METHODS OF SOLUTION OF DIFFERENTIAL EQUATIONS IN GENERAL RELATIVITY AND RELATED POTENTIAL PROBLEMS

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ABSTRACT

This thesis is primarily concerned with methods of obtaining solutions of differential equations which arise in various branches of mathematical physics. In general relativity the differential equations considered describe scalar fields on a fixed and unquantised background space. The Klein-Gordon equation exterior to a rotating black hole is examined with a physically realistic source term, representing the meson field of an infalling baryon, so generalising previous work. We discuss the possibility of soliton-like solutions of a class of non-linear Klein-Gordon equation exterior to a non-rotating black hole. We also examine from a physical viewpoint a known solution of the Einstein-Maxwell field equations describing the interior of an object supported against total gravitational collapse by an internal magnetic field. Techniques used previously in work on scalar meson fields are found to be inapplicable to these problems, but their usefulness has suggested their application to other differential equations. Accordingly, the Liouville-Green asymptotic method is used to obtain approximate eigenvalues and eigenfunctions of the Schrödinger equation with an anharmonic oscillator potential, and to investigate the propagation of electromagnetic waves in optical waveguides.

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CHAPTER 1

INTRODUCTION

In recent years a substantial amount of work has been carried out on the problem of external fields propagating in a fixed and unquantised background space. Such considerations seem a natural prerequisite for the understanding of quantised fields on a fixed background and, ultimately, for a fully quantised theory of gravity. In much of the work the background considered is the space exterior to a black hole, a region of spacetime from which, classically, nothing can escape to infinity. Since the investigations of Oppenheimer and his collaborators (1939), it has been known that there exists an upper limit on the mass of a neutron star. Many stars with masses greater than this value are observed, so that in the final stages of their evolution no forces due to degeneracy pressure can prevent total gravitational collapse to a singularity. Such singularities are shielded from the universe by an event horizon, the interior being the black hole. That singularities are always isolated in this way is the as yet unproven hypothesis of cosmic censorship, due to Penrose (1969).

An important feature of black hole physics is the existence of the so-called 'no-hair' theorem. This was originally a conjecture based on the work of Israel (1967), and was later proved by the contributions of Chase (1970), Carter (1971), Hawking (1972), Bekenstein (1972 a,b), Robinson (1974) and, more recently, Adler and Pearson

(1978). The theorem states that, independently of the initial conditions, a collapsing object will eventually reach a stationary state described by only three parameters, namely, charge, mass, and angular momentum. This state will be a member of the Kerr-Newman family of solutions of the field equations. The Kerr-Newman solutions therefore enjoy a special significance in the study of astrophysical processes near a black hole (see for example Bardeen et al, 1972), particularly in view of the recent method of testing observationally for a rotating black hole (Stark and Connors, 1977 a,b).

The 'no-hair' theorem also implies that if a baryon is allowed to fall into a black hole, since the final configuration is again a member of the Kerr-Newman family, then the meson field of the baryon must fall to zero as the source crosses the event horizon. Rowan and Stephenson (1976 a,b) first considered the case of the meson field of a baryon falling into a Schwarzschild black hole. The problem was treated quasi-statically, so that the baryon was assumed to be at rest at each stage of its infall. Subsequently, Rowan (1977) treated the problem of infall into a rotating black hole, with the baryon on the axis of rotation. The problem can again be treated quasi-statically, despite the existence of the ergosphere associated with the rotating black hole. This is a finite region, exterior to the event horizon, bounded by the so-called static limit, within which no particle may remain at rest. The static limit and the event horizon coincide at the poles, so that baryon infall along the axis encounters no portion of the ergosphere before reaching the event horizon. If the infall were along any path which passed through the ergosphere, it could not be treated quasi-statically. It is this problem which we discuss in Chapter 2. In Chapter 3, we consider the existence of solutions of non-linear equations describing self-interacting meson fields exterior to a non-rotating black hole.

In the work of Rowan and Stephenson referred to above, the Klein-Gordon equation describing the meson field was solved approximately over the whole range exterior to the black hole by an application of the Liouville-Green asymptotic method (see Olver, 1974). This technique may be applied to second order, linear differential equations in normal form, and is equivalent to the method of comparison equations briefly discussed by Berry and Mount (1972). An advantage of this method over the J.W.K.B. approximation, also discussed by Berry and Mount (see also Fröman and Fröman, 1965), is that it removes the need for matching at the turning points of the equation (see Appendix 1). The usefulness of the technique is demonstrated in this thesis by its application to the Schrödinger equation with an anharmonic oscillator potential in Chapter 5, and to a Schrödinger-type equation associated with propagation along a dielectric waveguide in Chapter 6. In both cases approximate eigenvalues are obtained after a small amount of computing, and at the very least, these could be used as the starting point for more sophisticated computing methods.

The Liouville-Green technique consists in the transformation of the differential equation of interest into another, with the same number of turning points, which is soluble in terms of known functions, either exactly or approximately. In the latter case the neglected term appears in closed form, in contrast to the J.W.K.B. approximation. Despite the extensive literature on propagation in dielectric waveguides (see Olshansky, 1979, and the references cited therein), the Liouville-Green technique appears not to have been used in this field.

CHAPTER 2

NON-STATIC NUCLEAR FORCES IN A KERR-NEWMAN BACKGROUND SPACE

1. Introduction

In a series of papers (Rowan and Stephenson, 1976 a,b, 1977 and Rowan, 1977), the Klein-Gordon equation for a massive scalar meson field has been examined in various background spaces by an application of the Liouville-Green asymptotic method (see Appendix 1). Rowan (1977) has extended the work of Rowan and Stephenson to the infall of an uncharged baryon down the axis of a charged, rotating black hole described by the Kerr-Newman metric and has shown that the field of the baryon source falls to zero as the source crosses the event horizon. By allowing the particle to move down the axis of rotation, Rowan was able to treat the infall as a series of quasi-static problems since the event horizon and the static limit coincide on the axis of rotation.

In this chapter, we extend this work to the infall of a baryon along a geodesic in the equatorial plane of the black hole. This requires that the source term be modified to a time-dependent one, since the tidal forces destroy the static situation. By solving the geodesic equations near the event horizon and using the solutions of the Klein-Gordon equation near the event horizon as found by Rowan and Stephenson (1977), we have again deduced that the field of the baryon falls to zero as the particle crosses the event horizon. It has not been possible to solve the basic differential equation over the whole range owing to the breakdown of the asymptotic method. The reason for this will emerge in the following analysis.

In sections 7 and 8, we extend to massive scalar fields the work of de Felice (1979) on massless scalar fields in a Kerr background space without an event horizon. We compare the Klein-Gordon equation in this case to that obtained in the Kerr black hole case.

2. Basic equations

We begin with the Klein-Gordon equation

$$(\Box^2 + \mu^2)\Phi = 4\pi f(t, r, \theta, \varphi) \quad , \tag{2.1}$$

where $\Phi = \Phi(t, r, \theta, \varphi)$ is the scalar field, $f(t, r, \theta, \varphi)$ represents a point source and μ is the inverse Compton wavelength of the meson associated with the field. In generally covariant form (2.1) is

$$\frac{1}{\sqrt{-g_4}}\frac{\partial}{\partial x^i}\left(\sqrt{-g_4}\, g^{ik}\frac{\partial \Phi}{\partial x^k}\right) + \mu^2\Phi$$
$$= 4\pi f(t, r, \theta, \varphi) \quad , \tag{2.2}$$

where g_4 is the determinant of the metric tensor g_{ik}. Together with the Kerr-Newman metric in Boyer-Lindquist coordinates describing a body of mass M, charge Q and angular momentum per unit mass a,

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$
$$- \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)d\varphi - a\,dt\right]^2 , \tag{2.3}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \quad , \quad \rho^2 = r^2 + a^2\cos^2\theta \; , \qquad (2.4)$$

equation (2.2) becomes

$$\left\{ \frac{\left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right]}{\Delta} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial r}\left(\Delta \frac{\partial}{\partial r}\right) \right.$$

$$- \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{(\Delta - a^2\sin^2\theta)}{\Delta\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\left. - \frac{2a\left[\Delta - (r^2+a^2)\right]}{\Delta}\frac{\partial^2}{\partial\varphi\partial t} + \rho^2\mu^2 \right\} \Phi = 4\pi\rho^2 f(t,r,\theta,\varphi) . \qquad (2.5)$$

Throughout, we take $c = G = 1$, and until section 7, we take $M^2 \geqslant a^2 + Q^2$ so that the metric (2.3) describes a black hole. Then Δ in (2.4) is zero at $r = M \pm \sqrt{M^2 - a^2 - Q^2}$, the larger root being the event horizon of the black hole.

Now write

$$\Phi = \sum_{l,m} \int d\omega \left(R_{lm\omega}(r) S_l^m(\theta) e^{im\varphi} e^{-i\omega t} \right) , \qquad (2.6)$$

where $S_l^m(\theta) = S_l^m\left(a\sqrt{\mu^2 - \omega^2}\,,\, \cos\theta\right)$ is the prolate spheroidal harmonic satisfying

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - a^2(\mu^2 - \omega^2)\cos^2\theta \right.$$

$$\left. + \lambda_{lm} - \frac{m^2}{\sin^2\theta} \right] S_l^m(\theta) = 0 \qquad (2.7)$$

and λ_{lm} is the eigenvalue corresponding to $S_l^m(\theta)$. Taking the normalisation

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \left| S_l^m(\theta) \right|^2 d\theta = 1 \qquad (2.8)$$

and substituting (2.6)-(2.8) into (2.5), we find that $R_{lm\omega}(r)$ satisfies

$$\left\{ \frac{d}{dr}\left(\Delta \frac{d}{dr}\right) + \frac{a^2m^2 + 2am\omega(Q^2 - 2Mr) + (r^2+a^2)^2\omega^2}{\Delta} - \lambda_{lm} - a^2\omega^2 - \mu^2r^2 \right\} R_{lm\omega}(r)$$

$$= -2\int_{-\infty}^{+\infty} dt\, e^{i\omega t} \int_0^{2\pi} d\varphi \int_0^{\pi} \rho^2 \sin\theta\, S_l^m(\theta) e^{-im\varphi} f(t,r,\theta,\varphi)\, d\theta. \tag{2.9}$$

3. The geodesic equations

We take the equations of motion along a geodesic in a Kerr-Newman background space (Misner, Thorne and Wheeler, 1973, p.899) and consider the case of motion confined to the equatorial plane of a black hole. The equations are

$$\left.\begin{aligned} r^2\frac{dr}{d\lambda} &= \sqrt{R} \\ r^2\frac{d\varphi}{d\lambda} &= -(aE - L_z) + \frac{aP}{\Delta} \\ r^2\frac{dt}{d\lambda} &= -a(aE - L_z) + \frac{(r^2+a^2)P}{\Delta} \end{aligned}\right\} \tag{3.1}$$

where

$$\left.\begin{aligned} P &= E(r^2+a^2) - aL_z \\ R &= P^2 - \Delta\left[\bar{\mu}^2r^2 + (L_z - aE)^2\right] \end{aligned}\right\} \tag{3.2}$$

and where $\bar{\mu}$ is the rest-mass of the baryon, and E and L_z are the energy at infinity and the angular momentum about the axis of rotation, respectively. Putting

$J = L_z - aE$, we get from (3.2)

$$\left.\begin{aligned} P &= Er^2 - aJ \\ R &= (Er^2 - aJ)^2 - \Delta(\bar{\mu}^2 r^2 + J^2) \end{aligned}\right\} . \tag{3.3}$$

From (3.1) using (3.3) we have

$$\frac{d\varphi}{dt} = \frac{J\Delta + a(Er^2 - aJ)}{aJ\Delta + (r^2 + a^2)(Er^2 - aJ)} \tag{3.4}$$

and

$$\frac{dr}{dt} = \frac{\Delta\left\{(Er^2 - aJ)^2 - \Delta(\bar{\mu}^2 r^2 + J^2)\right\}^{1/2}}{aJ\Delta + (r^2 + a^2)(Er^2 - aJ)} . \tag{3.5}$$

From (3.4), we see that an infalling particle cannot follow a radial path $\varphi =$ constant, even if $L_z = 0$, unless a is also zero.

We now confine our attention to motion near the event horizon $r = r_+ = M + \sqrt{M^2 - a^2 - Q^2}$ and assume that $a^2 + Q^2 \neq M^2$ so that $r_+ \neq r_- = M - \sqrt{M^2 - a^2 - Q^2}$. Putting

$$Mx = r - r_+ , \quad 2Md = r_+ - r_- , \tag{3.6}$$

we have

$$\Delta = M^2 x(x + 2d) . \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.4) and (3.5), we may expand the right-hand sides in powers of x to obtain

$$\frac{d\varphi}{dt} = \frac{a}{(r_+^2 + a^2)} + \alpha x + O(x^2) \tag{3.8}$$

and

$$\frac{dr}{dt} = M\frac{dx}{dt} = -\left(\frac{2dM^2 x}{(r_+^2 + a^2)} + \beta x^2\right) + O(x^3) , \tag{3.9}$$

where the constants α and β are given by

$$\alpha = \frac{2M}{(r_+^2 + a^2)^2(Er_+^2 - aJ)}\left(MJdr_+^2 - aEr_+^3 + a^2Jr_+\right) \qquad (3.10)$$

and

$$\beta = \frac{M^2}{(r_+^2+a^2)}\left[\frac{Md(4E^2r_+^3 - 4aJEr_+ - 2Md\bar{\mu}^2r_+^2 - 2MdJ^2)}{(Er_+^2 - aJ)^2}\right.$$

$$\left. - \frac{2Md(4Er_+^3 + 2a^2Er_+ - 2aJr_+ + 2aJMd)}{(r_+^2+a^2)(Er_+^2 - aJ)} + 1\right]. \qquad (3.11)$$

4. The source term

To get an explicit expression for $f(t, r, \theta, \varphi)$, we choose, following Persides (1974),

$$f(t, r, \theta, \varphi) = g\frac{1}{u^0}\delta^{(3)}(\underline{r} - \underline{r}') , \qquad (4.1)$$

where $u^0 = dt/ds$ along the trajectory of the particle $\underline{r}'(t) = (r'(t), \theta'(t), \varphi'(t))$ and g is the source strength. To calculate u^0, we first put (3.6) and (3.7), together with $\theta = \pi/2$, into the metric (2.3), obtaining

$$\left(\frac{ds}{dt}\right)^2 = \frac{M^2x(x+2d)}{(Mx+r_+)^2}\left(1 - a\frac{d\varphi}{dt}\right)^2 - \frac{(Mx+r_+)^2}{x(x+2d)}\left(\frac{dx}{dt}\right)^2$$

$$- \frac{1}{(Mx+r_+)^2}\left[\left\{(Mx+r_+)^2 + a^2\right\}\frac{d\varphi}{dt} - a\right]^2 . \qquad (4.2)$$

Then substituting (3.8) and (3.9) into (4.2), we see that to second order in x,

$$\left(\frac{1}{u^0}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \gamma x^2 , \qquad (4.3)$$

where the constant γ is given by

$$\gamma = \frac{(2M^2r_+^2 - 8dM^3r_+)}{(r_+^2 + a^2)^2} - \frac{(4adM^2\alpha + 2r_+^2\beta)}{(r_+^2 + a^2)}$$

$$- \frac{1}{r_+^2}\left[\frac{2Mar_+}{(r_+^2 + a^2)} + (r_+^2 + a^2)\alpha\right]^2 . \tag{4.4}$$

On substituting into (4.4) the expressions for α and β from (3.10) and (3.11), we find, after considerable algebra, that γ is given simply by

$$\gamma = \frac{4d^2M^4\bar{\mu}^2r_+^4}{(r_+^2 + a^2)^2(Er_+^2 - aJ)^2} , \tag{4.5}$$

so that

$$\frac{1}{u^0} = K(r - r_+) , \tag{4.6}$$

where

$$K = \frac{2dM\bar{\mu}r_+^2}{(r_+^2 + a^2)(Er_+^2 - aJ)} . \tag{4.7}$$

From (4.1), using (4.6), the source term can be written

$$f(t,r,\theta,\varphi) = gK(r - r_+)\delta(r - r_0(\varphi_0))\delta(\varphi - \varphi_0(t))\delta(\theta - \tfrac{\pi}{2}) \tag{4.8}$$

where, from (3.8) and (3.9),

$$\varphi_0(t) = \frac{at}{(r_+^2 + a^2)} \tag{4.9}$$

and

$$r_0(\varphi_0) = r_+ + \exp\left\{-\frac{(r_+ - r_-)}{a}\varphi_0\right\} . \tag{4.10}$$

Hence the right-hand side of (2.9) becomes, on substituting (4.8),

$$-2gK(r-r_+)\,\delta(r-r_0)\int_{-\infty}^{+\infty} e^{i\omega t}\,dt\int_0^{2\pi} d\varphi$$

$$\times\int_0^{\pi} \rho^2 \sin\theta\, S_l^m(\theta)\, e^{-im\varphi}\,\delta(\varphi-\varphi_0)\,\delta(\theta-\tfrac{\pi}{2})\,d\theta \quad . \qquad (4.11)$$

After performing the Θ-integration, (4.11) becomes

$$-2gKr^2(r-r_+)\,S_l^m(\tfrac{\pi}{2})\,\delta(r-r_0)\int_{-\infty}^{+\infty} e^{i\omega t}dt\int_0^{2\pi} e^{-im\varphi}\,\delta(\varphi-\varphi_0(t))\,d\varphi \,. \qquad (4.12)$$

Now from (4.9)

$$\int_{-\infty}^{+\infty} e^{i\omega t}\,dt\int_0^{2\pi} e^{-im\varphi}\,\delta(\varphi-\varphi_0(t))\,d\varphi$$

$$=\int_0^{2\pi} d\varphi\, e^{-im\varphi}\int_{-\infty}^{+\infty} e^{i\omega t}\,\delta\left(\varphi-\frac{at}{(r_+^2+a^2)}\right)dt$$

$$=\frac{2\pi}{a}(r_+^2+a^2) \qquad \text{for} \qquad \omega=\frac{ma}{(r_+^2+a^2)}\,, \qquad (4.13)$$

so that (4.12) becomes

$$-4\pi g\frac{K}{a}(r_+^2+a^2)r^2(r-r_+)\,S_l^m(\tfrac{\pi}{2})\,\delta(r-r_0)\,. \qquad (4.14)$$

Finally, equation (2.9) becomes

$$\left\{\frac{d}{dr}\left(\Delta\frac{d}{dr}\right)+\frac{a^2m^2+2am\omega(Q^2-2Mr)+(r^2+a^2)^2\omega^2}{\Delta}\right.$$

$$\left.-\lambda_{lm}-a^2\omega^2-\mu^2r^2\right\}R_{lm\omega}(r)$$

$$=-4\pi g\frac{K}{a}(r_+^2+a^2)r^2(r-r_+)\,S_l^m(\tfrac{\pi}{2})\,\delta(r-r_0)\,. \qquad (4.15)$$

5. The radial equation

Rowan and Stephenson (1977) have shown that after defining x and d by

$$Mx = r - r_+ \quad , \quad 2Md = r_+ - r_- \tag{5.1}$$

and writing

$$R_{lm\omega}(x) = Z(x)\left[x(x+2d)\right]^{-\frac{1}{2}} , \tag{5.2}$$

substitution of (5.1) and (5.2) into (4.15) leads to (for $r \neq r_0$)

$$\frac{d^2Z}{dx^2} + \left[M^2(\omega^2 - \mu^2) + \frac{1}{M^2}\left(\frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x+2d)^2} + \frac{D}{(x+2d)}\right)\right]Z = 0 \tag{5.3}$$

where A, B, C and D are constants given by

$$A = \frac{1}{4d^2}\Big[M^2d^2 + m^2a^2 + 2Q^2a\omega m - 4a\omega m M^2(d+1) + \omega^2Q^4 - 4M^2\omega^2Q^2(d+1) + 4M^4\omega^2(d+1)^2\Big], \tag{5.4}$$

$$B = \frac{1}{4d^3}\Big[4\omega^2M^4(d+1)^2(2d-1) - 4\omega^2M^2Q^2(d^2-1) + 4a\omega m M^2 - 2\mu^2M^4(d+1)^2d^2 - 2d^2(\omega^2a^2 + \lambda_{lm})M^2 - M^2d^2 - m^2a^2 - 2Q^2a\omega m - \omega^2Q^4\Big], \tag{5.5}$$

$$C = \frac{1}{4d^2}\Big[4\omega^2M^4(d-1)^2 + 4M^2\omega^2Q^2(d-1) + M^2d^2 + m^2a^2 + 2Q^2a\omega m + \omega^2Q^4\Big], \tag{5.6}$$

and

$$D = \frac{1}{4d^3}\Big[4\omega^2 M^4(2d+1)(d-1)^2 + 4\omega^2 M^2 Q^2(d^2-1)$$
$$-4a\omega m M^2 + 2\mu^2 M^4 d^2(d-1)^2 + 2d^2(\omega^2 a^2 + \lambda_{lm})M^2$$
$$+M^2 d^2 + m^2 a^2 + 2Q^2 a\omega m + \omega^2 Q^4\Big] \,. \qquad (5.7)$$

It has not been possible to solve the homogeneous equation (5.3) over the whole range $0<x<\infty$ due to the breakdown of the asymptotic method. This is due to the fact that the accuracy of the method depends on the existence of a large parameter in the differential equation which we do not necessarily have in (5.3), since ω may be close to or equal to μ . Also (5.3) will have, in general, four turning points (see Appendix 1), and we have no standard four-turning-point equation to which we can transform. However, we are chiefly interested in the behaviour of the solutions as $x\to 0$ (that is $r\to r_+$) and may use the solutions near the event horizon obtained by Rowan and Stephenson (1977). These are

$$R_{lm\omega}(r) \sim \begin{cases} R_{(1)}(r) = \dfrac{M}{\Delta^{1/2}}\, e^{-\left(\frac{\sqrt{F}}{M}\right)(r-r_+)}(r-r_+)^{\frac{1}{2}+\bar{m}} \\[2ex] R_{(2)}(r) = \dfrac{M}{\Delta^{1/2}}\, e^{-\left(\frac{\sqrt{F}}{M}\right)(r-r_+)}(r-r_+)^{\frac{1}{2}-\bar{m}} \end{cases} \qquad (5.8)$$

where

$$\left.\begin{aligned} \bar{m}^2 &= \frac{1}{4} - \frac{A}{M^2} \\ F &= M^2(\mu^2-\omega^2) - \frac{C}{4M^2 d^2} - \frac{D}{2M^2 d} \end{aligned}\right\} , \qquad (5.9)$$

provided $F \neq 0$. The case of $F = 0$ was treated separately and we will not repeat the solutions here.

We now integrate (4.15) across the singularity and impose continuity of $R_{lm\omega}(r)$ at $r = r_0$ to obtain

$$\Delta_0 \left\{ \frac{dR_{lm\omega}}{dr}\bigg|_{r_0+0} - \frac{dR_{lm\omega}}{dr}\bigg|_{r_0-0} \right\}$$

$$= -4\pi g \frac{K}{a}(r_+^2 + a^2) r_0^{\,2}(r_0 - r_+) S_l^m\left(\frac{\pi}{2}\right) , \qquad (5.10)$$

where $\Delta_0 = (r_0 - r_+)(r_0 - r_-)$. Then for r_0 near r_+, we have

$$R_{lm\omega}(r) = 4\pi g \frac{K}{a}(r_+^2 + a^2) r_0^{\,2}(r_0 - r_+) S_l^m\left(\frac{\pi}{2}\right)$$

$$\times \begin{cases} R_{(2)}(r_0) R_{(1)}(r) & (r_+ < r \leqslant r_0) \\ R_{(1)}(r_0) R_{(2)}(r) & (r_0 \leqslant r) \end{cases} \qquad (5.11)$$

where $R_{(1)}$ and $R_{(2)}$ are given by (5.8).

Now from (5.4) and (5.9)

$$\bar{m}^2 = -\frac{1}{4d^2M^2}\Big[m^2a^2 + 2Q^2 a\omega m - 4a\omega m M^2(d+1)$$

$$+ \omega^2 Q^4 - 4M^2\omega^2 Q^2(d+1) + 4M^4\omega^2(d+1)^2\Big]$$

$$= -\frac{1}{4d^2M^2}\left[ma + \omega Q^2 - 2\omega M^2(d+1)\right]^2 , \qquad (5.12)$$

and since this is negative, $\bar{m}$ is imaginary and we can write

$$\bar{m} = ib , \qquad (5.13)$$

where $i = \sqrt{-1}$ and

$$b = \frac{1}{2dM}\left| ma + \omega Q^2 - 2\omega M^2(d+1) \right| . \qquad (5.14)$$

Hence, from (5.8), it follows that $R_{(1)}$ and $R_{(2)}$ have the behaviour $(r-r_+)^{ib}$ and $(r-r_+)^{-ib}$, respectively, as $r \to r_+$. Alternatively, we could redefine $R_{(1)}$ and $R_{(2)}$ to be the solutions with behaviour $\cos(b\ln[r-r_+])$ and $\sin(b\ln[r-r_+])$, respectively, since

$$\begin{aligned}(r-r_+)^{\pm ib} &= \exp\{\pm ib\ln(r-r_+)\} \\ &= \cos(b\ln[r-r_+]) \pm i\sin(b\ln[r-r_+]) .\end{aligned}$$

Although both solutions exhibit increasingly frequent oscillations as $r \to r_+$, they are bounded.

If we now let $r_0 \to r_+$, we see from (5.7) that $R_{lm\omega}(r)$, for $r_0 < r$, tends to zero since $R_{(1)}(r_0)$ is bounded. Provided the series for Φ is uniformly convergent, as has been proved in the case of a Schwarzschild black hole (Rowan and Stephenson, 1976 a), then $\Phi \to 0$ as $r_0 \to r_+$. We note that Φ here is an expression for the scalar field near the event horizon since (5.8) gives solutions of (5.3) only for r near r_+.

6. Special cases

The case $a^2 + Q^2 = M^2$ must be considered separately since $d = 0$ and consequently from (3.9), the term of order x in dx/dt is zero. To simplify the algebra and to begin a connection with the following sections of this chapter, we consider the case of an extreme Kerr black hole, so that $a = M$, $Q = 0$ and $r_+ = r_- = M$. Expanding $d\varphi/dt$ to order x^2 and dx/dt to order x^4 and substituting into (4.2) with $d = 0$ and $a = M$, we find

$$\left(\frac{1}{u^0}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \bar{\mu}^2 \frac{M^2 x^4}{4(EM-J)^2} \tag{6.1}$$

to order x^4. Hence

$$\frac{1}{u^0} = \bar{\mu}\, \frac{(r-M)^2}{2M(EM-J)} \tag{6.2}$$

which can be used in place of (4.6). The homogeneous radial equation becomes

$$\frac{d^2 Z}{dx^2} + \left[M^2(\omega^2-\mu^2) + \frac{\bar{A}}{x} + \frac{\bar{B}}{x^2} + \frac{\bar{C}}{x^3} + \frac{\bar{D}}{x^4}\right] Z = 0 \tag{6.3}$$

where

$$R_{lm\omega}(x) = x^{-1} Z(x) \quad , \quad Mx = r - M \tag{6.4}$$

and the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are given by

$$\left.\begin{aligned} \bar{A} &= 4M^2\omega^2 - 2M^2\mu^2 \\ \bar{B} &= 7M^2\omega^2 - M^2\mu^2 - \lambda_{lm} \\ \bar{C} &= 8M^2\omega^2 - 4Mm\omega \\ \bar{D} &= 4M^2\omega^2 - 4Mm\omega + m^2 \end{aligned}\right\} . \tag{6.5}$$

The relation between ω and m is now, from (4.13),

$$\omega = \frac{m}{2M} \tag{6.6}$$

and on substituting (6.6) into (6.5) we find

$$\left.\begin{aligned}\bar{A} &= m^2 - 2M^2\mu^2 \\ \bar{B} &= \tfrac{7}{4}m^2 - M^2\mu^2 - \lambda_{lm} \\ \bar{C} &= \bar{D} = 0\end{aligned}\right\} . \tag{6.7}$$

Equation (6.3) now becomes

$$\frac{d^2Z}{dx^2} = \left(N - \frac{\bar{A}}{x} - \frac{\bar{B}}{x^2}\right)Z \ , \tag{6.8}$$

where

$$N = M^2\mu^2 - \frac{m^2}{4} \ . \tag{6.9}$$

After defining

$$\eta = 2\sqrt{N}\,x \tag{6.10}$$

and substituting (6.10) into (6.8), we see that (6.8) has solutions in terms of Whittaker functions (see Whittaker and Watson, 1927)

$$Z = M_{K,\pm q}(\eta) \ , \tag{6.11}$$

where

$$K = \frac{\bar{A}}{2\sqrt{N}} \ , \quad q^2 = \frac{1}{4} - \bar{B} \ . \tag{6.12}$$

The solutions (6.11) form two independent solutions of (6.8) provided $2q$ is not an integer. From the asymptotic behaviour of $M_{K,q}(\eta)$ as $\eta \rightarrow 0$ and the transformation (6.4), we have

$$R_{(1)}(r) \sim \frac{M}{\Delta^{1/2}}\, e^{-\left(\sqrt{N}/M\right)(r-M)}\,(r-M)^{\frac{1}{2}+q} \tag{6.13}$$

as $r \rightarrow M$. Again, $R_{lm\omega}(r)$ tends to zero for $r_0 < r$ since $R_{(1)}(r_0)$ either tends to zero if q is real or is bounded, as before, if q is imaginary.

For the case $N=0$, we get

$$Z = \begin{cases} x^{1/2} I_{\bar{\alpha}}(\bar{\beta} x^{1/2}) \\ x^{1/2} K_{\bar{\alpha}}(\bar{\beta} x^{1/2}) \end{cases} \tag{6.14}$$

where $I_{\bar{\alpha}}$, $K_{\bar{\alpha}}$ are the modified Bessel functions of order $\bar{\alpha}$ of the first and second kind, respectively, and

$$\bar{\alpha}^2 = 1 - 4\bar{B} \ , \ \bar{\beta}^2 = -4\bar{A} \ . \tag{6.15}$$

The radial solution for $r_0 < r$ has the behaviour

$$R_{(1)}(r) \sim \frac{M^{1/2}}{\Delta^{1/2}} (r - M)^{\frac{1}{2} + \frac{\bar{\alpha}}{2}} \tag{6.16}$$

as $r \to M$, and either tends to zero if $\bar{\alpha}$ is real or is bounded if $\bar{\alpha}$ is imaginary.

In all cases, therefore, we have the result that the scalar field Φ tends to zero as the source crosses the event horizon. Although, in general, the solutions of the homogeneous radial equation exhibit increasingly frequent oscillations as $r_0 \to r_+$, these are damped by a power of $(r_0 - r_+)$ arising from the source term.

In the next two sections, we examine the massive Klein-Gordon equation in the Kerr metric with $a > M$ and study the radial solutions near the surface $r = M$, where an event horizon forms when $a = M$.

7. The Kerr metric with $a > M$

The metric is (2.3) and (2.4), with $Q = 0$. We shall take $a > M$ but close enough to M so that

$$k^2 = a^2 - M^2 \tag{7.1}$$

can be considered small; for example $k/a \ll 1$. The work of de Felice (1979) was concerned with massless fields in this naked singularity background, and here we make an extension to massive fields, while following his approach.

First we take the radial equation (4.15) without the source term, putting $Q = 0$,

$$x = \frac{r - M}{k} \tag{7.2}$$

and

$$y(x) = (1 + x^2)^{1/2} R_{lm\omega}(x) \ . \tag{7.3}$$

Equation (4.15) then becomes

$$\left(\frac{d^2}{dx^2} + \frac{H(x)}{(1 + x^2)^2} \right) y(x) = 0 \ , \tag{7.4}$$

where

$$H(x) = \frac{\alpha_0}{k^2} + \frac{\alpha_1}{k} x + \alpha_2 x^2 + k\alpha_3 x^3 + k^2 \alpha_4 x^4 \tag{7.5}$$

and

$$\alpha_0 = \omega^2 (M^2 + a^2)^2 + a^2 m^2 - 4am\omega M^2$$
$$- k^2 (\lambda_{lm} + a^2\omega^2 + \mu^2 M^2 + 1) \ , \tag{7.6}$$

$$\alpha_1 = 4M\omega^2 (M^2 + a^2) - 4am\omega M - 2Mk^2\mu^2 , \tag{7.7}$$

$$\alpha_2 = 2\omega^2 (a^2 + 3M^2) - (\lambda_{lm} + a^2\omega^2 + a^2\mu^2) , \tag{7.8}$$

$$\alpha_3 = 4M\omega^2 - 2M\mu^2 \ , \tag{7.9}$$

$$\alpha_4 = \omega^2 - \mu^2 \ . \tag{7.10}$$

Equation (7.4) will again have four turning points, in general, and we encounter the same difficulties as in the black hole case regarding a solution over the whole range.

Putting

$$\Omega = \frac{a}{a^2 + M^2} \quad , \quad \beta = \frac{\omega}{\Omega} \tag{7.11}$$

into (7.6)-(7.10), the constants α_0, α_1, α_2, α_3 and α_4 can be written

$$\alpha_0 = a^2(\beta - m)^2 + k^2(2am\beta\Omega - \lambda_{lm} - 1 - a^2\omega^2 - \mu^2 M^2) , \tag{7.12}$$

$$\alpha_1 = 4\Omega^2 M(M^2 + a^2)\beta(\beta - m) - 2Mk^2\mu^2 , \tag{7.13}$$

$$\alpha_2 = 2\beta^2\Omega^2(a^2 + 3M^2) - (\lambda_{lm} + a^2\omega^2 + a^2\mu^2), \tag{7.14}$$

$$\alpha_3 = 4\beta^2\Omega^2 M - 2M\mu^2 , \tag{7.15}$$

$$\alpha_4 = \beta^2\Omega^2 - \mu^2 . \tag{7.16}$$

Two possible large parameters in (7.12)-(7.16) are μM, as in the work of Rowan and Stephenson, and λ_{lm} , where if l is large

$$\lambda_{lm} \sim l(l+1) = t^2 \tag{7.17}$$

(see Meixner and Schäfke, 1954). We shall consider each of these possibilities.

8. Bound states

(i) l large

In (7.4), we expand the term $H(x)(1 + x^2)^{-2}$, using (7.5), in powers of x and retain terms up to and including x^2 , so that

$$\frac{d^2y}{dx^2} + \left\{ \frac{\alpha_0}{k^2} + \frac{\alpha_1}{k} x + \left(\alpha_2 - \frac{2\alpha_0}{k^2}\right) x^2 \right\} y = 0 . \tag{8.1}$$

In contrast to a corresponding expansion of (5.3) near $x=0$, equation (8.1) has the appearance of a harmonic oscillator and accordingly, we look for bound states. Following de Felice we demand $\alpha_0 > 0$ in order to have travelling wave solutions across $x=0$. Further, we require that α_0/k^2 be finite in the limit of small k and

$$\frac{\alpha_0}{k^2 t^2} \ll 1 \quad . \tag{8.2}$$

From (7.12) and (7.17), these conditions imply

$$|\beta - m| \gtrsim \frac{kt}{a} \quad . \tag{8.3}$$

Now from (7.14),

$$\alpha_2 \sim -t^2 \tag{8.4}$$

so that, using (8.2),

$$\alpha_2 - \frac{2\alpha_0}{k^2} \sim -t^2 \quad . \tag{8.5}$$

Using the result (8.5), (8.1) becomes

$$\frac{d^2y}{dx^2} = \left(-\frac{\alpha_0}{k^2} - \frac{\alpha_1}{k}x + t^2x^2\right)y \quad . \tag{8.6}$$

The substitution

$$z = \left(x - \frac{\alpha_1}{2kt^2}\right)\sqrt{2t} \tag{8.7}$$

transforms (8.6) into

$$\frac{d^2y}{dz^2} = \left(\frac{1}{4}z^2 - E\right)y \quad , \tag{8.8}$$

where

$$E = \left(\frac{\alpha_0}{k^2} + \frac{\alpha_1^2}{4k^2t^2}\right)\frac{1}{2t} \quad . \tag{8.9}$$

As z increases, the parabolic cylinder function solution of (8.8) has a factor $e^{-z^2/4} z^{E-1/2}$ (see Abramowitz and Stegun, 1964, p.689). The exponential term behaves as $e^{-t(r-M)/2k}$ and hence severely damps the solution. We shall therefore take the boundary condition that the solution of equation (8.8) is zero at infinity, giving

$$E = n + \frac{1}{2} \quad , \tag{8.10}$$

where $n = 0,1,2,\ldots.$ Then from (8.9)

$$\frac{\alpha_0}{k^2} + \frac{\alpha_1^2}{4k^2t^2} = (2n+1)t \ . \tag{8.11}$$

Now from (7.12) and (7.13)

$$\alpha_0 = a^2(\beta - m)^2 - k^2t^2 + O(k^2) \tag{8.12}$$

and

$$\alpha_1 = 4\Omega^2 M(M^2+a^2)\beta(\beta-m) + O(k^2) \ . \tag{8.13}$$

To satisfy (8.3), we put

$$\beta = m \pm \left(\frac{kt}{a} + \epsilon\right) \quad , \tag{8.14}$$

where ϵ is a small positive quantity. From (8.12) and (8.13) we find that

$$\alpha_0 \sim 2akt\epsilon \tag{8.15}$$

and

$$\alpha_1 \sim \pm 4\Omega^2 M(M^2+a^2)m\frac{kt}{a} \ . \tag{8.16}$$

On substituting (8.15) and (8.16) into (8.11) we arrive at an equation for ϵ :

$$\frac{2at\epsilon}{k} = (2n+1)t - 4\Omega^4\frac{M^2}{a^2}(M^2+a^2)^2m^2 \ . \tag{8.17}$$

The dominant term on the right-hand side is $(2n+1)t$, so that

$$\epsilon = \frac{k}{a}\left(n + \frac{1}{2}\right) \tag{8.18}$$

giving, finally,

$$\beta = m \pm \frac{k}{a}\left(t + n + \frac{1}{2}\right) \tag{8.19}$$

to order k .

This is almost the result obtained by de Felice, despite an error in his expression for the eigenvalue E . The result (8.19) is seen to be consistent with the previous assumptions (8.2) and (8.3).

(ii) μM large

If the parameter μM is large then μa will also be large since a is close to M . The procedure is very similar to that in (i) above and involves only the replacement of t by either μa or μM. The result, to order k , is

$$\beta = m \pm \frac{k}{a}\left(\mu M + n + \frac{1}{2}\right) \tag{8.20}$$

We note that as k decreases ($a \rightarrow M$), the expressions for β tend to $\beta = m$. From (7.11), we see that this is equivalent to the result (6.6) which was obtained in the extreme Kerr black hole case, where $a = M$.

9. Discussion

The success of the Liouville-Green method when used to solve the radial equation for the massive scalar meson field (Rowan and Stephenson 1976 a,b, Rowan, 1977, see also Chapters 5 and 6 of this thesis) depends on the appearance of a large parameter in the differential equation. This was due to the non-zero rest-mass of the π -meson. When considering the Klein-Gordon equation in a Kerr-Newman background space, solutions of the radial equation over the whole range are known only in special cases (Rowan, 1977); the equation may no longer contain a large parameter and in general will have four turning points. Although in principle it would be possible to match solutions so as to cover the whole range, the complexity is prohibitive. Information about the solutions in regions of interest can nevertheless be obtained, as in this chapter.

CHAPTER 3

NON-LINEAR WAVE EQUATIONS IN CURVED BACKGROUND SPACES

1. Introduction

In Chapter 2, we were concerned with the Klein-Gordon scalar wave equation in curved background spaces, the solutions of which were related to the infall of baryons into black holes. In this chapter, we consider whether it is possible to have soliton-like solutions of the non-linear Klein-Gordon equation containing self-interaction terms.

The origin of this consideration is the work of Derrick (1964) who has shown that for a wide class of non-linear wave equations, there exist no stable time-independent solutions in two or more space dimensions, other than constant solutions. This result was established only in Minkowski space and is not applicable to the one-dimensional case where stable, non-constant solutions have been obtained (Enz, 1963). Accordingly, in the space exterior to a non-rotating black hole, we investigate the spherically symmetric case where the wavefunction depends only on the radial coordinate. We shall examine the non-linear Klein-Gordon equation in both infinite and finite background spaces in an attempt to generalise the results of Derrick.

2. Minkowski space

We begin by briefly reviewing the method and results of Derrick (1964) who, in Minkowski space, considers the non-linear equation

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = -\frac{1}{2} f'(\Phi) , \tag{2.1}$$

derivable from the variational principle

$$\delta \int \left\{ \left(\frac{\partial \Phi}{\partial t} \right)^2 - (\nabla \Phi)^2 - f(\Phi) \right\} d^n \underline{r}\, dt = 0 . \tag{2.2}$$

Here Φ is the scalar field in one time and n space dimensions, where $n = 1,2$ or 3, and primes denote differentiation with respect to Φ . For time-independent solutions, (2.2) can be written

$$\delta E = 0 \tag{2.3}$$

where

$$E = \int \left\{ (\nabla \Phi)^2 + f(\Phi) \right\} d^n \underline{r} . \tag{2.4}$$

A necessary condition for the solution Φ to be stable is

$$\delta^2 E \geqslant 0 , \tag{2.5}$$

for all possible variations of the wavefunction. It is sufficient therefore to find a particular variation of Φ which violates (2.5) in order to prove instability.

Derrick proceeded by defining

$$\Phi_\alpha(r) = \Phi(\alpha r) , \tag{2.6}$$

where α is an arbitrary constant and writing

$$E = I_1 + I_2 , \tag{2.7}$$

where

$$I_1 = \int (\nabla\Phi)^2 \, d^n\underline{r} \geqslant 0 \tag{2.8}$$

and

$$I_2 = \int f(\Phi) \, d^n\underline{r} \quad . \tag{2.9}$$

Then

$$E_\alpha = \int \left\{ (\nabla\Phi_\alpha)^2 + f(\Phi_\alpha) \right\} d^n\underline{r} \tag{2.10}$$

$$= \alpha^{2-n} I_1 + \alpha^{-n} I_2 \quad , \tag{2.11}$$

after changing the variable from $\underline{r}$ to $\alpha\underline{r}$. Hence

$$\left.\frac{dE_\alpha}{d\alpha}\right|_{\alpha=1} = (2-n)I_1 - nI_2 \tag{2.12}$$

and

$$\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = (2-n)(1-n)I_1 + n(n+1)I_2 \, . \tag{2.13}$$

Now (2.3) and (2.5) imply that

$$(2-n)I_1 = nI_2 \tag{2.14}$$

and

$$(2-n)(1-n)I_1 + n(n+1)I_2 \geqslant 0 \quad . \tag{2.15}$$

We now take $f(\Phi) \geqslant 0$, so that $I_2 \geqslant 0$ from (2.9). If $n = 1$, we see from (2.14) that $I_1 = I_2$ and that (2.15) is satisfied. If $n = 2$, then from (2.14) $I_2 = 0$ and so from (2.9) Φ is a constant subject to $f(\Phi) = 0$. If $n = 3$, then from (2.14) $I_1 = I_2 = 0$, since both I_1 and I_2 are positive, and we again have $f(\Phi) = 0$.

3. Reissner-Nordstrøm background space

We now consider the unbounded space exterior to a non-rotating black hole of mass M and charge Q described by the Reissner-Nordstrøm metric

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2$$
$$- r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 . \qquad (3.1)$$

We write (2.1) in covariant form as

$$\Box^2 \Phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i}\left(\sqrt{-g}\, g^{ik} \frac{\partial \Phi}{\partial x^k}\right) = -\frac{1}{2} f'(\Phi) , \qquad (3.2)$$

arising from the variational principle

$$\delta \int \left\{ g^{ik} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^k} - f(\Phi) \right\} \sqrt{-g} \, d^4x = 0 , \qquad (3.3)$$

where g is the determinant of the metric tensor g_{ik} . Using (3.1) and taking Φ to be a function of r only, we obtain the radial equation

$$\frac{1}{r^2} \frac{d}{dr}\left\{ (r^2 - 2Mr + Q^2) \frac{d\Phi}{dr} \right\} = \frac{1}{2} f'(\Phi) . \qquad (3.4)$$

In this case, the variational principle (3.3) is equivalent to

$$\delta E = 0 \qquad (3.5)$$

where the energy E of the Φ -field is given by

$$E = 4\pi \int_{r_+}^{\infty} \left\{ (r^2 - 2Mr + Q^2)\left(\frac{d\Phi}{dr}\right)^2 + f(\Phi) r^2 \right\} dr , \qquad (3.6)$$

and where $r_+ = M + \sqrt{M^2 - Q^2}$ $(Q^2 \leq M^2)$ is the event horizon of the black hole. Writing

$$I_1 = \int_{r_+}^{\infty} (r^2 - 2Mr + Q^2)\left(\frac{d\Phi}{dr}\right)^2 dr \qquad (3.7)$$

and

$$I_2 = \int_{r_+}^{\infty} f(\Phi) r^2 \, dr \, , \tag{3.8}$$

so that

$$E = 4\pi \left(I_1 + I_2 \right) \, , \tag{3.9}$$

we must require that I_1 and I_2 be finite.

In the original presentation of this work (Radmore and Stephenson, 1978), we proceeded by considering the variation (2.6) as in the work of Derrick. However, as was shown by Palmer (1979), this variation is not permissible since the variation in Φ is then non-zero at $r = r_+$. The following variation was proposed:

$$\Phi_\alpha(r) = \Phi \left[\alpha (r - r_+) + r_+ \right] \tag{3.10}$$

and we shall use (3.10) in this section. The variation is such that $\Phi_\alpha(r_+) = \Phi(r_+)$ and $\Phi_{\alpha=1}(r) = \Phi(r)$. In (3.6), we write

$$r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-) \tag{3.11}$$

where $r_- = M - \sqrt{M^2 - Q^2}$, and define

$$E_\alpha = 4\pi \int_{r_+}^{\infty} \left\{ (r - r_+)(r - r_-) \left(\frac{d\Phi_\alpha}{dr} \right)^2 + f(\Phi_\alpha) r^2 \right\} dr. \tag{3.12}$$

In (3.12), we change the variable of integration from r to x where

$$x = \alpha (r - r_+) + r_+ \, , \tag{3.13}$$

obtaining

$$\frac{E_\alpha}{4\pi} = \int_{r_+}^{\infty} \left\{ \frac{(x - r_+)}{\alpha} \left[\frac{(x - r_+)}{\alpha} + r_+ - r_- \right] \left(\frac{d\Phi}{dx} \right)^2 \alpha^2 \right.$$
$$\left. + f(\Phi) \left[\frac{(x - r_+)}{\alpha} + r_+ \right]^2 \right\} \frac{dx}{\alpha} \, . \tag{3.14}$$

Now replacing the dummy variable x by r , we find

$$\frac{E_\alpha}{4\pi} = \int_{r_+}^{\infty} \left\{ \frac{(r-r_+)^2}{\alpha}\left(\frac{d\Phi}{dr}\right)^2 + (r-r_+)(r_+-r_-)\left(\frac{d\Phi}{dr}\right)^2 \right.$$

$$\left. + \frac{(r-r_+)^2}{\alpha^3} f(\Phi) + \frac{2r_+}{\alpha^2}(r-r_+)f(\Phi) + \frac{r_+^2}{\alpha} f(\Phi) \right\} dr. \quad (3.15)$$

To determine the first variation, we differentiate (3.15) with respect to α , so that

$$\frac{1}{4\pi}\frac{dE_\alpha}{d\alpha} = -\int_{r_+}^{\infty} \left\{ \frac{(r-r_+)^2}{\alpha^2}\left(\frac{d\Phi}{dr}\right)^2 + \frac{3(r-r_+)^2}{\alpha^4} f(\Phi) \right.$$

$$\left. + 4\frac{r_+(r-r_+)}{\alpha^3} f(\Phi) + \frac{r_+^2}{\alpha^2} f(\Phi) \right\} dr. \quad (3.16)$$

From (3.5), we must have

$$\left.\frac{dE_\alpha}{d\alpha}\right|_{\alpha=1} = 0 \quad (3.17)$$

and from (3.16), this gives

$$\int_{r_+}^{\infty} \left\{ (r-r_+)^2\left(\frac{d\Phi}{dr}\right)^2 + f(\Phi)\, r(3r-2r_+) \right\} dr = 0. \quad (3.18)$$

A further differentiation of (3.16) leads to

$$\frac{1}{8\pi}\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = \int_{r_+}^{\infty} \left\{ (r-r_+)^2\left(\frac{d\Phi}{dr}\right)^2 \right.$$

$$\left. + f(\Phi)(6r^2 - 6rr_+ + r_+^2) \right\} dr \quad (3.19)$$

and using the result (3.18) to eliminate the integral in

(3.19) containing $\left(\frac{d\Phi}{dr}\right)^2$, we arrive at a concise expression for the second variation:

$$\frac{1}{8\pi}\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = \int_{r_+}^{\infty} f(\Phi)(3r-r_+)(r-r_+)\,dr\,. \tag{3.20}$$

A necessary condition for stability is therefore

$$\int_{r_+}^{\infty} f(\Phi)(3r-r_+)(r-r_+)\,dr \geqslant 0\,, \tag{3.21}$$

while for (3.18) to be satisfied we must have

$$\int_{r_+}^{\infty} f(\Phi)\,r\,(3r-2r_+)\,dr \leqslant 0\,. \tag{3.22}$$

We note here that the terms $(3r-r_+)(r-r_+)$ in (3.21) and $r(3r-2r_+)$ in (3.22) are both positive since $r > r_+$.

We now see that if $f(\Phi) \geqslant 0$ everywhere, so that the energy density has the property of being everywhere positive, then the only solution, from (3.22), is $\Phi = C$, a constant, subject to $f(C) = 0$. Furthermore if $f(\Phi) \leqslant 0$ everywhere, then the stability condition (3.21) is only satisfied by $f(\Phi)=0$. Hence if $f(\Phi)$ has constant sign, then the only solutions are the vacuum states given by $f(\Phi) = 0$. If no restriction is made on the sign of $f(\Phi)$, then no immediate conclusion can be reached.

4. Special cases of $f(\Phi)$

We now examine two special cases of $f(\Phi)$ for which (3.2) becomes

$$(\Box^2 \pm \mu^2)\Phi + \lambda\Phi^3 = 0 \tag{4.1}$$

where λ is positive.

With the positive sign of the μ^2 term in (4.1), we have from (3.2)

$$\frac{1}{2} f'(\Phi) = \mu^2 \Phi + \lambda \Phi^3 , \tag{4.2}$$

so that

$$f(\Phi) = \mu^2 \Phi^2 + \frac{\lambda}{2} \Phi^4 . \tag{4.3}$$

The constant of integration in (4.3) has been taken to be zero in order that the energy integral (3.6) be finite, the vacuum solution $\Phi = 0$ having zero energy. Since $f(\Phi)$ in (4.3) is everywhere positive, we have that the only solution is $\Phi = 0$, using the result of section 3. With the negative sign of the μ^2 term in (4.1), (3.2) gives

$$\frac{1}{2} f'(\Phi) = - \mu^2 \Phi + \lambda \Phi^3 , \tag{4.4}$$

so that

$$f(\Phi) = \frac{\lambda}{2} \left(\Phi^2 - \mu^2/\lambda \right)^2 , \tag{4.5}$$

which is the form of current interest in gauge theories. The constant of integration is chosen so that the vacuum solutions $\Phi = \pm \mu/\sqrt{\lambda}$ have zero energy. From (4.5), we again have the result that $f(\Phi)$ is everywhere positive, and the only solutions are $\pm \mu/\sqrt{\lambda}$.

5. Schwarzschild-de Sitter background space

As an example of a finite background space, consider the Schwarzschild metric generalised to a non-vanishing cosmological constant $\Lambda > 0$ (Lake, 1979)

$$ds^2 = h(r)\, dt^2 - \frac{1}{h(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 \tag{5.1}$$

where

$$h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 \quad . \tag{5.2}$$

From (3.2), the radial equation in this case becomes

$$\frac{1}{r^2}\frac{d}{dr}\left\{\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\frac{d\Phi}{dr}\right\} = \frac{1}{2}f'(\Phi) \quad . \tag{5.3}$$

Horizons occur where $h(r) = 0$ and putting $dh/dr = 0$ gives the position of the maximum of $h(r)$ as

$$r = r_0 = \left(\frac{3M}{\Lambda}\right)^{1/3} \quad , \tag{5.4}$$

and

$$h(r_0) = 1 - \left(9M^2\Lambda\right)^{1/3} \quad . \tag{5.5}$$

For $r > 0$, we will have two zeros of $h(r)$, $r = r_b$ and $r = r_c$, $r_b < r_0 < r_c$, provided $h(r_0) > 0$, which from (5.5) is equivalent to the condition

$$\Lambda < \frac{1}{9M^2} \quad . \tag{5.6}$$

We shall assume that (5.6) is satisfied so that the radial coordinate r has bounds $r_b \leqslant r \leqslant r_c$, and proceed as in section 3.

The energy E of the Φ-field is given by

$$E = 4\pi\int_{r_b}^{r_c}\left\{\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\left(\frac{d\Phi}{dr}\right)^2 + f(\Phi)r^2\right\}dr \, , \tag{5.7}$$

and we define

$$E_\alpha = 4\pi\int_{r_b}^{r_c}\left\{\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\left(\frac{d\Phi_\alpha}{dr}\right)^2 + f(\Phi_\alpha)r^2\right\}dr \, , \tag{5.8}$$

where

$$\Phi_\alpha(r) = \Phi\left[r + (\alpha - 1)\epsilon(r)\right] . \tag{5.9}$$

In (5.9), $\epsilon(r)$ is a bounded, differentiable function satisfying $\epsilon(r_b) = \epsilon(r_c) = 0$. This ensures that the variation in Φ at r_b and r_c is zero. Without loss of generality we take $\epsilon(r) > 0$ for $r_b < r < r_c$.

We now change the variable of integration in (5.8) from r to R where

$$R = r + (\alpha - 1)\epsilon(r) , \tag{5.10}$$

so that (5.8) becomes

$$\frac{E_\alpha}{4\pi} = \int_{r_b}^{r_c} \left\{ \left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\left(\frac{d\Phi}{dR}\right)^2 \frac{dR}{dr} + f(\Phi) r^2 \frac{dr}{dR} \right\} dR . \tag{5.11}$$

From (5.10)

$$\frac{dR}{dr} = 1 + (\alpha - 1)\frac{d\epsilon(r)}{dr} \tag{5.12}$$

and substituting (5.12) into (5.11), we obtain

$$\frac{E_\alpha}{4\pi} = \int_{r_b}^{r_c} \left\{ \left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\left(1 + [\alpha - 1]\frac{d\epsilon(r)}{dr}\right)\left(\frac{d\Phi}{dR}\right)^2 + f(\Phi)\frac{r^2}{\left(1 + [\alpha - 1]\frac{d\epsilon(r)}{dr}\right)} \right\} dR . \tag{5.13}$$

In the integrand of (5.13), r is defined implicitly as a function of α and R by (5.10). As before, we wish to differentiate (5.13) with respect to α and then put

$\alpha = 1$. We must therefore take account of the fact that r is a function of α . From (5.10), we have

$$0 = \frac{\partial r}{\partial \alpha} + \epsilon(r) + (\alpha - 1)\frac{d\epsilon(r)}{dr}\frac{\partial r}{\partial \alpha} , \qquad (5.14)$$

giving

$$\frac{\partial r}{\partial \alpha} = -\frac{\epsilon(r)}{\left[1 + (\alpha - 1)\frac{d\epsilon(r)}{dr}\right]} . \qquad (5.15)$$

In particular, since $r = R$ when $\alpha = 1$,

$$\left.\frac{\partial r}{\partial \alpha}\right|_{\alpha=1} = -\epsilon(R) . \qquad (5.16)$$

Now differentiating (5.13) with respect to α , we find

$$\begin{aligned}\frac{1}{4\pi}\frac{dE_\alpha}{d\alpha} = \int_{r_b}^{r_c} \Bigg[\Bigg[\Bigg\{ \left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\left(\frac{d\epsilon(r)}{dr} - \frac{\epsilon(r)(\alpha-1)\frac{d^2\epsilon(r)}{dr^2}}{\left[1 + (\alpha-1)\frac{d\epsilon(r)}{dr}\right]}\right) \\ - \epsilon(r)\left(2r - 2M - \frac{4\Lambda}{3}r^3\right)\Bigg\}\left(\frac{d\Phi}{dR}\right)^2 - f(\Phi)\Bigg\{\frac{2r\epsilon(r)}{\left[1 + (\alpha-1)\frac{d\epsilon(r)}{dr}\right]^2} \\ + \frac{r^2}{\left[1 + (\alpha-1)\frac{d\epsilon(r)}{dr}\right]^2}\left(\frac{d\epsilon(r)}{dr} - \frac{\epsilon(r)(\alpha-1)\frac{d^2\epsilon(r)}{dr^2}}{\left[1 + (\alpha-1)\frac{d\epsilon(r)}{dr}\right]}\right)\Bigg\}\Bigg]\Bigg] dR ,\end{aligned} \qquad (5.17)$$

so that

$$\begin{aligned}\frac{1}{4\pi}\left.\frac{dE_\alpha}{d\alpha}\right|_{\alpha=1} = \int_{r_b}^{r_c} \Bigg[\left(\frac{d\Phi}{dr}\right)^2 \Bigg\{ \left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\frac{d\epsilon}{dr} \\ - \epsilon\left(2r - 2M - \frac{4\Lambda}{3}r^3\right)\Bigg\} - f(\Phi)\Bigg\{2r\epsilon + r^2\frac{d\epsilon}{dr}\Bigg\}\Bigg] dr ,\end{aligned} \qquad (5.18)$$

where the dummy variable R has been replaced by r in (5.18). A further differentiation of (5.17) leads to

$$\frac{1}{8\pi}\frac{d^2E_\alpha}{d\alpha^2}\bigg|_{\alpha=1} = \int_{r_b}^{r_c}\Bigg[\left(\frac{d\Phi}{dr}\right)^2\left\{\epsilon^2(1-2\Lambda r^2)-\epsilon\frac{d^2\epsilon}{dr^2}\left(r^2-2Mr-\frac{\Lambda}{3}r^4\right)\right\}$$

$$+f(\Phi)\left\{\epsilon^2+4r\epsilon\frac{d\epsilon}{dr}+r^2\left(\frac{d\epsilon}{dr}\right)^2+r^2\epsilon\frac{d^2\epsilon}{dr^2}\right\}\Bigg]dr. \tag{5.19}$$

Again, R has been replaced by r and ϵ is understood to be $\epsilon=\epsilon(r)$.

From (5.18), the variational principle $\delta E=0$ is equivalent to

$$\int_{r_b}^{r_c}\Bigg[\left(\frac{d\Phi}{dr}\right)^2\left\{\epsilon\left(2r-2M-\frac{4\Lambda}{3}r^3\right)-\left(r^2-2Mr-\frac{\Lambda}{3}r^4\right)\frac{d\epsilon}{dr}\right\}$$

$$+q(r)f(\Phi)\Bigg]dr = 0, \tag{5.20}$$

where

$$q(r) = 2r\epsilon + r^2\frac{d\epsilon}{dr}, \tag{5.21}$$

while the stability condition $\delta^2E\geqslant 0$ is equivalent to

$$\int_{r_c}^{r_b}\Bigg[\left(\frac{d\Phi}{dr}\right)^2\left\{\epsilon^2(1-2\Lambda r^2)-\left(r^2-2Mr-\frac{\Lambda}{3}r^4\right)\epsilon\frac{d^2\epsilon}{dr^2}\right\}$$

$$+p(r)f(\Phi)\Bigg]dr \geqslant 0, \tag{5.22}$$

where

$$p(r) = \epsilon^2+4r\epsilon\frac{d\epsilon}{dr}+r^2\left(\frac{d\epsilon}{dr}\right)^2+r^2\epsilon\frac{d^2\epsilon}{dr^2}. \tag{5.23}$$

We now wish to consider the case of $f(\Phi) \geqslant 0$ as in sections 3 and 4. For any function ϵ satisfying $\epsilon(r_b) = \epsilon(r_c) = 0$ and $\epsilon(r) > 0$ for $r_b < r < r_c$, we have

$$\mu_b = \left.\frac{d\epsilon}{dr}\right|_{r=r_b} > 0 \quad , \quad \mu_c = \left.\frac{d\epsilon}{dr}\right|_{r=r_c} < 0 \, , \tag{5.24}$$

or either or both of μ_b, μ_c may be zero. Assuming first that (5.24) is satisfied, then from (5.21) we have

$$q(r_b) > 0 \quad , \quad q(r_c) < 0 \, . \tag{5.25}$$

If $\epsilon(r_b) = \mu_b = 0$, then as $r \to r_b + 0$, we put

$$\epsilon \sim A(r - r_b)^k \, , \tag{5.26}$$

where $A > 0$ and $k \geqslant 2$. Substituting $r = r_b + \delta$, where $0 < \delta \ll 1$, and (5.26) into (5.21) gives

$$q(r) \sim A r_b^2 k \delta^{k-1} > 0 \, . \tag{5.27}$$

Similarly, if $\epsilon(r_c) = \mu_c = 0$, then as $r \to r_c - 0$, we put

$$\epsilon \sim B(r_c - r)^l \tag{5.28}$$

where $B > 0$ and $l \geqslant 2$. Substituting $r = r_c - \lambda$, where $0 < \lambda \ll 1$, and (5.28) into (5.21) gives

$$q(r) \sim -B r_c^2 l \lambda^{l-1} < 0 \, . \tag{5.29}$$

In all cases therefore, $q(r)$ changes sign in the interval $r_b < r < r_c$. Hence, from (5.20), we can draw no conclusions as to the existence of non-constant solutions since at least one term in the integrand changes sign. Furthermore, for any ϵ, $p(r)$ contains positive terms and consequently, the right-hand side of (5.22) contains at least one positive term. We can infer, therefore, no violation of the stability condition (5.22).

6. Discussion

The method presented in this chapter is a convenient way of writing an identity involving Φ, once the differential equation for Φ has been specified. For consider the Reissner-Nordstrøm case where the differential equation is

$$\frac{1}{r^2}\frac{d}{dr}\left[(r-r_+)(r-r_-)\frac{d\Phi}{dr}\right] = \frac{1}{2}f'(\Phi) \ . \tag{6.1}$$

We derived the expression

$$J = \int_{r_+}^{\infty} (r-r_+)^2\left(\frac{d\Phi}{dr}\right)^2 dr + \int_{r_+}^{\infty} f(\Phi)\, r(3r-2r_+)\,dr = 0 \ . \tag{6.2}$$

To see that J is identically zero if (6.1) is satisfied we first integrate the second term in (6.2) by parts, obtaining

$$J = \int_{r_+}^{\infty} (r-r_+)^2\left(\frac{d\Phi}{dr}\right)^2 dr + \Big[f(\Phi)(r^3-r^2r_+)\Big]_{r_+}^{\infty} - \int_{r_+}^{\infty} (r^3-r^2r_+)f'(\Phi)\frac{d\Phi}{dr}\,dr \ . \tag{6.3}$$

The integrated term in (6.3) is zero at $r=r_+$ and also at ∞ by the requirement that the energy integral (3.6) is finite. Using (6.1), we eliminate $f'(\Phi)$ from (6.3) so that

$$J = \int_{r_+}^{\infty} (r - r_+)^2 \left(\frac{d\bar{\Phi}}{dr}\right)^2 dr$$

$$- \int_{r_+}^{\infty} 2(r - r_+) \frac{d\bar{\Phi}}{dr} \frac{d}{dr} \left\{ (r - r_+)(r - r_-) \frac{d\bar{\Phi}}{dr} \right\} dr . \qquad (6.4)$$

The second term in (6.4) may be expanded into two terms by partially carrying out the differentiation, and we find

$$J = \int_{r_+}^{\infty} (r - r_+)^2 \left(\frac{d\bar{\Phi}}{dr}\right)^2 dr - \int_{r_+}^{\infty} 2(r - r_+)^2 \left(\frac{d\bar{\Phi}}{dr}\right)^2 dr$$

$$- \int_{r_+}^{\infty} 2(r - r_+) \left(\frac{d\bar{\Phi}}{dr}\right) (r - r_-) \frac{d}{dr} \left\{ (r - r_+) \frac{d\bar{\Phi}}{dr} \right\} dr . \qquad (6.5)$$

The first and second terms in (6.5) are now combined and the third rewritten to give

$$J = -\int_{r_+}^{\infty} (r - r_+)^2 \left(\frac{d\bar{\Phi}}{dr}\right)^2 dr - \int_{r_+}^{\infty} (r - r_-) \frac{d}{dr} \left\{ \left[(r - r_+) \frac{d\bar{\Phi}}{dr} \right]^2 \right\} dr . \qquad (6.6)$$

Integrating the last term in (6.6) by parts, we are left with

$$J = - \left[(r - r_+)^2 (r - r_-) \left(\frac{d\bar{\Phi}}{dr}\right)^2 \right]_{r_+}^{\infty} , \qquad (6.7)$$

and this is zero, from the finiteness of the energy integral (3.6).

Similarly in the Schwarzschild-de Sitter case the differential equation is

$$\frac{1}{r^2}\frac{d}{dr}\left\{\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\frac{d\Phi}{dr}\right\} = \frac{1}{2}f'(\Phi) , \qquad (6.8)$$

while performing the first variation gave, in (5.20),

$$L = \int_{r_b}^{r_c}\left(\frac{d\Phi}{dr}\right)^2\left\{\epsilon\left(2r - 2M - \frac{4\Lambda}{3}r^3\right) - \left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\frac{d\epsilon}{dr}\right\}dr$$

$$+ K = 0 , \qquad (6.9)$$

where

$$K = \int_{r_b}^{r_c} f(\Phi)\left(2r\epsilon + r^2\frac{d\epsilon}{dr}\right)dr. \qquad (6.10)$$

Integrating (6.10) by parts and using $\epsilon(r_b) = \epsilon(r_c) = 0$ gives

$$K = -\int_{r_b}^{r_c} r^2\epsilon f'(\Phi)\frac{d\Phi}{dr}\,dr , \qquad (6.11)$$

and substituting for $f'(\Phi)$ in (6.11) from (6.8), we find

$$K = -\int_{r_b}^{r_c} 2\epsilon\frac{d\Phi}{dr}\frac{d}{dr}\left\{\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\frac{d\Phi}{dr}\right\}dr. \qquad (6.12)$$

Performing the differentiation in (6.12),

$$K = -\int_{r_b}^{r_c} 2\epsilon\left(2r - 2M - \frac{4\Lambda}{3}r^3\right)\left(\frac{d\Phi}{dr}\right)^2 dr$$

$$-\int_{r_b}^{r_c}\epsilon\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\frac{d}{dr}\left\{\left(\frac{d\Phi}{dr}\right)^2\right\}dr , \qquad (6.13)$$

and integrating the second term by parts gives

$$K = -\int_{r_b}^{r_c}\epsilon\left(2r - 2M - \frac{4\Lambda}{3}r^3\right)\left(\frac{d\Phi}{dr}\right)^2 dr$$

$$+\int_{r_b}^{r_c}\frac{d\epsilon}{dr}\left(r^2 - 2Mr - \frac{\Lambda}{3}r^4\right)\left(\frac{d\Phi}{dr}\right)^2 dr . \qquad (6.14)$$

On substituting (6.14) into (6.9), we see that L is identically zero, given that Φ satisfies (6.8).

The above manipulation indicates the difficulty one would have in deriving the identities (6.2) and (6.9) from the equations (6.1) and (6.8) without recourse to a variational principle.

7. Conclusions

We have seen in this chapter that an extension of the method of Derrick gives a convenient way of writing identities involving the solutions Φ of a class of non-linear Klein-Gordon equation, and of expressing the stability condition on Φ . For a spherically symmetric wavefunction in the infinite region exterior to a non-rotating black hole, the only solutions in certain cases of physical interest are the vacuum solutions. However, for a particular spatially finite background space, we can find nothing to preclude the existence of non-constant, stable, finite energy solutions. For other topologies, there may well exist non-trivial, stable solutions (Avis and Isham, 1978).

CHAPTER 4

MAGNETIC SUPPORT AGAINST GRAVITATIONAL COLLAPSE

1. Introduction

In a recent paper (Ardavan and Partovi, 1977), a static, axisymmetric interior solution of the Einstein-Maxwell equations was found. The solution describes the interior of an object whose mass exceeds the upper limit of the mass of a neutron star (see Oppenheimer and Volkoff, 1939, and Oppenheimer and Snyder, 1939), but is supported against total gravitational collapse to a black hole by internal magnetic stresses. In such an object, degeneracy pressure alone could not support collapse, and the authors therefore consider the case of negligible pressure, with the gravitational attraction balancing the magnetic forces.

In this chapter, we examine the solution from a physical viewpoint by applying the so-called strong energy condition (see Hawking and Ellis, 1973) to the energy-momentum tensor. It is found that the condition reduces to the requirement that the density of matter should be positive, and is therefore satisfied by the interior solution. Other aspects of the problem are briefly discussed.

2. The interior solution

We begin by briefly reviewing the derivation of the interior solution found by Ardavan and Partovi (1977).

In suitably chosen units ($G = c = 1$), the Einstein-

Maxwell equations with negligible pressure and non-zero matter density ρ are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \quad , \tag{2.1}$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda} F^{\lambda}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}\right) + \rho\, u_{\mu} u_{\nu} \ , \tag{2.2}$$

and

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad . \tag{2.3}$$

The authors consider the particular forms of the electromagnetic potential and four-velocity

$$A_{\mu} = (0, 0, 0, A(r, z)) , \quad u_{\mu} = (g_{00}^{1/2}, 0, 0, 0) \tag{2.4}$$

where g_{00} is the coefficient of dt^2 in the metric, which is taken to be

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)}(dr^2 + dz^2) - r^2 e^{-2\lambda} d\varphi^2 \ . \tag{2.5}$$

In (2.5), r , z and φ are cylindrical coordinates and λ and ν are functions of r and z .

The field equation from (2.1) which we shall need in this chapter is

$$\nabla^2 \lambda = \frac{e^{2\lambda}}{r^2} |\nabla A|^2 + 4\pi \rho\, e^{2(\nu-\lambda)} \ . \tag{2.6}$$

Also derivable from (2.1) are the equations

$$A_z\left(\nabla^2 A - \frac{2}{r} A_r + 2\nabla\lambda . \nabla A\right) + 4\pi \lambda_z r^2 \rho\, e^{2(\nu - 2\lambda)} = 0 \tag{2.7}$$

and

$$A_r\left(\nabla^2 A - \frac{2}{r} A_r + 2\nabla\lambda . \nabla A\right) + 4\pi \lambda_r r^2 \rho\, e^{2(\nu - 2\lambda)} = 0 \ , \tag{2.8}$$

where subscripts denote partial differentiation. Now (2.7) and (2.8) possess a non-trivial solution only if

$$\lambda_r A_z - A_r \lambda_z = 0 \quad , \tag{2.9}$$

so that A must be a function of λ . The authors make the particular choice

$$A' = e^{-\lambda} \quad , \tag{2.10}$$

where prime denotes differentiation with respect to λ , and eliminating ρ from (2.6) and (2.7), we obtain

$$(1 + r^2)\nabla^2\lambda - \frac{2}{r}\lambda_r = 0 \quad . \tag{2.11}$$

Writing $\lambda = U(r)V(z)$, with a separation constant k^2 , we find

$$\lambda = \sum_k a_k e^{ikz} u_k(r) \quad , \tag{2.12}$$

for $k \neq 0$, where the a_k are constants and $u_k(r)$ satisfies

$$\frac{d^2u}{dr^2} - \frac{(1-r^2)}{r(1+r^2)}\frac{du}{dr} - k^2 u = 0 \quad . \tag{2.13}$$

We note here that in the case $k = 0$, (2.13) has the solution

$$U(r) = \alpha \ln(1 + r^2) + \beta \quad , \tag{2.14}$$

while

$$V(z) = \gamma z + \delta \quad , \tag{2.15}$$

where α , β , γ and δ are arbitrary constants. The particular solution used by the authors was

$$\lambda(r,z) = u_k(r)\cos(kz) \quad . \tag{2.16}$$

From (2.6) and (2.10), the mass density ρ can be written

$$\rho = \frac{e^{2(\lambda-\nu)}}{4\pi}\left(\nabla^2\lambda - \frac{1}{r^2}|\nabla\lambda|^2\right) . \tag{2.17}$$

With λ given by (2.16), the density was found to be positive at the centre and falling to zero at a surface defined by

$$\nabla^2\lambda = \frac{1}{r^2}|\nabla\lambda|^2 . \tag{2.18}$$

This surface defines the boundary of the collapsed object. The exterior field equations can be written

$$\nabla^2\lambda - \frac{e^{2\lambda}}{r^2}|\nabla A|^2 = 0 \tag{2.19}$$

and

$$\frac{\partial}{\partial r}\left(\frac{e^{2\lambda}}{r}A_r\right) + \frac{\partial}{\partial z}\left(\frac{e^{2\lambda}}{r}A_z\right) = 0 . \tag{2.20}$$

A solution of (2.19) and (2.20) was not considered by the authors.

3. The energy-momentum tensor

Since the metric (2.5) is diagonal, the contravariant components $g^{\mu\nu}$ are given by

$$\left.\begin{aligned} g^{00} &= e^{-2\lambda} \\ g^{11} = g^{22} &= -e^{2(\lambda-\nu)} \\ g^{33} &= -\frac{e^{2\lambda}}{r^2} \end{aligned}\right\} , \tag{3.1}$$

while from (2.3) and (2.4), the only non-zero components of $F_{\mu\nu}$ are

$$\left.\begin{aligned} F_{13} = -F_{31} &= A_r \\ F_{23} = -F_{32} &= A_z \end{aligned}\right\} . \tag{3.2}$$

In order to calculate $T_{\mu\nu}$ from (2.2), we first note that, from (2.4), the term $\rho U_\mu U_\nu$ is non-zero only if $\mu = \nu = 0$, in which case

$$\rho U_0^{\,2} = \rho g_{00} = \rho e^{2\lambda} . \tag{3.3}$$

Next, we calculate the scalar

$$F^2 = F_{\lambda\sigma} F^{\lambda\sigma} = g^{\lambda\alpha} g^{\sigma\beta} F_{\alpha\beta} F_{\lambda\sigma} . \tag{3.4}$$

Using (3.1), we find

$$\begin{aligned} F^2 &= \sum_\lambda \sum_\sigma g^{\lambda\lambda} g^{\sigma\sigma} F_{\lambda\sigma}^2 \\ &= \frac{2}{r^2} e^{2(2\lambda - \nu)} \left(A_r^2 + A_z^2 \right) . \end{aligned} \tag{3.5}$$

Occurring in (2.2) is the tensor

$$C_{\mu\nu} = F_{\mu\lambda} F^{\lambda}{}_{\nu} . \tag{3.6}$$

Since the metric is diagonal,

$$C_{\mu\nu} = g^{\sigma\lambda} F_{\mu\lambda} F_{\sigma\nu} = \sum_\lambda g^{\lambda\lambda} F_{\mu\lambda} F_{\lambda\nu} . \tag{3.7}$$

From (3.1) and (3.2), the non-zero elements of $C_{\mu\nu}$ are

$$\left. \begin{aligned} C_{11} &= g^{33} F_{13} F_{31} = \frac{e^{2\lambda}}{r^2} A_r^2 \\ C_{12} = C_{21} &= g^{33} F_{13} F_{32} = \frac{e^{2\lambda}}{r^2} A_r A_z \\ C_{22} &= g^{33} F_{23} F_{32} = \frac{e^{2\lambda}}{r^2} A_z^2 \\ C_{33} &= g^{11} F_{31} F_{13} + g^{22} F_{32} F_{23} = e^{2(\lambda - \nu)} (A_r^2 + A_z^2) \end{aligned} \right\} . \tag{3.8}$$

From (2.2), (3.3), (3.5) and (3.8), the non-zero components of $T_{\mu\nu}$ are

$$\left. \begin{aligned} T_{00} &= \frac{1}{4\pi} \left\{ \frac{1}{2r^2} e^{2(3\lambda - \nu)} (A_r^2 + A_z^2) + 4\pi \rho e^{2\lambda} \right\} \\ T_{11} &= -T_{22} = \frac{e^{2\lambda}}{8\pi r^2} (A_r^2 - A_z^2) \end{aligned} \right\} \tag{3.9a}$$

$$\left.\begin{aligned} T_{12} = T_{21} &= \frac{e^{2\lambda}}{4\pi r^2} A_r A_z \\ T_{33} &= \frac{1}{8\pi} e^{2(\lambda-\nu)}(A_r^2 + A_z^2) \end{aligned}\right\} . \qquad (3.9b)$$

Using (2.10) to eliminate A and (2.6) to eliminate ρ from (3.9), we obtain

$$\left.\begin{aligned} T_{00} &= \frac{1}{4\pi} e^{2(2\lambda-\nu)}\left\{\nabla^2\lambda - \frac{1}{2r^2}(\lambda_r^2 + \lambda_z^2)\right\} \\ T_{11} = -T_{22} &= \frac{1}{8\pi r^2}(\lambda_r^2 - \lambda_z^2) \\ T_{12} = T_{21} &= \frac{1}{4\pi r^2}\lambda_r\lambda_z \\ T_{33} &= \frac{1}{8\pi} e^{-2\nu}(\lambda_r^2 + \lambda_z^2) \end{aligned}\right\} . \qquad (3.10)$$

Now

$$T^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} T_{\alpha\beta} , \qquad (3.11)$$

and using the contravariant components $g^{\mu\nu}$ given by (3.1) we finally obtain

$$\left.\begin{aligned} T^{00} &= \frac{1}{4\pi} e^{-2\nu}\left\{\nabla^2\lambda - \frac{1}{2r^2}(\lambda_r^2 + \lambda_z^2)\right\} \\ T^{11} = -T^{22} &= \frac{1}{8\pi r^2} e^{4(\lambda-\nu)}(\lambda_r^2 - \lambda_z^2) \\ T^{12} = T^{21} &= \frac{1}{4\pi r^2} e^{4(\lambda-\nu)}\lambda_r\lambda_z \\ T^{33} &= \frac{1}{8\pi r^4} e^{(4\lambda-2\nu)}(\lambda_r^2 + \lambda_z^2) \end{aligned}\right\} . \qquad (3.12)$$

4. Energy conditions

The strong energy condition as stated by Hawking and Ellis (1973) is that at any space-time point P ,

$$\tilde{T}^{00} > |\tilde{T}^{ab}| , \tag{4.1}$$

for each a and b , where $\tilde{T}^{\mu\nu}$ represents the components of the energy-momentum tensor referred to a local orthonormal basis at P . As presented by Hawking and Ellis (1973), this condition holds for all known physical situations and is believed to hold in general. It is equivalent to the demand of positive energy and that $S^b = T^{ab}V_a$ is a timelike vector for any timelike V_a .

Since the metric is diagonal, then at any point P , we already have an orthogonal set of axes. Defining a tetrad $\underline{e}_\mu$ by

$$g_{\mu\nu} = \underline{e}_\mu{}^a \underline{e}_{\nu a} \tag{4.2}$$

and

$$\omega^a = \underline{e}_\mu{}^a dx^\mu , \tag{4.3}$$

then the metric is formed by contraction with respect to the Minkowski metric η_{ab} ,

$$ds^2 = \omega^a \omega^b \eta_{ab} = g_{\mu\nu} dx^\mu dx^\nu . \tag{4.4}$$

The energy-momentum tensor transforms according to

$$\tilde{T}^{ab} = \underline{e}_\mu{}^a \underline{e}_\nu{}^b T^{\mu\nu} . \tag{4.5}$$

From (4.4) for a diagonal metric, we have that

$$\left.\begin{array}{l} \omega^0 = \sqrt{g_{00}}\, dx^0 , \ \omega^1 = \sqrt{-g_{11}}\, dx^1 , \\ \omega^2 = \sqrt{-g_{22}}\, dx^2 , \ \omega^3 = \sqrt{-g_{33}}\, dx^3 , \end{array}\right\} \tag{4.6}$$

so that, from (4.3),

$$\left.\begin{array}{l} \underline{e}_0{}^0 = \sqrt{g_{00}} , \quad \underline{e}_1{}^1 = \sqrt{-g_{11}} , \\ \underline{e}_2{}^2 = \sqrt{-g_{22}} , \quad \underline{e}_3{}^3 = \sqrt{-g_{33}} , \end{array}\right\} \tag{4.7}$$

Using the transformation (4.5), we find

$$\left.\begin{aligned}
\widetilde{T}^{00} &= g_{00}T^{00} = \frac{1}{4\pi}e^{2(\lambda-\nu)}\left\{\nabla^2\lambda - \frac{1}{2r^2}\left(\lambda_r^2+\lambda_z^2\right)\right\} \\
|\widetilde{T}^{11}| &= |g_{11}T^{11}| = |\widetilde{T}^{22}| = \frac{1}{8\pi r^2}e^{2(\lambda-\nu)}\left|\lambda_r^2-\lambda_z^2\right| \\
|\widetilde{T}^{12}| &= |\widetilde{T}^{21}| = \left|\sqrt{g_{11}g_{22}}\,T^{12}\right| = \frac{1}{4\pi r^2}e^{2(\lambda-\nu)}\left|\lambda_r\lambda_z\right| \\
|\widetilde{T}^{33}| &= |g_{33}T^{33}| = \frac{1}{8\pi r^2}e^{2(\lambda-\nu)}\left(\lambda_r^2+\lambda_z^2\right)
\end{aligned}\right\} . \tag{4.8}$$

Now since

$$\lambda_r^2 + \lambda_z^2 > \left|\lambda_r^2 - \lambda_z^2\right| \tag{4.9}$$

and

$$\lambda_r^2 + \lambda_z^2 > 2\left|\lambda_r\lambda_z\right| , \tag{4.10}$$

then, from (4.8),

$$|\widetilde{T}^{33}| > |\widetilde{T}^{11}| = |\widetilde{T}^{22}| \tag{4.11}$$

and

$$|\widetilde{T}^{33}| > |\widetilde{T}^{12}| . \tag{4.12}$$

Condition (4.1) therefore reduces to

$$\widetilde{T}^{00} > |\widetilde{T}^{33}| . \tag{4.13}$$

Using the components calculated in (4.8), (4.13) can be written

$$\frac{1}{4\pi}e^{2(\lambda-\nu)}\left\{\nabla^2\lambda - \frac{1}{r^2}\left(\lambda_r^2+\lambda_z^2\right)\right\} > 0 . \tag{4.14}$$

From (2.17), (4.14) is simply the condition $\rho > 0$, and the strong energy condition (4.1) is therefore satisfied by the particular interior solution found by Ardavan and Partovi (1977).

5. Discussion

The preceding analysis was based on a special solution of the field equations. For example, if a general relationship between A and λ is taken, in place of (2.10), of the form

$$A' = e^{-\lambda} f(\lambda) \quad , \tag{5.1}$$

then in place of (2.11), we would have the non-linear equation

$$\left(1 + \frac{r^2}{f^2}\right)\nabla^2\lambda - \frac{2}{r}\lambda_r + \frac{f'}{f}|\nabla\lambda|^2 = 0 \quad , \tag{5.2}$$

which would no longer be amenable to solution by separation of variables. In the absence of an exterior solution satisfying equations (2.19) and (2.20), it is not certain that the solution given by (2.10) and (2.16) could be matched at the boundary of the collapsed object. Indeed, if an infinite sum of the form (2.12) is used, it is no longer clear that there would be a solution of (2.18) defining the boundary. The exterior field equations have no simplifying feature of the form (5.1) and an analysis is much more difficult (Young and Bentley, 1975).

CHAPTER 5

THE SCHRÖDINGER EQUATION WITH AN ANHARMONIC OSCILLATOR POTENTIAL

1. Introduction

In a recent paper (Stephenson, 1977), the Liouville-Green technique (see Appendix 1) was used to obtain the eigenvalues of the Schrödinger equation with a radial Gaussian potential. Recent work on the anharmonic oscillator (Gillespie, 1976, Fung et al, 1978, Banerjee et al,1978) has led to computation and comparison of the eigenvalues of the Schrödinger equation. In view of the fact that the Liouville-Green technique and other so-called semi-classical methods are not as widely applied as they might be (Berry and Mount, 1972), and of the importance of the anharmonic oscillator potential in nuclear structure, quantum chemistry and quark confinement, we now use the same method for this potential. The eigenvalues obtained are compared with those found by direct means.

2. The basic transformation

Setting $2m = \hbar = 1$, the one-dimensional Schrödinger equation with an anharmonic oscillator potential $V = x^2 + x^4$ is

$$\frac{d^2\psi}{dx^2} = \left(-E + x^2 + x^4\right)\psi \quad , \tag{2.1}$$

where E is the energy, and the boundary conditions are

$\psi(\infty) = \psi(-\infty) = 0$. We make the Liouville-Green transformation (Olver, 1974)

$$x = x(\xi) \quad , \psi(x) = (\xi')^{-\frac{1}{2}} G(\xi) \ , \tag{2.2}$$

where primes denote differentiation with respect to x , so that (2.1) becomes

$$\frac{d^2 G}{d\xi^2} = \left(\frac{P(x)}{\xi'^2} + \Delta(x) \right) G \ , \tag{2.3}$$

where

$$P(x) = x^4 + x^2 - E \tag{2.4}$$

and

$$\Delta(x) = \frac{\xi'''}{2\xi'^3} - \frac{3\xi''^2}{4\xi'^4} \ . \tag{2.5}$$

When E is positive, $P(x)$ has two zeros $x = \pm x_0$ where

$$x_0 = \left\{ \frac{1}{2} \left[-1 + (1 + 4E)^{\frac{1}{2}} \right] \right\}^{\frac{1}{2}} \ , \tag{2.6}$$

these being the classical turning points.

The Liouville-Green technique consists in choosing $\xi(x)$ so that $\Delta(x)$ is a small bounded function and (2.3), with $\Delta(x)$ neglected, is soluble in terms of known functions. Two ways of achieving this will be presented. First, since (2.1) has two turning points, we may try to choose $\xi(x)$ so that, after neglecting $\Delta(x)$, (2.3) becomes the standard two-turning-point equation, namely the Weber equation

$$\frac{d^2 G}{d\xi^2} = \left(\frac{1}{4}\xi^2 - \lambda \right) G \ , \tag{2.7}$$

the solutions of which are the parabolic cylinder functions where λ is a parameter. Alternatively, since $P(x)$ depends only on x^2 , the wavefunctions $\psi(x)$ will be either even or odd functions and we can consider the problem for $x \geqslant 0$, applying the additional boundary condition that either $\psi(0)=0$ or $\psi'(0)=0$. In this case, since $P(x)$ has only one zero for $x \geqslant 0$, we may try to choose $\xi(x)$ so that (2.3) becomes the Airy equation

$$\frac{d^2G}{d\xi^2} = (\xi - a)G , \tag{2.8}$$

after neglecting $\Delta(x)$, where a is a parameter to be determined from the boundary conditions.

Both approaches lead to approximate eigenvalues and eigenfunctions (Olver, 1974).

3. The Weber equation method

With the choice

$$\xi'^2\left(\tfrac{1}{4}\xi^2 - \lambda\right) = P(x) \tag{3.1}$$

(2.3) becomes the Weber equation (2.7), if we neglect $\Delta(x)$. Assuming for the moment that this is justified, we find by integration of (3.1) that for $x \geqslant x_0$,

$$\begin{aligned}\tfrac{1}{2}\xi(\xi^2 - 4\lambda)^{\frac{1}{2}} - 2\lambda \ln\left|\xi + (\xi^2 - 4\lambda)^{\frac{1}{2}}\right| \\ + 2\lambda \ln(2\sqrt{\lambda}) = 2\int_{x_0}^{x} \{P(t)\}^{\frac{1}{2}} dt ,\end{aligned} \tag{3.2}$$

while between the turning points

$$\tfrac{1}{2}\xi(4\lambda - \xi^2)^{\frac{1}{2}} + 2\lambda \sin^{-1}\left(\frac{\xi}{2\sqrt{\lambda}}\right) = 2\int_{0}^{x} \{-P(t)\}^{\frac{1}{2}} dt . \tag{3.3}$$

The constants of integration have been chosen so that $\xi = 0$ when $x=0$ and $\xi = \pm 2\sqrt{\lambda}$ correspond to $x = \pm x_0$.

Putting $x = x_0$ in (3.3) we obtain

$$\lambda\pi = 2\int_0^{x_0} (E - t^2 - t^4)^{1/2}\, dt . \qquad (3.4)$$

The boundary conditions $\psi(\infty) = \psi(-\infty) = 0$ correspond to $G(\infty) = G(-\infty) = 0$ and bounded solutions of the Weber equation satisfying these conditions exist only if

$$\lambda = n + \frac{1}{2} , \qquad (3.5)$$

where $n = 0,1,2,\ldots$. Substituting (3.5) into (3.4) gives

$$\frac{\pi}{2}(n + \tfrac{1}{2}) = \int_0^{x_0} (E - t^2 - t^4)^{1/2}\, dt , \qquad (3.6)$$

which is the Bohr-Sommerfeld quantisation formula, on noticing that

$$\int_0^{x_0} (E - t^2 - t^4)^{1/2}\, dt = \frac{1}{2}\int_{-x_0}^{x_0} (E - t^2 - t^4)^{1/2}\, dt . \qquad (3.7)$$

Table 1.

n	Eigenvalue	Accurate Eigenvalue	Percentage error
0	1.2508	1.3924	10.17
1	4.5926	4.6488	1.21
2	8.6130	8.6550	0.49
3	13.1231	13.1568	0.26
4	18.0290	18.0576	0.16
5	23.2725	23.2974	0.11
6	28.8130	28.8353	0.077
7	34.6206	34.6408	0.058
8	40.6717	40.6904	0.046
9	46.9477	46.9650	0.037
10	53.4329	53.4491	0.03
20	127.6076	127.6178	0.008
30	214.7721	214.7797	0.0035
40	311.8254	311.8315	0.002
50	417.0512	417.0563	0.0012
100	1035.5422	1035.5442	0.0002

Using Simpson's rule and Newton iteration, the eigenvalues have been computed from (3.6) and in Table 1 are compared with accurate values calculated by Banerjee et al (1978) using scaled bases. The two sets of values are in close agreement, the accuracy increasing with increasing n .

We now examine the neglected term $\Delta(x)$. From (2.4) and (3.1) we have

$$\xi' = \left[\frac{(-E + x^2 + x^4)}{(\frac{1}{4}\xi^2 - \lambda)}\right]^{1/2}, \tag{3.8}$$

from which ξ'' and ξ''' can be calculated in terms of x and ξ and, using (2.5), $\Delta(x)$ can be written out explicitly as

$$\Delta(x) = \frac{(3\xi^2 + 8\lambda)}{64(\frac{1}{4}\xi^2 - \lambda)^2}$$

$$- \left(\frac{1}{4}\xi^2 - \lambda\right)\frac{[2E + (12E + 3)x^2 + 6x^4 + 8x^6]}{4(-E + x^2 + x^4)^3} . \tag{3.9}$$

At the turning points, although both terms in (3.9) diverge, we can show that $\Delta(x)$ tends to a finite limit, as follows:

From (3.8) we have

$$\xi'^2 = \frac{(-E + x^2 + x^4)}{(\frac{1}{4}\xi^2 - \lambda)} . \tag{3.10}$$

Now $x = x_0$ corresponds to $\xi = 2\sqrt{\lambda}$, so that both top and bottom of the right-hand side of (3.10) tend to zero as $x \rightarrow x_0$. We therefore use L'Hôpital's rule to evaluate the limit, giving

$$\lim_{x \to x_0} \xi'^2 = \lim_{x \to x_0} \frac{(2x + 4x^3)}{\frac{1}{2}\xi\xi'} , \qquad (3.11)$$

so that

$$L_1 = \lim_{x \to x_0} \xi' = \left(\frac{2x_0 + 4x_0^3}{\sqrt{\lambda}}\right)^{1/3} . \qquad (3.12)$$

Differentiating (3.10) leads to

$$2\xi'\xi'' = \frac{(4x + 8x^3 - \xi\xi'^3)}{2(\frac{1}{4}\xi^2 - \lambda)} , \qquad (3.13)$$

after elimination of the term $(-E + x^2 + x^4)$. By (3.12), both top and bottom in (3.13) tend to zero as $x \to x_0$. Writing

$$L_2 = \lim_{x \to x_0} \xi'' \qquad (3.14)$$

and taking the limit in (3.13) using L'Hôpital's rule gives

$$2L_1L_2 = \frac{(4 + 24x_0^2 - L_1^4 - 6\sqrt{\lambda}L_1^2L_2)}{2L_1\sqrt{\lambda}} , \qquad (3.15)$$

so that

$$L_2 = \frac{(4 + 24x_0^2 - L_1^4)}{10L_1^2\sqrt{\lambda}} . \qquad (3.16)$$

By a further differentiation of (3.13) and use of L'Hôpital's rule, a lengthy but straightforward calculation gives

$$L_3 = \lim_{x \to x_0} \xi''' = \frac{(48x_0 - 24\sqrt{\lambda}L_1L_2^2 - 9L_1^3L_2)}{14L_1^2\sqrt{\lambda}} . \qquad (3.17)$$

Since L_1 , L_2 and L_3 are non-zero and finite, then by (2.5), $\Delta(x)$ tends to a finite limit given by

$$\lim_{x \to x_0} \Delta(x) = \frac{L_3}{2L_1^3} - \frac{3}{4}\frac{L_2^2}{L_1^4} . \qquad (3.18)$$

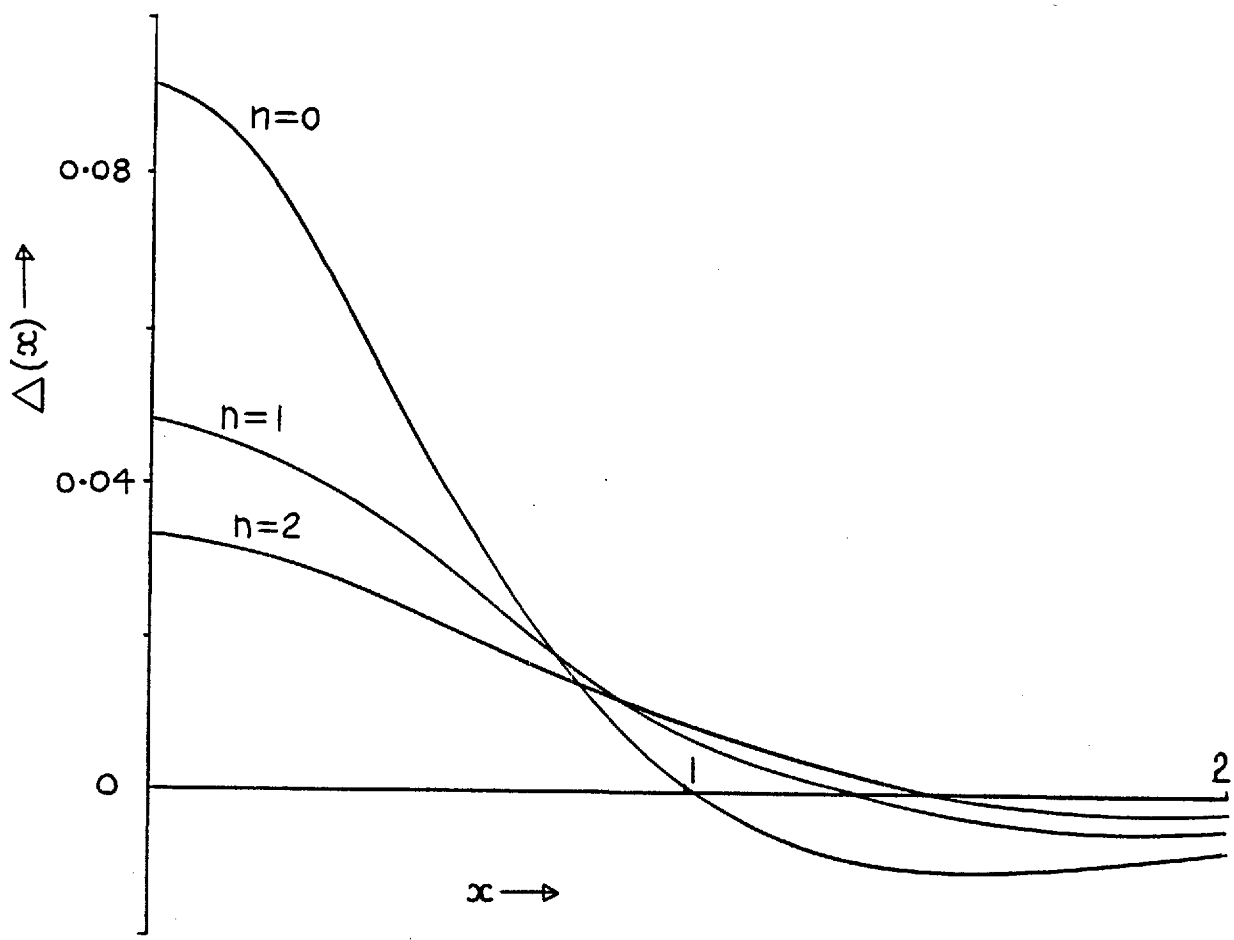
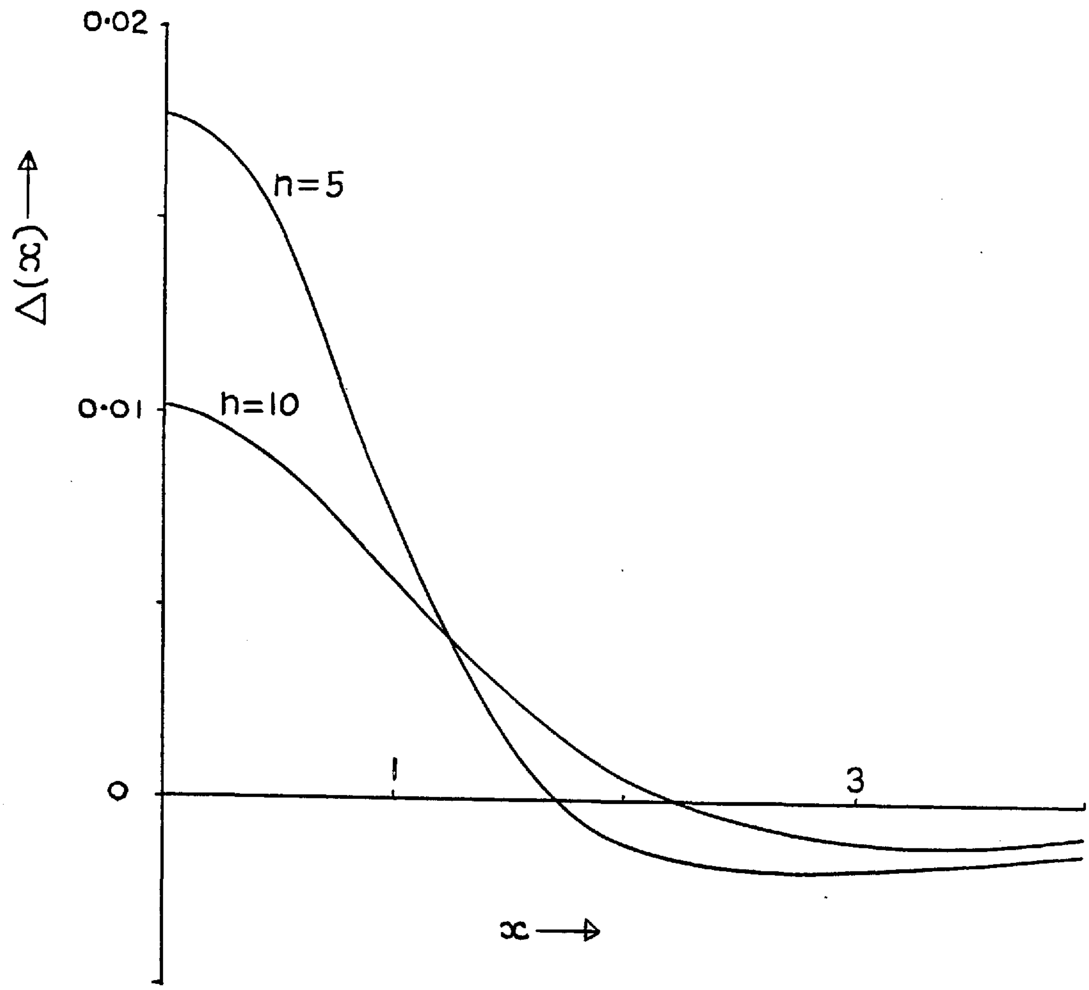


Figure 1. $\Delta(x)$ against x, for $n = 0, 1, 2$.



<u>Figure 2.</u> $\Delta(x)$ against x, for $n = 5, 10$.

The values of $\Delta(x)$ have been computed by first finding ξ for a given x from (3.2) or (3.3) and then substituting in (3.9), with the value at the turning point given by (3.18). The results are shown in Figures 1 and 2 for selected values of h and indicate that $\Delta(x)$ attains its absolute maximum at $x = 0$, this value decreasing with increasing h, and that $\Delta(x)$ is a small, bounded, slowly varying function.

4. The Airy equation method

Here we consider $x \geqslant 0$, and with the choice

$$\xi'^2(\xi - a) = P(x) , \tag{4.1}$$

(2.3) becomes the Airy equation (2.8) on neglecting $\Delta(x)$. We then find by integration of (4.1) that for $x \geqslant x_0$,

$$\frac{2}{3}(\xi - a)^{3/2} = \int_{x_0}^{x} \{P(t)\}^{1/2}\, dt , \tag{4.2}$$

the constant of integration being chosen so that $x = x_0$ corresponds to $\xi = a$. For $0 \leqslant x \leqslant x_0$, we have

$$\frac{2}{3}a^{3/2} - \frac{2}{3}(a - \xi)^{3/2} = \int_{0}^{x} \{-P(t)\}^{1/2}\, dt , \tag{4.3}$$

where $x = 0$ corresponds to $\xi = 0$. Substituting $x = x_0$ into (4.3), we obtain

$$\frac{2}{3}a^{3/2} = \int_{0}^{x_0} (E - t^2 - t^4)^{1/2}\, dt . \tag{4.4}$$

The required solution of (2.8) is the Airy function $Ai(\xi - a)$, since this satisfies the boundary condition $G(\infty) = 0$. We can now find the parameter a from the additional condition that either $G'(0) = 0$ or $G(0) = 0$, corresponding to even and odd wavefunctions respectively, since this condition implies that either $Ai'(-a) = 0$ or $Ai(-a) = 0$. Hence $-a$ is the position of either a turning point or a zero of the Airy function Ai. The values of a obtained from Abramowitz and Stegun (1964, p.478) were used to compute the eigenvalues using (4.4). The results are shown in Table 2 and compare favourably with accurate values.

Table 2.

n	a from $Ai'(-a)=0$	a from $Ai(-a)=0$	Eigenvalue	Accurate Eigenvalue
0	1.01879		1.0706	1.3924
1		2.33811	4.6573	4.6488
2	3.24820		8.5471	8.6550
3		4.08795	13.1605	13.1568
4	4.82010		17.9849	18.0576
5		5.52056	23.3000	23.2974
6	6.16331		28.7788	28.8353
7		6.78671	34.6428	34.6408
8	7.37218		40.6433	40.6904
9		7.94413	46.9666	46.9650
10	8.48849		53.4084	53.4491
11		9.02265	60.1310	60.1295
12	9.53545		66.9589	66.9950
13		10.04017	74.0371	74.0359
14	10.52766		81.2108	81.2435
15		11.00852	88.6115	88.6103
16	11.47506		96.0998	96.1296
17		11.93602	103.7966	103.7953
18	12.38479		111.5743	111.6018
19		12.82878	119.5454	119.5442

The connection between (3.6) and (4.4) can be seen by noting that the leading order term in the asymptotic expansion of a is

$$a \sim \left\{ \frac{3}{4}\pi \left(n + \frac{1}{2}\right) \right\}^{2/3} \tag{4.5}$$

where $n = 0,1,2,\ldots$.(see Abramowitz and Stegun, 1964, p.450).

The neglected term $\Delta(x)$ in this case is given by

$$\Delta(x) = \frac{5}{16(\xi - a)^2} - (\xi - a)\frac{[2E + (12E+3)x^2 + 6x^4 + 8x^6]}{4(-E + x^2 + x^4)^3}, \tag{4.6}$$

and we can again show that $\Delta(x)$ tends to a finite limit at the turning point $x = x_0$. By the process of differentiation and L'Hôpital's rule used in section 3, we obtain from (4.1)

$$K_1 = \lim_{x \to x_0} \xi' = (2x_0 + 4x_0^3)^{1/3}, \tag{4.7}$$

$$K_2 = \lim_{x \to x_0} \xi'' = \frac{(2 + 12x_0^2)}{5K_1^2}, \tag{4.8}$$

and

$$K_3 = \lim_{x \to x_0} \xi''' = \frac{12}{7K_1^2}(2x_0 - K_1K_2^2). \tag{4.9}$$

Finally, from (2.5),

$$\lim_{x \to x_0} \Delta(x) = \frac{3}{28K_1^5}(16x_0 - 15K_1K_2^2). \tag{4.10}$$

The results of computing $\Delta(x)$ for selected values of a are shown in Figures 3 and 4. Curves are labelled by the corresponding quantum number n (see Table 2).

5. Discussion

The method presented here depends on the initial choice of $\xi(x)$. Consider for example the Weber equation method. The exact relation between ξ and x is given by

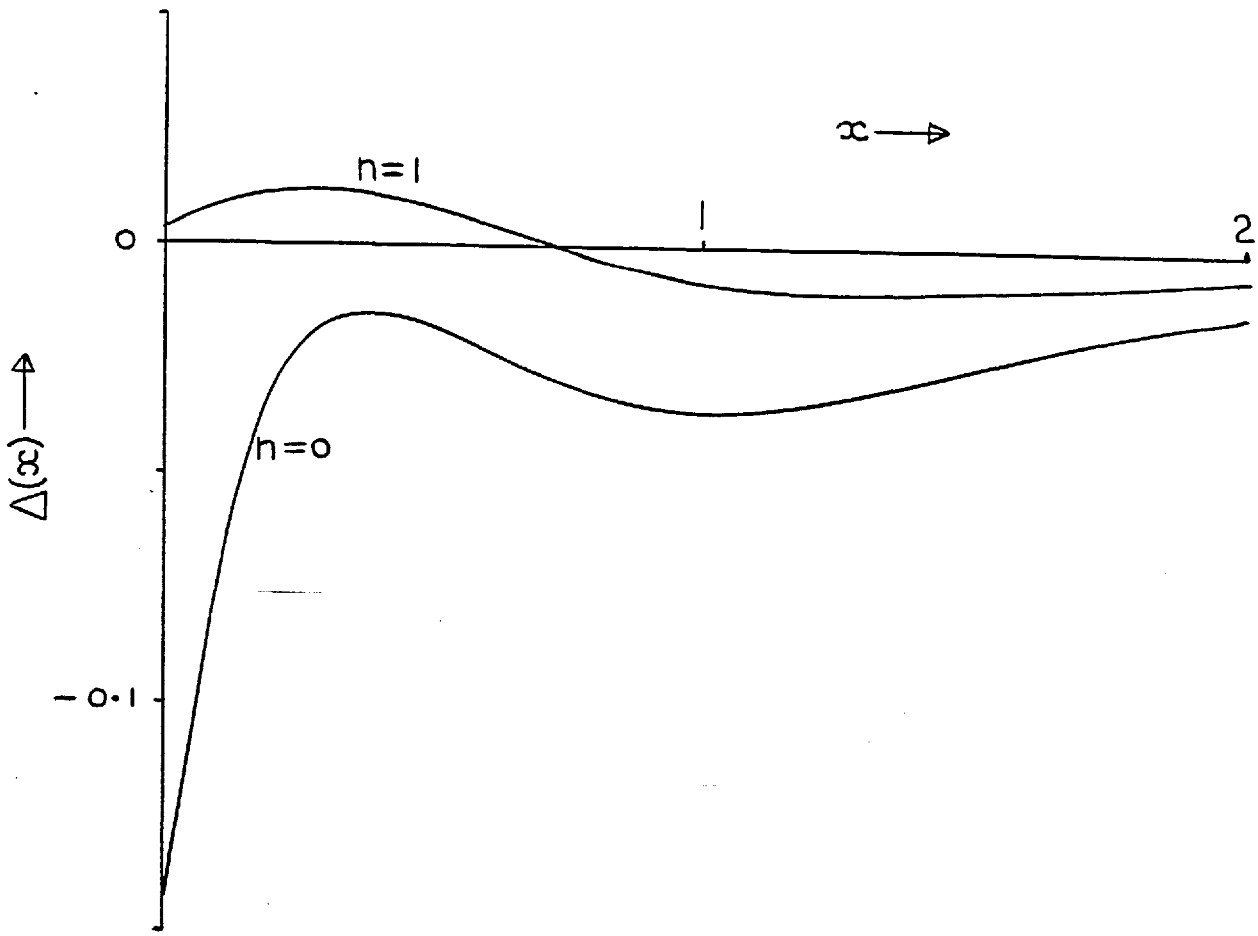


Figure 3. $\Delta(x)$ against x, for selected values of a ($n = 0, 1$).

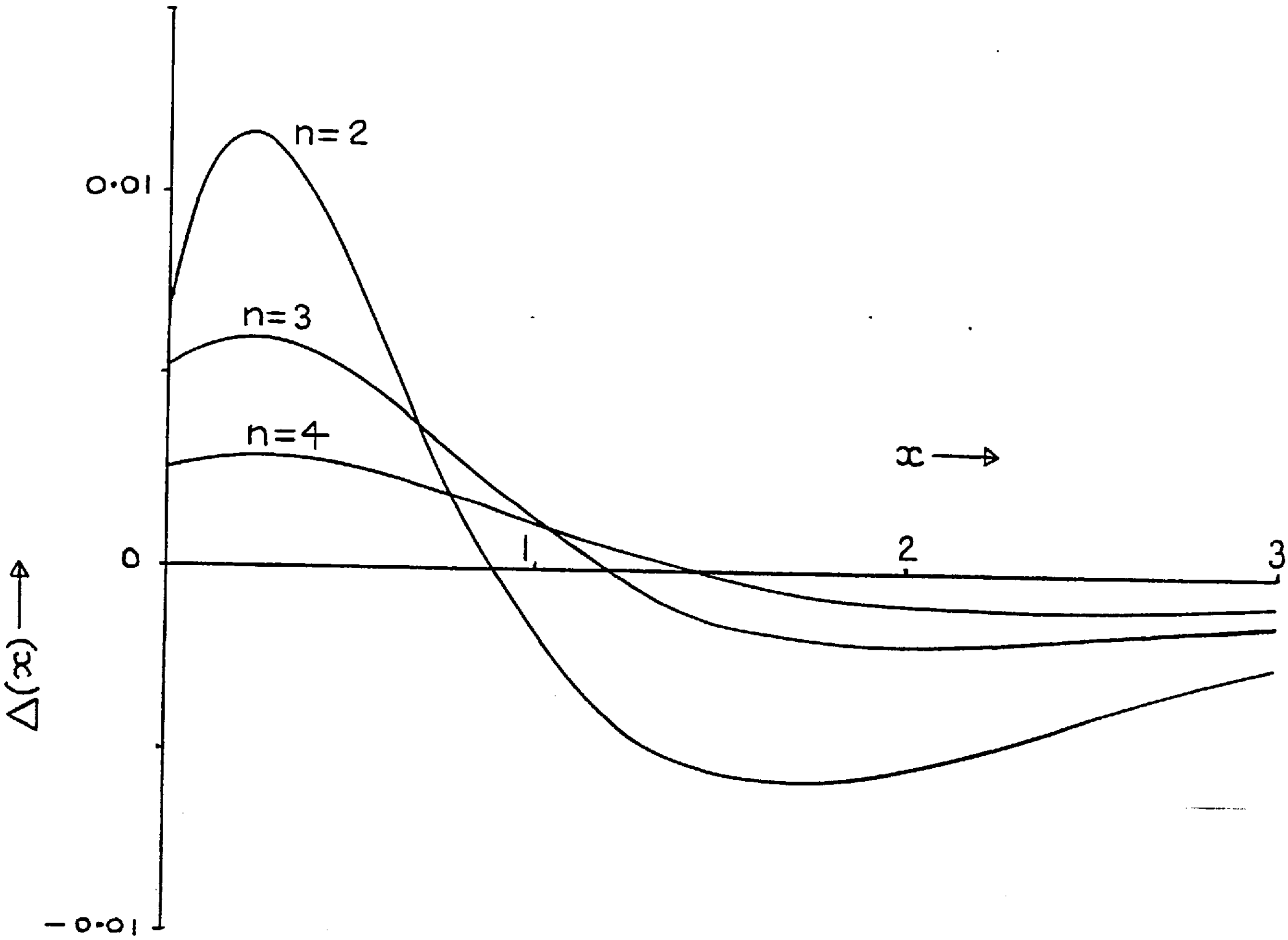


Figure 4. $\Delta(x)$ against x , for selected values of a ($n=2, 3, 4$).

$$\left(\frac{1}{4}\xi^2-\lambda\right)-\frac{P(x)}{\xi'^2}=\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}, \tag{5.1}$$

and on neglecting the right-hand side, we obtain (3.1). The next approximation would be

$$\left(\frac{1}{4}\xi^2-\lambda\right)-\frac{P(x)}{\xi'^2}=\Delta(x(\xi)), \tag{5.2}$$

from which we see that

$$\int_0^{x_0}\left\{-P(x)\right\}^{1/2}dx=\int_0^{\xi_0}\left\{\lambda-\frac{1}{4}\xi^2+\Delta(x(\xi))\right\}^{1/2}d\xi \tag{5.3}$$

where ξ_0 is given by

$$\lambda-\frac{1}{4}\xi_0^2+\Delta(x(\xi_0))=0. \tag{5.4}$$

Numerical calculations of (3.18) indicate that, except for the case $n=0$, $\Delta(x)$ is negative at the turning point $x=x_0$ (corresponding to $\xi=2\sqrt{\lambda}$), so that $\xi_0<2\sqrt{\lambda}$. Hence an upper bound for the right-hand side of (5.3) is

$$2\sqrt{\lambda}\left(\lambda+\Delta(0)\right)^{1/2}, \tag{5.5}$$

which from (5.3) gives an upper bound for the eigenvalues in this approximation. For upper and lower bounds derived using the J.W.K.B. approximation, see Birx and Houk, 1977.

The approximate eigenfunctions follow from (2.7) and the transformation (2.2). These solutions have been obtained from the equation

$$\frac{d^2G}{d\xi^2}=\left\{\frac{1}{4}\xi^2-\lambda+\Delta(x(\xi))\right\}G, \tag{5.6}$$

after neglecting $\Delta(x(\xi))$. At the turning point $\xi=2\sqrt{\lambda}$, $\Delta(x(\xi))$ is non-zero and therefore dominates over $\xi^2/4-\lambda$. However, near $\xi=2\sqrt{\lambda}$, we put

$$\epsilon y = \xi - 2\sqrt{\lambda} , \tag{5.7}$$

where $\epsilon \ll 1$ is a small parameter and y is a variable of order 1 . Substituting (5.7) into (5.6) gives

$$\frac{d^2G}{dy^2} = \left\{ \epsilon^2 \Delta\left(x(\epsilon y + 2\sqrt{\lambda})\right) + \epsilon^3 y\sqrt{\lambda} + \frac{1}{4}\epsilon^4 y^2 \right\} G . \tag{5.8}$$

Now $\Delta(x(\xi))$ tends to a finite limit as $\xi \longrightarrow 2\sqrt{\lambda}$, so that an expansion of $\Delta\left(x(\epsilon y + 2\sqrt{\lambda})\right)$ in powers of ϵ contains no inverse powers of ϵ . Taking ξ close enough to $2\sqrt{\lambda}$ so that $\epsilon\sqrt{\lambda} \ll 1$, we see that, although the term involving $\Delta(x(\xi))$ in (5.8) is of a lower order in ϵ than the remaining terms on the right-hand side, the dominant term is the second derivative of G . Equation (5.8), correct to order ϵ , is

$$\frac{d^2G}{dy^2} = 0 . \tag{5.9}$$

Hence, if we neglect $\Delta(x(\xi))$, the resulting equation is still correct to order ϵ near $\xi = 2\sqrt{\lambda}$.

In Appendix 2, we use a modification of the analysis of Titchmarsh (1961) to show that for large E , the error in the approximate eigenfunctions obtained by the Airy equation method is $O(E^{-1/2})$. In this case, ξ is known explicitly as a function of x from (4.2) and (4.3), whereas in the Weber equation case, only an implicit relation is known from (3.2) and (3.3), making a similar analysis much more difficult.

A wide class of potentials can be treated in a similar manner, using the methods presented in this chapter, for example, the interaction of the type $fx^2(1+gx^2)^{-1}$ (Mitra, 1978, and Kaushal, 1979).

CHAPTER 6

ELECTROMAGNETIC PROPAGATION IN OPTICAL WAVEGUIDES

1. Introduction

In view of recent advances in the manufacture of fibre optical waveguides and of their attractive properties, much work has been carried out on electromagnetic propagation in such fibres (see Olshansky, 1979, hereafter referred to as [1], and the references cited therein). The waveguide consists of a cladding region surrounding a cylindrically symmetric core, with the refractive index of the core greater than that of the cladding. Ideally the cladding is of infinite thickness. The core of radius a will be taken to lie along the positive z-axis, and the configuration of refractive index in the core and cladding will be referred to as the 'profile' of the waveguide.

Properties of propagation in the z-direction can be derived from the solution of a single differential equation, an eigenvalue problem for the propagation constants. Two problems of current interest [1] are the step-index profile and the parabolic index profile. In the first, the refractive index is taken to be n_1 in the core and n_2 in the cladding, assumed to be infinite in thickness, where n_1 and n_2 are constants satisfying $n_1 > n_2$. In the second, the infinite cladding again has a constant refractive index n_2, but the core has a refractive index which decreases parabolically with radial coordinate from n_1 at the centre to n_2 at the core-cladding interface. Previous approaches to these problems have had the disadvantage of resulting in transcendental equations for the eigenvalues involving, for example, Bessel functions. This is due to the matching condition of solutions of the basic differential equation at the core-cladding interface.

In this chapter, we derive approximate eigenvalues and eigenfunctions using the Liouville-Green technique (see Appendix 1) by replacing the index profiles of interest by close approximations which are, however, continuous and differentiable throughout the core and cladding. This removes the need for matching at the interface.

2. The basic differential equation

From Maxwell's equations, we have the following wave equations for the electric and magnetic field vectors $\underline{E}$ and $\underline{H}$ (see Born and Wolf, 1975):

$$\left.\begin{aligned} \nabla^2\underline{E} - \frac{\mu\varepsilon}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} + \nabla(\underline{E}.\nabla\ln\varepsilon) &= 0 \\ \nabla^2\underline{H} - \frac{\mu\varepsilon}{c^2}\frac{\partial^2\underline{H}}{\partial t^2} + (\nabla\ln\varepsilon)\times(\nabla\times\underline{H}) &= 0 \end{aligned}\right\}, \qquad (2.1)$$

where μ is the magnetic permeability, assumed constant, and ε is the dielectric permittivity of the medium. We shall take ε to be a function of position. Assuming time dependence of the form

$$\left.\begin{aligned} \underline{E}(\underline{r},t) &= \underline{E}_0(\underline{r})e^{i\omega t} \\ \underline{H}(\underline{r},t) &= \underline{H}_0(\underline{r})e^{i\omega t} \end{aligned}\right\} \qquad (2.2)$$

and putting

$$k = \frac{\omega}{c} \quad , \quad n^2(\underline{r}) = \mu\varepsilon(\underline{r}) \quad , \tag{2.3}$$

then from (2.1)

$$\left.\begin{aligned} \nabla^2\underline{E}_0 + k^2n^2(\underline{r})\underline{E}_0 + \frac{2}{\mu}\nabla\left(\underline{E}_0 \cdot \frac{1}{n}\nabla n\right) = 0 \\ \nabla^2\underline{H}_0 + k^2n^2(\underline{r})\underline{H}_0 + \frac{2}{\mu}\frac{1}{n}(\nabla n)\times(\nabla\times\underline{H}_0) = 0 \end{aligned}\right\} . \tag{2.4}$$

The standard procedure now is to neglect the terms in (2.4) containing ∇n . This is equivalent to assuming that the variation of the dielectric permittivity is small in distances of the order of the wavelength (see Sodha and Ghatak, 1977). We shall discuss this assumption, in connection with the particular problems investigated, later in this chapter. Neglecting the last terms of each of the equations in (2.4), we can write (2.4) in terms of a single partial differential equation

$$\nabla^2\Psi + k^2n^2(\underline{r})\Psi = 0 , \tag{2.5}$$

where $\Psi(\underline{r})$ represents any one of the field components.

We shall be concerned with problems having cylindrical symmetry, where n^2 is a function of the radial coordinate only. Expressing (2.5) in cylindrical polar coordinates r , θ and z , and writing

$$\Psi(r,\theta,z) = \psi(r)e^{i\beta z}\cos(\nu\theta + j) , \tag{2.6}$$

where j is an arbitrary constant, β is the propagation constant and $\nu = 0,1,2,.....$, we obtain the scalar equation

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}+\left(k^2n^2(r)-\beta^2-\frac{\nu^2}{r^2}\right)\psi = 0 \,. \tag{2.7}$$

It is equation (2.7) that is so extensively treated in the literature [1] , and which we shall consider in this chapter.

3. The step-index profile

In this case, $n^2(r)$ in (2.7) is given by

$$n^2(r) = \begin{cases} n_1^2 & (0 \leq r < a) \\ n_2^2 & (r > a) \end{cases} \,, \tag{3.1}$$

where n_1 and n_2 are constants satisfying $n_1 > n_2$ and a is the radius of the core. In both the core and cladding regions, equation (2.7) can be solved exactly in terms of Bessel functions. Satisfying continuity conditions at the core-cladding interface then gives the transcendental eigenvalue equation for the problem (see [1] and Gloge, 1971,1975). A simpler equation for the eigenvalues can be obtained by the Liouville-Green technique (see Appendix 1), as follows:

We first replace the step-function behaviour of $n^2(r)$ in (3.1) by a continuous function

$$n^2(r) = n_1^2 - \frac{1}{2}(n_1^2 - n_2^2)\left\{1 + \tanh[N(r-a)]\right\}, \tag{3.2}$$

in order to apply the method more readily. As the parameter N tends to infinity, the function $\tanh[N(r-a)]$ approaches a step-function of value -1 for $r < a$, and value $+1$ for $r > a$, and so (3.2) approaches (3.1). Substituting (3.2) together with the transformation

$$r = at \quad , \quad \psi = t^{-1/2} R(t) \tag{3.3}$$

into equation (2.7), we obtain

$$\frac{d^2R}{dt^2} = \left\{ a^2(\beta^2 - k^2 n_1^2) + \frac{a^2k^2}{2}(n_1^2 - n_2^2)\left[1 + \tanh\{\gamma(t-1)\}\right] + \frac{(\nu^2 - 1/4)}{t^2} \right\} R = 0 , \tag{3.4}$$

where $\gamma = Na$. Following Olshansky [1] and Gloge (1971), we define the parameters

$$\left.\begin{aligned} d &= \frac{(n_1^2 - n_2^2)}{2n_1^2} \\ V^2 &= 2dk^2n_1^2a^2 \\ b &= \frac{(\beta^2 - k^2n_2^2)}{2dk^2n_1^2} \end{aligned}\right\} . \tag{3.5}$$

Once n_1 , n_2 and the core radius a are specified, then the parameters d and V are fixed. The eigenvalue of the problem is taken as b , which once found gives the propagation constant β , from (3.5).

Rewriting (3.4), using (3.5), gives

$$\frac{d^2R}{dt^2} = \left\{ (b - \tfrac{1}{2})V^2 + \frac{V^2}{2}\tanh[\gamma(t-1)] + \frac{(\nu^2 - 1/4)}{t^2} \right\} R = 0 . \tag{3.6}$$

Applying to (3.6) the Liouville-Green transformation

$$t = t(\xi) \quad , \quad R(t) = \dot{\xi}^{-1/2} G(\xi) \quad , \tag{3.7}$$

where dots indicate differentiation with respect to t ,
we obtain

$$\frac{d^2G}{d\xi^2} = \left(\frac{\{(b-\frac{1}{2})V^2 + P(t)\}}{\dot{\xi}^2} + \Delta(t)\right)G\ , \tag{3.8}$$

where

$$P(t) = \frac{V^2}{2}\tanh\left[\gamma(t-1)\right] + \frac{(\nu^2 - \frac{1}{4})}{t^2}\ , \tag{3.9}$$

and

$$\Delta(t) = \frac{\dddot{\xi}}{2\dot{\xi}^3} - \frac{3}{4}\frac{\ddot{\xi}^2}{\dot{\xi}^4}\ . \tag{3.10}$$

For a range of values of b, the function $(b-\frac{1}{2})V^2 + P(t)$ in (3.8) will have two zeros, corresponding to two turning points of equation (3.6). We then wish to transform equation (3.8) into the standard two-turning-point equation, namely the Weber equation

$$\frac{d^2G}{d\xi^2} = \left(\frac{1}{4}\xi^2 - \lambda\right)G\ , \tag{3.11}$$

where λ is a parameter to be determined from the boundary conditions. Now Stephenson (1977), when considering the Schrödinger equation with a radial Gaussian potential, has shown that the choice

$$\dot{\xi}^2\left(\frac{1}{4}\xi^2 - \lambda\right) = \left(b - \frac{1}{2}\right)V^2 + P(t) \tag{3.12}$$

leads to a form of $\Delta(t)$ in (3.10) which is divergent as $t \to 0$. The correct transformation is

$$\dot{\xi}^2\left(\frac{1}{4}\xi^2 - \lambda\right) = \left(b - \frac{1}{2}\right)V^2 + Q(t)\ , \tag{3.13}$$

where

$$Q(t) = P(t) + \frac{1}{4t^2} = \frac{V^2}{2}\tanh\left[\gamma(t-1)\right] + \frac{\nu^2}{t^2}\ . \tag{3.14}$$

This is equivalent to making the Langer correction (see Langer, 1937) and is typical of problems which contain second order poles (see Rosenzweig and Kreiger, 1968). In accordance with the Liouville-Green technique, in equation (3.8) we neglect

$$F(t) = \Delta(t) - \frac{1}{4t^2\dot{\xi}^2} \quad . \tag{3.15}$$

Bounded solutions of the Weber equation (3.11) which vanish at infinity exist only if

$$\lambda = n + \frac{1}{2} \quad , \tag{3.16}$$

where $n = 0,1,2,\ldots$.. Taking the square root of (3.13) and integrating between the turning points gives

$$\int_{-2\sqrt{\lambda}}^{2\sqrt{\lambda}} \left(\lambda - \frac{1}{4}\xi^2\right)^{\frac{1}{2}} d\xi = \int_{t_1}^{t_2} \left\{\left(\frac{1}{2} - b\right)V^2 - Q(t)\right\}^{\frac{1}{2}} dt \quad , \tag{3.17}$$

where t_1 and t_2 are the two roots of $\left(\frac{1}{2} - b\right)V^2 - Q(t)$. Performing the integral on the left-hand side of (3.17) and using (3.16) gives

$$\pi\left(n + \frac{1}{2}\right) = \int_{t_1}^{t_2} \left\{\left(\frac{1}{2} - b\right)V^2 - Q(t)\right\}^{\frac{1}{2}} dt \quad . \tag{3.18}$$

For $V > \nu \geqslant 1$, the function $Q(t)$ is shown qualitatively in Figure 1.

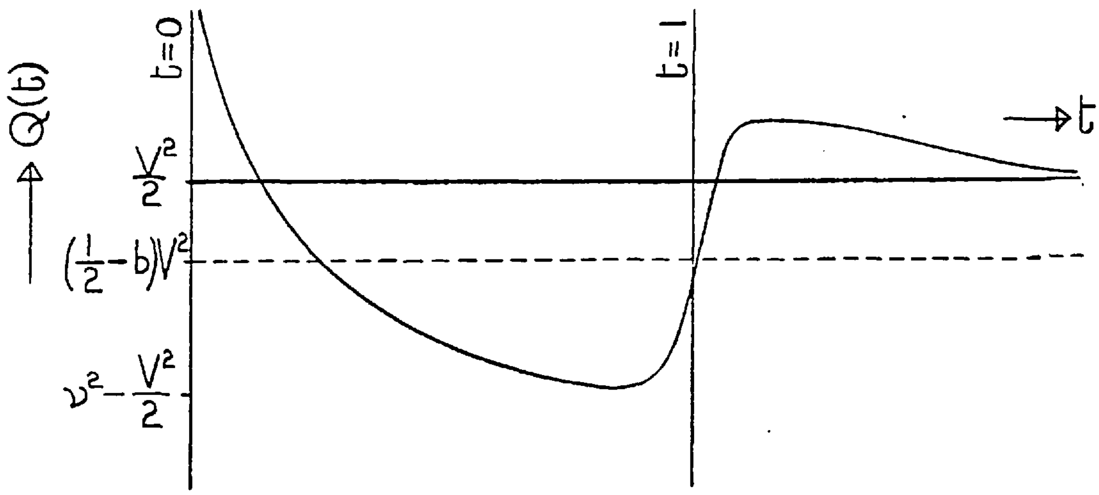


Figure 1.

In order to evaluate the integral in (3.18), we take the limit of large γ. The approximate minimum value of $Q(t)$ is $\nu^2 - V^2/2$, so that the condition on the eigenvalue b is (see Figure 1)

$$\nu^2 - \frac{V^2}{2} < \left(\frac{1}{2} - b\right)V^2 < \frac{V^2}{2} , \tag{3.19}$$

or equivalently,

$$0 < b < 1 - \frac{\nu^2}{V^2} . \tag{3.20}$$

As $\gamma \to \infty$, the larger root of the integrand of (3.18) approaches $t_2 = 1$, while the smaller root is given by

$$\left(\frac{1}{2} - b\right)V^2 = -\frac{V^2}{2} + \frac{\nu^2}{t_1^2} , \tag{3.21}$$

since the function $\tanh[\gamma(t-1)]$ approaches the value -1 for $t < 1$. Hence t_1 is given by

$$t_1 = \frac{\nu}{V\sqrt{1-b}} . \tag{3.22}$$

Using (3.14) in the limit $\gamma \to \infty$, (3.18) becomes

$$\pi(n + \tfrac{1}{2}) = \int_{t_1}^{1} \left\{\left(\frac{1}{2} - b\right)V^2 + \frac{V^2}{2} - \frac{\nu^2}{t^2}\right\}^{1/2} dt . \tag{3.23}$$

Performing the integral in (3.23) with the aid of the substitution $t = t_1 \sec\theta$ gives

$$\pi\left(n + \frac{1}{2}\right) = \nu\left(X - \tan^{-1} X\right) , \tag{3.24}$$

where

$$X = \frac{\sqrt{1 - t_1^2}}{t_1} . \tag{3.25}$$

It is now a straightforward matter to find X from (3.24), for example by Newton iteration, and then, using (3.22) and (3.25), b is found from

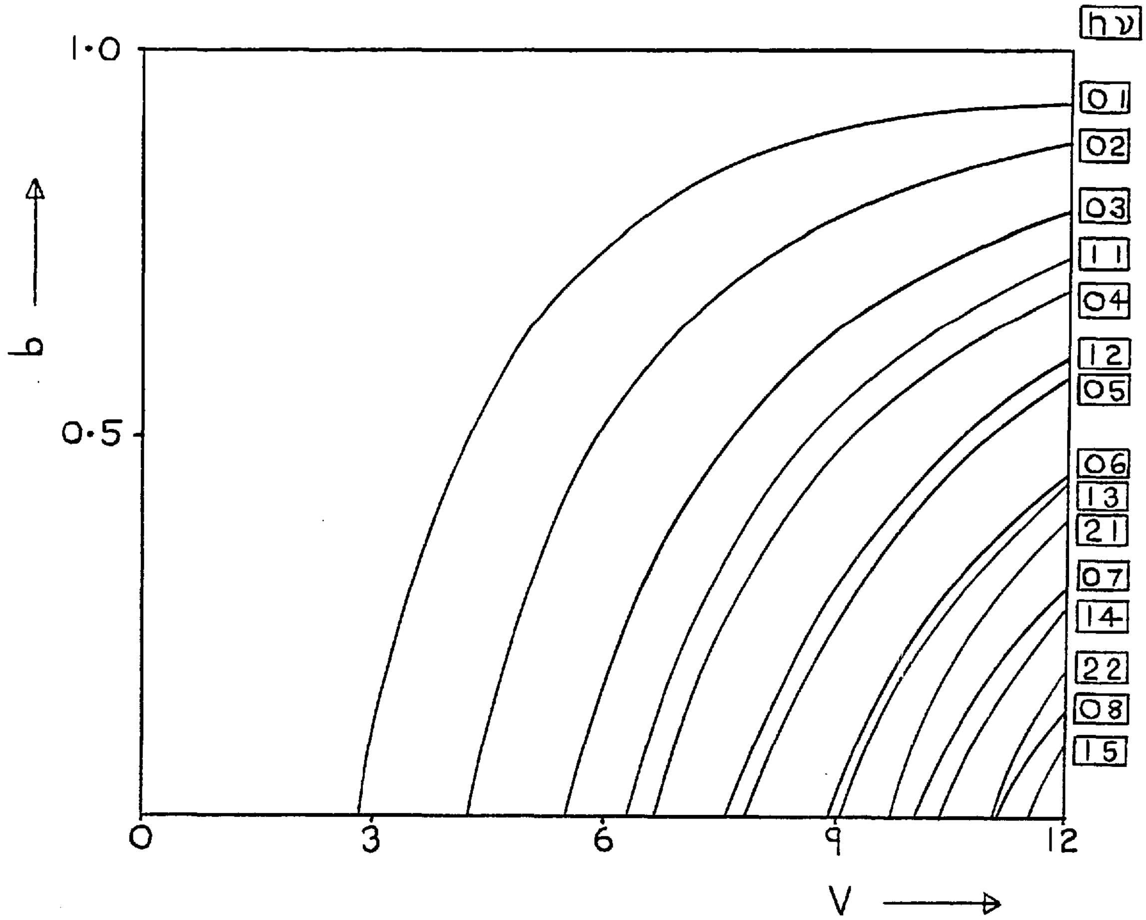


Figure 2. b against V for the step-index profile.

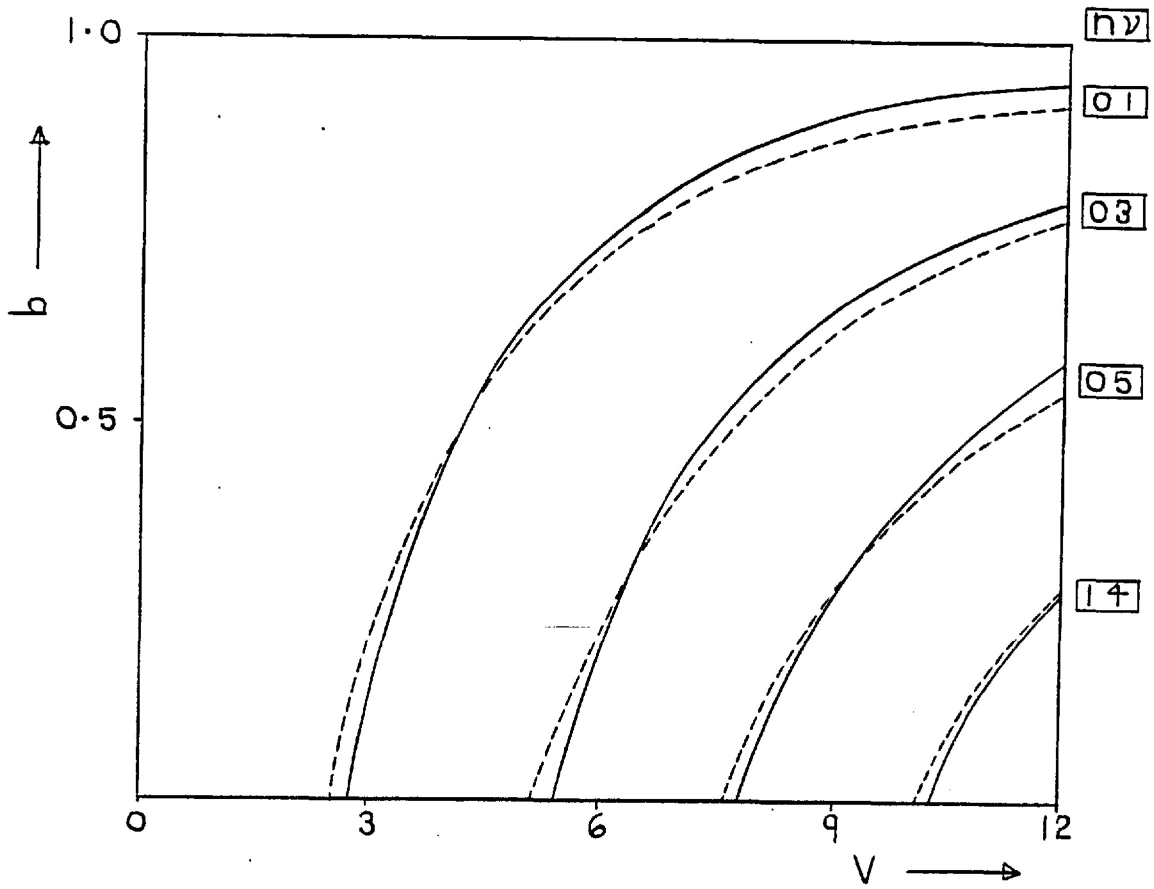


Figure 3. Selected results from figure 2 (——) compared with those given in [1] (-----).

$$b = 1 - \frac{\nu^2}{V^2}(1 + X^2) \; . \qquad (3.26)$$

The results of computing b using (3.24) and (3.26) are shown in Figure 2 for $V \leqslant 12$. In Figure 3, some of the results are compared with those given in [1]. The two sets of results are seen to be in excellent agreement.

4. The neglected term $F(t)$

We now examine the neglected term $F(t)$ for the step-index profile in the limit $\gamma \to \infty$. From (3.13) and (3.14) for $t < 1$, we have

$$\dot{\xi}^2\left(\tfrac{1}{4}\xi^2 - \lambda\right) = \left(b - \tfrac{1}{2}\right)V^2 - \frac{V^2}{2} + \frac{\nu^2}{t^2} \; . \qquad (4.1)$$

Integration of (4.1) gives

$$\frac{\xi}{4}\sqrt{4\lambda - \xi^2} + \lambda \sin^{-1}\left(\frac{\xi}{2\sqrt{\lambda}}\right) + \frac{\lambda\pi}{2}$$

$$= \nu\left\{\frac{\sqrt{t^2 - t_1^2}}{t_1} - \tan^{-1}\left(\frac{\sqrt{t^2 - t_1^2}}{t_1}\right)\right\} \qquad (4.2)$$

for $t_1 \leqslant t < 1$, while for $0 \leqslant t \leqslant t_1$,

$$\frac{\xi}{4}\sqrt{\xi^2 - 4\lambda} - \lambda \ln\left|\xi + \sqrt{\xi^2 - 4\lambda}\right| + \lambda \ln(2\sqrt{\lambda})$$

$$= \left\{\ln\left(\frac{t_1 - \sqrt{t_1^2 - t^2}}{t}\right) + \frac{\sqrt{t_1^2 - t^2}}{t_1}\right\} . \qquad (4.3)$$

The constants of integration have been chosen so that $t = t_1$ corresponds to $\xi = -2\sqrt{\lambda}$. For $t > 1$, we have

$$\dot{\xi}^2\left(\tfrac{1}{4}\xi^2 - \lambda\right) = \left(b - \tfrac{1}{2}\right)V^2 + \frac{V^2}{2} + \frac{\nu^2}{t^2} , \qquad (4.4)$$

so that

$$\frac{\xi}{4}\sqrt{\xi^2 - 4\lambda} - \lambda \ln\left|\xi + \sqrt{\xi^2 - 4\lambda}\right| + \lambda \ln(2\sqrt{\lambda})$$

$$= \nu\left\{\ln\left(\frac{\sqrt{t^2 + D^2} - D}{t}\right) - \ln\left(\sqrt{1 + D^2} - D\right)\right.$$

$$\left. + \frac{\sqrt{t^2 + D^2} - \sqrt{1 + D^2}}{D}\right\} , \qquad (4.5)$$

where $D^2 = \nu^2/bV^2$, and we have chosen $t = 1$ to correspond to $\xi = 2\sqrt{\lambda}$.

From the transformation (3.13) and use of L'Hôpital's rule, as in Chapter 5, we find

$$L_1 = \lim_{t \to t_p} \dot{\xi} = \left(\frac{\dot{Q}(t_p)}{\sqrt{\lambda}}\right)^{1/3} , \qquad (4.6)$$

$$L_2 = \lim_{t \to t_p} \ddot{\xi} = \frac{(2\ddot{Q}(t_p) - L_1^4)}{10\sqrt{\lambda}\, L_1^2} , \qquad (4.7)$$

and

$$L_3 = \lim_{t \to t_p} \dddot{\xi} = \frac{(2\dddot{Q}(t_p) - 24\sqrt{\lambda}\, L_1 L_2^2 - 9 L_1^3 L_2)}{14\sqrt{\lambda}\, L_1^2} , \qquad (4.8)$$

where t_p represents the position of a root of the function $(b - \frac{1}{2})V^2 + Q(t)$, corresponding to either $\xi = -2\sqrt{\lambda}$ or $\xi = +2\sqrt{\lambda}$. Then (4.6)-(4.8) give

$$\lim_{t \to t_p} F(t) = \frac{L_3}{2L_1^3} - \frac{3}{4}\frac{L_2^2}{L_1^4} - \frac{1}{4t_p^2 L_1^2} . \qquad (4.9)$$

At the smaller root $t_p = t_1$, $Q(t) \simeq -V^2/2 + \nu^2/t^2$ from (3.14), and since L_1 , L_2 and L_3 are all finite and non-zero at $t_p = t_1$, we find that $F(t)$ tends to a finite limit at $t = t_1$, given by (4.9). The larger root, lying close to $t = 1$, must be treated independently. In this case t_p is given by

$$\left(b - \tfrac{1}{2}\right)V^2 + \frac{V^2}{2}\tanh\left[\gamma(t_p - 1)\right] + \frac{\nu^2}{t_p^2} = 0. \qquad (4.10)$$

From (3.14),

$$\dot{Q}(t_p) = \frac{\gamma V^2}{2}\operatorname{sech}^2\left[\gamma(t_p - 1)\right] - \frac{2\nu^2}{t_p^3}, \qquad (4.11)$$

and so on for higher derivatives of $Q(t)$, required in (4.7) and (4.8). Then using (4.10) and the identity $\operatorname{sech}^2\theta = 1 - \tanh^2\theta$, we can eliminate all hyperbolic functions, to see that as $\gamma \to \infty$,

$$\dot{Q}(t_p) = O(\gamma), \quad \ddot{Q}(t_p) = O(\gamma^2), \quad \dddot{Q}(t_p) = O(\gamma^3). \qquad (4.12)$$

Then from (4.6)-(4.8),

$$L_1 = O(\gamma^{1/3}), \quad L_2 = O(\gamma^{4/3}), \quad L_3 = O(\gamma^{7/3}) \qquad (4.13)$$

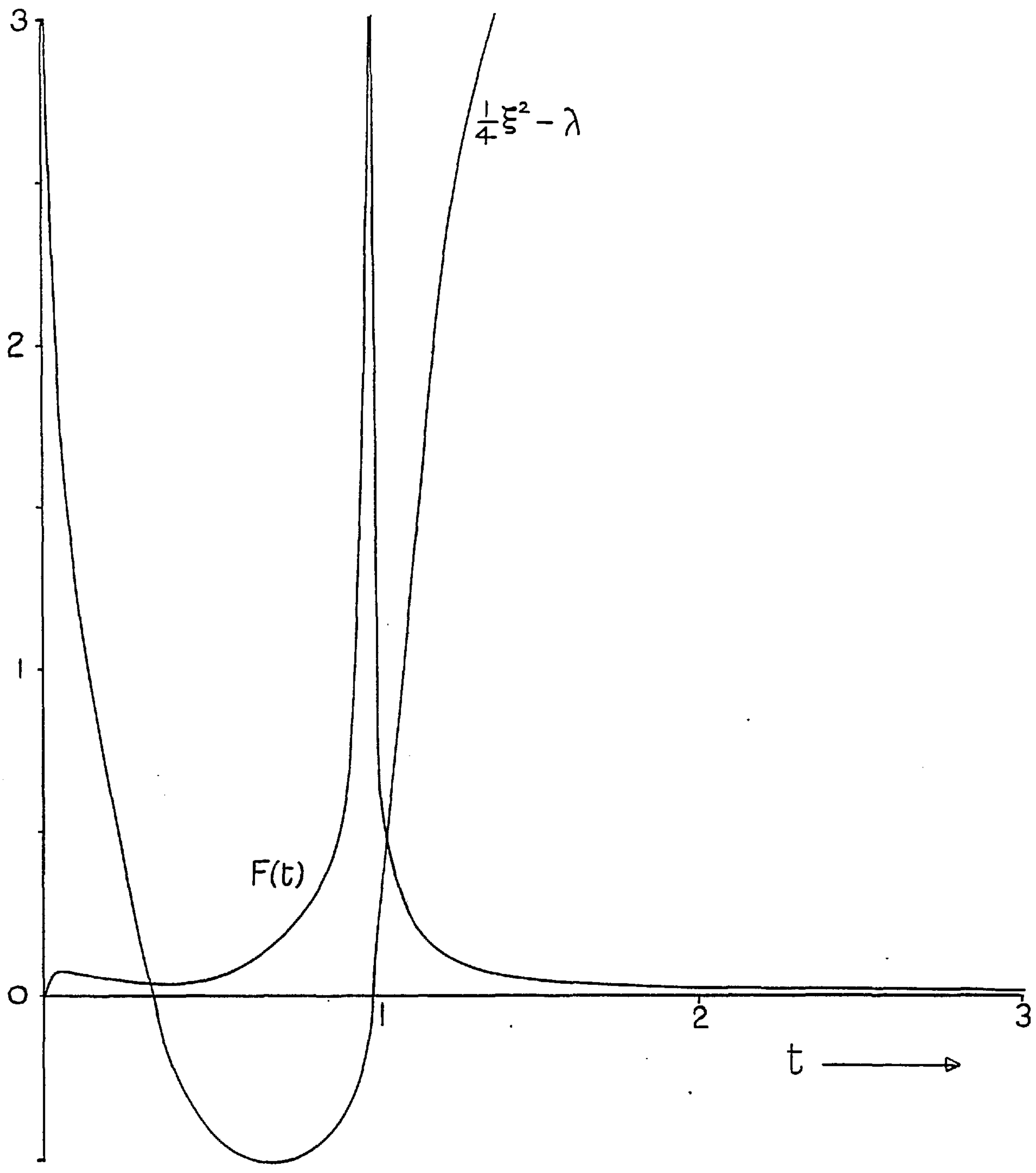
and finally from (4.9),

$$\lim_{t \to t_p} F(t) = O(\gamma^{4/3}), \qquad (4.14)$$

which tends to infinity with γ.

The results of plotting $F(t)$ against t for particular values of h, ν and V are shown in Figures 4 and 5. Also shown in each case for comparison is $\xi^2/4 - \lambda$ as a function of t, this being the term retained in equation (3.8).

From (3.8), the exact relationship between ξ and t is



<u>Figure 4.</u> $F(t)$ and $\frac{1}{4}\xi^2 - \lambda$ against t, for $n=0$, $\nu = 1$, $V = 8$.

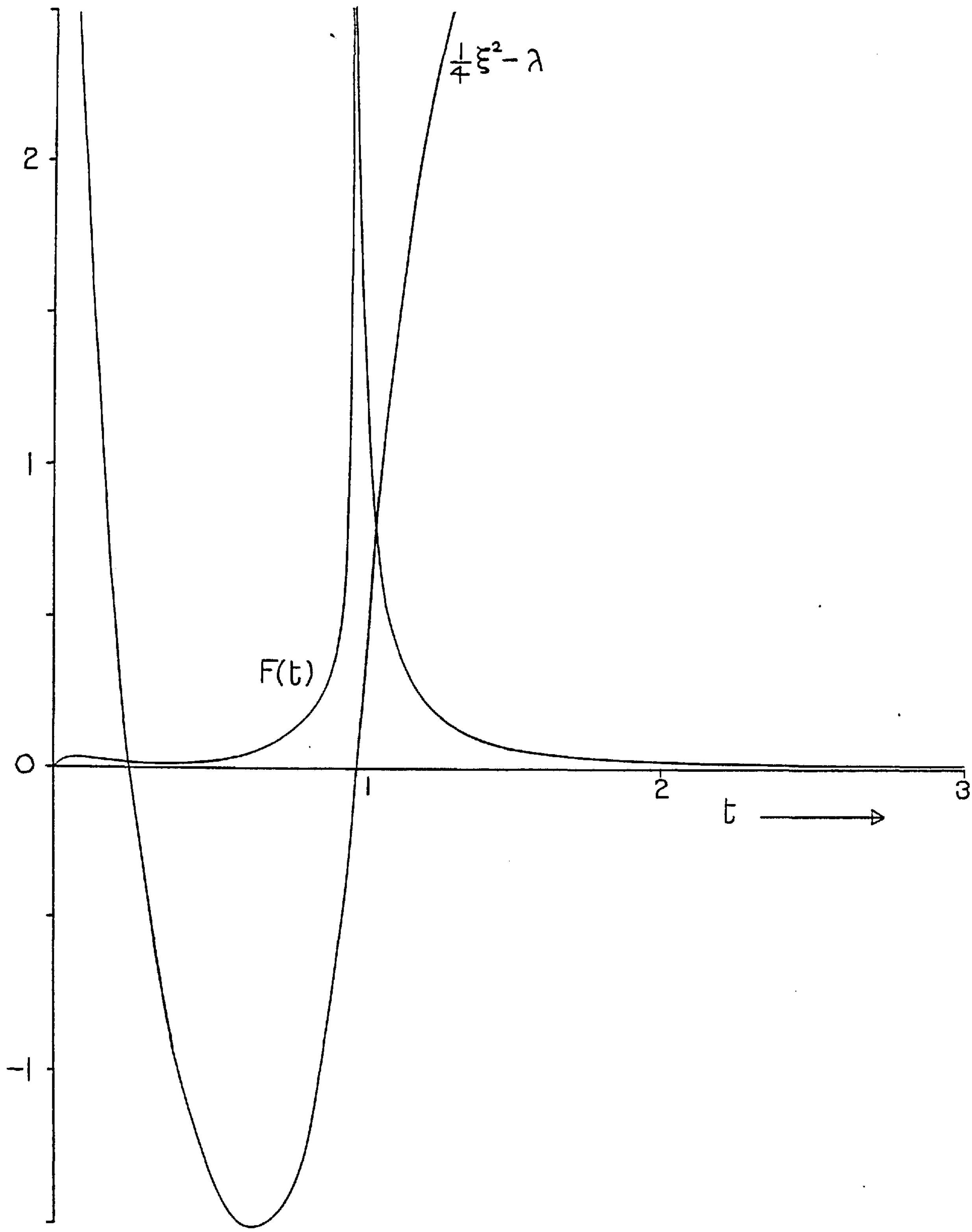


Figure 5. $F(t)$ and $\frac{1}{4}\xi^2 - \lambda$ against t, for $h = 1$, $\nu = 2$, $V = 9$.

$$\frac{\left(b-\frac{1}{2}\right)V^2 + Q(t)}{\dot{\xi}^2} + \frac{\dddot{\xi}}{2\dot{\xi}^3} - \frac{3}{4}\frac{\ddot{\xi}^2}{\dot{\xi}^4} - \frac{1}{4t^2\dot{\xi}^2}$$

$$= \left(\frac{1}{4}\xi^2 - \lambda\right) , \qquad (4.15)$$

so that, after the original choice (3.13) and neglecting $F(t)$, the next approximation would be

$$\left(b-\frac{1}{2}\right)V^2 + Q(t) = \dot{\xi}^2\left\{\frac{1}{4}\xi^2 - \lambda - F(t)\right\} . \qquad (4.16)$$

Integrating (4.16) between the turning points gives

$$\int_{t_1}^{t_2}\left\{\left(\frac{1}{2}-b\right)V^2 - Q(t)\right\}^{\frac{1}{2}} dt = \int_{\xi_1}^{\xi_2}\left\{\lambda - \frac{1}{4}\xi^2 + F(t(\xi))\right\}^{\frac{1}{2}} d\xi , \qquad (4.17)$$

where ξ_i $(i = 1,2)$ are the two solutions of

$$\lambda - \frac{1}{4}\xi_i^2 + F(t(\xi_i)) = 0 . \qquad (4.18)$$

It would appear from Figures 4 and 5 that, although $F(t) \longrightarrow \infty$ at $t = 1$ in the limit $\gamma \longrightarrow \infty$, the steepness and size of the function $\xi^2/4 - \lambda$ is such that ξ_2 will not differ greatly from $2\sqrt{\lambda}$ and a good approximation results from neglecting $F(t)$, due to its small size elsewhere.

Approximate eigenfunctions obtained from (3.13), (3.11) and the transformation (3.7) for large but finite values of γ are likely to be inaccurate in the region of rapid change of $n(r)$, due to neglecting the terms involving ∇n in the basic equations (2.4), as well as $F(t)$. Away from this region both ∇n and $F(t)$ are small.

5. The parabolic index profile

In this case, $n^2(r)$ in (2.7) is given by

$$n^2(r) = \begin{cases} n_1^2\left(1 - 2d\frac{r^2}{a^2}\right) & (r \leqslant a) \\ n_2^2 & (r \geqslant a) \end{cases} \tag{5.1}$$

where d is given by (3.5). In the core region $r \leqslant a$, (2.7) becomes

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \left(k^2 n_1^2 - 2k^2 n_1^2 d\frac{r^2}{a^2} - \beta^2 - \frac{\nu^2}{r^2}\right)\psi = 0. \tag{5.2}$$

With the transformation

$$r = at \quad , \quad \psi = t^{-1/2} R(t) \quad , \tag{5.3}$$

and defining b and V as before, by (3.5), equation (5.2) becomes

$$\frac{d^2R}{dt^2} = \left\{(b-1)V^2 + V^2t^2 + \frac{(\nu^2 - 1/4)}{t^2}\right\}R . \tag{5.4}$$

In [1], it was shown that for β in the range

$$n_2 k \leqslant |\beta| \leqslant n_1 k , \tag{5.5}$$

one is led to an eigenvalue equation and a finite number of guided mode solutions, with propagating electromagnetic waves in the core region. From (3.5), the condition (5.5) is equivalent to

$$0 \leqslant b \leqslant 1 , \tag{5.6}$$

and it is this eigenvalue problem we wish to consider here. Exact solutions to (5.4) have been given in [1], and for completeness, we give one particular form of these solutions. By further transformations of (5.4):

$$t = x^{1/2}, \quad R = \frac{u(x)}{x^{1/4}}, \quad \theta = Vx, \tag{5.7}$$

we obtain

$$\frac{d^2u}{d\theta^2} = \left(\frac{1}{4} - \frac{(1-b)V}{4\theta} + \frac{(\nu^2-1)}{4\theta^2} \right) u . \tag{5.8}$$

The solution of (5.8) that is well-behaved as $\theta \longrightarrow 0$ (corresponding to $r \rightarrow 0$) is

$$u(\theta) = M_{\frac{(1-b)V}{4}, \frac{\nu}{2}}(\theta) , \tag{5.9}$$

where $M_{K,\eta}(\theta)$ is the Whittaker function (Whittaker and Watson, 1927). From (5.7), the solution of (5.4) is then

$$R(t) = \frac{1}{t^{1/2}} M_{\frac{(1-b)V}{4}, \frac{\nu}{2}}(Vt^2) . \tag{5.10}$$

In a full treatment of the problem, the solution in the cladding region would have to be matched to (5.10), resulting in an eigenvalue condition involving Whittaker and Bessel functions. Previous approaches to this problem [1] have made the assumption that the core region extends to infinity in order to derive eigenvalues. With this assumption, the Whittaker function $M_{K,\eta}(\theta)$ is bounded as $\theta \rightarrow \infty$ only if

$$-\frac{1}{2} + K - \eta = n , \tag{5.11}$$

where $n = 0,1,2,\ldots..$ From (5.9), this gives

$$b = 1 - \frac{2}{V}(2n + \nu + 1) . \tag{5.12}$$

The eigenvalues are then restricted by the condition (5.6). The values of b calculated from (5.12) are shown in Figure 6 for $V \leqslant 12, \nu \geqslant 1$.

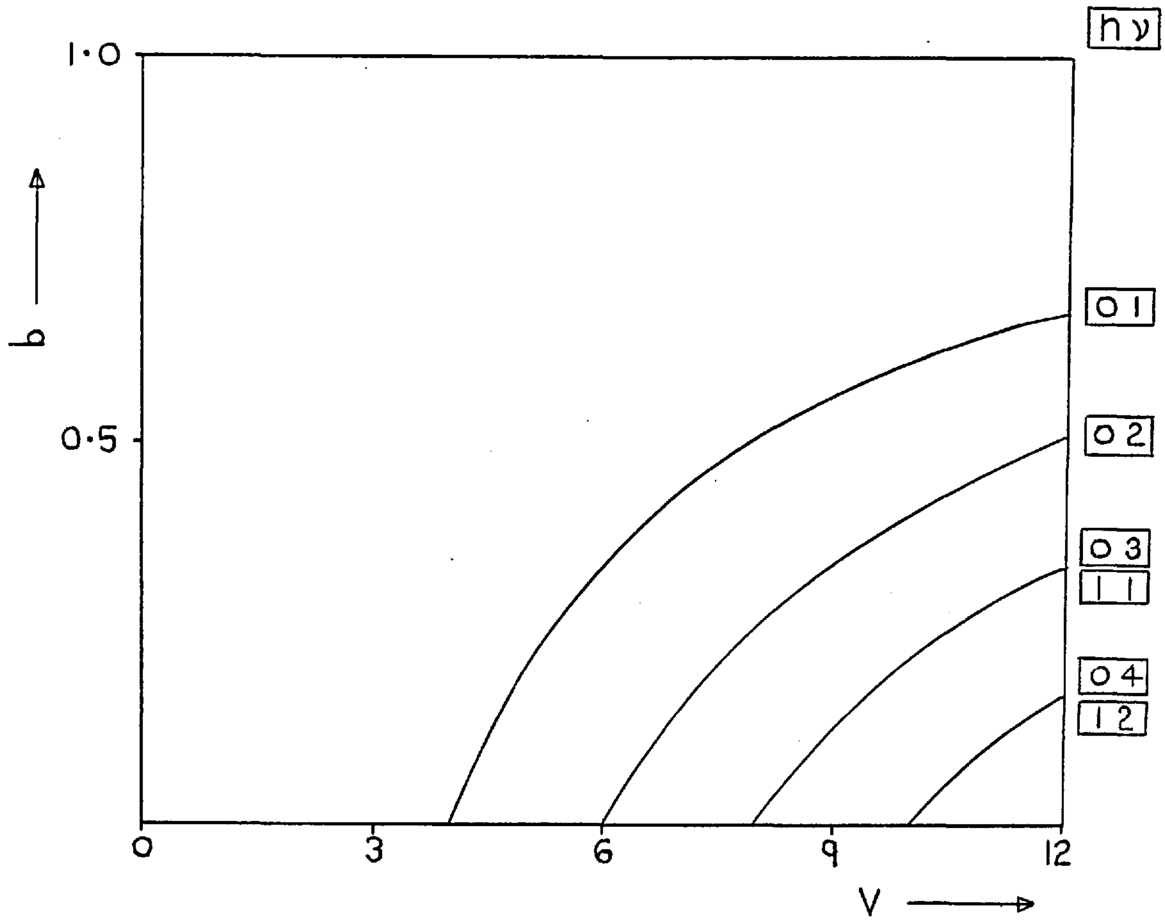


Figure 6. b against V , calculated from (5.12).

We note here that a J.W.K.B. analysis of equation (5.4), with the assumption of infinite core thickness, would give the eigenvalue condition

$$\pi\left(n+\frac{1}{2}\right) = \int_{t_1}^{t_2}\left\{(1-b)V^2 - V^2t^2 - \frac{\nu^2}{t^2}\right\}^{1/2} dt, \qquad (5.13)$$

where t_1 and t_2 are the roots of the integrand. With the aid of the substitution $t^2 = y$ in (5.13), the integral may be performed and the resulting equation solved for b. The result is again (5.12). A Liouville-Green analysis of (5.4) would give the same eigenvalue condition (5.13). Rosenzweig and Kreiger (1968) have discussed potentials possessing exact quantisation conditions of the type (5.13). One of these is the potential $At^2 + B/t^2$ appearing in equation (5.4).

Using the infinite core assumption, an essential feature of the permittivity, that it remains finite as $r \to \infty$, is lost. In order to see how the inclusion of this behaviour might affect the eigenvalues, we consider in the next section a modified profile which approximates to (5.1) but which is smooth for all r.

6. A modified profile

In order to approximate the profile (5.1), we consider

$$n_0^2(r) = n_1^2\left(1 - \frac{2\alpha r^2}{r^2 + q a^2}\right), \qquad (6.1)$$

where q is an as yet unspecified parameter. This profile has the correct limits as $r \to 0$ and $r \to \infty$, and has the advantage that the turning points of the differential equation (2.7) can be found analytically. As a method of choosing q , we set

$$\int_0^a n_0^2(r)\,dr = \int_0^a n^2(r)\,dr\,, \tag{6.2}$$

where $n^2(r)$ is given by (5.1). From (6.2), we then find

$$\int_0^a \frac{r^2}{(r^2+qa^2)}\,dr = \int_0^a \frac{r^2}{a^2}\,dr\,, \tag{6.3}$$

and performing the integrals gives

$$\sqrt{q}\,\tan^{-1}\left(\frac{1}{\sqrt{q}}\right) = \frac{2}{3}\,. \tag{6.4}$$

The approximate value of q found from (6.4) is

$$q = 0{\cdot}52\,. \tag{6.5}$$

Using (6.1), equation (2.7) becomes

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \left(k^2n_1^2 - \beta^2 - \frac{2k^2n_1^2dr^2}{(r^2+qa^2)} - \frac{\nu^2}{r^2}\right)\psi = 0. \tag{6.6}$$

As in the case of the step-index profile, we make the transformations

$$r = at\;,\quad \psi = t^{-1/2}R(t)\,. \tag{6.7}$$

Using (6.7), the definition (3.5) and the identity

$$\frac{V^2t^2}{(t^2+q)} = V^2 - \frac{V^2q}{(t^2+q)}\,, \tag{6.8}$$

equation (6.6) becomes

$$\frac{d^2R}{dt^2} = \left(bV^2 - \frac{V^2q}{(t^2+q)} + \frac{(\nu^2-1/4)}{t^2}\right)R\,. \tag{6.9}$$

We now follow the same procedure as in the case of the step-index profile. For certain values of b , equation (6.9) will have two turning points, and we attempt a transformation of (6.9) into the Weber equation (3.11).

The Liouville-Green transformation (3.7) together with the choice

$$\dot{\xi}^2\left(\frac{1}{4}\xi^2 - \lambda\right) = bV^2 - \frac{V^2 q}{(t^2+q)} + \frac{\nu^2}{t^2} \tag{6.10}$$

gives, from (6.9),

$$\frac{d^2G}{d\xi^2} = \left(\frac{1}{4}\xi^2 - \lambda + F(t)\right)G \;, \tag{6.11}$$

where

$$F(t) = \frac{\dddot{\xi}}{2\dot{\xi}^3} - \frac{3}{4}\frac{\ddot{\xi}^2}{\dot{\xi}^4} - \frac{1}{4t^2\dot{\xi}^2} \;. \tag{6.12}$$

In (6.10), we have again made the Langer correction in order to make $F(t)$ bounded as $t \to 0$. We now neglect $F(t)$, set $\lambda = n + \frac{1}{2}$ and integrate (6.10) between the turning points, giving the eigenvalue condition

$$\left(n + \frac{1}{2}\right)\pi = \int_{t_1}^{t_2} \left\{-bV^2 + \frac{V^2 q}{(t^2+q)} - \frac{\nu^2}{t^2}\right\}^{\frac{1}{2}} dt \;. \tag{6.13}$$

In (6.13), t_1 and t_2 are the roots of the integrand which, for $\nu \neq 0$, are given by

$$t_i^2 = \frac{1}{2bV^2}\left\{(1-b)V^2 q - \nu^2 \pm \sqrt{\left\{(1-b)V^2 q - \nu^2\right\}^2 - 4bV^2\nu^2 q}\right\} \tag{6.14}$$

for $i = 1, 2$. Now writing

$$S(t) = -bV^2 + \frac{V^2 q}{(t^2+q)} - \frac{\nu^2}{t^2} \;, \tag{6.15}$$

then $S(t) \to -\infty$ as $t \to 0$ and $S(t) \longrightarrow -bV^2$ as $t \to \infty$. In order to have two roots, the maximum value of $S(t)$ must therefore be positive. Setting $\dot{S}(t_0) = 0$ gives

$$t_0^2 = \frac{\nu q}{V\sqrt{q} - \nu} \tag{6.16}$$

and hence we must have

$$V\sqrt{q} > \nu . \qquad (6.17)$$

The maximum value of $S(t)$ is given by

$$S(t_0) = -bV^2 + \left(V - \frac{\nu}{\sqrt{q}}\right)^2 , \qquad (6.18)$$

and the condition on b for there to be two roots of $S(t)$, and for b to satisfy (5.6) is therefore

$$0 \leqslant b < \frac{(V\sqrt{q} - \nu)^2}{V^2 q} , \qquad (6.19)$$

with $\nu \geqslant 1$, since the right-hand term in (6.19) is less than unity, by (6.17).

The results of calculating b using (6.13) are shown in Figure 7 for $V \leqslant 12$, with the value of q given by (6.5). Alternative ways of choosing q could be considered. For example, taking the range of integration in (6.2) to be 0 to $3a/2$ gives a value $q \simeq 0 \cdot 3$, and a change in the larger eigenvalues of about 18% . Comparing these results with those of Figure 6, we see that, for the modified profile, the degenerate modes are split, and that a particular mode appears at a smaller value of V than in the case of the infinite parabolic profile.

By integration of (6.10), we find that between the turning points, $t_1 \leqslant t \leqslant t_2$,

$$\frac{1}{4}\xi(4\lambda - \xi^2)^{1/2} + \lambda \sin^{-1}\left(\frac{\xi}{2\sqrt{\lambda}}\right) + \lambda\frac{\pi}{2}$$

$$= \int_{t_1}^{t_2} \left\{-bV^2 + \frac{V^2 q}{(t^2+q)} - \frac{\nu^2}{t^2}\right\}^{1/2} dt , \qquad (6.20)$$

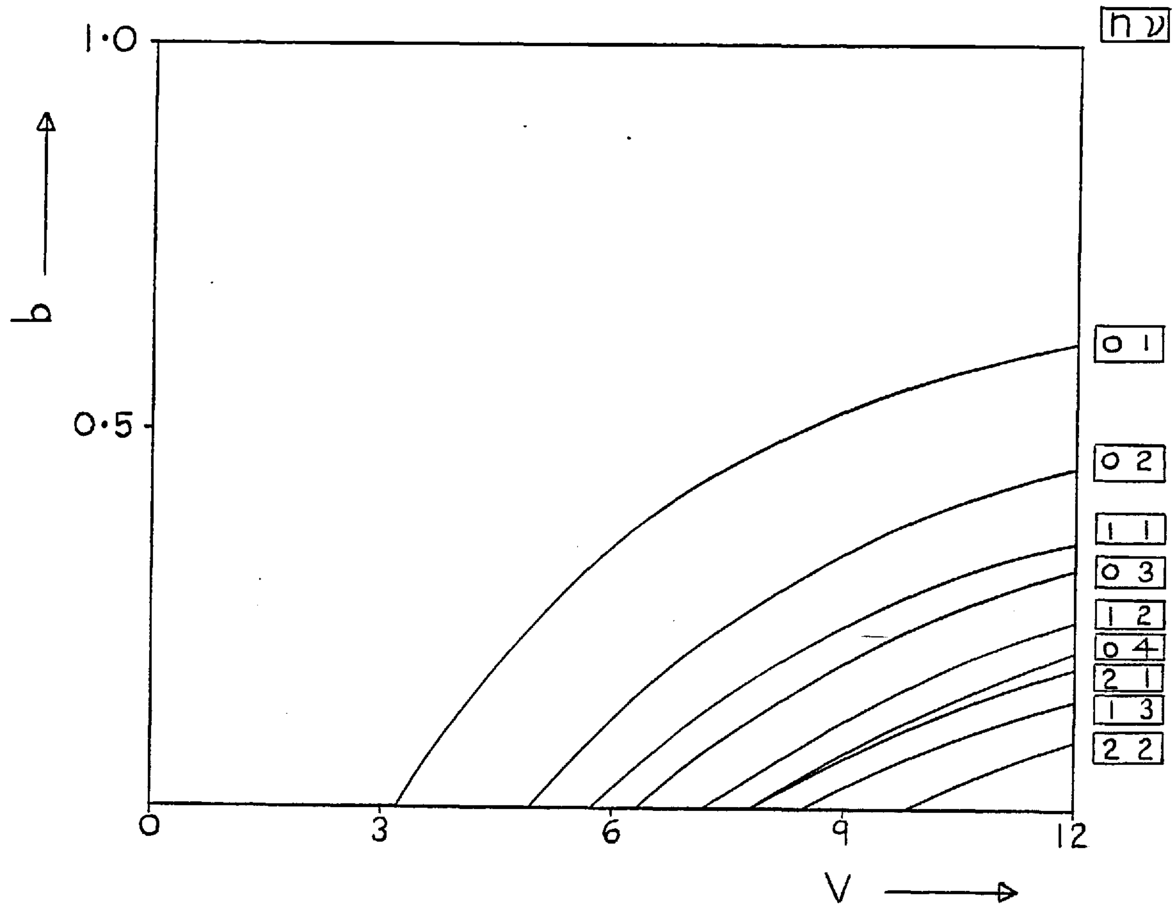


Figure 7. b against V , calculated from (6.13).

while outside the turning points,

$$\frac{1}{4}\xi(\xi^2-4\lambda)^{\frac{1}{2}} - \lambda\ln\left|\xi + (\xi^2-4\lambda)^{\frac{1}{2}}\right| + \lambda\ln(2\sqrt{\lambda})$$

$$= \begin{cases} -\displaystyle\int_t^{t_1}\left\{bV^2 - \frac{V^2q}{(t^2+q)} + \frac{\nu^2}{t^2}\right\}^{\frac{1}{2}}dt & (0\leqslant t\leqslant t_1) \\ \displaystyle\int_{t_2}^{t}\left\{bV^2 - \frac{V^2q}{(t^2+q)} + \frac{\nu^2}{t^2}\right\}^{\frac{1}{2}}dt & (t\geqslant t_2) \end{cases} \qquad (6.21)$$

where $\lambda = n+\frac{1}{2}$.

To calculate $F(t)$, we find the derivatives $\ddot{\xi}$ and $\dddot{\xi}$ from (6.10) and substitute into (6.12). At the turning points, we have the same results as (4.6)-(4.9) with $Q(t)$ replaced by $S(t)$. Then using the results (6.20) and (6.21), $F(t)$ can be calculated as a function of t for any chosen values of ν, n and V. The expressions involved are very lengthy, so we merely give, in Figure 8, the result of one particular calculation, that for $n = 1$, $\nu = 1$, $V = 8$. Other choices of the parameters indicate that the values obtained are typical for this problem.

7. Discussion

The results in section 6 could be improved upon if a closer approximation to the profile (5.1) could be found, in the same way as the step-index profile was approximated in section 3. In most physical problems, typical values of the parameter d are in the range $0{\cdot}01 - 0{\cdot}02$ (see [1]) and from (6.1), the term ∇n neglected in (2.4) is of order $d^{\frac{1}{2}}$. Other problems of interest include the

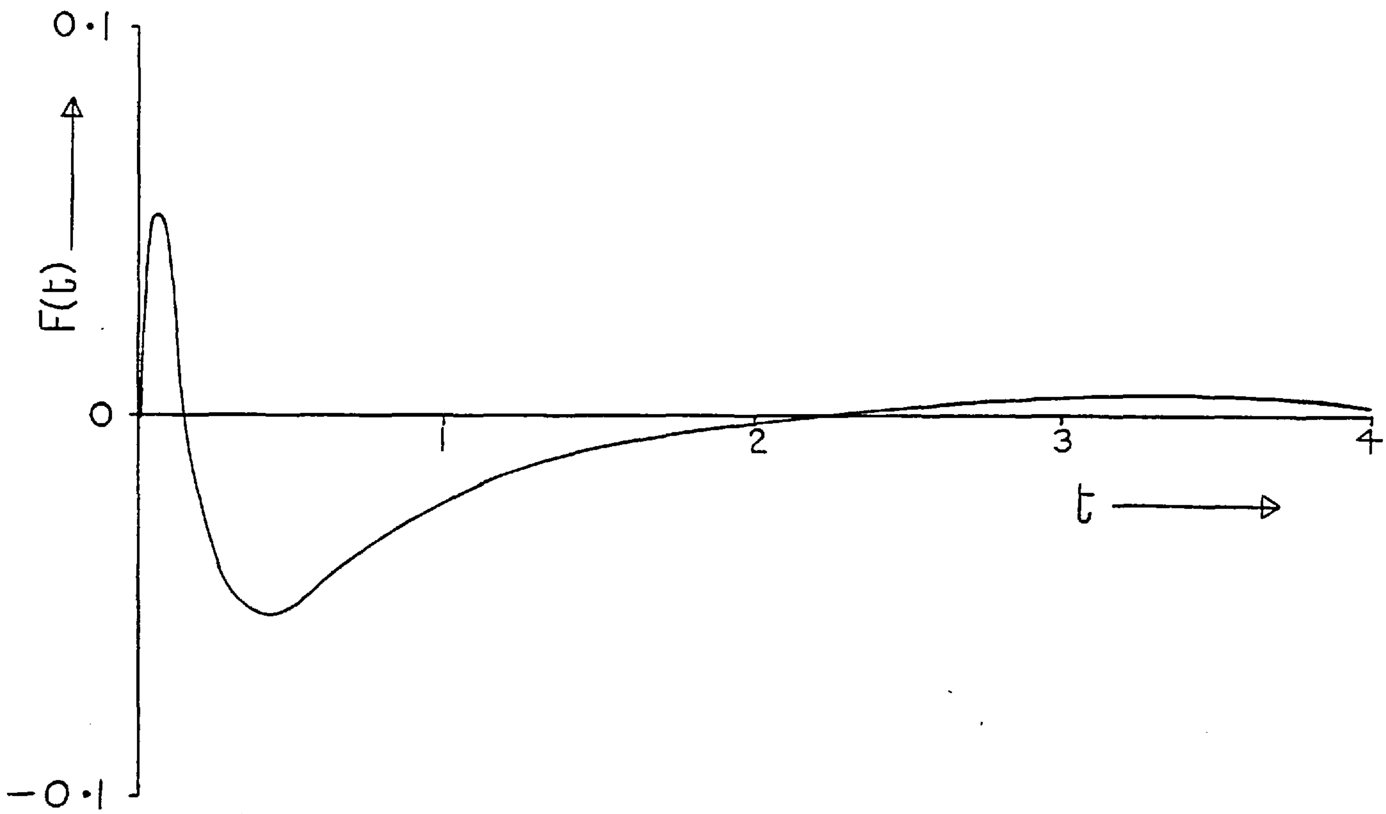


Figure 8. $F(t)$ against t, for the modified profile (6.1), with $n=1$, $\nu=1$, $V=8$.

modification of the parabolic profile to the behaviour $r^{2+\alpha}$, where α is a small parameter, and the consideration of an elliptical waveguide. The first is briefly discussed in Appendix 3, while the second results in a non-separable partial differential equation, for general profiles, and is therefore outside the scope of the present work.

APPENDIX 1

THE LIOUVILLE-GREEN TECHNIQUE

1. Transformation of differential equations

We begin with a second-order linear differential equation in normal form

$$\frac{d^2y}{dx^2} = f(x)y \quad . \tag{1.1}$$

The values of x for which $f(x)=0$ are called the turning points of the differential equation (1.1). After performing the Liouville-Green transformation (see Olver, 1974)

$$x = x(\xi) \quad , \quad G = (\xi')^{1/2} y \quad , \tag{1.2}$$

equation (1.1) becomes

$$\frac{d^2G}{d\xi^2} = \left\{ \frac{f(x)}{\xi'^2} + \Delta(x(\xi)) \right\} G \, , \tag{1.3}$$

where

$$\Delta(x(\xi)) = \frac{\xi'''}{2\xi'^3} - \frac{3}{4}\frac{\xi''^2}{\xi'^4} \quad . \tag{1.4}$$

Primes denote differentiation with respect to x .

We now try to choose the relationship between ξ and x so that (1.3) is soluble, exactly or approximately, in terms of known functions. If equation (1.1) has turning points, then we cannot choose $\xi'^2 = f(x)$ in (1.3), since $f(x)$ changes sign whereas ξ'^2 does not. In such cases, we choose a relationship of the type $h(\xi)\xi'^2 = f(x)$, for some $h(\xi)$ having as many zeros as $f(x)$ in the region of interest. This method of transformation is used in Chapters 5 and 6 to find approximate solutions of certain differential equations.

Since there are no standard second-order differential equations with more than two turning points, then, without resorting to matching, we can only apply the Liouville-Green technique to equations of the form (1.1) which have at most two turning points.

2. Exact solutions

For certain equations of the form (1.1), exact solutions can be quickly obtained by the Liouville-Green technique. We now give two simple examples.

Example 1.

Consider the equation

$$\frac{d^2y}{dx^2} = e^{nx}y \,, \tag{2.1}$$

where $n \neq 0, -\infty < x < \infty$. Using (1.2), equation (2.1) transforms to

$$\frac{d^2G}{d\xi^2} = \left\{ \frac{e^{nx}}{\xi'^2} + \Delta(x(\xi)) \right\} G \,, \tag{2.2}$$

where $\Delta(x(\xi))$ is given by (1.4). If we choose

$$\xi'^2 = e^{nx} \,, \tag{2.3}$$

then

$$\xi = \frac{2}{|n|} e^{nx/2} \,, \tag{2.4}$$

so that $0 < \xi < \infty$.

Calculating the higher derivatives of ξ from (2.3) and substituting into (1.4) gives

$$\Delta(x(\xi)) = -\frac{n^2}{16e^{nx}} \quad . \tag{2.5}$$

Now (2.5) may be expressed in terms of ξ from (2.4) as

$$\Delta(x(\xi)) = -\frac{1}{4\xi^2} \quad , \tag{2.6}$$

so that (2.2) becomes

$$\frac{d^2G}{d\xi^2} = \left(1 - \frac{1}{4\xi^2}\right)G \quad . \tag{2.7}$$

Finally, (2.7) has solutions

$$G(\xi) = \begin{cases} \xi^{\frac{1}{2}} I_0(\xi) \\ \xi^{\frac{1}{2}} K_0(\xi) \end{cases} \quad , \tag{2.8}$$

where I_0 and K_0 are the modified Bessel functions of order zero of the first and second kind respectively.

Using (2.4) and (1.2), we obtain

$$y(x) = \begin{cases} I_0\left(\frac{2}{|n|} e^{nx/2}\right) \\ K_0\left(\frac{2}{|n|} e^{nx/2}\right) \end{cases} \quad . \tag{2.9}$$

<u>Example 2.</u>

Consider the equation

$$\frac{d^2y}{dx^2} = x^n y \quad , \tag{2.10}$$

where $n \neq -2$, $x > 0$. Using (1.2), equation (2.10) transforms to

$$\frac{d^2G}{d\xi^2} = \left\{ \frac{x^n}{\xi'^2} + \Delta(x(\xi)) \right\} G \quad . \tag{2.11}$$

By the same procedure as in Example 1, the choice

$$\xi^{1/2} = x^n \tag{2.12}$$

gives

$$\xi = \frac{2}{|2+n|} x^{n/2+1} \tag{2.13}$$

and

$$\Delta(x(\xi)) = -\frac{n(n+4)}{4(n+2)^2\xi^2} . \tag{2.14}$$

Equation (2.11) becomes

$$\frac{d^2G}{d\xi^2} = \left\{ 1 - \frac{n(n+4)}{4(n+2)^2\xi^2} \right\} G , \tag{2.15}$$

and putting

$$2\xi = \theta \tag{2.16}$$

finally gives

$$\frac{d^2G}{d\theta^2} = \left\{ \frac{1}{4} - \frac{n(n+4)}{4(n+2)^2\theta^2} \right\} G . \tag{2.17}$$

Comparing (2.17) with the equation

$$\frac{d^2G}{d\theta^2} = \left\{ \frac{1}{4} - \frac{K}{\theta} + \frac{(m^2-1/4)}{\theta^2} \right\} G , \tag{2.18}$$

we find that two independent solutions of (2.17) are

$$G(\theta) = M_{0,\pm m}(\theta) , \tag{2.19}$$

where

$$m^2 - \frac{1}{4} = -\frac{n(n+4)}{4(n+2)^2} , \tag{2.20}$$

except if $2m$ is an integer, in which case

$$G(\theta) = \left\{ \begin{array}{l} M_{0,m}(\theta) \\ W_{0,m}(\theta) \end{array} \right. . \tag{2.21}$$

In (2.19) and (2.21), $M_{0,m}$ and $W_{0,m}$ are the Whittaker functions (Whittaker and Watson, 1927). From (2.20),

$$m = \frac{1}{|n+2|} \tag{2.22}$$

so that, from (2.12) and (1.2), one solution of (2.10) is

$$y(x) = \frac{1}{x^{n/4}} M_{0,\frac{1}{|n+2|}}\left(\frac{4}{|n+2|}x^{n/2+1}\right). \tag{2.23}$$

The second solution is

$$y(x) = \begin{cases} \dfrac{1}{x^{n/4}} M_{0,-\frac{1}{|n+2|}}\left(\dfrac{4}{|n+2|}x^{n/2+1}\right) & \left(\dfrac{2}{|n+2|} \neq \text{integer}\right) \\ \dfrac{1}{x^{n/4}} W_{0,\frac{1}{|n+2|}}\left(\dfrac{4}{|n+2|}x^{n/2+1}\right) & \left(\dfrac{2}{|n+2|} = \text{integer}\right) \end{cases} \tag{2.24}$$

We can also demonstrate that (2.23) reduces to known solutions in simple cases using the Kummer series (Whittaker and Watson, 1927)

$$M_{0,m}(u) = u^{m+1/2}\left(1 + \sum_{p=1}^{\infty} \frac{u^{2p}}{2^{4p}\,p!\,(m+1)\ldots\ldots(m+p)}\right), \tag{2.25}$$

valid provided m is not an integer. Consider for example the case of $n = 0$ in (2.10). From (2.23), one solution is

$$y(x) = M_{0,\frac{1}{2}}(2x) . \tag{2.26}$$

Using (2.25) we find

$$y(x) = x\left(1 + \sum_{p=1}^{\infty} \frac{(2x)^{2p}}{2^{4p}\,p!\left(\frac{1}{2}+1\right)\ldots\ldots\left(\frac{1}{2}+p\right)}\right)$$

$$= x\left(1 + \sum_{p=1}^{\infty} \frac{x^{2p}}{2^{p}\,p!\,(3)(5)\ldots\ldots(2p+1)}\right)$$

$$= \sum_{p=0}^{\infty} \frac{x^{2p+1}}{(2p+1)!}$$

$$= \sinh x \quad ,$$

which we recognize as a solution of $\dfrac{d^2y}{dx^2} = y$.

From the form of equation (2.15), we would expect to obtain a simple solution if $n = -4$. In this case, one solution is

$$y(x) = x M_{0,\frac{1}{2}}\left(\frac{2}{x}\right) . \tag{2.27}$$

Using (2.25), we find

$$y(x) = 1 + \sum_{p=1}^{\infty} \frac{\left(2/x\right)^{2p}}{2^{4p}\, p!\left(\frac{1}{2}+1\right)\cdots\cdots\left(\frac{1}{2}+p\right)}$$

$$= 1 + \sum_{p=1}^{\infty} \frac{1}{x^{2p}(2p+1)!}$$

$$= \sum_{p=0}^{\infty} \frac{1}{x^{2p}(2p+1)!}$$

$$= x \sinh\left(\frac{1}{x}\right) .$$

3. Non-linear differential equations

Generalising a result of Pinney (1950), Reid (1971) has noted that an exact solution of the non-linear differential equation

$$\frac{d^2y}{dx^2} + p(x)y = q_m(x)\, y^{1-2m} \tag{3.1}$$

is

$$y = \left[u^m + \frac{c}{(m-1)W^2} v^m \right]^{1/m}, \tag{3.2}$$

where u and v are two independent solutions of

$$\frac{d^2y}{dx^2} + p(x)y = 0 , \tag{3.3}$$

and where

$$W = uv' - vu' = \text{constant} , \tag{3.4}$$

$$q_m(x) = C\,(uv)^{m-2}, \tag{3.5}$$

and C is an arbitrary constant. This result can be obtained by a Liouville-Green transformation of (3.1).

Applying (1.2), and using (3.5), (3.1) becomes

$$\frac{d^2G}{d\xi^2} + \left(\frac{p(x)}{\xi'^2} - \frac{\xi'''}{2\xi'^3} + \frac{3}{4}\frac{\xi''^2}{\xi'^4} \right) G = c\,(uv\xi')^{m-2} G^{1-2m} . \tag{3.6}$$

Choosing

$$\xi' = \frac{1}{uv} , \tag{3.7}$$

we can calculate the derivatives ξ'' and ξ''', and on substituting into (3.6) we find

$$\frac{d^2G}{d\xi^2} = \frac{1}{4} W^2 G + cG^{1-2m} . \tag{3.8}$$

The particular form of $q_m(x)$ given by (3.5) is therefore such that (3.1) can be transformed to an equation with constant coefficients (3.8). By suitable choices of the constants of integration in (3.8), we should obtain the result (3.2). A first integral of (3.8) is

$$\left(\frac{dG}{d\xi}\right)^2 = \frac{1}{4}W^2G^2 + \frac{c}{(1-m)}G^{2-2m}, \tag{3.9}$$

on setting the constant of integration to zero. Separating the variables in (3.9) we obtain

$$\int \frac{dG}{G\sqrt{1+\frac{4c}{(1-m)W^2}G^{-2m}}} = \frac{1}{2}W\xi + A \tag{3.10}$$

where A is an arbitrary constant. Performing the integral in (3.10) gives

$$-\frac{1}{m}\ln\left|\frac{\sqrt{1+\frac{4c}{(1-m)W^2}G^{-2m}}-1}{G^{-m}}\right| = \frac{1}{2}W\xi + A. \tag{3.11}$$

Now, from (3.7),

$$\xi = \frac{1}{W}\ln\left(\frac{v}{u}\right), \tag{3.12}$$

and (3.11) becomes

$$\sqrt{1+\frac{4c}{(1-m)W^2}G^{-2m}} - 1 = D\left(\frac{u}{v}\right)^{m/2}G^{-m}, \tag{3.13}$$

where D is arbitrary. Solving (3.13) for G and using the transformation $y(x) = \sqrt{uv}\,G$, from (1.2) and (3.7), we obtain

$$y = \left[\alpha u^m + \frac{-c}{(1-m)W^2\alpha}v^{-m}\right]^{1/m}, \tag{3.14}$$

where $\alpha = -D/2$ is arbitrary. The choice $\alpha = 1$ gives the result of Reid, (3.2).

APPENDIX 2

THE ERROR IN THE APPROXIMATE EIGENFUNCTIONS CALCULATED IN CHAPTER 5

1. Preliminary calculations

In this appendix, we modify the method of Titchmarsh (1961) to find, for large eigenvalues, the order of the error in the eigenfunctions calculated by the Airy equation method in Chapter 5. We begin by reviewing some of the results of that chapter.

We write the basic equation as

$$\frac{d^2\psi}{dx^2} = (q - E)\psi \ , \tag{1.1}$$

where

$$q = q(x) = x^2 + x^4 \ , \tag{1.2}$$

and $0 \leqslant x < \infty$. Transforming (1.1) by the Liouville-Green method gave

$$\frac{d^2G}{d\xi^2} = \left\{ \xi - a + \Delta(x(\xi)) \right\} G \ , \tag{1.3}$$

where $\Delta(x(\xi))$ can be written

$$\Delta(x(\xi)) = (\xi - a)\left\{ \frac{5}{16(\xi-a)^3} + \frac{q''}{4(q-E)^2} - \frac{5q'^2}{16(q-E)^3} \right\}, \tag{1.4}$$

primes denoting differentiation with respect to x . The relationship between ξ and x ,

$$(\xi - a)\xi'^2 = (q - E) \ , \tag{1.5}$$

gives, on rewriting (4.2)-(4.4) of Chapter 5,

$$\left.\begin{aligned} \frac{2}{3}(\xi-a)^{3/2} &= \int_{x_0}^{x}\{q(t)-E\}^{1/2}\,dt \qquad (x \geqslant x_0) \\ \frac{2}{3}(a-\xi)^{3/2} &= \int_{x}^{x_0}\{E-q(t)\}^{1/2}\,dt \qquad (x \leqslant x_0) \end{aligned}\right\} \tag{1.6}$$

where

$$x_0^2 = \frac{1}{2}\left(\sqrt{1+4E} - 1\right). \tag{1.7}$$

Substituting

$$G(\xi) = c_1(\xi)Ai(\xi-a) + c_2(\xi)Bi(\xi-a) \tag{1.8}$$

into (1.3) gives, by the standard method of variation of parameters,

$$G(\xi) = \alpha Ai(\xi-a) - \pi\int_{\xi}^{\infty}\Delta(x(\bar{\xi}))G(\bar{\xi}) \times\left\{Ai(\bar{\xi}-a)Bi(\xi-a)-Bi(\bar{\xi}-a)Ai(\xi-a)\right\}d\bar{\xi}\,. \tag{1.9}$$

In (1.9), we change the variable of integration, using (1.5), to obtain

$$G(\xi(x)) = \alpha Ai(\xi(x)-a) - \pi\int_{x}^{\infty}\Delta(t)\,G(\xi(t))\left(\frac{q(t)-E}{\xi(t)-a}\right)^{1/2} \times\left\{Ai(\xi(t)-a)Bi(\xi(x)-a) - Bi(\xi(t)-a)Ai(\xi(x)-a)\right\}dt\,. \tag{1.10}$$

We now consider the quantity

$$I = \int_0^\infty |\Delta(x)| \left| \frac{q - E}{\xi(x) - a} \right|^{\frac{1}{2}} dx \quad . \tag{1.11}$$

Substituting the expression (1.4) for $\Delta(x)$ into (1.11) and splitting up the range of integration gives

$$I = \left(\int_0^{(1-d)x_0} + \int_{(1-d)x_0}^{x_0} + \int_{x_0}^{(1+d)x_0} + \int_{(1+d)x_0}^{\infty} \right) |\xi(x) - a|^{\frac{1}{2}}$$

$$\times |q - E|^{\frac{1}{2}} \left| \frac{5}{16(\xi(x) - a)^3} + \frac{q''}{4(q - E)^2} - \frac{5q'^2}{16(q - E)^3} \right| dx \tag{1.12}$$

$$= I_1 + I_2 + I_3 + I_4 \quad , \tag{1.13}$$

where d is in the range, say, $(0, \frac{1}{2}]$.

Now for large E , $x_0 \sim E^{\frac{1}{4}}$ from (1.7), while for large x , $q \sim x^4$, $q' \sim 4x^3$, and so on for higher derivatives. Expressing these results more formally, we have that, for each small $\epsilon > 0$, there exists $X(\epsilon)$ such that when $x > X(\epsilon)$,

$$\left. \begin{aligned} x^4 < \; & q < (1 + \epsilon) x^4 , \\ 4x^3 < \; & q' < 4(1 + \epsilon) x^3 , \\ 12x^2 < \; & q'' < 12(1 + \epsilon) x^2 . \end{aligned} \right\} \tag{1.14}$$

We will assume that E is large enough so that $(1-d)x_0 > X(\epsilon)$. Also, there exists $\mathcal{E}(\epsilon)$ such that when $E > \mathcal{E}(\epsilon)$,

$$(1 - \epsilon) E < x_0^4 < (1 + \epsilon) E \quad . \tag{1.15}$$

2. The behaviour of I for large E

We treat each I_j (j = 1 to 4) in (1.12) and (1.13) separately. In what follows, we will use A to represent a generic positive number which is $O(1)$.

(a) I_3 , $x_0 \leqslant x \leqslant (1+d)x_0$

Integrating by parts twice in the first equation of (1.6) gives

$$\frac{2}{3}(\xi - a)^{3/2} = \frac{2}{3}\frac{(q-E)^{3/2}}{q'}\left\{1 + \frac{2(q-E)q''}{5q'^2} - S\right\}, \quad (2.1)$$

where

$$S = \frac{2q'}{5(q-E)^{3/2}}\int_{x_0}^{x}(q(t)-E)^{5/2}q'(t)Q(t)\;dt \quad (2.2)$$

and

$$Q = \frac{(q'q''' - 3q''^2)}{q'^5}. \quad (2.3)$$

Now by the mean value theorem

$$(q-E) = (x - x_0)\,q'(\tau) \quad (2.4)$$

where $x_0 \leqslant \tau \leqslant x$. Since q' is a monotonically increasing function, replacing the right-hand side of (2.4) by its maximum value for $x_0 \leqslant x \leqslant (1+d)x_0$ gives

$$\begin{aligned}(q-E) &\leqslant \{(1+d)x_0 - x_0\}\,q'\{(1+d)x_0\} \\ &= dx_0 q'\{(1+d)x_0\} \\ &< 4dx_0(1+\epsilon)(1+d)^3 x_0^3 ,\end{aligned}$$

from (1.14). Therefore, from (1.15),

$$(q - E) < AdE\,. \tag{2.5}$$

From (1.14),

$$\frac{q''}{q'^2} < \frac{A}{x^4} < \frac{A}{x_0^4}$$

so that, from (1.15),

$$\frac{q''}{q'^2} < \frac{A}{E}\,. \tag{2.6}$$

Hence

$$\frac{2(q-E)q''}{5q'^2} < \frac{A(q-E)}{E} < Ad\,, \tag{2.7}$$

where, in the last inequality, we have used the result (2.5). A similar procedure applied to Q , defined by (2.3), gives

$$|Q| < \frac{A}{E^{1/4}}\,. \tag{2.8}$$

Substituting the result (2.8) into (2.2), and then using (1.14), we have the inequalities

$$|S| < \frac{Aq'}{(q-E)^{3/2}E^{1/4}}(q-E)^{7/2} < \frac{A(q-E)^2}{E^{1/4}}x_0^3\,.$$

Then from (1.15) and (2.5),

$$|S| < A\left(\frac{q-E}{E}\right)^2 < Ad^2\,. \tag{2.9}$$

To estimate the size of the integrand in (1.12), we use the relationship (2.1) to write

$$\frac{5}{16(\xi-a)^3} = \frac{5q'^2}{16(q-E)^3}\left\{1 + \frac{2(q-E)q''}{5q'^2} - S\right\}^{-2}. \tag{2.10}$$

By virtue of the inequalities (2.7) and (2.9), we perform a binomial expansion of (2.10), since the terms in large brackets decrease in order (for sufficiently small d). Hence

$$\frac{5}{16(\xi-a)^3} = \frac{5q'^2}{16(q-E)^3}\left\{1 - \frac{4(q-E)q''}{5q'^2} + \left(\frac{q-E}{E}\right)^2 O(1)\right\}. \tag{2.11}$$

Substituting (2.11) into the expansion for I_3 from (1.12) and (1.13) gives

$$I_3 = \frac{A}{E^2}\int_{x_0}^{(1+d)x_0} \left|\frac{\xi(x)-a}{q-E}\right|^{\frac{1}{2}} q'^2\, dx . \tag{2.12}$$

Finally, from (2.1),

$$\left|\frac{\xi(x)-a}{q-E}\right|^{\frac{1}{2}} = O(q'^{-\frac{1}{3}}) \quad , \tag{2.13}$$

and substituting (2.13) into (2.12) leads to

$$I_3 < \frac{A}{E^2}\int_{x_0}^{(1+d)x_0} q'^{\frac{5}{3}}\, dx$$

$$= O\left\{\frac{1}{E^2} x_0 (x_0^3)^{\frac{5}{3}}\right\} ,$$

using (1.14). Hence from (1.15),

$$I_3 = O\left(\frac{1}{E^{\frac{1}{2}}}\right) . \tag{2.14}$$

The integral I_2 with range $(1-d)x_0 \leqslant x \leqslant x_0$ can be treated in the same way as I_3 , and we obtain a corresponding result

$$I_2 = O\left(\frac{1}{E^{\frac{1}{2}}}\right) . \tag{2.15}$$

(b) I_4, $(1+d)x_0 \leqslant x < \infty$

From the definition of I_4 in (1.12) and (1.13), we obtain the following bound by replacing the negative term by its absolute value:

$$I_4 \leqslant \frac{5}{16}\int_{(1+d)x_0}^{\infty} \frac{(q-E)^{\frac{1}{2}}}{(\xi(x)-a)^{5/2}}\,dx + \frac{1}{4}\int_{(1+d)x_0}^{\infty} \frac{q''}{(q-E)^{3/2}}(\xi(x)-a)^{\frac{1}{2}}\,dx$$

$$+ \frac{5}{16}\int_{(1+d)x_0}^{\infty} \frac{q'^2}{(q-E)^{5/2}}(\xi(x)-a)^{\frac{1}{2}}\,dx\,. \qquad (2.16)$$

From the transformation (1.4), the first term in (2.16) can be rewritten and evaluated:

$$\frac{5}{16}\int_{(1+d)x_0}^{\infty} \frac{\xi'(x)}{(\xi(x)-a)^2}\,dx = \frac{5}{16}\frac{1}{\left\{\xi[(1+d)x_0]-a\right\}}\,. \qquad (2.17)$$

Now

$$\frac{(q-E)}{x^4} > \frac{(x^4-E)}{x^4} \qquad (2.18)$$

and the right-hand side of (2.18) is an increasing function of x and so is larger than its value at the lower end of the range $(1+d)x_0 \leqslant x < \infty$. Hence

$$\frac{(q-E)}{x^4} > \frac{(1+d)^4x_0^4 - E}{(1+d)^4x_0^4}$$

$$> \frac{(1-\epsilon)(1+d)^4E - E}{(1-\epsilon)(1+d)^4E}\,,$$

the last inequality being derived from (1.15). Provided d is chosen so that $(1-\epsilon)(1+d)^4 > 1$, then

$$q - E > Ax^4 . \tag{2.19}$$

Now expression (2.1) is valid for all $x > x_0$. In particular at $x = (1+d)x_0$,

$$\left\{ \xi\left[(1+d)x_0\right] - a \right\}^{3/2} > A \frac{\left\{ q\left[(1+d)x_0\right] - E \right\}^{3/2}}{q'\left[(1+d)x_0\right]} . \tag{2.20}$$

Using the results (2.19), (1.14) and (1.15), we find

$$\xi\left[(1+d)x_0\right] - a > \frac{Ax_0^4}{x_0^2} > AE^{1/2} . \tag{2.21}$$

Substituting (2.21) into (2.17), we have a bound for the first term in (2.16):

$$\frac{5}{16} \int_{(1+d)x_0}^{\infty} \frac{(q-E)^{1/2}}{(\xi(x) - a)^{5/2}} \, dx < \frac{A}{E^{1/2}} . \tag{2.22}$$

In order to bound the remaining terms in (2.16), we need a bound on $\left\{ \xi(x) - a \right\}^{1/2}$. First, we note that for $x > x_0$,

$$E < q = x^4 + x^2 < 2x^4 ,$$

so that

$$q - E < q + E < 2q < 4x^4 . \tag{2.23}$$

Substitution of (2.23) into the first expression of (1.6) gives

$$\frac{2}{3}(\xi(x) - a)^{3/2} < \int_{x_0}^{x} 2t^2 \, dt < \frac{2}{3} x^3 . \tag{2.24}$$

Hence, from (2.24),

$$\left\{\xi(x) - a\right\}^{\frac{1}{2}} < x . \tag{2.25}$$

Finally, the results (1.14), (2.19), (2.22) and (2.25) enable a bound on I_4 to be found from (2.16):

$$I_4 \leqslant A\left(\frac{1}{E^{\frac{1}{2}}} + \int_{(1+d)x_0}^{\infty} \frac{x^2 x}{(x^4)^{3/2}}\,dx + \int_{(1+d)x_0}^{\infty} \frac{x^6 x}{(x^4)^{5/2}}\,dx\right)$$

$$< A\left(\frac{1}{E^{\frac{1}{2}}} + \frac{1}{x_0^2}\right) .$$

Hence

$$I_4 = O\left(\frac{1}{E^{\frac{1}{2}}}\right) . \tag{2.26}$$

(c) I_1 , $0 \leqslant x \leqslant (1-d)x_0$

From (1.12), we bound I_1 by

$$I_1 \leqslant \frac{5}{16}\int_0^{(1-d)x_0} \frac{(E-q)^{\frac{1}{2}}}{(a-\xi(x))^{5/2}}\,dx + \frac{1}{4}\int_0^{(1-d)x_0} \frac{q''}{(E-q)^{3/2}}(a-\xi(x))^{\frac{1}{2}}\,dx$$

$$+ \frac{5}{16}\int_0^{(1-d)x_0} \frac{q'^2}{(E-q)^{5/2}}\left(a-\xi(x)\right)^{\frac{1}{2}}\,dx . \tag{2.27}$$

Now

$$\begin{aligned} E - q &> E - q\{(1-d)x_0\} \\ &= E - \left\{(1-d)^4 x_0^4 + (1-d)^2 x_0^2\right\} . \end{aligned} \tag{2.28}$$

Taking E large enough so that

$$(1-d)x_0 > 2$$

then

$$(1-d)^4 x_0^4 > 4(1-d)^2 x_0^2$$

and hence

$$(1-d)^4 x_0^4 + (1-d)^2 x_0^2 < \frac{5}{4}(1-d)^4 x_0^4 . \qquad (2.29)$$

Substituting (2.29) into (2.28) gives

$$E - q > E - \frac{5}{4}(1-d)^4 x_0^4$$

$$> E - \frac{5}{4}(1-d)^4(1+\epsilon)E$$

from (1.15). Therefore

$$E - q > AE , \qquad (2.30)$$

provided $\frac{5}{4}(1-d)^4(1+\epsilon) < 1$ (for example, this condition would give $d > 0.066$ for $\epsilon = 0.05$).

From the transformation (1.6) for $x < x_0$, we have

$$\frac{2}{3}(a - \xi(x))^{3/2} < \int_0^{x_0} \{E - q(t)\}^{1/2}\, dt$$

$$< E^{1/2} x_0$$

$$< AE^{3/4}$$

from (1.15), and so

$$(a - \xi(x))^{1/2} < AE^{1/4} . \qquad (2.31)$$

Also from (1.6), we find

$$\frac{2}{3}\left(a - \xi\left[(1-d)x_0\right]\right)^{3/2} = \int_{(1-d)x_0}^{x_0} \{E - q(t)\}^{1/2}\, dt$$

$$> AE^{1/2} x_0$$

$$> AE^{3/4} ,$$

using the result (2.30), and (1.15). Then

$$\frac{1}{\{a - \xi[(1-d)x_0]\}} < \frac{A}{E^{1/2}} . \tag{2.32}$$

The first term in (2.27) may be rewritten, from (1.5),

$$\frac{5}{16}\int_0^{(1-d)x_0} \frac{\xi'(x)}{\{a - \xi(x)\}^2}\, dx = \frac{5}{16}\left(\frac{1}{\{a - \xi[(1-d)x_0]\}} - \frac{1}{a}\right)$$

$$< \frac{5}{16}\,\frac{1}{\{a - \xi[(1-d)x_0]\}}$$

$$= O\left(\frac{1}{E^{1/2}}\right) , \tag{2.33}$$

from (2.32). Using (2.31), the third term of (2.27) has the upper bound

$$Aq_V'\{(1-d)x_0\}\,E^{1/4}\int_0^{(1-d)x_0} \frac{q_V'}{(E - q_V)^{5/2}}\, dx$$

$$< AE\left(\frac{1}{\{E - q_V[(1-d)x_0]\}^{3/2}} - \frac{1}{E^{3/2}}\right)$$

$$= O\left(\frac{1}{E^{1/2}}\right) , \tag{2.34}$$

where we have used (1.14) and (1.15) to replace the factor $q_V'\{(1-d)x_0\}$ and (2.30) to give the order of the terms.

The only remaining term is the second term of (2.27). Again using (2.31), this has the upper bound

$$\frac{1}{4}\int_0^{(1-d)x_0} \frac{q''}{(E-q)^{3/2}}\left(a-\xi(x)\right)^{1/2} dx < AE^{1/4}\int_0^{(1-d)x_0} \frac{q''}{(E-q)^{3/2}}\, dx.$$

Integrating the right-hand side by parts, we obtain

$$AE^{1/4}\left\{\left[\frac{q'}{(E-q)^{3/2}}\right]_0^{(1-d)x_0} - \frac{3}{2}\int_0^{(1-d)x_0} \frac{q'^2}{(E-q)^{5/2}}\, dx\right\}. \tag{2.35}$$

Finally, using (1.14), (1.15), (2.30) and the method of deriving (2.34), we see that (2.35) is $O(E^{-1/2})$.

The results of this section show that the integral defined by (1.11) satisfies

$$I = O\left(\frac{1}{E^{1/2}}\right) \tag{2.36}$$

for large values of E .

3. The error in the approximate eigenfunctions

By successive approximations in (1.10), we find that the first correction C to the basic solution $Ai(\xi(x)-a)$ is given by

$$C = -\pi\int_x^{\infty} \Delta(t)\left(\frac{q(t)-E}{\xi(t)-a}\right)^{1/2} K(x,t)\, dt \tag{3.1}$$

where

$$K(x,t) = Ai\left(\xi(t)-a\right)\Big\{Ai\left(\xi(t)-a\right)Bi\left(\xi(x)-a\right)$$
$$- Bi\left(\xi(t)-a\right)Ai\left(\xi(x)-a\right)\Big\}. \tag{3.2}$$

From (3.1),

$$|c| < m\pi \int_0^{\infty} |\Delta(t)| \left| \frac{q(t)-E}{\xi(t)-a} \right|^{\frac{1}{2}} dt \,, \tag{3.3}$$

where

$$m = \max_{0 \leqslant x \leqslant t < \infty} |K(x,t)| \,. \tag{3.4}$$

Writing

$$\left. \begin{aligned} u &= \xi(t) - a \\ v &= \xi(x) - a \end{aligned} \right\} , \tag{3.5}$$

then for $t \geqslant x$, we have $u \geqslant v$. As $E \to \infty$, then $a \to \infty$ (see Table 2, Chapter 5) and so $-\infty < v \leqslant u < \infty$. We express m, from (3.2), as

$$m = \max_{-\infty < v \leqslant u < \infty} |K| \,, \tag{3.6}$$

where

$$K = Ai(u)\left\{Ai(u)Bi(v) - Bi(u)Ai(v)\right\} \,. \tag{3.7}$$

In Figure 1, we plot $|K|$ as a function of u for selected values of v. For all values of u and v satisfying $v \leqslant u$, $|K|$ is a bounded function, which the numerical results indicate is less than unity.

Finally, from (3.3) and the result (2.36), we find that for large E, the solution $G(\xi)$ of (1.3) satisfies

$$G(\xi) = \alpha Ai(\xi - a) + O\left(\frac{m}{E^{\frac{1}{2}}}\right) \,. \tag{3.8}$$

As the Airy function $Ai(\xi - a) = Ai(v)$ in (3.8) tends to zero for large values of its argument $\xi - a = v$, then the corresponding value of m also tends to zero, from (3.6).

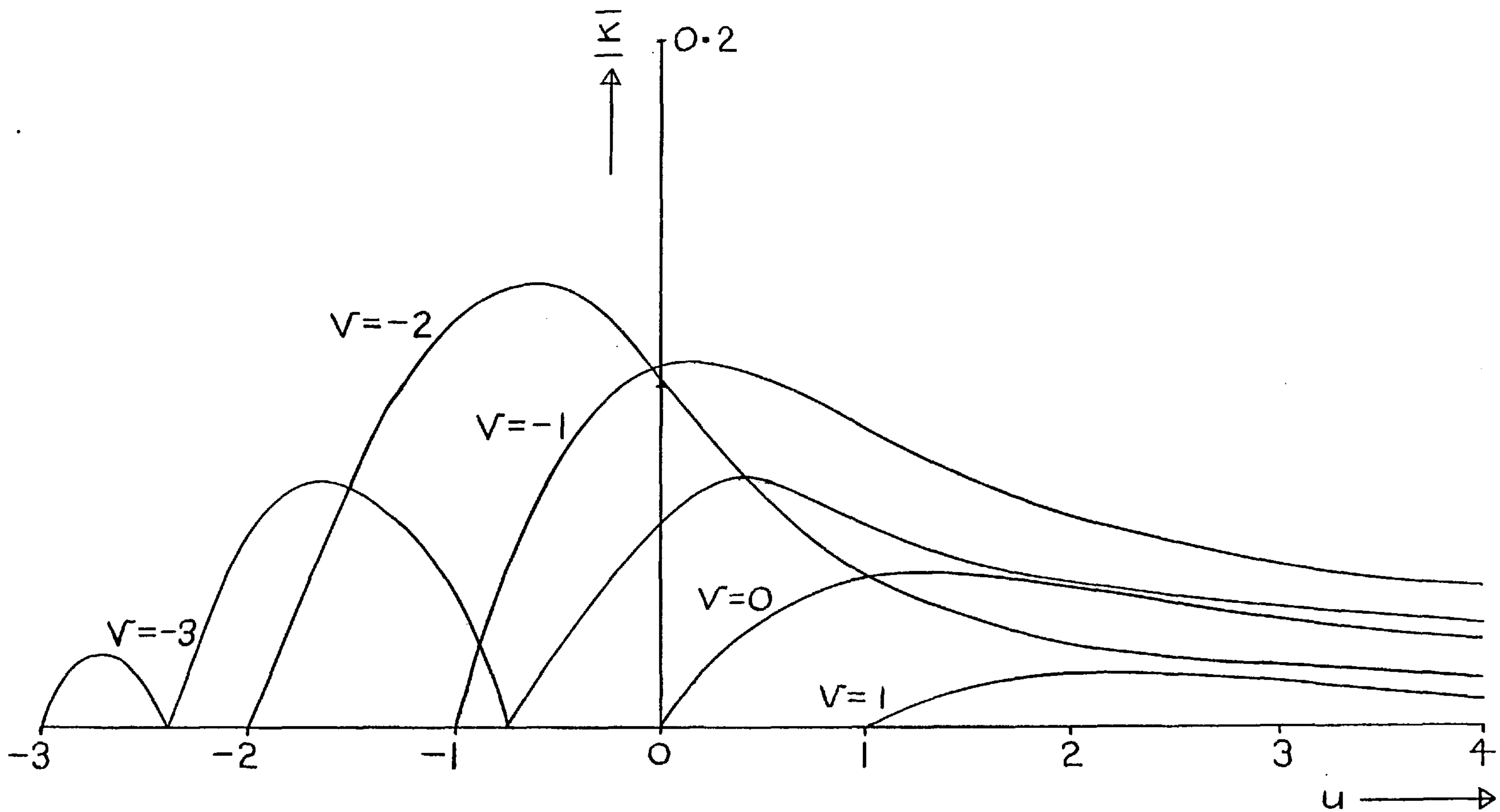


Figure 1. $|K|$ against u for selected values of ν.

APPENDIX 3

SMALL PERTURBATIONS FROM THE PARABOLIC REFRACTIVE INDEX PROFILE IN OPTICAL WAVEGUIDES

1. Introduction

In Chapter 6, we discussed the differential equation

$$\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} + \left(A - Br^2 - \frac{\nu^2}{r^2}\right)E = 0, \quad (1.1)$$

where the term Br^2 arose from the parabolic refractive index profile. Problems of current interest (Olshansky, 1979) are concerned with profiles in which the r^2 nature is modified to $r^{2+\alpha}$, where α is a small parameter. We expect that the solutions for the modified profile do not differ substantially from those for the parabolic profile, if α is sufficiently small.

An exact solution of equation (1.1) was presented in Chapter 6, but the modified equation

$$\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} + \left(A - Br^{2+\alpha} - \frac{\nu^2}{r^2}\right)E = 0, \quad (1.2)$$

does not appear to be exactly soluble in terms of known functions. Although we could find an approximate solution of (1.2) using the Liouville-Green method, we would not know if the difference between this solution and the exact solution for the parabolic profile was mainly due to the introduction of the parameter α or to the approximation method. Accordingly, we consider the somewhat unphysical case of $A = 0$, since equation (1.2) is then exactly soluble for all α.

2. Exact solutions

With $A = 0$, (1.2) becomes

$$\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} - \left(Br^{2+\alpha} + \frac{\nu^2}{r^2}\right)E = 0 . \tag{2.1}$$

We make the transformation

$$r = x^p , \quad E = x^{-1/2}\psi , \tag{2.2}$$

where p is a constant to be chosen. Using (2.2), equation (2.1) is transformed to

$$\frac{d^2\psi}{dx^2} = \left\{p^2 B x^\lambda + \frac{(p^2\nu^2 - 1/4)}{x^2}\right\}\psi , \tag{2.3}$$

where

$$\lambda = p(4+\alpha) - 2 . \tag{2.4}$$

Choosing $\lambda = 0$, so that

$$p = \frac{2}{(4+\alpha)} , \tag{2.5}$$

then (2.3) becomes

$$\frac{d^2\psi}{dx^2} = \left\{\frac{4B}{(4+\alpha)^2} + \frac{\left(\frac{4\nu^2}{(4+\alpha)^2} - \frac{1}{4}\right)}{x^2}\right\}\psi . \tag{2.6}$$

The solution of (2.6) that is well-behaved as $x \to 0$ is

$$\psi(x) = x^{1/2} I_{\frac{2\nu}{(4+\alpha)}}\left(\frac{2\sqrt{B}}{(4+\alpha)}x\right) , \tag{2.7}$$

where I_γ is the modified Bessel function of the first kind of order γ. The solution of (2.1) is then

$$E_\alpha(r) = m_\alpha I_{\frac{2\nu}{(4+\alpha)}}\left(\frac{2\sqrt{B}}{(4+\alpha)} r^{(4+\alpha)/2}\right) \tag{2.8}$$

the solution for $\alpha = 0$ being

$$E_0(r) = m_0 I_{\frac{\nu}{2}}\left(\frac{\sqrt{B}}{2} r^2\right) , \tag{2.9}$$

where m_α and m_0 are constants.

3. Numerical estimates

As a particular example, we consider the problem of fields confined to a core region of the waveguide $r \leqslant r_0$, with the values (from Olshansky, 1979)

$$\left.\begin{aligned} r_0 &= 30\ \mu m \\ \sqrt{B} &= 0{\cdot}0447 \\ \alpha &= -0{\cdot}024 \end{aligned}\right\} . \tag{3.1}$$

We apply the boundary condition that the field is equal to a fixed value at $r = r_0$, say $E(r_0) = C$. Then from (2.8),

$$\left.\begin{aligned} m_\alpha &= \frac{C}{I_{\frac{2\nu}{(4+\alpha)}}\left(\frac{2\sqrt{B}}{(4+\alpha)} r_0^{(4+\alpha)/2}\right)} \\ m_0 &= \frac{C}{I_{\frac{\nu}{2}}\left(\frac{\sqrt{B}}{2} r_0^2\right)} \end{aligned}\right\} . \tag{3.2}$$

To obtain some measure of the difference between the solution for $\alpha = 0$ and that with α given by (3.1), we consider

$$d(r) = E_0(r) - E_\alpha(r) . \tag{3.3}$$

Putting $d'(r_1) = 0$ to find the maximum gives

$$m_0 I'_{\frac{\nu}{2}}\left(\frac{\sqrt{B}}{2} r_1^2\right) = m_\alpha r_1^{\alpha/2} I'_{\frac{2\nu}{(4+\alpha)}}\left(\frac{2\sqrt{B}}{(4+\alpha)} r_1^{(4+\alpha)/2}\right), \tag{3.4}$$

where primes denote differentiation with respect to argument. Now with the values given by (3.1), the quantity $\sqrt{B}\, r_0^2/2 \simeq 20\cdot 115$, so that r_0 lies in the region of asymptotic behaviour of the modified Bessel function, where

$$I_\gamma(\theta) \sim \frac{e^\theta}{\sqrt{2\pi\theta}} \quad . \tag{3.5}$$

We look for a solution of (3.4) which is also in the asymptotic region. Using (3.1) and calculating all quantities in (3.2) and (3.4) from the relation (3.5), we find that a solution of (3.4) is

$$r_1 \simeq 28\cdot 44 \ \mu m \quad . \tag{3.6}$$

Since $\sqrt{B}\, r_1^2/2 \simeq 18\cdot 08$, then (3.6) is consistent with a solution in the asymptotic region.

With the value of r_1 given by (3.6) and again using (3.5), we find that $|d(r_1)|$ from (3.3) is about $8\cdot 4\%$ of the value $|E_0(r_1)|$.

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Non-static nuclear forces in a Kerr–Newman background space

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Abstract. In the Kerr–Newman background space, an explicit expression for the source term due to a particle moving along a geodesic near the event horizon in the equatorial plane of the black hole is found. This is used, together with the solutions of the Klein–Gordon equation near the event horizon (found elsewhere) to show that the meson field near the black hole vanishes as the source crosses the event horizon.

1. Introduction

In a recent series of papers (see Rowan and Stephenson 1976a, b, 1977 and Rowan 1977), the Klein–Gordon equation for a massive scalar meson field has been examined in various background spaces. Rowan (1977) has extended the work of Rowan and Stephenson to the in-fall of an uncharged baryon down the axis of a charged rotating black hole described by the Kerr–Newman metric, and has shown that the field of the baryon source falls to zero as the source crosses the event horizon. By allowing the particle to move down the axis of rotation, Rowan was able to treat the in-fall as a series of quasi-static problems since the event horizon and the static limit coincide on the axis of rotation.

In this paper we extend this work to the in-fall of a baryon along a geodesic in the equatorial plane of the black hole. This requires that the source term be modified to a time-dependent one, since the tidal forces inside the ergosphere destroy the static situation. By solving the geodesic equations near the event horizon and using the solution of the Klein–Gordon equation near the event horizon as found by Rowan and Stephenson (1977), we have again deduced that the field of the baryon falls off to zero as the particle crosses the event horizon. It has not been possible to solve the basic equation over the whole range owing to the breakdown of the uniform asymptotic method. The reason for this will emerge in the following analysis.

2. Basic equations

We start with the Klein–Gordon equation

$$(\Box^2+\mu^2)\Phi=4\pi f(t, r, \theta, \phi) \tag{2.1}$$

where Φ is the scalar field and $f(t, r, \theta, \phi)$ represents a point source. In generally covariant form (2.1) is

$$\frac{1}{\sqrt{-g_4}}\frac{\partial}{\partial x^i}\left((\sqrt{-g_4})g^{ik}\frac{\partial\Phi}{\partial x^k}\right)+\mu^2\Phi=4\pi f(t,r,\theta,\phi). \tag{2.2}$$

Together with the Kerr–Newman metric in Boyer–Lindquist coordinates

$$ds^2=\frac{\Delta}{\rho^2}(dt-a\sin^2\theta\, d\phi)^2-\frac{\sin^2\theta}{\rho^2}[(r^2+a^2)\,d\phi-a\,dt]^2-\frac{\rho^2}{\Delta}dr^2-\rho^2\,d\theta^2 \tag{2.3}$$

where

$$\Delta=r^2-2Mr+a^2+Q^2, \qquad \rho^2=r^2+a^2\cos^2\theta, \tag{2.4}$$

equation (2.2) becomes

$$\left[\frac{[(r^2+a^2)^2-\Delta a^2\sin^2\theta]}{\Delta}\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial r}\left(\Delta\frac{\partial}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{(\Delta-a^2\sin^2\theta)}{\Delta\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right.$$
$$\left.-\frac{2a[\Delta-(r^2+a^2)]}{\Delta}\frac{\partial^2}{\partial\phi\,\partial t}+\rho^2\mu^2\right]\Phi=4\pi\rho^2 f(t,r,\theta,\phi). \tag{2.5}$$

Write

$$\Phi=\sum_{l,m}\int d\omega\,(R_{lm\omega}(r)S_l^m(\theta)\,e^{im\phi}\,e^{-i\omega t}) \tag{2.6}$$

where $S_l^m(\theta)=S_l^m(a^2(\mu^2-\omega^2),\cos\theta)$ is the oblate spheroidal harmonic satisfying

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\lambda_{lm}-a^2(\mu^2-\omega^2)\cos^2\theta-\frac{m^2}{\sin^2\theta}\right]S_l^m(\theta)=0 \tag{2.7}$$

and λ_{lm} is the eigenvalue corresponding to $S_l^m(\theta)$. Taking the normalisation

$$\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta|S_l^m(\theta)|^2=1 \tag{2.8}$$

and substituting (2.6)–(2.8) into (2.5), we see that $R_{lm\omega}(r)$ satisfies

$$\left[\frac{d}{dr}\left(\Delta\frac{d}{dr}\right)+\frac{a^2m^2+2am\omega(Q^2-2Mr)+(r^2+a^2)^2\omega^2}{\Delta}-\lambda_{lm}-a^2\omega^2-\mu^2r^2\right]R_{lm\omega}(r)$$
$$=-2\int_{-\infty}^{\infty}dt\,e^{i\omega t}\int_0^{2\pi}d\phi\int_0^{\pi}\rho^2\sin\theta S_l^m(\theta)\,e^{-im\phi}f(t,r,\theta,\phi)\,d\theta. \tag{2.9}$$

3. The geodesic equations

We take the equations of motion along a geodesic in a Kerr–Newman background space (Misner *et al* 1973) and consider the case of motion confined to the equatorial plane of a black hole. The equations become

$$\begin{aligned} r^2\frac{\mathrm{d}r}{\mathrm{d}\lambda} &= \sqrt{R}, \\ r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} &= -(aE-L_z)+\frac{aP}{\Delta}, \\ r^2\frac{\mathrm{d}t}{\mathrm{d}\lambda} &= -a(aE-L_z)+\frac{(r^2+a^2)P}{\Delta}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} P &= E(r^2+a^2)-L_z a, \\ R &= P^2-\Delta[\bar{\mu}^2r^2+(L_z-aE)^2] \end{aligned} \tag{3.2}$$

and where $\bar{\mu}$ is the rest mass of the baryon, and E and L_z are the energy at infinity and the angular momentum about the axis of rotation, respectively.

Putting $G=L_z-aE$, we get from (3.2)

$$\begin{aligned} P &= Er^2-aG, \\ R &= (Er^2-aG)^2-\Delta(\bar{\mu}^2r^2+G^2). \end{aligned} \tag{3.3}$$

From (3.1) using (3.3) we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{G\Delta+a(Er^2-aG)}{aG\Delta+(r^2+a^2)(Er^2-aG)} \tag{3.4}$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\Delta[(Er^2-aG)^2-\Delta(\bar{\mu}^2r^2+G^2)]^{1/2}}{aG\Delta+(r^2+a^2)(Er^2-aG)}. \tag{3.5}$$

We now confine our attention to motion near the event horizon $r=r_+=M+[M^2-(a^2+Q^2)]^{1/2}$ and assume that $a^2+Q^2\neq M^2$ so that $r_+\neq r_-=M-[M^2-(a^2+Q^2)]^{1/2}$.

Putting

$$Mx=r-r_+, \qquad 2Md=r_+-r_- \tag{3.6}$$

we have

$$\Delta=M^2x(x+2d). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.4) and (3.5), we may expand the right-hand sides in powers of x to get

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{a}{(r_+^2+a^2)}+\alpha x+\mathrm{O}(x^2) \tag{3.8}$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}t}=M\frac{\mathrm{d}x}{\mathrm{d}t}=-\left(\frac{2dM^2x}{(r_+^2+a^2)}+\beta x^2\right)+\mathrm{O}(x^3), \tag{3.9}$$

taking the minus square root in (3.5), and where the constants α and β are given by

$$\alpha=\frac{2M}{(r_+^2+a^2)^2(Er_+^2-aG)}(MG\,\mathrm{d}r_+^2-aEr_+^3+a^2Gr_+) \tag{3.10}$$

and

$$\beta=\frac{M^2}{(r_+^2+a^2)}\left(\frac{Md(4E^2r_+^3-4aGEr_+-2Md\bar{\mu}^2r_+^2-2MdG^2)}{(Er_+^2-aG)^2}\right.$$
$$\left.-\frac{2Md(4Er_+^3+2a^2Er_+-2aGr_++2aGMd)}{(r_+^2+a^2)(Er_+^2-aG)}+1\right) \tag{3.11}$$

for $Er_+^2-aG\neq 0$.

4. The source term

To get an explicit expression for $f(t,r,\theta,\phi)$, we choose, following Persides (1974),

$$f(t,r,\theta,\phi)=g\frac{1}{u^0}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}') \tag{4.1}$$

where $u^0=\mathrm{d}t/\mathrm{d}s$ along the trajectory of the particle $\boldsymbol{r}'(t)=(r'(t),\theta'(t),\phi'(t))$ and g is the source strength. To calculate $1/u^0$ we first put (3.6) and (3.7), together with $\theta=\pi/2$, into the metric (2.3), obtaining

$$\left(\frac{ds}{\mathrm{d}t}\right)^2=\frac{M^2x(x+2d)}{(Mx+r_+)^2}\left(1-a\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2-\frac{1}{(Mx+r_+)^2}\left([(Mx+r_+)^2+a^2]\frac{\mathrm{d}\phi}{\mathrm{d}t}-a\right)^2$$
$$-\frac{(Mx+r_+)^2}{M^2x(x+2d)}M^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2. \tag{4.2}$$

Then substituting (3.8) and (3.9) into (4.2), we see that to the second order in x,

$$\left(\frac{1}{u^0}\right)^2=\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=\gamma x^2 \tag{4.3}$$

where the constant γ is given by

$$\gamma=\frac{(2M^2r_+^2-8dM^3r_+)}{(r_+^2+a^2)^2}-\frac{(4adM^2\alpha+2r_+^2\beta)}{(r_+^2+a^2)}-\frac{1}{r_+^2}\left(\frac{2Mar_+}{(r_+^2+a^2)}+(r_+^2+a^2)\alpha\right)^2. \tag{4.4}$$

On substituting into (4.4) the expressions for α and β from (3.10) and (3.11), we find, after considerable algebra, that γ is given simply by

$$\gamma=\frac{4d^2M^4\bar{\mu}^2r_+^4}{(r_+^2+a^2)^2(Er_+^2-aG)^2} \tag{4.5}$$

so that

$$\frac{1}{u^0}=K(r-r_+) \tag{4.6}$$

where

$$K=\frac{2dM\bar{\mu}r_+^2}{(r_+^2+a^2)(Er_+^2-aG)}. \tag{4.7}$$

From (4.1), using (4.6), the source term can be written

$$f(t, r, \theta, \phi)=gK(r-r_+)\delta(r-r_0(\phi_0))\delta(\phi-\phi_0(t))\delta(\theta-\tfrac{1}{2}\pi) \tag{4.8}$$

where, from (3.8) and (3.9),

$$\phi_0(t)=\frac{at}{(r_+^2+a^2)} \tag{4.9}$$

$$r_0(\phi_0)=r_++\exp\left(-\frac{(r_+-r_-)}{a}\phi_0\right). \tag{4.10}$$

Hence the right-hand side of (2.9) becomes on substituting (4.8)

$$-2gK(r-r_+)\delta(r-r_0)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\rho^2\sin\theta S_l^m(\theta)\,\mathrm{e}^{-\mathrm{i}m\phi}\delta(\phi-\phi_0)\delta(\theta-\tfrac{1}{2}\pi)\,\mathrm{d}\theta. \tag{4.11}$$

After performing the θ-integration (4.11) becomes

$$-2gKr^2(r-r_+)S_l^m(\tfrac{1}{2}\pi)\delta(r-r_0)\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}m\phi}\delta(\phi-\phi_0(t))\,\mathrm{d}\phi. \tag{4.12}$$

Now from (4.9)

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}m\phi}\delta(\phi-\phi_0(t))\,\mathrm{d}\phi$$

$$=\int_0^{2\pi}\mathrm{d}\phi\;\mathrm{e}^{-\mathrm{i}m\phi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\delta\left(\phi-\frac{at}{(r_+^2+a^2)}\right)\mathrm{d}t$$

$$=\frac{2\pi}{a}(r_+^2+a^2)\qquad \text{for } \omega=\frac{ma}{(r_+^2+a^2)} \tag{4.13}$$

so that (4.12) becomes

$$-4\pi g\frac{K}{a}(r_+^2+a^2)r^2(r-r_+)S_l^m(\tfrac{1}{2}\pi)\delta(r-r_0). \tag{4.14}$$

Finally equation (2.9) becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{a^2m^2+2am\omega(Q^2-2Mr)+(r^2+a^2)^2\omega^2}{\Delta}-\lambda_{lm}-a^2\omega^2-\mu^2r^2\right]R_{lm\omega}(r)$$

$$=-4\pi g\frac{K}{a}(r_+^2+a^2)r^2(r-r_+)S_l^m(\tfrac{1}{2}\pi)\delta(r-r_0). \tag{4.15}$$

5. The radial equation

Rowan and Stephenson (1977) have shown that after defining x and d by

$$Mx=r-r_+,\qquad 2Md=r_+-r_- \tag{5.1}$$

and writing

$$R_{lm\omega}(x)=Z(x)[x(x+2d)]^{-1/2}, \tag{5.2}$$

substitution of (5.1) and (5.2) into (4.15), leads to (for $r \neq r_0$)

$$\frac{d^2Z}{dx^2}+\left[M^2(\omega^2-\mu^2)+\frac{1}{M^2}\left(\frac{A}{x^2}+\frac{B}{x}+\frac{C}{(x+2d)^2}+\frac{D}{(x+2d)}\right)\right]Z=0 \tag{5.3}$$

where A, B, C and D are constants. It has not been possible to solve (5.3) over the whole range $0 \leq x < \infty$ due to the breakdown of the uniform asymptotic method. This is due to the fact that the method depends on the existence of a large parameter in the differential equation which we do not necessarily have in (5.3) since ω may be close to or equal to μ. However, we may use the solutions obtained by Rowan and Stephenson for $x \to 0$ (that is $r \to r_+$). These are

$$R_{lm\omega}(r) \sim \begin{cases} R_{(1)}(r)=\dfrac{M}{\Delta^{1/2}}\,\mathrm{e}^{-(F^{1/2}/M)(r-r_+)}\left(\dfrac{2F^{1/2}}{M}(r-r_+)\right)^{\frac{1}{2}+\bar{m}} \\[2ex] R_{(2)}(r)=\dfrac{M}{\Delta^{1/2}}\,\mathrm{e}^{-(F^{1/2}/M)(r-r_+)}\left(\dfrac{2F^{1/2}}{M}(r-r_+)\right)^{\frac{1}{2}-\bar{m}} \end{cases} \tag{5.4}$$

where

$$\begin{aligned} \bar{m}^2&=\frac{1}{4}-\frac{A}{M^2}, \\ F&=\left(M^2(\mu^2-\omega^2)-\frac{C}{4M^2d^2}-\frac{D}{2M^2d}\right) \end{aligned} \tag{5.5}$$

provided $F \neq 0$. The case of $F=0$ was treated separately and we will not repeat the solutions here.

We now integrate (4.15) across the singularity and impose continuity of $R_{lm\omega}(r)$ at $r=r_0$ to get

$$\Delta_0\left\{\frac{dR_{lm\omega}}{dr}\bigg|_{r_0+0}-\frac{dR_{lm\omega}}{dr}\bigg|_{r_0-0}\right\}=-4\pi g\frac{K}{a}(r_+^2+a^2)r_0^2(r_0-r_+)S_l^m(\tfrac{1}{2}\pi) \tag{5.6}$$

where $\Delta_0=(r_0-r_+)(r_0-r_-)$.

Then for r_0 near r_+ we have

$$R_{lm\omega}(r)=4\pi g\frac{K}{a}(r_+^2+a^2)r_0^2(r_0-r_+)S_l^m(\tfrac{1}{2}\pi)\begin{cases} R_{(2)}(r_0)R_{(1)}(r) & r_+ \leq r \leq r_0 \\ R_{(1)}(r_0)R_{(2)}(r) & r_0 \leq r \end{cases} \tag{5.7}$$

where $R_{(1)}$ and $R_{(2)}$ are given by (5.4). If we now let $r_0 \to r_+$ we see from (5.7) that $R_{lm\omega}(r)$ (for $r_0 \leq r$) tends to zero since $R_{(1)}(r_0)$ either tends to zero if $\bar{m}$ is real, or is bounded if $\bar{m}$ is complex. Provided the series for Φ is uniformly convergent, $\Phi \to 0$ as $r_0 \to r_+$. We note that Φ here is an expression for the scalar field near the event horizon since (5.4) are solutions of (5.3) only for r near r_+.

6. Special cases

The case where $a^2+Q^2=M^2$ must be considered separately since $d=0$ and consequently from (3.9) the term of order x in dx/dt is zero. To simplify the algebra, we consider the case of an extreme Kerr black hole, so that $a=M$ and $Q=0$. Expanding $d\phi/dt$ to order x^2 and dx/dt to order x^4 and substituting these into (4.2) with $d=0$ and $a=M$ we find

$$\left(\frac{1}{u^0}\right)^2=\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=\frac{\bar{\mu}^2M^2x^4}{4(EM-G)^2} \tag{6.1}$$

to order x^4. Hence

$$\frac{1}{u^0}=\frac{\bar{\mu}(r-M)^2}{2M(EM-G)} \tag{6.2}$$

which can be used in place of (4.6).

The homogeneous radial equation becomes

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}+\left(M^2(\omega^2-\mu^2)+\frac{\bar{A}}{x}+\frac{\bar{B}}{x^2}+\frac{\bar{C}}{x^3}+\frac{\bar{D}}{x^4}\right)Z=0 \tag{6.3}$$

where

$$R_{lm\omega}(x)=\frac{Z(x)}{x}, \qquad Mx=r-M \tag{6.4}$$

and the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are given by

$$\begin{aligned}
\bar{A}&=4M^2\omega^2-2M^2\mu^2\\
\bar{B}&=7M^2\omega^2-M^2\mu^2-\lambda_{lm}\\
\bar{C}&=8M^2\omega^2-4Mm\omega\\
\bar{D}&=4M^2\omega^2-4Mm\omega+m^2.
\end{aligned} \tag{6.5}$$

The relation between ω and m is now

$$\omega=m/2M \tag{6.6}$$

and on substituting (6.6) into (6.5), we find

$$\begin{aligned}
\bar{A}&=m^2-2M^2\mu^2\\
\bar{B}&=\tfrac{7}{4}m^2-M^2\mu^2-\lambda_{lm}\\
\bar{C}&=\bar{D}=0.
\end{aligned} \tag{6.7}$$

Equation (6.3) now becomes

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}=\left(N-\frac{\bar{A}}{x}-\frac{\bar{B}}{x^2}\right)Z \tag{6.8}$$

where

$$N=M^2\mu^2-\tfrac{1}{4}m^2. \tag{6.9}$$

After defining

$$\eta=2N^{1/2}x \tag{6.10}$$

and substituting (6.10) into (6.8) we see that (6.8) has solutions in terms of Whittaker functions

$$Z=M_{\kappa,\pm\bar{m}}(\eta) \tag{6.11}$$

where

$$\kappa = \frac{\bar{A}}{2\sqrt{N}}, \qquad \bar{m}^2 = \tfrac{1}{4} - \bar{B}. \tag{6.12}$$

For the case $N = 0$ we get

$$Z = \begin{cases} x^{1/2} I_{\bar{\alpha}}(\bar{\beta} x^{1/2}) \\ x^{1/2} K_{\bar{\alpha}}(\bar{\beta} x^{1/2}) \end{cases} \tag{6.13}$$

where $I_{\bar{\alpha}}$, $K_{\bar{\alpha}}$ are the modified Bessel functions of order $\bar{\alpha}$ of the first and second kind respectively and

$$\bar{\alpha}^2 = 1 - 4\bar{B}, \qquad \bar{\beta}^2 = -4\bar{A}. \tag{6.14}$$

7. Conclusions

The success of the Liouville-Green asymptotic method when used to solve the radial equation for a massive scalar meson field (Rowan and Stephenson 1976a, b, Rowan 1977), and also when applied to the Schrödinger equation with a Gaussian potential (Stephenson 1977), depended on the appearance of a large parameter in the differential equation. This was due, in the first case, to the non-zero rest mass of the π-meson. When considering the most general black hole, solutions of the radial equation over the whole range are known only in special cases (Rowan 1977, Linet 1977); the equation may no longer contain a large parameter and in general will have four turning points. Although in principle it would be possible to match the solutions in the five regions, these problems together with the complexity of the differential equations for the geodesics give rise to great difficulties in any further work in this direction.

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LETTER TO THE EDITOR

Non-linear wave equations in a curved background space

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Abstract. Derrick's theorem concerning the existence of soliton-like solutions of non-linear scalar wave equations in Minkowski space is extended to the curved background space exterior to a charged, non-rotating black hole.

In a recent series of papers (Rowan and Stephenson, 1976a, b, 1977, Rowan 1977, Radmore 1978) solutions of the Klein–Gordon scalar wave equation in curved background spaces were obtained using Liouville–Green techniques. These solutions were related to the infall of baryons into black holes. We now consider whether it is possible to have soliton-like solutions of the *non-linear* Klein–Gordon equation containing self-interaction terms in the space exterior to a charged, non-rotating black hole as described by the Reissner–Nordström metric. It is well-known (Derrick 1964) that if Φ is a scalar field in one time and D space dimensions satisfying the non-linear equation

$$\frac{\partial^2\Phi}{\partial t^2}-\nabla^2\Phi=-\tfrac{1}{2}f'(\Phi) \tag{1}$$

derivable from the variational principle

$$\delta\int[(\partial\Phi/\partial t)^2-(\nabla\Phi)^2-f(\Phi)]\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}t=0 \tag{2}$$

then for $D\geqslant 2$ and $f(\Phi)\geqslant 0$ the only non-singular time-independent solutions are the vacuum (or ground) states for which $f(\Phi)=0$. This result, however, was established only in Minkowski space and we now extend this work to the space exterior to a non-rotating black hole of mass m and charge e defined by the metric

$$\mathrm{d}s^2=\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)\mathrm{d}t^2-\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)^{-1}\mathrm{d}r^2-r^2\,\mathrm{d}\theta^2-r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{3}$$

We first write (1) in covariant form as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,g^{ik}\frac{\partial\Phi}{\partial x^k}\right)=-\tfrac{1}{2}f'(\Phi) \tag{4}$$

which arises from the variational principle

$$\delta\int\left(g^{ik}\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi}{\partial x^k}-f(\Phi)\right)\sqrt{-g}\,\mathrm{d}^4x=0. \tag{5}$$

Using (3) and taking Φ to be a function of r only, we obtain from (4) the radial equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left((r^2-2mr+e^2)\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)=\tfrac{1}{2}f'(\Phi). \tag{6}$$

In this case the variational principle (5) is equivalent to

$$\delta E=0, \tag{7}$$

where the energy E of the Φ field is given by

$$E=4\pi\int_{r_+}^{\infty}\left[(r^2-2mr+e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2+f(\Phi)r^2\right]\mathrm{d}r, \tag{8}$$

and where $r_+=m+\sqrt{(m^2-e^2)}$ $(e^2\leqslant m^2)$ is the event horizon of the black hole.

Writing

$$I_1=\int_{r_+}^{\infty}(r^2-2mr+e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r \tag{9}$$

and

$$I_2=\int_{r_+}^{\infty}f(\Phi)r^2\,\mathrm{d}r \tag{10}$$

so that

$$E=4\pi(I_1+I_2) \tag{11}$$

we must require that I_1 and I_2 converge.

We now define

$$\Phi_\alpha(r)=\Phi(\alpha r), \tag{12}$$

where α is an arbitrary constant and

$$E_\alpha=4\pi\int_{r_+}^{\infty}\left((r^2-2mr+e^2)\left(\frac{\mathrm{d}\Phi_\alpha}{\mathrm{d}r}\right)^2+f(\Phi_\alpha)r^2\right)\mathrm{d}r. \tag{13}$$

Then on changing the variable of integration from r to αr we have

$$\frac{E_\alpha}{4\pi}=\int_{\alpha r_+}^{\infty}(r^2-2m\alpha r+e^2\alpha^2)\frac{1}{\alpha}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r+\int_{\alpha r_+}^{\infty}f(\Phi)\frac{r^2}{\alpha^3}\,\mathrm{d}r. \tag{14}$$

Differentiation of (14) with respect to α gives

$$\frac{1}{4\pi}\frac{\mathrm{d}E_\alpha}{\mathrm{d}\alpha}=\int_{\alpha r_+}^{\infty}(-2mr+2e^2\alpha)\frac{1}{\alpha}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r+\int_{\alpha r_+}^{\infty}(r^2-2m\alpha r+e^2\alpha^2)\left(-\frac{1}{\alpha^2}\right)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r$$

$$+\int_{\alpha r_+}^{\infty}f(\Phi)\left(-\frac{3r^2}{\alpha^4}\right)\mathrm{d}r-\frac{r_+^3}{\alpha}f(\Phi)\bigg|_{r=\alpha r_+} \tag{15}$$

so that

$$\frac{1}{4\pi}\frac{\mathrm{d}E_\alpha}{\mathrm{d}\alpha}\bigg|_{\alpha=1}=-I_1-3I_2+I_3-r_+^3f(\Phi)\bigg|_{r=r_+} \tag{16}$$

where

$$I_3=\int_{r_+}^{\infty}(-2mr+2e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r. \tag{17}$$

Now from (7) we must have

$$\left.\frac{\mathrm{d}E_\alpha}{\mathrm{d}\alpha}\right|_{\alpha=1}=0 \tag{18}$$

which gives from (16)

$$3I_2=-I_1+I_3-r_+^3 f(\Phi)\Big|_{r=r_+}. \tag{19}$$

Similarly, differentiating (14) twice with respect to α and setting $\alpha=1$ we obtain

$$\begin{aligned}\frac{1}{4\pi}\left.\frac{\mathrm{d}^2E_\alpha}{\mathrm{d}\alpha^2}\right|_{\alpha=1}&=I_4-2I_3+2I_1+12I_2+4r_+^3 f(\Phi)\Big|_{r=r_+}\\&\quad+r_+(2mr_+-2e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\Big|_{r=r_+}-r_+^4\left[f'(\Phi)\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right]_{r=r_+}\end{aligned} \tag{20}$$

where

$$I_4=\int_{r_+}^{\infty}2e^2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r. \tag{21}$$

Now using (19) we eliminate I_2 from (20) to get

$$\frac{1}{4\pi}\left.\frac{\mathrm{d}^2E_\alpha}{\mathrm{d}\alpha^2}\right|_{\alpha=1}=I_4+2I_3-2I_1+r_+(2mr_+-2e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\Big|_{r=r_+}-r_+^4\left[f'(\Phi)\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right]_{r=r_+}. \tag{22}$$

From (6) we have

$$\left[f'(\Phi)\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right]_{r=r_+}=\frac{4(r_+-m)}{r_+^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\Big|_{r=r_+} \tag{23}$$

and substitution of (23) into (22) leads to

$$\frac{1}{4\pi}\left.\frac{\mathrm{d}^2E_\alpha}{\mathrm{d}\alpha^2}\right|_{\alpha=1}=I_4+2I_3-2I_1-2r_+(mr_+-e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\Big|_{r=r_+}. \tag{24}$$

Finally, inserting the expressions for I_1, I_3 and I_4 from (9), (17) and (21), equation (24) becomes

$$\frac{1}{8\pi}\left.\frac{\mathrm{d}^2E_\alpha}{\mathrm{d}\alpha^2}\right|_{\alpha=1}=\int_{r_+}^{\infty}(2e^2-r^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r-r_+(mr_+-e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\Big|_{r=r_+}. \tag{25}$$

A necessary condition for the solution of (6) to be stable is

$$\left.\frac{\mathrm{d}^2E_\alpha}{\mathrm{d}\alpha^2}\right|_{\alpha=1}\geqslant 0 \tag{26}$$

which from (25) is

$$\int_{r_+}^{\infty}(2e^2-r^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r-r_+(mr_+-e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\Big|_{r=r_+}\geqslant 0. \tag{27}$$

We can now establish a general result. If $f(\Phi) \geqslant 0$ then from (10)

$$I_2 \geqslant 0. \tag{28}$$

We also have from (9) and (17)

$$I_1 \geqslant 0, \qquad I_3 \leqslant 0. \tag{29}$$

On substituting (28) and (29) into (19) we see that we must have (since $f(\Phi) \geqslant 0$)

$$I_1 = I_3 = f(\Phi) = 0 \tag{30}$$

giving that the only solutions of (6) are those where Φ is a constant C satisfying $f(C) = 0$.

We now consider two special cases. Firstly, suppose that

$$\tfrac{1}{2} f'(\Phi) = \lambda \Phi^3 + \mu^2 \Phi, \qquad \lambda, \mu \text{ constant.} \tag{31}$$

Since then $f(\Phi) = \frac{1}{2}\lambda\Phi^4 + \mu^2\Phi^2$ is non-negative, (30) gives that (6) has only the trivial solution $\Phi = 0$. Secondly, suppose that

$$\tfrac{1}{2} f'(\Phi) = \lambda \Phi^3 - \mu^2 \Phi \tag{32}$$

which is the form of current interest in gauge theories. Then again

$$f(\Phi) = \tfrac{1}{2}\lambda[\Phi^2 - (\mu^2/\lambda)]^2 \tag{33}$$

is non-negative. The only solutions of (6) are therefore the vacuum states

$$\Phi = \pm\mu/\sqrt{\lambda}. \tag{34}$$

For compact spatial topologies there may well exist non-trivial stable vacuum solutions (Avis and Isham 1978).

Finally, if no restriction is made on the sign of $f(\Phi)$, then we may have non-constant, finite energy solutions of (6). If (27) is to hold for such solutions, we must have $2e^2 - r^2 > 0$ for some part of the range $r_+ \leqslant r < \infty$ since the second term in (27) is non-positive. This gives $\sqrt{2}e > r_+$ or

$$m^2 > e^2 > \tfrac{8}{9}m^2. \tag{35}$$

In particular (35) shows that there will be no such solutions in a Schwarzschild background space.

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The Schrödinger equation with an anharmonic oscillator potential

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Abstract. The Liouville–Green uniform asymptotic method is used to obtain approximate eigenvalues and eigenfunctions of the one-dimensional Schrödinger equation with an anharmonic oscillator potential. The term neglected in the basic differential equation, in accordance with the method, is studied in some detail.

1. Introduction

In a recent paper (Stephenson 1977), the Liouville–Green technique was used to obtain the eigenvalues of the Schrödinger equation with a radial Gaussian potential. Recent work on the anharmonic oscillator (e.g. Gillespie 1976, Fung *et al* 1978, Banerjee *et al* 1978) has led to computation and comparison of the eigenvalues of the Schrödinger equation. In view of the fact that the Liouville–Green technique and other so-called semi-classical methods are not as widely applied as they might be (Berry and Mount 1972), and of the importance of the anharmonic oscillator potential in nuclear structure, quantum chemistry and quark confinement, we now use the same method for this potential. The eigenvalues obtained are compared with those found by direct methods.

2. The basic transformation

Setting $2m = \hbar = 1$, the one-dimensional Schrödinger equation with an anharmonic oscillator potential $V = x^2 + x^4$ is

$$\frac{d^2\psi}{dx^2} = (-E + x^2 + x^4)\psi, \tag{2.1}$$

where E is the energy and the boundary conditions are $\psi(\infty) = \psi(-\infty) = 0$. We make the Liouville–Green transformation

$$x = x(\xi), \qquad \psi(x) = (\xi')^{-1/2}G(\xi), \tag{2.2}$$

where primes denote differentiation with respect to x, so that (2.1) becomes

$$d^2G/d\xi^2 = (P(x)/\xi'^2 + \Delta(x))G, \tag{2.3}$$

where

$$P(x) = x^4 + x^2 - E \tag{2.4}$$

and

$$\Delta(x)=\xi'''/2\xi'^3-3\xi''^2/4\xi'^4. \tag{2.5}$$

When E is positive, $P(x)$ has two zeros $x=\pm x_0$ where

$$x_0=\{[-1+(1+4E)^{1/2}]/2\}^{1/2}, \tag{2.6}$$

these being the classical turning points.

The Liouville–Green technique consists in choosing $\xi(x)$ so that $\Delta(x)$ is a small bounded function and (2.3), with $\Delta(x)$ neglected, is soluble in terms of known functions. Two ways of achieving this will be presented. First, since (2.1) has two turning points, we may try to choose $\xi(x)$ so that, after neglecting $\Delta(x)$, (2.3) becomes the standard two-turning-point equation, namely the Weber equation

$$\mathrm{d}^2G/\mathrm{d}\xi^2=(\xi^2/4-\lambda)G, \tag{2.7}$$

the solutions of which are the parabolic cylinder functions, where λ is a parameter. Alternatively, since $P(x)$ depends only on x^2, the wavefunctions $\psi(x)$ will be either even or odd functions and we can consider the problem for $x\geqslant 0$, applying the additional boundary condition that either $\psi(0)=0$ or $\psi'(0)=0$. In this case, since $P(x)$ has only one zero for $x\geqslant 0$, we may try to choose $\xi(x)$ so that (2.3) becomes the Airy equation

$$\mathrm{d}^2G/\mathrm{d}\xi^2=(\xi-a)G, \tag{2.8}$$

after neglecting $\Delta(x)$, where a is a parameter to be determined from the boundary conditions.

Both approaches lead to approximate eigenvalues and eigenfunctions (Olver 1974).

3. The Weber equation method

With the choice

$$\xi'^2(\xi^2/4-\lambda)=P(x), \tag{3.1}$$

(2.3) becomes the Weber equation (2.7), if we neglect $\Delta(x)$. Assuming for the moment that this is justified, we find by integration of (3.1) that for $x\geqslant x_0$,

$$\tfrac{1}{2}\xi(\xi^2-4\lambda)^{1/2}-2\lambda\ \ln|\xi+(\xi^2-4\lambda)^{1/2}|+2\lambda\ \ln(2\sqrt{\lambda})=2\int_{x_0}^{x}(P(t))^{1/2}\,\mathrm{d}t, \tag{3.2}$$

while between the turning points

$$\frac{\xi}{2}(4\lambda-\xi^2)^{1/2}+2\lambda\ \sin^{-1}\left(\frac{\xi}{2\sqrt{\lambda}}\right)=2\int_0^x(-P(t))^{1/2}\,\mathrm{d}t. \tag{3.3}$$

The constants of integration have been chosen so that $\xi=0$ when $x=0$ and $\xi=\pm2\sqrt{\lambda}$ correspond to $x=\pm x_0$. Putting $x=x_0$ in (3.3) we obtain

$$\lambda\pi=2\int_0^{x_0}(E-t^2-t^4)^{1/2}\,\mathrm{d}t. \tag{3.4}$$

The boundary conditions $\psi(\infty)=\psi(-\infty)=0$ correspond to $G(\infty)=G(-\infty)=0$ and bounded solutions of the Weber equation satisfying these conditions exist only if

$$\lambda = n+\tfrac{1}{2}, \tag{3.5}$$

where $n=0, 1, 2, \ldots$.

Substituting (3.5) into (3.4) gives

$$\frac{\pi}{2}(n+\tfrac{1}{2})=\int_0^{x_0}(E-t^2-t^4)^{1/2}\,\mathrm{d}t, \tag{3.6}$$

which is the Bohr–Sommerfeld quantisation formula, on noticing that

$$\int_0^{x_0}(E-t^2-t^4)^{1/2}\,\mathrm{d}t=\tfrac{1}{2}\int_{-x_0}^{x_0}(E-t^2-t^4)^{1/2}\,\mathrm{d}t. \tag{3.7}$$

Using (3.6), the eigenvalues have been computed and in table 1 are compared with accurate values calculated by Banerjee *et al* (1978) using scaled bases. The two sets of values are in close agreement, the accuracy increasing with increasing n.

Table 1. Eigenvalues computed using equation (3.6) are compared with accurate values calculated by Banerjee *et al* (1978) using scaled bases.

n	Eigenvalue	Accurate eigenvalue	Approximate percentage error
0	1·2508	1·3924	10·17
1	4·5926	4·6488	1·21
2	8·6130	8·6550	0·49
3	13·1231	13·1568	0·26
4	18·0290	18·0576	0·16
5	23·2725	23·2974	0·11
6	28·8130	28·8353	0·077
7	34·6206	34·6408	0·058
8	40·6717	40·6904	0·046
9	46·9477	46·9650	0·037
10	53·4329	53·4491	0·03
20	127·6076	127·6178	0·008
30	214·7721	214·7797	0·0035
40	311·8254	311·8315	0·002
50	417·0512	417·0563	0·0012
100	1035·5422	1035·5442	0·0002

We now examine the neglected term $\Delta(x)$. From (2.4) and (3.1) we have

$$\xi'=[(-E+x^2+x^4)/(\xi^2/4-\lambda)]^{1/2}, \tag{3.8}$$

from which ξ'' and ξ''' can be calculated in terms of x and ξ and, using (2.5), $\Delta(x)$ can be written out explicitly as

$$\Delta(x)=\frac{(3\xi^2+8\lambda)}{64(\xi^2/4-\lambda)^2}-(\xi^2/4-\lambda)\frac{[2E+(12E+3)x^2+6x^4+8x^6]}{4(-E+x^2+x^4)^3}. \tag{3.9}$$

At the turning points, although both terms in (3.9) diverge, we can show that $\Delta(x)$ tends to a finite limit, as follows:

Using L'Hôpital's rule in (3.8), we have

$$L_1 = \lim_{x \to x_0} \xi' = \left(\frac{2x_0 + 4x_0^3}{\sqrt{\lambda}} \right)^{1/3}. \tag{3.10}$$

By differentiation of (3.8) and use of L'Hôpital's rule, we find

$$L_2 = \lim_{x \to x_0} \xi'' = \frac{(4 + 24x_0^2 - L_1^4)}{10 L_1^2 \sqrt{\lambda}} \tag{3.11}$$

and

$$L_3 = \lim_{x \to x_0} \xi''' = \frac{(48x_0\sqrt{\lambda} - 24\lambda L_1 L_2^2 - 9\sqrt{\lambda} L_1^3 L_2)}{14\lambda L_1^2}. \tag{3.12}$$

L_1, L_2 and L_3 are non-zero and finite so that by (2.5), $\Delta(x)$ tends to a finite limit given by

$$\lim_{x \to x_0} \Delta(x) = \frac{L_3}{2L_1^3} - \frac{3}{4}\frac{L_2^2}{L_1^4}. \tag{3.13}$$

The values of $\Delta(x)$ have been computed by first finding ξ for a given x from (3.2) or (3.3) and then substituting in (3.9), with the value at the turning point given by (3.13). The results are shown in figures 1 and 2 for selected values of n and indicate that $\Delta(x)$ attains its absolute maximum at $x = 0$, this value decreasing with increasing n, and that $\Delta(x)$ is a small, bounded, slowly varying function.

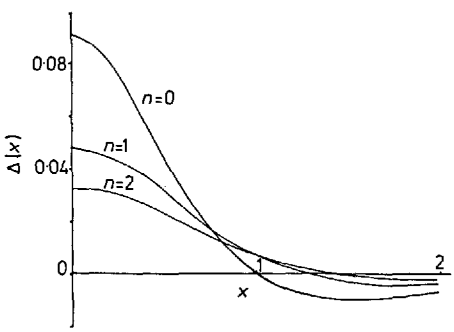

Figure 1. $\Delta(x)$ against x, for $n = 0, 1, 2$.

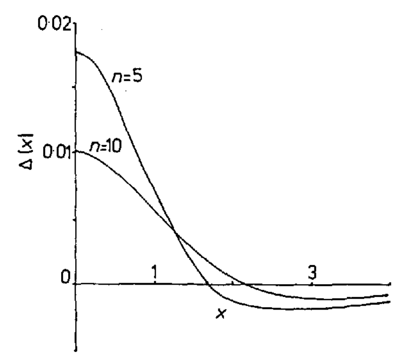

Figure 2. $\Delta(x)$ against x, for $n = 5, 10$.

4. The Airy equation method

Here we consider $x \geq 0$, and with the choice

$$\xi'^2(\xi - a) = P(x), \tag{4.1}$$

(2.3) becomes the Airy equation (2.8) on neglecting $\Delta(x)$. We then find by integration of (4.1) that for $x \geq x_0$,

$$\tfrac{2}{3}(\xi - a)^{3/2} = \int_{x_0}^{x} (P(t))^{1/2}\, dt, \tag{4.2}$$

the constant of integration being chosen so that $x = x_0$ corresponds to $\xi = a$. For $0 \leqslant x \leqslant x_0$, we have

$$\tfrac{2}{3}a^{3/2} - \tfrac{2}{3}(a-\xi)^{3/2} = \int_0^x (-P(t))^{1/2}\, \mathrm{d}t, \tag{4.3}$$

where $x = 0$ corresponds to $\xi = 0$. Substituting $x = x_0$ into (4.3), we obtain

$$\tfrac{2}{3}a^{3/2} = \int_0^{x_0} (E - t^2 - t^4)^{1/2}\, \mathrm{d}t. \tag{4.4}$$

The required solution of (2.8) is the Airy function $\mathrm{Ai}(\xi - a)$, since this satisfies the boundary condition $G(\infty) = 0$. We can now find the parameter a from the additional condition that either $G'(0) = 0$ or $G(0) = 0$ corresponding to even and odd wavefunctions respectively, since this condition implies that either $\mathrm{Ai}'(-a) = 0$ or $\mathrm{Ai}(-a) = 0$. Hence $-a$ is the position of either a turning point or a zero of the Airy function Ai. The values of a obtained from Abramowitz and Stegun (1964, p 478) were used to compute the eigenvalues using (4.4). The results are shown in table 2 and compare favourably with accurate values.

Table 2. Values of a obtained from Abramowitz and Stegun (1964) were used to compute the eigenvalues using equation (4.4).

n	a from $\mathrm{Ai}'(-a) = 0$	a from $\mathrm{Ai}(-a) = 0$	Eigenvalue	Accurate eigenvalue
0	1·01 879		1·0706	1·3924
1		2·33 811	4·6573	4·6488
2	3·24 820		8·5471	8·6550
3		4·08 795	13·1605	13·1568
4	4·82 010		17·9849	18·0576
5		5·52 056	23·3000	23·2974
6	6·16 331		28·7788	28·8353
7		6·78 671	34·6428	34·6408
8	7·37 218		40·6433	40·6904
9		7·94 413	46·9666	46·9650
10	8·48 849		53·4084	53·4491
11		9·02 265	60·1310	60·1295
12	9·53 545		66·9589	66·9950
13		10·04 017	74·0371	74·0359
14	10·52 766		81·2108	81·2435
15		11·00 852	88·6115	88·6103
16	11·47 506		96·0998	96·1296
17		11·93 602	103·7966	103·7953
18	12·38 479		111·5743	111·6018
19		12·82 878	119·5454	119·5442

The connection between (3.6) and (4.4) can be seen by noting that the leading order term in the asymptotic expansion of a is

$$a \sim [\tfrac{3}{4}\pi(n + \tfrac{1}{2})]^{2/3} \tag{4.5}$$

where $n = 0, 1, 2, \ldots$ (see Abramowitz and Stegun p 450).

The neglected term $\Delta(x)$ in this case is given by

$$\Delta(x)=\frac{5}{16(\xi-a)^2}-(\xi-a)\frac{[2E+(12E+3)x^2+6x^4+8x^6]}{4(-E+x^2+x^4)^3}, \tag{4.6}$$

and we can again show that $\Delta(x)$ tends to a finite limit at the turning point $x = x_0$. Using the results

$$K_1=\lim_{x\to x_0}\xi'=(2x_0+4x_0^3)^{1/3}, \tag{4.7}$$

$$K_2=\lim_{x\to x_0}\xi''=\frac{(2+12x_0^2)}{5K_1^2}, \tag{4.8}$$

$$K_3=\lim_{x\to x_0}\xi'''=\frac{12}{7K_1^2}(2x_0-K_1K_2^2), \tag{4.9}$$

we obtain from (2.5)

$$\lim_{x\to x_0}\Delta(x)=\frac{3}{28K_1^5}(16x_0-15K_1K_2^2). \tag{4.10}$$

The results of computing $\Delta(x)$ for selected values of a are shown in figures 3 and 4.

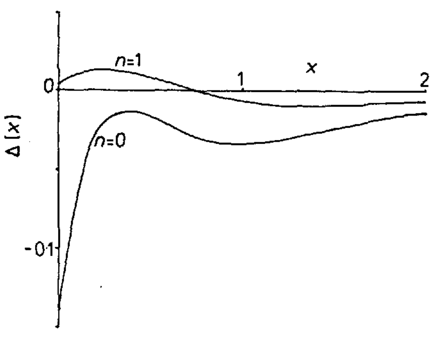

Figure 3. $\Delta(x)$ against x, for selected values of a $(n=0,1)$.

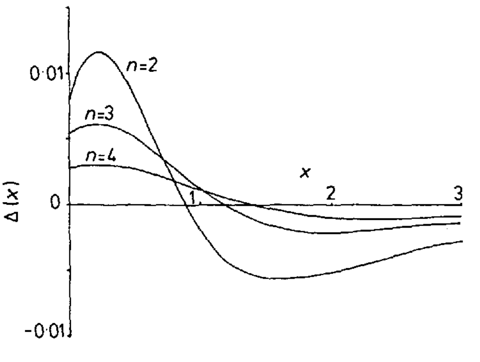

Figure 4. $\Delta(x)$ against x, for selected values of a $(n=2,3,4)$.

5. Discussion

The method presented here depends on the initial choice of $\xi(x)$. Consider for example the Weber equation method. The exact relation between ξ and x is given by

$$(\tfrac{1}{4}\xi^2-\lambda)-\frac{P(x)}{\xi'^2}=\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}, \tag{5.1}$$

and on neglecting the right-hand side, we obtain (3.1). The next approximation would then be

$$(\tfrac{1}{4}\xi^2-\lambda)-P(x)/\xi'^2=\Delta(x(\xi)), \tag{5.2}$$

from which we see that

$$\int_0^{x_0} (-P(x))^{1/2}\,\mathrm{d}x = \int_0^{\xi_0} \{\lambda - \tfrac{1}{4}\xi^2 + \Delta(x(\xi))\}^{1/2}\,\mathrm{d}\xi, \tag{5.3}$$

where ξ_0 is given by

$$\lambda - \tfrac{1}{4}\xi_0^2 + \Delta(x(\xi_0)) = 0. \tag{5.4}$$

Except for the case $n = 0$, $\Delta(x)$ is negative at the turning point $x = x_0$ (corresponding to $\xi = 2\sqrt{\lambda}$), so that $\xi_0 < 2\sqrt{\lambda}$. Hence an upper bound for the right-hand side of (5.3) is

$$2\sqrt{\lambda}(\lambda + \Delta(0))^{1/2}, \tag{5.5}$$

which, from (5.3), gives an upper bound for the eigenvalues in this approximation. For upper and lower bounds derived using the WKB approximation, see Birx and Houk 1977.

The approximate eigenfunctions follow from (2.7) or (2.8) and the transformation (2.2).

A wide class of potentials can be treated in a similar manner, for example the interaction of the type $\lambda x^2/(1+gx^2)$ (see Mitra 1978).

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