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NON-LINEAR QUANTUM MECHANICS AND SEMI-CLASSICAL FIELD THEORIES.

by

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This thesis is dedicated to

DARIUSH ASHOURI

in friendship and devotion.

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ABSTRACT

By constructing a Schrodinger picture action integral for the derivation of the dynamical equations of the semi-classical theory of gravity we attempt a systematic study of the possibility of leaving the gravitational field classical while quantizing all other physical fields. A consistent theory of this kind makes quantum mechanics implicitly non-linear. After formally transforming to the manifestly covariant Heisenberg picture we set up a perturbation scheme to discuss the renormalization of the theory.

Amongst other topics one of the central points discussed in this thesis is the observation that in a consistent semi-classical theory the dynamics of the unquantized fields enable us to remove the renormalization ambiguities by imposing physical conditions on the full model. In order to clarify this point we first study two simpler models in flat space-time. These are the models of two real scalar fields V and $\hat{\phi}$ where V is classical and $\hat{\phi}$ is quantized. The couplings are implemented through the terms $<\psi|\hat{\phi}^2|\psi>$ and $<\psi|\partial_\mu\hat{\phi}\partial^\mu\hat{\phi}|\psi>$ in the V-field equations. In both of these models we start with the derivation of the dynamical field equations consistently from a variational principle in the Schrodinger picture. However, in the course of renormalization we find it convenient to transform into the Heisenberg picture. Then we develop a perturbation theory and show that at each order one can fix the constant coefficients of renormalization counter-terms by imposing some plausible physical conditions on the full model. Finally we apply this idea to the semi-classical theory of gravity. The existence of an action integral for the coupled set of Einstein-Schrodinger equation accomodates in a natural way the purely geometrical renormalization counter terms, whose constant (infinite) coefficients are fixed once and for all by imposing physical conditions on the linearized Einstein equations.

PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, between October 1976 and September 1979 under the supervision of Professor T.W.B. Kibble.

Except where otherwise stated, this work is original and has not been submitted for a degree of this or any other university.

The author is extremely grateful to Professor T.W.B. Kibble for his patient supervision of this thesis and his permanent encouragement.

Parts of Chapters 3 and 4 have already been published in the form of two papers in collaboration with Professor T.W.B. Kibble and Professor T.W.B. Kibble and Dr. B.S. Kay respectively. I am grateful to them for permitting me to include these materials in this thesis.

I also acknowledge useful comments of my friends Mr. S. Rouhani and Dr. B.S. Kay.

To my wife, Fatemeh, goes special thanks not only for her help in the preparation of this thesis, but also in maintaining a happy home during the period leading to its completion.

The award of a scholarship from the Iranian Ministry of Science and Education is also acknowledged.

This thesis has been typed by Mavis whom I wish to thank for her thoroughness and perseverance.

NOTATIONS AND CONVENTIONS.

Throughout this thesis the space-time manifold denoted by M - is assumed to be a 4-dimensional pseudo-Riemannian manifold with signature (-,+,+,+) and the following convention of the Ricci tensor

$$R_{\mu\nu} \quad \partial_\alpha \Gamma^\alpha_{\mu\nu} - \ldots.$$

We have used the sign := to mean definition. The factors of 2 in the Fourier transforms have usually been absorbed in $d^n p$ and $\delta(p)$ according to the following definition

$$\bar{d}^n p \;:=\; \frac{1}{(2\pi)^n}\; d^n p$$

$$\bar{\delta}^n(p) \;:=\; (2\pi)^n\, \delta^n(p)$$

such that

$$\int \bar{d}^n p\, \bar{\delta}^n(p) \;=\; 1$$

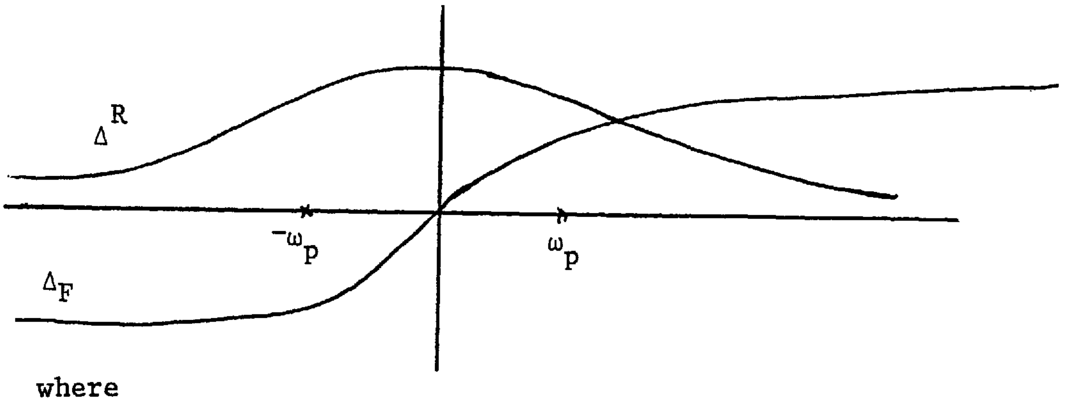
The flat Minkowskian metric is denoted by $\eta_{\mu\nu}$ and the Euclidean metric by $\delta_{\mu\nu}$

The different flat space Green's functions are defined by different sòlutions of

$$(\partial_\mu \partial^\mu - \mu^2)\Delta\,(x;x') \;=\; -\,\delta^4\,(x-x')$$

In this thesis we will only need the Feynman and the retarded Green's functions. These are defined by the following contour integrals.

$$\Delta\,(x-x') = \int d^4p \; \frac{e^{ip.(x-x')}}{p^2 + \mu^2}$$



where

$$\omega_p \;:=\; +\,(p^2 + \mu^2)^{\frac{1}{2}}$$

We have also made use of the following notations

E.T.C.	=	Equal Time Commutation.
w.r.t.	=	with respect to
r.h.s.	=	right hand side
l.h.s.	=	left hand side

The 3-dimensional vectors have usually been denoted as $\underline{A}$. The Heisenberg picture field operators and state vectors have always been written as $\hat{\phi}$ and $|\psi_o>$ respectively, whereas the free field operators are distinguished by a subscript o e.g. $\hat{\phi}_o$.

The expectation values like $<\psi_o|\; \hat{T}_{\mu\nu}|\psi_o>$ have sometimes been shortened to $<\hat{T}_{\mu\nu}>_{\psi_o}$ and when $|\psi_o> \;=\; |o>$ we have occasionally omitted the dependence on the state vector, e.g.

$$<\hat{T}_{\mu\nu}>_{|o>} \;\leftrightarrow\; <\hat{T}_{\mu\nu}>$$

or

$$\Delta^{|o>}(x,x') \quad\leftrightarrow\quad \Delta\,(x-x') \qquad \text{etc.}$$

CHAPTER I. INTRODUCTION

1.1 The Definition of a Semi-Classical Field Theory.

Consider a set of dynamically interacting fields. Assume a subset of these fields to satisfy the E.T.C. rules of quantum mechanics with their instantaneous physical states being described by a normalized vector in a Hilbert space. Let us also assume that the time evolution of these states is governed by Schrodinger equation while the dynamics of the remaining c-number fields are governed by classical field equations. Both the Schrodinger equation as well as the classical field equations are assumed to involve appropriate couplings of different fields contained in the set.

A mathematically consistent theory of this kind will be called a "semi-classical field theory",

Although there are several areas in physics where semi-classical theories are of practical (and phenomenological) interest[(1)], in this thesis we are mainly concerned with one specific example, namely, the semi-classical theory of gravity.

The motivation for such an interest is of course the vast number of unsurmountable obstacles encountered right at the beginning of any attempt to construct a quantum theory of gravity[(2)]. These difficulties which are of conceptual as well as technical nature are believed by some physicists to be stemming from the fundamental structural disparity between quantum mechanics and Einsteinian theory of gravity[(3)]. In this respect one should compare the flexible pseudo-Riemannian geometries of the Einsteinian theory of relativity with a priori fixed unit spheres of the Hilbert spaces of quantum mechanics. Some physicists (notably Mielnik) suggests that present day quantum theory still represents a relatively premature stage of development and lacks some essential evolutionary steps leading towards structural flexibility. For this reason they have the opinion that instead of modifying general relativity to fit quantum mechanics one should rather modify quantum mechanics to fit general relativity. One of the ways of performing these modifications is the convex set theoretical approach to quantum mechanics which we will briefly describe in the next section of this chapter.

Our aim in this thesis is however more modest than shattering the "foundations of quantum mechanics". There has been at least two decades of intensive work on subjecting Einsteinian gravitational law to the rules of quantum mechanics, but the progress has been meagre. All of these efforts have been done in the absence of any experimental or observational necessity. Indeed there is not a single physical phenomenon which begs the quantization of gravity for its explanation. As yet even the existence of classical gravitational waves, which would be equivalent to the Herzian waves of Maxwell's theory is only a theoretical possibility. Therefore in the absence of any observational evidence for the necessity of postulating the existence of the counterpart of the photon in the arena of gravity it is relevant to ask more seriously than what has been done before about the possibility of living in a half quantized and half classical world. This does not mean a simple reliance on the smallness of the Newtonian constant and neglecting the gravitational effects in the essentially quantum mechanical situations. It means on the contrary incorporating gravity in the quantum domain but leaving it classical. Our aim in this thesis is a consistent attempt in this direction.

From the outset we try to be as conservative as possible. We therefore assume that the usual Schrodinger picture description in terms of the time dependent normalizable state vectors and time independent operator observables holds true. The effect of the gravity field on quantum time evolution can be incorporated by formulating the Schrodinger picture in a curved space-time with a given metric tensor $g_{\mu\nu}$. As we show in Chapter 3, one can obtain the Schrodinger equation as well as the normalization condition of the state vector from a variational principle. It is interesting to note that if we now consider the $g_{\mu\nu}$ as a dynamical variable and add the usual Einstein action integral to the Schrodinger action, then extremization with respect to $g_{\mu\nu}$ of the total action will yield the Einstein field equations for $g_{\mu\nu}$ with the expectation value of $T_{\mu\nu}$ on its right hand side. The details of derivations will be given in Chapter 3. Here we only note that such a coupling of the "orthodox quantum mechanics" to the orthodox classical gravity has the important implication of making quantum mechanics non-orthodox. Indeed if we assume that Einstein field equations have been solved under some suitable boundary conditions, then the resultant $g_{\mu\nu}$ will be a function of the state vector, which when substituted in the Schrodinger equation will make it non-linear. We therefore conclude that if one adopts a description of gravity in terms of c-number fields and couples it dynamically to quantum mechanical time evolution law then the latter becomes implicitly non-linear.

1.2 *Non-linear quantum mechanics.*

Ever since the creation of quantum mechanics there have been attempts to put it on axiomatic basis. One of the universally accepted axioms is, of course, the superposition principle. In recent years there have been several attempts to go beyond these axioms and most of these have started by questioning the superposition principle.[(4-6)] This principle is directly related to the unitary structure of quantum time evolution law, and is usually regarded to be an exact law of nature. In this respect the Schrodinger's linear equation is an outstanding exception to all other linear laws which usually are regarded to be suitable approximations to more exact non-linear evolution laws.

The scrutiny of the superposition principle usually starts from speculations on the measurement processes in quantum mechanics[4]. An initial state which is a linear superposition of several states will not remain so after the act of observation is complete. Therefore the superposition principle is respected by the linear time-evolution law only as long as we do not perturb the system by subjecting it to observation. On the other hand in the axiomatic approach to quantum mechanics we insist on the unitary time-evolution precisely because we want to respect the superposition of states.

In this section we want to give a brief survey of some of the non-relativistic models of non-linear quantum mechanics. These models all share the property that although they violate the superposition principle of the orthodox quantum mechanics they still maintain its statistical interpretation which is embodied in the wave function Ψ.

a) *Mielnik's generalized Quantum Mechanics*[5].

Mielnik starts his criticism of the orthadox theory by noting that the whole body of observable properties derivable from the fundamental axioms of quantum mechanics are indeed contained in the geometry of the space of states of the system. This space (denoted henceforth by S), both in the case of classical as well as quantum systems, is a convex set, i.e. it has the property that any of its points p may be written as a linear combination $\sum_{i\varepsilon I}\lambda_i x_i$ with $0\leqslant \lambda_i \leqslant 1$; $\sum_{i\varepsilon I}\lambda_i = 1$ and $x_i \varepsilon S$ for $\forall_i \varepsilon I$. Here I is some index set. The points for which all $\lambda_i = 0$ except for one $i \varepsilon I$ are called the extremal points of S. Physically these points represent the pure states of the system while their convex combinations are the mathematical representatives of the mixed states.

In this language what distinguishes the classical statistical physics from the quantum mechanical one is the geometry of S. As

Mielnik asserts the possibility of a unique decomposition of a mixture into its constituent pure ensembles is an indication of the classical nature of the system. This implies (an ∞ dimensional) simplicial structure for the space S of the classical systems. Therefore a non-simplicial structure of S will be an expression of the non-classical nature of the physical system under consideration. In this case a mixture cannot be uniquely decomposed into its constituent pure components. In the case of the orthodox quantum mechanics S is, of course, the set of the density matrices acting on a separable Hilbert space H , i.e.

$$S_H = \{x \in L(H) \mid x = x^+ \geq 0 \ , T_r \, x = 1 \} \tag{1.1}$$

The pure states are represented by the projection operators $|\psi\rangle\langle\psi|$ where $|\psi\rangle$ is a normalized vector in H .

From what has been said Mielnik concludes that instead of being confined by the inflexible axioms of quantum logic to the unique structure S given above, let us abstract the most essential properties of this structure and then try to construct more generalized models which share these properties with the orthodox theory.

Supplementing this attitude with the assumption that the manifold of the pure states (henceforth denoted by Φ) spans the space S one only needs a way of constructing Φ . This space is, of course, the set of solutions of some equation of motion. Thus in Mielnik's theory one must necessarily be given a class of equations which describes the dynamics. Then the physical interpretation follows by investigating the geometrical properties of Φ and therefore those of S. In this respect Mielnik suggests the following non-linear generalizations of the one-particle Schrodinger equation as possible candidates for dynamical laws.

$$i\hbar \frac{\partial\psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2\psi + V|\psi|^2\psi \tag{1.2}$$

$$i\hbar \frac{\partial\psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2\psi + V\psi + U|\psi|^2\psi \tag{1.3}$$

$$i\hbar \frac{\partial\psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2 (|\psi|^2\psi) + V\psi \tag{1.4}$$

$$i\hbar \frac{\partial\psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2(|\psi|^2\psi) + V|\psi|^2\psi \tag{1.5}$$

In these equations V and U are external potentials.

Since all of these equations admit the existence of some conserved functional of ψ(i.e.$\int_\infty|\psi|^2 d^3x$ for eqn(1.2 to 3) and $\int_\infty|\psi|^4 d^3x$ for eqn. (1.4.5)), Mielnik assumed that one could simply apply the Born statistical interpretation of the orthodox quantum mechanics to these models. However, it was discovered independently by several physicists that due to the lack of scale invariance of these equations such an interpretation is indeed problematic[6-8]. Of course this does not invalidate the whole scheme developed by Mielnik. It only suggests that the examples given above are not suitable candidates for a generalization of the ordinary one particle Schrodinger equation.

b) *The Nonlinear Wave-Mechanics of Iwo Bialynicki-Birula and J. Mycielski*[6]

Mielnik arrives at the possibility of generalized quantum mechanics by scrutinizing the global mobility of the orthodox theory. The investigations of Bielynicki-Birula and J. Mycielski are less formal in character. They doubt the exactness of the linearity of one-particle non-relativistic Schrodinger equation . They therefore attempt to construct a non-linear equation which shares as much as possible the properties of the orthodox equation and reduces to it under suitable conditions. These restrictions immediately rule out the equations involving derivative non-linearity (e.g. of type 1.4 and 1.5) and leaves us with the following class of non-linear equations.

$$i\hbar \frac{\partial\psi}{\partial t} = [- \frac{\hbar^2}{2m} \nabla^2 + U + F(|\psi|^2)]\psi(r,t) \tag{1.6}$$

Here U is an external potential and $F(|\psi|^2)$ is assumed to be a real-valued function of its argument.

Although sharing some of the features of the orthodox theory, equations of class (1.6) do also exhibit a number of undesirable features. We already know that for a polynomial choice of F the equation will not be invariant under the scale transformation $\psi \to \lambda\psi$ where λ is a constant. It is also obvious that the scalar product of two solutions will not be preserved in time. In fact this seems to be the common property of all non-linear equations. This means that the transition amplitudes are not the suitable candidates to bridge the gap between the theoretical predictions and experimental data.

It is also observed by the above mentioned authors that the wave functions of stationary states with different frequencies are not orthogonal. They claim that this problem is impossible to solve for any choice of non-linearity. However, it is rather surprising that there is a unique choice of F which removes the obstacles associated with the scale invariance of the theory. This is the following logorithmic function.

$$F(\rho) = -b \log(\rho a^n) \tag{1.7}$$

where a and b are two arbitrary constants with the dimensionality of length and energy respectively. The interger n is the dimensionality of the configuration space. The constant a is of course of no immediate physical significance. (It can be redefined by adding a constant to U). The constant b on the other hand is a measure of non-linearity. If it is chosen to be a universal constant independent of the specificities of the physical system described by (1.6) then this equation will share all of the properties of the linear equation except for the violation of the superposition principle, lack of the time independent inner product and non orthogonality of the solutions corresponding to different frequencies.

Bialynicki-Birula and J. Mycielski show that the non-linear Schordinger equation with the logarithmic non linearity has soliton like solutions of the gaussian shape (which they call gaussons) in any number of dimensions. These solutions describe the wave packets of freely moving particles. Unlike the wave packets of the linear equation the gausson do not spread. From the existing agreement between the linear theory and the experiment they also find an upper limit of 4×10^{-10} eV for b which accounts for unobservability of the non-linearity.

c) *The Non-linear relativistic models of Kibble*[7].

The above mentioned upper limit for b has been obtained by comparing the results of the measurement with the theoretical predictions of quantum electrodynamics (e.g. Lamb shift, hyperfine splitting). This is already an indication of the fact that the non-linearities may be important only when the inter-particle interactions take place. Motivated by this, Kibble constructed a class of relativistic models of non-linear quantum mechanics. This construction proceeds by making the parameters of an orthodox theory state-dependent.

Consider a self-interacting real scalar field ϕ whose dynamics in the Schrodinger picture is described by the ordinary Schrodinger eqn.

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle \tag{1.8}$$

where

$$H = \frac{1}{2} \int d^3\underline{x} \, \{\pi^2(\underline{x}) + [\nabla\phi(\underline{x})]^2 + m^2\phi^2(\underline{x}) + \lambda\phi^4(\underline{x})\} \tag{1.9}$$

Here $\pi(\underline{x})$ and $\phi(\underline{x})$ are the time-independent canonical variables satisfying the E.T.C. rules

$$[\phi(\underline{x}),\pi(\underline{x}')] = i\,\delta^3(\underline{x}-\underline{x}') \tag{1.10}$$

Now let us make the following substitution

$$m^2\phi^2 + \lambda\phi^4 \rightarrow \langle f(\phi)\rangle_\psi + \langle h(\phi)\rangle_\psi\,\phi^2 + \langle g(\phi)\rangle_\psi\,\phi^4 \tag{1.11}$$

where f,h and g are local functions of the field $\phi(\underline{x})$ and the expectation values are defined in the usual way, for instance

$$\langle f(\phi)\rangle_\psi = \frac{\langle\psi|f(\phi)|\psi\rangle}{\langle\psi|\psi\rangle} \tag{1.12}$$

It is obvious that the substitution of (1.11)into (1.9) makes the equation (1.8) non linear. Furthermore, owing to the denominator in (1.12) the resulting non linear Schrodinger equation will be invariant under the scale transformations $|\psi\rangle \rightarrow \lambda|\psi\rangle$. Therefore the problems associated with the measurement theories of equations of the type (1.2) to (1.5) will not arise here.

The models constructed in this manner are clearly Lorentz invariant. Furthermore they have the peculiar property that even in the absence of the self-interaction terms (i.e. a choice like g = f = 0 and $h = m^2 + \phi^2(\underline{x})$) they will transform a single particle initial state into a many particle final state.

We leave a more detailed discussion of a subclass of these models to the next chapter.

1.3 *Conclusion of the Introduction.*

According to B. d'Espagnat the two pillars of the Copenhagen interpretation of quantum mechanics are the following: (c.f. B d'Espagnet in ref.4 p.p.251).

a) The working of measuring instruments must be accounted for in purely classical terms.

b) In general quantum systems should not even be thought of as possessing individual properties independently of the experimental arrangement.

Whatever view one may adopt in regard to the measurement theory in quantum mechanics, these two principles seem to be indispensible for a communicable interpretation of the results of the measurements. Thus quantum mechanics - unlike Einstein's theory of relativity - locates the classical physics not on its periphery (as a limitting theory), but right at its centre. It becomes meaningful only when it is viewed as a whole encompassing this centre. This is of course because of the singular role played by the human observer on any act of measurement. If an atomic system (i.e. a system capable of being in a superposition of several quantum states) could be used as a measuring device with no need for a final act of observation by a human observer then one would not need the above mentioned two principles. Such a hypothetical situation is, however, devoid of any practical scientific significance.

A semi-classical physics defined at the beginning of §1.1 may be viewed as an attempt in the direction of a dynamical integration of the classical physics into the quantum domain (or vice-versa). This becomes particularly more relevant when the classical part of the theory consists only of the gravitational field whose dynamics is described by the Einsteinian field equations for the metric of space-time. Einsteinian law of gravity is in a sense a dynamical theory of the measurement of the space-time intervals, a theory to which any scientific description of nature (physical theory) must ultimately be reduced for its comprehensibility and communicability by human observers. In this respect the two above mentioned principles may as well be regarded as the supporting pillars of the semi-classical theory of gravity. This might be the case at least as long as the gravitational field - unlike any other field in physics - describes (and is described by) the geometry of the space-time manifold. The plan of this thesis is as follows.

In Chapter 2 we set up a perturbative scheme to study a specific model of the class mentioned in §1.2.c. This model will be defined by settling $g = f = 0$ and $h = \frac{1}{2}(m^2 + \lambda\phi^2(\underline{x}))$ in eqn. (1.11). After deriving the resulting non-linear Schrodinger equation as well as the normalization condition of $|\psi>$ from a variational principle, we will introduce suitable renormalization counter terms whose constant coefficients will be fixed by imposing some physical conditions on the Schrodinger equation.

In Chapter 3 we will construct the action integral for the combined Schrodinger-Einstein field equations of the semi-classical theory of gravity. The remarkable result of this chapter is the following.

If for a self gravitating quantized field one insists on having Schrodinger's equation for the temporal changes of the normalized, $|\psi>$ then the metric tensor of the space-time manifold will <u>necessarily</u> satisfy the Einsteinian field equations with the expectation value of $T_{\mu\nu}$ on its right hand side.

In this chapter we will also discuss a qualitative comparison of semi-classical gravity with a full quantum theory of gravity.

Owing to the non-linear nature of Einsteinian field equations as well as the derivative coupling of gravity and the matter field it is rather difficult to start a direct study of the renormalization of the theory developed in Chapter 3. We therefore motivate the investigation of the renormalizability of the semi-classical theories by studying in Chapter 4, a simple model consisting of two interacting real scalar fields V and $\hat{\phi}$ where V is classical and $\hat{\phi}$ is quantized. After deriving the coupled set of dynamical equations from a variational principle in the Schrodinger picture we find it more convenient to discuss the renormalization theory in the Heisenberg picture. Therefore we transform into this picture. Then we develop a manifestly Lorentz covariant perturbative scheme and introduce counter-terms to cancel the infinities at each order of our perturbation theory.

The model studied in Chapter 4 does not involve any derivative couplings ; therefore it does not share all of the complications of the full semi-classical theory of gravity. In Chapter 5 we study a more complicated model this time with derivative coupling between V and $\hat{\phi}$ in a flat background Minkowskian space-time. We then follow the same program as in Chapter 4 i.e. we introduce counter-terms to remove the infinities order by order in a perturbation theory.

Now having done enough preliminary exercises in Chapters 4 and 5 we finally attempt a perturbative treatment of the full semi-classical theory of gravity. The existence of an action integral for the coupled set of Einstein-Schrodinger equation accommodates in a natural way the purely geometrical renormalization counter terms whose (infinite) coefficients are fixed once and then for all by imposing physical renormalization conditions on the linearized Einsteinian equations. Having renormalized the theory we investigate some of the qualitative features of the solutions of the linearized theory and prove that the solution $(M\ ,\ \eta_{\mu\nu}, |o\rangle)$ is unstable.

We conclude Chapter 6 by giving a unified summary of the results obtained in the main body of the thesis.

We relegate the details of calculations of Chapters 3-6 to the appendices at the end of the thesis. Some of the results of the Appendix A are original.

CHAPTER 2. RENORMALIZATION OF A RELATIVISTIC MODEL OF NON LINEAR SCHRODINGER EQUATIONS.

§2.1 *The action integral.*

Consider the following non-linear Schrodinger equation

$$i\frac{\partial}{\partial t}\,|\psi(t)\rangle = H_\psi\,|\psi(t)\rangle \tag{2.1a}$$

where H_ψ is defined by

$$H_\psi = \frac{1}{2}\int d^3\underline{x}[\pi^2(\underline{x}) + (\nabla\phi(\underline{x}))^2 + m^2\phi^2(\underline{x}) + \lambda\langle\phi^2(\underline{x})\rangle_\psi\,\phi^2(\underline{x})] \tag{2.1b}$$

$\pi(\underline{x})$ and $\phi(\underline{x})$ satisfy equations (1.10) and λ is a coupling constant. The expectation value $\langle\phi^2(\underline{x})\rangle_\psi$ is defined as in (1.12).

Equations (2.1) can of course be obtained from (1.9 - 1.11) by choosing $f(\phi) = 0$ $g = o$ and $h(\phi) = m^2 + \frac{\lambda}{2}\phi^2$.

Because of the invariance of the equation under the scale transformation $|\psi(t)\rangle \to \gamma|\psi(t)\rangle$ with a constant γ , we may consistently impose the normalization condition

$$\langle\psi(t)|\psi(t)\rangle = 1 \tag{2.2}$$

We notice that equations (2.1) together with condition (2.2) may be obtained from the following action integral

$$S[|\psi\rangle,\langle\psi|,\alpha] = \int_{t_o}^{t} dt\{\mathrm{Im}\,\langle\dot{\psi}(t)|\psi(t)\rangle - \langle\psi(t)|H_o|\psi\rangle + \alpha(t)(\langle\psi(t)|\psi(t)\rangle - 1)\}$$
$$- \frac{\lambda}{4}\int d^4x\,\langle\psi(t)|\phi^2(\underline{x})|\psi\rangle^2 \tag{2.3a}$$

where

$$H_o = \frac{1}{2}\int d^3\underline{x}\,(\pi^2(\underline{x}) + (\nabla\phi(\underline{x})^2 + m^2\phi^2(\underline{x})) \tag{2.3b}$$

$\alpha(t)$ is a Lagrange multiplier.

The independent variables in the action integral (2.3) are $<\psi|, |\psi>$ and α. The Euler-Lagrange equations for α will immediately yield the normalization condition (2.2). While the same equations for $<\psi|$ will result in the following

$$i \frac{\partial}{\partial t} |\psi(t)> = (H_\psi - \alpha(t))|\psi(t)> \tag{2.4}$$

where H_ψ is defined by (2.1b).

Equation (2.4) can of course easily be transformed into eqn. (2.1) by a time dependent phase transformation of $|\psi(t)>$. Thus α will not be determined by the equations of motion. This is an indication of the freedom in choosing the zero point of H_ψ.

The action integral (2.3) as well as the equations (2.1) and (2.2) derived from it have at best a formal value. They are ill defined as they involve the infinite quantities $<\psi(t)|\phi^2|\psi(t)>$. To render the theory finite we subtract constant infinites from (2.3). This is done by the following substitutions.

$$H_o \to H_o - c_1$$

$$\lambda\phi^2(\underline{x}) \to \lambda(\phi^2(\underline{x}) - c_2)$$

Then the modified action integral becomes

$$S[|\psi>,<\psi|,\alpha] = S + \Delta S \tag{2.5}$$

where S is given by (2.3) and ΔS is defined by

$$\Delta S = \int_{t_o}^{t} dt\langle\psi(t)|\psi(t)\rangle\, c_1 - \frac{\lambda}{4}\int d^4x \,\{-2c_2\langle\psi(t)|\phi^2(\underline{x})|\psi(t)\rangle + c_2^2\langle\psi(t)|\psi(t)\rangle\} \times$$

$$\times \quad \langle\psi(t)|\psi(t)\rangle$$

The Euler-Lagrange equation corresponding to the variable α in (2.5) will of course be the same as (2.2) while H_ψ in (2.1d) is modified into

$$\tilde{H}_\psi = (H_o - c_1) + \frac{\lambda}{2}\int d^4x \,\{\langle\psi(t)|(\phi^2(\underline{x}) - c_2)|\psi(t)\rangle\, (\phi^2(\underline{x}) - c_2)\} \quad (2.6)$$

In writing eqn. (2.6) we have made use of eqn. (2.2).

To fix the constants c_1 and c_2 we will consider the state dependent term of $\tilde{H}_\psi$ as an interaction term. In order to regard this term as a perturbation of the free Hamiltonian $H_o - c_1$ we also adopt the usual assumption that as $t \to -\infty$ we recover the free theory. Then the constant c_1 will fix the origin of the measurement of the energies of the free linear system. As usual we choose it to be

$$c_1 = \langle 0|H_o|0\rangle \quad (2.7)$$

where $|0\rangle$ is the vacuum state of the state independent Hamiltonian H_o. This choice of c_1 corresponds to the condition that $|0\rangle$ of the linear theory is independent of time.

In order to fix c_2 we make the further assumption that if at $t = -\infty$ the state of the non-linear theory corresponds to $|0\rangle$ of the linear theory then the same must be true for all $t > -\infty$. This condition will fix c_2

$$c_2 = \langle 0|\phi^2(\underline{x})|0\rangle \quad (2.8)$$

Then the renormalized H_ψ will be written as

$$\tilde{H}_\psi = :H_o: + \frac{\lambda}{2}\int d^4x \, \langle\psi(t)| :\phi^2(\underline{x}): |\psi(t)\rangle :\phi^2(\underline{x}): \qquad (2.9)$$

where : : stands for the usual definition of normal ordering, i.e.

$$:\phi^2: = \phi^2 - \langle 0|\phi^2|0\rangle$$

We notice that the choice of (2.7) and (2.8) for c_1 and c_2 automatically implies $\langle 0|H_{|0\rangle}|0\rangle = 0$.

§2.2 *Perturbation Theory.*

In this section we would like to develop a covariant perturbation theory for the non-linear Schrodinger equation (2.1a) with H_ψ defined according to (2.9). To this end we perform the usual transformation into the interaction picture, i.e. we define

$$|\psi_I(t)\rangle = e^{i:H_o:t} |\psi(t)\rangle \tag{2.10a}$$

and

$$\phi_o(x) = e^{i:H_o:t} \phi(\underline{x})\, e^{-i:H_o:t} \tag{2.10b}$$

$$\pi_o(x) = e^{i:H_o:t} \pi(\underline{x})\, e^{-i:H_o:t} \tag{2.10c}$$

Then as in the ordinary linear quantum theory ϕ_o will satisfy the free Klein-Gordon equation

$$(\partial_\mu \partial^\mu - \mu^2)\, \phi_o(x) = 0.$$

The state $|\psi_I(t)\rangle$ on the other hand will satisfy the following interaction-picture non-linear Schrodinger equation

$$i \frac{\partial}{\partial t} |\psi_I(t)\rangle = \frac{\lambda}{2} \int d^3\underline{x}\, \langle\psi_I(t)|:\phi_o^2(x):|\psi_I(t)\rangle :\phi_o{}^2(x):|\psi_I(t)\rangle \tag{2.11}$$

We notice that the transformation (2.10a) will leave the Schrodinger picture vacuum state invariant. This is necessary for the correct transformation of the normal ordering of the operators.

Our objective is to find a solution of eqn (2.11) which has the following form

$$|\psi_I(t)\rangle = \sum_{n=o}^{\infty} \lambda^n |\psi_n(t)\rangle$$

If we insert this into (2.11) and equate to zero the coefficients of different powers of λ we get a recursive set of equations (c.f.2.15) whose first term is the following

$$|\dot{\psi}_o(t)> \; = \; 0 \tag{2.12a}$$

This implies that $|\psi_o>$ is a const. Since under our assumption the interaction is switched off asymptotically as $t \to -\infty$ therefore the consistency requires that

$$|\psi_n(t)> \xrightarrow[t \to -\infty]{} o \quad \forall_n > 0 \tag{2.12b}$$

Thus the equations

$$i \sum_{n=o}^{\infty} \lambda^n \; |\dot{\psi}_n(t)> \; = \; \frac{\lambda}{2} \sum_{p_1,p_2,p_3=o}^{\infty} \lambda^{p_1+p_2+p_3} \int d^3\underline{x} <\psi_{p_1}(t)| :\phi_o^2(x): |\psi_{p_2}(t)>: \phi_o^2(x): |\psi_{p_3}(t)> \tag{2.13}$$

must be solved under the initial conditions (2.12). To solve these equations up to an arbitrary order in λ we must identify the coefficients of the same powers of λ in both sides of (2.13). To this end we must convert the product of three sums on the r.h.s. of (2.13) into a single sum. This can be done by making use of the following identity

$$\sum_{p_1=o}^{\infty} \sum_{p_2=o}^{\infty} \cdots \sum_{p_n=o}^{\infty} \lambda^{p_1+\dots+p_n} u_{p_1 \dots p_n} =$$

$$\sum_{q_n=o}^{\infty} \lambda^{q_n} \cdot \sum_{r_n=o}^{q_n} \; \sum_{r_{n-1}=o}^{q_n-r_n} \; \sum_{r_{n-2}=o}^{q_n-r_n-r_{n-1}} \cdots \sum_{r_2=o}^{q_n-r_n-r_{n-1}\dots r_3} U_{q_n-r_n-r_{n-1}\dots r_2, r_2, r_3 \dots r_n}$$

Now if we define

$$\left| U_{p_1p_2p_3}(t)\right> = \int d^3\underline{x} <\psi_{p_1}(t)| :\phi_o^2(x) :|\psi_{p_2}(t)>:\phi_o^2(x):|\psi_{p_3}(t)>$$

Then by making use of (2.14) for n=3 we immeidately get

$$\sum_{p_1=o}^{\infty} \sum_{p_2=o}^{\infty} \sum_{p_3=o}^{\infty} \lambda^{p_1+p_2+p_2} |U_{p_1p_2p_3}(t)> = \sum_{q_3=o}^{\infty} \lambda^{q_3} \sum_{r_3=o}^{q_3} \sum_{r_2=o}^{q_3-r_3}$$

$$| Uq_3-r_3-r_2,r_2,r_3 >$$

Upon substitution of this into the r.h.s. of (2.13) we get

$$|\dot{\psi}_{n+1}(t) >= \frac{1}{2} \sum_{s=o}^{n} \sum_{r=o}^{n-s} \int d^3\underline{x}<\psi_{n-s-r}(t)|: \phi_o^2(x)|\psi_r(t) >: \phi_o^2(x):|\psi_s(t)> \tag{2.15}$$

Thus starting from an initial state $|\psi_o>$ we can now solve for $|\psi_n(t)>$ in a recursive manner e.g.

$$i|\dot{\psi}_1(t)>= \frac{1}{2} \int d^3\underline{x} <\psi_o|: \phi_o^2(x):|\psi_o>: \phi_o^2(x):|\psi_o> \tag{2.16a}$$

$$\begin{aligned} i|\dot{\psi}_2(t)>= \frac{1}{2} \int d^3\underline{x}\{&<\psi_1(t)|:\phi_o^2(x):|\psi_o>:\phi_o^2(x):|\psi_o> + \\ &<\psi_o|:\phi_o^2(x): |\psi_1(t)>:\phi^2(x)|\psi_o> + \\ &<\psi_o|:\phi_o^2(x):|\psi_o>: \phi_o^2(x):|\psi_1(t)> \end{aligned} \tag{2.16b}$$

similarly for $|\psi_3>$,$|\psi_4>$, etc.

It is rather obvious from (2.15) that if we begin with a single particle initial state $|\psi_o\rangle$, then in the course of time it will evolve into a multiparticle final state. For example if we restrict our attention only up to the first order correction $|\psi_1(t)\rangle$ then we will have non-zero transition amplitudes into single particle and three particle final states, whereas the second order correction $|\psi_2(t)\rangle$ would also involve non-vanishing amplitudes for the five particle final states, and so on.

Obviously one may choose $|\psi_o\rangle$ to be a multiparticle initial state and ask questions about the mutual scattering of these particles. For example if it is chosen to represent a two particle initial state then the first order correction will involve non - zero transition amplitudes to the vacuum, two particle and four particle final states.

In many ways the non-linear quantum mechanics studied in this chapter manifests similar features to the ordinary $\lambda\phi^4$ interaction. The basic difference between the two theories is of course the linearity versus non-linearity of the quantum time evolution law. Because of the linearity the single particle initial states in the $\lambda\phi^4$ theory must be stable otherwise the energy conservation law will be violated. The instability of the single particle initial states of the non-linear quantum mechanics does not, of course, violate any conservation law. It is rather a consequence of the violation of the superposition principle.

In this chapter we dealt with the non-linear Schrodinger equation only in the Schrodinger picture. At the moment the interaction like picture seems to be the nearest we can get to a covariant treatment of these theories. Due to the fundamental role played by the state $|\psi(t)\rangle$ it is rather natural to formulate the non-linear theories in the Schrodinger picture. It is however desirable to have a manifestly covariant Heisenberg picture description of these theories. This is as yet an unsolved problem.

Remark: It is of course possible to transform formally into the Heisenberg picture by the following operator

$$\theta_\psi(t) = T \exp [i \int_{-\infty}^{t} H_\psi \, dt]$$

where H is defined by (2.9). This transformation is however not a linear one. The resulting Heisenberg equations of motion will be equivalent to

$$(\partial_\mu \partial^\mu + \mu^2) \hat{\phi}(x) = \lambda \langle\psi_o| :\hat{\phi}^2(x): |\psi_o\rangle \hat{\phi}(x)$$

where $|\psi_o\rangle$ is the time independent Heisenberg state. Unfortunately this equation cannot be obtained from a variational principle which incorporates the dynamics of $|\psi_o\rangle$. It also differs from the usual Heisenberg equation in that it involves the state $|\psi_o\rangle$.

CHAPTER 3. THE SEMI-CLASSICAL THEORY OF GRAVITY

§3.1 Introduction

In this chapter we will derive the dynamical equations of the semi-classical theory of gravity.

Since the time the physicists started to realize the conceptual difficulties of subjecting the Einsteinian law of gravity to the rules of quantum mechanics, it was occasionally suggested that the gravitational field must be exempted from these rules[10]. Indeed Møller purposed a theory in which the r.h.s. of Einstein's field equations involved the expectation value of $T_{\mu\nu}$ of the quantized fields[9]. This theory however did not provide any dynamical law describing the time evolution of the quantum states. Therefore the whole consistency of such proposals was questionable. At best, some physicists regarded this theory as a kind of approximation to a fully quantized theory of gravity.

One of the difficulties in proving the consistency of the semi-classical theory of gravity is the lack of an Heisenberg picture action integral which yields the covariant (quantized) matter field equations as well as the Einstein's field equations with $\langle\psi|T_{\mu\nu}|\psi\rangle$ on its r.h.s. In fact this seems to be a general characteristic of the theories with non-linear quantum time evolution law. A consistent theory of semi-classical gravity would undoubtedly involve an implicit non-linearity in its quantum time evolution.

In all of the non-linear models of quantum mechanics constructed so far the wave function ψ has played a prominent role. Certainly it was the investigation of the Schrodinger equation which led the physicists to the recognition of the linear Hilbertian nature of the space of states in quantum mechanics. Therefore it is logically understandable that an attempt to go beyond the frontiers of the linearity should again be carried out in the Schrodinger picture. On the other hand as we saw in the previous chapter at least one of the models belonging to the large class of Kibble's non-linear models can be derived from a variational principle. Indeed this is the case for all of the models which admit the existence of a conserved energy functional. For example. the model with logarithmic non-linearity (c.f. eqn. 1.6-7) may be derived from the following action integral

$$S[\psi^*,\psi]= \int d^4x \; \{\mathrm{Im}\; \dot{\psi}^*(t,\underset{\sim}{x})\psi(t,\underset{\sim}{x})-\psi^*(t,\underset{\sim}{x})(-\frac{h^2}{2m}\nabla^2+ U- b\ln a|\psi|^2+b)\psi(t,\underset{\sim}{x}) \tag{3.1}$$

These examples may be considred as evidence to the possibility that if the semi-classical theory of gravity is viewed from the standpoint of the non-linearity of its quantum mechanics, then its appropriate treatment must be undertaken in the Schrodinger picture rather than in the Heisenberg's. The problem with such an approach is, of course, the lack of manifest general covariance of the whole construction, although the theory, if it is consistent will be independent of the choice of the coordinates in the space-time manifold. This aspect of the problem is not specific to our theory. Even in flat space-time any Lorentz invariant field theory - when formulated in the Schrodinger picture will lack in manifest covariance.

In the sequal we will need the Schrodinger picture only to write down the action integral. After deriving the dynamical equations we will carry out the actual computations in the Heisenberg picture.

We will formulate the theory only for a real scalar quantum field ϕ. It may be possible to extend this formalism - with appropriate modifications - to the fields with higher spins. We will also make use freely of the material contained in the appendix A.

§3.2 *Field theory in the Schrodinger picture.*

To start with we assume that the metric $g_{\mu\nu}$ is given. We then consider a real scalar field ϕ propagating in our fixed globally hyperbolic space-time manifold . As usual the dynamics of this field can be described by giving the Lagrangian density L . We will take L to be given by

$$L\phi = -\frac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)) \tag{3.2a}$$

$V(\phi)$ may be taken to be

$$V(\phi) = m^2\phi^2 + \frac{1}{12}\lambda\,\phi^4 \tag{3.2b}$$

where m and λ are constant real numbers.

In the Heisenberg picture the field equations for ϕ are given by variations of the operator action integral

$$S_\phi = \int L_\phi d^4x \tag{3.3a}$$

The corresponding energy-momentum tensor may be defined by

$$\frac{1}{2}\sqrt{-g}\;T_{\mu\nu}(x) = \frac{\delta\,S_\phi}{\delta g^{\mu\nu}(x)} = \frac{\partial L_\phi(x)}{\partial g^{\mu\nu}(x)} \tag{3.3b}$$

although of course this is ill-defined until a regularization scheme has been adopted. We will discuss this in Chapter 6.

Our objective is to pass from this manifestly covariant description to a Schrodinger picture. In this latter picture with each simultaneity surface $\sigma(t)$, defined by eq. (A.1), one associates a state vector $|\psi(t)>$ and the canonical operators ϕ and π satisfy the equal time commutation rule

$$[\phi(\xi,t)\ ,\pi(\xi,t)] = i\ \delta^3(\xi,\xi') \quad \xi,\ \xi' \varepsilon\ \sigma(t) \tag{3.4}$$

Assuming that the two pictures coincide on a surface $\sigma(t_o)$ then the transition between them can be achieved by applying a "unitary" transformation $U(t,t_o)$ satisfying the equation

$$i\,\frac{\partial}{\partial t}\ U(t,t_o) = H(t)\ U(t,t_o) \tag{3.5}$$

where the surface dependent Hamiltonian is given by [c.f. A.4 and A.5]

$$H(t) = -\int_{\sigma(t)} d\sigma_\mu\ T^\mu_\nu\ \dot{x}^\nu \tag{3.6}$$

Upon performing this transformation we obtain a transformed vector

$$|\psi(t)> = U(t,t_o)|\psi_o> \tag{3.7}$$

which satisfies the Schrodinger equation

$$i\ |\dot{\psi}(t)> = H(t)|\psi\ (t)> \tag{3.8}$$

Correspondingly, the transformed operators $U(t,t_o)\ \ \phi(\underline{x})\ \ U(t,t_o)$ and $U(t,t_o)\ \ \pi(\underline{x})\ \ U^{-1}(t,t_o)$ become independent of t, functions of the intrinsic coordinate $\vec{\xi}$ only.

It must be emphasised that the eqn. (3.5) has at best a formal significance. In general there is no unitary operator satisfying this equation, and indeed H(t) may not exist. Nonetheless, the derivation has a heuristic value, and we shall pursue it further. Later we return to the question of how to give the resulting formalism a precise mathematical meaning.

We note also that the transformation to the Schrodinger picture depends not only on the slicing of space-time, but also on the parametrization of the slices. A time-dependent change of intrinsic coordinates leads to a different schrodinger picture.

The Schrodinger equation (3.8) can also be written in a local form. To do this we note that the variation $\delta|\psi(t)\rangle$ is the sum of infinitisimal variations, i.e.

$$\delta|\psi\rangle = \int_{t}^{t+\delta t} dt' \int_{\sigma(t')} d^3\xi\ \dot{x}^{\mu}(\xi) \frac{\delta}{\delta x^{\mu}(\xi)} |\psi\rangle$$

or

$$|\dot{\psi}\rangle = \frac{\delta|\psi\rangle}{\delta t} = \int_{\sigma(t)} d^3\xi\ \dot{x}^{\mu}(\xi) \frac{\delta}{\delta x^{\mu}(\xi)} |\psi\rangle \tag{3.9}$$

Substituting from (3.6) and (3.9)into(3.8) we get

$$i \frac{\delta}{\delta x^{\mu}} |\psi\rangle = - \gamma^{\frac{1}{2}}\, n_{\mu}\, T^{\mu}_{\nu}\, (x)|\psi\rangle \tag{3.10}$$

This equation enables us to determine $|\psi\rangle$ associated with the surface σ (t + δt) provided we know the one associated with σ(t). As in flat space-time the equation (3.8) leaves the phase of $|\psi\rangle$ undetermined. This of course corresponds to the freedom in choosing the zero point of H or what amounts to the same, in the conservation of the norm of $|\psi\rangle$.

§3.3 *The action integral.*

It is obvious that if our problem was to study the dynamics of the ϕ-field in a given background $g_{\mu\nu}$ field there would be no need to discuss the Schrodinger picture formalism. In fact because of its lack of manifest covariance, it could be very inappropriate to do so, knowing that we already have the manifestly covariant Heisenberg picture at our disposal.

Our main problem, however, is to describe the dynamics of the classical $g_{\mu\nu}$ field as well as that of the quantized ϕ-field. We do this first by noting that the eqn (3.8) together with the normalization condition of $|\psi\rangle$ can by obtained from the following action integral.

$$S_\psi = \int dt \,\{\mathrm{Im}\,\langle\dot{\psi}|\psi\rangle - \langle\psi|H|\psi\rangle + \alpha(t)\,(\langle\psi|\psi\rangle - 1)\} \tag{3.11}$$

Here H is given by eq. (3.6) and $\alpha(t)$ is a Lagrange multiplier. The independent variables are $\langle\psi|$, $|\psi\rangle$ and $\alpha(t)$. Requiring that the first order variations of S_ψ with respect to these variables must vanish we get

$$i|\dot{\psi}\rangle = (H(t) - \alpha(t))|\psi(t)\rangle \tag{3.12a}$$

$$\langle\psi|\psi\rangle - 1 = 0 \tag{3.12b}$$

Note that the Euler-Lagrange equations leave α undetermined. This is the reflection of the freedom in choosing the phase of $|\psi(t)\rangle$. Physically of course, (3.12a) is equivalent to (3.8) because an overall phase in the state is unobservable. In practice we may remove the arbitrariness by a suitable choice of the zero point of H.

Next, we consider the dynamics of the $g_{\mu\nu}$ field by augmenting S_ψ with the Einstein action, i.e. we consider the action integral

$$S = S_\psi + Sg \tag{3.13}$$

where Sg is given by eq. (A.38) with the following choice of the parameters

$$A = \frac{1}{16\pi G_N} \qquad X = 1, \quad B = C = 0 = \Lambda \tag{3.14}$$

where G_N is the Newton's constant. Now we consider S as a functional of $\langle\psi|$, $|\psi\rangle$, α and $g_{\mu\nu}$. Extremization of S with respect to $\langle\psi|$ and α will, of course, yield equations (3.12), while the equation

$$0 = \frac{\delta S}{\delta g^{\mu\nu}(x)} = \frac{\delta S_\psi}{\delta g^{\mu\nu}(x)} + \frac{\delta Sg}{\delta g^{\mu\nu}(x)} \tag{3.15}$$

gives us the Einstein field equations

$$G_{\mu\nu} = - 8\pi\, G_N \langle\psi(t)|\; T_{\mu\nu}(x)\; |\psi(t)\rangle \tag{3.16}$$

To see this we note from eq. (3.11) that

$$\frac{\delta S_\psi}{\delta g^{\mu\nu}(x)} = - \frac{\delta}{\delta g^{\mu\nu}(x)} \int dt \langle\psi|H|\psi\rangle$$

Now if we insert from equation (A.37b) into the right hand side of this equation we get :

$$\frac{\delta S_\psi}{\delta g^{\mu\nu}(x)} = \frac{1}{2}\sqrt{-g}\, \langle\psi(t)|\; T_{\mu\nu}(x)|\psi\;(t)\rangle \tag{3.17}$$

On the other hand if we substitute from (3.14) into eqn (A.51) we get

$$\frac{\delta Sg}{\delta g^{\mu\nu}(x)} = \frac{\sqrt{-g}}{16\pi G_N} \quad G_{\mu\nu} \tag{3.18}$$

Insertion of (3.17) and (3.18) into (3.15) immediately yields (3.16). Therefore we derived the Schrodinger equation (3.12) as well as Einstein equations (3.16) from a single action integral (3.13).

It is interesting to note that in order to write the action integral (3.13) we must work in the Schrodinger picture. At the moment it seems to be impossible to construct a Heisenberg picture version of this variational principle.

However, after deriving the equations of motion we may reverse the order of the transformation (3.7) and retain the Heisenberg picture version of the theory. If we do this, then the state vector $|\psi_o>$ will be time independent but now the time dependent field operator ϕ will satisfy the convariant field equation.

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\ g^{\mu\nu} \partial_\nu \hat{\phi}) - \frac{1}{2} \frac{\partial V}{\partial \hat{\phi}} = 0 \tag{3.19}$$

To check this, we remember that in the Heisenberg picture the time evolution of the dynamical variable $\hat{\phi}$ and $\hat{\pi}$ is described by the Heisenberg's equation of motion i.e.

$$\dot{\hat{\phi}} = i[H,\hat{\phi}] \tag{3.20a}$$

$$\dot{\hat{\pi}} = i[H,\hat{\pi}] \tag{3.20b}$$

Upon substitution from eqn (A.33) into these equations we get

$$\dot{\phi}(x) = i\int d^3x' \sqrt{-g}(x') \left(\frac{1}{2} \gamma^{-1}(x')[\hat{\pi}^2(x'), \hat{\phi}(x)] - \right.$$

$$\gamma^{-\frac{1}{2}}(x')\ n^s(x')\ [\hat{\phi}_s(x')\hat{\pi}(x'), \hat{\phi}(x)] +$$

$$\left. \frac{1}{2}\gamma^{rs}[\hat{\phi}_\gamma\ \hat{\phi}_s, \hat{\phi}] + \frac{1}{2}[V, \hat{\phi}] \right)$$

Now we make use of the equal time commutation rule (3.4) to get

$$\dot{\hat{\phi}}(x) = i\int d^3x' \sqrt{-g}(x') \Big(\tfrac{1}{2}\gamma^{-1}(x') \quad (-2\, i\delta^3(x,x')\,\hat{\pi}(x')) -$$

$$\gamma^{-\frac{1}{2}}(x')n^s(x')\,\hat{\phi}_s(x') \quad (-i\delta^3(x,x'))\Big)$$

or

$$\dot{\hat{\phi}}(x) = \sqrt{-g} \quad (\gamma^{-1}\hat{\pi} - \gamma^{-\frac{1}{2}}\, n^s\hat{\phi}_s)$$

or

$$\hat{\pi}(x) = \frac{\gamma}{\sqrt{-g}}\dot{\hat{\phi}} + \gamma^{\frac{1}{2}}\, n^s\,\hat{\phi}_s,$$

From eqn. (A.15) and (A.14) we can write

$$\gamma = \frac{-g}{N^2} = (-g)\,(-g^{oo})$$

$$\gamma^{\frac{1}{2}}\, n^s = \frac{\sqrt{-g}}{N} \quad (-N\, g^{so})$$

Therefore

$$\begin{aligned}\hat{\pi}(x) &= \sqrt{-g} \quad (-g^{oo}\dot{\hat{\phi}}) + \sqrt{-g}\,(-g^{os}\,\hat{\phi}_s) \\ &= -\sqrt{-g}\; g^{\mu o}\,\hat{\phi}_{,\mu}\end{aligned} \tag{3.21}$$

This is the same as in (A.27).

Similarly we insert for H from (A.33) into equation (3.20b) to get

$$\dot{\hat{\pi}}(x) = i\int d^3x' \sqrt{-g}(x') \Big(-\gamma^{-\frac{1}{2}}(x')\; n^s(x')\,[\hat{\phi}_s(x'),\hat{\pi}(x)]\,\hat{\pi}(x') +$$

$$\tfrac{1}{2}\,\gamma^{rs}(x')[\hat{\phi}_r(x')\hat{\phi}_s(x'),\hat{\pi}(x)] + \tfrac{1}{2}\,[V(\phi),\hat{\pi}(x)]\Big)$$

From eqn (3.5) we can write

$$[\,\hat{\phi}_s(x'),\hat{\pi}(x)\,] = i\;\delta^3_{,s'}\;(x',x)$$

$$[\hat{\phi}_r(x')\hat{\phi}_s(x'),\hat{\pi}(x)] = i\;(\hat{\phi}_r(x')\delta^3_{s'}(x',x) + \hat{\phi}_s(x')\delta^3_{,r'}(x',x))$$

$$[V(\hat{\phi}(x')),\hat{\pi}(x)] = i\,\frac{\partial V}{\partial\phi(x')}\;\delta^3(x',x)$$

If we insert these results into the right hand side of $\dot{\hat{\pi}}$ and integrate the terms involving the derivatives of δ^3-functions by parts we get

$$\dot{\hat{\pi}}(x) = (-\sqrt{-g}\,\gamma^{-\frac{1}{2}}\, n^s \pi)_{,s} + (\sqrt{-g}\,\gamma^{\gamma s}\hat{\phi})_{,r,s} - \frac{1}{2}\sqrt{-g}\,\frac{\partial V}{\partial \hat{\phi}(x)}$$

Now we make use of equation (A.11), (A.14), (A.5) and (3.21) to get

$$- \partial_o(\sqrt{-g}\, g^{\mu o}\hat{\phi}_{,\mu}) = (-\sqrt{-g}\, n^{\mu}\hat{\phi}_{\mu} n^s)_{,s} + [\sqrt{-g}(n^r n^s + g^{rs})\hat{\phi}_{,r}]_{,s} - \frac{1}{2}\sqrt{-g}\,\frac{\partial V}{\partial \phi(x)}$$

$$= -(\sqrt{-g}\, n^s n^o \phi_{,o})_{,s} + (\sqrt{-g}\, g^{rs}\phi_{,\gamma})_{,s} - \frac{1}{2}\sqrt{-g}\,\frac{\partial V}{\partial \phi}$$

$$= (\sqrt{-g}\, g^{so}\,\phi_{,o})_{,s} + (\sqrt{-g}\, g^{rs}\phi_{,\gamma})_{,s} - \frac{1}{2}\sqrt{-g}\,\frac{\partial V}{\partial \phi}$$

$$= (\sqrt{-g}\, g^{\mu s}\phi_{,\mu})_{,s} - \frac{1}{2}\sqrt{-g}\,\frac{\partial V}{\partial \phi}$$

i.e.

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\,\hat{\phi}_{,\nu}) - \frac{1}{2}\sqrt{-g}\,\frac{\partial V}{\partial \hat{\phi}} = 0$$

Therefore we established equation (3.19)

In this transformation from the Schrodinger picture to Heisenberg picture the Einstein equation retains the forms (3.16) but of course the states are now time independent and the energy-momentum operator $T_{\mu\nu}$ instead of being given by the Schrodinger picture expression (A.31) now is given by the Heisenberg picture eqn (A.25).

Bianchi's identities require

$$\langle\psi_o|T^{\mu\nu}|\psi_o\rangle_{;\nu} = 0 \qquad (3.22)$$

which are an expression of the general covariance of the theory.

One of the chief virtues of the derivation from an action principle is that these conditions are guaranteed. They can, of course, be directly verified in the Heisenberg picture. In the Schrodinger picture, however, because of the time dependence of the states, they are non-trivial. Nevertheless they can also be checked by direct computation in the Schrodinger picture[11].

§3.4. *Incorporation of Explicit Non-linearities.*

The Schrodinger equation (3.12) is formally linear in $|\psi\rangle$ if it is considered in conjunction with the Einstein field equation (3. 16) then it becomes intrinsically non-linear.

If the non-linearity of the quantum time evolution is admitted then, at least in principle, there is no reason in not adding to the action integral (3.13) terms which incorporate it explicitly. Under the restriction that the field equation must not be higher than second order, then the most general invariant integral which we can add to (3.13) is the following

$$S^{n\ell} = \int d^4x\sqrt{-g}\ (F_1(\langle f_1(\phi)\rangle_\psi) + RF_2\ (\langle f_2(\phi)\rangle_\psi)) \tag{3.23}$$

where

$$\langle f_i(\phi)\rangle_\psi := \langle\psi(t)|f_i(\phi)|\psi(t)\rangle \quad i=1,2$$

and F_1; F_2 are some suitably chosen functions of the expectation values $\langle\psi|f_1|\psi\rangle$, $\langle\psi|f_2|\psi\rangle$ respectively. f_1 and f_2 are invariant functions of the field operator ϕ. In the absence of gravity the F_1 term will generate the non-linear model studied in the previous chapter.

From the action principle

$$0 = \frac{\delta S}{\delta\langle\psi|} = \frac{\delta}{\delta\langle\psi|}\ (S_g + S_\psi + S_{n,\ell}) \tag{3.24}$$

we get the explicitly non-linear Schrodinger equation

$$i\ |\dot{\psi}\rangle \ = (H - \alpha(t))|\psi(t)\rangle - \int_{\sigma(t)} d^3\xi\sqrt{-g}\,[F_1'(\langle f_1(\phi)\rangle_\psi)\ f_1(\phi) + R\ F_2'(\langle f_2(\phi)\rangle_\psi) f_2(\phi)]|\psi\ (t)\rangle \tag{3.25}$$

Here H is given as before by eqn (A.33) and prime over $F_{1,2}$ denotes a partial derivative.

To find the modified Einstein field equations we note that

$$\frac{\delta}{\delta g^{\mu\nu}_{(x)}} \int d^4x \sqrt{-g}\, F_1(<f_1(\phi)_\psi) = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} F_1(<f_1(\phi)>_\psi) \tag{3.26}$$

while to evaluate the variational derivative of the second term in (3.23) we make use of (A.38) with the following choice of the parameters.

$$A = 1 \qquad X = F_2(<f_2(\phi)>_\psi) \quad , \quad B = C = 0 = \Lambda \tag{3.27}$$

Since now X is independent of $g_{\mu\nu}$ therefore inserting (3.27) into (A.51) yields

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \int d^4x \sqrt{-g}\, R\, F_2\, (<f_1(\phi)>_\psi) = \sqrt{-g}\, [G_{\mu\nu}F_2 - (F_{2;\mu;\nu} - g_{\mu\nu}F_{2;}{}^{\lambda}{}_{;\lambda})]$$

Thus the variational principle

$$0 = \frac{\delta S}{\delta g^{\mu\nu}(x)} = \frac{\delta}{\delta g^{\mu\nu}(x)} (S_g + S_\psi + S_{n\ell})$$

will yield the following Einstein field equations.

$$\frac{1}{16\pi G_N} G_{\mu\nu} + \frac{1}{2} <\psi(t)|T_{\mu\nu}(x)|\psi(t)> - \frac{1}{2} g_{\mu\nu}F_1(<f_1(\phi)>_\psi) +$$
$$G_{\mu\nu}\, F_2 - (F_{2;\mu;\nu} - g_{\mu\nu}F_{2;}{}^{\lambda}{}_{;\lambda}) = 0 \tag{3.28}$$

Notice that the effect of one term on the left hand side of (3.28) can be thought of as a state-dependent change in the gravitational constant G_N. The equation can be written in the alternative form

$$G_{\mu\nu} = \frac{-8\pi G_N}{1+16\pi G_N F_2} \left[\langle T_{\mu\nu} \rangle_\psi - g_{\mu\nu} F_1 - 2\, F_{2;\mu;\nu} + 2\, g_{\mu\nu} F_{;\ \ ;\lambda}^{\ \lambda} \right] \qquad (3.29)$$

Because the term (3.23) added to the action integral is the integral of a scalar density, the conditions (3.22) are preserved. Now, of course, it is the right-hand side of (3.29) which has vanishing covariant divergence.

§3.5 *Interaction-like picture.*

The operator H in equation (3.25) is given by eqn (A.33) therefore it depends on the choice of the space like surface $\sigma(t)$. For this reason one tends to believe that the description based on the equation (3.25) can hardly be given a rigorous mathematical meaning. On the other hand we saw in section (3.3) that in the absence of the explicit non-linearity one could transform into manifestly covariant Heisenberg picture. In this section we want to explore yet another picture which is akin to the usual Tomonaga-Schwinger picture of the tranditional quantum mechanics. This picture is particularly useful in the presence of the explicit non-linearities.

The transition between the Schrodinger and interaction picture is implemented by the same operator U satisfying eqn (3.5). If we apply this transformation to the state vector $|\psi(t)\rangle$ satisfying eqn (3.25) we obtain a vector $|\psi(t)\rangle^{I}$ satisfying

$$i|\dot{\psi}(t)\rangle^{I} = H^{I}_{\psi}\,|\psi(t)\rangle^{I} \tag{3.30}$$

where

$$H^{I}_{\psi} = -\int_{\sigma(t)} d^{3}\xi\sqrt{-g}\,\left(F_{1}'(\langle f(\hat{\phi})\rangle_{\psi I}\; f_{1}(\hat{\phi}) + RF_{2}'(\langle f_{2}(\hat{\phi})\rangle_{\psi I})f_{2}(\hat{\phi})\right) \tag{3.31}$$

The field operator $\hat{\phi}$ appearing in H^{I}_{ψ} will of course be time dependent and satisfies covariant field equation (3.19).

Since the operator H^{I}_{ψ} does not contain the derivatives of the field operator ϕ it is possible to write the equation (3.30) in a local form which does not manifestly involve the unit normal n^{μ}. But first we try to write this equation in a form similar to (3.10). To do this we make use of equations (3.9), (A.15) and (A.7a) to get

$$i\int_{\sigma(t)} d^3\xi \dot{x}^\mu(\xi) \frac{\delta}{\delta x^\mu(\xi)} |\psi>^I = \int d^3\xi \; \dot{x}^\mu n_\mu \gamma^{\frac{1}{2}} \left[F_1'(<f_1>)f_1(\phi) + RF_2'(<f_2>)f_2\right] |\psi>^I$$

or

$$i \frac{\delta}{\delta x^\mu(\xi)} |\psi>^I = n_\mu \gamma^{\frac{1}{2}} \left[F_1'(<f_1>)f_1 + RF_2'(<f_1>)f_2\right] |\psi>^I \qquad (3.32)$$

This is not exactly the form which we wanted it to be. To bring this to the manifestly covariant Schwinger-Tomonaga form we introduce an element of the world volume $\delta\sigma(x)$ enclosed between the two surfaces $\sigma(t)$ and $\sigma(t+\delta t)$ at the point $\xi\epsilon\sigma(t)$. This is done by adopting the following definition for $\frac{\delta}{\delta\sigma(x)}$

$$\int_{\sigma(t)} d^3\xi\sqrt{-g} \; \frac{\delta}{\delta\sigma(\xi)} |\psi>^I = \int_{\sigma(t)} d^3\xi \; \dot{x}^\mu \frac{\delta}{\delta x^\mu(\xi)} |\psi>^I = |\dot{\psi}>$$

Then multiplying both sides of (3.32) by $\dot{x}^\mu$ and integrating over the surface $\sigma(t)$ we get

$$i \int_{\sigma(t)} d^3\xi\sqrt{-g} \; \frac{\delta}{\delta\sigma(\xi)} |\psi>^I = \int d^3\xi \; \dot{x}^\mu \, n_\mu \gamma^{\frac{1}{2}} \left[F_1'f_1(\phi) + RF_2'f_2\right] |\psi>^I$$

$$= -\int d^3\xi\sqrt{-g} \left[F_1'f_1(\phi) + RF_2'f_2\right] |\psi>^I$$

or equivalently,

$$i \frac{\delta}{\delta\sigma(\xi)} |\psi>^I = - \left[F_1'(<f_1(\phi)>_{\psi I}) \; f_1(\phi) + RF_2'(<f_2(\phi)>)f_2\right] |\psi>^I \qquad (3.33)$$

The right-hand side of this equation is independent of any attributes of the space-like surface σ. So is the left hand side. As in the case of Schwinger-Tomonaga equation of traditional quantum mechanics we believe that equation (3.33) exhibits the coordinate independence better than the Schrodinger equation (3.25).

The right-hand side of the Einstein equation (3.29) will retain its form. We must only remember that now the states are given by the solutions of (3.33) and the field operators $\hat{\phi}$ by the solutions of (3.19).

Strictly speaking all of the equations written in this chapter, as they stand, are meaningless. This is because of the appearance in $T_{\mu\nu}$, H, and H^I_ψ , the products of the field operators and the products of their derivatives at the same point of space-time. As it is well known these products have always infinite expectation values. Unless these infinities are removed the theory is ill defined. In the next chapters we will talk about the renormalization of semi-classical field theories. But the actual discussion of removing the infinities of the semi-classical gravity will be postponed until the last chapter.

We hope that the introduction of the extra terms like (3.23) might improve the divergence behaviour of the theory. It is also conceivable that these terms may affect the singularities of the solutions of classical Einstein equations. The reason for this latter hope stems from the following observation.

If the state $|\psi\rangle$ in eqn. (3.29) is a very populated quantum state then the expectation values like $\langle\psi|T_{\mu\nu}|\psi\rangle$ would be of order N where N is an estimate of the number of quanta in the state $|\psi\rangle$. Now if the functions F_1 and F_2 are chosen to be quartic in $|\psi\rangle$, say, (as is the case in Kibble's non-linear quantum mechanics), then $F_{1,2}$ will be proportional to N^2. For large values of N one may neglect the contribution of $\langle T_{\mu\nu}\rangle_\psi$. Then the right hand side of (3.29) will be independent of N. This independence of N is at least one qualitative difference between the theories involving the explicit non-linearities and the one described by the eqn. (3.16).

§3.6 *Comparison with fully quantized theory.*

As mentioned above the consistency of a semi-classical theory of gravity will imply a non-linear time evolution law for the quantum states of the system. (This non-linearity is of course intrinsic and has nothing to do with the F_1 and F_2 terms which we introduced in the preceeding section). The only way of removing this non-linearity and restoring the superposition principle of the orthodox quantum mechanics is to quantize gravity as well as the matter fields. In this section we will summarize some of the techniques of doing this.

There are several approaches to quantum gravity[2]. The one which is mostly used in the current investigations is the path integral formalism[12]. This is particularly suitable if the object to be quantized is $g_{\mu\nu}$. In this approach one writes a transition amplitude for the quantum jumps from a given initial 3-geometry (γ_1,σ_1) and field configuration ϕ_1 on σ_1 to a final 3-geometry (γ_2,σ_2) and field configuration ϕ_2 on σ_2. As in Appendix A γ denotes the induced metric on the 3-dimensional space-like surface σ. Then according to Feynman's path integral prescription such an amplitude can be written as a sum over all field configurations which take the appropriate values on the boundary surfaces σ_1 and σ_2, i.e.

$$\langle\gamma_2,\sigma_2,\phi_2|\gamma_1,\sigma_1,\phi_1\rangle = N\int d[g_{\mu\nu}]\, d[\phi] \quad e^{-iS\,[g,\phi]} \tag{3.34}$$

Here N is a normalization factor, $d\,[g]$, $d[\phi]$ represent some measure of integration in the field configuration spaces and $S\,[g,\phi]$ is the action integral of the gravitational field interacting with the real scalar matter field ϕ. The integral is taken over all fields which have the given values on σ_1 and σ_2.

The integral (3.34) as it stands is ill defined. This is because of several reasons which we enumerate in the following:

i) For the real action $S[g,\phi]$ the path integral will oscillate and will not converge. To remove this difficulty in flat space-time, one usually changes t to $-i\tau$, where τ is real. Then $-iS$ transforms into $-S$ and therefore the integral will be exponentially damped. To get physical results one must of course at the end of the calculations analytically continue back to the real Minkowskian time.

Although this procedure is problematic in the curved space-time but one usually adopts it. Hawking has some arguments for its justification(12).

ii) Not all of the $g_{\mu\nu}$ - field configurations in (3.34) are physically independent. The field configurations which are related to each other by general coordinate transformations must be counted only once. To do this one usually adds a gauge fixing term to S which picks up a chosen class of co-ordinate systems and thereby forbids us from doing general coordinate transformations.

iii) As is well known from the theory of abelian and non-abelian gauge fields the zero rest mass of these fields makes some of their components physically insignificant. For example in the case of $g_{\mu\nu}$ field only two out of ten components are physically relevant. In order to quantize these theories consistently (i.e. preserve the unitarity) one must isolate the physical components and keep the unphysical ones out of the game at all stages of the calculations. The universally accepted technique for doing this is to add a new term to S in eqn. (3.34), called the ghost term. This term represents the action integral for "fictitious vector particles" which propagate only in the internal closed loops and never appear on the external legs.

iv) After doing all of the modifications enumerated in (i) - (iii) above, the quantity defined by eqn. (3.34) is still infinite. The analogous quantity for other non-abelian gauge field theories in the absence of gravity is also infinite. However for these theories the infinity can be rendered finite by introducing a finite number of renormalization counter terms. The effect of these counter-terms is to renormalize the "bare" parameters of the original action integral.

Unfortunately the theory of gravity based on (3.34) (in which the pure gravitational part may be given by the Einstein action) is not renormalizable. This is infact one of the reasons which justifies our attempt in studying a universe in which everything is quantized except for gravity. Although even in the semi-quantized universe the problem of renormalization is not resolved in the traditional sense, nevertheless as we shall see below the theory will behave considerably better as far as the divergences are concerned.

Having mentioned the basic reasons of ambiguities of (3.34) we will assume that it is meaningful as it stands and proceed to its formal manipulations.

We notice that if our theory is going to have a well defined classical limit then the substantial contribution to the path integral

(3.34) must come from those field configurations which correspond to the solutions of the classical field equations. Let us write the classical action integral of the system in the following form

$$S\,[g,\phi] = S_g\,[g] + S_\phi\,[g,\phi] \tag{3.35}$$

Then assuming that $g^c_{\mu\nu}$ and ϕ^c are local extrema of this action we may write

$$g_{\mu\nu} = g^c_{\mu\nu} + \hat{g}_{\mu\nu} \qquad \phi = \phi^c + \hat{\phi} \tag{3.36}$$

where $\hat{g}$ and $\hat{\phi}$ are some quantum perturbations of the classical background. Now we may expand the quantum action $S[g,\phi]$ in a Taylor series about g^c and ϕ^c

$$S\,[g,\phi] = S\,[g^c,\phi^c] + S_2\,[\hat{g},\hat{\phi}] + \text{higher-order terms} \tag{3.37}$$

Here $S_2[\hat{g},\hat{\phi}]$ is quadratic in the perturbations $\hat{g},\hat{\phi}$. The approximation scheme in which the terms higher than second order are ignored is called alternatively as the one-loop, stationary phase or WKB approximation. Practically all of our useful and relatively precise understanding of quantum gravity (ignoring the supergravity theories) is limited to this level of approximation.

Perhaps it is the appropriate place to mention the simple fact that in our semi-classical theory - in the absence of self-interacting matter fields - the one loop approximation will coincide with the exact theory. On the other hand it is a well known fact that one needs only a finite number of subtractions to render the one-loop theory finite.

Now we introduce a new functional $\bar{S}\,[g_c,\phi^c,\hat{g},\hat{\phi}]$ by the following definition :

$$S_2[\hat{g},\hat{\phi}] + \text{higher order terms} := \bar{S}[g_c,\phi^c,\hat{g},\hat{\phi}] + \int d^4x' \left.\frac{\delta S_g[g]}{\delta g_{\sigma\tau}(x')}\right|_{g=g^c} \hat{g}_{\sigma\tau}(x') \qquad (3.38a)$$

Then (3.37) can be written as follows :

$$S[g,\phi] = S_g[g^c] + S_\phi[g^c,\phi^c] + \int d^4x' \left.\frac{\delta S_g[g]}{\delta g_{\sigma\tau}(x')}\right|_{g=g^c} \hat{g}_{\sigma\tau}(x) + \bar{S}[g^c,\phi^c,\hat{g},\hat{\phi}]$$

(3.38b)

If we keep $\hat{g}$ and $\hat{\phi}$ fixed and vary g and ϕ then we get

$$\frac{\delta S}{\delta g_{\mu\nu}(x)} = 0 = \frac{\delta Sg[g^c]}{\delta g^c_{\mu\nu}(x)} + \frac{\delta S\phi[g^c,\phi^c]}{\delta g^c_{\mu\nu}(x)} + \frac{\delta}{\delta g^c_{\mu\nu}(x)} \int d^4x' \left.\frac{\delta S_g[g]}{S g_{\sigma\tau}(x')}\right|_{g=g^c} \hat{g}_{\sigma\tau}(x') +$$

(3.39a)

$$+ \frac{\delta\bar{S}[g^c,\phi^c,g,\phi]}{\delta g^c_{\mu\nu}} .$$

$$\frac{\delta S}{\delta\phi} = \frac{\delta S\phi[g^c,\phi^c]}{\delta\phi^c} + \frac{\delta\bar{S}\,[g^c,\phi^c,g,\phi]}{\delta\phi^c} = 0 \qquad (3.39b)$$

We require that g^c and ϕ^c to be solutions of

$$\frac{\delta S_g\,[g^c]}{\delta g^c_{\mu\nu}(x)} + \frac{\delta S\phi[g^c,\phi^c]}{\delta g^c_{\mu\nu}(x)} = 0 \qquad (3.40a)$$

and

$$\frac{\delta S\phi[g^c,\phi^c]}{\delta\phi^c} = 0. \qquad (3.40b)$$

Assuming as usual

$$Sg[g^c] = \frac{1}{16\pi G_N} \sqrt{-g^c}\ R^c \tag{3.41}$$

and defining the classical energy momentum tensor by (3.36) then (3.40a) becomes

$$G^c_{\mu\nu} = -8\pi\ G_N\ T^c_{\mu\nu}\ (\phi^c, \phi^c) \tag{3.42}$$

Now by introducing the notations

$$-\sqrt{-g}^c\ F^{\mu\nu\sigma\tau}\ \hat{g}_{\sigma\tau}(x) := \frac{\delta}{\delta g^c_{\mu\nu}(x)} \int d^4x' \left.\frac{\delta Sg[g]}{\delta g_{\sigma\tau}(x')}\right|_{g=g^c} \hat{g}_{\sigma\tau}(x') \tag{3.43a}$$

and

$$-\frac{1}{2}\sqrt{-g}^c\ \hat{T}^{\mu\nu}(x) := \frac{\delta}{\delta g^c_{\mu\nu}(x)}\ \bar{S}\ [g^c, \phi^c, \hat{g}, \hat{\phi}] \tag{3.43b}$$

one obtains from (3.39)

$$F^{\mu\nu\sigma\tau}\ \hat{g}_{\sigma\tau}(x) = -\frac{1}{2}\ \hat{T}^{\mu\nu}(x) \tag{3.44a}$$

and

$$\frac{\delta S}{\delta\phi} = 0 \tag{3.44b}$$

$F^{\mu\nu\sigma\tau}$ is by definition a differential operator which involves only g_c, while $\hat{T}^{\mu\nu}$ is a function of g_c, ϕ^c, $\hat{g}$ and $\hat{\phi}$. The lowering and raising of the indices are carried out by $g^c_{\mu\nu}$ and its inverse $g^{\mu\nu}_c$ respectively. One may interpret the eqn (3.44) as describing the propagation of the quantum fields $\hat{g}_{\mu\nu}$ and $\hat{\phi}$ in the background gravitational field $g^c_{\mu\nu}$ generated by ϕ^c via (3.42). The quantized fields interact not only with each other but also with ϕ^c and the energy of this interaction is taken into account by the ϕ^c dependence of $\hat{T}^{\mu\nu}$ on the right hand side of (3.44a).

If we regard $\hat{g}_{\mu\nu}$ and $\hat{\phi}$ not as linear quantum field operators as we did above - but as merely some corrections to the classical path defined by g^c and ϕ^c, then we can insert (3.38b) into (3.34) and integrate over all possible deviations from the classical configurations. The resulting generating functional will be a functional of g_c and ϕ^c. It can be successively differentiated to yield the Green's functions for a theory in which the external lines are labelled by the fields g^c, ϕ^c and in the internal loops one has the circulation of the quantized fields $\hat{g}$ and $\hat{\phi}$ (as well as the ghost particles to guarantee the unitarity). If we require this functional to be stationary with respect to the first order variations of g^c and ϕ^c and assume at the same time that these fields do satisfy the equations (3.40) then we get

$$F^{\mu\nu\sigma\tau} < \hat{g}_{\sigma\tau}(x) > = -\frac{1}{2} < \hat{T}^{\mu\nu}(x) > \tag{3.45a}$$

$$< \frac{\delta S}{\delta \phi} > = 0. \tag{3.45b}$$

where the Schwinger average of any observable A is defined by

$$< A > := \frac{N\int d[\hat{g}]\, d[\hat{\phi}]\, A\, e^{iS}}{Z[g^c,\phi^c]} \tag{3.46}$$

Equation (3.45a) looks rather like our semi-classical equation (3.16). However this similarity is only formal as the content of (3.45) is by no means different from that of (3.44) and therefore in contrast to the semi-classical theory here the quantum superposition principle is not violated. Now it is clear that the effect of taking into account the contribution from gravitons as well as the matter loops in the right hand side of (3.45a) is the restoration of the linearity of the quantum time evolution law of the states. However, this is done in the expense of the non-renormalizability of the theory. On the other hand if we dispense with the superposition principle of the quantum states, then we obtain a theory with no graviton loops on the right hand side of (3.45a) and therefore less violent divergences.

We will return to further discussions of this point in the final chapter.

CHAPTER 4.

THE SIMPLEST SEMI-CLASSICAL MODEL (The $\lambda V\langle\phi^2\rangle$ theory).

§4.1 *Introduction.*

The theory developed in the preceeding chapter involves the highly divergent quantity $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle$ on the r.h.s. of the Einstein field equations. The divergent structure of this quantity, or rather that of $\langle o,out|T_{\mu\nu}|o,in\rangle$ has been under an intensive study in recent years. However, most of these studies have been done in a fixed background space time and all of them have led to the conclusion that the subtraction of infinities of this matrix element is ambiguous. We will postpone the proper study of this point to the final chapter. In the mean time we will try to show that if the dynamics of the background field is taken into account then the ambiguities can be removed.

In this chapter we examine this idea on the simplest of all semi-classical field theories, namely the theory of two interacting real scalar fields V and ϕ in which V is left classical while ϕ is quantized. The space-time is assumed to be flat Minkowskian. As in semi classical gravity here also the dynamics can be derived from a variational principle, which of necessity is formulated in the Schrodinger picture. We will assume that V and ϕ have a non-derivative interaction. Thus the only similarity of the $\lambda V\langle\phi^2\rangle$ model and the semiclassical theory of gravity is the non-linearity of quantum mechanics in both of them. In the next chapter we will study a model which is more akin to the semi-classical gravity.

The renormalization of the theory will be carried out by introducing counter-terms into the action integral. The constant infinite coefficients of these counter-terms will be fixed by imposing two physical conditions on the full model.

§4.2. *The action integral.*

We begin with the Schrodinger picture action integral

$$S = S_\psi + S_V + S_{\psi V} \tag{4.1}$$

where S_ψ is the action integral yielding the Schrodinger equation for the free real scalar field ϕ, i.e.

$$S_\psi = \int dt \quad \{\mathrm{Im} <\dot{\psi}|\psi> - <\psi|H_o|\psi> + \alpha(t) \; (<\psi(t)\;|\psi(t)> -1)\} \tag{4.2a}$$

with H_o defined as usual by

$$H_o = \frac{1}{2}\int d^3\underline{x} \; (\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2) \tag{4.2b}$$

and $\alpha(t)$ is a Lagrange multiplier.

Similarly S_V is the action integral for the real scalar classical field V, i.e.

$$S_V = -\frac{1}{2}\int d^4x \quad (\partial_\mu V\partial^\mu V + m^2V^2) \tag{4.3}$$

and $S_{V\psi}$ is the interaction term which we take to be

$$S_{V\psi} = -\frac{\lambda}{2}\int d^4x \;\; V(x) <\psi(t)|\; \phi^2|\psi\;(t)> \tag{4.4}$$

Here λ is a coupling constant. One must regard S as a functional of $<\psi|$, $|\psi>$, α and V. The Schrödinger picture field operators $\phi(\underline{x})$ and $\pi(\underline{x})$ are assumed to satisfy the canonical commutation relations

$$[\phi(\underline{x}),\; \pi(\underline{x}')] = i\delta^3(\underline{x} - \underline{x}') \tag{4.5}$$

$$[\phi(\underline{x}),\; \phi(\underline{x}')] \;=\; 0 \;=\; [\pi(\underline{x}),\; \pi(\underline{x}')] \tag{4.5b}$$

By requiring S to be stationary with respect to the first order variations of the independent variables $\langle\psi|$, α and V one gets the following equations

$$i|\dot{\psi}\rangle = (H_o + H_V)|\psi(t)\rangle - \alpha(t)|\psi(t)\rangle \tag{4.6a}$$

$$\langle\psi(t)|\psi(t)\rangle - 1 = 0 \tag{4.6b}$$

$$(\partial_\mu\partial^\mu - m^2)\, V = \frac{\lambda}{2}\langle\psi(t)|\phi^2(\underline{x})|\psi(t)\rangle \tag{4.6c}$$

In equation (4.6a) the interaction Hamiltonian H_V is defined by

$$H_V(t) := \frac{\lambda}{2}\int d^3\underline{x}\, V(\underline{x},t)\, \phi^2(\underline{x}) \tag{4.7}$$

We will absorb α in the definition of a zero point of H and thereby omit it from the Schrodinger equation (4.6a).

If we ignore the eqn (4.6c) for a moment then equation (4.6a) is the Schrödinger equation for a quantum real scalar field ϕ interacting with a fixed external source V. We make use of this observation to define a 'unitary' operator

$$U(t,t_o) = T\, \exp\left[-i\int_{t_o}^{t} H(t')\, dt'\right] \tag{4.8}$$

where T denotes a chronological product and $H = H_o + H_V(t)$.

Now we can define a Heisenberg picture state vector

$$|\psi_o\rangle = U^{-1}(t,t_o)|\psi(t)\rangle \tag{4.9a}$$

and Heisenberg picture field operators $\hat{\phi}(x)$, $\hat{\pi}(x)$

$$\hat{\phi}(x) = U^{-1}(t,t_o)\ \phi(\underline{x})\ U(t,t_o) \tag{4.9b}$$

$$\hat{\pi}(x) = U^{-1}(t,t_o)\pi(x)\ U(t,t_o) \tag{4.9c}$$

The time evolution of $\hat{\phi}$ and $\hat{\pi}$ will be given by Heisenberg's equations of motion

$$\dot{\hat{\phi}} = i\,[H,\hat{\phi}\,] \tag{4.10a}$$

$$\dot{\hat{\pi}} = i\,[H,\hat{\pi}\,] \tag{4.10b}$$

If we insert for H and make use of (4.5) then we can immediately show that eqns. (4.10) imply the following manifestly Lorentz covariant field equation for the quantized field ϕ

$$(\partial_\mu \partial^\mu - \mu^2 - \lambda V(x))\ \phi(x) = 0 \tag{4.11a}$$

Now we require that the field V which had been assumed to be known and fixed - to be given by the solutions of the eqn (4.6c) i.e.

$$(\partial_\mu \partial^\mu - m^2)\ V(x) = \frac{\lambda}{2} <\psi_o|\ \phi^2(x)|\psi_o> \tag{4.11b}$$

The Heisenberg state vector $|\psi_o>$ will of course satisfy the constraint eqn. (4.6b). The right hand side of equation (4.11b) is - as usual - infinite and one of our main problems is to render it finite in an unambiguous way. We will show in the following that in the absence of a dynamical eqn. like (4.11b) for the V-field there is no unambiguous way of rendering the quantities like $<\psi_o|\hat{\phi}^2(x)|\psi_o>$ finite. The same remark applies equally well to the propagation of a quantized field in a fixed background gravitational field. In other words although in a fixed $g_{\mu\nu}$ background field one can isolate the infinities of $<\hat{T}_{\mu\nu}>$ in an unambiguous way, the elimination of these infinities is always ambiguous up to finite additive transformations of the renormalization parameters. We will clarify this point in the sequal. In order to do this first we will have to develop a perturbation theory for formal handling of our equations.

§4.3. *The perturbation theory.*

Let us assume for the time being that V is a given external field. Then assuming that at $t \to -\infty$ the field $\hat{\phi}$ approaches the free field $\hat{\phi}_o$ one may write the equation (4.11a) in the following integral form

$$\hat{\phi}(x) = \hat{\phi}_o(x) - \lambda \int d^4y \, \Delta^R(x-y,\mu) \, V(y)\hat{\phi}(y) \tag{4.12}$$

This form of the equation incorporates the initial condition at $t = -\infty$. The initial field $\hat{\phi}_o$ satisfies the free Kelin-Gordon equation, while the retarded Green's function $\Delta^R(x-y,\mu)$ is a solution of the inhomogenous equation, i.e.

$$(\partial_\mu \partial^\mu - \mu^2) \, \Delta^R(x-y,\mu) = -\delta^4(x-x'), \tag{4.13}$$

We are mainly interested in the perturbation expansion of the diagonal matrix elements $<\psi_o|\hat{\phi}^2(x)|\psi_o>$. To achieve this we define the following two point function

$$\Phi^{\psi_o}(x,x') := \frac{1}{4} <\phi_o |\{ \hat{\phi}(x), \hat{\phi}(x')\}|\psi_o >, \tag{4.14}$$

Here $\{\hat{\phi}(x), \hat{\phi}(x')\}$ denotes the anticommutator of $\hat{\phi}(x)$ and $\hat{\phi}(x')$. We know that as long as $x \neq x'$ the two point function $\Phi^{\psi_o}(x,x')$ is finite and well defined. It is only in the coincidence limit $x \to x'$ that we recover $<\psi_o |\hat{\phi}^2(x)|\psi_o>$ and the infinities associated with it. Therefore in order for our expressions in all of the intermediate steps to make sense we will develop a perturbative scheme for order by order treatment of $\Phi^{\psi_o}(x,x')$ rather than $<\psi_o|\hat{\phi}^2(x)|\psi_o>$ and then only after isolation of the infinities we will let $x \to x'$. This procedure is called "regularization". A prescription for the elimination of the infinities will be called "renormalization".

If we substitute for $\hat{\phi}(x)$ from (4.12) into (4.14) we get

$$\Phi^{\psi_o}(x,x') = \Phi_o^{\psi_o}(x,x') - \lambda\int d^4y\ \Delta^R(x-y,\mu)\ V(y)\ \Phi^{\psi_o}(y,x') \qquad (4.15)$$

where $\Phi_o^{\psi_o}(x,x')$ is defined by

$$\Phi_o^{\psi_o}(x,x') := \frac{1}{4}\langle\psi_o|\{\hat{\phi}_o(x),\ \hat{\phi}(x')\}|\psi_o\rangle\ , \qquad (4.16)$$

The equation (4.15) incorporates the initial conditions only w.r.t. one of the variables, namely, x. It shows that for all finite values of t' as $t \to -\infty$ then $\Phi^{\psi_o}(x,x')$ approaches $\Phi_o^{\psi_o}(x,x')$. Now in order to incorporate the initial conditions w.r.t. the x' variable as well we substitute for $\hat{\phi}(x')$ from (4.12) into (4.16). Thus we get

$$\Phi_o^{\psi_o}(x,x') = \Phi_{oo}^{\psi_o}(x,x') - \lambda\int d^4y\ \Delta^R(x'-y,\mu)\ V(y)\ \Phi_o^{\psi_o}(x,y), \qquad (4.17)$$

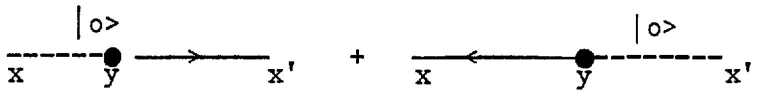
Here the zeroth order solution $\Phi_{oo}^{\psi_o}(x,x')$ is defined by

$$\Phi_{oo}^{\psi_o}(x,x') := \frac{1}{4}\ \langle\psi_o|\{\hat{\phi}_o(x)\ ,\ \hat{\phi}_o(x')\}|\psi_o\rangle\ , \qquad (4.18)$$

This two point function defines the initial value fo $\Phi^{\psi_o}(x,x')$ as t and t' both approach $-\infty$. Thus our perturbation expansion for (4.14) must be carried out in two steps. Starting with a given initial value defined by (4.18) we can solve (4.17) up to an arbitrary order in λ . In the second step we can substitute the solutions of (4.17) into (4.15) and iterate up to the desired order. These steps may be summarized conveniently by introducing the following set of diagrammatic notations

§4.4 *The Renormalization conditions.*

Until now we had assumed that V is a given background field. It is clear that if this was truly the situation it would be meaningless to talk about the counter term (4.23). This counter term is useful precisely because we are going to functionally differentiate it w.r.t. V which would be impossible for a fixed external V. Thus in the absence of the dynamics of the V-field the best one can do is to arbitrarily subtract the infinite graphs (4.22). However this can be done in an infinite number of different ways each leading to a different finite remainder . To see this point more clearly let us consider the following two graphs which would lead to the infinite loop graphs of (4.22b) when $\sigma \to 0$,



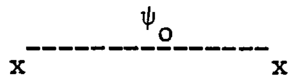
$$=$$

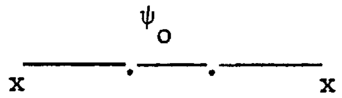
$$- \frac{\lambda}{4} \int d^4y \, V(y) \{ \Delta^R(x'-y,\mu) \, \Delta(x-y,\mu) + \Delta^R(x-y,\mu) \Delta(x'-y,\mu) \} =$$

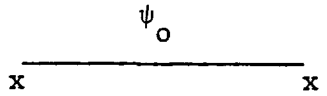
$$- \frac{\lambda}{4} \int d^4y \, \bar{d}^4q \, \bar{d}^4p \, \bar{d}^4q \, \frac{\bar{\delta}(k^2+\mu^2) \, \tilde{V}(q) e^{iq.y}}{p^2 + \mu^2} \{ e^{ip.(x'-y)+ik.(x-y)} + e^{ip.(x-y)+ik(x}$$

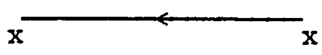
$$= - \frac{\lambda}{4} \int \bar{d}^4q \, \tilde{V}(q) \int \bar{d}^4k \, \frac{\bar{\delta}(k^2+\mu^2)}{q^2-2k.q} \left\{ e^{ix'.(q-k)+ix.k} + e^{ix(q-k)+ix'.k} \right\}$$

$$= - \frac{\lambda}{4} \int \bar{d}^4q \, e^{i\bar{x}.q} \, \tilde{V}(q) \int \bar{d}^4k \, \frac{\bar{\delta}(k^2+\mu^2)}{q^2-2k.q} \left\{ e^{i\sigma.(k-q/2)} + e^{-i\sigma(k-q/2)} \right\}$$

$\Phi_{oo}^{\psi_o}(x,x')$  (4.19a)

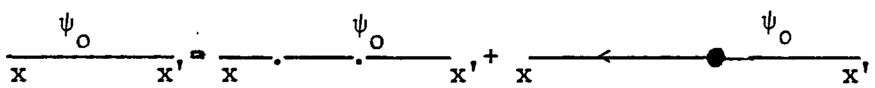
$\Phi_o^{\psi_o}(x,x')$  (4.19b)

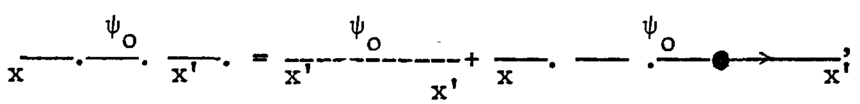
$\Phi^{\psi_o}(x,x')$  (4.19c)

$-\lambda\Delta^R(x-y,\mu)$  (4.19d)

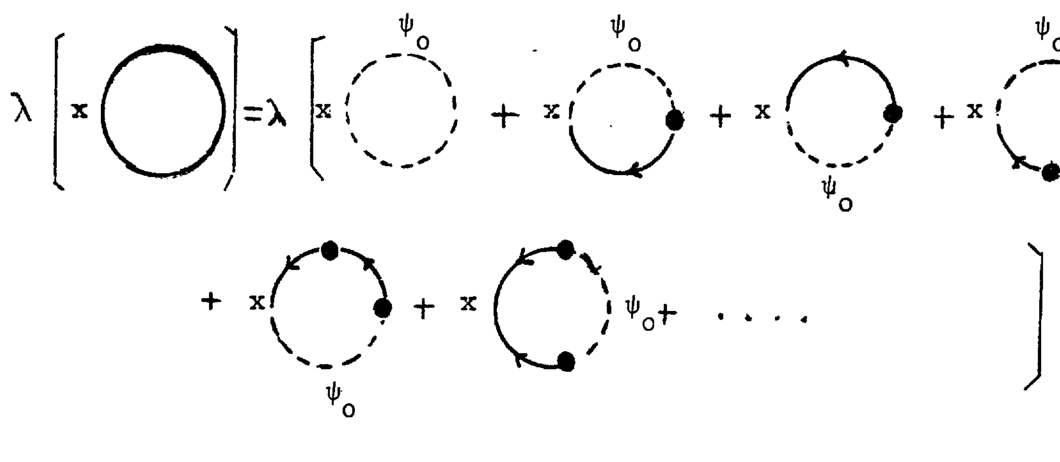
$V(x)$  (4.19e)

Note the symbol indicating dependence on the quantum state ψ_o. For technical reasons it is more convenient to associate the powers of the coupling constant λ with the "propagators" Δ^R rather than with the vertices. The arrow on Δ^R indicates the flow of time from past into the future. With the help of these conventions we can write equations (4.15) and (4.17) in the following form

 (4.20a)

 (4.20b)

In diagrammatic notation the r.h.s. of eqn (4.11b) will simply be given by λ multiplied by the coincidence limit of (4.20a), i.e.

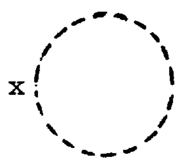
 (4.21)

To analyse the divergence structure of these loops we may choose $|\psi_o\rangle$ to be the vacuum of the free field $\hat{\phi}_o$. Then as is evident from eqn. (4.18) the dashed line on the r.h.s. of (4.21) will be associated with $\frac{1}{4}$ of $\Delta(x-y,\mu)$, where

$$\Delta(x-y,\mu) = \langle o|\{\, \phi_o(x) \,,\, \phi_o(x')\}|\, o \rangle$$

$$= \int \bar{d}^4k \;\; \bar{\delta}(k^2+\mu^2) \; e^{i.k(x-y)}$$

Therefore a naive power counting indicates that only diagrams up to one V-insertion are infinite, all the rest being finite. These diagrams are given by



$$|o\rangle = \frac{1}{4} \int_{\sigma\to o} \bar{d}^4k\bar{\delta}(k^2+\mu^2)\; e^{ik.\sigma} \,, \qquad (4.22a)$$

$$= -\frac{\lambda}{4} \int_{\sigma\to o} d^4y \;\; \Delta^R(x-y,\mu)\; V(y)\; \Delta(x'-y,\mu) \qquad (4.22b)$$

where

$$\sigma := x-x'.$$

Hence in order to make the theory finite we must introduce the following counter action

$$\Delta S = -\int d^4x\{\, \frac{1}{2}\, \delta m^2 \;V^2(x) \;\; + \lambda\; \delta\,\Phi\, V(x)\} \,, \qquad (4.23)$$

Here δm^2 and $\delta\Phi$ are constant (infinite) numbers, where $\bar{x} := \frac{x+x'}{2}$.

Clearly in the limit of $\sigma \to o$ the integral

$$I(q,\sigma) := \int d^4k \, \frac{\theta(k^2+\mu^2)}{q^2-2k.q} \left(e^{i\sigma.(k-q/_2)} + e^{-i\sigma.(k-q/_2)}\right) \tag{4.24a}$$

becomes infinite. Because of the Lorentz invariance in this limit it can only be a function of q^2. In fact if we make use of the δ-function to carry out the k^o-integration we get

$$I(q,\sigma) = \int \frac{d^3\underline{k}}{2\omega_{\underline{k}}} \left\{ \frac{1}{q^2-2k.q} \left(e^{i\sigma.(k-q/_2)} + e^{-i\sigma.(k-q/_2)}\right) + \right.$$

$$\left. \frac{1}{q^2-2\bar{k}.q} \left(e^{i\sigma.(\bar{k}-q/_2)} + e^{-i\sigma.(\bar{k}.q/_2)}\right) \right\}$$

where

$$\bar{k}^\mu = (-k^o,\underline{k}) = (-\omega_k,\underline{k}).$$

By changing $\underline{k}$ into $-\underline{k}$ in the second bracket of $I(q,\sigma)$ we get

$$I(q,\sigma) = \int \frac{d^3\underline{k}}{2\omega\underline{k}} \frac{1}{q^4-4(q.k)^2}\{ 2\, q^2(e^{ik.\sigma}+ e^{-ik\sigma}) \cos\frac{\sigma.q}{2} -$$

$$4iq.k(e^{ik.\sigma}- e^{-ik.\sigma}) \sin \frac{\sigma.q}{2}\} =$$

$$4\int \frac{d^3\underline{k}}{2\omega_{\underline{k}}} \frac{1}{q^4-4(qk)^2} \{ q^2 \cos k.\sigma \cos \frac{\sigma.q}{2} + 2q.k \sin k.\sigma \sin \frac{\sigma.q}{2} \} ,$$

In the limit of $\sigma \to o$ the second term inside the bracket will not contribute anything. Thus

$$I(q,\sigma) = 4q^2 \int \frac{d^3\underline{k}}{2\omega_{\underline{k}}} \frac{\cos k.\sigma}{q^4-4(q.k)^2} \tag{4.24b}$$

Now if we choose a frame in which $q = (q^o\ \underset{\sim}{o})$ then we get

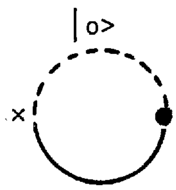
$$I(q,\sigma) = \sum_{n=o}^{\infty} (q^2+m^2)^n \ I_n(\sigma) \ , \tag{4.24c}$$

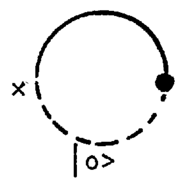
where

$$I_n = \frac{1}{n!} \frac{d}{d(q^2)^n} I(q^2,\sigma)\Big|_{q^2=-m^2} \tag{4.24d}$$

$$= 4(-1)^n \int \frac{d^3\underline{k}}{2\omega_{\underline{k}}} \frac{\cos k.\sigma}{(-m^2+4\omega_{\underline{k}}^2)^{n+1}} \qquad n = 0,1,2,3,\ldots\ldots ,$$

We see that as $\sigma \to o$ only I_o becomes infinite and all other terms are finite. Thus we obtain the following equation



$+$


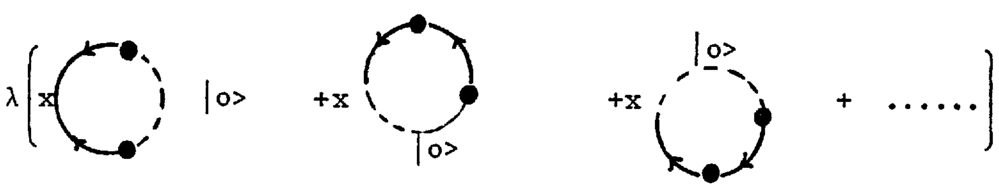
$$= -\frac{\lambda}{4} \int_{\sigma\to o} d^4q \ e^{iq.x} \tilde{V}(q) \ I_o(\sigma) \ - \ \frac{\lambda}{4} \sum_{n=1}^{\infty} I_n \int d^4q \ e^{iq.x} \ (q^2+m^2)^n \ \tilde{V}(q), \tag{4.25}$$

We see that if V was a fixed background field the subtraction of any combination of the r.h.s. of (4.25) which includes the first term would make the eqn. (4.21) finite (We assume that the constant infinity (4.22a) has already been eliminated). Since all of these subtractions must necessarily include the first term of the r.h.s. of (4.25) therefore their difference will be finite. This ambiguity will thus be reflected in the ambiguity of the finite remainders. In the absence of a dynamical equation for the V- field there does not seem to be any way of resolving this ambiguity. However, when the V-field satisfies the eq. (4.11b) then we can demand it to fulfil certain physical conditions. Let us re-write the equation (4.11b) and in doing this let us also take into account the contribution of (4.23). Hence if we insert from (4.11b) we get the following modified V-field equation

$$(\partial_\mu\partial^\mu - m^2)\, V(x) = \frac{\lambda}{4} \int_{\sigma\to o} \bar{d}^4k\ \tilde{\delta}(k^2+\mu^2)\, e^{ik.\sigma} + \lambda\delta\Phi +$$

$$(\delta m^2 - \frac{\lambda^2}{4}\ I_o(\sigma\to o))\ V(x) \ -$$

$$\frac{\lambda^2}{4} \sum_{n=1}^{\infty} I_n \int \bar{d}^n q\ \ e^{iq.x}\ (q^2+m^2)\tilde{V}(q) \ +$$

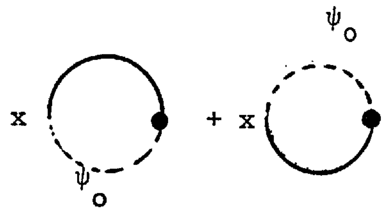


(4.26)

Before going any further we note that if instead of $|o\rangle$ we had considered any arbitrary normalizable $|\psi_o\rangle$ then we would get the same infinite parts but different finite parts. Thus in the first line of (4.26) we would get an additional finite term equal to

$$\psi_o \ - \frac{1}{4} \int_{\sigma\to o} \bar{d}^4k\ \tilde{\delta}\ (k^2 + p^2)\ e^{ik.\sigma} \ ,$$

and in the third line I_n must be replaced by $I_n^{\psi_o}$ defined by



$$+ \frac{\lambda}{4} V(x) I_o(\sigma \to o) ,$$

With these qualifications we can assume that the state on the r.h.s. of (4.26) is an arbitrary normalizable $|\psi_o\rangle$ rather than $|o\rangle$. This equation involves two arbitrary parameters δm^2 and $\delta\phi$ which may be fixed by imposing two physical conditions on the solutions of the equation. We therefore demand all of the solutions of the V-field equation to satisfy the following two conditions.

i). We require that when the initial V-field is $V_o = o$ and $|\psi_o\rangle$ is the vacuum state $|o\rangle$ of the quantum field $\hat{\phi}_o$ then nothing happens, i.e. $V(x) = 0$ and $\hat{\phi} = \hat{\phi}_o$. This requires that

$$\delta\Phi = - \frac{1}{4} \int_{\sigma\to o} d^4k \; \delta(k^2 + \mu^2) \; e^{ik.\sigma} , \tag{4.27}$$

ii) We again choose $|\psi_o\rangle = |o\rangle$ but take the initial V_o-field to be a non-zero solution of the Klein-Gordon equation. Now the r.h.s. of (4.26) no longer vanishes so that V is not equal to V_o. Its Fourier transform

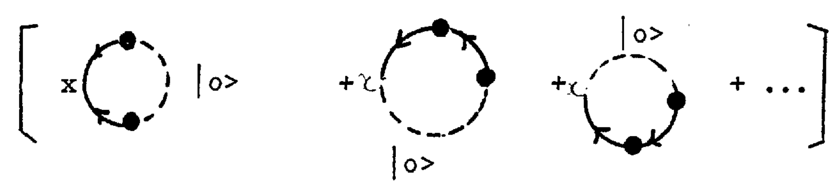
$$\tilde{V}(q) = \int d^4x \;\; e^{-iq.x} \; V(x),$$

is not confined, as $V_o(q)$ is, to the mass shell. However, we may require that $-q^2=m^2$ be a solution of the Fourier transform of (4.26). Then this condition - after we have substituted (4.27) into (4.26) - will require that

$$\delta m^2 = \frac{\lambda^2}{4} \quad I_o \ (\sigma \rightarrow o) \ , \tag{4.28}$$

Therefore we will have the following renomalized V-field equation for the initial $|o>$ state

$$(\partial_\mu \partial^\mu - m^2) V(x) = - \frac{\lambda^2}{4} \sum_{n=1}^{\infty} I_n \int d^4q \ e^{iq.x} \ (q^2+m^2)^n \ \tilde{V}(q) \ +$$

λ 

(4.29a)

Here I_n's are given by $\sigma \rightarrow o$ limit of (2.24d), i.e.

$$I_n = 4 \ (-1)^n \int \frac{d^3\underline{k}}{2\omega_{\underline{k}}} \ \frac{1}{(-m^2+4\omega_{\underline{k}}^2)^{n+1}} \qquad n = 1,2,3,... \tag{4.29b}$$

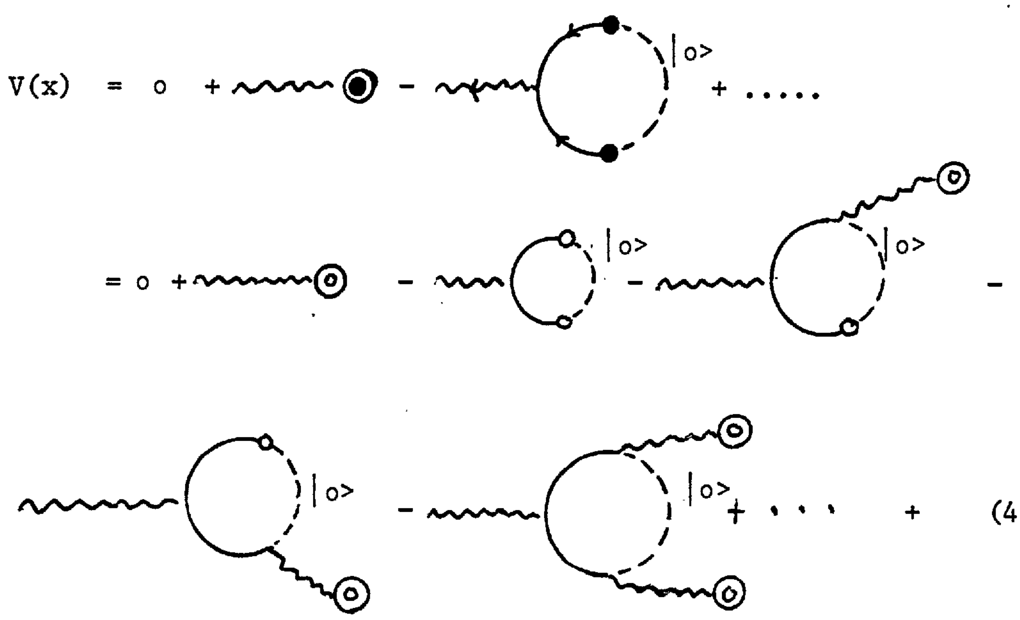
The equation (4.29a) can now be solved order by order in a power series of λ . By introducing the following additional diagrammatic notions

$$\frac{\lambda}{4} \sum_{n=1}^{\infty} I_n \int d^4q \ e^{iq.x} \ (q^2+m^2)^n \ \tilde{V}(q) \ = \odot \ , \tag{4.30a}$$

$$- \lambda \ \Delta^R(x-y,m) \quad = \quad x \rightsquigarrow y \ , \tag{4.30b}$$

$$V_o \quad = \quad o \ , \tag{4.30c}$$

We may write (4.29a) in the following integral form which incorporates the initial value V_o of the V-field



(4.31)

Provided we know the initial V-field, i.e. V_o, which is assumed to satisfy the free Kelin-Gordon equation with mass m then the series (4.31) will yield V to any desired order in λ. Having thus specified V we can then substitute it into (4.12) or equivalently into the Schrodinger equation (4.6a) and obtain a quantum mechanical problem of interaction of the quantum field ϕ with the given classical field V. Our theory then is ready to answer physical questions about the system. We may for example consider an initial state in which V vanishes asymptotically as $t \to -\infty$ while $|\psi_o>$ is some normalized many particle state and ask for the probability that the system will be found in some designated set of states in the distant future. This is a meaningful question provided that V also vanishes in some suitable sense as $t \to +\infty$, so that free "out" states exist (an assumption which can and of course must be checked). It is straightforward to compute the "scattering amplitude" $<\chi,\text{out}\,|\psi_o,\text{in}>$ for any designated process. As in chapter two here also the non-linear dependence of $<\chi,\text{out}\,|\psi_o,\text{in}>$ on $|\psi_o,\text{in}>$ will cause the single particle states to be unstable.

CHAPTER 5: A SEMI-CLASSICAL MODEL WITH DERIVATIVE COUPLING.

§5.1. *Introduction.*

The $\lambda V\langle\hat{\phi}^2\rangle$ model which we considered in the fourth chapter does not exhibit the real difficulties inherent in the semi-classical theory of gravity. These difficulties have essentially two inter-related sources:

i) The object of the prime importance in the semi-classical theory of gravity is $\langle\psi_o|\hat{T}_{\mu\nu}|\psi_o\rangle$. This symmetric second rank tensor appears in a natural way in our variational principle and it involves the product of the derivatives of the field operators at the same point of space-time. As is well known these products and hence their expectation values are ill defined.

ii) The $\lambda V\ \langle\phi^2\rangle$ model does not share the intrinsic non-linearity of the gravitational field equations. This non-linearity is a reflection of universality of the gravitational coupling. This difficulty however is only a technical one. In fact in both chapters five and six the linearized theories will suffice to clarify our techniques and our viewpoint.

We will deal with semi-classical gravity proper in the next chapter. In this chapter we would like to construct a model which exhibits only the first of the above mentioned aspects. In constructing this model the requirement of simplicity will be our only guide. We therefore begin with flat space-time with the Minkowskian metric $\eta_{\mu\nu}$ and consider two real scalar fields V and ϕ. We also assume that the propagation of the ϕ-field is governed by quantum mechanical rules whereas the field V is left classical. Then one of the simplest Lorentz invariant couplings between the two fields which involves the derivatives of the ϕ-field is the following

$$(\partial_\mu\partial^\mu - m^2)\, V(x) = \frac{\lambda}{2} <\psi_o|\partial_\mu \hat{\phi}(x)\partial^\mu \hat{\phi}(x)|\psi_o> +$$

$$\lambda\, (\mu^2 + \lambda\, V(x)) <\psi_o| \hat{\phi}^2(x)|\psi_o> \tag{5.1}$$

The r.h.s. of this equation has been written in the Heisenberg picture. In the next section we will see that there is infact a Schrodinger picture action integral yielding a V-field equation which reduces to (5.1) in the Heisenberg picture. There we shall also see why we have preferred this particular coupling rather than the seemingly simpler one in which the second term on the r.h.s. of (5.1) is absent. The corresponding Schrodinger (or Heisenberg) equation governing the dynamics of the $\hat{\phi}$-field is considerably more complicated than the one which we had for the $\lambda V <\hat{\phi}^2>$ model. Nevertheless we shall see that as far as the elimination of the infinities are concerned our programme is again applicable.

§5.2 *The equations of motion.*

We consider the semi-classical theory of two real scalar fields V and $\hat{\phi}$ whose dynamics is described by the following Schrodinger picture action integral

$$S[|\psi>,<\psi|,\alpha\ ,\ V\ ]=\int dt\{\ \mathrm{Im}\ <\dot{\psi}|\psi> \ -\ <\psi|H|\psi>\ +\ \ \alpha(t)(<\psi(t)|\psi(t)>\ -1)\}-$$
$$\frac{1}{2}\int d^4x\ (\partial_\mu V\partial^\mu V\ + m^2\ V^2) \tag{5.1a}$$

where H is defined by

$$H = \frac{1}{2}\int d^3\underline{x}[\ \frac{\pi^2(\underline{x})}{1+\lambda V(x)}\ +\ (1+\lambda V(x))(\nabla\phi(\underline{x}))^2\ +\ (1+\lambda V(x))^2\mu^2\phi^2(\underline{x})\ ] \tag{5.1b}$$

Here λ is a coupling constant. The reason for this particular choice of H will become clear shortly.

With considerations similar to those of §4.2 we can easily get the following equations of motion

$$i\ \frac{\partial}{\partial t}|\ \dot{\psi}(t)> \ =\ (H - \alpha(t))|\psi(t)> \tag{5.2a}$$

$$(\partial_\mu\partial^\mu - m^2)\ V(x) = \frac{\lambda}{2}\ <\psi(t)|\{\frac{-\pi^2(\underline{x})}{(1+\lambda V(x))^2}\ +\ (\nabla\phi(\underline{x}))^2\ +\ 2\mu^2(1+\lambda V(x))\phi^2(\underline{x})\}|\psi(t)> \tag{5.2b}$$

Now we would like to transform into the Heisenberg picture. Along similar lines to those of (§3.3) we can easily show that the Heisenberg field $\hat{\phi}$ satisfies the following covariant equation

$$(\partial_\mu\partial^\mu - \mu^2)\hat{\phi}\ \ -\ \lambda\mu^2 V\phi\ +\frac{\lambda}{1+\lambda V}\ \partial_\mu V\ \partial^\mu\hat{\phi}\ =\ 0 \tag{5.3a}$$

In this picture the r.h.s. of the V-field equation also becomes manifestly Lorentz invariant, i,e.

$$(\partial\partial_{\mu}^{\mu}-m^2)V(x) = \frac{\lambda}{2} <\psi_o|\partial_{\mu}\hat{\phi}(x)\partial^{\mu}\hat{\phi}(x)|\psi_o> +$$

$$\lambda\mu^2(1+\lambda V(x))<\psi_o|\hat{\phi}^2(x)|\psi_o> \tag{5.3b}$$

It is worth mentioning that if we introduce a field $g_{\mu\nu}$ by the définition

$$g_{\mu\nu} := \alpha\eta_{\mu\nu}, \tag{5.4a}$$

with

$$\alpha := 1 + \lambda V, \tag{5.4b}$$

then equation (5.3a) may be written in the following form

$$\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\; g^{\mu\nu}\partial_{\nu}) - \mu^2 \hat{\phi} = 0 \tag{5.5}$$

This is of course the main reason for the choice (5.1b) of the Hamiltonian. For a fixed V-field this equation may be derived from the usual Lagrangian density

$$L = -\frac{1}{2}\sqrt{-g}\;(g^{\mu\nu}\partial_{\mu}\hat{\phi}\partial_{\nu}\hat{\phi} + \mu^2\hat{\phi}^2), \tag{5.6}$$

In this form it is easy to discuss the divergence structure of the theory Infact it has been established that for a real scalar field operat or $\hat{\phi}$ whose dynamics in a given background $g_{\mu\nu}$ -field is given by (5.5) the most general counter-term which is needed for the elimination of the infinities has the following form.[14]

$$L = -\sqrt{-g}\;(\Lambda + AR + BR^2 + CR_{\mu\nu}R^{\mu\nu}) \tag{5.7}$$

where Λ,A,B and C are (infinite) constants and the R's are the curvature tensors associated with the metric $g_{\mu\nu}$[c.f. eqn A(59))] . However for a metric of the form (5.4) the $\int d^4x \sqrt{-g}\, R^2$ and $\int d^4x \sqrt{-g}\, R_{\mu\nu}R^{\mu\nu}$ differ only by a constant. (This is because of the vanishing of the Weyl tensor). Thus we need only the following

$$\Delta L \;=\; -\sqrt{-g}\;(\Lambda + AR + BR^2) \tag{5.8}$$

Up to now we have assumed thatthe V-field is given. However in order to fix the regularized forms of the constants Λ,A and B we will have to use the dynamic character of this field. This is indeed in conformity with our general idea. Thus first we must develop a perturbative scheme for our model. In principle we can do this exactly in the same way as we did in the preceeding chapter. However in practice it is more convenient to deal with the Feynman propagator rather than the retarded one. It is also worth mentioning that in dealing with the gravitational field it is more suitable to employ a regularization scheme which respects the general coordinate invariance of the theory. Although the covariant point splitting technique does have this property it nevertheless gives rise to direction dependent terms, which in order to be got rid of one must arbitrarily average over all directions[15]. The dimensional regularization scheme of 'tHooft and Veltman, on the other hand, have the property that it employs only one regularization paramater $\varepsilon = 2 - \frac{n}{2}$,with n being the space-time dimension, rather than four components of σ^μ which are needed in the covariant point splitting technique. (Here σ^μ is the tangent vector to the geodesic line connecting the neighbouring points x and x' and its length is half of the geodesic distance between the two points)[15]. For these reasons we will use the dimensional regularization in this and in the next chapter. To this end we define a two point function

$$\Delta^{\psi_o}(x-x') = i\langle\psi_o|T(\hat{\phi}(x)\,\hat{\phi}(x')|\psi_o\rangle \quad , \tag{5.9}$$

notice that when $|\psi_o\rangle = |o,in\rangle$ then Δ^{ψ_o} reduces to the exact Feynman propagator relative to the in-in vacuum states. As long as $x\neq x'$ this function is well defined and in both variables x and x' it satisfies the following inhomogenous equation

$$\left[\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\;g^{\mu\nu}\partial_\nu) - \mu^2\right]\Delta^{\psi_o}(x,x') = \frac{-\delta^4(x,x')}{\sqrt{-g(x)}}, \tag{5.10}$$

where, for the model studied in this chapter $g_{\mu\nu}$ is given by eqn. (5.4) If we substitute for $g_{\mu\nu}$ in terms of V and multiply both sides by $\sqrt{-g}$ we get the following equation

$$(\partial_\mu\partial^\mu-\mu^2)\Delta^{\psi_o}(x,x') = -\delta^4(x-x') - \lambda V(x)(\partial_\mu\partial^\mu-\mu^2)\Delta^{\psi_o}(x,x') + \lambda\mu^2 V(x)(1+\lambda V(x))\Delta^{\psi_o}(x,x') - \lambda\partial_\mu V(x)\partial^\mu\Delta^{\psi_o}(x,x') \;, \tag{5.11}$$

In the next chapter we will discuss a general way of expanding $\Delta^{\psi_o}(x,x')$ in terms of the zeroth order solution $\Delta_o^{\psi_o}(x,x')$ defined by

$$\Delta_o^{\psi_o}(x,x') := i\langle\psi_o|T\;(\hat{\phi}_o(x)\;\hat{\phi}_o(x'))|\psi_o\rangle \tag{5.12}$$

For the time being we will discuss only the linearized theory and take $|\psi_o\rangle$ to be the "in" vacuum $|o\rangle$. We will require that the first order solution of (5.11) approach the free Feynman propagator $\Delta_F(x-x')$ as both t and $t'\to-\infty$. Under this condition equation (5.11) may be written in the following form

$$\Delta(x,x') = \Delta_F(x-x') - \int d^4y\;\Delta^F(x-y)\;H(y)\;\Delta^F(x'-y) + \int d^4y\;\Delta^{(-)}(x-y)\;H(y)\,\Delta^{(-)}(x'-y), \tag{5.13}$$

here H(y) is the following linearized operator

$$H(y) := -\lambda V(y)\ (\partial_\mu \partial^\mu - \mu^2) + \lambda\mu^2 V(y) - \lambda \frac{\partial}{\partial y^\mu}(V(y))\ \frac{\partial}{\partial y^\mu}\ , \qquad (5.14)$$

and $\Delta^{(-)}$ is the negative freauency function. (In the next chapter we will give a general formula which reduces to (5.13) in the linearized theory). It is obvious that (5.13) satisfies the linearized form of (5.11) w.r.t. both of its arguments and approaches $\Delta_F(x-x')$ as t and $t' \to -\infty$.

We will be interested in the coincidence limit of $\Delta(x-x')$ and $\partial_\mu \partial^{\mu'} \Delta(x-x')$. It is not difficult to check that in this limit the term involving $\Delta^{(-)}$ is finite and it is only the first two terms of (5.13) which become infinite. By substituting the Fourier transforms of the $\Delta^{(-)}$, i.e.

$$\Delta^{(-)}(x-y) = i \int \bar{d}^4p\ \theta(p^o)\ \bar{\delta}\ (p^2+\mu^2)\ e^{-ip(x-y)}, \qquad (5.15)$$

we can show that

$$\Delta_h(x-x') := \int d^4y\ \Delta^{(-)}(x-y)\ H(y)\ \Delta^{(-)}\ (x'-y) =$$

$$- \lambda \int \bar{d}^4q\ \tilde{V}(q) \int \bar{d}^4p\ \ e^{-ix.p + ix'.(p+q)}\{\theta(p^o)\ \theta(-p^o-q^o)$$

$$\bar{\delta}.(p^2+\mu^2)\ \bar{\delta}\ (q^2+ 2p.q)\ (\mu^2 - \frac{q^2}{2})\}\ . \qquad (5.16a)$$

(The subscript h on Δ_h is to remind us that this function is a solution of the homogeneous equation). In writing (5.16 a)we made use of the δ-functions to substitute $-\mu^2$ for p^2 and $\frac{-q^2}{2}$ for p.q.

From (5.16a)we may easily evaluate

$$\eta^{\lambda\sigma} \frac{\partial}{\partial x^{\lambda}} \frac{\partial}{\partial x'^{\sigma}} \Delta_{h}(x,x') = \lambda \int d\!\!\!{}^{-}{}^4q\, \tilde{V}(q) \int d\!\!\!{}^{-}{}^4p\; e^{-ix.p+ix'.(p+q)}\{\theta(p^o)$$

$$\theta(-p^o-q^o)\delta\!\!\!{}^{-}(p^2+\mu^2)\delta\!\!\!{}^{-}(q^2+2p.q)(\mu^4 - \frac{q^4}{4})\}\ , \qquad (5.16b)$$

Now we can let x=x' in (5.16). Then we will need to evaluate the following integral

$$I(q^2) = \int d\!\!\!{}^{-}{}^4p\{\theta(p^o)\theta\,(-p^o-q^o)\ \delta\!\!\!{}^{-}(p^2+\mu^2)\delta\!\!\!{}^{-}\,(q^2+2p.q)\}\ ,$$

$$= \int \frac{d\!\!\!{}^{-}{}^4p}{2\,\omega_{\underline{p}}}\ \{\ \delta\!\!\!{}^{-}(p^o-\omega_{\underline{p}})\ \theta(-\omega_{\underline{p}}-q^2)\delta\!\!\!{}^{-}\,(q^2+2p.q)\}\ , \qquad (5.17)$$

where $\omega_{\underline{p}} := +\,(\underline{p}^2+\mu^2)^{\frac{1}{2}}$

To calculate this integral we choose a frame in which $q = (q^o,0)$. Then

$$\theta(-\omega_{p}-q^o)\ \delta(q^2+2p.q) = \frac{1}{2\omega_{p}}\ \delta\ (q^o + 2\omega)\ ,$$

and we get

$$I(q^2) = \frac{1}{4\pi}\ \frac{1}{|q^o|}\ \left(\frac{(q^o)^2}{4} - \mu^2\right)^{\frac{1}{2}}\ \theta(-q^o-2\mu)$$

$$= \frac{1}{4\pi}\ \left(\frac{1}{4} + \frac{\mu^2}{q^2}\right)^{\frac{1}{2}}\ \theta(-q^2-4\mu^2)\ \theta(-q^o) \qquad (5.18)$$

If we substitute (5.18) into (5.16) we get

$$\Delta_h(x) = -\lambda \int d\!\!\!{}^{-}{}^4q\ e^{iq.x}\tilde{V}(q)\ (\mu^2 - \frac{q^2}{2})I(q^2)\ , \qquad (5.19a)$$

$$\eta^{\lambda\sigma} \frac{\partial}{\partial x^{\lambda}} \frac{\partial}{\partial x'^{\sigma}} \Delta_{h}(x,x')\Big|_{x=x'} = \lambda\int d\!\!\!{}^{-}{}^4q\ e^{iq.x}\ \tilde{V}(q)(\mu^4 - \frac{q^4}{4})\ I(q^2) \qquad (5.19b)$$

Unlike these finite quantities the contribution of the first two terms of (5.13) - denoted by Δ_{inhomo} - is infinite. Indeed if we substitute the Fourier expansions of Δ^F in $\Delta_{inhomog}$ we get

$$\Delta_{inh}(x,x') = \int đ^4 p \; \frac{e^{ip.(x-x')}}{p^2+\mu^2} -$$

$$\lambda \int đ^4 q\, \tilde{V}(q) \int đ^4 p \; e^{ix.p-ix'.(p-q)} \frac{2\mu^2+p.(p-q)}{(p^2+\mu^2)[(p-q)^2+\mu^2]}, \tag{5.20a}$$

Similarly

$$\eta^{\lambda\sigma} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x'^\sigma} \Delta_{inh}(x,x') = \int đ^4 p \; \frac{p^2 \, e^{ip.(x-x')}}{p^2+\mu^2} -$$

$$\lambda \int đ^4 q\, \tilde{V}(q) \int đ^4 q \; e^{ix.p-ix'.(p-q)} \frac{p.(p-q)[2\mu^2+p.(p-q)]}{(p^2+\mu^2)[(p-q)^2+\mu^2]}, \tag{5.20b}$$

Clearly eqns (5.20) become ill defined when $x=x'$. To give them meaning first we must regularize the integrals. To this end first we make use of the Feynman identity

$$\frac{1}{ab} = \int_0^1 d\alpha \; \frac{1}{[a\,\alpha+(1-\alpha)b]^2}, \tag{5.21a}$$

with $a = (p-q)^2 + \mu^2$ and $b = p^2+\mu^2$, (5.21b)

to write

$$\int d^4 p \; \frac{f(p)}{(p^2+\mu^2)[(p-q)^2+\mu^2]} = \int_0^1 d\alpha \; d^4 p \; \frac{f(p)}{(p^2+2k.p+M^2)^2},$$

where

$$k := -\alpha q \quad \text{and} \quad M^2 := \alpha q^2 + \mu^2 \tag{5.21c}$$

then we apply the technique of dimensional regularization whereby we substitute $p^o = -ip^4$ and then make the space-time dimension an arbitrary complex number n. This amounts to the following

$$đ^4 p \to -i\, đ^n p \tag{5.21d}$$

Having gone through all of the steps (5.21a-21d) (and making use of the formulae in Appendix B), the equations (5.20) may be written in the following form

$$\Delta^{reg}_{inh}(x) = ic +$$

$$\frac{\lambda\Gamma(\varepsilon)}{16\pi^2} \int đ^n q\, e^{iq.x}\, \tilde{V}(q) \int_0^1 d\alpha\, \frac{\{q^2[\alpha^2(3+\varepsilon)-\alpha(3+\varepsilon)]-\varepsilon\mu^2)\}}{[\mu^2-q^2(\alpha^2-\alpha)]^{\varepsilon}}, \tag{5.22a}$$

$$\eta^{\lambda\sigma} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x'^\sigma} \Delta^{reg}_{inh}(x-x')\Big|_{x=x'} = -i\mu^2 c +$$

$$\frac{\lambda\Gamma(\varepsilon)}{16\pi^2} \int đ^n q\, e^{iq.x} \tilde{V}(q) \int_0^1 d\alpha\, \frac{\{a(\alpha)q^4+b(\alpha)\mu^2 q^2-\mu^4\}}{[\mu^2-q^2(\alpha^2-\alpha)]^{\varepsilon}}, \tag{5.22b}$$

Here $\varepsilon := 2-\frac{n}{2}$ is the regularization parameter. The new symbols a,b and c are defined by

$$c := -\int đ^n p\, \frac{1}{p^2+\mu^2} = \frac{\Gamma(\varepsilon)}{16\pi^2}\, \mu^2[1+\varepsilon(1-\log\mu^2)], \tag{5.23a}$$

$$a(\alpha) := \alpha^4(10+6\varepsilon)+\alpha^3(-20-12\varepsilon)+\alpha^2\left(\frac{21}{2}+\frac{13}{2}\varepsilon\right) - \alpha\frac{1+\varepsilon}{2}, \tag{5.23b}$$

$$b(\alpha) = \alpha^2(-6-6\varepsilon) + \alpha(6+6\varepsilon) - \frac{1+\varepsilon}{2}, \tag{5.23c}$$

Except for the first terms in eqns (5.22) which are constant infinities, in the V-dependent terms it is only maximum up to the q^4-terms which are infinite. To see this and to write (5.22) in a form which is more suitable for our renormalization prescription we expand these equations in a power series of (q^2+m^2). In doing this we make use of the following approximation

$$[\mu^2-q^2(\alpha^2-\alpha)]^{\varepsilon}=1+\varepsilon\log[\mu^2-q^2(\alpha^2-\alpha)]+\theta(\varepsilon^2), \tag{5.24a}$$

We also introduce the expansion coefficients $f_j(\alpha)$ through

$$\log[\mu^2-q^2(\alpha^2-\alpha)]=\sum_{j=o}^{\infty} f_j(\alpha)\quad(q^2+m^2)^j, \tag{5.24b}$$

where

$$f_o(\alpha) := \log[\mu^2+m^2(\alpha^2-\alpha)], \tag{5.24c}$$

Now with some straightforward algebraic calculations we can show that

$$-i\Delta_{inh}^{reg}(x) = c + c_o\lambda V - i\lambda\int d^nq\, e^{iq.x}\,\tilde{V}(q)\{c_1(q^2+m^2) + c_2(q^2+m^2)^2\} + \sigma(x), \tag{5.25a}$$

and

$$-i\,\eta^{\lambda\sigma}\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x'^\sigma}\Delta_{inh}^{reg}(x,x')\Big|_{x=x'} = -\mu^2 c + d_o\lambda V - i\lambda\int d^nq\, e^{iq.x}\,\tilde{V}(q)\quad\{d_1(q^2+m^2)+d_2(q^2+m^2)^2\} + \hat{\sigma}(x) \tag{5.25b}$$

The infinite numbers c_i and d_i are given in (B-7) to (B-12). The finite functions $\sigma(x)$ and $\hat{\sigma}(x)$ are defined by .

$$\sigma(x) = \frac{-3\lambda\Gamma(\varepsilon+1)}{16\pi^2} \int d^4q\, e^{iq.x}\, \tilde{V}(q) \int_0^1 d\alpha(\alpha^2-\alpha) \sum_{j=2}^{\infty} (q^2+m^2)^{j+1} [\, f_j(\alpha) - m^2 f_{j+1}(\alpha)\,], \tag{5.26a}$$

$$\hat{\sigma}(x) = \frac{-\lambda\Gamma(\varepsilon+1)}{16\pi^2} \int d^4q\, e^{iq.x}\, \tilde{V}(q) \int_0^1 d\alpha \sum_{j=1}^{\infty} (q^2+m^2)^{j+2} [\, a(\alpha)\, f_j(\alpha) + (\mu^2 b(\alpha) - 2m^2 a(\alpha))\, f_{j+1}(\alpha) + (m^4 a(\alpha) - m^2\mu^2 b(\alpha) - \mu^4) f_{j+2}(\alpha)\,], \tag{5.26b}$$

Finally we define $<o|\hat{\phi}^2(x)|o>^{reg}$ through

$$< o|\, \hat{\phi}^2(x)\,|\, o>^{reg} := -i\, \Delta^{reg}(x,x')\Big|_{x=x'}, \tag{5.27a}$$

similarly,

$$< o|\partial_\mu\hat{\phi}(x)\,\partial^\mu\hat{\phi}(x) >^{reg} := -i\,\eta^{\lambda\sigma}\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\sigma}\,\Delta^{reg}(x,x')\Big|_{x=x'} \tag{5.27b}$$

where

$$\Delta^{reg}(x,x') = \Delta^{reg}_{inh}(x,x') + \Delta_h(x,x'), \tag{5.28}$$

In substituting from (5.19) and (5.28) we must remember first to analytically continue back the finite part of Δ^{reg}_{inh} into the 4-dimensional Minkowskian space-time then add it with Δ_h.*

Now we substitute (5.27) as well as the contribution of the counter term (5.8) into the r.h.s. of the V-field equantion. Thus the modified V-field equation will be the following: (the contribution of the curvature terms have been calculated in (A.59) to (A.60)).

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* [On the question of reality of the regularized $<o|\hat{\phi}^2(x)|o>^{reg}$ and $<o|\partial_\mu\hat{\phi}(x)\partial^\mu\hat{\phi}(x)|o>^{reg}$ see §4 of Appendix B.]

$$(\partial_\mu\partial^\mu - m^2)V(x) = \lambda\{-\mu^2\frac{c}{2} + (\mu^2 c + 2\Lambda)(1+\lambda V) +$$

$$[(\mu^2 c_o + \frac{d_o}{2}) + m^2(-3A+18Bm^2) + g_o]\lambda V -$$

$$i\lambda\int d\!\!{}^-{}^n q\, e^{iq.x}\tilde{V}(q)\,[\,(q^2+m^2)^2(\mu^2 c_2 + \frac{d_2}{2} + 18B + g_2) +$$

$$(q^2+m^2)(\mu^2 c_1 + \frac{d_1}{2} + 3A - 36m^2 B + g_1)] + \Phi(x)\} \qquad (5.29)$$

where the finite numbers g_o, g_1 and g_2 are defined by

$$g_i = \frac{d}{d(q^2)^i}\tilde{T}(q^2)\Big|_{q^2=-m^2} \qquad i=o,1,2,$$

with $\tilde{T}(q^2)$ given by

$$\tilde{T}(q^2) = -i\int d^4x\, e^{-iq.x}\left(\mu^2\Delta_h(x,x') + \frac{1}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\mu}\Delta_h(x,x')\right)\Big|_{x=x'}$$

and the finite function $\Phi(x)$ is given by

$$\Phi(x) = \mu^2\sigma(x) + \frac{1}{2}\hat{\sigma}(x) -$$

$$i\int d\!\!{}^-{}^4 q\, e^{iq.x}\{\tilde{T}(q^2) - \sum_{i=o}^{2} g_i(q^2+m^2)^i\}, \qquad (5.30)$$

with σ and $\hat{\sigma}$ defined by (5.26)

§5.3 *The renormalization conditions.*

As stated several times before, in our program, in order to give an unambiguous expression for the finite remainders we fix the regularized renormalization parameters by imposing physical conditions on equation (5.29). In principle any three conditions may be imposed on the solutions of this equation. In practice however we choose the most convenient ones, e.g.

i) We demand that the coefficient of $(q^2+m^2)^2$ to vanish.

ii) If the initial V-field satisfies the free Klein-Gordon equation then the coefficient of (q^2+m^2) in the Fourier expansion of (5.29) must be -1.

iii) If the initial V-field is zero and the initial quantum state is $|o\rangle$ so must they be for all future times.

As we see all of these three conditions - particularly the last one - depend on the dynamic character of the V-field. For a fixed background field these conditions would be meaningless.

The first condition immediately implies that

$$B = -\frac{1}{18}\left(\mu^2 c_2 + \frac{d_2}{2} + g_2\right) \tag{5.31a}$$

The second condition plus equation (5.31a) yield

$$A = -\frac{1}{3}\left[\mu^2 c_1 + \frac{d_1}{2} + 2m^2\left(\mu^2 c_2 + \frac{d_2}{2} + g_2\right) + g_1\right] \tag{5.31b}$$

In order to implement the third condition we first evaluate the pole parts of A and B. We do this in order to show that the coefficient of λV in the 2nd line of (5.29) is of the form $-\mu^2\frac{c}{2} + \delta m^2$, where δm^2 is finite. Therefore the non-derivative <u>infinite</u> terms in (5.29) occur only through the combinations $(1+\lambda V)$, which is obtained from $\delta\sqrt{g} = \Phi(1+\lambda V)^2$ in the counter action. This is of course essential for the renormalizability of the theory. Otherwise we would have needed the non-polynomial counter-terms $\sum_{n=0}^{\infty} \Lambda_n V^n$, for the determination of whose constant coefficients we would have required an infinite number of physical conditions.

Since c_2 and g_2 are finite the pole part of B is the same as that of $\frac{1}{36}\ d_2$, which may easily be calculated from eqn (B.13) and (5.23) to be

$$p.p[B] = \frac{-1}{36}\ p.p\left[d_2\right] = \frac{-1}{36} \quad \frac{\Gamma(\varepsilon)}{16\pi^2} \times \frac{1}{4}\ , \tag{5.32a}$$

similarly

$$p.p[A] = \frac{1}{3}\ \frac{\Gamma(\varepsilon)}{16\pi^2} \quad \frac{+\mu^2}{4} \tag{5.32b}$$

Therefore the pole part of the coefficient of λV in the 2nd line of (5.29) will b

$$p.p[\ \mu^2 c_o + \frac{d_o}{2} + m^2(-3A+18m^2B)+g_o] = \frac{-\mu^4}{2} \quad \frac{\Gamma(\varepsilon)}{16\pi^2}\ ,$$

which is the same as $-\mu^2/_2\ p.p\ [\ c\ ]$. Thus if we make use of (5.31) in eqn. (5.29) we get

$$(\partial_\mu\partial^\mu - m^2)V = \lambda\{\ (\ \mu^2\ \frac{c}{2} + 2\Lambda)(1+\lambda V) + \delta m^2\lambda V + \Phi(x)\ \} \tag{5.33}$$

Here δm^2 is a finite number and is given by

$$\delta m^2 := (\mu^2 c_o + \frac{d_o}{2}) + m^2(-3A-18Bm^2) + g_o + \frac{\mu^2 c}{2}\ ,$$

Now we can impose the third renormalization condition on (5.33). This will imply

$$\Lambda = -\mu^2\ \frac{c}{4}\ , \tag{5.34a}$$

with

$$p.p[\Lambda] = \frac{-\mu^4}{4} \quad \frac{\Gamma(\varepsilon)}{16\pi^2}\ , \tag{5.34b}$$

§5.4 *Comparison with the results of 'tHooft.*

'tHooft has given a master formula which removes the infinities of the class of the models whose dynamics is described by the following Lagrangian density.(19)

$$L = \sqrt{-g}\,\{ -\tfrac{1}{2} (\partial_\mu \phi_i + N^{ij}_\mu \phi_j)\, g^{\mu\nu} (\partial_\nu \phi_i + N^{ik}_j \phi_k) + \tfrac{1}{2} \phi_i X_{ij} \phi_j \}, \tag{5.35a}$$

where $N^{ij}_\mu = - N^{ji}_\mu$ and $X_{ij} = X_{ji}$.

Note that L is bilinear in ϕ. The i,j are the internal indices. 'tHooft argues that all one-loop infinities as $n \to 4$ (or $\varepsilon \to 0$) are absorbed by the counter-Lagrangian

$$\Delta L = \frac{1}{8\pi^2 (n-4)} \sqrt{-g}\ \mathrm{Tr}\,\{ \frac{1}{24} Y^{\mu\nu} Y_{\mu\nu} + \frac{1}{4} X^2 - \frac{1}{12} RX + \frac{1}{120} R_{\mu\nu} R^{\mu\nu} I + \frac{1}{240} R^2 I \} \tag{5.35b}$$

where

$T_r\ I$ = number of fields

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + N_\mu N_\nu - N_\nu N_\mu$$

Now if we compare (5.35b) with (5.8) then we can make the following identifications

$$N = 0 \qquad Y = 0, \quad X = -\mu^2$$

If we also use the fact that for the metric (5.4) we have

$$\int d^4x \sqrt{-g}\, (R_{\mu\nu} R^{\mu\nu} - \tfrac{1}{3} R^2) = \text{total divergence}$$

then (5.35b) becomes

$$\Delta L = \sqrt{-g}\,\{\Lambda' + A'R + B'R^2\} \tag{5.36}$$

where

$$\Lambda' = \frac{1}{8\pi^2(n-4)}\,\frac{\mu^4}{4} = \frac{-\mu^4\Gamma(\varepsilon)}{4\times 16\pi^2} \tag{5.37a}$$

$$A' = \frac{1}{8\,\pi(n-4)}\,\frac{\mu^2}{12} = \frac{-\mu^2\Gamma(\varepsilon)}{12\times 16\pi^2} \tag{5.37b}$$

$$B' = \frac{1}{8\,\pi(n-4)}\,\frac{1}{24\times 6} = \frac{-\Gamma(\varepsilon)}{4\times 36\times 16\pi^2} \tag{5.37c}$$

Recalling that the definition of our Ricci tensor $R_{\mu\nu}$ differs by a minus sign from that of 'tHooft's we observe that (5.37) is exactly the same as the pole parts of Λ, A and B - given by eqn (5.34b), (5.32b) and (5.32a) respectively.

Thus the two counter Lagrangians (5.8) and(5.35a)differ only by finite quantities. This finite difference is actually unambiguously fixed by our renormalization conditions.

CHAPTER 6. THE RENORMALIZATION OF SEMI-CLASSICAL THEORY OF GRAVITY.

§6.1 *Introduction*

The model studied in Chapter 5 - if fully quantized - is clearly non-renormalizable. We saw however that leaving one of the fields classical permits us to eliminate the infinities by introducing a finite number of renormalization counterterms. This observation is of course of a paramount importance for the semi-classical theory of gravity.

In recent years there has been an extensive study of the propagation of the linear quantum fields in a fixed background space-time. All of these studies have indicated that in order to give the theory a meaning one must add the counterterm (5.7) to the action integral. Apart from the fact that for a fixed $g_{\mu\nu}$ field such a procedure is meaningless, as we saw in the last two chapters in the absence of the dynamics of the $g_{\mu\nu}$-field the finite remainders are also ambiguous.

In our theory however the existence of the action integral (3.13) and therefore the dynamical character of $g_{\mu\nu}$ solves both of these problems simultaneously. Thus the renormalized action

$$S^{ren} = Sg + S_{\psi} + \int d^4x \, \Delta L \tag{6.0}$$

with ΔL given by (5.7) will lead to the renormalized Einstein-field equation. By imposing four renormalization conditions on this equation we will be able to fix the regularized values of the parameters, Λ, A, B and C unambiguously and thereby obtain uniquely defined finite renormalized field equations.

The procedure will be exactly as in Chapter 5. Only the computations will be somewhat more involved.

§6.2 *The regularization of* $\langle\psi_o|\hat{T}_{\mu\nu}|\psi_o\rangle$

In recent years there has been considerable amount of interest in the matrix elements of $\hat{T}_{\mu\nu}$ -operators of the quantized fields propagating in a given back-ground space-time. These matrix elements are infinite and in order for them to be meaningfully handled their infinities must be isolated by some regularization scheme and then removed by a renormalization prescription. For a given quantized field the singular structure of the matrix elements of $\hat{T}_{\mu\nu}$ is independent of the choice of states although the finite parts depend on them(14). In order to study the divergence structure of these matrix elements it has proved technically simpler to work with $\langle \text{out};o|\hat{T}_{\mu\nu}|\text{in},o\rangle$ rather than with any other; and therefore all of the regularization schemes developed so far have been so constructed that they only yield the finite part of this particular matrix element. Of course if our purpose was just to investigate the nature of the counter-terms to be introduced for the elimination of infinities this would be quite sufficient. However, in the semi-classical theory of gravity an off-diagonal matrix element on the right hand side of the Einstein's equation in general would lead to complex solutions for $g_{\mu\nu}$ which from a physical point of view are very undesirable. On the other hand in our theory-based on a variational principle - we are forced to have the diagonal matrix elements of $\hat{T}_{\mu\nu}$ as the source of the gravitational field. This calls for a systematic treatment of the infinite as well as the finite parts of these quantities.

Let us consider the two point function $\Delta^{\psi_o}(x,x')$ defined by eqn (5.9). As in Chapter 5 we will define the $\langle\psi_o|\hat{\phi}^2(x)|\psi_o\rangle^{reg}$ and $\langle\psi_o|\partial_\mu\hat{\phi}(x)\,\partial_\nu\hat{\phi}(x)|\psi_o\rangle^{reg}$ through

$$\langle\psi_o|\hat{\phi}^2(x)|\psi_o\rangle^{reg} := [-i\Delta^{\psi_o}(x,x')]^{reg}_{x=x'} \quad , \tag{6.1a}$$

$$\langle\psi_o|\partial_\mu\hat{\phi}(x)\partial_\nu\hat{\phi}(x)|\psi_o\rangle^{reg} := \left[-i\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}\Delta^{\psi_o}(x,x')\right]^{reg}_{x=x'}, \tag{6.1b}$$

Having got these regularized quantities we can easily construct $\langle\psi_o|\hat{T}_{\mu\nu}|\psi_o\rangle$. Thus the main problem is to evaluate $\Delta^{\psi_o}(x,x')$ and then regularize it. For a real field satisfying the eqn (5.5) this function will satisfy the inhomogeneous equation (5.10). Our objective is to solve this equation with the condition that as t and t' $\to -\infty$ then $\Delta^{\psi_o}(x,x')$ approaches $\Delta^{\psi_o}_o(x,x')$ defined by (5.12). To this end we choose a class of coordinate systems in which the following conditions are satisfied

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}) = 0, \quad \nu\,;\,o,1,2,3 \tag{6.2}$$

We also introduce a new field $h^{\mu\nu}(x)$ through

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x), \tag{6.3a}$$

We may also write

$$\sqrt{-g}(x) = 1 + \frac{1}{2}h(x), \tag{6.3b}$$

We remark that in the linearized theory h(x) will be given by

$$h(x) = \eta_{\mu\nu}h^{\mu\nu}(x), \tag{6.3c}$$

In general however h will be a complicated function of $h^{\mu\nu}(x)$ which we do not need to specify. If we substitute from (6.2),(6.3a) and (6.3b) into (5.10) then this equation may be written in the following integral form

$$\Delta^{\psi_o}(x,x') = \Delta^{\psi_o}_o(x,x') - \int d^4y\,\Delta^R(x-y)\,H(y)\,\Delta^{\psi_o}(y,x') \tag{6.4a}$$

where H(y) is defined by

$$H(y) := -\frac{1}{2}h(y)\ (\partial_\mu\partial^\mu - \mu^2) + (1+\frac{1}{2}h(y))h^{\mu\nu}(y)\,\partial_\mu\partial_\nu \tag{6.4b}$$

Equation (6.4a) incorporates the initial condition w.r.t. the x variable. However as it stands it is not symmetric w.r.t. the interchange of x and x'. Thereforeit does not satisfy the eqn (5.10) w.r.t. its x' variable. On the other hand if we symmetrize this equation w.r.t. the interchange of x and x' then it will not satisfy (5.10) in neither of its variables. However, we may force it to satisfy this equation by symmetrising and adding a new term to it, i.e. if we write instead of (6.4a) the following equation

$$\begin{aligned}\overset{\psi}{\Delta}{}^{o}(x,x') = \Delta_o^{\psi o}(x,x') - \int d^4y\, \{&\Delta^R(x-y)\ H(y)\ \overset{\psi}{\Delta}{}^{o}(y,x') + \\ &\Delta^R(x'-y)\ H(y)\ \overset{\psi}{\Delta}{}^{o}(y,x) + \\ &\Delta^R(x-y)\ F^{\psi o}(y)\ \Delta^R(x'-y)\ \},\end{aligned} \tag{6.5a}$$

then we can choose F(y) such that $\overset{\psi}{\Delta}{}^{o}(x,x')$ meets all of our demands. Before going any further let us write (6.4a) in a slightly different form. We do this first by inserting (6.3a) and (6.3b) into (6.2) to get

$$\partial_\mu\ H^{\mu\nu} = 0, \tag{6.6a}$$

where

$$H^{\mu\nu} := \frac{1}{2}\eta^{\mu\nu}h - H^{\mu\nu} - \frac{1}{2}\ h\, h^{\mu\nu}, \tag{6.6b}$$

Next we substitute (6.4b) into (6.4a). After doing some integration by parts and making use of (6.6) we get

$$\overset{\psi_o}{\Delta}(x,x') = \overset{\psi_o}{\Delta_o}(x,x') - \int d^4y \{ \tfrac{1}{2}\mu^2 \Delta^R(x-y)\, h(y) \overset{\psi_o}{\Delta}(y,x') +$$

$$\frac{\partial}{\partial y^\mu} \Delta^R(x-y)\, H^{\mu\nu} \frac{\partial}{\partial y^\nu} \overset{\psi_o}{\Delta}(y,x')\} \quad ,$$

Now we symmetrize the expression under the integral sign w.r.t. the interchange of x and x' and add the additional terms. This yields

$$\overset{\psi_o}{\Delta}(x,x') = \overset{\psi_o}{\Delta_o}(x,x') - \int d^4y \{ \tfrac{1}{2}\mu^2 \Delta^R(x-y)\, h(y) \overset{\psi_o}{\Delta}(y,x') +$$

$$\frac{\partial}{\partial y^\mu} \Delta^R(x-y) H^{\mu\nu} \frac{\partial}{\partial y^\nu} \overset{\psi_o}{\Delta}(y,x') +$$

$$\frac{1}{2}\mu^2 \Delta^R(x'-y)\, h(y)\, \overset{\psi_o}{\Delta}(y,x) +$$

$$\frac{\partial}{\partial y^\mu} \Delta^R(x'-y)\, H^{\mu\nu}(y) \frac{\partial}{\partial y^\nu} \overset{\psi_o}{\Delta}(y,x) +$$

$$\Delta^R(x-y)\, \overset{\psi_o}{F}(y)\, \Delta^R(x'-y)\} \,, \tag{6.7}$$

where the unknown function $\overset{\psi_o}{F}$ is to be determined from the condition that (6.7) satisfies (5.10) w.r.t. x and x'. Thus if we act on (6.7) by $(\partial_\mu \partial^\mu{}_x - \mu^2)$, say, we obtain the following

$$\Delta^R(x'-x)\, \overset{\psi_o}{F}(x) = \int d^4y \{ \frac{\mu^2}{2} \Delta^R(x'-y) h(y)\, (\partial_x{}^2 - \mu^2) \overset{\psi_o}{\Delta}(y,x) +$$

$$\frac{\partial}{\partial y^\mu} \Delta^R(x'-y)\, H^{\mu\nu}(y) \frac{\partial}{\partial y^\nu} (\partial_x^2 - \mu^2) \overset{\psi_o}{\Delta}(y,x)\} \,, \tag{6.8}$$

Hence the perturbative solution for $\overset{\psi_o}{\Delta}(x,x')$ must be obtained in the following order. First we substitute the zeroth order solutions $\overset{\psi_o}{\Delta_o}$

of eqn (6.7) into (6.8) to get the first order solution for F then we replace it into (6.7) to obtain the first order solution for Δ^{ψ_o} which in turn must be put into (6.8) to yield the second order solution . These steps may be repeated up to any arbitrary order. It is worth mentioning that here we are dealing with perturbation expansion for Δ^{ψ_o}. At each stage of this expansion we may in turn expand $g_{\mu\nu}$ in a power series of some parameter, the Newtonian constant, say. In solving the equations of the semi-classical theory of gravity these two expansions must of course go hand-in-hand.

Now as an example (and for use in the next section) let us choose $|\psi_o\rangle = |o,in\rangle$ and calculate the $\Delta^{\psi_o}(x,x')$ up to lowest non-trivial order.

The lowest order solutions is of course trivial. It is the free Feynman propagator $\Delta_F(x-x')$. If we substitute this into (6.8) we get

$$\Delta^R(x-x')\ F(x) = \frac{-\mu^2}{2}\Delta^R(x'-x)\ h(x) + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\Delta^R(x'-x)\ H^{\mu\nu}(x),$$

To get the next order solution this expression as well as Δ_F for Δ^{ψ_o} must be replaced into (6.7). If in doing this we also make use of the identity

$$\Delta^R(x-y) = \Delta^F(x-y) - \Delta^{(-)}(x-y)$$

where $\Delta^{(-)}$ is the negative frequency function defined by (5.15) then after some algebraic manipulations we get

$$\Delta^{|o>}_{(1)}(x,x') = \Delta_F(x-x') -$$

$$\int d^4y \, \{ \frac{\mu^2}{2} \Delta^F(x-y)\, h(y)\, \Delta^F(x'-y) +$$

$$\frac{\partial}{\partial y^\mu} \Delta^F(x-y)\, H^{\mu\nu}(y) \frac{\partial}{\partial y^\nu} \Delta^F(x'-y) -$$

$$\frac{\mu^2}{2} \Delta^{(-)}(x-y)\, h(y)\, \Delta^{(-)}(x'-y) -$$

$$\frac{\partial}{\partial y^\mu} \Delta^{(-)}(x-y)\, H^{\mu\nu}(y) \frac{\partial}{\partial y^\nu} \Delta^{(-)}(x'-y)\} , \tag{6.9}$$

By introducing the function $\Delta_h(x,x')$ through

$$\Delta_h(x,x') := \int d^4y \{ \frac{\mu^2}{2} \Delta^{(-)}(x-y)\, h(y)\, \Delta^{(-)}(x'-y) +$$

$$\frac{\partial}{\partial y^\mu} \Delta^{(-)}(x-y)\, H^{\mu\nu}(y) \frac{\partial}{\partial y^\nu} \Delta^{(-)}(x'-y) \} \tag{6.10}$$

we may write (6.9) in the following form

$$\Delta^{|o>}_{(1)}(x,x') = \Delta_{inh}(x,x') + \Delta_h(x,x') \tag{6.11}$$

The subscript h on Δ_h is to remind us that it satisfies the homogeneous Klein-Gordon equation in either of its variables, i.e.

$$(\partial_\mu \partial^\mu - \mu^2)\Delta_h = 0.$$

Thus the Δ_h term in (6.11) is just to ensure the right boundary conditions satisfied by $\Delta^{|o>}_{(1)}(x,x')$

§6.3 *The linearized theory.*

The linearized theory is an approximation in which h is given by $\eta_{\mu\nu}h^{\mu\nu}$ and $H^{\mu\nu}$ is given by

$$H^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu} h - h^{\mu\nu} \tag{6.12a}$$

In this approximation any term involving second or a higher power of h will be ignored. Thus $g_{\mu\nu}$, the inverse to $g^{\mu\nu}$, will be given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6.12b}$$

where

$$h_{\mu\nu} := \eta_{\mu\lambda}\eta_{\nu\sigma}\, h^{\lambda\sigma} \tag{6.12c}$$

From now on we will restrict our attention only to this approximation. What we want to do in the remaining of this chapter is to study the general structure of solutions of the linearized Einstein's equations. To this end the first task is evidently to have an unambiguous expression for $<\hat{T}_{\mu\nu}>$. Therefore first we must renormalize the theory.

As mentioned in the beginning of the previous section as far as the divergent structure of $<\psi_o|\hat{T}_{\mu\nu}|\psi_o>$ is concerned the choice of the state $|\psi_o>$ is immaterial. However, it is the finite part which we are interested in; and that depends on the choice of $|\psi_o>$. We shall choose $|\psi_o>$ to be $|o,in>$.

Let us first calculate the contribution of Δ_h to $<\hat{T}_{\mu\nu}>_o$. If we substitute from (5.15) and (6.12a) into (6.10) we get

$$\Delta_h(x,x') = \int d^4q\, \tilde{h}^{\lambda\sigma}(q) \int d^4p \quad e^{-ix.p + ix'.(p+q)} \{\theta(p^o)\delta(p^2+\mu^2)$$

$$\theta(-p^o-q^o)\delta\,(q^2+2q.p)\, p_\lambda\, p_\sigma \quad \} \tag{6.13a}$$

From this we easily get

$$\partial_\mu \partial'_\nu \Delta_h(x,x') = \int d^4q\, \tilde{h}^{\lambda\sigma}(q) \int d^4p \quad e^{-ix.p + ix'.(p+q)} \tag{6.13b}$$

$$\{\theta(p^o)\ \delta(p^2+\mu^2)\theta(-p^o-q^o)\delta(p^2+2p.q)p_\lambda p_\sigma(p_\mu p_\nu + p_\mu q_\nu)\ \} \quad ,$$

Now we may let $x \to x'$. In this limit the p-integrals in (6.13) can easily be evaluated. Let us consider (6.13a). On the basis of Lorentz invariance the p-integral of (6.13a) can only have the following form

$$I_{\lambda\sigma} := \int d^4p\ \theta(p^o)\delta(p^2+\mu^2)\theta(-p^o-q^o)\delta(q^2+2p.q)\ p_\lambda p_\sigma =$$

$$a_1(q)\ \frac{q_\lambda q_\sigma}{q^2} \quad + \quad a_2(q)\ \eta_{\lambda\sigma}$$

where a_1 and a_2 are invariant functions of q. These functions can be determined by forming the following tensor contractions

$$\eta^{\lambda\sigma}\ I_{\lambda\sigma} = -\ \mu^2\ I(q)$$

$$= a_1(q) \ + 4a_2(q),$$

$$\frac{q^\lambda q^\sigma}{q^2} \quad I_{\lambda\sigma} = \frac{q^2}{4} \quad I(q)$$

$$= a_1(q) \ + \ a_2(q),$$

where I(q) has been defined by (5.17) and evaluated in (5.18). Thus we obtain the following value for a_1 and a_2

$$a_1(q) = \frac{1}{3} (\mu^2 + q^2) I(q),$$

$$a_2(q) = \frac{-1}{3} (\mu^2 + \frac{q^2}{4}) I(q)$$

Upon substitution of a_1 and a_2 into $I_{\lambda\sigma}$ and the resultant expression into (6.13a) we get

$$\Delta_h(x) = \frac{1}{6} \int d^4q \, e^{iq.x} \tilde{h}(q) \, (-\mu^2 + \frac{q^2}{2}) \, I(q) \, , \tag{6.14}$$

On the other hand the p-integral in (6.13b) - after being symmetrized w.r.t. the interchange of $\mu \leftrightarrow \nu$ - may be written in the following form

$$\begin{aligned} I_{\mu\nu,\lambda\sigma} := & \int d^4p \{\theta(p^o) \, \delta(p^2+\mu^2) \, \theta(-p^o - q^o) \quad \times \\ & \delta(q^2+2q.p) \, p_\lambda p_\sigma (p_\mu p_\nu + \frac{p_\mu q_\nu + p_\nu q_\mu}{2})\} \\ = & \; a_1 \, \eta_{\mu\nu} \, \eta_{\lambda\sigma} + a_2 \, (\eta_{\mu\lambda} \, \eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}) + \\ & a_3 \frac{q_\mu q_\nu}{q^2} \eta_{\lambda\sigma} + a_4 \frac{q_\lambda q_\sigma}{q^2} \eta_{\mu\nu} + \\ & \frac{1}{q^2} a_5 \, (q_\nu q_\sigma \, \eta_{\mu\lambda} + q_\mu q_\lambda \, \eta_{\nu\sigma} + q_\nu q_\lambda \, \eta_{\mu\sigma} + \\ & q_\mu q_\sigma \eta_{\nu\lambda}) + \\ & a_6 \frac{q_\mu q_\nu q_\lambda q_\sigma}{q^4} \, , \end{aligned} \tag{6.15}$$

where $a_1 \ldots . a_6$ are invariant functions of q. To determine these functions we evaluate the following tensor contractions.

$$\eta^{\mu\nu}\ \eta^{\lambda\sigma}\ I_{\mu\nu\lambda\sigma} = \mu^2\ (\mu^2 + \frac{q^2}{2})\ \ I(q)\ \ =$$

$$16a_1 + 8a_2 + 4a_3 + 4a_4 + 4a_5 + a_6\ ,$$

$$\eta^{\mu\lambda}\ \eta^{\nu\sigma}\ I_{\mu\nu\lambda\sigma} = \mu^2\ (\mu^2 + \frac{q^2}{2})\ I(q)\ \ =$$

$$= 4a_1 + 20\ a_2 + a_3 + a_4 + 10\ a_5 + a_6\ ,$$

$$\frac{q^\mu q^\nu}{q^2}\ \eta^{\lambda\sigma}\ \ I_{\mu\nu\lambda\sigma} = \frac{1}{4}\ \mu^2 q^2 I(q)\ \ =$$

$$= 4a_1 + 2a_2 + 4a_3 + a_4 + 4a_5 + a_6$$

$$\frac{q^\lambda q^\sigma}{q^2}\ \eta^{\mu\nu}\ \ I_{\mu\nu\lambda\sigma} = \frac{-q^2}{4}\ \ (\mu^2 + \frac{q^2}{2})\ I(q)\ \ =$$

$$= \ 4a_1 + 2a_2 + a_3 + 4a_4 + 4a_5 + a_6\ ,$$

$$\frac{q^\mu q^\lambda}{q^2}\ \eta^{\nu\sigma}\ I_{\mu\nu\lambda\sigma} = \frac{-q^4}{16}\ \ I(q)\ \ =$$

$$= a_1 + 5a_2 + a_3 + a_4 + 7a_5 + a_6,$$

$$\frac{q^\mu q^\nu q^\lambda q^\sigma}{q^4}\ \ \ I_{\mu\nu\lambda\sigma}\ \ = \frac{-q^4}{16}\ \ \ I(q)\ \ =$$

$$= \ a_1 + 2a_2 + a_3 + a_4 + 4a_5 + a_6\ ,$$

These equations can easily be solved to yield the following

$$a_1 = a_2 = -a_5 \ = \frac{1}{15}\ (\mu^2 + \frac{q^2}{4})^2\ I(q)$$

$$a_3 = \frac{1}{15} \left(\mu^2 + \frac{q^2}{4}\right) (-\mu^2 + q^2)\, I(q) \quad ,$$

$$a_4 = \frac{1}{15} \left(\mu^2 + \frac{q^2}{4}\right) \left(-\mu^2 - \frac{3}{2} q^2\right) I(q)$$

$$a_6 = \frac{1}{5} \left(\mu^2 + \frac{q^2}{4}\right)^2 I(q^2) - \frac{q^4}{16}\, I(q)$$

After substituting $a_1 \ldots . a_6$ into (6.15) and then putting $I_{\mu\nu\lambda\sigma}$ into (6.13b) we get

$$\partial_\mu \partial'_\nu \Delta_h(x,x')\Big|_{x=x'} = \frac{1}{15} \int d^4q\, e^{iq.x}\, I(q) \left\{ \left(\mu^2 + \frac{q^2}{4}\right) \times \right.$$

$$\left[\frac{\eta_{\mu\nu}}{2} (\mu^2 - q^2)\, \tilde{h}(q) + \frac{q_\mu q_\nu}{q^2} \left(- \frac{3}{2} \mu^2 + \frac{7}{8} q^2\right) \tilde{h}(q) - \right.$$

$$\left. \left. \frac{15}{32} q_\mu q_\nu q^2 \frac{1}{\mu^2 + \frac{q^2}{4}}\, \tilde{h}(q) + 2 \left(\mu^2 + \frac{q^2}{4}\right) \tilde{h}_{\mu\nu}(q) \right] \right\} \tag{6.16}$$

To calculate the contribution of Δ_h to $< o|\hat{T}_{\mu\nu}|o>$, we substitute (6.14) and (6.16) into the following*

$$<o|\ \hat{T}_{\mu\nu}(x)|o>_h = -i \text{ limit} \left\{ - \frac{\partial}{\partial x^{(\mu}} \frac{\partial}{\partial x'^{\nu)}} + \right.$$

$$\left. \frac{1}{2} \eta_{\mu\nu} \left(\eta^{\lambda\sigma} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x'^\sigma} + \mu^2\right) \right\} \Delta_h(x,x') \quad ,$$

-----

* On the question of the reality of $<o|\ \hat{T}_{\mu\nu}(x)\ |o>_h$ see §4 of Appendix B.

From which we get

$$< o|\,\hat{T}_{\mu\nu}(x)|o>_h \;=\; \frac{-i}{15}\int d^4q\; e^{iq.x}\; I(q)\;\{\,-\frac{q_\mu q_\nu}{q^2}(\mu^2+\frac{q^2}{4})(-\frac{3}{2}\mu^2+\frac{7}{8}\,q^2)\tilde{h}(q)$$

$$-\frac{15}{32}\;q_\mu q_\nu q^2\;\tilde{h}(q)\;+\;2\;(\mu^2+\frac{q^2}{4})^2\;\tilde{h}_{\mu\nu}\;(q)\}$$

$$-i\;\frac{\eta_{\mu\nu}}{10}\int d^4q\;e^{iq.x}\;I(q)\;\;\tilde{h}(q)\;\;(\frac{\mu^4}{3}+\frac{\mu^2q^2}{6}+\frac{q^4}{8})\;, \qquad (6.17).$$

In a similar manner we can calculate the contribution of Δ_{inh} to $<o|T_{\mu\nu}|o>$. We denote this by $<o|\hat{T}_{\mu\nu}|o>_{inh}$. The result of a relatively long calculation is the following

$$<o|\hat{T}_{\mu\nu}(x)|o>_{inh} \;=\; \frac{\Gamma(\varepsilon)}{16\pi^2}\;\;\frac{\mu^4}{4}\,[1+\varepsilon(\frac{3}{2}\;-\log\mu^2)\,]g_{\mu\nu}\;+$$

$$i\int d^nq e^{iq.x}\,\tilde{h}^{\lambda\sigma}(q)\;\{q^4[\,-A_{\mu\nu\lambda\sigma}+\frac{1}{2}\,\delta_{\mu\nu}(B^{\gamma}{}_{\gamma\lambda\sigma}+\mu^2E\delta_{\lambda\sigma}\,)]$$

$$-\;q_\mu q_\nu A_{\lambda\sigma}+q^2\,[-B_{\mu\nu\lambda\sigma}\;+\frac{1}{2}\delta_{\mu\nu}(B^{\gamma}_{\gamma\lambda\sigma}\;+\mu^2E\delta_{\lambda\sigma}\,)]$$

$$-\;\Phi_{\mu\nu}(x)\;\;+\;\frac{\delta_{\mu\nu}}{2}\,[\mu^2\Phi(x)\;\;+\Phi^{\gamma}_{\gamma}\;\;(x)\,]\;, \qquad (6.18)$$

Here $\varepsilon = 2 - \frac{n}{2}$ with n as the spacetime dimension. The constant tensors A's and B' are defined in §3 of Appendix B. The finite function $\Phi_{\mu\nu}$ is defined by

$$\Phi_{\mu\nu}(x) = -\int d^4q \; e^{iq.x} \, \tilde{h}^{\lambda\sigma}(q) \int_o^1 d\alpha \; D_{\mu\nu\lambda\sigma}(q^2,\alpha) \; , \tag{6.19a}$$

where

$$D_{\mu\nu\lambda\sigma}(q^2,\alpha) := -\varepsilon \sum_{j=1}^{\infty} (q^2)^j \{ f_j(\alpha) \, [(q^2)^2 \, a_{\mu\nu\lambda\sigma}^{(\alpha)} +$$

$$q^2 q_\mu q_\nu a_{\lambda\sigma}^{(\alpha)}] \; +$$

$$f_{j+1}(\alpha) \, [(q^2)^2 \; b_{\mu\nu\lambda\sigma}^{(\alpha)} + q^2 \, q_\mu q_\nu b_{\lambda\sigma}(\alpha)] \; +$$

$$f_{j+2}(\alpha) \quad (q^2)^2 \; c_{\mu\nu\lambda\sigma} \Big\} \tag{6.19b}$$

The $f_j(\alpha)$ in this expression are defined by (5.24b-c) with $m^2=0$. The tensors a's and b's are also defined in Appendix B. Similarly for the finite function $\Phi(x)$ we have

$$\Phi(x) = -\frac{1}{16\pi^2} \int d^4q \; e^{iq.x} \, \tilde{h}^{\lambda\sigma}(q) \int_o^1 d\alpha \sum_{j=1}^{\infty} (q^2)^{j+2}$$

$$[\; f_{j+1}(\alpha) \quad (\alpha^2 - \frac{3\alpha}{2} + \frac{1}{2}) - \frac{\mu^2}{2} \; f_{j+2}(\alpha)] \; , \tag{6.20}$$

Now we are in a position to calculate the total $<o|\hat{T}_{\mu\nu}(x)|o>$

$$<o|\hat{T}_{\mu\nu}(x) \; |o>^{reg} = \; <o|\hat{T}_{\mu\nu}(x)|o>^{reg}_{inh} + <o|\hat{T}_{\mu\nu}(x)|o>^{reg}_{h} \tag{6.21}$$

§6.4 *The renormalization.*

The eqn. (6.21) must be inserted into the modified Einstein field equations. These equations are obtained from the extremization of (6.0) w.r.t. $g_{\mu\nu}$. With the help of eqn. (A.38) and (A.51) we get

$$\frac{\delta S^{ren}}{\delta g^{\mu\nu}} = 0 = \frac{1}{16\pi G_N} G_{\mu\nu} + \frac{1}{2} <o| \hat{T}_{\mu\nu}|o>^{reg} -$$

$$[\frac{-\Lambda}{2} g_{\mu\nu} + AG_{\mu\nu} +$$

$$(B + \frac{C}{2}) \delta_{\lambda\sigma} \partial_\mu \partial_\nu \partial^2 h^{\lambda\sigma}(x) -$$

$$(B \delta_{\mu\nu}\delta_{\lambda\sigma} + \frac{C}{4} \Delta_{\mu\nu\lambda\sigma}) \partial^4 h^{\lambda\sigma}(x)]$$

or equivalently [recall that $h_{\mu\nu}(x) = -i\int d^n q\, e^{iq.x} \tilde{h}_{\mu\nu}(q)$]

$$0 = g_{\mu\nu}\{\frac{\Lambda}{2} + \frac{\Gamma(\varepsilon)}{16\pi^2} \frac{\mu^4}{8} [1 + \varepsilon (\frac{3}{2} - \log\mu^2)]\} +$$

$$\{(\frac{1}{16\pi G_N} - A) (- \frac{1}{4} \Delta_{\mu\nu\lambda\sigma} + \frac{1}{2} \delta_{\mu\nu}\delta_{\lambda\sigma})+$$

$$\frac{1}{2} [- B_{\mu\nu\lambda\sigma} + \frac{1}{2} \delta_{\mu\nu}(B^\gamma{}_{\gamma\lambda\sigma} + \mu^2 E\, \delta_{\lambda\sigma})] \partial^2 h^{\lambda\sigma}(x)$$

$$+ [(B \delta_{\mu\nu}\delta_{\lambda\sigma} + \frac{C}{4} \Delta_{\mu\nu\lambda\sigma}) +$$

$$\frac{1}{2} [- A_{\mu\nu\lambda\sigma} + \frac{1}{2}\delta_{\mu\nu}(A^\gamma{}_{\gamma\lambda\sigma} + A_{\lambda\sigma} + \mu^2 D\, \delta_{\lambda\sigma})] \partial^4 h^{\lambda\sigma}(x)$$

$$- \frac{1}{2} \Phi_{\mu\nu}(x) + \frac{\delta_{\mu\nu}}{4} [\mu^2 \Phi(x) + \Phi^\gamma{}_\gamma (x)] + \frac{1}{2} <o|\hat{T}_{\mu\nu}(x) |o>_h \qquad (6.22)$$

We mention that since $<o|\hat{T}_{\mu\nu}|o>_h$ involves I(q) - which together with all of its derivatives vanish at $q^2 = 0$, therefore this term will only contribute to the finite part of $<o|\hat{T}_{\mu\nu}(x)|o>$. The renormalization parameters Λ, A, B and C may be fixed as in Chapter 4 and 5, namely by requiring that all solutions of the linearized Einstein equations to satisfy four physical restrictions.

i) As before the first condition is imposed by demanding that if we start from an initially flat spacetime and the vacuum state of the $\hat{\phi}$-field then nothing should happen, i.e. the spacetime should remain flat and the quantum state should remain vacuum. This implies

$$\Lambda = - \frac{\Gamma(\varepsilon)}{16\pi^2} \frac{\mu^4}{4} [\Lambda + \varepsilon(\tfrac{3}{2} - \log\mu^2)] , \tag{6.23}$$

ii) The second condition will essentially say that the coupling constant G_N in equation (6.22) is the true Newtonian constant. This is implemented by requiring that if the initial $h^{\lambda\sigma}$ satisfies the free field equation

$$\frac{1}{16\pi G_N} (- \tfrac{1}{4} \Delta_{\mu\nu\lambda\sigma} + \tfrac{1}{2} \delta_{\mu\nu}\delta_{\lambda\sigma})\partial^2 h^{\lambda\sigma}(x) = 0, \tag{6.24}$$

then the coefficient of $\partial^2 h^{\lambda\sigma}$ in eqn (6.22) should always be the same as in (6.24). This immediately implies the following

$$A(- \tfrac{1}{4} \Delta_{\mu\nu\lambda\sigma} + \tfrac{1}{2} \delta_{\mu\nu}\delta_{\lambda\sigma}) - \tag{6.25}$$
$$\tfrac{1}{2} [- B_{\mu\nu\lambda\sigma} + \tfrac{1}{2} \delta_{\mu\nu} (B^{\gamma}{}_{\gamma\lambda\sigma} + \mu^2 E\delta_{\lambda\sigma})] = 0 ,$$

The solution of this equation yields A.

iii) and iv) The coefficients of $\partial_\mu \partial_\nu \partial^2 h^{\lambda\sigma}$ and $\partial^4 h^{\lambda\sigma}$ of course can not be determined by renormalizing any of the physical parameters of the original action integral. Originally our hope was that by an appropriate choice of the coefficients of these two terms we could perhaps get rid of some of the undesirable features of the theory. However as we shall see at the end of this chapter the worst of these, namely the existence of solutions which grow exponentially in time can not be avoided by any choice of these coefficients. For the time being we assume that they are given by the following.

$$(B + \frac{C}{2})\, \delta_{\lambda\sigma} - \frac{1}{2} A_{\lambda\sigma} = \frac{1}{2} \xi\, \delta_{\lambda\sigma} \,, \tag{6.26a}$$

$$\begin{aligned}(B\, \delta_{\mu\nu}\delta_{\lambda\sigma} + \frac{C}{4} \Delta_{\mu\nu\lambda\sigma}) & - \\ \frac{1}{2} [- A_{\mu\nu\lambda\sigma} + \frac{1}{2} \delta_{\mu\nu} (A^{\gamma}{}_{\gamma\lambda\sigma} + A_{\lambda\sigma} + \mu^2 D\, \delta_{\lambda\sigma})] & \\ = \frac{\eta}{2} \Delta_{\mu\nu\lambda\sigma} &,\end{aligned} \tag{6.26b}$$

where ξ and η are some fixed real finite members. These two equations must be solved for B and C.

It is not hard to see that the equations (6.23), (6.25) and (6.26) yield the same pole parts for Λ,A,B and C as those of 'tHooft in eqn (5.36b). Thus the difference between the two renormalization counterterms is a finite local function of the space-time point x.

If we now substitute from (6.23) - (6.26) into (6.22) we get the following renormalized linearized gravitational equation

$$G_{\mu\nu}(x) = -8\pi G_N <o|\hat{T}_{\mu\nu}(x)|o>^{ren} \tag{6.27a}$$

where

$$\begin{aligned} <o|\hat{T}_{\mu\nu}(x)|o>^{ren} = & -\Phi_{\mu\nu}(x) + \\ & \tfrac{1}{2}\eta_{\mu\nu}[\mu^2\Phi(x) + \eta^{\lambda\sigma}\Phi_{\lambda\sigma}(x)] + \\ & <o|\hat{T}_{\mu\nu}(x)|o>_h + \\ & \xi\partial_\mu\partial_\nu\partial^2 h(x) + \eta\,\Delta_{\mu\nu\lambda\sigma}\,\partial^4 h^{\lambda\sigma}(x), \end{aligned} \tag{6.27b}$$

here $\Phi_{\mu\nu}, \Phi(x)$ and $<o|\hat{T}_{\mu\nu}(x)|o>_h$ are respectively given by eqns (6.19a), (6.20) and (6.17). We notice that the sums over j in eqn. (6.19a) and (6.20) may be simplified. This is done by making use of the following idenity

$$\sum_{j=k}^{\infty} (q^2)^j f_j(\alpha) = \log\frac{\mu^2 - q^2(\alpha^2-\alpha)}{\mu^2} - \sum_{j=1}^{k-1} (q^2)^j f_j(\alpha),$$

If we also change the Feynman parameter into $x := \alpha - \frac{1}{2}$ then after some straightforward but lengthy calculation we get the following Fourier transform of (6.27b).

$$\frac{1}{2}\, q^2 \tilde{h}_{\mu\nu}(q) - \frac{1}{4}\, \eta_{\mu\nu} q^2 \tilde{h}(q) = -\, 8\pi G_N \langle \tilde{T}_{\mu\nu}(q) \rangle_h \; -$$

$$\frac{G_N}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \,\log\left[\left(1 + \frac{q^2}{4\mu^2}\right) - \frac{q^2}{\mu^2}\, x^2 \right] \{ q^4 \left[-\frac{1}{4} \left(x^4 - \frac{x^2}{2} + \frac{1}{16} \right) \tilde{h}_{\mu\nu} + \right.$$

$$\eta_{\mu\nu} \tilde{h} \left(\frac{7}{4} x^4 - \frac{33}{16} x^2 - \frac{27}{64} \right) + q^2 q_\mu q_\nu \tilde{h} \, (-2x^4 + 2x^2) \, +$$

$$q^2 \mu^2 \left[\frac{3}{4} \left(\frac{x^2}{2} - \frac{1}{8} \right) \tilde{h}_{\mu\nu} + \eta_{\mu\nu} \tilde{h} \left(-x^2 + \frac{1}{8} \right) \right] +$$

$$\mu^2 q_\mu q_\nu \tilde{h} \left(\frac{3}{2} x^2 - \frac{1}{8} \right) - \frac{\mu^4}{4} \tilde{h}_{\mu\nu} \} -$$

$$\frac{G_N}{2\pi} \left\{ \left(\frac{1}{80} + 2\eta \right) q^4 \tilde{h}_{\mu\nu} + \left(\eta - \frac{1}{80} \right) q^4 \eta_{\mu\nu} \tilde{h} \; + \right.$$

$$\left. \left(\frac{1}{120} + \xi \right) q_\mu q_\nu q^2 \tilde{h} + \frac{1}{24}\, q^2 \mu^2\, \tilde{h}_{\mu\nu} \right\} \qquad (6.28)$$

where $\langle \tilde{T}_{\mu\nu}(q) \rangle_h$ is the Fourier transform of $\langle o | T_{\mu\nu}(x) | o \rangle_h$ given by (6.17) . We mention that because of the factor I(q) in $\langle \tilde{T}_{\mu\nu}(q) \rangle_h$ this function vanishes at $q^2 = 0$. For this reason we see clearly that $q^2 = 0$ and $\tilde{h}_{\mu\nu} = 0$ and satisfying the gauge condition

$$q_\mu \tilde{h}^{\mu\nu}(q) = \frac{1}{2}\, q^\nu \tilde{h}\,(q) \;,$$

is a solution of (6.28). Physically these are the gravitational waves propagating with the velocity of light. It is not hard to see that almost all of the results obtained by Horowitz[16] for the coupling of classical gravity and quantum Maxwell field can infact be deduced from our equation (6.28). In particular we will check the existence of solutions which grow exponentially in time. These solutions correspond to $q = (\omega, o)$ where ω is a complex number. For these solutions the contribution of $\langle \tilde{T}_{\mu\nu}(q) \rangle_h$ again does vanish. If we substitute from

$$\tilde{G}_{\mu\nu}(q) = \frac{1}{2} q^2 \tilde{h}_{\mu\nu}(q) - \frac{1}{4} \eta_{\mu\nu} q^2 \tilde{h}(q) ,$$

for $\tilde{h}_{\mu\nu}$ in terms of $\tilde{G}_{\mu\nu}$ in eqn (6.28) then we can easily check that subject to the linearized Bianchi identities

$$\tilde{G}^{o\nu}(\omega) = 0 \qquad \nu = o,1,2,3,$$

the only consistent way of satisfying (6.28) with a non-zero $\tilde{G}_{\mu\nu}$ and $\omega = o$ leads to $\tilde{G}(q) := \eta^{\mu\nu}\tilde{G}_{\mu\nu}(q) = 0$.. Thus eqn (6.28) reduces to:

$$\frac{2\pi}{G_N}\omega^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \log\left[1 - \frac{\omega^2}{4\mu^2} + \frac{\omega^2}{\mu^2} x^2\right]\{ \frac{x^4}{2} +$$

$$\frac{x^2}{4} - \frac{1}{32}) \quad \omega^4 - \omega^2\mu^2 (x^2 - \frac{1}{4}) - \frac{\mu^4}{2} \} +$$

$$(\frac{1}{40} + \eta)\,\omega^4 - \frac{1}{12}\,\omega^2\mu^2 , \tag{6.29}$$

To investigate the question of existence of solutions to this equation first we carry out the x-integration to get

$$\frac{2\pi}{G_N}\omega^2 = \{ (\frac{23}{450} + \eta)\ \omega^4 - \frac{4}{45}\mu^2\omega^2 + \frac{8}{15}\mu^4 +$$

$$4\sqrt{\frac{4\mu^2-\omega^2}{-\omega^2}} \log\left(\sqrt{1-\frac{\omega^2}{4\mu^2}} + \frac{1}{2}\sqrt{\frac{-\omega^2}{\mu^2}}\right)\left(\frac{-\omega^4}{60} + \frac{2}{15}\omega^2\mu^2 - \frac{4}{15}\mu^4\right) \} \tag{6.30}$$

For a pure imaginary ω satisfying $\frac{\omega^2}{\mu^2} \gg 1$ one may approximate the logarithmic term by $\frac{1}{2} \log \frac{\omega^2}{\mu^2}$. Then if we neglect $\omega^2\mu^2$ and μ^4 terms on the r.h.s. of (6.30) we get

$$\frac{2\pi}{G_N}\ \omega^2 \simeq \omega^4\ \{(\frac{23}{450} + \eta) - \frac{1}{30}\ \log \frac{-\omega^2}{\mu^2}\}\ ,$$

It is rather evident that for any choice of η this equation has a solution for negative ω^2. It is also interesting to note that this equations has exactly the same structure as the stability equation of Horowitz (c.f. eqn 32 of ref. 16).

CONCLUSION.

In the first half of this thesis I tried to show that if one insists on integrating the classical Einsteinian theory of gravity into Schrodinger's quantum mechanics then the time evolution law of the quantum state of the system becomes implicitly non-linear. Once this was admitted then we obtained the coupled set of Schrodinger + Einstein field equation from a single variational principle. The action integral of this variational principle also has been used not only to incorporate the explicit non-linearities of the type introduced in §3.5 but also to accommodate the renormalization counterterms which are necessary to eliminate the infinities of the theory. We also saw in §3.6 that if one wishes to restore the linearity of the quantum time evolution law one must quantize gravity as well as the matter fields with which it interacts.

Having renormalized the theory in Chapter 6 now we may employ it to make physical predictions.

One of the areas in which this theory may be of considerable physical significance is cosmology. In fact the preliminary investigations on the problem of back-reaction on the metric of space-time of the particles created by a back-ground gravitational field out of the vacuum state of a quantized matter field have already been able to account - at least partially - for the present day homogeniety and isotropy of the large scale structure of the universe[17].

However most of the investigations of this kind have employed the semi-classical theory as a sort of approximation to a fully yet undiscovered quantum theory of gravity. It is principally for this reason that until now attention has been usually confined to the gravitational "vacuum polarization" effects. Comparatively there has been very little study of the semi-classical theory of gravity in which the quantum state is anything different from the vacuum. In all of these investigations the main object to be used is the matrix elements of $\hat{T}_{\mu\nu}$.

In the last ten years or so there has been a considerable amount of work on one particular matrix element of $\hat{T}_{\mu\nu}$ - namely $\langle o,out|T_{\mu\nu}|o,in\rangle$. These studies usually have been directed towards the isolation of the infinite structure of this quantity, the structure which one expects to be independent of the choice of the quantum state. However, if one is to avoid complex solutions to the classical Einsteinian equations one must use the diagonal matrix elements of $T_{\mu\nu}$ as the source term. Furthermore regarding the semi-classical theory as a theory in its own right (i.e. not as an approximation to a fully quantized theory) would demand a general technique of handling an arbitrary diagonal matrix element of the type $\langle\psi_o|T_{\mu\nu}|\psi_o\rangle$ in which $|\psi_o\rangle$ is an arbitrary normalizable Heisenberg state. We developed such a technique in §6.2 . This enables us to investigate the perturbative solutions of the theory for a wide range of choices of the initial state $|\psi_o\rangle$. We did infact obtain some qualitative results regarding the weak field limit in which $|\psi_o\rangle$ has been chosen to be $|o,in\rangle$. The outcome of these studies was that $(M, \eta_{\mu\nu}, |o\rangle\ )$ is an unstable solution of the theory. For a massive real scalar field - with no self-interaction except via its own gravitational field - this unstability occurs at very high frequencies. This might be an indication of the fact that our theory is essentially a theory of comparatively large space-time intervals. There is infact a wide spread belief that at the space-time intervals of the order of Planck length ($L^* = \frac{hG}{c^3})^{\frac{1}{2}} = 1.616\ 10^{-33}$cm) and Planck time ($T^* = (\frac{hG}{c^5})^{\frac{1}{2}} = 5391$ x 10^{-44} sec) the quantum effects of gravity do become important. However it could also indicate that the semi-classical theory of gravity has no weak field limit. After all being a Bose field it is only in the limit of intensely populated states that one expects the would be quantum theory of gravity to satisfy Bohr's correspondence principle, namely to be approximated by a classical field. In this case the results of the weak field approximation will certainly be doubtful.

One of the qualitative differences of the classical theory of gravity + quantum matter fields with a fully quantized theory is the instability versus stability of the single particle states. In a hypothetical situation in which the state of the universe consists of a single particle surrounded by its own gravitational field if the wave function of the particle becomes sharply localized at a particular time it can generate a strong enough gravitational field to produce additional particles. This phenomenon - if it happens - would be strictly due to the inherent non-linearity of the quantum time evolution. In other words in a fully quantized theory of gravity + matter fields one can not generate a many particle final state out of a single particle initial state merely by subjecting the particle to the influence of its own gravitational field. This is because of the stability of the momentum eigenstates and hence by superposition principle the stability of any single particle state obtained from their linear combination.

The rate of such a particle production can be expanded in a power series of the Newtonian coupling constant G_N . In this expansion the leading term will be the zeroth order term which will correspond to the orthodox linear quantum mechanics. The non-linearity of quantum mechanics will only appear in the coefficients of higher powers of G_N . Thus the smallness of G_N may account for the unobservability of such a non-linearity.

One other problem of cardinal importance which may be attacked in the framework of our perturbative expansion for $\langle\psi_o|T_{\mu\nu}|\psi_o\rangle$ is the possibility of a solution $(M, g_{\mu\nu}, |\psi_o\rangle)$ for which

$< \psi_o|\hat{T}_{\mu\nu}|\psi_o >$ does not satisfy the Hawking - Penrose energy conditions. In fact the preliminary studies of L. Parker and S. Fulling conducted on a self gravitating massive scalar field have shown that for an appropriate choice of $|\psi_o>$ such that the metric of M is restricted to be of the Robertson-Walker form

$$ds^2 = dt^2 - R^2(t) \sum_{i,j=1}^{3} S_{ij}(x^1,x^2,x^3)\, dx^i dx^j$$

- where $S_{ij}(x^1,x^2,x^3)$ is the (fixed) metric of a 3-sphere-one can obtain a solution R(t) which possesses the remarkable feature that the system does not exhibit the classical gravitational collapse but rather 'bounces off' the singularity of R=0 with the radius R(t) achieving a minimum of the Compton wavelength of the massive scalar particle. It is interesting to note that for a pion field this would mean a radius of 10^{-13} cm which is much greater than the Planck length 10^{-33} cm.

However, there are several technical ambiguities in the renormalization procedure of Fulling and Parker. It would be interesting to carry out an analogous kind of investigation in the framework of our approach.

Finally it might also be interesting to investigate the contribution of the explicit non-linearities introduced in §3.4.

APPENDIX A

In this appendix we collect all of the geometrical properties of the space time manifold which are essential for the derivation of the equations in Chapters 3 and 6.

Although the method of the presentation of the material in Sections A.1 and A.2 is rather new the results are, however, well known.

The material of the sections A.3, A.4 and A.5 are (so far as I know) original.

The section A.6 contains only well known results.

APPENDIX A

§A.1 *3+1 slicing of the space-time manifold.*

We shall assume that the space-time continuum is 4-dimensional globally hyperbolic manifold with signature (-,+,+,+). Let the metric $g_{\mu\nu}$ of M be given. Then the equations (A.1) given below will define a family $\sigma(t)$ of 3-dimensional space-like surfaces.

$$x^\mu = x^\mu(\xi^1,\xi^2,\xi^3,t) \tag{A.1a}$$

$$g_{\mu\nu}\, n^\mu n^\nu = -1 \tag{A.1b}$$

and

$$n_\mu\, x^\mu_{,r} = 0 \qquad r = 1,2,3 \tag{A.1c}$$

where

$$x^\mu{}_{,r} := \frac{\partial x^\mu}{\partial \xi^r} \tag{A.1d}$$

Here x^μ are the coordinate functions on a domain of M and ξ^μ, for a fixed t are the intrinsic coordinates on the surface $\sigma(t)$. We also introduce the lapse function N and the shift vector N by means of the following equations

$$\left.\frac{dx^\mu}{dt}\right|_{(\xi^r)\,=\,\mathrm{const}} := \dot{x}^\mu = Nn^\mu + N^r x^\mu_{,r} \tag{A.2}$$

The definitions (A.1) will give $\sigma(t)$ an induced metric geometry which enables us to decompose any tensor field on M into normal and tangential components to σ. Indeed for a fixed t we can write

$$ds^2 = g_{\mu\nu} dx^\mu\, dx^\nu = \gamma_{rs}\ d\xi^r\, d\xi^s$$

where

$$\gamma_{rs} = g_{\mu\nu}\, x^\mu{}_{,r}\, x^\nu{}_{,s} \qquad r,s = 1,2,3 \tag{A.3}$$

is the induced metric on the surface $\sigma(t)$.

For any tangent vector $A \epsilon T_x(M)$ we define the tangent and the normal components by

$$A^\mu = A_\perp n^\mu + x^\mu_{,r} A^r \tag{A.4a}$$

where

$$A_\perp = - n_\mu A^\mu \tag{A.4b}$$

and

$$A_r = A_\mu x^\mu_{,r} \tag{A.4c}$$

Since A_μ is a covariant vector field on the 3-dimensional manifold $\sigma(t)$ Therefore we can use the γ^{sr}, the inverse of the metric γ_{rs} to raise its index, i.e.

$$A^r = \gamma^{rs} A_s \tag{A.4d}$$

The inner product of any two tangent vector $A,B \epsilon T_x (M)$ can be written

$$A.B = g^{\mu\nu} A_\mu B_\nu = -A_\perp B_\perp + A_r B^r \tag{A.5}$$

Inserting from eqn (A.4b) and (A.4c) for $A_\perp, B_\perp$ and A_r, B_r we get

$$g^{\mu\nu} = -n^\mu n^\nu + \gamma^{rs} x^\mu_{,r} x^\nu_{,s} \tag{A.6}$$

Expressions similar to (A.4b) and (A.4c) can also be written for N and N^r which are introduced through equation (A.2).

$$N = -n_\mu \dot{x}^\mu \tag{A.7a}$$

$$N_r = g_{\mu\nu} \dot{x}^\mu x^\nu_{,r} \qquad N^r = \gamma^{rs} N_s \tag{A.7b}$$

Note that $\dot{x}^\mu$ is <u>not</u> a vector field on M, but (N^r) defines a vector field on the surface $\sigma(t)$.

§A.2 *Standard parameterization*

One way of parameterizing the surfaces $\sigma(t)$ is to write the equation (A.1a) in the following form:

$$\begin{cases} \xi^r = x^r & r = 1,2,3 \\ t = x^o \end{cases} \tag{A.8a}$$

Then

$$\dot{x}^o = 1 \quad \text{and} \quad \dot{x}^r = 0 \quad r = 1,2,3 \tag{A.8b}$$

and

$$x^o{}_{,r} = 0 \quad \text{and} \quad x^s{}_{,r} = \delta^s_r \tag{A.8c}$$

Inserting from equations (A.8) into eqn. (A.3) and (A.7) yields

$$\gamma_{rs} = g_{rs} \tag{A.9a}$$

$$N = -n_o \tag{A.9b}$$

$$N_r = g_{or} \tag{A.9c}$$

To evaluate γ^{rs} we make use of the following identity

$$g^{\mu\nu} g_{\nu\lambda} = g^{\mu o} g_{o\lambda} + g^{\mu p} g_{p\lambda} = \delta^\mu{}_\lambda \tag{A.10a}$$

We once let $\mu = r$ and $\lambda = s$ then

$$\begin{aligned} g^{ro} g_{so} + g^{rp} g_{sp} = \delta^r_s &= \gamma^{rp} \gamma_{ps} \\ &= \gamma^{rp} g_{ps} \end{aligned} \tag{A.10b}$$

Nexttime we let $\mu = 0, \lambda = s$ then we get

$$g^{oo} g_{os} + g^{op} g_{ps} = 0 \quad g_{so} = - \frac{g^{op} g_{ps}}{g^{oo}} \tag{A.10c}$$

Upon insertion from (A.10c) into (A.10b) we get

$$\gamma^{rs} = \frac{g^{ro} g^{so}}{-g^{oo}} + g^{rs} \tag{A.11}$$

Now we want to determine the components of the unit normal n in terms of $g_{\mu\nu}$. If we substitute from (A.8c) into (A.1c) then we get

$$n_r = 0 \qquad \text{(A.12a)}$$

Insertion of this result into (A.1b) yields

$$n_o = \frac{-1}{\sqrt{-g^{oo}}}$$

[We take the -ve root to agree with the Minkowski M, where n = (-1,o)]

Therefore n_μ is given by :

$$n_\mu = \left(\frac{-1}{\sqrt{-g^{oo}}} , o \right) \qquad \text{(A.12b)}$$

By raising the index of n_μ we get

$$n^\mu = g^{\mu o} n_o = \frac{-g^{\mu o}}{\sqrt{-g^{oo}}} \qquad \text{(A.12c)}$$

Upon insertion from (A.9c) and (A.11) into $N^r = \gamma^{rs} N_s$ we get

$$N^r = g^{rp} g_{op} - \frac{g^{ro} g^{po} g_{op}}{g^{oo}}$$

Now we make use of the eqn. (A.10a) with the values of $\mu = r$ and $\lambda = 0$ to get

$$g^{rp} g_{op} = -g^{ro} g_{oo}$$

By substituting this result into the previous equation we get

$$N^r = - \frac{g^{ro}}{g^{oo}} \left(g^{oo} g_{oo} + g^{po} g_{op} \right) = \frac{g^{ro}}{-g^{oo}} \qquad \text{(A.13)}$$

By comparing the equations (A.9b), (A.12) and (A.13) we obtain the following relations between n^μ and (N, N^r)

$$n^{\mu} = \left(\frac{1}{N}\ ,\ -\frac{N^{\mu}}{N}\right) \tag{A.14a}$$

$$n_{\mu} = (-N,\ 0) \tag{A.14b}$$

$$g^{\mu o} = \frac{n^{\mu}}{n_o} = \frac{-n^{\mu}}{N} \tag{A.14c}$$

Finally from the definition

$$g^{oo} = \frac{\text{cofactor of } g_{oo}}{g} = \frac{\det g_{rs}}{g} = \frac{\det\gamma_{rs}}{g} = \frac{\gamma}{g}$$

we get

$$\sqrt{-g} = \frac{\gamma^{\frac{1}{2}}}{\sqrt{-g^{oo}}}$$

i.e.

$$\sqrt{-g} = N\sqrt{\gamma} \tag{A.15}$$

In the sequal we will only use this standard representation.

§A.3 *The variation of $g_{\mu\nu}$.*

To obtain the results of chapter 3 we must vary $g_{\mu\nu}$ while keeping $\dot{x}^{\mu}$ and x^{μ} fixed. This will induce variations in n^{μ}, n_{μ}, and γ_s, Now we attempt to evaluate these variations. The final aim of these calculations is to obtain eq (A.37).

The variations of $g_{\mu\nu}$ will be subjected to the condition that they preserve the validity of equations (A.1b) and (A.1c). Then

$$0 = \delta(g_{\mu\nu}\, n^{\mu}n^{\nu}) = \delta g_{\mu\nu}n^{\mu}n^{\nu} + 2g_{\mu\nu}n^{\mu}\delta n^{\nu}$$

i.e.

$$n_{\mu}\ \delta n^{\mu} = -\frac{1}{2}\, n^{\mu}n^{\nu}\delta g_{\mu\nu} \tag{A.16}$$

On the other hand from equation (A6) we derive

$$\delta g^{\mu\nu} = -\delta n^{\mu}n^{\nu} - n^{\mu}\,\delta n^{\nu} + \delta\gamma^{rs}\ x^{\mu}_{,\gamma}\, x^{\nu}{}_{,\gamma}$$

By multiplying both orders of this relation by n_{μ} and making use of (A.1b-c) we get

$$\delta n^{\nu} = n_{\mu}\delta g^{\mu\nu} + n_{\mu}\ \delta n^{\mu}\, n^{\nu} \tag{A.17}$$

Substitution of

$$\delta g^{\mu\nu} = -g^{\mu\lambda}\, g^{\nu\sigma}\, \delta g_{\lambda\sigma} \tag{A.18}$$

and (A.16) into (A.17) yields

$$\delta n^{\nu} = -n^{\lambda}\ (g^{\nu\sigma} + \tfrac{1}{2}\ n^{\nu}n^{\sigma})\ \ \delta g_{\lambda\sigma} \tag{A.19}$$

By making use of

$$n_{\lambda} = n^{\nu}\, g_{\nu\lambda}$$

we get

$$\delta n_\lambda = \delta g_{\lambda\nu} n^\nu + g_{\lambda\nu} \delta n^\nu$$

We substitute from eqn. (A.18) for δn^ν to get

$$\delta n_\lambda = -\frac{n_\lambda}{2} n^\gamma n^\nu \delta g_{\gamma\nu} \tag{A.19}$$

Multiplying both sides of (A.19) by x^λ yields δN (c.f. eqn (A.7a)

$$\delta N = -\delta n_\lambda \dot{x}^\lambda$$

i.e.

$$\delta N = -\frac{N}{2} n^\gamma n^\nu \delta g_{\gamma\nu} \tag{A.20}$$

Now we subject both sides of equation (A.15) to the variations of $g_{\mu\nu}$, then we get

$$\frac{1}{2}\sqrt{-g}\; g^{\gamma\nu}\,\delta g_{\gamma\nu} = \delta N\,\gamma^{\frac{1}{2}} + \frac{1}{2} N\,\gamma^{-\frac{1}{2}}\delta\gamma$$

Upon insertion for $\sqrt{-g}$ from eqn. (A.15) and for δN from eqn (A.20) we get

$$\delta\gamma = \gamma(g^{\gamma\nu} + n^\gamma n^\nu)\,\delta g_{\gamma\nu} \tag{A.21}$$

The eqn (A.3) immediately yields

$$\delta\gamma_{rs} = x^\mu_{,r}\, x^\nu_{,s}\, \delta g_{\mu\nu} \tag{A.22}$$

and substituting this result into $\delta(\gamma_{rs}\gamma^{st}) = 0$ gives us

$$\delta\gamma^{rs} = -\gamma^{rp}\gamma^{sq}\, x^\mu_{,p}\, x^\nu_{,q}\, \delta g_{\mu\nu} \tag{A.23}$$

§A.4 *The energy-momentum tensor in terms of the covariant variables.*

Consider a real scalar field ϕ whose dynamics on $\mathbb{M}$ may be obtained from an action principle with the following lagrangian density

$$L = -\frac{1}{2}\sqrt{-g}\ (\nabla_\mu\phi\nabla^\mu\phi + V(\phi)) \tag{A.24}$$

Here $V(\phi)$ is a function of ϕ only. The energy-momentum tensor may be defined by

$$-\frac{1}{2}\sqrt{-g}\ T^{\mu\nu} = \frac{\partial}{\partial g_{\mu\nu}(x)}$$

i.e.;

$$T^{\mu\nu} = -\nabla^\mu\phi\nabla^\nu\phi + \frac{1}{2}g^{\mu\nu}(\nabla_\lambda\phi\nabla^\lambda\phi + V(\phi)) \tag{A.25}$$

The canonical conjugate to ϕ is defined as usual by

$$\pi = \frac{\partial L}{\partial\dot{\phi}} \tag{A.26}$$

where

$$\dot{\phi} = \frac{\partial\phi}{\partial t}$$

In what follows we will use the standard parameterization defined by eqn. (A.8) . Then inserting from (A.24) into (A.26) yields

$$\pi = -\sqrt{-g}\ g^{\mu o}\nabla_\mu\phi \tag{A.27}$$

By substituting from (A.14c) for $g^{\mu o}$ and (A.15) for $\sqrt{-g}$ we get

$$\pi = -\gamma^{\frac{1}{2}}\phi_\perp \tag{A.28}$$

where $\phi_\perp$ is defined by

$$\phi_\perp = -n^\mu\nabla_\mu\phi$$

Applying the formula (A.4a) to the vector $\nabla^\mu\phi$ yields

$$\nabla^\mu\phi = \phi_\perp n^\mu + \phi^{\ r} x^\mu_{\ ,r} \tag{A.29}$$

Here ϕ^r is given by eqn (A.4c) i.e.

$$\phi^r = \gamma^{rs}\,\phi_s$$

and

$$\phi_s = \nabla_\mu \phi\; x^\mu_{,s} = \partial_\mu \phi\; x^\mu_{,s} = \frac{\partial\phi}{\partial x^s}$$

Substituting from (A.28) into (A.29) expresses $\nabla^\mu\phi$ in terms of the Hamiltonian variables π and ϕ, i.e.

$$\nabla^\mu\phi = -\gamma^{-\frac{1}{2}}\,\pi n^\mu + \phi^r x^\mu_{,r} \tag{A.30}$$

Now in order to express $T^{\mu\nu}$ in terms of π and ϕ we substitute $\nabla^\mu\phi$ into eqn (A.25) and then we make use of Eqn. (A.1b) and (A.1c). Then we get the following

$$T^{\mu\nu} = -\gamma^{-1}(n^\mu n^\nu + \tfrac{1}{2}g^{\mu\nu})\pi^2 + \gamma^{-\frac{1}{2}}\phi^r(n^\mu x^\nu_{,r} + n^\nu x^\mu_{,r})\pi - (x^\mu_{,r}x^\nu_{s} - \tfrac{1}{2}g^{\mu\nu}\gamma_{rs})\phi^r\phi^s + \frac{g^{\mu\nu}}{2}V(\phi) \tag{A.31}$$

The Hamiltonian is defined by

$$H = -\int_{\sigma(t)} d\sigma_\mu T^\mu_\nu\, \dot{x}^\nu \tag{A.32}$$

Here $d\sigma_\mu := n_\mu \gamma^{\frac{1}{2}}\, d^3\xi$ is a surface element in the normal direction to $\sigma(t)$ For the standard parameterization defined by eqn. (A.8) eqn. (A.32) reduces to

$$H = \int_{\overset{o}{x}=\text{const}} d^3x\,\sqrt{-g}\;T_o{}^o$$

Upon substitution into $T^o{}_o = g_{o\lambda}\,T^\lambda_o$ from eqn (A.31) and then substituting the result into H we get

$$H = \int d^3x\sqrt{-g}\;(\tfrac{1}{2}\gamma^{-1}\,\pi^2 - \gamma^{-\frac{1}{2}}\phi_s\, n^s\pi + \tfrac{1}{2}\,\gamma^{rs}\,\phi_r\phi_s + \tfrac{1}{2}\,V(\phi)) \tag{A.33}$$

In deriving the 2nd term inside bracket we have made use of the following two identifications

$$g_{\rho\lambda}\, x^{\lambda}_{,\gamma} = g_{o\gamma}$$

$$\gamma^{rs} n^{o} g_{o\mu} = - n^{s}$$

§A.5 Variation of H w.r.t. the variations of $g_{\mu\nu}$

It is obvious that if we keep ϕ and π fixed and let $g_{\mu\nu}$ vary then H will also vary. This variation of H is calculated as follows:

$$\delta H = \int d^3x\delta\sqrt{-g}\ \left(\frac{1}{2}\gamma^{-1}\pi^2 - \gamma^{-\frac{1}{2}}\phi_s n^s\pi + \frac{1}{2}\gamma^{rs}\phi_r\phi_s + \frac{1}{2}V(\phi)\right) +$$

$$\int d^3x\ \sqrt{-g}\left[-\frac{1}{2}\delta\gamma\gamma^{-2}\pi^2 + \frac{1}{2}(\delta\gamma\gamma^{-3/2}\ n^s - 2\gamma^{-\frac{1}{2}}\delta n^s)\phi_s\pi + \frac{1}{2}\delta\gamma^{rs}\phi_r\phi_s\right)$$

Now we can substitute δn^s, $\delta\gamma$ and $\delta\gamma^{rs}$ from eqn (A.18), A.21) and (A.23) respectively [we also insert $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\ g^{\mu\nu}\delta g_{\mu\nu}$], then we get

$$\delta H = \frac{1}{2}\int d^3x\sqrt{-g}\ g^{\mu\nu}\delta g_{\mu\nu}\left(\frac{1}{2}\gamma^{-1}\pi^2 - \gamma^{-\frac{1}{2}}\phi_s\ n^s\pi + \frac{1}{2}\gamma^{rs}\ \phi_r\phi_s + \frac{1}{2}V(\phi)\right) +$$
$$\frac{1}{2}\int d^3x\sqrt{-g}\ [-\gamma^{-1}\pi^2(g^{\mu\nu}+n^\mu n^\nu) + \gamma^{-\frac{1}{2}}((g^{\mu\nu}+n^\mu n^\nu)n^s + 2n^\mu(g^{s\nu}+\frac{1}{2}n^s n^\nu))\phi_s\pi +$$
$$\gamma^{rp}\ \gamma^{sq}\ x^\mu_{,p}x^\nu_{,q}\phi_r\phi_s]\delta g_{\mu\nu}$$

$$= \frac{1}{2}\int d^3x\sqrt{-g}\ [-\gamma^{-1}\ (\frac{1}{2}g^{\mu\nu} + n^\mu n^\nu)\pi^2 + 2\gamma^{-\frac{1}{2}}\ n^\mu(g^{s\nu}+n^\nu n^s)\phi_s\pi +$$
$$(\frac{1}{2}\gamma^{rs}\ g^{\mu\nu} - \gamma^{rp}\ \gamma^{sq}\ \ x^\mu_{,p}x^\nu_{,q})\phi_r\phi_s + \frac{1}{2}g^{\mu\nu}V(\phi)]\ \delta g_{\mu\nu} \tag{A.34}$$

Now by making use of eqn (A.6) we can write the bracket in the coefficient of $\phi_r\phi_s$ in the following form

$$g^{s\nu} + n^s n^\nu = \gamma^{rp}x^s_{\ ,r}\ x^\nu_{\ ,p} = \gamma^{rp}\delta^s_{\ p}\ x^\nu_{\ ,r}$$
$$= \gamma^{rs}x^\nu_{\ ,r}$$

This expression must be multiplied by n^μ and then symmetrized w.r.t. μ and ν (because $\delta g_{\mu\nu}$ is symmetric w.r.t. the interchange of μ and ν). Therefore the coefficient of $\phi_s\pi$ in equation (A.34) will have the following form

$$\gamma^{-\frac{1}{2}}\,\gamma^{rs}(n^{\mu}x^{\nu}_{,r} + n^{\nu}x^{\mu}_{,r})\delta g_{\mu\nu} \tag{A.35}$$

Similarly we manipulate the term in eqn (A.34) which involves the term $\phi_r\phi_s$

$$(\tfrac{1}{2}\,\gamma^{rs}\,g^{\mu\nu} - \gamma^{rp}\,\gamma^{sq}\,x^{\mu}_{,p}\,x^{\nu}_{,q})\,\phi_r\phi_s =$$

$$\frac{1}{2}\,g^{\mu\nu}\gamma_{rs}\,\phi^r\phi^s - \phi^p\phi^q\,x^{\mu}_{,p}\,x^{\nu}_{,q} =$$

$$= -\,(\,x^{\mu}_{,r}\,x^{\nu}_{,s} - \frac{1}{2}\,g^{\mu\nu}\gamma_{rs})\,\phi^r\phi^s \tag{A.36}$$

Insertions of (A.35) and (A.36) into (A.34) yields

$$\delta H = \frac{1}{2}\int d^3x\sqrt{-g}\,[-\gamma^{-1}(\,\frac{1}{2}\,g^{\mu\nu} + n^{\mu}n^{\nu})\,\pi^2 + \gamma^{-\frac{1}{2}}\,\gamma^{rs}(n^{\mu}x^{\nu}_{,r} + n^{\nu}x^{\mu}_{,r})\phi_s\pi -$$

$$(x^{\mu}_{,r}\,x^{\nu}_{,s} - \frac{1}{2}\,g^{\mu\nu}\gamma_{rs})\,\phi^r\phi^s + \frac{1}{2}\,g^{\mu\nu}V(\phi)]\,\delta g_{\mu\nu}$$

Now comparing the term inside bracket with eqn (A.31) gives us the following

$$\delta H = \frac{1}{2}\int d^3x\,\sqrt{-g}\,\,T^{\mu\nu}\delta g_{\mu\nu} \tag{A.37a}$$

If we insert for $\delta g_{\mu\nu} = -g_{\mu\alpha}\,g_{\nu\beta}\,\delta g^{\alpha\beta}$ we get

$$\delta H = -\,\frac{1}{2}\int d^3x\sqrt{-g}\,T_{\mu\nu}\,(x)\,\delta g^{\mu\nu}(x) \tag{A.37b}$$

This equation plays a central role in deriving the Einstein field equations in the semi-classical gravity.

§A.6 *The higher order action integrals.*

In this section we will derive some mathematical formula which are needed in chapters 5 and 6.

The most general functional of $g_{\mu\nu}$ which is needed in our future treatment is the following

$$S = \int d^4x \sqrt{-g} \{\Lambda + ARX + BR^2 + CR_{\mu\nu}R^{\mu\nu}\} \tag{A.38}$$

Here Λ,A,B and C are some constant numbers. X is a scalar function of the space-time point x which may or may not be a function of $g_{\mu\nu}$

$R = g^{\mu\nu}R_{\mu\nu}$ and $R_{\mu\nu}$ is defined by

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^{\alpha} - \Gamma_{\mu\alpha,\nu}^{\alpha} + \Gamma_{\beta\ \alpha}^{\ \alpha}\Gamma_{\mu\nu}^{\beta} - \Gamma_{\beta\ \nu}^{\ \alpha}\Gamma_{\mu\ \alpha}^{\ \beta} \tag{A.39}$$

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) \tag{A.40}$$

The comma denotes partial derivative. Under a small variation $\delta g^{\alpha\beta}$ of the metric tensor S changes by the following

$$\delta S = \int d^4x\, \delta\sqrt{-g}\{\Lambda + ARX + BR^2 + CR_{\mu\nu}R^{\mu\nu}\} + \tag{A.41}$$

$$\int d^4x \sqrt{-g}\, \{A\,(\delta RX + R\delta X) + 2BR\,\delta R + C\,\delta(g^{\mu\lambda}g^{\nu\sigma}R_{\mu\nu}R_{\lambda\sigma})$$

In order to calculate δS we calculate the variation of each term individually.

First we note that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\alpha\beta}\delta g^{\alpha\beta} \tag{4.42}$$

and

$$\delta\Gamma_{\mu\nu}^{\ \ \lambda} = \frac{1}{2}\,\delta g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) \ +$$

$$\frac{1}{2}\ g^{\lambda\alpha}((\delta g_{\mu\alpha})_{,\nu} + (\delta g_{\nu\alpha})_{,\mu} - (\delta g_{\mu\nu})_{,\alpha}) \qquad \text{(A.43)}$$

Or upon insertion from (A.40) and by making use of

$$\delta g^{\lambda\alpha} = -\ g^{\lambda\gamma} g^{\alpha\sigma}\ \delta g_{\gamma\sigma} \qquad \text{(A.44)}$$

eqn (A.43) becomes

$$\delta\Gamma_{\mu\nu}^{\ \ \lambda} = -g^{\lambda\gamma}\ \Gamma_{\mu\nu}^{\sigma}\ \delta g_{\gamma\sigma} + \frac{1}{2} g^{\lambda\alpha}\ [(\delta g_{\mu\alpha})_{,\nu} + (\delta g_{\mu\alpha})_{,\mu} - (\delta g_{\mu\nu})_{,\alpha}]$$

The expression inside bracket can be written in a covariant tensor form

$$\delta\Gamma_{\mu\nu}^{\ \ \lambda} = \frac{1}{2}\ g^{\lambda\alpha}\ (\delta g_{\mu\alpha})_{;\nu} + (\delta g_{\nu\alpha})_{;\mu} - (\delta g_{\mu\nu})_{;\alpha} \qquad \text{(A.45a)}$$

This expression clearly shows that although $\Gamma_{\mu\nu}^{\lambda}$ is not a tensor but $\delta\Gamma_{\mu\nu}^{\ \ \lambda}$ is. If we contract the λ and ν indices in $\delta\Gamma_{\mu\nu}^{\ \ \lambda}$ we get :

$$\delta\Gamma_{\mu\lambda}^{\ \ \lambda} = \frac{1}{2}\ \ g^{\lambda\alpha}\ (\delta g_{\lambda\alpha})_{;\mu} \qquad \text{(A.45b)}$$

Similarly the eqn. (A.39) gives us the following

$$\delta R_{\mu\nu} = \delta\Gamma_{\mu\nu,\alpha}^{\ \ \alpha} - \delta\Gamma_{\mu\alpha,\nu}^{\ \ \alpha} + \delta\Gamma_{\beta\alpha}^{\ \ \alpha}\ \Gamma_{\mu\nu}^{\ \ \beta} + \Gamma_{\beta\alpha}^{\ \ \alpha}\ \delta\Gamma_{\mu\nu}^{\ \ \beta} - \delta\Gamma_{\beta\nu}^{\ \ \alpha}\ \Gamma_{\mu\alpha}^{\ \ \beta} + \Gamma_{\beta\nu}^{\ \ \alpha}\ \delta\Gamma_{\mu\alpha}^{\ \ \beta}$$

Since $\delta\Gamma_{\mu\nu}^{\ \ \alpha}$ is a tensor we can write this eqn in the following form.

$$\delta R_{\mu\nu} = (\delta\Gamma_{\mu\nu}^{\ \ \alpha})_{;\alpha} - (\delta\Gamma_{\mu\alpha}^{\ \ \alpha})_{;\nu} \qquad \text{(A.46)}$$

Now we go back to the equation (A.41) and evaluate each term separately,

We start with

$$\delta I_1 := \int d^4x \sqrt{-g}\ \ X\delta R =$$

$$\int d^4x \sqrt{-g}\ \ X\ \{\delta g^{\alpha\beta} R_{\alpha\beta} + g^{\mu\nu}\ \delta R_{\mu\nu}\}$$

Upon insertion for $\delta R_{\mu\nu}$ from eqn. (1.46) we get

$$\delta I_1 = \int d^4x \sqrt{-g} X\{\delta g^{\alpha\beta} R_{\alpha\beta} + g^{\mu\nu}(\delta\Gamma_{\mu\nu}{}^{\alpha})_{;\alpha} - g^{\mu\nu}(\delta\Gamma_{\mu\alpha}{}^{\alpha})_{;\nu}\}$$

By making use of the fact that

$$(g^{\mu\nu})_{;\alpha} = 0 \tag{A.47}$$

One can integrate the terms involving the covariant derivatives by part. This is done by noting that

$$\sqrt{-g}\ X\ \ g^{\mu\nu}(\delta\Gamma_{\mu\nu}^{\ \alpha})_{;\alpha} = (\sqrt{-g}\ X\ g^{\mu\nu}\delta\Gamma_{\mu\nu}^{\ \alpha})_{;} - \sqrt{-g}\ \ g^{\mu\nu}X_{;\alpha}\ \delta\Gamma_{\mu\nu}{}^{\alpha}$$

$$= \sqrt{-g}\ (X\ g^{\mu\nu}\delta\Gamma_{\mu\nu}^{\ \alpha})_{;\alpha} - \sqrt{-g}\ \ g^{\mu\nu}X_{;\alpha}\ \delta\Gamma_{\mu\nu}^{\ \alpha}$$

$$= \sqrt{-g}\ \ \frac{1}{\sqrt{-g}}\ \partial_{\alpha}(\sqrt{-g}\ X\ g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}) - \sqrt{-g}\ \ g^{\mu\nu}X_{;\alpha}\ \delta\Gamma_{\mu\nu}{}^{\alpha}$$

$$= -\partial_{\alpha}\ (\sqrt{-g}\ X\ g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}) - \sqrt{-g}\ \ g^{\mu\nu}X_{;\alpha}\ \delta\Gamma_{\mu\nu}{}^{\alpha}$$

The integral of the first term over the 4-volume can be transformed into a surface integral and assuming that all of the variations vanish on the boundary surfaces we get

$$\delta I_1 = \int d^4x \sqrt{-g}\, \{\delta g^{\alpha\beta} X R_{\alpha\beta} - g^{\mu\nu}(X;_{\alpha}\delta\Gamma^{\alpha}_{\mu\nu} - X;_{\nu}\delta\Gamma^{\alpha}_{\mu\alpha})\}$$

Now we insert for $\delta\Gamma$'s from eqn (A.45) and carry out the covariant integration by parts to get

$$\delta I_1 = \int d^4x \sqrt{-g}\, \{\delta g^{\alpha\beta} R_{\alpha\beta} X + \frac{1}{2} g^{\mu\nu} [X;^{\lambda};_{\nu}(\delta g_{\mu\lambda}) + X;^{\lambda};_{\mu}(\delta g_{\nu\lambda}) -$$

$$X;^{\lambda};_{\lambda}\delta g_{\mu\nu} - g^{\lambda\alpha} X;_{\nu};_{\mu}\delta g_{\lambda\alpha} \}$$

Finally by making use of eqn (A.44) we get

$$\delta I_1 = \int d^4x \sqrt{-g}\, X\delta R$$

$$= \int d^4x \sqrt{-g}\, \{X R_{\alpha\beta} - (X;_{\alpha};_{\beta} - X;^{\lambda};_{\lambda}\, g_{\alpha\beta})\}\, \delta g^{\alpha\beta} \tag{A.48}$$

If we let X=R in eqn (A.48) then we get

$$\int d^4x \sqrt{-g}\; R\delta R = \int d^4x \sqrt{-g}\, \{R R_{\alpha\beta} - (R;_{\alpha};_{\beta} - g_{\alpha\beta} R;^{\lambda};_{\lambda})\}\delta g^{\alpha\beta} \tag{A.49}$$

In a similar manner we can calculate

$$\int d^4x \sqrt{-g}\, \delta(g^{\mu\lambda} g^{\nu\sigma} R_{\mu\nu} R_{\lambda\sigma}) = \tag{A.50}$$

$$2\int d^4x \sqrt{-g}\{R_{\alpha}{}^{\sigma} R_{\beta\sigma} - \frac{1}{2}(2R^{\mu}{}_{\alpha;\beta;\mu} - R_{\alpha\beta};^{\lambda};_{\lambda} - g_{\alpha\beta} R^{\mu\nu};_{\mu\nu})\}\, \delta g^{\alpha\beta}$$

Upon substituting from (A.42), (A.48), (A.49) and (A.50) into (A.41) we get

$$\delta S = \int d^4x \sqrt{-g}\, \{A\, [X\, G_{\alpha\beta} - (X;_{\alpha};_{\beta} - g_{\alpha\beta} X;^{\lambda};_{\lambda}) + R \frac{\delta X}{\delta g^{\alpha\beta}_{(x)}}] - \frac{1}{2}\Lambda g_{\alpha\beta} +$$

$$2B\;\; R G_{\alpha\beta} - (R;_{\alpha};_{\beta} - g_{\alpha\beta} R;^{\lambda};_{\lambda}) + \frac{1}{4} g_{\alpha\beta} R^2 \;\; +$$

$$C\,[-\tfrac{1}{2}\,g_{\alpha\beta}R_{\mu\nu}R^{\mu\nu} + 2R_{\nu}{}^{\sigma}R_{\beta\sigma} - 2R^{\mu}{}_{(\alpha;\beta);\mu} + R_{\alpha\beta;}{}^{\lambda}{}_{;\lambda} + g_{\alpha\beta}R^{\mu\nu}{}_{;\mu\nu}]\}\,\delta g^{\mu\beta}$$

(A.51)

where the Einstein tensor $G_{\alpha\beta}$ is defined by

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}R \qquad \text{(A.52)}$$

Now we investigate the necessary and sufficient condition for the quantity

$$I = \int d^4x\,\sqrt{-g}\;F(x) \qquad \text{(A.53)}$$

to be invariant under general coordinate transformations on M .
Assume that $\delta g^{\alpha\beta}$ is caused by an infinitesimal coordinate transformation

$$x^{\alpha} \to x^{\alpha} + \xi^{\alpha}(x) \qquad \text{(A.53)}$$

then

$$\delta I = \int d^4x\,\sqrt{-g}\,[-\frac{1}{2}\,g_{\alpha\beta}F + \frac{\delta F}{\delta g^{\alpha\beta}(x)}\,]\,\delta g^{\alpha\beta} \qquad \text{(A.54)}$$

Under the transformation (A.53) one gets

$$\delta g^{\alpha\beta} = \xi^{\alpha;\beta} + \xi^{\beta;\alpha}$$

Upon insertion in (A.54) and integrating by parts we get

$$\delta I = \int d^4x\,\sqrt{-g}\,[-\frac{1}{2}\,g_{\alpha\beta}F;^{\alpha} + (\frac{\delta F}{\delta g^{\alpha\beta}(x)})\,;^{\alpha}\,]\,\xi^{\beta}$$

Since, ξ^{β} are arbitrary therefore the necessary and sufficient condition for $\delta I=0$ is

$$[-\frac{1}{2}g_{\alpha\beta}F + (\frac{\delta F}{\delta g^{\alpha\beta}(x)})]_{;}^{\alpha} \equiv 0 \tag{A.55}$$

If F=R , then equation (A.48) implies that (let X=1)

$$\frac{\delta R}{\delta g^{\alpha\beta}(x)} \Rightarrow R_{\alpha\beta}$$

and the identity (A.55) becomes

$$(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R)_{;}{}^{\alpha} = G_{\alpha\beta;}{}^{\alpha} \equiv 0 \tag{A.56}$$

Similarly if we take $R = R^2$ then (A.55) becomes

$$[RG_{\alpha\beta} - (R_{;\alpha;\beta} - g_{\alpha\beta}R_{;}{}^{\lambda}{}_{;\lambda}) + \frac{1}{4}g_{\alpha\beta}R^2]_{;}{}^{\alpha} \equiv 0 \tag{A.57}$$

and if we assume $F = R_{\mu\nu}R^{\mu\nu}$ then the same identity requires

$$[-\frac{1}{2}g_{\alpha\beta}R_{\mu\nu}R^{\mu\nu} + 2R_{\alpha}{}^{\sigma}R_{\beta\sigma} - 2R^{\mu}{}_{\alpha;\beta;\mu} + R_{\alpha\beta;}{}^{\lambda}{}_{;\lambda} + g_{\alpha\beta}R^{\mu\nu}{}_{;\mu\nu}]_{;}{}^{\alpha} \equiv 0 \tag{A.58}$$

Finally if $F=\Lambda+AXR + BR^2 + CR_{\mu\nu}R^{\mu\nu}$ then it is the covariant divergence of the expression inside the bracket in (A.51) which vanishes identically,

We notice that when $g_{\mu\nu} = \alpha\eta_{\mu\nu}$ then because of the vanishing of the Weyl tensor we get $\int d^4x\sqrt{-g}\,R^2 = 3\int d^4x\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu}$ + total div, where

$$R = \alpha^{-2}[\frac{3}{2}\alpha^{-1}\alpha_{,\lambda}\alpha^{,\lambda} - 3\partial_{\mu}\partial^{\lambda}\alpha] \tag{A.59}$$

and

$$\alpha^{\lambda} := \eta^{\lambda\mu}\partial_{\mu}\alpha$$

Then upon inserting $c = 0$ and $\delta g^{\alpha\beta} = -\alpha^{-2}\delta\alpha\,\eta^{\alpha\beta}$ in (A.51) we get (we also put X=1)

$$\delta S = -\int d^4x\; \delta\alpha\{A\,\eta^{\alpha\beta}G_{\alpha\beta} - \frac{\Lambda}{2}\,\eta^{\alpha\beta}g_{\alpha\beta} +$$

$$2B\,[RG_{\alpha\beta}\eta^{\alpha\beta} - \eta^{\alpha\beta}(R\,;\alpha;\beta - g_{\alpha\beta}R\,;^{\lambda}\,;\lambda\,) + \frac{1}{4}\eta^{\alpha\beta}g_{\alpha\beta}R^2]\,\}$$

$$= -\int d^4x\;\delta\alpha\{A\alpha\;G^{\alpha}{}_{\alpha} - 2\Lambda\alpha +$$

$$2B\,[\alpha RG^{\beta}{}_{\beta} - \alpha(R;^{\beta};\beta - 4R;^{\beta};\lambda) + \alpha R^2]\}$$

$$= \int d^4x\;\delta\alpha\{2\Lambda\alpha + A\alpha\,R - 6B\;R;^{\beta};\beta\}\;.$$

$$R;\beta^{\;\beta} = \frac{1}{\sqrt{-g}}\,\partial_\beta\,(\sqrt{-g}\;\;g^{\beta\gamma}\;R;_{\gamma})$$

$$= \alpha^{-2}\;\partial_\beta(\,\alpha\;\eta^{\beta\gamma}\partial_\gamma\;R)$$

$$= \alpha^{-2}\;\alpha,^{\gamma}\;R,_{\gamma} + \alpha^{-1}\partial_\mu\partial^\mu R$$

$$\delta S = \int d^4x\;\{2\Lambda\alpha + A\,\alpha R - 6B\,(\,\alpha^{-2}\,\alpha^{\gamma}R,_{\gamma} + \alpha^{-1}\partial_\mu\partial^\mu R\}\delta\alpha \qquad (A.60)$$

APPENDIX B

§B.1 *The n-dimensional integrals.*

In Chapters 5 and 6 we have made use of the dimensional regularization scheme of 't'Hooft and Veltman.[13] In evaluating various integrals of these chapters we have employed the following formulae:

$$\int \bar{d}^n p \, \frac{1}{(p^2+2k.p+M^2)^\alpha} = \frac{1}{(4\pi)^{n/2}} \, \frac{1}{(M^2-k^2)^{\alpha-n/2}} \, \frac{\Gamma(\alpha-\frac{n}{2})}{\Gamma(\alpha)} \, , \tag{B.1}$$

$$\int \bar{d}^n p \, \frac{p_\mu}{(p^2+2k.p+M^2)^\alpha} = \frac{1}{(4\pi)^{n/2}} \, \frac{1}{(M^2-k^2)^{\alpha-n/2}} \, \frac{\Gamma(\alpha-\frac{n}{2})}{\Gamma(\alpha)} \, (-k_\mu), \tag{B.2}$$

$$\int \bar{d}^n p \, \frac{p_\mu p_\nu}{(p^2+2k.p+M^2)^\alpha} = \frac{1}{(4\pi)^{n/2}} \, \frac{1}{(M^2-k^2)^{\alpha-n/2}} \, \frac{1}{\Gamma(\alpha)} \times \tag{B.3}$$

$$\left\{ \Gamma(\alpha-\tfrac{n}{2}) k_\mu k_\nu + \frac{\delta_{\mu\nu}}{2} \Gamma(\alpha-1-\tfrac{n}{2}) \, (M^2-k^2) \right\} ,$$

$$\int \bar{d}^n p \, \frac{p_\mu p_\nu p_\lambda}{(p^2+2k.p+M^2)^\alpha} = \frac{1}{(4\pi)^{n/2}} \, \frac{1}{(M^2-k^2)^{\alpha-n/2}} \, \frac{1}{\Gamma(\alpha)} \times$$

$$\left\{ -\Gamma(\alpha-\tfrac{n}{2}) \, k_\mu k_\nu k_\lambda - \Gamma(\alpha-1-\tfrac{n}{2}) \, \tfrac{1}{2} \, (\delta_{\mu\nu} k_\lambda + \delta_{\mu\lambda} k_\nu + \delta_{\mu\lambda} k_\mu)(M^2-k^2) \right\} , \tag{B.4}$$

$$\int \bar{d}^n p \, \frac{p_\mu p_\nu p_\lambda p_\sigma}{(p^2+2k.p+M^2)^\alpha} = \frac{1}{(4\pi)^{n/2}} \, \frac{1}{(M^2-k^2)^{\alpha-n/2}} \, \frac{1}{\Gamma(\alpha)} \times$$

(B.5)

$$\{ \Gamma(\alpha-\tfrac{n}{2}) \, k_\mu k_\nu k_\lambda k_\sigma + \tfrac{1}{2} \, \Gamma(\alpha-1-\tfrac{n}{2}) [\delta_{\mu\nu} k_\lambda k_\sigma +$$

$$\delta_{\mu\lambda} k_\nu k_\sigma + \delta_{\nu\lambda} k_\mu k_\sigma + \delta_{\mu\sigma} k_\lambda k_\nu + \delta_{\lambda\sigma} k_\nu k_\mu +$$

$$\delta_{\nu\sigma} k_\mu k_\lambda] \, (M^2-k^2) + \tfrac{1}{4} \, \Gamma(\alpha-2-\tfrac{n}{2}) \Delta_{\mu\nu\lambda\sigma} \, (M^2-k^2)^2 \, \} ,$$

where

$$\Delta_{\mu\nu\lambda\sigma} := (\delta_{\mu\nu}\delta_{\lambda\sigma} + \delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda}) \tag{B.6}$$

§B.2 *APPENDIX TO CHAPTER V.*

The coefficients c's and d's in (5.25) are defined by the following :

$$c = -\int d^n p \; \frac{1}{p^2+\mu^2} = \frac{\Gamma(\varepsilon)}{16\pi^2}\,\mu^2\,[1 + \varepsilon(1-\log\mu^2)] \,, \tag{B.7}$$

$$c_o = \frac{\Gamma(\varepsilon)}{16\pi^2}\int_o^1 d\alpha(1 - \varepsilon f_o(\alpha))\,\{-m^2[\alpha^2(3+\varepsilon) - \alpha(3+\varepsilon)] - \varepsilon\mu^2\} \,, \tag{B.8}$$

$$c_1 = \frac{\Gamma(\varepsilon)}{16\pi^2}\int_o^1 d\alpha[1 - \varepsilon(f_o(\alpha) + m^2 f_1(\alpha))[\alpha^2(3+\varepsilon) - (3+\varepsilon)] \,, \tag{B.9}$$

$$c_2 = \frac{-3\Gamma(\varepsilon+1)}{16\,\pi^2}\int_o^1 d\alpha(f_1(\alpha) - m^2 f_2(\alpha))\,(\alpha^2-\alpha), \tag{B.10}$$

Notice that as $\varepsilon \to o$, $c, c_o, c_1 \to \infty$ but $c_2 < \infty$.

$$d_o = \frac{\Gamma(\varepsilon)}{16\pi^2}\int_o^1 d\alpha\,(1-\varepsilon f_o(\alpha))\,(m^4 a(\alpha) - m^2\mu^2 b(\alpha) - \mu^4), \tag{B.11}$$

$$d_1 = \frac{\Gamma(\varepsilon)}{16\pi^2}\int_o^1 d\alpha\{a(\alpha)\;[-2m^2 + \varepsilon\, m^2(2\, f_o(\alpha) - m^2 f_1(\alpha))]$$

$$+ \mu^2 b(\alpha)\,[1 - \varepsilon(f_o(\alpha) - m^2 f_1(\alpha)) + \varepsilon\mu^4\, f_1(\alpha)]\} \,, \tag{B.12}$$

$$d_2 = \frac{\Gamma(\varepsilon)}{16\pi^2}\int_o^1 d\alpha\,\{a(\alpha)[\,1 - \varepsilon(f_o(\alpha) - 2m^2 f_1(\alpha) +$$

$$m^4 f_2(\alpha)] + \varepsilon\mu^2 b(\alpha)\,[-\,f_1(\alpha) + m^2 f_2(\alpha)] +$$

$$\varepsilon\mu^4\, f_2(\alpha)\} \,, \tag{B.13}$$

§B.3 *APPENDIX TO CHAPTER VI.*

The formula (6.) have been obtained from insertion of $-i\Delta^{reg}_{inh}$ into

$$\langle o|\ \hat{T}_{\mu\nu}(x)\ |o\rangle^{reg}_{inh} := -i\lim_{x\to x'} \{[-\frac{\partial}{\partial x^{\mu}}\ \frac{\partial}{\partial x'^{\mu}} + \frac{1}{2}\eta_{\mu\nu}(\eta^{\lambda\sigma}\frac{\partial}{\partial x^{\lambda}}\ \frac{\partial}{\partial x^{\rho}} + \mu^2)]\ \Delta^{reg}_{inh}(x,x')\} , \tag{B.14}$$

It is not hard to see that

$$-i\ \Delta^{reg}_{inh}(x) = \frac{\Gamma(\varepsilon)}{16\pi^2}\ \mu^2\ [1+\varepsilon(1-\log\mu^2)] + i\int d^nq\ e^{iq.x}\ \tilde{h}(q)\ [Dq^4 + E\ q^2] + \Phi(x) , \tag{B.15}$$

where $\Phi(x)$ has been defined by (6.) and

$$D := \frac{-\varepsilon\Gamma(\varepsilon)}{16\pi^2}\int_0^1 d\alpha\ [\ f_1(\alpha)\ (\alpha^2 - \tfrac{3}{2}\alpha + \tfrac{1}{2}) - \frac{\mu^2}{2}\ f_2(\alpha)] , \tag{B.16}$$

$$E := \frac{\Gamma(\varepsilon)}{16\pi^2}\int_0^1 d\alpha\{(1-\varepsilon\log\mu^2)[\alpha^2\ (1+\tfrac{\varepsilon}{2}) - \tfrac{\alpha}{2}\ (3+\varepsilon) + \tfrac{1}{2}\ ] + \tfrac{\varepsilon}{2}\ f_1(\alpha)\ \mu^2\} , \tag{B.17}$$

Similarly one can show that

$$-\ i\ \frac{\partial}{\partial x^{\mu}}\ \frac{\partial}{\partial x'^{\nu}}\ \Delta^{reg}_{inh}(x-x')\Big|_{x=x'} =$$

$$\frac{-\Gamma(\varepsilon)}{16\pi^2}\ \frac{\mu^4}{4}\ [1+\varepsilon(\tfrac{3}{2} - \log\mu^2)]\ g_{\mu\nu} +$$

$$b_{\lambda\sigma}(\alpha) \quad := \quad \frac{\Gamma(\varepsilon)}{16\pi^2} \quad \mu^2\delta_{\lambda\sigma}[-\tfrac{3}{2}\,\alpha^2 + 2\alpha - \tfrac{1}{2}\,](1+\varepsilon), \tag{B.25}$$

$$c_{\mu\nu\lambda\sigma} \quad := \quad \frac{\Gamma(\varepsilon)}{16\pi^2} \quad \frac{\mu^4}{8}\,\Delta_{\mu\nu\lambda\sigma} \quad (1 + \tfrac{3}{2}\varepsilon), \tag{B.26}$$

We notice that as $\varepsilon \to o$ all of these coefficients become infinite. However, we also remark that

$$\int_o^1 d\alpha\; b_{\lambda\sigma}(\alpha) \quad = \quad o, \tag{B.27}$$

§B.4 THE REALITY OF $i\,\Delta(x,x')\big|_{x\to x'}$

In this section we want to show that the regularized quantities such as (5.27) are indeed real functions of x. We need to consider only the following generic form of the two point function $\Delta^{\psi_o}(x,x')$

$$\Delta^{\psi}(x,x') = \Delta_F(x-x') - \int d^4y\; H(y)\{\Delta^F(x-y)\,\Delta^F(x'-y) - \Delta^{(-)}(x-y)\cdot\Delta^{(-)}(x'-y)\} \tag{B.28}$$

It must be mentioned that if the real function H did involve any derivative operator - such as (6.4b) - we first carry out some appropriate integration by parts to bring it to a form similat to (B.28). see e.g. (6.9) . The Feynman propagator $\Delta_F(x-x')$ when regularized according to $d^4p \to -id^np$ becomes

$$\Delta_F(x-x') \xrightarrow[x\to x']{} -i\int d^np\,\frac{1}{p^2+\mu^2} \tag{B.29}$$

Therefore $i\;\Delta_F^{reg}(o)$ is real.

Now consider the integrand of the second term in (B.28). By making use of the identity

$$\Delta^F(x-y) = -\,\theta(x-y)\,\Delta^{(+)}(x-y) + \theta(y-x)\,\Delta^{(-)}(x-y), \tag{B.30}$$

we envisage the following three possibilities

i) The region for which $y > x$ and $y > x'$. Then

$$\Delta^F(x-y)\,\Delta^F(x'-y) - \Delta^{(-)}(x-y)\,\Delta^{(-)}(x'-y) \equiv 0.$$

ii) The region defined by $x > x' > y$ ½ Then

$$\Delta^F(x-y)\,\Delta^F(x'y) = \Delta^{(+)}(x-y)\,\Delta^{(+)}(x'-y) ,$$

therefore

$$\Delta^F(x-y)\,\Delta^F(x'-y) - \Delta^{(-)}(x-y)\,\Delta^{(-)}(x'-y) =$$

$$2i \, \mathrm{Im}\, \Delta^{(+)}(x-y)\,\Delta^{(+)}(x'-y) \qquad ,$$

where we made use of the following

$$\Delta^{(-)}(x-y) = [\,\Delta^{(+)}(x-y)\,]^*$$

* denoting a complex conjugate

Thus

$$i\,[\,\Delta^F(x-y)\,\Delta^F(x'-y) - \Delta^{(-)}(x-y)\,\Delta^{(-)}(x'-y)]$$

is real.

iii) Finally the region $x > y > x'$ which will not contribute anything in the limit of $x \to x'$. Thus the quantity $i\,\Delta(x,x')\Big|_{x \quad x'}$ is either zero or real in any point of space-time.

Let us introduce the following notations

$$F(x,x') := i\int d^4y\ H(y)\ \ \Delta^F(x-y)\,\Delta^F(x'-y)\ -$$

$$\Delta^{(-)}\ (x-y)\ \ \Delta^{(-)}\ \ (x'-y) \tag{B.31}$$

$$G(x,x') := i\int d^4y\ \ \Delta^{(-)}(x-y)\ \Delta^{(-)}\ \ (x'-y)\ H(y)$$

We want to show that as $x \to x'$, $G^{reg}(x,x')$ is real. If we insert F and G into (B.31) we get

$$F(x,x') + G(x,x') = i\int d^4y\ H(y)\Delta^F(x-y)\Delta^F(x'-y), \tag{B.32}$$

when $x \to x'$ the r.h.s. will give the following regularized integral

$$i\int d\!\!\!{}^-{}^n q\ e^{iq.x}\ \ \tilde{H}(q)\ \ K(q^2) \tag{B.33}$$

where

$$K(q^2) := \int_0^1 d\ \alpha\ \int d\!\!\!{}^-{}^n p\ \frac{1}{(p^2+2k.p + M^2)^2}$$

with $k = -\alpha q$ and $M^2 = \mu^2 + \alpha q^2$. Because of the reality of $H(x)$ we have $H^*(q) = H(-q)$ therefore if we continue (B.33) back to the physical Minkowskian space-time we get the following real quantity

$$-\int d\!\!\!{}^-{}^4 q\ e^{iq.x}\ \ \tilde{H}(q)\ \ K(q^2).$$

Since we have already proved that $F(x,x')\big|_{x \to x'}$ is real therefore $G(x,x')$ must be real.

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