ASPECTS OF METRIC-TORSION THEORIES OF GRAVITATION

by

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ABSTRACT

This thesis consists of two parts. Part I begins with a description of Weyl's (1929) concept of gauge invariance as applied to electrodynamics, along with its generalisation to non-abelian symmetries by Yang and Mills (1954).

A generalisation of Einstein's (1915) theory of general relativity, due principally to Cartan (1922), Sciama (1962) and Kibble (1961) is then reviewed, both in its geometrical and physical aspects.

It is then shown that upon trying to incorporate gauge fields into metric-torsion theories, inconsistencies arise leading to loss of gauge invariance. A recently suggested solution for a consistent coupling of torsion to electrodynamics is then described and a generalisation to non-abelian gauge fields is put forward.

Part II studies the role of variational principles and lagrangians in metric-torsion theories of gravity. The concept of Invariant Variational Principle (IVP) is described. The usefulness of IVP's is detailed through the example of a second order lagrangian, $L(g,\partial g,\partial\partial g)$ in the metric g. Three identities are derived and it is shown how they can be used to reduce the Euler-Lagrange field equations to a simple form. The method is generalised to metric-torsion theories of gravitation by application to a lagrangian of the form $L(g,\partial g,\partial\partial g,s,\partial s)$, where s is the torsion. Having simplified the field equations for this lagrangian, the Construction of lagrangians for metric-torsion theories is studied. In particular, it is shown that Einstein's principle of taking his lagrangian to be linear in the curvature when generalised to metric-torsion theories, does not lead simply to the ECSK lagrangian, but allows an additional pseudoscalar term.

Finally, some consequences of incorporating this additional term into the ECSK lagrangian are illustrated by coupling torsion to the Dirac and Proca fields.

PREFACE

The work presented in this thesis was carried out in the Department of Mathematics, Imperial College, London, on a part-time basis from October 1977 to September 1978 and on a full-time basis between October 1978 and October 1980 under the supervision of Dr. Patrick Dolan. Except where otherwise stated, this work is original and has not been submitted for a degree of this or any other University.

Copies of some subsidiary material, not related to the main topic of this thesis, have also been bound-in at the end of this volume.

Thoughts on thanks and appreciation bring forth many names, foremost among them being Dr. Patrick Dolan, to whom I owe a deep debt of gratitude, not only for introducing me to the problem of torsion in general relativity, but for his constant guiding support and encouragement through these three long years.

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This thesis is dedicated to my parents.

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INTRODUCTION

Einstein's theory of general relativity incorporates macroscopic gravitational phenomena successfully into a geometrical theory based on Riemannian geometry (by a Riemannian geometry, we shall mean a four dimensional, smooth manifold having a symmetric metric tensor with signature (-2) defined on it). However, it has not been very well tested at the microscopic level. It is with this in mind, that one studies metric-torsion theories of gravity - to extend general relativity into the microphysical realm. But, is it necessary to introduce new geometrical ideas for the extension?

The new quantity that comes into play in elementary particle physics is the concept of intrinsic spin of a particle. General relativity incorporates the concept of orbital angular momentum in its definition of the stress-energy-momentum tensor. Is it not possible to simply generalise this orbital angular momentum to total angular momentum by the addition of spin? One could, but then the symmetrisation procedure that is carried out on the energy-momentum tensor in general relativity generally nullifies the effects of spin. Therefore, it seems simpler and indeed, more natural, to introduce a new geometrical entity that couples to the intrinsic spin of matter fields analogous to the coupling of the curvature to the energy-momentum of matter fields in general relativity. The torsion tensor having 24 independent components, is such a geometrical quantity. In the Einstein-Cartan-Sciama-Kibble (ECSK) theory, it is this torsion tensor which couples to the canonical spin tensor of matter fields. Just as the energy-momentum tensor is defined as the variational derivative of the matter lagrangian with respect to the metric tensor, for metric-torsion theories, the canonical spin tensor is defined as the variational derivative of the matter lagrangian with respect to the Contortion tensor. This Contortion tensor, also having 24 independent components, is a linear combination of the torsion tensor and may be used to describe the deviation of the new geometry (which we shall call Riemann-Cartan geometry) from Riemannian.

Riemannian geometry allows a unique (Christoffel) symmetric connection.

Riemann-Cartan geometry postulates an asymmetric connection, defining its antisymmetric part to be the torsion tensor. The contortion tensor is the difference between the asymmetric connection and the Christoffel connection, and is hence said to describe the deviation from Riemannian geometry. From the field equations for the Einstein-Cartan-Sciama-Kibble theory, derived in Chapter I, we shall see that it differs from general relativity only in the presence of spinning matter. This is due to the fact that the field equation relating torsion to the spin tensor is algebraic, i.e. a zeroth order differential equation, implying that whenever the spin tensor is zero (no spinning matter), so is the torsion. This fundamental deficiency of the theory is reflected in the fact that within the limits of present technology, the two theories, general relativity and the Einstein-Cartan-Sciama-Kibble theory are experimentally indistinguishable.

In Part I of this thesis, we shall study a particular matter field interaction with torsion, which allows the possibility of distinguishing metric-torsion theories of gravity from general relativity. The matter fields we shall consider are gauge fields, hence we begin Chapter I with an introduction to gauge theories. In gauge theories, one first considers the invariance of a matter field lagrangian in flat space-time under the action of some global, finite-dimensional Lie group, G. In enlarging this invariance to independent transformations of G for each point in space-time, "compensating" gauge fields have to be introduced.

The electromagnetic potential is shown to be such a gauge field, of the group U(1) (Or, the group of unitary matrices in one dimension). The discussion is extended to non-abelian Lie groups (Lie groups, whose elements commute under the group multiplication law are said to be abelian groups, while those that do not have a commutative group multiplication law are said to be non-abelian groups. As an example, U(1) is an abelian group, for its elements are simply 1×1 unitary matrices, while the group U(2), containing 2×2 unitary matrices, is a non-abelian group, for the simple reason that n×n matrices do not commute (n≠1) in general). We see that it is necessary to introduce a collection of gauge fields, one for each generator of the group. Hence the gauge field is labelled not only by the space-time coordinate index, but by a group index. This additional group index on the gauge fields will be seen to give rise to a non-linearity in their lagrangian.

We continue in Chapter I by first giving a brief introduction to Riemann-Cartan geometry. We go on to describe the physics of the Einstein-Cartan-Sciama-Kibble theory, by outlining the arguments for taking its lagrangian to be the curvature scalar of the underlying Riemann-Cartan geometry and deriving the corresponding field equations.

In Chapter II, we first show that the only effect of general relativity when coupled to gauge fields(let us call this the E-G.F. coupling), is the addition of a factor of $\sqrt{-g}$ to the lagrangian of the gauge fields (g denotes the determinant of the symmetric metric of Riemann-Cartan space-time). However, when we attempt to couple torsion to gauge fields (let us call this the T-G.F. coupling), we find that gauge invariance is lost. It is then argued that loss of gauge invariance should not even be considered; for all the present day successes of elementary particle physics are attributable to the fact that they are generally based on the formalism of gauge theories. Another possibility is to give up the coupling of torsion to gauge fields (T-G.F. coupling). This would be carried out by coupling gauge fields to gravity through the torsionless Christoffel connection (E-G.F. coupling), while coupling all other matter fields to torsion through the full asymmetric connection (i.e., Riemann-Cartan minimal coupling). This however, is rather ad hoc, and a novel suggestion (which modifies the usual form of the gauge covariant derivative), recently put forward,is described. All the above problems are removed by this solution, and allows the coupling of torsion to electromagnetic fields. The novel feature of this solution is that torsion takes a special form; it comes from the gradient of a scalar field. Another feature of this solution is that it allows for the first time, a restricted form of dynamic (or propagating) torsion within the confines of a theory that takes its lagrangian to be linear in the curvature.

We also show that a generalisation of this solution to arbitrary non-abelian gauge fields necessitates a modification to the field strength tensor in addition to the modification of the gauge covariant derivative. It is found that the special form of torsion, allowing the electromagnetic field to couple to metric torsion theories of gravity is carried through to the non-abelian case. Very briefly, we explain why this happens.

In Part II of the thesis, lagrangians and variational principles for metric-torsion theories are studied. Our attitude to variational principles is outlined in Chapter III as follows. A physical field is described by a set of field variables (e.g. the components of a metric tensor), and we assume that the field equations governing the behaviour of the field are identical with the Euler-Lagrange equations of the given problem in the calculus of variations. The action integral in the calculus of variations is supposed to be invariant under general coordinate transformations; this implies that the corresponding integrand (the lagrangian), is a scalar density. For any given type of physical field variable, this is augmented by an additional assumption concerned with invariance properties. This assumption specifies the transformation properties of the field variables under general coordinate transformations.

These two invariance requirements, taken together, severely restrict the classes of admissible lagrangians and hence the type of acceptable field equations. The two invariance requirements, along with the assumption that the field equations are identical to the Euler-Lagrange equations, are collectively called an Invariant variational principle (IVP).

The restrictions on the lagrangians are expressed in terms of some identities which must be satisfied by the lagrangians and their derivatives. By applying the IVP to the problem of a second order lagrangian in the metric tensor (a lagrangian is said to be of nth order whenever it depends on partial derivatives of at least some of the field variables with respect to the space-time coordinates, up to and including the nth order), we derive three identities that the lagrangian along with its derivatives, satisfies. The third identity, in this example, is a remarkable one, for it highlights the well known theorem from Riemannian geometry, that any invariant function of the metric and its first two derivatives, can be expressed in terms of the Riemannian curvature tensor. Indeed, this identity goes much further, in that, it demonstrates quite clearly how the given function is to be expressed in terms of the curvature tensor. The restriction on the type of acceptable field equations is illustrated by reducing the Euler-Lagrange equation for the metric example.

As our aim in this thesis is to study various aspects of metric-torsion theories, we generalise the above procedure by applying it to a lagrangian of second order in the metric tensor, and containing no higher than first derivatives of the torsion tensor. The basic reason for considering lagrangians containing at least first derivatives of the torsion is that they may allow the possibility of propagating torsion, i.e., the field equation for torsion may be a differential equation of at least the first order. No higher than first derivatives are taken for simplicity only, there is no loss of generality. Once again, the restrictions on admissible lagrangians is expressible in the form of three identities that the lagrangian along with its derivatives must satisfy. The third identity tells us that any invariant function depending on the metric, the first two derivatives of the metric, the torsion tensor and its first derivatives can be expressed in terms of the curvature tensor, the torsion tensor and its derivatives. As the expressions for the three identities are large, we illustrate the restrictions on the type of acceptable equations by reducing the Euler-Lagrange equation for a lagrangian which depends only on the torsion along with the metric and its first two derivatives. The lagrangian for the Einstein-Cartan-Sciama-Kibble theory is of this type. Restricting to such lagrangians means the loss of propagating torsion. This is not very important here, as our sole motivation for taking a reduced lagrangian is simplicity in illustrating the restrictions on the type of field equations brought out by the identities.

Chapter IV points out first, that if we require the lagrangian of a metric-torsion theory to be linear in the curvature, then we are allowed the addition of a pseudo-scalar term to the Einstein-Cartan-Sciama-Kibble lagrangian. This term, fortunately vanishes identically in general relativity due to the cyclic symmetry on the Riemannian curvature tensor (or Riemann-Christoffel tensor). Allowing the additional term, the field equations are derived, showing that as expected, the torsion field equation is again algebraic (i.e., torsion does not propagate).

In order to observe the effects of this additional pseudo-scalar-parity violating-term, we couple the theory to the Dirac spinor field. We find, however, that the only effect of the additional term is to reduce the strength of the existing parity-violating interactions in the ECSK-Dirac theory. This is easily understood, since the parity violating interaction term in the ECSK theory when coupled to the Dirac field arises due basically to the Dirac algebra, leading to total antisymmetry of the spin angular momentum tensor, and hence to the total antisymmetry of the contortion tensor. While the additional term that we motivate leads, manifestly, to a totally antisymmetric contribution to the contortion tensor. We then discuss the Proca (or massive Maxwell) field, and show there that we do indeed have a parity violating effect, which in principle, would enable us to experimentally prove or disprove the existence of torsion by observing the motion of massive elementary particles carrying spin 1.

The thesis ends with some Conclusions and discussion.

PART I

GAUGE FIELDS AND TORSION

"Symmetry as narrow or as wide as you may define its meaning, is one idea by which man through the ages tried to comprehend and create order, beauty and perfection".

Herman Weyl.

CHAPTER I

PRINCIPLES OF GAUGE THEORIES

AND

METRIC-TORSION THEORIES OF GRAVITATION

GAUGE THEORIES

§1. *The abelian theory*

The fundamental notion of gauge invariance is a rather simple generalisation of the concept of a continuous space-time symmetry of a lagrangian. We shall illustrate the idea by deriving the Maxwell lagrangian for electrodynamics as a local gauge theory of the abelian group U(1). Suppose $L(\phi,\partial\phi)$ describes a theory for a zeroth rank tensor field $\phi(x)$. Let us impose the following invariance on $L(\phi,\partial\phi)$:

$$\phi(x) \rightarrow \phi'(x) = e^{i\varepsilon}\,\phi(x) \qquad (1.1.1)$$

and

$$\partial_\mu\phi(x) \rightarrow \{\partial_\mu\phi(x)\}' = e^{i\varepsilon}\,\partial_\mu\phi(x) \qquad (1.1.2)$$

where ε is an arbitrary constant.

The group of transformations (1.1.1) is the group of unitary transformations in one dimension, U(1).

Because ε is a constant, the transformations (1.1.1) are called global gauge transformations.

Throughout the rest of this dissertation, we shall not be dealing with Conservation laws and therefore we shall not demonstrate here that the invariance of $L(\phi,\partial\phi)$ under (1.1.1) leads to current conservation which is simply the electric current conservation law /1/. Instead we shall now define local gauge transformations.

Suppose in (1.1.1), we allow ε to become a function on the space-time i.e., $\varepsilon \rightarrow \varepsilon(x)$. Then (1.1.1) becomes

$$\phi(x) \rightarrow \phi'(x) = e^{i\varepsilon(x)}\phi(x) \qquad (1.1.3)$$

however,

$$\partial_\mu \phi(x) \rightarrow \{ \partial_\mu \phi(x) \}' = e^{i\varepsilon(x)} \partial_\mu \phi(x) + i\, e^{i\varepsilon(x)} \{\partial_\mu \varepsilon(x)\} \phi(x) \tag{1.1.4}$$

These transformations are called local gauge transformations. Notice that the second term in (1.1.4), the inhomogeneous term, "breaks" the invariance of $L(\phi,\partial\phi)$.

Clearly, invariance of $L(\phi,\partial\phi)$ under (1.1.3) will be assured if a new vector field is introduced into the partial derivative such that its transformation law under (1.1.3) acts to cancel the inhomogeneous term in (1.1.4).

With this in mind, a new derivative, called the gauge covariant derivative, D_μ is defined;

$$D_\mu = \partial_\mu - iqA_\mu \tag{1.1.5}$$

where q is an arbitrary coupling parameter which will be identified with electric charge. Then, replacing $\partial_\mu\phi$ in $L(\phi,\partial\phi)$ by $D_\mu\phi$, invariance of $L(\phi, D_\mu\phi)$ will be ensured if we require, in accordance with (1.1.2),

$$D_\mu\phi \rightarrow \{D_\mu\phi\}' = e^{i\varepsilon(x)} D_\mu\phi \tag{1.1.6}$$

Suppose that under, (1.1.3), $A_\mu \rightarrow A'_\mu$, then (1.1.6) gives

$$(\partial_\mu - iq A'_\mu)\, \phi'(x) = e^{i\varepsilon(x)} (\partial_\mu - iq A_\mu)\, \phi(x) \tag{1.1.7}$$

or,

$$(\partial_\mu - iq A'_\mu)\, e^{i\varepsilon(x)} \phi(x) = e^{i\varepsilon(x)} (\partial_\mu - iq A_\mu) \phi(x) \tag{1.1.8}$$

Simplifying, we find,

$$A'_\mu = A_\mu - \frac{i}{q} (\partial_\mu e^{i\varepsilon(x)})\, e^{-i\varepsilon(x)} \tag{1.1.9}$$

The infinitesimal form of this transformation is

$$A'_\mu = A_\mu + \frac{1}{q} \partial_\mu \varepsilon(x) \tag{1.1.10}$$

So the new "gauge potential" A_μ must transform inhomogeneously, like a connection in order for the lagrangian $L(\phi, D\phi)$ to be invariant under the local gauge transformation (1.1.3):

The rule (1.1.5) for the gauge covariant derivative is also known as minimal coupling.

Having introduced the new field A_μ, and its coupling to the matter field $\phi(x)$, we must consider possible kinetic energy and mass terms coupling A_μ to itself.

Observing the transformation laws for A_μ, (1.1.10), it is easy to see that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.1.11}$$

is invariant under (1.1.10), i.e.

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \tag{1.1.12}$$

$$= \partial_\mu(\partial_\nu \varepsilon) - \partial_\nu(\partial_\mu \varepsilon) \tag{1.1.13}$$

$$= 0 . \tag{1.1.14}$$

Hence the scalar $F_{\mu\nu}F^{\mu\nu}$ is an invariant. In fact, because we are dealing with four-dimensional space-time, we have one other scalar, namely,

$$F_{\mu\nu} {}^* F^{\mu\nu} \tag{1.1.15}$$

where

$${}^*F^{\mu\nu} := \frac{1}{4} \varepsilon^{\mu\nu\rho\pi} F_{\rho\pi} \tag{1.1.16}$$

and $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor density. However, we find that (1.1.15) is a total divergence quantity, so the kinetic energy term for the potential A_μ is taken as

$$L_{EM} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{1.1.17}$$

The numerical factor is chosen for convenience in the field equation. A mass term for A_μ is not possible because it breaks gauge invariance. Therefore, the electromagnetic field, the photon is a massless gauge field. Indeed, this massless property is quite general for gauge fields as we shall see in the next section, when we generalise the above analysis to non-abelian gauge groups.

§2. *The non abelian theory*.

The generalisation of local gauge invariance to non-abelian groups was first studied by Yang and Mills (1954), who sought to explain the conservation of isotopic spin by using the non-abelian group SU(2). In this section, we shall describe their idea for a general non-abelian compact unitary group, G. The basic reason for dealing only with compact groups is that all non-trivial irreducible continuous unitary representations of non-compact (indeed of locally compact) groups are infinite dimensional. Since elementary particles are assumed to be irreducible continuous unitary representations of symmetry groups, it is more sensible to deal with compact groups, for which we know that every irreducible continuous representation is linearly equivalent to a unitary, and hence finite dimensional representation.

Let us suppose our gauge group is G, with generators T_i satisfying the following Lie algebra,

$$[T_i, T_j] = iC_{ijk}\, T_k \quad , \quad i,j,k = 1,\ldots,\dim G \qquad (1.2.1)$$

where C_{ijk} are the structure constants of the algebra. Throughout we shall take the representation matrices also to be $(T_i)_{jk}$. A collection of scalar fields $\phi_i(x)$ transforms according to

$$\phi_i(x) \to \phi'_i(x) \;=\; e^{iT.\varepsilon}\phi_i(x) \qquad (1.2.2)$$

where $$T.\varepsilon \;:=\; T_i\varepsilon_i \quad , \; i= 1,\ldots,\dim G. \qquad (1.2.3)$$

and the ε_i are arbitrary constants.

In what follows, we shall suppress matrix (group) indices for the most part. We shall also write the transformation law, (1.2.2) as

$$\phi(x) \rightarrow \phi'(x) = U(\varepsilon)\phi(x) \tag{1.2.4}$$

if we impose the invariance (1.2.4) on a lagrangian for the multiplet $\phi_i(x)$, $L(\phi_i, \partial_\mu \phi_i)$, we have

$$\partial_\mu \phi(x) \rightarrow \{\partial_\mu \phi(x)\}' = U(\varepsilon) \quad \partial_\mu \phi(x) \tag{1.2.5}$$

As before, the transformations (1.2.4) are called global gauge transformations. Suppose we make them local, i.e., let $\varepsilon_i \rightarrow \varepsilon_i(x)$ then we still have

$$\phi(x) \rightarrow \phi'(x) = U(\varepsilon(x))\phi(x) \tag{1.2.6}$$

However, we no longer have (1.2.5), instead,

$$\begin{aligned} \partial_\mu \phi(x) \rightarrow \{\partial_\mu \phi(x)\}' &= U(\varepsilon(x)) \ \{\partial_\mu \phi(x)\} \\ &+ \{\partial_\mu U(\varepsilon(x))\} \ \phi(x) \end{aligned} \tag{1.2.7}$$

so that $L(\phi_i, \partial_\mu \phi_i)$ is not invariant under the extended, local gauge transformations. In imposing local gauge invariance on $L(\phi_i, \partial_\mu \phi_i)$ we must as before, introduce new compensating gauge potentials $A_{\mu i}$, one for each generator of the group. This is done by defining a new derivative, the gauge covariant derivative;

$$D_\mu = \partial_\mu - ig\, A_\mu . T \tag{1.2.8}$$

where g is a generalisation of the electric charge to non-abelian theories. Throughout the rest of this section, we shall assume the following notation,

$$A_\mu = A_\mu . T = A_{\mu i} T_i \tag{1.2.9}$$

similarly for other fields.

The lagrangian $L(\phi_i, D_\mu \phi_i)$ will clearly be gauge invariant under (1.2.6) if we have

$$D_\mu \phi(x) \to \{D_\mu \phi(x)\}' = U(\varepsilon(x)) \quad D_\mu \phi(x) \tag{1.2.10}$$

or,

$$\partial_\mu \phi'(x) - ig\, A'_\mu \phi'(x) = U(\varepsilon(x))\{\partial_\mu \phi(x) - ig\, A_\mu \phi(x)\} \tag{1.2.11}$$

Using (1.2.6),

$$\{\partial_\mu U(\varepsilon)\} \phi(x) - ig\, A'_\mu U(\varepsilon)\phi(x) = - ig\, U(\varepsilon)\, A_\mu \phi(x) \tag{1.2.12}$$

or,

$$A'_\mu = U(\varepsilon)\, A_\mu U^{-1}(\varepsilon) - \frac{i}{g}\{\partial_\mu U(\varepsilon)\}\, U^{-1}(\varepsilon). \tag{1.2.13}$$

Therefore, we see that the new gauge potentials that have been introduced, transform inhomogeneously under a local gauge transformation.

Having introduced new fields into the theory, possible kinetic energy and mass terms must be considered for these fields. However, before we do that, let us note that as the multiplet fields $A_{\mu i}$ also transform under a representation of the gauge group G, we must check to see if the group property holds for $A_{\mu i}$. i.e., if we have

$$A'_\mu = U(\varepsilon)\, A_\mu\, U^{-1}(\varepsilon) - \frac{i}{g}\{\partial_\mu U(\varepsilon)\}\, U^{-1}(\varepsilon) \tag{1.2.14}$$

and $$A''_\mu = U(\eta)\, A'_\mu\, U^{-1}(\eta) - \frac{i}{g}\{\partial_\mu U(\eta)\}\, U^{-1}(\eta), \tag{1.2.15}$$

Can we find a parameter ξ, such that $U(\xi) = U(\eta)\, U(\varepsilon)$, and

$$A''_\mu = U(\xi)\, A_\mu U^{-1}(\xi) - \frac{i}{g}\{\partial_\mu U(\xi)\}\, U^{-1}(\xi). \tag{1.2.16}$$

It is quite a trivial matter to check that the group property does indeed hold and we shall not give it here.

A mass term for the gauge potentials, $A_{\mu i}$ is not possible as is easily seen from the non-gauge invariant lagrangian,

$$L_{mass} = \frac{1}{2} m^2 A_{\mu i} A^{\mu i} , \qquad (1.2.17)$$

due, essentially to the inhomogeneous term in (1.2.13).

As for the kinetic energy term, because the gauge potentials $A_{\mu i}$ carry a group index, there will be self-interactions among them and the kinetic term L_o, cannot have the simple form it did in the electrodynamic example. In fact, we must have the field strength tensor $F_{\mu\nu}$ defined by :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu , A_\nu], \qquad (1.2.18)$$

where, in accordance with (1.2.9) we have

$$F_{\mu\nu} \equiv F_{\mu\nu i} T_i \qquad (1.2.19)$$

and,

$$[A_\mu , A_\nu]^i = A_{\mu j} A_{\nu k} [T_j , T_k]^i \qquad (1.2.20)$$

$$= A_{\mu j} A_{\nu k} \; i C^i{}_{jk} . \qquad (1.2.21)$$

In total analogy with electrodynamics, we take the kinetic energy term to be

$$L_o = - \frac{1}{4} F_{\mu\nu} . F^{\mu\nu}, \qquad (1.2.22)$$

for this to be invariant however, we must have

$$F'_{\mu\nu i} = F_{\mu\nu i} + C_{ijk} \varepsilon^j F_{\mu\nu}{}^k , \qquad (1.2.23)$$

or

$$F'_{\mu\nu} = U(\varepsilon) F_{\mu\nu} U^{-1}(\varepsilon) , \qquad (1.2.24)$$

i.e. the $F_{\mu\nu}$ must transform covariantly under a gauge transformation. So the total lagrangian for a set of scalar fields interacting with a set of non-abelian gauge fields is

$$L = L_o + L (\phi, (\partial_\mu - ig A_\mu . T)\phi) \qquad (1.2.25)$$

Of course, as before, the four-dimensionality of space-time allows the existence of one other invariant,

$$F_{\mu\nu i} \, {}^*F^{\mu\nu}{}_i , \qquad (1.2.26)$$

where ${}^{*}F^{\mu\nu}_{i}$ is the dual tensor of $F_{\mu\nu i}$ and is defined by

$$ {}^{*}F^{\mu\nu}_{i} = \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma i} \; . \qquad (1.2.27) $$

As in the abelian example, the invariant (1.2.26) is a total divergence term, and is ignored, except when dealing with compact manifolds, where surface effects cannot be thrown away. This completes our introduction to gauge theories. Notice that the non-linearity arising from the quadratic term in (1.2.18) is akin to the non-linearity in the Riemann-Christoffel tensor of general relativity. These two sections on gauge theories have been taken for the most part, from the excellent review by E.S. Abers and B.W. Lee /1/ .

METRIC-TORSION THEORIES OF GRAVITY

§3. *The geometry.*

Metric-torsion theories of gravity are a generalisation of Einstein's general theory of relativity and are based on a simple extension of Riemannian geometry, a geometry which we shall call Riemann-Cartan geometry. The difference between the two geometries appears in the differentiable structure, in the definition of a connection on the manifold. It is well known that in Riemannian geometry, the connection is symmetric and is such that

$$\nabla_\nu(\{\ \})\, g_{\mu\rho} = 0 \qquad (1.3.1)$$

where $g_{\mu\nu}$ is a symmetric metric defined on the manifold and $\nabla_\mu(\{\ \})$ is defined by its action on an arbitrary vector field as follows:

$$\nabla_\mu(\{\ \})A_\nu = \partial_\mu A_\nu - \{{}^{\ \sigma}_{\mu\ \nu}\}\, A_\sigma \qquad (1.3.2)$$

and

$$\nabla_\mu(\{\ \})A^\nu = \partial_\mu A^\nu + \{{}^{\ \nu}_{\mu\ \sigma}\}A^\sigma\ . \qquad (1.3.3)$$

The property of symmetry and (1.3.1) yields a unique connection, the Christoffel connection, determined completely by the metric and its first derivatives,

$$\{{}^{\ \sigma}_{\mu\ \nu}\} = \frac{1}{2}\, g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \qquad (1.3.4)$$

$g^{\sigma\rho}$ is the inverse of $g_{\sigma\rho}$,

$$g^{\mu\nu}\, g_{\nu\sigma} = \delta^\mu_\sigma\ . \qquad (1.3.5)$$

Cartan /2/ generalised Riemann's geometry by simply not imposing symmetry on the connection symbols. The resulting antisymmetric part, a tensor, he called the torsion tensor,

$$S_{\mu\nu}{}^\sigma = \frac{1}{2}\,(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu})\ . \qquad (1.3.6)$$

It just happens that (1.3.1) is automatically satisfied in Riemannian geometry, it is Ricci's lemma. Upon going to general relativity, the property (1.3.1) acquires great significance, it allows space-time to be locally Minkowskian, i.e., locally, the laws of special relativity hold good. As special relativity is such a well tested theory, equation (1.3.1) seems a very necessary assumption for any theory of gravity based on a geometrical framework. Taking Ricci's lemma over into Riemann-Cartan geometry, we have the postulate of metricity,

$$\nabla_\mu g_{\nu\rho} = 0 \tag{1.3.7}$$

where $g_{\mu\nu}$ is, as before, a symmetric metric tensor with inverse $g^{\mu\nu}$, and ∇_μ is defined by

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu} A_\sigma \tag{1.3.8}$$

and

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma} A^\sigma \tag{1.3.9}$$

The ordering of indices on $\Gamma^\mu_{\nu\sigma}$ is important for this geometry, and we take the convention that the differentiating index is the first index on $\Gamma^\mu_{\nu\sigma}$. Just as one can derive (1.3.4) from the requirement of symmetry on $\{{}^{\ \mu}_{\nu\ \sigma}\}$ and (1.3.1), we can derive the explicit form of $\Gamma^\mu_{\nu\sigma}$ by using (1.3.6) and (1.3.7). We have,

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0 \tag{1.3.10}$$

permuting indices $(\mu\nu\rho)$, we have

$$\partial_\nu g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} g_{\rho\sigma} = 0 \tag{1.3.11}$$

and

$$\partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma} = 0 \tag{1.3.12}$$

(1.3.10) + (1.3.11) − (1.3.12) implies

$$(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) - (\Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\mu})\, g_{\rho\sigma}$$

$$+ (\Gamma^\sigma_{\rho\mu} - \Gamma^\sigma_{\mu\rho})\, g_{\nu\sigma} + (\Gamma^\sigma_{\rho\nu} - \Gamma^\sigma_{\nu\rho})\, g_{\mu\sigma} = 0 \qquad (1.3.13)$$

multiplying throughout by $\frac{1}{2}\, g^{\alpha\rho}$,

$$\frac{1}{2}\, g^{\alpha\rho}(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}) - \frac{1}{2}\,(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu})$$

$$+ \frac{1}{2}\,(\Gamma^\sigma_{\rho\mu} - \Gamma^\sigma_{\mu\rho}) g^{\alpha\rho} g_{\nu\sigma} + \frac{1}{2}\,(\Gamma^\sigma_{\rho\nu} - \Gamma^\sigma_{\nu\rho}) g^{\alpha\rho} g_{\mu\sigma} = 0 \,. \qquad (1.3.14)$$

Using equations (1.3.4) and (1.3.6),

$$\{{}_{\nu}{}^{\alpha}{}_{\mu}\} - \frac{1}{2}\,(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu}) + S_{\rho\mu}{}^{\sigma}\, g^{\alpha\rho} g_{\nu\sigma}$$

$$+ S_{\rho\nu}{}^{\sigma}\, g^{\alpha\rho} g_{\mu\sigma} = 0 \,. \qquad (1.3.15)$$

Now any geometrical object, $\Gamma^\sigma_{\mu\nu}$ can be broken up into its symmetric and antisymmetric parts,

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\,(\Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\mu}) + \frac{1}{2}\,(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}) \qquad (1.3.16)$$

with the help of (1.3.6) we can write

$$\frac{1}{2}\,(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu}) = \Gamma^\alpha_{\mu\nu} - S_{\mu\nu}{}^{\alpha} \qquad (1.3.17)$$

substituting (1.3.17) into (1.3.15) we finally have,

$$\Gamma^\alpha_{\mu\nu} = \{{}_{\mu}{}^{\alpha}{}_{\nu}\} + S_{\mu\nu}{}^{\alpha} + S^{\alpha}{}_{\mu\nu} + S^{\alpha}{}_{\nu\mu} \qquad (1.3.18)$$

or,
$$\Gamma^\alpha_{\mu\nu} = \{{}_{\mu}{}^{\alpha}{}_{\nu}\} + S_{\mu\nu}{}^{\alpha} - S_{\nu}{}^{\alpha}{}_{\mu} + S^{\alpha}{}_{\mu\nu} \,. \qquad (1.3.19)$$

Conventionally, at this point one defines a new tensor, called the contortion, $K_{\mu\nu.}{}^{\alpha}$ having 24 independent components just as the torsion tensor, to describe the "deviation" from Riemannian geometry ,

$$K_{\mu\nu.}{}^{\alpha} = -S_{\mu\nu.}{}^{\alpha} + S_{\nu.\mu}{}^{\alpha} - S^{\alpha}{}_{.\mu\nu} \qquad (1.3.20)$$

So that

$$\Gamma^{\alpha}_{\mu\nu} = \{{}^{\alpha}_{\mu\ \nu}\} - K_{\mu\nu.}{}^{\alpha} \qquad (1.3.21)$$

Note that because the antisymmetry of the contortion tensor is on the last two indices,

$$\Gamma^{\alpha}_{(\mu\nu)} = \{{}^{\alpha}_{\mu\ \nu}\} - K_{(\mu\nu)}{}^{\alpha} \ , \ K_{(\mu\nu).}{}^{\alpha} \neq 0 \qquad (1.3.22)$$

(round brackets denote symmetrisation), with the consequence that K depends on the metric and torsion while the torsion tensor is a priori independent of the metric.

The Riemann-Cartan curvature tensor $R_{\mu\nu\rho\sigma}(\Gamma)$ is defined in analogy with that of the Riemann-Christoffel tensor $R_{\mu\nu\rho\sigma}(\{\ \})$ by

$$R_{\mu\nu\rho.}{}^{\sigma}(\Gamma) = \partial_{\mu}\Gamma^{\sigma}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\sigma}_{\nu\alpha}\Gamma^{\alpha}_{\mu\rho} \qquad (1.3.23)$$

As in Riemannian geometry, we have antisymmetry on the first two indices of $R_{\mu\nu\rho.}{}^{\sigma}(\Gamma)$, through the definition. We also have antisymmetry on the last two indices of $R_{\mu\nu\rho.}{}^{\sigma}(\Gamma)$ due to metricity. However, we have no symmetry on the pairs of indices $(\mu\nu)$ and $(\rho\sigma)$, as the connection is no longer symmetric, i.e.

$$R_{\mu\nu\rho\sigma}(\Gamma) \neq R_{\rho\sigma\mu\nu}(\Gamma) \ . \qquad (1.3.24)$$

Hence in a Riemann-Cartan geometry, the Ricci tensor,

$$R_{\mu\nu}(\Gamma) := R_{\sigma\mu\nu.}{}^{\sigma}(\Gamma) \qquad (1.3.25)$$

remains the only essential contraction of the curvature tensor. Because of (1.3.24), the Ricci tensor is asymmetric in general. The Ricci scalar of a Riemann-Cartan geometry,

$$R(\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma) \qquad (1.3.26)$$

while the Einstein tensor is given by,

$$G_{\mu\nu}(\Gamma) = R_{\mu\nu}(\Gamma) - \frac{1}{2} R(\Gamma) g_{\mu\nu} \qquad (1.3.27)$$

and is in general asymmetric.

The Ricci scalar, $R(\Gamma)$ can be decomposed into its Riemannian and non-Riemannian parts as follows (by choosing a normal coordinate system in which $\{{}^{\sigma}_{\mu\;\nu}\} = 0$);

$$R(\Gamma) = R(\{\}) + \partial^{\rho} K_{\sigma\rho.}{}^{\sigma} - g^{\nu\rho}\partial_{\sigma} K_{\nu\rho.}{}^{\sigma} + K_{\sigma\alpha.}{}^{\sigma} K^{.\nu\alpha}_{\nu} - K_{\nu\alpha.}{}^{\sigma} K^{.\nu\alpha}_{\sigma} \qquad (1.3.28)$$

while the Ricci tensor decomposes into

$$R_{\nu\rho}(\Gamma) = R_{\nu\rho}(\{\}) + \partial_{\nu} K_{\sigma\rho.}{}^{\sigma} - \partial_{\sigma} K_{\nu\rho.}{}^{\sigma} + K_{\sigma\alpha.}{}^{\sigma} K_{\nu\rho.}{}^{\alpha} - K_{\nu\alpha.}{}^{\sigma} K_{\sigma\rho.}{}^{\alpha} \qquad (1.3.29)$$

The Bianchi identities on the Riemann-Christoffel tensor upon going to Riemann-Cartan geometry, generalise to

$$R_{[\mu\nu\rho].}{}^{\sigma}(\Gamma) = 2\nabla_{[\mu} S_{\nu\rho].}{}^{\sigma} - 4 S_{[\mu\nu.}{}^{\alpha} S_{\rho]\alpha.}{}^{\sigma} \qquad (1.3.30)$$

and

$$\nabla_{[\alpha} R_{\mu\nu]\rho.}{}^{\sigma}(\Gamma) = 2 S_{[\alpha\mu.}{}^{\beta} R_{\nu]\beta\rho.}{}^{\sigma}(\Gamma), \qquad (1.3.31)$$

(square brackets denote antisymmetrisation). Having described the geometry as far as is required for this thesis, we shall, in the next section describe the physics behind metric-torsion theories by writing down the lagrangian for the Einstein-Cartan-Sciama-Kibble (ECSK) theory, and deriving the corresponding field equations.

We shall now complete this section on the geometric framework of metric-torsion theories of gravity by giving a geometrical picture of torsion / 3 /. Viewing the torsion tensor as a vector valued operator, operating on two vectors, $\underline{u}$, $\underline{v}$, we have,

$$T(\underline{u},\underline{v}) = \nabla_{\underline{u}}\underline{v} - \nabla_{\underline{v}}\underline{u} - [\underline{u},\underline{v}], \qquad (1.3.32)$$

where $[\underline{u},\underline{v}]$ denotes the Lie bracket. In the picture below, $\{\underline{u}(\varepsilon_o)\}_{//}$ denotes the parallely transported vector field. Similarly for $\underline{v}(\lambda_o)$.

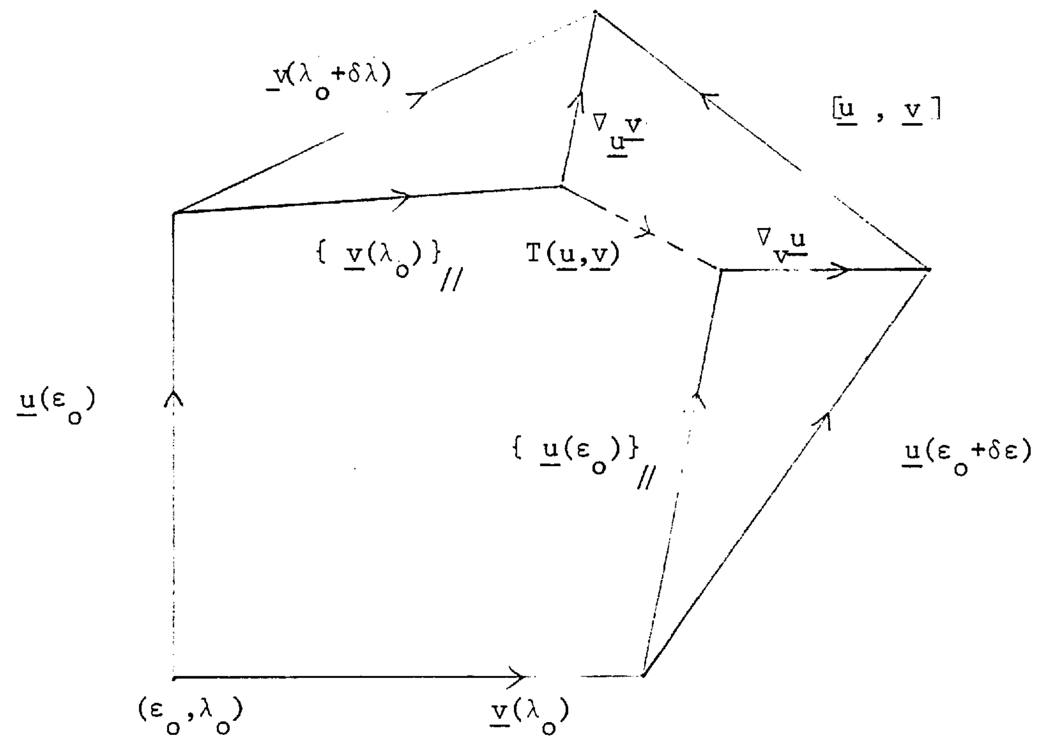

FIG. 1

This figure shows torsion as the wrecker of parallelograms. Under parallel transport, the vectors $\underline{u},\underline{v}$ are given an additional "twist" by the torsion field. We note that even if the vectors $\underline{u},\underline{v}$ commute, i.e., $[\underline{u},\underline{v}] = 0$, the parallelogram is broken.

METRIC-TORSION THEORIES OF GRAVITY.

§4. *Physics of Einstein-Cartan-Sciama-Kibble (ECSK) Theory.*

Having briefly described the geometry of metric-torsion theories of gravity, we come now to the question of constructing and describing a metric-torsion theory of gravity. In this section we shall describe the simplest possible generalisation of general relativity, the ECSK theory. As the ECSK theory is also the simplest possible metric-torsion theory, we shall, for later purposes, take the ECSK theory to be the prototype of metric-torsion theories of gravity.

Cartan's /2/ idea of a relation between spin and torsion may be supported by the following argument /4/. In special relativity, we have the inhomogeneous Lorentz group as the isometry group of space-time. The Lie algebra of this group has two basic invariants which are interpreted as the mass and intrinsic spin of elementary particles. The inhomogeneous Lorentz group is a semi-direct product of the group of translations in four dimensions and the homogeneous Lorentz group of rotations. Mass arises as the invariant related to the translational part and spin with the rotational part of the isometry group. In a classical field theory, mass is taken to correspond to the canonical energy-momentum tensor while spin corresponds to a canonical spin tensor. Einstein's theory of general relativity expresses a dynamical relation between the energy-momentum and curvature tensors. If a theory of gravity is considered to be a generalisation of the special theory of relativity, one would like to have a dynamical relation between the spin tensor and any allowed geometrical entity analogous to curvature. Having introduced the torsion tensor, we have such a possibility, by coupling torsion to the canonical spin tensor, we shall have the desired relation. If we have a lagrangian L, for a matter field in general relativity, then the definition of the dynamical energy-momentum tensor is,

$$\sqrt{-g}\, T^{\mu\nu} := 2 \frac{\delta L}{\delta g_{\mu\nu}} \tag{1.4.1}$$

where $g(<0)$ denotes the determinant of the metric tensor $g_{\mu\nu}$. The introduction of torsion (equivalently, contortion) allows us to introduce a dynamical definition of spin in a straightforward manner :

$$\sqrt{-g}\, \tau_{\sigma}^{\cdot\nu\mu} := \frac{\delta L}{\delta K_{\mu\nu\cdot}{}^{\sigma}} \tag{1.4.2}$$

$\frac{\delta}{\delta g}$ and $\frac{\delta}{\delta K}$ denote variational derivatives with respect to the metric and contortion tensors respectively.

In constructing lagrangians for metric-torsion theories, in particular, for the ECSK theory, it is best to start with Einstein's theory and build upon it. The lagrangian for a matter field ψ in Minkowski space-time reads (suppressing all indices on the matter field),

$$L\,(\psi, \partial\psi) \tag{1.4.3}$$

In coupling to Einstein's general relativity, one uses the minimal coupling principle,

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} \text{and } \partial_{\mu} \rightarrow \nabla_{\mu}(\{\ \}) \tag{1.4.4}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The total lagrangian for the gravitational and matter field is,

$$L\,(\psi, \nabla_{\mu}(\{\ \})\psi) + \sqrt{-g}\, R\,(\{\}) \tag{1.4.5}$$

As this principle works so well for the macroscopic theory it is advisable to retain as much of this as possible. As such, the corresponding, total lagrangian for the matter field coupled to ECSK gravity is taken as,

$$L\,(\psi, \nabla_{\mu}\psi) + \sqrt{-g}\, R(\Gamma) , \tag{1.4.6}$$

and the torsion is said to be minimally coupled.

The field equations are obtained by variations with respect to the independent variables $(\psi, g_{\mu\nu}, S_{\mu\nu.}{}^{\sigma})$ or $(\psi, g_{\mu\nu}, K_{\mu\nu.}{}^{\sigma})$. However, since the 24 components of torsion are a priori independent of the metric, we shall take variations with respect to $(\psi, g_{\mu\nu}, S_{\mu\nu.}{}^{\sigma})$; We have,

$$\frac{\delta L}{\delta \psi} = 0 \tag{1.4.7}$$

$$\frac{\delta L}{\delta g_{\mu\nu}} = - \frac{\delta \sqrt{-g}\, R(\Gamma)}{\delta g_{\mu\nu}} \tag{1.4.8}$$

and

$$\frac{\delta L}{\delta S_{\mu\nu}{}^{\sigma}} = - \frac{\delta \sqrt{-g}\, R(\Gamma)}{\delta S_{\mu\nu}{}^{\sigma}} \tag{1.4.9}$$

Provisionally, defining

$$\sqrt{-g}\ \mu_{\sigma}{}^{\mu\nu} = \frac{\delta L}{\delta S_{\mu\nu}{}^{\sigma}} \tag{1.4.10}$$

and noting equation (1.3.20), we shall have,

$$\mu^{\mu\nu\rho} = - \tau^{\mu\nu\rho} + \tau^{\nu\rho\mu} - \tau^{\rho\mu\nu} \tag{1.4.11}$$

or equivalently,

$$\tau^{\mu\nu\rho} = \mu^{[\nu\mu]\rho} \tag{1.4.12}$$

Using equations (1.4.12), (1.4.10), (1.4.1) and (1.4.2) we can write the field equations (1.4.8) and (1.4.9) as follows :

$$T^{\mu\nu} = - \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}\, R(\Gamma)}{\delta g_{\mu\nu}} \tag{1.4.13}$$

and

$$\tau^{\mu\nu\sigma} = - \frac{1}{2\sqrt{-g}}\, g^{\alpha[\mu} \frac{\delta \sqrt{-g}\, R\ (\Gamma)}{\delta S_{\nu]\sigma.}{}^{\alpha}} \tag{1.4.14}$$

These field equations can be reduced to the following simple form ,

$$G^{\mu\nu}(\Gamma) = \Sigma^{\mu\nu} \qquad (1.4.15)$$

and

$$T^{\mu\nu\rho} = \tau^{\mu\nu\rho} \quad . \qquad (1.4.16)$$

where $G^{\mu\nu}(\Gamma)$ is the asymmetric Einstein tensor defined in equation (1.3.27) and $\Sigma^{\mu\nu}$ is an asymmetric energy-momentum tensor involving $T^{\mu\nu}$ and $\tau^{\mu\nu\rho}$. The quantity $T^{\mu\nu\rho}$ is a new combination of torsion tensors, and is given the name of modified torsion tensor. It is defined as

$$T_{\mu\nu\cdot}{}^{\sigma} := S_{\mu\nu\cdot}{}^{\sigma} + 2\,\delta^{\sigma}_{[\mu} \quad S_{\nu]\alpha\cdot}{}^{\alpha} \qquad (1.4.17)$$

This completes our introduction to metric-torsion theories of gravitation, in particular to the ECSK theory. (See the review of Hehl et. al. /4/). In the next chapter, we shall attempt to couple gauge fields to torsion and we will see that inconsistencies arise. A solution for the case of electrodynamics is described and a generalisation to non-abelian fields is given. The essential result is that torsion is of a special form, and the gauge coupling constants become space-time dependent functions.

CHAPTER II

ROLE OF GAUGE FIELDS IN GRAVITATIONAL THEORIES

§1. *Introduction of gauge fields into general relativity.*

We have seen in the last chapter that introducing matter fields into gravity is achieved simply through the minimal coupling procedure by letting $\eta_{\mu\nu} \to g_{\mu\nu}(x)$ and $\partial_\mu \to \nabla_\mu$ in the matter field lagrangian.

The lagrangian for a gauge field in Minkowski space-time invariant under a group G, whose generators, T_i, $i=1,\ldots,\dim G$ satisfy

$$[T_i, T_j] = iC_{ijk} T_k \tag{2.1.1}$$

is simply

$$L = -\frac{1}{4} F_{\mu\nu}^{\ i} F^{\mu\nu}_{\ \ i} \tag{2.1.2}$$

with $F_{\mu\nu}^{\ i} = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\, C^i_{\ jk} A_\mu^j A_\nu^k$ (2.1.3)

So, coupling to gravity modifies the lagrangian to

$$L_G = -\frac{1}{4}\sqrt{-g}\ \tilde{F}_{\mu\nu}^{\ i}\ \tilde{F}^{\mu\nu}_{\ \ i} \tag{2.1.4}$$

with $\tilde{F}_{\mu\nu}^{\ i} = \nabla_\mu(\{\})A_\nu^i - \nabla_\nu(\{\})A_\mu^i + g\, C^i_{\ jk} A_\mu^j A_\nu^k$ (2.1.5)

Remembering that

$$\nabla_\mu(\{\})A_\nu = \partial_\mu A_\nu - \{{}^{\ \sigma}_{\mu\ \nu}\} A_\sigma \ , \tag{2.1.6}$$

we see that due to symmetry of $\{{}^{\ \sigma}_{\mu\ \nu}\}$ in $(\mu\nu)$, $F_{\mu\nu}^{\ i}$ remains unchanged after coupling to general relativity. The only modification is to the lagrangian, through the incorporation of $\sqrt{-g}$, and the gauge structure of the fields $F_{\mu\nu}^{\ i}$ is unaffected.

$$\tilde{F}_{\mu\nu}^{\ i} = F_{\mu\nu}^{\ i} = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\, C^i_{\ jk} A_\mu^j A_\nu^k \tag{2.1.7}$$

One might be tempted to say that such a result should have been expected on the grounds that the minimal coupling procedure as applied to gauge fields is quite distinct from minimal coupling as applied to the gravitational fields. One involves parallel transfer on the physical space-time one is studying, while the other has parallel transfer defined on an "internal" group space.

We shall see in the next section, that such an expectation is false when applied to metric-torsion theories due basically to the existence of torsion as the "wrecker of infinitesimal parallelograms" - see Fig. 1, Chapter I.

§2. *Generalisation to metric torsion theories and loss of gauge invariance.*

As before, coupling $L = -\frac{1}{4} F_{\mu\nu}^{\ \ i} F^{\mu\nu}_{\ \ i}$ to gravity modifies the lagrangian to

$$L_G = -\frac{1}{4} \tilde{F}_{\mu\nu}^{\ \ i} \tilde{F}^{\mu\nu}_{\ \ i} \sqrt{-g} \tag{2.2.1}$$

with

$$\tilde{F}_{\mu\nu}^{\ \ i} = \nabla_\mu A^i_\nu - \nabla_\nu A^i_\mu + g\, C^i_{\ jk} A^j_\mu A^k_\nu \tag{2.2.2}$$

For metric-torsion theories of gravity,

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu} A_\sigma \tag{2.2.3}$$

so that

$$\begin{aligned}\nabla_\mu A^i_\nu - \nabla_\nu A^i_\mu &= -\Gamma^\sigma_{\mu\nu} A^i_\sigma + \Gamma^\sigma_{\nu\mu} A^i_\sigma + \partial_\mu A^i_\nu - \partial_\nu A^i_\mu \\ &= (\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu}) A^i_\sigma + \partial_\mu A^i_\nu - \partial_\nu A^i_\mu \\ &= 2\, S_{\nu\mu.}^{\ \ \ \sigma} A^i_\sigma + \partial_\mu A^i_\nu - \partial_\nu A^i_\mu \end{aligned} \tag{2.2.4}$$

Therefore

$$\tilde{F}_{\mu\nu}^{\ \ i} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\, C^i_{\ jk} A^j_\mu A^k_\nu + 2 S_{\nu\mu.}^{\ \ \ \sigma} A^i_\sigma \tag{2.2.5}$$

So the field strength tensor $F_{\mu\nu}^{\ \ i}$ is modified when coupled to torsion, and we have no grounds to assume $\tilde{F}_{\mu\nu}^{\ \ i}$ is still gauge covariant.

Under a gauge transformation, we have equation (1.2.13), the infinitesimal form of which is,

$$\delta A_{\mu i} = -\frac{1}{g}\partial_\mu \varepsilon_i - C_{ikj}\,\varepsilon_k\,A_{\mu j} \qquad (2.2.6)$$

and we also know that

$$F_{\mu\nu}{}^{i} = \partial_\mu A^i_\nu - \partial_\nu A_\mu{}^{i} + g\; C^i{}_{jk}\,A^j_\mu\,A^k_\nu \qquad (2.2.7)$$

under a gauge transformation changes covariantly, i.e.

$$\delta F_{\mu\nu}{}^{i} = C^i{}_{jk}\varepsilon^j\,F_{\mu\nu}{}^{k} \qquad (2.2.8)$$

The behaviour of $\tilde{F}_{\mu\nu}{}^{i}$ under a gauge transformation can therefore be described as

$$\delta\tilde{F}_{\mu\nu}{}^{i} = \delta F_{\mu\nu}{}^{i} + 2\;S_{\nu\mu.}{}^{\sigma}\delta A^i_\sigma$$

$$= C^i{}_{jk}\;\varepsilon^j F_{\mu\nu}{}^{k} - \frac{2}{g}\;S_{\nu\mu.}{}^{\sigma}\partial_\sigma\varepsilon^i - 2S_{\nu\mu.}{}^{\sigma}C^i{}_{kj}\varepsilon_k A_{\sigma j} \qquad (2.2.9)$$

Hence,

$$\delta\tilde{F}_{\mu\nu}{}^{i} \neq C^i{}_{jk}\varepsilon^j\,\tilde{F}_{\mu\nu}{}^{k}\;. \qquad (2.2.10)$$

and therefore the gauge invariance of the lagrangian L_G is ruined by allowing non-zero torsion /5/ .

We come up against this problem of either not having torsion or abandoning gauge invariance. However, we have seen in the last chapter, that metric-torsion theories of gravity, while differing very little from general relativity offer great hope of carrying gravity into the microphysical realm, and incorporating intrinsic spin into the gravitational interaction. As the concept of intrinsic spin is purely quantum mechanical /6/ , it would seem to suggest that if ever a quantum theory of gravity is found, it should contain some form of spin-gravity interaction. Clearly if one has a quantum theory of gravity, the limit of zero gravity should be the usual quantum mechanics with its concept of intrinsic spin, when considering particle interactions with gravity (as opposed to field interactions). Metric-torsion theories of gravity might therefore be simply the classical limit of this quantum theory of gravity, showing up a remnant spin-gravity interaction in the form of torsion-spin interactions.

Another possibility that suggests itself is the abandoning of gauge invariance. With the great successes of electrodynamics as a gauge theory of the U(1) group, since its inception in 1929 by Hermann Weyl, and with the recent achievements of Salam and Weinberg /7 / in unifying the electromagnetic force with the weak force (responsible for radioactive decay of nuclei), through the use of gauge theories (specifically the group SU(2)×U(1)) suggests that it would be foolish to throw away such an inspiring formalism with no equally viable alternative at hand.

So, perhaps one can keep torsion and gauge invariance, but allow only non-gauge fields to couple to torsion, i.e. for gauge fields we should carry out the minimal coupling procedure with the Christoffel symbols only, while all other matter fields would be minimally coupled to the full Riemann-Cartan geometry through the asymmetric connection /5/ .

However, such an alternative is rather unsatisfactory, for, if torsion is to couple to the spin of matter fields, it is rather ad hoc to disallow spin one gauge fields from coupling to torsion while at the same time allowing massive spin one fields, like the proca field to couple to torsion. Therefore, we need to reassess the situation.

In the next section we shall discuss an alternative suggestion of S. Hojman, M. Rosenbaum, M.P. Ryan and L.C. Shepley /8/ which overcomes all of the above objections for electrodynamics.

§3. *Coupling of torsion to electrodynamics.*

In this section we shall couple torsion to the U(1) gauge field, the electromagnetic field by using a modification of the gauge minimal coupling procedure suggested by ref. /8/ .

In order to exhibit clearly what is happening, we shall write out explicitly, the lagrangian for a massless complex (charged) scalar field as /1/

$$L_\phi = - \sqrt{-g}\ \partial_\mu \phi^* \partial^\mu \phi \ . \qquad (2.3.1)$$

The lagrangian is clearly invariant under the global gauge transformation

$$\phi \rightarrow e^{i\Lambda}\phi \quad , \quad \Lambda = \text{constant} \quad , \qquad (2.3.2)$$

$$\partial_\mu \phi \rightarrow e^{i\Lambda}\partial_\mu \phi \qquad (2.3.3)$$

however, it is not invariant under local gauge transformations, for while

$$\phi \rightarrow \phi' = e^{i\Lambda(x)}\phi \quad , \quad \Lambda = \Lambda(x) \ . \qquad (2.3.4)$$

$$\partial_\mu \phi \not\rightarrow e^{i\Lambda(x)}\partial_\mu \phi \quad . \qquad (2.3.5)$$

Requiring invariance of L_ϕ under local gauge transformations imposes a compensating gauge potential $A_\mu(x)$ which is normally introduced by redefining the derivative operator ∂_μ:

$$\partial_\mu \rightarrow \partial_\mu - igA_\mu \qquad (2.3.6)$$

This, we have seen in Chapter I is the normal procedure. Incorporating the modification suggested in /8/, we redefine the derivative operator ∂_μ to be

$$\partial_\mu \rightarrow D_\mu := \partial_\mu \ - ig\ b_\mu^\alpha\ A_\alpha \ , \qquad (2.3.7)$$

where the function $b_\mu{}^\alpha$ will in general be a function of space-time but not of A_μ .

Invariance of L_ϕ under (2.3.3) with ∂_μ replaced by D_μ will be assured if the transformation of D_μ is given as

$$D_\mu\phi \rightarrow D'_\mu\phi' = e^{i\Lambda(x)} D_\mu\phi . \qquad (2.3.8)$$

or,

$$\partial_\mu\phi' - igb_\mu{}^\alpha A'_\alpha\phi' = e^{i\Lambda(x)}\partial_\mu\phi - ige^{i\Lambda(x)} b_\mu{}^\alpha A_\alpha\phi ,$$

$$\partial_\mu(e^{i\Lambda(x)}\phi) - ige^{i\Lambda(x)} b_\mu{}^\alpha A'_\alpha\phi$$

$$= e^{i\Lambda(x)}\partial_\mu\phi - ig\, e^{i\Lambda(x)} b_\mu{}^\alpha A_\alpha\phi ,$$

and,

$$e^{ig\Lambda(x)}\phi\, \partial_\mu\Lambda(x) - ige^{i\Lambda(x)} b_\mu{}^\alpha A'_\alpha\phi$$

$$= -ige^{i\Lambda(x)} b_\mu{}^\alpha A_\alpha\phi . \qquad (2.3.9)$$

Therefore we have

$$b_\mu{}^\alpha A'_\alpha = -\frac{i}{g}\partial_\mu\Lambda(x) + b_\mu{}^\alpha A_\alpha . \qquad (2.3.10)$$

or, defining $C_\mu{}^\alpha$ to be the inverse of $b_\mu{}^\alpha$:

$$C_\mu{}^\alpha b_\alpha{}^\sigma = \delta_\mu{}^\sigma, \qquad (2.3.11)$$

$$A'_\alpha = A_\alpha - \frac{i}{g} C_\alpha{}^\mu \partial_\mu\Lambda(x) \qquad (2.3.12)$$

This is the modified transformation law for the electromagnetic potential.

The field strength tensor of electrodynamics is simply

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \qquad (2.3.13)$$

So, coupling to torsion modifies this to

$$\tilde{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \qquad (2.3.14)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + 2S_{\nu\mu.}{}^{\sigma} A_\sigma \ .$$

We must solve for $C_\mu{}^\alpha$ by requiring gauge invariance of $\tilde{F}_{\mu\nu}$ under the transformation (2.3.12):

$$\delta\tilde{F}_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu + 2S_{\nu\mu.}{}^{\sigma} \delta A_\sigma$$

$$= \partial_\mu(-\frac{i}{g} C_\nu{}^\alpha \partial_\alpha \Lambda) - \partial_\nu(-\frac{i}{g} C_\mu{}^\alpha \partial_\alpha \Lambda) + 2S_{\nu\mu.}{}^{\sigma}(-\frac{i}{g} C_\sigma{}^\alpha \partial_\alpha \Lambda)$$

$$= -\frac{i}{g}\partial_\mu C_\nu{}^\alpha \partial_\alpha \Lambda - \frac{i}{g} C_\nu{}^\alpha \partial_\mu \partial_\alpha \Lambda + \frac{i}{g}\partial_\nu C_\mu{}^\alpha \partial_\alpha \Lambda + \frac{i}{g} C_\mu{}^\alpha \partial_\nu \partial_\alpha \Lambda$$

$$- \frac{2i}{g} S_{\nu\mu.}{}^{\sigma} C_\sigma{}^\alpha \partial_\alpha \Lambda \qquad (2.3.15)$$

$$= 0 \ .$$

or,

$$(-\frac{i}{g}\partial_\mu C_\nu{}^\alpha + \frac{i}{g}\partial_\nu C_\mu{}^\alpha - \frac{2i}{g} S_{\nu\mu}{}^{\sigma} C_\sigma{}^\alpha)$$

$$+ (-\frac{i}{g} C_\nu{}^\alpha \delta_\mu{}^\beta + \frac{i}{g} C_\mu{}^\alpha \delta_\nu{}^\beta) \partial_\beta \partial_\alpha \Lambda \qquad (2.3.16)$$

$$= 0 \ .$$

This equation must hold for arbitrary parameter Λ, so we have the two equations,

$$C_\mu{}^{(\alpha} \delta_\nu{}^{\beta)} - C_\nu{}^{(\alpha} \delta_\mu{}^{\beta)} = 0 \qquad (2.3.17)$$

and

$$\partial_\nu C_\mu{}^\alpha - \partial_\mu C_\nu{}^\alpha - 2S_{\nu\mu.}{}^{\sigma} C_\sigma{}^\alpha = 0 \qquad (2.3.18)$$

to solve.

The round brackets on indices α,β in equation (2.3.17) denote symmetrisation.

Carrying out the symmetrisation, we obtain

$$C_\mu^{\ \alpha}\,\delta_\nu^{\ \beta} + C_\mu^{\ \beta}\delta_\nu^{\ \alpha} - C_\nu^{\ \alpha}\delta_\mu^{\ \beta} - C_\nu^{\ \beta}\delta_\mu^{\ \alpha} = 0 \ . \qquad (2.3.19)$$

Tracing over indices ν and β , we find

$$4C_\mu^{\ \alpha} + C_\mu^{\ \alpha} - C_\mu^{\ \alpha} - C_\beta^{\ \beta}\,\delta_\mu^{\ \alpha} = 0 \qquad (2.3.20)$$

or,

$$C_\mu^{\ \alpha} = \frac{1}{4}\,C_\beta^{\ \beta}\,\delta_\mu^{\ \alpha} \qquad (2.3.21)$$

we can write this as

$$C_\mu^{\ \alpha} = f(x)\,\delta_\mu^{\ \alpha}, \qquad (2.3.22)$$

where $f(x) = \frac{1}{4}C_\beta^{\ \beta}$ is an arbitrary function of the space-time.

Substitution into the second equation, equation (2.3.18) gives

$$\partial_\nu f(x)\delta_\mu^{\ \alpha} - \partial_\mu f(x)\delta_\nu^{\ \alpha} - 2S_{\nu\mu}^{\ \ \sigma} f(x)\delta_\sigma^{\ \alpha} = 0 \qquad (2.3.23)$$

or,

$$2S_{\nu\mu.}^{\ \ \ \sigma} = \delta_\mu^{\ \sigma}\,\partial_\nu \ln f(x) - \delta_\nu^{\ \sigma}\partial_\mu \ln f(x)\ . \qquad (2.3.24)$$

Discarding the singular solution (when $f(x) = 0$), we see that the requirement that as $S_{\mu\nu.}^{\ \ \ \sigma} \to 0$, the covariant derivative defined in equation (2.3.7) reduce to the usual definition, i.e. that $b_\mu^{\ \alpha} \to \delta_\mu^{\ \alpha}$ allows us to parameterise the function $f(x)$ as an exponential.

$$f(x) = e^{\psi(x)} \qquad (2.3.25)$$

The field $\psi(x)$ is a scalar field which serves to define the torsion field $S_{\mu\nu}^{\ \ \sigma}$ through a gradient operation,

$$2S_{\nu\mu.}^{\ \ \ \sigma} = \delta_\mu^{\ \sigma}\partial_\nu\psi(x) - \delta_\nu^{\ \sigma}\,\partial_\mu\psi(x) \qquad (2.3.36)$$

so that $\psi(x)$ acts as a potential for the torsion field.

So we have managed to couple a restricted form of torsion to the electromagnetic field by modifying the usual minimal coupling procedure for the gauge field. Notice also that because this torsion is given as the gradient of a scalar field, we can construct a theory of gravitation which allows a dynamical theory of torsion, where the torsion field is able to propagate and is non-zero in the absence of matter fields. This is a clear departure from the Einstein-Cartan-Sciama-Kibble theory (which was introduced in Chapter I) in which torsion is not allowed to propagate and is zero in the absence of matter fields.

The question we must ask ourselves now, is whether this analysis can be extended to quite general, non-abelian gauge fields. This is an important issue, for the modern theory of elementary particles views the electromagnetic field, not as an entirely independent gauge field, with the other elementary forces needing additional non-abelian gauge fields, but rather, the electromagnetic field is to be part of a large set of non-abelian gauge fields /1/ . In the next section we will show that indeed, it is possible to generalise the above analysis to non-abelian gauge fields, provided we also simultaneously generalise the non-abelian gauge field strength.

§4, *Coupling of torsion to non-abelian gauge fields.*

Taking the gauge group to be G, with dimG generators T_i satisfying a Lie algebra

$$[T_i,T_j] = iC_{ijk}T_k \qquad (2.4.1)$$

with structure constants C_{ijk}, we have seen in Chapter I that the usual gauge covariant derivative is,

$$D_\mu = \partial_\mu - ig\,(T.A_\mu) \qquad (2.4.2)$$

Let us generalise this to /9/

$$D_\mu = \partial_\mu - ig\, b_\mu{}^\alpha\, (T.A_\alpha) \qquad (2.4.3)$$

where $b_\mu{}^\alpha$ is a function of the space-time and not of the $A_\alpha{}^i$ in total analogy with the generalisation in §3. To find the transformation law for the potential $A_\alpha{}^i$, we require that under

$$\psi \rightarrow U(\varepsilon(x)) \quad , \qquad (2.4.4)$$

$$D_\mu\psi \rightarrow U(\varepsilon(x))\, D_\mu\psi \qquad (2.4.5)$$

i.e.,

$$D_\mu\psi \rightarrow D'_\mu\psi' = U(\varepsilon)\, \{\partial_\mu\psi - ig\, b_\mu{}^\alpha (T.A_\alpha)\psi\} \qquad (2.4.6)$$

or

$$\{\partial_\mu\psi' - ig\, b_\mu{}^\alpha (T.A'_\alpha)\psi'\} = U(\varepsilon)\partial_\mu\psi - ig\, b_\mu{}^\alpha U(\varepsilon)(T.A_\alpha)\psi \qquad (2.4.7)$$

or,

$$(\partial_\mu U)\psi - ig\, b_\mu{}^\alpha (T.A'_\alpha)U\psi = -\, ig\, b_\mu{}^\alpha U(T.A_\alpha)\psi \ , \qquad (2.4.8)$$

and, $$ig\, b_\mu{}^\alpha (T.A'_\alpha)\, U = \partial_\mu U + ig\, b_\mu{}^\alpha U(T.A_\alpha) \qquad (2.4.9)$$

defining $C_\mu{}^\alpha$ to be the inverse of $b_\mu{}^\alpha$,

$$(T.A'_\alpha) = -\frac{i}{g}\, C_\alpha{}^\mu (\partial_\mu U)\, U^{-1} + U(T.A_\alpha)U^{-1} . \qquad (2.4.10)$$

At this stage we must first check to see if the product rule still holds, i.e;

If $$(T.A'_\alpha) = -\frac{i}{g}\, C_\alpha{}^\mu (\partial_\mu U(\varepsilon))\, U^{-1}(\varepsilon) + U(\varepsilon)\ (T.A_\alpha)\ U^{-1}(\varepsilon) \qquad (2.4.11)$$

and

$$(T.A''_\alpha) = -\frac{i}{g}\, C_\alpha{}^\mu (\partial_\mu\, U(\eta))\, U^{-1}(\eta) + U(\eta)(T.A'_\alpha)U^{-1}(\eta) , \qquad (2.4.12)$$

then we must show that there exists a transformation $U(\xi)$ such that

$$(T.A''_{\alpha}) = -\frac{i}{g} C_{\alpha}^{\ \mu}(\partial_{\mu}U(\xi))\, U^{-1}(\xi) + U(\xi)(T.A_{\alpha})U^{-1}(\xi) \tag{2.4.13}$$

with $$U(\xi) = U(\eta)\, U(\varepsilon) \tag{2.4.14}$$

only if such a group property holds, are we allowed to go to an infinitesimal transformation. This property is a manifestation of the Lie algebra that the group satisfies. In appendix II(A), it is proved that this property is indeed satisfied for this modified gauge covariant derivative.

From equation (2.1.3) we see that the field strength tensor for non-abelian gauge fields is

$$F_{\mu\nu}^{\ \ i} = \partial_{\mu} A_{\nu}^{i} - \partial_{\nu} A_{\mu}^{i} + g\, C^{i}_{\ jk}\, A_{\mu}^{j}\, A_{\nu}^{k} . \tag{2.4.15}$$

Coupling this to torsion leads to

$$\tilde{F}_{\mu\nu}^{\ \ i} = \partial_{\mu} A_{\nu}^{i} - \partial_{\nu} A_{\mu}^{i} + g\, C^{i}_{\ jk} A_{\mu}^{j}\, A_{\nu}^{k} + 2\, S_{\nu\mu.}^{\ \ \ \sigma}\, A_{\sigma}^{i} \tag{2.4.16}$$

remembering that

$$[T_j, T_k] = i\, C^{i}_{\ jk} T_i \quad , \tag{2.4.17}$$

we can write $\tilde{F}_{\mu\nu}^{\ \ i}$ as ;

$$T_i \tilde{F}_{\mu\nu}^{\ \ i} = \partial_{\mu}\, T_i A_{\nu}^{i} - \partial_{\nu} T_i A_{\mu}^{i} + g\, C_{jk}^{\ \ i} T_i A_{\mu}^{j}\, A_{\nu}^{k} + 2S_{\nu\mu.}^{\ \ \ \sigma} A_{\sigma}^{i} T_i$$

$$= \partial_{\mu} T.A_{\nu} - \partial_{\nu}\, T.A_{\mu} - ig\, [T_j A_{\mu}^{j}, T_k A_{\nu}^{k}]^{i}\, T_i + 2S_{\nu\mu.}^{\ \ \ \sigma} A_{\sigma} .T . \tag{2.4.18}$$

For the rest of this Chapter, except where otherwise stated, we shall take the convention that all gauge potentials and field strengths have associated with them a generator of the group G although not explicitly written, e.g.

$$\tilde{F}_{\mu\nu} \equiv \tilde{F}_{\mu\nu}^{\ \ i}\, T_i \quad \text{and} \quad A_{\mu} \equiv A_{\mu}^{i}\, T_i \quad . \tag{2.4.19}$$

Then, eqn. (2.4.18) can be written

$$\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] + 2\, S_{\nu\mu}{}^{\sigma} A_\sigma \qquad (2.4.20)$$

Now the question we must answer is, can we by using the transformation law for A_μ given in eqn. (2.4.10), consistently solve for $C_\mu{}^\alpha$ and $S_{\nu\mu}{}^\sigma$ by requiring $\tilde{F}_{\mu\nu}$ to be gauge covariant?

The answer turns out to be / 9/ (see also appendix II(B)) trivial. We find that the modified gauge covariant derivative when used to require gauge covariance of (2.4.20) gives the trivial result that

$$b_\mu{}^\alpha = \delta_\mu{}^\alpha \text{ and } \quad S_{\nu\mu.}{}^\sigma = 0. \qquad (2.4.21)$$

Therefore we cannot couple non-zero torsion through the modified gauge covariant derivative to the unmodified $F_{\mu\nu}$. It is this last statement, that we have used the unmodified $F_{\mu\nu}$ in the case of non-abelian gauge fields that gives us the solution. Let us modify the non-abelian field strength $F_{\mu\nu}$ to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - igB_\mu{}^\alpha B_\nu{}^\beta [A_\alpha, A_\beta] \qquad (2.4.22)$$

and now couple this to torsion, while still retaining the modified covariant derivative. We have introduced the two arbitrary functions

$$b_\mu{}^\alpha = b_\mu{}^\alpha(x) \quad \text{and} \quad B_\mu{}^\alpha = B_\mu{}^\alpha(x)\,. \qquad (2.4.23)$$

Coupling this modified $F_{\mu\nu}$ to torsion, we find

$$\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\, B_\mu{}^\alpha B_\nu{}^\beta\, [A_\alpha, A_\beta] + 2\, S_{\nu\mu.}{}^\sigma A_\sigma\,. \qquad (2.4.24)$$

Under a gauge transformation we suppose

$$\tilde{F}_{\mu\nu} \to \tilde{F}'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig\, B_\mu{}^\alpha B_\nu{}^\beta\, [A'_\alpha, A'_\beta] + 2S_{\nu\mu.}{}^\sigma A'_\sigma\,. \qquad (2.4.25)$$

From equation (2.4.10) we have that under a gauge transformation,

$$A_\mu \to A'_\mu = -\frac{i}{g} C_\mu{}^\alpha (\partial_\alpha U) U^{-1} + U A_\mu U^{-1} \qquad (2.4.26)$$

where, as before, $C_\mu{}^\alpha$ denotes the inverse of $b_\mu{}^\alpha$.

Hence,

$$\begin{aligned}\tilde{F}'_{\mu\nu} = & -\frac{i}{g}\partial_\mu \{C_\nu{}^\alpha(\partial_\alpha U)U^{-1}\} + \partial_\mu\{UA_\nu U^{-1}\} \\ & + \frac{i}{g}\partial_\nu\{C_\mu{}^\alpha(\partial_\alpha U) U^{-1}\} - \partial_\nu\{U A_\mu U^{-1}\} \\ & - ig\, B_\mu{}^\alpha B_\nu{}^\beta[-\frac{i}{g} C_\alpha{}^\sigma(\partial_\sigma U)U^{-1} + U A_\alpha U^{-1}, -\frac{i}{g} C_\beta{}^\rho(\partial_\rho U)U^{-1}+UA_\beta U^{-1}] \\ & - \frac{2i}{g} C_\sigma{}^\alpha(\partial_\alpha U)U^{-1}S_{\nu\mu\cdot}{}^\sigma + U\, 2S_{\nu\mu\cdot}{}^\sigma A_\sigma U^{-1} \qquad (2.4.27)\end{aligned}$$

$$\begin{aligned} = & U\{\partial_\mu A_\nu - \partial_\nu A_\mu - ig\, B_\mu{}^\alpha B_\nu{}^\beta[A_\alpha, A_\beta] + 2S_{\nu\mu.}{}^\sigma A_\sigma\} U^{-1} \\ & + (\partial_\mu U) A_\nu U^{-1} - (\partial_\nu U)A_\mu U^{-1} + UA_\nu(\partial_\mu U^{-1}) - U A_\mu(\partial_\nu U^{-1}) \\ & - \frac{i}{g}(\partial_\mu C_\nu{}^\alpha) (\partial_\alpha U) U^{-1} + \frac{i}{g} (\partial_\nu C_\mu{}^\alpha) (\partial_\alpha U) U^{-1} \\ & - \frac{i}{g} C_\nu{}^\alpha(\partial_\mu\partial_\alpha U) U^{-1} + \frac{i}{g} C_\mu{}^\alpha(\partial_\nu\partial_\alpha U)U^{-1} \\ & - \frac{i}{g} C_\nu{}^\alpha(\partial_\alpha U) (\partial_\mu U^{-1}) + \frac{i}{g} C_\mu{}^\alpha(\partial_\alpha U) (\partial_\nu U^{-1}) \\ & - ig\, B_\mu{}^\alpha B_\nu{}^\beta[-\frac{i}{g} C_\alpha{}^\sigma(\partial_\sigma U) U^{-1}, -\frac{i}{g} C_\beta{}^\rho(\partial_\rho U)U^{-1}] \\ & - ig\, B_\mu{}^\alpha B_\nu{}^\beta [-\frac{i}{g} C_\alpha{}^\sigma(\partial_\sigma U)U^{-1}, U A_\beta U^{-1}] \\ & - ig\, B_\mu{}^\alpha B_\nu{}^\beta [U A_\alpha U^{-1}, -\frac{i}{g} C_\beta{}^\sigma(\partial_\sigma U)U^{-1}] \\ & - \frac{2i}{g} C_\sigma{}^\alpha(\partial_\alpha U) U^{-1} S_{\nu\mu.}{}^\sigma . \qquad (2.4.28)\end{aligned}$$

Gauge covariance of $\tilde{F}_{\mu\nu}$ is simply

$$\tilde{F}'_{\mu\nu} = U \tilde{F}_{\mu\nu} U^{-1}. \qquad (2.4.29)$$

So, requiring gauge covariance of $\tilde{F}_{\mu\nu}$ tells us that

$$(\partial_\sigma U)\{ \delta_\mu{}^\sigma A_\nu - \delta_\nu{}^\sigma A_\mu \}U^{-1} + U\{A_\nu \delta_\mu{}^\sigma - A_\mu \delta_\nu{}^\sigma\} (\partial_\sigma U^{-1})$$

$$- \frac{i}{g} (\partial_\sigma U)\{\partial_\mu C_\nu{}^\sigma - \partial_\nu C_\mu{}^\sigma\} U^{-1} - \frac{i}{g} (\partial_\sigma \partial_\alpha U) \{ C_\nu{}^\alpha \delta_\mu{}^\sigma - C_\mu{}^\alpha \delta_\nu{}^\sigma\}U^{-1}$$

$$+ \frac{i}{g} (\partial_\alpha U)\{ C_\mu{}^\alpha \delta_\nu{}^\sigma - C_\nu{}^\alpha \delta_\mu{}^\sigma\} (\partial_\sigma U^{-1})$$

$$- ig\, B_\mu{}^\alpha B_\nu{}^\beta\{- \frac{1}{g^2} C_\alpha{}^\sigma C_\beta{}^\rho (\partial_\sigma U)U^{-1}(\partial_\rho U)U^{-1} + \frac{1}{g^2} C_\alpha{}^\rho C_\beta{}^\sigma (\partial_\sigma U)U^{-1}(\partial_\rho U)U^{-}\}$$

$$- ig\, B_\mu{}^\alpha B_\nu{}^\beta \{-\frac{i}{g} C_\alpha{}^\sigma (\partial_\sigma U) A_\beta U^{-1} + \frac{i}{g} U A_\beta C_\alpha{}^\sigma U^{-1}(\partial_\sigma U) U^{-1}\}$$

$$- ig\, B_\mu{}^\alpha B_\nu{}^\beta\{- \frac{i}{g} U A_\alpha C_\beta{}^\sigma U^{-1}(\partial_\sigma U) U^{-1} + \frac{i}{g} C_\beta{}^\sigma (\partial_\sigma U) A_\alpha U^{-1}\}$$

$$- \frac{2i}{g} C_\sigma{}^\alpha (\partial_\alpha U) U^{-1} S_{\nu\mu.}{}^\sigma$$

$$= 0 , \qquad (2.4.30)$$

or, remembering that

$$UU^{-1} = U^{-1}U = I , \qquad (2.4.31)$$

implies

$$(\partial_\mu U) U^{-1} = - U(\partial_\mu U^{-1}) \qquad (2.4.32)$$

or

$$U^{-1} (\partial_\mu U) U^{-1} = -\partial_\mu U^{-1} \qquad (2.4.33)$$

we find

$$(\partial_\sigma U)\{\delta_\mu{}^\sigma A_\nu - \delta_\nu{}^\sigma A_\mu - \frac{i}{g}\partial_\mu C_\nu{}^\sigma + \frac{i}{g}\partial_\nu C_\mu{}^\sigma - B_\mu{}^\alpha B_\nu{}^\beta C_\alpha{}^\sigma A_\beta$$

$$+ B_\mu{}^\alpha B_\nu{}^\beta C_\beta{}^\sigma A_\alpha - \frac{2i}{g} C_\alpha{}^\sigma S_{\nu\mu.}{}^\alpha \} U^{-1}$$

$$+ U\{ A_\nu \delta_\mu{}^\sigma - A_\mu \delta_\nu{}^\sigma - B_\mu{}^\alpha B_\nu{}^\beta A_\beta C_\alpha{}^\sigma + B_\mu{}^\alpha B_\nu{}^\beta A_\alpha C_\beta{}^\sigma\}(\partial_\sigma U^{-1})$$

$$- \frac{i}{g} (\partial_\sigma \partial_\alpha U)\{ C_\nu{}^\alpha \delta_\mu{}^\sigma - C_\mu{}^\alpha \delta_\nu{}^\sigma \} U^{-1}$$

$$+ \frac{i}{g}(\partial_\sigma U)\{ C_\mu{}^\sigma \delta_\nu{}^\rho - C_\nu{}^\sigma \delta_\mu{}^\rho - B_\mu{}^\alpha B_\nu{}^\beta C_\alpha{}^\sigma C_\beta^\rho + B_\mu{}^\alpha B_\nu{}^\beta C_\beta{}^\sigma C_\alpha{}^\rho\}(\partial_\rho U^{-1})$$

$$= 0. \qquad (2.4.34)$$

So, the solution we are seeking is a simultaneous solution to the following set of four equations :

$$C_\nu^{(\alpha}\delta_\mu{}^{\sigma)} - C_\mu^{(\alpha}\ \delta_\nu{}^{\sigma)} = 0 \qquad (2.4.35)$$

$$C_\mu{}^\sigma \delta_\nu{}^\rho - C_\nu{}^\sigma \delta_\mu{}^\rho + B_\mu{}^\alpha B_\nu{}^\beta C_\alpha{}^\rho C_\beta{}^\sigma - B_\mu{}^\alpha B_\nu{}^\beta C_\alpha{}^\sigma C_\beta{}^\rho = 0 \qquad (2.4.36)$$

$$A_\nu \delta_\mu{}^\sigma - A_\mu \delta_\nu{}^\sigma + B_\mu{}^\alpha B_\nu{}^\beta A_\alpha C_\beta{}^\sigma - B_\mu{}^\alpha\ B_\nu{}^\beta A_\beta C_\alpha{}^\sigma = 0 \qquad (2.4.37)$$

and

$$\delta_\mu^\sigma A_\nu - \delta_\nu{}^\sigma A_\mu + B_\mu{}^\alpha B_\nu{}^\beta C_\beta{}^\sigma A_\alpha - B_\mu{}^\alpha\ B_\nu{}^\beta C_\alpha{}^\sigma A_\beta$$
$$+ \frac{i}{g}\partial_\nu C_\mu{}^\sigma - \frac{i}{g}\partial_\mu C_\nu{}^\sigma - \frac{2i}{g} C_\alpha{}^\sigma S_{\nu\mu}{}^\alpha = 0. \qquad (2.4.38)$$

The first of these, equations (2.4.35) is identical to that obtained in the electrodynamic case, and its solution is

$$C_\mu{}^\alpha = f(x)\delta_\mu{}^\alpha \qquad (2.4.39)$$

where f(x) is an arbitrary function of the space-time. Substituting this solution into equation (2.4.36) yields;

$$f(x)\delta_\mu{}^\sigma\delta_\nu{}^\rho - f(x)\delta_\nu{}^\sigma\delta_\mu{}^\rho + f^2(x)\ B_\mu{}^\rho B_\nu{}^\sigma - f^2(x)\ B_\mu{}^\sigma B_\nu{}^\rho = 0$$

or

$$(2.4.40)$$

$$f(x)\ \delta_\mu{}^{[\sigma}\ \delta_\nu{}^{\rho]} - f^2(x) B_\mu{}^{[\sigma} B_\nu{}^{\rho]} = 0 \qquad (2.4.41)$$

where the square brackets denote anti-symmetrisation. So we have the reduced equation

$$f(x)\ \delta_\mu{}^\sigma \delta_\nu{}^\rho - f^2(x)\ \ B_\mu{}^\sigma B_\nu{}^\rho = 0, \qquad (3.4.42)$$

tracing over indices ν and ρ

$$4\delta_\mu{}^\sigma - f(x) B_\mu{}^\sigma B_\nu{}^\nu = 0 \ . \qquad (2.4.43)$$

Taking a further trace over μ and σ gives

$$B_\mu{}^\mu B_\nu{}^\nu = \frac{16}{f(x)} \qquad (2.4.44)$$

or,

$$B_\nu{}^\nu = \pm \frac{4}{\sqrt{f(x)}} \ . \qquad (2.4.45)$$

Substitution of this back into equation (2.4.42) finally allows us to write

$$B_\mu{}^\alpha = \pm \frac{1}{\sqrt{f(x)}} \ \delta_\mu{}^\alpha \ . \qquad (2.4.46)$$

Equation (2.4.37), after using the solutions for $C_\mu{}^\alpha$ and $B_\mu{}^\alpha$ is

$$A_\nu \delta_\mu{}^\sigma - A_\mu \delta_\nu{}^\sigma + A_\mu \delta_\nu{}^\sigma - A_\nu \delta_\mu{}^\sigma = 0 \qquad (2.4.47)$$

i.e., an identity.

While equation (2.4.38) is

$$A_\nu \delta_\mu{}^\sigma - A_\mu \delta_\nu{}^\sigma + A_\mu \delta_\nu{}^\sigma - A_\nu \delta_\mu{}^\sigma + \frac{i}{g}\delta_\mu{}^\sigma \partial_\nu f(x) - \frac{i}{g}\delta_\nu{}^\sigma \partial_\mu f(x)$$
$$- \frac{2i}{g} S_{\nu\mu.}{}^\sigma \ f(x) = 0, \qquad (2.4.48)$$

or,

$$2\, S_{\nu\mu.}{}^\sigma = \delta_\mu{}^\sigma \partial_\nu \ln f(x) - \delta_\nu{}^\sigma \partial_\mu \ln f(x) \ . \qquad (2.4.49)$$

Once again, discarding the singular solution ($f(x) = 0$), and requiring that in the limit $S_{\mu\nu}{}^\sigma \to 0$, we have $b_\mu{}^\alpha \to \delta_\mu{}^\alpha$ and $B_\mu{}^\alpha \to \delta_\mu{}^\alpha$ in order for the modified gauge covariant derivative and the modified gauge field strength tensor given in equations (2.4.10) and (2.4.22) respectively reduce to their unmodified counterparts, tells us that $f(x)$

is an everywhere non-zero, positive valued function. Therefore, we parameterise it in the form

$$f(x) = e^{\psi(x)} . \qquad (2.4.50)$$

In terms of $\psi(x)$, we can write the modified gauge field strength as,

$$\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig \; e^{-\psi(x)} [A_\mu , A_\nu] + 2 \, S_{\nu\mu.}{}^{\sigma} A_\sigma \qquad (2.4.51)$$

with

$$S_{\nu\mu}{}^{\sigma} = \delta_\mu{}^\sigma \partial_\nu \psi(x) - \delta_\nu{}^\sigma \partial_\mu \psi(x). \qquad (2.4.52)$$

Thus, we have accomplished what we started out to do. The remarkable fact is that the special type of torsion that was found in §3 in coupling to electrodynamics is carried through to the non-abelian case. Torsion is still determined by a scalar field, $\psi(x)$ acting as a potential for the torsion tensor field / 9 / .

This is easily understood, if one remembers that the gauge potentials A_μ are simply connection symbols on a principle bundle with its structure group being the gauge group. While the field strength tensor is nothing but the curvature tensor of these connection symbols. In generalising the gauge covariant derivative, one may imagine that the gauge coupling parameter has been allowed to become a function on the space-time. If one evaluates the curvature tensor of such a covariant derivative, some additional terms containing partial derivatives of the logarithm of the coupling function arise. In coupling to torsion, we equate these additional terms with the torsion tensor, and hence the torsion tensor is always of the form (2.4.52).

PART II

VARIATIONAL PRINCIPLES AND LAGRANGIANS

"Whenever any action occurs in nature, the quantity of action employed by this change is the least possible"

Pierre Moreau de Maupertuis (1746)

CHAPTER III

INVARIANT VARIATIONAL PRINCIPLES

§1. *Invariant Variational Principles*

At the time Einstein proposed field equations for his theory of general relativity /10/ , Hilbert was preoccupied with an axiomatisation of physics, having declared that "physics is much too difficult for the physicists". It was this that led him to propose an elegant derivation of Einstein's field equations through variational principles. Through this work too, he stimulated the work of Klein and Nöther. Klein's work culminated in an extensive study of "the differential laws for the conservation of momentum and energy in Einstein's theory of gravitation" using the theory of invariants, while Nöther's work led to her well known theorems relating continuous symmetries of the classical equations of motion to conservation laws for the lagrangian from which they are derived.

The variational principle itself consists of writing down an action integral for a lagrangian that depends on the field quantities one is dealing with :

$$I = \int L \, d^4x \tag{3.1.1}$$

The variational integral is said to be of the nth order whenever the integrand depends on partial derivatives of at least some of the field functions with respect to the space-time coordinates, upto and including the nth order.

One then seeks field equations for the dependent quantities, for the solutions of which, the action integral assumes extreme values.

However, in this classical treatment, it is usually assumed that the dependent field functions are unaffected by coordinate transformations of the type

$$\bar{x}^j = \bar{x}^j(x^i) \tag{3.1.2}$$

under which the action integral I is taken to be invariant.

In many physical applications, this assumption is not justified,for instance, the classical electromagnetic field is described by a *vector field* satisfying field equations derivable from a variational principle.

In the study of invariant variational principles /11/, in addition to requiring the field functions to transform according to their tensorial/spinorial character, we impose a further condition on the action integral by requiring it to be invariant under the transformation (3.1.2). This invariance implies that L must be a scalar density of weight 1. if we denote our field functions as ψ with indices suppressed, then we must have

$$\bar{L}\ (\bar{x}^j,\ \bar{\psi}, \partial_j\ \bar{\psi}, \partial_k \partial_j\ \bar{\psi}) = B\ \ L(x^j, \psi, \partial_j \psi, \partial_k \partial_j \psi) \tag{3.1.3}$$

under the coordinate transformation (3.1.2).

Where L has been taken to be of the 2nd order, and

$$B = \left| \frac{\partial x^j}{\partial \bar{x}^i} \right| . \tag{3.1.4}$$

We shall also assume that L satisfies the appropriate Euler-Lagrange equations and that these equations are identical to the field equations satisfied by the dependent functions. Clearly these two invariance requirements may be expected to impose severe restrictions on the form of L and on the field equations that L satisfies, i.e. the Euler-Lagrange equations. In the next section, we shall show that this is indeed the case for Einstein's theory by taking a 2nd order lagrangian in the metric. We shall see that one has to consider 2nd order lagrangians, as one of the restrictions on L will be that one cannot have a non-zero first order L . In addition we shall derive three identities for L/11/.

§2. *IVP's in general relativity.*

Einstein's theory is based on a four-dimensional Riemannian manifold endowed with a symmetric, covariant metric tensor of the second rank, and having signature (+,-,-,-) . The lagrangian is chosen through physical arguments to be linear in the second derivatives of g_{ij} and taken to contain no higher than second derivatives of g_{ij} . For our purposes, we shall consider a lagrangian

$$L = L(g_{ij}, g_{ij,k}, g_{ij,k\ell}) \qquad (3.2.1)$$

where a comma denotes partial derivatives, and a semi-colon denotes covariant derivatives with respect to the symmetric Christoffel connection.

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} \; ; \; g_{ij,k\ell} = \frac{\partial^2 g_{ij}}{\partial x^k \partial x^\ell} \qquad (3.2.2)$$

We shall also assume that $g_{ij;k} = 0$, so that we have a local Minkowski structure . We have seen in the last section, that the requirement of invariance under

$$\bar{x}^j = \bar{x}^j(x^i) \qquad (3.2.3)$$

of

$$I = \int d^4x\, L \qquad (3.2.4)$$

tells us that

$$\bar{L}(\bar{g}_{ij}, \bar{g}_{ij,k}, \bar{g}_{ij,k\ell}) = \left| \frac{\partial x^j}{\partial \bar{x}^i} \right| L(g_{ij}, g_{ij,k}, g_{ij,k\ell}) \qquad (3.2.5)$$

Remembering that g_{ij} is a 2nd rank covariant tensor field, we have, under (3.2.3),

$$\bar{g}_{ij} = B^a{}_i\, B^b{}_j\, g_{ab} \qquad (3.2.6)$$

where

$$B^a_i = \frac{\partial x^a}{\partial \bar{x}^i} \quad , \quad B^a_{ij} = \frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j} \quad , \text{ etc.} \tag{3.2.7}$$

Similarly for $\overline{g_{ij,k}}$ and $\overline{g_{ij,k\ell}}$.

From equation (3.2.5) we notice that the right hand side is independent of $B^i_{h\ell m}$, so we have

$$\frac{\partial \bar{L}}{\partial B^i_{h\ell m}} = 0 \tag{3.2.8}$$

or,

$$\frac{\partial \bar{L}}{\partial \bar{g}_{hk}} \frac{\partial \bar{g}_{hk}}{\partial B^i_{npq}} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell}} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{npq}} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell m}} \frac{\partial \overline{g_{hk,\ell m}}}{\partial B^i_{npq}} = 0 \; . \tag{3.2.9}$$

Let us define

$$\Lambda^{hk} = \frac{\partial L}{\partial g_{hk}} \quad , \quad \Lambda^{hk,\ell} = \frac{\partial L}{\partial g_{hk,\ell}} \quad \text{and} \quad \Lambda^{hk,\ell m} = \frac{\partial L}{\partial g_{hk,\ell m}} \; . \tag{3.2.10}$$

So we can write equation (3.2.9) in the form;

$$\bar{\Lambda}^{hk} \frac{\partial \bar{g}_{hk}}{\partial B^i_{npq}} + \bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{npq}} + \bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{npq}} = 0 \; . \tag{3.2.11}$$

In Appendix III (B), it is shown that this reduces to

$$\bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{npq}} = 0 \tag{3.2.12}$$

because of the transformation laws for $\bar{g}_{hk,\ell}$ and $\bar{g}_{hk}$.

We also show there that a further reduction in equation (3.2.12) leads to,

$$\bar{\Lambda}^{kn,pq} + \bar{\Lambda}^{kp,qn} + \bar{\Lambda}^{kq,pn} = 0, \qquad (3.2.13)$$

which, by noting that $\Lambda^{ij,k\ell} = \Lambda^{ji,k\ell} = \Lambda^{ij,\ell k}$, can be used to yield the required identity in the form,

$$\bar{\Lambda}^{hk,\ell m} = \bar{\Lambda}^{\ell m,hk} \quad . \qquad (3.2.14)$$

From Appendix III (A), the transformation law for $\bar{\Lambda}^{hk,\ell m}$ is that of a tensor density of weight 1 and contravariant rank 4, so we can remove the "bars" on equation (3.2.14) and write the first identity, satisfied by L as

$$\Lambda^{hk,\ell m} = \Lambda^{\ell m,hk} \quad . \qquad (3.2.15)$$

§2.1 *The second identity.*

The right hand side of equation (3.2.5) is also independent of $B^i{}_{pq}$ so we have

$$\frac{\partial \bar{L}}{\partial B^i{}_{pq}} = 0 \qquad (3.2.16)$$

or,

$$\frac{\partial \bar{L}}{\partial \bar{g}_{hk}} \frac{\partial \bar{g}_{hk}}{\partial B^i{}_{pq}} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell}} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i{}_{pq}} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell m}} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i{}_{pq}} = 0 \qquad (3.2.17)$$

As before,

$$\bar{\Lambda}^{hk} \frac{\partial \bar{g}_{hk}}{\partial B^i{}_{pq}} + \bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i{}_{pq}} + \bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i{}_{pq}} = 0 \qquad (3.2.18)$$

At this point, we notice that although $\Lambda^{hk,\ell m}$ is a tensor density, the same is not true of Λ^{hk} and $\Lambda^{hk,\ell}$ (c.f. Appendix III(A)).
Hence, in order to write our second and third identities in a tensorial fashion, we must construct tensorial analogues of Λ^{hk} and $\Lambda^{hk,\ell}$.

This is done in Appendix III (A), and we denote the corresponding tensor densities by Π^{hk} and $\Pi^{hk,\ell}$ respectively.

In Appendix III(B), we use the following two simplifications in deriving the identities.

(i) Because equation (3.2.5) holds for coordinate transformations of the form (3.2.6), it is true, in particular, that (3.2.5) holds for transformations of the form;

$$\bar{x}^i = x^i \quad , \text{ i.e. } B^i_{\ j} = \delta^i_{\ j} \quad \text{and} \quad B^i_{\ jk} = 0, \text{ etc.}$$

(ii) Because our metric satisfies $g_{ij;k} = 0$, we can choose our local coordinates to be such that the Christoffel connection symbols vanish, i.e. that our coordinates be gaussian normal coordinates. Returning to equation (3.2.18), we see that it can be reduced to

$$\bar{\Pi}^{pk,q} + \bar{\Pi}^{qk,p} = 0. \tag{3.2.19}$$

As before, this is a tensorial equation and therefore if it holds in one frame, it holds true in any frame. Hence,

$$\Pi^{pk,q} + \Pi^{qk,p} = 0 \ . \tag{3.2.20}$$

In Appendix III(B), we show that (3.2.20) implies

$$\Pi^{pq,k} = 0 \tag{3.2.21}$$

remembering that $\Pi^{pk,q}$ is symmetric in indices (pk) . This is a most remarkable identity. Suppose for the moment that L depends solely on the g_{hk} and their first derivatives. Then, by equation (3.2.10) we have

$\Lambda^{hk,\ell m} = 0$. According to the definition of $\Pi^{pq,k}$ (c.f. Appendix III(A)),

$$\Pi^{hk,\ell} = \Lambda^{hk,\ell} + \Gamma^{\ell}_{jm} \Lambda^{hk,jm} + 2\Gamma^{k}_{jm} \Lambda^{hj,\ell m} + 2\Gamma^{h}_{jm} \Lambda^{kj,\ell m} , \qquad (3.2.22)$$

so the quantities $\Pi^{hk,\ell}$ reduce to the partial derivatives $\Lambda^{hk,\ell}$, which now vanish by (3.2.21). We therefore have :

There does not exist a scalar density $L(g_{hk}, g_{hk,\ell})$ *depending on the* g_{hk} *and their first derivatives only.*

Because of this amazing identity, we are forced to consider second order variational problems when faced with a field function which is a symmetric metric tensor.

§2.2 *The third identity.*

For the third and final identity, we return to equation (3.2.5) and differentiate it with respect to the quantities B^i_j. Remembering that if A^i_j denotes the inverse of B^i_j,

$$A^i_j \, B^j_k = \delta^i_k , \qquad (3.2.23)$$

then

$$\frac{\partial B}{\partial B^i_j} = B \, A^j_i , \qquad (3.2.24)$$

we find,

$$BL\,A^j_i = \frac{\partial \bar{L}}{\partial \bar{g}_{hk}} \frac{\partial \bar{g}_{hk}}{\partial B^i_j} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell}} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_j} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell m}} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_j} \qquad (3.2.25)$$

or,

$$B\,LA^j_i = \bar{\Lambda}^{hk} \frac{\partial \bar{g}_{hk}}{\partial B^i_j} + \bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_j} + \bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_j}$$

(3.2.26)

After a large amount of tedious calculations, this equation is reduced in Appendix III (B) to ;

$$\frac{1}{2} L \delta^j{}_i = \Lambda^{jk} g_{ik} + \frac{4}{3} R_{iqp\ell}(\{\}) \Lambda^{jp,q\ell} \tag{3.2.27}$$

or in terms of tensorial quantities,

$$\frac{1}{2} L \delta^j{}_i = \Pi^{jk} g_{ik} + \frac{2}{3} R_{ikm\ell}(\{\}) \Lambda^{jm,k\ell} \quad . \tag{3.2.28}$$

This is the required third identity, where $R_{ikm\ell}(\{\})$ is the Riemann-Christoffel curvature tensor.

§2.3. *Reduction of Euler-Lagrange equations for* L*(g,∂g,∂∂g).*

We have just the one field function, g_{ij} , and the Euler-Lagrange equations for

$$L = L(g_{ij}, g_{ij,k}, g_{ij,k\ell}) \tag{3.2.29}$$

read

$$E^{ij}(L) = \frac{d}{dx^k}\left\{\frac{\partial L}{\partial g_{ij,k}} - \frac{d}{dx^\ell}\frac{\partial L}{\partial g_{ij,k\ell}}\right\} - \frac{\partial L}{\partial g_{ij}} = 0 \tag{3.2.30}$$

or,

$$E^{ij}(L) = \Lambda^{ij,k}{}_{,k} - \Lambda^{ij,k\ell}{}_{,\ell,k} - \Lambda^{ij} = 0 \tag{3.2.31}$$

The relations

$$G = \Lambda^{ij,k\ell} h_{ij;k;\ell} + \Pi^{ij,k} h_{ij;k} + \Pi^{ij} h_{ij} \tag{3.2.32}$$

and

$$G = \Lambda^{ij,k\ell} h_{ij,k,\ell} + \Lambda^{ij,k} h_{ij,k} + \Lambda^{ij} h_{ij} \tag{3.2.33}$$

from Appendix III (A) can be used to derive some surprising results for (3.2.31);

We first note that

$$h_{ij,k}\,\Lambda^{ij,k} = (h_{ij}\,\Lambda^{ij,k})_{,k} = h_{ij}\,\Lambda^{ij,k}{}_{,k} \tag{3.2.34}$$

and

$$h_{ij,k\ell}\,\Lambda^{ij,k\ell} = (h_{ij,k}\,\Lambda^{ij,k\ell})_{,\ell} - h_{ij,k}\,\Lambda^{ij,k\ell}{}_{,\ell}$$

$$= (h_{ij,\ell}\Lambda^{ij,k\ell})_{,k} - (h_{ij}\Lambda^{ij,k\ell}{}_{,\ell})_{,k} + h_{ij}\Lambda^{ij,k\ell}{}_{,\ell k}$$

(3.2.35)

so that (3.2.33) is expressible as

$$G = -h_{ij}\,E^{ij}(L) + [h_{ij}\,\Lambda^{ij,k} + h_{ij,\ell}\Lambda^{ij,k\ell} - h_{ij}\,\Lambda^{ij,k\ell}{}_{,\ell}]_{,k}\ . \tag{3.2.36}$$

We can repeat the calculations, by replacing the partial derivatives by covariant derivatives and the Λ's by their tensorial counterparts, allowing us to rewrite (3.2.32) as

$$G = -h_{ij}\,\{-\Lambda^{ij,k\ell}{}_{;k;\ell} + \Pi^{ij,k}{}_{;k} - \Pi^{ij}\} + [h_{ij}\Pi^{ij,k} + h_{ij;\ell}\Lambda^{ij,k\ell}$$

$$- h_{ij}\,\Lambda^{ij,k\ell}{}_{;\ell}]_{;k}\ . \tag{3.2.37}$$

The quantities in square brackets are clearly the components of a tensor density of weight 1 and contravariant rank 1. Now, by the rules of tensor calculus, the divergence of a contravariant vector density is an invariant density, therefore we can replace the covariant derivative of the square bracket (with respect to x^k), by a partial derivative.

Having done this, we shall now demonstrate that the quantities in square brackets in equations (3.2.36) and (3.2.37) are equivalent. To see this, we have

$$h_{ij}\,\Pi^{ij,k} + h_{ij;\ell}\Lambda^{ij,k\ell} - h_{ij}\,\Lambda^{ij,k\ell}{}_{;\ell}$$

$$= h_{ij}\{\Lambda^{ij,k} + \Lambda^{\alpha j,k\ell}\Gamma^{i}_{\ell\alpha} + 2\Lambda^{\alpha i,k\ell}\Gamma^{j}_{\ell\alpha} + \Lambda^{ij,\ell\alpha}\Gamma^{k}_{\ell\alpha}\}$$

$$+ (h_{ij,\ell}\Lambda^{ij,k\ell} - \Lambda^{ij,k\ell}\Gamma^{\alpha}_{\ell i}\,h_{\alpha j} - \Lambda^{ij,k\ell}\Gamma^{\alpha}_{\ell j}\,h_{i\alpha})$$

$$- h_{ij}(\Lambda^{ij,k\ell}{}_{,\ell} + \Gamma^{i}_{\ell\alpha}\Lambda^{\alpha j,k\ell} + \Gamma^{j}_{\ell\alpha}\Lambda^{i\alpha,k\ell} + \Gamma^{k}_{\ell\alpha}\,\Lambda^{ij,\alpha\ell})\ . \qquad (3.2.38)$$

Using the symmetry of h_{ij} ,

$$h_{ij}\,\Pi^{ij,k} + h_{ij;\ell}\Lambda^{ij,k\ell} - h_{ij}\,\Lambda^{ij,k\ell}{}_{;\ell}$$

$$= h_{ij}\{\Lambda^{ij,k} + 4\ \Gamma^{i}_{\ell\alpha}\Lambda^{\alpha j,k\ell} + \Lambda^{ij,\ell\alpha}\Gamma^{k}_{\ell\alpha} - 2\Gamma^{\ i}_{\ell\alpha}\,\Lambda^{\alpha j,k\ell}$$

$$- \Lambda^{ij,k\ell}{}_{,\ell} - 2\Gamma^{i}_{\ell\alpha}\Lambda^{\alpha j,k\ell} - \Gamma^{k}_{\ell\alpha}\,\Lambda^{ij,\ell\alpha}\} + h_{ij,\ell}\Lambda^{ij,k\ell}$$

$$= h_{ij}\,\Lambda^{ij,k} + h_{ij,\ell}\Lambda^{ij,k\ell} - h_{ij}\,\Lambda^{ij,k\ell}{}_{,\ell}\ . \qquad (3.2.39)$$

So we have, by subtracting equation (3.2.36) from (3.2.37) that

$$h_{ij}\,\{E^{ij}(L) - (-\Lambda^{ij,k\ell}{}_{;k;\ell} + \Pi^{\ ij,k}{}_{;k} - \Pi^{ij})\} = 0 \qquad (3.2.40)$$

and since h_{ij} is an arbitrary symmetric tensor, we have the identity

$$E^{ij}(L) = -\Pi^{ij} + \Pi^{ij,k}{}_{;k} - \Lambda^{ij,k\ell}{}_{;k;\ell}\ . \qquad (3.2.41)$$

This clearly shows that the $E^{ij}(L\ )$ are the components of a symmetric tensor density of weight 1 and contravariant rank 2. We further have, from the second identity, (3.2.21) that

$$E^{ij}(L) = -\Pi^{ij} - \Lambda^{ij,k\ell}{}_{;k;\ell}, \tag{3.2.42}$$

while from the third identity, in equation (3.2.28),

$$E^{ij}(L) = -\frac{1}{2}Lg^{ij} + \frac{2}{3}R^{j}{}_{.k\ell m}(\{\})\,\Lambda^{i\ell,km} - \Lambda^{ij,k\ell}{}_{;k;\ell}\,. \tag{3.2.43}$$

So we have the remarkable result, that in order to obtain the field equations for a second order variational problem, one simply needs to evaluate $\Lambda^{ij,k\ell}$.

Indeed, from (3.2.42) and (3.2.43) it is a trivial matter to show that

$$E^{ij}(L)_{;j} = 0, \tag{3.2.44}$$

i.e., the Euler-Lagrange expressions of *any* such lagrangian density are divergence free.

This result tells us that *any* gravitational theory based on a lagrangian of the form $L(g,\partial g,\partial\partial g)$, with g_{ij} being a symmetric, second rank, metric tensor will lead to what we might term automatic conservation laws. That the energy-momentum tensor is divergence free - hence conserved - because of geometrical identities. In the next section, we shall generalise the above formalism to metric-torsion theories, and derive three identities, for a second order lagrangian in the metric and torsion containing no higher than first derivatives in the torsion tensor.

§3. *IVP's in metric-torsion theories of gravity.*

In this section, we shall generalise the previous procedure of using invariant principles to the case of Riemann-Cartan geometry. We shall take the field functions to be the symmetric metric tensor g_{ij} and the torsion tensor $S_{ij}{}^{k}$ defined as

$$S_{ij.}{}^{k} = \frac{1}{2}(\Gamma^{k}{}_{ij} - \Gamma^{k}{}_{ji}), \tag{3.3.1}$$

where $\Gamma_{ij}^{\cdot\cdot k}$ is an asymmetric connection defined on the space-time. Since we have seen in the last section that whenever we need to deal with a variational problem involving a symmetric metric tensor, we must go to a second order problem, we shall take the lagrangian in this example to be of the form:

$$L = L(g_{ij}, g_{ij,k}, g_{ij,k\ell}, S_{ijk}, S_{ijk,\ell}) . \tag{3.3.2}$$

There is no particular reason for including the first derivatives of $S_{ijk}^{\cdot\cdot}$ in L. We cannot expect a result stating that "there does not exist a scalar density that depends on the metric, its first two derivatives and the torsion", because a counter example exists in the ECSK lagrangian itself. Note that after the removal of total divergence terms from the ECSK lagrangian, we are left with a (non-zero) lagrangian of the form $L(g,\partial g,\partial\partial g,S)$ (cf. Chapter I).

Requiring invariance of the action integral

$$I = \int L \, d^4x \tag{3.3.3}$$

under coordinate transformations of the form,

$$\bar{x}^i = \bar{x}^i (x^j), \tag{3.3.4}$$

we note that L must be a scalar density, i.e.,

$$\bar{L}(\bar{g}_{ij}, \bar{g}_{ij,k}, \bar{g}_{ij,k\ell}, \bar{S}_{ijk}, \bar{S}_{ijk,\ell}) = B\,L(g_{ij}, g_{ij,k}, g_{ij,k\ell}, S_{ijk}, S_{ijk,\ell}) \tag{3.3.5}$$

where, as before, $B = \left|\frac{\partial x^i}{\partial \bar{x}^j}\right|$ is the jacobian of the transformation (3.3.4).

In this section, a comma will denote partial derivatives as before, while a semi-colon is used to denote covariant differentiation with respect to the asymmetric connection Γ^i_{jk}.

Now, since S_{ijk} is a tensor,

$$\bar{S}_{ijk} = B^a_{\ i} B^b_{\ j} B^c_{\ k} S_{abc} \qquad (3.3.6)$$

and

$$\bar{S}_{ijk,\ell} = B^a_{\ i\ell} B^b_{\ j} B^c_{\ k} S_{abc} + B^a_{\ i} B^b_{\ j\ell} B^c_{\ k} S_{abc}$$

$$+ B^a_{\ i} B^b_{\ j} B^c_{\ k\ell} S_{abc} + B^a_{\ i} B^b_{\ j} B^c_{\ k} S_{abc,d} B^d_{\ \ell} \qquad (3.3.7)$$

where,

$$B^a_{\ i} = \frac{\partial x^a}{\partial \bar{x}^i} \quad , \quad B^a_{\ ij} = \frac{\partial}{\partial \bar{x}^j} B^a_{\ i} \text{ , etc.}$$

We shall need the following definitions

$$\Lambda^{ij} = \frac{\partial L}{\partial g_{ij}} \quad , \quad \Lambda^{ij,k} = \frac{\partial L}{\partial g_{ij,k}} \quad , \quad \Lambda^{ij,k\ell} = \frac{\partial L}{\partial g_{ij,k\ell}}$$

and

$$M^{ijk} = \frac{\partial L}{\partial S_{ijk}} \quad , \quad M^{ijk,\ell} = \frac{\partial L}{\partial S_{ijk,\ell}} \quad . \qquad (3.3.8)$$

From the work in §2, we know that $\Lambda^{ij,k\ell}$ is a tensor density, and we define Π^{ij} and $\Pi^{ij,k}$ to be the tensorial counterparts of Λ^{ij} and $\Lambda^{ij,k}$ respectively.

Before going on to derive the identities for L , we shall demonstrate that $M^{ijk,\ell}$ is a tensor density and then construct a tensorial counterpart for M^{ijk}, which we shall denote by N^{ijk}. To show that $M^{ijk,\ell}$ is a tensor density, we simply construct its transformation law under (3.3.4).

Now,

$$\frac{\partial \bar{L}}{\partial S_{abc,d}} = \frac{\partial \bar{L}}{\partial \bar{S}_{ijk,\ell}} \frac{\partial \bar{S}_{ijk,\ell}}{\partial S_{abc,d}} \qquad (3.3.9)$$

while from (3.3.7) we have

$$\frac{\partial \bar{S}_{ijk,\ell}}{\partial S_{abc,d}} = B^a{}_i \, B^b{}_j \, B^c{}_k \, B^d_\ell \tag{3.3.10}$$

so that,

$$\frac{\partial \bar{L}}{\partial S_{abc,d}} = \bar{M}^{ijk,\ell} \; B^a{}_i \, B^b{}_j \quad B^c{}_k \, B^d_\ell \quad . \tag{3.3.11}$$

Finally, from (3.3.5), we find

$$\frac{\partial \bar{L}}{\partial S_{abc,d}} = B \frac{\partial L}{\partial S_{abc,d}} = B \; M^{abc,d} \quad . \tag{3.3.12}$$

Therefore, we have, using $B^i{}_j \, A^j{}_k = \delta^i{}_k$, (3.3.13)

$$\bar{M}^{ijk,\ell} = B \, A^i{}_a \, A^j{}_b \, A^k{}_c \, A^\ell{}_d \, M^{abc,d} \tag{3.3.14}$$

the transformation law for a tensor density of weight 1 and contravariant rank 4.

To construct a tensorial counterpart for M^{ijk}, we shall use the same indirect method used in Appendix III (A) to construct Π^{ik} and $\Pi^{ik,\ell}$. Let Q_{ijk} be a totally arbitrary tensor field having the same symmetries as the torsion tensor S_{ijk}.

We define

$$G = M^{ijk} \, Q_{ijk} + M^{ijk,\ell} Q_{ijk,\ell} \quad . \tag{3.3.15}$$

It is an easy task to show that G is a scalar density,

$$\bar{G} = BG \; . \tag{3.3.16}$$

We must now find quantities N^{ijk} such that G can be expressed as

$$G = N^{ijk} \, Q_{ijk} + M^{ijk,\ell} Q_{ijk;\ell} \quad . \tag{3.3.17}$$

We have,

$$Q_{ijk;\ell} = Q_{ijk,\ell} - \Gamma_{\ell i}^{\alpha} Q_{\alpha jk} - \Gamma_{\ell j}^{\alpha} Q_{i\alpha k} - \Gamma_{\ell\ k}^{\ \alpha} Q_{ij\alpha} \tag{3.3.18}$$

so that

$$G = (N^{abc} - \Gamma_{\ell\ i}^{\ a} M^{ibc,\ell} - \Gamma_{\ell j}^{b} M^{ajc,\ell} - \Gamma_{\ell\ k}^{\ c} M^{abk,\ell}) Q_{abc}$$

$$+ \ M^{ijk,\ell} \ Q_{ijk,\ell} \ . \tag{3.3.19}$$

Identifying suitably symmetrised coefficients of Q_{abc} and $Q_{ijk,\ell}$ with those in (3.3.15), we have

$$N^{ijk} = M^{ijk} + \Gamma_{\ell a}^{i} M^{ajk,\ell} + \Gamma_{\ell a}^{j} M^{iak,\ell} + \Gamma_{\ell a}^{k} M^{ija,\ell} \tag{3.3.20}$$

and $$M^{ijk,\ell} = M^{ijk,\ell} \ . \tag{3.3.21}$$

Confirming our proof above that $M^{ijk,\ell}$ is a tensorial quantity. To demonstrate that N^{ijk} is indeed a tensor density, we note from (3.3.16) and (3.3.14) that

$$(G - M^{ijk,\ell} Q_{ijk;\ell}) \tag{3.3.22}$$

is a scalar density, hence it follows that

$$N^{ijk} Q_{ijk} \tag{3.3.23}$$

is a scalar density.

But our assumption was that Q_{ijk} was a third rank tensor with the same symmetries as the torsion tensor, S_{ijk}.

By the quotient theorem of tensor calculus, we therefore have the result that N^{ijk} is a tensor density of weight 1 and of contravariant rank 3.

We also have the a posteriori result that M^{ijk} is not a tensor density.

Having constructed all the required tensorial quantities, we can go on to derive the first identity.

We have the right hand side of (3.3.5) being independent of $B^i{}_{jk\ell}$, so that

$$\frac{\partial\bar{L}}{\partial B^i{}_{jk\ell}} = 0, \tag{3.3.24}$$

or,

$$\frac{\partial\bar{L}}{\partial\bar{g}_{ij}}\frac{\partial\bar{g}_{ij}}{\partial B^a{}_{bcd}} + \frac{\partial\bar{L}}{\partial\bar{g}_{ij,k}}\frac{\partial\bar{g}_{ij,k}}{\partial B^a{}_{bcd}} + \frac{\partial\bar{L}}{\partial\bar{g}_{ij,k\ell}}\frac{\partial\bar{g}_{ij,k\ell}}{\partial B^a{}_{bcd}} + \frac{\partial\bar{L}}{\partial\bar{S}_{ijk}}\frac{\partial\bar{S}_{ijk}}{\partial B^a{}_{bcd}} +$$

$$+ \frac{\partial\bar{L}}{\partial\bar{S}_{ijk,\ell}}\frac{\partial\bar{S}_{ijk,\ell}}{\partial B^a{}_{bcd}} = 0. \tag{3.3.25}$$

Looking at the transformation equations of g, ∂g,$\partial\partial g$, S and ∂S,we see that (3.3.25) reduces to

$$\frac{\partial\bar{L}}{\partial\bar{g}_{ij,k\ell}}\frac{\partial\bar{g}_{ij,k\ell}}{\partial B^a{}_{bcd}} = 0 \tag{3.3.26}$$

or,

$$\bar{\Lambda}^{ij,k\ell}\frac{\partial\bar{g}_{ij,k\ell}}{\partial B^a{}_{bcd}} = 0 \quad . \tag{3.3.27}$$

But this is simply equation (3.2.12), and there it was shown that it led to the identity

$$\Lambda^{kn,pq} + \Lambda^{kp,qn} + \Lambda^{kq,pn} = 0, \tag{3.3.28}$$

and was further reduced to

$$\Lambda^{ij,k\ell} = \Lambda^{k\ell,ij} \quad . \tag{3.3.29}$$

So this first identity is simply carried over from the second order metric variational problem.

§3.1 *The second identity.*

Going back to equation (3.3.5), using independence of the right hand side on $B^a_{\ bc}$ we have

$$\frac{\partial \bar{L}}{\partial B^a_{\ bc}} = 0 \qquad (3.3.30)$$

or,

$$\bar{\Lambda}^{ij} \frac{\partial \bar{g}_{ij}}{\partial B^a_{\ bc}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial B^a_{\ bc}} + \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial B^a_{\ bc}} + \bar{M}^{ijk} \frac{\partial \bar{S}_{ijk}}{\partial B^a_{bc}} +$$

$$+ \bar{M}^{ijk,\ell} \frac{\partial \bar{S}_{ijk,\ell}}{\partial B^a_{\ bc}} = 0. \qquad (3.3.31)$$

Since the first three terms are carried through from §2.1, we have equation (3.3.31) reducing to

$$g_{a\alpha} B^\alpha_{\ j} \, (\bar{\Lambda}^{bj,c} + \bar{\Lambda}^{cj,b}) + 2 g_{a\alpha} B^\alpha_{\ jk} \, \bar{\Lambda}^{jb,ck} + 2 g_{a\alpha} B^\alpha_{jk} \, \bar{\Lambda}^{jc,bk}$$

$$+ \quad 2 g_{a\alpha,\beta} \, B^\beta_k B^\alpha_{\ j} \, \bar{\Lambda}^{jb,ck} + 2 g_{a\alpha,\beta} B^\alpha_{\ j} \, B^\beta_{\ k} \, \bar{\Lambda}^{jc,bk}$$

$$+ \quad g_{\alpha\beta,a} \, B^\beta_{\ k} \, B^\alpha_{\ j} \, \bar{\Lambda}^{kj,bc}$$

$$+ \quad \bar{M}^{ijk,\ell} \, (B^\beta_j B^\gamma_k \, S_{\alpha\beta\gamma} \, \delta^\alpha_a \delta^b_i \delta^c_\ell \; + \; B^\alpha_i B^\gamma_k S_{\alpha\beta\gamma} \delta^\beta_a \; \delta^b_j \delta^c_\ell$$

$$+ \; B^\alpha_{\ i} \, B^\beta_j \, S_{\alpha\beta\gamma} \, \delta^\gamma_a \delta^b_k \, \delta^c_\ell \,) \; = \; 0 \quad . \qquad (3.3.32)$$

This holds for all coordinate transformations of the form (3.3.4), in particular, we can choose the transformation such that

$$B^i_{\ j} = \delta^i_{\ j} \quad \text{and} \quad B^i_{\ jk} = 0, \text{ etc.} \tag{3.3.33}$$

For such a transformation, (3.3.32) reduces to

$$g_{aj}(\bar{\Lambda}^{bj,c} + \bar{\Lambda}^{cj,b}) + 2g_{aj,k}\,\bar{\Lambda}^{jb,ck} + 2g_{aj,k}\,\bar{\Lambda}^{jc,bk}$$
$$+ g_{jk,a}\,\bar{\Lambda}^{kj,bc} + \bar{M}^{bjk,c}\,S_{ajk} + \bar{M}^{jbk,c}\,S_{jak} + \bar{M}^{kjb,c}\,S_{kja} = 0 \,. \tag{3.3.34}$$

Except for the two terms involving $\Lambda^{ij,k}$, equation (3.3.34) is tensorial, so we must substitute for $\Lambda^{ij,k}$ in terms of the tensor density $\Pi^{ij,k}$.

In Appendix III(A), we have shown that

$$\Pi^{ij,k} = \Lambda^{ij,k} + 2\Lambda^{\alpha j,k\ell}\Gamma^i_{\ell\alpha} + 2\Lambda^{\alpha i,k\ell}\Gamma^j_{\ell\alpha} + \Lambda^{ij,\alpha\ell}\Gamma^k_{\ell\alpha} \tag{3.3.35}$$

so that substitution into (3.3.34) yields,

$$g_{aj}\,(\bar{\Pi}^{bj,c} + 2\bar{\Lambda}^{\alpha j,c\ell}\bar{\Gamma}^{\,b}_{\ell\alpha} + 2\bar{\Lambda}^{\alpha b,c\ell}\bar{\Gamma}^j_{\ell\alpha} + \bar{\Lambda}^{bj,\alpha\ell}\bar{\Gamma}^c_{\ell\alpha}$$
$$+ \bar{\Pi}^{cj,b} + 2\bar{\Lambda}^{\alpha j,b\ell}\bar{\Gamma}^c_{\ell\alpha} + 2\bar{\Lambda}^{\alpha c,b\ell}\bar{\Gamma}^j_{\ell\alpha} + \bar{\Lambda}^{cj,\alpha\ell}\bar{\Gamma}^b_{\ell\alpha})$$
$$+ 2g_{aj,k}\,\bar{\Lambda}^{jb,ck} + 2g_{aj,k}\,\bar{\Lambda}^{jc,bk} + g_{jk,a}\,\bar{\Lambda}^{kj,bc} + 2\bar{M}^{bjk,c}\,S_{ajk}$$
$$+ \bar{M}^{kjb,c}\,S_{kja} = 0. \tag{3.3.36}$$

We have postulated earlier that metric-torsion theories are based on the sound physical assumption that a local Minkowskian tangent space exists. This was expressed through the requirement of metricity,

$$g_{ij;k} = 0 \tag{3.3.37}$$

which allows us to choose a local coordinate system in which the partial derivatives of the metric, and the Christoffel symbols can be set to zero.

Choosing such a gaussian normal coordinate system, we can write equation (3.3.36) as

$$g_{aj}(\bar{\Pi}^{bj,c} + \bar{\Pi}^{cj,b}) + g_{aj}(2\,\bar{\Lambda}^{\alpha j,c\ell} + \bar{\Lambda}^{cj,\alpha\ell})\,\bar{K}_{\ell\alpha}{}^{b}$$

$$+ \quad 2g_{aj}(\bar{\Lambda}^{\alpha b,c\ell} + \bar{\Lambda}^{\alpha c,b\ell})\bar{K}_{\ell\alpha}{}^{j} + g_{aj}\,(2\bar{\Lambda}^{\alpha j,b\ell} + \bar{\Lambda}^{bj,\alpha\ell})\bar{K}_{\ell\alpha}{}^{c}$$

$$+ \quad 2\bar{M}^{bjk,c}S_{ajk} + \bar{M}^{kjb,c}\,S_{kja} = 0, \tag{3.3.38}$$

where we have used the fact that, in a Riemann-Cartan space-time

$$\Gamma^{i}{}_{jk} = \{{}_{j}{}^{i}{}_{k}\} - K_{jk}{}^{i}\ . \tag{3.3.39}$$

$K_{ij}{}^{k} = -K_{i}{}^{k}{}_{j}$ being the contortion tensor.

Finally, since the identity (3.3.38) is a tensorial equation, we can remove the "bars" on the tensors.

It is in this second identity, that we see the first complexities arising in allowing non-zero torsion to be present. We will also see in the next section, that these complexities multiply when deriving the third identity and there are no miraculous reductions leading to general results.

§3.2. *The third identity*.

For the third identity, we take the derivatives of equation (3.3.5) with respect to $B^{a}{}_{b}$,

$$\frac{\partial \bar{L}}{\partial B^{a}{}_{b}} = B\,A^{b}{}_{a} \tag{3.3.40}$$

and,

$$\frac{\partial \bar{L}}{\partial B^a{}_b} = \bar{\Lambda}^{ij} \frac{\partial \bar{g}_{ij}}{\partial B^a{}_b} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial B^a{}_b} + \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial B^a{}_b} +$$

$$+ \bar{M}^{ijk} \frac{\partial \bar{S}_{ijk}}{\partial B^a{}_b} + \bar{M}^{ijk,\ell} \frac{\partial \bar{S}_{ijk,\ell}}{\partial B^a{}_b} \tag{3.3.41}$$

from (3.3.6) we have,

$$\bar{M}^{ijk} \frac{\partial \bar{S}_{ijk}}{\partial B^a{}_b} = \bar{M}^{bjk} B^\beta_j \, B^\gamma{}_k \, S_{a\beta\gamma} + \bar{M}^{ibk} B^\alpha_i B^\gamma_k \, S_{\alpha a\gamma} + \bar{M}^{ijb} \, B^\alpha_i \, B^\beta_j \, S_{\alpha\beta a} \tag{3.3.42}$$

while (3.3.7) yields,

$$\bar{M}^{ijk,\ell} \frac{\partial \bar{S}_{ijk,\ell}}{\partial B^a{}_b}$$

$$= \bar{M}^{ibk,\ell} B^\alpha_{i\ell} B^\gamma_k \, S_{\alpha a\gamma} + \bar{M}^{ijb,\ell} \, B^\alpha_{i\ell} B^\beta_j \, S_{\alpha\beta a} + \bar{M}^{bjk,\ell} B^\beta_{j\ell} B^\gamma_k \, S_{a\beta\gamma}$$

$$+ \bar{M}^{ijb,\ell} B^\beta_{j\ell} B^\alpha_i \, S_{\alpha\beta a} + \bar{M}^{bjk,\ell} B^\gamma_{k\ell} B^\beta_j S_{a\beta\gamma} + \bar{M}^{ibk,\ell} B^\gamma_{k\ell} B^\alpha{}_i \, S_{\alpha a\gamma}$$

$$+ \bar{M}^{bjk,\ell} B^\beta_j \, B^\gamma{}_k \, B^\delta_\ell \, S_{a\beta\gamma,\delta} + \bar{M}^{ibk,\ell} B^\alpha_i \, B^\gamma_k \, B^\delta_\ell \, S_{\alpha a\gamma,\delta}$$

$$+ \bar{M}^{ijb,\ell} B^\alpha_i \, B^\beta{}_j \, B^\delta_\ell \, S_{\alpha\beta a,\delta} + \bar{M}^{ijk,b} \, B^\alpha{}_i B^\beta{}_j B^\gamma{}_k \, S_{\alpha\beta\gamma,a} \, . \tag{3.3.43}$$

Noting that the first three terms of equation (3.3.41) carry over from the example in §2, we can, by taking the coordinate transformation (3.3.4) to be such that

$$B^i{}_j = \delta^i{}_j \quad , \quad B^i{}_{jk} = 0, \quad \text{etc,} \tag{3.3.44}$$

write (3.3.41) as,

$$L\delta^b_{\ a} = \Lambda^{bj} g_{aj} + \Lambda^{jb} g_{aj} + \Lambda^{bj,k} g_{aj,k} + \Lambda^{jb,k} g_{ja,k}$$

$$+ \Lambda^{jk,b} g_{jk,a} + \Lambda^{bj,k\ell} g_{aj,k\ell} + \Lambda^{jb,k\ell} g_{ja,k\ell} + \Lambda^{jk,b\ell} g_{jk,a\ell}$$

$$+ \Lambda^{jk,\ell b} g_{jk,\ell a} + M^{bjk} S_{ajk} + M^{jbk} S_{jak} + M^{jkb} S_{jka}$$

$$+ M^{bjk,\ell} S_{ajk,\ell} + M^{jbk,\ell} S_{jak,\ell} + M^{jkb,\ell} S_{jka,\ell}$$

$$+ M^{jk\ell,b} S_{jk\ell,a} \ . \qquad (3.3.45)$$

The non-tensorial terms involving $\Lambda^{ij,k}$ will vanish when we go to a local gaussian normal coordinate system, in which $g_{ab,c} = 0$. Hence, in order to obtain a tensorial equation from (3.3.45), we need only substitute for Λ^{ij} and M^{ijk}, their tensorial counterparts, Π^{ij} and N^{ijk} respectively, before going to gaussian normal coordinates. As can be seen from Appendix III (A), the expression for Π^{ij} in the presence of non-zero torsion, is not reduced to simply Λ^{ij} on going to gaussian normal coordinates. So, we shall only write down the final expression for the third identity that one obtains upon going to such a coordinate system. We find,

$$L\delta^b_{\ a} = 2\Pi^{bj} g_{aj} + \frac{4}{3} R_{ak\alpha\ell}(\{\})\ \Lambda^{b\alpha,k\ell}$$

$$+ 2\Pi^{\alpha j,k} K_{k\alpha}^{\ \ b} g_{aj} + 2\Pi^{\alpha b,k} K_{k\alpha a} + 2\,\Pi^{b\alpha,k} K_{k\alpha a}$$

$$+ 2\,\Pi^{j\alpha,k} K_{k\alpha}^{\ \ b} g_{aj} + 4\Lambda^{\alpha j,k\ell} K_{k\alpha.}^{\ \ \ \beta} K_{\ell\beta.}^{\ \ \ b} g_{aj}$$

$$- 4\Lambda^{\alpha b,k\ell} K_{\ell a}^{\ \ \beta} K_{k\alpha\beta} + 8\Lambda^{\alpha\beta,k\ell} K_{\ell\beta a} K_{k\alpha}^{\ \ b}$$

$$+ 4\Lambda^{\alpha j,k\ell} K_{k\alpha\ \ ;\ell}^{\ \ b} g_{aj} + 4\Lambda^{\alpha b,k\ell} K_{k\alpha a;\ell}$$

$$-2\, M^{bjk,\ell} K_{\ell a}{}^{\alpha} S_{\alpha jk}$$

$$+ 2M^{bjk,\ell} S_{ajk;\ell} + M^{jkb,\ell} S_{jka;\ell} + M^{jk\ell,b}\, S_{jk\ell;a} + 2M^{\alpha jk,\ell} K_{\ell\alpha}{}^{b} S_{ajk}$$

$$- M^{jkb,\ell} K_{\ell a}{}^{\alpha} S_{jk\alpha} - M^{jk\ell,b} K_{aj}{}^{\alpha} S_{\alpha k\ell} - M^{jk\ell,b} K_{ak}{}^{\alpha} S_{j\alpha\ell}$$

$$- M^{jk\ell,b} K_{a\ell}{}^{\alpha} S_{jk\alpha} + M^{jk\alpha,\ell} K_{\ell\alpha}{}^{b} S_{jka} + N^{jkb}\, S_{jka}$$

$$+ 2\, N^{bjk}\, S_{ajk} \,. \tag{3.3.46}$$

This is the tensorial form of the third identity for metric-torsion theories with a lagrangian of the form $L(g,\partial g,\partial\partial g, S,\partial S)$. It has been written in such a manner that the limit of zero-torsion can be seen immediately. We note that, as expected, the limit of zero-torsion leads to the third identity obtained in the example given in §2, equation (3.2.30).

Of course, in (3.3.46), we could, by combining some more terms, write it in terms of the full Riemann-Cartan curvature tensor rather than the Riemann-Christoffel curvature tensor, but as no new insights are gained by doing this, we shall be content with the equation as it stands.

In the next section, we shall reduce the Euler-Lagrange field equations obtained from $L(g,\partial g,\partial\partial g, S,\partial S)$ for g_{ij} and $S_{ij}{}^{k}$ as much as possible.

§3.3 *Reduction of the Euler-Lagrange equations of* $L(g,\partial g,\partial\partial g, S,\partial S)$.

Since we have two sets of independent field functions, namely, the metric tensor and the torsion tensor, we shall have two Euler Lagrange field equations. The field equation for the metric is simply,

$$E^{ij}(L) = \frac{d}{dx^k}\left\{\frac{\partial L}{\partial g_{ij,k}} - \frac{d}{dx^\ell}\frac{\partial L}{\partial g_{ij,k\ell}}\right\} - \frac{\partial L}{\partial g_{ij}} = 0 . \tag{3.3.47}$$

This equation is identical to that derived in §2.3. Hence we can borrow the tensorial form of $E^{ij}(L)$ that has been obtained there, while noting that the covariant derivatives will now be with respect to the asymmetric connection with non-zero torsion. The tensorial form of the equation is ;

$$E^{ij}(L) = -\Pi^{ij} - \Lambda^{ij,k\ell}{}_{;k\ell} + \Pi^{ij,k}{}_{;k} = 0 . \tag{3.3.48}$$

In the previous example, this equation was further simplified to an expression containing derivatives of $\Lambda^{ij,k\ell}$ as the only unknown quantitaties due, mainly to the result given in equation (3.2.23). This reduction is no longer possible. So, rather than try to "simplify" this equation by substitution of $\Pi^{ij,k}{}_{;k}$ and Π^{ij} , through the second and third identities, we note that indeed after substitution, we shall have a field equation, requiring only the evaluation of $\Lambda^{ij,k\ell}$, $\Pi^{ij,k}$, $M^{ijk,\ell}$ and N^{ijk} to obtain the explicit form of the equation for any given lagrangian. Further, we shall see now that by combining the two, metric and torsion field equations, one reduces the requirement, to evaluation of only $\Lambda^{ij,k\ell}$, $\Pi^{ij,k}$ and $M^{ijk,\ell}$.

The second field equation that we have is for the torsion field;

$$E^{ijk}(L) = \frac{d}{dx^{\ell}} \left\{ \frac{\partial L}{\partial S_{ijk,\ell}} \right\} - \frac{\partial L}{\partial S_{ijk}} = 0 \qquad (3.3.49)$$

or,

$$E^{ijk}(L) = M^{ijk,\ell}{}_{,\ell} - M^{ijk} = 0 \; . \qquad (3.3.50)$$

In order to write this in a tensorial form, we note first, that

$$Q_{ijk,\ell}M^{ijk,\ell} = (Q_{ijk}M^{ijk,\ell})_{,\ell} - Q_{ijk}\, M^{ijk,\ell}{}_{,\ell} \qquad (3.3.51)$$

where, Q_{ijk} is a tensor, having the symmetries of the torsion tensor. Going back to equation (3.3.15), we can write

$$G = (Q_{ijk}M^{ijk,\ell})_{,\ell} + M^{ijk}Q_{ijk} - Q_{ijk}M^{ijk,\ell}{}_{,\ell} \qquad (3.3.52)$$

$$= (Q_{ijk}M^{ijk,\ell})_{,\ell} - (M^{ijk,\ell}{}_{,\ell} - M^{ijk})\, Q_{ijk} \qquad (3.3.53)$$

or, $$G = (Q_{ijk}M^{ijk,\ell})_{,\ell} - E^{ijk}(L)\, Q_{ijk}. \qquad (3.3.54)$$

Similarly, we can write equation (3.3.17) as

$$G = (Q_{ijk}\, M^{ijk,\ell})_{;\ell} + Q_{ijk}(N^{ijk} - M^{ijk,\ell}{}_{;\ell}) \qquad (3.3.55)$$

so that, comparison of coefficients of Q_{ijk} in equations (3.3.55) and (3.3.54) yields

$$E^{ijk}(L) = M^{ijk,\ell}{}_{;\ell} - N^{ijk} \; . \qquad (3.3.56)$$

We have shown earlier that G is a scalar density, we also note that because $M^{ijk,\ell}$ is a tensor density, while Q_{ijk} is a tensor, their product is a tensor density and hence the covariant divergence of the product is a scalar density. We have therefore that

$$G - (Q_{ijk} M^{ijk,\ell})_{;\ell} \qquad (3.3.57)$$

is a scalar density. Hence

$$Q_{ijk}(N^{ijk} - M^{ijk,\ell}{}_{;\ell}) \qquad (3.3.58)$$

is a scalar density. Since Q_{ijk} is a tensor, by the quotient theorem, it follows that

$$E^{ijk}(L) = M^{ijk,\ell}{}_{;\ell} - N^{ijk} \qquad (3.3.59)$$

is a tensor density of weight 1 and contravariant rank 3.

§3.4. *A simple example.*

To show that evaluation of the field equations will involve only partial derivatives of a given lagrangian with respect to the field quantities, we shall substitute the third identity into the field equations for a simple lagrangian of the form $L(g,\partial g,\partial\partial g,S)$. Since no derivatives of torsion are present, we immediately have

$$\frac{\partial L}{\partial S_{ijk,\ell}} = M^{ijk,\ell} = 0 \qquad (3.3.60)$$

The second field equation (3.3.59) reduces to,

$$N^{ijk} = 0. \qquad (3.3.61)$$

Using these two equations in the third identity, (3.3.46), we shall have,

$$L\delta^b_a = 2\Pi^{bj} g_{aj} + \frac{4}{3} R_{ak\alpha\ell}(\{\}) \Lambda^{b\alpha,k\ell} + 2\Pi^{\alpha j,k} K_{k\alpha}{}^b g_{aj}$$

$$+ 2\Pi^{\alpha b,k} K_{k\alpha a} + 2\Pi^{b\alpha,k} K_{k\alpha a} + 2\Pi^{j\alpha,k} K_{k\alpha}{}^b g_{aj}$$

$$+ 4\Lambda^{\alpha j,k\ell}K_{k\alpha}{}^{\beta}K_{\ell\beta}{}^{b}g_{aj} - 4\Lambda^{\alpha b,k\ell}K_{\ell a}{}^{\beta}K_{k\alpha\beta} + 8\Lambda^{\alpha\beta,k\ell}K_{\ell\beta a}K_{k\alpha}{}^{b} \cdot$$

$$+ 4\Lambda^{\alpha j,k\ell}K_{k\alpha}{}^{b}{}_{;\ell}g_{aj} + 4\Lambda^{\alpha b,k\ell}K_{k\alpha a;\ell} \qquad (3.3.62)$$

or,

$$L\, g^{bc} = 2\Pi^{bc} + \frac{4}{3} R^{c}{}_{.k\alpha\ell}(\{\})\Lambda^{b\alpha,k\ell} + 4\Pi^{\alpha c,k}K_{k\alpha}{}^{b}$$

$$+ 4\Pi^{\alpha b,k}K_{k\alpha}{}^{c}$$

$$+ 4\Lambda^{\alpha c,k\ell}K_{k\alpha}{}^{\beta}K_{\ell\beta}{}^{b} + 4\Lambda^{\alpha b,k\ell}K_{\ell\beta}{}^{c}K_{k\alpha}{}^{\beta}$$

$$+ 8\Lambda^{\alpha\beta,k\ell}K_{\ell\beta}{}^{c}K_{k\alpha}{}^{b} + 4\Lambda^{\alpha c,k\ell}K_{k\alpha}{}^{b}{}_{;\ell} \qquad (3.3.63)$$

$$+ 4\Lambda^{\alpha b,k\ell}K_{k\alpha}{}^{c}{}_{;\ell} \cdot$$

Substitution into (3.3.48) yields

$$2E^{bc}(L) = -L\, g^{bc} + \frac{4}{3} R^{c}{}_{k\alpha\ell}(\{\})\Lambda^{b\alpha,k\ell} + 4\Pi^{\alpha c,k}K_{k\alpha}{}^{b}$$

$$+ 4\Pi^{\alpha b,k}K_{k\alpha}{}^{c} + 4\Lambda^{\alpha c,k\ell}K_{k\alpha}{}^{\beta}K_{\ell\beta}{}^{b} + 4\Lambda^{\alpha b,k\ell}K_{\ell\beta}{}^{c}K_{k\alpha}{}^{\beta}$$

$$+ 8\Lambda^{\alpha\beta,k\ell}K_{\ell\beta}{}^{c}K_{k\alpha}{}^{b} + 4\Lambda^{\alpha c,k\ell}K_{k\alpha;\ell}{}^{b}$$

$$+ 4\Lambda^{\alpha b,k\ell}K_{k\alpha.;\ell}{}^{c} - \Lambda^{bc,k\ell}{}_{;k\ell} + \Pi^{bc,k}{}_{;k}$$

$$= 0, \qquad (3.3.64)$$

while the second equation is

$$E^{ijk}(L) = N^{ijk} = M^{ijk} = \frac{\partial L}{\partial S_{ijk}} = 0 \ . \qquad (3.3.65)$$

We see clearly from these equations that, given a lagrangian, we need only evaluate partial derivatives of the lagrangian in order to obtain the explicit form of the field equations. It is interesting to note that similar work has been carried out by P. von der Heyde /12/ , who has derived a general form of the lagrangian for metric-torsion theories by imposing conditions on the form of the field equations.

CHAPTER IV

HOW LINEAR IS LINEAR?

OR

THE ART OF CONSTRUCTING LAGRANGIANS

§1. *Lagrangians for general relativity.*

In the last Chapter we have seen how one can exploit the invariance of the action functional

$$S = \int L\, d^4x \qquad (4.1.1)$$

under general coordinate changes to derive identities satisfied by the lagrangian density L. We have further shown how we can use these identities to reduce the Euler-Lagrange equations of L into a simple form involving only partial derivatives of L with respect to its arguments, especially in the case of general relativity.

In this chapter, we shall study the general criteria used to write down lagrangian densities for a theory. In particular, we shall show that the criterion used by Einstein for his lagrangian, that it be linear in the curvature, when extended to the ECSK theory does not yield the usual lagrangian chosen for the theory. Instead, we find that we can include an additional term of the form $\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\Gamma)$ involving the totally antisymmetric Levi-Civita tensor density in four-dimensions and the full curvature tensor of the Riemann-Cartan geometry.

We shall begin in this section by outlining some of the important points that led Einstein to his lagrangian for general relativity,

Einstein /10/ having adopted Riemannian geometry and the absolute differential calculus of Ricci and Levi-Civita, to describe the phenomenon of gravitation, argues for a non-degenerate metric;

"Should $\sqrt{-g}$ vanish at a spot in the four-dimensional continuum then it signifies that a finite coordinate volume there corresponds to an infinitesimal "natural" volume. This, however, may not be so anywhere, and therefore the sign of g cannot change. Following special relativity, we shall assume that g always has a finite negative value. This represents a hypothesis about the physical nature of the continuum under consideration and at the same time a determination of the choice of coordinates. If, however, $\sqrt{-g}$ is always positive and finite, then it is obvious that a posteriori the choice of coordinates can be made such that this quantity is equal to 1".

While concerning his field equations,

$$G_{ij} = R_{ij} - \frac{1}{2} Rg_{ij} = \kappa T_{ij} , \qquad (4.1.2)$$

he writes /10/

"It must be pointed out that only a minimum arbitrariness is connected with these equations. For, other than R_{ij}, there is no tensor of the second rank connected with it which can be constructed from the g_{ij} and their derivatives, which does not contain higher than second derivatives, and is linear in them".

From this, it is usually stated that Einstein's lagrangian must be linear in the second derivatives of the metric. The main reason for requiring linearity in the second derivatives of the metric, is that field equations for the metric would then be of the second order. In recent times, however, it has been shown that linearity in the second derivatives of g_{ij} is not a necessary assumption /11/. Indeed, David Lovelock /13/ has shown that assuming the field equation is of the form

$$G_{ij} = \kappa T_{ij} , \qquad (4.1.3)$$

being symmetric in (ij) and containing only second derivatives of g_{ij} leads, by requiring conservation of T_{ij}, i.e.

$$T_{ij};j = 0. \qquad (4.1.4)$$

uniquely to the lagrangian

$$L = \sqrt{-g} \;\; R(\{\}) + \sqrt{-g}\, \Lambda , \qquad (4.1.5)$$

where Λ is a cosmical constant.

Having established that to describe general relativity, one must use the Riemann tensor, and the lagrangian is indeed taken to be simply

$$\sqrt{-g}\, R\,(\{\})\ , \qquad (4.1.6)$$

the question we have to ask ourselves now, is, is this the only scalar, linear in the curvature, that one can construct in Riemannian geometry?

The answer is yes. For, although, we implicitly have the possibility of constructing the following scalar,

$$\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}(\{\})\ , \qquad (4.1.7)$$

we remember that the Riemannian curvature tensor satisfies a set of Bianchi identities, one of them is /14/

$$R_{\mu\nu\rho\sigma}(\{\}) + R_{\mu\rho\sigma\nu}(\{\}) + R_{\mu\sigma\nu\rho}(\{\}) = 0 \qquad (4.1.8)$$

which can also be written as :

$$\varepsilon^{\lambda\nu\rho\sigma} R_{\mu\nu\rho\sigma}(\{\}) = 0\ . \qquad (4.1.9)$$

Hence, the additional term that might have contributed to the lagrangian, vanishes by virtue of one of the Bianchi identities.

We shall show in the next section, that a reduction in the symmetries of the curvature tensor in generalising the geometry from Riemannian to Riemann-Cartan, allows us to add the additional term

$$\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\ (\Gamma) \qquad (4.1.10)$$

to the usual ECSK lagrangian.

§2. *Generalisation to metric-torsion theories /15/.*

We have seen in Chapter I that the curvature tensor of a Riemann-Cartan geometry is defined by

$$R_{ijk}{}^{\ell}(\Gamma) = \partial_i \Gamma_{jk}{}^{\ell} - \partial_j \Gamma_{ik}{}^{\ell} + \Gamma_{im}^{\ell}\Gamma_{jk}{}^{m} - \Gamma_{jm}{}^{\ell}\Gamma_{ik}{}^{m} . \quad (4.2.1)$$

The definition of the curvature tensor is such that it is antisymmetric in the first two indices. In general this is the only symmetry on the curvature tensor that one has. If, however, we demand metricity, i.e. that the geometry be locally Minkowskian, we gain the additional symmetry

$$R_{ijk\ell}(\Gamma) = -R_{ij\ell k}(\Gamma) . \quad (4.2.2)$$

For metric-torsion theories, these are the only symmetries one has on the curvature tensor. In contrast, in Einstein's general relativity, symmetry of Christoffel connection symbols gives rise to a third symmetry

$$R_{ijk\ell}(\{\}) = R_{k\ell ij}(\{\}) . \quad (4.2.3)$$

In Chapter I, it was seen that the lagrangian for the ECSK theory was taken to be

$$L_{ECSK} = \sqrt{-g}\, R\,(\Gamma) \quad (4.2.4)$$

on the grounds that it was the absolute minimal deviation from Einstein's lagrangian. However, in §1 we saw that, potentially, Einstein's lagrangian can be written as

$$L_E = \sqrt{-g} \;\; R(\{\}) + \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \;\; (\{\}) \quad (4.2.5)$$

with the Bianchi identity allowing us to equate the second term to zero.

In the case of Riemann-Cartan geometry, however, we see that the Bianchi identities take the form (equations (13.30) and (13.31)).

$$R_{[\mu\nu\rho]}{}^{\sigma}(\Gamma) = 2\nabla_{[\mu}S_{\nu\rho]}{}^{\sigma} - 4\, S_{[\mu\nu}{}^{\alpha}S_{\rho]\alpha}{}^{\sigma} \qquad (4.2.6)$$

and

$$\nabla_{[\alpha}R_{\mu\nu]\rho}{}^{\sigma}(\Gamma) = 2S_{[\alpha\mu}{}^{\beta}R_{\nu]\beta\rho}{}^{\sigma}(\Gamma) \ . \qquad (4.2.7)$$

We realise that in generalising Einstein's lagrangian to metric-torsion theories of gravity, one must in reality, generalise L_E given in equation (4.2.5).

The Bianchi identity from equation (4.2.6) tells us that while

$$\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\{\}) = 0, \qquad (4.2.8)$$

$$\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\Gamma) \neq 0. \qquad (4.2.9)$$

Let us write the lagrangian for our metric-torsion theory as

$$L = L_{ECSK} + L_A$$

where

$$L_{ECSK} = \sqrt{-g}\ R(\Gamma) \qquad (4.2.10)$$

and

$$L_A = \frac{p}{2}\,\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\Gamma) \ . \qquad (4.2.11)$$

The coupling parameter, p, determines the relative strengths of the gravitational forces obtained from L_A and L_{ECSK}. We note that $\varepsilon^{\mu\nu\rho\sigma}$ is a pseudo-tensor density, and changes sign upon a coordinate transformation of the form ;

$$\underline{x} \to -\underline{x} \ ; \ t \to t \ , \ \text{where} \quad x^{\mu} = (\underline{x}, t), \qquad (4.2.12)$$

i.e. the term involving $\varepsilon^{\mu\nu\rho\sigma}$ is a parity violating term, and the coupling parameter, p, governs the strength of these parity violating interactions.

§3. *The Generalised lagrangian.*

Having outlined the reasons for incorporating the additional, parity violating term into the usual ECSK lagrangian, we shall now go on to simplify the total lagrangian.

$$L = L_{ECSK} + L_A$$
$$= \sqrt{-g}\ R(\Gamma) + \frac{p}{2}\,\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\Gamma)\ . \qquad (4.3.1)$$

In chapter I we saw that the scalar curvature R(Γ) can be decomposed into the scalar curvature constructed from the Christoffel symbols and the 24 component contortion tensor.

So that we can write

$$L_{ECSK} = \sqrt{-g}\ R(\{\}) + K_{\sigma\alpha}{}^{\sigma}K_{\nu}{}^{\nu\alpha} - K_{\nu\alpha}{}^{\sigma}K_{\sigma}{}^{\nu\alpha}$$
$$+ \partial^{\rho}K_{\sigma\rho}{}^{\sigma} - \partial_{\sigma}(g^{\nu\rho}K_{\nu\rho}{}^{\sigma})\ . \qquad (4.3.2)$$

The terms involving derivatives of K can be discarded, as they are pure divergence terms and hence will not contribute to the field equations (we are ignoring any surface effects which might arise when dealing with compact manifolds).

From the Bianchi identity for Riemann-Cartan geometries, given in equation (4.2.6), we can write

$$\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\Gamma) = 2\ \varepsilon^{\mu\nu\rho\sigma}(K_{\mu\alpha\sigma}K_{\nu\rho}{}^{\alpha} + \partial_{\nu}K_{\mu\rho\sigma}) \qquad (4.3.3)$$

where again, the second term can be ignored as it gives rise to a total divegence term,

$$\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu K_{\mu\rho\sigma} = \partial_\nu(\varepsilon^{\mu\nu\rho\sigma}K_{\mu\rho\sigma}) \quad . \tag{4.3.4}$$

As expected, we can see that contortion appears quadratically in the total lagrangian, with no derivatives of the contortion appearing, after the total divergence terms are removed. Indeed, this is a general restriction for theories incorporating lagrangians linear in the curvature. This in turn tells us that as in the usual ECSK theory, the field equations for torsion will be algebraic, i.e. they will be zeroth order differential equations, and hence will not allow propagating torsion.

We note at this point, that from the contortion tensor, one can construct only three scalars quadratic in the contortion, namely

$$K_1 = K_{\sigma\alpha}{}^{\sigma}K_{\nu}{}^{\nu\alpha}, K_2 = K_{\nu\alpha}{}^{\sigma}K_{\sigma}{}^{\nu\alpha} \text{ and } K_3 = K_{\nu\sigma}{}^{\lambda}K^{\nu\sigma}{}_{\lambda} \ . \tag{4.3.5}$$

The interesting feature of our lagrangian is that the simple requirement of linearity in the curvature picks out two of these three scalars namely, K_1 and K_2.

At this stage, one may say that an equally valid approach to construct a lagrangian for the torsion would be to write down linear combinations of all the possible scalars, quadratic in the contortion $K_{\mu\nu}{}^{\sigma}$ and simply add these to Einstein's lagrangian L_E. Clearly such an approach is unsatisfactory, since each term would necessitate an associated coupling parameter. In general, this would mean the introduction of at least three parameters. In appendix IV(A), we show that taking into account the possibility of allowing $\varepsilon^{\mu\nu\rho\sigma}$ to enter the lagrangian increases the number of scalars, quadratic in the contortion, that one can include in the lagrangian to five. In the same appendix, it is also shown how further complications arise if one tries to incorporate all possible scalars quadratic in the first derivatives of contortion. In view of this, it seems much simpler, and perhaps more natural, to restrict the total lagrangian to be linear in the curvature, and consider only the scalars obtained from $R_{\mu\nu\rho\sigma}(\Gamma)$ through all possible contractions.

§4. *Field Equations for the New Lagrangian.*

We have seen in Chapter I that variations with respect to the torsion $S_{\mu\nu}{}^{\sigma}$ are more fundamental in the sense that torsion is a priori independent of the metric. However, the variations with respect to the torsion are equally as good as variations with respect to the contortion tensor K. As we have written out the new lagrangian, equations (4.3.2) and (4.3.3) in terms of the contortion tensor, it will be easier for the purposes of this chapter, to take variations of the action with respect to the contortion tensor, K.

The action functional for the new lagrangian is;

$$S = \int L_{ECSK} + L_A \quad d^4x$$

or,

$$S = \int \left\{ \sqrt{-g} \quad R(\{\}) \quad + K_{\sigma\alpha}{}^{\sigma} K_{\nu}{}^{\nu\alpha} - K_{\nu\alpha}{}^{\sigma} K_{\sigma}{}^{\nu\alpha} \quad + p \ \varepsilon^{\mu\nu\rho\sigma} K_{\mu\alpha\sigma} K_{\nu\rho}{}^{\alpha} \right\} d^4x \ . \tag{4.4.1}$$

The field equations obtained by variations with respect to the metric are simply

$$\sqrt{-g} \ \{R_{\mu\nu}(\{\}) - \frac{1}{2} R(\{\}) \ g_{\mu\nu}\} + \frac{\sqrt{-g}}{2}\{g_{\mu\nu} g^{\rho\sigma} - g^{\rho}{}_{\mu} \ g^{\sigma}_{\nu} - g^{\rho}{}_{\nu} g^{\sigma}{}_{\mu}\}\{K_{\rho\alpha}{}^{\beta} K_{\beta\sigma}{}^{\alpha} - K_{\alpha\beta}{}^{\alpha} \ K_{\rho\sigma}{}^{\beta}\}$$

$$+ p \ (\varepsilon^{\rho\sigma\alpha}{}_{\mu} K_{\rho\beta\nu} K_{\sigma\alpha}{}^{\beta} + \varepsilon^{\rho\sigma\alpha}{}_{\nu} K_{\rho\beta\mu} K_{\sigma\alpha}{}^{\beta}) = 0. \tag{4.4.2}$$

while variations with respect to the contortion tensor yield

$$\sqrt{-g}\,\{K^{\mu\nu\rho} + K^{\rho\mu\nu} - K_{\sigma}{}^{\sigma\rho}\, g^{\nu\mu} - K_{\sigma}{}^{\mu\sigma}\, g^{\nu\rho}$$

$$+ 2p\,(\eta^{\nu\rho\sigma\alpha}\, K_{\sigma}{}^{\mu}{}_{\alpha} + \eta^{\nu\mu\sigma\alpha}\, K_{\sigma\alpha}{}^{\rho}\,)\} = 0. \qquad (4.4.3)$$

The quantity $\eta^{\mu\nu\rho\sigma}$, intróduced into (4.4.3) in a tensor constructed out of the tensor density $\varepsilon^{\mu\nu\rho\sigma}$ it satisfies the following properties,

$$\eta^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}}\,\varepsilon^{\mu\nu\rho\sigma} \qquad \eta_{\mu\nu\rho\sigma} = \sqrt{-g}\;\varepsilon_{\mu\nu\rho\sigma} \qquad (4.4.4)$$

and we have,

$$\eta^{\mu\nu\lambda\sigma}\;\eta_{\mu\alpha\beta\gamma} = -\delta^{\nu\lambda\sigma}_{\alpha\beta\gamma}$$

$$\eta^{\mu\nu\lambda\sigma}\;\eta_{\mu\nu\beta\gamma} = -2\delta^{\lambda\sigma}_{\beta\gamma} \qquad (4.4.5)$$

$$\eta^{\mu\nu\lambda\sigma}\;\eta_{\mu\nu\lambda\gamma} = -6\delta^{\sigma}_{\gamma}$$

and, $\eta^{\mu\nu\lambda\sigma}\;\eta_{\mu\nu\lambda\sigma} = -24$,

where the tensor $\delta^{\mu\nu\lambda\ldots}_{\alpha\beta\gamma\ldots}$ is a generalised Kronecker symbol obeying the following rules:

If $\mu,\nu,\lambda,\ldots.$ are all different and $\alpha,\beta,\gamma,\ldots.$ are obtained from them by a certain permutation, then its value is +1 or -1 depending on whether the permutation $\left(\begin{smallmatrix}\mu\nu\lambda\ldots\\ \alpha\beta\gamma\ldots\end{smallmatrix}\right)$ is even or odd. In the remaining cases it is equal to zero.

At this point we shall give a rather simple and quite general argument for the vanishing of contortion in the vacuum , if one begins with a lagrangian which contains the contortion in a non-dynamic manner, i.e. it does not contain second or higher derivatives of the contortion, or equivalently, terms quadratic in the derivatives of the contortion /15/.

Now, the Euler-Lagrange equations for a lagrangian not containing any derivatives of the contortion fields, reads

$$\frac{\partial L}{\partial K_{\mu\nu}{}^{\sigma}} = 0 \ , \qquad (4.4.6)$$

and hence will be an algebraic equation for K. In principle, this equation can be solved for K, and this solution must then be expressible in terms of the other quantities in the theory. In our present example, we have at our disposal, $g_{\mu\nu}, g_{\mu\nu,\alpha}$, $g_{\mu\nu,\alpha\beta}$ and $\varepsilon_{\mu\nu\alpha\beta}$. Since contraction of indices always removes indices on a pairwise basis, no third rank tensor can be constructed from $g_{\mu\nu}, g_{\mu\nu,\alpha\beta}$ and $\varepsilon_{\mu\nu\alpha\beta}$. Thus, $g_{\mu\nu,\alpha}$ must enter each term of the expression for K. But, because of our assumption of metricity, $g_{\mu\nu;\alpha} = 0$, we can always choose, locally, a coordinate system in which $g_{\mu\nu,\alpha} = 0$. Therefore, K will vanish in this coordinate system, and by virtue of its tensorial character, in all coordinate systems. It is for this reason that all matter-free metric-torsion theories exhibiting an algebraic field equation for the torsion are identical to the matter-free theory of general relativity.

To begin to see the effects of torsion, one must therefore consider the coupling of matter fields to torsion.

In the next section we shall couple our metric-torsion theory, using the generalised lagrangian, to the Dirac spinor field and we shall see that the ECSK theory itself predicts parity violating effects for the Dirac field.

§5. *Coupling to Matter Fields : The Dirac Field.*

In the last section we have seen that the general structure of metric-torsion theories of gravitation allows us to include an additional term in the action of the ECSK theory, while still keeping the lagrangian linear in the curvature. In this section we shall couple the torsion field arising from this new lagrangian, to the Dirac spin-½ field; the lagrangian for the Dirac field in flat space-time is

$$L_D = (\partial_i \bar{\psi})\, \gamma^i \psi - \bar{\psi}\, \gamma^i\, (\partial_i \psi) - m\bar{\psi}\psi \qquad (4.5.1)$$

The γ^μ are the Dirac matrices, satisfying the following algebra:

$$\{\gamma^i, \gamma^j\} := \gamma^i \gamma^j + \gamma^j \gamma^i = 2\eta^{ij} \qquad (4.5.2)$$

and it is spanned by the sixteen independent elements,

$$\gamma_i\ , \gamma_5\ , \gamma_{[k}\ \gamma_{j\,]}\ , \gamma_j \gamma_5\ , 1, \text{ where } \gamma_5 = \gamma_o \gamma_1 \gamma_2 \gamma_3\ .$$

In minimally coupling to gravity, the Dirac algebra is generalised by simply replacing η^{ij}, the Minkowski metric,by g^{ij} while, the lagrangian is modified to

$$L_D = e\ \{(\nabla_i \bar{\psi})\gamma^i \psi - \bar{\psi}\gamma^i\ (\nabla_i \psi) - m\bar{\psi}\psi\ \}, \qquad (4.5.3)$$

where $e = \det(e_i^{\ \alpha})$, and $e_i^{\ \alpha}$,$\alpha = 1,2,3,4$ is a tetrad chosen at each point of the space-time. The index α labels the four linearly independent tetrad vectors.

It is necessary to introduce local orthonormal tetrads when dealing with spinor fields for the simple reason that while tensorial fields are representations of the group of general coordinate transformations, spinorial fields are representations of the Lorentz group. Therefore, when dealing with spinorial quantities incurved space-time, we are compelled to introduce at each point of the space-time, a tangent Minkowski space-time via the tetrads $e_i^{\ \alpha}$. The basics of anholonomic orthonormal tetrads and their relationship to the Dirac field are explained in appendix IV(B).

In the same appendix, the covariant derivative of a spinor is defined to be

$$\nabla_\alpha \bar{\psi} = \partial_\alpha \bar{\psi} - \frac{1}{4} \Gamma_{\alpha\beta\gamma} \bar{\psi} \gamma^\gamma \gamma^\beta \tag{4.5.4}$$

with
$$\Gamma_{\alpha\beta\gamma} = -\Omega_{\alpha\beta\gamma} + \Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta} - K_{\alpha\beta\gamma} \tag{4.5.5}$$

and
$$\nabla_\alpha \psi = \partial_\alpha \psi - \frac{1}{4} \Gamma_{\alpha\beta\gamma} \gamma^\beta \gamma^\gamma \psi \tag{4.5.6}$$

while the torsionless theory has

$$\nabla_\alpha(\{\})\psi := \partial_\alpha \psi + \frac{1}{4} (\Omega_{\alpha\beta\gamma} - \Omega_{\beta\gamma\alpha} + \Omega_{\gamma\alpha\beta}) \gamma^\beta \gamma^\gamma \psi . \tag{4.5.7}$$

In terms of $\nabla_\alpha(\{\})$, the lagrangian L_D is

$$L_D = e \{(\nabla_\alpha(\{\})\bar{\psi})\gamma^\alpha \psi - \bar{\psi} \gamma^\alpha (\nabla_\alpha(\{\})\psi) - m\bar{\psi}\psi \}$$
$$- \frac{1}{2} e K_{\alpha\beta\gamma} \bar{\psi} \gamma^{[\alpha|} \gamma^\beta \gamma^{|\gamma]} \psi . \tag{4.5.8}$$

The spin-angular momentum $\tau_{ij}^{\ \ k}$, of the Dirac field is given by

$$e \tau_k^{\ ji} = \frac{\delta L_D}{\delta K_{ij}^{\ \ k}} . \tag{4.5.9}$$

Equation (4.5.8) immediately gives

$$\tau_{k}{}^{ji} = -\frac{1}{2} e_{k\gamma} e^{i}_{\alpha} e^{j}_{\beta} \bar{\psi} \gamma^{[\alpha|} \gamma^{\beta} \gamma^{|\gamma]} \psi \qquad (4.5.10)$$

or, equivalently,

$$\tau^{\gamma\beta\alpha} = -\frac{1}{2} \bar{\psi} \gamma^{[\alpha|} \gamma^{\beta} \gamma^{|\gamma]} \psi \quad . \qquad (4.5.11)$$

We can use the Dirac algebra to simplify this expression /16/.
we have,

$$\gamma^{\alpha} \gamma^{\beta} \gamma^{\gamma} = \gamma^{[\alpha} \gamma^{\beta} \gamma^{\gamma]} + g^{\alpha\beta}\gamma^{\gamma} + g^{\beta\gamma} \gamma^{\alpha} - g^{\gamma\alpha}\gamma^{\beta} \qquad (4.5.12)$$

or,

$$\gamma^{[\alpha|} \gamma^{\beta} \gamma^{|\gamma]} = \gamma^{[\alpha} \gamma^{\beta} \gamma^{\gamma]} \quad . \qquad (4.5.13)$$

Hence, we have that the spin-angular momentum tensor is totally antisymmetric:

$$\tau^{[\alpha\beta\gamma]} = -\frac{1}{2} \bar{\psi} \gamma^{[\alpha} \gamma^{\beta} \gamma^{\gamma]} \psi \quad . \qquad (4.5.14)$$

Now the torsion field equation for the total lagrangian

$$L = L_{ECSK} + L_{A} + L_{D} \qquad (4.5.15)$$

is,

$$\frac{\partial L_{ECSK}}{\partial K_{ij}{}^{k}} + \frac{\partial L_{A}}{\partial K_{ij}{}^{k}} + \frac{\partial L_{D}}{\partial K_{ij}{}^{k}} = 0 \quad . \qquad (4.5.16)$$

From the expressions for L_{ECSK} and L_{A}, we have that

$$\tau_{k}{}^{ji} + p\,(\varepsilon^{i\nu\lambda}{}_{k}\, K_{\nu\lambda}{}^{j} + \varepsilon^{i\nu\lambda j}\, K_{\nu k\lambda})$$

$$+ T_{k}{}^{ji} = 0 \quad , \qquad (4.5.17)$$

where $T_k^{\ ji}$ is the modified torsion tensor defined in Chapter I . From the Dirac algebra, we have the relation

$$\gamma_5\gamma^{\alpha} = \frac{1}{3!}\varepsilon^{\alpha\beta\gamma\delta} \quad \gamma_\beta \ \gamma_\gamma \ \gamma_\delta \ . \tag{4.5.18}$$

So that equation (4.5.14) takes the form

$$\tau^{[\alpha\beta\gamma]} = -3\bar{\psi} \ \varepsilon^{\alpha\beta\gamma\delta}\gamma_5 \ \gamma_\delta \ \psi \ , \tag{4.5.19}$$

while the field equation for the torsion, equation (4.5.17) can be solved to yield,

$$K_{\alpha\beta\gamma} = \frac{2 \ \tau_{\gamma\alpha\beta}}{(1+p^2)} - \frac{6p \ g_{\gamma[\alpha} \ \varepsilon_{\beta]ijk}\tau^{ijk}}{(1 + p^2)} \tag{4.5.20}$$

which clearly shows that -

(i) The ECSK theory allows a parity violating term when coupled to the Dirac field, which, as we mentioned in the introduction, is due to the Dirac algebra and the total antisymmetry of the spin angular momentum tensor and is well illustrated by equations (4.5.18) and (4.5.19).

(ii) Although equation (4.5.20) shows an additional term which vanishes as $p\rightarrow 0$, substitution of this expression into the Dirac field equation shows quite clearly that the effect of the additional parity violating term in the lagrangian is simply to alter the strength of the parity violating effect in the Dirac field equation. We have /19/,

$$\gamma^{\alpha}\,\nabla_{\alpha}(\{\})\psi + \frac{3}{8(1+p^{2})}\,(\bar{\psi}\gamma_5\gamma^{\beta}\psi)\;\gamma_5\gamma_{\beta}\psi \;+ m\psi = 0\,. \qquad (4.5.21)$$

In the next section, we shall couple our new lagrangian to the Proca field lagrangian and derive an explicit parity violating term that is not present in the ECSK-Proca coupling.

§6. *Coupling to matter fields : The Proca field.*

We have seen in the last section that coupling of torsion to the Dirac field gave rise to a parity-violating term in the field equations which persisted even in the absence of the new parity-violating term that we have added to the ECSK lagrangian i.e., that the usual ECSK lagrangian when added to the Dirac lagrangian gives rise to a parity violating term in the field equations. We also saw that this was basically due to the fact that the spin-angular momentum tensor for the Dirac field is a totally antisymmetric quantity. In this section, we shall carry out a similar analysis for the Proca-or massive Maxwell-field and show that here we have a new parity violating term in the field equations which is not present in the coupling of L_{ECSK} to the Proca field. The lagrangian for the Proca field in flat Minkowski space-time is

$$L_p = -\frac{1}{4}\,G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\,m^2\,A_{\mu}A^{\mu}\,, \qquad (4.6.1)$$

with

$$G_{\mu\nu} := \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\,. \qquad (4.6.2)$$

Introducing torsion through minimal coupling modifies $G_{\mu\nu}$ to $B_{\mu\nu}$;

$$B_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + 2A_{\sigma}S_{\nu\mu}{}^{\sigma} \qquad (4.6.3)$$

and,

$$B_{\mu\nu} = G_{\mu\nu} + 2\,S_{\nu\mu}{}^{\sigma}\,A_{\sigma} \qquad (4.6.4)$$

In terms of $B_{\mu\nu}$, L_p can be written /15/,

$$L_p = \sqrt{-g}\ \left(-\frac{1}{4} B_{\mu\nu}B^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu\right)$$

$$= -\frac{\sqrt{-g}}{4}\ G_{\mu\nu}G^{\mu\nu} - \frac{\sqrt{-g}}{2}\ G_{\mu\nu}A_\sigma S^{\nu\mu\sigma} - \frac{\sqrt{-g}}{2}\ G^{\mu\nu}A_\sigma S_{\nu\mu}{}^{\sigma}$$

$$- \sqrt{-g}\ \ A_\rho S_{\nu\mu}{}^{\rho}\ A_\sigma S^{\nu\mu\sigma} + \frac{\sqrt{-g}}{2} m^2 A_\mu A^\mu \qquad (4.6.5)$$

or,

$$L_p = -\frac{\sqrt{-g}}{4}\ G_{\mu\nu}G^{\mu\nu} + \frac{\sqrt{-g}}{2}\ m^2\ A_\mu A^\mu\ - \sqrt{-g}\ G^{\mu\nu}S_{\nu\mu}{}^{\sigma}A_\sigma$$

$$- \sqrt{-g}\ S_{\nu\mu}{}^{\rho}\ S^{\nu\mu\sigma}\ A_\rho A_\sigma\ . \qquad (4.6.6)$$

The spin-angular momentum of a matter field is defined by

$$\sqrt{-g}\ \ \tau_k{}^{ji} = \frac{\delta L_p}{\delta K_{ij}{}^{k}} \qquad (4.6.7)$$

Since L_p contains only $S_{\nu\mu}{}^{\sigma}$, but no derivatives of $S_{\nu\mu}{}^{\sigma}$,

$$\frac{\delta L_p}{\delta K_{ij}{}^{k}} = \frac{\partial L_p}{\partial K_{ij}{}^{k}}$$

$$= -\sqrt{-g}\ G^{\mu\nu}A_\sigma\ \frac{\partial S_{\nu\mu}{}^{\sigma}}{\partial K_{ij}{}^{k}}\ -\sqrt{-g}\ \ S_{\nu\mu}{}^{\rho}A_\rho A_\sigma\ \frac{\partial S^{\nu\mu\sigma}}{\partial K_{ij}{}^{k}}$$

$$- \sqrt{-g}\ \ S^{\nu\mu\sigma}A_\rho A_\sigma\ \frac{\partial S_{\nu\mu}{}^{\rho}}{\partial K_{ij}{}^{k}}$$

$$= -\sqrt{-g}\ \ G^{\mu\nu}A_\sigma\ \frac{\partial S_{\nu\mu}{}^{\sigma}}{\partial K_{ij}{}^{k}}\ - 2\ \sqrt{-g}\ \ S^{\nu\mu\sigma}\ A_\rho A_\sigma\ \frac{\partial S_{\nu\mu}{}^{\rho}}{\partial K_{ij}{}^{k}}$$

$$= -\frac{1}{2}\ \sqrt{-g}\ G^{ij}\ A_k + \frac{1}{2}\ \sqrt{-g}\ \ G^{ji}\ A_k$$

$$-\sqrt{-g}\ S^{ji\sigma}A_k A_\sigma + \sqrt{-g}\ S^{ij\sigma}A_k A_\sigma$$

or,

$$\frac{\delta L_p}{\delta K_{ij}{}^k} = -\sqrt{-g}\ G^{ij}A_k + 2\sqrt{-g}\ S^{ij\ell}A_k A_\ell \ . \qquad (4.6.8)$$

Therefore, we have the result that the spin-angular momentum of the Proca field is given by ,

$$\tau_k{}^{ij} = -G^{ij}A_k + 2\,S^{ij\ell}A_k A_\ell \ . \qquad (4.6.9)$$

The lagrangian for the total system is

$$L = L_{ECSK} + L_A + L_p$$

and, again the field equations obtained by variations with respect to $K_{ij}{}^k$ are simply

$$\frac{\partial L}{\partial K_{ij}{}^k} = 0 \qquad (4.6.10)$$

or,
$$\frac{\partial L_{ECSK}}{\partial K_{ij}{}^k} + \frac{\partial L_A}{\partial K_{ij}{}^k} + \frac{\partial L_p}{\partial K_{ij}{}^k} = 0, \qquad (4.6.11)$$

from equations (4.3.2), we see that

$$-\frac{\partial L_{ECSK}}{\partial K_{ij}{}^k} = \sqrt{-g}\ (K_k{}^{ij} + K^j{}_k{}^i) - (K_{\ell k}{}^\ell g^{ij} + K_\ell{}^{\ell j} g^i_k) \qquad (4.6.12)$$

while, writing

$$L_A = p\sqrt{-g}\ \eta^{\mu\nu\lambda\beta}\ g_{\beta\rho}\ K_{\mu\sigma}{}^\rho\ K_{\nu\lambda}{}^\sigma, \qquad (4.6.13)$$

we obtain,

$$\frac{\partial L_A}{\partial K_{ij}{}^{k}} = p\sqrt{-g}\,(\eta^{i\nu\lambda}{}_{k}K_{\nu\lambda}{}^{j} + \eta^{i\nu\lambda j}K_{\nu k\lambda}) \; . \tag{4.6.14}$$

Substitution of equations (4.6.9), (4.6.12) and (4.6.14) yields the desired field equation for the spin-angular momentum

$$\tau_k{}^{ij} = -p\;(\eta^{i\nu\lambda}{}_{k}K_{\nu\lambda}{}^{j} + \eta^{i\nu\lambda j}K_{\nu k\lambda})$$
$$- (K_k{}^{ij} + K^{j}{}_{k}{}^{i}) + (g^{ij}K_{\ell k}{}^{\ell} + g^{i}{}_{k}K_{\ell}{}^{\ell j}) \; . \tag{4.6.15}$$

The field equation for the Proca field reads

$$\nabla_\rho G_\mu{}^\rho + 2\nabla_\rho\,(A_\sigma S_\mu{}^{\rho\sigma}) + m^2 A_\mu = 0 \quad . \tag{4.6.16}$$

In order to exhibit a parity violating effect in the Proca field equation, we must solve equation (4.6.15) for the 24 components of the torsion in terms of the spin-angular momentum. Then, substitution into (4.6.16) will reveal the desired term..

The process for solving equation (4.6.15) is simple but laborious (see appendix IV(C)) . Let us first write out (4.6.15) in a suggestive form by using the modified torsion tensor, it is,

$$T_{kji} = \tau_{kji} + p\eta^{\nu\lambda}{}_{i\sigma}K_{\nu\lambda\rho}(\delta^\sigma_k\delta^\rho_j - \delta^\sigma_j\delta^\rho_k) \; . \tag{4.6.17}$$

Now it is quite a simple matter to show that the solution to this equation is,

$$S_{kji} = \frac{1}{(1+8p^2)}\{\tau_{\alpha\beta i}\;(\delta^\alpha_k\delta^\beta_j + p\eta^{\alpha\beta}{}_{kj})$$
$$- \frac{1}{2}\,g_{ij}\;(\tau_{k\ell}{}^{\ell} + p\,\eta_k{}^{\gamma\alpha\beta}\tau_{\alpha\beta\gamma})$$
$$+ \frac{1}{2}\;g_{ik}\;(\tau_{j\ell}{}^{\ell} + p\,\eta_j{}^{\gamma\alpha\beta}\tau_{\alpha\beta\gamma}\,)\,\} \; . \tag{4.6.18}$$

Here we see that the coefficients of the term containing $\eta^{ijk\ell}$, are non-zero, hence,when (4.6.18) is substituted into the field equation for the Proca field, we shall have parity violating interaction terms present. That the usual lagrangian for the ECSK theory does not contain these parity violating terms when coupled to the Proca field /5/, is easily seen: In (4.6.18), the limit p=o does not contain any parity violating terms.

CONCLUSIONS AND DISCUSSION

What can we conclude from the last four chapters? Firstly, in the introductory chapter, Chapter I, we saw how gauge theories were defined, in terms of what might be called an "internal symmetry". We also saw how gauge fields were introduced as connections on the Lie (symmetry) group manifold, in total analogy with the definition of Christoffel symbols on space-time, in particular, we saw that electromagnetism could be derived as a gauge theory. Metric-torsion theories, in particular the ECSK theory was seen to be a simple and natural generalisation of Einstein's general relativity, when attempting to extend the gravitational phenomenon to the microphysical realm of elementary particle physics.

An immediate, surprising problem arose with this programme of extending general relativity into microphysics when we attempted to couple gauge fields to the new torsion field of metric-torsion theories. This coupling, we saw in Chapter II, was possible only if gauge invariance is given up. As gauge theories are a very important part of present day elementary particle physics, it was argued that one must resist loss of gauge invariance to the last! One alternative to this loss of gauge invariance, suggested by /5/ is to couple gauge fields to the torsionless, Christoffel connection. This is allowed because of the generalised geometry; we have the freedom to choose either of two connections which can now be defined on the manifold, namely, the Christoffel connection and the full, asymmetric connection.

Of course, as explained in the introduction, this defeats the purpose of introducing torsion into general relativity. S. Hojman et, al. /8/, determined to couple torsion to electrodynamics, modified the gauge covariant derivative of electrodynamics, thus introducing a new variable, which was then determined by requiring gauge invariance. This, we saw, led to two consequences. One was to make the new variable into a scalar field and the second was to restrict the possible types of torsion that could couple to gauge fields. Indeed, torsion was required to be essentially, the gradient of the new scalar field. An added bonus was pointed out, namely that the torsion being the gradient of a scalar field, could be used to construct a metric-torsion theory that allowed for the first time, propagating torsion (albeit in a restricted form through propagation of the scalar field), within the confines of a linear R theory.

However, the job of coupling torsion to gauge fields was only half done. For, according to modern theories of elementary particle physics, the electromagnetic field is not the only gauge field. Instead, the electromagnetic field is just one of a large number of gauge fields that are introduced in order to explain the elementary forces. With this in mind, we set out to generalise the procedure of Hojman et. al. /8/ to non-abelian gauge theories /9/.

It was shown that a generalisation of the non-abelian gauge covariant derivative in blind analogy with /8/, led us back to the torsionless example of general relativity upon requiring gauge invariance. A successful generalisation was carried out by modifying the gauge field strength tensor of non-abelian gauge fields in addition to modifying the non-abelian gauge covariant derivative. One curious aspect of this generalisation was that even though two new variables were introduced, in the final analysis, we were left with only one scalar field. Furthermore, torsion was restricted to be of a special form, the gradient of the scalar field. That is, the special form of torsion derived in the electrodynamic example was carried through to the non-abelian case.

This effect was briefly explained as being due to the fact that the two modifications were equivalent to allowing the gauge coupling parameters, or gauge charges to become space-time functions. This would mean for example, that two electrons would repel each other with a force that depends on their position in space-time. In particular, it would mean that the electromagnetic energy of an atom of say, Gold would be different from that of an atom of say, aluminium. Indeed, W.T.Ni (Physical Review D19 (1979) 2260) has shown that if the ECSK theory is taken to be the correct theory of gravitation and if torsion is coupled to electrodynamics in the form suggested in Chapter II and ref./8/ then the equivalence principle would hold up to $\sim 10^{-7}\underline{\nabla}U$ (where U is the newtonian potential and $\underline{\nabla}$ denotes the gradient), i.e., the accelerations suffered by Gold and aluminium atoms in the earth's gravity field would differ by $\sim 10^{-7}\underline{\nabla}U$. However, we know that the equivalence principle has been tested experimentally to 12 orders of magnitude, $10^{-12}\underline{\nabla}U$ (see for example, V.B. Braginsky and V.I. Panov, JETP 34 (1972) 463). So we are forced to abandon the coupling of torsion to gauge fields. Rather than throwing away torsion completely, it may be better to accept the suggestion of /5/ and couple torsion to matter fields other than gauge fields, and leave gauge fields to couple to the Christoffel connection as in general relativity. We need not be totally disheartened by this result, for we know that since 1952, several workers in solid state physics, particularly, Kondo and Bilby, Bullough and Smith (see e.g. refs. in /5/) have taken up Riemann-Cartan geometry to describe the theory of continuous dislocations of crystals. So we may hope that although the theory of Chapter II cannot be defended as a description of space-time, it may lead to new effects in the continuous dislocation theory of crystals.

Chapter III was devoted to a new form of variational principles. The conventional variational principle, studies small variations of an action integral over four-volume. The integrand is taken to be an invariant function of the field variables and their partial derivatives. It is well known that most physical field variables have specific geometrical properties. e.g., the components of a vector field form the electromagnetic potential, while the gravitational field is described by a metric tensor, whose components transform like those of a second rank symmetric tensor under general coordinate transformations. The conventional variational principle is insensitive to the transformation properties of the field variables. Invariant variational principles are defined to remove this defect in the conventional variational principle. The technique, when applied to a metric variational problem, by taking a lagrangian of the form $L(g,\partial g,\partial\partial g)$ imposed restrictions on the form of admissible lagrangians in requiring the lagrangian and its partial derivatives to satisfy three identities. It was shown that some of these identities were remarkable extensions of well known theorems in Riemannian geometry.

Despite the expected complexities in the calculations, the invariant variational principle was generalised to metric-torsion theories by applying the method to a lagrangian of the form $L(g,\partial g,\partial\partial g, s,\partial s)$. S being the torsion tensor. Once again, the restrictions on the form of admissible lagrangians were derived in the form of three identities satisfied by the lagrangian and its partial derivatives. Due to the large number of terms in the identities, we showed how the method could be applied to derive field equations by deriving them for a simplified lagrangian of the form $L(g,\partial g,\partial\partial g,s)$. No generality is lost in doing this, except of course, the theory would not allow propagating torsion. This is an important feature of any metric-torsion theory, for without it, we have the result that in the vacuum (absence of matter), it is equivalent to general relativity.

Lastly in Chapter IV, a new metric-torsion theory, allowing a parity-violating interaction was put forward. There, we saw that one could add an additional, pseudo-scalar term to the ECSK lagrangian while still keeping it linear in the curvature. Because of this linearity in the curvature, by virtue of the field equations, this theory is equivalent to general relativity, in the absence of any matter fields. Some suggestions to alleviate this problem are as follows :

(i) Use the special form of torsion, derived in Chapter II as an ansatz to obtain a restricted form of dynamic, propagating torsion .

(ii) Add terms to the ECSK lagrangian that would allow propagating torsion. These terms could simply be either second derivatives in the torsion, or quadratics in the first derivatives of the torsion.

This second possibility was discussed, and ruled out on the grounds that each additional term should, in general, carry an arbitrary coupling constant, analogous to the Newtonian constant. The parity-violating effects were explicitly exhibited for two matter fields, the Dirac and Proca field.

The parity-violation terms in the Proca field equations are the ones that should be looked at if one wishes to check them experimentally, for they offer the possibility of not only establishing the existence or non-existence of torsion, but offer the possibility of distinguishing between the ECSK theory and the theory we have put forward.

In this thesis, we have concerned ourselves with the purely classical aspects of metric-torsion theories of gravity. It is to be hoped that the results contained herein will find some significance in a quantum theory of gravity.

APPENDICES

APPENDIX II (A)

In this appendix we shall prove that the group property on $A_\mu (=A^i_\mu T_i)$ holds, i.e. if under a gauge transformation $U(\varepsilon)$, A_μ is transformed into A'_μ and if under a further gauge transformation $U(\eta)$, A'_μ is transformed into A''_μ, then, can we find a parameter ξ, such that the gauge transformation $U(\xi) = U(\eta)\, U(\varepsilon)$ carries A_μ into A''_μ ?

Clearly this is an important group property, for without it, we cannot, beginning with series of infinitesimal gauge transformations, build up a finite gauge transformation.

From equations (2.4.11) and (2.4.12) we have;

$$A'_\alpha = -\frac{i}{g} C_\alpha{}^\mu (\partial_\mu U(\varepsilon))\, U^{-1}(\varepsilon) + U(\varepsilon)\, A_\alpha U^{-1}(\varepsilon) \tag{A.1}$$

and
$$A''_\alpha = -\frac{1}{g} C_\alpha{}^\mu (\partial_\mu U(\eta))\, U^{-1}(\eta) + U(\eta) A'_\alpha\, U^{-1}(\eta) \tag{A.2}$$

Upon substitution for A'_α we obtain

$$A'_\alpha = -\frac{-i}{g} C_\alpha{}^\mu (\partial_\mu U(\eta))\, U^{-1}(\eta)$$
$$+ U(\eta)\{ -\frac{i}{g} C_\alpha{}^\mu (\partial_\mu U(\varepsilon))\, U^{-1}(\varepsilon) + U(\varepsilon)\, A_\alpha U^{-1}(\varepsilon)\}\, U^{-1}(\eta) \tag{A.3}$$

$$= -\frac{i}{g} C_\alpha{}^\mu \{(\partial_\mu U(\eta))\, U(\varepsilon) + U(\eta)\partial_\mu\, U(\varepsilon)\}\ \ U^{-1}(\varepsilon)\, U^{-1}(\eta)$$
$$+ U(\eta)\, U(\varepsilon)\, A_\alpha U^{-1}(\varepsilon)\, U^{-1}(\eta) \tag{A.4}$$

or,

$$A''_\alpha = -\frac{i}{g} C_\alpha{}^\mu \partial_\nu \{U(\eta)\, U(\varepsilon)\}\ \ \{U(\eta)\, U(\varepsilon)\}^{-1}$$

$$+\{U(\eta)\, U(\varepsilon)\}\, A_\alpha \{U(\eta) U(\varepsilon)\}^{-1} \tag{A.5}$$

we have used the relations that $U^{-1}(\varepsilon)\, U^{-1}(\eta) = \{U(\eta) U(\varepsilon)\}^{-1}$ (A.6) and,

$$(\partial_\mu U(\eta))\, U(\varepsilon) \;+\; U(\eta)\, \partial_\mu U(\varepsilon) = \partial_\mu\{U(\eta)\, U(\varepsilon)\} \;. \qquad (A.7)$$

Defining $U(\xi) \;:=\; U(\eta)\, U(\varepsilon)$,

we have,

$$A''_\alpha \;=\; -\frac{i}{g}\, C_\alpha{}^\mu (\partial_\mu U(\xi))\; U^{-1}(\xi) + U(\xi) A_\alpha U^{-1}(\xi) \qquad (A.8)$$

Therefore, we see that altering the gauge covariant derivative has not spoiled the underlying Lie algebra structure, and we can carry through the analysis as if nothing had changed.

APPENDIX II (B)

As mentioned in §4, we shall prove in this appendix, that coupling of torsion through the modified gauge covariant derivative,

$$D_\mu = \partial_\mu - ig\, b_\mu^{\ \alpha}\,(T.A_\alpha) \ , \tag{B.1}$$

to the unmodified field strength tensor,

$$\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu , A_\nu] + 2\, S_{\nu\mu .}^{\ \ \ \sigma} A_\sigma \tag{B.2}$$

leads to the trivial result that

$$b_\mu^{\ \alpha} = \delta_\mu^{\ \alpha} \text{ and } S_{\nu\mu .}^{\ \ \ \sigma} = 0 \quad . \tag{B.3}$$

From equation (2.4.10), remembering the notation that

$$A_\mu \equiv A_\mu^i\, T_i \quad , \tag{B.4}$$

we have that under a gauge transformation, A_μ transforms as

$$A'_\alpha = -\frac{i}{g}\, C_\alpha^{\ \mu}(\partial_\mu U)\, U^{-1} + U\, A_\alpha\, U^{-1} \tag{B.5}$$

Suppose, under this transformation, $\tilde{F}_{\mu\nu} \to \tilde{F}'_{\mu\nu}$, then

$$\tilde{F}'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig\,[A'_\mu , A'_\nu] + 2\, S_{\nu\mu}^{\ \ \sigma} A'_\sigma \tag{B.6}$$

$$= -\frac{i}{g}\partial_\mu\{C_\nu^{\ \alpha}(\partial_\alpha U)U^{-1}\} + \partial_\mu\{U A_\nu U^{-1}\}$$

$$+ \frac{i}{g}\partial_\nu\{C_\mu^{\ \alpha}(\partial_\alpha U)U^{-1}\} - \partial_\nu\{U A_\mu U^{-1}\}$$

$$- ig\,[-\frac{i}{g}C_\mu^{\ \alpha}(\partial_\alpha U)U^{-1} + UA_\mu U^{-1}),(-\frac{i}{g}C_\nu^{\alpha}(\partial_\alpha U)U^{-1} + UA_\nu U^{-1})]$$

$$- \frac{2i}{g}\, C_\sigma^{\ \alpha}(\partial_\alpha U)U^{-1}\, S_{\nu\mu}^{\ \ \sigma} + 2U\, A_\sigma\, U^{-1}\, S_{\nu\mu}^{\ \ \sigma} . \tag{B.7}$$

or,

$$\tilde{F}'_{\mu\nu} = U\{\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] + 2S_{\nu\mu}{}^{\sigma}.A_\sigma\}U^{-1}$$

$$- \frac{i}{g}(\partial_\mu C_\nu{}^{\alpha})(\partial_\alpha U)U^{-1} - \frac{i}{g} C_\nu{}^{\alpha}\partial_\mu\{(\partial_\alpha U)U^{-1}\} + \frac{i}{g}(\partial_\nu C_\mu{}^{\alpha})(\partial_\alpha U)U^{-1}$$

$$+ \frac{i}{g} C_\mu{}^{\alpha}\partial_\nu\{(\partial_\alpha U)U^{-1}\} + (\partial_\mu U) A_\nu U^{-1} - (\partial_\nu U) A_\mu U^{-1}$$

$$+ U A_\nu \partial_\mu U^{-1} - U A_\mu \partial_\nu U^{-1} - \frac{2i}{g} C_\sigma{}^{\alpha}(\partial_\alpha U)U^{-1} S_{\nu\mu}{}^{\sigma}$$

$$- ig[-\frac{i}{g} C_\mu{}^{\alpha}(\partial_\alpha U)U^{-1}, -\frac{i}{g} C_\nu{}^{\alpha}(\partial_\alpha U)U^{-1}] \tag{B.8}$$

$$- ig[-\frac{i}{g} C_\mu{}^{\alpha}(\partial_\alpha U)U^{-1}, UA_\nu U^{-1}] - ig[UA_\mu U^{-1}, -\frac{i}{g} C_\nu{}^{\alpha}(\partial_\alpha U)U^{-1}]$$

We must determine $C_\mu{}^{\alpha}$ by requiring $\tilde{F}_{\mu\nu}$ to transform gauge covariantly, as the preserving of gauge invariance is our primary concern. The statement of gauge covariance is simply

$$\tilde{F}'_{\mu\nu} = U \tilde{F}_{\mu\nu} U^{-1} \tag{B.9}$$

From equation (B8), we must therefore have,

$$\begin{aligned}
&- \frac{i}{g}(\partial_\alpha U)\{\partial_\mu C_\nu{}^{\alpha} - \partial_\nu C_\mu{}^{\alpha} + 2C_\sigma{}^{\alpha} S_{\nu\mu}{}^{\sigma}\}U^{-1} \\
&+ \frac{i}{g}(\partial_\sigma\partial_\alpha U)\{C_\mu{}^{\alpha}\delta_\nu{}^{\sigma} - C_\nu{}^{\alpha}\delta_\mu{}^{\sigma}\}U^{-1} \\
&+ \frac{i}{g}(\partial_\alpha U)\{C_\mu{}^{\alpha}\delta_\nu{}^{\sigma} - C_\nu{}^{\alpha}\delta_\mu{}^{\sigma}\}(\partial_\sigma U^{-1}) \\
&+ (\partial_\sigma U)\{A_\nu\delta_\mu{}^{\sigma} - A_\mu\delta_\nu{}^{\sigma}\}U^{-1} + U\{A_\nu\delta_\mu{}^{\sigma} - A_\mu\delta_\nu{}^{\sigma}\}(\partial_\sigma U^{-1}) \\
&- \frac{i}{g} C_\mu{}^{\alpha} C_\nu^{\beta}(\partial_\alpha U)(\partial_\beta U^{-1}) + \frac{i}{g} C_\nu{}^{\alpha} C_\mu^{\beta}(\partial_\alpha U)(\partial_\beta U^{-1}) \\
&- C_\mu{}^{\alpha}(\partial_\alpha U) A_\nu U^{-1} - C_\mu{}^{\alpha} U A_\nu(\partial_\alpha U^{-1}) \\
&+ C_\nu{}^{\alpha} U A_\mu(\partial_\alpha U^{-1}) + C_\nu{}^{\alpha}(\partial_\alpha U) A_\mu U^{-1} \\
&= 0.
\end{aligned} \tag{B.10}$$

where we have used the following results:

Since $U(\epsilon)$ is a unitary matrix operator, we have

$$UU^{-1} = U^{-1}U = I, \tag{B.11}$$

differentiating with respect to x^{μ},

$$(\partial_{\mu}U)\, U^{-1} = - U(\partial_{\mu}U^{-1}) \tag{B.12}$$

and,

$$(\partial_{\mu}U^{-1}) = - U^{-1}(\partial_{\mu}U)\, U^{-1} \tag{B.13}$$

Equation (B.10) can be further simplified, by collecting terms, to

$$\begin{aligned}
&- \frac{i}{g}(\partial_{\alpha}U)\{\partial_{\mu}C_{\nu}{}^{\alpha}-\partial_{\nu}C_{\mu}{}^{\alpha}+ 2C_{\sigma}{}^{\alpha}S_{\nu\mu}{}^{\sigma}- ig\, C_{\mu}{}^{\alpha}A_{\nu} + ig\, C_{\nu}{}^{\alpha}A_{\mu} \\
&\qquad + ig\delta_{\mu}{}^{\alpha} A_{\nu} - ig\, \delta_{\nu}{}^{\alpha}A_{\mu}\}\, U^{-1} \\
&+ \frac{i}{g} (\partial_{\sigma}\partial_{\alpha}U)\{ C_{\mu}{}^{\alpha}\delta_{\nu}{}^{\sigma}- C_{\nu}{}^{\alpha}\delta_{\mu}{}^{\sigma}\}U^{-1} \\
&+ \frac{i}{g} (\partial_{\alpha}U)\, \{C_{\mu}{}^{\alpha}\delta_{\nu}{}^{\sigma}- C_{\nu}{}^{\alpha}\delta_{\mu}{}^{\sigma}- C_{\mu}{}^{\alpha}C_{\nu}{}^{\sigma} + C_{\nu}{}^{\alpha}C_{\mu}{}^{\sigma}\}\, (\partial_{\sigma}U^{-1}) \\
&+ U\, \{A_{\nu}\delta_{\mu}{}^{\sigma} - A_{\mu}\delta_{\nu}{}^{\sigma}+ C_{\nu}{}^{\sigma}A_{\mu} - C_{\mu}{}^{\sigma}A_{\nu}\}\, (\partial_{\sigma}U^{-1}) \\
&= 0.
\end{aligned} \tag{B.14}$$

So, we must solve for the following four simultaneous equations

$$\partial_{\mu}C_{\nu}{}^{\alpha}- \partial_{\nu}C_{\mu}^{\alpha}+2C_{\sigma}{}^{\alpha}S_{\nu\mu}{}^{\sigma}+igA_{\mu}(C_{\nu}{}^{\alpha}-\delta_{\nu}{}^{\alpha})+ig\, A_{\nu}(\delta_{\mu}{}^{\alpha}- C_{\mu}{}^{\alpha}) = 0 \tag{B.15}$$

$$C_{\mu}^{(\alpha}\delta_{\nu}{}^{\sigma)} - C_{\nu}^{(\alpha}\, \delta_{\mu}{}^{\sigma)} = 0 \tag{B.16}$$

$$C_{\mu}{}^{\alpha}\, (\delta_{\nu}{}^{\sigma}- C_{\nu}{}^{\sigma}) + C_{\nu}{}^{\alpha}\, (C_{\mu}{}^{\sigma}- \delta_{\mu}{}^{\sigma}) = 0 \tag{B.17}$$

and

$$A_{\nu}\, (\delta_{\mu}{}^{\sigma} - C_{\mu}{}^{\sigma}) + A_{\mu}(C_{\nu}{}^{\sigma} - \delta_{\nu}{}^{\sigma}) = 0 \tag{B.18}$$

Equation (B.15) is a complex equation, so its solution is obtained by equating its real and imaginary parts separately, to zero. We then see that equation (B.18) is contained in (B.15). Noting that equation (B.16) is identical to the one obtained in the electrodynamic example allows us to write down its solution immediately as

$$C_\mu{}^\alpha = f(x)\delta_\mu{}^\alpha. \tag{B.19}$$

Substitution into (B.17) reveals that either

$$f(x) = 0, \quad \text{or } f(x) = 1 \tag{B.20}$$

The solution $f(x) = 0$ is singular (remember that $C_\mu{}^\alpha$ is the inverse of $b_\mu{}^\alpha$), and we discard it, leaving us with the trivial solution that

$$C_\mu{}^\alpha = \delta_\mu{}^\alpha \quad \text{and} \quad b_\mu{}^\alpha = \delta_\mu{}^\alpha \tag{B.21}$$

The real part of equation (B.15) is

$$\partial_\mu C_\nu{}^\alpha - \partial_\nu C_\mu{}^\alpha + 2C_\sigma{}^\alpha S_{\nu\mu}{}^\sigma = 0 \tag{B.22}$$

or,

$$S_{\nu\mu}{}^\alpha = 0, \tag{B.23}$$

after using equation (B.21).

While equation (B.18), or equivalently, the imaginary part of equation (B.15) is satisfied identically due to equation (B.21). So we see that in the blind generalisation of the gauge covariant derivative from electrodynamics to non-abelian gauge fields, we have lost something. In reality, we have not gone far enough in the generalisation, leaving us with nothing but the torsionless case of general relativity /9/ .

APPENDIX III (A)

Construction of tensorial quantities /11/.

The lagrangian scalar density that we have, satisfies the following transformation law,

$$\bar{L}(\bar{g}_{ij},\bar{g}_{ij,k},\bar{g}_{ij,k\ell}) = B\, L(g_{ij}.g_{ij,k},g_{ij,k\ell}) \tag{A.1}$$

under $\bar{x}^i = \bar{x}^i\,(x^j)$, (A.2)

where $B = \left| \dfrac{\partial x^i}{\partial \bar{x}^j} \right|$. (A.3)

we shall also define $B^i_{\ j} = \dfrac{\partial x^i}{\partial \bar{x}^j}$, $B^i_{\ jk} = \dfrac{\partial}{\partial \bar{x}^k} B^i_{\ j}$, etc. (A.4)

Since one of the important features of IVP's is the exploitation of the tensorial properties of the field functions, the transformation rules for g_{ij}, $g_{ij,k}$ and $g_{ij,k\ell}$ are found to be;

$$\bar{g}_{hk} = B^a_{\ h}\, B^b_{\ k}\, g_{ab} \, . \tag{A.5}$$

$$\bar{g}_{hk,\ell} = \frac{\partial\, \bar{g}_{hk}}{\partial \bar{x}^\ell} = B^a_{\ h\ell} B^b_{\ k}\, g_{ab} + B^a_{\ h}\, B^b_{\ k\ell} g_{ab} + B^a_h\, B^b_{\ k} g_{ab,c} B^c_\ell \tag{A.6}$$

and

$$\bar{g}_{hk,\ell m} = \frac{\partial}{\partial \bar{x}^m}(B^a_{\ h\ell}B^b_{\ k}g_{ab}) + \frac{\partial}{\partial \bar{x}^m}(B^a_{\ h}B^b_{\ k\ell}g_{ab}) + \frac{\partial}{\partial \bar{x}^m}(B^a_{\ h}B^b_{\ k}g_{ab,c}B^c_\ell) . \tag{A.7}$$

Let us also define

$$\Lambda^{ij} = \frac{\partial L}{\partial g_{ij}} \quad , \Lambda^{ij,k} = \frac{\partial L}{\partial g_{ij,k}} \quad , \Lambda^{ij,k\ell} = \frac{\partial L}{\partial g_{ij,k\ell}} \tag{A.8}$$

and their tensorial counterparts will be denoted by Π^{ij}, $\Pi^{ij,k}$ and $\Lambda^{ij,k\ell}$ respectively. The last, because $\Lambda^{ij,k\ell}$ is already a tensor density. We shall now show that Λ^{ij} and $\Lambda^{ij,k}$ are not tensor densities while $\Lambda^{ij,k\ell}$ is, by evaluating their transformation laws. Differentiate (A1) with respect to $g_{ab,cd}$, $g_{ab,c}$ and g_{ab},

$$B\Lambda^{ab,cd} = \frac{\partial \bar{L}}{\partial \bar{g}_{ij,k\ell}} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab,cd}} \qquad (A.9)$$

or,

$$B\Lambda^{ab,cd} = \bar{\Lambda}^{ij,k\ell} B^a_i B^b_j B^c_k B^d_\ell \qquad (A.10)$$

having used (A.7),

Equation (A.10) demonstrates clearly that $\Lambda^{ij,k\ell}$ is a tensor density of weight 1 and contravariant rank 4.

We also have

$$B\Lambda^{ab,c} = \frac{\partial \bar{L}}{\partial \bar{g}_{ij,k\ell}} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab,c}} + \frac{\partial \bar{L}}{\partial \bar{g}_{ij,k}} \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} \qquad (A.11)$$

or,

$$B\Lambda^{ab,c} = \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab,c}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} \qquad (A.12)$$

we need go no further, for we see that this first term in equation (A.12) spoils that tensorial character of $\Lambda^{ab,c}$.

Similarly, we have

$$B\Lambda^{ab} = \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} + \bar{\Lambda}^{ij} \frac{\partial \overline{g_{ij}}}{\partial g_{ab}}$$

(A.13)

Here, the first two terms spoil the tensorial character of Λ^{ij}.

There was a purpose in deriving equations (A.10), (A.12) and (A.13).

We shall need the equations in order to derive their tensorial forms. The method we shall use is an indirect one, and is based on the definition of a scalar quantity G,

$$G = \Lambda^{ij,k\ell} h_{ij,k\ell} + \Lambda^{ij,k} h_{ij,k} + \Lambda^{ij} h_{ij}, \qquad (A.14)$$

where the h_{ij} are components of an arbitrary second rank tensor having the same symmetries as the metric tensor. This quantity, G, is a scalar density :

$$BG = B\Lambda^{ab,cd} h_{ab,cd} + B\Lambda^{ab,c} h_{ab,c} + B\Lambda^{ab} h_{ab}, \qquad (A.15)$$

or,

$$BG = \bar{\Lambda}^{ij,k\ell}\left(\frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab,cd}} h_{ab,cd} + \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} h_{ab,c} + \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} h_{ab} \right)$$

$$+ \bar{\Lambda}^{ij,k}\left(\frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} h_{ab,c} + \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} h_{ab} \right)$$

$$+ \bar{\Lambda}^{ij}\left(\frac{\partial \bar{g}_{ij}}{\partial g_{ab}} h_{ab} \right) \qquad (A.16)$$

From equations (A.5), (A.6) and (A.7), it is easy to see that the terms in square brackets in equation (A.16) are simply $\bar{h}_{ij,k\ell}$ $\bar{h}_{ij,k}$ and $\bar{h}_{ij}$ respectively.

We have therefore,

$$BG = \bar{G}, \qquad (A.17)$$

i.e., G is a scalar density.

Now what is required is the construction of quantities Π^{ij}, $\Pi^{ij,k}$ so that G can be written as

$$G = \Lambda^{ij,k\ell} h_{ij;k\ell} + \Pi^{ij,k} h_{ij;k} + \Pi^{ij} h_{ij} \qquad \text{(A.18)}$$

where the semi-colon denotes covariant differentiation. As we shall be needing Π^{ij} and $\Pi^{ij,k}$, later on for the non-zero torsion example, we shall evaluate Π^{ij} and $\Pi^{ij,k}$ here, without assuming any symmetry on the connection symbols $\Gamma^{i}{}_{jk}$. In reality, for the metric example, we always have the Christoffel symbols in mind. We know that

$$h_{ij;k} = h_{ij,k} - \Gamma^{\alpha}_{ki} h_{\alpha j} - \Gamma^{\alpha}_{kj} h_{i\alpha} \qquad \text{(A.19)}$$

Similarly for $h_{ij,k\ell}$.

Substitution into (A.18) yields

$$G = \Pi^{ij} h_{ij} + \Pi^{ij,k} h_{ij,k} - \Pi^{ij,k} \Gamma^{\alpha}_{ki} h_{\alpha j} - \Pi^{ij,k} \Gamma^{\alpha}_{kj} h_{i\alpha}$$

$$+ \Lambda^{ij,k\ell} h_{ij,k\ell} - \Lambda^{ij,k\ell} \Gamma^{\alpha}_{\ell i} h_{\alpha j,k} - \Lambda^{ij,k\ell} \Gamma^{\alpha}_{\ell j} h_{i\alpha,k} - \Lambda^{ij,k\ell} \Gamma^{\alpha}_{\ell k} h_{ij,\alpha}$$

$$-\Lambda^{ij,k\ell} \Gamma^{\alpha}_{ki} h_{\alpha j,\ell} - \Lambda^{ij,k\ell} \Gamma^{\alpha}_{kj} h_{i\alpha,\ell} + \Lambda^{ij,k\ell} \Gamma^{\beta}_{\ell k} \Gamma^{\alpha}_{\beta i} h_{\alpha j} + \Lambda^{ij,k\ell} \Gamma^{\beta}_{\ell i} \Gamma^{\alpha}_{k\beta} h_{\alpha j}$$

$$+ \Lambda^{ij,k\ell} \Gamma^{\beta}_{\ell j} \Gamma^{\alpha}_{ki} h_{\alpha\beta} + \Lambda^{ij,k\ell} \Gamma^{\beta}_{\ell k} \Gamma^{\alpha}_{\beta j} h_{i\alpha} + \Lambda^{ij,k\ell} \Gamma^{\beta}_{\ell j} \Gamma^{\alpha}_{k\beta} h_{i\alpha}$$

$$+ \Lambda^{ij,k\ell} \Gamma^{\beta}_{\ell i} \Gamma^{\alpha}_{kj} h_{\beta\alpha} - \Lambda^{ij,k\ell} \Gamma^{\alpha}_{ki,\ell} h_{\alpha j} - \Gamma^{\alpha}_{kj,\ell} h_{i\alpha} \Lambda^{ij,k\ell} \qquad \text{(A.20)}$$

Collecting together terms, and equating coefficients of h_{ij} and $h_{ij,k}$ after suitable symmetrisations, we find that

$$\Pi^{ij,k} = \Lambda^{ij,k} + 2\Lambda^{\alpha j,k\ell}\,\Gamma^{i}_{\ell\alpha} + 2\,\Lambda^{\alpha i,k\ell}\Gamma^{j}_{\ell\alpha} + \Lambda^{ij,\alpha\ell}\,\Gamma^{k}_{\ell\alpha} \qquad \text{(A.21)}$$

while

$$\Pi^{ij} = \Lambda^{ij} + \Pi^{\alpha j,k}\,\Gamma^{i}_{k\alpha} + \Pi^{\alpha i,k}\,\Gamma^{j}_{k\alpha} - \Lambda^{\alpha j,k\ell}\Gamma^{\beta}_{\ell k}\,\Gamma^{i}_{\beta\alpha}$$

$$- \Lambda^{\alpha i,k\ell}\Gamma^{\beta}_{\ell k}\,\Gamma^{j}_{\beta\alpha} - 2\Lambda^{\alpha\beta,k\ell}\Gamma^{j}_{\ell\beta}\,\Gamma^{i}_{k\alpha} - \Lambda^{\alpha j,k\ell}\,\Gamma^{\beta}_{\ell\alpha}\,\Gamma^{i}_{k\beta}$$

$$- \Lambda^{\alpha i,k\ell}\Gamma^{\beta}_{\ell\alpha}\,\Gamma^{j}_{k\beta} + \Lambda^{\alpha j,k\ell}\,\Gamma^{i}_{k\alpha,\ell} + \Lambda^{\alpha i,k\ell}\,\Gamma^{j}_{k\alpha,\ell} \qquad \text{(A.22)}$$

where the symmetries of $\Lambda^{ij,k\ell}$ have been used and also the symmetry of $\Pi^{ij,k}$ in indices (ij).

The proof that the quantities $\Pi^{ij,k}$ and Π^{ij}, given in equations (A.21) and (A.22) are tensorial is quite easy. We have that G, as defined by equation (A.18) is a scalar density.

We also know that

$$\Lambda^{ij,k\ell}\,h_{ij;k\ell} \qquad \text{(A.23)}$$

is a scalar density, so we have

$$G - \Lambda^{ij,k\ell}h_{ij;k\ell} \qquad \text{(A.24)}$$

being a scalar density. Hence

$$\Pi^{ij}h_{ij} + \Pi^{ij,k}\,h_{ij;k} \qquad \text{(A.25)}$$

are components of a scalar density, and by a simple generalisation of the quotient theorems of tensor calculus, we have that Π^{ij} is a tensor density of weight 1 and contravariant rank 2, while $\Pi^{ij,k}$ is a tensor density of rank 3.

APPENDIX III (B).

Construction of identities for $L = L(g,\partial g,\partial\partial g)$ /11/.

We have,

$$\bar{L}(\bar{g}_{ij},\bar{g}_{ij,k},\bar{g}_{ij,k\ell}) = BL(g_{ij},g_{ij,k},g_{ij,k\ell}) \qquad (B.1)$$

with

$$\bar{g}_{hk} = B^a_h\, B^b_k\, g_{ab} , \qquad (B.2)$$

$$\bar{g}_{hk,\ell} = B^a_{j\ell}\, B^b{}_k\, g_{ab} + B^a_h\, B^b_{k\ell}\, g_{ab} + B^a_h\, B^b_k\, g_{ab,c} B^c_\ell \qquad (B.3)$$

and

$$\bar{g}_{hk,\ell m} = \frac{\partial}{\partial \bar{x}^m} (B^a_{h\ell} B^b{}_k g_{ab}) + \frac{\partial}{\partial \bar{x}^m} (B^a_h B^b_{k\ell} g_{ab}) + \frac{\partial}{\partial \bar{x}^m} (B^a_h B^b_k B^c_\ell g_{ab,c})$$

(B.4)

The first identity.

The right hand side of equation (B.1) is independent of B^i_{npq} so, differentiating (B.1) with respect to B^i_{npq}

$$\frac{\partial \bar{L}}{\partial \bar{g}_{hk}} \frac{\partial \bar{g}_{hk}}{\partial B^i_{npq}} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell}} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{npq}} + \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell m}} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{npq}} = 0 \qquad (B.5)$$

Inspection of equations (B.2) and (B.3) shows that $\bar{g}_{hk}$ and $\bar{g}_{hk,\ell}$ are independent of B^i_{npq} , so we have

$$\bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{npq}} = 0 \qquad (B.6)$$

Explicitly writing out (B.4), it is easy to show that

$$\frac{\partial \bar{g}_{hk,\ell m}}{\partial B^{i}_{npq}} = \delta^a_i \delta^n_h \delta^p_\ell \delta^q_m B^b_k g_{ab} + B^a_h \delta^b_i \delta^n_k \delta^p_\ell \delta^q_m g_{ab} \qquad (B.7)$$

so that (B.6) can be written as

$$\left\{ \bar{\Lambda}^{nk,pq} g_{ib} B^b{}_k + \bar{\Lambda}^{hn,pq} B^b_h g_{ib} \right\}^{(npq)} = 0 \qquad (B.8)$$

or,

$$\left\{ (\bar{\Lambda}^{nk,pq} + \bar{\Lambda}^{kn,pq}) \; g_{ib} B^b{}_k \right\}^{(npq)} = 0 \qquad (B.9)$$

This is true for arbitrary $B^b{}_k$, so, in particular, it is true for $B^b{}_k = \delta^b{}_k$.

We therefore have

$$\left\{ \bar{\Lambda}^{nk,pq} + \bar{\Lambda}^{kn,pq} \right\}^{(npq)} = 0 \qquad (B.10)$$

Also, since this is a tensorial equation, we can remove the "bars"

$$\left\{ \Lambda^{nk,pq} + \Lambda^{kn,pq} \right\}^{(npq)} = 0 \qquad (B.11)$$

where we have used the notation $[\;\;]^{(npq)}$, to denote symmetrisation in indices (npq) because $B^i{}_{npq}$ is totally symmetric in (npq).

So, we have

$$\Lambda^{nk,pq} + \Lambda^{kn,pq} + \Lambda^{pk,qn} + \Lambda^{kp,qn} + \Lambda^{qk,pn} + \Lambda^{kq,pn} = 0 \qquad (B.12)$$

or,

$$\Lambda^{kn,pq} + \Lambda^{kp,qn} + \Lambda^{kq,np} = 0 \qquad (B.13)$$

upon using the symmetries

$$\Lambda^{ij,k\ell} = \Lambda^{ji,k\ell} = \Lambda^{ij,\ell k} \quad . \tag{B.14}$$

From equation (B.13) ,

$$\Lambda^{kn,pq} = -\Lambda^{kp,qn} - \Lambda^{kq,np} \tag{B.15}$$

$$= \Lambda^{pn,kq} + \Lambda^{pq,nk} + \Lambda^{qn,kp} + \Lambda^{qp,kn} \tag{B.16}$$

$$= 2\Lambda^{pq,nk} + \Lambda^{np,kq} + \Lambda^{nq,kp} \tag{B.17}$$

$$= 2\,\Lambda^{pq,nk} - \Lambda^{nk,qp} \tag{B.18}$$

therefore,

$$\Lambda^{nk,pq} = \Lambda^{pq,nk} \tag{B.19}$$

This is the first identity that was written down in §2.

<u>The second identity</u>.

In deriving the second of the three identities, we use the fact that equation (B.1) is independent of $B^i_{\ pq}$. We have,

$$\frac{\partial \bar{L}}{\partial \bar{g}_{hk}} \frac{\partial \bar{g}_{hk}}{\partial B^i_{\ pq}} \quad \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell}} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{\ pq}} \quad \frac{\partial \bar{L}}{\partial \bar{g}_{hk,\ell m}} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{\ pq}} = 0 \ , \tag{B.20}$$

looking at the functional form of equation (B.2) we are left with

$$\bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{pq}} + \bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{\ pq}} = 0 \tag{B.21}$$

Using equation (B.3), we find

$$\bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{pq}} = \bar{\Lambda}^{pk,q} B^b_k g_{ib} + \bar{\Lambda}^{kp,q} B^b_k g_{ib}, \qquad (B.22)$$

while equation (B.4) yields

$$\bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{pq}} = g_{ib}\bar{\Lambda}^{pk,qm} B^b_{km} + \bar{\Lambda}^{pk,qm} B^c_m B^b_k g_{ib,c}$$

$$+ \bar{\Lambda}^{hp,\ell q} B^b_{h\ell} g_{ib} + \bar{\Lambda}^{pk,\ell q} B^b_{k\ell} g_{ib} + \bar{\Lambda}^{hp,qm} B^b_{hm} g_{ib}$$

$$+ \bar{\Lambda}^{hp,qm} B^b_h B^c_m g_{ib,c} + \bar{\Lambda}^{pk,\ell q} B^b_k B^c_\ell g_{ib,c}$$

$$+ \bar{\Lambda}^{hp,\ell q} B^b_h B^c_\ell g_{ib,c} + \bar{\Lambda}^{hk,pq} B^a_h B^b_k g_{ab,i} \qquad (B.23)$$

Substituting (B.22) and (B.23) into (B.21) and simplifying, we obtain

$$g_{ib} B^b_k (\bar{\Lambda}^{pk,q} + \bar{\Lambda}^{qk,p}) + 2g_{ib} B^b_{km} \bar{\Lambda}^{kp,qm} + 2g_{ib} B^b_{km} \bar{\Lambda}^{kq,pm}$$

$$+ 2g_{ib,c} B^c_m B^b_k \bar{\Lambda}^{kp,qm} + 2g_{ib,c} B^b_k B^c_m \bar{\Lambda}^{kq,pm}$$

$$+ g_{cb,i} B^c_m B^b_k \bar{\Lambda}^{mk,pq} = 0 \qquad (B.24)$$

(after symmetrisation in (pq) indices due to symmetry of B^i_{pq} in (pq)).

Equation (B.24) is true for all B^i_j, in particular, it is true for $B^i_j = \delta^i_j$, so that

$$g_{ik} (\bar{\Lambda}^{pk,q} + \bar{\Lambda}^{qk,p}) + 2g_{ik,m} \bar{\Lambda}^{kp,qm} + 2g_{ik,m} \bar{\Lambda}^{kq,pm}$$

$$+ g_{mk,i} \bar{\Lambda}^{mk,pq} = 0 \qquad (B.25)$$

From Appendix III(A), equation (A.21) we have

$$\Pi^{ij,k} = \Lambda^{ij,k} + 2\Lambda^{\alpha j,k\ell}\,\Gamma^{i}_{\ell\alpha} + 2\Lambda^{\alpha i,k\ell}\,\Gamma^{j}_{\ell\alpha} + \Lambda^{ij,\alpha\ell}\,\Gamma^{k}_{\ell\alpha} \tag{B.26}$$

so that in a gaussian normal coordinate system,

$$\Pi^{ij,k} = \Lambda^{ij,k} \tag{B.27}$$

remembering that in this example, $\Gamma^{i}_{jk} = \{{}^{\;i}_{j\;k}\}$.

Evaluating equation (B.25) in a gaussian normal coordinate system, we obtain

$$g_{ik}\,(\bar{\Pi}^{pk,q} + \bar{\Pi}^{qk,p}) = 0 \tag{B.28}$$

But this is a tensor equation,removing the "bars",

$$\Pi^{ij,k} + \Pi^{kj,i} = 0 \tag{B.29}$$

or,

$$\Pi^{ij,k} = -\Pi^{kj,i} = -\Pi^{jk,i} = \Pi^{ik,j} = \Pi^{ki,j} = -\Pi^{ji,k}, \tag{B.30}$$

so that,

$$\Pi^{ij,k} = 0 \tag{B.31}$$

In (B.30), we have repeatedly used (B.29) and the symmetry property

$$\Pi^{ij,k} = \Pi^{ji,k} \tag{B.32}$$

Equation (B.32) is the second identity we set out to prove, and we have shown, that indeed, equation (3.2.20) does imply equation (3.2.21) in § 2.1

The third identity.

To obtain this last identity, we differentiate equation (B.1) with respect to $B^i_{\ j}$,

$$\frac{\partial \bar{L}}{\partial B^i_j} = B\, A^j_{\ i} L, \tag{B.33}$$

where, $B^i_j\, A^j_{\ k} = \delta^i_{\ k}$, (B.34)

therefore,

$$\frac{\partial B}{\partial B^i_{\ j}} = B\, A^j_{\ i}\ . \tag{B.35}$$

Now,

$$\frac{\partial \bar{L}}{\partial B^i_{\ j}} = \bar{\Lambda}^{hk} \frac{\partial \bar{g}_{hk}}{\partial B^i_{\ j}} + \bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{\ j}} + \bar{\Lambda}^{hk,\ell m} \frac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_{\ j}} \tag{B.36}$$

Equation (B.2) and (B.3) imply

$$\bar{\Lambda}^{hk} \frac{\partial \bar{g}_{hk}}{\partial B^i_{\ j}} = \bar{\Lambda}^{jk} B^b_{\ k} g_{ib} + \bar{\Lambda}^{kj} B^b_{\ k}\, g_{ib} \tag{B.37}$$

and

$$\bar{\Lambda}^{hk,\ell} \frac{\partial \bar{g}_{hk,\ell}}{\partial B^i_{\ j}} = \bar{\Lambda}^{kj,\ell} B^b_{\ k\ell} g_{ib} + \bar{\Lambda}^{jk,\ell} B^b_{\ k\ell} g_{ib} + \bar{\Lambda}^{jk,\ell} B^b_{\ k} B^c_\ell\, g_{ib,c}$$

$$+ \bar{\Lambda}^{kj,\ell} B^b_{\ k} B^c_\ell\, g_{ib,c} + \bar{\Lambda}^{\ell k,j} B^b_{\ k}\, B^c_\ell\, g_{cb,i} \tag{B.38}$$

while equation (B.4) allows us to evaluate $\Lambda^{hk,\ell m} \dfrac{\partial \bar{g}_{hk,\ell m}}{\partial B^i_j}$.

However, as this term is rather cumbersome, we shall not write it down, except to say that after substitution into (B.36) and using (B.33) we can write

$$
\begin{aligned}
BL\, A^j_{\ i} = {} & (\bar\Lambda^{jk} + \bar\Lambda^{kj})\, B^b_{\ k}\, g_{ib} + (\bar\Lambda^{kj,\ell} + \bar\Lambda^{jk,\ell})\, B^b_{\ k\ell}\, g_{ib} \\
& + (\bar\Lambda^{jk,\ell} + \bar\Lambda^{kj,\ell})\, B^b_{\ k} B^c_{\ell}\, g_{ib,c} + \bar\Lambda^{\ell k,j}\, B^b_{\ k}\, B^c_{\ell}\, g_{cb,i} + B^b_{\ h\ell m}\, \bar\Lambda^{hj,\ell m}\, g_{ib} \\
& + B^a_{\ h\ell}\, \bar\Lambda^{hj,\ell m}\, g_{ai,c} B^c_{\ m} + B^a_{\ h\ell} B^b_{\ k}\, \bar\Lambda^{hk,\ell j} g_{ab,i} + B^b_{\ k\ell m}\, \bar\Lambda^{jk,\ell m} g_{ib} \\
& + B^b_{\ k\ell} \bar\Lambda^{jk,\ell m} g_{ib,c} B^c_{\ m} + B^b_{\ k\ell} B^a_{\ h}\, \bar\Lambda^{hk,\ell j}\, g_{ab,i} + B^a_{\ hm}\, \bar\Lambda^{hj,\ell m}\, B^c_{\ell}\, g_{ai,c} \\
& + B^a_{\ hm}\, B^b_{\ k}\, \bar\Lambda^{hk,jm}\, g_{ab,i} + B^b_{\ km}\, \bar\Lambda^{jk,\ell m}\, B^c_{\ell}\, g_{ib,c} + B^b_{\ km}\, B^a_{\ h}\, \bar\Lambda^{hk,jm} g_{ab,i} \\
& + B^c_{\ell m}\, \bar\Lambda^{jk,\ell m} B^b_{\ k} g_{ib,c} + B^c_{\ell m} B^a_{\ h}\, \bar\Lambda^{hj,\ell m} g_{ai,c} + \bar\Lambda^{jk,\ell m} B^b_{\ k} B^c_{\ell} B^d_{\ m} g_{ib,cd} \\
& + \bar\Lambda^{hj,\ell m} B^a_{\ h} B^c_{\ell}\, B^d_{\ m} g_{ai,cd} + \bar\Lambda^{hk,jm} B^a_{\ h} B^b_{\ k}\, B^d_{\ m} g_{ab,id} + \bar\Lambda^{hk,\ell j} B^a_{\ h} B^b_{\ k} B^c_{\ell} g_{ab,ci}
\end{aligned}
\tag{B.39}
$$

As it stands, this is not a nice expression!, but we can, as before simplify by taking

$$
B^i_{\ j} = \delta^i_{\ j}\ , \quad A^j_{\ i} = \delta^j_{\ i}\ , \text{ etc} \tag{B.40}
$$

Equation (B.39) then reduces to

$$
\begin{aligned}
L\delta^j_i = {} & (\bar\Lambda^{jk} + \bar\Lambda^{kj})\, g_{ik} + (\bar\Lambda^{jk,\ell} + \bar\Lambda^{kj,\ell})\, g_{ik,\ell} + \bar\Lambda^{\ell k,j}\, g_{\ell k,i} \\
& + \bar\Lambda^{jk,\ell m} g_{ik,\ell m} + \bar\Lambda^{hj,\ell m}\, g_{hi,\ell m} + \bar\Lambda^{hk,jm}\, g_{hk,im} + \bar\Lambda^{hk,\ell j} g_{hk,\ell i}
\end{aligned}
\tag{B.41}
$$

We can simplify further, by taking our coordinate system to be gaussian normal. We then have,

$$L\delta^j_{\ i} = (\Lambda^{jk} + \Lambda^{kj})\, g_{ik} + \Lambda^{jk,\ell m}\, g_{ik,\ell m} + \Lambda^{kj,\ell m}\, g_{ik,\ell m}$$

$$+ \Lambda^{\ell k,jm}\, g_{\ell k,im} + \Lambda^{\ell k,mj}\, g_{\ell k,mi} \quad . \tag{B.42}$$

The presence of second derivatives of the metric tensor in equation (B.42) tells us that some function of the Riemann-Christoffel tensor is going to come in.

In a gaussian normal coordinate system, the Riemann-Christoffel curvature tensor is given by

$$2R_{i\ell km}(\{\}) = g_{ik,\ell m} + g_{\ell m,ik} - g_{\ell k,im} - g_{im,\ell k} \tag{B.43}$$

From equation (B.13), we have that

$$\Lambda^{jk,m\ell} + \Lambda^{j\ell,mk} = -\Lambda^{jm,k\ell} , \tag{B.44}$$

multiplying throughout by $g_{ik,\ell m}$,

$$g_{ik,\ell m}\, \Lambda^{jk,m\ell} + g_{ik,\ell m}\, \Lambda^{j\ell,mk} = -g_{ik,\ell m}\, \Lambda^{jm,k\ell} \tag{B.45}$$

or since $g_{ik,\ell m}$ is symmetric in (ℓm),

$$g_{ik,\ell m}\, \Lambda^{jk,m\ell} = -2\, g_{ik,\ell m}\, \Lambda^{jm,k\ell} \tag{B.46}$$

$$= -2\, g_{im,\ell k}\, \Lambda^{jk,m\ell} \quad . \tag{B.47}$$

Similarly, we have

$$g_{\ell m,ik}\, \Lambda^{jk,m\ell} = -2\, g_{\ell k,im}\, \Lambda^{jk,m\ell} \tag{B.48}$$

Putting together equations (B.43), (B.47) and (B.48), we obtain,

$$2\Lambda^{jk,m\ell}R_{i\ell km}(\{\}) = \Lambda^{jk,m\ell}(g_{ik,\ell m} + g_{\ell m,ik})$$

$$- \Lambda^{jk,m\ell}(g_{\ell k,im} + g_{im,\ell k}) \tag{B.49}$$

or, using (B.47) and (B.48)

$$\frac{4}{3}\Lambda^{jk,m\ell}R_{i\ell km}(\{\}) = \Lambda^{jk,m\ell}(g_{ik,\ell m} + g_{\ell m,ik}) \tag{B.50}$$

In equation (B.42) we have the following terms:

$$\Lambda^{jk,\ell m}\, g_{ik,\ell m} + \Lambda^{\ell k,mj}\, g_{\ell k,mi}$$

$$+ \Lambda^{kj,\ell m}\, g_{ik,\ell m} + \Lambda^{\ell k,jm}\, g_{\ell k,im}$$

or, $\qquad \Lambda^{jk,\ell m}\, g_{ik,\ell m} + \Lambda^{jk,\ell m}\, g_{\ell m,ik}$

$$+ \Lambda^{kj,\ell m}\, g_{ik,\ell m} + \Lambda^{jk,\ell m}\, g_{\ell m,ik} \quad ,$$

where we have used the identity $\Lambda^{ij,k\ell} = \Lambda^{k\ell,ij}$ derived earlier. Therefore, we can rewrite equation (B.42) as

$$L\,\delta^j_{\ i} = (\Lambda^{jk} + \Lambda^{kj})\, g_{ik} + 2\Lambda^{jk,\ell m}(g_{ik,\ell m} + g_{\ell m,ik}) \tag{B.51}$$

Substituting from equation (B.50),

$$L\,\delta_i^{\ j} = (\Lambda^{jk} + \Lambda^{kj})g_{ik} + \frac{8}{3}\Lambda^{jk,m\ell}\, R_{i\ell km}(\{\}) \ . \tag{B.52}$$

From appendix III(A), equation (A.22) we have, in a gaussian normal coordinate system,

$$\Pi^{ij} = \Lambda^{ij} + \Lambda^{\alpha j,k\ell}\Gamma^{i}_{k\alpha,\ell} + \Lambda^{\alpha i,k\ell}\Gamma^{j}_{k\alpha,\ell} \tag{B.53}$$

remembering that in this example $\Gamma^{i}{}_{jk} = \{j{}^{i}k\}$.

Also, in gaussian normal coordinates,

$$\Gamma^{k}_{pq,\ell} = \{p{}^{k}q\}_{,\ell} = \frac{1}{2}\, g^{km}(g_{mp,q\ell} + g_{mq,p\ell} - g_{pq,m\ell}), \tag{B.54}$$

So we have that

$$\Lambda^{\alpha j,k\ell}\,\Gamma^{i}{}_{k\alpha,\ell} + \Lambda^{\alpha i,k\ell}\Gamma^{j}{}_{k\alpha,\ell}$$

$$= \frac{1}{2}\{g^{im}\Lambda^{\alpha j,k\ell}(g_{mk,\alpha\ell} + g_{m\alpha,k\ell} - g_{k\alpha,m\ell})$$

$$+ g^{im}\,\Lambda^{\alpha i,k\ell}(g_{mk,\alpha\ell} + g_{m\alpha,k\ell} - g_{k\alpha,m\ell})\} \tag{B.55}$$

With the use of equation (B.47), (B.53) can be reduced, after using (B.55), to

$$\Pi^{ij} = \Lambda^{ij} + \frac{1}{4}\, g^{im}\,\Lambda^{\alpha j,k\ell}\,(g_{mk,\alpha\ell} + g_{\alpha\ell,mk})$$

$$+ \frac{1}{4}\, g^{jm}\,\Lambda^{\alpha i,k\ell}(g_{mk,\alpha\ell} + g_{\alpha\ell,mk}). \tag{B.56}$$

Equation (B.50) allows us to write this as

$$\Pi^{ij} = \Lambda^{ij} + \frac{1}{4}\, g^{im}\,\Lambda^{\alpha j,k\ell}\, R_{m\ell\alpha k}(\{\})\,\frac{4}{3}$$

$$+ \frac{1}{4}\, g^{im}\,\Lambda^{\alpha i,k\ell}\, R_{m\ell\alpha k}(\{\})\,\frac{4}{3} \tag{B.57}$$

or,

$$\Pi^{ij} = \Lambda^{ij} + \frac{1}{3}\Lambda^{\alpha j,k\ell} R^{i}{}_{\ell\alpha k}(\{\}) + \frac{1}{3} \Lambda^{\alpha i,k\ell} R^{j}{}_{\ell\alpha k}(\{\}) \tag{B.58}$$

Finally, substituting (B.58) into equation (B.52),

$$\begin{aligned} L\delta_{i}{}^{j} = g_{ik}\Big(& \Pi^{jk} - \frac{1}{3} \Lambda^{\alpha k,m\ell} R^{j}{}_{\ell\alpha m}(\{\}) - \frac{1}{3} \Lambda^{\alpha j,m\ell} R^{k}{}_{\ell\alpha m}(\{\}) \\ & + \Pi^{kj} - \frac{1}{3} \Lambda^{\alpha j,m\ell} R^{k}{}_{\ell\alpha m}(\{\}) - \frac{1}{3} \Lambda^{\alpha k,m\ell} R^{j}{}_{\ell\alpha m}(\{\})\Big) \\ & + \frac{8}{3} \Lambda^{jk,m\ell} R_{i\ell km}(\{\}), \end{aligned} \tag{B.59}$$

but now, we note the Π^{ij} is symmetric in (ij), so, symmetrising,

$$L\delta_{i}{}^{j} = g_{ik}\left(2\Pi^{jk} - \frac{4}{3}\Lambda^{\alpha j,m\ell} R^{k}{}_{\ell\alpha m}(\{\}) + \frac{8}{3} \Lambda^{\alpha j,m\ell} R^{k}{}_{\ell\alpha m}(\{\})\right) \tag{B.60}$$

Therefore, we have

$$\frac{1}{2} L g^{ij} = \Pi^{ij} + \frac{2}{3} \Lambda^{\alpha j,m\ell} R^{i}{}_{\ell\alpha m}(\{\}) \tag{B.61}$$

which was the identity we set out to prove.

APPENDIX IV (A)

The possible scalars quadratic in contortion one can construct in the usual ECSK theory, we have shown in the text to be three ;

$$K_1 = K_{\sigma\alpha}{}^{\sigma} K_{\nu}{}^{\nu\alpha} , \quad \text{(A1)}$$

$$K_2 = K_{\nu\alpha}{}^{\sigma} K_{\sigma}{}^{\nu\alpha} , \quad \text{(A2)}$$

and

$$K_3 = K_{\nu\sigma}{}^{\lambda} K^{\nu\sigma}{}_{\lambda} . \quad \text{(A3)}$$

Upon allowing the possibility of including the pseudoscalar density $\varepsilon^{\mu\nu\rho\sigma}$, we have four additional terms, quadratic in the contortion tensor, that can be allowed. They are,

$$J_1 = \varepsilon^{\mu\nu\rho\sigma} K_{\mu\alpha\sigma} K_{\nu\rho}{}^{\alpha} \quad \text{(A4)}$$

$$J_2 = \varepsilon^{\mu\nu\rho\sigma} K_{\alpha}{}^{\alpha}{}_{\mu} K_{\nu\rho\sigma} \quad \text{(A5)}$$

$$J_3 = \varepsilon^{\mu\nu\rho\sigma} K_{\alpha\mu\nu} K_{\rho\sigma}{}^{\alpha} \quad \text{(A6)}$$

and,

$$J_4 = \varepsilon^{\mu\nu\rho\sigma} K_{\alpha\mu\nu} K^{\alpha}{}_{\rho\sigma} \quad \text{(A7)}$$

So it seems that one can add seven terms, quadratic in the contortion tensor to the Einstein lagrangian, to form a new lagrangian for a metric-torsion theory. This would in general, necessitate the introduction of seven arbitrary parameters, governing the strength of each of the interactions. However, closer examination of equations (A.4) to (A.7) reveals that there are identities among them /19/:

$$J_3 + 4J_1 + 4J_2 = 0 \tag{A8}$$

and

$$J_4 + 2J_1 + J_2 = 0 \tag{A9}$$

These identities reduce the number of possible linearly independent scalars to five. It still is not a satisfactory situation, for we have six coupling parameters in the theory, including the Newtonian gravitational constant.

The situation gets even worse if we now try to build a theory with propagating torsion. The least requirement on the lagrangian for such a theory, is simple dependence on quadratics in the derivatives of the contortion. We have no physical reason for picking one such scalar over any other. The proper valid procedure in constructing such a lagrangian is to write down all such possible scalars, and incorporate them into the lagrangian, taking note to add a coupling parameter to the theory for each of the scalars put into the lagrangian. The situation now is ludicrous, for we have sixteen (!) possible scalars, as illustrated below: We can write

$$L_K = \sum_{i=1}^{16} \frac{1}{16\pi G_i} K_i , \tag{A10}$$

where

$$
\begin{aligned}
K_1 &= K^{\alpha\beta\lambda}{}_{;\sigma} K_{\lambda\alpha\beta}{}^{;\sigma} & K_9 &= K^{\beta\sigma\lambda}{}_{;\alpha} K^{\alpha}{}_{\sigma\lambda;\beta} \\
K_2 &= K_{\alpha}{}^{\alpha\lambda}{}_{;\sigma} K_{\beta\lambda}{}^{\beta;\sigma} & K_{10} &= K^{\beta\sigma\lambda}{}_{;\alpha} K_{\sigma}{}^{\alpha}{}_{\lambda;\beta} \\
K_3 &= K^{\alpha\beta\lambda}{}_{;\sigma} K_{\alpha\beta\lambda}{}^{;\sigma} & K_{11} &= K^{\sigma\alpha\lambda}{}_{;\alpha} K_{\sigma}{}^{\beta}{}_{\lambda;\beta} \\
K_4 &= K^{\alpha\beta\sigma}{}_{;\alpha} K_{\lambda\sigma}{}^{\lambda}{}_{;\beta} & K_{12} &= K^{\sigma\beta\lambda}{}_{;\alpha} K_{\sigma}{}^{\alpha}{}_{\lambda;\beta} \\
K_5 &= K^{\beta\alpha\sigma}{}_{;\alpha} K_{\lambda\sigma}{}^{\lambda}{}_{;\beta} & K_{13} &= K^{\sigma\alpha\lambda}{}_{;\alpha} K_{\lambda}{}^{\beta}{}_{\sigma;\beta} \\
K_6 &= K^{\sigma\alpha\beta}{}_{;\alpha} K_{\lambda\sigma}{}^{\lambda}{}_{;\beta} & K_{14} &= K^{\sigma\beta\lambda}{}_{;\alpha} K_{\lambda}{}^{\alpha}{}_{\sigma;\beta} \\
K_7 &= K^{\alpha\sigma\lambda}{}_{;\alpha} K^{\beta}{}_{\sigma\lambda;\beta} & K_{15} &= K^{\sigma\alpha}{}_{\sigma;\alpha} K^{\lambda\beta}{}_{\lambda;\beta} \\
K_8 &= K^{\alpha\sigma\lambda}{}_{;\alpha} K_{\sigma}{}^{\beta}{}_{\lambda;\beta} & K_{16} &= K^{\sigma\beta}{}_{\sigma;\alpha} K^{\lambda\alpha}{}_{\lambda;\beta}
\end{aligned}
\tag{A11}
$$

Allowing for $\varepsilon^{\mu\nu\rho\sigma}$ in such a theory would make things even worse, so clearly the approach we have adopted is a reasonable one. For not only do we allow for $\varepsilon^{\mu\nu\rho\sigma}$, but we also, through requiring linearity in the curvature tensor, pick out only one additional scalar, quadratic in the torsion, namely J_1.

APPENDIX IV (B)

Elements of anholonomic tetrad.

In a Riemann-Cartan geometry, just as in Riemannian geometry we are compelled to introduce a (pseudo-) orthonormal basis of four vectors $\underline{e}_\alpha$, the greek index $\alpha = 1,2,3,4$ labels the tetrad, at each point of the space-time as anholonomic coordinates.

In component form,

$$\underline{e}_\alpha = e_\alpha{}^i \, \partial_i \ , \tag{B1}$$

while the dual basis of one-forms is,

$$\theta^\alpha = e^\alpha{}_i \ dx^i \ . \tag{B2}$$

Because the tetrad is taken to be (pseudo-) orthonormal, we have the relations

$$e^\alpha{}_i \, e_\alpha{}^j = \delta^j_i \ , \qquad e^\beta{}_i \, e_\alpha{}^i = \delta_\alpha{}^\beta \tag{B3}$$

and

$$g_{ij} = e^\alpha{}_i \, e^\beta{}_j \, g_{\alpha\beta} \tag{B4}$$

among the components $e^\alpha{}_i$ and their reciprocals $e_\alpha{}^i$. The $g_{\alpha\beta}$ are components of the Minkowski metric tensor;

$$e_\alpha{}^i = g_{\alpha\beta} g^{ij} \, e^\beta{}_j \ ; \ g_{\alpha\beta} = \mathrm{diag}\,(-1,-1,-1,+1) \tag{B5}$$

The object of anholonomity, $\Omega^\gamma{}_{\alpha\beta}$ is defined by

$$\Omega^\gamma{}_{\alpha\beta} = e_\alpha{}^i e_\beta{}^j \, \partial_{[i} e^\gamma{}_{j]} ; \quad \Omega_{\alpha\beta\gamma} = g_{\gamma\delta} \, \Omega^\delta{}_{\alpha\beta} \tag{B6}$$

The covariant derivative

Suppose we have a matter field ψ which, under an infinitesimal Lorentz transformation δx^γ , behaves as follows :

$$\delta\psi = \partial_\beta \, (\delta x^\gamma) f_\gamma{}^\beta \psi \ , \tag{B7}$$

$f_\gamma{}^\beta$ is an operator determined by the Lorentz group. The covariant derivative of ψ is then defined as

$$\nabla_\alpha \psi = \partial_\alpha \psi + \Gamma^\gamma{}_{\alpha\beta} f_\gamma{}^\beta \psi \tag{B8}$$

Remembering that

$$\Gamma^i{}_{jk} = \{j{}^i{}k\} - K_{jk}{}^i \tag{B9}$$

and using (B3), (B5), the connection $\Gamma^\gamma{}_{\alpha\beta}$ can be expressed in anholonomic coordinates as

$$\Gamma_{\alpha\beta\gamma} = g_{\gamma\delta}\, \Gamma_\alpha{}^\delta{}_\beta = -\Omega_{\alpha\beta\gamma} + \Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta} - K_{\alpha\beta\gamma} \tag{B10}$$

where

$$K_{\alpha\beta\gamma} := e_\alpha{}^i\, e_\beta{}^j\, e_\gamma{}^k\, K_{ijk} \tag{B11}$$

equation (B10) gives rise to the following symmetries

$$\Gamma_{\alpha(\beta\gamma)} = 0 \tag{B12}$$

$$\Gamma_{[\alpha\beta\gamma]} = -\Omega_{[\alpha\beta\gamma]} - K_{[\alpha\beta\gamma]} \tag{B13}$$

and we also have

$$g^{\beta\gamma}\, \Gamma_{\beta\gamma\alpha} = -2\Omega^\beta{}_{\alpha\beta} + K_{\beta\alpha}{}^\beta \; . \tag{B14}$$

If ψ is a spinor field, we have from /16/ , that

$$f_{\alpha\beta} = \frac{1}{4}\, \gamma_{[\alpha}\gamma_{\beta]} \tag{B15}$$

So that the covariant derivative of a spinor field is given by

$$\nabla_\alpha \psi = \partial_\alpha \psi - \frac{1}{4} \Gamma_{\alpha\beta\gamma} \gamma^\beta \gamma^\gamma \psi \tag{B16}$$

since the Dirac adjoint is defined by $\bar{\psi} = \psi^* \gamma_o$, we have

$$\nabla_\alpha \bar{\psi} = \partial_\alpha \bar{\psi} - \frac{1}{4} \Gamma_{\alpha\beta\gamma} \bar{\psi} \gamma^\gamma \gamma^\beta \tag{B17}$$

This completes our rather brief introduction to tetrads. Most of this appendix is contained in Hehl and Datta /16/, also Hehl et al /5/.

APPENDIX IV (C)

In this appendix, we shall show how one can solve equation (4.6.17);

$$T_{kji} = \tau_{kji} + p\eta^{\nu\lambda}{}_{i\sigma}K_{\nu\lambda\rho}(\delta^{\sigma}_{k}\delta^{\rho}_{j} - \delta^{\sigma}_{j}\delta^{\rho}_{k}) \tag{C1}$$

to obtain equation (4.6.18).

We first note that,

$$2p\eta^{\nu\lambda}{}_{i\sigma}K_{\nu\lambda\rho}\,\delta^{\sigma\rho}_{kj}$$

$$= 2p\,\eta^{\nu\lambda}{}_{i\sigma}\delta^{\sigma\rho}_{kj}\,(S_{\lambda\rho\nu} - S_{\nu\lambda\rho} - S_{\rho\nu\lambda}) \tag{C2}$$

$$= -2p\eta^{\nu\lambda}{}_{i\sigma}\delta^{\sigma\rho}_{kj}\,S_{\nu\lambda\rho} \tag{C3}$$

from antisymmetry of $\eta^{\nu\lambda}{}_{i\sigma}$, which implies

$$\eta^{\nu\lambda}{}_{i\sigma}\,S_{\lambda\rho\nu} = -\eta^{\nu\lambda}{}_{i\sigma}\,S_{\nu\rho\lambda}\quad, \tag{C4}$$

and antisymmetry of $S_{\nu\rho\lambda}$ in its first two indices further implies

$$\eta^{\nu\lambda}{}_{i\sigma}\,S_{\lambda\rho\nu} = \eta^{\nu\lambda}{}_{i\sigma}S_{\rho\nu\lambda}\quad. \tag{C5}$$

Substitution of (C3) into (C1) yields ,

$$T_{kji} = \tau_{kji} + 2p\,\eta^{\nu\lambda}{}_{\sigma i}\delta^{\sigma\rho}_{kj}\,S_{\nu\lambda\rho}\quad. \tag{C6}$$

Multiplying (C6) by $\eta^{kj\alpha\beta}$ and simplifying, we find that

$$\eta^{\nu\lambda}{}_{\sigma i}\eta^{kj\alpha\beta}\,\delta^{\sigma\rho}_{kj}\,S_{\nu\lambda\rho} = -2\,T^{\alpha\beta}{}_{i} \tag{C7}$$

so that substitution back into equation (C6) multiplied by $\eta^{kj\alpha\beta}$ gives :

$$\eta^{kj\alpha\beta}T_{kji} = \tau_{kji}\,\eta^{kj\alpha\beta} - 4p\, g^{k\alpha}g^{j\beta}T_{kji} \ . \qquad \text{(C8)}$$

Further multiplication of (C8) by $\eta_{\alpha\beta\rho\sigma}$ gives, after simplification,

$$T^{\alpha\beta}{}_{i} - 2p\,\eta^{kj\alpha\beta}T_{kji} = \tau^{\alpha\beta}{}_{i} \ , \qquad \text{(C9)}$$

from (C8) we have

$$2p\eta^{kj\alpha\beta}T_{kji} = 2p\,\tau_{kji}\eta^{kj\alpha\beta} - 8p^2\, g^{k\alpha}g^{j\beta}T_{kji} \qquad \text{(C10)}$$

substituting into (C9) yields :

$$T^{\alpha\beta}{}_{i} + 8p^2\, g^{k\alpha}g^{j\beta}T_{kji} = \tau^{\alpha\beta}{}_{i} + 2p\tau_{kji}\,\eta^{kj\alpha\beta} \ . \qquad \text{(C11)}$$

Therefore we have finally,

$$S_{kji} + g_{ki}S_{j\ell}{}^{\ell} - g_{ji}S_{k\ell}{}^{\ell} = \frac{\tau_{\alpha\beta i}}{(1+8p^2)}\left(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2p\eta^{kj\alpha\beta}\right) \qquad \text{(C12)}$$

with

$$S_{j\ell}{}^{\ell} = -\frac{1}{2(1+8p^2)}\left[\tau_{k\ell}{}^{\ell} + 2p\,\eta_{k}{}^{i\alpha\beta}\tau_{\alpha\beta i}\right] \qquad \text{(C13)}$$

Together (C12) and (C13) yield the desired equation, namely equation (4.6.18).

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INVARIANT DEDUCTION OF THE GRAVITATIONAL EQUATIONS FROM THE PRINCIPLE OF HAMILTON

by Attilio Palatini

Rendiconti del Circolo Matematico di Palermo

10 August 1919, 43, 203-212

[Translation by Roberto Hojman and Chandrasekher Mukku]

TRANSLATOR'S NOTE

In this translation of Palatini's article, we have tried to adhere as closely as possible to the original, not only as regards the original text, but also the choice of english equivalents for technical expressions. It should be noted that Palatini not only uses superscripts for contravariant indices but appends round brackets to them. This is not to be confused with the modern use of round brackets - denoting symmetrization. $\{{jk \atop i}\}$ is the historical form of the Christoffel symbols. In keeping with the summation convention, they are nowadays written as $\{{i \atop jk}\}$. We retain the historical form. To avoid sources of confusion, we have introduced extra labelling of equations. These are with greek indices. We would like to take this opportunity to thank Professors P.G. Bergmann and V. De Sabbata for their kind hospitality at Erice.

INTRODUCTION

It is already well known that in the general theory of relativity, physical space is characterized by a quadratic differential form (that mixes space and time)

$$ds^2 = \sum_{ij} g_{ij}\, dx^i\, dx^j \qquad (1)$$

in the differentials of the four co-ordinate variables $x_0 = t, x_1, x_2, x_3$, whose coefficients g_{ij} are the gravitational potentials of Einstein. The discriminant of (1) - essentially negative - will be denoted by (g).

The mutual interdependence of all physical phenomena and the geometric nature of the space is completely determined by the ten gravitational equations

$$G_{ik} - \left(\frac{1}{2} G + \lambda\right) g_{ik} = -\kappa \, T_{ik} \quad , \tag{2}$$

where $G_{ik} = \sum_{0\,h}^{3} \{ih,hk\}$ is the Riemann curvature tensor; $G = \sum_{0\,ik}^{3} G_{ik} \, g^{(ik)}$ is the mean curvature of the four-dimensional space (1); T_{ik} is the energetic tensor that is determined from all the elements - stresses, quantity of motion, energy density and flux - that characterize the physical phenomena; κ and λ are two universal constants.

After these gravitational equations were discovered by Einstein, efforts were made to derive them from a variational principle just as one derives the equations of Lagrange from Hamiltonian's principle in ordinary mechanics.

This goal was reached by Einstein himself, establishing a new Hamiltonian principle that was made precise by Hilbert and Weyl [1].

However the procedures followed by these authors do not conform to the spirit of the absolute differential calculus, because in deriving the invariant equations, one must use non-invariant formulae.

My aim is to reach the same goal, while preserving the invariance of all the formulae at every step. In doing this, I will take advantage of the results obtained in my note: "On the foundations of the absolute differential calculus" (see the preceding note in this volume; Rend. Circ. Mat. Palermo, Vol.43, 1919); Hereafter referred to as N.

1. FUNDAMENTAL POSTULATE

We begin by introducing with Hilbert, the following fundamental postulate: The laws of physics depend on a unique, universal function H having the following properties:

(a) It is invariant with respect to general co-ordinate transformations;

(b) It depends on the gravitational potentials $g^{(ik)}$ and on the corresponding Christoffel and Riemann symbols and

(c) it depends on the elements that characterize the physical phenomena.

However, we have no a priori knowledge of the explicit form of the universal function H and must therefore introduce some hypotheses.

From the point of view of the synthesis of all physical phenomena, it is convenient to suppose that

$$H = G + L + 2\lambda \quad ,$$

where λ is a universal constant, G (the mean curvature of the four-dimensional space) is a term that contains the information and characterizes the influence of the space-time on the behaviour of the phenomena, and L is a term that includes all the manifestations of physical origin except those that are intimately related to the structure of space-time itself.

2. STRUCTURE OF THE FUNCTION L. REDUCED MECHANICAL SCHEME

From the speculative point of view, it seems desirable to attribute to all these manifestations (direct or indirect) an electromagnetic origin (as should be the case for the luminous and thermodynamic phenomena). The expression for L should depend in a complicated way on the parameters fixing the electromagnetic state of the system, and the gravitational equations should not be isolated from those governing the behaviour of all the other phenomena.

Having in mind the possibility of adopting the study to concrete cases, it is convenient to limit oneself to the consideration of the gravitational field by itself and to collect everything that arises from the set of physical phenomena (excluding gravitation), into a specific function of position and time, precisely in an energetic tensor T_{ik}.

A similar situation is found in ordinary mechanics when wishing for instance, to study the motion in a conservative field, of a material point on a frictional surface, the energetic analysis of the phenomena (that might lead one to consider the thermal aspects of the problem when taking into account the heat dissipated due to friction) is replaced by introducing a position dependent non-conservative frictional force.

In the usual mechanical approach, taking into account a whole set of circumstances (giving rise to loss of kinetic energy) would be impossible, or at least quite laborious to analyse with profit. Instead, one is led to consider as given, forces that are not derivable from a potential. In the same way, in the Einsteinian scheme, it is satisfactory to study, instead of L (whose precise expression should depend on a whole on a set of phenomena, making its explicit study impossible or undesirable) the tensor T_{ik}.

By using T_{ik}, an appropriate matter Lagrangian *)

$$L = \kappa \sum_{ik\,0}^{3} T_{ik}\, g^{(ik)} \qquad (3)$$

can be constructed so as to lead us to the gravitational equations. κ denotes a universal constant of homogeneity. The given elements of the tensor T_{ik} are not to be considered as independent of the $g^{(ik)}$, instead, one should take the products $\sqrt{g}\, T_{ik} = \mathfrak{T}_{ik}$ (which constitute the so-called tensor of volume associated to the tensor T_{ik}).

3. PRINCIPLE OF HAMILTON

With these assumptions, and taking the form of the universal function to be

$$H = G + L + 2\lambda$$

with

$$L = \kappa \sum_{ik\,0}^{3} T_{ik}\, g^{(ik)} ,$$

we want to show that the gravitational equations follow from the variational principle

$$\delta \int_S H\, dS = 0 \quad . \qquad (4)$$

*) For such a purpose one can again invoke the mentioned analogy with classical mechanics, by noting that from Hamilton's variational principle $\delta \int (T + U)\, dt = 0$ [T kinetic energy, U potential] is valid for the case of conservative forces, one can go to the generalized principle, valid for any force with components X_i $(i = 1,2,3)$ by substituting for U the linear expression $\sum_{i\,1}^{3} X_i x_i$ and assuming $\delta X_i = 0$. The expression (3) for L is in a sense the analogue of $\sum_{i\,1}^{3} X_i x_i$.

Here S denotes an arbitrary region of the four-dimensional space-time and δ denotes a variation with respect to the potentials $g^{(ik)}$ with the condition that $\delta g^{(ik)}$ (and their first and second derivatives) vanish on the boundary of S.

Before proceeding with the proof of our proposition, it is necessary to establish some preliminary formulae.

4. PRELIMINARY FORMULAE

Variation of the Christoffel symbols: Let us begin with the identities *)

$$\frac{\partial g_{nk}}{\partial x_j} - \sum_p \left[\{{nj \atop p}\} g_{pk} + \{{kj \atop p}\} g_{np} \right] = 0 \quad , \qquad (5)$$

essentially expressing the well-known lemma of Ricci. They can be easily verified by using the expressions for the Christoffel symbols of the second kind.

With the above definition of δ, we write $\delta g_{nk} = e_{nk}$ and applying δ to (5) one gets

$$\frac{\partial e_{nk}}{\partial x_j} - \sum_p \left(\{{nj \atop p}\} e_{pk} + \{{kj \atop p}\} e_{np} \right) - \sum_p \left(g_{pk}\delta\{{nj \atop p}\} + g_{np}\delta\{{kj \atop p}\} \right) = 0.$$

The first two terms constitute the covariant derivative of the system e_{nk} (cf. formula (14) of N for the particular case $m = 2$), therefore

$$e_{nk|j} = \sum_p \left(g_{pk}\delta\{{nj \atop p}\} + g_{np}\delta\{{kj \atop p}\} \right) \quad . \qquad (\alpha)$$

Permuting k with j and then h with j, and summing up the two equations thus obtained and then subtracting (α) one gets

$$\eta_{hkj} = \sum_p g_{pj}\delta\{{hk \atop p}\} \quad , \qquad (6)$$

where

$$\eta_{nkj} = \frac{1}{2}\left(e_{nj|k} + e_{kj|n} - e_{nk|j} \right) \quad .$$

*) In this section, summation indices have no limits, so that all considerations here, will be valid not only for the four-dimensional ds^2 of Einstein, but for any ds^2.

Multiplying (6) with $g^{(ij)}$ and summing over j, we immediately obtain

$$\delta\left\{ {hk \atop i} \right\} = \eta_{hk}^{(i)} , \tag{7}$$

where

$$\eta_{hk}^{(i)} := \sum_j g^{(ij)} \eta_{hkj} . \tag{8}$$

Variation of Riemann symbols and explicit expression for G: Eq.(8) immediately reveals $\eta_{hk}^{(j)}$ to be a mixed system, twice covariant and once contravariant.

From the fundamental formula that defines the covariant derivative of a mixed system (cf. formula (13) of N) we get

$$\eta_{hk|j}^{(i)} = \frac{\partial \eta_{hk}^{(i)}}{\partial x_j} - \sum_\ell \left[\left\{ {hj \atop \ell} \right\} \eta_{\ell k}^{(i)} + \left\{ {kj \atop \ell} \right\} \eta_{h\ell}^{(i)} - \left\{ {\ell j \atop i} \right\} \eta_{hk}^{(\ell)} \right] . \tag{9}$$

Let us now consider the Riemann symbols of the second kind

$$\{hi,kj\} = \frac{\partial}{\partial x_j} \left\{ {hk \atop i} \right\} - \frac{\partial}{\partial x_k} \left\{ {hj \atop i} \right\} + \sum_\ell \left[\left\{ {hk \atop \ell} \right\} \left\{ {\ell j \atop i} \right\} - \left\{ {hj \atop \ell} \right\} \left\{ {\ell k \atop i} \right\} \right]$$

and act on them with the symbol δ. Having in mind Eq.(7) one finds

$$\delta\{hi,kj\} = \frac{\partial \eta_{hk}^{(i)}}{\partial x_j} - \frac{\partial \eta_{hj}}{\partial x_k}$$

$$+ \sum_\ell \left[\eta_{hk}^{(\ell)} \left\{ {\ell j \atop i} \right\} + \eta_{\ell j}^{(i)} \left\{ {hk \atop \ell} \right\} - \eta_{hj}^{(\ell)} \left\{ {\ell k \atop i} \right\} - \eta_{\ell k}^{(i)} \left\{ {hj \atop \ell} \right\} \right] .$$

Applying (9) [adding and subtracting $\sum_\ell \eta_{\ell h}^{(i)} \left\{ {kj \atop \ell} \right\}$ from the right-hand sides] we get

$$\delta\{hi,kj\} = \eta_{hk|j}^{(i)} - \eta_{hj|k}^{(i)} .$$

Since $G_{hj} = \sum_k \{hk,kj\}$, it follows that

$$\delta G_{hj} = \sum_k \left[\eta_{hk|j}^{(k)} - \eta_{hj|k}^{(k)} \right] .$$

Therefore, for the variation of the mean curvature

$$G = \sum_{ik} G_{ik} \, g^{(ik)}$$

one has

$$\delta G = \sum_{ik} G_{ik} \, \delta g^{(ik)} + \sum_{ihk} g^{(ik)} \left\{ \eta^{(h)}_{ih|k} - \eta^{(h)}_{ik|h} \right\} \quad . \qquad (10)$$

Defining

$$i^{(k)} := \sum_{ih} \left\{ g^{(ik)} \eta^{(h)}_{ih} - g^{(ih)} \eta^{(k)}_{ih} \right\} \quad ,$$

it can be immediately verified that

$$\sum_{ihk} g^{(ik)} \left\{ \eta^{(h)}_{ih|k} - \eta^{(h)}_{ik|h} \right\} = \sum_{k} i^{(k)}_{|k}$$

then by virtue of formula (17) of N, Eq.(10) can be written as

$$\delta G = \sum_{ik} G_{ik} \, \delta g^{(ik)} + \frac{1}{\sqrt{g}} \sum_{k} \frac{\partial(\sqrt{g}\, i^{(k)})}{\partial x_k} \quad . \qquad (11)$$

5. DEDUCTION OF THE GRAVITATIONAL EQUATIONS

Defining $d\omega := dx_0 \, dx_1 \, dx_2 \, dx_3$, one has

$$ds = \sqrt{g} \, d\omega$$

and (4) can be written as

$$\delta \int_S \left\{ (G + 2\lambda) \sqrt{g} + \kappa \sum_{0\,ik}^{3} \mathfrak{T}_{ik} \, g^{(ik)} \right\} d\omega = 0 \quad ,$$

or, remembering that $\mathfrak{T}_{ik}$ should be regarded as being independent of $g^{(ik)}$

$$\int_S \left\{ \delta G \sqrt{g} + (G+2\lambda) \, \delta\sqrt{g} + \kappa \sum_{0\,ik}^{3} \mathfrak{T}_{ik} \, \delta g^{(ik)} \right\} d\omega = 0 \quad . \qquad (12)$$

Now

$$\delta \sqrt{g} = \sum_{ik} \frac{\partial \sqrt{g}}{\partial g^{(ik)}} \, \delta g^{(ik)} \quad ;$$

but as is well known

$$\frac{\partial \sqrt{g}}{\partial g^{(ik)}} = -\frac{1}{2} \sqrt{g}\; g_{ik} \; .$$

Therefore

$$\delta \sqrt{g} = -\frac{1}{2} \sqrt{g} \sum_{ik} g_{ik} \, \delta g^{(ik)} \; .$$

Let us now substitute this into Eq.(12), and use (11) for δG. Also, writing

$$\int_S \sum_{0}^{3}{}_{k} \frac{\partial(\sqrt{g}\; i^{(k)})}{\partial x_k} \, d\omega = \sum_{0}^{3}{}_{k} \int_S \frac{\partial(\sqrt{g}\; i^{(k)})}{\partial x_k} \, d\omega$$

allows one to use Green's lemma to convert the volume integral into a surface integral. Consequently, the integral vanishes by virtue of the expression for $i^{(k)}$ and the assumption that the variation of the potentials and their derivatives vanish on the boundary of S .

So one is left with

$$\int_S \sum_{0}^{3}{}_{ik} \left\{ G_{ik} - \left(\frac{1}{2} G + \lambda\right) g_{ik} + \kappa \, T_{ik} \right\} \delta g^{(ik)} \, dS = 0 \; .$$

Given the arbitrariness of S and $\delta g^{(ik)}$, the usual prescription gives

$$G_{ik} - \left(\frac{1}{2} G + \lambda\right) g_{ik} = -\kappa \, T_{ik} \; . \qquad (\beta)$$

Thus, the gravitational equations have been derived from the variational principle while keeping the calculations invariant throughout.

6. DIFFERENTIAL CONDITIONS FROM CONSERVATION PRINCIPLES

Let us recall that the elements of the energetic tensor T_{ik} are open to a simple physical interpretation; stresses, density and energy flux [2], and we should not forget that such a tensor is constructed from all physical phenomena except gravitation. It then follows that the so-called conservation theorems must hold, that is to say for each material system considered, and for each of its elementary portions, the components of the external force applied to the system and the power density (rate of energy transferred to the system from external sources) must vanish. In other words, the T_{ik} components constitute a double system with vanishing divergence. This can be expressed, in the notation of the absolute differential calculus, as

$$\sum_{0}^{3}{}_{k} \; T_{ik}^{(k)} = 0 \quad .$$

If we now denote by A_{ik}, the left-hand side of Eqs.(β), then we must have

$$\sum_{0}^{3}{}_{k} \; A_{ik}^{(k)} = 0 \quad . \tag{13}$$

One might be led to imagine that these relations between the g_{ij}'s, impose a restriction on the possible forms of ds^2, characterizing the Einstein manifold.

However, it is easy to prove that Eqs.(13) are satisfied identically. In order to prove this, one may use the same methods that allowed us to deduce the gravitational equations, following a criterion already indicated by Weyl [3].

Under a change of variables, the parameters x_0, x_1, x_2, x_3 are substituted by new ones related to the old ones by

$$x_i' = x_i + \xi^{(i)}, \quad i = 0,1,2,3 \quad , \tag{14}$$

where $\xi^{(i)}$ denote four arbitrary infinitesimal functions of x_0, x_1, x_2, x_3 and constitute a simple contravariant system.

Let us now determine the variations δg_{ik} suffered by the coefficients of the fundamental form

$$ds^2 = \sum_{0}^{3}{}_{k} \; g_{ik} \, dx_i dx_k$$

under the transformations (14).

Subjecting ds^2 to variation, it is found that

$$\delta ds^2 = \sum_{0}^{3}{}_{ikj} \left\{ \frac{\partial g_{ik}}{\partial x_i} \xi^{(j)} + 2g_{ij} \frac{\partial \xi^{(j)}}{\partial x_k} \right\} dx_i dx_k \quad ,$$

or, by defining ξ_i to be the reciprocal elements of the elements $\xi^{(i)}$, i.e.

$$\xi^{(j)} = \sum_{0}^{3} g^{(ij)} \xi_i \quad ,$$

$$\delta ds^2 = 2 \sum_{0}^{3}{}_{ik} \left\{ \frac{\partial \xi_i}{\partial x_k} - \sum_{0}^{3}{}_{j} \{^{ik}_{j}\} \xi_j \right\} dx_i \, dx_k \quad .$$

The term in parenthesis is immediately recognized as the covariant derivative of the system ξ_i and therefore we can rewrite δds^2 as

$$\delta ds^2 = 2 \sum_{0}^{3}{}_{ik} \xi_{i|k} \, dx_i \, dx_k \quad ,$$

or, using symmetry,

$$\delta ds^2 = \sum_{0}^{3}{}_{ik} (\xi_{i|k} + \xi_{k|i}) \, dx_i \, dx_k \quad .$$

Therefore for the variations δg_{ik}, one gets

$$\delta g_{ik} = \xi_{i|k} + \xi_{k|j} \quad . \qquad (\gamma)$$

To get the variation of the reciprocal elements $g^{(ik)}$, one uses the following identity:

$$\sum_{0}^{3}{}_{p} g^{(ip)} g_{qp} = \varepsilon_{iq} \quad .$$

Applying the symbol δ to this identity, we find

$$\sum_{0}^{3}{}_{p} \delta g^{(ip)} g_{qp} + \sum_{0}^{3}{}_{p} g^{(ip)} \delta g_{qp} = 0 \quad ,$$

or, multiplying by $g^{(kq)}$ and summing over the q index

$$\delta g^{(ik)} = - \sum_{0}^{3}{}_{pq} g^{(ip)} g^{(kq)} \delta g_{pq}$$

and finally (γ) gives

$$\delta g^{(ik)} = - \sum_{0}^{3}{}_{pq} \, g^{(ip)} \, g^{(kq)} \, (\xi_{p|q} + \xi_{q|p}) \quad . \qquad (15)$$

We now consider the expression $I = \int_S (G + \lambda) \, dS$ (where G and λ are defined above). I is an invariant under any change of variables, in particular under the transformation (14).

One then deduces that the variation δI that I suffers under the transformation (14) must vanish, i.e.

$$\delta I = \delta \int_S (G + \lambda) \, dS = 0.$$

Proceeding as in Sec.5 one gets

$$\int_s \sum_{0}^{3}{}_{ik} \left\{ G_{ik} - (\tfrac{1}{2} G + \lambda) \, g_{ik} \right\} \delta g^{(ik)} \, dS$$

$$= \int_S \sum_{0}^{3}{}_{ik} A_{ik} \, \delta g^{(ik)} \, dS = 0 \quad .$$

Substituting the expression for $\delta g^{(ik)}$ from Eq.(15), and noting that A_{ik} is a symmetric system, one obtains

$$\int_S \sum_{0}^{3}{}_{ikpq} A_{ik} \, g^{(ip)} \, g^{(kq)} \, \xi_{p|q} \, dS = 0 \quad .$$

Integrating by parts and using the formula (23) established in N

$$\int_S \sum_{0}^{3}{}_{ikpq} A_{ik|q} \, g^{(ip)} g^{(kq)} \xi_p dS = \int_S \sum_{0}^{3}{}_{ik} A_{ik}^{(k)} \xi^{(i)} \, dS = 0 \ .$$

Since the region of integration S and the functions $\xi^{(i)}$ are arbitrary, one concludes that

$$\sum_{0}^{3}{}_{k} A_{ik}^{(k)} = 0 \quad .$$

Padova, August 1919.

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QUANTUM GRAVITATIONAL EFFECTS IN THE LABORATORY*

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Summary

Some interesting consequences of the effects of gravitation and finite temperature on quantum field theory are presented which have important implications for experimental high energy physics and the status of the "No-Hair" Conjecture for black holes. We point out two consequences for laboratory situations in high energy physics which disprove the usual assertion that quantum gravitational effects are only important at planckian energies. The first of these is that beams of particles in circular accelerators cannot be cooled to below a certain temperature determined simply by the accelerator's radius, while the second shows that spontaneously broken gauge symmetries may be restored by quantum gravitational effects. We end by describing briefly circumstances under which these effects might have a bearing on the "No-Hair" conjecture.

Two parallel sets of investigations have been carried out in the last few years to study the effects of gravitation and temperature on quantum field theory. One set of investigations initiated by Khirznits and Linde[1] has considered what happens when a system of particles described by a spontaneously broken local gauge invariant quantum field theory is placed in a heat bath or strong electric or magnetic fields[2]. The authors of refs. (1) and (2) have found that gauge symmetries which are spontaneously broken at zero temperature via the Higgs-Kibble mechanism (for example, those of the Salam-Weinberg electroweak theory) may be restored at sufficiently high temperatures, or in sufficiently strong electric or magnetic environments, and they have calculated the critical temperatures and fields at which such restoration would take place.

The basic idea of this approach is that at finite temperatures (or field strengths) the effective potential of the theory picks up terms of the type $+T^2\phi^2$ (where T is the temperature and ϕ is the Higgs-Kibble scalar field). For sufficiently high temperatures, this term becomes larger than the negative $(\text{mass})^2\phi^2$ term which drives the symmetry breaking in the zero temperature theory. As a consequence, the scalar field ϕ becomes a real physical particle degree of freedom and the symmetry is restored.

Parallel to the study of these effects, several authors[3] have carried out a study of the effects of gravitation and space-time topology on quantum field theory. A number of interesting results have been obtained but the two which concern us in this essay are outlined below. Firstly, it has been shown that an observer accelerating uniformly through empty Minkowski space-time appears to find himself in a heat bath at a temperature given by

$$T = \frac{\hbar a}{2\pi K c} \simeq 10^{-20} \text{ a Kelvin} \qquad (1)$$

where $\hbar$ is Planck's constant, a is the acceleration, k is Boltzmann's constant and c is the velocity of light.

In order to illustrate this let us consider a uniformly accelerating observer in Minkowski space-time. If we assume that an inertial observer and the accelerating observer use the same transition amplitudes to describe objectively the same processes, it can be shown that the free Feynman propagator for the inertial observer, when translated into the accelerating observer's frame, is identical with that of a free finite temperature propagator with the relationship between the acceleration and the temperature being that given by (1).

This result can be understood on the basis of quantum gravitational effects (through non-simply connected topologies) in flat Minkowski space-time. To try and understand how this arises, let us use coordinates (t, x, y, z) and (τ, ξ, y, z) to describe the inertial and accelerating observers respectively. If, for simplicity, we assume that the accelerating observer moves in the (τ, ξ) plane with a constant uniform acceleration a, then his world-line is given by the hyperbola $\xi = \frac{1}{a}$ with asymptotes $\xi = 0$. The coordinate transformation from the inertial to the accelerating observer's frame reads

$$x = \xi \cosh a\tau , \qquad t = \xi \sinh a\tau .$$

In contrast to the inertial observer, the accelerating observer has a very restricted range of vision. The surface $x = |t|$ forms an event horizon, and any signals sent from the origin O, after $t = 0$ never reach the accelerating observer. It is the existence of this event horizon which causes the space-time to seem multiply connected when the two observers translate themselves into euclidean coordinates $(t \rightarrow it, \tau \longrightarrow i\tau)$ with periodic complex time coordinates, and leads to the above-mentioned thermal effect.

Secondly, by considering quantum fields in the exterior region of a black-hole, Hawking has shown that when a star collapses to a black-hole, the formation of the event horizon around the singularity enables the black-hole to absorb one of a pair of virtual particles created just outside the horizon, thus leaving its partner, which is now a real particle, free to travel to an arbitrarily large affine distance from the horizon. This continuous process is observed asymptotically as a net flux of radiation, and after all transient effects which arise during the collapse die out, the left-over radiation has been shown to be that which would be produced by a hot body at a temperature given by

$$kT = \frac{\hbar K}{2\pi c} \qquad (2)$$

where K is the surface gravity of the black-hole. Thus, a black-hole can be considered to be a black-body radiating at a temperature T given by (2).

Both the above results may be understood mathematically by noting that spacetimes with event horizons are periodic in an appropriate time coordinate with an imaginary period. The Green's functions of a quantum field theory in such a spacetime are, therefore, also periodic in imaginary time. Coupled with the observation that the thermal Green's functions of a field theory at a finite temperature T also possess this property, one arrives at the result that field theories in spacetimes with event horizons may be considered to be in thermal equilibrium at some finite temperature.

All that follows is based essentially on the interplay between the various effects we have discussed briefly above. We will now describe a couple of laboratory situations in which it might be possible to detect effects of quantum gravitation.

The first observation we wish to make concerns the recent attempts being made at CERN and other high energy particle physics laboratories to cool particle beams in accelerators. We shall show that equation (1) implies a lower bound to the extent to which such a cooling can be achieved. It is clear that a bunch of relativistic elementary particles going round at a constant velocity v($\approx$ c, the velocity of light) in a circular accelerator of radius r experience a uniform acceleration a, given by

$$a \simeq \frac{c^2}{r} .$$

We see, therefore, that such a bunch of relativistic elementary particles would find themselves in a heat bath at temperature $T \simeq \hbar c/2\pi kr$. Since this temperature is due simply to their acceleration, it would be impossible for accelerator beams to be cooled to temperatures below this lower bound. This bound does not apply, of course, to linear accelerators.

In order to remove any doubts as to whether such effects are "real", it would perhaps be helpful to show that such observer dependent effects are already very familiar. Indeed, it is only natural to expect such observer dependent effects in general relativity when one remembers that in special relativity one has a similar situation arising due to the effect of time dilation. This is illustrated beautifully by the experimental verification of time dilation effects through measurement of the lifetimes of a μ-meson at rest, and in motion in the laboratory. The results of such experiments show clearly that a μ-meson that is stationary in the laboratory decays at a much faster rate than one which is travelling at a speed reasonably close to that of light. This observer dependence arises in special relativity through requiring equivalence of all inertial observers. In contrast, general relativity requires equivalence of all observers, inertial and non-inertial, and thus gives rise to the effects we are considering in this essay.

The second effect that we shall now discuss concerns the concept of symmetry restoration, which we have outlined earlier, but with the added significance that the restoration will now be due to quantum gravitational effects. Let us consider the situation illustrated schematically in Fig. 1.

If we introduce a set of relativistic, charged particles, the interactions of which are described by a spontaneously broken gauge theory, into a region containing an extremely high magnetic field, then they will all experience an acceleration, $\underline{a}$, perpendicular to the plane defined by the directions of $\underline{B}$ and $\underline{v}$, the velocity of the particles, given by

$$\underline{a} = \frac{q}{m} \underline{v} \times \underline{B},$$

where q and m are the charge and mass of the particle respectively. Assuming that $\underline{v}$ is perpendicular to $\underline{B}$ and is close to c in magnitude, we obtain for $\underline{a}$ the value $a \simeq \frac{qcB}{m}$. However, equation (1) tells us that such a bunch of particles will experience a heat bath of temperature

$$T \simeq \frac{\hbar qB}{2\pi km} = \left(\frac{\hbar}{2\pi k}\right) \frac{qB}{m} .$$

Assuming, for simplicity, that such a bunch of particles is composed of electrons, we obtain the result that $a \simeq 5 \times 10^{19}$ B. So that the temperature for this set of electrons would be $T \simeq 0.5$ B.

Now, if one combines this information with the knowledge that the symmetry of the Salam-Weinberg theory is restored at temperatures of $O(10^{15})$ Kelvin, we see that magnetic fields of strength around 10^{15} Tesla would suffice for restoring the Salam-Weinberg theory. Comparison of the data obtained from an experiment of the type illustrated in Fig. 1 in the presence and absence of $\underline{B}$ would allow us to determine whether such a restoration has taken place, and whether the accelerating observer does indeed see a heat bath at temperature T given by (1) much as the observations of the lifetime of the μ-meson allowed us to vindicate the time-dilation effect of special relativity. It is encouraging to note that experiments involving such strong fields have already been suggested by Salam and Strathdee in ref. 2.

We will now go on to study the possible relationship of the effects described above to the "No-Hair" Conjecture[5] for black-holes. It will be shown that they allow a possible mechanism for transcending the "No-Hair" Conjecture in the quantum regime. For this purpose, let us consider a black-hole in thermal equilibrium with a heat bath at temperature T, and let us introduce into the heat bath a system of particles interacting through some spontaneously broken gauge fields, e.g. SU(2) x U(1), while maintaining thermal equilibrium[4]. This means that if the mass of the black-hole is sufficiently small, the corresponding temperature will be sufficiently large to allow the initial spontaneously broken gauge symmetry to be restored and the corresponding gauge fields become long range due to their masslessness. We further obtain conserved charges, apart from those associated with electromagnetism. This means that the interacting particles we are considering will have associated with them conserved gauge charges and the corresponding Gauss law for the system. The existence of Gauss' law immediately raises the possibility for the black-hole to carry the gauge charge if the system of interacting particles falls through its event horizon. Let us take the example of $SU(2) \times U(1)$. The restoration temperature for this gauge group is $\sim 10^{15}$ Kelvin. Taking the black-hole to be of the Schwarzschild type, the mass can be found from (2) to be $\sim 10^{8}$ kg. So as long as the interacting particles have Compton wavelengths less than the size of the black-hole (i. e. its Schwarzschild radius), the possibility of transcending the "No-Hair" Conjecture exists.

It is known[6] that small primordial black-holes possibly formed by fluctuations in the early universe, with masses $\sim 10^{11}$ kg, would just decay away through Hawking radiation (with a characteristic spectrum) within the present age of the universe. It is found[7] that for electrically charged primordial black-holes, fluctuations in the charge will cause the average emission rate for charged particles to be lower than that for similar uncharged particles. Coupled with the arguments presented above for the transcendance of the "No-Hair" Conjecture, it is clear that the emission rate will be further reduced (after the mass of the black-hole reaches $\sim 10^{8}$ kg) due to the accumulation and subsequent fluctuations of the new gauge charges acquired by the decaying black-hole. This, we suggest, will lead primordial black-holes not to an explosive death but rather to a slow, "quiet" death.

So we see that in principle it is possible to transcend the "No-Hair" Conjecture. However, it remains to be seen if the arguments can be extended to more realistic situations, as in stellar collapse, for example to form a black-hole.

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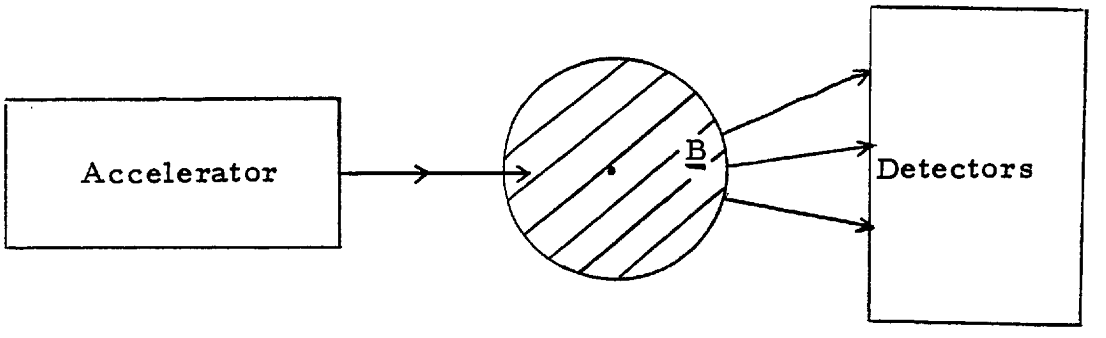

Fig. 1

Schematic experiment to demonstrate symmetry restoration through acceleration and temperature effects. The shaded region contains a magnetic field directed perpendicular to the plane of the paper. For large $\underline{B}$, the motion of particles entering the shaded region will be confined to it and subsequent decay products are observed by detectors.