SOLITONS IN SUPERSYMMETRIC MODELS

BY

SHAHIN ROUHANI

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Theoretical Physics Group
Imperial College of Science and Technology
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ABSTRACT

The superalgebra and its relation to the soliton mass is investigated.

Using a modified version of Dirac's method, for singular lagrangians, the supersymmetry algebra of the supersymmetric CP^n model is derived and found to contain a central charge, composed of the topological charge and a field dependent U(1) transformation. This modified algebra leads to a lower bound on the mass of the soliton of this model.

Such bounds are common to a large class of models which admit soliton solutions. However in the supersymmetric models the mass formula of the self-dual solitons survives quantization. The controversy concerning this phenomenon is resolved using functional integral methods and care was taken to preserve the supersymmetry of the model, at all stages of regularization.

Finally the correspondence between constraints and confinement is discussed.

PREFACE

The work described in this thesis was carried out in the Theoretical Physics Group, Imperial College, London between October 1977 and September 1980 under the supervision of Dr. D. Olive. Except where otherwise stated the work is original and has not been submitted for a degree of this or any other university.

I would like to thank Dr. D. Olive for guidance and encouragement and all the other members of the theory group for numerous discussions.

Special thanks are due to my parents without whose support all this would not have been possible.

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NOTATION

$$\frac{\partial}{\partial x_\mu} = \partial_\mu \qquad , \qquad \not{p} = \gamma^\mu p_\mu$$

Two dimensions $\quad \gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

metric $\quad g_{\mu\nu} = \mathrm{diag}(+,-)$

$\epsilon^{01} = +1$

Three dimensions $\quad \gamma^2 = i\gamma^0\gamma^1 \qquad \gamma^0$ and γ^1 as above

metric $\quad g_{\mu\nu} = \mathrm{diag}(+,-,-)$

$\epsilon^{012} = +1$

CHAPTER 1

INTRODUCTION

1.1 Prelude

The conceptual problems of point particles, physical entities without any dimensions, render an extended model of material particles a very attractive one. Some particles are indeed thought to be extended, but as bound states of point particles, at this moment there is no consistent picture of nature, which admits a fundamental constituent which is extended, nevertheless some progress has taken place towards an extended particle picture. The soliton is interesting and promising but it has short-comings; if one believes in non-derivative interactions the soliton can only exist in 2-dimensions, its 4-dimensional analogue has to be a magnetic monopole which has not yet been observed. However solitons point the way for a future theory which will hopefully be of more physical interest. Also in other branches of physics, solitons have observable effects |11|, in fact a soliton in the shape of a canal wave can be observed with the naked eye |10|, thus making the soliton, interesting enough to be studied.

A recent development in particle physics has been the introduction of supersymmetry, a symmetry between bosons and fermions . This will be discussed in following sections. It is the purpose of this thesis to investigate some of the properties of the models which are both supersymmetric and possess soliton solutions, such models exhibit interesting properties.

One such property is the effect that the solitons have on the superalgebra, the symmetry algebra of the supersymmetric models is modified in the presence of solitons to admit a central charge |29|. The central charge is related to the topological charge of the soliton thus leading to a relation between the mass and the topological charge of the soliton |29| .

In chapter three we shall see how the superalgebra is modified in the case of the supersymmetric CP^n model. In this case the algebra already contains a central charge independent of the soliton, but the presence of the soliton introduces an extra term into the central charge. The extra term being proportional to the topological charge of the soliton. In sections two and three of the introduction I shall review some of the concepts that have been introduced such as the soliton, the topological charge and the central charge of superalgebras.

The other property which will be discussed concerns the relation between the mass and the charge of the soliton already mentioned. It was claimed |29,35| that this relation remains unchanged upon quantization, this remarkable cancellation of quantum corrections was not however left uncontested, a recent paper by Schonfeld |29| claims that such corrections do not vanish and there is indeed a contribution due to quantization to the mass. In Chapter 6 I shall resolve this controversy by using functional methods. Unfortunately it is only possible to do this calculation up to the first order in perturbation expansion due to the complexity of the methods involved, thus the complete answer will not be given to this question, only the first loop corrections can we say vanish.

In chapter two Dirac's formalism and its generalization for singular lagrangians are discussed. This formalism is used for the treatment of the CP^n lagrangian. I shall prove in this chapter, that the Dirac bracket, for a lagrangian with mixed grassmann and C number variables always exists.

Chapter 4 is devoted to a digression which comes out of the CP^n lagrangian and is not related to the concept of solitons. The notion of confinement is formulated by utilizing constraints of singular lagrangians.

In Chapter 5 I shall review and develop the definition of the path integral for fermions which is then used in Chapter 6.

1.2 Soliton

Let us start with the definition of the soliton, instead of the old technical definition of a soliton |11| let us adopt a definition given by Coleman |12| , and call a non-dissipative, non-sigular and finite energy solution of a field theory, a soliton. Let us consider two such examples in 1 + 1 dimensions. Consider the lagrangian density;

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi \;-\; V(\phi) \qquad 1.2.1$$

The energy density is;

$$T_{00} = \frac{1}{2}\,\dot{\phi}^2 + \frac{1}{2}\,|\nabla\phi|^2 + V(\phi) \qquad 1.2.2$$

Let us consider a time independent solution $\phi_s(x)$, the mass of this solution is;

$$M_s = \int_{-\infty}^{+\infty} dx\,\left[\, \frac{1}{2}\,|\nabla\phi|^2 + V(\phi)\,\right] \qquad 1.2.3$$

but we also know from the equation of motion that;

$$-\nabla^2\phi + V'(\phi) = 0$$

or;

$$\frac{1}{2}\,|\nabla\phi|^2 = V(\phi) + C$$

thus if C = 0

$$M_s = 2\int_{-\infty}^{+\infty} dx\; V(\phi) \qquad 1.2.4$$

Equation (1.2.4) is a very useful one which will be used many times in this work. As specific example consider the potential;

$$V(\phi) = -\frac{1}{2} m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + \frac{m^4}{4\lambda}$$

$$= \frac{m^4}{4\lambda}\left(1 - \frac{\lambda}{m^2}\phi^2\right)^2 \qquad 1.2.5$$

The stationary solution is;

$$\phi_s(x) = \frac{m}{\sqrt{\lambda}} \tanh \frac{m(x-a)}{\sqrt{2}} \qquad 1.2.6$$

where 'a' can take any real value. This solution is known as the $\lambda\phi^4$ "kink". The energy of this solution is given by (1.2.4)

$$M_s = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} \qquad 1.2.7$$

Another well known field theory with soliton solutions is the Sine-Gordon model; The potential for this model is;

$$V(\phi) = \frac{m^4}{\lambda}\left(\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right) \qquad 1.2.8$$

The time independent solution is;

$$\phi_s(x) = \frac{4m}{\sqrt{\lambda}} \tan^{-1}\left(e^{mx}\right) \qquad 1.2.9$$

The mass of the soliton is;

$$M_s = 8m^3/\lambda \qquad 1.2.10$$

An interesting property of the soliton is the translational invariance of it which gives rise to a zero mode in the equation of stability. The stability equation is;

$$\left[\Box + V''(\phi_s)\right] f(x,t) = 0 \qquad 1.2.11$$

where f is a perturbation about the given solution $\phi_s(x)$, it is evident that one solution of (1.2.11) is the first derivative of $\phi_s(x)$

that is $f(x) = \frac{d\,\phi_s(x)}{dx}$ which is time independent. This arises because the stationary solution can be shifted to move its centre of mass from x to $x + \delta x$, thus if $\phi(x)$ is a solution so is $\phi(x + \delta x)$ therefore a taylor expansion immediately indicates that

$$\phi(x+\delta x) = \phi(x) + \phi'(x)\,\delta x + \cdots \tag{1.2.12}$$

$\phi'(x)$ will satisfy the equation (1.2.1).
Now if $f(x,t) = \sum_n e^{iw_n} fn(x)$ then (1.2.11) becomes

$$\left[-\nabla^2 + V''(\phi_s)\right] f_n(x) = W_n^2\, f_n(x) \tag{1.2.13}$$

it is sufficient for stability that w_n be real, that is $w_n^2 \geq 0$ but $\frac{d}{dx}\phi_s(x)$ is a solution with $w_0 = 0$ and it does not have any nodes (in the case of the "kink" and the sine-gordon solitons) therefore it is the ground state, thus all the other w_n are positive [13]. We therefore see that the solitons are stable. In fact there is a fundamental reason for their stability which is related to the topological properties of the model, which brings us to the notion of topological current and changes.

A given finite energy solution must tend to the minima of the potential at spatial infinity at all times; therefore associated with each finite energy solution is a map ϕ_∞ which maps the spatial infinity into the minima of the potential, now if the potential has more than one minimum the set of maps $\{\phi_\infty\}$ can be divided into distinct classes where different classes are not connected by continuous transformations. Therefore once a solution is given with a particular asymptotic behaviour, evolution in time cannot change its asymptotic behaviour, thus it seems that there exists a conserved quantity associated with that solution; in fact called the topological charge is the index for the homotopy classes of $\phi(x)$. The topological charge can be written as;

$$Q = \phi(+\infty) - \phi(-\infty) \tag{1.2.14}$$

For the $\lambda\phi^4$ theory with minima $\phi(\pm\infty) = \pm\frac{m}{\sqrt{\lambda}}$ Q is either zero or $\frac{2m}{\sqrt{\lambda}}$ for a kink or $-\frac{2m}{\sqrt{\lambda}}$ for an anti kink. In the case of sine-gordon model $\phi(\pm\infty) = \pm\frac{2Nm\pi}{\sqrt{\lambda}}$, where one can have a soliton or anti-soliton with any arbitrary charge N. The existence of a conserved quantity suggests a conserved current whose time component is a density for this charge. This current is

$$J^{\mu} = \epsilon^{\mu\nu}\partial_{\nu}\phi(x,t) \qquad \epsilon^{01} = +1 \tag{1.2.15}$$

where

$$Q = \int_{-\infty}^{+\infty} dx\, J^{0}(x,t) = \phi(+\infty,t) - \phi(-\infty,t) \tag{1.2.16}$$

Evidently a soliton with a non zero topological charge will be prevented from decaying by virtue of its topological charge.

Finally let us consider the possibility of having similar solutions in higher dimensions. Unfortunately we encounter a discouraging theorem here due to Derrik [45]. Assume that there are no derivative interactions. The energy of a configuration can be written as;

$$H[\phi] = T[\phi] + V[\phi] \tag{1.2.17}$$

where the two functionals, the kinetic energy $T[\phi]$ and the potential energy $V[\phi]$ are both positive. For a static solution, $H[\phi]$ must be stationary with respect to any arbitrary field variation in particular the scale transformations $x \to \lambda x$. Now such a transformation in D space dimensions and one time dimension results in the following hamiltonian

$$H_{\lambda}[\phi] = \lambda^{D-2}\, T[\phi] + \lambda^{D}\, V[\phi] \tag{1.2.18}$$

differentiating with respect to λ and setting $\lambda = 1$ gives:

$$\left.\frac{\partial H}{\partial \lambda}\right|_{\lambda=1} = 0 = (D-2)\, T[\phi] + D\, V[\phi] \tag{1.2.19}$$

which cannot be satisfied for $D > 2$ and for $D = 2, V[\phi]$ has to vanish. More elaborate arguments exclude the case of several fields as well |46| and we are lead to consider gauge theories in which case finite energy solutions can be found with an asymptotic magnetic field which resembles the field of a magnetic monopole |47-48|.

1.3 Supersymmetry

In mordern physics symmetry principles have assumed a major role, not only do they have a practical use in deriving the lagrangians of different interactions but also the laws of nature should be understood as consequences of symmetries; such as the conservation laws. The primary symmetry is of course the Poincaré invariance, arising from the structure of space-time. Another kind of symmetry, that is one between different species of particles is also thought to exist, referred to as internal symmetry. Although it is not yet clear what is the complete internal symmetry group of nature but it must include the groups U(1), of the electromagnetism and SU(3) of quantum chromodynamics. An intriguing question is; can one construct a model where the symmetry is not a direct sum of the space-time symmetry and the internal symmetry? The answer was given in two parts, first, no, |39| one cannot construct a model which is invariant under a nontrivial fusion of the Poincaré group and a compact lie group of internal symmetries, and later, yes, it can be done if one admits supersymmetries |36|. The other aspect of supersymmetries which is unusual, if not radical is that it sets up a symmetry between bosons and fermions. The starting point of the introduction of the supersymmetries was in fact an attempt to unify these two classes of particles with apparently diverse properties |38, 40-43|. An example of a supertransformation is:

$$\delta\phi = i\bar{\epsilon}\psi$$

$$\delta\psi = -\partial\!\!\!/\phi\epsilon \qquad 1.3.1$$

where ϕ is a complex scalar field, ψ a complex spinor and ϵ an infinitesimal spinor. The algebra of these transformations closes in two dimensional space-time (with the aid of the equation of motion) if ϵ_α are taken to be anticommuting numbers or more precisely grassmann variables. The Noether current associated with these transformations is known as the supercurrent, it is a spin $\frac{3}{2}$ object which is conserved if the equations of motion are satisfied. The zeroth component of this current, the supercharge, is the generatorsof supertransformations. The supercharge, Q_α and the generator of the Poincaré group (P_μ, $M_{\mu\nu}$) form the extension of the Poincaré group where Q_α commutes with the momenta and transforms like a spinor under the lorentz group. The anticommutator of two supercharges is however given by:

$$\{Q_\alpha , Q_\beta\} = -(\gamma^\mu C)_{\alpha\beta} P_\mu \qquad 1.3.2$$

Where C is the charge conjucation matrix. Here we have encountered the peculiarity of having to introduce anticommutators as well as commutators into the algebra therefore the symmetry algebra is really a graded algebra leading to a graded lie group of symmetry. The details of supersymmetric field theories can be found in a number of reviews |37|, here I only intend to discuss the concept of central charge |36| which is essential to what follows in later chapters.

Let Q_α^L be n majorana spinors, representing the supercharges of the model, and J_i be the generators of the lie algebra of G, the group of internal symmetries, which can be assumed to be a direct sum of a semi-simple and an abelian part . The commutator of J_i and Q_α^L being:

$$[J_i , Q_\alpha^L] = i S_i^{LM} Q_\alpha^M \qquad 1.3.3$$

The Jacobi identity concerning J_i, J_k and Q_α^L then shows that the matrices S_i^{LM} form a representation of J_i. In this section I shall use the Jacobi identity frequently and shall denote it by (J_i, J_k, Q_α^L) say for the case just discussed, the identity for a superalgebra is given by (2.3.8)(c). The supercharges Q^L_α are assumed to commute with the momenta P_μ and transform like spinors under the lorentz group, it was shown by Haag, Loupazanski and Sohinus |36| that one can admit supercharges which do not commute with the momenta only if one assumes conformally invariant space-times. Thus with the internal symmetry group being disjoint from the Poincaré group the only bracket to be determined is that of the supercharges:

$$\{ Q_\alpha^L , Q_\beta^M \} = \delta^{LM} (\gamma^\mu \gamma^0)_{\alpha\beta} P_\mu + Z^{LM}_{\alpha\beta} \qquad 1.3.4$$

where I have used the majorana representation of gamma matrices thus $\left\{ C_{\alpha\beta} = -\gamma^0_{\alpha\beta} \right\}$. Anticipating that momenta appear on the right hand side of (1.3.4) I have included them, what remains to be done is to determine what $Z^{LM}_{\alpha\beta}$ can be.

Consider the Jacobi identity (P, Q, Q), this leads to

$$[P_\mu , Z^{LM}_{\alpha\beta}] = 0 \qquad 1.3.5$$

therefore $Z^{LM}_{\alpha\beta}$ can only contain the generators of the lie algebra, J_i, also $Z^{LM}_{\alpha\beta}$ can only be expanded in terms of $\gamma^0_{\alpha\beta}$ and $(\gamma^5\gamma^0)_{\alpha\beta}$ where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ in 4-dimensions, now in 4-dimensions and the majorana representation both γ^0 and $\gamma^5\gamma^6$ are antisymmetric and since the left hand side of (1.3.4) is symmetric under the exchange of α and β and simultaneously L and M it follows that $Z^{LM}_{\alpha\beta}$ should be antisymmetric under the exchange of L and M or α and β. Thus in four dimensions one must have more than one supercharge for a nonzero $Z^{LM}_{\alpha\beta}$ hence one cannot have a $Z^{LM}_{\alpha\beta}$ charge without an internal symmetry group. In 3-dimensional space-time; the only gamma matrix left for $Z^{LM}_{\alpha\beta}$ is γ^0 whic is antisymmetric in the majorana representation, thus in 3-dimensions as well, one must have an internal symmetry group before $Z^{LM}_{\alpha\beta}$ could exist. However, in 2-dimensions the matrix corresponding to

$\gamma^5\gamma^0$ is in fact $i\gamma^1$ which is symmetric in the majorana representation so one can in principle have a term on the right hand side of (1.3.4) apart from the momenta and lorentz rotations, without having a group of internal symmetries. Although this argument rests upon the use of majorana representation but this does not disqualify it since (1.3.4) must hold in all of the representations of the gamma matrices.

Now consider the Jacobi identity for three supercharges $(Q^L_\alpha, Q^M_\beta, Q^P_\gamma)$, this leads to:

$$[Q^P_\gamma, Z^{LM}_{\alpha\beta}] = [Z^{PL}_{\gamma\alpha}, Q^M_\beta] - [Q^L_\alpha, Z^{PM}_{\gamma\beta}] \qquad 1.3.5.1$$

since Q^L_α are independent,(1.3.5.1) can only hold if each of the brackets vanish independently, therefore

$$[Z^{LM}_{\alpha\beta}, Q^P_\gamma] = 0 \qquad 1.3.6$$

Next, consider the Jacobi identity for $(Z^{LM}_{\alpha\beta}, Q^P_\gamma, Q^N_\gamma)$ this, using (1.3.6) leads to:

$$[Z^{LM}_{\alpha\beta}, Z^{PN}_{\gamma\delta}] = 0 \qquad 1.3.7$$

Finally the Jacobi identity (J, Z, Q) results in

$$[Q^N_\gamma, [J_i, Z^{LM}_{\alpha\beta}]] = 0 \qquad 1.3.8$$

therefore $[J_i, Z^{LM}_{\alpha\beta}]$ must be a linear sum of the $Z^{LM}_{\alpha\beta}$ this result together with (1.3.7) shows that $Z^{LM}_{\alpha\beta}$ form an abelian ideal of the lie algebra of internal symmetries . But $\mathcal{L}$ is a direct sum of a semi-simple algebra and an abelian one, $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, but $\mathcal{L}_1$ being semi-simple does not have any abelian ideals, thus $Z^{LM}_{\alpha\beta}$ lies entirely in $\mathcal{L}_2$, therefore $Z^{LM}_{\alpha\beta}$ commutes with J_i as well. We therefore see that $Z^{LM}_{\alpha\beta}$ form a centre in the superalgebra hence known by the name central charges.

CHAPTER 2

SINGULAR LAGRANGIANS

2.1 Introduction

All the fundamental physical models have singular lagrangians, that is, the canonical variables are not independent, for instance any gauge theory is described by a singular lagrangian. Therefore it is necessary to have a formalism to deal with such lagrangians. This was initially done by Dirac|14|, for a bosonic system, and later generalized to include fermions|17|. In section 2 I shall review Dirac's method (for an extensive review see|16|) and the generalization of it in section 3.

2.2 Dirac's Method for Singular Lagrangians.

Consider a system described by the lagrangian $L(q_i, \dot{q}_i)$, $i = 1, \ldots, n$, the momenta conjugate to q_i are defined by:

$$P_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.2.1}$$

If all the momenta are independent, (2.2.1) can be solved to give P_i as functions of $\dot{q}_i$ and vice versa. However in general P_i need not be independent in which case the determinant $\left|\frac{\partial P_i}{\partial \dot{q}_j}\right|$ vanishes. This in turn implies that there exists a number of expressions relating q_i and P_i:

$$\chi^{\alpha}(q_i, P_i) \approx 0 \qquad \alpha = 1, \cdots, V \tag{2.2.2}$$

The sign "≈ 0" throughout this chapter is read "weakly zero", to mean that the primary constraints $\chi^\alpha \approx 0$, can be set equal to zero only after all of the poisson brackets have been worked out. The set of constraints 2.2.2 rendered the hamiltonian ambiguous since one can add any linear combination of χ^α,s to the hamiltonian, so let us define the total hamiltonian to be:

$$H_T = H_c + C^{\alpha}(t)\,\chi_{\alpha}(q,p) \qquad 2.2.3$$

where H_c is the canonical hamiltonian, and $C^{\alpha}(t)$ are to be determined. Consistency requires (2.2.2) to be time independent, that is the poisson brackets of χ^{α} with the total hamiltonian have to vanish. Calculating these poisson brackets may result either in new constraints, in which case the new constraint is to be commuted with H_T and the process repeated until no more new constraints are generated, or it may result in expressions for the $C^{\alpha}(t)$. A third possibility also exists, where one obtains an inconsistent equation in which case the lagrangian under consideration is inconsistent and we have to abandon it.

Once this process is completed we are left with some additional constraints to (2.2.2) called "secondary" constraints so let us add them to the set (2.2.2)

$$\chi^{\alpha} \approx 0 \qquad \alpha = 1, \cdots, V, V+1, \cdots, k \qquad 2.2.4$$

Now let us introduce the concept of a first class constraint.

If a constraint Γ has a weakly zero poisson bracket with the rest of the constraints (2.2.4) Γ is called "first class" so the set (2.2.4) decomposes into two classes, the first and second class constraints, the two classes do not in general coincide with the primary and secondary divisions. The division of the constraints into first and second class is somewhat fundamental since the first class constraints are in fact generators of the "gauge transformations" to see this let us write:

$$H_T = H_c + \chi_2^{\alpha} C_{\alpha}^{2}(t) + \chi_1^{\alpha} C_{\alpha}^{1}(t) \qquad 2.2.5$$

where the sum over the constraints, in (2.2.3) has been broken into the first class constraints χ_1^{α} and second class constraints χ_2^{α}, now the

Poisson bracket of χ^β a general constraint with H_T is

$$0 \approx [\chi^\beta, H_T] = [\chi^\beta, H_c] - [\chi_2^\alpha, \chi^\beta]\, C_\alpha^2(t) \qquad 2.2.6$$

as is evident, $C_\alpha^1(t)$ will remain undetermined, that is we can use arbitrary functions of time $C_\alpha^1(t)$ in (2.2.5) and the dynamics of the system remains unaltered. Furthermore the variation of a dynamical variable g is given by H_T that is

$$\delta_1 g = [g, H_T] = [g, \chi_2^\alpha]\, C_\alpha^2(t)\, \delta t + [g, H_c]\, \delta t + [g, \chi_1^\alpha]\, c_\alpha^1\, \delta t$$

$$\delta_2 g = [g, H_T] = [g, \chi_2^\alpha]\, C_\alpha^2(t)\, \delta t + [g, H_c]\, \delta t + [g, \chi_1^\alpha]\, b_\alpha^1\, \delta t$$

or

$$\Delta g = (\delta_1 - \delta_2)\, g = (c_\alpha^1 - b_\alpha^1)\, \delta t\, [g, \chi_1^\alpha]$$

$$= \epsilon_\alpha(t)\, [g, \chi_1^\alpha] \qquad 2.2.7$$

where in the second line we have used different functions of time for the coefficients of the first class constraints, we are allowed to do so because the coefficients of the first class constraints are arbitrary functions of time. We can now see that expression (2.2.7) is indeed an infinitesimal gauge transformation.

The classical treatment of singular lagrangians now seems to be complete except for one point: The first class constraints which are secondary have not been added to the hamiltonian, therefore it is not clear wether they are generators of gauge transformations or not. In the case of electrodynamics or the CP^n model (see chapter 3) the first class constraint which is secondary is included in the canonical hamiltonian, thus is a generator. Dirac conjectures [14] that these constraints should always be added to the canonical hamiltonian, however this conjecture has not been proved and some doubts have been raised about it [44].

The quantization of the singular lagrangians follows the same pattern as the canonical quantization, but the constraints must be understood as follows:

$$\chi^{\alpha} | \psi > = 0 \qquad 2.2.8$$

where $|\psi>$ is any arbitrary state; however (2.2.8) leads to inconsistency if χ^{α} is a second class constraint, since if A and B are two second class constraints we have

$$[A, B] | \psi > = 0 \qquad 2.2.9$$

but the left hand side of (2.2.9) does not vanish. Therefore we must remove the second class constraints before quantization by either solving the constraint equations and removing the extra degrees of freedom or using Dirac brackets. These brackets are formed as follows, consider the matrix $C_{\alpha\beta}$

$$C_{\alpha\beta} = [\chi^{\alpha}, \chi^{\beta}] \qquad \alpha = 1, \ldots, k \qquad 2.2.10$$

if this matrix were singular then there would exist a set of non zero coefficients d_{α} such that:

$$d_{\alpha} [\chi^{\alpha}, \chi^{\beta}] = 0$$

or

$$[d_{\alpha} \chi^{\alpha}, \chi^{\beta}] = 0 \qquad 2.2.11$$

that is a first class constraint; $d_\alpha \chi^\alpha$ can be formed from the members of the set χ^α, taking this first class constraint out of the set, we are then left with a smaller set which may then form a non-singular $C_{\alpha\beta}$. Thus taking out all of the first class constraints and then forming the matrix $C_{\alpha\beta}$, we can find an inverse for it. Now define the Dirac bracket:

$$[A,B]^* = [A,B] - [A,\chi^\alpha]\, C^{-1}_{\alpha\beta}\, [\chi^\beta, B] \qquad 2.2.12$$

Clearly $[A,\chi^\alpha]^* = 0$, for any arbitrary operator A. Therefore one can set the second class constraints strongly equal to zero and use Dirac brackets in conjunction with the hamiltonian

$$H_T = H_c + C'_\alpha(t)\, \Gamma^\alpha \qquad 2.2.13$$

Where Γ^α are first class constraints.

A useful property of the Dirac bracket is that it can be defined iteratively[16]. If the set of constraints is too large causing C to become very large, therefore difficult to invert, one can define a Dirac bracket using a subset of the constraints and then define the final Dirac bracket using the rest of the constraints and the previous Dirac bracket. Let us divide the set of constraint into two parts, then:

$$\chi_\alpha \approx 0 \qquad \alpha = 1, \ldots, \mu, \mu+1, \ldots, \nu$$

$$C_{\alpha\beta} = [\chi_\alpha, \chi_\beta] \qquad \alpha, \beta = 1, \ldots, \mu$$

$$[A,B]^*_1 = [A,B] - [A,\chi_\alpha]\, C^{-1}_{\alpha\beta}\, [\chi_\beta, B]$$

2.2.14

Now we can use $[\,,]^*_1$ with the set of the constraints to define the final bracket:

$$[A,B]^* = [A,B]_1^* - [A,\chi_\alpha]_1^* D_{\alpha\beta}^{-1} [\chi_\beta, B]_1^*$$

$$D_{\alpha\beta} = [\chi_\alpha, \chi_\beta]_1^* \qquad \alpha, \beta = \mu+1, \ldots, \nu$$

2.2.15

Clearly now $[A, \chi_\alpha]^* = 0$ for any χ_α .

We are now ready for quantization, where the correspondence is achieved by letting the commutator to be proportional to the Dirac bracket of the operators:

$$\hat{A}\hat{B} - \hat{B}\hat{A} = i\hbar [A,B]^* \qquad 2.2.16$$

The generalization of these ideas to field theory is straight forward. The matrix $C_{\alpha\beta}$ in this case becomes space dependent, thus the inverse of it is understood as follows:

$$\int d^3x \; C_{\alpha\beta}(y,x) \, C^{-1}_{\beta\delta}(x,z) = \delta_{\alpha\delta} \, \delta(y-z) \qquad 2.2.17$$

note that the constraints are time independent and we need not integrate over time.

2.3 Generalization of Dirac's Method

The generalization of the method described in section 2.2 to include fermions is necessary if we wish to study any fermionic system, since the Dirac Lagrangian is singular. The first step inevitably is to have a consistent dynamics of grassmann variables. This has been done by Martin|5|, Berezin and Marinov|15| and Casalbuoni|17| , here I shall review the necessary points through an example:

Let us consider the following Lagrangian:

$$\mathcal{L} = \frac{i}{2} \sum_{\alpha=1}^{N} \xi_\alpha \dot{\xi}_\alpha - V(\xi_\alpha) \qquad 2.3.1$$

Where ξ_α are real grassmann variables, the equation of motion is:

$$i \dot{\xi}_\alpha - \frac{\partial V}{\partial \xi_\alpha} = 0 \qquad 2.3.2$$

Now to obtain a hamiltonian formulation let us define the conjugate momenta similarly to the usual mechanics:

$$P_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\xi}_\alpha} \approx -i/_2\, \xi_\alpha \qquad 2.3.3$$

the equations (2.3.3) are constraints, so we are quickly lead to consider the case of singular Lagrangians but let us define the canonical hamiltonian first:

$$H_c = \dot{\xi}_\alpha P_\alpha - \mathcal{L} \qquad 2.3.4$$

note that it is necessary to define H_c with the velocities to the left of the momenta, otherwise H_c would not be independent of velocities. Now the Hamiltonian equations read:

$$\dot{P}_\alpha = -\frac{\partial H_c}{\partial \xi_\alpha} \qquad \dot{\xi}_\alpha = -\frac{\partial H_c}{\partial P_\alpha} \qquad 2.3.5$$

clearly they do not produce the right equations of motion, the reason is that instead of H_c we must use the total hamiltonian, we shall come to this point later.

Of course a lagrangian in general may depend on a grassmann and ordinary variables at the same time so we must define a graded Poissson bracket for such a dynamics. Let us use the notation that the grade of a variable A, is a where a = 0 if A is even and a = 1 if A is odd. Now the graded poisson bracket is given by

$$[A,B]_G = \sum_i \left(\frac{\partial A}{\partial q_i} B\frac{\overleftarrow{\partial}}{\partial P_i} - (-1)^{ab} \frac{\partial B}{\partial q_i} A\frac{\overleftarrow{\partial}}{\partial P_i} \right) \qquad 2.3.6$$

Where the right derivative $\frac{\partial}{\partial x}$, is the same as left derivative for the commuting variables but not for grassmann variables. Using (2.3.6) the equations of motion (2.3.5) can be rewritten as

$$\dot{\xi}_\alpha = [\xi_\alpha, H]_G \quad , \quad \dot{P}_\alpha = [P_\alpha, H]_G \qquad 2.3.7$$

The bracket (2.3.6) forms a graded algebra |15| that is it satisfys the following properties:

(a) $[A, B]_G = -(-1)^{ab} [B, A]_G$

(b) $[AB, C]_G = A[B, C]_G + (-1)^{bc} [A, C]_G B$

(c) $(-1)^{ac}[A, [B, C]_G]_G + (-1)^{ab}[B, [C, A]_G]_G + (-1)^{cb}[C, [A, B]_G]_G = 0$ 2.3.8

Let us now turn to the question of constraints, define the total hamiltonian.

$$H_T = H_c + \sum_\alpha a^\alpha (P_\alpha + i/_2\, \xi_\alpha) \qquad 2.3.9$$

where the constraints (2.3.3) have been added to the canonical hamiltonian using grassmann coefficients a^α , similar to the section 2.2. Consistency requires the brackets of the constraints with H_T to vanish leading to expressions for a^α ,

$$i\, a^\alpha = \frac{\partial V}{\partial \xi_\alpha} \qquad 2.3.10$$

Clearly now the right equation of motion results using H_T instead of H_C in (2.3.7) or (2.3.5) therefore this procedure seems to produce the desired results. The constraints $P_\alpha + \frac{i}{2}\xi_\alpha \approx 0$ are second class and if we wish to quantize this model we should remove them, therefore we come to the question of graded Dirac brackets. A direct generalization of the equation (2.2.12) to the graded case [17,18] works perfectly well but it is not obvious why it should. In general the matrix $C_{\alpha\beta}$ is graded that is $C_{\alpha\beta}$ contains both odd and even elements, thus the Dirac bracket may not satisfy the conditions (2.3.8) in general, but we shall see that $C_{\alpha\beta}$ has exactly the form required for (2.3.8) to be satisfied. But let us first calculate $C_{\alpha\beta}$ in the simple case of (2.3.3) constraints. Note that $C_{\alpha\beta}$ must be symmetric and not antisymmetric [20] as in the bosonic case.

$$C_{\alpha\beta} = [\chi_\alpha, \chi_\beta]_G = i\,\delta_{\alpha\beta} \qquad 2.3.11$$

The inverse of $C_{\alpha\beta}$ is easily found leading to the following Dirac bracket:

$$[\xi_\alpha, \xi_\beta]^* = -i\,\delta_{\alpha\beta} \qquad 2.3.12$$

This bracket is symmetric as it should be and the quantum version of it is the familiar bracket used in the quantization of the Dirac field.

The matrix $C_{\alpha\beta}$ will always admit an inverse if all the first class constraints are extracted out of the set of second class constraints, the fact that $C_{\alpha\beta}$ is graded does not make a difference here, since the condition of singularity of $C_{\alpha\beta}$ is:

$$\sum_\alpha e_\alpha\, [\chi_\alpha, \chi_\beta]_G \approx 0 \qquad 2.3.13$$

which implies that the first class constraint $\sum_\alpha e_\alpha \chi_\alpha$ has not been taken out of the set of the second class constraints.

The proof of (2.3.13) for graded matrices is not as easy as the bosonic matricies since the existence of an inverse is a more stringent condition on a graded matrix, because not only should the determinant be non-zero but it must also not be nilpotent. To prove that (2.3.13) is necessary and sufficient for $C_{\alpha\beta}$ to be singular let us first note that it is sufficient, because if (2.3.13) holds and $C^{-1}_{\alpha\beta}$ exists it follows that $e_\alpha = 0$. Now let us write the matrix C as follows:

$$C_{\alpha\beta} = A_{\alpha\beta} - B_{\alpha\beta} \qquad 2.3.14$$

where all of the nilpotent elements of C are gathered in B. If A^{-1} exists then the inverse of C can be constructed by iteration as follows, let

$$C^{-1} = A^{-1} + \delta$$

then

$$C^{-1}C = I = I - A^{-1}B + \delta C$$

or

$$\begin{aligned} \delta &= A^{-1}BC^{-1} = A^{-1}B(A^{-1} + \delta) \\ &= A^{-1}\sum_{r=1}^{\infty}(BA^{-1})^r \end{aligned}$$

2.3.15

Now since B is nilpotent so is BA^{-1} therefore the sum is convergent and δ exists. Thus if C is singular A should also be singular, hence we can find the vector ν_α, made up of C-number elements such that,

$$A_{\alpha\beta}\, \mathcal{V}^{g}_{\beta} = 0 \qquad 2.3.16$$

now if one forms the vector $e_\alpha = \pi_i\, \xi_i\, \nu_\alpha$, where ξ_i are the generators of the grassmann algebra G_n over which $C_{\alpha\beta}$ is defined, we shall have

$$C_{\alpha\beta}\, e_\beta = \prod_{i=1}^{n} \xi_i\, A_{\alpha\beta}\, \mathcal{V}^{g}_{\beta} + B_{\alpha\beta}\, \mathcal{V}^{g}_{\beta} \prod_{i=1}^{n} \xi_i \qquad 2.3.17$$
$$= 0$$

Where $B_{\alpha\beta}\, \pi_i\, \xi_i$ vanishes because $B_{\alpha\beta}$ are nilpotent. Therefore (2.3.13) is also necessary.

Let us now check the conditions (2.3.8) for the Dirac bracket, to do so I must assume some symmetry property for $C^{-1}_{\alpha\beta}$ otherwise the question cannot be addressed at all, so let us choose

$$C^{-1}_{\alpha\beta} = -(-1)^{t_{\alpha\beta}}\, C^{-1}_{\beta\alpha} \qquad 2.3.18$$

where $t_{\alpha\beta}$ can be zero or one. Now one has

$$[A, B]^{*} = [A, B]_C - [A, \chi_\alpha]_C\, C^{-1}_{\alpha\beta}\, [\chi_\beta, B]_C$$
$$= -(-1)^{ab}\left\{[B, A]_C - (-1)^{g_{\alpha\beta}}[B, \chi_\alpha]\, C^{-1}_{\alpha\beta}\, [\chi_\beta, A]\right\}$$

2.3.15

where $$g_{\alpha\beta} = ab + ak_\alpha + bk_\beta + (b + k_\beta)\, q_{\alpha\beta} + (a + k_\alpha)(q_{\alpha\beta} + b + k_\beta) + t_{\alpha\beta}$$

and $q_{\alpha\beta}$ is the grade of $C^{-1}_{\alpha\beta}$, k_α is the grade of χ_α . Now if the Dirac bracket is to satisfy property (2.3.8) (a) $g_{\alpha\beta}$ must equal zero modulo 2, for all values of α and β, independent of a and b, thus one arrives at:

$$k_\alpha + k_\beta + q_{\alpha\beta} = 0$$

$$k_\alpha k_\beta + q_{\alpha\beta}(k_\alpha + k_\beta) + t_{\alpha\beta} = 0 \qquad 2.3.19$$

thus $t_{\alpha\beta} = 0$ if $k_\alpha = k_\beta = 0$ and $t_{\alpha\beta} = 1$ otherwise, therefore if the set of constraints is arranged such that: $\alpha = 1, \ldots, n$ correspond to odd constraints and $\alpha = n+1, \ldots, m$ to even constraints; then (2.3.19) implies that: C^{-1} should have the following form:

$$C^{-1} = \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{m} \end{array} \begin{array}{c} \\ \end{array}\left[\begin{array}{cc} M & Q \\ Q^T & T \end{array}\right]\begin{array}{l} \}n \\ \}m \end{array} \qquad 2.3.20$$

this form of C^{-1} is also adequate for the Dirac bracket to satisfy the other relations of (2.3.8)|17|. Surprisingly the matrix $C_{\alpha\beta}$ does indeed give rise to such a matrix provided it is not singular: The matrix C has the following form by definition:

$$C = \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{m} \end{array} \left[\begin{array}{cc} S & U \\ -U^t & A \end{array}\right]\begin{array}{l} \}n \\ \}m \end{array}$$

where S is symmetric, A antisymmetric and they are even, the matrix U is made up of odd elements. The inverse of $C_{\alpha\beta}$ can then be written as

$$C^{-1} = \left[\begin{array}{cc} M & Q \\ N & T \end{array}\right] \qquad 2.3.21$$

then one obtains the following relations:

(a) $MS - QU^t = I_n$ (b) $MU + QA = 0$

(c) $NS - TU^t = 0$ (d) $NU + TA = I_m$

2.3.22

(a) $SM + UN = I_n$ (b) $-U^tM + AN = 0$

(c) $SQ + UT = 0$ (d) $-U^tQ + AT = I_m$

2.3.23

Where I_n is the unit nxn matrix using (2.3.15)(b) and (a) we find:

$$M = [S + UA^{-1}U^t]^{-1} \tag{2.3.24}$$

which shows that M is symmetric also (2.3.23)(b) and (2.3.22)(b) imply that:

$$N = Q^t \tag{2.3.25}$$

Furthermore (2.3.23)(d) and (2.3.22)(d) imply that T is antisymmetric, therefore the inverse of C has the following form:

$$C^{-1} = \begin{pmatrix} M & Q \\ Q^t & T \end{pmatrix} \tag{2.3.26}$$

where M is symmetric and T anti-symmetric therefore C^{-1} does have the desired form.

Before closing this chapter let us note that quantization is effected by setting a correspondence between the graded commutator of two operators and the Dirac bracket of the dynamical variables:

$$\hat{A}\hat{B} - (-1)^{ab}\hat{B}\hat{A} = i\hbar\,[A,B]^* \tag{2.3.27}$$

CHAPTER 3

THE CENTRAL CHARGE IN THE SUPERSYMMETRIC CP^{n-1} MODEL

3.1 Introduction

The CP^n model was primarily constructed in relation with supergravity by Cremmer and Scherk [21] and later studied extensively because of instanton solutions and confinement [24, 25, 26, 28]. Instantons are classical solutions which make the euclidean action stationary, they also have a topological quantum number associated with them, thus one would expect the supersymmetric CP^n model to have a modified superalgebra [29]. However, in an euclidean theory all the components of space enter on an exactly equal footing, consequently there exists an ambiguity in the choice of the time axis and thus the canonical momenta. In addition, the charge density of a current cannot be identified since all the components of a current can be continuously rotated into each other. These difficulties render the explicit calculation of the brackets of the algebra of an euclidean theory impossible. Therefore I shall consider the supersymmetric CP^n over a Minkowski space-time. The models with instanton solutions where constructed over a 2-dimensional euclidean space, therefore I need a 3-dimensional space-time, with the metric diagonal and given by $\mathcal{J}_{\mu\nu} = \mathrm{diag}(+1, -1, -1)$, then the instanton solutions of the two-dimensional euclidean model correspond to time independent solitons of the three-dimensional model which minimise the hamiltonian.

3.2 The Model [21-27]

The supersymmetric CP^n model is a particular case of a general class of supersymmetric models of Kählerian manifolds [27]. The complex projective space CP^{n-1} is defined by the equivalence classes of the complex vectors $\phi_a = (\phi_1, \ldots, \phi_n)$ which satisfy [31]

$$\sum_{a=1}^{n} \phi_a^* \phi_a = K \; , \; \phi_a' \equiv \phi_a \text{ if } \phi_a' = \lambda \phi_a \; , \; |\lambda| = 1 \qquad 3.2.1$$

where K is a positive real number. The fields $\phi_a(x)$ are maps from the space-time into the CP^{n-1} manifold, the constant K is not dimensionless therefore it is misleading to set it equal to one. The lagrangian of this model is given by,[26] .

$$\mathcal{L} = \sum_{a=1}^{n} \left[D_\mu \phi_a^* D^\mu \phi_a + i/_2 \bar{\psi}_a \not{D} \psi_a - i/_2 D_\mu \bar{\psi}_a \gamma^\mu \psi_a + \frac{1}{4K} (\bar{\psi}_a \psi_a)^2 \right] \qquad 3.2.2$$

apart from the constraint (3.2.1), the fields also satisfy the following constraints:

$$\phi_a^* \psi_a \approx \bar{\psi}_a \phi_a \approx 0 \qquad 3.2.3$$

The covariant derivative $D_\mu \phi_a(x)$ $\mu = 0, 1, 2$ is constructed using an auxiliary vector field $A_\mu(x)$ given by:

$$A_\mu(x) = \frac{i}{2eK} \left(\phi_a^* \partial_\mu \phi_a - \phi_a \partial_\mu \phi_a^* - i \bar{\psi}_a \gamma_\mu \psi_a \right)$$
$$D_\mu = \partial_\mu + ie A_\mu \qquad 3.2.4$$

The lagrangian density (3.2.2) is invariant under local U(1) transformations:

$$\phi_a(x) \to e^{ie\Lambda(x)} \phi_a(x) \; , \; \psi_a(x) \to e^{ie\Lambda(x)} \psi_a(x) \qquad 3.2.5$$

Note that although the lagrangian density is invariant under local U(1) transformations, and it contains a vector field U(1) but it does not contain the kinetic term for $A_\mu(x)$ thus the equation of motion for $A_\mu(x)$ does not involve any derivatives and it can be used to eliminate A_μ as given by (3.2.4). This lagrangian density is also invariant under complex supertransformations,

$$\delta \phi_a = i \bar{\epsilon} \psi_a$$

$$\delta \psi_a = -\not{D} \phi_a \epsilon - \frac{i}{2K} \phi_a \bar{\psi}_b \psi_b \epsilon \qquad 3.2.6$$

where ϵ is an infinitesimal, complex, grassman spinor but in proving the supersymmetry of the lagrangian density we shall find it necessary to use the constraints (3.2.3) and (3.2.1) therefore we must first check that the set of constraints does not enlarge upon supertransformations, this is also important when one comes to treat the constraints as outlined in the last chapter. Since these constraints are ones which are imposed from outside they must be added to the lagrangian density using Lagrange multipliers $\alpha(x)$ and $\theta(x)$:

$$\mathcal{L}_T = \mathcal{L}(x) + \alpha(x)(\phi_a^* \phi_a - K) + \bar{\theta} \psi_a \phi_a^* + \bar{\psi}_a \theta \phi_a \qquad 3.2.7$$

now the constraints arise from the equations for conjugate momenta of $\alpha(x)$ and $\theta(x)$ for example:

$$\frac{\partial \mathcal{L}_T}{\partial \dot{\alpha}(x)} = P_{\alpha(x)} \approx 0 \qquad 3.2.8$$

then the commutation relation of $P_\alpha(x)$ and the total hamiltonian gives rise to a secondary constraint, $\phi_a^* \phi_a(x) - K \approx 0$, therefore we can use $\mathcal{L}_T$ instead of $\mathcal{L}(x)$ and proceed with the method outlined in chapter 2. Nevertheless if $\mathcal{L}_T$ is to be used instead of $\mathcal{L}(x)$ we must now prove that $\mathcal{L}_T(x)$ is supersymmetric rather than $\mathcal{L}(x)$, thus:

$$\delta \mathcal{L}_T(x) = \delta \mathcal{L} + \delta \alpha (\phi_a^* \phi_a - K) + \alpha [\delta (\phi_a^* \phi_a - K)] + \delta \theta \bar{\psi}_a \phi_a + \theta [\delta (\bar{\psi}_a \phi_a)] + \cdots$$

3.2.9

where $\delta\mathcal{L}_T$ vanishes if the R.H.S. of (3.2.9) is a linear combination of the constraints, we shall see that $\delta\mathcal{L}$ is a linear combination of the constraints, $\delta\alpha$ and $\delta\theta$ are coefficients of constraints therefore what remains to do is to show that the variation of the constraints is itself a linear combination of the constraints. Let us begin by $\phi_a^* \phi_a - K \approx 0$,

$$\delta(\phi_a^* \phi_a - K) = -i\bar{\psi}_a \epsilon \phi_a + i\phi_a^* \bar{\epsilon}\psi_a \approx 0 \qquad 3.2.10$$

and

$$\delta(\phi_a^* \psi_a) = -i\bar{\psi}_a \epsilon \psi_a - \phi_a^* \not{\partial}\phi_a \epsilon - \frac{i}{2}\bar{\psi}_b \psi_b \epsilon$$

$$= \left(\tfrac{i}{2}\bar{\psi}_a \gamma_\mu \psi_a - \phi_a^* \partial_\mu \phi_a - iKA_\mu\right)\gamma^\mu \epsilon$$

$$\approx 0$$

3.2.11

where we have Fierz transformed and used (3.2.4). Let us now look at the variation of the lagrangian density (3.2.1). Note that we need not consider the variation of $A_\mu(x)$ because its equation of motion is algebraic therefore $\frac{\delta\mathcal{L}_0}{\delta A_\mu} = 0$ is in fact equivalent to (3.2.4) thus

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\psi}\delta\psi + \frac{\partial L}{\partial A_\mu}\delta A_\mu$$

$$= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi$$

3.2.12

using (3.2.4), in other words $[\delta, D_\mu] = 0$ where δ is the supertransformation. Therefore we have:

$$
\begin{aligned}
\delta\mathcal{L} = {} & i\bar{\epsilon} D_\mu \psi_a D^\mu \phi_a^* + i\partial_\mu \bar{\epsilon}\psi_a D^\mu \phi_a^* - i D_\mu \phi_a D^\mu \bar{\psi}_a \epsilon \\
& - i D_\mu \phi_a \bar{\psi}_a \gamma^\mu \epsilon - i\bar{\epsilon} \not{D}\phi_a^* \not{D}\psi_a - \tfrac{1}{2K}\bar{\epsilon}\phi_a \bar{\psi}_b \psi_b \not{D}\psi_a \\
& + i D_\mu \bar{\psi}_a \gamma^\mu \not{D}\phi_a \epsilon - \frac{1}{2K} D_\mu \bar{\psi} \gamma^\mu \phi_a \bar{\psi}_b \psi_b \epsilon \\
& + \tfrac{i}{2}\partial_\mu (\bar{\psi}\gamma^\mu \delta\psi - \delta\bar{\psi}\gamma^\mu \psi) \\
& - \tfrac{1}{2K}\left(\bar{\psi}_a \left(\not{D}\phi_a \epsilon + \tfrac{i}{2K}\phi_a \bar{\psi}_c \psi_c \epsilon\right)\right) \bar{\psi}_b \psi_b \\
& - \tfrac{1}{2K}\left(\bar{\epsilon}\not{D}\phi_a^* + \tfrac{i}{2K}\phi_a^* \bar{\psi}_b \psi_b\right)\psi_a \bar{\psi}_c \psi_c
\end{aligned}
$$

3.2.13

now noting that the product of more than four spinor fields vanishes, and neglecting the total divergence we have:

$$
\begin{aligned}
\delta\mathcal{L} = {} & i\bar{\epsilon} D_\mu \psi_a D^\mu \phi_a^* - i D_\mu \bar{\psi}_a \epsilon D^\mu \phi_a + i\partial_\mu \bar{\epsilon}\psi_a D^\mu \phi_a \\
& - i D_\mu \phi_a \bar{\psi}_a \partial^\mu \epsilon - i\bar{\epsilon}\gamma^\mu \gamma^\nu D_\nu \psi_a D_\mu \phi_a^* \\
& + i D_\mu \bar{\psi}_a \gamma^\mu \gamma^\nu \epsilon D_\nu \phi_a - \tfrac{1}{2K} D_\mu \bar{\psi}_a \gamma^\mu \epsilon \phi_a \bar{\psi}_b \psi_b \\
& - \tfrac{1}{2K}\bar{\epsilon}\not{D}\psi_a \phi_a^* \bar{\psi}_b \psi_b - \frac{\bar{\psi}_b \psi_b}{2K} \bar{\psi}_a \not{D}\phi_a \epsilon \\
& - \frac{\bar{\psi}_b \psi_b}{2K} \bar{\epsilon}\not{D}\phi_a^* \psi_a
\end{aligned}
$$

now use

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma_\rho$$

3.2.14

resulting in:

$$
\begin{aligned}
\delta \mathcal{L} = \; & i\partial_\mu \bar{\epsilon}\, \psi_a D^\mu \phi_a^* - i D_\mu \phi_a \bar{\psi}_a \partial^\mu \epsilon \\
& + \bar{\epsilon}\, \epsilon^{\mu\nu\rho} \gamma_\rho D_\nu \psi_a D_\mu \phi_a^* - D_\mu \bar{\psi}_a \epsilon^{\mu\nu\rho} \gamma_\rho \epsilon D_\nu \phi \\
= \; & i\partial_\mu \bar{\epsilon}\, \psi_a \partial^\mu \phi_a^* + e\, \partial_\mu \bar{\epsilon}\, A^\mu \psi_a \phi_a^* \\
& - \partial_\nu \bar{\epsilon}\, \epsilon^{\mu\nu\rho} \gamma_\rho \psi_a \partial_\mu \phi_a^* + i e\, \partial_\nu \bar{\epsilon}\, \epsilon^{\mu\nu\rho} \gamma_\rho A_\mu \psi_a \phi_a^* \\
& + 2\pi i\, \eta^\rho \bar{\epsilon} \gamma_\rho \psi_a \phi_a^* \quad + \quad h.c. \\
\approx \; & i\partial_\mu \bar{\epsilon}\, J^\mu \quad + h.c.
\end{aligned}
$$

3.2.14

Where I have used the notation $n^{\rho} = \frac{1}{2\pi}\ {}^{\rho\mu\nu}\ \partial_{\mu} A_{\nu}$, we shall see that n^{ρ} is the topological current of this model, therefore if ϵ is not position dependent $\delta\mathcal{L}$ is the sum of a total divergence and a term proportional to the constraints, curiously the topological current appears in the variation of the lagrangian density, the reason for which is not clear to me.

The supercurrent evidently from (3.2.14) is

$$
J^\mu = i\, \partial_\nu \phi_a^*(x) \gamma^\nu \gamma^\mu \psi_a(x) \qquad 3.2.15
$$

The supercharge obtained from this supercurrent does not produce the right supertransformation, this is due to the fact that the system is constrained and the Dirac bracket of the supercharge does give the desired supertransformation, but this current is gauge invariant although it does not appear to be so, because of the constraints (3.2.3).

3.3 The Superalgebra

The product law of the algebra is a Dirac bracket therefore the complete set of constraints has to be calculated and the matrix $C_{ij} = [K_i, K_j]$ be inverted before one can look at the structure of this algebra. So let us start with the lagrangians density:

$$\mathcal{L}(x) = D_\mu \phi_a^*(x)\, D^\mu \phi_a(x) + \tfrac{i}{2}\bar{\psi}_a \not{D} \psi_a - \tfrac{i}{2}\overline{D_\mu \psi_a}\gamma^\mu \psi_a$$
$$+ \frac{1}{4K}(\bar{\psi}_a \psi_a)^2 + \alpha(x)(\phi_a^* \phi_a - K) + \bar{\theta}\psi_a \phi_a^* + \bar{\psi}_a \theta \phi_a \qquad 3.3.1$$

Assuming that $A_\mu(x)$ and the lagrange multipliers α and θ are independent degrees of freedom the following relations for the canonical momenta are obtained:

a) $$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \approx 0 \qquad 3.3.2$$

b) $$P_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \approx 0$$

c) $$P_\theta(x) = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \approx 0$$

d) $$\Pi_a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = D_0 \phi_a^* \quad , \quad \Pi_a^* = D_0 \phi_a$$

e) $$P_{\psi_a^\dagger} = -\tfrac{i}{2}\psi_a \quad , \quad P_{\psi_a} = -\tfrac{i}{2}\psi_a^\dagger$$

3.3.2

The set of constraints (3.3.2)(e) are the typical second class constraints of spinor fields (as in (2.3.3) but here we have one constraint per point of space) which can be removed by defining Dirac brackets iteratively, and since there exists a correspondence between Dirac brackets and the commutator or anticommutator of the operators we shall adopt the following anticommutator for the spinor fields,

$$\left\{ \psi_\alpha^a(x),\ \psi_\beta^{b}(x)^{\dagger} \right\} = \delta_{ab}\, \delta_{\alpha\beta}\, \delta(x-y) \qquad 3.3.3$$

and the following commutation relations are adopted for the other fields:

a) $[\phi_a(x),\ \Pi_b(y)] = i\,\delta_{ab}\,\delta(x-y)$

b) $[A_\mu(x),\ \Pi_\nu(x)] = i\, g_{\mu\nu}\, \delta(x-y)$

c) $[\alpha(x),\ P_\alpha(y)] = i\, \delta(x-y)$

d) $\{\theta_\beta(x),\ P_{\theta_\gamma}(y)\} = i\,\delta_{\beta\gamma}\,\delta(x-y)$ 3.3.4

The hamiltonian density is given by:

$$H(x) = \Pi_a \Pi_a^* + |D_j\phi|^2 + \tfrac{i}{2}\overline{D_j\psi_a}\,\gamma^j\psi_a - \tfrac{i}{2}\bar\psi_a\gamma^j D_j\psi_a$$
$$-\frac{1}{4K}(\bar\psi_a\psi_a)^2 + iA_0(\phi_a^*\Pi_a^* - \phi_a\Pi_a - i\bar\psi_a\gamma^0\psi_a) \qquad 3.3.5$$

where the index j runs over the space dimensions only. The total hamiltonian density is:

$$H_T(x) = H(x) + b^\mu(x)\,\Pi_\mu(x) + \dot\alpha P_\alpha + \dot\theta P_\theta$$
$$+ \dot\theta^\dagger P_{\theta^\dagger} - \alpha(x)(\phi_a^*\phi_a - K)$$
$$- \bar\psi_a\theta\,\phi_a - \bar\theta\psi_a\,\phi_a^* \qquad 3.3.6$$

where the primary constraints (3.4.2) (a, b, c) have been added to the hamiltonian density. Now:

a) $[\Pi_0, H_T] \Rightarrow \phi_a^* \Pi_a^* - \phi_a \Pi_a - i\bar{\psi}_a \gamma^0 \psi_a \approx 0$

b) $[\Pi_i, H_T] \Rightarrow \phi_a D_i \phi_a^* - \phi_a^* D_i \phi_a + i\bar{\psi}_a \gamma_i \psi_a = j_i \approx 0$

c) $[P_\alpha, H_T] \Rightarrow i(\phi_a^* \phi_a - K) \approx 0$

d) $[P_\theta, H_T] \Rightarrow i\bar{\psi}_a \phi_a \approx 0$

e) $[P_\theta^\dagger, H_T] \Rightarrow -i\phi_a^* \psi_a \approx 0$

3.3.7

A number of secondary constraints have arisen which we must commute with the hamiltonian.

a) $[\phi_a^* \Pi_a^* - \phi_a \Pi_a - i\bar{\psi}_a \gamma^0 \psi_a, H_T] \approx 0$

b) $[j_i, H_T] \Rightarrow i\Pi_a D_i \phi_a - i\phi_a^* D_i \Pi_a^* - i\Pi_a^* D_i \phi_a^* + i\phi_a D_i \Pi_a$

$$- 2 b_i K - \tfrac{1}{2} D_j \bar{\psi}_a \gamma^j \gamma^0 \gamma_i \psi_a - \tfrac{1}{2} \bar{\psi}_a \gamma_i \gamma^0 \gamma^j D_j \psi_a$$

$$\approx 0$$

or

$$b_i(x) = \frac{i}{2K} \Pi_a D_i \phi_a - \frac{i}{2K} \phi_a^* D_i \Pi_a^* - \frac{i}{2K} \Pi_a^* D_i \phi_a^* + \frac{i}{2K} \phi_a D_i \Pi_a$$

$$\frac{i}{4K} D_j \bar{\psi} \gamma^j \gamma^0 \gamma_i \psi - \frac{1}{4K} \bar{\psi} \gamma_i \gamma^0 \gamma^j D_j \psi$$

$$= \frac{\partial}{\partial t} A_i(x)$$

c) $[\phi_a^* \phi_a - K, H_T] \Rightarrow (\phi_a^* \Pi_a^* + \phi_a \Pi_a) \approx 0$

d) $\left[\psi_a \phi_a^* , H_T\right] \Rightarrow \quad i\Pi_a \psi_a - i/_2 \phi_a^* D_i(\gamma^0\gamma^i\psi_a)$

$$-i/_2\, \phi_a^* \gamma^0\gamma^i D_i \psi_a - K\gamma^0\theta \approx 0$$

or $\qquad \theta \approx -\frac{i}{K}\, \phi_a^* \not{D}\, \psi_a$

e) $[\psi_a\phi_a, H_T]$ is the complex conjugate of (d) this results in the conjugate of the constraint, leading to:

$$\bar{\theta} \approx \frac{i}{K} D_\mu \bar{\psi}_a \gamma^\mu \phi_a \qquad 3.3.8$$

The constraints (3.3.8) (d) and (ë) when commuted with the hamiltonian give rise to expressions for $\dot{\theta}$ and $\dot{\theta}^+$ similar to (3.4.8) (b), therefore the last constraint to consider is (3.3.8) (c)

$$[\phi_a^* \Pi_a^* + \phi_a \Pi_a , H_T] \Rightarrow \qquad 2i\, \Pi_a \Pi_a^* - i\, \phi_a^* D_j D^j \phi^a$$

$$- i\phi_a D_j D^j \phi^{*a} - 2Ki\, \alpha(x) \quad \approx 0$$

or,

$$\alpha(x) \approx \frac{1}{2K}\left(2\Pi_a \Pi_a^* - \phi_a^* D_j D^j \phi_a - \phi_a D_j D^j \phi_a^*\right) \qquad 3.3.9$$

The constraint (3.3.9) likewise, when commuted with H_T produces an expression for $\dot{\alpha}(x)$, the set of constraints has thus been exhausted. The classification is rather easy, $\pi_0 \approx 0$ is clearly a first class constraint since no constraint involves the zeroth component of the vector field and the constraint $\phi_a^* \pi_a^* - \phi_a\pi_a - i\psi_a\gamma^0\psi_a \approx 0$ is in fact the generator of the U(1) transformation therefore it commutes with any gauge invariant operator, but all the constraints are gauge invariant, therefore the first class constraints are :

a) $\Pi_0 \approx 0$

b) $\phi_a^* \Pi_a^* - \phi_a \Pi_a - i \bar{\psi}_a \gamma^0 \psi_a \approx 0$ 3.3.10

and the rest are second class. Let me index the constraints by $K_\ell, \ell = 1, \ldots, 20$

$$K_1 = P_\alpha$$

$$K_{2,3} = P_\theta$$

$$K_{4,5} = P_{\theta^+}$$

$$K_{6,7} = \Pi_i \qquad i = 1,2$$

$$K_8 = \alpha(x) - \frac{1}{K}\left(\Pi_a \Pi_a^* + |D_i \phi|^2\right)$$

$$K_{9,10} = \theta + i/_K \ \phi_a^* \not{D} \psi_a$$

$$K_{11,12} = \bar{\theta} - i/_K \ D_\mu \bar{\psi} \gamma^\mu \phi_a$$

$$K_{13,14} = \frac{i}{eK} \phi_a^* D_i \phi_a - \frac{i}{eK} \phi_a D_i \phi_a^* + \frac{1}{eK} \bar{\psi}_a \gamma_i \psi_a$$

$$K_{15,16} = \psi_a^\dagger \phi_a$$

$$K_{17,18} = \phi_a^* \psi_a$$

$$K_{19} = \phi_a^* \phi_a - K$$

$$K_{20} = \phi_a^* \Pi_a^* + \phi_a \Pi_a$$

3.3.11

This large set of second class constraints results in a cumbersome matrix $C_{ij} = [K_i^*, K_j]$, however an unexpected simplification comes to the rescue, the matrix C has the following form:

$$C = \begin{bmatrix} 0 & -iI_7 & 0 \\ \eta_7 & L & M \\ 0 & N & B \end{bmatrix} \qquad 3.3.12$$

where I_7 is the unit 7 x 7 matrix and η_7 a diagonal matrix, such that $\eta_7 \eta_7^* = I$. The unexpected development is that the inverse of the 6 x 6 matrix B appears in the inverse of C, that is C^{-1} is given by,

$$C^{-1} = \begin{bmatrix} Y & \eta_7^* & T \\ iI_7 & 0 & 0 \\ U & 0 & B^{-1} \end{bmatrix} \qquad 3.3.13$$

The fact that B^{-1} appears in C^{-1} surrounded by the rows of zeros reduces the effort of calculation in working out the bracket of the supercharges. The supercharge, given by (3.2.15) is

$$Q_\alpha = \int d^2x \, (i\pi_a \psi_a + i\gamma^i\gamma^0 \partial_i \phi_a^* \psi_a) \qquad 3.3.14$$

The supercharge commutes with the constraints K_1 to K_7 therefore the Dirac bracket of the supercharges can be expanded as follows, (where star denotes Dirac bracket)

$$\{Q_\alpha, Q_\beta^\dagger\}^* = \{Q_\alpha, Q_\beta^\dagger\} - \int d^2x\, d^2y \sum_{i,j=1}^{20} [Q_\alpha, K_i(x)]\, C_{ij}^{-1}(x,y) [K_j(y), Q_\beta^\dagger]$$

$$= \{Q_\alpha, Q_\beta^\dagger\} - \int d^2x\, d^2y \sum_{i,j=15}^{20} [Q_\alpha, K_i(x)]\, B_{ij}^{-1}(x,y) [K_j(y), Q_\beta^\dagger]$$

3.3.15

where I have also used the fact that rows 7 to 14 are zeros, from column 7 onwards! Therefore all we need is the inverse of the matrix B, the matrix B is

$$B(x,y) = \begin{bmatrix} 0 & 0 & K & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ K & 0 & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2iK \\ 0 & 0 & 0 & 0 & -2iK & 0 \end{bmatrix} \delta(x-y) \qquad 3.3.16$$

thus the inverse is easily found to be:

$$B^{-1}(x,y) = \begin{bmatrix} 0 & 0 & 1/K & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/K & 0 & 0 \\ 1/K & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i/2K \\ 0 & 0 & 0 & 0 & -i/2K & 0 \end{bmatrix} \delta(x-y) \qquad 3.3.17$$

The first point to check is the transformation law of the spinor filed ψ_a:

$$\delta\psi_a = i\,[\bar{Q}\epsilon, \psi_a(x)]^*$$

$$= -\gamma^0\epsilon\,\Pi_a^* - \gamma^i\epsilon\,\partial_i\phi_a - i\int d^2z\,d^2\omega\,[\bar{Q}\epsilon, \phi_c^*\psi_c]\frac{\delta(z-\omega)}{K}[\bar{\phi}_b\phi_b, \psi_a]$$

$$= -\gamma^0\epsilon\,\Pi_a^* - \gamma^i\epsilon\,\partial_i\phi_a - i\frac{\phi_a}{K}\left(-\bar{\psi}_b\epsilon\,\psi_b + i\gamma^i\epsilon\,\phi_b^*(x)\,\partial_i\phi_b\right)$$

Now Fierz reshuffling and using (3.2.4) results in the right answer:

$$\delta\psi_a = i[\bar{Q}\epsilon, \psi_a]^* = -\gamma^\mu D_\mu\phi_a\epsilon \quad -\frac{i}{2K}\phi_a\,\bar{\psi}_b\psi_b\,\epsilon \qquad 3.3.18$$

We can now calculate the Dirac bracket of the supercharges. The only non-vanishing terms on the right hand side of (3.4.15) are:

$$\{Q_\alpha, Q_\beta^\dagger\}^* = \{Q_\alpha, Q_\beta^\dagger\} - \int\frac{d^2z}{K}\{Q_\alpha, \phi_a\psi_a^\dagger(z)\}\{\phi_b^*\psi_b(z), Q_\beta^\dagger\} \qquad 3.3.19$$

The second term of (3.4.19) is:

$$= -\int \frac{d^2x}{K}\Big[\delta_{\alpha\beta}\left(-e^2K^2A_jA^j + \tfrac{1}{4}(\bar{\psi}_a\psi_a)^2\right)$$

$$+i(\gamma^j\gamma^0)_{\alpha\beta}\left[eKA_j(\phi_a^*\Pi_a^* - \phi_a\Pi_a - i\bar{\psi}_a\gamma_0\psi_a\right]$$

$$+\tfrac{i}{2}(\bar{\psi}_a\psi_a)\gamma^0_{\alpha\beta}\left(\phi_a^*\Pi_a^* - \phi_a\Pi_a - i\bar{\psi}_a\gamma_0\psi_a\right)\Big]$$

3.3.20

Note that the coefficient of $\gamma^0_{\alpha\beta}$ is a first class constraint, it cannot be set equal to zero. The other term in (3.3.19) is:

$$\{Q_\alpha, Q^\dagger_\beta\} = \int d^2x\Big[\delta_{\alpha\beta}\left(\Pi_a\Pi_a^* + |\partial_i\phi_a|^2 + \tfrac{i}{2}\partial_i\bar{\psi}\gamma^i\psi_a - \tfrac{i}{2}\bar{\psi}_a\gamma^i\partial_i\psi_a\right)$$

$$+(\gamma^j\gamma^0)_{\alpha\beta}\left(\Pi_a\partial_j\phi_a - \tfrac{i}{2}\partial_j\bar{\psi}_a\gamma^0\psi_a + \tfrac{i}{2}\bar{\psi}_a\gamma_0\partial_j\psi_a + \Pi_a^*\partial_j\phi_a^*\right)$$

$$+\gamma^0_{\alpha\beta}\left(i\epsilon^{ij}\partial_i\phi^*\partial_j\phi + \tfrac{1}{2}\epsilon^{ij}(\partial_i\bar{\psi}\gamma_j\psi + \bar{\psi}\gamma_j\partial_i\psi)\right)$$

$$+\tfrac{i}{2}\gamma^j_{\alpha\beta}\,\partial_j(\bar{\psi}_a\psi_a)\Big]$$

3.3.21

The last term in (3.3.21) is a total divergence and vanishes if $\psi_a\psi_a$ vanishes at infinity, since in this work the asymptotic behaviour of the fermion fields is assumed trivial, this term can be neglected. But the coefficient of $\gamma^0_{\alpha\beta}$ cannot be neglected although it is a total divergence, the asymptotic behaviour of the $\phi_a(x)$ is non-trivial. The final expression for the anticommutator of the supercharges is:

$$\{Q_\alpha, Q^\dagger_\beta\}^* = (\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu + \gamma^0_{\alpha\beta}(Z+t) \qquad 3.3.22$$

where

(a) $t = eK \int d^2x \; \epsilon^{ij} \; \partial_i A_j(x)$

(b) $Z = -\frac{i}{2eK} \int d^2x \; \bar{\psi}_a(x) \psi_a(x) \left(\phi_a^* \Pi_a^* - \phi_a \Pi_a - i \bar{\psi}_a \gamma_0 \psi_a \right)$

3.3.23

Alternatively (3.3.22) can be written in terms of two majorana supercharges Q^L_α L = 1,2 defined by the real and imaginary parts of Q_α .

$$\left\{ Q^L_\alpha , Q^M_\beta \right\} = 2\, \delta^{LM} (\gamma^\mu \gamma^0)_{\alpha\beta} P_\mu + i \epsilon^{LM} \gamma^0_{\alpha\beta} (Z + t) \qquad 3.3.24$$

this result of course depends on the fact that:

$$\left\{ Q_\alpha , Q_\beta \right\}^* \approx 0 \qquad 3.3.23$$

3.4 The topological and the central charge

If the scalar fields $\phi_a(t,r)$ tend to a constant at spatial infinity independent of time:

$$\phi_a(t,r) \xrightarrow[|r| \to \infty]{} g(\theta)\, \phi^\infty_a \quad , \quad |g(\theta)| = 1 \qquad \phi^{\infty *}_\alpha \phi^\infty_\alpha = K \qquad 3.4.1$$

then $\phi_a(t,r)$, in effect, maps the compactified space into CP^{n-1}. In other words $\phi_a(t,r)$ is mapping the sphere into CP^{n-1}, the homotopy classes of this map is denoted by $\Pi_2(CP^{n-1})$ which is the group of integers |22|. The integer associated with each map $\phi_a(t,r)$ is given by

$$N = \frac{eK}{2\pi} \int d^2x \; \epsilon^{ij} \; \partial_i A_j \qquad 3.4.2$$

where A_j is determined by (3.2.4). One can also construct a current η^μ whose charge density is related to N,

$$\eta^{\mu} = \epsilon^{\mu\nu\rho}\,\partial_{\nu} A_{\rho} \qquad\qquad \partial_{\mu}\eta^{\mu} = 0 \qquad \mu = 0,1,2 \tag{3.4.3}$$

If the integral of the zeroth component of this current is interpreted as a topological charge, associated with the configuration $\phi_a(t,r)$ then one obtains the following relation:

$$g = \int d^2x\, \eta^{0}(x) = \frac{2\pi N}{eK} \tag{3.4.4}$$

which results in the quantization of the topological and the "electric" charge:

$$eg = \frac{2\pi N}{K} \tag{3.4.5}$$

Furthermore one can find a lower bound for the hamiltonian by noting that:

$$\left| D_i \phi_a \pm i \epsilon_{ij} D^j \phi_a \right|^2 \geqslant 0$$

or

$$|D_i \phi_a|^2 \geqslant \left| i\, \epsilon_{ij}\, D^i\phi\, D^j\phi^* \right| \tag{3.4.6}$$

now if $\partial_0\phi_a = 0$ and $\psi_a = 0$ (3.4.6) can be interpreted as:

$$H = \int d^2x\, H(x) \geqslant 2\pi |N| \tag{3.4.7}$$

and the equality is achieved when

$$D_i \phi_a = \overset{+}{(-)}\, i\, \epsilon_{ij} D^j \phi_a \tag{3.4.8}$$

The (anti) self-duality conditions (3.4.8) can be solved to obtain exact n-soliton solutions |22|. When $\psi \neq 0$ a different bound onthe mass can be obtained using the superalgebra of the model |29|. Consider the right hand side of (3.3.24) as a 4 x 4 matrix, it is a positive definite matrix which means in the rest frame:

$$M \geqslant |Z + t| \tag{3.4.9}$$

where M is the mass of the soliton, and equality is achieved when some of the eigenvalues are zero, in other words when there exists a linear relationship among the supercharges. In fact if the fields are self-dual the supercharges obey the following relation in the rest frame:

$$Q_\alpha^1 + i\left(\gamma^0 Q^2\right)_\alpha = 0 \qquad 3.4.10$$

which follows directly from the expressions for Q_α and the self duality expression (3.4.8). The relation (3.4.10) can be generalized to an arbitrary frame:

$$(z+t)\ Q_\alpha^1 + i\left(\not{P} Q^2\right)_\alpha = 0 \qquad 3.4.11$$

this relation suggests that a 4-dimensional majorana spinor can be formed by:

$$S_\alpha = \begin{pmatrix} Q_\alpha^1 \\ Q_\alpha^2 \end{pmatrix} \qquad 3.4.12$$

Then using 4-dimensional gamma matrices Γ^A, A = 0, 1, 2, 3:

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \qquad \Gamma^3 = \begin{pmatrix} -iI_2 & 0 \\ 0 & iI_2 \end{pmatrix} \qquad 3.4.13$$

and a four dimensional momentum $P_A = (P_\mu, z + t)$ equation (3.4.11) can be formulated in four dimensions as:

$$\Gamma^A P_A S = 0 \qquad 3.4.14$$

The relation (3.4.14) has been obtained by Olive [30] for other models with soliton solutions.

This relation suggests the existance of a massless 4-dimensional theory which reduces to the 3-dimensional model by the compactification [34] of the third dimension of space. The supersymmetric CP^n in 4-dimensions, first constructed by Cremmer and Scherk [21] is in fact the right model. This model is described by the following lagrangian density:

$$\mathcal{L}(x) = (\mathcal{D}_A \phi_a)^* (\mathcal{D}^A \phi_a) + i/2\, \overline{\chi}_a \not{\mathcal{D}} \chi_a \qquad 3.4.15$$

where χ_a are n-majorana spinors satisfying:

$$\phi_a^* \phi_a = K$$

$$\tilde{\phi}_a \chi_a = 0, \quad \tilde{\phi}_a = \frac{1}{2}\left[\phi_a - \phi_a^* + \Gamma^5(\phi_a + \phi_a^*)\right] \qquad 3.4.16$$

The covariant derivatives are given by:

$$D_A \phi_a = \partial_A \phi_a + ie B_A \phi_a$$

$$D_A \chi_a = \partial_A \chi_a - ie \Gamma^5 B_A \chi_a \qquad 3.4.17$$

where

$$B_A = \frac{1}{2Ke}\left(i \phi_a^* \partial_A \phi_a - i \phi_a \partial_A \phi_a^* - \overline{\chi}_a \Gamma_A \Gamma^5 \chi_a \right) \qquad 3.4.18$$

The lagrangian density is invariant under the following transformations:

$$\delta \phi_a = ie \Lambda(x)\, \phi_a$$

$$\delta \chi_a = -ie \Lambda(x) \Gamma^5 \chi_a \qquad 3.4.19$$

and the supertransformations:

3.4.20

As before it is possible to show that the set of constraints (3.4.16) is left invariant by the transformation (3.4.20) |21| . The supersymmetry in four dimensions is simple but enlarges to an N(2) supersymmetry in three dimensions on compactification. This is in accordance with the result obtained by Zumino |27| . From (3.4.15) we can arrive at (3.2.2) by the following identification:

$$\chi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_a + \psi_a^* \\ i\psi_a^* - i\psi_a \end{pmatrix} \qquad 3.4.21$$

and compactifying the third axis of space. Then the superalgebra of the 4-dimensional model should transform into the superalgebra of the 3-dimensional model. Therefore the third component of the 4-momentum should produce the same transformation as the central charge:

$$\left[P_3 , \phi_a(x) \right] = i D_3 \phi_a = e B_3 \phi_a \qquad 3.4.22$$

but from 3.4.18 we have:

$$B_3 = \frac{1}{2eK} \left(- \bar{\chi}_a \Gamma_3 \Gamma^5 \chi_a \right)$$

$$= \frac{1}{eK} \bar{\psi}_a \psi_a \qquad \Gamma^5 = i \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3$$

3.4.23

Therefore the operator Z is exactly the gauge transformation required to make the fourth component covariant.

3.5 Summary

As was expected the superalgebra of the CP^n model does contain a term due to the existence of solitons, which is directly proportional to the topological charge; however the central charge of the algebra is not entirely composed of this topological charge but it also includes a field dependent U(1) transformation. Since the CP^n model in three dimensions has an extended supersymmetry it is not surprising that it should have a central charge. The three dimensional model can be obtained by compactifying the third dimension of a four dimensional model, but the four dimensional version is symmetric only under simple supersymmetry, thus the central charge has to arise out of the third component of the momentum, hence we saw (3.4.22) that the third component does indeed produce the required U(1) transformation.

CHAPTER 4

CONSTRAINTS AND CONFINEMENT

4.1 Introduction

The CP^n model has received most of the attention devoted to it because it confines its constituents, that is to say the fundamental fields ϕ_a, in terms of which the lagrangian is written, interact with each other with a potential which increases with distance [26]. The arguments leading to this conclusion involve complicated arguments using functional integrals. But the canonical approach provides a simple insight into this property of the CP^n model, furthermore, it points the way for a more realistic model with quarks and gluons in 4-dimensions which would have confining properties as well. This chapter is devoted to this question.

4.2 Confinement and CP^n

In chapter 2 we saw that a consistent quantization requires the second class constraints be removed using Dirac brackets but the first class constraints remain and we thus have to require all physical states to be annihilated by first class constraints. This is similar to Q.E.D. where the longitudinal photon is decoupled by requiring physical states to be annihilated by a first class constraint [14]. The first class constraints of CP^n are $\pi_o \approx o$, $\phi^*_a \pi^*_a - \phi_a \pi_a - i\psi_a \gamma_o \psi_a \approx o$, thus the conditions physical states have to satisfy are:

a) $\Pi_o|\text{Physical}\rangle = o$

$$\text{b)}\ (\phi^*_a \Pi^*_a - \phi_a \Pi_a - i\bar{\psi}_a \gamma_o \psi_a)|\text{Physical}\rangle = 0 \qquad 4.2.1$$

The condition (4.2.1) states that all the physical states with "electric charge" are unphysical, in other words confined - therefore the fundamental particles of the fields ϕ_a and ψ_a will be confined since they have an electric charge, but their bound states with zero total charge will not be confined. The reference to "electric charge" is however to be qualified since the lagrangian density does not contain a kinetic term for the gauge field, therefore no electric flux either. Here the word "electric" only signifies the U(1) nature of the local invariance of the lagrangian density.

The condition (4.2.1)(a) does not however mean that the fundamental excitations of ϕ_a and ψ_a can not be involved in virtual processes, on the contrary as the constraint of Q.E.D. does not prohibit the existence of virtual longitudinal photon, there is no reason why a virtual ϕ_a should not exist.

The first class constraint (4.2.1)(a) has its origin in the auxiliary vector field $A\mu(x)$, defined by (3.2.4), therefore we may be able to obtain similar results if a model is considered over 4-dimensional space time which has non-abelian local invariance, realized through auxiliary fields.

4.3 Colour Confinement

The generalized version of the CP^n model|32| involves k x n complex scalar fields : Z_a^α which transform under the global group U(n) and the local group (U(k), α = 1, ..., k and a = 1, ..., n, and the fields are subject to the constraint:

$$Z_a^{\alpha *} \; Z_a^\beta = \delta^{\alpha\beta} \tag{4.3.1}$$

The lagrangian density describing the model is:

$$\mathcal{L} = (D_\mu Z_a^\alpha)^* D^\mu Z_a^\alpha \tag{4.3.2}$$

$$D_\mu Z_a^\alpha = \partial_\mu Z_a^\alpha + i\, A^{\alpha\beta} Z_a^\beta$$

where the matrix gauge potential $A_\mu^{\alpha\beta}$ is given in terms of the fields Z_a^α by virtue of its equation of motion

$$\frac{\partial \mathcal{L}}{\partial A_\mu^{\alpha\beta}} = -i\left(Z_a^\beta\right)^* D_\mu Z_a^\alpha + i\, Z_a^\alpha \left(D_\mu Z_a^\beta\right)^* = 0$$

or,

$$A_\mu^{\alpha\beta} = \frac{i}{2}\left(Z_a^{\beta *} \partial_\mu Z_a^\alpha - Z_a^\alpha \partial_\mu Z_a^{\beta *} \right) \qquad 4.3.3$$

The fields Z_a^α undergo two different transformations a global U(n) transformation V_{ab}:

$$Z_a^\alpha \mapsto V_{ab} Z_b^\alpha \qquad 4.3.4$$

and a local U(k) transformation $k < n$:

$$Z_a^\alpha \longmapsto U_{(x)}^{\alpha\beta} \; Z_a^\beta \qquad 4.3.5$$

Evidently the transformation (4.3.4) leaves $A_\mu^{\alpha\beta}$ invariant but the transformation (4.3.5) does transform $A_\mu^{\alpha\beta}$, according to (4.3.3) :

$$A_\mu \rightarrow i\, \partial_\mu U \, U^\dagger + U A_\mu U^\dagger \qquad 4.3.6$$

which is the right transformation to keep $(D_\mu Z_a)^\alpha$ covariant, the process leading to the first class constraints is identical to the CP^n case except that the equations are profilerated with indices in this case, so let me just quote the first class constraints

a) $\Pi_0^{\alpha\beta} \approx 0$

b) $$Q^{\alpha\beta} = i\left(Z_a^{*\alpha} \Pi_a^{*\beta} - Z_a^\alpha \Pi_a^\beta \right) \approx 0 \qquad 4.3.7$$

where $Q^{\alpha\beta} \underset{\sim}{\sim} o$ are also the k^2 generators of $U(k)$, therefore the physical condition:

$$Q^{\alpha\beta} | \text{Physical} \rangle = 0 \qquad 4.3.8$$

implies that all physical states are "colourless". The particles Z_a^{α} and $A_{\mu}^{\alpha\beta}$ are thus confined and only their bound states which are colourless are observable. In this model the fermions are not included but that is not too difficult to achieve, it can be done using $k \times n$ complex spinors $\psi_a^{\alpha}(x)$ which transform similar to Z_a^{α} under $U(n)$ and $U(k)$. However, the addition of fermions intorduces four fermions interactions into the lagrangian density, therefore the theory becomes non-renormalizable in 4-dimensions. The way out may be to couple the fermions in a supersymmetric way |33| to the scalars Z_a^{α} , and hope that the cancellations which occur in supersymmetric models may help to make the model renormalizable. However this model is fairly complicated and requires further work.

CHAPTER 5

FERMIONIC PATH INTEGRAL

5.1 Introduction

The path integral formulation of quantum mechanics provides a very convenient frame work for a number of approximation methods. However there exists a certain amount of confusion with regards to the definition of the path integral and how it is to be performed particularly in the case of fermions. Although there are a few articles in the literature [2,3,4,5,7,8,15], they do not all obtain the same results, consequently I would like to obtain the necessary results here, which shall be used in the following chapter.

Since the path integral provides an expression for the transition matrix elements in terms of the classical action, it is necessary to have a classical mechanics for the system under consideration, but in the case of the fermions a "physical" classical dynamics does not exist because the action, on the classical path, vanishes, thus can never be much bigger than plank's constant, and the classical limit is not attained. Nevertheless the non-existence of classical dynamics does not ban a path integral formulation as long as a formal lagrangian for the fermions can be given. Let us consider the lagrangian describing a fermionic oscillator:

$$\mathcal{L}(t) = i/_2 (\xi^* \dot{\xi} - \dot{\xi}^* \xi) - \omega \xi^* \xi \qquad 5.1.1$$

where ξ is a complex grassmann variable. This lagrangian is singular (see chapter 2) and upon the removal of the constraints, using Dirac brackets, one finds that the phase space is described by the pair (ξ,ξ^*). The striking similarity between this lagrangian and the holomorphic representation of the harmonic oscillator [1] indicates that a similar approach to quantization of (5.1.1) should be taken [2].

Therefore let us first review the holomorphic representation.

5.2 The Holomorphic Representation

The familiar harmonic oscillator lagrangian is:

$$\mathcal{L}(t) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \qquad 5.2.1$$

now upon setting $Z = \frac{1}{\sqrt{2\omega}}(\dot{q} - i\omega q)$, (5.2.1) can be written

$$\mathcal{L}(t) = \frac{i}{2}\left(z^*\dot{z} - \dot{z}^* z\right) - \omega z^* z \qquad 5.2.2$$

the two expressions (5.2.2) and (5.2.1) differ by a total derivative, hence they result in the same dynamics, the conjugate momenta are:

$$P_z = \frac{\partial L}{\partial \dot{z}} = \frac{i}{2} z^* \qquad 5.2.3$$

$$P_{z^*} = \frac{\partial L}{\partial \dot{z}^*} = -\frac{i}{2} z \qquad 5.2.4$$

The pair of second class constraints (5.2.3 and 4) can be removed leading to the Dirac bracket,

$$[z, z^*] = -i \qquad 5.2.5$$

Now the hamiltonian takes the form:

$$H = \frac{\omega}{2}(z z^* + z^* z) \qquad 5.2.6$$

where the product z * z is symmetrized. Following the rules of the canonical quantization, we replace z by the operator $\hat{a}$ whose eigen-values are the complex numbers z, and the commutation relation ($\hbar = 1$)

$$[\hat{a}, \hat{a}^+] = 1 \qquad 5.2.7$$

is obtained using (5.2.5). The hamiltonian operator is given by

$$\hat{H} = \omega \hat{a}^{+} \hat{a} + \omega/2 \qquad 5.2.8$$

we can now verify the following commutation relations:

$$[\hat{a}, \hat{H}] = \omega \hat{a} \quad , \quad [\hat{a}^{+}, \hat{H}] = -\omega \hat{a}^{+} \qquad 5.2.9$$

establishing $\hat{a}$ as the annihilation operator and $\hat{a}$ as the creation operator, acting on the energy eigenstates $|n\rangle$,

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^{+}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$5.2.10$$

leading to

$$\hat{H}\,|n\rangle = (n+\tfrac{1}{2})\,\omega\,|n\rangle \qquad 5.2.11$$

The ground state is characterized by

$$\hat{a}\,|0\rangle = 0 \qquad 5.2.12$$

leading to a compact expression for the n-particle state:

$$|n\rangle = \frac{1}{\sqrt{n!}} \{\hat{a}^{+}\}^{n}\,|0\rangle \qquad 5.2.13$$

The eigenstates of the operator $\hat{a}$ can be expressed in terms of the basis vectors $|n\rangle$

$$|z\rangle = e^{z\,\hat{a}^{+}}\,|0\rangle \qquad 5.2.14$$

Clearly,

$$\hat{a}\,|z\rangle = z\, e^{z\hat{a}^+}\,|0\rangle = z\,|z\rangle \qquad 5.2.15$$

where z can take all the complex values. The eigenstates of $\hat{a}^+$ can be found by conjugating 5.2.16:

$$\langle z^*| = \langle 0|\, e^{\hat{a}\, z^*} \qquad 5.2.16$$

Note that the two vectors defined by (5.2.16) and (5.2.14) are not orthogonal:

$$\langle z'^*|z\rangle = \langle 0|\, e^{\hat{a}\, z'^*}\, e^{z\hat{a}^+}\,|0\rangle = e^{zz'} \qquad 5.2.17$$

An arbitrary state $|f\rangle$ can be represented by an analytic function of z, given by:

$$\langle z|f\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,\langle n|f\rangle \qquad 5.2.18$$

This representation is accompanied by the following representation of the bracket(5.1.7):

$$\hat{a} \equiv \frac{d}{dz} \qquad \hat{a}^+ \equiv z \qquad 5.2.18.1$$

where z is the argument of the function on which $\hat{a}$ and $\hat{a}^+$ are acting and since z can take complex values, we may have also written $\hat{a} \equiv \frac{d}{dz^*}$ and $\hat{a}^+ = z^*$, depending on the function under consideration. We must now define an inner product under which, $\hat{a}$ and $\hat{a}^+$ are conjugate :

$$\langle f|g\rangle = (f,g) = \int d\mu(z)\, \overline{f(z)}\, g(z) \qquad 5.2.19$$

where

$$d\mu(z) = \left[\frac{dz^* dz}{\pi}\right] e^{-z^* z}$$

and we understand the product $[dz^*dz]$ as d x dy where z = x + iy. The scalar product (5.2.19) is positive definite, this can be seen by noting that,

$$\int \left[\frac{d z^* dz}{\pi}\right] e^{-z^* z} \; z^{*n} \; z^m$$

$$= \frac{1}{\pi} \int_0^\infty r\,dr \int_0^{2\pi} d\theta \;\; r^{n+m} \; e^{-i\theta(n-m)} \; e^{-r^2}$$

$$= \delta_{n,m} \;\; n!$$

5.2.20

To check the adjointness of $\hat{a}$ and $\hat{a}^+$, let us write:

$$(f, \hat{a}^+ g) = \int d\mu(z) \; \overline{f(z)} \; z \; g(z)$$

$$= \int \left[\frac{dz^* dz}{\pi}\right] \overline{f(z)} \left(-\frac{d}{dz^*} \left(e^{-z^* z} g(z)\right)\right)$$

$$= \int \left[\frac{dz^* dz}{\pi}\right] \overline{\frac{d f(z)}{dz}}$$

$$= (\hat{a} f, g)$$

5.2.21

Where we have used the Cauchy-Rieman condition and the fact that the exponential factor rapidly damps any analytic function at infinity.

Let us next consider the completeness relation for the states $|z\rangle$. From the scalar product 5.2.19 we can write:

$$\langle f | g \rangle = \int d\mu(z) \; \overline{\langle z | f \rangle} \; \langle z | g \rangle$$

$$= \int d\mu(z) \; \langle f | z^* \rangle \langle z | g \rangle$$

leading to the completeness relation:

$$\int d\mu(z) \; |z^*\rangle \langle z| = 1$$ 5.2.22

or alternatively, if we consider the analytic functions of z*

$$\int d\mu(z^*)\, |z\rangle\langle z^*| = 1 \qquad 5.2.23$$

The two measures used in (5.1.23) and (5.1.22) are of course identical. We therefore see that despite (5.2.17) the states |z> do satisfy a completeness relations, this is caused by the fact that $e^{z'z^*}$ acts like a delta function. To see this consider:

$$f(z) = \langle z|f\rangle = \int d\mu(z')\, \overline{\langle z'|z^*\rangle}\, f(z')$$

$$= \int d\mu(z')\, e^{zz^{*\prime}}\, f(z') \qquad 5.2.24$$

Thus $e^{zz_i^*}$ when integrated with the measure (5.1.19) acts like a delta function.

Finally before deriving the path integral for this system; let us obtain a trace formula in terms of the states |z> since this will indicate the form of the transition element which is to be calculated.

Using the energy eigenstates |n> the trace of an operator U is given by:

$$\mathrm{tr}\, U = \sum_n \langle n|U|n\rangle \qquad 5.2.25$$

now inserting the completeness relation (5.2.23) and noting the property (5.1.24) we are lead to:

$$\mathrm{tr}\, U = \int d\mu(z)\, \langle z^*|U|z\rangle \qquad 5.2.26$$

Therefore note that the diagonal elements of the evolution operator correspond to <z*, t"|z, t'> which we shall consider in the next section .

5.3 The path integral in the Holomorphic Representation

We wish to obtain an expression in terms of the classical action for the expression $\langle z'^* \; t''|z', \; t'\rangle$, to do so we shall divide the interval $t'' - t'$ into N equal parts ϵ, such that:

$$t''-t' = N\epsilon \qquad t_{i+1} - t_i = \epsilon \qquad t_0 = t' \qquad t_N = t'' \qquad 5.3.1$$

then insert the completeness relation 5.3.23 (N-1) times in between the transition element $\langle z'^*, t''|z, t'\rangle$, to give:

$$\langle z'^*, t''|z, t'\rangle = \int \prod_{i=1}^{N-1} d\mu(z_i) \langle z'^*, t_N | z_{N-1} t_{N-1}\rangle \langle z^*_{N-1}, t_{N-1}| \cdots$$
$$\cdots \langle z_1^*, t_1 | z, t_0\rangle$$

5.3.2

We now consider the elements:

$$\langle z^*_{i+1}, t_{i+1} | z_i, t_i\rangle = \langle z^*_{i+1} | e^{-iH\epsilon} | z_i\rangle$$
$$\cong e^{z^*_{i+1} z_i - iH(z^*_{i+1}, z_i)\epsilon}$$

5.3.3

Thus (5.2.2) becomes:

$$\langle z'^*, t''|z, t'\rangle \cong \int \prod_{i=1}^{N-1} \left[\frac{dz_i^* dz_i}{\pi}\right] \exp\left\{-\sum_{i=1}^{N-1} z_i^* z_i + \sum_{i=0}^{N-1} (z^*_{i+1} z_i - iH\epsilon)\right\}$$

$$= \int \prod_{i=1}^{N-1} \left[\frac{dz_i^* dz_i}{\pi}\right] \exp\left\{ \tfrac{1}{2} z_0^* z_0 + \tfrac{1}{2} z_N^* z_N + i\sum_{i=0}^{N-1} \left[i\left(z^*_{i+1} \frac{(z_{i+1} - z_i)}{\epsilon}\right) \right.\right.$$
$$\left.\left. - \left(\frac{z^*_{i+1} - z_i^*}{\epsilon}\right) z_i - H(z^*_{i+1}, z_i)\right]\epsilon \right\}$$

5.3.4

Now let $N \to \infty$ and $\epsilon \to 0$ such that $N\epsilon = t'' - t'$ [1]

$$\langle z'^*, t'' | z, t' \rangle = \int \prod_t \left[\frac{dz^*(t)\,dz(t)}{\pi} \right] \exp\left\{ \tfrac{1}{2} z^*(t')\, z(t') + \tfrac{1}{2} z^*(t'')\, z(t'') + i \int_{t'}^{t''} dt\, \mathcal{L}(t) \right\} \qquad 5.3.5$$

where $\mathcal{L}(t)$ is the lagrangian (5.2.2), minus the zero point energy. The extra term in the exponent is due tothe fact that at t = t' and t = t" we have fixed the value of non-commuting operators, and z*(t') cannot be interpreted as the complex conjugate of z(t') since they both cannot be fixed simultaneously|2| . This unusual property becomes clear when we try to evaluate the integral (5.3.5). To evaluate (5.3.5) let us expand about a classical solution:

$$z(t) = z\, e^{-i\omega(t-t')} + w(t)$$

$$z^*(t) = z'\, e^{-i\omega(t''-t)} + w^*(t) \qquad 5.3.6$$

where w(t) signifies quantum corrections about the classical solution.

It is clear from the set (5.3.6) that the boundary conditions z(t') = z and z*(t") = z'* are observed and that z(t) and z*(t) are not complex conjugates of each other!

Now:

$$U(z'^*, z, t''-t') = \langle z'^*, t'' | z, t' \rangle$$

$$= e^{z'^* z\, e^{-i\omega(t''-t')}}\; U(0, 0, t''-t') \qquad 5.3.7$$

The trace of the evolution operator is obtained, using (5.3.25):

$$\mathrm{tr}\, U = \int d\mu(z)\, e^{z^* z\, e^{-i\omega(t''-t')}}\; U(0,0, t''-t')$$

$$= \frac{1}{1 - e^{-i\omega(t''-t')}}\; U(0,0, t''-t') \qquad 5.3.8$$

To evaluate U(0, 0, t"-t') we shall use expression (5.3.4) with the boundary conditions $z_0 = 0$, $z_N^* = 0$ and note that each integration over z_i is a gaussian of the form:

$$\int \left[\frac{dz_1^* dz_1}{\pi}\right] e^{-z_1^* z_1 + z_2^* z_1 - i\omega z_2^* z_1 \epsilon} = 1 \qquad 5.3.9$$

thus the only contribution from U(0, 0, t"-t') is $e^{-iw(t''-t')}$ since the hamiltonian used in (5.3.3) contains a zero point energy. The final result for the trace is

$$\mathrm{tr}\, U = \frac{e^{-\frac{i}{2}\omega(t''-t')}}{1 - e^{-i\omega(t''-t')}} \qquad 5.3.10$$

Note that we did not need to impose any periodicity conditions on the paths over which we integrated.

The above results can easily be extended to a system containing a number of oscillators each having a different frequency. Let the phase space coordinates of each oscillator be given by z_ℓ, $\ell=1, \ldots, N$, then (5.3.5) is generalized to |1| :

$$\langle \{z_\ell'^*\}, t'' | \{z_\ell\}, t' \rangle =$$

$$\int \prod_{t,\ell} \left[\frac{dz_\ell^*(t)\, dz_\ell(t)}{\pi}\right] \exp\left\{ \frac{1}{2} \sum_\ell \left(z_\ell^*(t') z_\ell(t') + z_\ell^*(t'') z_\ell(t'') \right) + i \int_{t'}^{t''} dt\, \mathcal{L}(t) \right\}$$

5.3.11

where the lagrangian is simply the sum of the individual lagrangians. The trace of evolution operator for this system is simply the product of the individual oscillator result (5.2.10), therefore:

$$\mathrm{tr}\, U = \prod_{\ell=1}^{N} \frac{e^{-\frac{i}{2}\omega_\ell(t''-t')}}{1 - e^{-i\omega_\ell(t''-t')}} \qquad 5.3.12$$

Formula (5.3.12) paves the way for a quantum field theory. Let us consider the simplest case, that of single real scalar field $\phi(x)$ and its conjugate momentum $\pi(x)$. Consider an orthonormal set of functions $f_n(x)$ which satisfy:

$$\int d^3x\ f_n(x)\ f_m(x) = \delta_{nm}$$

$$\sum_m f_m(x)\ f_m(x') = \delta(x-x')$$

$$(-\nabla^2 + m^2)\ f_n(x) = \omega_n^2\ f_n(x)\ . \qquad 5.3.13$$

The index n is not discrete, as is implied by (5.3.13), but it can be made discrete by quantizing in a box. Now the variables z_n are defined by:

$$Z_n(t) = \frac{1}{\sqrt{2\omega_n}} \int d^3x\ f_n(x)\left[\Pi(x) - i\omega_n\,\phi(x)\right] \qquad 5.3.14$$

Consequently the Lagrangian for the field decomposes into an infinite sum of harmonic oscillator lagrangians, thus the formalism which has already been developed can be used, setting $N=\infty$ |1|. The transition element is given by (5.3.11) and the partition function is an infinite product:

$$tr\ U = \prod_n \frac{e^{-i/2\,\omega_n(t''-t')}}{1-e^{i\omega_n(t''-t')}} \qquad 5.3.15$$

The product (5.3.15) is in fact ill-defined when w_n becomes a continuous variable. These results can also be written in terms of the field $V(x)$ defined by:

$$V(t,\underline{x}) = \sum_n f_n(x)\ Z_n(t) \qquad 5.3.16$$

Then the transition element becomes; |1|

$$U(V^{*}_{final}, V_{initial}, t''-t') =$$

$$= \int D[V(x)] \; exp\left\{ \tfrac{1}{2} \int d^3x \left(V^*(t',x)V(t',x) + V^*(t'',x)V(t'',x) \right) + i\int_{t'}^{t''} dt \int d^3x \left[\tfrac{i}{2}(V^*\partial_t V - V\partial_t V^*) - \mathcal{H}(V^*, V) \right] \right\}$$

5.2.17

where $\mathcal{H}$ is the Hamiltonian density expressed in terms of V^* and V.

5.4 The Fermionic Oscillator

We can now proceed with the quantization of a fermionic oscillator |5,4| following the setps of sections (5.2) and (5.3). However in this case we shall consider the oscillator ensemble straight away since the anti-commutivity of the grassmann variables may cause new effects which could be overlooked in generalizing the single oscillator results. Therefore the lagrangian to consider is:

$$\mathcal{L}(t) = \sum_{\ell=1}^{N} \left[\tfrac{i}{2}\left(\xi^*_\ell \dot{\xi}_\ell - \dot{\xi}^*_\ell \xi_\ell \right) - \omega_\ell \, \xi^*_\ell \xi_\ell \right] \qquad 5.4.1$$

The momenta conjugate to ξ_ℓ and ξ^*_ℓ are:

$$P_\ell = \frac{\partial \mathcal{L}}{\partial \dot{\xi}_\ell} = -\tfrac{i}{2}\, \xi^*_\ell \, , \quad P^*_\ell = \frac{\partial \mathcal{L}}{\partial \dot{\xi}^*_\ell} = -\tfrac{i}{2}\, \xi_\ell \qquad 5.4.2$$

The pair of second class constraints (5.4.1) can be removed using generalized Dirac brackets (see chapter 2) to yield:

$$\left\{ \xi_\ell \, , \, \xi^*_k \right\} = -i \, \delta_{\ell k} \qquad 5.4.3$$

Canonical quantization (see chapter 2) now replaces (5.4.2) by the following anticommutator: (h=1)

$$\left\{ \hat{\xi}_\ell , \hat{\xi}_k^+ \right\} = \delta_{\ell k} \qquad 5.4.3$$

where $\hat{\xi}_\ell$ is an operator with eigenvalue ξ_ℓ. The hamiltonian is given by:

$$\hat{H} = \sum_{\ell=1}^{N} \omega_\ell \hat{\xi}_\ell^+ \xi_\ell - \tfrac{1}{2} \sum_{\ell=1}^{N} \omega_\ell \qquad 5.4.4$$

The usual representation of (5.4.3) is by pauli matrices |17| but to be able to derive the path integral formalism we have to construct a representation which admits grassmann eigenvalues in which case, hermiticity is not the only condition an observable has to satisfy but it must also have real numbers as eigenvalues.

Then the operators whose eigenvalues are odd or even members of the grassmann algebra will be called odd or even operators. The odd grassmann variables anticommute with odd operators, for instance if is a grassmann variable,

$$\eta_k \hat{\xi}_\ell = - \hat{\xi}_\ell \eta_k \qquad 5.4.5$$

As a consequence of this modification, odd operators which do not anticommute with each other cannot be diagonalized simultaneously. let us now work out the energy eigenstates, the following brackets designate $\hat{\xi}_\ell$ as annihilation operator and $\hat{\xi}_\ell^+$ as creation operator

$$\left[\hat{\xi}_\ell , \hat{H} \right] = \omega \hat{\xi}_\ell \quad , \quad \left[\hat{\xi}_\ell^+ , \hat{H} \right] = - \omega \hat{\xi}_\ell^+ \qquad 5.4.6$$

starting with, |o> the ground state, which satisfies

$$\xi_\ell |0\rangle = 0$$
$$\hat{H}\ |0\rangle = \left(-\tfrac{1}{2}\sum_\ell \omega_\ell\right) |0\rangle \qquad 5.4.7$$

we can construct the energy states by repeated application of the creation operators $\hat{\xi}_\ell^+$,

$$|\alpha, \beta, \gamma, \cdots\rangle = \hat{\xi}_\alpha^+ \hat{\xi}_\beta^+ \hat{\xi}_\gamma^+ \cdots |0\rangle$$
$$\hat{H}\ |\alpha, \beta, \gamma, \cdots\rangle = \left(\omega_\alpha + \omega_\beta + \omega_\gamma + \cdots - \tfrac{1}{2}\sum_\ell \omega_\ell\right) |\alpha, \beta, \gamma, \cdots\rangle$$

5.4.8

The order of indices α, β, γ ... in the ket is crucial since the states are antisymmetric with respect to interchange of indices:

$$|\alpha, \beta\rangle = -\,|\beta, \alpha\rangle \qquad 5.4.9$$

evidently as a consequence of (5.4.9) $|\alpha, \alpha\rangle$ vanishes. Now we can construct the eigenstates of the annihilation operator by expanding it in terms of the energy eigenstates defined by (5.4.8), it turns out to be similar to (5.2.14)

$$|\eta\rangle = e^{-\sum_\ell \eta_\ell \hat{\xi}_\ell^+} |0\rangle \qquad 5.4.10$$

The fermionic coherent state (5.3.10) has been constructed by Montonen[6], Ohnuki and Kashiwa[4]. The left eigenstate of $\hat{\xi}_\ell^+$ can be constructed by conjugating expression (5.3.10), leading to

$$\langle 0|\, e^{-\sum_\ell \hat{\xi}_\ell \eta_\ell^*} = \langle \eta^*| \qquad 5.4.11$$

where I have used η to represent the set of N grassmann variables $y = (\eta_1, \eta_2, \ldots)$. This construction corresponds to a representation of the algebra, (5.4.3) by $\hat{\xi}_\ell \equiv \frac{\partial}{\partial \xi_\ell}$, $\hat{\xi}_\ell^+ \equiv \xi_\ell$, one can also construct left eigenstates of $\hat{\xi}_\ell$ and right eigenstates of $\hat{\xi}_\ell^+$ |4| but that corresponds to a different representation of the algebra (5.4.3) where $\hat{\xi}_\ell$ is represented by ξ_ℓ, therefore the two representations should not be mixed. Wave functions can be defined as functions of grassmann variables, $f(\eta)$ by;

$$f(\eta) = \langle \eta | f \rangle \qquad 5.4.12$$

also an inner product for $f(\eta)$ exists such that it is positive definite, and under which ξ_ℓ and $\frac{\partial}{\partial \xi_\ell}$ are conjugate |2| ;

$$\langle f | g \rangle = (f, g) = \int d\mu(\eta)\, \overline{f(\eta)}\, g(\eta)$$

$$d\mu(\eta) = \prod_{\alpha=1}^{N} d\eta_\alpha d\eta_\alpha^* \; e^{-\sum_{\alpha=1}^{N} \eta_\alpha \eta_\alpha^*} \qquad 5.4.13$$

To check that ξ_ℓ and $\frac{\partial}{\partial \xi_\ell}$ are conjugate to each other under (5.4.13) let us consider the following

$$\begin{aligned}(f_1, \hat{\xi}_\ell f_2) &= \int \prod_{\alpha=1}^{N} d\eta_\alpha d\eta_\alpha^* \, e^{-\sum_\alpha \eta_\alpha \eta_\alpha^*} \, \overline{f_1(\eta)} \frac{\partial}{\partial \eta_\ell} f_2(\eta) \\
&= \int \prod_{\alpha=1}^{N} d\eta_\alpha d\eta_\alpha^* \, \overline{f_1(\eta)} \frac{\partial}{\partial \eta_\ell} \left[e^{-\sum_\alpha \eta_\alpha \eta_\alpha^*} f_2(\eta) \right] \\
&\quad + \int \prod_{\alpha=1}^{N} d\eta_\alpha d\eta_\alpha^* \, \overline{f_1(\eta)} \, \eta_\ell^* \, e^{-\sum_\alpha \eta_\alpha \eta_\alpha^*} f_2(\eta) \\
&= (\hat{\xi}_\ell^\dagger f_1, f_2)\end{aligned}$$

5.4.14

where we have used $\int d\xi \frac{\partial}{\partial \xi} x(\xi) = 0$ for a grassmann variable ξ.

The following competeness relations can be obtained using (5.4.13)

$$\langle f|g\rangle = \int d\mu(\eta) \langle f|\eta^*\rangle\langle\eta|g\rangle$$

thus

$$\int d\mu(\eta)\ |\eta^*\rangle\langle\eta| = 1$$

5.4.15

or alternatively if f and g are defined over η^*

$$\int d\mu(\eta^*)\ |\eta\rangle\langle\eta^*| = 1$$

$$d\mu(\eta^*) = \prod_{\alpha=1}^{N} d\eta_\alpha^* d\eta_\alpha\ e^{-\sum_\alpha \eta_\alpha^*\eta_\alpha} \qquad 5.4.16$$

Note that in constrast to the bosonic case where the two measures (5.2.23) and (5.2.24) where identical, here the two measures (5.4.16) and (5.4.15) are not identical.

Similar to the bosonic case the coherent states $|\eta\rangle$ are not orthogonal in the usual sense. Since

$$\langle\xi^*|\eta\rangle = \langle 0|\ e^{-\sum_\alpha \hat{\xi}_\alpha \xi_\alpha^*}\ e^{-\sum_\alpha \eta_\alpha \hat{\xi}_\alpha}\ |0\rangle$$

$$= e^{-\eta_\alpha \xi_\alpha^*}$$

5.4.17

but when integrated with the measure $d\mu(\eta)$ (5.4.17) acts like a delta function:

$$f(\eta) = \int d\mu(\xi)\ \overline{\langle\xi|\eta^*\rangle}\ f(\xi) \qquad 5.4.18$$

The final point to discuss before we can derive the path integral is the trace formula in the ξ basis.The trace of an "even" operator U is given in terms of the energy eigenstates $|\{\alpha\}\rangle$

$$\mathrm{trace}\,U = \sum_{\{\alpha\}} \langle \{\alpha\} \,|\, U \,|\, \{\alpha\} \rangle \tag{5.4.19}$$

where the general energy eigenstate is represented by $|\{\alpha\}\rangle$, and the sum is over all possible sets $\{\alpha\}$ excluding the permutations. Now inserting the completeness relation (5.4.16), we obtain

$$tr\,U = \sum_{\{\alpha\}} \int d\mu(\eta^*)\,d\mu(\xi^*)\,\langle \{\alpha\}|\eta\rangle\langle\eta^*|U|\xi\rangle\langle\xi^*|\{\alpha\}\rangle \tag{5.4.20}$$

now commuting the wave functions $\langle\{\alpha\}|\eta\rangle$ through $\langle\eta^*|U|\xi\rangle$; all the wave functions with an even number of indices do not change sign, but those with an odd number of indices change from $|\xi\rangle$ to $|-\xi\rangle$ and $\langle\eta^*|$ to $\langle-\eta^*|$, then with a change of variables from ξ to $-\xi$ we get the result,

$$\begin{aligned} tr\,U &= \int d\mu(\eta^*)\,d\mu(\xi^*)\,\langle\eta^*|U|\xi\rangle\, e^{\sum_\alpha \eta_\alpha \xi_\alpha^*} \\ &= \int d\mu(\xi^*)\,\langle -\xi^*|U|\xi\rangle \end{aligned} \tag{5.4.21}$$

The calculation can be easily done for N = 3, where the wave functions are

$$\begin{array}{llll} \langle\eta|0\rangle = 1 & \langle\eta|1\rangle = -\eta_1 & \langle\eta|1,2\rangle = -\eta_2\eta_1 & \langle\eta|123\rangle = -\eta_3\eta_2\eta_1 \\ & \langle\eta|2\rangle = -\eta_2 & \langle\eta|1,3\rangle = -\eta_3\eta_1 & \\ & \langle\eta|3\rangle = -\eta_3 & \langle\eta|2,3\rangle = -\eta_3\eta_2 & \end{array} \tag{5.4.22}$$

Thus (5.3.21) for N = 3 becomes:

$$tr\, U = \int d\mu(\eta^*)\, d\mu(\xi^*) \Big\{ \langle \eta^* | U | \xi \rangle + \eta_1 \langle \eta^* | U | \xi \rangle \xi_1$$

$$+ \eta_2 \langle \eta^* | U | \xi \rangle \xi_2 + \eta_3 \langle \eta^* | U | \xi \rangle \xi_3$$

$$+ \eta_1 \eta_2 \langle \eta^* | U | \xi \rangle \xi_2^* \xi_1^* + \eta_1 \eta_2 \langle \eta^* | U | \xi \rangle \xi_3^* \xi_1^*$$

$$+ \eta_2 \eta_3 \langle \eta^* | U | \xi \rangle \xi_3^* \xi_2^*$$

$$+ \eta_1 \eta_2 \eta_3 \langle \eta^* | U | \xi \rangle \xi_3^* \xi_2^* \xi_1^*$$

or

$$tr\, U = \int d\mu(\eta^*)\, d\mu(\xi^*) \Big\{ \langle \eta^* | U | \xi \rangle + \langle -\eta^* | U | \xi \rangle (\eta_1 \xi_1^* + \eta_2 \xi_2^* + \eta_3 \xi_3^*)$$

$$+ \langle \eta^* | U | \xi \rangle (\eta_1 \eta_2 \xi_2^* \xi_1^* + \eta_1 \eta_3 \xi_3^* \xi_1^* + \eta_2 \eta_3 \xi_3^* \xi_2^*)$$

$$+ \langle -\eta | U | -\xi \rangle (\eta_1 \eta_2 \eta_3\, \xi_3^* \xi_2^* \xi_1^*)$$

5.4.23

Changing the variable from η to $-\eta$ and ξ to $-\xi$ in the second and third terms of the R.H.S. of (5.4.23) produces the desired result:

$$tr\, U = \int d\mu(\eta^*)\, d\mu(\xi^*) \langle \eta^* | U / \xi^* \rangle \, e^{\sum_{\alpha=1}^{3} \eta_\alpha \xi_\alpha^*}$$

$$= \int d\mu(\xi^*) \langle -\xi^* | U | \xi \rangle$$

5.4.24

This result was also obtained by Ohnuki and Kashiwa [4] where the trace of the operator is given by summing over all elements $\langle -\xi^* | U | \xi \rangle$ rather than $\langle \xi^* | U | \xi \rangle$ as was the case for the bosonic oscillator. This minus sign plays an important role in obtaining the right result for the partition function, but it cannot be interpreted as an antiperiodic boundary condition for the fermions.

5.5 The Path Integral for Fermions

We wish to evaluate the transition matrix element: $\langle\eta^*, t''|\xi,t'\rangle$, $t''>t'$, to do so we shall adopt the same method used in section (5.3) and divide the interval $t''-t'$ into K small parts each of duration ϵ

$$t''-t'=K\epsilon \qquad t_{i+1}-t_i=\epsilon \qquad t''>t_{K-1}>\cdots t_1>t' \tag{5.5.1}$$

then using the completeness relation; (5.4.16)

$$\langle\eta^*,t''/\xi,t'\rangle \cong \int\prod_{i=1}^{K-1}d\mu(\xi_i^*)\langle\eta^*,t''|\xi_{K-1},t_{K-1}\rangle\cdots$$
$$\cdots\langle\xi_1^*,t_1|\xi,t'\rangle \tag{5.5.2}$$

now for small ϵ, we have

$$\langle\xi_{i+1}^*,t_{i+1}/\xi_i,t_i\rangle \cong e^{\xi_{i+1}^*\xi_i-iH(\xi_{i+1}^*,\xi_i)\epsilon} \tag{5.5.3}$$

Therefore (5.5.2) results in

$$\langle\eta^*,t''|\xi,t'\rangle=\int\prod_{i=1}^{K-1}d\mu(\xi_i^*)\;\exp\left\{\sum_{i=0}^{K-1}\left[\xi_{i+1}^*\xi_i-iH(\xi_{i+1},\xi_i)\epsilon\right]\right\} \tag{5.5.4}$$

where $\xi_0=\xi(t')$ and $\xi_K^*=\eta^*(t'')$, now expanding the measure and rearranging, we finally obtain;

$$U(\eta^*,\xi,t''-t')=\int\prod_{i=1}^{K-1}\prod_{\alpha=1}^{N}d\xi_{i,\alpha}^*d\xi_{i\alpha}\;\exp\left\{\tfrac{1}{2}\sum_{\alpha}(\xi_{0,\alpha}^*\xi_{0\alpha}+\xi_{K,\alpha}^*\xi_{K,\alpha})\right\}$$

$$\times\exp\left\{i\sum_{i=0,\alpha=1}^{K-1,N}\left[\tfrac{i}{2}\,\xi_{i+1,\alpha}^*\left(\frac{\xi_{i+1,\alpha}-\xi_{i,\alpha}}{\epsilon}\right)-\tfrac{i}{2}\left(\frac{\xi_{i+1,\alpha}^*-\xi_{i,\alpha}^*}{\epsilon}\right)\xi_{i,\alpha}-H(\xi_{i+1,\alpha}^*,\xi_i)\right]\epsilon\right\} \tag{5.5.5}$$

now let $k\to\infty, \epsilon\to 0$ such that $K\epsilon = t''-t'$, expression 5.5.5 leads to

$$U(\eta^*, \xi, t''-t') = \int \prod_{t,\alpha} d\xi_\alpha^*(t)\, d\xi_\alpha(t)\, \exp\Big\{ \tfrac{1}{2}\sum_\alpha \xi_\alpha^*(t')\xi_\alpha(t') + \tfrac{1}{2}\sum_\alpha \xi_\alpha^*(t'')\xi_\alpha(t'') + i\int_{t'}^{t''} \mathcal{L}(t)\, dt \Big\} \qquad 5.5.6$$

where $\mathcal{L}(t)$ is the lagrangian given by (5.4.1) minus the zero point energy. Note that $\xi_\alpha^*(t')$ and $\xi_\alpha(t')$ can not be fixed simultaneously since they are the eigenvalues of two non-anticommuting operators [2]. To evaluate the partition function we shall follow the same approach as in section (5.3) and expand $\xi(t)$ about a classical solution with the desired asymptotic behaviour,

$$\xi_\ell(t) = \xi_\ell e^{-i\omega_\ell(t-t')} + \xi_\ell'(t) \quad , \quad \xi_\ell^*(t) = -\xi_\ell^* e^{-i\omega_\ell(t''-t)} + \xi_\ell^{*\prime}(t) \qquad 5.5.7$$

Clearly $\xi_\ell(t)$ and $\xi_\ell^*(t)$ can not be complex conjugates of each other because of the boundary conditions, furthermore one can not say that the paths are antiperiodic. Inserting (5.5.7) into (5.5.6) leads to

$$U(-\xi^*, \xi, t''-t') = e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha e^{-i\omega_\alpha(t''-t')}}\, U(0,0,t''-t') \qquad 5.5.8$$

performing the integration results in:

$$\mathrm{tr}\, U = \prod_{\alpha=1}^{N} \left(1 + e^{-i\omega_\alpha(t''-t')}\right) U(0,0,t''-t',\omega_\alpha) \qquad 5.5.9$$

The gaussian integral $U(0,0, t''-t', w_\alpha)$ can be evaluated in a manner similar to the bosonic case resulting,

$$U(0,0,t''-t',\omega_\alpha) = e^{\frac{i}{2}\omega_\alpha(t''-t')} \qquad 5.5.10$$

which is the zero point energy contribution. The complete answer for the partition function is:[19]

$$\operatorname{tr} U = \prod_\alpha e^{\frac{i}{2}\omega_\alpha (t''-t')} \left[1 + e^{-i\omega_\alpha (t''-t')} \right] \qquad 5.5.11$$

Note that in this calculation too we did not need to impose anti-periodicity on fermionic paths, therefore the periodicity conditions usually imposed on the paths[9] appear to have only a heuristic value.

The field theoretic representation is somewhat simpler in this case, let us assume the existance of an orthonormal basis $\psi_n(x)$ such that,

$$\int d^3x \; \bar{\psi}_m(\underline{x}) \, \gamma^0 \, \psi_n(x) = \delta_{m,n}$$

$$\left(i\gamma^j \partial_j - m \right) \psi_n(x) = -\gamma^0 \omega_n \psi_n(x) \qquad 5.5.12$$

Where j runs over the space indices. Now expanding $\psi(x)$ in terms of the basis $\psi_n(x)$;

$$\psi(x) = \sum_n \xi_n(t) \, \psi_n(\underline{x}) \qquad 5.5.13$$

expresses the Dirac lagrangian in terms of an infinite sum of fermionic oscillators; thus the results (5.5.6) can be carried over to the field theoretic case;

$$\langle \text{final} | \, e^{-iH(t''-t')} \, | \text{initial} \rangle = \int D[\psi^+(x) \psi(x)]$$

$$\times \; \exp\left\{ \frac{1}{2} \int d^3x \left(\bar{\psi}(t',\underline{x}) \, \psi(t',x) + \bar{\psi}(t'',x) \, \psi(t'',x) \right) + i \int_{t'}^{t''} dt \int d^3x \left[\frac{1}{2} \bar{\psi} (i\not{\partial} - m) \psi - \frac{1}{2} (i \partial_\mu \bar{\psi} \gamma^\mu + m) \psi \right] \right\} \qquad 5.5.14$$

5.6 The Super Oscillator

A supersymmetric lagrangian for a bosonic and fermionic oscillator can be found:

$$\mathcal{L}(t) = \tfrac{i}{2}\,\xi^{*}\dot{\xi} - \tfrac{i}{2}\,\dot{\xi}^{*}\xi + \tfrac{i}{2}\,z^{*}\dot{z} - \tfrac{i}{2}\,\dot{z}^{*}z - \omega(\xi^{*}\xi + z^{*}z) \qquad 5.6.1$$

where ξ is a grassmann and z a complex variable. The lagrangian (5.6.1) is invariant under the transformations:

$$\begin{pmatrix} z' \\ \xi' \end{pmatrix} = V \begin{pmatrix} z \\ \xi \end{pmatrix} \qquad 5.6.2$$

where V is a unitary matrix with grassmann variables for its anti-diagonal elements. The functional integral for this system is simply the product of (5.3.5) and (5.5.6) with N = 1, which is invariant under (5.6.2). Since the zero point energies of the two oscillators have different signs they cancel each other and the trace of the evolution operator for (5.6.1) is:

$$\mathrm{tr}\, U = \frac{1 + e^{-i\omega(t''-t')}}{1 - e^{-i\omega(t''-t')}} = 1 + 2\sum_{n=1}^{\infty} e^{-in\omega(t''-t')} \qquad 5.6.3$$

CHAPTER 6

SOLITON MASSES

6.1 Introduction

An intriguing property of the supersymmetric models with soliton solutions is that the classical mass of the soliton is exact to any order of perturbation expansion. There is however a certain amount of controversy concerning this phenomenon. Formal arguments given by Olive and Witten [29] indicate that, if the quantized theory observes the supersymmetry of the classical version then the mass of the self-dual soliton is equal to its classicalmass. In support of this claim, D'Adda, Di Vecchia and Horsley [35] have demonstrated that the one loop correction to the classical mass vanishes. Their argument rest upon the semi-classical approximation [51]

$$M = M_{cl} + \frac{\hbar}{2}\left(\sum \omega_B - \sum \omega_F \right) \qquad 6.1.1$$

where $w_B(w_F)$ are the Boson (Fermion) eigenfrequencies of the excitations about the soliton, and the vacuum energy vanishes due to supersymmetry [50,42]. Since the number of Fermion and Boson models are equal and $w_B = w_F$ around a self-dual solution it follows that $M = M_{cl}$. However the counter argument put forward by Schonfeld [51] is that upon regularization of the infinite sum over the frequencies a finite remainder is obtained, which is independent of the form of the potential.

But a non-vanishing contribution to the mass implies that the quantum theory does not respect supersymmetry since there is no reason for this effect, it appears that the non-zero correction to the mass is the by-product of the regularisation scheme. Indeed one can show that the inequality of w_B and w_F or that of the number of modes (both obtained by Schonfeld) lead to a breakdown of supersymmetry. However one may argue that regularization inevitably breaks supersymmetry, for instance quantizing in a box of finite length can lead to a breakdown of supersymmetry since the superalgebra relates the supercharges to translations and a system in a finite box is not translationally invariant. But box quantization does not necessarily break supersymmetry, indeed supersymmetric boundary conditions were employed in the theory of dual strings [52].

In the presence of a soliton some of the supersymmetry is lost and only a subset of the full symmetry remains valid [29] which does not contain translations, and it is this remnant symmetry which is responsible for the vanishing of quantum corrections.

6.2 Supersymmetric Solitons in Two Dimensions

Let us consider the supersymmetric extension of the models described in section 1.2 . The supersymmetric version of (1.2.1) is [53]

$$\mathcal{L} = \frac{1}{2}\left\{ \partial_\mu\phi\partial^\mu\phi - V(\phi)^2 + \bar{\psi}(i\not\partial - V'(\phi))\psi \right\} \qquad 6.2.1$$

where the potential $V(\phi)$ here is proportional to the square root of the potential of (1.2.1) and $V'(\phi) = \frac{dV}{d\phi}$. This lagrangian is invariant under the transformations;

$$\delta\phi = \bar{\epsilon}\psi$$

$$\delta\psi = (-i\not\partial\phi - V(\phi))\epsilon \qquad 6.2.2$$

The super current is;

$$J^\mu = \left[\not\partial\phi + iV(\phi)\right]\gamma^\mu\psi \qquad 6.2.3$$

The equations of motion are:

$$\Box\phi + V(\phi)V'(\phi) + V''(\phi)\bar{\psi}\psi = 0$$

$$i\not{\partial}\psi - V'(\phi)\psi = 0$$

6.2.4

The set of equations (6.2.4) is satisfied if, $\psi = 0$, $\partial_0\phi = 0$ and

$$\frac{d\phi}{dx} = \pm V(\phi)$$

6.2.5

Thus this lagrangian admits the self-dual solution. Olive and Witten |29| observed that if one derives the commutation relation between two supercharges by using (6.2.3) a different algebra is obtained to the algebra derived from (6.2.2). The modified algebra contains an extra term which commutes with all the generators of the algebra hence named a "central charge". In terms of the Chiral components of Q, the supercharge, the commutation relations are; |29|

$$Q_1^2 = P_+ \qquad Q_2^2 = P_-$$

$$\{Q_1, Q_2\} = 2\int_{\phi(-\infty)}^{\phi(+\infty)} d\phi\, V(\phi) = T$$

6.2.6

where $P_\pm = P_0 \pm P_1$

Clearly T vanishes if $\phi(-\infty) = \phi(+\infty)$ but when $\phi(x)$ does not have the same value at the two spatial infinities it must be a topologically non-trivial soliton, however T does not count the number of solitons. Now we can see that for a solution which satisfies $\frac{d\phi}{dx} = V(\phi)$ we have,

$$(1 + i\gamma^1)Q = 0$$

or

$$Q_1 - Q_2 = 0$$

6.2.7

But we can derive a relation for the mass of the soliton using 6.2.7, since

$$2 P_o = (Q_1 - Q_2)^2 + T$$

or

$$M_c = \frac{1}{2} |T| \qquad 6.2.8$$

where M_c is the classical mass and the absolute value of T allows for anti-soliton as well. Now if one could construct a soliton state $|S\rangle$ say, (6.2.7) would be interpreted as an operator relation;

$$(\hat{Q}_1 - \hat{Q}_2) |s\rangle = 0 \qquad 6.2.9$$

and from (6.2.8) we have;

$$M = \langle s | P_o | s \rangle = \tfrac{1}{2} |T| = M_c$$

However this reasoning assumes the quantum theory to respect the supersymmetry. Conversely, a non-vanishing correction to the mass of the self-dual soliton, indicates that the quantization has been carried out in a way which breaks the supersymmetry.

6.3 The Quantum Corrections

Let us expand the action about a self-dual solution $\phi_c(x)$

$$S^{(2)} = \int d^2x \, \frac{1}{2} \Big\{ -\eta \left(\Box + V'(\phi_c)^2 + V''(\phi_c) V(\phi_c) \right) \eta + \bar{\psi} \left(i\partial\!\!\!/ - V'(\phi_c) \right) \psi \Big\} \qquad 6.3.1$$

where $y = \phi - \phi_c(x)$. The term linear in y vanishes, because $\phi_c(x)$ is a stationary point of action, and we have neglected terms higher than quadratic. The quadratic action (6.3.1) is the source of one-loop corrections to the soliton mass so let us investigate some of its properties. The action (6.3.1) is supersymmetric under the following transformations;

$$\delta\eta = \bar{\epsilon}\psi$$
$$\delta\psi = (-i\not{\partial} - V'(\phi_c))\eta\epsilon$$

6.3.2

if ϕ_C satisfies;

$$\frac{d\phi_c(x)}{dx} = \pm V(\phi_c(x)) \qquad 6.3.3$$

and the parameter of transformation should satisfy;

$$(1 \pm i\gamma^1)\epsilon = 0 \qquad 6.3.4$$

Assuming the positive sign in (6.3.3) and (6.3.4) the transformations can also be written as;

$$\delta\eta = \bar{\epsilon}\psi^+$$
$$\delta\psi^+ = -i\gamma^0\epsilon\partial_0\eta$$
$$\delta\psi^- = D\eta\epsilon$$

6.3.5

where

$$D = \frac{d}{dx} - V'(\phi_c(x))$$
$$i\gamma^1\psi^+ = \pm\psi^\pm$$

6.3.6

The transformations (6.3.5) closes, using the equation of motion; $D^+ = -i\gamma^0\partial_0\psi^-$. The algebra of (6.3.5) is

$$Q_-^2 = 2P_0 - T$$
$$[Q_-, P_0] = [Q_-, T] = [P_0, T] = 0$$

6.3.7

which is a subset of (2.2.6) and of course to be able to know about T one has to know the full algebra (6.2.6). Therefore translations in space are no longer related to the conserved supercharge and box quantization will not necessarily break the supersymmetry of (6.3.1). The boundary conditions on the ends of a box of length L which are left invariant by (6.3.5) are;

$$D\eta\Big|_{x=0,L} = D\psi^{+}\Big|_{x=0,L} = 0$$

$$\psi^{-}\Big|_{x=0,L} = \theta^{-} \qquad 6.3.8$$

where θ^{-} is an arbitrary spinor which satisfies $i\,\gamma'\,\theta^{-} = -\theta^{-}$. It is not difficult to see that (6.3.8) is left invariant by (6.3.5).

With these boundary conditions we can now proceed to decompose the fields of (6.3.1) into their eigenmodes.

The equations governing the excitation modes are;

$$\text{(a)}\quad \left[-\frac{d^2}{dx^2} + V'(\phi_c)^2 + V''(\phi_c)\,V(\phi_c)\right]\eta_k = (\omega_k^B)^2\,\eta_k$$

$$\text{(b)}\quad \left[i\gamma'\frac{d}{dx} - V'(\phi_c)\right]\psi_k = -\gamma^0\omega_k^F\,\psi_k \qquad 6.3.9$$

Multiplying (6.3.9)(b) by $-i\,\gamma'\,\frac{d}{dx} - V'(\phi_c)$ and decomposing ψ^k into $\psi_k^{\pm} = \pm\, i\gamma'\,\psi_k^{\pm}$ we get;

$$\text{(a)}\quad \bar{D}D\,\eta_k = \omega_k^2\,\eta_k$$

$$\text{(b)}\quad \bar{D}D\,\psi_k^{+} = \omega_k^2\,\psi_k^{+}$$

$$\text{(c)}\quad D\bar{D}\,\psi_k^{-} = \omega_k^2\,\psi_k^{-}$$

6.3.10

Where $\bar{D}$ is the adjoint of the operator D. The difference in the spectrum of ψ_k^- and ψ_k^+ is the zero modes [54]. In the case of a single soliton there is only one zero mode as described in section (1.2). Since ψ_k^+ and y_k satisfy the same differential equation and identical boundary conditions they must have identical spectrums, thus $w_k^B = w_k^F$.

At this point it is clear from (6.1.1) that the corrections to the soliton mass vanish, since the spectrum is discrete and we can impose an ultraviolet cut-off, then every term in (6.1.1) will be well behaved and the cut-off can be let to tend to infinity without causing any problems. However for the sake of completeness let us carry out the decomposition of the fields in terms of $_k$ and ψ_k;

$$\eta(x,t) = \sum_{k\geq 0} a_k(t)\, \eta_k(x)$$

$$\psi(x,t) = \sum_k \xi_k(t)\, \psi_k(x) \qquad 6.3.11$$

where $\xi_k^*(t) = \xi_{-k}(t)$ are grassmann variables. Now the eigenfunctions η_k and ψ_k are orthonormal sets since $\bar{D}D$ is a self-adjoint operator, thus inserting (6.3.11) into (6.3.1) results in;

$$S^{(2)} = \frac{1}{2}\int_0^T dt \left\{ \sum_{k>0} \left(\dot{a}_k^2 - \omega_k^2 a_k^2 + i\xi_k^* \dot{\xi}_k - i\dot{\xi}_k^* \xi_k - 2\omega_k \xi_k^* \xi_k \right) \right.$$
$$\left. + \frac{1}{2}\left(\dot{a}_0^2 + i\,\xi_0 \dot{\xi}_0 \right) \right.$$

$$6.3.12$$

Note that the boundary conditions (6.3.8) do not exclude the zero mode. The quadratic action $S^{(2)}$ is supersymmetric if and only if the fermionic and the bosonic frequencies are identical, and the supertransformations are;

$$\delta a_k = i\xi_k \epsilon - i\epsilon \xi_k^*$$

$$\delta \xi_k = \dot{a}_k \epsilon - i\omega_k a_k \epsilon \qquad 6.3.13$$

Let us now look at the trace of the evolution operator for the lagrangian density (6.2.1);

$$tr\left(e^{-iHT}\right) = \int d(\phi_0,\psi_0) \int D[\phi,\psi]\; e^{iS[\phi,\psi]} \qquad 6.3.14$$

Where we have integrated over all end points of the paths over which the path integration is performed. Now $S\{\phi,\psi\}$ is expanded about the classical solution $\phi_c(x)$ and all the terms higher than quadratic are dropped, then we are left with

$$tr\; e^{-iHT} \approx e^{-iM_cT} D(T) \qquad 6.3.15$$

Where D(T) is trace of the evolution operator of the quadratic action (6.3.1) and M_c is the classical mass of the soliton [55]. The factor D(T) represents the one loop corrections to the propagator and it can be calculated using the results of chapter 5.

The result obtained in section (5.6) can be extended to the case of an ensemble of super oscillators giving rise to

$$D(T) = \prod_{k>0}^{\Lambda} \left[1 + 2 \sum_{n=1}^{\infty} e^{-in\omega_k T} \right] \qquad 6.3.16$$

where the product is over a discrete spectrum of frequencies since the length of the box, L, is finite and the ultra-violet cut-off Λ keeps the product finite. We thus see that there is no zero point energy contribution, thus there are no corrections to the soliton mass. We can now let Λ and L tend to infinity and this result will not be affected.

6.4 The Supersymmetric CP^n Soliton

We can now look at the corrections to the mass of the CP^n soliton. This model was described in section 3.2. Following the calculations of the last section what we must demonstrate is that, first there exists a remnant supersymmetry which leaves the soliton invariant and when the field is shifted to $\eta^a = \phi^a - \phi^a_c\,(x,y)$ the resulting quadratic lagrangian density remains supersymmetric under it, secondly the equations of the excitation modes are identical for fermions and bosons, and finally that there exists supersymmetric boundary conditions.

The self-dual soliton $\phi^a_c(x,y)$ is static and satisfies;

$$D_i\,\phi^a(x,y) = i\,\epsilon_{ij}\,D^j\,\phi^a(x,y) \qquad i,j = 1,2 \qquad 6.4.1$$

where we have chosen the positive sign in (3.4.8). Now if $\phi^a_c\{(x)y\}$ satisfies (6.4.1) and $\psi^a = 0$ the solution $\phi^a_c(x,y)$ is left invariant by the supertransformations (3.2.6) since;

$$\begin{aligned}\delta\psi^a &= -\gamma^i D_i\,\phi^a_c(x,y)\,\epsilon \\ &= -\tfrac{1}{2}\left(\gamma^i - i\,\epsilon^{ij}\gamma_j\right) D_i\phi^a_c\,\epsilon \\ &= 0 \qquad\qquad \text{if} \quad (1+\gamma^0)\,\epsilon = 0\end{aligned} \qquad 6.4.2$$

Now let us expand (3.2.2) about $\phi^a_c(x)$, but note that the constraints (3.2.1) and (3.2.3) must first be added to the lagrangian density using two lagrange multipliers α and θ, then the quadratic action is given by;

$$S^{(2)} = \int d^2x \left\{ - \bar{\eta}_a D_\mu D^\mu \eta_a + \alpha(\phi_c(x)) \bar{\eta}_a \eta_a + \tfrac{i}{2} \bar{\psi} \not{D} \psi - \tfrac{i}{2} D_\mu \bar{\psi} \gamma^\mu \psi \right\} \qquad 6.4.3$$

Where the covariant derivative and α are evaluated at the background field $\phi^a = \phi^a_c(x)$ and $\psi = 0$. The quadratic action (6.4.3) is invariant under the following transformations:

$$\delta \eta^a = i \bar{\epsilon} \psi^a$$

$$\delta \psi^a = - \not{D} \eta^a \epsilon \qquad 6.4.4$$

if the spinor ϵ satisfies the equation;

$$(1 + \gamma^0) \epsilon = 0 \qquad 6.4.5$$

To show the invariance of (6.4.3), it is enough to note that α is given in terms of $\phi^a_c(x)$ by:

$$\alpha = \frac{1}{K} \phi^a D_\mu D^\mu \phi^a$$

$$= - \epsilon_{ij} \partial^i A^j (\phi^a_c(x)) \qquad 6.4.6$$

Where the dependence of A_μ on ϕ^a is given by (3.2.4). Therefore the first requirement is satisfied. Now let us write the stability equations:

(a)

$$(- D_i D^i + \alpha) \eta^a_k = - \omega_k^2 \eta^a_k$$

(b)

$$i \gamma^j D_j \psi^a_k(x) = - \omega_k \gamma^0 \psi^a_k(x) \qquad 6.4.7$$

multiplying (6.4.7) (b) by $i\gamma^j D_j$ and resolving ψ_k^a into $\psi_k^{\pm a} = \frac{1}{2}(1 \pm \gamma^0)\psi_k^a$, we get,

$$\text{(a)} \quad (-D_i D^i + \epsilon^{ij} \partial_i A_j)\, \psi_k^{+a} = -\omega_k^2\, \psi_k^{+a}$$

$$\text{(b)} \quad (-D_i D^i - \epsilon^{ij} \partial_i A_j)\, \psi_k^{-a} = -\omega_k^2\, \psi_k^{-a}$$

6.4.8

In terms of the operator $V_i = \frac{1}{\sqrt{2}}(-D_i + i\epsilon_{ij}D^j)$ and its adjoint $\overline{V}_i$ the equations (6.4.8) and (6.4.7) can be rewritten:

$$\overline{V}_i\, V_i\, \eta_k^a = \omega_k^2\, \eta_k^a$$

$$\overline{V}_i\, V_i\, \psi_k^{-a} = \omega_k^2\, \psi_k^{-a}$$

$$V_i\, \overline{V}_i\, \psi_k^{+a} = \omega_k^2\, \psi_k^{+a}$$

6.4.9

Clearly ψ_t^{-a} and ${}_k^a$ satisfy the same equation thus will have the same spectrum, and the difference in the spectrum of ψ_{ic}^{+a} and ψ_k^{-a} are just the zero modes |54|. The equations (6.4.9) admit a range of different zero modes arising from scale invariance, translational invariance, etc. however since the zero modes do not contribute to the poles of the propagator they do not affect the quantum corrections.

Finally, to find the right boundary conditions let us write (6.4.4) as follows:

$$\delta\eta = i\bar{\epsilon}\psi^-$$

$$\delta\psi^- = -\gamma^0 \partial_0 \eta\, \epsilon$$

$$\delta\psi^+ = \frac{1}{\sqrt{2}}\gamma^i \epsilon\, V_i \eta$$

6.4.10

Note that since the soliton is static $A_o(x) = 0$.

It is clear from (6.4.10) that the supertransformation leaves the following boundary conditions invariant;

$$V_i\, \eta^a \Big|_{y,x=0,L} = V_i\, \psi^{-a} \Big|_{x,y=0,L} = 0$$

$$\psi^{+a} \Big|_{x,y=0,L} = \xi^{a+} \qquad 6.4.11.$$

where ξ_a^+ is an arbitrary spinor which satisfies $\gamma^o \xi_a^+ = \xi_a^+$. Hence the quantum corrections to the mass of the multi-soliton solutions of CP^n vanish .

6.5 Conclusions

A classical model may have several quantum versions, indeed if one could think of the classical limit at the limit of $\hbar$ tending to zero any operator which is multiplied by $\hbar$ can be added to the algebra of the quantum model and the classical limit remains unchanged. A prescription for removing this ambiguity is to preserve as much of the symmetry of the classical model as possible in the process of quantization.

The choice of boundary conditions (6.4.11) is one such occasion where a different set of boundary conditions would have led to a non-zero correction to the soliton mass hence a different quantum model.

The exactness of the classical mass of the soliton appears to be a universal property of the supersymmetric models with topological charge, we also know that solitons are stable due to the existance of the topological charge. The inter-relation between topological charge and the quantization of solitons is perhaps worthy of some further inquiry.

SUMMARY

We have seen that the Dirac's method for singular lagrangians can be extended to the lagrangians which contain both C numbers and grassmann variables. The Dirac bracket for such a dynamics can be defined and it was proved that in the general case the Dirac bracket exists and possesses the right symmetry properties.

Using this extended scheme we treated the supersymmetric CP^n model. In deriving the superalgebra of the model, we found that the algebra contains a central charge, which is composed of two parts a field dependent U(1) transformation which is the central charge of the O(2) extended supersymmetry, and a term proportional to the topological charge of the configuration. The central charge can be understood as the extra component of the momentum of a four dimensional model which reduces to the 2 + 1 dimensional CP^n model on compactification of the extra dimension.

The central charge leads to a bound on the mass which is saturated for self-dual solutions, resulting in a mass relation for the soliton reminiscent of all the models with topologically non-trivial solutions.

The other question addressed was the quantum corrections to the soliton mass. The corrections were found to vanish for solitons which obey a "self-duality" condition, that is a first order differential equation instead of the second order differential equation of motion. The soliton is not invariant under the full extended Poincaré group but only under a sub-group of it, however, this remnant symmetry is adequate to guarantee the vanishing of the corrections, if the regularization is carried out in a supersymmetric fashion . Although we must not forget that the methods used in chapter six are semi-classical and a fully quantized model of solitons may have some very different features but the semi-classical methods can be trusted to give some indications of the properties of the model , and this result stands a good chance of persisting in a fully quantized version.

APPENDIX

THE GRASSMANN ALGEBRAS [3]

The grassmann algebra G_n has n-generators ξ_ℓ, $\ell = 1, \ldots, n$, where

$$\xi_\ell \xi_k + \xi_k \xi_\ell = 0 \qquad \text{A.1}$$

since A.1 is true for all k and ℓ it implies that $\xi_\ell^2 = 0$. Any element $g \in G_n$ may be represented by a finite sum of homogenous monomials;

$$g(\xi) = \sum_{r=0}^{n} \sum_{\{k\}} g_r^{k_1 \cdots k_r} \xi_{k_1} \cdots \xi_{k_r} \qquad \text{A.2}$$

where $g\{k\}_r$ can be real or complex numbers. The set of elements for which only terms with even n are present are called "even" and those with odd r, named "odd". Complex conjugation can be defined over G_n as follows

$$(g^*)^* = g \; , \; (g_1 g_2)^* = g_2^* g_1^* \; , \; (\alpha g)^* = \alpha^* g^* \qquad \text{A.3}$$

where α is a complex number. The reversing of order when conjugating, is necessary to keep the modulus real, that is

$$(g^* g)^* = g^* g \qquad \text{A.4}$$

A real grassmann algebra is one with $\xi_\ell^* = \xi_\ell$. Two kinds of derivatives can be introduced a left and a right derivative;

$$\frac{\overrightarrow{\partial}}{\partial \xi_\ell} (\xi_{k_1} \cdots \xi_{k_r}) = \delta_{\ell k_1} \xi_{k_2} \cdots \xi_{k_r} - \delta_{\ell k_2} \xi_{k_1} \xi_{k_3} \cdots \xi_{k_r} + \cdots$$

$$(\xi_{k_1} \cdots \xi_{k_r}) \frac{\overleftarrow{\partial}}{\partial \xi_\ell} = \delta_{k_r, \ell} \xi_{k_1} \xi_{k_2} \cdots \xi_{k_{r-1}} - \delta_{\ell, k_{r-1}} \xi_{k_1} \xi_{k_2} \cdots \xi_{k_{r-2}} \xi_{k_r} + \cdots$$

A.5

in other words $\frac{\vec{\partial}}{\partial \xi_\ell}$ and $\frac{\overleftarrow{\partial}}{\partial \xi_\ell}$ satisfy (A.1).

Integration can be introduced but it will not be the inverse of derivation but equivalent to derivation:

$$\int d\xi_\ell \, \xi_k = \delta_{\ell k} \qquad \text{A.6}$$

Finally an integral similar to the Gaussian integral, can be done; [3]

$$\int d\xi_1 \cdots d\xi_n \, e^{a_{ij}\xi_i \xi_j} = \left[\det(2a_{ij})\right]^{1/2} \qquad \text{A.7}$$

A_{ij} is an anti-symmetric matrix thus this result holds for n even only.

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