469

PARAMETER JUMP DETECTION IN STOCHASTIC DYNAMICAL SYSTEMS

by

Robert Colin Rogers

A thesis submitted for the degree of

Doctor of Philosophy

January 1980

Department of Computing and Control

Imperial College of Science and Technology

University of London

PARAMETER JUMP DETECTION IN STOCHASTIC DYNAMICAL SYSTEMS

R. C. Rogers

ABSTRACT

In this thesis the problem of the detection of parameter jumps in stochastic systems is considered. Previous work on the detection of disorders in a stochastic process when the jump time has an exponential probability distribution is extended to give optimal detection rules for a class of dynamical systems having autoregressive dynamics. This leads to a sub-optimal approach to parameter jump detection for more complicated linear systems, related to approaches proposed in a number of applications oriented papers.

The methods considered here are appropriate when parameter values before and after the jump time are known although the jump time itself is unknown. In order to relax these requirements a study is made of the performance of detection rules when the parameters jump to a different value to that designed for. The results obtained lead to the identification of a set of parameter values to which, with some restrictions, a jump is detected on average at least as quickly as in the design case. These results are obtained in a stronger form in the case of first order autoregressive systems.

It is suggested that these results may enable a detection rule having near optimal properties (in a minimax sense) to be designed, if only a set of possible post-jump parameter values is specified.

CONTENTS

ACKNOWLEDGMENTS

I would like to thank my supervisor, Dr. M.H.A.Davis, for his guidance throughout this work, and for introducing me to stochastic systems. I am particularly grateful to him for finding time to see me during brief visits to London while spending a year in the United States. I must also thank Dr. J.M.C.Clark and other members of the Department staff for their help at this time.

The encouragement given to me by fellow research students, in particular Katsuhiko Yamashita and John Hutcheson, as well as a number of friends outside the College was very important to me. I must also include my family for their understanding during this period.

For most of the duration of this work I was supported by the Science Research Council.

ERRATA

Equation (2.3.16) on page 22: $P:(y_u : u \geqslant t_o) \mapsto \tau^P$

Equation (3.1.7) on page 38: Left hand side should be $h^*(\tilde{\pi},\tilde{v})$

Equation (3.5.7) on page 62: $\alpha_t = \alpha' \in A' \subset \bigcup_{i=1}^{j} A^{\beta_i}$, $j < \infty$

Equation (4.2.9) on page 73:

$$S_1 \triangleq \begin{cases} \inf\{S : (S,y) \in \Gamma\} & \text{if } \Gamma \neq \emptyset \\ +\infty & \text{if } \Gamma = \emptyset \end{cases}$$

Page 81: Reference to [22, Theorem 1.1] should be to [18, Theorem 1.3]

Page 83: $P(y^2_{\hat{T}} \geqslant \frac{\theta}{-2a_c} | Y_c)$ should be $P(y^2_{\hat{T}} \geqslant \frac{\theta^2}{-2a_c} | Y_o)$
(two occurances)

Page 99: Reference to [22, Theorem 1.1] should be to [18, Theorem 1.3]

Page 111, End of proof: "The result of the Theorem follows from Lemma 4.5"

Page 131, Final equation in proof: Right hand side should be

$$E'(\hat{C}_{t_c}(t_o) | R_{t_c} = r_p)$$

NOTATION

General Notation

$\mathcal{R}$	The set of real numbers	
$\mathcal{N}$	The set of integers	
y_t	The observation process	
$\tilde{\tau}$	A detection time	
$\tilde{\tau}_{t_o}$	A detection time such that $\tilde{\tau}_{t_o} \geq t_o$	
τ	The optimal detection time	
τ_{t_o}	Optimal detection time if $\tau \not< t_o$	
$\mathcal{P}$	Policy mapping $(y_u : u \geq 0) \mapsto \tilde{\tau}$	
$\mathcal{Y}_t$	The σ-field generated by $(y_u : u \leq t)$	
$\mathcal{Y}_t^R$	An enlarged σ-field permitting randomized detection rules	
t_j	Jump time of parameters	
λ	Parameter of distribution of t_j	
$C(\tilde{\tau})$	Cost function	see (2.2.1)
$K(\tilde{\tau})$	Cost function	see (2.2.2)
$C_{t_o}(\tilde{\tau}_{t_o})$		see (4.1.6)
$K_{t_o}(\tilde{\tau}_{t_o})$		see (4.1.7)
c	Delay weighting coefficient in $C(\tilde{\tau})$, $K(\tilde{\tau})$	
Q	Cost function	see (2.2.7)
π_t	Value of $P(t \geq t_j \mid \mathcal{Y}_t)$ evaluated under the assumption of parameters jumping to design values	
v_t	State vector of system (2.5.6)	
n	Dimension of v_t	
W_t	A Wiener process (input noise process)	
m	Dimension of W_t	
R_t	$\ln(\pi_t/(1-\pi_t))$	

$h^*(\cdot,\cdot)$		see (3.1.7)
S_t	Process related to π_t	see (3.1.19)
$\pi(\cdot,\cdot)$	Function such that $\pi_t=\pi(S_t,v_t)$	
γ	The stopping boundary (in the appropriate space)	
$h(\cdot,\cdot)$		see (3.1.22)
$S_\gamma(v)$	$\inf\{S:h(S,v)\geq 0\}$	
$R_\gamma(v)$	$\inf\{R:h^*(\frac{1}{1+\exp(-R)},v)\geq 0\}$	

$a \wedge b$ denotes $\min(a,b)$, $a \vee b$ denotes $\max(a,b)$

The abbreviation s.t. is sometimes used for "such that".

Notation used in Chapter 4

α, a_o, β_t	Parameters of system (4.0.1) (Note: $a_o<0$)	
$\sigma(S,y)$	$-\lambda+(\lambda+c)\pi(S,y)$	see (4.1.10)
S_c	$\ln(\lambda/(-(\alpha+1)a_o+\lambda))$	see (4.2.1)
N,P,Q	Regions of (S,y) space	see (4.2.5)
θ	Common boundary of P and Q	
S_1		see (4.2.9)
y_1		see (4.2.10)
$\bar{S}_c$	$\ln(\frac{\lambda}{-(\alpha+1)a_o}) - \frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o}$	
$\bar{y}_c^2$	$\frac{2\lambda-(3\alpha+1)a_o}{(\alpha^2-1)a_o^2}$	
$r(y)$		see (4.3.2)

In Section 4.3 symbols with a bar correspond to those above in the context of the first modified problem. Similarly symbols with stars are used for the second modified problem.

In Section 4.4

y_t^o, y_t^* are defined in (4.4.2),(4.4.3)

$t_j^o, t_j^*, \hat{t}$ are defined in (4.4.5) to (4.4.8)

$K_{t_o}^o(\tilde{\tau}_{t_o}), K_{t_o}^*(\tilde{\tau}_{t_o})$ are defined in (4.4.9),(4.4.10)

Notation used in Chapter 5

P^1, E^1	Denotes probability and expectation given that parameters jump to "design" values	
P^2, E^2	Probability and expectation given that parameters jump to "non-design" values	
ζ_t	Process associated with transient effects in sub-optimal detection rules	see (5.1.5b)
M^i, h^i, g^i		see (5.1.8) to (5.1.10)
τ^c, γ_c	The superscript/subscript indicates dependence of τ, γ on the coefficient c	
$\bar{C}_{t_o}(\cdot)$	Modified cost function used in proofs of Lemmas 5.2 and 5.3	see (5.1.20)
$Q, \lVert\cdot\rVert^*$		see (5.1.29)
r_ρ	Bound for $R_{\gamma_1}(v)$, $\lVert v\rVert^* \leq \rho$	see Lemma 5.3
t_s	$\inf\{t \geq t_j : R_t \geq \ln\lambda, \lVert v_t\rVert^* \leq \rho\}$	see (5.1.44)
$\hat{R}_t^c$	Process defined such that $\hat{R}_t^c \leq R_t$ $\forall t \leq \hat{\tau}^c \geq \tau^c$ if (5.1.5) holds	see (5.1.46)
σ_1, σ_2	Defined in (5.1.65),(5.1.66)	
$\hat{\sigma}$	Defined in (5.1.64)	
μ_t	Process related to R_t when (5.1.4) holds	see (5.1.67)

CHAPTER 1

INTRODUCTION

1.1. The detection of parameter jumps in stochastic dynamical systems has been the subject of a number of recent papers. The problem may involve the detection either of failures in control systems or simply of changes in mode of operation of a system whose state is being tracked. Examples are most numerous in the aerospace field, particularly in inertial navigation where effective detection procedures may enable reduced redundancy levels to be employed.

Two main approaches have been proposed for the case of linear systems considered here. The first involves the application of statistical tests to the innovations process generated by a Kalman Filter designed with pre-jump parameter values. In the case of discrete time systems the innovations process until a jump takes place will be a sequence of independent normal random variables. A chi squared test used to check this property should, therefore, be able to identify when a parameter change occurs. This method is simple and requires no assumptions about the post-jump dynamics. However, this means that it does not take advantage of all the information available. In particular other approaches might be able to distinguish better between external variation in the statistical properties of noise entering the system and parameter jumps.

The second approach, which is the one of interest here, uses a-priori knowledge of the system struture to recognise behaviour typical of a parameter jump. Here, however, it is generally necessary to know in advance the values of the system parameters following a jump.

Unfortunately, except in simple cases, attempts to contruct detection schemes which are in some sense optimal lead to infinite dimensional filtering problems and so are not feasible. However, several approximations have been proposed which in many cases should give near optimal performance.

Because of these difficulties work on parameter jump detection methods has been split into theoretically complete investigations of simple problems, in continuous time, and practical studies mainly involving discrete time systems in which proposed schemes are justified largely by simulation. In this thesis optimal detection rules are derived for a wider class of continuous time systems (systems with an autoregressive structure) than previously considered.

The requirement that post-jump parameter values be known in advance is a major restriction. In order to relax this it seems appropriate to consider the robustness of detection rules: that is their performance if the system parameters jump to values other than those designed for. Robustness is considered in detail here and a possible strategy for effective detection is outlined in the case where only a set of possible post-jump parameter values is specified.

1.2 Organisation of Thesis

In Section 2.1 the parameter jump detection problem is introduced as a special case of the disorder problem for stochastic processes. Suitable cost functions are proposed and a-priori assumptions are discussed in Section 2.2. General properties of these formulations are given In sections 2.3 and 2.4 previous theoretical work on parameter jump detection in the case of systems with trivial dynamics is described. In Section 2.5 the problems encountered in trying to extend these results to more complicated systems are demonstrated and practical approaches to this problem are described.

Section 3.1 introduces optimal detection rules for a special class of system (autoregressive dynamics). Some properties are obtained for use in later chapters. In Section 3.2 an approach due to Kushner is applied to the problem of synthesizing an optimal detection rule. A simplified approach is described in Section 3.3 and in Section 3.4 a natural sub-optimal approach (related to previously proposed discrete time schemes) is suggested for use with more general linear systems. Finally the investigation of the robustness of detection rules is motivated in Section 3.5 and a possible approach described for the detection of jumps where post-jump parameter values are only known to be in a given set.

In Chapter 4 the robustness of detection rules for first order autoregressions is investigated. Roughly speaking, the results obtained show that optimal or near optimal detection rules will detect "larger" than designed for jumps at least as quickly on average. This was not previously entirely obvious as is suggested in the discussion at the beginning of the chapter. In the case in which the robustness property is only obtained for a near-optimal detection rule a bound is established on the expected performance degredation using this. This is done in Section 4.4.

In Chapter 5 the robustness properties of detection rules designed for more general systems is investigated. For the optimal, or, where this is not implementable, the sub-optimal detection rule proposed in Section 3.4 a set of post-jump parameter values is characterized such that the expected detection time is not increased, at least if a coefficient in the cost fuction is sufficiently small. This restriction corresponds to typical detection times being long compared to system time constants. Section 5.1 develops the robustness theory while in Section 5.2 its application is considered.

1.3 Original Contributions

In Chapter 2 previously published results are reformulated in the form appropriate here. In Section 3.1 the construction of the optimal detection rule is orginal, though the construction of Lemmas 3.3 and 3.4 is inspired by Shiryaev [12]. Lemma 3.1 is an application of a result in [17]. The use of non-linear filtering is inspired by Davis [14]. The formulation of the detection problems in terms of the time differentiable process S_t (equation (3.1.19)) is original, and it is this which enables the application of results in [16] to the synthesis problem in Section 3.2. Section 3.4 is related to approaches listed in [4] for discrete time problems. The discussion in Section 3.5 is original.

Chapters 4 and 5 are entirely original (Lemma 5.7 has been obtained independently : no previous derivation of this result has been found).

CHAPTER 2

DISORDER AND PARAMETER JUMP DETECTION PROBLEMS

In this chapter the parameter jump detection problem is introduced as a special case of the disorder problem for stochastic processes [e.g. 1,11]. The a-priori assumptions used later concerning the time of the jump are discussed, and various cost functions are defined and their properties investigated. The detection of disorders in a class of systems having trivial dynamics is discussed, and a summary given of the results of [1,2,3]. Finally, practical approaches given in [4] to the detection of parameter jumps in more general systems are described and some difficulties outlined.

2.1 The disorder problem for stochastic processes

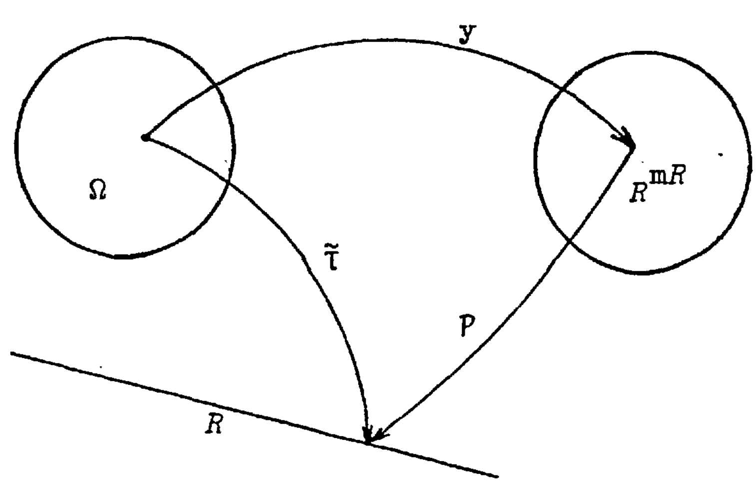


Figure 2.1.1

Consider a probability space (Ω,F,P) on which is defined a process $y_t \in R^m$ $\forall t$, and a random variable $t_j \geq 0$. The process y_t is interpreted as undergoing a change of regime (a disorder) at the time t_j.

$\mathcal{Y}_t$ is the σ-field generated by $(y_u : u\le t)$.

$\tilde{\tau}$ is a $\mathcal{Y}_t^R$-stopping time, interpreted as the time at which the change of regime is "detected" (possibly falsely) observing y_t. Here $\mathcal{Y}_t^R$ is a σ-field generated by $(y_u : u\le t)$ together, possibly, with other random variables independent of t_j and y_s $\forall s$. The introduction of $\mathcal{Y}_t^R$ enables randomized stopping rules to be considered.

Since $\tilde{\tau}$ is a stopping time, for any t_o, given $(y_u, u\le t_o)$ and that $\tilde{\tau}\ge t_o$, there is a (possibly randomized) map or <u>policy</u> P so that

$$P : (y_u, u\ge t_o) \mapsto \tilde{\tau} \tag{2.1.1}$$

The performance of a detection scheme for the "disorder" occuring at time t_j is usually measured by its success in achieving the conflicting objectives of quick detection and infrequent false alarms while no disorder exists. In some formulations of the problem an a-priori distribution is assumed for t_j, while in others this is avoided by a suitable definition of optimality, or by using a liklihood formulation. Usually when an a-priori distribution is assumed for t_j it is the exponential distribution $P(t\ge t_j)=1-e^{-\lambda t}$ for some $\lambda>0$. This greatly simplifies the problem because of the property

$$P(t+u\ge t_j \mid u<t_j) = P(t\ge t_j) \quad \text{for } t\ge 0.$$

<u>2.2 Formulations of the disorder problem</u>

<u>a) With a-priori information about t_j</u>

In this case the performance of a detection rule may be measured by its expected cost. Several possible cost functions are given here, but as is shown they are inter-related.

1) The cost function $C(\tilde{\tau})$ is defined as [2,11]

$$C(\tilde{\tau}) = I(\tilde{\tau}<t_j) + c(\tilde{\tau}-t_j).I(\tilde{\tau}>t_j) \quad c>0, \quad t_j\geq 0 \tag{2.2.1}$$

where $\tilde{\tau}$ is a $\mathcal{Y}_t^R$-stopping time.

The use of this enlarged σ-field enables randomized stopping rules to be considered. With this cost function a fixed cost is paid if there is a false alarm, while if there is a disorder before $\tilde{\tau}$ a cost proportional to the detection delay is incurred. Note that only one detection attempt is allowed and if this is a false alarm the test terminates. Since, unless c is small so that long delays are permitted, the probability of a false alarm is likely to be nearly one, an optimal detection rule is likely to give an expected cost very close to that of stopping at time zero (i.e. 1).

2) The cost function $K(\tilde{\tau})$ is defined as [11]

$$K(\tilde{\tau}) = -\lambda\tilde{\tau} + (\lambda+c)(\tilde{\tau}-t_j).I(\tilde{\tau}>t_j) \quad c>0, \quad t_j\geq 0 \tag{2.2.2}$$

where $\tilde{\tau}$ is a $\mathcal{Y}_t^R$-stopping time.

Here there is a reward of λ/unit time while the process is allowed to continue uninterupted, but a penalty of (λ+c)/unit time after the disorder occurs. The main interest of this formulation is its relation to $C(\tilde{\tau})$ which is used in chapters 3 and 4. This result was established in [11].

Lemma 2.1

If $t_j\geq 0$ is distributed such that

$$P(t\geq t_j | t_j>0, \mathcal{Y}_o) = 1-e^{-\lambda t} \quad t\geq 0 \tag{2.2.3}$$

$$E(K(\tilde{\tau})|\mathcal{Y}_o) = E(C(\tilde{\tau})|\mathcal{Y}_o) - P(t_j>0|\mathcal{Y}_o) \tag{2.2.4}$$

for any $\mathcal{Y}_t$-stopping time $\tilde{\tau}>0$

Proof

$$P(\tilde{\tau}\geq t_j|\mathcal{Y}_o) = \int_0^\infty \lim_{\delta\to 0} \tfrac{1}{\delta}P(t_j\in(u,u+\delta]|u<\tilde{\tau},\mathcal{Y}_o).P(u<\tilde{\tau}|\mathcal{Y}_o)du + P(t_j=0|\mathcal{Y}_o)$$

$$= \int_0^\infty \lambda P(u<t_j|u<\tilde{\tau},\mathcal{Y}_o).P(u<\tilde{\tau}|\mathcal{Y}_o)du + P(t_j=0|\mathcal{Y}_o)$$

$$= \lambda E(\int_0^\infty I(u<t_j)I(u<\tilde{\tau})du|\mathcal{Y}_o) + P(t_j=0|\mathcal{Y}_o)$$

$$= \lambda E(\tilde{\tau}\wedge t_j|\mathcal{Y}_o) + P(t_j=0|\mathcal{Y}_o) \qquad (2.2.5)$$

But $\quad E(K(\tilde{\tau})|\mathcal{Y}_o) = E(C(\tilde{\tau})|\mathcal{Y}_o) - P(\tilde{\tau}<t_j|\mathcal{Y}_o) - \lambda E(\tilde{\tau}\wedge t_j|\mathcal{Y}_o)$

from (2.2.1) and (2.2.2).

So $\quad E(K(\tilde{\tau})|\mathcal{Y}_o) = E(C(\tilde{\tau})|\mathcal{Y}_o) - 1 + P(\tilde{\tau}\geq t_j|\mathcal{Y}_o) - \lambda E(\tilde{\tau}\wedge t_j|\mathcal{Y}_o)$

Then using (2.2.5), (2.2.4) follows. □

It follows that if the conditions of Lemma 2.1 are satisfied and an optimal stopping time τ exists such that

$$E(K(\tau)|\mathcal{Y}_o) \leq E(K(\tilde{\tau})|\mathcal{Y}_o) \quad \forall\ \mathcal{Y}_t^R\text{-stopping times } \tilde{\tau}$$

then this is also optimal in the sense of the cost function $C(\tilde{\tau})$.

3) A further cost function is now introduced which is appropriate if the detection procedure does not terminate with a false alarm. The situation of interest here is the following: The output of a system is observed and a sequence of alarm times $\tilde{\tau}^1<\tilde{\tau}^2<\tilde{\tau}^3<\cdots<\tilde{\tau}^N$ is generated, where

$$N \triangleq \inf\{i:\tilde{\tau}^i\geq t_j\} \qquad (2.2.6)$$

For each alarm a fixed cost is incurred, and there is a further cost proportional to the detection delay $(\tilde{\tau}^N-t_j)$.

The cost $\qquad Q = N + d(\tilde{\tau}^N-t_j) \qquad d>0 \qquad (2.2.7)$

This might be interpreted as an inspection cost following each alarm, together with a cost propotional to the detection delay. This formulation is proposed in [3]. The following Lemma establishes a relationship between this situation and that corresponding to (2.2.1).

Lemma 2.2

Suppose that for each $t,u>0$, conditioning on the events $\tilde{\tau}^i+t \le \tilde{\tau}^{i+1}$ and $t_j = \tilde{\tau}^i+u$ $(\tilde{\tau}^o \triangleq 0)$, and on $\mathcal{Y}_{\tilde{\tau}^i}$. $y_{\tilde{\tau}^{i+1}}$ is identically distributed for $i=0,1,\cdots,N$.

Also $P(t \ge t_j|\mathcal{Y}_o) = 1 - e^{-\lambda t}$, $\lambda>0$.

Suppose τ is a stopping time which minimizes $E(C(\tilde{\tau})|\mathcal{Y}_o)$, where $C(\cdot)$ is defined in (2.2.1) with

$$c = d/Q^o \tag{2.2.8}$$

and

$$Q^o \triangleq \inf_{\{\tilde{\tau}^i\}} E(Q|\mathcal{Y}_o) \tag{2.2.9}$$

Let P be the (possibly randomized) map defined by (see(2.1.1))

$$P:(y_u:u\ge 0) \mapsto \tau \tag{2.2.10}$$

Then a sequence of stopping times which minimizes $E(Q|\mathcal{Y}_o)$ is defined by

$$P:(y_u:u\ge\tau^i) \mapsto \tau^{i+1} - \tau^i \quad i=0,1,\cdots,N-1 \tag{2.2.11}$$

Proof

Under the conditions of the Lemma, minimization of the expectation of $(Q-i)$ conditioned on $(\tilde{\tau}^i<t_j)$ is an identical problem to the minimization of the expectation of Q. Therefore only $\{\tilde{\tau}^i\}$ defined by (2.2.11) for some policy $\tilde{P}$ need be considered, since the same stopping rule should be used following each false alarm.

For $\varepsilon>0$ arbitrarily small, $\exists\hat{P}$ such that when this is used to generate $\{\tilde{\tau}^i\}$

$$E(Q-n|\tilde{\tau}^n<t_j,y_o) = E(Q|y_o) = Q^o+\varepsilon \quad \forall n, \tag{2.2.12}$$

Now suppose $\tilde{\tau}^1$ is generated by a policy $\bar{P}$ and $\tilde{\tau}^2,\cdots,\tilde{\tau}^n$ by $\hat{P}$. Then

$$\begin{aligned} E(Q|y_o) &= 1 + E(d(\tilde{\tau}^1-t_j)I(\tilde{\tau}^1>t_j)|y_o) \\ &\qquad + (Q^o+\varepsilon)P(\tilde{\tau}^1<t_j|y_o) \\ &= 1 + Q^oE(C(\tilde{\tau}^1)|y_o) + \varepsilon P(\tilde{\tau}^1<t_j|y_o) \end{aligned} \tag{2.2.13}$$

where the parameter c is given in (2.2.8)

If $\bar{P}=\hat{P}$, $E(Q|y_o) = Q^o+\varepsilon$, so $1+Q^oE(C(\tilde{\tau}^1)|y_o) \leq Q^o+\varepsilon$

If $\bar{P}$ is defined by (2.2.10), $Q^oE(C(\tilde{\tau}^1)|y_o)$ is minimized so again

$$1 + Q^oE(C(\tilde{\tau}^1)|y_o) \leq Q^o + \varepsilon$$

As ε is arbitrarily small,

$$1 + Q^oE(C(\tilde{\tau}^1)|y_o) \leq Q^o \tag{2.2.14}$$

Now choose $\hat{P}$ also to be the policy defined by (2.2.10), and ε to be the appropriate value in (2.2.12). From (2.2.13), using (2.2.12) and (2.2.14)

$$\begin{aligned} Q^o+\varepsilon &= 1 + Q^oE(C(\tilde{\tau}^1)|y_o) + \varepsilon P(\tilde{\tau}^1<t_j|y_o) \\ &\leq Q^o + \varepsilon P(\tilde{\tau}^1<t_j|y_o) \end{aligned}$$

Since $P(\tilde{\tau}^1<t_j|y_o)<1$, it follows that $\varepsilon=0$, and optimality of $\hat{P}$ follows from (2.2.12). □

Remark

If y_t is of the form

$$dy_t = (\alpha+\beta I(t\geq t_j))dt + dW_t \tag{2.2.15}$$

where W_t is a Wiener process, then there is a one to one mapping relating y_t to the process $\hat{y}_t$ where

$$\hat{y}_t = y_t - y_{\tilde{\tau}^i} \quad \forall t \in (\tilde{\tau}^i, \tilde{\tau}^{i+1}], \; i=0,1,\cdots \quad (\tilde{\tau}^0 \triangleq 0) \qquad (2.2.16)$$

$\hat{y}_t$ satisfies the conditions of Lemma 2.2, which then defines the optimal detection rule for cost Q, if a solution exists for the formulation (2.2.1). Alternatively if y_t is generated by a more complicated stochastic system, and at each alarm time the state of the system is reset to y_o, Lemma 2.2 again holds. As is argued later, the effect of the initial condition $y_{\tilde{\tau}^i}$ may not be very important in practice.

4) An alternative approach proposed by Shiryaev [1] is to minimize the expected delay time in detecting a disorder, $E((\tilde{\tau}-t_j)I(\tilde{\tau}>t_j)|\mathcal{Y}_o)$ while constraining the maximum permitted false alarm probability, $P(\tilde{\tau}<t_j|\mathcal{Y}_o)$. This is refered to in [2,12] as the "Variational Formulation".

In the situation described above, if the conditions of Lemma 2.2 hold and $\{\tilde{\tau}^i\}$ is a sequence of stopping times defined by (2.2.11) for some $\tilde{P}$, it follows that

$$P(\tilde{\tau}^1<t_j|\mathcal{Y}_o) = P(\tilde{\tau}^2<t_j|\tau^1<t_j,\mathcal{Y}_o) = \cdots$$
$$\cdots P(\tilde{\tau}^{m+1}<t_j|\tilde{\tau}^m<t_j,\mathcal{Y}_o) = p, \text{say} \qquad (2.2.17)$$

Then $$E(N-1|\mathcal{Y}_o) = p(1+p(1+p(\cdots\cdots))) = \frac{p}{1-p} \qquad (2.2.18)$$

Therefore constraining the false alarm probability is equivalent to constraining the expected number of false alarms (N-1).

<u>b) With no a-priori information about t_j</u>

1) In [1], Shiryaev proposes an approach which avoids the need for a-priori information about t_j. The mean delay time in detecting a disorder is minimized while the mean time

between false alarms with no disorder present is constrained to be no less than a given value. In the case considered, the solution to this problem turns out to be a limiting case of the solution to the formulation (2.2.1) as $\lambda \to 0$ (see section 2.4).

2) Willsky and others [4,7,8] have proposed approaches based on likelihood ratios in which no explicit assumption is made about the distribution of t_j. A single parameter is then chosen to balance false alarm frequency and detection delay.

2.3 Observation processes without dynamics

Disorder problems have been investigated both where the process y_t is a counting process [10,11], and where y_t is a process related to $I(t \geq t_j)$ with additive noise. The second case is of most interest here. In this section results concerning the situation

$$dy_t = rI(t \geq t_j) + dW_t \qquad 0<r<\infty, \tag{2.3.1}$$

W_t a Wiener process independent of t_j

are discussed. The distribution

$$P(t \geq t_j | t_j > 0, Y_0) = 1 - e^{-\lambda t} \tag{2.3.2}$$

is assumed, and except where explicitly stated, $P(t_j = 0 | Y_0)$ is taken to be zero.

Defining $\quad M_t = I(t \geq t_j) - \int_0^t \lambda(1 - I(u \geq t_j))du \qquad (2.3.3)$

M_t is a Martingale (this follows from the proof of Lemma 2.2 for example). Then, as in [14], the non-linear filtering equations (see Appendix 1) may be applied to the equations

$$dI(t \geq t_j) = \lambda(1 - I(t \geq t_j))dt + dM_t \tag{2.3.4}$$

$$dy_t = rI(t \geq t_j)dt + dW_t \tag{2.3.5}$$

to obtain

$$\pi_t = P(t \ge t_j | \mathcal{Y}_t) = E(I(t \ge t_j) | \mathcal{Y}_o) \qquad (2.3.6)$$

$$d\pi_t = \lambda(1-\pi_t)dt + r\pi_t(1-\pi_t)d\nu_t \qquad (2.3.7)$$

where the innovations process ν_t (a Wiener process) is defined by

$$d\nu_t = dy_t - \pi_t r dt \qquad (2.3.8)$$

It is sufficient to consider optimal detection rules with cost function (2.2.2) since Lemma 2.1 implies these are optimal with cost function (2.2.1).

Let t_o be an arbitrary stopping time and define

$$q(\tilde{\pi}) = \inf_{\tilde{\tau}_{t_o}} E(-\lambda(\tilde{\tau}_{t_o} - t_o) + (\lambda+c)\int_{t_o}^{\tilde{\tau}_{t_o}} \pi_u du | \mathcal{Y}_{t_o})\Big|_{\pi_{t_o}=\tilde{\pi}} \qquad (2.3.9)$$

Note that because of the form of (2.3.2) and the Markov property of π_t, q is only a function of $\tilde{\pi}$.

Define $\quad \tau = \inf\{t \ge 0 : q(\pi_t) \ge 0\} \qquad (2.3.10)$

τ is the optimal stopping time with cost function $C(\tilde{\tau})$ or $K(\tilde{\tau})$ as shown below.

For any $\mathcal{Y}_t^R$-stopping time $\tilde{\tau} \ge 0$

$$E(K(\tilde{\tau})|\mathcal{Y}_o) = E(K(\tilde{\tau}\wedge\tau)|\mathcal{Y}_o) + E[E(-\lambda(\tilde{\tau}-\tau)+(\lambda+c)\int_\tau^{\tilde{\tau}} \pi_u du | \mathcal{Y}_\tau, \tau) I(\tilde{\tau} \ge \tau) | \mathcal{Y}_o]$$

and $\quad E(-\lambda(\tilde{\tau}-\tau)+(\lambda+c)\int_\tau^{\tilde{\tau}} \pi_u du | \mathcal{Y}_\tau, \tau) \ge 0 \quad$ for $\tilde{\tau} \ge \tau$

by definition of τ (2.3.10) and of $q(\cdot)$ (2.3.9).

Therefore $\quad E(K(\tilde{\tau}\wedge\tau)|\mathcal{Y}_o) \le E(K(\tilde{\tau})|\mathcal{Y}_o) \qquad (2.3.11)$

Also $\quad E(K(\tilde{\tau}\vee\tau)|\mathcal{Y}_o) = E(K(\tilde{\tau})|\mathcal{Y}_o) + E[E(-\lambda(\tau-\tilde{\tau})+(\lambda+c)\int_{\tilde{\tau}}^{\tau} \pi_u du | \mathcal{Y}_{\tilde{\tau}}) I(\tau \ge \tilde{\tau}) | \mathcal{Y}_o]$

and $\quad E(-(\tau-\tilde{\tau})+(\lambda+c)\int_{\tilde{\tau}}^{\tau} \pi_u du | \mathcal{Y}_{\tilde{\tau}}) \le 0 \quad$ if $\tau \ge \tilde{\tau}$

by definition of τ (2.3.10).

Therefore $$E(K(\tilde{\tau}\vee\tau)|\mathcal{Y}_o) \leq E(K(\tilde{\tau})|\mathcal{Y}_o) \quad (2.3.12)$$

Since $\tilde{\tau}$ here is an arbitrary stopping time, (2.3.11) and (2.3.12) together imply

$$E(K(\tau)|\mathcal{Y}_o) = E(K(\tau\wedge[\tau\vee\tilde{\tau}])|\mathcal{Y}_o)$$
$$\leq E(K(\tilde{\tau})|\mathcal{Y}_o) \quad (2.3.13)$$

This shows the required optimality of τ. Note that τ is a $\mathcal{Y}_t$-stopping time, that is, it is not randomized.

Next it is shown that τ is the first crossing time of a threshold value by π_t. For t_o an arbitrary stopping time

define $$K_{t_o}(\tilde{\tau}_{t_o}) = -\lambda(\tilde{\tau}_{t_o}-t_o) + (\lambda+c)(\tilde{\tau}_{t_o}-t_o\vee t_j)I(\tilde{\tau}_{t_o}\geq t_j)$$
$$\text{for } \tilde{\tau}_{t_o}\geq t_o \quad (2.3.14)$$

Then $$q(\pi_{t_o}) = E(K_{t_o}(\tau)|\mathcal{Y}_{t_o},\tau\geq t_o) = E(K_{t_o}(\tau)|\pi_{t_o},\tau\geq t_o) \quad (2.3.15)$$

from (2.3.9). $\mathcal{Y}_{t_o}$ may be replaced in this way since π_t is a Markov process.

If $\pi_{t_o}=\tilde{\pi}$, say and $\tau\geq t_o$, τ is given by some policy (see section 2.1)

$$P:(y_u:u\leq t_o) \nrightarrow \tau^P \quad (2.3.16)$$

Suppose this policy is used in fact when $\pi_{t_o}=\hat{\pi}$. Then $\exists\tilde{\Phi},\tilde{\Psi}$ such that

$$E(K_{t_o}(\tau^P)|\pi_{t_o}=\hat{\pi}) = \tilde{\Phi}.\hat{\pi} + \tilde{\Psi}.(1-\hat{\pi}) \quad (2.3.17)$$

i.e. $\tilde{\Phi}= E(K_{t_o}(\tau^P)|t_j\leq t_o)$; $\tilde{\Psi}=E(K_{t_o}(\tau^P)|t_j>t_o)$

$\tilde{\Phi}\geq 0$ from (2.3.14). As $q(\pi)\leq 0$ (consider $\tilde{\tau}=t_o$ in (2.2.2)) it follows from figure 2.3.1 that $\tilde{\Psi}\leq 0$.

Also by definition,

$$q(\hat{\pi}) \leq \tilde{\Phi}.\hat{\pi} + \tilde{\Psi}.(1-\hat{\pi}) \quad \forall\hat{\pi}\in[0,1]$$

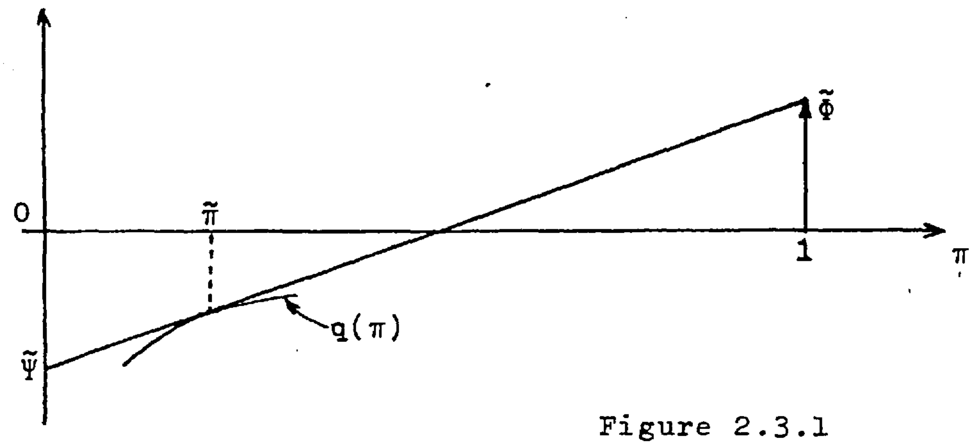


Figure 2.3.1

Let $\quad \tilde{\pi} \triangleq \sup\{\pi : q(\pi)<0\} - \varepsilon, \quad \varepsilon>0$

Then $\quad q(\hat{\pi}) \le \tilde{\Phi}\hat{\pi} + \tilde{\Psi}(1-\hat{\pi}) \le \tilde{\Phi}\tilde{\pi} + \tilde{\Psi}(1-\tilde{\pi}) = q(\tilde{\pi})<0, \quad \hat{\pi}\le\tilde{\pi}$

for values of ε chosen arbitrarily small.

Therefore

$$q(\hat{\pi}) < 0 \quad \forall\hat{\pi}<\sup\{\pi : q(\pi)<0\} \tag{2.3.18}$$

Now let $\tilde{\pi}=\sup\{\pi : q(\pi)<0\}$, and suppose $q(\tilde{\pi})<0$

By definition of $\tilde{\pi}$, $\exists\varepsilon>0$ sufficiently small so that

$$0 \le q(\tilde{\pi}+\varepsilon) \le \tilde{\Phi}\tilde{\pi} + \tilde{\Psi}(1-\tilde{\pi}) + \varepsilon(\tilde{\Phi}-\tilde{\Psi}) = q(\tilde{\pi}) + \varepsilon(\tilde{\Phi}-\tilde{\Psi}) < 0$$

This contradiction implies that $q(\tilde{\pi})=0$, and together with (2.3.10) and (2.3.18) it follows that

$$\tau = \inf\{t : \pi_t \ge \pi_\gamma\} \quad \text{for some } \pi_\gamma \in [0,1]. \tag{2.3.19}$$

Note that, from Lemma 2.1, τ is an optimal detection time with both cost functions $K(\tilde{\tau})$ and $C(\tilde{\tau})$.

<u>Disorders of unknown magnitude</u>.

Up to now it has been assumed that the dynamics of the system are known before and after the occurence of a disorder. Here the situation

$$dy_t = \rho I(t \ge t_j)dt + dW_t \qquad \rho \ge r > 0 \tag{2.3.20}$$

is considered.

Suppose the detection rule discussed above is implemented, which is optimal if $\rho=r$. Then the process π_t is independent of ρ up to time t_j and

$$d\pi_t = \lambda(1-\pi_t)dt + \pi_t(1-\pi_t)r(\rho-\pi_t r)dt + r\pi_t(1-\pi_t)dW_t \qquad t\geq t_j \tag{2.3.21}$$

Let $R_t = \ln(\pi_t/(1-\pi_t))$. Then

$$dR_t = \lambda(1+e^{-R_t})dt + r(\rho-\tfrac{1}{2}r)dt + rdW_t \tag{2.3.22}$$

by Itô's differentiation rule.

By monotonicity it follows from (2.3.19) that

$$\tau = \inf\{t:R_t\geq R_\gamma\} \quad \text{for some } R_\gamma\in\mathcal{R}$$

For a given sample path of W_t, let $\tau^{\tilde{\rho}}$ be the stopping time τ if $\rho=\tilde{\rho}$. Then from (2.3.22) $\tau^\rho\leq\tau^r$ and so

$$(\tau^\rho-t_j)I(\tau^\rho>t_j) \leq (\tau^r-t_j)I(\tau^r>t_j)$$

But the event $(\tau^\rho<t_j)$ is independent of ρ, so that from (2.2.1)

$$E(C(\tau)|\mathcal{Y}_o,\rho=\tilde{\rho}) \leq E(C(\tau)|\mathcal{Y}_o,\rho=r), \quad \tilde{\rho}\geq r$$

i.e.

$$\tau = \arg\min_{\tilde{\tau}}\max_{\rho\geq r} E(C(\tilde{\tau})|\mathcal{Y}_o) \tag{2.3.23}$$

$\tilde{\tau}$ a $\mathcal{Y}_t^R$-stopping time.

This also holds with $C(\tilde{\tau})$ replaced by $K(\tilde{\tau})$.

2.4 Analysis of the disorder problem without dynamics.

In this section some published results on the disorder problem are briefly described [1,2,3]. It is assumed that (2.3.2) holds. The problem of interest is the determination of the threshold value π_γ in (2.3.19).

Define

$$f(\tilde{\pi}) \triangleq E(I(\tau<t_j) + c(\tau-t_j\vee t_o)I(\tau>t_j)|\pi_{t_o}=\pi,\tau\geq t_o) \tag{2.4.1}$$

(c as in (2.2.1))

Note that since π_t is Markov, and from (2.3.19), $f(\tilde{\pi})$ is independent of the value of t_o.

From (2.3.7)

$$d\pi_t = (1-\pi_t)dt + r\pi_t(1-\pi_t)d\nu_t$$

where ν_t is a Wiener process. Therefore using Itô's differentiation rule,

$$df_t = [\lambda(1-\pi_t)\frac{df(\tilde{\pi})}{d\tilde{\pi}}\bigg|_{\tilde{\pi}=\pi_t} + \tfrac{1}{2}\pi_t^2(1-\pi_t)^2 r^2 . \frac{d^2 f(\tilde{\pi})}{d\tilde{\pi}^2}\bigg|_{\tilde{\pi}=\pi_t}]dt$$

$$+ \pi_t(1-\pi_t)r\frac{df(\tilde{\pi})}{d\tilde{\pi}}\bigg|_{\tilde{\pi}=\pi_t} d\nu_t \qquad (2.4.2)$$

if $f(\cdot)$ is sufficiently smooth.

But from (2.4.1)

$$\frac{d}{du}E(f(\pi_u)|\pi_t,\tau\geq t)\bigg|_{u=t} = -c\pi_t \qquad (2.4.3)$$

Taking expectations conditioned on π_t in (2.4.2), and equating with $-c\pi_t$ gives

$$\lambda(1-\pi)\frac{df(\pi)}{d\pi} + \tfrac{1}{2}\pi^2(1-\pi)^2 r^2 . \frac{d^2 f(\pi)}{d\pi^2} = -c\pi \qquad \pi<\pi_\gamma \qquad (2.4.4)$$

and of course

$$f(\pi) = (1-\pi) \quad \text{for } \pi\geq\pi_\gamma \qquad (2.4.5)$$

since in (2.4.1) $\tau=t_o$ in this case.

Assuming in addition that

$$\frac{df(\tilde{\pi})}{d\tilde{\pi}}\bigg|_{\tilde{\pi}=\pi_\gamma^-} = \frac{df(\tilde{\pi})}{d\tilde{\pi}}\bigg|_{\tilde{\pi}=\pi_\gamma^+} = -1 \qquad (2.4.6)$$

(the so-called smooth pasting condition) the function $f(\pi)$ is uniquely defined by the equations (2.4.4) & (2.4.5).

Now $\quad f(\tilde{\pi}) = E(C(\tau)|\pi_o=\tilde{\pi},Y_o)$

from (2.2.1) and (2.4.1), so that from Lemma 2.1 it follows

that $\quad q(\tilde{\pi}) = f(\tilde{\pi}) - (1-\tilde{\pi})$

with $q(\tilde{\pi})$ defined in (2.3.9).

It follows that

$$\pi_\gamma = \inf\{\tilde{\pi} : f(\tilde{\pi}) \geq 1-\tilde{\pi}\} \tag{2.4.7}$$

Using this approach, Shiryaev [2] deduces that π_γ is the unique solution of

$$\pi_\gamma = \frac{\lambda}{\lambda+\frac{1}{2}hr^2}$$
$$\int_h^\infty \frac{e^{-z}(z+2\lambda/r^2)}{z^{(2+2\lambda/r^2)}}dz = \frac{r^2e^{-h}}{2c}\, h^{(-2\lambda/r^2)} \tag{2.4.8}$$

The necessary assumptions concerning the smoothness of $f(\pi)$ for $\pi \leq \pi_\gamma$ are justified in [12].

<u>Other formulations</u>

a) In [1], Shiryaev shows that optimal detection rules for the "Variational formulation" (see section 2.2) of the problem (2.3.1) are also solutions to the above formulation based on cost function $C(\tilde{\tau})$ for some choice of $c>0$. With this formulation an acceptable false alarm probability is fixed and a detection rule chosen to minimize the expected delay time $(\tilde{\tau}\vee t_j - t_j)$. For this particular problem the threshold value is given simply by

$$\pi_\gamma = 1 - (\text{acceptable false alarm probability})$$

He also deduces that for $D(\alpha,\lambda)$ the infimum of expected delay times conditioning on $\tilde{\tau} \geq t_j$

$$D(\alpha,\lambda) = \inf_{\tilde{\tau}} E(\tilde{\tau}-t_j \mid \tilde{\tau} \geq t_j) \tag{2.4.9}$$

where the infimum is over $\mathcal{Y}_t^R$-stopping times such that $P(\tilde{\tau}<t_j) \leq \alpha > 0$, and where λ is the parameter in the distribution for t_j (2.3.2), then

$$D(\alpha,\lambda) \to \frac{2}{r^2}[\exp(2t/r^2)(-Ei(-2T/r^2)) - 1 + \frac{2T}{r^2}\int_0^\infty \exp(-2Tz/r^2)\frac{\ln(1+z)}{z}dz] \quad (2.4.10)$$

as $\alpha \to 1$, $\lambda \to 0$ such that $\frac{1-\alpha}{\lambda} = T$ (fixed)

Here, $-Ei(-y) = e^{-y}\int_0^\infty \frac{e^{-z}}{y+z}dz$

and T is the limiting value as $\alpha \to 1$, $\lambda \to 0$ of the mean time between false alarms with no disorder present if the detection procedure is used repeatedly.

In [1] Shiryaev shows that, with some restrictions, this is the best expected delay time that may be achieved by a stopping rule having mean time between false alarms not less than T. Using this formulation the need for an a-priori distribution for t_j is avoided.

b) Bather [3] considers the multi-stage problem of minimizing $E(Q|\mathcal{Y}_o)$ where Q is defined in (2.2.7). Using a similar approach to that described at the beginning of this section, he deduces that for his problem the optimal solution is to stop (for an "inspection") at each time that $\pi_t = \pi_\gamma$ (the process π_t being reset to zero each time a false alarm occurs) where

$$\pi_\gamma = \frac{a}{1+a}$$
$$\frac{1}{a} = 2\int_0^a x^{-2\lambda-1}\exp(2\lambda/x)\int_0^x y^{2\lambda}\exp(-2\lambda/y)dydx \quad (2.4.11)$$

2.5 Detection of disorders in systems with dynamics

The problems considered in the previous section were straightforward due to the simple nature of the observation process and the resulting Markov property of the process $\pi_t = P(t \geq t_j|\mathcal{Y}_t)$. More complicated problems arise when considering systems with non-trivial dynamics.

Evaluation of the posterior probability of a disorder

First, the usual state-space model is considered.

$$dx_t = A_t x_t dt + q_t dt + G_t dV_t$$
$$dy_t = H_t x_t dt + dZ_t \qquad (2.5.1)$$

$x_t, q_t \in R^n \ \forall t; \ y_t, Z_t \in R^m \ \forall t$

V_t, Z_t are independent Wiener processes, independent of t_j

$$A_t = A^0,\ G_t = G^0,\ H_t = H^0,\ q_t = q_t^0 \quad \forall t < t_j \geq 0 \qquad (2.5.2)$$

$$A_t = A^1,\ G_t = G^1,\ H_t = H^1,\ q_t = q_t^1 \quad \forall t \geq t_j \qquad (2.5.3)$$

Here $A^0, G^0, H^0, A^1, G^1, H^1$ are constant matrices and q_t^0 & q_t^1 are control processes known to the observer.

The a-priori distribution $P(t \geq t_j | t_j > 0) = 1 - e^{-\lambda t}$ is assumed, and t_j is independent of x_o, y_o. $P(t_j \geq 0)$ is known.

Then as before,

$$M_t \triangleq I(t \geq t_j) - \lambda \int_0^t (1 - I(t \geq t_j)) du \qquad (2.5.4)$$

is a Martingale, and so the process

$$E\left(\begin{bmatrix} I(t \geq t_j) \\ x_t \end{bmatrix} \middle| \mathcal{Y}_t \right)$$

may be generated using the non-linear filtering equations (Appendix 1) with (2.5.4),(2.5.1). Note that

$$\pi_t \triangleq P(t \geq t_j | \mathcal{Y}_t) = E(I(t \geq t_j) | \mathcal{Y}_t)$$

Then

$$d\pi_t = \lambda(1 - \pi_t)dt + [E_t(H^1 x_t I(t \geq t_j)) - E_t[H^1 x_t I(t \geq t_j) + H^0 x_t I(t < t_j)]\pi_t]^T d\nu_t \qquad (2.5.5)$$

where $E_t(\cdot) = E(\cdot | \mathcal{Y}_o)$ and ν_t is the innovations process. In order to use this expression to generate π_t it is necessary to have the estimates $E_t(x_t I(t \geq t_j))$ & $E_t(x_t I(t < t_j))$.

If x_t is $\mathcal{Y}_t$ measurable, (2.5.5) provides a feasible approach to the evaluation of π_t. Otherwise the non-linear filtering equations must be applied again to obtain the necessary estimates, but this in turn requires further estimates to be provided. In fact

$$E_t(x_t I(t\geq t_j)),\ E_t(x_t x_t^T I(t\geq t_j)),\ E_t(x_t x_t^T x_t I(t\geq t_j)),\cdots\cdots$$

$$E_t(x_t I(t<t_j)),\ E_t(x_t x_t^T I(t<t_j)),\ E_t(x_t x_t^T x_t I(t<t_j)),\cdots\cdots$$

are required- that is infinite sequences of estimates. A natural approach would be to truncate these sequences in some way. This is discussed in [13], but it is not clear how it should be done.

A class of system for which π_t may be obtained by finite dimensional filtering has the following form

$$dv_t = \begin{bmatrix} J & B \\ D_t & F_t \end{bmatrix} v_t dt + \begin{bmatrix} u_t \\ z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ I_m \end{bmatrix} dW_t \tag{2.5.6}$$

Observations $y_t = [0 : I_m] v_t$

v_t is an n dimensional process (n>m)

J is an (n-m)×(n-m) constant matrix, B a constant matrix

$D_t = D^0$, $F_t = F^0$, $z_t = z_t^0$ (D^0,F^0 constant matrices, z_t^0 a known process) $\forall t<t_j$

$D_t^1 = D^1$, $F_t = F^1$, $z_t = z_t^1$ (D^1,F^1 constant matrices, z_t^1 a known process) $\forall t\geq t_j$

W_t is an m dimensional Wiener process

u_t is a (n-m) dimensional known process

Again t_j is distributed so that (2.5.4) holds, and is independent of W_t and of v_o.

v_o is assumed given so that v_t is $\mathcal{Y}_t$-measurable, since

$$dv_t = \begin{bmatrix} J & B \\ 0 & 0 \end{bmatrix} v_t \, dt + \begin{bmatrix} u_t \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ I_m \end{bmatrix} dy_t \qquad (2.5.7)$$

A particular example of such systems is the autoregression

$$dv_t = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ \hline & & r^T & & \end{bmatrix} v_t \, dt + \begin{bmatrix} u_t \\ \hline z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix} dW_t \qquad (2.5.8)$$

$$r_t = r^0, \quad z_t = z_t^0 \quad \forall t < t_j$$

$$r_t = r^1, \quad z_t = z_t^1 \quad \forall t \geq t_j$$

where u_t, z_t^0, z_t^1 are known.

In addition it is shown in chapter 3 how a natural sub-optimal approach to the detection problem for the system (2.5.1) may be constructed based on (2.5.6).

Optimal stopping rules

In order to construct an optimal stopping time, in the sense of the cost functions defined in section 2.2, it is necessary to have some a-priori information about the controls $u_t, z_t^0, z_t^1, q_t^0, q_t^1$ in (2.5.1) or (2.5.6). For simplicity, only the case in which these take constant, known values is considered. With system (2.5.6) for example, $(\pi, v)_t$ is then a Markov process. In chapter 3, the corresponding optimal stopping rule is developed.

Approaches to the detection of disorders in general systems

Although in many cases it is not possible to construct optimal detection rules for disorders occuring in dynamical systems (because this involves infinite dimensional filtering as described above) several practical approaches have been proposed [4,5,9 for example]. The problem is of some practical interest, especially in the aerospace and inertial navigation fields [5,8,9]. Mostly this work concerns the discrete time version of the problem, and since this clarifies the way in which the infinite dimensional filtering problem arises a first order example is given here.

Consider the system

$$x_{k+1} = ax_k + (b+\delta I(k\geq k_j)) + w_k$$
$$y_k = x_k + v_k \tag{2.5.9}$$

where x_k, y_k are scalar processes

$a,b,\delta\in R$ are constant, $\delta\neq 0$, $|a|<1$

$x_o \sim N(\hat{x}_o, r_o)$

w_k, v_k are sequences of normal independent zero mean random variables such that $Ew_k^2=Ev_k^2=1$ $\forall k$

k_j (the time of appearance of the disorder) is independent of w_k, v_k $\forall k$ and of x_o

$$P(k\geq k_j) = 1-\lambda^k \tag{2.5.10}$$

By Kalman filtering the a-posteriori distribution of the state x_k, conditioned on observations $y_0,\cdots,y_{k-1}$ and the event $k_j=i$ may be obtained.

let

$$\hat{x}^{(i)}_{k|k-1} = E(x_k|k_j=i, y_1, y_2, \cdots, y_{k-1})$$
$$r_k = E([x_k-\hat{x}^{(i)}_{k|k-1}]^2|k_j=i,y_1,y_2,\cdots,y_{k-1}) \tag{2.5.11}$$

In this example, r_k is independent of i.

Then if $$y_k = \hat{x}^{(i)}_{k|k-1} + \nu^{(i)}_k$$

$\nu^{(k_j)}_k$ is a sequence of independent normal random variables of zero mean and variance $1+r_k$.

Defining $$P^{(i)}_k = P(k=i|y_1,y_2,\cdots,y_k) \tag{2.5.12}$$

then $$P^{(i)}_k = f_k(\nu^{(i)}_k)\cdot\frac{1}{N}\cdot P^{(i)}_{k-1} \tag{2.5.13}$$

using Bayes' Theorem, where $f_k(\cdot)$ is the probability density function associated with the distribution $N(0,r_k+1)$, and N is a normalizing factor eliminated by imposing the condition $\sum_{i=1}^{\infty} P_k^{(i)} = 1$.

At each time step k, $\nu_k^{(i)}$ will have the same value $\forall i>k$. However, for each $i\leq k$, it will be necessary to use (2.5.13) separately to obtain $P_k^{(i)}$.

Then $$P(k_j\leq k|y_1,y_2,\cdots,y_k) = \sum_{i=1}^{\infty} P_k^{(i)} \tag{2.5.14}$$

The computational load of evaluating this increases linearly with time k, as does the memory requirement. Since k_j is unbounded, implementation of an "optimal" detection rule involving the disorder probability would require an infinitely powerful computer.

However (in this case)

$$\nu_k^{(i)} - \nu_k^{(j)} = a(\nu_{k-1}^{(i)} - \nu_{k-1}^{(j)}) \qquad |a|<1$$

so $$\nu_k^{(i)} - \nu_k^{(j)} \to 0 \quad \text{as } k\to\infty$$

Therefore it is reasonable to suppose that a good sub-optimal policy could be constructed by approximating $\nu_k^{(i)}$ as k-i becomes large in such a way that only a finite number of terms need be updated independently.

Much of the work reported in the survey paper [4] deals with methods of approximating jump probabilities (or equivalently liklihoods) by exploiting this type of structure. Many of the contributions which have appeared on failure detection problems in practical situations deal with sudden jumps in the system state rather than in the parameters [e.g. 5,6]. The filtering problems which arise are then similar, but there is an important difference in that the evidence of such a disorder in the observations will not continue indefinately. Because of this, the performance of detection schemes is then often discussed in terms of "missed alarms" rather than of delay times. Other contributions [e.g. 7,8] deal with "sensor" and "actuator" failures which are permanent and correspond more closely to the problem considered here.

In [5,6] for example, state estimates and jump probabilities corresponding to jump times long before the current time are "fused" into a single representative value. Disorders may only be considered to occur at intervals of several sampling periods. In[7] sequential probability ratio tests are used repeatedly to test the hypothesis that a disorder is present. The possibility that the disorder appeared at any time other than the start of one of these tests is ignored.

Simulations carried out on the various approaches suggested in [4] indicates good performance in the particular situations for which they were proposed. Also in [4] the issue of the robustness of these detection rules is indicated as requiring further investigation. In chapters 4 and 5 these aspects are considered.

A simple approach [4, ref 24] to avoiding excessive complexity is to use a single state estimate for all possible disorder times k_j before the current time, based on "steady-state" Kalman filtering for the post-jump system model. This is reported to work well, and seems a natural approach where detection times are typically long compared with the system time constants - an inevitable situation when trying to detect small jumps in parameters without too many false alarms.

In [8] a sub-optimal solution to a problem similar to one considered by Shiryaev [1] (the discrete time version of the situation described in section 2.3) is proposed.

Davis, in [14] looks at a continuous time problem similar to that described in this section. He considers an approximation to the infinite dimensional filtering equations which involves using for $E_t(x_t)$ in equation (2.5.5) the value calculated assuming $t<t_j$. The approximation seems reasonable if it is expected that detection times will be typically small compared with system time constants.

In this chapter, results are given concerning the existence and properties of optimal detection rules in the case of systems with dynamics of the form(2.5.6). In addition, it is shown how the methods of [16] may be used to generate these detection rules.

A natural suboptimal approach is suggested which avoids the need for extensive computation at the design stage. The increase in the expected cost when using this detection rule is discussed.

An approach to the detection of disorders in the more general system(2.5.1) is also suggested. This is related to the methods proposed in [4 ref.24] for discrete time systems.

The problem of detecting parameter jumps to unknown values is considered briefly, and the study in chapters 4 & 5 of the robustness of detection rules designed for known disorders is motivated.

3.1 Optimal detection rules

The first part of this section follows the arguments of section 2.3, but for y_t generated by a more complicated stochastic differential equation. Because of this, π_t is no longer a Markov process. The cost function $K(\tilde{\tau})$ defined in (2.2.2) is used, but Lemma 2.1 relates this to the cost $C(\tilde{\tau})$ when the usual distribution for t_j holds.

In order to show that the optimal stopping time is the first time of entry of the process (π_t, v_t) into a closed set (Theorem 3.1) it is necessary to derive a continuity result. To do this, an approximating problem is considered in which there are only a finite number of possible values for the optimal stopping time. This enables a dynamic programming approach to be used (Lemma 3.3).

Some results needed in later chapters are given in Theorem 3.2. In addition, it is shown in Definition (3.1.19) how if certain conditions are satisfied the process π_t may be replaced by one which is generated by an ordinary differential equation in the definition of the optimal detection rule.

The problem of interest here is that of the system defined in (2.5.6) with a more general a-priori distribution for the jump time t_j. This generalization is useful in chapter 4.

$$dv_t = \begin{bmatrix} J & B \\ D_t & F_t \end{bmatrix} v_t dt + \begin{bmatrix} u \\ z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ I_m \end{bmatrix} dW_t \tag{3.1.1}$$

Observations: $y_t = [0 : I_m] v_t$

v_t is an n dimensional process $(n \geq m)$

J is an $(n-m)\times(n-m)$ constant matrix, B is constant

$D_t = D^0,\ F_t = F^0,\ z_t = z^0$ $(D^0, F^0, z^0$ constant) $\forall t < t_j$

$D_t = D^1,\ F_t = F^1,\ z_t = z^1$ $(D^1, f^1, z^1$ constant) $\forall t \geq t_j$

W_t is an m dimensional Wiener process

u is a constant (n-m) dimensional vector

$t_j \geq 0$ is a random variable such that

$$dI(t \geq t_j) = \rho(v_t)(1 - I(t \geq t_j))dt + dM_t \tag{3.1.2}$$

where M_t is a Martingale orthoganal to W_t and $\rho(.)$ is a bounded non-negative function with bounded derivative.

N.B. Unless otherwise stated it is assumed that $P(t_j = 0 | \mathcal{Y}_0) = 0$. y_0 is assumed given, so that v_t is $\mathcal{Y}_t$-measurable.
The cost function considered is that given in (2.2.2).

i.e. $K(\tilde{\tau}) = -\lambda\tilde{\tau} + (\lambda+c)(\tilde{\tau}-t_j).I(\tilde{\tau}\geq t_j)$ (3.1.3)

$\tilde{\tau}$ a $\mathcal{Y}_t^R$-stopping time.

Define $K_{t_o}(\tilde{\tau}_{t_o}) \triangleq -\lambda(\tilde{\tau}_{t_o}-t_o) + (\lambda+c)(\tilde{\tau}_{t_o}-t_j \vee t_o).I(\tilde{\tau}_{t_o} > t_j)$

$\tilde{\tau}_{t_o} \geq t_o$ a $\mathcal{Y}_t^R$ -stopping time, t_o an arbitrary stopping time. (3.1.4)

Lemma 2.1 shows that a detection rule which is optimal with cost $K(\tilde{\tau})$ is also optimal with cost $C(\tilde{\tau})$ (2.1.1).

Using the non-linear filtering equations (Appendix 1) for $\pi_t = E(I(t\geq t_j)|\mathcal{Y}_t)$ gives (3.1.5)

$$d\pi_t = \rho(v_t)(1-\pi_t)dt + \pi_t(1-\pi_t)\{[D^1-D^0;F^1-F^0]v_t+z^1-z^0\}d\nu_t$$

$$\begin{aligned} \text{where } d\nu_t &= dy_t - \{[D^0F^0]v_t+z^0\}(1-\pi_t)dt \\ &\quad - \{[D^1F^1]v_t+z^1\}\pi_t dt \\ &= dW_t + (I(t\geq t_j)-\pi_t)\{[D^1-D^0:F^1-F^0]v_t+z^1-z^0\}dt \end{aligned}$$ (3.1.6)

ν_t the innovations process is a Wiener process.

<u>Lemma 3.1</u>

$(\pi,v)_t$ is uniquely defined given $(\nu_u;u\leq t)$, $(\pi,v)_o$

$(\pi,v)_t$ is a Feller process and therefore a strong Markov process.

<u>Proof</u>

From (3.1.5)

$$d\begin{bmatrix} v \\ \pi \end{bmatrix}_t = b(\pi_t,v_t)dt + \sigma(\pi_t,v_t)d\nu_t \qquad \text{where } \pi\epsilon[0,1]$$

$$\text{and} \quad b(\pi,v) = \left[\begin{array}{c} \begin{bmatrix} J & B \\ D^0 & F^0 \end{bmatrix} v + \begin{bmatrix} u \\ z^0 \end{bmatrix} + \pi\left\{\begin{bmatrix} 0 & 0 \\ D^1-D^0 & F^1-F^0 \end{bmatrix} v + \begin{bmatrix} 0 \\ z^1-z^0 \end{bmatrix}\right\} \\ \hline \lambda(1-\pi) \end{array}\right]$$

$$\sigma(\pi,v) = \left[\begin{array}{c} 0 \\ I_m \\ \hline \pi(1-\pi)\{[D^1-D^0 : F^1-F^0]v + z^1-z^0\} \end{array}\right]$$

For this proof, if $M \epsilon R^{n\times m}$, $\|M\| \triangleq \sqrt{\left(\sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}^2\right)}$

$\exists K<\infty$ s.t.

$$\| b(\pi,v)\| \le K\left(1+\left\|\begin{bmatrix} v \\ \pi \end{bmatrix}\right\|\right) \quad ; \quad \| \sigma(\pi,v)\| \le K\left(1+\left\|\begin{bmatrix} v \\ \pi \end{bmatrix}\right\|\right)$$

and for any $N>0$, $\exists K_N<\infty$ s.t.

$$\left.\begin{array}{l} \| b(\pi,v) - b(\pi',v')\| \le K_N\left\|\begin{bmatrix} v \\ \pi \end{bmatrix} - \begin{bmatrix} v' \\ \pi' \end{bmatrix}\right\| \\ \| \sigma(\pi,v) - \sigma(\pi',v')\| \le K_N\left\|\begin{bmatrix} v \\ \pi \end{bmatrix} - \begin{bmatrix} v' \\ \pi' \end{bmatrix}\right\| \end{array}\right. \quad \text{for } \left\|\begin{bmatrix} v \\ \pi \end{bmatrix}\right\|, \left\|\begin{bmatrix} v' \\ \pi' \end{bmatrix}\right\| \le N$$

Then [17, Theorem 5.2.2] gives the uniqueness of $(\pi,v)_t$ given $(\nu_u, u\le t)$, $(\pi,v)_0$

[17, Theorem 5.3.6] gives the Feller and Strong Markov property of $(\pi,v)_t$ □

<u>Definition</u>

For an arbitrary stopping time t_0

$$h^*(\pi,v) \triangleq \inf_{\tilde{\tau}_{t_0}\ge 0} E(K_{t_0}(\tilde{\tau}_{t_0})|\pi_{t_0}=\tilde{\pi}, v_{t_0}=\tilde{v}) \qquad (3.1.7)$$

where $\tilde{\tau}_{t_0}$ is a Y_t^R-stopping time

Then $h^*(\pi,v) = \inf_{\tilde{\tau}_{t_0}\ge 0} E(-\lambda(\tilde{\tau}_{t_0}-t_0) + \int_{t_0}^{\tilde{\tau}_{t_0}} (c+\lambda)\pi_u du | Y_{t_0})$

from (3.1.4).

Note that h^* is independent of the value of t_0 chosen.

Define $\tau_{t_0} \triangleq \inf\{t\ge t_0 ; h^*(\pi_t, v_t)\ge 0\}$ (3.1.8)

<u>Lemma 3.2</u>

$$E(K_{t_0}(\tau_{t_0})|Y_{t_0}) \le E(K_{t_0}(\tilde{\tau}_{t_0})|Y_{t_0}) \quad \forall\ Y_t^R\text{-stopping times } \tilde{\tau}_{t_0}\ge t_0 \qquad (3.1.9)$$

Proof (c.f. section 2.3)

$$E(K_{t_o}(\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o}) = E(K_{t_o}(\tilde{\tau}_{t_o}\wedge\tau_{t_o})|\mathcal{Y}_{t_o})$$

$$+ E[E(-\lambda(\tilde{\tau}_{t_o}-\tau_{t_o}) + (\lambda+c)\int_{\tau_{t_o}}^{\tilde{\tau}_{t_o}} \pi_u du|\mathcal{Y}_{\tau_{t_o}}).I(\tau_{t_o}\le\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o}]$$

But $$E[-\lambda(\tilde{\tau}_{t_o}-\tau_{t_o}) + (\lambda+c)\int_{\tau_{t_o}}^{\tilde{\tau}_{t_o}} \pi_u du|\mathcal{Y}_{\tau_{t_o}}] \ge 0 \quad \text{if } \tilde{\tau}_{t_o}\ge\tau_{t_o}$$

by the definition of τ_{t_o} (3.1.8).

Therefore $$E(K_{t_o}(\tilde{\tau}_{t_o}\wedge\tau_{t_o})|\mathcal{Y}_{t_o}) \le E(K_{t_o}(\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o}) \quad (3.1.9)$$

Also, $$E(K_{t_o}(\tilde{\tau}_{t_o}\vee\tau_{t_o})|\mathcal{Y}_{t_o}) = E(K_{t_o}(\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o})$$

$$+ E[E(-\lambda(\tau_{\tilde{\tau}_{t_o}}-\tilde{\tau}_{t_o}) + (\lambda+c)\int_{\tilde{\tau}_{t_o}}^{\tau_{\tilde{\tau}_{t_o}}} \pi_u du|\mathcal{Y}_{\tilde{\tau}_{t_o}}).I(\tau_{t_o}\ge\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o}]$$

But $$E[-\lambda(\tau_{\tilde{\tau}_{t_o}}-\tau_{t_o}) + (\lambda+c)\int_{\tilde{\tau}_{t_o}}^{\tau_{\tilde{\tau}_{t_o}}} \pi_u du|\mathcal{Y}_{\tilde{\tau}_{t_o}}] \le 0 \text{ if } \tau_{t_o}\ge\tilde{\tau}_{t_o}$$

by definition (3.1.8)

So $$E(K_{t_o}(\tilde{\tau}_{t_o}\vee\tau_{t_o})|\mathcal{Y}_{t_o}) \le E(K_{t_o}(\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o}) \quad (3.1.10)$$

Combining (3.1.9) and (3.1.10) (since $\tilde{\tau}_{t_o}$ is an arbitrary stopping time)

$$E(K_{t_o}(\tau_{t_o})|\mathcal{Y}_{t_o}) = E(K_{t_o}(\tau_{t_o}\wedge[\tau_{t_o}\vee\tilde{\tau}_{t_o}])|\mathcal{Y}_{t_o})$$

$$\le E(K_{t_o}(\tilde{\tau}_{t_o})|\mathcal{Y}_{t_o}) \qquad \square$$

It follows from Lemma 3.2 that only non-randomized stopping times need be considered, i.e. τ_{t_o} is a $\mathcal{Y}_t$-stopping time. Since $K(\tilde{\tau})=K_o(\tilde{\tau})$, the optimal stopping time for the cost function (3.1.3) is

$$\tau \triangleq \inf\{t:h^*(\pi_t,v_t)\ge 0\} \quad (3.1.11)$$

It follows from (3.1.8) that $\tau=\tau_{t_o}$ if $\tau\ge t_o$ (3.1.12)

Also $h^*(\tilde{\pi},\tilde{v}) = E(K_{t_o}(\tau_{t_o})|\pi_{t_o}=\tilde{\pi},v_{t_o}=\tilde{v})$ from (3.1.7)

(3.1.13)

Definitions

$$K^N_{t_o}(\tilde{\tau}_{t_o}) \triangleq -\lambda(\tilde{\tau}_{t_o} \wedge N\Delta - t_o) + (\lambda+c)(\tilde{\tau}^+_{t_o} - t^-_j \vee t_o).I(\tilde{\tau}^+_{t_o} > t_j) \tag{3.1.14}$$

where, if $\Lambda \triangleq \{i\Delta : i=0,1,\cdots\infty\}$, $\Delta>0$

$t^+ \triangleq \inf\{u\in\Lambda : u\geq t\}$

$t^- \triangleq t^+ - 1$

$\tilde{\tau}_{t_o} \geq t_o$ is a $\mathcal{Y}^R_{t_o}$-stopping time

λ, c, t_o as before

$$h^N(\tilde{\pi},\tilde{v},i) \triangleq \inf_{\tilde{\tau}_{t_o}\geq t_o} E(K^N_{t_o}(\tilde{\tau}_{t_o}) \mid \pi_{t_o}=\tilde{\pi}, v_{t_o}=\tilde{v}, t_o=i\Delta), \quad i=0,1,2,\cdots \tag{3.1.15}$$

Lemma 3.3

$h^N(\tilde{\pi},\tilde{v},i)$ is continuous in $\tilde{\pi},\tilde{v}$ for each $i=0,1,2,\cdots$

Proof

First, from (3.1.14), $h^N(\tilde{\pi},\tilde{v},i)=0$ for $i\geq N$

In (3.1.15) only stopping times taking values in Λ need be considered since $K^N_{t_o}(\tau^+_{t_o}) \leq K^N_{t_o}(\tau_{t_o})$

It follows that $h^N(\tilde{\pi},\tilde{v},i)=\hat{h}^N(\tilde{\pi},\tilde{v},i)$ defined by $\hat{h}^N(\tilde{\pi},\tilde{v},N)=0$

and
$$\hat{h}^N(\tilde{\pi},\tilde{v},i) = \min\{0, E(\hat{h}^N(\pi_{(i+1)\Delta}, v_{(i+1)\Delta}, i+1) -\lambda+(\lambda+c)\pi_{(i+1)\Delta} \mid \pi_{i\Delta}=\tilde{\pi}, v_{i\Delta}=\tilde{v})\}$$

since otherwise a stopping rule giving lower expected cost is provided by

$$\tau_{t_o} = \inf\{t\in\Lambda : \hat{h}^N(\pi_t, v_t, t/\Delta)\geq 0\}$$

So if $h^N(\tilde{\pi},\tilde{v},i+1)$ is continuous in $\tilde{\pi},\tilde{v}$, so is $h^N(\tilde{\pi},\tilde{v},i)$, using the Feller property (Lemma 3.1). Note that for each i, h^N is bounded above by zero and below by $-N\lambda\Delta$. The required result now follows by induction. □

Lemma 3.4

$h^N(\tilde{\pi},\tilde{v},0) \downarrow h^*(\tilde{\pi},\tilde{v})$ as $N\Delta\to\infty$, $\Delta\to 0$ (3.1.16)

Proof

Firstly, from (3.1.4) & (3.1.14)

$$K_o^N(\tilde{\tau}_o) \geq K_o(\tilde{\tau}_o) \quad \forall\ \mathcal{Y}_t^R\text{-stopping times } \tilde{\tau}_o$$

Next, $K_o^N(\tilde{\tau}_o) = K_o^\infty(\tilde{\tau}_o) + \lambda(\tilde{\tau}_o - \tilde{\tau}_o \wedge (N\Delta))$

and by (3.1.4) & (3.1.14)

$$K_o^\infty(\tilde{\tau}_o) \leq K_o(\tilde{\tau}_o) + 2(\lambda+c)\Delta$$

So
$$E(K_o^N(\tilde{\tau}_o)|\pi_o=\tilde{\pi},v_o=\tilde{v}) \leq E(K_o(\tilde{\tau}_o)|\pi_o=\tilde{\pi},v_o=\tilde{v}) + E(\lambda(\tilde{\tau}_o-\tilde{\tau}_o\wedge(N\Delta))|\pi_o=\tilde{\pi},v_o=\tilde{v}) + 2(\lambda+c)\Delta \quad (3.1.17)$$

Set $\tilde{\tau}_o=\tau_o$ defined in (3.1.8). Note that by optimality of τ_o and since $E(t_j|\pi_o=\tilde{\pi},v_o=\tilde{v})<\infty$,

$$E(\lambda(\tau_o-\tau_o\wedge(N\Delta))|\pi_o=\tilde{\pi},v_o=\tilde{v}) \downarrow 0 \text{ as } N\Delta\to\infty$$

Therefore from (3.1.17)

$$E(K_o^N(\tau_o)|\pi_o=\tilde{\pi},v_o=\tilde{v}) \downarrow E(K_o(\tau_o)|\pi_o=\tilde{\pi},v_o=\tilde{v})$$

as $N\Delta\to\infty$, $\Delta\to 0$.

(3.1.16) follows from the definition of h^N and (3.1.13). □

Theorem 3.1

The set $\{(\pi,v):h^*(\pi,v)\geq 0\}$ is closed.

Proof

Suppose $(\pi^i,v^i)\in\{(\pi,v):h^*(\pi,v)\geq 0\}$ $\forall i\in\mathcal{N}^+$

and (π^i,v^i) has limit point $(\bar{\pi},\bar{v})$.

Then Lemma 3.3 implies that $h^N(\bar{\pi},\bar{v},0)\geq 0$ $\forall N,\Delta$

But from Lemma 3.4

$$h^N(\bar{\pi},\bar{v},0) \to h^*(\bar{\pi},\bar{v}) \quad \text{as } N\Delta\to\infty,\ \Delta\to 0$$

establishing the theorem. □

Definitions

$R_t \triangleq \ln(\pi_t/(1-\pi_t))$ (3.1.18)

Suppose (F^1-F^0) is symmetric in (3.1.1) and let

$x_t \triangleq [I_{n-m}:0]v_t, \quad y_t \triangleq [0:I_m]v_t$

In this case,

$$S_t \triangleq \ln\left[\frac{\pi_t}{1-\pi_t}\right] - y_t^T(D^1-D^0)x_t - \tfrac{1}{2}y_t^T(F^1-F^0)y_t$$
$$-y_t^T(z^1-z^0) \qquad (3.1.19)$$

Using Itô's differentiation rule gives

$$\frac{dS_t}{dt} = \frac{\lambda}{\pi_t} - \tfrac{1}{2}g_t^T g_t - g_t^T\{[D^0:F^0]v_t + z^0\} - \tfrac{1}{2}\sum_{i=1}^{m}(F^1-F^0)_{ii} \qquad (3.1.20)$$

where $g_t \triangleq [D^1-D^0:F^1-F^0]v_t + z^1-z^0$ (3.1.21)

Since there is a one to one correspondance between (S,v) and (π,v) under which any solution of (3.1.6) & (3.1.19) is mapped into a solution of (3.1.6) & (3.1.5), it follows from Lemma 3.1 that (3.1.6) & (3.1.19) has a unique solution.

This provides a simpler implementation for a stopping rule, since no stochastic integral need be evaluated to obtain S. To avoid handling infinite initial values ($\pi_o=0 \Rightarrow S_o=-\infty$) the process $U_t \triangleq 1/(1+e^{-S_t})$ could be used instead of S_t. Note that if m=1, F^1-F^0 is trivially symmetric. This condition will also be satisfied in other problems considered later. If F^1-F^0 is not symmetric it is not in general possible to make a transformation of this sort.

If S_t is defined

$$h(S,\tilde{v}) \triangleq h^*(\pi(S,\tilde{v}),\tilde{v}) \qquad (3.1.22)$$

where $\pi(S,\tilde{v})$ is defined so that $\pi_t=\pi(S_t,v_t)$ (see(3.1.19)) i.e.

$$\pi(\delta,\tilde{v}) = \frac{1}{1 + \exp[-\delta-\tilde{y}^T(D^1-D^0)\tilde{x}-\frac{1}{2}\tilde{y}^T(F^1-F^0)\tilde{y}-\tilde{y}^T(z^1-z^0)]}$$

where $\tilde{x}=[I_{n-m}:0]\tilde{v}$; $\tilde{y}=[0:I_m]\tilde{v}$ (3.1.23)

Theorem 3.2

$h^*(\pi,v)$ is a non-decreasing function of π for fixed v.

$h^*(\pi,v)$ is continuous for fixed v (except possibly at $\pi=0$)

Proof

Consider an arbitrary fixed value of v, $\tilde{v}$, and stopping time t_o

Let
$$\Phi_{\tilde{\pi}} = E(K_{t_o}(\tau_{t_o})|t_j\le t_o,\pi_{t_o}=\tilde{\pi},v_{t_o}=\tilde{v})$$
$$\Psi_{\tilde{\pi}} = E(K_{t_o}(\tau_{t_o})|t_j> t_o,\pi_{t_o}=\tilde{\pi},v_{t_o}=\tilde{v}) \qquad (3.1.24)$$

i.e. $\Phi_{\tilde{\pi}}$ is the expected cost of using the policy $P:(v_u,u\ge t_o) \to \tau^P_{t_o}$ (see section 2.1) which is optimal if $\pi_{t_o}=\tilde{\pi}$, conditioned on $(t_j\le t_o)$, while $\Psi_{\tilde{\pi}}$ is that conditioned on $(t_j>t_o)$.

Then $h^*(\tilde{\pi},v) = \Phi_{\tilde{\pi}}.\tilde{\pi} + \Psi_{\tilde{\pi}}.(1-\tilde{\pi})$ (3.1.25)

Let $\hat{h}^*_{\tilde{\pi}}(\hat{\pi},\tilde{v})$ be the expected cost of using this policy if in fact $\pi_{t_o}=\hat{\pi}$

$$h^*(\hat{\pi},\tilde{v}) \triangleq E(K_{t_o}(\tau^P_{t_o})|\pi_{t_o}=\hat{\pi},v_{t_o}=\tilde{v})$$
$$= \Phi_{\tilde{\pi}}.\hat{\pi} + \Psi_{\tilde{\pi}}.(1-\hat{\pi}) \qquad (3.1.26)$$

By optimality $h^*(\pi,\tilde{v}) \le \hat{h}^*_{\tilde{\pi}}(\pi,\tilde{v}) \quad \forall\pi\in[0,1]$ (3.1.27)

Also $h^*(\tilde{\pi},\tilde{v}) = \hat{h}^*_{\tilde{\pi}}(\tilde{\pi},\tilde{v}) \le 0$ as $K_{t_o}(t_o)=0$

and $\Phi_{\tilde{\pi}}\ge 0$ since $t_j\le t_o \Rightarrow K_{t_o}(\tilde{\tau})\ge 0 \quad \forall\tilde{\tau}\ge t_o$ (see (3.1.4))

This implies (see Figure 3.1.1) that

$$h^*(\pi,\tilde{v}) \le h^*(\tilde{\pi},\tilde{v}) \quad \forall\pi<\tilde{\pi}$$

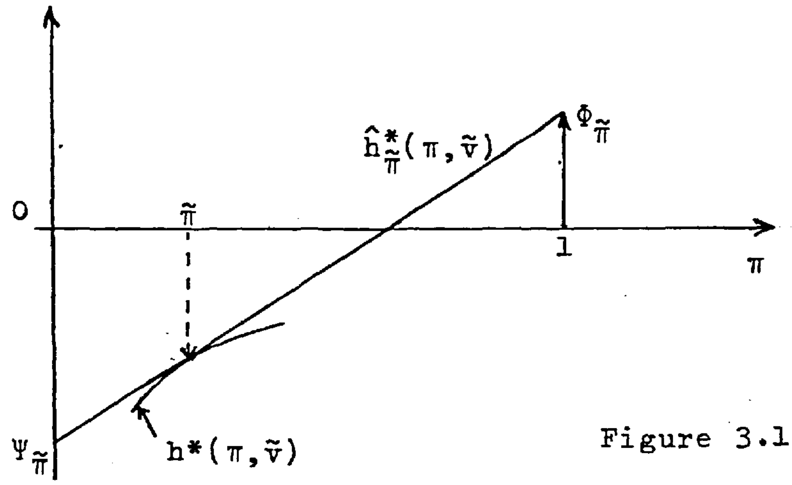


Figure 3.1.1

So $h^*(\pi,\tilde{v})$ is non-increasing with <u>decreasing</u> π at $\pi=\tilde{\pi}$. Since $\tilde{\pi}$ is arbitrary, $h^*(\pi,\tilde{v})$ is non-increasing with <u>decreasing</u> π $\forall \pi\epsilon[0,1]$. This proves the first part of the theorem.

Next, suppose $h^*(\pi,\tilde{v})$ is discontinuous in π for some $\tilde{v}$ and $\pi>0$. Then $\exists\ \pi_1,\pi_2>0$ such that

$$h^*(\pi_1,\tilde{v}) > h^*(\pi_2,\tilde{v}) + \delta$$

for some $\delta>0$ (fixed) where π_1,π_2 may be chosen such that $|\pi_1 - \pi_2|<\varepsilon$ for any $\varepsilon>0$. (3.1.28)

Since $h^*(\pi_2,\tilde{v}) = \hat{h}^*_{\pi_2}(\pi_2,\tilde{v})$

$$h^*(\pi_1,\tilde{v}) > \hat{h}^*_{\pi_2}(\pi_2,\tilde{v}) + \delta \qquad (3.1.29)$$

Choose π' s.t. $0<\pi'\leq\min(\pi_1,\pi_2)$

From Figure 3.1.1, $\Phi_{\pi'}\geq \Phi_{\pi_2}$, $\Psi_{\pi'}\leq \Psi_{\pi_2}$ $\forall\pi'\leq\pi_2$

because $\hat{h}^*_{\pi'}(\pi_2,\tilde{v}) \geq h^*(\pi_2,\tilde{v})$

So $$0 \leq \frac{d\hat{h}^*_{\pi_2}(\pi,\tilde{v})}{d\pi} \leq \Phi_{\pi'} - \Psi_{\pi'} < \infty \qquad \forall\pi_2\geq\pi'>0$$

Therefore

$$h^*(\pi_1,\tilde{v}) \leq \hat{h}^*_{\pi_2}(\pi_1,\tilde{v}) \leq \hat{h}^*_{\pi_2}(\pi_2,\tilde{v}) + \varepsilon.(\Phi_{\pi'}-\Psi_{\pi'})$$

for $\pi_2,\pi_1\geq\pi'$, from (3.1.27) & (3.1.28)

Comparing this with (3.1.29) gives a contradiction since ε is arbirarily small while $\delta>0$ is fixed. So $h^*(\pi,\tilde{v})$ is continuous in π for $\pi>0$. □

Corollary 3.2.1

From Theorem 3.2, $\exists$ a function $\pi_\gamma(v)$, $v\in R^n$ s.t.

$$h^*(\pi,v) \geq 0 \quad \forall \pi\geq\pi_\gamma(v)$$

Therefore $$\tau_{t_o} = \inf\{t\geq t_o:\pi_t\geq\pi_\gamma(v_t)\} \tag{3.1.30}$$

□

Corollary 3.2.2

When S_t is defined, $h(S,v)$ is a non-decreasing function of S for fixed v.

$h(S,v)$ is continuous in S for fixed v (except possibly at $S=-\infty$).

$$\tau_{t_o} = \inf\{t\geq t_o:h(S_t,v_t)\geq 0\}$$

$$= \inf\{t\geq t_o:S_t\geq S_\gamma(v_t)\} \tag{3.1.31}$$

where $S_\gamma(\tilde{v})$ is defined so that $\pi_\gamma(\tilde{v})=\pi(S_\gamma(\tilde{v}),\tilde{v})$ □

Definition

The stopping boundary γ^R is defined as

$$\gamma^R \triangleq \{(R,v):h^*(\frac{1}{1+\exp(-R)},v)\geq 0\} \cap \tag{3.1.32}$$

$$\text{closure}\{(R,v):h^*(\frac{1}{1+\exp(-R)},v)<0\}$$

and if S_t is defined (3.1.33)

$$\gamma^S \triangleq \{(S,v):\ h(S,v)\geq 0\} \cap \text{closure}\{(S,v):h(S,v)<0\}$$

The superscripts are usually ommited as the appropriate definition is clear.

Note that

$$\tau_{t_o} = \inf\{t\geq t_o:(R,v)_t\in\gamma^R\}$$

$$\text{if } R_{t_o}\leq R_\gamma(v_{t_o}) = \ln(\pi_\gamma(v_{t_o})/(1-\pi_\gamma(v_{t_o})))$$

where R_t is defined by (3.1.18). A similar result applies for γ^S.

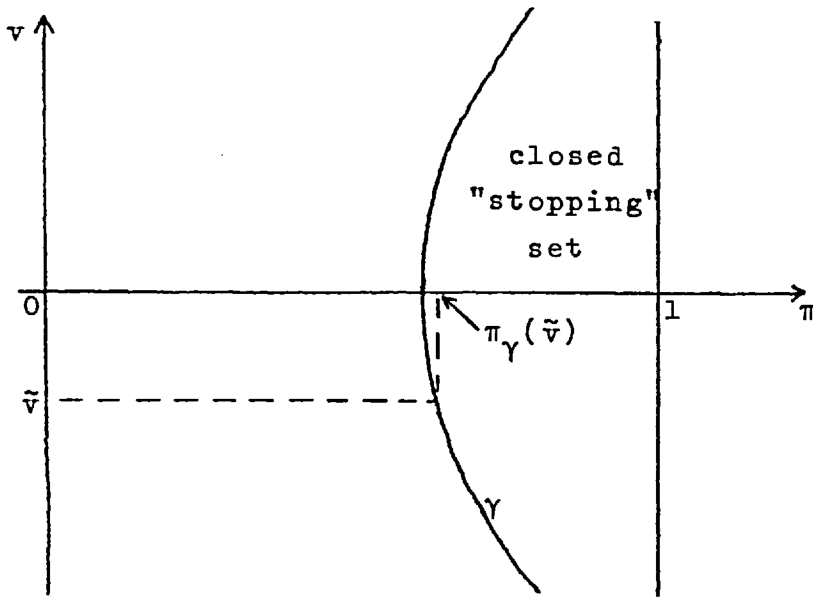


Figure 3.1.2
1-dimensional
example

3.2 Determination of the stopping boundary

A natural approach to the determination of the stopping boundary γ would be to consider a sequence of approximations of the form (3.1.14). It follows from Lemma 3.4 that

$$h^N(\pi,v,0) \downarrow h^*(\pi,v) \quad \text{as } N\Delta\to\infty, \Delta\to 0.$$

Evaluation of $h^*(\pi,v)$ would enable the "stopping set" to be determined using the equation (3.1.11). Difficulties might arise however in the solution of the approximating problems by the dynamic programming approach of Lemma 3.3. Firstly it is not clear how best to construct a grid of points in the state space of $(\pi,v)_t$ so that an approximating finite state process may be constructed. Secondly, a rigorous proof of the convergence of the solutions as the grid size is reduced might be complicated. Thirdly, a great deal of computation would be involved, as a two stage approximation is used.

In [16] a more direct approach is proposed to the solution of optimal stopping problems. This involves the solution of corresponding problems for an approximating sequence of finite-state Markov processes. However in this case the time between successive state transitions of the approximating process varies dependent on its current state. In this way the first two difficulties mentioned above are overcome, and while the problem still requires considerable computation, one stage of approximation is avoided. Certain conditions do need to be satisfied, but this is possible at least when S_t is defined (see (3.1.19)).

For the remainder of this chapter $\rho(v)$ is set equal to λ.

<u>The approximation</u>

Let the process $X_t \in R^{n+1}$ $\forall t$ satisfy

$$dX_t = f(X_t)dt + \sigma(X_t)dV_t \tag{3.2.1}$$

where V_t is m dimensional Wiener process and $f(\cdot)$, $\sigma(\cdot)$ are R^{n+1} and $(n+1)\times m$ matrix valued functions on R^{n+1}, respectively, satisfying the uniform Lipschitz condition

$$\begin{aligned} &\|f(X)\| \le K(1+\|X\|), && \|\sigma(X)\| \le K(1+\|X\|) \\ &\|f(X)-f(X')\| \le K\|X-X'\|, && \|\sigma(X)-\sigma(X')\| \le K\|X-X'\| \end{aligned} \tag{3.2.2}$$

for some $K<\infty$, where $\|\sigma(\cdot)\|$ is defined

by $$\|\sigma(x)\| \triangleq \surd\left(\sum_{i=1}^{n+1} \sum_{j=1}^{m} \sigma(x)_{ij}^2\right)$$

Let $a(\cdot)=\sigma(\cdot)\sigma(\cdot)^T$, and suppose that (3.2.3)

$$a_{ii}(x) \ge \sum_{\substack{j=1 \\ j\neq i}}^{n+1} |a_{ij}(x)| \qquad \forall x,\ i=1,2,\cdots,n+1$$

Let $k(\cdot)$ and $b(\cdot)$ be bounded continuous real valued functions on R^n and

$$k(x) \ge k_o \qquad \text{for some } k_o>0 \tag{3.2.4}$$

Define $$R(x,\tilde{\tau}) = E_x[\int_o^{\tilde{\tau}} k(X_s)ds + b(X_{\tilde{\tau}})] \tag{3.2.5}$$

where $E_x(\cdot)=E(\cdot|X_o=x)$, and $\tilde{\tau}$ is a stopping time of the σ-field generated by $(X_s : s \le t)$ and, possibly, other random variables independent of V_s $\forall s$.

Also, $$E_x(\tilde{\tau}) < \infty \qquad (3.2.6)$$

Now let ξ_i^h, $h>0$ be a Markov chain with state-space

$$\{x \in R^{n+1} : x = \sum_{i=1}^{n+1} j(i).he_i\} \qquad h>0$$

$\{e_i\}$ an orthonormal basis for R^{n+1}, $j(i)$ integer for $i=1,\cdots n+1$

$$P(\xi_{i+1}^h = y \mid \xi_i^h = x) = p^h(x,y) = \bar{Q}_h(x,y)/Q_h(x) \qquad (3.2.7)$$

where $$\bar{Q}_h(x, x\pm he_i) = a_{ii}(x) - \sum_{\substack{j=1 \\ j\neq i}}^{n+1} |a_{ij}(x)| + h.f^{\pm}(x)$$

$$\left.\begin{aligned} \bar{Q}_h(x, x+he_i \pm he_j) &= a_{ij}^{\pm}(x), \quad i\neq j \\ \bar{Q}_h(x, x-he_i \pm he_j) &= a_{ij}^{\mp}(x), \quad i\neq j \end{aligned}\right\} \qquad (3.2.8)$$

$$\bar{Q}_h(x,y) = 0 \text{ for other } y$$

$$Q_h(x) = 2\sum_i a_{ii}(x) - \sum_{\substack{i,j \\ i\neq j}} |a_{ij}(x)| + h\sum_i |f_i(x)| \qquad (3.2.9)$$

Here for $r \in R$, $r^+ = r.I(r>0)$, $r^- = -r.I(r<0)$

$$\Delta t^h(x) \triangleq h^2/Q_h(x) \qquad (3.2.10)$$

If ℓ^h is a (integer valued) stopping time for the σ-field generated by $(\xi_j, j\le i)$ and, possibly other random variables independent of ξ_j $\forall j$, and $E_x(\ell^h) < \infty$ $\forall x \in R^{n+1}$ where $E_x(\cdot) = E(\cdot \mid \xi_o = x)$

then $$R^h(x,\ell^h) \triangleq E_x\Big[\sum_{i=0}^{\ell^h - 1} k(\xi_i^h).\Delta t^h(\xi_i^h) + b(\xi_{\ell^h}^h)\Big] \qquad (3.2.11)$$

$$V^h(x) \triangleq \inf_{\ell^h} R^h(x,\ell^h) \qquad (3.2.12)$$

<u>Theorem 3.3</u>

For each $x \in R^{n+1}$, $V^h(x) \downarrow \inf_{\tilde{\tau}} R(x,\tilde{\tau})$ as $h \downarrow 0$

where the infimum is over all stopping times satisfying (3.2.6).

This result is part of [16,Theorem 8.2.4]. □

To apply this result to the detection problem in the case where S_t is defined (see (3.1.19)), X_t is identified with $\begin{bmatrix} v \\ U \end{bmatrix}_t$, where $U_t = \frac{1}{1+\exp(-S_t)}$, x with $\begin{bmatrix} v \\ U \end{bmatrix}$.

Then

$$\frac{dU_t}{dt} = U_t(1-U_t)[\frac{\lambda}{\pi_t} - \tfrac{1}{2}g_t^T g_t - g_t^T([D^0:F^0]v_t+z^0) \\ \tfrac{1}{2}\sum_{i=1}^{m}(F^1-F^0)_{ii}] \qquad (3.2.13)$$

with g_t defined in (3.1.21). $\frac{dU_t}{dt}=0$ if $\pi_t=0$.

Equations (3.1.1) and (3.2.13) do not have the uniform Lipschitz property, but if $r<\infty$, K may be found so (3.2.2) does hold if $\|v\|,\|v'\|\le 2r$.

Since it is in any case necessary to bound the state-space of the process X_t in some way so that V^h in Theorem 3.3 may be evaluated, an arbitrary modification to (3.1.1) may be made for $\|v\|> 2r$ so that (3.2.2) holds.

Set $\quad k(X) = \alpha\lambda + (c-\alpha\lambda)\pi \qquad (3.2.14)$

and $\quad b(X) = (1+\alpha)(1-\pi)(1-I(\|v\|\ge r)\frac{(\|v\|-r)\wedge\delta}{\delta}) \qquad 0<\delta\le r \qquad (3.2.15)$

where $\quad \pi = \pi(\ln(U/(1-U)),v) \quad$ (see(3.1.23))

and $\quad \alpha\in(0,c/\lambda)$ so that (3.2.4) is satisfied.

Theorem 3.3 now states that $V^h(x)\to\inf_{\tilde{\tau}} R(x,\tilde{\tau})$ at each point in the state space of ξ_i^h defined by (3.2.7). The restriction (3.2.6) is unimportant since stopping times for X_t having infinite expectation are trivially non-optimal. Since $b(X)=0$ if $\|v\|= r+\delta$, the process ξ_i^h will stop before leaving the set on which $\|v\|\le r+\delta$ if the optimal rule is used, so that $V^h(x)$ may be evaluated over only a finite number of values of x.

From (3.1.2)

$$E_x(\pi_{\tilde{\tau}}-\pi_o) = E_x(\lambda\tilde{\tau}-\lambda\int_o^{\tilde{\tau}} \pi_s ds)$$

for any $\mathcal{Y}_t$-stopping time $\tau\geq 0$

Then from (3.2.5) it follows by addition that

$$R(x,\tilde{\tau}) = E_x[-\lambda\tilde{\tau} + (\lambda+c)\int_o^{\tilde{\tau}} \pi_u du - q(\pi_{\tilde{\tau}},v_{\tilde{\tau}})] + (1+\alpha)(1-\pi_o) \quad (3.2.16)$$

where $q(\pi,v) \triangleq I(\|v\|\geq r)\frac{(\|v\|-r)\wedge\delta}{\delta}(1-\pi_o)$

Therefore

$$V^h(x) - (1+\alpha)(1-\pi_o) \to \inf_{\tilde{\tau}} E_x[K(\tilde{\tau})-q(\pi_{\tilde{\tau}},v_{\tilde{\tau}})] \text{ as } h\downarrow 0 \quad (3.2.17)$$

Now a) $0\leq q(\pi,v)\leq 1 \quad \forall\pi,v$

b) $q(\pi_{\tilde{\tau}},v_{\tilde{\tau}})=0 \quad \text{if } \|v_{\tilde{\tau}}\|\leq r$

If $P_x(\|v_{\tilde{\tau}}\|>r) \not\to 0$ as $r\to\infty$, then $E_x(\tilde{\tau})\to\infty$ as $r\to\infty$. From (3.1.3) this implies that $E_x(K(\tilde{\tau}))\to\infty$, so that a better stopping time exists in the infimum of (3.2.17). Hence only $\tilde{\tau}$ such that

c) $P_x(\|v_{\tilde{\tau}}\|>r)\to 0 \quad \text{as } r\to\infty$

need be considered.

From (a),(b)&(c) it follows that

$$\inf_{\tilde{\tau}} E_x[K(\tilde{\tau})-q(\pi_{\tilde{\tau}},v_{\tilde{\tau}})] \to \inf_{\tilde{\tau}} E_x K(\tilde{\tau}) = h^*(\pi_o,v_o)\Bigg|_{\begin{bmatrix}v\\U\end{bmatrix}_o=x}$$

$$\text{as } r\to\infty \quad (3.2.18)$$

$V^h(x)$ may be evaluated by dynamic programming, assuming an artificial horizon.

Define $V^h(x,N) = b(x)$

$$V^h(x,i) = \min\{[\sum_y p^h(x,y)V^h(y,i+1) + k(x)\Delta t^h(x)],b(x)\} \qquad i=0,1,\cdots,N$$

then $V^h(x,0) \to V^h(x)$ as $N\to\infty$

This is discussed, for example, in [18,chapter 7].

Finally, note that it is stated in [16] that the above results hold if $b(x)$ is replaced by

$$(1+\alpha)(1-\pi)$$

and the process is forced to stop at t if $\|v_t\|=r$. This seems a more natural approach as $h^*(\pi,v)$ is likely to be nearer $1-\pi$ than zero. The requirement that $\alpha>0$ in (3.2.14) and (3.2.15) is probably unnecessary in this application since in any case the optimal stopping time has finite expectation (see [16]). However this is not proved.

Once the function $h^*(\pi,v)$ has been evaluated in this way, the stopping boundary γ may be identified by making the appropriate co-ordiate changes and using the definition (3.1.32) or (3.1.33).

Although it has not been explicitly assumed ,the system (3.1.1) would need to be stable at all times (eigenvalues of $\begin{bmatrix} J & B \\ D_t & F_t \end{bmatrix}$ strictly negative $\forall t$) to avoid the need to consider large values of r.

Remark

If the formulation (2.2.7)is used, with, say, v_{τ^i} reset to zero for $i=1,\cdots,N$ so that the conditions of Lemma 2.2 are satisfied, the optimal detection rule could be obtained as above, with c determined itteratively as the solution of

$$cE(C(\tau)|\pi_o=0,v_o=0) = d \qquad (3.2.19)$$

where $$E(C(\tau)|\pi_o=0,v_o=0) = h^*(0,0) + 1$$

from Lemma 2.1.

As the expectation in (3.2.19) is non-decreasing with c, from (2.2.1), this equation has a unique solution.

Alternatively a more direct approach might be considered, in which a finite state version of this formulation is constructed. [3] considers this problem with observations of the simpler form (2.3.1).

In practice, the requirement that v_t be reset at each false alarm time is unlikely to be important if typical inter-alarm times are long compared to the system time constants. In that case the effect of these "initial conditions" of $(\pi,v)_t$ would usually become insignificant before the stopping boundary was approached.

Examples

In the folowing examples α in (3.2.14), (3.2.15) was taken to be zero (see comments above). By making suitable transformations to the state space of $(\pi,v)_t$ a more flexible grid system was used. Forced stopping was employed for $\|v_t\| \geq r$. In each case the effect of this on the stopping boundary shape was checked by considering both the case in which terminal cost zero and $1-\pi$ is paid if the process reaches this boundary before γ. The estimates of π_γ obtained in this way are upper and lower bounds respectively for that which would be obtained without this artificial boundary. In the examples here, the same stopping boundary is obtained in both cases.

The system considered was

$$\begin{aligned} dy_t &= ay_t dt + dW_t \qquad t<t_j \\ dy_t &= by_t dt + dW_t \qquad t\geq t_j \\ P(t\geq t_j|\mathcal{Y}_o) &= 1-e^{-\lambda t} \\ t_j &\text{ independent of } W_t \end{aligned} \tag{3.2.20}$$

In an attempt to reduce the computation required a two stage procedure was used. Initial itterations use a coarse grid size, and then the spacing of the grid points is halved and further itterations carried out.

EXAMPLE 1

$a=-1$, $b=0$, $\lambda=0.01$, $c=0.1$

200 itterations with coarse grid

200 itterations with full grid

```
  y                                                      πγ(y)
6.00  • γ • • • • • • • • • • • • • • • • • • •          1.00
5.66  • γ • • • • • • • • • • • • • • • • • • •          1.00
5.32  • γ • • • • • • • • • • • • • • • • • • •          1.00
5.00  • γ • • • • • • • • • • • • • • • • • • •          1.00
4.69  • γ • • • • • • • • • • • • • • • • • • •          1.00
4.39  • γ • • • • • • • • • • • • • • • • • • •          1.00
4.11  • γ • • • • • • • • • • • • • • • • • • •          1.00
3.83  • γ • • • • • • • • • • • • • • • • • • •          0.99
3.56  • γ • • • • • • • • • • • • • • • • • • •          0.97
3.29  • γ • • • • • • • • • • • • • • • • • • •          0.92
3.04  • γ • • • • • • • • • • • • • • • • • • •          0.84
2.79  • • • γ • • • • • • • • • • • • • • • • •          0.90
2.55  • • • • γ • • • • • • • • • • • • • • • •          0.87
2.32  • • • • • γ • • • • • • • • • • • • • • •          0.83
2.09  • • • • • • • γ • • • • • • • • • • • • •          0.83
1.86  • • • • • • • • γ • • • • • • • • • • • •          0.79
1.64  • • • • • • • • • γ • • • • • • • • • • •          0.76
1.43  • • • • • • • • • • γ • • • • • • • • • •          0.74
1.22  • • • • • • • • • • γ • • • • • • • • • •          0.68
1.01  • • • • • • • • • • • γ • • • • • • • • •          0.67
0.81  • • • • • • • • • • • γ • • • • • • • • •          0.63
0.60  • • • • • • • • • • • γ • • • • • • • • •          0.59
0.40  • • • • • • • • • • • γ • • • • • • • • •          0.57
0.20  • • • • • • • • • • • γ • • • • • • • • •          0.55
0.00  • • • • • • • • • • • γ • • • • • • • • •          0.55
      0       0.2     0.4     0.6     0.8     1.0
                                          U →
```

Figure 3.2.1

Note that the stopping boundary is symmetric about the $y=0$ axis so only positive y need be considered.

EXAMPLE	a	b	λ	c	"coarse itterations"	"fine itterations"
2	-1	0.8	0.01	0.1	180	160
3	-1	1.3	0.01	0.1	180	160
4	-1	2.0	0.01	0.1	360	160
5	-1	1.3	0.005	0.05	500	300

Below the numbers 2 to 5 are used to mark points on the corresponding stopping boundary. In case of co-incidence the lowest number is shown.

```
  y
5.34   . 2 . . . . . . . . . . . . . . . . . . 3
5.03   . 2                                     3
4.73   . 2                                     3
4.45   .   2                                   3
4.17   .   2                                   3
3.91   .   2                                   3
3.66   .   2                                   3
3.41   .   2                                 3 4
3.18   .   2                               3 5 4
2.95   .     2                         3   5   4
2.74   .     2                     3   5       4
2.53   .     2                 3   5           4
2.33   .     2               3   5             4
2.13   .     2           3   5                 4
1.94   .     2         3   5                   4
1.76   .     2       3   5                     4
1.58   .     2     3   5                   4   .
1.41   .     2     3 5                   4     .
1.24   .     2   3 5                 4         .
1.08   .     2   3 5             4             .
0.92   .     2 3 5             4               .
0.76   .     2 3 5         4                   .
0.61   .     2 3 5       4                     .
0.45   .     2 5       4                       .
0.30   .     2 5     4                         .
0.15   .     2 5   4                           .
0.00   . . . 2 5 . 4 . . . . . . . . . . . . . .

       0                         U →           1
```

Figure 3.2.2

The relationship between a,b and the stopping boundary shape is further investigated in Chapter 4.

Note that the relationship between U and π is not the same for each of the examples above: however it is the same for examples 3 and 5 since a,b have the same values in this case. The effect of reducing λ and c while keeping their ratio unchanged is a slight shift of the boundary to the right.

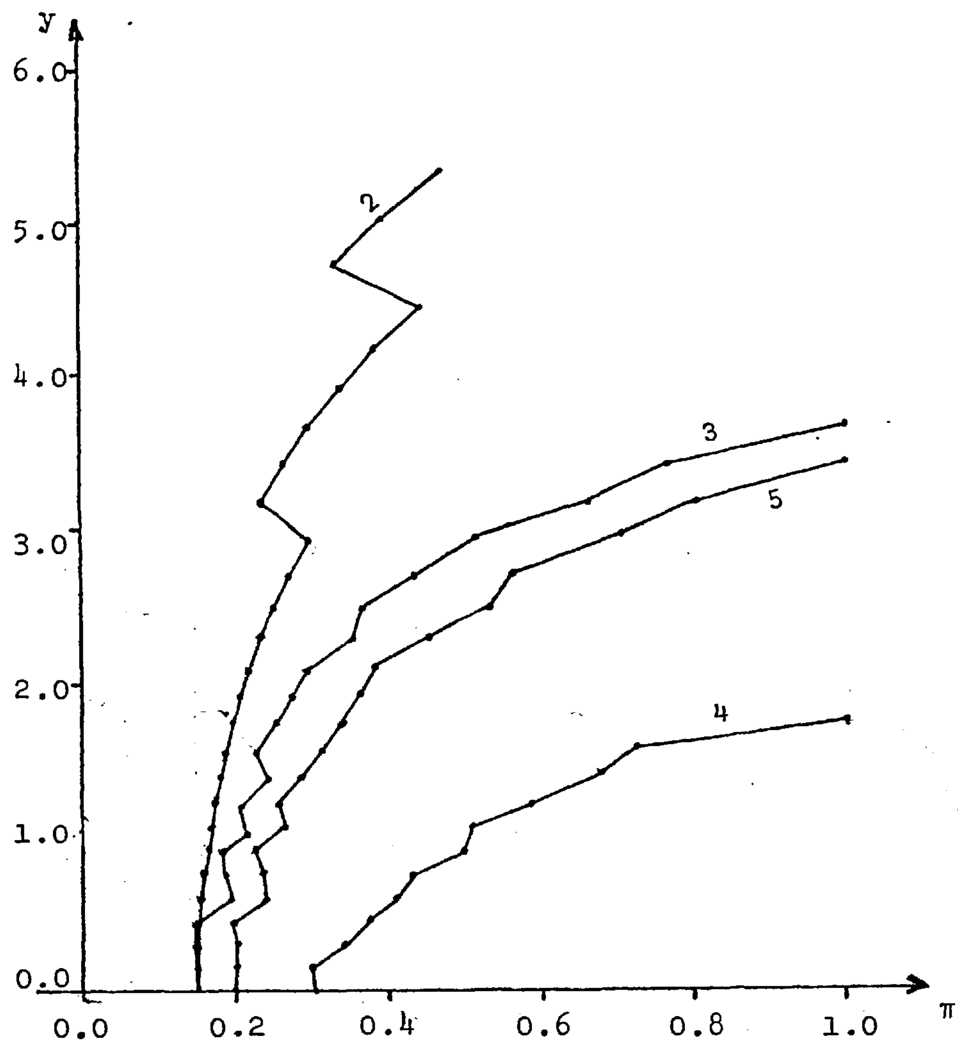


Figure 3.2.3

In Figure 3.2.3 the stopping boundaries are plotted in π, y space. For large y the discretization of U becomes a problem, especially in the case of example 2. Note that for the three largest values of y considered the estimate of π_γ in this case takes its smallest possible non-zero value. However, the standard deviation of y_t is no more than 0.625 both before and after the jump so that this may not be too important.

3.3 Simplified detection rules

As is clear from the previous section, the determination of the optimal stopping boundary γ involves considerable computation, especially for systems of high order. It seems worthwhile therefore to consider the performance of a class of simpler detection rules, for example

$$\tilde{\tau} = \inf\{t:\pi_t \geq \hat{\pi}\} \quad \text{for some } \hat{\pi}\in(0,1) \qquad (3.3.1)$$

Unfortunately no concrete results could be obtained for this problem. The reason why, in general, the optimal stopping rule is not of the form (3.3.1) is that the ammount of information about $I(t\geq t_j)$ given by observations (3.1.1) at time t depends on the value of v_t. If the value of v_t is such that little new information is expected to be available in the near future it is more attractive to stop immediately than otherwise. If considerable information is expected, the possibility of incurring delay costs while waiting would be more acceptable.

Values of v_t much more than the slowest system time constant in the future are largely independent of the current value. If the "jump" in the parameters which is to be detected is small so that typically much longer periods of observation are needed to detect it, use of a stopping rule from the class (3.3.1) should be possible without a large increase in expected cost.

3.4 Detection schemes for general systems

In section 2.5 the following problem was introduced.

$$dx_t = A_t x_t dt + q_t dt + G_t dV_t \tag{3.4.1}$$

$$dy_t = H_t x_t dt + dZ_t \tag{3.4.2}$$

where $x_t \in R^N$, $y_t \in R^m$ $\forall t$

V_t, Z_t are independent Wiener processes, independent of t_j

$P(t \geq t_j) = 1-e^{-\lambda t}$, t_j independent of x_o, y_o

$A_t=A^0$, $q_t=q^0$, $G_t=G^0$, $H_t=H^0$ $\quad \forall t<t_j$

$A_t=A^1$, $q_t=q^1$, $G =G^1$, $H_t=H^1$ $\quad \forall t \geq t_j$

where $A^0, q^0, G^0, H^0, A^1, q^1, G^1, H^1$ are constant matrices and vectors.

A^0, A^1 have strictly negative eigenvalues

As discussed in chapter 2, it is not in general possible to generate $\pi_t = P(t \geq t_j | \mathcal{Y}_t)$ with a finite dimensional filter, and so there is no realizable optimal detection rule.

A natural sub-optimal approach is given here, following the discrete time versions suggested by Chien [4,ref 24]. This involves the use of a "steady-state Kalman filter" designed for the system (3.4.1),(3.4.2) with post-jump $(A_t=A^1, q_t=q^1, G_t=G^1, H_t=H^1)$ parameters.

Suppose that an a-priori distribution for x_o is given, $x_o \sim N(\hat{x}_o, Q_o)$.

Define $\hat{x}_t^i$ as the Kalman filter estimate of x_t for the system

$$\begin{aligned} dx_t &= A^i x_t dt + q^i dt + G^i dV_t \\ dy_t &= H^i x_t dt + dZ_t \end{aligned} \tag{3.4.3}$$

where $x_o \sim N(\hat{x}_o, Q^0)$ if i=0; $x_o \sim N(\hat{x}_o, Q^1)$ if i=1

and Q^i is the "steady-state" error covariance matrix associated with the estimate $\hat{x}^i_t$, i.e. it is the unique positive semi-definite solution of

$$0 = G^i G^{i^T} - Q^i H^{i^T} H^i Q^i + A^i Q^i + Q^i A^{i^T} \qquad (3.4.4)$$

Then

$$\begin{aligned} d\hat{x}^i_t &= (A^i - Q^i H^{i^T} H^i)\hat{x}^i_t dt + q^i dt + Q^i H^{i^T} dy_t \\ \hat{x}^i_o &= \hat{x}_o \qquad \text{for } i=0,1 \end{aligned} \qquad (3.4.5)$$

Note $A^i - Q^i H^{i^T} H^i$ has strictly negative eigenvalues for i=0,1. This is because A^i has this property (see e.g.[21, Chapter 12]).

If r_t denotes the Kalman filter estimate of x_t when t_j is known a-priori, W_t is a Wiener process in the equation

$$dy_t = I(t<t_j)H^0 r_t dt + I(t \geq t_j)H^1 r_t dt + dW_t \qquad (3.4.6)$$

Now suppose that instead of (3.4.2), y_t is generated by

$$dy_t = I(t<t_j)H^0 \hat{x}^0_t dt + I(t \geq t_j)H^1 \hat{x}^1_t dt + dW_t \qquad (3.4.7)$$

where $\hat{x}^0_t, \hat{x}^1_t$ satisfy (3.4.5). In this situation the following equation is satisfied

$$d\begin{bmatrix} \hat{x}^0_t - Q^0 H^{0^T} y_t \\ \hat{x}^1_t - Q^1 H^{1^T} y_t \\ y_t \end{bmatrix} = \begin{bmatrix} A^0 - Q^0 H^{0^T} H^0 & 0 & (A^0 - Q^0 H^{0^T} H^0) Q^0 H^{0^T} \\ 0 & A^1 - Q^1 H^{1^T} H^1 & (A^1 - Q^1 H^{1^T} H^1) Q^1 H^{1^T} \\ L^0_t & L^1_t & F_t \end{bmatrix} .$$

$$\begin{bmatrix} \hat{x}^0_t - Q^0 H^{0^T} y_t \\ \hat{x}^1_t - Q^1 H^{1^T} y_t \\ y_t \end{bmatrix} dt + \begin{bmatrix} q^0 \\ q^1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} dW_t \qquad (3.4.8)$$

where $L^0_t = H^0 I(t<t_j)$, $L^1_t = H^1 I(t \geq t_j)$

$F_t = H^0 Q^0 H^{0^T} I(t<t_j) + H^1 Q^1 H^{1^T} I(t \geq t_j)$

This equation has the form (3.1.1) so an optimal detection rule may be constructed for the situation (3.4.7). Note that F_t is symmetric so that a process S_t may be defined as in section 3.1 (equation (3.1.19)).

The sub-optimal detection rule proposed is that which is optimal where (3.4.7) holds instead of (3.4.2). Comparing (3.4.6) and (3.4.7), note that y_t is the same in either case for $t \le t_j$. For $t \ge t_j$, $\hat{x}^1_t$ and r_t satisfy

$$du_t = (A^1 - M_t H^{1^T} H^1) u_t dt + q^1 dt + M_t H^{1^T} dy_t$$

$$dy_t = H^1 u_t dt + dW_t$$

where, as $t - t_j$ increases M_t tends to Q^1 in each case.

The differences involve transient effects at time t_j. In Lemma 5.8 it is verified that, (where (3.4.2) holds)

$$E(\|r_t - \hat{x}^1_t\| \mid t_j, r_{t_j}, \hat{x}^1_{t_j}) \le a(r_{t_j}, x^1_{t_j}) e^{-b.(t-t_j)} \quad \forall t \ge t_j$$

for some $a(\cdot,\cdot) < \infty$, $b > 0$ such that

$$E(a(r_{t_j}, \hat{x}^1_{t_j}) \mid t_j) \le d < \infty \quad \forall t_j \quad \text{for some } d$$

Because the differences between the actual system and that for which the detection rule is optimal are limited to transient effects it seems likely that near optimal performance is attained in the case where detection times are typically long compared with system time constants.

3.5 Detection of parameter jumps to unknown values

The optimal and sub-optimal detection rules considered so far in this chapter require a-priori knowledge of the system parameters after the disorder has appeared. If only a set of possible values is specified a more complicated problem arises.

Suppose y_t is generated by a system with dynamics specified by a parameter $\alpha_t \epsilon A$. As usual, suppose $P(t \geq t_j) = 1 - e^{-\lambda t}$, and let $\alpha_t = \alpha^0 \ \forall t < t_j$.

For $t \geq t_j$ $$\alpha_t = \alpha^1 \epsilon A^1 \subset A \tag{3.5.1}$$

where α^1 is not known a-priori.

In order to define the expected cost $E(C(\tilde{\tau})|\mathcal{Y}_o)$, (2.1.1) it is necessary to assume an a-priori distribution for α^1 over A^1. Then to generate $\pi_t = P(t \geq t_j | \mathcal{Y}_t)$, it is in general necessary to evaluate the a-posteriori distribution of α^1 at all times t. If A^1 is finite, this may be feasible, although it increases the complexity of the problem. Otherwise, an infinite dimensional problem is encountered.

An alternative formulation for this problem involves the minimization of the expected cost assuming that the parameter α^1 will always take the least favourable value in A^1.

A $\mathcal{Y}_t^R$-stopping time is required which minimizes

$$\max_{\alpha^1 \epsilon A^1} E(C(\tilde{\tau})|\mathcal{Y}_o, \alpha^1) \tag{3.5.2}$$

Min-max formulations of this sort have been investigated for a number of sequential and non-sequential decision problems, and the solution is characteristically the optimal solution to the previous formulation where a "least favourable" a-priori distribution is assumed for the unknown parameter [18]. A simple example illustrates this for the disorder problem considered here.

Suppose $A^1 = \{\beta,\delta\} \subset A$ (3.5.3)

Define $F = \{x\epsilon R^2 : x_1 = E(C(\tilde{\tau})|Y_o, \alpha^1=\beta), x_2 = E(C(\tilde{\tau})|Y_o, \alpha^1=\delta)$
for some Y_t^R-stopping time $\tilde{\tau}\}$ (3.5.4)

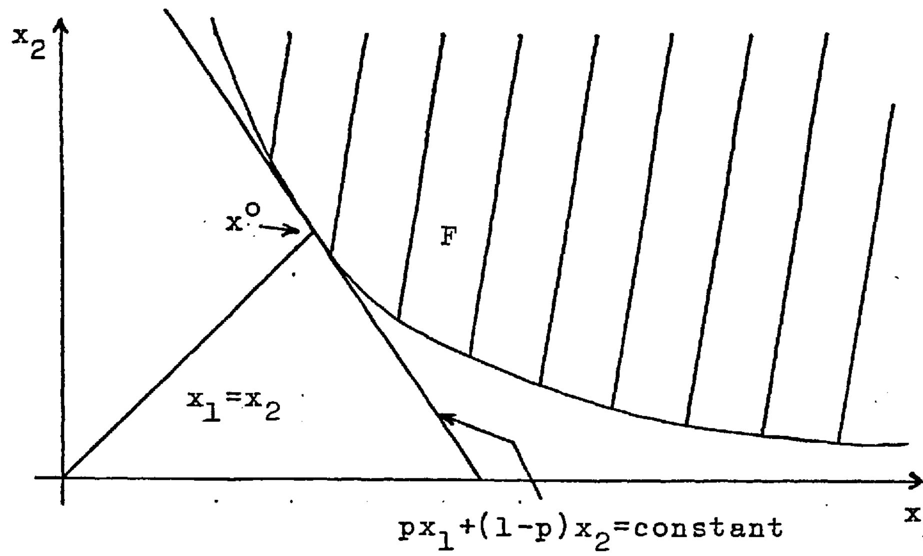


Figure 3.5.1

The convexity of F is assured since randomized stopping rules are allowed [20]. In the example, Figure 3.5.1, the min-max solution with cost (3.5.2) is the stopping time corresponding to the point x^o, since $x_1^o=x_2^o \leq \max(x_1,x_2)\ \forall x\epsilon F$. However, x^o is also a solution to the problem

$$\text{minimize } E(C(\tilde{\tau})|Y_o) \qquad (3.5.5)$$
$$\text{given } P(\alpha^1=\beta) = p_1,\ P(\alpha^1=\delta) = 1-p_1 \quad p_1\epsilon[0,1]$$

where p_1 is defined by the tangent to F at x^o in figure 3.5.1. Note that the stopping time corresponding to x^o gives the same expected cost for all p, since $x_1^o=x_2^o$. Therefore the a-priori distribution for α^1 assumed in (3.5.5) is least favourable in the sense that for any other value of p the expected cost may be made at least as small by using the stopping rule corresponding to x_o.

A second example shows that for certain parameter values δ, the optimal solution in the sense of (3.5.2) may be just that which is optimal if $\alpha^1=\beta$ w.p.1.

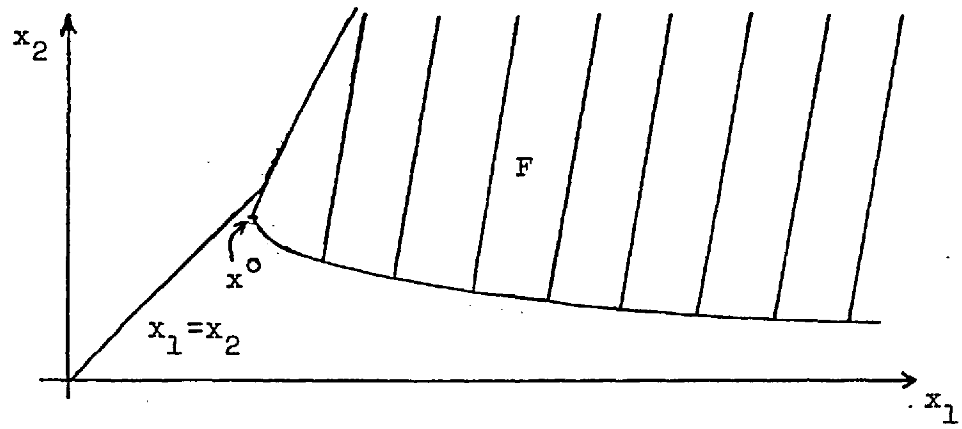


Figure 3.5.2

In this case, $\min\{\max(x_1,x_2):x\epsilon F\} = \min\{x_1:x\epsilon F\}$.

Define $A^{\beta}=\{\delta\epsilon A:x_2^o\leq x_1^o\}$ where x is defined as above (3.5.6)

It seems of interest to investigate the form of the set A^{β} associated with optimal detection rules designed for parameter jumps $\alpha_t=\alpha^0$, $t<t_j$, $\alpha_t=\beta$, $t\geq t_j$. A practical approach to the more general problem introduced in this section might then be to implement independently a finite number of such detection rules, such that the union of the corresponding sets A^{β} contains A^1. Considering systems of form (3.1.1) where α_t is a vector composed of the elements of D_t,F_t,z_t, suppose it is known that following a disorder at time t_j,

$$\alpha_t = \alpha^1\epsilon A^1\subset \bigcup_{i=1}^{j} A^{\beta_i}, \quad j<i. \tag{3.5.7}$$

Here A^{β_i} is the set of parameter points defined as in (3.5.6) with $\beta=\beta_i$.

Choose probabilities p_i; $i=1,\cdots,j$; $\sum_{i=1}^{j} p_i=1$ to maximize

$$\min_{\tilde{\tau}} E(C(\tilde{\tau})|\mathcal{Y}_o), \quad \tilde{\tau} \text{ a } \mathcal{Y}_t^R\text{-stopping time} \tag{3.5.8}$$

where $\alpha^1=\beta_i$ with probability p_i, $i=1,\cdots,j$

A min-max detection rule, where α^1 is restricted to $\{\beta_1,\cdots,\beta_j\}$ is also a solution to this problem, as previously argued in the j=2 case.

Now $$dI(t\geq t_j,\alpha^1=\beta_i) = \lambda p_i(1-\sum_{i=1}^{j} I(t\geq t_j,\alpha^1=\beta_i))dt + dM_{i_t}$$

$$i=1,\cdots,j \tag{3.5.9}$$

where M_t is a Martingale. Using the non-linear filtering equations (c.f. Appendix 1) with the observation process (3.1.1) as usual gives

$$d\pi_t^i = \lambda p_i(1-\sum_{k=1}^{j} \pi_t^k)dt + (\pi_t^i g_t^i - \pi_t^k \sum_{k=1}^{j} \pi_t^k g_t^k)^T d\nu_t$$

where $$g_t^i = ([D_t:F_t]v_t+z_t)\Big|_{\alpha_t=\beta_i} - ([D^0:F^0]v_t+z^0)$$

and $$\pi_t^i = P(t\geq t_j,\alpha_t=\beta_i|\mathcal{Y}_t) \qquad i=1,\cdots,j$$

$$d\nu_t = dy_t - ([D^0:F^0]v_t+z^0)dt - \sum_{k=1}^{j} \pi_t^k g_t^k dt \tag{3.5.10}$$

An optimal solution to the problem (3.5.8), $\hat{\tau}$, may be constructed using these processes (c.f. section 3.1); assuming it is also the unique optimal solution it is the min-max solution for $\alpha^1\in\{\beta_1,\beta_2,\cdots,\beta_j\}$.

In the case that the processes π_t^k are relatively insensitive to disorders of type $\alpha^1=\beta_i$, $i\neq k$,

i.e. $\sum_{\substack{k=1\\k\neq i}}^{j} \pi_t^k$ does not significantly increase following such a disorder,

π_t^i, $i=1,\cdots,j$ might be reasonably approximated by $\tilde{\pi}_t^i$.

satisfying

$$d\tilde{\pi}_t^i = \lambda p_i(1-\tilde{\pi}_t^i)dt + (\tilde{\pi}_t^i g_t^i - (\tilde{\pi}_t^i)^2 g_t^i)d\tilde{v}_t^i \qquad (3.5.11)$$

where $d\tilde{\pi}_t^i = dy_t - ([D^0:F^0]v_t + z^0)dt - \tilde{\pi}_t^i g_t^i dt$

This seems feasible because of the way that $\beta_1, \cdots \beta_j$ have been chosen.

But $\tilde{\pi}_t^i$ is just the probability, given Y_t, of a disorder of type $\alpha^1 = \beta_i$ with a-priori distribution for t_j

$$P(t \geq t_j) = 1-\exp(-\lambda p_i t)$$

This suggests that implementation of independent detection rules for α^1 taking each value in $\{\beta_1, \beta_2, \cdots \beta_j\}$ with corresponding parameters $\lambda p_1, \lambda p_2, \cdots, \lambda p_j$ could give performance close to that of the min-max approach for $\alpha^1 \epsilon \{\beta_1, \cdots, \beta_j\}$. Since however each of these detection rules gives no higher expected cost for all $\alpha^1 \epsilon A^{\beta i}$ than for $\alpha^1 = \beta_i$, the resulting approach should be close to min-max for $\alpha^1 \epsilon A^1 \subset \bigcup_{i=1}^{j} A^{\beta i}$.

In the following chapters the robustness properties of detection rules designed for known post-disorder parameter values is investigated, and sets of parameter points are found having properties similar to those of the sets $A^{\beta i}$ above. This robustness information is therefore of interest in the design of more complex schemes.

An example of this is given in chapter 5 (example 1 section 5.2).

CHAPTER 4

ROBUSTNESS OF DETECTION RULES: FIRST ORDER AUTOREGRESSIONS

This chapter is concerned with the robustness of optimal detection rules for systems with very simple dynamics: first order autoregressions. In this case a more complete analysis is possible than for the more complicated systems considered in the next chapter. Some structural results are obtained concerning the process $(S,y)_t$, and the shape of the stopping boundary γ introduced in the previous chapter.

The problem of interest here is this

$$dy_t = k_t a_o y_t dt + dW_t, \qquad a_o<0 \qquad (4.0.1)$$

where y_t is a scalar process

W_t is a scalar Wiener process

$$P(t\geq t_j|t_j>0) = 1-e^{-\lambda t}, \ \lambda>0 \qquad (4.0.2)$$

$t_j\geq 0$ is independent of W_t and y_o

$k_t = 1 \ \forall t<t_j$

τ is the optimal $\mathcal{Y}_t^R$-stopping time derived in Chapter 3 with the cost function

$$C(\tilde{\tau}) = I(\tilde{\tau}<t_j) + c(\tilde{\tau}-t_j)I(\tilde{\tau}>t_j) \qquad c>0 \qquad (4.0.3)$$

for the case

$$k_t = \alpha\geq -1/3 \ \forall t\geq t_j \quad \alpha\in R \qquad (4.0.4)$$

The response of the stopping rule is to be investigated for the case

$$k_t = \beta_t \ \forall t\geq t_j \qquad (4.0.5)$$

N.B. Except when explicitly stated, the notation $P(\cdot),E(\cdot)$ in this chapter refers to probability and expectation given that (4.0.4) holds.

Also, except where explicitly stated, $P(t_j=0)=0$.

Discussion

π_t denotes the a-posteriori probability that the disorder has occured by time t. Bearing in mind the note above,

$$\pi_t = P(t \geq t_j | Y_t)$$

Then

$$d\pi_t = \lambda(1-\pi_t)dt + \pi_t(1-\pi_t)(\alpha-1)a_o y_t[dy_t-(1+\pi_t(\alpha-1))a_o y_t dt] \quad (4.0.6)$$

by (3.1.5).

Therefore, if in fact $k_t=\beta \; \forall t \geq t_j$, β constant

$$d\pi_t = \lambda(1-\pi_t)dt + \pi_t(1-\pi_t)(\alpha-1)a_o^2 y_t^2(\beta-1-\pi_t(\alpha-1))dt + \pi_t(1-\pi_t)(\alpha-1)a_o y_t dW_t \quad \forall t \geq t_j \quad (4.0.7)$$

In the case of the system considered in Chapter 2, (2.3.20) it is immediately clear that larger than designed for parameter jumps result in π_t increasing more quickly. However, in (4.0.7) the second term which is positive and involves β also involves the random process y_t^2. As β increases, the mean value of y_t^2 tends to zero for $t>t_j$, which would appear to slow down the growth of π_t. In fact substituting the mean value of y_t^2, $t>t_j$ into the second term in (4.0.7) gives

$$\tfrac{1}{2}\pi_t(1-\pi_t)(\alpha-1)(-a_o)[1-\frac{1+\pi_t(\alpha-1)}{\beta}]dt \quad (4.0.8)$$

which does increase with β for $\beta>\alpha>1$, though it is bounded as $\beta\to\infty$.

In addition the contribution of the third term in (4.0.7) is likely to be less important for β large since the mean value of y_t^2 is reduced. This could have some effect on the first crossing times of the stopping boundary.

In section 5.5 an example is given of a system for which the behaviour described above does appear to destroy the robustness property of the optimal detection scheme.

It is also possible to demonstrate another way in which a jump in k_t to $\beta>\alpha>1$ for $t\geq t_j$ might not be detected as quickly as the design case disorder. Define $Q_t \triangleq \pi_t/(1-\pi_t)$ so that, from Itô's differentiation rule applied to (4.0.7)

then $$dQ_t = \lambda(1+Q_t)dt + Q_t[(\alpha-1)(\beta-1)a_o^2 y_t^2 dt+(\alpha-1)a_o y_t dW_t] \qquad t\geq t_j \qquad (4.0.9)$$

Suppose $\pi_{t_j}\simeq 0 \Rightarrow Q_{t_j}\simeq 0$, and $y_{t_j}^2$ is large. Also $\alpha>1$.

$$y_t^2 = e^{2\beta a_o(t-t_j)} y_{t_j}^2 + 2e^{2\beta a_o(t-t_j)} y_{t_j}\int_{t_j}^t e^{2\beta a_o(t_j-u)} dW_u + [\int_{t_j}^t e^{2\beta a_o(t-u)} dW_u]^2 \qquad (4.0.10)$$

Approximate y_t^2 by its initial condition response component (since $y_{t_j}^2$ is large)

$$y_t^2 \simeq e^{2\beta a_o(t-t_j)} y_{t_j}^2 \qquad (4.0.11)$$

Substituting for y_t^2 in (4.0.9), and again assuming that y_t^2 is large enough to dominate the contribution of the term $Q_t(\alpha-1)a_o y_t dW_t$

$$dQ_t \simeq \lambda(1+Q_t)dt + Q_t[\lambda+(\alpha-1)(\beta-1)a_o^2 y_{t_j}^2 . e^{2\beta a_o(t-t_j)}]dt \qquad t>t_j \qquad (4.0.12)$$

This has solution

$$Q_t \simeq \lambda\int_{t_j}^t \exp[\lambda(t-u) + \frac{-a_o(\alpha-1)}{2}.\frac{\beta-1}{\beta}.y_{t_j}^2 . e^{2\beta a_o(u-t_j)}(1-e^{2\beta a_o(t-t_j)})]du$$

$$\leq \lambda\int_{t_j}^t \exp[\lambda(t-u) + \frac{-a_o(\alpha-1)}{2}.y_{t_j}^2 . e^{2\beta a_o(u-t_j)}]du \qquad (4.0.13)$$

As β increases the second term tends to zero for each u (note that $a_o<0$) tending to decrease Q_t. In this way it seems that for small π_{t_j} and large $y_{t_j}^2$ quicker detection might occur with $k_t=\alpha\ \forall t\geq t_j$ rather than $k_t=\beta>\alpha\ \forall t\geq t_j$. Of course this also depends on other factors such as the stopping boundary shape. The structural results in this chapter clarify these aspects.

4.1 Preliminaries

The detection rule

Applying the results of section 3.1 to the system (4.0.1) the optimal detection rule for $k_t=\alpha\ \forall t\geq t_j$ is

$$\tau = \inf\{t: S_t \geq S_\gamma(y)\} \tag{4.1.1}$$

where $$S_t = \ln(\pi_t/(1-\pi_t)) - \tfrac{1}{2}(\alpha-1)a_o y_t^2 \tag{4.1.2}$$

by (3.1.19) and γ is the stopping boundary in the state space of the Markov process $(S,y)_t$.

$$S_\gamma(y) \triangleq \inf\{S:(S,y)\in\gamma\}$$

$S_\gamma(y)$ is defined for all y, (possibly infinite valued).

From (3.1.20)

$$\frac{dS_t}{dt} = \lambda(1+e^{-S_t-\frac{1}{2}(\alpha-1)a_o y_t^2}) - \tfrac{1}{2}(\alpha^2-1)a_o^2 y_t^2 - \tfrac{1}{2}(\alpha-1)a_o \tag{4.1.3}$$

As before

$$\tau_{t_o} \triangleq \inf\{t\geq t_o: S_t \geq S_\gamma(y_t)\} \tag{4.1.4}$$

so that $\tau=\tau_{t_o}$ if $\tau\geq t_o$.

Note that by Lemma 2.1 the stopping time τ is also optimal in the problem of minimizing the expectation of the cost

$$K(\tilde{\tau}) = -\lambda\tilde{\tau} + (\lambda+c)(\tilde{\tau}-t_j)I(\tilde{\tau}>t_j) \tag{4.1.5}$$

$\tilde{\tau}$ a $\mathcal{Y}_t^R$-stopping time, λ as in (4.0.2)

Define $C_{t_o}(\tilde{\tau}_{t_o}) \triangleq I(\tilde{\tau}_{t_o}<t_j) + c(\tilde{\tau}_{t_o}-t_j \vee t_o)I(\tilde{\tau}_{t_o}>t_j)$ (4.1.6)

and as before

$$K_{t_o}(\tilde{\tau}_{t_o}) \triangleq -\lambda(\tilde{\tau}_{t_o}-t_j) + (\lambda+c)(\tilde{\tau}_{t_o}-t_j \vee t_o)I(\tilde{\tau}_{t_o}>t_j) \quad (4.1.7)$$

for $\tilde{\tau}_{t_o} \geq t_o$ a $\mathcal{Y}_t^R$-stopping time, t_o an arbitrary time

These correspond to the cost "incurred after time t_o", if $\tilde{\tau} \geq t_o$. If t_o is a stopping time

$$E_{(S,y)_{t_o}} C_{t_o}(\tilde{\tau}_{t_o}),\ E_{(S,y)_{t_o}} K_{t_o}(\tilde{\tau}_{t_o})$$

are minimized for $\tilde{\tau}_{t_o}=\tau_{t_o}$.

Note that $C(\tau)=C_o(\tau_o)$, $K(\tau)=K_o(\tau_o)$.

Outline of the robustness argument

The cases $\alpha\in[-1/3,1)$ and $\alpha\in(1,\infty)$ are treated separately. The case $\alpha<-1/3$, for which the system would be unstable after time t_j, cannot be handled since one of the structural properties required does not then hold.

For the case $\alpha\in[-1/3,1)$ in order to prove the robustness result, Theorem 4.2, it is first necessary to show the function $S_\gamma(y)$ is non-increasing with y^2. This is done by considering the sample path properties of the Markov process $(S,y)_t$ and decomposing its state space into three regions in which special properties apply. A partial result, concerning the shape of the part of the stopping boundary γ lying in two of these regions is given in Lemma 4.1. It is more difficult to extend this result to the third region. This is done in Theorem 4.1, for which Lemma 4.2 provides a necessary preliminary result.

The robustness result holds for disorders occuring after a $\mathcal{Y}_t$-stopping time t_c. This should be typically very small and an assesssment of this is given in Table 4.2.1.

For the case $\alpha\epsilon(1,\infty)$ the situation is more complicated. Robustness is proved for a detection rule which is optimal for a slightly modified problem, using the previous arguments. It is suggested that this indicates the near robustness of the true optimal detection rule. Finally, a (not necessarily tight) upper bound is established for the increase in expected cost resulting from the use of the guaranteed robust sub-optimal approach.

Notes

$$\pi(S,y) \triangleq \frac{1}{1+\exp(-S-\frac{1}{2}(\alpha-1)a_0y^2)} \qquad (4.1.8)$$

so that $\pi(S_t,y_t)=\pi_t$

$$h(\tilde{S},\tilde{y}) = E_{(\tilde{S},\tilde{y})}K(\tau) = E_{(\tilde{S},\tilde{y})}(-\lambda\tau+(\lambda+c)\int_0^\tau\pi(S_u,y_u)du)$$

from (3.1.13),(3.1.22) (4.1.9)

So $$h(\tilde{S},\tilde{y}) = E_{(\tilde{S},\tilde{y})}\int_0^\tau\sigma(S_u,y_u)du \qquad (4.1.10)$$

where $\sigma(S,y) \triangleq -\lambda + (\lambda+c)\pi(S,y)$ (4.1.11)

Note that

a) $h(S_t,y_t) = E_{(S,y)_t}K_t(\tau_t) = E_{(S,y)_t}\int_t^{\tau_t}\sigma(S_u,y_u)du$

from (4.1.7) (4.1.12)

b) $h(S,y) \leq 0$ (4.1.13)

since by optimality of τ, $E_{(S,y)}K(\tau) \leq E_{(S,y)}K(0) = 0$

c) From (3.1.11) $\tau=\inf\{t:h(S_t,y_t)=0\}$ (4.1.14)

therefore $h(S,y) < 0 \quad S<S_\gamma(y)$ (4.1.15)

$h(S,y) = 0 \quad S\geq S_\gamma(y)$ (4.1.16)

d) $\sigma(S_t,y_t)<0 \Rightarrow \tau_t>t$, since otherwise if $\tau_t=t$

$$E_{(S,y)_t} K_t(\inf\{u\ge t:\sigma(S_u,y_u)\ge 0\})$$

$$= E_{(S,y)_t} \int_t^{\inf\{u\ge t:\sigma(S_u,y_u)\ge 0\}} \sigma(S_u,y_u)\,du$$

$$< 0 = E_{(S,y)_t} K_t(\tau_t)$$

which is impossible since τ_t is optimal.

Therefore $\sigma(S,y) \ge 0$ if $(S,y)\epsilon\gamma$ (4.1.17)

e) Setting $\rho(y)=\lambda$ in (3.1.2), Theorems 3.1 and 3.2 hold. In particular, $h(S,y)$ is continuous in S (except, possibly, at $S=-\infty$) and non-decreasing in S.

f) $h(S,y) = h(S,-y)$, $S_\gamma(y) = S_\gamma(-y)$ by symmetry. (4.1.18)

The cases $-1/3\le\alpha<1$ & $\alpha>1$ in (4.0.4) are now considered separately

4.2 The $\alpha\epsilon[-1/3,1)$ case

First some definitions are given.

Define $S_c \triangleq \ln[\frac{\lambda}{-(\alpha+1)a_o+\lambda}]$ (4.2.1)

Let $\frac{dS}{dt}(\tilde{S},\tilde{y}) \triangleq \left.\frac{dS_t}{dt}\right|_{\substack{S_t=\tilde{S}\\ y_t=\tilde{y}}}$

$$= \lambda(1+e^{-\tilde{S}-\frac{1}{2}(\alpha^2-1)a_o\tilde{y}^2}) - \tfrac{1}{2}(\alpha^2-1)a_o^2\tilde{y}^2 - \tfrac{1}{2}(\alpha-1)a_o$$

(4.2.2)

Then $\frac{dS}{dt}(S,y) \ge 0 \quad \forall S\le S_c,\ \forall y$ (4.2.3)

This only holds for $\alpha\ge-1/3$

Also $\frac{dS}{dt}(S,y)$ is monotonically increasing in y^2 for $S\ge S_c$ (4.2.4)

Note here that $\frac{dS}{dt}(S,y)=\frac{dS}{dt}(S,-y)$.

S_c as defined in (4.2.1) is the smallestvalue such that (4.2.4) holds.

The state space of the process $(S,y)_t$ is now decomposed into three disjoint sets

$$N \triangleq \{(S,y):S<S_c\}$$

$$P \triangleq \{(S,y):\frac{dS}{dt}(S,y)\geq 0, S\geq S_c\} \qquad (4.2.5)$$

$$Q \triangleq \{(S,y):\frac{dS}{dt}(S,y)<0\}$$

Also $\theta \triangleq P \cap \text{closure}(Q)$ (4.2.6)

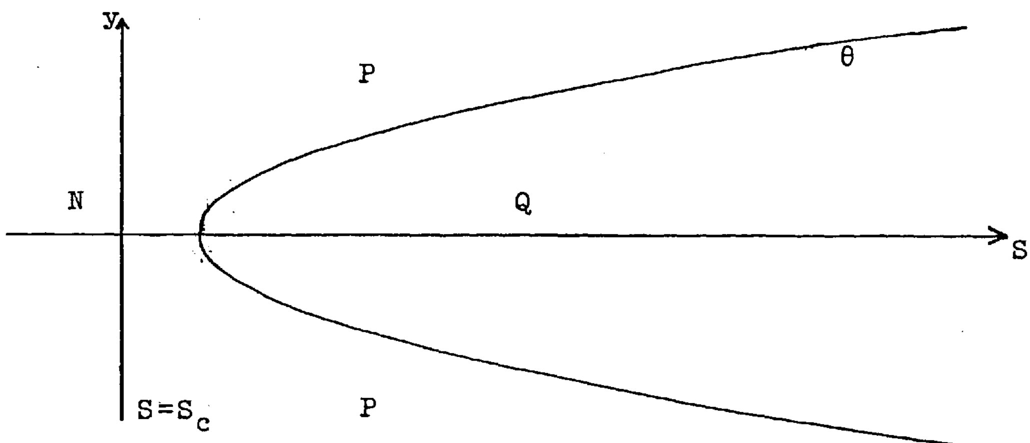


Figure 4.2.1

Define $t_c \triangleq \inf\{t:(S,y)_t \epsilon P\cup Q\}$ (4.2.7)

Since $\frac{dS}{dt}(S_c,y)\geq 0$ $\forall y$, it follows that

$$(S,y)_t \epsilon P\cup Q \quad \forall t\geq t_c \qquad (4.2.8)$$

<u>Lemma 4.1</u>

$S_\gamma(y)$ is non-increasing with increasing y^2 $\forall y$ such that $(S_\gamma(y),y)\epsilon P\cup N$

<u>Proof</u>

If the Lemma is not true there exists $y'\geq 0$ such that

for $S'= S_\gamma(y')$, $(S',y') \epsilon P\cup N$

and $S_\gamma(y)$ is strictly increasing with increasing y at $y=y'$.

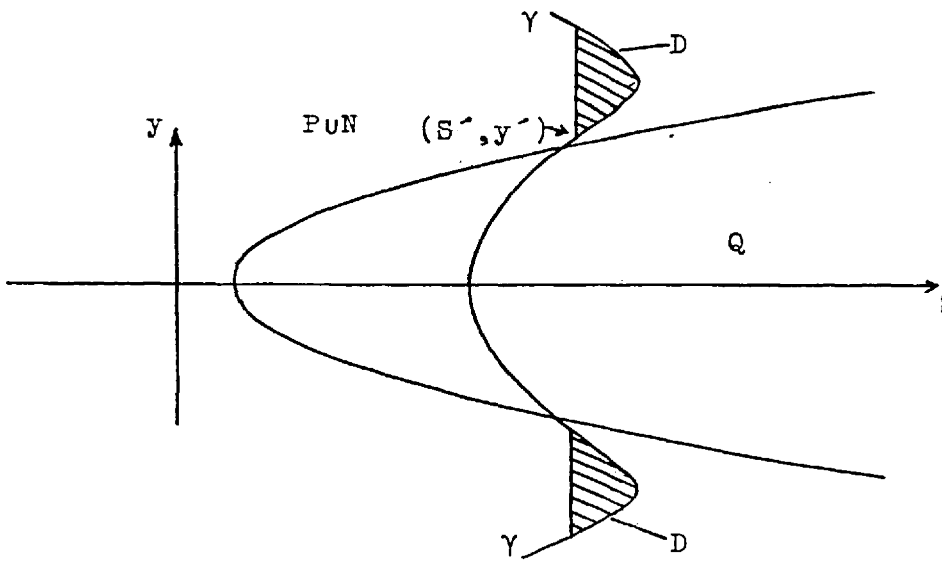


Figure 4.2.2

Then if $D \triangleq \{(S,y): S\epsilon[S',S_\gamma(y)], y \geq y'\}$

$D\backslash\gamma$ is non-empty (γ is the boundary of the closed stopping set).

Choose $(S,y)_{t_o} \epsilon D\backslash\gamma \Rightarrow (S,y)_t \epsilon D \quad \forall t \epsilon [t_o, \tau_{t_o}]$

since $\frac{dS}{dt}(S',\tilde{y}) \geq 0 \quad \forall \tilde{y} \geq y'$

[because a) $(S',y') \epsilon N \Rightarrow (S',\tilde{y}) \epsilon N$

b) $(S',y') \epsilon P \Rightarrow \frac{dS}{dt}(S',\tilde{y}) \geq \frac{dS}{dt}(S',y') \geq 0$

by (4.2.4)]

$\sigma(S,y) \geq 0 \ \forall (S,y) \epsilon D$, since $\sigma(S',y') \geq 0$ by (4.1.17) and σ is increasing with S and with y^2 from (4.1.11).

Therefore $\quad h(S_{t_o},y_{t_o}) = E_{(S,y)_{t_o}} \int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du \geq 0$

But this contradicts (4.1.15), since $S_{t_o} < S_\gamma(y_{t_o})$. □

Definition

Let $\Gamma \triangleq \{(S,y) \epsilon Q \cap \gamma : \exists (S',y') \epsilon Q \cap \gamma \text{ with } S' > S, y'^2 > y^2\}$

then $$S_1 \triangleq \begin{cases} \inf\{S:(S,y)\epsilon\Gamma\} & \text{if } \Gamma \not\epsilon \Phi \\ +\infty & \text{if } \Gamma \epsilon \Phi \end{cases} \qquad (4.2.9)$$

(see over)

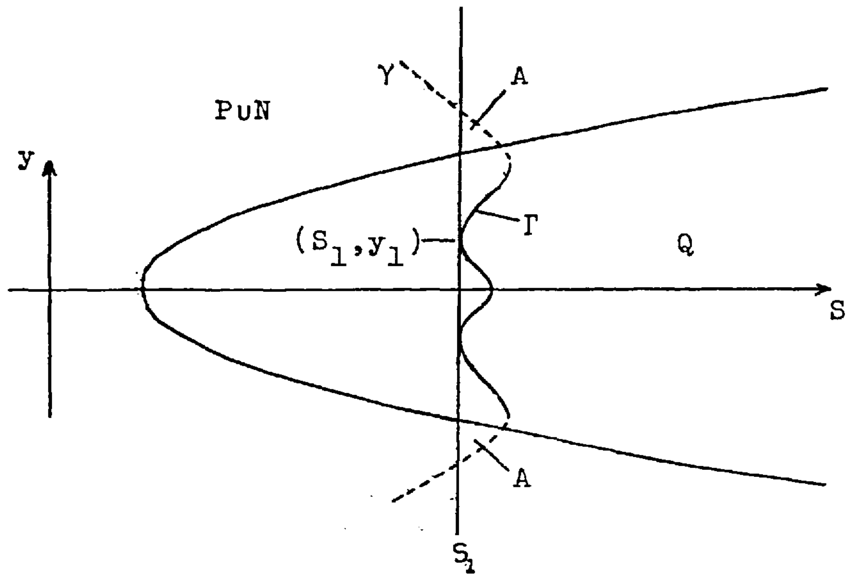


Figure 4.2.3

If $S_1<\infty$ choose $y_1 \triangleq \inf\{y\geq 0:(S_1,y)\in\gamma\}$ (4.2.10)

$(S_1,y_1)\in\Gamma$ since the stopping set is closed (Theorem 3.1).

Note that $S_\gamma(y)$ is non-increasing with increasing y^2 $\forall y$ st $S_\gamma(y)<S_1$ by (4.2.9) and Lemma 4.1.

Lemma 4.2

If $S_1<\infty$, $h(S,y)$ is non-decreasing with increasing y^2 for $(S,y)\in P$, $S\in[S_1,S_\gamma(y))$:(i.e. in the sets "A" in Figure 4.2.3)

Proof

Suppose the Lemma is not true.

Then $\exists$ $S_2\geq S_1$, $y_2>y_3>0$ such that

$$S_2<S_\gamma(y_2), \quad S_2<S_\gamma(y_3)$$

$$(S_2,y_2),(S_2,y_3)\in P$$

& $$h(S_2,y_2) < h(S_2,y_3) \tag{4.2.11}$$

$D' \triangleq \{(S,y):y\geq y_3, S\in[S_2,S_\gamma(y)]\}$

(see Figure 4.2.4)

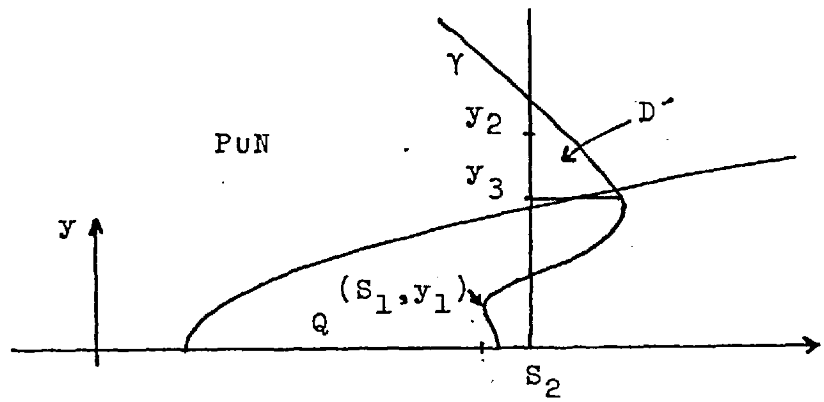


Figure 4.2.4

Suppose $(S,y)_{t_o} = (S_2,y_2)$. Then the process $(S,y)_t$ leaves D' either across γ or across the line $y=y_3$, since $\frac{dS}{dt}(S_2,y)\geq 0 \;\forall y\geq y_3$ by (4.2.4).

Define $t_1 = \inf\{t\geq t_o : y_t = y_3\}$

$$\int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du = \int_{t_o}^{\tau_{t_o}\wedge t_1} \sigma(S_u,y_u)du + \int_{\tau_{t_o}\wedge t_1}^{\tau_{t_o}} \sigma(S_u,y_u)du$$

The first term on the right is positive or zero, as $\sigma(S,y)\geq 0 \quad \forall (S,y)\in D'$. This is because $\sigma(S_1,y_1)\geq 0$ by (4.1.17) and σ is increasing with S and y^2 from (4.1.11).

As $\tau_{t_o}\wedge t_1$ is a $\mathcal{Y}_t$-stopping time

$$\begin{aligned} h(S_2,y_2) &= E_{(S,y)_{t_o}} \int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du \\ &\geq E_{(S,y)_{t_o}}[E(\int_{\tau_{t_o}\wedge t_1}^{\tau_{t_o}} \sigma(S_u,y_u)du \,|\, \mathcal{Y}_{\tau_{t_o}\wedge t_1})] \\ &\geq E_{(S,y)_{t_o}}[h(S_{\tau_{t_o}\wedge t_1}, y_{\tau_{t_o}\wedge t_1}) \,|\, \tau_{t_o}\geq t_1] \\ &\geq h(S_2,y_3) \end{aligned} \tag{4.2.12}$$

The second inequality is because if $\tau_{t_o}<t_1$, $h(S_{\tau_{t_o}\wedge t_1}, y_{\tau_{t_o}\wedge t_1}) = 0$ by (4.1.16).

For $\tau_{t_o}\geq t_1$, $S_{\tau_{t_o}\wedge t_1}\geq S_2$, $y_{\tau_{t_o}\wedge t_1}=y_3$.

Then by Corollary 3.2.2 $h(S_{\tau_{t_o}\wedge t_1}, y_{\tau_{t_o}\wedge t_1}) \geq h(S_2,y_3)$

which establishes the third inequality. But (4.2.12) contradicts (4.2.11). □

Theorem 4.1

$S_\gamma(y)$ is non-increasing with increasing y^2

Proof

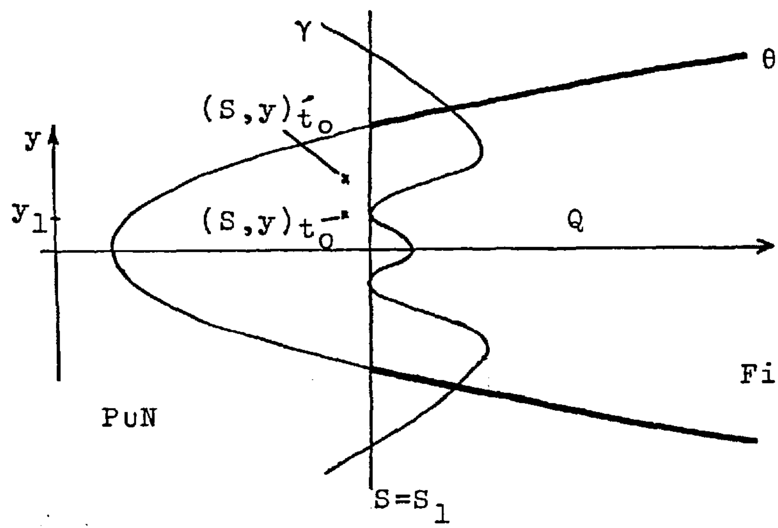


Figure 4.2.5

Suppose the Theorem is not true. Then by Lemma 4.1 and (4.2.9), $S_1<\infty$.

y_1 exists and is defined in (4.2.10).

Let $y'>y_1$ be chosen so $S_\gamma(y')>S_\gamma(y_1)$. (4.2.9) gaurantees that such a y' exists.

Let $(S,y)_t$, $(S,y)'_t$ both be solutions of (3.1.6) and (4.1.3)

i.e. $dy_t = (1+(\alpha-1)\pi(S_t,y_t))a_o y_t dt + d\nu_t$

$$\frac{dS_t}{dt} = \lambda(1+e^{-S_t-\frac{1}{2}(\alpha-1)a_o y_t{}^2}) - \tfrac{1}{2}(\alpha^2-1)a_o^2 y_t^2 - \tfrac{1}{2}(\alpha-1)a_o \qquad (4.2.13)$$

with the innovations process ν_t the same in both cases but with

$$(S,y)_{t_o}=(S_1-\varepsilon,y_1), \qquad (S,y)'_{t_o}=(S_1-\varepsilon,y') \qquad (4.2.14)$$

Here ε is chosen so that $S_c\le S_1-\varepsilon<S_1$. This is possible since $S_1\le S_c$ contradicts Lemma 4.1.

S'_t, y'_t are defined such that $(S'_t, y'_t) = (S,y)'_t$

Note that $(y_t : t \geq t_o)$ and $(y'_t : t \geq t_o)$ both generate the same σ-field $\mathcal{Y}_t$ (both processes may be reconstructed given ν_t - see Lemma 3.1). In this proof all probabilities and expectations are conditioned on the initial conditions (4.2.14).

The following $\mathcal{Y}_t$-stopping times are defined

$$t_1 \triangleq \inf\{t \geq t_o : y'^2_t = y_t^2\} \tag{4.2.15}$$

$$t_2 \triangleq \inf\{t \geq t_o : (S,y)_t \in \theta, S_t \geq S_1\} \tag{4.2.16}$$

$$\tau_{t_o} \triangleq \inf\{t \geq t_o : (S,y)_t \in \gamma\} \tag{4.2.17}$$

$$\tau'_{t_o} \triangleq \inf\{t \geq t_o : (S,y)'_t \in \gamma\} \tag{4.2.18}$$

Note that (4.2.17) is equivalent to (4.1.4) in this case. t_2 is the first time $(S,y)_t$ crosses the thick line in Figure 4.2.5.

Also $(S,y)_t \in P \cup Q \quad \forall t \geq t_o$ (c.f. (4.2.8)), and by (4.2.4)

$$\frac{dS}{dt}(\tilde{S},\hat{y}) \geq \frac{dS}{dt}(\tilde{S},\tilde{y}) \text{ if } \hat{y}^2 \geq \tilde{y}^2, \quad (\tilde{S},\tilde{y}) \in P \cup Q \tag{4.2.19}$$

A preliminary result is now established.

Suppose $\tau_{t_o} < t_1$, $\tau_{t_o} < t_2$.

Then since $\frac{dS}{dt}(S,y) < 0$ in Q and by definition of t_2,

$(S,y)_t \in P \cup \{(S,y) : S < S_1\} \quad \forall t \leq \tau_{t_o}$ (see Figure 4.2.6).

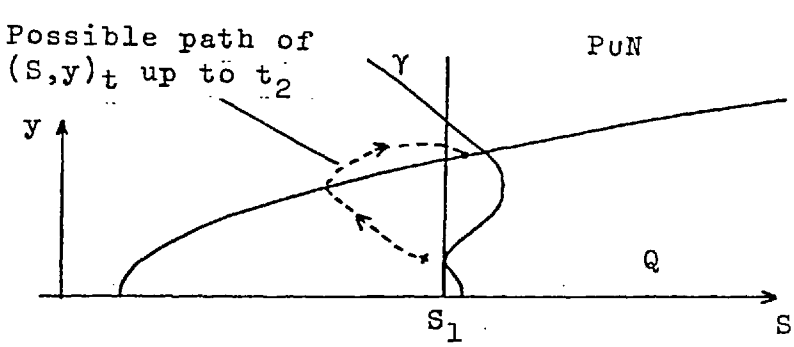


Figure 4.2.6

As $\tau_{t_o}<t_1$, $y_t'^2>y_t^2$ $\forall t\le\tau_{t_o}$. Then from (4.2.19) $S_t'\ge S_t$ $\forall t\le\tau_{t_o}$.

Since , from Lemma 4.1 and (4.2.9), $S_\gamma(\tilde{y})$ is non-increasing with $\tilde{y}^2$ for $\tilde{y}^2\ge y^2_{\tau_{t_o}}$ (see Figure 4.2.6)

$$S'_{\tau_{t_o}} \ge S_{\tau_{t_o}} = S_\gamma(y_{\tau_{t_o}}) \ge S_\gamma(y'_{\tau_{t_o}})$$

so that $\tau'_{t_o}\le\tau_{t_o}$.

Therefore $\tau_{t_o}<t_1$, $\tau_{t_o}<t_2 \Rightarrow \tau'_{t_o}\le\tau_{t_o}$ (4.2.20)

The following events are defined

$$\begin{aligned} A &\triangleq \{\omega: t_1\le\min(t_2,\tau'_{t_o})\} \\ B &\triangleq \{\omega: t_2<\min(t_1,\tau'_{t_o})\} \\ C &\triangleq \{\omega: \tau'_{t_o}\le t_2, \tau'_{t_o}<t_1\} \end{aligned} \qquad (4.2.21)$$

A,B,C are disjoint, and $\omega\in A\cup B\cup C$ w.p.1.

Each event is now considered separately.

EVENT A

If $\omega\in A$, $t_1\le t_2$, $t_1\le\tau'_{t_o}$. By (4.2.20) it follows that $\tau_{t_o}\ge t_1$.

Also $y_t'^2\ge y_t^2$, $S_t'\ge S_t$ $\forall t\le t_1$.

Since $\sigma(S,y)$ increases with S and y^2, from (4.1.11)

$$\int_{t_o}^{t_1} \sigma(S'_u,y'_u)du \ge \int_{t_o}^{t_1} \sigma(S_u,y_u)du$$

Also, since $y'^2_{t_1}=y^2_{t_1}$ and $S'_{t_1}\ge S_{t_1}$, $h(S'_{t_1},y'_{t_1}) \ge h(S_{t_1},y_{t_1})$ from Corollary 3.2.2.

$$\begin{aligned} \text{Then}\quad & E[\int_{t_o}^{\tau'_{t_o}} \sigma(S'_u,y'_u)du - \int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du \mid \omega\in A] \\ &= E[\int_{t_o}^{t_1}(\sigma(S'_u,y'_u)-\sigma(S_u,y_u))du \mid \omega\in A] \\ &\quad + E[h(S'_{t_1},y'_{t_1})-h(S_{t_1},y_{t_1}) \mid \omega\in A] \ge 0 \end{aligned} \qquad (4.2.22)$$

since A is a $\mathcal{Y}_{t_1}$-measurable event, and from (4.1.12).

EVENT B

If $\omega \in B$, $t_2 < t_1$, $t_2 < \tau'_{t_o}$

If $\tau_{t_o} < t_2$, (4.2.20) gives a contradiction. Therefore $\tau_{t_o} \geq t_2$.

Since $t_2 < t_1$, as before, $y'^2_t \geq y_t^2$, $S'_t \geq S_t$ $\forall t \leq t_2$.

$$E[\int_{t_o}^{\tau'_{t_o}} \sigma(S'_u,y'_u)du - \int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du | \omega \in B]$$

$$= E[\int_{t_o}^{t_2} (\sigma(S'_u,y'_u) - \sigma(S_u,y_u))du | \omega \in B]$$

$$+ E[h(S'_{t_2},y'_{t_2}) - h(S_{t_2},y_{t_2}) | \omega \in B]$$

The first term on the right is positive or zero by the properties of σ.

$h(S'_{t_2},y'_{t_2}) \geq h(S_{t_2},y'_{t_2}) \geq h(S_{t_2},y_{t_2})$

where the first inequality is from Corollary 3.2.2 since $S'_{t_2} \geq S_{t_2}$, and the second inequality is from Lemma 4.2 using $S_{t_2} \geq S_1$, $y'^2_{t_2} \geq y^2_{t_2}$.

Therefore $E[\int_{t_o}^{\tau'_{t_o}} \sigma(S'_u,y'_u)du - \int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du | \omega \in B] \geq 0$

(4.2.23)

EVENT C

If $\omega \in C$, $\tau'_{t_o} \leq t_2$, $\tau'_{t_o} < t_1$.

From (4.2.20) $\tau_{t_o} < t_1$, $\tau_{t_o} < t_2 \Rightarrow \tau'_{t_o} \leq \tau_{t_o}$, so that $\tau_{t_o} < \tau'_{t_o}$ leads to a contradiction if $\omega \in C$.

Therefore $\tau_{t_o} \geq \tau'_{t_o}$.

$y'^2_t \geq y_t^2$, $S'_t \geq S_t$ $\forall t \leq \tau'_{t_o}$.

$$E[\int_{t_o}^{\tau'_{t_o}} \sigma(S'_u,y'_u)du - \int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du | \omega \in C]$$

$$= E[\int_{t_o}^{\tau'_{t_o}} (\sigma(S'_u,y'_u) - \sigma(S_u,y_u))du | \omega \in C] + E[-h(S_{\tau'_{t_o}},y_{\tau'_{t_o}}) | \omega \in C]$$

$$\geq 0 \qquad (4.2.24)$$

COMPLETION OF PROOF

From (4.2.22),(4.2.23) & (4.2.24)

$$E[\int_{t_o}^{\tau'_{t_o}} \sigma(S'_u,y'_u)du-\int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du|\omega\epsilon F] \geq 0$$

for F=A,B,C.

Therefore $h(S'_{t_o},y'_{t_o}) - h(S_{t_o},y_{t_o})$

$$= E[\int_{t_o}^{\tau'_{t_o}} \sigma(S'_u,y'_u)du-\int_{t_o}^{\tau_{t_o}} \sigma(S_u,y_u)du] \geq 0$$

i.e. $h(S_1-\varepsilon,y') \geq h(S_1-\varepsilon,y_1)$

Now as $\varepsilon\downarrow 0$, $h(S_1-\varepsilon,y_1)\to 0$ by Corollary 3.2.2 and since $(S_1,y_1)\epsilon\gamma$

So $\lim_{\varepsilon\downarrow 0} h(S_1-\varepsilon,y')\geq 0$. By continuity of h with S (Corollary 3.2.2) $h(S_1,y')\geq 0 \Rightarrow (S_1,y')\epsilon\gamma$.

But y' was chosen so that $S_1<S_\gamma(y')$ which gives a contradiction. □

The response of the detection rule is now investigated for $k_t=\beta_t\ \forall t\geq t_j$ in (4.0.1).

<u>Theorem 4.2</u>

$$E(\tau_{t_j}-t_j|(S,y)_{t_j},t_j,k_t=\beta_t\ \forall t\geq t_j)$$

$$\leq E(\tau_{t_j}-t_j|(S,y)_{t_j},t_j,k_t=\alpha\ \forall t\geq t_j)$$

if $\beta_t\leq\alpha\ \forall t\geq t_j\geq t_c$ where $\alpha\epsilon[-1/3,1)$.

<u>Proof</u>

Suppose $\beta_t\leq\alpha\ \forall t\geq t_j\geq t_c$

Define y_t^β such that

$$dy_t^\beta = \beta_t a_o y_t^\beta dt + dW_t^\beta \qquad t\geq t_j \tag{4.2.25}$$
$$y_{t_j}^\beta = y_{t_j}$$

where W_t^β is a Wiener process.

Define y_t^α such that

$$dy_t^\alpha = \alpha a_o y_t^\alpha dt + dW_t^\alpha \quad t \geq t_j$$
$$y_{t_j}^\alpha = y_{t_j} \tag{4.2.26}$$

From Itô's differentiation rule, if $x_t^\beta \triangleq (y_t^\beta)^2$, $x_t^\alpha \triangleq (y_t^\alpha)^2$

$$x_{t_j}^\beta = x_{t_j}^\alpha$$
$$dx_t^\beta = (2\beta_t a_o x_t^\beta + 1)dt + 2\surd(x_t^\beta).dV_t^\beta \tag{4.2.27}$$
$$dx_t^\alpha = (2\alpha a_o x_t^\alpha + 1)dt + 2\surd(x_t^\alpha).dV_t^\alpha$$

where

$$V_t^\beta = \int_{t_j}^t J(y_u^\beta)dW_u^\beta$$
$$\qquad t \geq t_j$$
$$V_t^\alpha = \int_{t_j}^t J(y_u^\alpha)dW_u^\alpha$$

$$J(x) = +1 \text{ if } x \geq 0$$
$$\quad -1 \text{ if } x < 0$$

V_t^α, V_t^β are then Wiener processes. Suppose that W_t^α, W_t^β are chosen so that $V_t^\alpha = V_t^\beta = V_t$. Then, by [22, Theorem 1.1]

$$x_t^\beta \geq x_t^\alpha \quad \forall t \geq t_j \tag{4.2.28}$$

Now define S_t^α, S_t^β so that $S_{t_j}^\alpha = S_{t_j}^\beta = S_{t_j}$ and (S_t^α, x_t^α) & (S_t^β, x_t^β) satisfy

$$\frac{d\tilde{S}_t}{dt} = \lambda(1+e^{-\tilde{S}_t - \frac{1}{2}(\alpha-1)a_o\tilde{x}_t}) - \tfrac{1}{2}(\alpha^2-1)a_o^2\tilde{x}_t - \tfrac{1}{2}(\alpha-1)a_o$$

As $t_j \geq t_c$ $S_t^\alpha, S_t^\beta \geq S_c$ $\forall t \geq t_j$. $\frac{d\tilde{S}}{dt}$ is an increasing function of $\tilde{x}$ for given $\tilde{S} \geq S_c$. Therefore

$$S_t^\beta \geq S_t^\alpha \quad \forall t \geq t_j \tag{4.2.29}$$

Now define

$$\tau^\alpha = \inf\{t \geq t_j : S_t^\alpha \geq S_\gamma(y_t^\alpha)\}$$
$$\tau^\beta = \inf\{t \geq t_j : S_t^\beta \geq S_\gamma(y_t^\beta)\}$$

Then $S^{\beta}_{\tau\alpha} \geq S^{\alpha}_{\tau\alpha} = S_{\gamma}(y^{\alpha}_{\tau\alpha}) \geq S_{\gamma}(y^{\beta}_{\tau\alpha})$

The final inequality follows from (4.2.28), noting that $x_t \hat{=} y^2_t$, and Theorem 4.1. Therefore

$$\tau^{\beta} \leq \tau^{\alpha}$$

The result of the Theorem now follows because of the way in which y^{α}_t, y^{β}_t, τ^{α}, τ^{β} have been defined. □

Since $E_{(S,y)_o}[C(\tau)|k_t=\beta_t \forall t \geq t_j \geq t_c] = P_{(S,y)_o}(\tau<t_j|t_j \geq t_c)$

$$+ E_{(S,y)_o}[E(\tau_{t_j}-t_j|(S,y)_{t_j},t_j,k_t=\beta_t \forall t \geq t_j)I(\tau \geq t_j)|t_j \geq t_c] \tag{4.2.30}$$

and the event $(\tau<t_j)$ and $(S,y)_{t_j}$ are independent of β_t, it follows that

$$E_{(S,y)_o}[C(\tau)|k_t=\beta_t \forall t \geq t_j \geq t_c]$$

$$\leq E_{(S,y)_o}[C(\tau)|k_t=\alpha \forall t \geq t_j \geq t_c] \tag{4.2.31}$$

if $\beta_t \leq \alpha$ $\forall t$. This also holds with $C(\tau)$ replaced by $K(\tau)$ or Q (see section 2.2).

Remark

A similar result would apply if the simplified stopping boundary discussed in section 3.3 was used.

It is not easy to be precise about the time t_c in this case. However it is possible to get an idea of the value of the probability that $t_j<t_c$ as follows.

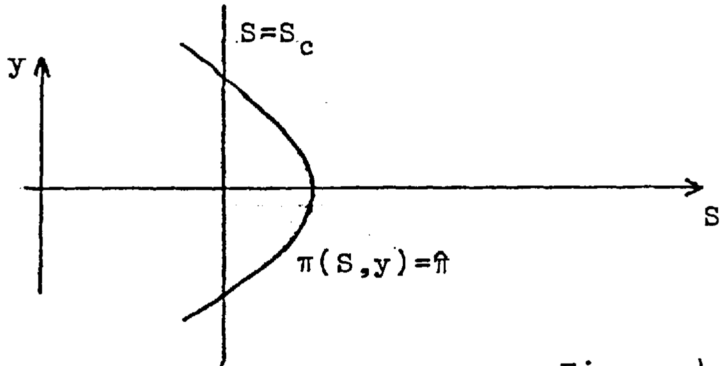


Figure 4.2.7

Let $\hat{t} \triangleq \inf\{t:\pi_t \geq \hat{\pi}\}$ $\hat{\pi}\in[0,1)$

If $y_{\hat{t}}^2 \leq \frac{2}{-(\alpha-1)a_o} \ln\left(\frac{\lambda}{-(a+1)a_o+\lambda}\cdot\frac{1-\hat{\pi}}{\hat{\pi}}\right) = \frac{\theta^2}{-2a_o}$ say (4.2.32)

then from (4.1.2) and (4.2.1)

$$S_{\hat{t}} \geq S_c \quad \text{i.e.} \quad \hat{t} \geq t_c$$

Therefore

$$P(t_c \geq t_j | Y_o) \leq P(\hat{t} \geq t_j | Y_o) + P(y_{\hat{t}}^2 \geq \frac{\theta^2}{-2a_o} | Y_o)$$

$$\leq \hat{\pi} + P(y_{\hat{t}}^2 \geq \frac{\theta}{-2a_o} | Y_o)$$

Now (4.2.32) may be interpreted as

$$|y_{\hat{t}}| \leq \theta[\text{"steady-state" pre-jump standard deviation of } y_t]$$

Presumably $P(y_{\hat{t}}^2 \geq \frac{\theta}{-2a_o} | Y_o) \to 0$ as θ increases, so $\hat{\pi}$ gives a tentative upper bound to $P(t_c \geq t_j | Y_o)$. Below some approximate values are given ($\lambda/(-a_o)$ assumed small).

α	$\hat{\pi}$ ($\theta=2$)	$\hat{\pi}$ ($\theta=3$)
-1/3	$5.691\lambda'$	$30.128\lambda'$
0.0	$2.718\lambda'$	$9.488\lambda'$
0.4	$1.301\lambda'$	$2.755\lambda'$
0.8	$0.679\lambda'$	$1.307\lambda'$

where $\lambda' = \lambda/(-a_o)$.

As λ' would normally be very small, so would the probability that $t_c \geq t_j$.

4.3 The $\alpha\epsilon(1,\infty)$ case

In order to use arguments similar to those of section 4.2 in this case it would be necessary to find some value of S, S_c such that $\frac{dS}{dt}(\tilde{S},\tilde{y})$ as defined in (4.2.2) decreases with $\tilde{y}^2$ for all $\tilde{S}\geq S_c$. However the contribution of the exponential term in (4.2.2) destroys this property for large $\tilde{y}^2$ whatever value is chosen for $\tilde{S}_c$. The situation is shown in Figure 4.3.1 below.

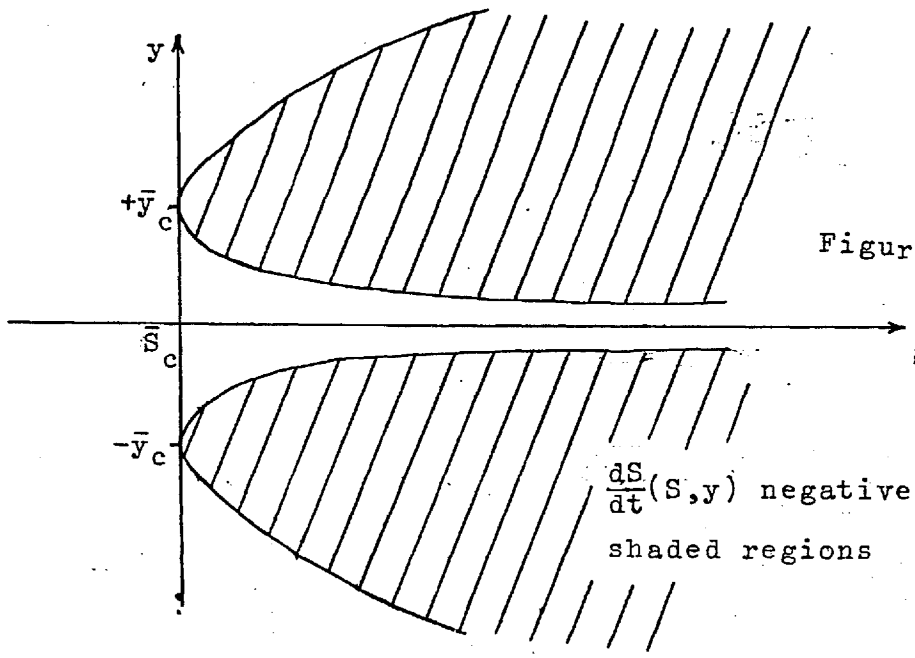


Figure 4.3.1

where $\bar{S}_c=\ln\left(\frac{\lambda}{-(\alpha+1)a_o}\right) - \frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o}$

$$\bar{y}_c^2 = \frac{2\lambda-(3\alpha+1)a_o}{(\alpha^2-1)a_o^2}$$

By modifying the a-priori distribution of t_j, making a disorder less likely to occur while y_t^2 is large, this problem may be avoided. The optimal detection rule for this new problem is guaranteed to be robust, in the sense that the expected detection time for a disorder is not increased if in fact $k_t=\beta_t\geq\alpha$ $\forall t\gg t_j$ in (4.0.1). Since $\bar{y}_c^2$ is large it should also be near-optimal in the original situation.

Alternatively it could be argued that the true optimal detection rule should be "near-robust". In section 4.4 an upper bound is derived for the increase in expected cost due to the use of the guaranteed robust detection rule.

In this section modified versions of the problem are investigated, and the appropriate robustness results obtained following closely the approach of section 4.2.

First Modified Problem

The system defined in (4.0.1) is considered but with the random variable t_j defined so that

$$dI(t \geq t_j) = r(y_t)(1-I(t \geq t_j))dt + d\bar{M}_t \qquad (4.3.1)$$

where $\bar{M}_t$ is a Martingale and

$$r(y) = \lambda \quad \forall y \text{ st } y^2 \leq \bar{y}_c^2 \triangleq \frac{2\lambda-(3\alpha+1)a_o}{(\alpha^2-1)a_o^2}$$

$$= \frac{\frac{1}{2}(\alpha^2-1)a_o^2y^2 + \frac{1}{2}(\alpha-1)a_o}{1 + \exp(-\bar{S}_c-\frac{1}{2}(\alpha-1)a_oy^2)} \quad \forall y \text{ st } y^2 > \bar{y}_c^2 \qquad (4.3.2)$$

$$\bar{S}_c \triangleq \ln\left(\frac{\lambda}{-(\alpha+1)a_o}\right) - \frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o} \qquad (4.3.3)$$

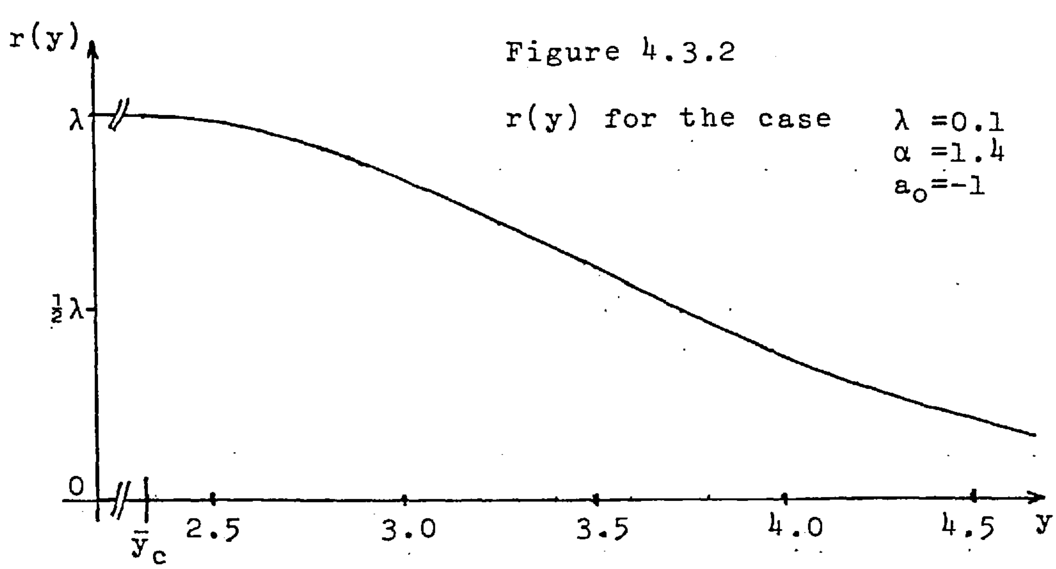


Figure 4.3.2

r(y) for the case $\lambda = 0.1$, $\alpha = 1.4$, $a_o = -1$

$\bar{P}$ and $\bar{E}$ denote probability and expectation respectively given that t_j satisfies (4.3.1), and, except when explicitly stated, that $\bar{P}(t_j=0|\mathcal{Y}_o)=0$ and that $k_t=\alpha\ \forall t\geq t_j$.

Then
$$\lim_{\delta\to 0}\frac{1}{\delta}\bar{P}(t_j\in(t,t+\delta)|t_j>t,y_t=\tilde{y}) = r(\tilde{y}) \qquad (4.3.4)$$

Using the non-linear filtering equations (Appendix 1) as before, if $\bar{\pi}_t=\bar{P}(t\geq t_j|\mathcal{Y}_t)$

$$d\bar{\pi}_t = r(y_t)(1-\bar{\pi}_t)dt + \bar{\pi}_t(1-\bar{\pi}_t)(\alpha-1)a_o y_t d\bar{\nu}_t \qquad (4.3.5)$$

$$dy_t = (1+(\alpha-1)\bar{\pi}_t)a_o y_t dt + d\bar{\nu}_t \qquad (4.3.6)$$

$\bar{\nu}_t$ is a Wiener process (the innovations process)

Note that increments of $\bar{M}_t$ are orthogonal to W_t.

As before, $\bar{R}=\ln(\frac{\bar{\pi}}{1-\bar{\pi}})$, $\bar{S}=\bar{R}-\frac{1}{2}(\alpha-1)a_o y^2$ (4.3.7)

Then
$$\frac{d\bar{S}_t}{dt} = r(y_t)(1+e^{-\bar{S}_t-\frac{1}{2}(\alpha-1)a_o y_t^2})$$
$$- \tfrac{1}{2}(\alpha^2-1)a_o^2 y_t^2 - \tfrac{1}{2}(\alpha-1)a_o \qquad (4.3.8)$$

Because of the definitions (4.3.2),(4.3.3)

if $\frac{d\bar{S}}{dt}(\tilde{S},\tilde{y}) = \left.\frac{d\bar{S}_t}{dt}\right|_{\substack{\bar{S}_t=\tilde{S}\\ y_t=\tilde{y}}}$ then $\frac{d\bar{S}}{dt}(\tilde{S},\tilde{y})\geq 0\ \forall\tilde{S}<\bar{S}_c,\ \forall\tilde{y}$ (4.3.9)

and $\frac{d\bar{S}}{dt}(\tilde{S},\tilde{y})$ is a non-increasing function of $\tilde{y}^2$ for fixed $\tilde{S}\geq\bar{S}_c$. (4.3.10)

In fact $\frac{d\bar{S}}{dt}(\tilde{S},\tilde{y}) = \frac{dS}{dt}(\tilde{S},\tilde{y})$ for $\tilde{y}^2\leq\bar{y}_c^2$ (4.3.11)

Here $\frac{dS}{dt}$ is as introduced in section 4.2.

The existence of a $\mathcal{Y}_t$-stopping time $\bar{\tau}$ which minimizes $\bar{E}_{(\bar{S},y)_o}K(\tilde{\tau})$ follows from Lemma 3.2 with $\rho(y)=r(y)$, and the existence of the stopping boundary $\bar{\gamma}$ from Theorem 3.2.

Continuity in $\bar{S}$ and non-increasing in $\bar{S}$ properties of

$$\bar{h}(\bar{S},y) \triangleq \bar{E}_{(\bar{S},y)}K(\bar{\tau})$$

follow similarly from Corollary 3.2.2.

Then $$\bar{\tau}_{t_o} \triangleq \inf\{t\geq t_o:\bar{h}(\bar{S}_t,y_t)\geq 0\} \qquad (4.3.12)$$

$$= \inf\{t\geq t_o:\bar{S}_t\geq\bar{S}_{\bar{\gamma}}(y_t)\}$$

where $$\bar{S}_{\bar{\gamma}}(y) \triangleq \inf\{\bar{S}:(\bar{S},y)\in\bar{\gamma}\} \qquad (4.3.13)$$

and $$\bar{\tau}=\bar{\tau}_{t_o} \text{ if } \bar{\tau}\geq t_o \qquad (4.3.14)$$

From (4.1.5)

$$\bar{h}(\bar{S},y) = \bar{E}_{(\bar{S},y)}\int_0^{\bar{\tau}}\bar{\sigma}(\bar{S}_u,y_u)du \qquad (4.3.15)$$

where $$\bar{\sigma}(\bar{S},y) \triangleq -\lambda +(\lambda+c)\bar{\pi}(\bar{S},y) \qquad (4.3.16)$$

and $$\bar{\pi}(\bar{S},y) \triangleq \frac{1}{1 + \exp(-\bar{S}-\frac{1}{2}(\alpha-1)a_o y^2)} \qquad (4.3.17)$$

so that $\bar{\pi}_t=\bar{\pi}(\bar{S}_y,y_t)$

As in section 4.1

a) $$\bar{h}(\bar{S}_t,y_t) = \bar{E}_{(\bar{S},y)_t}K_t(\bar{\tau}_t) = \bar{E}_{(\bar{S},y)_t}\int_t^{\bar{\tau}_t}\bar{\sigma}(\bar{S}_u,y_u)du \qquad (4.3.18)$$

b) $$\bar{h}(\bar{S},y) \leq 0 \quad \forall(\bar{S},y) \qquad (4.3.19)$$

c) $$\bar{\sigma}(\bar{S}_t,y_t)<0 \Rightarrow \bar{\tau}_t>t, \text{ so that } \bar{\sigma}(\bar{S},y)\geq 0 \text{ if } (\bar{S},y)\in\bar{\gamma} \qquad (4.3.20)$$

d) $$\bar{h}(\bar{S},y) = \bar{h}(\bar{S},-y), \quad \bar{S}_{\bar{\gamma}}(y) = \bar{S}_{\bar{\gamma}}(-y) \qquad (4.3.21)$$

Next it is shown that $\bar{S}_{\bar{\gamma}}(y)$ is non-increasing with <u>decreasing</u> y^2. The argument used follows closely that used in section 4.2.

As before the state-space of the process $(\bar{S},y)_t$ is divided into three disjoint sets.

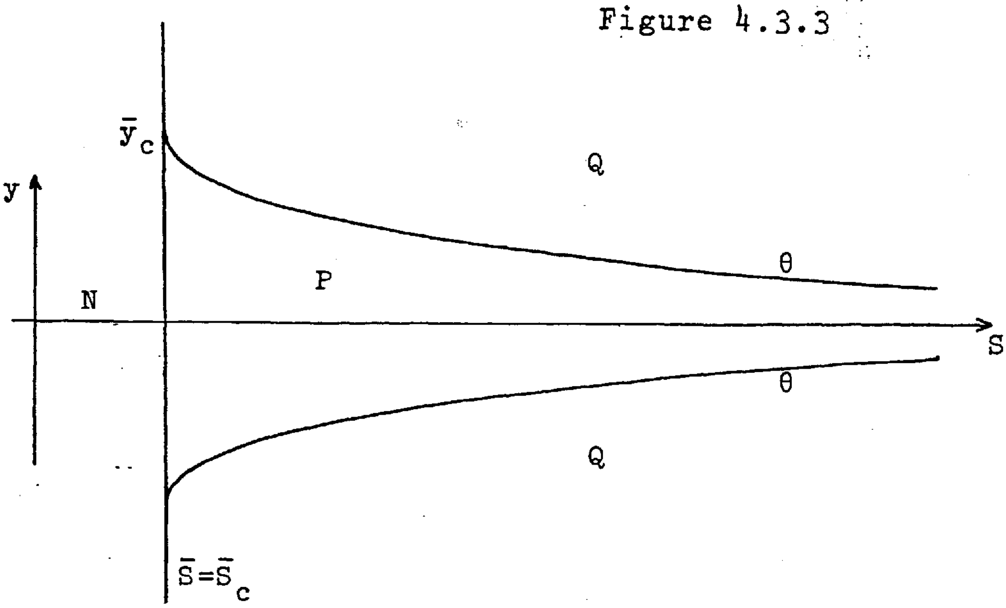
$$N \triangleq \{(\bar{S},y):\bar{S}<\bar{S}_c\}$$

$$P \triangleq \{(\bar{S},y):\frac{d\bar{S}}{dt}(\bar{S},y)\geq 0,\bar{S}\geq\bar{S}_c\} \quad (4.3.22)$$

$$Q \triangleq \{(\bar{S},y):\frac{d\bar{S}}{dt}(\bar{S},y)<0,\bar{S}\geq\bar{S}_c\}$$

$$\theta \triangleq P\cap \text{closure}(Q) \quad (4.3.23)$$

Figure 4.3.3



Define $\bar{t}_c = \inf\{t:(\bar{S},y)_t\in P\cup Q\}$ (4.3.24)

Since $\frac{d\bar{S}}{dt}(\bar{S}_c,y)\geq 0\ \forall y$, it follows that $(\bar{S},y)_t\in P\cup Q\ \forall t\geq\bar{t}_c$.

(4.3.25)

Note that $(\bar{S}_c,y)\in P\ \forall y$.

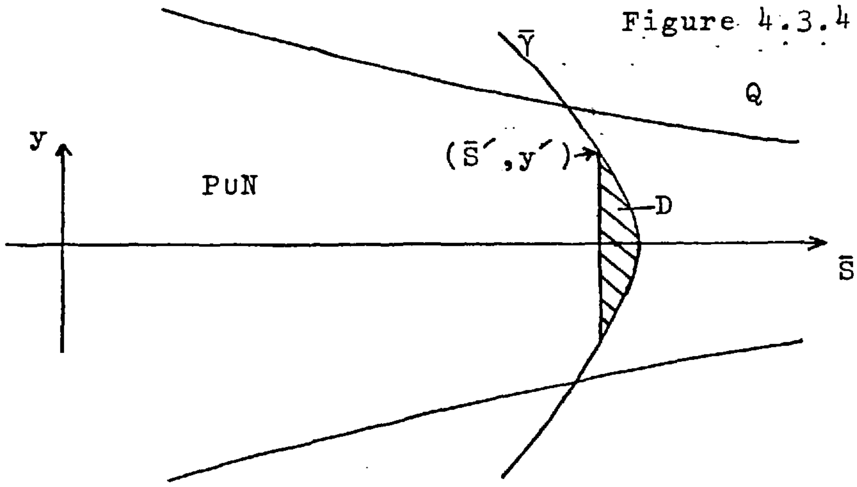
<u>Lemma 4.3</u>

$\bar{S}_{\bar{\gamma}}(y)$ is non-increasing with decreasing y^2 $\forall y$ such that $(\bar{S}_{\bar{\gamma}}(y),y)\in P\cup N$.

<u>Proof</u> (similar to proof of Lemma 4.1)

If the Lemma is not true $\exists y'>0$ such that for $\bar{S}'=\bar{S}_{\bar{\gamma}}(y')$, $(\bar{S}',y')\in P\cup N$ and $\bar{S}_{\bar{\gamma}}(y)$ is strictly increasing with <u>decreasing</u> y at $y=y'$.

Figure 4.3.4



Then if $D \triangleq \{(\bar{S},y):\ \bar{S}\in[\bar{S}',\bar{S}_\gamma(y)],\, y^2\leq y'^2\}$

$D\backslash\bar{\gamma}$ is non-empty ($\bar{\gamma}$ is the boundary of the closed stopping set).

Choose $(\bar{S},y)_{t_o}\in D\backslash\bar{\gamma} \;\Rightarrow\; (\bar{S},y)_t\in D \quad \forall t\in[t_o,\bar{\tau}_{t_o}]$

since $\frac{d\bar{S}}{dt}(\bar{S}',\tilde{y})\geq 0 \quad \forall\tilde{y}\in[-y',y']$

[because a) $(\bar{S}',y')\in N \Rightarrow (\bar{S}',\tilde{y})\in N$

b) $(\bar{S}',y')\in P \Rightarrow \frac{d\bar{S}}{dt}(\bar{S}',\tilde{y})\geq\frac{d\bar{S}}{dt}(\bar{S}',y')\geq 0$ by (4.3.10)]

$\bar{\sigma}(\bar{S},y)\geq 0\ \forall(\bar{S},y)\in D$, since $\bar{\sigma}(\bar{S}',y')\geq 0$ by (4.3.20) and $\bar{\sigma}$ is increasing with $\bar{S}$ and decreasing with y^2 by (4.3.16).

Therefore $\quad \bar{h}(\bar{S}_{t_o},y_{t_o}) = \bar{E}_{(\bar{S},y)_{t_o}}\int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)\,du \geq 0$

But this contradicts (4.3.12) since $\bar{S}_{t_o}<\bar{S}_{\bar{\gamma}}(y_{t_o})$. □

<u>Definition</u>

Let $\quad \Gamma \triangleq \{(\bar{S},y)\in Q\cap\bar{\gamma}: \exists(\bar{S}',y')\in Q\cap\bar{\gamma} \text{ with } \bar{S}'>\bar{S}, y'^2<y^2\}$

then $$\bar{S}_1 \triangleq \begin{cases} \inf\{\bar{S}:(\bar{S},y)\in\Gamma\} & \text{if } \Gamma\notin\emptyset \\ +\infty & \text{if } \Gamma\in\emptyset \end{cases} \qquad (4.3.26)$$

(see over)

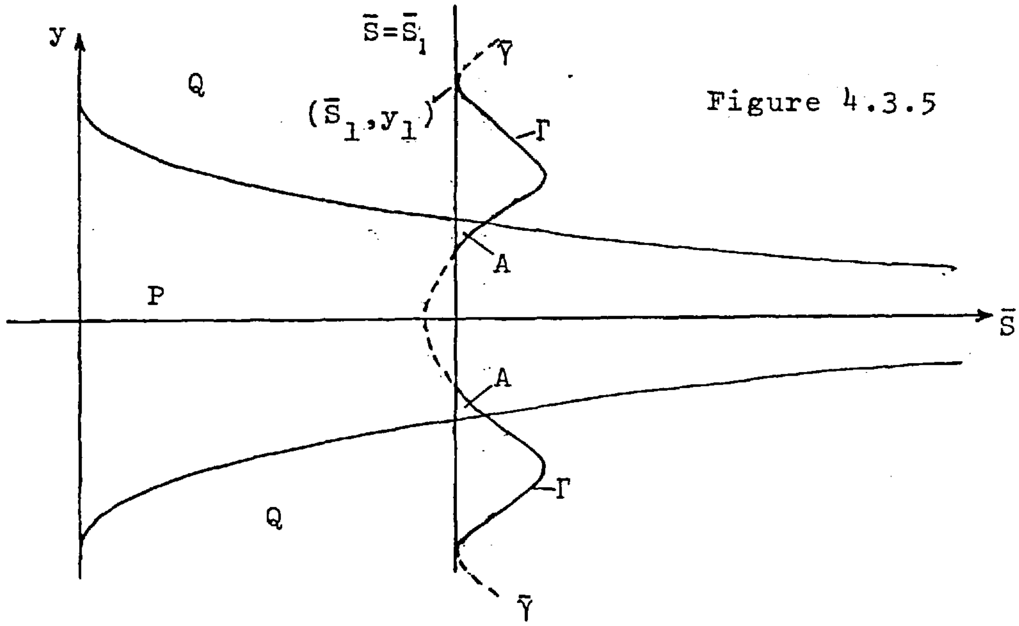


Figure 4.3.5

If $\bar{S}_1<\infty$ choose $y_1 \triangleq \sup\{y\geq 0:(\bar{S}_1,y)\in\bar{\gamma}\}$ (4.3.27)

$(\bar{S}_1,y_1)\in\Gamma$ since the stopping set is closed (Theorem 3.1).

$y_1<\infty$ as $\bar{S}_1<\bar{S}_{\bar{\gamma}}(y)$ for y^2 large since $\bar{\sigma}(\bar{S},y)\downarrow-\lambda$ as $y^2\to\infty$.

Note that $\bar{S}_{\bar{\gamma}}(y)$ is non-increasing with decreasing y^2 $\forall y$ st $\bar{S}_{\bar{\gamma}}(y)<\bar{S}_1$ by (4.3.26) and Lemma 4.3.

Lemma 4.4

If $\bar{S}_1<\infty$, $\bar{h}(\bar{S},y)$ is non-increasing with increasing y^2 for $(\bar{S},y)\in P$, $\bar{S}\in[\bar{S}_1,\bar{S}_{\bar{\gamma}}(y))$ (i.e. $(\bar{S},y)$ in sets "A", Fig 4.3.5)

Proof (similar to proof of Lemma 4.2)

Suppose the Lemma is not true.

Then $\exists$ $\bar{S}_2\geq\bar{S}_1$, $y_3>y_2\geq 0$ such that

$$\bar{S}_2<\bar{S}_{\bar{\gamma}}(y_2), \quad \bar{S}_2<\bar{S}_{\bar{\gamma}}(y_3)$$

$$(\bar{S}_2,y_2),(\bar{S}_2,y_3)\in P$$

& $$\bar{h}(\bar{S}_2,y_2) < \bar{h}(\bar{S}_2,y_3) \quad (4.3.28)$$

$D' \triangleq \{(\bar{S},y):y^2\leq y_3^2,\bar{S}_2\in[\bar{S}_2,\bar{S}_{\bar{\gamma}}'(y)]\}$

(see Figure 4.3.6)

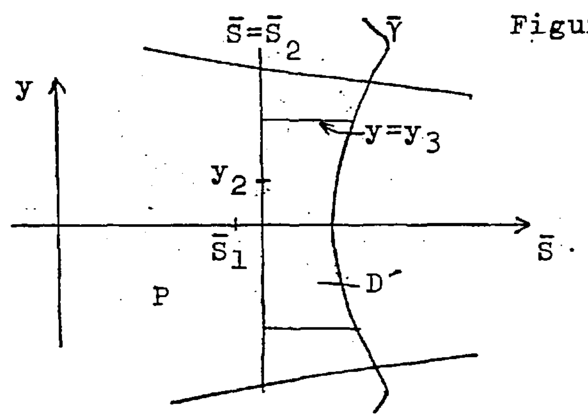


Figure 4.3.6

Suppose $(\bar{S},y)_{t_o} = (\bar{S}_2,y_2)$. Then the process $(\bar{S},y)_t$ leaves D' either across $\bar{Y}$ or across the lines $y=y_3$ or $y=-y_3$.

$\frac{d\bar{S}}{dt}(\bar{S}_2,y) \geq 0 \ \forall y$ st $y^2 \leq y_3^2$ by (4.3.10).

Define $t_1 = \inf\{t \geq t_o : y_t = \pm y_3\}$

$$\int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du = \int_{t_o}^{\bar{\tau}_{t_o}\wedge t_1} \bar{\sigma}(\bar{S}_u,y_u)du + \int_{\bar{\tau}_{t_o}\wedge t_1}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du$$

The first term on the right is positive or zero, as $\bar{\sigma}(\bar{S},y) \geq 0 \ \forall (\bar{S},y) \in D'$. This is because $\bar{\sigma}(\bar{S}_1,y_1) \geq 0$ by (4.3.20) and $\bar{\sigma}$ is increasing with $\bar{S}$ and decreasing with y^2 from (4.3.16).

As $\bar{\tau}_{t_o}\wedge t_1$ is a $\mathcal{Y}_t$-stopping time

$$\bar{h}(\bar{S}_2,y_2) = \bar{E}_{(\bar{S},y)_{t_o}} \int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du$$

$$\geq \bar{E}_{(\bar{S},y)_{t_o}} [\bar{E}(\int_{\bar{\tau}_{t_o}\wedge t_1}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du \,|\, \mathcal{Y}_{\bar{\tau}_{t_o}\wedge t_1})]$$

$$\geq \bar{E}_{(\bar{S},y)_{t_o}} [\bar{h}(\bar{S}_{\bar{\tau}_{t_o}\wedge t_1}, y_{\bar{\tau}_{t_o}\wedge t_1}) \,|\, \bar{\tau}_{t_o} \geq t_1]$$

$$\geq \bar{h}(\bar{S}_2,y_3) \qquad (4.3.29)$$

The second inequality is because if $\bar{\tau}_{t_o} < t_1$,

$\bar{h}(\bar{S}_{\bar{\tau}_{t_o}\wedge t_1}, y_{\bar{\tau}_{t_o}\wedge t_1}) = 0$ by (4.3.18).

For $\bar{\tau}_{t_o} \geq t_1$, $\bar{S}_{\bar{\tau}_{t_o}\wedge t_1} \geq \bar{S}_2$, $y_{\bar{\tau}_{t_o}\wedge t_1} = \pm y_3$

Then by Corollary 3.2.2 $\bar{h}(\bar{S}_{\bar{\tau}_{t_o}\wedge t_1}, y_{\bar{\tau}_{t_o}\wedge t_1}) \geq \bar{h}(\bar{S}_2,y_3)$

which establishes the third inequality. But (4.3.29) contradicts (4.3.28). □

Theorem 4.3

$\bar{S}_{\bar{\gamma}}(y)$ is non-increasing with decreasing y^2

Proof (similar to proof of Theorem 4.1)

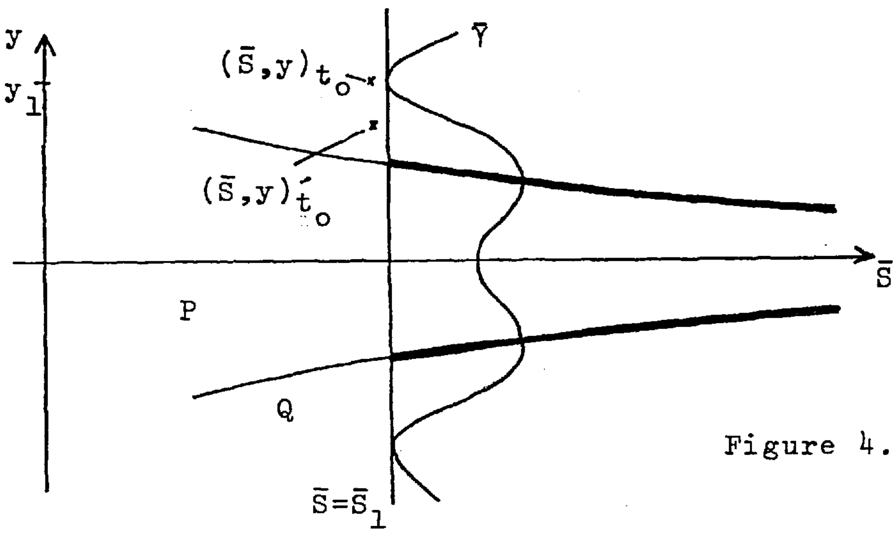


Figure 4.3.7

Suppose the Theorem is not true. Then by Lemma 4.3 and (4.3.26) $\bar{S}_1<\infty$.

y_1 exists and is defined in (4.3.27). $y_1<\infty$.

$y'\in[0,y_1)$ is chosen so $\bar{S}_{\bar{\gamma}}(y')>\bar{S}_{\bar{\gamma}}(y_1)$. (4.3.26) gaurantees that such a y' exists.

Let $(\bar{S},y)_t$, $(\bar{S},y)'_t$ both be solutions of (4.3.6) and (4.3.8)

i.e. $dy_t = (1+(\alpha-1)\bar{\pi}(\bar{S}_t,y_t))a_o y_t dt + d\bar{\nu}_t$

$$\frac{d\bar{S}_t}{dt} = r(y_t)(1+e^{-\bar{S}_t-\frac{1}{2}(\alpha-1)a_o y_t^2}) - \tfrac{1}{2}(\alpha^2-1)a_o^2 y_t^2 - \tfrac{1}{2}(\alpha-1)a_o \qquad (4.3.30)$$

with the innovations process $\bar{\nu}_t$ the same in both cases but with

$$(\bar{S},y)_{t_o}=(\bar{S}_1-\varepsilon,y_1), \qquad (\bar{S},y)'_{t_o}=(\bar{S}_1-\varepsilon,y') \qquad (4.3.31)$$

Here ε is chosen so that $\bar{S}_c\le\bar{S}_1-\varepsilon<\bar{S}_1$. This is possible since $\bar{S}_1\le\bar{S}_c$ contradicts Lemma 4.3.

$\bar{S}'_t$, y'_t are defined such that $(\bar{S}'_t, y'_t) = (\bar{S}, y)'_t$

Note that $(y_t : t \geq t_o)$ and $(y'_t : t \geq t_o)$ both generate the same σ-field $\mathcal{Y}_t$ (both processes may be reconstructed given $\bar{\nu}_t$ - see Lemma 3.1). In this proof all probabilities and expectations are conditioned on the initial conditions (4.3.31).

The following $\mathcal{Y}_t$-stopping times are defined

$$t_1 \triangleq \inf\{t \geq t_o : y'^2_t = y_t^2\} \qquad (4.3.32)$$

$$t_2 \triangleq \inf\{t \geq t_o : (\bar{S}, y)_t \in \theta, \bar{S}_t \geq \bar{S}_1\} \qquad (4.3.33)$$

$$\bar{\tau}_{t_o} \triangleq \inf\{t \geq t_o : (\bar{S}, y)_t \in \bar{\gamma}\} \qquad (4.3.34)$$

$$\bar{\tau}'_{t_o} \triangleq \inf\{t \geq t_o : (\bar{S}, y)'_t \in \bar{\gamma}\} \qquad (4.3.35)$$

Note that (4.3.34) is equivalent to (4.3.12) in this case. t_2 is the first time $(\bar{S}, y)_t$ crosses the thick line in Figure 4.3.7.

Also $(\bar{S}, y)_t \in P \cup Q \quad \forall t \geq t_o$ (c.f. (4.3.25)), and by (4.3.10)

$$\frac{d\bar{S}}{dt}(\tilde{S}, \hat{y}) \geq \frac{d\bar{S}}{dt}(\tilde{S}, \tilde{y}) \text{ if } \hat{y}^2 \leq \tilde{y}^2, \quad (\tilde{S}, \tilde{y}) \in P \cup Q \qquad (4.3.36)$$

A preliminary result is now established.

Suppose $\bar{\tau}_{t_o} < t_1$, $\bar{\tau}_{t_o} < t_2$.

Then since $\frac{d\bar{S}}{dt}(\bar{S}, y) < 0$ in Q and by definition of t_2,

$(\bar{S}, y)_t \in P \cup \{(\bar{S}, y) : \bar{S} < \bar{S}_1\} \quad \forall t \leq \bar{\tau}_{t_o}$ (see Figure 4.3.8).

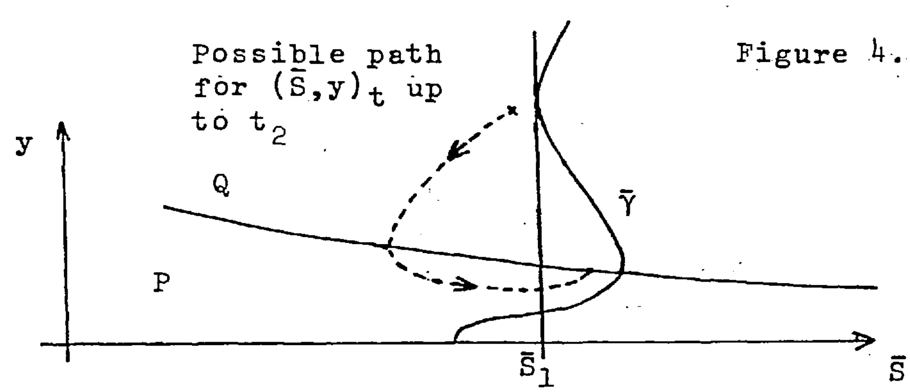


Figure 4.3.8

As $\bar{\tau}_{t_o}<t_1$, $y_t'^2<y_t^2$ $\forall t\le\bar{\tau}_{t_o}$. Then from (4.3.10) $\bar{S}_t'\ge\bar{S}_t$ $\forall t\le\bar{\tau}_{t_o}$.

Since , from Lemma 4.3 and (4.3.26) $\bar{S}_\gamma(\tilde{y})$ is non-increasing with-decreasing $\tilde{y}^2$ for $\tilde{y}^2<y^2_{\bar{\tau}_{t_o}}$ (see Figure 4.3.8).

$$\bar{S}'_{\bar{\tau}_{t_o}} \ge \bar{S}_{\bar{\tau}_{t_o}} = \bar{S}_\gamma(y_{\bar{\tau}_{t_o}}) \ge \bar{S}_\gamma(y'_{\bar{\tau}_{t_o}})$$

so that $\bar{\tau}'_{t_o}\le\bar{\tau}_{t_o}$.

Therefore $\bar{\tau}_{t_o}<t_1$, $\bar{\tau}_{t_o}<t_2 \Rightarrow \bar{\tau}'_{t_o}\le\bar{\tau}_{t_o}$ (4.3.37)

The following events are defined

$$\begin{aligned} A &\triangleq \{\omega: t_1\le\min(t_2,\bar{\tau}'_{t_o})\} \\ B &\triangleq \{\omega: t_2<\min(t_1,\bar{\tau}'_{t_o})\} \\ C &\triangleq \{\omega: \bar{\tau}'_{t_o}\le t_2, \bar{\tau}'_{t_o}<t_1\} \end{aligned} \qquad (4.3.38)$$

A,B,C are disjoint, and $\omega\in A\cup B\cup C$ w.p.1.

Each event is now considered separately.

EVENT A

If $\omega\in A$, $t_1\le t_2$, $t_1\le\bar{\tau}'_{t_o}$. By (4.3.37) it follows that $\bar{\tau}_{t_o}\ge t_1$.

Also $y_t'^2\le y_t^2$, $\bar{S}_t'\ge\bar{S}_t$ $\forall t\le t_1$.

Since $\bar{\sigma}(\bar{S},y)$ increases with $\bar{S}$ and decreases with y^2

$$\int_{t_o}^{t_1} \bar{\sigma}(\bar{S}_u',y_u')du \ge \int_{t_o}^{t_1} \bar{\sigma}(\bar{S}_u,y_u)du$$

Also, since $y_{t_1}'^2=y_{t_1}^2$ and $\bar{S}'_{t_1}\ge\bar{S}_{t_1}$, $\bar{h}(\bar{S}'_{t_1},y'_{t_1}) \ge \bar{h}(\bar{S}_{t_1},y_{t_1})$ from Corollary 3.2.2.

$$\begin{aligned} \text{Then}\quad & \bar{E}[\int_{t_o}^{\bar{\tau}'_{t_o}} \bar{\sigma}(\bar{S}_u',y_u')du - \int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du \,|\, \omega\in A] \\ &= \bar{E}[\int_{t_o}^{t_1}(\bar{\sigma}(\bar{S}_u',y_u')-\bar{\sigma}(\bar{S}_u,y_u))du \,|\, \omega\in A] \\ &\quad + \bar{E}[\bar{h}(\bar{S}'_{t_1},y'_{t_1})-\bar{h}(\bar{S}_{t_1},y_{t_1}) \,|\, \omega\in A] \ge 0 \end{aligned} \qquad (4.3.39)$$

since A is a $\mathcal{Y}_{t_1}$-measurable event, and from (4.3.18).

EVENT B

If $\omega\epsilon B$, $t_2<t_1$, $t_2<\bar{\tau}'_{t_o}$

If $\bar{\tau}_{t_o}<t_2$, (4.3.37) gives a contradiction. Therefore $\bar{\tau}_{t_o}\geq t_2$.

Since $t_2<t_1$, as before, $y_t'^2<y_t^2$, $\bar{S}_t'\geq\bar{S}_t$ $\forall t\leq t_2$.

$$\bar{E}[\int_{t_o}^{\bar{\tau}'_{t_o}} \bar{\sigma}(\bar{S}_u',y_u')du-\int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du|\omega\epsilon B]$$

$$= \bar{E}[\int_{t_o}^{t_2}(\bar{\sigma}(\bar{S}_u',y_u')-\bar{\sigma}(\bar{S}_u,y_u))du|\omega\epsilon B]$$

$$+ \bar{E}[\bar{h}(\bar{S}'_{t_2},y'_{t_2})-\bar{h}(\bar{S}_{t_2},y_{t_2})|\omega\epsilon B]$$

The first term on the right is positive or zero by the properties of $\bar{\sigma}$.

$\bar{h}(\bar{S}'_{t_2},y'_{t_2}) \geq \bar{h}(\bar{S}_{t_2},y'_{t_2}) \geq \bar{h}(\bar{S}_{t_2},y_{t_2})$

where the first inequality is from Corollary 3.2.2 since $\bar{S}'_{t_2}\geq\bar{S}_{t_2}$, and the second inequality is from Lemma 4.4 using $\bar{S}_{t_2}\geq\bar{S}_1$, $y'^2_{t_2}<y^2_{t_2}$.

Therefore $\bar{E}[\int_{t_o}^{\bar{\tau}'_{t_o}} \bar{\sigma}(\bar{S}_u',y_u')du-\int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du|\omega\epsilon B] \geq 0$

(4.3.40)

EVENT C

If $\omega\epsilon C$, $\bar{\tau}'_{t_o}\leq t_2$, $\bar{\tau}'_{t_o}<t_1$.

From (4.3.37) $\bar{\tau}_{t_o}<t_1$, $\bar{\tau}_{t_o}<t_2$ => $\bar{\tau}'_{t_o}\leq\bar{\tau}_{t_o}$, so that $\bar{\tau}_{t_o}<\bar{\tau}'_{t_o}$ leads to a contradiction if $\omega\epsilon C$.

Therefore $\bar{\tau}_{t_o}\geq\bar{\tau}'_{t_o}$.

$y_t'^2<y_t^2$, $\bar{S}_t'\geq\bar{S}_t$ $\forall t\leq\bar{\tau}'_{t_o}$.

$$\bar{E}[\int_{t_o}^{\bar{\tau}'_{t_o}} \bar{\sigma}(\bar{S}_u',y_u')du-\int_{t_o}^{\bar{\tau}_{t_o}} \bar{\sigma}(\bar{S}_u,y_u)du|\omega\epsilon C]$$

$$= \bar{E}[\int_{t_o}^{\bar{\tau}'_{t_o}}(\bar{\sigma}(\bar{S}_u',y_u')-\bar{\sigma}(\bar{S}_u,y_u))du|\omega\epsilon C] + \bar{E}[-\bar{h}(\bar{S}_{\bar{\tau}'_{t_o}},y_{\bar{\tau}'_{t_o}})|\omega\epsilon C]$$

$$\geq 0 \qquad (4.3.41)$$

COMPLETION OF PROOF

From (4.3.39), (4.3.40), (4.3.41)

$$\bar{E}[\int_{t_o}^{\bar{\tau}'t_o} \bar{\sigma}(\bar{S}_u, y_u)du - \int_{t_o}^{\bar{\tau}t_o} \bar{\sigma}(\bar{S}_u, y_u)du | \omega\epsilon F] \geq 0$$

for F=A, B, C

Therefore $\bar{h}(\bar{S}'_{t_o}, y'_{t_o}) - \bar{h}(\bar{S}_{t_o}, y_{t_o})$

$$= \bar{E}[\int_{t_o}^{\bar{\tau}'t_o} \bar{\sigma}(\bar{S}'_u, y'_u)du - \int_{t_o}^{\bar{\tau}t_o} \bar{\sigma}(\bar{S}_u, y_u)du] \geq 0$$

i.e. $\bar{h}(\bar{S}_1-\varepsilon, y') \geq \bar{h}(\bar{S}_1-\varepsilon, y_1)$

Now as $\varepsilon \downarrow 0$, $\bar{h}(\bar{S}_1-\varepsilon, y_1) \to 0$ by Corollary 3.2.2, and because $(\bar{S}_1, y_1)\epsilon\bar{\gamma}$.

So $\lim_{\varepsilon\downarrow 0} \bar{h}(\bar{S}_1-\varepsilon, y') \geq 0$. By continuity of $\bar{h}$ with $\bar{S}$ (Corollary 3.2.2)

$$\bar{h}(\bar{S}_1, y') \geq 0 \Rightarrow (\bar{S}_1, y')\epsilon\bar{\gamma}.$$

But y' was chosen so that $\bar{S}_1 < \bar{S}_\gamma(y')$ which gives a contradiction. □

<u>Second modified problem</u>

Before proceeding to investigate robustness a slightly different version of the problem is introduced. Here y_t is still generated by (4.0.1) but the random variable t_j is defined such that

$$dI(t \geq t_j) = [I(S^*_t < \bar{S}_c)\lambda + I(S^*_t \geq \bar{S}_c)r(y_t)](1 - I(t \geq t_j))dt + dM^*_t \qquad (4.3.42)$$

where M^*_t is a Martingale

$r(y)$ is defined in (4.3.2)

$\bar{S}_c$ is defined in (4.3.3)

and S^*_t is defined by

$$S_o^* = \ln\left(\frac{\pi_o^*}{1-\pi_o^*}\right) - \tfrac{1}{2}(\alpha-1)a_o y_o^2 \qquad \text{where } \pi_o^* = P(t_j=0|\mathcal{Y}_o)$$

$$\frac{dS_t^*}{dt} = [I(S_t^*<\bar{S}_c)\lambda + I(S_t^*\geq\bar{S}_c)r(y_t)](1+e^{-S_t^*-\frac{1}{2}(\alpha-1)a_o y_t^2})$$

$$- \tfrac{1}{2}(\alpha^2-1)a_o^2 y_t^2 - \tfrac{1}{2}(\alpha-1)a_o \tag{4.3.43}$$

P^*, E^* denote probability and expectation respectively given that t_j satisfies (4.3.42), and, unless explicitly stated, that $P^*(t_j=0|\mathcal{Y}_o)=0$ and that $k_t=\alpha\ \forall t\geq t_j$ in (4.0.1).

Then
$$\lim_{\delta\downarrow 0} \frac{1}{\delta}P^*(t_j\in(t,t+\delta)|t_j>t, S_t^*, y_t) = \lambda \text{ if } S_t^*<\bar{S}_c$$

$$= r(y_t) \text{ if } S_t^*\geq\bar{S}_c \tag{4.3.44}$$

Using as before the non-linear filtering equations (Appendix 1) if $\pi_t^* \triangleq P^*(t\geq t_j|\mathcal{Y}_t)$

$$d\pi_t^* = [I(S_t^*<\bar{S}_c)\lambda + I(S_t^*\geq\bar{S}_c)r(y_t)](1-\pi_t^*)(\alpha-1)a_o y_t d\nu_t^* \tag{4.3.45}$$

$$dy_t = [1+(\alpha-1)\pi_t^*]a_o y_t dt + d\nu_t^* \tag{4.3.46}$$

ν_t^* is a Wiener process (the innovations process).

$R^* \triangleq \ln\left(\frac{\pi^*}{1-\pi^*}\right)$

It turns out that $S_t^* = R_t^* - \tfrac{1}{2}(\alpha-1)a_o y_t^2$ (4.3.47)

Note that $S_t^*=S_t\ \forall t<t_c$, where S_t is defined by (4.1.3) and $t_c \triangleq \inf\{t: S_t\geq\bar{S}_c\}$. (4.3.48)

Also $S_t^*\geq\bar{S}_c\ \ \forall t\geq t_c$, since $\frac{dS^*}{dt}(\bar{S}_c, y)\geq 0\ \forall y$ (4.3.49)

and $\frac{dS^*}{dt}(\tilde{S},\tilde{y}) = \frac{d\bar{S}}{dt}(\tilde{S},\tilde{y})\ \ \forall\tilde{S}\geq\bar{S}_c,\ \ \forall y$ (4.3.50)

For $(S^*,y)_{t_o}=(\bar{S},y)_{t_o}=(\tilde{S},\tilde{y})$, $\nu_t^*=\bar{\nu}_t$ $\forall t\geq t_o$, $\tilde{S}\geq\bar{S}_c$

(4.3.6),(4.3.8) and (4.3.46),(4.3.43) have identical solutions for $t\geq t_o$.

$$\text{Therefore}\quad E^*_{(\tilde{S},\tilde{y})}K_{t_o}(\tilde{\tau}_{t_o}) = E^*_{(\tilde{S},\tilde{y})}[-\lambda\tilde{\tau}_{t_o}+(\lambda+c)\int_{t_o}^{\tilde{\tau}_{t_o}}\pi_u^* du]$$

$$= \bar{E}_{(\tilde{S},\tilde{y})}[-\lambda\tilde{\tau}_{t_o}+(\lambda+c)\int_{t_o}^{\tilde{\tau}_{t_o}}\bar{\pi}_u du] = \bar{E}_{(\tilde{S},\tilde{y})}K_{t_o}(\tilde{\tau}_{t_o})$$

for any $\mathcal{Y}_t$-stopping time $\tilde{\tau}_{t_o}\geq t_o$ (4.3.51)

Therefore the optimal detection time $\tau^*_{t_o}$, in the sense of the expected cost $E^*_{(S^*,y)_{t_o}}K_{t_o}(\tilde{\tau}_{t_o})$, satisfies

$$\tau^*_{t_o} = \inf\{t\geq t_o: S_t^*\geq\bar{S}_{\bar{\gamma}}(y_t)\},\quad \text{if } t_o\geq t_c \tag{4.3.52}$$

<u>Note</u>

If $-(\alpha+1)a_o\geq c$, as would be expected, $\tau^*_{t_o}=\inf\{t\geq t_o:S_t^*\geq\bar{S}_{\bar{\gamma}}(y_t)\}$

for $t_o<t_c$ too, since then $-\lambda+(\lambda+c)\pi_t^*<0$ if $S_t^*<\bar{S}_c \Rightarrow \tau^*\geq t_c$, from (4.1.7) (c.f. (4.1.17)).

The robustness result is now derived.

<u>Theorem 4.4</u>

$$E(\tau^*_{t_j}-t_j|(S^*,y)_{t_j},t_j,k_t=\beta_t\ \forall t\geq t_j)$$

$$\leq E(\tau^*_{t_j}-t_j|(S^*,y)_{t_j},t_j,k_t=\alpha\ \forall t\geq t_j)$$

if $\beta_t\geq\alpha>1$ $\forall t\geq t_j\geq t_c$

<u>Proof</u> (similar to proof of Theorem 4.2)

Suppose $\beta_t\geq\alpha$ $\forall t\geq t_j\geq t_c$

Define y_t^α such that

$$dy_t^\alpha = \alpha a_o y_t^\alpha dt + dW_t^\alpha \qquad t\geq t_j$$
$$y_{t_j}^\alpha = y_{t_j} \tag{4.3.53}$$

where W_t^α is a Wiener process. Define y_t^β such that

$$dy_t^\beta = \beta_t a_o y_t^\beta dt + dW_t^\beta \qquad t \geq t_j$$
$$y_{t_j}^\beta = y_{t_j} \qquad (4.3.54)$$

From Itô's differentiation rule, if $x_t^\alpha = (y_t^\alpha)^2$, $x_t^\beta = (y_t^\beta)^2$

$$x_{t_j}^\alpha = x_{t_j}^\beta$$
$$dx_t^\alpha = (2\alpha a_o x_t^\alpha + 1)dt + 2\surd(x_t^\alpha).dV_t^\alpha \qquad (4.3.55)$$
$$dx_t^\beta = (2\beta_t a_o x_t^\beta + 1)dt + 2\surd(x_t^\beta).dV_t^\beta$$

where $V_t^\alpha = \int_{t_j}^t J(y_u^\alpha)dW_u^\alpha$

$V_t^\beta = \int_{t_j}^t J(y_u^\beta)dW_u^\beta \qquad t \geq t_j$

$J(x) = +1$ if $x \geq 0$

$\qquad -1$ if $x < 0$

V_t^α, V_t^β are then Wiener processes. Suppose that W_t^α, W_t^β are chosen so that $V_t^\alpha = V_t^\beta = V_t$. Then, by [22, Theorem 1.1]

$$x_t^\beta \leq x_t^\alpha \quad \forall t \geq t_j \qquad (4.3.56)$$

Now define S_t^α, S_t^β so that $S_{t_j}^\alpha = S_{t_j}^\beta = S_{t_j}^*$ and (S_t^α, y_t^β) & (S_t^β, y_t^β) satisfy

$$\frac{d\tilde{S}^*}{dt} = r(\tilde{y}_t)(1+e^{-\tilde{S}_t^* - \frac{1}{2}(\alpha-1)a_o\tilde{y}_t^2}) - \tfrac{1}{2}(\alpha^2-1)a_o^2\tilde{y}_t^2 - \tfrac{1}{2}(\alpha-1)a_o$$

As $t_j \geq t_c$ $S_t^\alpha, S_t^\beta \geq \bar{S}_c$ $\forall t \geq t_j$. $\frac{dS^*}{dt}$ is a decreasing function of $\tilde{y}^2$ for given $\tilde{S}^* \geq \bar{S}_c$. Therefore from (4.3.56)

$$S_t^\beta \geq S_t^\alpha \quad \forall t \geq t_j \qquad (4.3.57)$$

Now define $\quad \tau^\alpha = \inf\{t \geq t_j : S_t^\alpha \geq \bar{S}_{\bar{\gamma}}(y_t^\alpha)\}$

$\qquad\qquad\qquad \tau^\beta = \inf\{t \geq t_j : S_t^\beta \geq \bar{S}_{\bar{\gamma}}(y_t^\beta)\}$

then $\quad S_{\tau^\alpha}^\beta \geq S_{\tau^\alpha}^\alpha = \bar{S}_{\bar{\gamma}}(y_{\tau^\alpha}^\alpha) \geq \bar{S}_{\bar{\gamma}}(y_{\tau^\alpha}^\beta)$

The final inequality follows from (4.3.56), noting that $x_t \triangleq y_t^2$, and Theorem 4.3.

Therefore $\tau^\beta \le \tau^\alpha$

The result of the Theorem now follows because of the way in which y_t^α, y_t^β, τ^α, τ^β have been defined. □

It follows as in section 4.2 that if $\beta_t \ge \alpha\ \forall t$

$$E_{(S,y)_o}[C(\tau^*)|k_t=\beta_t \forall t\ge t_j, t_j\ge t_c]$$

$$\le E_{(S,y)_o}[C(\tau^*)|k_t=\alpha \forall t\ge t_j, t_j\ge t_c] \tag{4.3.58}$$

This also holds with $C(\tau^*)$ replaced by $K(\tau^*)$.
Note that the distribution of the time t_j at which the disorder occurs, specified by the notation E, $\bar{E}$ or E^* is irrelevant in (4.3.58) because of the conditioning in Theorem 4.4. The robustness of the detection rule τ^* is established regardless of this distribution.

In this case it is possible to say something about t_c, defined in (4.3.48).
Since $S_t = \ln(\pi_t/(1-\pi_t)) - \frac{1}{2}(\alpha-1)a_o y_t^2$

and from (4.3.48)

$$t_c \le \hat{\tau} = \inf\{t:\pi_t \ge \frac{\lambda}{\lambda-(\alpha+1)a_o\exp(\frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o})}\}$$

Therefore $P_{(S,y)_o}(t_j\le t_c) \le P_{(S,y)_o}(t_j\le\hat{\tau}) = E_{(S,y)_o}\pi_{\hat{\tau}}$

$$\le \frac{\lambda}{\lambda-(\alpha+1)a_o\exp(\frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o})} = \rho, \text{ say} \tag{4.3.59}$$

since t_c is a $\mathcal{Y}_t$-stopping time.

Also $E_{(S,y)_o}\pi_{t_c} = E_{(S,y)_o}\int_o^{t_c}\lambda(1-\pi_u)du$ from (4.0.2)

so that $\rho\ge\lambda(1-\rho)E_{(S,y)_o}t_c$.

$$\text{Therefore } E_{(S,y)_o}t_c \le \frac{\rho}{\lambda(1-\rho)} = \frac{1}{-(\alpha+1)a_o\exp(\frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o})} \tag{4.3.60}$$

If $\lambda/(-a_o)$ is small, as would be expected, this leads to the following approximate values for the upper bounds given in (4.3.59) and (4.3.60).

α	upper bound for $E_{(S,y)_o} t_c$	upper bound for $P_{(S,y)_o}(t_j<t_c)$
1.1	$1.326/(-a_o)$	$1.326\lambda'$
1.2	$1.293/(-a_o)$	$1.293\lambda'$
1.4	$1.231/(-a_o)$	$1.231\lambda'$
1.7	$1.146/(-a_o)$	$1.146\lambda'$
2.0	$1.070/(-a_o)$	$1.070\lambda'$

where $\lambda' \triangleq \lambda/(-a_o)$.

$P_{(S,y)_o}(t_j<t_c)$ is then typically small.

4.4 The sub-optimal detection rule $\alpha>1$

Theorem 4.4 provides a robustness result for the detection rule τ^*, which is valid regardless of the distribution of t_j. τ^* is the optimal detection time if t_j is distributed according to (4.3.42). Under this distribution the probability density

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}P^*(t_j\epsilon(t,t+\delta)|t_j>t,y_t=y)$$

$$= \lim_{\delta\downarrow 0} \frac{1}{\delta}P(t_j\epsilon(t,t+\delta)|t_j>t,y_t=y) = \lambda$$

for $y^2\leq\bar{y}_c^2$, while it is reduced if $y^2\geq\bar{y}_c^2$ (4.4.1)

The disorder is less likely to occur while $y_t^2>\bar{y}_c^2$. However $\bar{y}_c$ is typically several times the standard deviation of y_t $t\leq t_j$, so that most of the time $y_t^2\leq\bar{y}_c^2$. In table 4.4.1 values of the probability that the disorder is delayed are given if $y_o\sim N(0,-\frac{1}{2a_o})$ and t_j is distributed according to (4.0.2). The low values obtained, together with the fact that τ and τ^* minimize the expected values of $K(\tilde{\tau})$ for their respective cases suggest that the properties of τ will be similar to those of τ^*. In particular, it is likely that (4.3.52) holds for τ as well as τ^*, even if the result corresponding to Theorem 4.4 does not hold for every $(S,y)_{t_j}$.

Here the increase in the expected cost resulting from the use of the detection time τ^* with its guaranteed robustness properties is investigated, where (4.0.2) holds.

In order to do this the following situation is considered:

$$dy_t^o = [1+(\alpha-1)I(t\geq t_j^o)]a_o y_t^o dt + dV_t \ , \quad y_o^o=y_o \quad (4.4.2)$$

$$dy_t^* = [1+(\alpha-1)I(t\geq t_j^*)]a_o y_t^* dt + dW_t \ , \quad y_o^*=y_o \quad (4.4.3)$$

where V_t, W_t are Wiener processes such that $V_t=W_t \ \forall t\leq\hat{t}$;
$(V_t-V_{\hat{t}})$ and $(W_t-W_{\hat{t}})$ are independent for $t\geq\hat{t}$ (4.4.4)

$t_j^*\geq 0$ is defined so that

$$dI(t\geq t_j^*) = [\lambda+(r(y_t^*)-\lambda)I(t\geq t_c)](1-I(t\geq t_j^*))dt + dM_t^*$$

M_t^* a Martingale, $P(t_j^*=0|\mathcal{Y}_o)=0$ (4.4.5)

$\hat{t}\geq 0$ is defined so that

$$dI(t\geq\hat{t}) = (\lambda-r(y_t^*))I(t\geq t_c)(1-I(t\geq\hat{t}))dt + d\hat{M}_t \quad (4.4.6)$$

$\hat{M}_t$ a Martingale, $P(\hat{t}=0|\mathcal{Y}_o)=0$

M_t^* and $\hat{M}_t$ are orthogonal.

Here t_c is defined as in (4.3.48) with y_t taken as y_t^* and S_t generated by (4.1.3).

Then

$$t_j^o \triangleq t_j^* \wedge \hat{t} \quad (4.4.7)$$

so $$dI(t\geq t_j^o) = \lambda(1-I(t\geq t_j^o))dt + dM_t^o \qquad (4.4.8)$$

where $dM_t^o = I(t\geq t_j^o)(dM_t^* + d\hat{M}_t)$ w.p.1, i.e. M_t^o is a Martingale and $P(t\geq t_j^o | \mathcal{Y}_o) = 1-e^{-\lambda t}$.

The observation process y_t from which the processes S_t, S_t^* are generated using (4.1.3) and (4.3.43) respectively is either equal to y_t^o for all t or it is equal to y_t^* for all t.

P^o, E^o are defined as probability and expectation given that $y_t = y_t^o$ $\forall t\geq 0$.

P^*, E^* are defined as probability and expectation given that $y_t = y_t^*$ $\forall t\geq 0$.

$$K_{t_o}^o(\tilde{\tau}_{t_o}) \triangleq -\lambda(\tilde{\tau}_{t_o} - t_o) + (\lambda+c)(\tilde{\tau}_{t_o} - t_j^o \vee t_o) I(\tilde{\tau}_{t_o} > t_j^o)$$

$$\text{with } K^o \triangleq K_o^o \qquad (4.4.9)$$

$$K_{t_o}^*(\tilde{\tau}_{t_o}) \triangleq -\lambda(\tilde{\tau}_{t_o} - t_o) + (\lambda+c)(\tilde{\tau}_{t_o} - t_j^* \vee t_o) I(\tilde{\tau}_{t_o} > t_j^*)$$

$$\text{with } K^* \triangleq K_o^* \qquad (4.4.10)$$

for $\tilde{\tau}_{t_o}$ a $\mathcal{Y}_t$-stopping time st $\tilde{\tau}_{t_o} \geq t_o$ ($\mathcal{Y}_t$ generated by $(y_u : u\leq t)$)

Then minimizing the expectation of $K_{t_o}^o(\tilde{\tau}_{t_o})$ with observations $y_t = y_t^o$ is the original problem of sections 4.0 and 4.1 (t_j^o distributed as t_j in (4.0.2)).

$$\text{i.e. } E_{(S,y)_{t_o}} K_{t_o}(\tilde{\tau}_{t_o}) = E^o[K_{t_o}^o(\tilde{\tau}_{t_o}) | (S,y)_{t_o}]$$

and this is minimized by $\tau_{t_o}^o = \tau_{t_o} \triangleq \inf\{t\geq t_o : S_t \geq S_\gamma(y_t)\}$.

Similarly observations $y_t = y_t^*$ and cost $K_{t_o}^*(\tilde{\tau}_{t_o})$ correspond to the "second modified problem" defined by (4.0.1) and (4.3.42).

$$\text{i.e. } E^*_{(S^*,y)_{t_o}} K_{t_o}(\tilde{\tau}_{t_o}) = E^*[K_{t_o}^*(\tilde{\tau}_{t_o}) | (S^*,y)_{t_o}]$$

and this is minimized by $\tau_{t_o}^* = \inf\{t\geq t_o : S_t^* \geq \bar{S}_{\bar{\gamma}}(y_t)\}$

under the assumption

Assumption 4.4.1

$$-(\alpha+1)a_o \geq c$$

This assumption requires that the weighting given to delays in detection in the cost function (4.0.3) or (4.1.5) does not force the delays to be typically of the same order as the system time constants. In applications this seems likely.

Outline of the argument

Lemmas 4.6 and 4.8 are concerned with the expected detection delay using τ^* in detecting a disorder in observations y_t^o at $t_j^o=\hat{t}<t_j^*$. In order to achieve a reasonable upper bound the delay time is considered in two parts using different methods in each case.

Using Lemma 4.5, Theorem 4.5 then establishes a bound on the expected cost of using detection rule τ^*, with observations y^o, to detect a disorder occuring at t_j^o. To simplify the analysis two assumptions are made which should hold in any practical situation.

The bound is evaluated and values given in Table 4.4.1. Lemma 4.7 provides a technical result.

Lemma 4.5

$$E^o[K^o(\tau^o)|\mathcal{Y}_o] \geq E^*[K^*(\tau^*)|\mathcal{Y}_o]$$

Proof

Note: S_o, S_o^* are $\mathcal{Y}_o$-measurable ($S_o=S_o^*=-\infty$).

Suppose $y_t=y_t^*$. Then a random variable $\tilde{t}$ may be generated distributed as $\hat{t}$, by using (4.4.6).

Generate $\hat{y}_t$ st $\hat{y}_t = y_t^* \quad \forall t\leq\tilde{t}$

$$d\hat{y}_t = \alpha a_o \hat{y}_t dt + d\hat{V}_t \quad \forall t\geq\tilde{t}$$

for some independent Wiener process $\hat{V}_t$.

Then observations of $\hat{y}_t$ are statistically indistinguishable from observations of y_t^o for given t_j^*. Since $\hat{y}_t$ may be generated given $(y_u^*, u \leq t)$, it follows from the optimality of τ^* that

$$E^o[K^*(\tau^o)|\mathcal{Y}_o] \geq E^*[K^*(\tau^*)|\mathcal{Y}_o]$$

Since $t_j^o \leq t_j^*$, $K^o(\tau^o) \geq K^*(\tau^o)$ from (4.4.9),(4.4.10) and the result of the Lemma follows. □

The following definition is made.

$\hat{\mathcal{Y}}_t$ denotes the σ-field generated by$(I(\omega \geq \hat{t}), y_u : u \leq t)$

Lemma 4.6

∃ a $\hat{\mathcal{Y}}_t$-stopping time $\tau_x \geq \hat{t}$ such that

$$\pi^*_{\tau_x} = P^*(\tau_x \geq t_j^* | \hat{\mathcal{Y}}_{\tau_x}) \geq \frac{1}{1+\exp(-x+\frac{(\alpha-1)^2}{4\alpha(\alpha-1)})} \qquad (4.4.11)$$

$$y^2_{\tau_x} \leq \frac{1}{-(\alpha+1)a_o} \leq \bar{y}_c^2 \qquad (4.4.12)$$

$$E^o(\tau_x - \hat{t} | \hat{\mathcal{Y}}_{\hat{t}}) \leq \frac{4\alpha(x - \bar{S}_c - \frac{\alpha^2-1}{4\alpha} a_o y_{\hat{t}}^2)}{-(\alpha-1)^2 a_o} \qquad (4.4.13)$$

for any $x \geq \bar{S}_c$.

Proof

Define $U_t \triangleq S_t^* + \frac{\alpha^2-1}{4\alpha} a_o y_t^2$ (4.4.14)

If $y_t = y_t^o$ ∀t, then $\forall t \geq \hat{t} \geq t_c$ (so that, from (4.4.2), $dy_t = \alpha a_o y_t dt + dV_t$)

$$dU_t = r(y_t)(1+e^{-S_t^* - \frac{1}{2}(\alpha-1)a_o y_{\hat{t}}^2})dt - \frac{(\alpha-1)^2}{4\alpha} a_o dt + \frac{\alpha^2-1}{2\alpha} a_o y_t dV_t \qquad (4.4.15)$$

by (4.3.43).

Now define $\hat{U}_t$ so that $\hat{U}_{\hat{t}} = \bar{S}_c + \frac{\alpha^2-1}{4\alpha}a_o y_{\hat{t}}^2 \le \bar{S}_c$

and $d\hat{U}_t = dU_t - r(y_t)(1+e^{-S_t^*-\frac{1}{2}(\alpha-1)a_o y_t^2})dt$

$$= -\frac{(\alpha-1)^2}{4\alpha}dt + \frac{\alpha^2-1}{2\alpha}a_o y_t dV_t \qquad (4.4.16)$$

Then $\hat{U}_t \le U_t \ \forall t \ge \hat{t}$, since $S_{\hat{t}}^* \ge \bar{S}_c$ and from (4.4.16) $(r(y)\ge 0)$.

Suppose $x \ge \bar{S}_c \ge \hat{U}_{\hat{t}}$ is fixed, and define

$$\tau_x \triangleq \inf\{t \ge \hat{t} : \hat{U}_t \ge x\} \qquad (4.4.17)$$

For fixed $T > \hat{t}$

$$E^o(\hat{U}_T - \hat{U}_{\hat{t}} | \mathcal{Y}_{\hat{t}}) = E^o(\hat{U}_{\tau_x \wedge T} - \hat{U}_{\hat{t}} | \mathcal{Y}_{\hat{t}}) + E^o(\hat{U}_T - \hat{U}_{\tau_x \wedge T} | \mathcal{Y}_t)$$

Since $\hat{U}_{\tau_x \wedge T} \le x$, it follows from (4.4.16) that

$$-\frac{(\alpha-1)^2}{4\alpha}a_o(T-\hat{t}) \le x - \hat{U}_{\hat{t}} + E^o([T-\tau_x \wedge T][-\frac{(\alpha-1)^2}{4\alpha}a_o] | \mathcal{Y}_{\hat{t}})$$

Therefore

$$E^o(\tau_x \wedge T - \hat{t} | \mathcal{Y}_{\hat{t}}) \le -\frac{4\alpha(x-\hat{U}_{\hat{t}})}{(\alpha-1)^2 a_o} \qquad \forall T > \hat{t}$$

i.e. $$\int_0^T P^o(\tau_x - \hat{t} > u | \mathcal{Y}_{\hat{t}})du \le -\frac{4\alpha(x-\hat{U}_{\hat{t}})}{(\alpha-1)^2 a_o} \qquad \forall T > \hat{t}$$

$$\Rightarrow \quad E^o(\tau_x - \hat{t} | \mathcal{Y}_{\hat{t}}) \le -\frac{4\alpha(x-\hat{U}_{\hat{t}})}{(\alpha-1)^2 a_o}$$

Using (4.4.16) yields (4.4.13).

Now $\frac{d}{dt}[\hat{U}_t - \frac{\alpha^2-1}{4\alpha}a_o y_t^2] = -\frac{1}{2}(\alpha-1)a_o - \frac{1}{2}(\alpha^2-1)a_o^2 y_t^2$, $t \ge \hat{t}$ (4.4.18)

from (4.4.16),(4.4.2).

So $\frac{d}{dt}[\hat{U}_t - \frac{\alpha^2-1}{4\alpha}a_o y_t^2] < 0$, if $y_t^2 > \frac{1}{-(\alpha+1)a_o}$

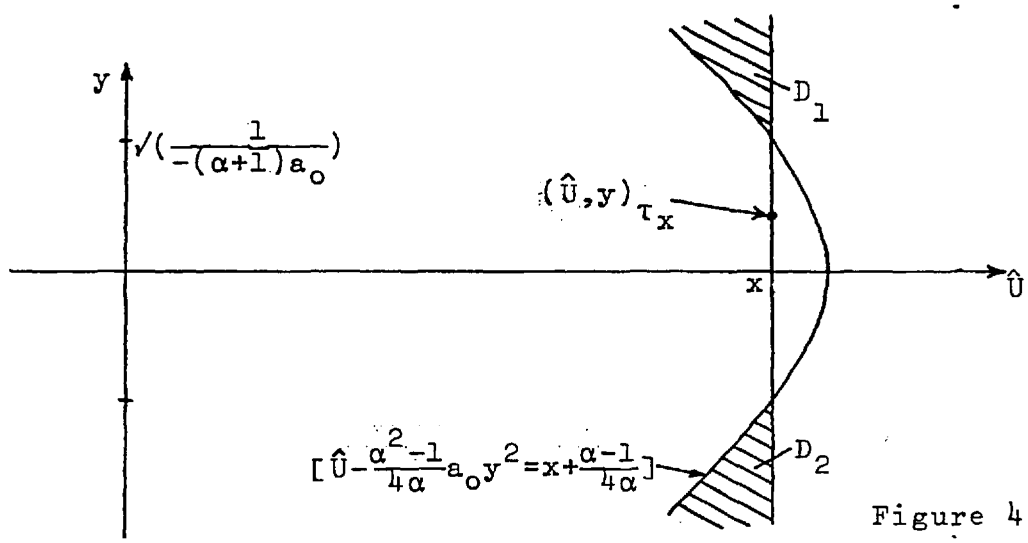


Figure 4.4.1

So the process $(\hat{U},y)_t$ cannot enter the (open) sets D_1,D_2 for $t<\tau_x$ (see Figure 4.4.1).

Therefore $y^2_{\tau_x} \le \frac{1}{-(\alpha+1)a_o}$ which is (4.4.12).

Finally, $U_{\tau_x} \ge \hat{U}_{\tau_x} = x$, and from (4.4.14) and (4.3.47)

$$R^*_{\tau_x} = U_{\tau_x} + \frac{(\alpha-1)^2}{4\alpha}a_o y^2_{\tau_x}$$

Using (4.4.12) this yields (4.4.11). (Note that $\pi^*_t \triangleq P^*(t\ge t^*_j|\mathcal{Y}_t) = P^*(t\ge t^*_j|\hat{\mathcal{Y}}_t)$ $\forall t$ by definition of $t^*_j, \hat{t}$ (4.4.5),(4.4.6).) □

Lemma 4.7

For any $\hat{\mathcal{Y}}_t$-stopping time t_o

$$E^*(t^*_j - t_o | t^*_j > t_o, \hat{\mathcal{Y}}_{t_o}) \le \frac{1}{\lambda\theta} - \frac{1}{a_o}\left(\left|\frac{y_{t_o}}{\bar{y}_c}\right| \vee 1\right)$$

where $\theta = P(X\in[-\bar{y}_c,\bar{y}_c])$; $X\sim N(e^{-1}\bar{y}_c, -\frac{1}{2a_o})$ (4.4.19)

Proof

Conditioning on t^*_j, $\hat{\mathcal{Y}}_{t_o}$, $y_t = y^*_t$ $\forall t$, $y_t \sim N(\mu_t,\sigma^2_t)$

where $t_1 = t_o - \frac{1}{a_o} - \frac{1}{a_o}\ln(|y_{t_o}/\bar{y}_c| \vee 1)$

$$|\mu_t| \le e^{-1}\bar{y}_c \ ; \ \sigma^2_t \le -1/2a_o \quad \forall t\ge t_1 \tag{4.4.20}$$

from (4.4.3) and since $\alpha>1$.

From (4.4.5)

$$P^*(t\geq t_j^*|t_j^*>t_o,\hat{y}_{t_o}) \geq E^*(\int_{t_o}^{t} \lambda I(y_u^2\leq\bar{y}_c^2)I(u<t_j^*)du|t_j^*>t_o,\hat{y}_{t_o})$$

$$\geq E^*(\int_{t_o}^{t} \lambda P^*(y_u^2\leq\bar{y}_c^2|t_j^*>u,\hat{y}_{t_o})I(u<t_j^*)du|t_j^*>t_o,\hat{y}_{t_o})$$

i.e. if $P^*(u\geq t_j^*|t_j^*>t_o,\hat{y}_{t_o})=q_u$

$$q_t \geq \int_{t_1}^{t} \lambda\theta(1-q_u)du \qquad (4.4.21)$$

from (4.4.20), with θ as defined in (4.4.19).

Now $\quad E^*(t_j^*-t_o|t_j^*>t_o,\hat{y}_{t_o}) = \int_{t_o}^{\infty}(1-q_u)du \leq t_1-t_o+1/\lambda\theta$

the final inequality following from (4.4.21).

Then the result of the Lemma follows from the definition of t_1, since

$$(1+\ln(|^{y}t_o/\bar{y}_c|\vee 1)) \leq |^{y}t_o/\bar{y}_c|\vee 1 \qquad \square$$

<u>Lemma 4.8</u>

$$E^o(\tau_{\hat{t}}^*-\hat{t}|\hat{y}_{\hat{t}}) \leq (\frac{1}{\theta}-\frac{\lambda}{a_o})\frac{1}{c}[1+\exp(-x+\frac{(\alpha-1)^2}{4\alpha(\alpha+1)})]$$

$$+\frac{4\alpha(x-\bar{S}_c-\frac{\alpha^2-1}{4\alpha}a_o y_{\hat{t}}^2)}{-(\alpha-1)^2 a_o}$$

where $x>\bar{S}_c$

and $\quad \tau_{\hat{t}}^* \triangleq \inf\{t\geq\hat{t}:S_t^*\geq\bar{S}_{\bar{\gamma}}(y_t)\}$

<u>Proof</u>

At any $\hat{y}_{t_o}$-stopping time t_o such that $y_{t_o}^2\leq\bar{y}_c^2$

$$E^*[\lambda(\tau_{t_o}^*\wedge t_t^*-t_o)I(\tau_{t_o}^*\wedge t_j^*\geq t_o)|\hat{y}_{t_o}] \leq \frac{1}{\theta}-\frac{\lambda}{a_o}$$

from Lemma 4.7.

Since $K_{t_o}^*(\tau_{t_o}^*) = -\lambda(\tau_{t_o}^*\wedge t_j^*-t_o)I(\tau_{t_o}^*\wedge t_j^*\geq t_o)$

$$+\ c(\tau_{t_o}^*-t_j^*\vee t_o)I(\tau_{t_o}^*>t_j^*)$$

and by optimality $E^*(K_{t_o}^*(\tau_{t_o}^*)|\hat{y}_{t_o})\leq 0$ (as $K_{t_o}^*(t_o)=0$).

it follows that

$$E^*[c(\tau^*_{t_o}-t^*_j \vee t_o)I(\tau^*_{t_o}>t^*_j)|\hat{\mathcal{Y}}_{t_o}] \le \frac{1}{\theta} - \frac{\lambda}{a_o}$$

Now $\quad c(\tau^*_{t_o}-t^*_j \vee t_o)I(\tau^*_{t_o}>t^*_j)\ge 0 \quad$ and $\pi^*_{t_o}=P^*(t^*_j\le t_o|\hat{\mathcal{Y}}_{t_o})$

so $\quad E^*[c(\tau^*_{t_o}-t^*_j \vee t_o)I(\tau^*_{t_o}>t^*_j)|\hat{\mathcal{Y}}_{t_o},t^*_j\le t_o]\pi^*_{t_o} \le \frac{1}{\theta} - \frac{\lambda}{a_o}$

i.e. $$E^*(\tau^*_{t_o}-t_o|\hat{\mathcal{Y}}_{t_o},t^*_j\le t_o) \le \left(\frac{1}{\theta} - \frac{\lambda}{a_o}\right)\frac{1}{c\pi^*_{t_o}} \qquad (4.4.22)$$

From (4.4.2),(4.4.3) y_t evolves in the same way $\forall t\ge t_o\ge t^*_j$ whether $y_t=y^*_t$ or $y_t=y^o_t$. So

$$E^o(\tau^*_{t_o}-t_o|\hat{\mathcal{Y}}_{t_o},t^*_j\le t_o) \le \left(\frac{1}{\theta} - \frac{\lambda}{a_o}\right)\frac{1}{c\pi^*_{t_o}}$$

$$\Longrightarrow \quad E^o(\tau^*_{t_o}-t_o|\hat{\mathcal{Y}}_{t_o},\hat{t}\le t_o) \le \left(\frac{1}{\theta} - \frac{\lambda}{a_o}\right)\frac{1}{c\pi^*_{t_o}} \qquad (4.4.23)$$

since $(S^*,y)_t$ is a Markov process given $t\ge t^o_j=\hat{t}\wedge t^*_j$, $y_u=y^o_u$ $\forall u$ and $(S^*,y)_{t_o}$ is $\hat{\mathcal{Y}}_{t_o}$-measurable.

For $t_o=\tau_x$, defined in Lemma 4.6, noting that $\hat{t}\le\tau_x$ by definition of τ_x, then

$$E^o(\tau^*_{\tau_x}-\tau_x|\hat{\mathcal{Y}}_{\tau_x}) \le \left(\frac{1}{\theta} - \frac{\lambda}{a_o}\right)\frac{1}{c\pi^*_{\tau_x}}$$

and from (4.4.11),(4.4.13), the result of the Lemma follows since $\tau^*_{\tau_x}\ge\tau^*_{\hat{t}}$. □

<u>Assumption 4.4.2</u>

$$\frac{(\alpha-1)^2}{4\alpha(\alpha+1)} > \ln\left(\frac{4\lambda\alpha c\theta}{(\alpha+1)(\alpha-1)^2 a_o^2}\right)$$

For λ small, as would be expected, this will hold.

The bound given in Lemma 4.8 is approximately minimized by choosing

$$x = \ln\left(\frac{-(\alpha-1)^2 a_o}{4\alpha c\theta}\right) + \frac{(\alpha-1)^2}{4\alpha(\alpha+1)} \ge \bar{S}_c \qquad (4.4.24)$$

if Assumption 4.4.2 holds. Then

$$E^o(\tau^*_{\hat{t}}-\hat{t}\,|\,\hat{y}_{\hat{t}}) \le \frac{-4\alpha}{(\alpha-1)^2 a_o}\ln\left(\frac{(\alpha-1)^2(\alpha+1)a_o^2}{4\alpha c\theta\lambda}\right)$$

$$+ \left(\frac{1}{\theta}-\frac{\lambda}{a_o}\right)\left(\frac{1}{c}+\frac{4\alpha\theta}{-(\alpha-1)^2 a_o}\right)$$

$$+ \left(\frac{\alpha+1}{\alpha-1}\right)y_{\hat{t}}^2 \qquad = \varepsilon(y_{\hat{t}}) \text{ say.} \qquad (4.4.25)$$

Theorem 4.5

Given Assumptions 4.4.1 and 4.4.2

$$E^o(C^o(\tau^*)\,|\,y_o) - E^o(C^o(\tau^o)\,|\,y_o)$$

$$= E^o(K^o(\tau^*)\,|\,y_o) - E^o(K^o(\tau^o)\,|\,y_o)$$

$$\le E[(\frac{1}{\theta}-\frac{\lambda}{a_o}\,|y_{t_j^o}/\bar{y}_c| + c\varepsilon(y_{t_j^o}))I(t_j^o=\hat{t})\,|\,y_o]$$

where $\varepsilon(\cdot)$ is defined in (4.4.25).

Proof

The first equality follows from Lemma 2.1

If $t_j^o \ne \hat{t}$, then $t_j^o = t_j^*$ and $K^o_{t_j}(\tau^*_{t_j^o}) = K^*(\tau^*_{t_j^o})$

where $\tau^*_{t_j^o}$ is defined by (4.3.44).

Therefore $\quad E^o(K^o(\tau^*)\,|\,t_j^o\ne\hat{t},y_o,\tau^*\ge t_j^o) = E^*(K^*(\tau^*)\,|\,t_j^o\ne\hat{t},y_o,\tau^*\ge t_j^o)$

So $\quad E^o(K^o(\tau^*)\,|\,y_o,\tau^*\ge t_j^o) - E^*(K^*(\tau^*)\,|\,y_o,\tau^*\ge t_j^o)$

$$= E^o[K^o(\tau^*)I(t_j^o=\hat{t})\,|\,y_o,\tau^*\ge t_j^o] - E^*[K^*(\tau^*)I(t_j^o=\hat{t})\,|\,y_o,\tau^*\ge t_j^o]$$

$$= E^o[K^o_{t_j^o}(\tau^*)I(t_j^o=\hat{t})\,|\,y_o,\tau^*\ge t_j^o] - E^*[K^*_{t_j^o}(\tau^*)I(t_j^o=\hat{t})\,|\,y_o,\tau^*\ge t_j^o] \qquad (4.4.26)$$

The second equality follows from (4.4.9),(4.4.10), and since $(\tau^*\ge t_j^o$ when $y_t=y_t^o) \Leftrightarrow (\tau^*\ge t_j^o$ when $y_t=y_t^*)$ by (4.4.4).

Now $E^*[K^*_{t^o_j}(\tau^*)I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j]$

$$\geq E^*[-\lambda E^*(t^*_j-\hat{t}|t^*_j>\hat{t},\hat{\mathcal{Y}}_{\hat{t}})I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j]$$

by (4.4.10). Note that $(t^o_j=\hat{t})\Rightarrow(t^*_j\geq\hat{t})$.

Then from Lemma 4.7, as $t^o_j=\hat{t} \Rightarrow y^2_{t^o_j}\geq\bar{y}^2_c$

$$E^*[K^*_{t^o_j}(\tau^*)I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j]$$
$$\geq E^*[(-\frac{1}{\theta}+\frac{\lambda}{a_o}\cdot|^{y}t^o_j/\bar{y}_c|)I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j] \leq 0 \tag{4.4.27}$$

Next,

$$E^o[K^o_{t^o_j}(\tau^*)I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j] = cE^o[(\tau^*_{t^o_j}-\hat{t})I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j]$$
$$\leq cE^o[\varepsilon(y_{\hat{t}})I(t^o_j=\hat{t})|\mathcal{Y}_o,\tau^*\geq t^o_j] \geq 0 \tag{4.4.28}$$

where $\varepsilon(\cdot)$ is defined in (4.4.25). This may be done since

$$E^o(\tau^*_{\hat{t}}-\hat{t}|\hat{\mathcal{Y}}_{\hat{t}}) = E^o(\tau^*_{\hat{t}}-\hat{t}|\hat{\mathcal{Y}}_{\hat{t}},t^o_j=\hat{t})$$

as $(S^*,y)_t$ is a Markov process given $t\geq\hat{t}\geq t^o_j$, if $y_u=y^o_u$ $\forall u$.

Note that $(S^*,y)_t$ is the same $\forall t\leq t^o_j$ whether $y_u=y^*_u$ $\forall u$ or $y_u=y^o_u$ $\forall u$.

Therefore $P^o(\tau^*\geq t^o_j|\mathcal{Y}_o) = P^*(\tau^*\geq t^o_j|\mathcal{Y}_o)$

(4.4.29)

and $E^o(K^o(\tau^*)|\mathcal{Y}_o,\tau^*<t^o_j) = E^*(K^*(\tau^*)|\mathcal{Y}_o,\tau^*<t^o_j)$

as $t^o_j\leq t^*_j$ (see (4.4.9),(4.4.10)).

So from (4.4.26) substituting with (4.4.27),(4.4.28) it follows that

$$E^o(K^o(\tau^*)|\mathcal{Y}_o) - E^*(K^*(\tau^*)|\mathcal{Y}_o)$$
$$\leq E[(\frac{1}{\theta}-\frac{\lambda}{a_o}|^{y}t^o_j/\bar{y}_c| + c\varepsilon(y_{\hat{t}}))I(t^o_j=\hat{t})|\mathcal{Y}_o]$$

where the superscripts on E on the right hand side are dropped since irrelevant. The result of the Theorem follows.

□

A further assumption is made to facilitate the evaluation of the bound provided by Theorem 4.5.

<u>Assumption 4.4.3</u>

$$y_o \sim N(0, -\frac{1}{2a_o})$$

This corresponds to y_t having achieved the "steady-state" distribution for the system with no disorder.

Since t_j^o is independent of the noise processes V_t, W_t,

$$y_{t_j^o} \sim N(0, -\frac{1}{2a_o})$$

$$\begin{aligned} \text{Now } P(t_j^o = \hat{t} \mid y_{t_j^o} = \tilde{y}) &= 0 \quad \text{for } \tilde{y}^2 \le \bar{y}_c^2 \\ &= 1 - r(\tilde{y})/\lambda \quad \text{for } \tilde{y}^2 > \bar{y}_c^2 \quad \text{(from (4.4.5) and (4.4.8))} \\ &\le 1 - \frac{\frac{1}{2}(\alpha^2-1)a_o^2\tilde{y}^2 + \frac{1}{2}(\alpha-1)a_o}{(\lambda-(\alpha+1)a_o)\exp(\frac{2\lambda-(3\alpha+1)a_o}{2(\alpha+1)a_o})} \exp(\tfrac{1}{2}(\alpha-1)a_o\tilde{y}^2) \\ &= 1 - \rho(\tilde{y}) \quad \text{say} \end{aligned}$$

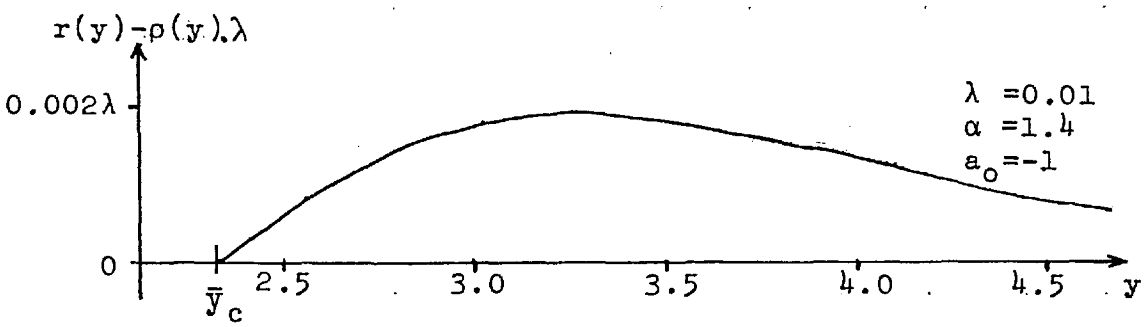


Figure 4.4.2

$$\text{Therefore} \quad P(t_j^o = \hat{t}) \le 2\sqrt{(\frac{-a_o}{\pi})} \cdot \int_{\bar{y}_c}^{\infty} (1-\rho(y)) e^{a_o y^2} dy = p \text{ say}$$

Integration by parts yields

$$p = P(y_{t_j^o}^2 > \bar{y}_c^2) - \sqrt{(\frac{3\alpha+1}{\alpha^2-1})} \cdot \frac{\alpha-1}{\lambda'+\alpha+1} \exp(-\frac{2\lambda'+3\alpha+1}{\alpha^2-1}) \cdot \sqrt{(\frac{1}{\pi})}$$

where $\lambda' \triangleq \lambda/(-a_o)$. Also define $c' \triangleq c/(-a_o)$.

Values of p are given in Table 4.4.1. for λ' small ($\lim_{\lambda' \to 0} p$).

From Theorem 4.5

$$E^{o}C^{o}(\tau^{*}) - E^{o}C^{o}(\tau^{o}) \leq$$

$$\Big[\frac{1}{\theta} + \frac{4\alpha c'}{(\alpha-1)^2}\cdot\ln\Big(\frac{(\alpha-1)^2(\alpha+1)}{4\alpha c'\theta\lambda'}\Big) - \frac{5\alpha-1}{(\alpha-1)^2}\cdot c'$$

$$+ \Big(\frac{1}{\theta} + \lambda'\Big)\Big(1+ \frac{4\alpha\theta c'}{(\alpha-1)^2}\Big)\Big]p$$

$$+ \frac{2\lambda'}{\bar{y}_c}\sqrt{\Big(\frac{-a_o}{\pi}\Big)}\cdot\int_{\bar{y}_c}^{\infty} y(1-\rho(y))e^{a_o y^2}dy$$

$$+ 2\frac{\alpha+1}{\alpha-1}c'\cdot(-a_o)\sqrt{\Big(\frac{-a_o}{\pi}\Big)}\cdot\int_{\bar{y}_c}^{\infty} y^2(1-\rho(y))e^{a_o y^2}dy$$

Again integrating by parts gives

$$E^{o}C^{o}(\tau^{*}) - E^{o}C^{o}(\tau^{o}) \leq \hat{\varepsilon} \triangleq$$

$$\Big[\frac{1}{\theta} + \frac{4\alpha c'}{(\alpha-1)^2}\ln\Big(\frac{(\alpha-1)^2(\alpha+1)}{4\alpha c'\theta\lambda'}\Big) - \frac{5\alpha-1}{(\alpha-1)^2}\cdot c'$$

$$+ \Big(\frac{1}{\theta} + \lambda'\Big)\Big(1+ \frac{4\alpha\theta c'}{(\alpha-1)^2}\Big)\Big]p$$

$$+\Big[\lambda'\sqrt{\Big(\frac{\alpha^2-1}{2\lambda'+3\alpha+1}\Big)}\cdot\frac{\lambda'(\alpha-1)+(\alpha-1)^2}{\lambda'(\alpha+1)+(\alpha+1)^2} + c'\sqrt{\Big(\frac{2\lambda'+3\alpha+1}{\alpha^2-1}\Big)}\cdot\frac{(\alpha-1)\lambda'+\alpha^2-3\alpha+2}{(\alpha-1)(\lambda'+\alpha+1)}\Big]$$

$$\times\sqrt{\Big(\frac{1}{\pi}\Big)}\exp\Big(-\frac{2\lambda'+3\alpha+1}{\alpha^2-1}\Big)$$

$$+ \frac{(\alpha+1)c'}{2(\alpha-1)}\cdot P(y_{t_j^o}^2 > \bar{y}_c^2) - \frac{\sqrt{\big(\frac{1}{\alpha+1}\big)}c'}{\lambda'+\alpha+1}\exp\Big(\frac{2\lambda'+3\alpha+1}{2(\alpha+1)}\Big)P(X^2 > \bar{y}_c^2)$$

where $X \sim N\big(0, \frac{-1}{(\alpha+1)a_o}\big)$.

Values of this bound are given in Table 4.4.1.

Note that these are likely to be very pessimistic (higher than necessary). The steps leading to (4.4.30) are one cause of this.

TABLE 4.4.1

α	$\lim_{\lambda' \to 0} p$
1.3	0.32×10^{-5}
1.4	0.25×10^{-4}
1.5	0.94×10^{-4}
1.7	0.68×10^{-3}
2.0	0.29×10^{-2}

α	λ'	c'	$\hat{\varepsilon}$
1.2	0.01	0.1	0.000006
1.2	0.00001	0.1	0.000014
1.2	0.00001	0.01	0.000002
1.2	0.00001	0.001	0.000000
1.2	0.00000001	0.1	0.000022
1.2	0.00000001	0.01	0.000003
1.2	0.00000001	0.001	0.000000
1.4	0.01	0.1	0.000664
1.4	0.00001	0.1	0.001293
1.4	0.00001	0.01	0.000199
1.4	0.00001	0.001	0.000070
1.4	0.00000001	0.1	0.001929
1.4	0.00000001	0.01	0.000262
1.4	0.00000001	0.001	0.000077
1.7	0.01	0.1	0.008467
1.7	0.00001	0.1	0.014602
1.7	0.00001	0.01	0.002894
1.7	0.00001	0.001	0.001537
1.7	0.00000001	0.1	0.020781
1.7	0.00000001	0.01	0.003511
1.7	0.00000001	0.001	0.001599
2.0	0.01	0.1	0.026867
2.0	0.00001	0.1	0.042779
2.0	0.00001	0.01	0.010605
2.0	0.00001	0.001	0.006907
2.0	0.00000001	0.1	0.058808
2.0	0.00000001	0.01	0.012208
2.0	0.00000001	0.001	0.007067

In section 2.2 a possibly more realistic formulation of the failure detection problem is proposed (2.2.7). It is shown that subject to the conditions of Lemma 2.2 the optimal detection rule following each false alarm is $\tau=\tau^o$, with an appropriate choice of c in (4.0.3).

Suppose following each false alarm y_t is "reset" so that $y_t \sim N(0,-\frac{1}{2a_o})$ as in Assumption 4.4.3 (probably unimportant in practice if λ & c are much smaller than $-a_o$ so that the inter-alarm time is typically long compared to the system time constants).

As in Lemma 4.5 it may be shown that

$$E^o(K^o_{t_o}(\tau^o_{t_o})|(S,y)_{t_o}=(\tilde{S},\tilde{y}))$$
$$\geq E^*(K^*_{t_o}(\tau^*_{t_o})|(S^*,y)_{t_o}=(\tilde{S},\tilde{y})) \qquad (4.4.30)$$

Therefore from Corollary 3.2.2, since $S^*_t \leq S_t$ $\forall t$ by (4.3.42)

$$0 > h(S_t,y_t) \geq h(S^*_t,y_t) \geq E^*(K^*(\tau^*_t)|(S^*,y)_t) \qquad \forall t<\tau^o$$

$$\Rightarrow \qquad \tau^* \geq \tau^o \text{ since } K^*(\tau^*_t)=0 \text{ if } \tau^*_t=t. \qquad (4.4.31)$$

Let

$$q^o=P^o(\tau^o<t^o_j) \qquad g^o=E^o((\tau^o-t^o_j)|\tau^o\geq t^o_j)$$
$$q^*=P^o(\tau^*<t^o_j) \qquad g^*=E^o((\tau^*-t^o_j)|\tau^*\geq t^o_j) \qquad (4.4.32)$$

From (4.4.31) $q^* \leq q^o$

The difference between the expectation of cost Q defined in (2.2.7) using τ^* and τ^o following each false alarm is

$$\frac{q^*}{1-q^*} + g^* - \frac{q^o}{1-q^o} - g^o$$

$$\leq \frac{q^*}{1-q^o} + \frac{1-q^*}{1-q^o}g^* - \frac{q^o}{1-q^o} - g^o$$

$$\leq \frac{1}{1-q^o}[q^* + (1-q^*)g^* - q^o - (1-q^o)g^o]$$

$$\leq \frac{1}{1-q^o}[E^oK^o(\tau^*) - E^oK^o(\tau^o)] = \frac{\hat{\varepsilon}}{1-q^o}$$

where $\hat{\varepsilon}$ is as in Table 4.4.1.

Now if Q^o is the expected optimal cost with cost function Q it follows from (2.2.7) that $Q^o \geq 1/(1-q^o)$. Therefore the increase in expected cost using the detection rule τ^* following each false alarm is no greater than $\hat{\varepsilon}Q^o$.

CHAPTER 5

ROBUSTNESS OF DETECTION RULES: GENERAL CASE

In this chapter the behaviour of the optimal detection rule given in Chapter 3 for the problem (2.5.6) is investigated, for the case where the disorder is different from that designed for. The result obtained is interpreted in two ways. Firstly a robustness result is obtained for auto-regressive systems of general order. This specifies a set of post-jump parameter values such that, if c (the delay weighting coefficient in cost function $C(\tilde{\tau})$) is chosen sufficiently small, the expected cost is no greater than in the design case.

The second interpretation concerns the detection of disorders in the more general situation discussed in Section 3.4, where a natural sub-optimal approach is suggested. This approach is in fact the optimal detection rule for a related problem in which additional transient effects occur following the occurence of a disorder. The result of this chapter characterizes a set of post-jump parameter values for the system (3.4.1) such that the expected cost is no greater than for the problem for which the proposed detection rule is optimal, again if c is sufficiently small.

It is suggested that the restriction on the value of c may be interpreted as a requirement that the average detection delay following a disorder be long compared to system time constants.

5.1 The robustness result

Problem formulation

The following situation is considered

$$dv_t = \begin{bmatrix} J & B \\ D_t & F_t \end{bmatrix} v_t dt + \begin{bmatrix} 0 \\ z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ I_m \end{bmatrix} dW_t + \begin{bmatrix} 0 \\ \zeta_t \end{bmatrix} dt$$

Observations: $y_t = [0:I_m]v_t$ (5.1.1)

v_t is an n-dimensional process $(n \geq m)$

J is an $(n-m)\times(n-m)$ constant matrix

B is a $(n-m)\times m$ constant matrix

$D_t = D^0, F_t = F^0, z_t = z^0$ $(D^0, F^0, z^0$ constant$)$ $\forall t < t_j \geq 0$

$D_t = \bar{D}$, $F_t = \bar{F}$, $z_t = \bar{z}$ $(\bar{D}, \bar{F}, \bar{z}$ constant$)$ $\forall t \geq t_j$

$P(t \geq t_j) = 1 - e^{-\lambda t}$, t_j independent of v_o (5.1.2)

W_t is a Wiener process independent of t_j.

$\zeta_t = 0$ $\forall t < t_j$

Cost function: $C(\tilde{\tau}) = I(\tilde{\tau} < t_j) + c(\tilde{\tau} - t_j) I(\tilde{\tau} > t_j)$ (5.1.3)

$\tilde{\tau}$ a $\mathcal{Y}_t^R$-stopping time

In this chapter v_o is assumed known, so that v_t is $\mathcal{Y}_t$-measurable.

An optimal detection rule is designed for the case:

$\bar{D} = D^1$, $\bar{F} = F^1$, $\bar{z} = z^1$

$\zeta_t = 0$ $\forall t$ (5.1.4)

The notation P^1, E^1 throughout this chapter denotes probability and expectation respectively given that (5.1.4) holds, i.e. the disorder which occurs is the design case.

The notation P^2, E^2 denotes probability and expectation given that

$\bar{D} = D^2$, $\bar{F} = F^2$, $\bar{z} = z^2$ (5.1.5a)

ζ_t is a random variable such that ζ_t is independent of $W_{t+u} - W_t$ for $\forall u \geq 0$ and

$$E^2(\|\zeta_t\|^2 | t_j, v_{t_j}) \le \alpha(v_{t_j}).e^{-\beta(t-t_j)} \tag{5.1.5b}$$

for some $\alpha(\cdot)$, $\beta>0$ independent of t_j and such that

$$E^2(\alpha(v_{t_j})|t_j) \text{ uniformly bounded } \forall t_j$$

The introduction of ζ_t will enable sub-optimal detection rules to be considered in Section 5.3.

The detection rule

From Section 3.1, if $\pi_t \triangleq P^1(t \ge t_j | \mathcal{Y}_t)$

$$d\pi_t = \lambda(1-\pi_t)dt + \pi_t(1-\pi_t)([D^1-D^0:F^1-F^0]v_t+z^1-z^0)^T d\nu_t \tag{5.1.6}$$

where $$d\nu_t = dy_t - ([D^0:F^0]v_t+z^0)(1-\pi_t)dt$$
$$- ([D^1:F^1]v_t+z^1)\pi_t dt \tag{5.1.7}$$

Defining $R_t = \ln(\pi_t/(1-\pi_t))$ and

$$M^i \triangleq [D^1-D^0:F^1-F^0]^T[D^i-\tfrac{1}{2}D^1-\tfrac{1}{2}D^0:F^i-\tfrac{1}{2}F^1-\tfrac{1}{2}F^0] \tag{5.1.8}$$

$$h^{iT} \triangleq [z^1-z^0]^T[D^i-\tfrac{1}{2}D^1-\tfrac{1}{2}D^0:F^i-F^1-\tfrac{1}{2}F^0]$$
$$+ [z^i-\tfrac{1}{2}z^1-\tfrac{1}{2}z^0]^T[D^1-D^0:F^1-F^0] \tag{5.1.9}$$

$$g^i \triangleq [z^1-z^0]^T[z^i-\tfrac{1}{2}z^1-\tfrac{1}{2}z^0] \tag{5.1.10}$$

for $i=0,1,2$

$$dR_t = \lambda(1+e^{-R_t})dt + (v_t^T M^i v_t + h^i v_t^T + g^i)dt$$
$$+ ([D^1-D^0:F^1-F^0]v_t+z^1-z^0)^T dW_t$$
$$+ ([D^1-D^0:F^1-F^0]v_t+z^1-z^0)^T \zeta_t dt \tag{5.1.11}$$

where $i=0$ if $t<t_j$; $i=1$ if (5.1.4) holds, $i=2$ if (5.1.5) holds for $t \ge t_j$.

As discussed in Section 3.1 the optimal stopping time τ for the design case (5.1.4) will be of the form

$$\tau^c = \inf\{t:(R,v)_t \in \gamma_c\} \tag{5.1.12}$$

where γ_c is a stopping boundary in the state-space of the process $(R,v)_t$. The index c is used to indicate the dependence of the stopping boundary on the weighting coefficient in the cost function $C(\bar{\tau})$ (see (5.1.3)).

τ^c may also be expressed as

$$\tau^c = \inf\{t:R_t \geq R_{\gamma_c}(v_t)\} \qquad (5.1.13)$$

where $\quad R_{\gamma_c}(v) = \inf\{R:(R,v)\epsilon\gamma_c\}$

<u>Outline of the robustness argument</u>

The ideas leading to Theorem 5.1 are briefly introduced. The probability of a false alarm is independent of the system behaviour after the jump time, since then $\tau<t_j$ and τ is a $\mathcal{Y}_t$-stopping time. Only the delay time is affected by the actual form of the disorder.

The delay time is the time taken for the process $(R,v)_t$ to move from its value at time t_j to the stopping boundary γ_c.

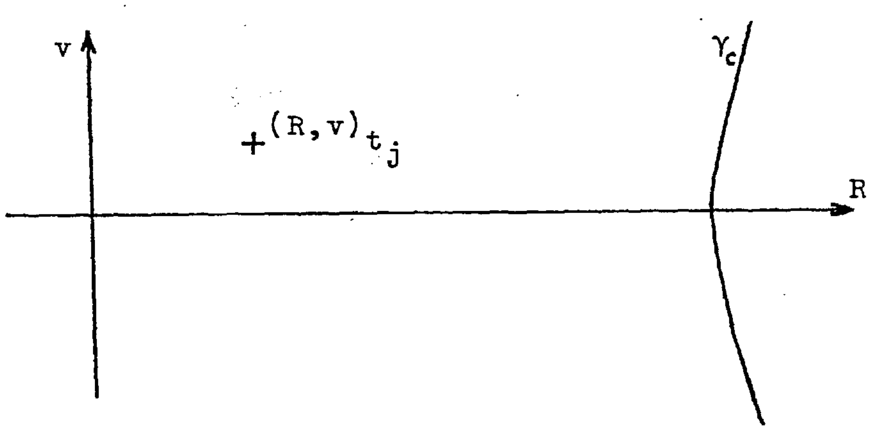


Figure 5.1.1

As $c\downarrow 0$, γ_c moves to the right in the diagram above, as longer delay times may then be tolerated in order that the false alarm probability may be reduced. For R_t large, the term λe^{-R_t} in equation (5.1.11) becomes small. Therefore, for small c,the expected delay times are closely related to the mean values under disorder conditions $(t>t_j)$ of

$$\lambda + v_t^T M^i v_t + h^{iT} v_t + g^i \qquad i=1,2$$

(see (5.1.11)).

Lemma 5.1 provides the necessary result which bounds the effect of the λe^{-R_t} term. It is shown that

$$\int_{\inf\{t:R_t \geq \ln\lambda\}}^{\infty} \lambda e^{-R_u}\, du < \infty$$

by using a probabilistic argument based on the properties of the process $\pi_t = \frac{1}{1+\exp(-R_t)}$. Because of this the bound obtained is very weak, since no account is taken of the actual dynamics of π_t: the proof of the lemma would also be valid if π_t was the jump probability based only on a-priori information, in which case the integral would be expected to take larger values. This is not disastrous if only a qualitative result is required as in Theorem 5.1. However if a quantative result is needed, this together with uncertainty about the boundary shape are major problems.

Lemmas 5.2 and 5.3 describe the evolution of the stopping boundary as $c \downarrow 0$. Lemma 5.2 shows that

$$\ln\lambda - \ln c \leq R_{\gamma_c}(v) \leq R_{\bar{\gamma}_1}(v) - \ln c$$

for some function $R_{\bar{\gamma}_1}(v)$. Here the first inequality is an immediate property of the cost function used. The second inequality is obtained by considering a modified cost function for which the appropriate stopping boundary, defined by $R_{\bar{\gamma}_c}(v)$, retains its shape while being moved to the right as c tends to zero.

Lemma 5.3 is needed to show that, for $\|v\|^* \leq \rho$ there is a finite upper bound r_ρ for $R_{\bar{\gamma}_1}(v)$. Here ρ is an arbitrary positive real number and $\|v\|^*$ is the norm of v projected onto a sub-space of R^n. Using this definition instead of norm v allows a generalization to be made so that Theorem 5.1 may be applied to sub-optimal detection rules in Section 5.3. The proof of Lemma 5.3 involves the construction of an observation process which, up to some stopping time, carries more information than y_t. Since with this observation process it is optimal to stop if $R_t \geq r_\rho < \infty$, the same must be true with observations y_t since then the expected benefit of waiting for further observations is less.

To establish the results of Theorem 5.1 which gives a condition under which $\exists c_m$ such that $\forall c \in (0, c_m]$

$$E^2 C(\tau^c) \leq E^1 C(\tau^c)$$

a lower bound on the detection time is considered for the "E^1 case" (i.e. where (5.1.4) holds) and an upper bound is considered for the "E^2 case" ((5.1.5) holds). These are briefly discussed here.

In the E^2 case a process $\hat{R}_t^c$ is considered related to R_t, but such that, at times of interest, $\hat{R}_t^c \leq R_t$. This process satisfies (5.1.46) which is similar to (5.1.11) except that the contribution of the λe^{-R_t} term is removed. Also it is arranged that $\hat{R}_t^c$ cannot cross the r_ρ level while $\|v_t\|^* > \rho$. This means that at the first time $\hat{R}_t^c$ crosses the r_ρ level it is certain that R_t has reached the stopping boundary by the results of Lemmas 5.2 and 5.3. An equation involving the expectation of this time is established in the proof of Theorem 5.1. The laborious proof of Lemma 5.4 is necessary to verify that certain terms in the equation corresponding to transient effects are finite. An upper bound is obtained for the expected detection time which is linearly increasing with -lnc.

In the E^1 case the first time the process R_t reaches the level $\ln\lambda - \ln c$ is considered. At this time R_t cannot have reached the stopping boundary. Lemma 5.1 is used to bound the effect of the λe^{-R_t} term. It is found that the expected detection time again increases linearly with $-\ln c$ but, if the conditions are satisfied, more quickly than in the E^2 case. The result follows from this.

Lemma 5.1

$$\varepsilon_\lambda \triangleq E^1\left(\int_{\inf\{t:R_t \geq \ln\lambda\}}^{\infty} \lambda e^{-R_u}.du\right) < \infty$$

Proof

Define $\alpha(\bar{\pi},u,\hat{\pi},\hat{v}) = P^1(\exists t \geq t_o+u \text{ st } \pi_t \leq \bar{\pi} | \pi_{t_o}=\hat{\pi}, v_{t_o}=\hat{v})$

for $u>0$, $\bar{\pi}>\hat{\pi}$ (5.1.14)

Note $R_t = \ln(\pi_t/(1-\pi_t))$, $\pi_t = P^1(t \geq t_j | \mathcal{Y}_t)$.

Let τ^m be the $\mathcal{Y}_t$-stopping time

$$\tau^m = \inf\{t: t \geq t_o+u, \pi_t \leq \bar{\pi}\} \quad (5.1.15)$$

($\tau^m = \infty$ if no such time t exists)

Then
$$P^1(t_j \leq \tau^m | \mathcal{Y}_{t_o}) = E^1[P^1(t_j \leq \tau^m | \mathcal{Y}_{\tau^m}) | \mathcal{Y}_{t_o}]$$
$$\leq 1.(1-\alpha(\bar{\pi},u,\pi_{t_o},v_{t_o})) + \bar{\pi}\alpha(\bar{\pi},u,\pi_{t_o},v_{t_o}) \quad (5.1.16)$$

However, from the a-priori distribution of t_j, (5.1.2)

$$P^1(t_j \leq \tau^m | \mathcal{Y}_{t_o}) \geq P^1(t_j \leq t_o+u | \mathcal{Y}_{t_o}) = (1-\pi_{t_o})(1-e^{-\lambda u}) + \pi_{t_o}$$

Comparing this with (5.1.16) gives

$$\alpha(\bar{\pi},u,\pi_{t_o},v_{t_o}) \leq e^{-\lambda u}.\exp(\bar{R}-R_{t_o}) \quad (5.1.17)$$

where $\bar{R} = \ln(\bar{\pi}/(1-\bar{\pi}))$.

If $T(\bar{\pi}) \triangleq \sup\{t: \pi_t \leq \bar{\pi}\}$ (5.1.17)

then
$$E^1(T(\bar{\pi}) | \pi_{t_o}, v_{t_o}) = \int_o^\infty \alpha(\bar{\pi},u,\pi_{t_o},v_{t_o})du + t_o$$
$$\leq \frac{1}{\lambda}\exp(\bar{R}-R_{t_o}) + t_o \quad (5.1.18)$$

Now define $R^{(i)}=R_{t_o}+i$, $i+0,1,2,\cdots$, and $\pi^{(i)}=1/(1+e^{-R^{(i)}})$.

Let $t^{(i)}=T(\pi^{(i)})$, $i=1,2,3,\cdots$.

Then $E^1(t^{(1)}-t_o|R_{t_o},v_{t_o}) \le \frac{e}{\lambda}$ from (5.1.18) ($e=\exp(1)$).

Replacing t_o by s in (5.1.18)

$$E^1(t^{(i)}-s|R_s=R^{(i-1)},v_s) \le \frac{e}{\lambda}$$

for any $\mathcal{Y}_t$-stopping time $s\ge t_o$, $i=1,2,3,\cdots$

As $t^{(i-1)}=\sup\{s:R_s=R^{(i-1)}\}$ it follows that

$$E^1(t^{(i)}-t^{(i-1)}|R_{t_o},v_{t_o}) \le \frac{e}{\lambda} \quad i=1,2,3 \quad (t^{(0)}\triangleq t_o) \tag{5.1.19}$$

Then
$$E^1(\int_{t_o}^{\infty}\lambda e^{-R_u}.du|R_{t_o},v_{t_o})$$
$$\le \sum_{i=1}^{\infty} E^1(t^{(i)}-t^{(i-1)}|R_{t_o},v_{t_o}).\lambda e^{-R_{t_o}-(i-1)}$$
$$\le e^{-R_{t_o}}\sum_{i=1}^{\infty} e^{2-i} < \infty \quad \text{for } R_{t_o}>-\infty$$

Setting $t_o=\inf\{t:R_t\ge\ln\lambda\}$ gives the required result. □

Definition

The cost function $\bar{C}_{t_o}(\tilde{\tau}_{t_o})$ is defined for $\mathcal{Y}_t^R$-stopping times $\tilde{\tau}_{t_o}\ge t_o$ as

$$\bar{C}_{t_o}(\tilde{\tau}_{t_o}) = I(\tilde{\tau}_{t_o}<t_j) + c(\tilde{\tau}_{t_o}-t_o)I(t_o\ge t_j) \tag{5.1.20}$$

where c is as in (5.1.3), and t_o some fixed time.

Note that $\bar{C}_{t_o}(\tilde{\tau}_{t_o}) \le C_{t_o}(\tilde{\tau}_{t_o}) \quad \forall\tilde{\tau}_{t_o}\ge t_o$ (5.1.21)

This new cost function is useful in the investigation of the evolution of γ_c as $c\downarrow 0$.

Lemma 5.2

Define $R_{\bar{\gamma}_c}(\tilde{v}) \triangleq \inf\{R: \frac{1}{1+e^R} \le \inf_{\tilde{\tau}_{t_o}} E^1(\bar{C}_{t_o}(\tilde{\tau}_{t_o})|R_{t_o}=R, v_{t_o}=\tilde{v})\}$

(5.1.22)

Then

$$\ln\lambda - \ln c \le R_{\gamma_c}(v) \le R_{\bar{\gamma}_c}(v) = R_{\bar{\gamma}_1}(v) - \ln c \quad \forall v \in R^n$$

The final equality is interpreted as meaning $R_{\bar{\gamma}_c}(v) = \infty$ if $R_{\bar{\gamma}_1}(v) = \infty$

Proof

By definition (5.1.22) $\exists \delta < \delta_1$ for any $\hat{v} \in R^n$, $\delta_1 > 0$ such that

$$\frac{1}{1+\exp(R_{\bar{\gamma}_c}(\hat{v})+\delta)} \le \inf_{\tilde{\tau}_{t_o}} E^1(\bar{C}_{t_o}(\tilde{\tau}_{t_o})|R_{t_o}=R_{\bar{\gamma}_c}(\hat{v})+\delta, v_{t_o}=\hat{v})$$

Note that if $R_{\bar{\gamma}_c}(\hat{v}) = \infty$, δ may be chosen as zero, since the right-hand side is non-negative while the left is zero.

Therefore, by (5.1.21) and (5.1.3)

$$E^1(C_{t_o}(t_o)|R_{t_o}=R_{\bar{\gamma}_c}(\hat{v})+\delta, v_{t_o}=\hat{v}) = \frac{1}{1+\exp(R_{\bar{\gamma}_c}(\hat{v})+\delta)}$$

$$\le \inf_{\tilde{\tau}_{t_o}} E^1(C_{t_o}(\tilde{\tau}_{t_o})|R_{t_o}=R_{\bar{\gamma}_c}(\hat{v})+\delta, v_{t_o}=\hat{v}) \quad (5.1.23)$$

$$\Rightarrow R_{\bar{\gamma}_c}(\hat{v}) \ge R_{\gamma_c}(\hat{v}) \quad (5.1.24)$$

since (5.1.23) implies that $\tau^c_{t_o} = t_o$ if $R_{t_o} = R_{\bar{\gamma}_c}(\hat{v})+\delta$, $v_{t_o} = \hat{v}$ with δ arbitrarily small (see(5.1.13)).

Next consider the evolution of $R_{\bar{\gamma}_c}(v)$ as c varies.

Let P denote a (possibly randomized) policy (see Section 2.1) mapping observations of y_t, $t \ge t_o$ into a stopping-time $\tau^P \ge t_o$.

$$P: (y_u : u \ge t_o) \mapsto \tau^P \ge t_o$$

Define $\alpha_c^P = E^1(\bar{C}_{t_o}(\tau^P) \mid t_o \geq t_j, v_{t_o} = \hat{v})$

$$\beta_c^P = E^1(\bar{C}_{t_o}(\tau^P) \mid t_o < t_j, v_{t_o} = \hat{v}) \qquad (5.1.25)$$

From (5.1.20)

$$\alpha_c^P = c\alpha_1^P \text{ and } \beta_c^P = \beta_1^P$$

So $$E^1(\bar{C}_{t_o}(\tau^P) \mid R_{t_o}, v_{t_o} = \hat{v}) = c\pi_{t_o}\alpha_1^P + (1-\pi_{t_o})\beta_1^P \qquad (5.1.26)$$

where as usual $\pi_{t_o} = \dfrac{1}{1+\exp(-R_{t_o})}$.

$$\inf_{\tilde{\tau}_{t_o}} E^1(\bar{C}_{t_o}(\tilde{\tau}_{t_o}) \mid R_{t_o}, v_{t_o} = \hat{v}) - (1-\pi_{t_o})$$

$$= \inf_P [c\pi_{t_o}\alpha_1^P + (1-\pi_{t_o})(\beta_1^P - 1)]$$

$$= (1-\pi_{t_o}) \inf_P [c.\frac{\pi_{t_o}}{1-\pi_{t_o}}.\alpha_1^P + \beta_1^P - 1]$$

$R_{\bar{\gamma}_c}(\hat{v})$ is the infimum of R_{t_o} such that the right-hand side is zero or positive by (5.1.22). But then, unless $\pi_{\bar{\gamma}_c}(\hat{v}) = 1$ where

$$\pi_{\bar{\gamma}_c}(\hat{v}) = \frac{1}{1+\exp(-R_{\bar{\gamma}_c}(\hat{v}))}$$

it follows that

$$\frac{c\pi_{\bar{\gamma}_c}(\hat{v})}{1-\pi_{\bar{\gamma}_c}(\hat{v})}$$

is independent of c,

i.e. $\ln c + R_{\bar{\gamma}_c}(\hat{v})$ is constant.

$$\Rightarrow \quad R_{\bar{\gamma}_c}(\hat{v}) = R_{\bar{\gamma}_1}(\hat{v}) - \ln c \qquad (5.1.27)$$

It remains to show that $\ln\lambda - \ln c \leq R_{\gamma_c}(v) \quad \forall v \in R^n$.

If this is not so, $\exists\ \tilde{\pi} < \frac{\lambda}{\lambda+c}$, $\tilde{v}$ such that if $\pi_{t'} = \tilde{\pi}$, $v_{t'} = \tilde{v}$ it is optimal to stop at t' (if $\tilde{\tau} \not> t'$) when minimizing the expectation of $C(\tilde{\tau})$

i.e. $$1-\tilde{\pi} \le \inf_{\tilde{\tau}_{t_o}} E^1(C_{t_o}(\tilde{\tau}_{t_o})|\pi_{t_o}=\tilde{\pi}, v_{t_o}=\tilde{v})$$

$$\Longleftrightarrow \quad 0 \le \inf_{\tilde{\tau}_{t_o}} E^1(K_{t_o}(\tilde{\tau}_{t_o})|\pi_{t_o}=\tilde{\pi}, v_{t_o}=\tilde{v}) \tag{5.1.28}$$

by Lemma 2.1.

But from definition (2.2.2)

$$E^1(K_{t_o}(\inf\{t:\pi_t \ge \tfrac{\lambda}{\lambda+c}\})|\pi_{t_o}=\tilde{\pi}, v_{t_o}=\tilde{v})$$

$$= P^1\int_{t_o}^{\inf\{t:\pi_t \ge \frac{\lambda}{\lambda+c}\}} [-\lambda+(\lambda+c)\pi_u]du < 0$$

contradicting (5.1.28).

The Lemma is now established. □

<u>Assumption</u>

It is assumed that there exists a sub-space of the state-space R^n of the process v_t such that, if Qv_t denotes the projection of v_t, and

$$\|v\|^* = \|Qv\| \tag{5.1.29}$$

then a) $(R,Qv)_t$ is a Markov process

b) the system (5.1.1) is stable in the sense that

$$E^i(\|v_t\|^{*2}|v_{t_o},t_j) \to r^i \quad \forall t_o, v_{t_o}, t_j \text{ as } t\to\infty$$

for some $r^i<\infty$, $i=1,2$ (5.1.30)

The reason for the introduction of the projection Q is to facilitate the treatment of sub-optimal detection rules for systems of form (3.4.1). These may be put in the form of (5.1.1) by enlarging the state space. Then however (5.1.30) would not hold if the usual norm of v was used.

Lemma 5.3

For any $\rho>0$ $\exists r_\rho<\infty$ such that

$$R_{\bar{\gamma}_1}(v) \le r_\rho \quad \forall v \text{ st } \|v\|^* \le \rho$$

Therefore from Lemma 5.2

$$R_{\gamma_c}(v) \le r_\rho - \ln c \quad \forall v \text{ st } \|v\|^* \le \rho$$

Proof

Let $\tau^\rho \triangleq \inf\{t\ge t_o: \|q_t\|^*=\rho+\varepsilon;\ \exists t', q_{t'} \text{ st } t_o\le t'<t \ \&\ \|q_{t'}\|^*\le\rho;$
q_u is a solution of (5.1.7) $\forall u\in[t',t]\}$ (5.1.31)

for some $\varepsilon>0$. t_o is as in (5.1.20).

For any given sample path of W_t, τ^ρ is then the first possible exit time of a process evolving as v_t from the set $\{x\in R^n: \|x\|^*\le\rho+\varepsilon\}$, given that it started at <u>some</u> time $t'\in[t_o,\tau^\rho)$ in the set $\{x\in R^n: \|x\|^*\le\rho\}$.

Then $\exists\bar{\varepsilon}>0$ such that

$$\min\{E^1(\tau^\rho-t_o|t_j\le t_o), E^1(\tau^\rho-t_o|t_j>t_o)\} > \bar{\varepsilon} \tag{5.1.32}$$

The process η_t, $t\ge t_o$, is defined as follows:

$$d\eta_t = I(t\ge t_j)k + dW_t \tag{5.1.33}$$

where $k\in R^m$ is a constant vector such that

$$k_i = \sup_{\|v\|^*\le\rho+\varepsilon} \left|([D^1-D^0:F^1-F^0]v + z^1-z^0)_i\right| \tag{5.1.34}$$

Here, for $x\in R^m$, x_i is the i^{th} component.

The cost $\hat{C}_{t_o}(\tilde{\tau}_{t_o})$ is defined so that, for $\tilde{\tau}_{t_o}\ge t_o$

$$\hat{C}_{t_o}(\tilde{\tau}_{t_o}) = I(\tilde{\tau}_{t_o}<\min\{t_j,\tau^\rho\}) + c(\min\{\tilde{\tau}_{t_o},\tau^\rho\}-t_o)I(t_o\ge t_j) \tag{5.1.35}$$

Then $\hat{C}_{t_o}(\tilde{\tau}_{t_o}) \le \bar{C}_{t_o}(\tilde{\tau}_{t_o}) \quad \forall\tilde{\tau}_{t_o}\ge t_o$ (5.1.36)

and $\hat{C}_{t_o}(\tilde{\tau}_{t_o}) = \bar{C}_{t_o}(\tilde{\tau}_{t_o})$ for $\tilde{\tau}_{t_o}<\tau^\rho$ (5.1.37)

For $\tilde{\tau}_{t_o} \geq \tau^{\rho}$, $\hat{C}_{t_o}(\tilde{\tau}_{t_o})$ is independent of $\tilde{\tau}_{t_o}$.

Now

$$\inf\{E^1 \hat{C}_{t_o}(\tilde{\tau}_{t_o}) : \tilde{\tau}_{t_o} \text{ a } H_t^R\text{-stopping time}\} \qquad (5.1.38)$$

$$\leq \inf\{E^1 \hat{C}_{t_o}(\tilde{\tau}_{t_o}) : \tilde{\tau}_{t_o} \text{ a } Y_t^R\text{-stopping time}\}$$

where H_t^R is the σ-field generated by $(y_u : u \leq t_o) \& (\eta_u : u \in [t_o, t])$ and, possibly, additional random variables independent of t_j, W_t.

This may be justified as follows:

Given η_t and V_t, an independent Wiener process, generate $\hat{v}_t$, $t \in [t_o, \tau^{\rho}]$ using

$$d\hat{v}_t = \begin{bmatrix} A & B \\ D^0 & F^0 \end{bmatrix} \hat{v}_t dt + \begin{bmatrix} 0 \\ z^0 \end{bmatrix} dt$$

$$+ \left(\begin{bmatrix} 0 & 0 \\ D^1 - D^0 & F^1 - F^0 \end{bmatrix} \hat{v}_t + \begin{bmatrix} 0 \\ z^1 - z^0 \end{bmatrix} \right) \otimes \begin{bmatrix} d0 \\ k^{-1} \otimes d\eta_t \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ I_m \end{bmatrix} \alpha_t \otimes dV_t \qquad (5.1.39)$$

$$\hat{v}_{t_o} = v_{t_o}$$

Here, k^{-1} is a vector such that $[k^{-1}]_i = \frac{1}{k_i}$;

$$a \otimes b = \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_m b_m \end{bmatrix}; \qquad (\alpha_t)_i = \sqrt{\left(1 - \frac{1}{k_i^2}([D^1 - D^0 : F^1 - F^0]\hat{v}_t + z^1 - z^0)_i^2\right)}$$

Then $\hat{v}_t$ is statistically indistinguishable from v_t for given v_{t_o}, t_j, as may be seen from (5.1.1), (5.1.33) and (5.1.34). So with observations η_t a stopping rule may be constructed which has the same expected cost as any given Y_t^R-stopping time, in the sense of cost fuction $\hat{C}_{t_o}(\tilde{\tau}_{t_o})$, which justifies (5.1.38).

Suppose now that η_t is observed from time t_o istead of y_t and that $\|v_{t_o}\|^* \leq \rho$. For some finite value of R_{t_o} it is not optimal to continue until τ^ρ w.p.1 since for R_{t_o} sufficiently large (so $1-\pi_{t_o}$ is small as $\pi_{t_o} = \frac{1}{1+\exp(-R_{t_o})}$)

$$E^1(\hat{C}(t_o)|R_{t_o}, v_{t_o}) = 1-\pi_{t_o} < E^1(\hat{C}(\tau^\rho)|R_{t_o}, v_{t_o}) \geq \pi_{t_o} c\bar{\varepsilon} > 0$$

from (5.1.32).

Defining

$$\hat{C}_{t_o+u}(\tilde{\tau}) \triangleq I(\tilde{\tau}<\min\{t_j, \tau^\rho\}) + c(\min\{\tilde{\tau}, \tau^\rho\}-t_o-u)I(t_o \geq t_j)$$
$$\text{for } \tilde{\tau} \geq t_o+u \tag{5.1.40}$$

(c.f. (5.1.35)) it follows that for some $r_\rho < \infty$, $u \geq 0$

$$1- \frac{1}{1+\exp(-r_\rho)} = E^1(\hat{C}_{t_o+u}(t_o+u)|R_{t_o+u}=r_\rho, \eta_{t_o+u})$$
$$\leq \inf_{\tilde{\tau} \geq t_o+u} E^1(\hat{C}_{t_o+u}(\tilde{\tau})|R_{t_o+u}=r_\rho, \eta_{t_o+u}) \tag{5.1.41}$$

where $\tilde{\tau}$ is a η_t^R-stopping time. Otherwise there would allways be a better policy than stopping before τ^ρ, since $R_{t_o+u} < \infty$ $\forall u < \infty$.

But if $u>0$, it is also optimal to stop at t_o if $R_{t_o}=r_\rho$, since

$$E^1(\hat{C}_{t_o}(t_o)|R_{t_o}=r_\rho) = E^1(\hat{C}_{t_o+u}(t_o+u)|R_{t_o+u}=r_\rho, \eta_{t_o+u})$$
$$= 1- \frac{1}{1+\exp(-r_\rho)} \tag{5.1.42}$$

and
$$\inf_{\tilde{\tau}_{t_o} \geq t_o} E^1(\hat{C}_{t_o}(\tilde{\tau}_{t_o})|R_{t_o}=r_\rho)$$
$$\geq \inf_{\tilde{\tau} \geq t_o+u} E^1(\hat{C}_{t_o+u}(\tilde{\tau})|R_{t_o+u}=r_\rho, \eta_{t_o+u}) \tag{5.1.43}$$

($\tilde{\tau}, \tilde{\tau}_{t_o}$ H_t^R-stopping times)

from the definition of $\hat{C}_{t_o+u}$, (5.1.40).

So if $R_{t_o}=r_\rho$ it is optimal to stop at t_o with observations η_t and cost $\hat{C}_{t_o}(\tilde{\tau}_{t_o})$. But then by (5.1.38) it is optimal to stop at t_o with observations y_t and cost $\hat{C}_{t_o}(\tilde{\tau}_{t_o})$ if $R_{t_o}=r_\rho$.

Finally (5.1.36), (5.1.37) imply that it is optimal to stop at t_o with observations y_t and cost $\bar{C}_{t_o}(\tilde{\tau}_{t_o})$ if $R_{t_o}=r_\rho$, since

$$E^1(\bar{C}_{t_o}(t_o)|R_{t_o}=r_\rho, v_{t_o}) = E^1(C_{t_o}(t_o)|R_{t_o}=r_\rho)$$

Therefore $R_{\bar{\gamma}_c}(v) \le r_\rho \quad \forall v \text{ st } \|v\|^* \le \rho$. □

The results of Lemmas 5.2 and 5.3 may be illustrated (for n=1) as follows.

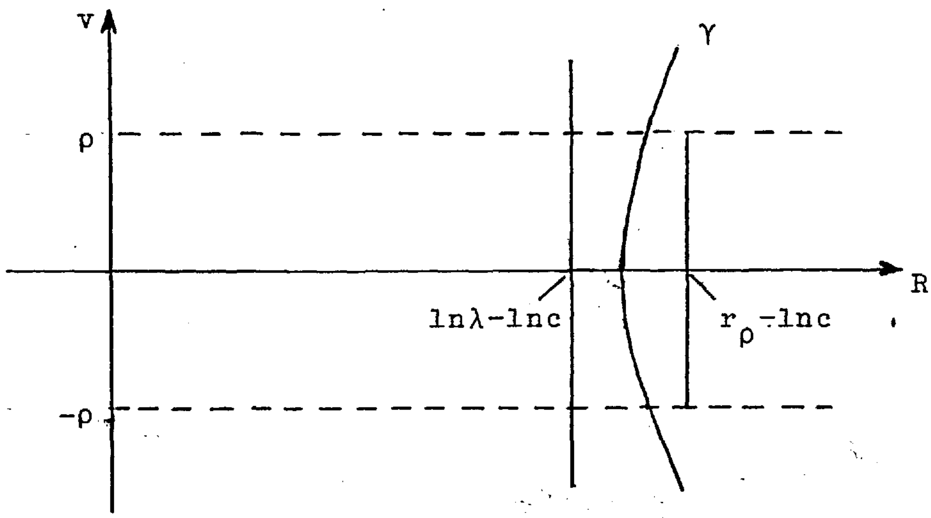


Figure 5.1.2

The evolution of γ_c as $c\downarrow 0$ is described by these two results.

<u>Definitions</u>

$$t_s \triangleq \inf\{t \ge t_j : R_t \ge \ln\lambda, \|v_t\|^* \le \rho\} \tag{5.1.44}$$

Let M^+ and M^- be finite n×n matrices chosen such that

$$v^T M^+ v \ge 0, \quad v^T M^- v \le 0 \qquad \forall v \in R^n$$

and $$v^T M^2 v = v^T(M^+ + M^-)v \qquad \forall v \in R^n \tag{5.1.45}$$

(M^2 defined in (5.1.8))

Let $[x]^+ \triangleq xI(x \ge 0)$ and $[x]^- \triangleq xI(x \le 0) \quad \forall x \in R$

The scalar process $\hat{R}_t^c$ is defined such that

$$\hat{R}_t^c = R_t \quad \forall t \le t_s$$

$$d\hat{R}_t^c = (v_t^T M^- v_t + [h^{2^T} v_t]^- + [g^2]^-)dt$$
$$+ [([D^1-D^0:F^1-F^0]v_t + z^1-z^0)^T \zeta_t]^- dt$$
$$+ (\lambda + v_t^T M^+ v_t + [h^{2^T} v_t]^+ + [g^2]^+)I(\|v_t\|^* \le \rho)dt$$
$$+ ([D^1-D^0:F^1-F^0]v_t + z^1-z^0)^T dW_t . I(R_t < r_\rho - \ln c)$$

$$\forall t \ge t_s, \quad \rho > 0 \tag{5.1.46}$$

Here r_ρ is as in Lemma 5.3 and c as in $C(\tilde{\tau})$, (5.1.3).

Remarks

$\hat{R}_t^c$ has been defined to have certain properties required in the proof of Theorem 5.1, when (5.1.5) holds as assumed in the following.

Since, while $R_t < r_\rho - \ln c$, $d(\hat{R}_t^c - R_t)$ is negative (see (5.1.11)), $\hat{R}_t^c$ is less than or equal to R_t up to the first time that $\hat{R}_t^c \ge r_\rho - \ln c$.

Then supposing that $R_{t_s} = \hat{R}_{t_s}^c < r_\rho - \ln c$ it follows that at the first time that $\hat{R}_t^c \ge r_\rho - \ln c$, $R_t \ge r_\rho - \ln c$. Because of the way (5.1.46) has been set up it follows that $\hat{R}_t^c \le r_\rho - \ln c$ until

$$\hat{\tau}_c \triangleq \inf\{t : \hat{R}_t^c \ge r_\rho - \ln c, \|v_t\|^* \le \rho\} \tag{5.1.47}$$

The following results are required later. As usual

$$\tau^c = \inf\{t : R_t \ge R_{\gamma_c}(v_t)\}$$

a) $\hat{\tau}^c \ge \tau^c$ (5.1.48)

from the above argument and Lemma 5.3

b) $\hat{R}_t^c \le r_\rho - \ln c \quad$ if $t_s < t \le \hat{\tau}^c$ (5.1.49)

by (5.1.44),(5.1.47) and above

Lemma 5.4

$\exists\hat{\sigma}\in R$ independent of c such that

$$E^2(\hat{R}^c_{t_s+T}-\hat{R}^c_{t_s}|t_j,v_{t_j}) = \hat{\sigma}T+\varepsilon \quad \text{for } T\geq 0$$

where $\quad \varepsilon\geq-\hat{\varepsilon}-q(v_{t_j})>-\infty, \quad Eq(v_{t_j})<\infty, \quad q(\cdot)\geq 0$

and $\quad E^2(\hat{R}^c_t-\hat{R}^c_{t_1}|t_1,t_1\geq t_s,v_{t_1}) = \hat{\sigma}(t-t_1)+\varepsilon \quad \text{for } t\geq t_1$

where $\quad \varepsilon\leq\hat{\varepsilon}<\infty$

Proof (superscripts ² on M,h,g are omitted)

(5.1.5) is assumed to hold throughout the following.

First consider the process

$$L_t \triangleq \hat{R}^c_t-\hat{R}^c_{t_s} - \int_{t_s}^t [([D^1-D^0:F^1-F^0]v_u+ z^1-z^0)^T\zeta_u]^- du$$

$$\geq \hat{R}^c_t-\hat{R}^c_{t_s} \qquad \text{for } t\geq t_s \quad (5.1.50)$$

Then L_t satisfies

$$dL_t = (v_t^T M^- v_t+[h^T v_t]^-+[g]^-)dt$$

$$+ (\lambda+v_t^T M^+ v_t+[h^T v_t]^+ +[g]^+)I(\|v_t\|^*\leq\rho)dt$$

$$+ ([D^1-D^0:F^1-F^0]v_t+ z^1-z^0)^T dW_t . I(R_t<r_\rho-\ln c) \qquad t\geq t_s$$

(5.1.51)

by (5.1.46) and (5.1.50), and (L_t,v_t) is a Markov process for $t\geq t_s$.

Let $(L',v')_t$, $(L'',v'')_t$ be solutions of (5.1.1),(5.1.51) for $t\geq t_1\geq t_s$ with the same sample path of W_t for $t\geq t_1$ in each case, but with initial conditions

$$(L',v')_{t_1} = (\bar{L},\bar{v}') \qquad (5.1.52)$$

$$(L'',v'')_{t_1} = (\bar{L},v'') \qquad (5.1.53)$$

for some fixed $\bar{L}\in R$, $v',v''\in R^n$ s.t. $\|v'\|^*,\|v''\|^*<\infty$

By (5.1.1) and assumption (5.1.30)

$$\|v'_t-v''_t\|^* \leq Ne^{-K(t-t_1)} \quad \text{for some } N,K\in(0,\infty),\ t\geq t_1$$

(5.1.54)

Define

$T_1 = \int_{t_1}^{t} (v_u'^T M^- v_u' - v_u''^T M^- v_u'')du$

$T_2 = \int_{t_1}^{t} ([h^T v_u']^- - [h^T v_u'']^-)du$

$T_3 = \int_{t_1}^{t} (v_u'^T M^+ v_u' - v_u''^T M^+ v_u'') I(\|v_u'\|^* \le \rho) I(\|v_u''\|^* \le \rho)du$

$T_4 = \int_{t_1}^{t} ([h^T v_u']^+ - [h^T v_u'']^+) I(\|v_u'\|^* \le \rho) I(\|v_u''\|^* \le \rho)du$

$T_5 = -\int_{t_1}^{t} (\lambda + v_u'' M^+ v_u'' + [h^T v_u'']^+ + [g]^+) I(\|v_u'\|^* > \rho) I(\|v_u''\| \ge \rho)$

$T_6 = \int_{t_1}^{t} (\lambda + v_u' M^+ v_u' + [h^T v_u']^+ + [g]^+) I(\|v_u'\|^* \le \rho) I(\|v_u''\| < \rho)$

Then $\quad \Delta_t \triangleq E^2(L_t' - L_t'') = \sum_{i=1}^{6} E^2 T_i \qquad (5.1.55)$

E^2T_i, $i=1,2,\cdots,6$ are uniformly bounded $\forall t \ge t_1 \ge t_j$, for each v', v'', as shown below:

E^2T_1: see (5.1.1)

E^2T_2: $|[h^T v_u']^- - [h^T v_u'']^-| \le \|h\|^* . \|v_u' - v_u''\|^* \le \|h\|^* . Ne^{-K(t-t_1)}$

(see definition (5.1.29) and (5.1.11) to see that $h^T v = h^T Qv$, $v^T Mv = (Qv)^T MQv$)

E^2T_3: $(v^T M^+ v)$ has bounded gradient in $\{v: \|v\|^* \le \rho\}$, and $\|v_u' - v_u''\|^* \le Ne^{-K(t-t_1)}$

E^2T_4: as for E^2T_3

E^2T_5: $(\lambda + v^T M^+ v + [h^T v]^+ + [g]^+)$ is bounded for $\|v\|^* \le \rho$

$\|v_u'\|^* > \rho$, $\|v_u''\|^* \le \rho \Rightarrow \|v_u'\|^* \in (\rho, \rho + Ne^{-K(u-t_1)})$

Since the p.d.f of v_u' is bounded $\forall t \ge t_1$,

$E^2 \int_{t_1}^{t} I(\|v_u'\|^* \in (\rho, \rho + Ne^{-K(u-t_1)}))du$ is uniformly bounded as $t \to \infty$.

E^2T_6: as for E^2T_5

So $\exists\ \bar{\varepsilon}_L^-(\cdot,\cdot)$ and $\bar{\varepsilon}_L^+(\cdot,\cdot)$ such that, from (5.1.55)

$$\infty < -\bar{\varepsilon}_L^-(v', v'') \le \Delta_t \le +\bar{\varepsilon}_L^+(v', v'') < \infty \quad \forall t \ge t_1 \qquad (5.1.56)$$

Since by assumption (5.1.30), $\mathcal{Q}v_t$ is an (asymptotically) stationary process, and by (5.1.51), it follows that $\exists\hat{\sigma}\in R, v''$ such that

$$\frac{1}{t-t_1}E^2(L''_t-\bar{L}) = \hat{\sigma} \qquad (5.1.57)$$

where L''_t is defined by (5.1.53). With v'' chosen in this way, it follows from (5.1.56) that

$$E^2(L_t-L_{t_1}|t_1,t_1\geq t_s,v_{t_1}) = \hat{\sigma}(t-t_1) + \varepsilon \qquad (5.1.58)$$

where $-\infty < -\bar{\varepsilon}_L^-(v_{t_1},v'') \leq \varepsilon \leq \bar{\varepsilon}_L^+(v_{t_1},v'') < \infty$

Since $N,K\in(0,\infty)$ may be chosen in (5.1.54) so that this holds $\forall v'=v'_{t_1}$ such that $\|v'\|^*\leq\rho$, with v'' as above,

$$\hat{\varepsilon} \triangleq \sup_{\|v'\|^*\leq\rho} \bar{\varepsilon}_L^-(v',v'') \vee \bar{\varepsilon}_L^+(v',v'') < \infty \qquad (5.1.59)$$

Now $E^2(L_t-L_{t_1}|t_1,t_1\geq t_s,v_{t_1}) \leq E^2(L_t-L_{\tau^\rho\wedge t}|t_1,t_1\geq t_s,v_{t_1})$
(5.1.60)

where $\tau^\rho\triangleq\inf\{t\geq t_1:\|v_t\|^*\leq\rho\}$, by (5.1.50),(5.1.46), so using (5.1.58)

$$E^2(L_t-L_{t_1}|t_1,t_1\geq t_s,v_{t_1})$$

$$\leq E^2(\hat{\sigma}(t-\tau^\rho\wedge t)+\bar{\varepsilon}_L^+(v_{\tau^\rho\wedge t},v'')|t_1,t_1\geq t_s,v_{t_1})$$

$$\leq \hat{\sigma}(t-t_1) + \hat{\varepsilon} \qquad (5.1.61)$$

So in (5.1.58) in fact $\varepsilon\leq\hat{\varepsilon}<\infty$ irrespective of v_{t_1}.

It remains to relate these results to the process $\hat{R}_t^c$ through (5.1.50). From (5.1.5b) and Assumption (5.1.30), it follows that

$$0 \leq E^2(\|[D^1-D^0:F^1-F^0]v_t+ z^1-z^0\|.\|\zeta_t\||t_j,v_{t_j}) \leq r(v_{t_j})e^{-\frac{\beta}{2}(t-t_j)}$$
(5.1.62)

for some function $r(\cdot)$ s.t. $E^2(r(v_{t_j})|t_j)$ is uniformly bounded $\forall t_j$.

Therefore

$$-\infty < -q(v_{t_j})$$
$$\leq E^2\left(\int_{t_j}^{\infty} [([D^1-D^0 : F^1-F^0]v_u + z^1-z^0)^T \zeta_u]^- du \,|\, t_j, v_{t_j}\right) \leq 0 \qquad (5.1.63)$$

for some function $q(\cdot)$ such that $E(q(v_{t_j})|t_j)$ is uniformly bounded $\forall t_j$.

So using (5.1.58)

$$E^2(\hat{R}^c_{t_s+T}-\hat{R}^c_{t_s}|t_j, v_{t_j}) \geq \hat{\sigma}T + \varepsilon \quad \text{for } T\geq 0$$

where $\varepsilon \geq -\hat{\varepsilon}-q(v_{t_j})$, since $\|v_{t_s}\|^* \leq \rho$ by definition of t_s and (5.1.59) implies $\bar{\varepsilon}_L^-(v_{t_s}, v'') \leq \hat{\varepsilon}$.

Also $\quad E^2(\hat{R}^c_t-\hat{R}^c_{t_1}|t_1, t_1\geq t_s, v_{t_1}) = \hat{\sigma}(t-t_1) + \varepsilon \quad$ for $t\geq t_1$

where $\varepsilon \leq \hat{\varepsilon} < \infty$ by (5.1.61) and (5.1.62). t_1 is a stopping time for W_t.

This establishes the results of the Lemma. □

It follows from Lemma 5.4 that (5.1.64)

$$\hat{\sigma} = \lim_{T\to\infty} \frac{1}{T}E^2\int_{t_j}^{t_j+T} (v_u^T M^- v_u + [h^{2T} v_u]^- + [g^2]^- + (\lambda + v_u^T M^+ v_u + [h^{2T}v_u]^+ + [g^2]^+) I(\|v_u\|^* \leq \rho))du$$

<u>Definitions</u>

Define $\quad \sigma_2 \triangleq \lim_{T\to\infty} \frac{1}{T}E^2\int_{t_j}^{t_j+T} (\lambda + v_u^T M^2 v_u + h^{2T} v_u + g^2)du \qquad (5.1.65)$

Then $\hat{\sigma}\to\sigma_2$ as $\rho\to\infty$.

Define $\quad \sigma_1 \triangleq \lim_{T\to\infty} \frac{1}{T}E^1\int_{t_j}^{t_j+T} (\lambda + v_u^T M^1 v_u + h^{1T} v_u + g^1)du \qquad (5.1.66)$

Also

$$\mu_t \triangleq \int_{t_j}^{t} (\lambda + v_u^T M^1 v_u + h^{1T} v_u + g^1)du + \int_{t_j}^{t} ([D^1-D^0 : F^1-F^0]v_u + z^1-z^0)^T dW_u \quad \forall t\geq t_j \qquad (5.1.67)$$

Then $\sigma_1 = \lim_{T\to\infty} \frac{1}{T}E^1 \mu_{t_j+T}$.

Note that from (5.1.11) if (5.1.4) holds

$$R_t - R_{t_j} = \mu_t + \int_{t_j}^{t} \lambda e^{-R_u} du \qquad (5.1.68)$$

Also, $\lambda + x^T M^1 x + h^{1T} x + g^1 > 0 \quad \forall x \in R^n$ (5.1.69)

by definition of M^1, h^1, g^1.

<u>Lemma 5.5</u>

$E^1(\mu_t | \mu_{t_1}, v_{t_1}, t_1 \geq t_j) - \mu_{t_1} = \sigma_1(t - t_1) + \delta$

for W_t-stopping time t_1, where $\infty < -\bar{\delta} \leq \delta \leq \hat{\delta}(v_{t_1}) < \infty$

for some $\hat{\delta}(\cdot), \bar{\delta}$; $E(\hat{\delta}(v_{t_j})) < \infty$.

<u>Proof</u>

Follows from (5.1.1),(5.1.2) and the property (5.1.69) □

<u>Lemma 5.6</u>

$E^2(t_s - t_j | t_j) \leq a < \infty \quad \forall t_j \geq 0$, for some a independent of c.

<u>Proof</u>

$$\begin{aligned} &\text{Define } t^{(0)} \triangleq \inf\{t \geq t_j : \|v_t\|^* \leq \rho\} \\ &\text{and } \quad t^{(i)} \triangleq \inf\{t \geq t^{(i-1)} + \Delta : \|v_t\|^* \leq \rho\} \quad i = 1, 2, \cdots \end{aligned} \qquad (5.1.70)$$

for some fixed $\Delta > 0$.

$\exists a_o < \infty$ such that $E^2(t^{(0)} - t_j | t_j) < a_o \; \forall t_j$ (5.1.71)

from (5.1.1) and (5.1.2).

Let the process R_t^* evolve as R_t (i.e. R_t^* satifies (5.1.5) and (5.1.11)) for $t \in [t^{(i-1)}, t^{(i)})$, i=1,2,$\cdots$ but with

$$R^*_{t(i)} = -\infty \quad (\pi^*_{t(i)} = 0), \quad i = 1, 2, \cdots \qquad (5.1.72)$$

Define L_t^* such that $L^*_{t(i)} = -\infty$ and for $t \in [t^{(i-1)}, t^{(i)})$ i=1,2,$\cdots$

$$dL_t^* = \lambda(1 + e^{-L_t^*})dt + (v_t^T M^2 v_t + h^{2T} v_t + g^2)dt + ([D^1 - D^0 : F^1 - F^0]v_t + z^1 - z^0)^T dW_t \qquad (5.1.73)$$

Comparing (5.1.11) and (5.1.73) it follows that

$$|L_t^*-R_t^*| \le \int_{t(i-1)}^{t} \|[D^1-D^0:F^1-F^0]v_u+ z^1-z^0\|^*.\|\zeta_u\|^* du$$

where $t\in[t^{(i-1)},t^{(i)})$. (5.1.74)

Now let $p_i \triangleq P^2(\lim_{t\uparrow t^{(i)}} L_t^*\ge \ln\lambda+\varepsilon \mid \nexists j<i \text{ s.t. } \lim_{t\uparrow t^{(i)}} R_t^*\ge\ln\lambda)$

for some fixed $\varepsilon>0$. (5.1.75)

Then $\lim_{i\to\infty} p_i <0$, $p_i>0\ \forall i$, so that $\exists\bar{p}$ and

$$p_i \ge \bar{p} > 0\ \forall i \qquad (5.1.76)$$

Define $\tilde{N} \triangleq \inf\{i:\lim_{t\uparrow t^{(i)}} R_t^*\ge\ln\lambda\}$ (5.1.77)

Note that $\exists\delta<\infty$ s.t. $E^2(t^{(i+1)}-t^{(i)}|t_s>t^{(i)},t_j) \le \delta$

$i=0,1,2,\cdots$ from (5.1.1) (5.1.78)

Now from (5.1.5b) $E^2(\|\zeta_u\|^2|t_j) \le \alpha e^{-\beta(t-t_j)}$ $\alpha<\infty,\beta>0,t\ge t_j$

Therefore

$$E^2(\int_{t_j}^{\infty} \|[D^1-D^0:F^1-F^0]v_u+z^1-z^0\|^*.\|\zeta_u\|^* du|t_j) < \infty$$

(5.1.79)

$\exists\tilde{\alpha}<\infty$, $\tilde{\beta}<1$ so that

$$E^2(\int_{t^{(i)}}^{t^{(i+1)}} \|[D^1-D^0:F^1-F^0]v_u+ z^1-z^0\|^*.\|\zeta_u\|^* du|t_j)$$

$$\le \tilde{\alpha}.\tilde{\beta}^i \quad i=0,1,2,\cdots$$

since otherwise (5.1.79) would be contradicted. Therefore

$$E^2(\lim_{t\uparrow t^{(i)}} |L_t^*-R_t^*| |t_j) \le \tilde{\alpha}.\tilde{\beta}^i \quad \text{from (5.1.74)}$$

So $P^2(\lim_{t\uparrow t^{(i)}} (L_t^* -R_t^*)>\varepsilon|t_j) \le \frac{1}{\varepsilon}.\tilde{\alpha}.\tilde{\beta}^i$ (5.1.80)

Then $P^2(\tilde{N}>i|t_j)-P(\tilde{N}>i-1|t_j) \le -\bar{p}.P^2(\tilde{N}>i-1|t_j) + \frac{1}{\varepsilon}.\tilde{\alpha}.\tilde{\beta}^i$

(5.1.81)

from (5.1.76),(5.1.77)&(5.1.80).

$$P^2(\bar{N}>i|t_j) \leq (1-\bar{p})^{i-1} + \frac{\tilde{\alpha}}{\varepsilon}.\sum_{j=2}^{i} \tilde{\beta}^j(1-\bar{p})^{i-j}$$

$$\leq (1-\bar{p})^{i-1} + \frac{\tilde{\alpha}}{\varepsilon}.[\tilde{\beta}^{i+1}-(1-\bar{p})^{i-1}\tilde{\beta}^2]/[\tilde{\beta}-(1-\bar{p})]$$

$$\leq \hat{\alpha}.\hat{\beta}^i \quad \text{for some } \hat{\alpha}<\infty, \hat{\beta}<1 \qquad (5.1.82)$$

Since by definitions (5.1.44) and (5.1.77) and also by (5.1.70), $t^{(\bar{N})}>t_s$

$$P^2(t_s>t^{(i)}|t_j) \leq \hat{\alpha}.\hat{\beta}^i$$

As $\quad E^2(t_s-t_j|t_j) \leq \sum_{i=1}^{\infty} P^2(t_s>t^{(i)}|t_j).\delta + a_o$

from (5.1.78) and (5.1.71)

$$E^2(t_s-t_j|t_j) \leq \sum_{i=1}^{\infty} \hat{\alpha}.\hat{\beta}^i.\delta + a_o < a \text{ say, where } a<\infty. \quad \square$$

Recall the following definitions:

$$\sigma_1 \triangleq \lim_{T\to\infty} \frac{1}{T}E^1\int_{t_j}^{t_j+T}(\lambda+v_u^TM^1v_u+h^{1T}v_u+g^1)du$$

$$\sigma_2 \triangleq \lim_{T\to\infty} \frac{1}{T}E^1\int_{t_j}^{t_j+T}(\lambda+v_u^TM^2v_u+h^{2T}v_u+g^2)du$$

$$\tau^c \triangleq \inf\{t:R_t\geq R_{\gamma_c}(v_t)\}$$

Theorem 5.1

If $\sigma_2>\sigma_1$, $\exists c_m$ such that $\forall c\in(0,c_m]$

$$E^2C(\tau^c) \leq E^1C(\tau^c)$$

Proof

Consider $c\leq 1$.

Suppose that $\sigma_2>\sigma_1$ and choose ρ in (5.1.46) so that $\hat{\sigma}>\sigma_1$.

Define $\hat{\tau}^c \triangleq \inf\{t:\hat{R}_t^c\geq r_\rho-\ln c, \|v_t\|^*\leq\rho\}$ (5.1.83)

where r_ρ is defined as in Lemma 5.3 Note that $r_\rho\geq\ln\lambda$.

If (5.1.5) holds, then $\hat{\tau}^c\geq\tau^c$ (see (5.1.48)) and

because of Lemmas 5.2 and 5.3, for $c\leq 1$

$$\tau^c\geq t_j \Rightarrow \hat{\tau}^c\geq t_j \Rightarrow \hat{\tau}^c\geq t_s \qquad (5.1.84)$$

by (5.1.83) and (5.1.44).

Now choose $T>0$. Then

$$E^2(\hat{R}^c_{T+t_s}-R_{t_s}|t_j\leq\tau^c) = E^2(\hat{R}^c_{T+t_s}-\hat{R}^c_{\hat{\tau}^c\wedge(T+t_s)}|t_j\leq\tau^c)$$

$$+ E^2(\hat{R}^c_{\hat{\tau}^c\wedge(T+t_s)}-R_{t_s}|t_j\leq\tau^c)$$

(5.1.85)

From (5.1.49) and (5.1.83).

$$\hat{R}^c_{\hat{\tau}^c\wedge(T+t_s)}-R_{t_s} \leq \max(0,r_\rho-\ln c-R_{t_s}) \qquad (5.1.86)$$

Also $R_{t_s}\geq\ln\lambda$ by (5.1.44).

By Lemma 5.4, from (5.1.85)

$$\hat{\sigma}T+\varepsilon_1 \leq \hat{\sigma}(T-E^2((\hat{\tau}^c-t_s)\wedge T)|t_j\leq\tau^c)) + \varepsilon_2 + \max(0,r_\rho-\ln c-\ln\lambda)$$

(5.1.89)

where $\varepsilon_1\geq-\hat{\varepsilon}-E^2(q(v_{t_j})|t_j\leq\tau^c)$

$\varepsilon_2\leq\hat{\varepsilon}<+\infty$

So $E^2[(\tau^c-t_s)\wedge T|t_j\le\tau^c] \le E^2[(\hat{\tau}^c-t_s)\wedge T|t_j\le\tau^c]$

$$\le [r_\rho-\ln c-\ln\lambda+2\hat{\varepsilon}+E^2(q(v_{t_j})|t_j\le\tau^c)]/\hat{\sigma}$$

$$\le [-\ln c+k_2+E^2(q(v_{t_j})|t_j\le\tau^c)]/\hat{\sigma}$$

where $k_2<\infty$ is independent of c.

i.e. $\int_0^T P^2(\tau^c-t_s\le u|t_j\le\tau^c)du \le [-\ln c+k_2+E^2(q(v_{t_j})|t_j\le\tau^c)]/\hat{\sigma}$

$c\le 1$, $\forall T>0$. Therefore

$$E^2(\tau^c-t_s|t_j\le\tau^c) \le [-\ln c+k_2+E^2(q(v_{t_j})|t_j\le\tau^c)]/\hat{\sigma} \quad , c\le 1 \tag{5.1.90}$$

Next,

$$E^1(\mu_{t_j+T}|t_j\le\tau^c) = E^1(\mu_{t_j+T}-\mu_{(t_j+T)\wedge\tau^c}|t_j\le\tau^c) + E^1(\mu_{(t_j+T)\wedge\tau^c}|t_j\le\tau^c) \tag{5.1.91}$$

where μ_t is defined in (5.1.67). From Lemma 5.5

$$\sigma_1 T+\delta_1 \ge \sigma_1(T-E^1((\tau^c-t_j)\wedge T|t_j\le\tau^c)) + \delta_2 + E^1(\mu_{(t_j+T)\wedge\tau^c}|t_j\le\tau^c) \tag{5.1.92}$$

where $\delta_1 = E^1(\hat{\delta}(v_{t_j})|t_j\le\tau^c)$

$\delta_2 \ge -\bar{\delta}>-\infty$

Next, the last term in (5.1.92) is investigated.

Firstly, as $T\to\infty$, $\pi_{t_j+T}\to 1$ w.p.1 if (5.1.4) holds. Otherwise

$P^1(t_j<\infty)=\lim_{t\to\infty} E^1\pi_t \ne 1$ which contradicts (5.1.2)

Therefore $R_{t_j+T}\to\infty$ w.p.1 as $T\to\infty$, which implies that

$R_{(t_j+T)\wedge\tau^c} \ge \ln\lambda-\ln c$ for T sufficiently large, w.p.1,

by Lemma 5.2.

Let $t_\lambda \triangleq \inf\{t:R_t\ge\ln\lambda\}$ (5.1.93)

$$E^1(R_{[(t_j+T)\wedge\tau^c]\vee t_\lambda}|t_j\le\tau^c) \to \ln\lambda-\ln c+\varepsilon,\ \varepsilon>0 \text{ as } T\to\infty \tag{5.1.94}$$

t_λ is introduced here to ensure that the expectation is well defined.

Secondly

$$E^1(R_{t_j \vee t_\lambda} | t_j \le \tau^c)$$
$$\le E^1[\int_{t_\lambda}^{t_j \vee t_\lambda} (\lambda + v_u^T M^0 v_u + h^{0T} v_u + g^0 + \lambda e^{-R_u}) du | t_j \le \tau^c] + \ln\lambda$$
$$\le E^1(\lambda t_j | t_j \le \tau^c) + E^1(\int_{t_\lambda}^{t_j \vee t_\lambda} \lambda e^{-R_u} du | t_j \le \tau^c) + \ln\lambda \tag{5.1.95}$$

by (5.1.11), since $v_u^T M^0 v_u + h^{0T} v_u + g^0 \le 0 \quad \forall u$ (from the definitions of M^0, h^0, g^0).

Now from (5.1.67), (5.1.69)

$$E^1(\mu_{(t_j+T)\wedge\tau^c} | t_j \le \tau^c)$$
$$\ge E^1(\int_{t_j \vee t_\lambda}^{[(t_j+T)\wedge\tau^c]\vee t_\lambda} (\lambda + v_u^T M^1 v_u + h^{1T} v_u + g^1) du | t_j \le \tau^c)$$
$$\ge E^1(R_{[(t_j+T)\wedge\tau^c]\vee t_\lambda} - R_{t_j \vee t_\lambda} - \int_{t_j \vee t_\lambda}^{\infty} \lambda e^{-R_u} du | t_j \le \tau^c) \tag{5.1.96}$$

Then from (5.1.94), (5.1.95) & (5.1.96) substituted into (5.1.92)

$$E^1(\tau^c - t_j | t_j \le \tau^c)$$
$$\ge \frac{1}{\sigma_1}[-\bar{\delta} - E^1(\hat{\delta}(v_{t_j}) + \lambda t_j + \int_{\inf\{t: R_t \ge \ln\lambda\}}^{\infty} \lambda e^{-R_u} du | t_j \le \tau^c) - \ln c] \tag{5.1.97}$$

Note that $P^2(t_j \le \tau^c) = P^1(t_j \le \tau^c) = E^1 \pi_{\tau^c} \ge \frac{\lambda}{\lambda + c}$ by Lemma 5.2, and that $I(t_j \le \tau^c)$, v_{t_j} are the same for a given path of W_t irrespective of whether (5.1.4) or (5.1.5) holds.

Then from (5.1.90) and (5.1.97) it follows that

$$E^2(\tau^c - t_s | t_j \le \tau^c) - E^1(\tau^c - t_j | t_j \le \tau^c) \le (\frac{1}{\sigma_1} - \frac{1}{\hat{\sigma}})\ln c + k_2/\hat{\sigma} + \bar{\delta}/\sigma_1$$
$$+ E[\frac{1}{\hat{\sigma}} q(v_{t_j}) + \frac{1}{\sigma_1}(\hat{\delta}(v_{t_j}) + \lambda t_j + \varepsilon_\lambda)]\frac{\lambda + c}{\lambda} \tag{5.1.98}$$

From Lemmas 5.1, 5.4, 5.5 the expectation on the right is finite. Therefore $\exists c_m > 0$ such that

$$E^2(\tau^c - t_s | t_j \le \tau^c) - E^1(\tau^c - t_j | t_j \le \tau^c) \le -a$$

$\forall c < c_m$, a as in Lemma 5.6.

As $P^1(t_j \le \tau^c) = P^2(t_j \le \tau^c)$ and from Lemma 5.6, then

$$E^2((\tau^c - t_j) I(t_j \le \tau^c)) - E^1((\tau^c - t_j) I(t_j \le \tau^c)) \le 0 \quad \forall c \le c_m$$

The result of the Theorem now follows from (5.1.3). □

Remarks

Theorem 5.1 does not specify the value of c_m. In the proof, a lower bound for τ^c in the case where (5.1.4) holds is compared to an upper bound for τ^c in the case where (5.1.5) holds. These bounds are very weak, especially with respect to the λe^{-R_t} term in (5.1.11). The contribution of this after time t_j is completely neglected in one case. The result is that in the proof of the Theorem very small values of c need to be considered.

The arguments given in the outline at the beginning of the section suggest that c need only be sufficiently small so that detection times are typically long compared with system time constants. Also it is likely that necessity holds in Theorem 5.1 as well as sufficiency. To improve the results a more quantative approach seems necessary.

If the system (5.1.1) becomes unstable following a parameter jump it is unclear whether the Theorem holds because of the effect of the shape of the boundary γ.

5.2 Robustness for autoregressive systems

The problem of interest here is that described in Section 3.1.

$$d\tilde{y}_t = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hline & & r_t^T & \end{bmatrix} \tilde{y}_t dt + \begin{bmatrix} u \\ \\ z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} dW_t \qquad (5.2.1)$$

where $\tilde{y}_t \epsilon R^n \ \forall t$, $u \epsilon R^{n-1}$ is constant

W_t is a scalar Wiener process independent of t_j

$$P(t \geq t_j) = 1-e^{-\lambda t} \qquad (5.2.2)$$

and
$$\begin{aligned}(r_t, z_t) &= (r^0, z^0) \ \forall t < t_j \\ &= (\bar{r}, \bar{z}) \quad \forall t \geq t_j\end{aligned} \qquad (5.2.3)$$

where $r^0 \epsilon R^n$ is constant, and $z^0 \epsilon R$ is constant (known).

$\tilde{y}_o$ is known, so that $\tilde{y}_t$ is Y_t-measurable, where observations $y_t = [00 \cdots 01]\tilde{y}_t$.

The optimal detection rule (see Section 3.1) is implemented, in the sense of the cost function (5.1.3), for the case where

$$(\bar{r}, \bar{z}) = (r^1, z^1) \qquad (5.2.4)$$

$r^1 \epsilon R^n$ constant, and $z^1 \epsilon R$ constant (known).

P^1, E^1 denotes probability and expectation given (5.2.4) holds. P^2, E^2 denotes probability and expectation given

$$(\bar{r}, \bar{z}) = (r^2, z^2) \qquad (5.2.5)$$

$r^2 \epsilon R^n$ constant, and $z^2 \epsilon R$ constant (known).

(5.2.1) is strictly stable for $r_t = r^i$, $i=0,1,2$

This is a special case of the problem of Section 5.1, such that

$$v_t = \tilde{y}_t \ ; \quad \zeta_t = 0 \ \forall t \qquad (5.2.6)$$

$$[D^i : F^i] = r^{i^T}, \quad i=0,1,2 \tag{5.2.7}$$

So
$$M^i = (r^1-r^0)(r^i-\tfrac{1}{2}r^1-\tfrac{1}{2}r^0)^T$$
$$h^i = (z^1-z^0)(r^i-\tfrac{1}{2}r^1-\tfrac{1}{2}r^0) + (z^i-\tfrac{1}{2}z^1-\tfrac{1}{2}z^0)(r^1-r^0)$$
$$g^i = (z^1-z^0)(z^i-\tfrac{1}{2}z^1-\tfrac{1}{2}z^0) \tag{5.2.8}$$

for $i=0,1,2$.

Let Q^i be the steady-state covariance matrix of the state vector $\tilde{y}_t$ in (5.2.1) with $(r_t, z_t) = (r^i, z^i)$ i.e Q^i is the unique positive definite solution of

$$\begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ \hline & r^{i^T} & & \end{bmatrix} Q + Q \begin{bmatrix} 0 & & & \\ 1 & \ddots & & r^i \\ 0 & & 1 & \end{bmatrix} + \begin{bmatrix} 0 & & 0 \\ \hline 0 & & 1 \end{bmatrix} = 0 \tag{5.2.9}$$

and let q^i be the steady state mean value of the state vector $\tilde{y}_t$, i.e.

$$q^i = - \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ \hline & r^{i^T} & & \end{bmatrix}^{-1} \begin{bmatrix} u \\ \hline z^i \end{bmatrix} \tag{5.2.10}$$

From (5.2.8), (5.1.66) and from (5.1.65)

$$\sigma_1 = \lambda + \tfrac{1}{2}(r^1-r^0)^T(Q^1+q^1q^{1^T})(r^1-r^0) + (z^1-z^0)(r^1-r^0)^Tq^1 + \tfrac{1}{2}(z^1-z^0)^2 \tag{5.2.11}$$

$$\sigma_2 = \lambda + (r^1-r^0)^T(Q^2+q^2q^{2^T})(r^2-\tfrac{1}{2}r^1-\tfrac{1}{2}r^0) + (z^1-z^0)(r^2-\tfrac{1}{2}r^1-\tfrac{1}{2}r^0)^Tq^2 + (z^2-\tfrac{1}{2}z^1-\tfrac{1}{2}z^0)(r^1-r^0)q^2 + (z^1-z^0)(z^2-\tfrac{1}{2}z^1-\tfrac{1}{2}z^0) \tag{5.2.12}$$

Then from Theorem 5.1, if $\sigma_2>\sigma_1$ $\exists c_m>0$ such that $\forall c\in(0,c_m]$

$$E^2C(\tau^c) \le E^1C(\tau^c) \tag{5.2.13}$$

It is therefore possible to characterize a set of disordered parameter values for the system(5.2.1), $\{(r^2,z^2):\sigma^2>\sigma^1\}$ such that the expected cost of using the detection rule designed assuming (5.2.4) holds is not increased, with c sufficiently small. The remarks following Theorem 5.1 discuss the restrictions on the value of c.

Although from the argument of Section 5.1 it appears likely that $E^2C(\tau^c)>E^1C(\tau^c)$ if $\sigma_2<\sigma_1$, c small, this has not been proved.

Example 1

$$d\tilde{y}_t = \begin{bmatrix} 0 & 1 \\ & r_t^T \end{bmatrix} \tilde{y}_t dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dW_t \qquad (5.2.14)$$

$$r^0 = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \quad r^1 = \begin{bmatrix} -4+\delta_1 \\ -3+\delta_2 \end{bmatrix}, \quad r^2 = \begin{bmatrix} -4+\rho_1 \\ -3+\rho_2 \end{bmatrix}$$

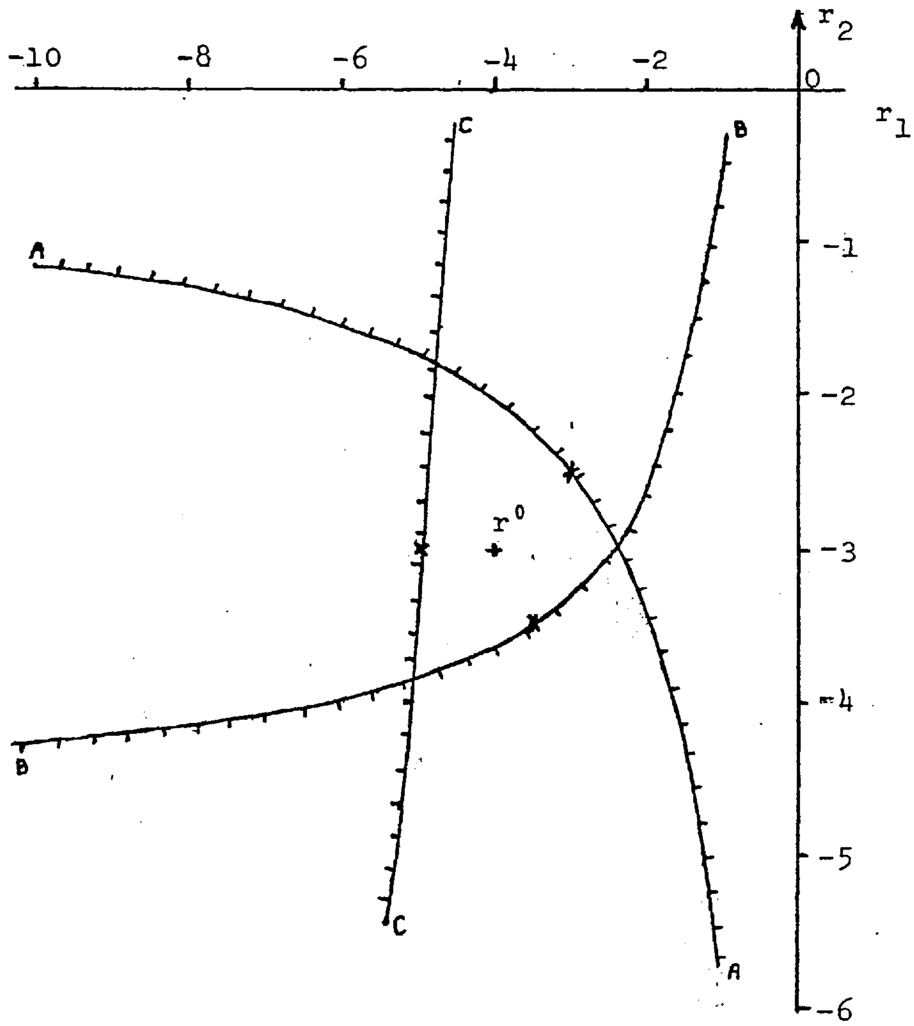
$$Q^0 = \begin{bmatrix} 1/24 & 0 \\ 0 & 1/6 \end{bmatrix} \quad Q^1 = \begin{bmatrix} \frac{1}{2(4-\delta_1)(3-\delta_2)} & 0 \\ 0 & \frac{1}{2(3-\delta_2)} \end{bmatrix}$$

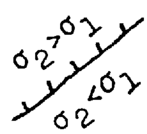
$$Q^2 = \begin{bmatrix} \frac{1}{2(4-\rho_1)(3-\rho_2)} & 0 \\ 0 & \frac{1}{2(3-\rho_2)} \end{bmatrix}$$

$$q^0=q^1=q^2=0$$

Figure 5.2.1 illustrates sets of r^2 parameter values such that $\sigma_2>\sigma_1$, for various choices of δ_1,δ_2.
Note that Theorem 5.1 only applies if (5.2.1) is stable for $r_t=r^2$ (i.e. $r_1\leq 0$, $r_2<0$ where $r^2=[r_1,r_2]^T$).

Figure 5.3.1





$\sigma_2=\sigma_1$ contour

AA : $\delta_1=1.0$ $\delta_2=0.5$
BB : $\delta_1=0.5$ $\delta_2=-0.5$
CC : $\delta_1=-1.0$ $\delta_2=0.0$

Figure 5.3.1 indicates how a detection rule of the type described in Section 3.5 might be constructed to detect jumps to unknown parameter values by combining the three "known jump" detection rules above.

Example 2

A second example is given to illustrate the discussion in Section 4.0 concerning the first order autoregression case.

Suppose

$$dy_t = r_t a_o y_t dt + bdt + dW_t$$

$$r^0=1 : r^1=\alpha : r^2=\beta \quad , \quad \beta>\alpha>1$$

Then $Q^1 = \dfrac{1}{-2\alpha a_o} \quad : \quad Q^2 = \dfrac{1}{-2\beta a_o}$

and from (5.2.11), (5.2.12)

$$\sigma_1 = \lambda + \tfrac{1}{2}(\alpha-1)^2 a_o^2\left(\frac{1}{-2\alpha a_o} + \frac{b^2}{\alpha^2 a_o^2}\right)$$

$$\sigma_2 = \lambda + (\alpha-1)(\beta-\tfrac{1}{2}\alpha-\tfrac{1}{2})a_o^2\left(\frac{1}{-2\beta a_o} + \frac{b^2}{\beta^2 a_o^2}\right)$$

Then $\sigma_2<\sigma_1$ if

$$\beta > \frac{\tfrac{1}{2}(\alpha+1)\alpha b^2/(-a_o)}{\tfrac{1}{2}(\alpha-1)b^2/(-a_o) - \tfrac{1}{4}\alpha(\alpha+1)} > 1$$

The robustness property of Chapter 4 appears to break down (assuming necessity in Theorem 5.1 as previously discussed) if b^2 is sufficiently large.

5.3 Robustness for general systems

In this section, Theorem 5.1 is applied to the problem described in Section 3.4.

$$\begin{aligned} dx_t &= A_t x_t dt + q_t dt + G_t dV_t \\ dy_t &= H_t x_t dt + dZ_t \end{aligned} \qquad (5.3.1)$$

where $x_t \in R^N$, $y_t \in R^m$ $\forall t$

V_t, Z_t are independent Wiener processes, independent of t_j

$$P(t \geq t_j) = 1 - e^{-\lambda t} \qquad (5.3.2)$$

$$\begin{aligned} &A_t = A^0, \; q_t = q^0, \; G_t = G^0, \; H_t = H^0 \quad \forall t < t_j \\ &A_t = \bar{A}, \; q_t = \bar{q}, \; G_t = \bar{G}, \; H_t = \bar{H} \quad \forall t \geq t_j \end{aligned} \qquad (5.3.3)$$

where $A^0, q^0, G^0, H^0, \bar{A}, \bar{q}, \bar{G}, \bar{H}$ are constant matrices and vectors. A^0, $\bar{A}$ have strictly negative eigenvalues.

The innovations formulation

Suppose x_o has a-priori distribution $N(r_o, Q_o)$ where Q_o is a covariance matrix.
For given t_j, $r_t \triangleq E(x_t | \mathcal{Y}_t)$ satisfies the Kalman Filtering equations

$$\begin{aligned} dr_t &= A^0 r_t dt + q^0 dt + Q_t H^{0T} d\nu_t \quad \forall t < t_j \\ dr_t &= \bar{A} r_t dt + \bar{q} dt + Q_t \bar{H}^T d\nu_t \quad \forall t \geq t_j \end{aligned} \qquad (5.3.4)$$

$$\dot{Q}_t = G^0 G^{0T} - Q_t H^{0T} H^0 Q_t + A^0 Q_t + Q_t A^{0T} \quad \forall t < t_j \qquad (5.3.5)$$

$$\dot{Q}_t = \bar{G}\bar{G}^T - Q_t \bar{H}^T \bar{H} Q_t + \bar{A} Q_t + Q_t \bar{A}^T \quad \forall t \geq t_j \qquad (5.3.6)$$

$$d\nu_t = dy_t - I(t < t_j) H^0 r_t dt - I(t \geq t_j) H^1 r_t dt \qquad (5.3.7)$$

ν_t is a Wiener process

Define Q^i as the asymptotic solution of (5.3.6) for

$$\bar{A} = A^i, \; \bar{K} = k^i, \; \bar{G} = G^i, \; \bar{H} = H^i \quad i = 0, 1, 2. \qquad (5.3.8)$$

$\hat{x}_t^i$ is defined as the Kalman Filter estimate for x_t assuming (x_t, y_t) satisfy

$$dx_t = A^i x_t dt + q^i dt + G^i dV_t$$
$$dy_t = H^i x_t dt + dZ_t$$

where the covariance of $(x_o - \hat{x}_o^i)$ is Q^i, the asymptotic solution of the corresponding Ricatti equation (A^i assumed negative definite) $i=0,1,2$ (5.3.9)

i.e. $$d\hat{x}_t^i = (A^i - Q^i H^{iT} H^i)\hat{x}_t^i dt + q^i dt + Q^i H^{iT} dy_t, \quad \hat{x}_o^i = r_o$$

Note that since $A^i - Q^i H^{iT} H^i$ has strictly negative eigenvalues, if y_t is actually generated by (5.3.1) then the covariance of $\hat{x}_t^i$ is uniformly bounded $\forall t \geq 0$, for any $x_o, \hat{x}_o^i$.

In Section 3.4 a natural sub-optimal approach to detection of a disorder in (5.3.1) is discussed for say $\bar{A}=A^1$, $\bar{k}=k^1$, $\bar{G}=G^1$, $\bar{H}=H^1$. This involves the estimates $\hat{x}_t^0$ and $\hat{x}_t^1$. Here, the robustness of this approach is investigated. First, some preliminary results are required so that Theorem 5.1 may be applied.

<u>Assumption</u> (5.3.10)

For simplicity it is assumed that $Q_o = \text{cov}(r_o - x_o) = Q^0$.

Note that then $\hat{x}_t^0 = r_t \quad \forall t \leq t_j$.

<u>Lemma 5.7</u>

In equation (5.3.6) if $\bar{A}=A^2$, $\bar{k}=k^2$, $\bar{G}=G^2$, $\bar{H}=H^2$

$$\dot{Q}_t = G^2 G^{2T} - Q_t H^{2T} H^2 Q_t - A^2 Q_t - Q_t A^{2T} \quad t \geq t_j$$

Then if $Q_{t_j} = Q^0$ $\exists \tilde{\alpha} < \infty$ $\tilde{\beta} > 0$ such that

$$\| Q_t - Q^2 \| \leq \tilde{\alpha} e^{-\tilde{\beta}(t - t_j)} \quad \forall t \geq t_j$$

Note

Here $\|M\| \triangleq \sup_{\substack{x\in R^N \\ \|x\|=1}} \|Mx\|$ for $M\in R^{N\times N}$ (5.3.11)

Proof

Consider the system

$$dx_t = A^2 x_t dt + q^2 dt + G^2 dV_t$$
$$dy_t = H^2 x_t dt + dZ_t \quad t\geq t_j \tag{5.3.12}$$

The associated Kalman Filter is (for t_j known) (5.3.13)

$$d\hat{x}_t = (A^2 - Q_t H^{2T} H^2)\hat{x}_t dt + Q_t H^{2T} dy_t + q^2 dt, \quad t\geq t_j$$

and Q_t satisfies the Ricatti equation in the statement of the Theorem.

Since $A^2 - Q^2 H^{2T} H^2$ has strictly negative eigenvalues and $Q_t \to Q^2$ as $t\to\infty$, $\exists \tilde{t}, \tilde{\beta} > 0$ such that

$$\text{max eigenvalue of } (A^2 - Q_t H^{2T} H^2) \leq -\tilde{\beta} \quad \forall t\geq \tilde{t} < \infty \tag{5.3.14}$$

From (5.3.12) and (5.3.13), if $\varepsilon_t = \hat{x}_t - x_t$

$$d\varepsilon_t = (A^2 - Q_t H^{2T} H^2)\varepsilon_t + Q_t H^{2T} dZ_t - G^2 dV_t \tag{5.3.15}$$

The following Kalman estimates of x_t are defined for the system (5.3.12).

$\hat{x}_t^{(0)}$: estimate of x_t assuming $\hat{x}_{t_j}^{(0)} \sim N(x_{t_j}, Q^0)$

$\hat{x}_t^{(1)}$: estimate of x_t assuming $\hat{x}_{t_j}^{(1)} \sim N(x_{t_j}, Q^0 + \Delta)$

where $\Delta\geq 0$ is chosen so that $Q^0 + \Delta \geq Q^2$

$\hat{x}_t^{(2)}$: estimate of x_t assuming $\hat{x}_{t_j}^{(2)} \sim N(x_{t_j}, Q^2)$

Here $C\geq D$ means $C-D$ is positive semi-definite. $C>D$, $C\leq D$ and $C<D$ are defined correspondingly.

$$\varepsilon_t^{(i)} \triangleq \hat{x}_t^{(i)} - x_t \;, \quad i=0,1,2 \tag{5.3.16}$$

$$E(\varepsilon_t^{(0)}\varepsilon_t^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0+\Delta)) \geq E(\varepsilon_t^{(1)}\varepsilon_t^{(1)T}|\varepsilon_{t_j}^{(1)}\sim N(0,Q^0+\Delta))$$
$$\geq E(\varepsilon_t^{(2)}\varepsilon_t^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^2)) \qquad (5.3.17)$$

The first inequality holds because of the optimality of $\hat{x}^{(1)}$. The second inequality holds because of the optimality of $\hat{x}^{(2)}$ and because $Q^0+\Delta \geq Q^2$.

Now let $\tilde{x}_t' = \hat{x}_t^{(0)}$ where $\hat{x}_{t_j}^{(0)} \sim N(x_{t_j}, Q^0)$

and $\tilde{x}_t'' = \hat{x}_t^{(0)}$ where $\hat{x}_{t_j}^{(0)} = \tilde{x}_{t_j}' + \delta$, $\delta \sim N(0,\Delta)$, independent r.v.

(5.3.18)

Then $\quad \tilde{x}_{t_j}'' \sim N(x_{t_j}, Q^0+\Delta) \qquad (5.3.19)$

Define $\tilde{\varepsilon}_t' = \tilde{x}_t' - x_t$, $\tilde{\varepsilon}_t'' = \tilde{x}_t'' - x_t$ so $\tilde{\varepsilon}_{t_j}'' - \tilde{\varepsilon}_{t_j}' = \delta$

From (5.3.15)

$$d(\varepsilon_t'' - \varepsilon_t') = (A^2 - Q_t H^{2T} H^2)(\tilde{\varepsilon}_t'' - \tilde{\varepsilon}_t')dt \qquad (5.3.20)$$

where Q_t is the covariance matrix appropriate to the estimate $\hat{x}^{(0)}$. Therefore

$$E[(\tilde{\varepsilon}_t'' - \tilde{\varepsilon}_t')(\tilde{\varepsilon}_t'' - \tilde{\varepsilon}_t')^T] \leq I.\tilde{\gamma}e^{-\tilde{\beta}(t-t_j)} \qquad (5.3.21)$$

for some $\tilde{\gamma} < \infty$.

So $E(\varepsilon_t''\varepsilon_t''^T) - E(\varepsilon_t'\varepsilon_t'^T) \leq I.\tilde{\gamma}e^{-\tilde{\beta}(t-t_j)}$ by the independence of δ in (5.3.18) and therefore

$$E(\varepsilon_t^{(0)}\varepsilon_t^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0+\Delta))$$
$$- E(\varepsilon_t^{(0)}\varepsilon_t^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0)) \leq I\tilde{\gamma}e^{-\tilde{\beta}(t-t_j)}$$

Using (5.3.17)

$$E(\varepsilon_t^{(2)}\varepsilon_t^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^2)) - E(\varepsilon^{(0)}\varepsilon^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0))$$
$$\leq I.\tilde{\gamma}e^{-\tilde{\beta}(t-t_j)} \qquad (5.3.22)$$

Also $\quad E(\varepsilon_t^{(2)}\varepsilon_t^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^0+\Delta))$

$$\geq E(\varepsilon^{(1)}\varepsilon_t^{(1)T}|\varepsilon_{t_j}^{(1)}\sim N(0,Q^0+\Delta))$$

$$\geq E(\varepsilon_t^{(0)}\varepsilon_t^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0)) \qquad (5.3.23)$$

The final inequality holds by the optimality of $\hat{x}^{(0)}$ and because $Q^0+\Delta\geq Q^0$. Since $Q^0+\Delta\geq Q^2$ it may be shown (see the argument of (5.3.18) tc (5.3.21)) that

$$E(\varepsilon^{(2)}\varepsilon^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^0+\Delta)) - E(\varepsilon_t^{(2)}\varepsilon_t^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^2))$$
$$\leq I.\hat{\gamma}e^{-\tilde{\beta}(t-t_j)} \qquad (5.3.24)$$

Therefore from (5.3.23)

$$E(\varepsilon_t^{(2)}\varepsilon_t^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^2)) - E(\varepsilon_t^{(0)}\varepsilon_t^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0))$$
$$\geq -I.\hat{\gamma}e^{-\tilde{\beta}(t-t_j)} \qquad (5.3.25)$$

But by definition of $\varepsilon^{(2)}$ (see(5.3.16)) and of Q^2

$$E(\varepsilon_t^{(2)}\varepsilon^{(2)T}|\varepsilon_{t_j}^{(2)}\sim N(0,Q^2)) = Q^2$$

and the covariance matrix Q_t satisfies

$$Q_t = E(\varepsilon_t^{(0)}\varepsilon^{(0)T}|\varepsilon_{t_j}^{(0)}\sim N(0,Q^0)) \text{ if } Q_{t_j}=Q^0$$

Therefore from (5.3.22) and (5.3.24)

$$-I.\hat{\gamma}.e^{-\tilde{\beta}(t-t_j)} \leq Q^2-Q_t \leq I.\tilde{\gamma}e^{-\tilde{\beta}(t-t_j)} \quad \text{if} \quad Q_{t_j}=Q^0$$

i.e. $$\sup_{\|r\|\leq 1} |r^T(Q_t-Q^2)r| \leq \max(\hat{\gamma},\tilde{\gamma})e^{-\tilde{\beta}(t-t_j)} \qquad (5.3.26)$$

Since Q_t-Q^2 is symmetric $\exists M\in R^{N\times N}$ such that $Q_t-Q^2=M^TM$. Therefore

$$\|Q_t-Q^2\| = \sup_{\|r\|,\|s\|\leq 1} r^TM^TMs = \sup_{\|r\|\leq 1} |r^TM^TMr|$$

The result of the Lemma now follows from (5.3.26). □

Lemma 5.8

If r_t is the Kalman filter estimate of x_t defined in (5.3.4); (x_t, y_t) are generated by (5.3.1) with $\bar{A}=A^2$, $\bar{q}=q^2$, $\bar{G}=G^2$, $\bar{H}=H^2$; $\hat{x}_t^0, \hat{x}_t^2$ are defined as in (5.3.9), $\hat{x}_o^0, \hat{x}_o^2$ known a priori, then

$\exists a(\cdot,\cdot)<\infty$ and $b>0$ such that

$$E(\|r_t-\hat{x}_t^2\|^2 | t_j, \hat{x}_{t_j}^0, \hat{x}_{t_j}^2) \le a(\hat{x}_{t_j}^0, \hat{x}_{t_j}^2) e^{-b(t-t_j)} \quad \forall t \ge t_j$$

$$E(a(\hat{x}_{t_j}^0, \hat{x}_{t_j}^2) | t_j) \le d < \infty \quad \forall t_j \text{ for some } d$$

Proof

From (5.3.9) and (5.3.4) with $\bar{A}=A^2$, $\bar{q}=q^2$, $\bar{G}=G^2$, $\bar{H}=H^2$

$$d(r_t-\hat{x}_t^2) = (A^2-Q^2H^{2T}H^2)(r_t-\hat{x}_t^2)dt + (Q_t-Q^2)H^{2T}d\nu_t \quad \forall t \ge t_j$$

where Q_t is the solution of (5.3.6) with $Q_{t_j}=Q^0$.

Note that $r_{t_j}=\hat{x}_{t_j}^0$.

Let $\delta_t \triangleq r_t-\hat{x}_t^2 \quad \forall t \ge t_j$ and $M \triangleq A^2-Q^2H^{2T}H^2$ (5.3.27)

Let $\bar{\beta}$, $\bar{\alpha}$ be such that $\|e^{Mt}x\| \le \bar{\alpha}e^{-\bar{\beta}t}.\|x\| \quad \forall x \in R^N$, $t \ge 0$, $\bar{\beta}>0$. (5.3.28)

$$\delta_t = e^{M(t-t_j)}\delta_{t_j} + \int_{t_j}^t e^{M(t-u)}(Q_u-Q^2)H^{2T}d\nu_u$$

Therefore

$$E(\delta_t\delta_t^T | t_j, \hat{x}_{t_j}^0, \hat{x}_{t_j}^2) = e^{M(t-t_j)}\delta_{t_j}\delta_{t_j}^T e^{M^T(t-t_j)}$$

$$+ \int_{t_j}^t e^{M(t-u)}(Q_u-Q^2)H^{2T}H^2(Q_u-Q^2)e^{M^T(t-u)}du$$

So
$$E(\delta_t\delta_t^T | t_j, \hat{x}_{t_j}^0, \hat{x}_{t_j}^2) \le \bar{\alpha}^2 e^{-2\bar{\beta}(t-t_j)}\|\hat{x}_{t_j}^0-\hat{x}_{t_j}^2\|^2$$

$$+ \bar{\alpha}^2\|H^{2T}H^2\|\tilde{\alpha}^2\int_{t_j}^t \exp(-2\tilde{\beta}(u-t_j)-2\bar{\beta}(t-u))du$$

where Lemma 5.7 has been used to bound $\|Q_u-Q^2\|$.

Choosing $\bar{\beta}$ so that $\tilde{\beta}\neq\bar{\beta}$.

$$E(\delta_t\delta_t^T|t_j,\hat{x}^0_{t_j},\hat{x}^2_{t_j}) \leq \tilde{\alpha}^2 e^{-2\bar{\beta}(t-t_j)}\|\hat{x}^0_{t_j}-\hat{x}^2_{t_j}\|^2$$

$$+\frac{\|H^{2T}H^2\|\tilde{\alpha}^2\bar{\alpha}^2}{2(\bar{\beta}-\tilde{\beta})}.[e^{-2\tilde{\beta}(t-t_j)}-e^{-2\bar{\beta}(t-t_j)}] \qquad (5.3.29)$$

Since for any random variable $u\in R^N$

$$E(\|u\|^2) = \text{trace } E(uu^T) \leq N\|E(uu^T)\|$$

Choose b as $2\min(\bar{\beta},\tilde{\beta})$

and $a(\hat{x}^0_{t_j},\hat{x}^2_{t_j})$ as $N.[\|x_{t_j}-x_{t_j}\|^2 + \frac{\|H^{2T}H^2\|\tilde{\alpha}^2}{2(\bar{\beta}-\tilde{\beta})}]$

The result of the Lemma is now established, since $(A^2-Q^2H^{2T}H^2)$ has strictly negative eigenvalues. □

Application of Theorem 5.1

From (5.3.7) if y_t is generated by (5.3.1) with $\bar{A}=A^2$, $\bar{q}=q^2$, $\bar{G}=G^2$, $\bar{H}=H^2$, then

$$dy_t = I(t<t_j)H^0r_tdt + I(t\geq t_j)H^2r_tdt + d\nu_t \qquad (5.3.30)$$

Let ζ_t be defined by

$$dy_t = I(t<t_j)H^0\hat{x}^1_tdt + I(t\geq t_j)H^2\hat{x}^2_tdt + d\nu_t + \zeta_tdt \qquad (5.3.31)$$

From Lemma 5.8, and since $\hat{x}^0_t=r_t$ $\forall t\leq t_j$

$$\zeta_t = 0 \quad \forall t<t_j$$

$$E(\|\zeta_t\|^2|t_j,\hat{x}^0_{t_j},\hat{x}^2_{t_j}) \leq \|H^2\|^2a(\hat{x}^0_{t_j},\hat{x}^2_{t_j})e^{-b(t-t_j)} \quad \forall t\geq t_j$$

$$\leq \infty \qquad (5.3.32)$$

From (5.3.9) it follows that

$$d(\hat{x}^i_t-Q^iH^{iT}y_t) = (A^i-Q^iH^{iT}H^i)(\hat{x}^i_t-Q^iH^{iT}y_t)dt + q^idt$$

$$+ (A^i-Q^iH^{iT}H^i)Q^iH^{iT}y_tdt$$

so if $v^i \triangleq \hat{x}^i - Q^iH^{iT}y_t$

$$d\begin{bmatrix} v^0 \\ v^1 \\ v^2 \\ y \end{bmatrix}_t = \begin{bmatrix} A^0-Q^0H^{0T}H^0 & 0 & & (A^0-Q^0H^{0T}H^0)Q^0H^{0T} \\ 0 & A^1-Q^1H^{1T}H^1 & & (A^1-Q^1H^{1T}H^1)Q^1H^1 \\ 0 & 0 & A^2-Q^2H^{2T}H^2 & (A^2-Q^2H^{2T}H^2)Q^2H^{2T} \\ L_t^0 & L_t^1 & L_t^2 & F_t \end{bmatrix} \cdot$$

$$\begin{bmatrix} v^0 \\ v^1 \\ v^2 \\ y \end{bmatrix}_t dt + \begin{bmatrix} q^0 \\ q^1 \\ q^2 \\ \zeta_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \cdot d\nu_t \quad (5.3.33)$$

where
$$L_t^0 = H^0 I(t<t_j)$$
$$L_t^1 = 0$$
$$L_t^2 = H^2 I(t\geq t_j)$$
$$F_t = H^0Q^0H^{0T}I(t<t_j) + H^2Q^2H^{2T}I(t\geq t_j)$$

The sub-optimal detection scheme proposed in Section 3.4 for the problem (5.3.1) when $\bar{A}=A^1$, $\bar{q}=q^1$, $\bar{G}=G^1$, $\bar{H}=H^1$ is that which is optimal for detecting the disorder described by (5.3.33) with

$$L_t^0 = H^0 I(t<t_j) \qquad F_t = H^0Q^0H^{0T}I(t<t_j)$$
$$L_t^1 = H^1 I(t\geq t_j) \qquad \qquad + H^1Q^1H^{1T}I(t\geq t_j)$$
$$L_t^2 = 0 \qquad \zeta_t = 0 \ \forall t$$

The system (5.3.33) has the same form as (5.1.1) with

$$D^0=[H^0 \ 0 \ 0]; \ D^1=[0 \ H^1 \ 0]; \ D^2=[0 \ 0 \ H^2]$$
$$F^0=H^0Q^0H^{0T} \ ; \ F^1=H^1Q^1H^{1T} \ ; \ F^2=H^2Q^2H^{2T} \quad (5.3.34)$$

The requirements (5.1.5b) are satisfied by Lemma 5.8 Assumption (5.1.30) holds. Theorem 5.1 then specifies a set of disordered parameter points (A^2,q^2,G^2,H^2) such that the expected cost is no greater, for c small, than that when the scheme is used to detect the disorder for which it is optimal.

Example

Consider the following system

$$dx_t = a_t x_t dt + g_t dV_t$$

$$dy_t = x_t dt + dZ_t$$

where x_t, y_t are scalar processes

V_t, Z_t are independent scalar Wiener processes, independent of t_j

$$a_t = -2, \quad g_t = 1 \quad \forall t<t_j$$

$$a_t = \bar{a}, \quad g_t = \bar{g} \quad \forall t \geq t_j$$

$$P(t \geq t_j) = 1-e^{-\lambda t}$$

$$x_o \sim N(\hat{x}_o, -2+\sqrt{5})$$

A (sub-optimal) detection scheme is implemented for the case

$$\bar{a} = -3, \quad \bar{g} = 1$$

Suppose the actual post-jump parameters are $\bar{a}=a^2, \bar{g}=g^2$.

From (5.3.1) and (5.3.24)

$$D^0=[1\ 0\ 0]\ ;\ D^1=[0\ 1\ 0]\ ;\ D^2=[0\ 0\ 1]$$

From (5.1.66)

$$\sigma_1 = \lim_{T\to\infty} \frac{1}{T}E^1\int_{t_j}^{t_j+T} (\lambda+v_u^T M^1 v_u)du$$

$$= \lim_{T\to\infty} \frac{1}{T}E^1\int_{t_j}^{t_j+T} (\lambda+\tfrac{1}{2}(\hat{x}_u^1-\hat{x}_u^0)^2)du \quad \text{by } (5.1.11)$$

and from (5.1.65)

$$\sigma_2 = \lim_{T\to\infty} \frac{1}{T}E^2\int_{t_j}^{t_j+T} (\lambda+v_u^T M^2 v_u)du$$

$$= \lim_{T\to\infty} \frac{1}{T}E^2\int_{t_j}^{t_j+T} (\lambda+\tfrac{1}{2}(\hat{x}_u^2-\hat{x}_u^0)^2-\tfrac{1}{2}(\hat{x}_u^2-\hat{x}_u^1)^2)du$$

by (5.1.11)

Here E^1 denotes expectation given that the disorder is that for which the detection rule is optimal, and E^2 denotes expectation given the disorder ~~is~~ the actual one defined above.

$$d\hat{x}_t^0 = -\sqrt{5}.\hat{x}_t^0 dt + (-2+\sqrt{5})dy_t$$

$$d\hat{x}_t^1 = -\sqrt{10}.\hat{x}_t^1 dt + (-3+\sqrt{10})dy_t$$

$$d\hat{x}_t^2 = -\sqrt{(a^{2\,2}+g^{2\,2})}.\hat{x}_t^2 dt + (a^2+\sqrt{(a^{2\,2}+g^{2\,2})})dy_t$$

$$\hat{x}_o^0=\hat{x}_o^1=\hat{x}_o^2=\hat{x}_o$$

This leads to $\sigma_1 \simeq 0.00131 + \lambda$

If $a^2=-2=a^0$, $g^2=\sqrt{(2/3)}$, then $\sigma_2 \simeq 0.00046 + \lambda$

The conditions of Theorem 5.1 are not satisfied: although this is only a sufficiency result, from the argument at the end of Section 5.1 it is conjectured that necessity also holds. In this case, the above disorder would not be detected as quickly as the design case disorder. This is of interest, since with this choice of a^2, g^2

$$\lim_{T\to\infty} E^2(x_t^{\,2}) = \lim_{T\to\infty} E^1(x_t^{\,2})$$

Hence the detection rule is capable of rejecting transient effects due to decreases in the externally generated noise covariance, and picks out output paths corresponding to changes in the dynamics of the system.

The case $g^2=g^1=1$ was also investigated. Figure 5.3.2 shows that the response of the detection scheme for small c improves if $a^2<-3$, i.e. the jump is larger than that designed for. In this case, a robustness property is exhibited.

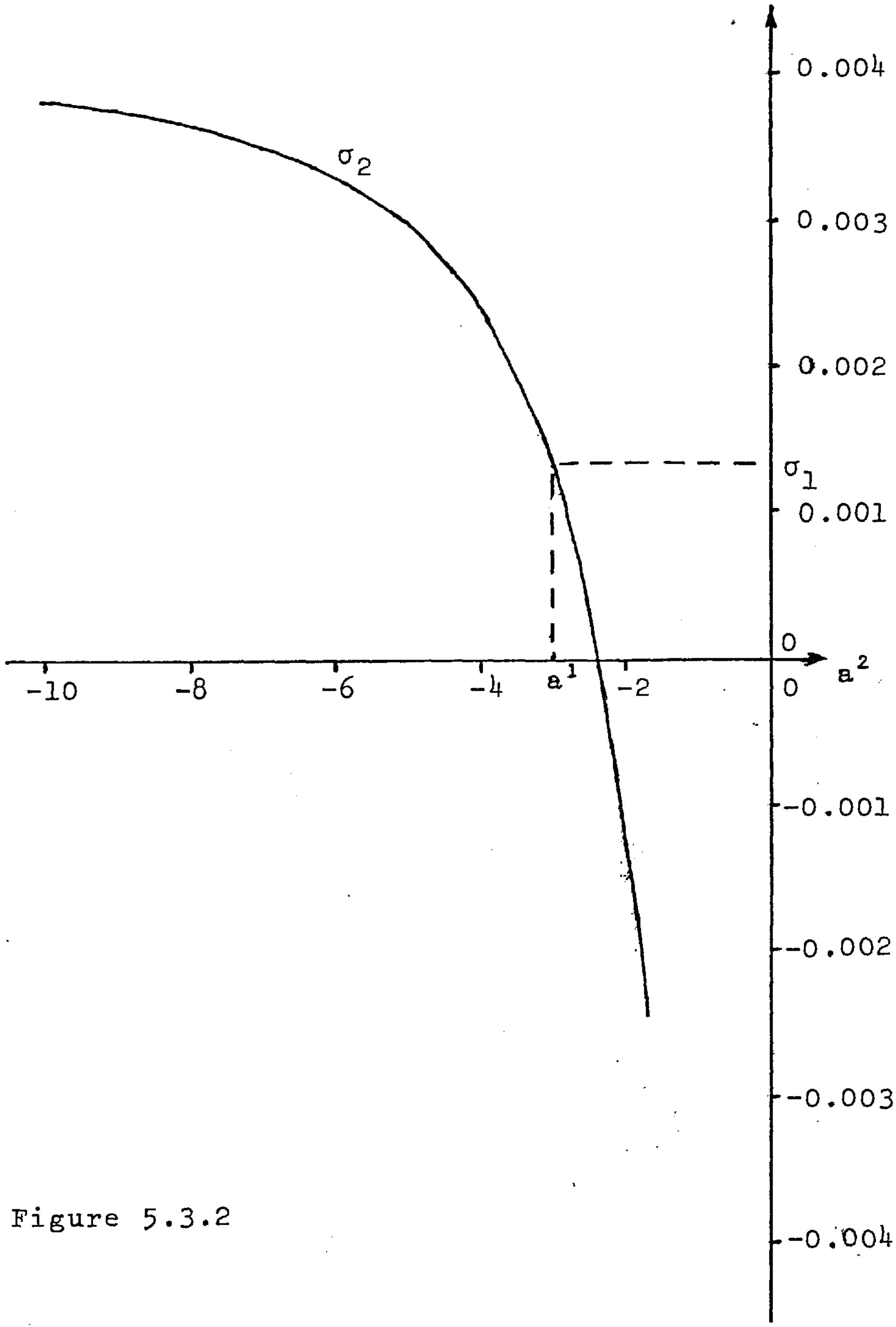


Figure 5.3.2

CHAPTER 6 CONCLUSIONS

6.1 The work presented in this thesis has two main objectives. Firstly some results are given on detection rules for systems with simple dynamics which extend those previously available. Also a number of results concerning the Baysian formulation of the detection problem are collected in Chapter 2, concerning the relationship between different cost functions.

It is hoped that this may help to bridge the gap between practical and theoretical studies. The sub-optimal approach for general systems proposed in Section 3.4 follows naturally from the optimal schemes discussed in earlier sections.

Secondly, the restriction on the formulation of Section 3.1 or 3.4 is obvious in that previous knowledge of the post-jump parameter is necessary. The robustness studies of Chapters 4 and 5 go some way towards the possibility of constructing effective detection rules with less precise advance information. Chapter 4 gives a detailed study of the first order autoregression case, which in fact has a fairly complicated structure.

Chapter 5 deals with more general systems, and provides a result which is felt should be useful in practical situations. The theory is however, somewhat incomplete and might be capable of some refinement.

6.2 Outstanding points for further research

It would be of interest to investigate the effect of initial conditions in the construction of detection rules with cost function Q in Section 2.2. It is felt that providing λ is small this should not be important (see the remark in Section 3.2) and this would enable the theory developed using costs $C(\tilde{\tau})$ and $K(\tilde{\tau})$ to be applied to this problem. Alternatively it might be possible to make a similar study of detection rules with cost Q directly.

b) The importance of the stopping boundary shape needs further investigation. It seems likely that it would be important to have a correctly shaped boundary if extremely quick detection was required. However, if this was not the case (more attention being attatched to the reduction of false alarms), the computationally demanding problem of generating the boundary shape would probably not be worthwhile except in simple cases. Even in the former case a method of approximating the boundary shape other than with a straight line in (π,v) space might be found to be satisfactory. No real progress was made in investigating these questions here.

c) Although the sub-optimal stopping rule of Section 3.4 seems to be a natural approach when c is small, it would be useful to have some quantative information on the increase in expected cost due to using this approach. It might be possible to obtain some information on this by considering the process $R_t = \ln(\pi_t/(1-\pi_t))$.

d) A more complete result on the robustness of detection rules for general systems than that obtained in Chapter 5 is desirable. It would be useful to obtain a guide to the value of c_m in Theorem 5.1 which corresponds to each parameter point.

e) If no progress is possible on point (d) above, it would be of interest to reconsider the way in which the exponential term in (5.1.11) is handled in Theorem 5.1. The remarks at the end of Section 5.1 explain how the present approach is rather unsatisfactory. Also it should be possible to prove necessity as well as sufficiency in Theorem 5.1.

f) Finally it would be of interest to investigate the relationship between the parameter sets characterized in Theorem 5.1 and the corresponding system structure. It might be possible then to use the ideas of Section 3.5 to construct near min-max detection rules.

APPENDIX NON-LINEAR FILTERING

In this Appendix, the necessary result of non-linear filtering theory, as applied to the evaluation of the probability π_t is stated. This approach follows [14] and the filtering result is taken from [15].

Suppose y_t is a stochastic process, and that $\mathcal{Y}_t$ is the σ-field generated by $(y_u, u \leq t)$. Also $y_t \in R^m \ \forall t$.

Suppose that t_j is a random variable such that $t_j \geq 0$ and

$$\lim_{\delta \to 0} P(t_j \in (t, t+\delta] \mid t_j > t, \mathcal{Y}_t) = g_t$$

where g_t is a $\mathcal{Y}_t$-measurable process.

Define $M_t \triangleq I(t \geq t_j) - \int_0^t g_u . (1 - I(u \geq t_j)) du \quad , \ t \geq u$

and let $\mathcal{M}_t$ denote the σ-field generated by $(M_u, u \leq t)$.

Then

$$E(M_{t+s} \mid \mathcal{M}_t, \mathcal{Y}_o) = P(t_j \in [0, t] \mid \mathcal{M}_t, \mathcal{Y}_o) - E\left(\int_0^{t_j \wedge t} g_u du \mid \mathcal{M}_t, \mathcal{Y}_o\right)$$

$$+ P(t_j \in [t, t+s] \mid \mathcal{M}_t, \mathcal{Y}_o) - E\left(\int_{t_j \wedge t}^{t_j \wedge (t+s)} g_u du \mid \mathcal{M}_t, \mathcal{Y}_o\right)$$

$$= M_t + 0$$

by definition of g_u.

Therefore, if $\mathcal{Y}_o$ is given a priori, M_t is a Martingale.

Now suppose that y_t satisfies

$$dy_t = f_t dt + dW_t$$

where f_t is measurable with respect to the σ-field generated by $(I(t \geq t_j), y_u : u \leq t)$, and W_t is an m-dimensional Wiener process. $M_t W_t$ is a Martingale, and from [15: Theorem 4.1]

if $\quad \pi_t \triangleq P(t \geq t_j \mid \mathcal{Y}_t) = E(I(t \geq t_j) \mid \mathcal{Y}_t)$

then

$$\pi_t = \pi_o + \int_0^t g_u.(1-\pi_u)du$$
$$+ \int_0^t (E^u(f_u I(t \geq t_j)) - E^u(f_u)\pi_u)^T d\nu_u$$

where $E^u(\cdot) = E(\cdot | y_u)$

and $\quad \nu_u = y_u - \int_0^u E_s f_s ds$

In addition, ν_u is a Wiener process.

REFERENCES

[1] A.N. Shiryaev; "On optimal methods in quickest detection problems" - Theory of Probability and Its Applications, vol.8 (1963) p.22.

[2] A.N. Shiryaev; "Some exact formulas in a disorder problem" - Theory of Probability and Its Applications, vol.10 (1965) p.348.

[3] J.A. Bather; "On a quickest detection problem" - Annals of Mathematical Statistics, vol.38 (1967) p.711.

[4] A.S. Willsky; "A survey of design methods for failure detection in dynamic systems" - Automatica, vol.12 (1976) p.601.

[5] A.S. Willsky, J.J. Deyst Jr., B.S. Crawford; "Two self test methods applied to an inertial system Problem" - Journal of Spacecraft and Rockets, vol.12 (1975) p.434.

[6] A.S. Willsky, J.J. Deyst Jr., B.S. Crawford; "Adaptive filtering and self test methods for failure detection and compensation" - Proceedings of the 1974 J.A.C.C., Austin, Texas.

[7] P.M. Newbold, Y-C. Ho; "Detection of changes in the characteristics of a Gauss-Markov process" - I.E.E.E. Transactions on Aerospace and Electronic Systems, vol.4 (1968) p.707.

[8] T-T. Chien, M.B. Adams; "A sequential failure detection technique and its application" - I.E.E.E. Transactions on Automatic Control, vol.21 p.750.

[9] M. Adams, K-P Dunn, C.S. Greene, W.H. Lee, N.R. Sandell Jr., I. Segall, A.S. Willsky; "The stochastic control of the F-8C aircraft using the multiple model adaptive control (MMAC) method" - Proceedings of the 1975 I.E.E.E. Conference on Decision and Control, Houston, Texas December 1975.

[10] C.B. Wan, M.H.A. Davis; "The general point process disorder problem" - I.E.E.E. Transactions on Information Theory, vol.23 (1977) p.538

[11] M.H.A. Davis; "A note on the Poisson disorder problem" - Banach Center Publications, vol.1

[12] A.N. Shiryaev; "Statistical Sequential Analysis" - Translations of Mathematical Monographs, vol.38 American Mathematical Society, Providence, Rhode Island, 1973.

[13] H.J. Kushner; "Approximations to optimal filters" - I.E.E.E. Transactions on Automatic Control, vol.12 (1967) p.546.

[14] M.H.A. Davis; "The applications of non-linear filtering to fault detection in linear systems" - I.E.E.E. Transactions on Automatic Control, vol.20 (1975) p.257.

[15] M. Fujisaki, G. Kallianpur, H. Kunita; "Stochastic differential equations for the non-linear filtering problem" - Osaka Journal of Mathematics, vol.9 (1972) p.19.

[16] H.J. Kushner; Probability Methods for Approximations in Stochastic Control and for Elliptic Equations - Mathematics in Science and Engineering vol.129, Academic Press 1977.

[17] A. Friedman; Stochastic Differential Equations and Applications, vol.1 - Probability and Mathematical Statistics, vol.28, Academic Press London 1975.

[18] T. Yamada; "On a comparison theorem for solutions of stochastic differential equations and its application" - Journal of Mathematics of Kyoto University, vol.13 (1973) p.497.

[19] H.J. Kushner, A.J. Kleinman; "Accelerated procedures for the solution of discrete Markov control problems" - I.E.E.E. Transactions on Automatic Control, vol.16 (1971) p.147.

[20] A. Wald; Statistical Decision Functions - John Wiley & Sons, New York 1950.

[21] W.M. Wonham; Linear Multivariable Control, a Geometric Approach - Springer-Verlag Berlin 1974.