level, this being the usual practice in constructing unified gauge models. Using these tree level constraints one then tries to determine the bound which higher loops impose on the tree level hierarchy. In the particular example considered here this would mean taking $2m_2^2 \simeq m_1^2$, in addition to $\Delta f_3 = 0$.

It is worth commenting, however, that had δf_3 been equal to $\frac{3\alpha^2}{2}(2m_1^2/m_2^2 - 1)$ rather than as in (13), the upper bound on R would have been $O(\alpha^{-1})$, since we have chosen from the outset to work with $m_1^2 \gg m_2^2$. It is interesting to pose the question whether there is any deep reason for δf_3 being as in (13). If not, then it should be possible to construct a model in which the one-loop effects limit R to be $O(\alpha^{-1})$. Nothing is to be gained, of course, by working with $m_2^2 \gg m_1^2$, since what we call m_1 or m_2 is a matter of choice.

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COULOMB-LIKE SOLUTIONS FOR SU(2) GAUGE FIELDS COUPLED TO STRONG GRAVITY

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ABSTRACT

Coulomb like solutions of the classical field equations for SU(2) gauge fields coupled to the strong gravity metric in the f-g theory of Isham, Salam, and Strathdee, are obtained in the limit that weak gravity is neglected.

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1. INTRODUCTION

The two tensor f-g theory of Isham, Salam, and Strathdee [1] was proposed some time ago to describe the gravitational interactions of hadrons and leptons through a gravitational analogue of the vector meson dominance hypothesis for hadron electrodynamics. The theory is based on a Lagrangian principle and uses, apart from the Einstein-Hilbert expressions for each of the two spin-2 fields, a generally covariant mixing term which provides the analogue of the $\rho^c - \gamma$ direct coupling between the f and g spin-2 fields. This mixing term is also required to give the f-meson a mass in the linearised limit of the theory and is, therefore, just the generally covariant form of the well known Pauli-Fierz (P-F) expression [2] for a massive spin-2 field.*

The Lagrangian density for the simple "matter-free" theory may thus be written symbolically as

$$\mathcal{L}_S = \mathcal{L}_{EH}(g_{\mu\nu}) + \mathcal{L}_{EH}(f_{\mu\nu}) + \mathcal{L}_{PF} \tag{1}$$

where

$$\mathcal{L}_{EH}(g_{\mu\nu}) = -\frac{1}{K_g^2}\sqrt{-g}\, R(g_{\mu\nu}) \tag{2}$$

involving the Newtonian coupling constant,

$$\mathcal{L}_{EH}(f_{\mu\nu}) = -\frac{1}{K_f^2}\sqrt{-f}\, R(f_{\mu\nu}) \tag{3}$$

involving the strong coupling constant, and

$$\mathcal{L}_{PF} = -\frac{M^2}{4K_f^2}(-g)^u(-f)^v \phi^{\alpha\beta}\phi^{\sigma\tau} g_{\alpha\sigma\beta\tau} \tag{4}$$

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* The theory is invariant under general co-ordinate transformations acting simultaneously on both metrics. It is for this reason that a generally covariant form of the P-F expression can be written.

involving the mass M of the massive spin-2 f meson. In the last expression we have used $\phi^{\alpha\beta}$ to denote the combination $f^{\alpha\beta} - g^{\alpha\beta}$ and $g_{\alpha\sigma\beta\tau}$ to denote $(g_{\alpha\sigma} g_{\beta\tau} - g_{\alpha\beta} g_{\sigma\tau})$. The parameters U and V appearing in (4) are constants restricted such that their sum equals 1/2 to ensure that $\mathcal{L}'_{PF}$ has the correct tensor density weight.

It should be pointed out, however, that the choice of the mixing term is by no means unique.[3] Indeed, several other choices were already suggested in ref.1. and different motivations led Salam and Strathdee[4] to propose a model Lagrangian combining Yang-Mills fields with tensor fields for the dynamical generation of masses of all particles involved.* For the present we shall restrict our attention to the mixing term given in (4).

With the recent discovery of spherically symmetric solutions[6] to the classical field equations of f-g theory interest in the theory has been revived and several physical applications have been proposed. It has been suggested[7] that f-gravity black holes might represent hadrons and that inside hadrons the geodesics associated with the f-metric may provide a clue to understanding confinement in hadron physics. It is also argued that Hawking radiation concepts applied to strong gravity may provide an explanation of the thermal spectrum in E_T observed in high energy collisions. These ideas have been applied also in the context of f-g theory to black hole evaporation and some astrophysical situations.[8]

At the same time the continuing experimental successes of both the Salam-Weinberg Electroweak theory and the strong interaction theory of QCD show that any successful description of the fundamental interactions

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* The work of Boulware and Deser[5] shows that the addition of a P-F type mass term leads to the appearance of an additional (ghost) scalar degree of freedom in f-g theory. Ref. 4 argued that using the following f-g Lagrangian

$$\mathcal{L} = \mathcal{L}_{EH}(g_{\mu\nu}) + \mathcal{L}_{EH}(f_{\mu\nu}) + \frac{r^2}{4}\sqrt{-f}\, \phi^{\mu\kappa} \phi^{\nu\lambda} \vec{F}_{\mu\nu} \cdot \vec{F}_{\kappa\lambda}$$

where the $\vec{F}_{\mu\nu}$ are the Yang-Mills field strengths, one might give mass to $f_{\mu\nu}$ and the gauge fields while avoiding the ho t of nature must incorporate non-abelian gauge fields. It is interesting, therefore, to couple SU(2) gauge fields to strong gravity and attempt to obtain solutions of the classical field equations of the resulting theory.

In the next section we write out the complete Lagrangian we shall use and obtain the field equations. Section 3 is devoted to looking for solutions in the limit that weak gravity is neglected. In the last section we conclude with a brief disucussion.

II LAGRANGIAN AND FIELD EQUATIONS

Before writing the Lagrangian we shall consider, it is worthwhile giving, briefly, the physical picture employed for constructing the Lagrangian for any given system in f-g theory - it not being clear which metric one should use for the various fields under consideration.

The implication of the f-g hypothesis that the Einstein graviton g, and some mixture of the known, massive, strongly interacting, spin-2 particles represents a complete analogue of the ρ°-photon scheme in vector meson dominance models of hadron electrodynamics is that the graviton interacts directly with leptons, but only indirectly - through the f-g mixing term - with hadronic matter. This idea is implemented at the Lagrangian level by postulating that in all hadronic matter parts of the Lagrangian one must use the strong metric to contract indices while all leptonic matter parts are constructed as usual with the $g_{\mu\nu}$.

Taking the SU(2) gauge fields to belong to the hadronic world the Lagrangian density for our system may be written as

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_{YM} \tag{5}$$

where

$$\mathcal{L}_{YM} = -\frac{1}{4}\sqrt{-f}\, f^{\mu\alpha} f^{\nu\beta} F^i_{\mu\nu} F^i_{\alpha\beta} \tag{6}$$

and

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + e\,\epsilon^{ijk} A^j_\mu A^k_\nu \tag{7}$$

where e is the gauge coupling and ϵ^{ijk} are the structure constants for SU(2).

In (5) $\mathcal{L}_S$ is as in (1) with

$$R(g_{\mu\nu}) = g^{\nu\sigma} R^{\mu}{}_{\nu\mu\sigma} \tag{8}$$

where

$$R^{\mu}{}_{\nu\lambda\sigma} = \Gamma^{\mu}_{\nu\sigma,\lambda} - \Gamma^{\mu}_{\nu\lambda,\sigma} + \Gamma^{\mu}_{\tau\lambda}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\tau\sigma}\Gamma^{\tau}_{\nu\lambda} \tag{9}$$

and $$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma}\right) \tag{10}$$

and similarly for R(f) constructed using everywhere $f_{\mu\nu}$ instead of $g_{\mu\nu}$ in the last three expressions.

We obtain the field equations by requiring stationarity of the action

$$S = \int \mathcal{L}\, d^4x = S_{EH}(g_{\mu\nu}) + S_{EH}(f_{\mu\nu}) + S_{PF} + S_{YM} \tag{11}$$

under infinitesimal variations of the fields appearing in the action. The changes due to the first two terms in (11) are well known, while the changes in S_{P-F} and S_{YM} are given by

$$\delta S_{PF} = \frac{M^2}{4K_f^2}\int (-g)^u (-f)^v d^4x \left\{ \phi^{\alpha\beta}\left[\phi^{\sigma\tau}\left[g_{\alpha\sigma\beta\tau}\left(u g_{\mu\nu}\delta g^{\mu\nu} + v f_{\mu\nu}\delta f^{\mu\nu}\right) + \right.\right.\right.$$
$$\left.\left.\left. + 2 g_{\alpha\mu} g_{\beta\sigma\nu\tau}\,\delta g^{\mu\nu}\right] + 2 g_{\mu\sigma\nu\tau}\,\delta\phi^{\mu\nu}\right]\right\} \tag{12}$$

and $$\delta S_{YM} = -\frac{1}{2}\int d^4x \left\{ \sqrt{-f}\left(\vec{F}^{\alpha}{}_{\mu}\cdot\vec{F}_{\alpha\nu} - \frac{1}{4} f_{\mu\nu}\vec{F}_{\alpha\beta}\cdot\vec{F}^{\alpha\beta}\right)\delta f^{\mu\nu} + \right.$$
$$\left. + 2\left[\left(\sqrt{-f}\,\vec{F}^{\lambda\sigma}\right)_{,\sigma} - e\sqrt{-f}\,\vec{A}_{\sigma}\times\vec{F}^{\sigma\lambda}\right]\cdot\delta\vec{A}_{\lambda}\right\} \tag{13}$$

where in (13) we have used the usual three-dimensional vector notation. We may now easily write the field equations for $g_{\mu\nu}$, $f_{\mu\nu}$, and the gauge fields $\vec{A}_\mu$:

$$R^{(g)}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R(g) = \frac{\kappa_g^2 M^2}{4\kappa_f^2}\left(\frac{f}{g}\right)^{v} \phi^{\alpha\beta}\left[\phi^{\sigma\tau}(u g_{\mu\nu} g_{\alpha\sigma\beta\tau} + 2 g_{\alpha\mu} g_{\beta\sigma\nu\tau}) + 2 g_{\mu\alpha\nu\beta}\right]; \tag{14}$$

$$R^{(f)}_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} R(f) = \frac{M^2}{4}\left(\frac{g}{f}\right)^{u} \phi^{\sigma\tau}\left[v\,\phi^{\alpha\beta} f_{\mu\nu} g_{\alpha\sigma\beta\tau} - 2 g_{\mu\sigma\nu\tau}\right] - \\ - \frac{\kappa_f^2}{2}\left(\vec{F}^{\alpha}{}_{\mu}\cdot\vec{F}_{\alpha\nu} - \tfrac{1}{4} f_{\mu\nu}\vec{F}_{\alpha\beta}\cdot\vec{F}^{\alpha\beta}\right); \tag{15}$$

and

$$\left(\sqrt{-f}\,\vec{F}^{\lambda\sigma}\right)_{,\sigma} - e\sqrt{-f}\,\vec{A}_\sigma \times \vec{F}^{\sigma\lambda} = 0 \tag{16}$$

Equations (14) and (15) may be written in the standard form

$$"G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}"$$

if we define the g and f energy momentum tensors as

$$T^{g}_{\mu\nu} = \frac{M^2}{2\kappa_f^2}\left(\frac{f}{g}\right)^{v} \phi^{\alpha\beta}\left[\phi^{\sigma\tau}(u g_{\mu\nu} g_{\alpha\sigma\beta\tau} + 2 g_{\alpha\mu} g_{\beta\sigma\nu\tau}) + 2 g_{\mu\alpha\nu\beta}\right], \tag{17}$$

and

$$T^{f}_{\mu\nu} = \frac{M^2}{2\kappa_f^2}\left(\frac{g}{f}\right)^{u} \phi^{\sigma\tau}\left[v\phi^{\alpha\beta} f_{\mu\nu} g_{\alpha\sigma\beta\tau} - 2 g_{\mu\sigma\nu\tau}\right] - \\ - \left(\vec{F}^{\alpha}{}_{\mu}\cdot\vec{F}_{\alpha\nu} - \tfrac{1}{4} f_{\mu\nu}\vec{F}_{\alpha\beta}\cdot\vec{F}^{\alpha\beta}\right). \tag{18}$$

In the next section we shall look for static spherically symmetric solutions to these equations in the limit that weak gravity is neglected. We shall leave to the discussion the question of obtaining exact spherically symmetric solutions to the three coupled set of equations (14) and (16).

III SOLUTIONS

Setting $\kappa_g^2 \to 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$ we look for solutions for $f_{\mu\nu}$ and the gauge fields A^i of the form*:

$$f_{\mu\nu}\,dx^\mu dx^\nu = C\,dt^2 - 2D\,dt\,dr - A\,dr^2 - B\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{19}$$

and $$A_\mu^i = W(r)\,\delta_\mu^0\,\delta_3^i \tag{20}$$

The inverse of $f_{\mu\nu}$ is given by

$$f^{\mu\nu}\partial_\mu\partial_\nu = \frac{A}{\Delta}\partial_t^2 - \frac{2D}{\Delta}\partial_t\partial_r - \frac{C}{\Delta}\partial_r^2 - \frac{1}{B}\left(\partial_\theta^2 + \sin^{-2}\theta\,\partial_\varphi^2\right) \tag{21}$$

Following Salam and Strathdee, we shall exchange D for Δ as the choice of variable where $\Delta = AC + D^2 > 0$. A, B, C, D, and W are all functions of r only. After a tedious calculation one can show that for this choice of $f_{\mu\nu}$, the only non-vanishing components of the Ricci tensor are:**

$$R_{00} = -\frac{C}{D}R_{01} = \frac{C}{2\Delta}\left(C'' + \frac{B'}{B}C' - \frac{C'}{2}\frac{\Delta'}{\Delta}\right) \tag{22}$$

$$R_{11} = \frac{B'^2}{2B^2} - \frac{B''}{B} - \frac{A}{2\Delta}\left(C'' + \frac{B'}{B}C' - \frac{C'}{2}\frac{\Delta'}{\Delta}\right) + \frac{B'}{2B}\frac{\Delta'}{\Delta} \tag{23}$$

$$R_{22} = \sin^{-2}\theta\, R_{33} = 1 - \frac{C}{2\Delta}\left(B'' + \frac{B'C'}{C} - \frac{B'\Delta'}{2\Delta}\right) \tag{24}$$

where the prime denotes differentiation with respect to r. Note that in the $K_g^2 = 0$ limit $g_{\mu\nu} = \eta_{\mu\nu}$ implies that (14) is automatically satisfied, so that we have only to consider equations (15) and (16) with $g_{\mu\nu}$ being replaced everywhere by $\eta_{\mu\nu}$.

The next task is to work out the various non-vanishing components of $T_{\mu\nu}$ for $g_{\mu\nu} = \eta_{\mu\nu}$. Before doing this note that with our ansatz for A_μ^i the only non-vanishing components of $\vec{F}_{\mu\nu}$ are

$$F_{10}^{3} = -F_{01}^{3} = W(r) \tag{25}$$

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* We work in spherical polar co-ordinates so that $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-r^2,-r^2\sin^2\theta)$

** Since we are working in the limit that weak gravity is neglected we shall drop all labelling used so far to separate quantities referring to the f or g metric.

Using this we find that

$$T_{00} = 2\beta\left\{\left(3 - \frac{2r^2}{B}\right)\left[\frac{C}{\Delta}(A+C)r - 1\right] + rC\left[\frac{r^2}{B}\left(6 - \frac{r^2}{B}\right) - 6\right] + \frac{C}{\Delta}(1-r)\right\} + \frac{3}{2}\frac{C}{\Delta}W'^2 \tag{26}$$

$$T_{01} = 2\beta\left\{\frac{rD}{\Delta}(A+C)\left(\frac{2r^2}{B} - 3\right) + rD\left[6 - \frac{r^2}{B}\left(6 - \frac{r^2}{B}\right)\right] + \frac{D}{\Delta}(r-1)\right\} - \frac{3}{2}\frac{D}{\Delta}W'^2 \tag{27}$$

$$T_{11} = 2\beta\left\{\left(\frac{2r^2}{B} - 3\right)\left[\frac{A}{\Delta}(A+C)r - 1\right] + rA\left[6 - \frac{r^2}{B}\left(6 - \frac{r^2}{B}\right)\right] + \frac{A}{\Delta}(r-1)\right\} - \frac{3}{2}\frac{A}{\Delta}W'^2 \tag{28}$$

and

$$T_{22} = 2\beta\left\{rB\left[\frac{1}{\Delta}(A+C)\left(\frac{2r^2}{B} - 3\right) + \frac{r^2}{B}\left(\frac{r^2}{B} - 6\right) + \frac{1}{\Delta} + 6\right] - \frac{r^2(A+C)}{\Delta} + r^2\left(3 - \frac{r^2}{B}\right)\right\} - \frac{B}{2}\frac{W'^2}{\Delta} \tag{29}$$

where

$$\beta = \frac{M^2}{2k^2}\left(\frac{\psi}{\rho}\right)^{u}$$

It is now immediately obvious from the ten equations (15) that the (02), (03), (12), (13) and (23) component equations are automatically satisfied. Further, it is clear from (19) and (22) that

$$DG_{00} + CG_{01} = 0 \tag{30}$$

From (15) it then follows that

$$DT_{00} + CT_{00} = 2\beta D\left(\frac{2r^2}{B} - 3\right) = 0 \tag{31}$$

So that either D = 0 or $B = \frac{2}{3}r^2$. Salam and Strathdee [6] refer to the latter as Class I and the former as Class II type solutions. We shall consider Class I solutions with $D \neq 0$ and $B = \frac{2}{3}r^2$. Substituting this into the (00),(01), (11), and (22) components equations of (15) we obtain

$$G_{00} \equiv \frac{3C}{2r^2} - \frac{C^2}{r\Delta}\left(\frac{1}{r} + \frac{C'}{C} - \frac{\Delta'}{\Delta}\right) = \frac{k^2}{2}T_{00} \tag{32}$$

$$G_{11} \equiv -\frac{3A}{2r^2} + \frac{CA}{r\Delta}\left(\frac{1}{r} + \frac{C'}{C} - \frac{\Delta'}{\Delta}\right) - \frac{\Delta'}{r\Delta} = \frac{k^2}{2}T_{11} \tag{33}$$

$$G_{01} \equiv -\frac{3D}{2r^2} + \frac{CD}{r\Delta}\left(\frac{1}{r} + \frac{C'}{C} - \frac{\Delta'}{\Delta}\right) = \frac{\kappa^2}{2} T_{01} \tag{34}$$

and

$$G_{22} \equiv \frac{r}{3\Delta}\left[rC'' + \left(2 - \frac{r}{2}\frac{\Delta'}{\Delta}\right)C' - C\frac{\Delta'}{\Delta}\right] = \frac{\kappa^2}{2} T_{22} \tag{35}$$

With $\beta = \frac{\varepsilon}{3} r^2$, the components of $T_{\mu\nu}$ become

$$T_{00} = -\frac{C}{D} T_{01} = -\frac{C}{A} T_{11} = C\beta\left[\frac{3v}{2} + \frac{2}{\Delta}(1-v)\right] + \frac{2}{2}\frac{C}{\Delta} W'^2 \tag{36}$$

and (37)

$$T_{22} = \beta\left[\left(\frac{2}{3\Delta} - \frac{1}{2}\right) r^2 v - \frac{2r^2}{\Delta}(1-v) + 3r^2\right] - \frac{r^2}{3}\frac{W'^2}{\Delta}$$

Now we note from (36) that

$$A T_{01} - D T_{11} = 0 \tag{38}$$

Equations (33) and (34) then imply that

$$-\frac{D}{r}\frac{\Delta'}{\Delta} = 0 \tag{39}$$

and since $D \neq 0$ we obtain the result that

$$\Delta = \Delta_0 = \text{const.} \tag{40}$$

Let us now consider the field equations for the gauge fields $\vec{A}_\mu$. With our ansatz equations (16) reduce to

$$W'' + \left[\frac{2}{r} + \frac{\Delta'}{2\Delta}\right] W' = 0 \tag{41}$$

which, for $\Delta' = 0$ gives

$$\left(r^2 W'\right)' = 0 \tag{42}$$

This integrates at once to yield

$$W(r) = \frac{\alpha_0}{r} + \alpha_1 \tag{43}$$

where α_0 and α_1 are constants of integration. Putting all this into (32) gives the following equation for C

$$C' + \frac{C}{r} = \frac{3}{2}\frac{\Delta_0}{r} + \frac{9}{2}\lambda\Delta_0 r - \frac{3}{2}\omega\frac{\Delta_0}{r^3} \tag{44}$$

where, for convenience we have introduced the symbols λ and ω which denote the following combinations:

$$\lambda = -\frac{\beta\kappa^2}{9}\left[\frac{3v}{2} + \frac{2(1-v)}{\Delta_0}\right] \tag{45}$$

$$\omega = \frac{\kappa^2\alpha_0^2}{2\Delta_0} \tag{46}$$

Equation (44) can be integrated easily to give

$$C(r) = \frac{3\Delta_0}{2}\left(1 + \frac{\mu_0}{r} + \frac{\omega}{r^2} + \lambda r^2\right) \tag{47}$$

where μ_0 is another arbitrary integration constant. Only A(r) remains to be determined . For this we use the only remaining equation, viz equation (35), which gives, after a little algebra,

$$A(r) = -\frac{3\Delta_0}{2}\left(-\frac{4}{9\Delta_0} + \frac{\mu_0}{r} + \frac{\omega}{r^2} + \frac{4\alpha_0^2}{9\beta\Delta_0}\frac{1}{r^4} + \lambda r^2\right) \tag{48}$$

Along with $B = \frac{2}{3}r^2$, equations (40), (43), (47), and (48) represent the complete set of solutions to the classical equations of f-g theory coupled to a strongly interacting SU(2) set of gauge fields in the absence of weak gravity.

IV DISCUSSION

We have obtained the analogue of the Salam-Strathdee solution [6] in the presence of a set of strongly interacting SU(2) gauge fields. The solution reduces in the absence of the gauge fields to the solutions obtained in ref. 6. The gauge fields do introduce the extra feature of the $\frac{1}{r^2}$ dependence in C and the $\frac{1}{r^4}$ dependence in A.

Salam and Strathdee [9] have shown by studying the Klein-Gordon equation in the $f_{\mu\nu}$ background found by them that a scalar hadron is confined. It would be interesting to see what modifications occur to this picture when we use the solution for $f_{\mu\nu}$ we have found as the background. Particularly interesting would be the changes in the energy levels and wave-functions obtained by Salam and Strathdee [9] for their case.

One might also attempt to obtain the exact solutions taking into account the effects of weak gravity in the manner of Isham and Storey.[6] It is simple enough to supplement (19) and (20) with an ansatz for $g_{\mu\nu}$ of the form:

$$g_{\mu\nu}\,dx^\mu dx^\nu = J\,dt^2 - K\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

and attempt to solve the complete set of equations (14) - (16). A preliminary effort in this direction does, however, seem to indicate that the ansatz for $f_{\mu\nu}$, $g_{\mu\nu}$, and A^i_μ we have used is not consistent with the field equations. This would suggest that exact spherically symmetric solutions to the complete set of equations do not exist. This peculiar situation seems to arise from the fact that whereas hadrons contribute to $T^f_{\mu\nu}$ their presence is not felt at all by the $g_{\mu\nu}$ part of the field equations. In the case of the "matter-free" theory, of course, the f-g mixing term contributed to both $T^f_{\mu\nu}$ and $T^g_{\mu\nu}$ in such a manner as to allow a consistent spherically symmetric ansatz which permitted an exact spherically symmetric solution.

We have already mentioned that it might be worthwhile looking at another Lagrangian proposed in ref. 4. which combines Yang-Mills fields with f-g theory in a different manner to that considered in this paper. It turns out that this Lagrangian does not suffer from the problem mentioned above - at least insofar as the gauge fields are concerned. It may be possible, therefore, to obtain exact spherically symmetric solutions in this case. We hope to be able to consider this problem in the future.

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QUANTUM GRAVITATIONAL EFFECTS IN THE LABORATORY*

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Summary

Some interesting consequences of the effects of gravitation and finite temperature on quantum field theory are presented which have important implications for experimental high energy physics and the status of the "No-Hair" Conjecture for black holes. We point out two consequences for laboratory situations in high energy physics which disprove the usual assertion that quantum gravitational effects are only important at planckian energies. The first of these is that beams of particles in circular accelerators cannot be cooled to below a certain temperature determined simply by the accelerator's radius, while the second shows that spontaneously broken gauge symmetries may be restored by quantum gravitational effects. We end by describing briefly circumstances under which these effects might have a bearing on the "No-Hair" conjecture.

Two parallel sets of investigations have been carried out in the last few years to study the effects of gravitation and temperature on quantum field theory. One set of investigations initiated by Khirznits and Linde[1] has considered what happens when a system of particles described by a spontaneously broken local gauge invariant quantum field theory is placed in a heat bath or strong electric or magnetic fields[2]. The authors of refs. (1) and (2) have found that gauge symmetries which are spontaneously broken at zero temperature via the Higgs-Kibble mechanism (for example, those of the Salam-Weinberg electroweak theory) may be restored at sufficiently high temperatures, or in sufficiently strong electric or magnetic environments, and they have calculated the critical temperatures and fields at which such restoration would take place.

The basic idea of this approach is that at finite temperatures (or field strengths) the effective potential of the theory picks up terms of the type $+T^2\phi^2$ (where T is the temperature and ϕ is the Higgs-Kibble scalar field). For sufficiently high temperatures, this term becomes larger than the negative $(\text{mass})^2\phi^2$ term which drives the symmetry breaking in the zero temperature theory. As a consequence, the scalar field ϕ becomes a real physical particle degree of freedom and the symmetry is restored.

Parallel to the study of these effects, several authors[3] have carried out a study of the effects of gravitation and space-time topology on quantum field theory. A number of interesting results have been obtained but the two which concern us in this essay are outlined below. Firstly, it has been shown that an observer accelerating uniformly through empty Minkowski space-time appears to find himself in a heat bath at a temperature given by

$$T = \frac{\hbar a}{2\pi K c} \simeq 10^{-20} \text{ a Kelvin} \qquad (1)$$

where $\hbar$ is Planck's constant, a is the acceleration, k is Boltzmann's constant and c is the velocity of light.

In order to illustrate this let us consider a uniformly accelerating observer in Minkowski space-time. If we assume that an inertial observer and the accelerating observer use the same transition amplitudes to describe objectively the same processes, it can be shown that the free Feynman propagator for the inertial observer, when translated into the accelerating observer's frame, is identical with that of a free finite temperature propagator with the relationship between the acceleration and the temperature being that given by (1).

This result can be understood on the basis of quantum gravitational effects (through non-simply connected topologies) in flat Minkowski space-time. To try and understand how this arises, let us use coordinates (t, x, y, z) and (τ, ξ, y, z) to describe the inertial and accelerating observers respectively. If, for simplicity, we assume that the accelerating observer moves in the (τ, ξ) plane with a constant uniform acceleration a, then his world-line is given by the hyperbola $\xi = \frac{1}{a}$ with asymptotes $\xi = 0$. The coordinate transformation from the inertial to the accelerating observer's frame reads

$$x = \xi \cosh a\tau \;, \qquad t = \xi \sinh a\tau \;.$$

In contrast to the inertial observer, the accelerating observer has a very restricted range of vision. The surface $x = |t|$ forms an event horizon, and any signals sent from the origin O, after $t = 0$ never reach the accelerating observer. It is the existence of this event horizon which causes the space-time to seem multiply connected when the two observers translate themselves into euclidean coordinates $(t \rightarrow it, \tau \rightarrow i\tau)$ with periodic complex time coordinates, and leads to the above-mentioned thermal effect.

Secondly, by considering quantum fields in the exterior region of a black-hole, Hawking has shown that when a star collapses to a black-hole, the formation of the event horizon around the singularity enables the black-hole to absorb one of a pair of virtual particles created just outside the horizon, thus leaving its partner, which is now a real particle, free to travel to an arbitrarily large affine distance from the horizon. This continuous process is observed asymptotically as a net flux of radiation, and after all transient effects which arise during the collapse die out, the left-over radiation has been shown to be that which would be produced by a hot body at a temperature given by

$$kT = \frac{\hbar K}{2\pi c} \tag{2}$$

where K is the surface gravity of the black-hole. Thus, a black-hole can be considered to be a black-body radiating at a temperature T given by (2).

Both the above results may be understood mathematically by noting that spacetimes with event horizons are periodic in an appropriate time coordinate with an imaginary period. The Green's functions of a quantum field theory in such a spacetime are, therefore, also periodic in imaginary time. Coupled with the observation that the thermal Green's functions of a field theory at a finite temperature T also possess this property, one arrives at the result that field theories in spacetimes with event horizons may be considered to be in thermal equilibrium at some finite temperature.

All that follows is based essentially on the interplay between the various effects we have discussed briefly above. We will now describe a couple of laboratory situations in which it might be possible to detect effects of quantum gravitation.

The first observation we wish to make concerns the recent attempts being made at CERN and other high energy particle physics laboratories to cool particle beams in accelerators. We shall show that equation (1) implies a lower bound to the extent to which such a cooling can be achieved. It is clear that a bunch of relativistic elementary particles going round at a constant velocity v($\approx$ c, the velocity of light) in a circular accelerator of radius r experience a uniform acceleration a, given by

$$a \simeq \frac{c^2}{r} \,.$$

We see, therefore, that such a bunch of relativistic elementary particles would find themselves in a heat bath at temperature $T \simeq \hbar c/2\pi k r$. Since this temperature is due simply to their acceleration, it would be impossible for accelerator beams to be cooled to temperatures below this lower bound. This bound does not apply, of course, to linear accelerators.

In order to remove any doubts as to whether such effects are "real", it would perhaps be helpful to show that such observer dependent effects are already very familiar. Indeed, it is only natural to expect such observer dependent effects in general relativity when one remembers that in special relativity one has a similar situation arising due to the effect of time dilation. This is illustrated beautifully by the experimental verification of time dilation effects through measurement of the lifetimes of a μ-meson at rest, and in motion in the laboratory. The results of such experiments show clearly that a μ-meson that is stationary in the laboratory decays at a much faster rate than one which is travelling at a speed reasonably close to that of light. This observer dependence arises in special relativity through requiring equivalence of all inertial observers. In contrast, general relativity requires equivalence of all observers, inertial and non-inertial, and thus gives rise to the effects we are considering in this essay.

The second effect that we shall now discuss concerns the concept of symmetry restoration, which we have outlined earlier, but with the added significance that the restoration will now be due to quantum gravitational effects. Let us consider the situation illustrated schematically in Fig. 1.

If we introduce a set of relativistic, charged particles, the interactions of which are described by a spontaneously broken gauge theory, into a region containing an extremely high magnetic field, then they will all experience an acceleration, $\underline{a}$, perpendicular to the plane defined by the directions of $\underline{B}$ and $\underline{v}$, the velocity of the particles, given by

$$\underline{a} = \frac{q}{m} \underline{v} \times \underline{B},$$

where q and m are the charge and mass of the particle respectively. Assuming that $\underline{v}$ is perpendicular to $\underline{B}$ and is close to c in magnitude, we obtain for $\underline{a}$ the value $a \simeq \frac{qcB}{m}$. However, equation (1) tells us that such a bunch of particles will experience a heat bath of temperature

$$T \simeq \frac{\hbar qB}{2\pi km} = \left(\frac{\hbar}{2\pi k}\right) \frac{qB}{m} .$$

Assuming, for simplicity, that such a bunch of particles is composed of electrons, we obtain the result that $a \simeq 5 \times 10^{19}$ B. So that the temperature for this set of electrons would be $T \simeq 0.5$ B.

Now, if one combines this information with the knowledge that the symmetry of the Salam-Weinberg theory is restored at temperatures of $O(10^{15})$ Kelvin, we see that magnetic fields of strength around 10^{15} Tesla would suffice for restoring the Salam-Weinberg theory. Comparison of the data obtained from an experiment of the type illustrated in Fig. 1 in the presence and absence of $\underline{B}$ would allow us to determine whether such a restoration has taken place, and whether the accelerating observer does indeed see a heat bath at temperature T given by (1) much as the observations of the lifetime of the μ-meson allowed us to vindicate the time-dilation effect of special relativity. It is encouraging to note that experiments involving such strong fields have already been suggested by Salam and Strathdee in ref. 2.

We will now go on to study the possible relationship of the effects described above to the "No-Hair" Conjecture[5] for black-holes. It will be shown that they allow a possible mechanism for transcending the "No-Hair" Conjecture in the quantum regime. For this purpose, let us consider a black-hole in thermal equilibrium with a heat bath at temperature T, and let us introduce into the heat bath a system of particles interacting through some spontaneously broken gauge fields, e. g. SU(2) x U(1), while maintaining thermal equilibrium[4]. This means that if the mass of the black-hole is sufficiently small, the corresponding temperature will be sufficiently large to allow the initial spontaneously broken gauge symmetry to be restored and the corresponding gauge fields become long range due to their masslessness. We further obtain conserved charges, apart from those associated with electromagnetism. This means that the interacting particles we are considering will have associated with them conserved gauge charges and the corresponding Gauss law for the system. The existence of Gauss' law immediately raises the possibility for the black-hole to carry the gauge charge if the system of interacting particles falls through its event horizon. Let us take the example of $SU(2) \times U(1)$. The restoration temperature for this gauge group is $\sim 10^{15}$ Kelvin. Taking the black-hole to be of the Schwarzschild type, the mass can be found from (2) to be $\sim 10^{8}$ kg. So as long as the interacting particles have Compton wavelengths less than the size of the black-hole (i. e. its Schwarzschild radius), the possibility of transcending the "No-Hair" Conjecture exists.

It is known[6] that small primordial black-holes possibly formed by fluctuations in the early universe, with masses $\sim 10^{11}$ kg, would just decay away through Hawking radiation (with a characteristic spectrum) within the present age of the universe. It is found[7] that for electrically charged primordial black-holes, fluctuations in the charge will cause the average emission rate for charged particles to be lower than that for similar uncharged particles. Coupled with the arguments presented above for the transcendance of the "No-Hair" Conjecture, it is clear that the emission rate will be further reduced (after the mass of the black-hole reaches $\sim 10^{8}$ kg) due to the accumulation and subsequent fluctuations of the new gauge charges acquired by the decaying black-hole. This, we suggest, will lead primordial black-holes not to an explosive death but rather to a slow, "quiet" death.

So we see that in principle it is possible to transcend the "No-Hair" Conjecture. However, it remains to be seen if the arguments can be extended to more realistic situations, as in stellar collapse, for example to form a black-hole.

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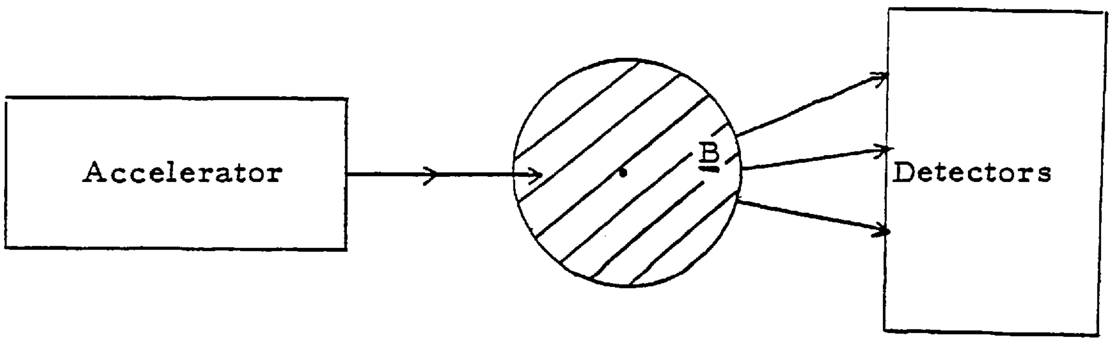

Fig. 1

Schematic experiment to demonstrate symmetry restoration through acceleration and temperature effects. The shaded region contains a magnetic field directed perpendicular to the plane of the paper. For large $\underline{B}$, the motion of particles entering the shaded region will be confined to it and subsequent decay products are observed by detectors.