UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY

THE ELASTOPLASTIC ANALYSIS OF PLANAR FRAMES FOR LARGE DISPLACEMENTS BY MATHEMATICAL PROGRAMMING

by

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A thesis submitted for the degree of Doctor of Philosophy in the Faculty of Engineering of the University of London

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January 1979

ABSTRACT

The object of the present work is to establish a unified formulation suitable for the deterministic analysis of planar elastoplastic skeletal structures undergoing finite deformations and/or arbitrarily large displacements, induced by quasi-static actions, from which formulations for the kinematically non-linear analysis of elastic and rigid-plastic frames are obtainable by simple specialization.

Systematic procedures to formulate and solve the problem are incorporated by discretizing the structure into a finite number of repetitive building elements. The alternative processes through which such elements can be assembled are exhausted by interpreting the discretized structure as a directed graph; their assemblage is implemented through fully automated procedures deriving from a physical interpretation of connectivity theory concepts.

A governing system featuring symmetry is obtained by preserving duality in the exact descriptions of equilibrium and compatibility of the nodal and mesh fundamental substructures, and reciprocity in the causality relations, derived from the analysis of a three-degree of freedom elastoplastic finite-element.

Four alternative methods for kinematically non-linear elastoplastic analysis, namely deformation, incremental, perturbation and asymptotic analysis, are provided. Each of the four alternative methods is described in four alternative formulations: nodal-stiffness, nodal-flexibility, mesh-stiffness and mesh-flexibility.

Every one of the four alternative formulations generates, when processed through Kuhn-Tucker equivalence theory, a pair of primal-dual mathematical programs, leading to the discrete representation of variational principles.

A unified treatment of problems in uniqueness and stability and of plastic unstressing and critical behaviour is presented.

A brief survey of existing formulations and procedures and a comparison with present analytical results are made.

Computational procedures are given and numerical results are obtained and compared with results proposed in published works in kinematically non-linear elastic, elastoplastic and rigid-plastic analysis.

ACKNOWLEDGEMENTS

The work described in this volume was carried out during the years 1975-1978 in the Systems and Mechanics Section, Civil Engineering Department, Imperial College of Science and Technology.

The sponsorship provided by the Instituto Nacional de Investigação Científica, Portugal, is gratefully acknowledged.

The success of my application for a scholarship was certainly a direct consequence of the prompt and effective support conveyed by Professors E. Cardoso, J. F. Lobo Fialho and E. R. Arantes e Oliveira.

To the late Professor Lobo Fialho, at the time chairing the Theory of Structures Cathedra, I owe the encouragement to persevere in my studies in structural mechanics, an interest he helped cultivate further by using his influence in effecting my appointment to an assistentship. I shall always remember Professor Lobo Fialho as an inspiring mentor and friend.

I am also particularly indebted to Professor Arantes e Oliveira; his guidance, interest and enthusiasm on the progress of the research work being carried out, only made possible through his continued support for the yearly renewal of my scholarship, are sincerely acknowledged.

Dr. J. Munro, Reader in Civil Engineering Systems and Optimization, very kindly presented me with the much appreciated opportunity to pursue my studies as a research student at the Section he heads; once again I express my gratitude for his help, support and incentive in my attempts to improve my understanding of his contemporaneous reading of structural mechanics.

I am deeply indebted to my advisor Dr. D. L. Smith, Lecturer in Civil Engineering, Systems and Mechanics Section, Imperial College. In the best pedagogical tradition, Dr. Smith patiently made his knowledge fully accessible, dedicating particular attention to the methodology for attaining and developing the knowledge acquired.

Among those who directly contributed to this work, my gratitude therefore goes mainly to Dr. Smith, to whom I promise a continued discipleship.

I also wish to thank my friends in the Systems and Mechanics research group, Dr. P. Alexa and Messers A. M. A. da Fonseca, J. A. S. Appleton, E. L. Mello and U. R. Madi, for a fruitful exchange of ideas and for their help in various ways.

I must also mention my friend Mr. A. M. R. da Silva; his advice and help in the preparation of some of the drawings were much appreciated.

My family's stay in England and my own would not have been so pleasant had I not met Dr. S. H. Perry, an excellent, most helpful friend.

I am profoundly grateful to my late mother and to my father. My attempts of retribution have been totally subdued by the extent of their dedication.

I must apologize to my dear wife Cristina and son Francisco for the very hard times I made them endure.

Without my wife's dedicated support, cheerful encouragement and endless patience, I would not have been able to reach this point in a confortable state of almost undented sanity.

Fatherhood and thesis-writing intersect in an empty, exhausted set, a point fully proven by my four-month old son; if it was not for my ever-starving, ever-awaken, ever-demanding, charming son, to bring this work to an end would not have been such an exhilarating experience.

J. A. TEIXEIRA DE FREITAS

C O N T E N T S

This volume contains a total of 409 pages.

CHAPTER ONE

INTRODUCTION

1.1 OBJECT AND OBJECTIVES OF THE RESEARCH: THE THESIS

Traditional in civil engineering is to build what can be designed safely with the minimum effort: is there, therefore, a need for works concerned with

OBJECT: The deterministic analysis of planar elasto-plastic skeletal structures undergoing large displacements, induced by quasi-static actions.

implying the costly solution of highly non-linear systems? Assuming there is, as the extensive list of contributions seems to suggest, is yet another formulation justifiable?

To accompany and sustain the continued evolution of social aspirations entertained throughout the ages, designers have been increasingly requested by their communities to venture in the production of ever more efficient structures. In order to substantiate present-day ambitions, and constrained by the urgency of using resources more effectively, to explore the non-linear phase in the behaviour of structures has already become a necessity in many applications.

With few exceptions, the intellectual incentive to formulate problems in non-linear structural mechanics under increasingly relaxed hypotheses, only recently was rewarded by the possibility of actually implementing the solutions so devised.

A survey of the published works on non-linear structural analysis thus instigated, reveals a multiplicity of approaches, methods and procedures to materialize commonly shared concepts and objectives.

Two or three decades ago, linear structural mechanics was in a comparable state of uneven, disconcerted development. Foreseeing the advent of more efficient means of numerical implementation, efforts were then developed to establish formulations leading to a synthetic, general theory.

Of the proposed theories, it is perhaps in Smith's (1974) presentation where more clearly suggested and consistently explored are the four fundamental ingredients responsible for the unification achieved, namely, structural discretization, substitution of structures by graphs, static-kinematic duality and mathematical programming. The primary motive behind the decision to undertake the research work soon to be presented, was to convey those concepts into the kinematically non-linear field, in order to establish a formulation suitable for a subsequent development into a unified theory on non-linear structural analysis; thus the research

OBJECTIVES:

1. Incorporate systematic procedures to formulate and solve the problem through the discretization of the structure into a finite number of repetitive building elements.

2. Exhaust the alternative processes through which such elements can be assembled by interpreting the discretized structure as a directed graph.

3. Implement a governing system featuring symmetry by preserving reciprocity in the causality relations, and duality in the descriptions of equilibrium and

compatibility.

4. Complement through mathematical programming theory the resulting discrete representation with a variational interpretation.

The concomitancy of the aforementioned objectives differentiates, thus justifying, the formulation being offered for consideration, and prompted the

THESIS: Mathematical Programming is a particularly appropriate theory for encoding and solving problems in kinematically non-linear structural analysis, synthetizing the benefits of both discrete and variational approaches in a unified mathematical formalism.

1.2 BASIC CONCEPTS AND METHODOLOGY

An engineering structure, being an orderly interconnection of (SUBSTRUCTURES formed by) parts or ELEMENTS into a meaningful whole, is essentially a SYSTEM.

The analysis of the system requires the use of information about its elements as well as the knowledge about the interaction between its components. When these two factors are taken into consideration, the response of the system to given inputs or ACTIONS can be determined.

If the system is such that its response can be determined through an automated assemblage of the relations governing the behaviour of the constituent subsystems, the response of each of which can be defined through the analysis of a single or restricted number of typical elements, then the formulation supporting the system analysis gains in unity, generality and computational viability.

A systems approach to structural mechanics suggests therefore a mathematical model simulating the response of the structural system formed by the combination of two INDEPENDENT sets of algebra: one, VECTORIAL, developing from the geometric-mechanic properties and characterizing the behaviour of the constituent elements and of those elements when forming a substructure; the other, BOOLEAN, implementing the connectivity properties and thus regulating the procedure for the assemblage of the substructures to gain the structure anew.

Pursuing such an approach, in the presentation to follow the structure is first resolved into its simplest elements to which a specific type of connectivity can be associated, the FUNDAMENTAL SUBSTRUCTURES. The conditions for the static equilibrium (STATICS) between the forces applied to the selected substructure and the developing stress-resultants are then derived, as well as, and INDEPENDENTLY, the conditions for the kinematic compatibility (KINEMATICS) between the strain-resultants and the displacements suffered by selected points, the movement of which is sufficient to characterize the rigid-body motion of the substructure. The description of Statics and Kinematics thus obtained can be EXACT, as no assumptions concerning the magnitude of the variables involved need to be made.

Next, the substructure is decomposed into its constituent fundamental FINITE-ELEMENTS embodying the mechanical characteristics of the structural material. The CONSTITUTIVE RELATIONS may then be derived by establishing the causality relations between the stress-resultants applied to the typical element and the corresponding strain-resultants. An idealized LINEAR ELASTIC-NONLINEAR PLASTIC response is adopted to simulate, as opposed to represent, the actual structural material behaviour.

In most of the inumerous engineering formulations in finite-element non-linear structural analysis, the theoretical development ceases at the next stage of recovering the structural behaviour by establishing an appropriate procedure to assemble the elemental governing relations; way is then given to the equally important aspect of search for a convenient numerical implementation technique.

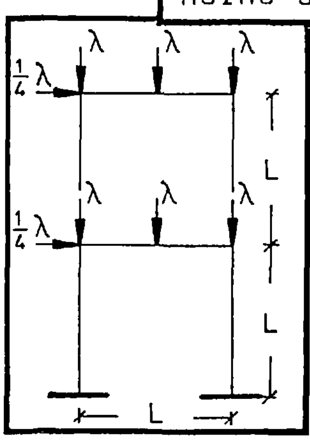
In the approach to be suggested, the system governing the behaviour of the structure, instead of being left to be numerically processed with or without the assistance of a pre-defined variational principle, is, as soon as derived, processed through MATHEMATICAL PROGRAMMING EQUIVALENCE THEORY. A formulation entirely generated from first-principles of mechanics, namely equilibrium, compatibility and causality, is thus allowed to develop naturally into the associated variational principles; the dimorphism in traditional structural analysis whereby variational and finite methods are kept apart in an atmosphere of near-rivalry is thus made inconsequent, the privileges of the two COMPLEMENTARY approaches being preserved in a UNIFIED formulation. Secured is the possibility, characteristic of the first-principle based finite formulations, of permanent and localized control of the hypotheses introduced, the implications of which may then be consistently appraised; also secured is the prerogative of the variational descriptions which enrich the theoretical scope of the formulation by facilitating and synthetizing the physical interpretation of the relevant phenomena.

Finally, use is made of MATHEMATICAL PROGRAMMING ALGORITHMS to implement in an efficient manner the numerical solution of the problem.

1.3 STRUCTURAL IDEALIZATION

An engineering structure is a highly complex system, the analysis of which has to be based on a simplified model, for instance and as herein, a deterministic mathematical model, designed to simulate within minimum accuracy requirements the response of the actual structural system to given external actions. Three phases of structural idealization can usually be distinguished in the process of establishing the sought mathematical model.

Horne's (1963) two-storey one-bay planar frame



GEOMETRIC PROPERTIES: The length, cross-sectional area, and second moment of area are constant and the same for all beams and columns.
MECHANICAL PROPERTIES: Beams and columns have an idealized linearly elastic-perfectly plastic flexural behaviour.
EXTERNAL ACTIONS: The frame is subject to quasi-static, conservative, proportionally increasing loads. The foundation is rigid, with unlimited capacity for absorbing stresses in every direction of the plane.

FIGURE 1.1

In the first phase the relevant mechanical and geometrical properties are summarized, as well as the nature and distribution of the external actions, and the connectivity properties graphically represented, as illustrated in the figure above for a simple two-storey one-bay planar frame.

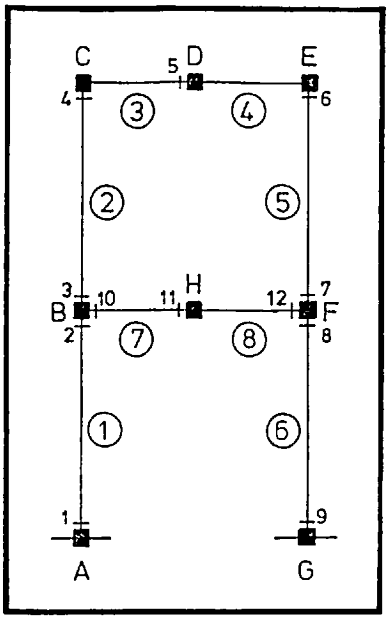


FIGURE 1.2

When, as in the present case, emphasis is placed upon finite methods, the next phase is concerned with DISCRETIZING the structure, the objective being to identify the constituent parts wherein stress and strain flow continuously.

This information is added to the graphic model, as in Fig.1.2, by positioning RIGID NODES at connections between three or more members and at points where the geometrical and/or mechanical properties change, where mechanical release devices exist, and where concentrated forces and/or couples are applied.

Whenever the structural representation is not unduly affected, curved members should be approximated by a set of straight members, and members with continuously varying cross-sectional properties by a set of members of different but constant cross-sections; distributed loads will be lumped into concentrated loads.

As the development of plasticity is assumed confined to certain discrete sections of the structural members, where extensible plastic hinges are allowed to develop, a CRITICAL SECTION will be positioned at the end of each member incident with a node. The number of critical sections can be reduced if stress-interaction effects are not accounted for in the characterization of the plastic capacities of the members, and/or if mechanical release devices exist, preventing the yield stress to be attained.

As the developing stresses and strains define vector fields, it is necessary to refer the graphic model to a global system of reference and to associate each member with a positive direction, as in Fig.1.3 where an additional rigid member was added to simulate the support offered by the foundation; such is the third phase of structural idealization.

The graphic model may now be considered as a set of orientated lines or branches connecting any two points, verteces or nodes, that is, a DIRECTED GRAPH.

The graph can be disconnected using two different types of

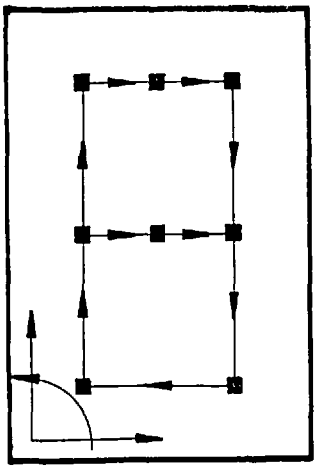


FIGURE 1.3

REPETITIVE ELEMENTS, branches (or MEMBERS) limited by nodes, and rings (or MESHES), that is, connected subgraphs in every vertex of which there are incident exactly two branches. The graph can be rebuilt either by incidence of the branches at the nodes (NODAL CONNECTIVITY) or by incidence of the rings at their contours (MESH CONNECTIVITY), as illustrated in Fig.1.4.

Consider now any of the deformed forms of the idealized structure, illustrated in Fig.5.25. Each of these new graphic models, upon which a kinematically non-linear analysis has to be based, can still be developed into a directed graph. In order to quantify the phenomena a structural analysis is concerned with, in essence the changes of form and the accompanying variations in the load-carrying capacity of the structure, a certain algebra has to be associated with this new directed graph; a possible approach for the derivation of such a mathematical model is summarized below.

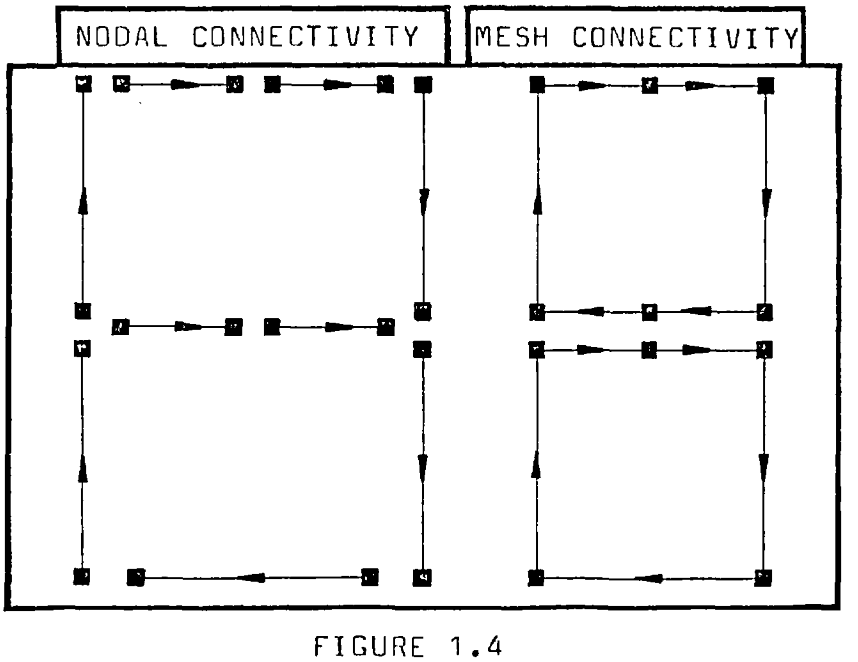


FIGURE 1.4

1.4 SUMMARY OF THE THESIS

Chapter Two is concerned with establishing the exact relations governing static equilibrium and kinematic compatibility. By analyzing two distinct fundamental substructures dissected from the structure, the nodal and mesh substructures, two alternative and complementary descriptions of Statics and Kinematics are presented. The selected fundamental substructures are the single branch and the generalized polygonal ring illustrated in Fig.1.5; they are simultaneously simple to analyze and sufficiently generic to be re-connected into a planar skeletal structure of arbitrary geometry.

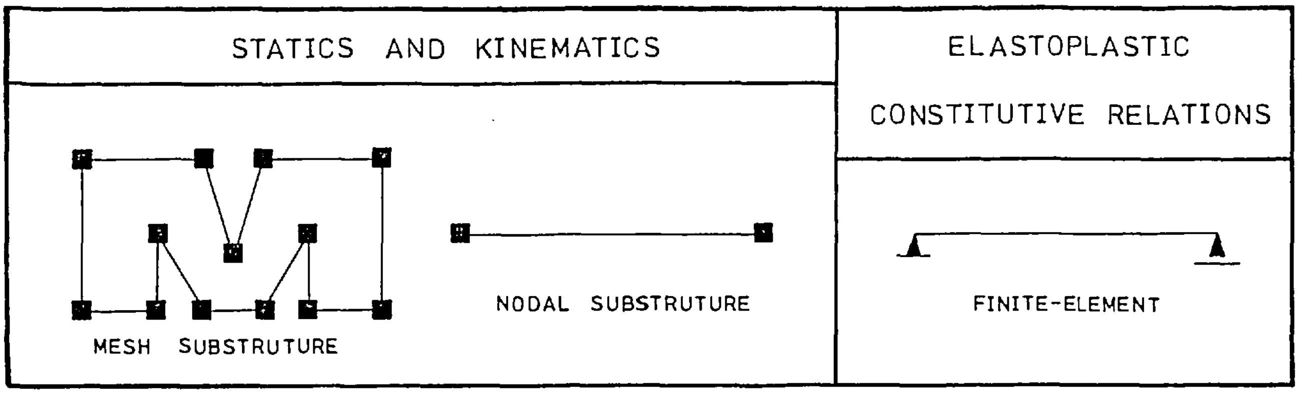


FIGURE 1.5

The derivation of the compatibility conditions, expressed through kinematic variables only, is exclusively based on geometrical considerations. The equilibrium conditions, performed on the deformed substructure and equally derived from first principles, involves both static and kinematic variables.

The static and kinematic descriptions so derived are neither linear nor do they represent dual transformations. To recover these two aspects, the latter of which has proved essential for the unity of the theory of linear structural mechanics, the equilibrium conditions are replaced by an equivalent system, still exact but explicitly linear, by introducing additional forces wherein the non-linearities generated by the Statics dependence on Kinematics are concentrated. The kinematic relations are then replaced by an equivalent set, again exact and explicitly linear, designed to recover Static-Kinematic Duality. The process generates, in a natural manner, additional kinematic variables, in the definitions of which the kinematic non-linearities are synthetized.

The finite description of Statics and Kinematics so obtained are implicitly non-linear. The corresponding incremental descriptions, in terms of finite incremental variables, as opposed to infinitesimal rates of variation, are then derived and treated by a standard perturbation technique, which replaces each of the implicitly non-linear equations by an infinite system of recursive, thoroughly linear equations.

Each of the aforementioned formulations, finite, in terms of arbitrarily large variables, (finite-) incremental and perturbed descriptions, are subsequently used to develop three of the four alternative methods of analysis considered in the present work; namely, deformation, incremental and perturbation analysis. The fourth method is designed to implement the asymptotic analysis of systems with a kinematically trivial, statically non-trivial initial response.

After extending the mesh and nodal descriptions to include the (non-linear) effects of internal mechanical release devices, the discussion on Statics and Kinematics is concluded by recovering the Principle of Virtual Work, interpreted herein as the variational representation of Static-Kinematic Duality.

In Chapter Three the causality relations associating the member stress-resultants with the corresponding strain resultants are derived through a first principle based analysis of the three-degree of freedom elastoplastic finite-element represented in Fig.1.5.

The complexity of the behaviour of such a simple beam element is primarily caused by the mechanics of the development of plasticity, the problem becoming tractable only if restrictive hypotheses are introduced. In this work the hypothesis of lumped plasticity holds. Furthermore the critical sections, where plastic strains are restricted to develop, are required to coincide with the element end-sections; plastic strains are thus hindered of flowing within the element. As a consequence of the above mentioned hypotheses, the maximum axial stress at interior sections is required to remain within the elastic range, allowing for the separation of the strain field into a continuous field of elastic strains, flowing along the beam, and a discrete field of plastic strains developing at its end-sections; thus the separation of the study of the elastoplastic constitutive relations into elasticity and plasticity, dealt with in sections 3.1 and 3.2, respectively.

The descriptions of Statics and Kinematics of the three-degree of freedom elastic beam are fed into the constitutive relations, corrected to include a measure of the shear deformation effects, and the resulting differential governing equation is solved by a standard perturbation technique. The elastic solutions are then interpreted and cast in two alternative formats, stiffness and flexibility, suitable to implement discrete structural analysis.

The several mathematical plasticity theories which have been proposed can be divided into two groups, according to whether the basic relationships connect stress and strains or stress and strain rates. The results provided by these theories, respectively known as deformation theories and flow theories, may only coincide in the absence of plastic unstressing. In either case, the first step is to decide on the yield criterion, the rule defining which combination of equilibrated stresses will cause yield; the next step is to impose the kinematic compatibility condition for the fully plastified cross-section, or, in the parlance of the theories of plasticity, to characterize the flow rule. Following the methodology the early works in plasticity adopted, the yield condition (Statics) and the flow rule (Kinematics) are herein treated separately. The possibility of deriving the flow rule from the yield condition is offered by the concept of plastic potential; in the terminology we adopt, this is understood as a relation of duality between the descriptions of the static and kinematic phases of plasticity. The plasticity relations are completed when the static and kinematic variables are connected through an association condition, wherein the essential difference between deformation and flow theory resides.

The elastoplastic constitutive relations are presented in four alternative formats suiting the methods of analysis to be considered, deformation, incremental, perturbation and asymptotic analysis.

The information supplied by the previous two Chapters is collected and re-arranged in Chapter Four in order to establish a resulting system of relations intended to represent an appropriate mathematical model of the structure under analysis.

Having previously numbered and oriented the fundamental substructures forming the structure, the elemental elastoplastic constitutive relations are simply grouped, arranging the causality operators in block-diagonal matrices and the stress- and strain-resultant vectors in super-vectors, according to the selected numbering sequence. Connectivity theory is called upon to assemble the static and kinematic descriptions, the objective being to establish the proper path guiding the flow of stress and strain developing in the structure.

In the present work, wherein Statics and Kinematics are assembled separately, the process of assemblage is designed to suit the type of substructure one considers the structure is formed of.

If the structure is interpreted as an assemblage of nodal substructures, continuity of displacements at the nodes is secured first and, by resorting to the Principle of Work Invariance, and thus automatically satisfying nodal equilibrium, the nodal description of Statics is assembled next.

The proposed method for assembling the mesh description of Statics and Kinematics uses a regional cycle basis. The complementary solution of Statics is easily assembled by superimposing the stresses flowing along branches common to incident meshes. To assemble the particular solution, the applied forces are first assigned to the constituint mesh substructures, transmitting next the flow of stress they generate along a selected path of incident meshes. Using the Principle of Work Invariance, Kinematics is assembled by satisfying continuity of the flow of strains.

In either of the formulations, nodal and mesh, Static-Kinematic Duality, at structure level, emerges as a direct consequence of the duality forced into the substructure relations through the use of additional forces and deformations. The Principle of Virtual Work is again interpreted as the variational representation of the relations of duality existing between the descriptions of Statics and Kinematics of the structure.

The fundamental conditions characterizing the behaviour of elastoplastic structures under large displacements are consistently combined in Chapter Five in order to generate four alternative descriptions for the systems governing the structural response; the nodal-stiffness, nodal-flexibility, mesh-stiffness and mesh-flexibility formulations.

Following the usual procedure in mathematical programming theory of structural analysis, each of the resulting governing systems is identified as a Kuhn-Tucker problem, the associated mathematical programs being derived next through the application of equivalence theory.

The resulting mathematical programs are physically interpreted and analyzed through mathematical programming theory.

As the structure governing system defines configurations which are simultaneously statically and kinematically admissible, the role of Kuhn-Tucker Equivalence will prove to be to separate that system into two distinct problems wherein static and kinematic admissibility are enforced independently. The extremization of the objective functions of the associated mathematical programs become the criteria of selecting among all statically (kinematically) admissible states, the correct static (kinematic) field or fields; the variational principles of kinematically nonlinear elastoplastic analysis are thus recovered.

Condition for uniqueness of solution are established and multiple solutions qualitatively investigated; an interpretation of Drucker's stability criteria is also included.

After a brief description of the algorithms used in the solution of illustrative examples, two special occurrences in the behaviour of elastoplastic structures, namely plastic unstressing and limit and bifurcation points, are analyzed and numerical procedures for identifying and solving such situations presented.

The alternative descriptions for the elastoplastic governing system are specialized in the latter part of Chapter Five for the analysis of elastic and rigid-plastic structures. The associated mathematical programs are then derived and interpreted folllowing the procedure adopted in the elastoplastic analysis.

Chapter Five ends with a brief comparative study of the behaviour a structure presents when elastic, elastoplastic and rigid-plastic constitutive relations are assumed.

Ideas for possible extension of the formulation to be suggested and on improvements it is susceptible of, within the envisaged scope of the proposed study, are briefly summarized in Chapter Six.

Named throughout the presentation, where and whenever relevant, are the simplificative hypotheses introduced, as well as the points of contact with related results presented in the literature.

The imense number of published works on the many areas of knowledge a kinematically non-linear elastoplastic analysis by mathematical programming has to rely on, prohibits an exhaustive survey of proposed formulations; thus the limited number of referenced contributions summarized in the latter part of this work.

The conviction that such a summary would certainly fail to mention other important contributions to the proposed area of study, has always deeply concerned the author. If the injustices thus perpetrated can be partially undertoned by sincere appology, the damage their ignorance has caused is irreparably reflected in the presentation soon to follow.

1.5 ORIGINAL FEATURES

To the author's knowledge, the following features of this thesis are original:

I. Derivation from first principles of mechanics of the exact mesh and nodal descriptions of Statics and Kinematics at substructure level, preserving Static-Kinematic Duality, from which the Principle of Virtual Work results.

II. Accurate and unified finite-element description, in both stiffness and flexibility formats, of the elastic constitutive relations, inter-relating the stability and bowing functions and extending their definitions to include the effect of axial deformability and a measure of the shear deformation effects.

III. Fully automated procedure, based on a physical interpretation of connectivity theory concepts, of the mesh (and nodal) descriptions of Statics and Kinematics which, although dealt with independently, emerge as dual transformations from which results the Principle of Virtual Work for structures undergoing arbitrarily large deformations and/or displacements.

IV. Unified formulation for the kinematically non-linear analysis of elastoplastic systems,

featuring

1) Four alternative methods of analysis, namely deformation, incremental, perturbation and asymptotic analysis, each described in
2) Four alternative formulations, nodal-stiffness, nodal-flexibility, mesh-stiffness and mesh-flexibility, generating
3) Four pairs of primal-dual mathematical programs obtained through the
4) Application of Kuhn-Tucker Equivalence Theory,
   i) leading to the discrete representation of variational principles, and
   ii) allowing for a unified treatment of problems in uniqueness and stability and of plastic unstressing and critical behaviour

from which

5) Unified formulations for the kinematically non-linear analysis of elastic and rigid-plastic systems result by simple specialization

and using

6) Physically interpreted system analysis procedures and mathematical programming algorithms to implement the numerical solution of the relevant problems in non-linear structural analysis.

C H A P T E R   T W O

STATICS AND KINEMATICS

OF THE FUNDAMENTAL SUBSTRUCTURES

A skeletal structure, when interpreted as a directed graph, can be thought of as the assemblage of two different types of fundamental substructures, depending on the inherent connectivity properties; the NODAL substructure, the line segment or branch of the orientated graph together with its end points or vertices, and the MESH substructure, a connected subgraph such that on every vertex there are incident exactly two branches.

The branch or MEMBER, a common component of both nodal and mesh substructures, represents from the structural point of view the centroidal locus idealization of a prismatic beam and embodies its mechanical properties.

Let us consider one such member m in both its initial and deformed configurations, as in Fig.2.1, in order to present some of the terminology and conventions adopted in setting up the static and kinematic descriptions at substructure level.

The member is referred to a global system of co-ordinates $\underset{\sim}{x}^*$; in its initial undeformed position, the distance between the two critical sections limiting the member defines the MEMBER LENGTH L_m and the initial MEMBER INCLINATION α_m is measured relatively to the axis x_3^* in the positive sense of x_1^*. The member is ORIENTATED positively from critical section 1 to critical section 2.

When the structure is loaded, the members deform elastoplastically, the development of plasticity being restricted to the critical sections where EXTENSIBLE PLASTIC HINGES may form.

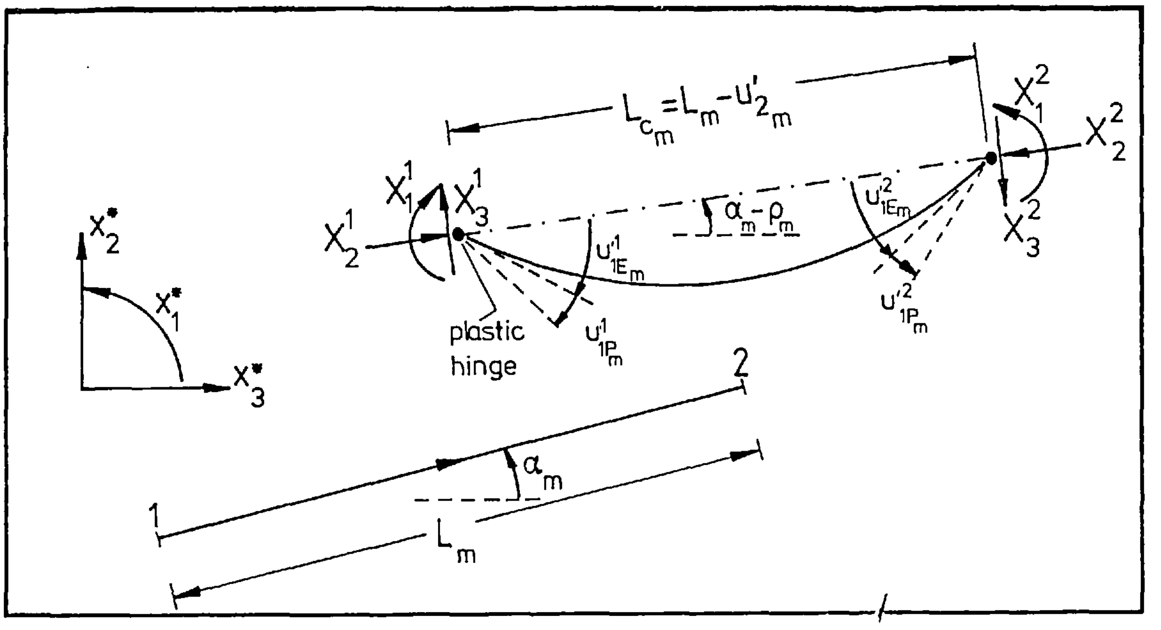


FIGURE 2.1

By MEMBER CHORD we understand the line joining the critical sections; the distance between those two points defines the MEMBER CHORD LENGTH L_{c_m} and the relative rotation between the initial and final chord positions, measured positively in the negative sense of x_1^*, is the MEMBER CHORD ROTATION ρ_m.

The stress-resultants at critical sections 1 and 2 are measured positively as indicated in Fig.2.1 following the usual engineering beam theory convention, except that the axial and shear forces are now parallel and perpendicular, respectively, to the chord of the deformed and displaced member.

As the direct effects of the deformation by shear are not considered, three variables are sufficient to characterize the deformation of the member; the axial shortenning u'_{2_m}, and the rotations $u'^1_{1_m}$ and $u'^2_{1_m}$ at the critical sections.

The axial shortening, defined as the difference between the initial length and the chord length, contains both the effects of axial deformation, caused by the varying axial stress-resultant along the deformed member, and shortening due to flexure.

The rotations are measured from the chord to the direction of the neighbouring node in the sense of the associate bending moment. The rotation at critical section i, $u'^{i}_{1_m}$, is the sum of the elastic rotation $u'^{i}_{1E_m}$, measured from the chord to the tangent of the member at critical section i, and the plastic rotation $u'^{i}_{1P_m}$, measured from that tangent to the direction of the node.

The results in Chapter Three will show that these three parameters of deformation are sufficient to define the displacement, relative to the chord, suffered by any point of the beam.

A typical substructure dissected from the deformed structure will be analyzed in order to establish two fundamental sets of relationshios, one defining the compatibility condition regulating deformation and displacement components, the other defining the existing state of equilibrium between applied forces and the stress-resultants developed.

The derivation of the compatibility condition, expressed through kinematic variables only, will be exclusively based on geometrical considerations. The equilibrium conditions, performed on the deformed substructure and equally derived from first principles, will involve both static and kinematic variables.

The Statics and Kinematics descriptions so derived will neither be linear nor represent dual transformations.

To recover these two aspects, the equilibrium conditions will be replaced by an equivalent set, still exact but explicitly linear, by introducing additional forces in such a way that equilibrium can be performed on the substructure in its initial, undeformed configuration.

Next the kinematic relations are replaced by an equivalent set, again exact and explicitly linear, which is designed to recover Static-Kinematic Duality. The process generates in a natural manner additional kinematic variables which are then subject to a physical interpretation.

Naturally, the process could be reversed by "linearizing" Kinematics first and re-defining Statics next by enforcing Static-Kinematic Duality.

The dependence of Statics on Kinematics is concentrated in the definition of the additional forces.

As no reference at all is made to the member constitutive relations, the (equilibrated) static variables and the (compatible) kinematic variables need not, at this stage, be associated through any cause-effect relationship.

The above procedure will first be applied to continuous (in the sense that no internal releases exist) nodal and mesh substructures.

The finite descriptions of Statics and Kinematics so obtained are implicitly non-linear. The corresponding incremental descriptions are then derived and treated by a standard perturbation technique in order to eliminate the auxiliary variables containing the non-linearities of the problem. Thus the description emerges as an infinite system of recursive linear equations.

After a brief reference to alternative formulations presented in the literature, the nodal and mesh descriptions are extended to include the effect of internal releases.

The discussion on Statics and Kinematics of the fundamental substructures is concluded when the Principle of Virtual Work is recovered and interpreted as the variational representation of Static-Kinematic Duality.

2.1 NODAL DESCRIPTION OF STATICS AND KINEMATICS

Consider the two-storey portal frame represented in Fig. 5.25 in both its initial and deformed configurations, and assume that one of its members, say member m, together with the limiting nodes, is disconnected from the structure at both stages.

The objective is two-fold; to establish the condition of compatibility between the variables describing the movement of the nodes and the deformations of the member, and to find the condition of equilibrium between the forces at the nodes and the stress-resultants developing in the deformed member.

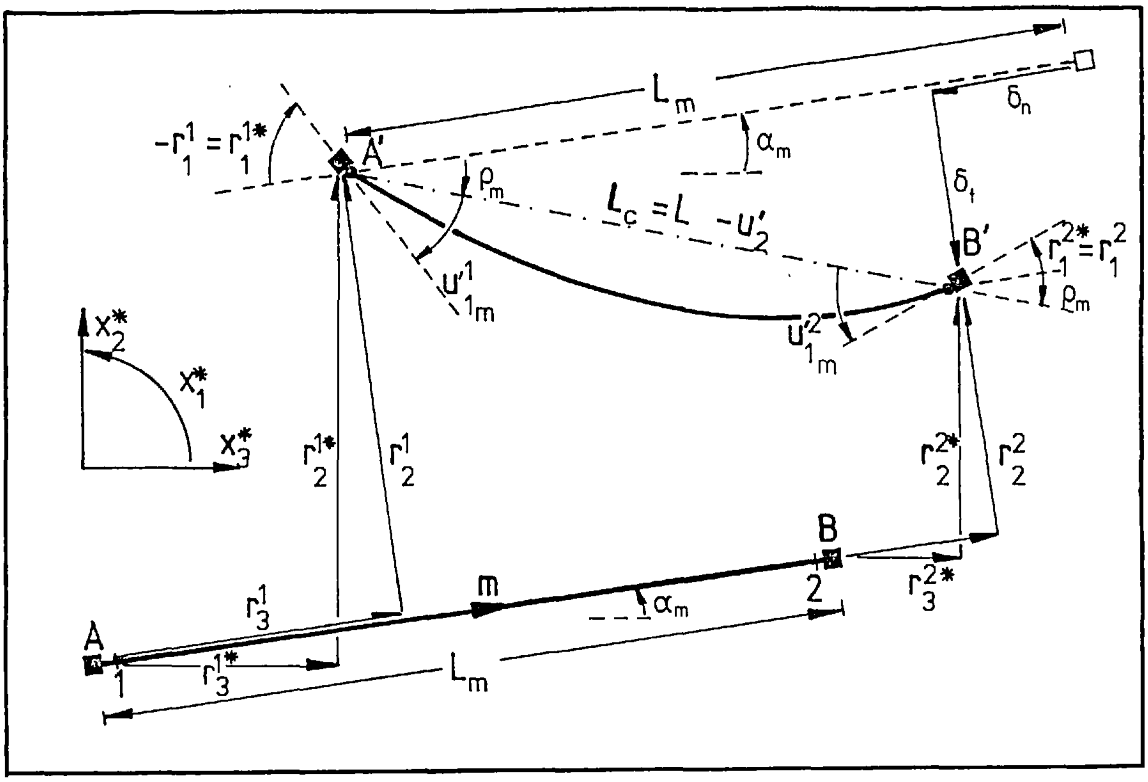


FIGURE 2.2

2.1.1 The Exact Non-linear Relations

The initial position of the member, referred to the system axes $\underset{\sim}{x}^*$, may be defined by the pair of its end co-ordinates $\underset{\sim}{x}_m^i$ (i=1,2), collected in $\underset{\sim}{x}_m^*$, and if $\underset{\sim}{r}_m^*$ represents the NODAL DISPLACEMENTS, referred to the same axes, as in Fig. 2.2, the final position of the member ends can be described by the sum

$$\underset{\sim}{x}_m^* + \underset{\sim}{r}_m^* = \begin{bmatrix} \underset{\sim}{x}^{1*} \\ \hline \underset{\sim}{x}^{2*} \end{bmatrix}_m + \begin{bmatrix} \underset{\sim}{r}^{1*} \\ \hline \underset{\sim}{r}^{2*} \end{bmatrix}_m$$

Let

$$u'^{1}_{1_m},\ u'^{2}_{1_m} \text{ and } u'_{2_m}$$

be the end-rotations measured to the chord and the chord shortening, respectively, and let these variables representing the MEMBER DEFORMATIONS be collected in $\underset{\sim}{u}'_m$.

The objective in Kinematics is to establish the condition of compatibility between the variables representing the relative displacement of the body and those describing its deformation.

Let $\underset{\sim}{r}_m$ represent the member end displacements referred to the member axes at its initial position, which can be obtained through an orthogonal transformation operating on the member end displacements referred to the system axes:

$$\underset{\sim}{r}_m = \underline{O}_m \, \underset{\sim}{r}^*_m \qquad (2.1.1)$$

where $\underline{O}_m$ is the block-diagonal matrix

$$\underline{O}_m = \left[\begin{array}{c|c} \underline{L} & \cdot \\ \hline \cdot & \underline{L} \end{array}\right]_m$$

and

$$\underline{L} = \left[\begin{array}{cc|c} 1 & \cdot & \cdot \\ \cdot & \cos\alpha & -\sin\alpha \\ \cdot & \sin\alpha & \cos\alpha \end{array}\right]_m$$

It is convenient to collect from $\underset{\sim}{r}_m$ a set of four auxiliary variables $\underset{\sim}{r}'_m$

$$\underset{\sim}{r}'_m = \underline{I}' \, \underset{\sim}{r}_m \qquad (2.1.2)$$

where the incidence matrix $\underline{I}'$ is defined by

$$\underline{I}' = \left[\begin{array}{ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{array}\right]$$

The auxiliary displacements $\underset{\sim}{r}'_m$ can be identified with the INTERMEDIATE DISPLACEMENTS mentioned in Jennings (1968). Herein, and for reasons that will become apparent, the components r'_2 and r'_3 will be termed ADDITIONAL FORCE DISPLACEMENTS, $\underset{\sim}{\delta}_{\pi_m}$

$$\underset{\sim}{\delta}_{\pi_m} = \begin{bmatrix} \delta_n \\ \delta_t \end{bmatrix}_m = \begin{bmatrix} r'_3 \\ r'_2 \end{bmatrix}_m \tag{2.1.3}$$

If ρ_m is the member chord rotation, then from Fig.2.2 and dropping subscripts

$$\tan\rho = \frac{\delta_t}{L-\delta_n} = \frac{r'_2}{L-r'_3} \tag{2.1.4}$$

where L is the original length of the member. The components of deformation may now be expressed in terms of the intermediate displacements as

$$u'^{\,1}_1 = -r'_1 - \arctan\frac{r'_2}{L-r'_3} \tag{2.1.5a}$$

$$u'^{\,2}_1 = r'_4 + \arctan\frac{r'_2}{L-r'_3} \tag{2.1.5b}$$

$$u'_2 = L - \left[(r'_2)^2 + (L-r'_3)^2\right]^{\frac{1}{2}} \tag{2.1.5c}$$

or in matrix form

$$\underset{\sim}{u}'_m = \underset{\sim}{D}_m\, \underset{\sim}{r}'_m \tag{2.1.6}$$

where
$$\underset{\sim}{D}_m = \begin{bmatrix} -1 & d_{12} & d_{13} & \cdot \\ \cdot & d_{22} & d_{23} & 1 \\ \cdot & d_{32} & d_{33} & \cdot \end{bmatrix}_m$$

it being totally unnecessary to express analytically the functionals d_{ij} ($i=1,2,3$; $j=2,3$). However, if the usual assumptions of linear Kinematics were to be enforced, the non-linear operator $\underline{D}_m$ would reduce to

$$\underline{D}_m = \begin{bmatrix} -1 & -1/L & \cdot & \cdot \\ \cdot & 1/L & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}_m$$

The compatibility condition between the variables describing the member deformation and its displacement can now be obtained by eliminating the intermediate displacements in equation (2.1.6) through equation (2.1.2) and the member end displacements $\underset{\sim}{r}'_m$ through equation (2.1.1) yielding

KINEMATICS
$\underset{\sim}{u}'_m = \underline{K}_m \; \underset{\sim}{r}^*_m$

(2.1.7)

where the functional matrix $\underline{K}_m$ is defined by the triple product

$$\underline{K}_m = \underline{D}_m \; \underline{I}' \; \underline{D}_m \qquad (2.1.8)$$

Consider now the free-body diagram of the deformed member m as represented in Fig.2.3.

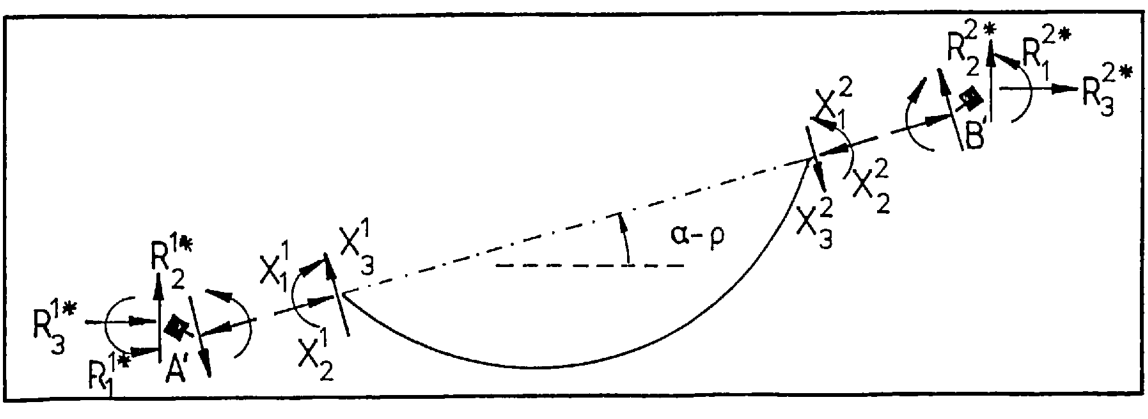


FIGURE 2.3

Let $$\underset{\sim}{R}^{*}_{m} = \begin{bmatrix} \underset{\sim}{R}^{1*} \\ \underset{\sim}{R}^{2*} \end{bmatrix}_m \quad \text{and} \quad \underset{\sim}{X}_m = \begin{bmatrix} \underset{\sim}{X}^{1} \\ \underset{\sim}{X}^{2} \end{bmatrix}_m$$

represent, respectively, the NODAL FORCES referred to the system axes $\underset{\sim}{x}^{*}_{m}$, and the STRESS-RESULTANTS at the critical sections of the member. The sign convention adopted and the ordering of their elements is shown in Fig.2.2; X^i_1, X^i_2 and X^i_3 represent the positive bending moment, axial and shear stress-resultants at critical section i.

The six stress-resultants are related through three equilibrium conditions. Selecting

$$\underset{\sim}{X}'_m = \begin{bmatrix} X^1_1 \\ X^2_1 \\ X_2 \end{bmatrix}_m = \begin{bmatrix} X^1_1 \\ X^2_1 \\ X^1_2 \end{bmatrix}_m$$

as the INDEPENDENT STRESS-RESULTANTS, in accordance with the variables previously chosen to represent the member deformation, $\underset{\sim}{u}'_m$, the internal equilibrium condition may be expressed as

$$\underset{\sim}{X}_m = \underline{H}_m \, \underset{\sim}{X}'_m \qquad (2.1.9)$$

where $$\underline{H}_m = \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \end{bmatrix}_m$$

and $$\underline{H}_{1_m} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ -1/L_c & 1/L_c & \cdot \end{bmatrix}_m \quad ; \quad \underline{H}_{2_m} = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ -1/L_c & 1/L_c & \cdot \end{bmatrix}_m$$

L_{c_m} being the chord length

$$L_{c_m} = L_m - u'_{2_m}$$

The nodal forces and stress-resultants at the critical sections can be related by imposing equilibrium at the nodes:

$$\underset{\sim}{R}^*_m = \underline{\mathbb{O}}_m \, \underset{\sim}{X}_m \tag{2.1.10}$$

where

$$\underline{\mathbb{O}}_m = \left[\begin{array}{c|c} \underline{O}' & \cdot \\ \hline \cdot & -\underline{O}' \end{array}\right]_m$$

and

$$\underline{O}'_m = \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & \cos(\alpha-\rho) & \sin(\alpha-\rho) \\ \cdot & -\sin(\alpha-\rho) & \cos(\alpha-\rho) \end{bmatrix}_m$$

The stress-resultants $\underset{\sim}{X}_m$ may now be eliminated in equation (2.1.10) through equation (2.1.9)

STATICS
$\underset{\sim}{R}^*_m = \underline{S}_m \, \underset{\sim}{X}'_m$

(2.1.11)

where

$$\underline{S}_m = \underline{\mathbb{O}}_m \, \underline{H}_m \tag{2.1.12}$$

After performing the products in equations (2.1.8) and (2.1.12) the definitions for the kinematic and static operators turn out to be, respectively

$$\underline{K}_m = \left[\begin{array}{c|c|c|c|c|c} -1 & c\cdot d_{12}+s\cdot d_{13} & -s\cdot d_{12}+c\cdot d_{13} & \cdot & -c\cdot d_{12}-s\cdot d_{13} & s\cdot d_{12}-c\cdot d_{13} \\ \cdot & c\cdot d_{22}+s\cdot d_{23} & -s\cdot d_{22}+c\cdot d_{23} & 1 & -c\cdot d_{22}-s\cdot d_{23} & s\cdot d_{22}-c\cdot d_{23} \\ \cdot & c\cdot d_{32}+s\cdot d_{33} & -s\cdot d_{32}+c\cdot d_{33} & \cdot & -c\cdot d_{32}-s\cdot d_{33} & s\cdot d_{32}-c\cdot d_{33} \end{array}\right]_m$$

(2.1.13)

$$\underline{S}^T_m = \left[\begin{array}{c|c|c|c|c|c} -1 & -c'/L_c & s'/L_c & \cdot & c'/L_c & -s'/L_c \\ \cdot & c'/L_c & -s'/L_c & 1 & -c'/L_c & s'/L_c \\ \cdot & s' & c' & \cdot & -s' & -c' \end{array}\right]_m$$

(2.1.14)

where $\quad s_m = \sin\alpha_m \quad , \quad c_m = \cos\alpha_m$

and $\quad s'_m = \sin(\alpha_m - \rho_m) \quad , \quad c'_m = \cos(\alpha_m - \rho_m)$

If the assumptions of linear analysis were to be adopted the above operators would reduce to

$$\underline{A}_m = \left[\begin{array}{ccc|ccc} -1 & -\cos\alpha/L & \sin\alpha/L & \cdot & \cos\alpha/L & -\sin\alpha/L \\ \cdot & \cos\alpha/L & -\sin\alpha/L & 1 & -\cos\alpha/L & \sin\alpha/L \\ \cdot & \sin\alpha & \cos\alpha & \cdot & -\sin\alpha & -\cos\alpha \end{array}\right]_m \qquad (2.1.15)$$

and Statics and Kinematics associated with member m could be expressed through the following dual transformations:

LINEAR ANALYSIS	
STATICS	KINEMATICS
$\underset{\sim}{R}^*_m \doteq \underline{A}^T_m \; \underset{\sim}{X}'_m$	$\underset{\sim}{u}'_m = \underline{A}_m \; \underset{\sim}{r}^*_m$
NODAL DESCRIPTION	

The linearized form $\underline{A}_m$ of the kinematic operator $\underline{K}_m$ can be found in most of the works dealing with skeletal structures, as Livesley (1964), Zienkiewicz and Cheung (1964), Spillers (1972), Smith (1974), Gallagher (1975) and others.

2.1.2 The Exact Explicitly Linear Dual Relations

If Statics is to be replaced by an equivalent linear form fundamented on equilibrium performed on the undeformed member, the first step is to introduce forcibly in equation (2.1.11) the linear operator $\underline{A}^T_m$. Hence

$$\underset{\sim}{R}^*_m = (\underline{S}_m + \underline{A}^T_m - \underline{A}^T_m)\; \underset{\sim}{X}'_m$$

or, rearranging

$$\underset{\sim}{R}^*_m = \underline{A}^T_m \, \underset{\sim}{X}'_m - \underset{\sim}{f}^*_m \tag{2.1.16}$$

where the auxiliary member end forces, represented in Fig.2.4, are defined by

$$\underset{\sim}{f}^*_m = (\underline{A}_m - \underline{S}_m) \, \underset{\sim}{X}'_m \tag{2.1.17}$$

or more explicitly

$$\underset{\sim}{f}^*_m = \left[\begin{array}{c|c|c} \cdot & \cdot & \cdot \\ \hline -\frac{\cos\alpha}{L}+\frac{\cos(\alpha-\rho)}{L_c} & \frac{\cos\alpha}{L}-\frac{\cos(\alpha-\rho)}{L_c} & \sin\alpha-\sin(\alpha-\rho) \\ \hline \frac{\sin\alpha}{L}-\frac{\sin(\alpha-\rho)}{L_c} & -\frac{\sin\alpha}{L}+\frac{\sin(\alpha-\rho)}{L_c} & \cos\alpha-\cos(\alpha-\rho) \\ \hdashline \cdot & \cdot & \cdot \\ \hline \frac{\cos\alpha}{L}-\frac{\cos(\alpha-\rho)}{L_c} & -\frac{\cos\alpha}{L}+\frac{\cos(\alpha-\rho)}{L_c} & -\sin\alpha+\sin(\alpha-\rho) \\ \hline -\frac{\sin\alpha}{L}+\frac{\sin(\alpha-\rho)}{L_c} & \frac{\sin\alpha}{L}-\frac{\sin(\alpha-\rho)}{L_c} & -\cos\alpha+\cos(\alpha-\rho) \end{array}\right]_m \left[\begin{array}{c} X_1^1 \\ \hline X_1^2 \\ \hline X_2 \end{array}\right]_m$$

The above definition, basically involving two independent variables, suggests the replacement of the end forces $\underset{\sim}{f}^*_m$ by the statically equivalent set of ADDITIONAL FORCES $\underset{\sim}{\pi}_m$, represented in Fig.2.5, and defined by

$$\underset{\sim}{f}^*_m = \underline{A}^T_{\pi_m} \, \underset{\sim}{\pi}_m \tag{2.1.18}$$

where

$$\boxed{\underline{A}_{\pi_m} = \left[\begin{array}{ccc|ccc} \cdot & \sin\alpha & \cos\alpha & \cdot & -\sin\alpha & -\cos\alpha \\ \cdot & \cos\alpha & -\sin\alpha & \cdot & -\cos\alpha & \sin\alpha \end{array}\right]_m} \tag{2.1.19}$$

Substituting equation (2.1.17) into equation (2.1.18) and solving, the additional forces can be expressed as non-linear functions of displacements and deformations directly proportional to the independent stress-resultants $\underset{\sim}{X}'_m$:

ADDITIONAL FORCES

$$\begin{bmatrix} \pi_n \\ \hline \pi_t \end{bmatrix}_m = \left[\begin{array}{c|c|c} \frac{\sin\rho}{L_c} & -\frac{\sin\rho}{L_c} & 1-\cos\rho \\ \hline -\frac{1}{L}+\frac{\cos\rho}{L_c} & \frac{1}{L}-\frac{\cos\rho}{L_c} & \sin\rho \end{array}\right]_m \begin{bmatrix} X_1^1 \\ \hline X_1^2 \\ \hline X_2 \end{bmatrix}_m \qquad (2.1.20)$$

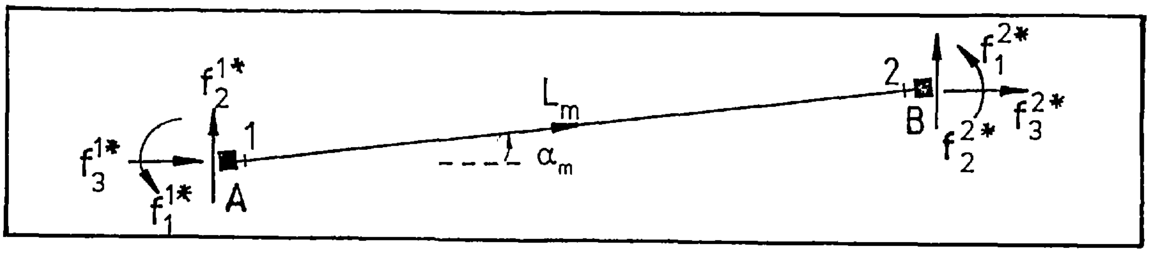


FIGURE 2.4

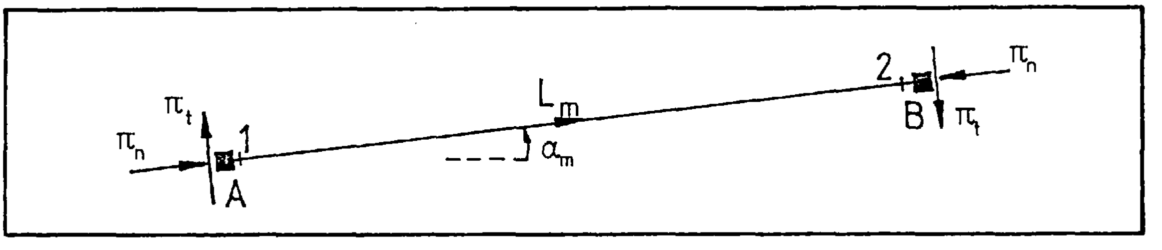


FIGURE 2.5

The additional forces may now be included in the member equilibrium condition (2.1.16)

$$\underset{\sim}{R}_m^* = \left[\underset{\sim}{A}^T \;\vdots\; \underset{\sim}{A}_\pi^T\right]_m \begin{bmatrix} \underset{\sim}{X}' \\ -\underset{\sim}{\pi} \end{bmatrix}_m$$

and if duality is to be preserved, the kinematic transformation must take the following form

$$\begin{bmatrix} \underset{\sim}{k}^1 \\ \underset{\sim}{k}^2 \end{bmatrix}_m = \begin{bmatrix} \underset{\sim}{A} \\ \hline \underset{\sim}{A}_\pi \end{bmatrix}_m \underset{\sim}{r}_m^* \qquad \begin{matrix}(2.1.21a) \\ (2.1.21b)\end{matrix}$$

it being necessary now to identify the variables $\underset{\sim}{k}_m^i$.

Similarly to what was done when dealing with Statics, let equation (2.1.7) be re-written as

$$\underset{\sim}{u}'_m = (\underline{K}_m - \underline{A}_m + \underline{A}_m)\ \underset{\sim}{r}^*_m$$

or

$$\underset{\sim}{u}'_m + (\underline{A}_m - \underline{K}_m)\ \underset{\sim}{r}^*_m = \underline{A}_m\ \underset{\sim}{r}^*_m$$

The first set of kinematic relations (2.1.21) is recovered by introducing the ADDITIONAL DEFORMATIONS $\underset{\sim}{u}'_{\pi_m}$ defined by

$$\underset{\sim}{u}'_{\pi_m} = (\underline{A}_m - \underline{K}_m)\ \underset{\sim}{r}^*_m \qquad (2.1.22)$$

yielding

$$\underset{\sim}{k}_m^1 = \underset{\sim}{u}'_m + \underset{\sim}{u}'_{\pi_m}$$

From equations (2.1.1) and (2.1.22)

$$\underset{\sim}{u}'_{\pi_m} = \begin{bmatrix} -\frac{1}{L}-d_{12} & -d_{13} \\ \frac{1}{L}-d_{22} & -d_{23} \\ -d_{32} & 1-d_{33} \end{bmatrix} \begin{bmatrix} r_2^1-r_2^2 \\ r_3^1-r_3^2 \end{bmatrix}$$

or, in terms of the intermediate displacements $\underset{\sim}{r}'_m$

$$\underset{\sim}{u}'_{\pi_m} = \begin{bmatrix} -1/L & \cdot \\ 1/L & \cdot \\ \cdot & 1 \end{bmatrix}_m \begin{bmatrix} r'_2 \\ r'_3 \end{bmatrix}_m - \begin{bmatrix} d_{12} & d_{13} \\ d_{22} & d_{23} \\ d_{32} & d_{33} \end{bmatrix}_m \begin{bmatrix} r'_2 \\ r'_3 \end{bmatrix}_m$$

Substituting equation (2.1.6) above

$$\underset{\sim}{u}'_{\pi_m} = \begin{bmatrix} -\frac{r'_2}{L} - ({u'_1}^1 + r'_1) \\ \frac{r'_2}{L} - ({u'_1}^2 - r'_4) \\ r'_3 - u'_2 \end{bmatrix}_m$$

and using equations (2.1.5), (2.1.4) and (2.1.3), the definition of the additional deformations simplifies to

ADDITIONAL DEFORMATIONS

$$\begin{bmatrix} u'^{\,1}_{1\pi} \\ u'^{\,2}_{1\pi} \\ u'_{2\pi} \end{bmatrix}_m = \begin{bmatrix} -\dfrac{\delta_t}{L} + \rho \\ \dfrac{\delta_t}{L} - \rho \\ \delta_n - u'_2 \end{bmatrix}_m \qquad \begin{matrix} (2.1.23a) \\ (2.1.23b) \\ (2.1.23c) \end{matrix}$$

Substituting equation (2.1.1) into equation (2.1.21b) and performing the multiplication, the last set of kinematic relations yields

$$\underset{\sim}{k}^2_m = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & -1 & \cdot \end{bmatrix} \underset{\sim}{r}_m$$

or, from equations (2.1.2) and (2.1.3)

$$\underset{\sim}{k}^2_m = \underset{\sim}{\delta}_{\pi_m}$$

Statics and Kinematics may now be summarized in the following explicitly linear form:

STATICS	KINEMATICS
(2.1.24) $\underset{\sim}{R}^*_m = \begin{bmatrix} \underset{\sim}{A}^T & \underset{\sim}{A}_\pi^T \end{bmatrix}_m \begin{bmatrix} \underset{\sim}{X}' \\ -\underset{\sim}{\pi} \end{bmatrix}_m$	$\begin{bmatrix} \underset{\sim}{u}' + \underset{\sim}{u}'_\pi \\ \underset{\sim}{\delta}_\pi \end{bmatrix}_m = \begin{bmatrix} \underset{\sim}{A} \\ \underset{\sim}{A}_\pi \end{bmatrix}_m \underset{\sim}{r}^*_m$ (2.1.25a) (2.1.25b)
NODAL DESCRIPTION	

The above equations, besides being explicitly linear, exhibit a contragredient transformation or in an alternative terminology, they represent a dual transformation. As the variables are static and kinematic, the above duality is termed STATIC-KINEMATIC DUALITY, following Munro (1974). The dual correspondence between static and kinematic variables is summarized below:

DUAL CORRESPONDENCE		
STATIC VARIABLE	KINEMATIC VARIABLE	
$\underset{\sim}{X}'_m$	$\underset{\sim}{u}'_m + \underset{\sim}{u}'_{\pi_m}$	(2.1.26a)
$\underset{\sim}{R}^*_m$	$\underset{\sim}{r}^*_m$	(2.1.26b)
$\underset{\sim}{\pi}_m$	$\underset{\sim}{\delta}_{\pi_m}$	(2.1.26c)

2.1.3 Incremental Analysis

The treatment to which Statics and Kinematics were subject in the above was aimed at "clearing" the transformations of the non-linear terms by concentrating them in new, artificial variables defined in such a way that the equivalent linear and dual systems remained exact.

The genuine improvement was in fact the possibility of enforcing Static-Kinematic Duality since the system remains implicitly non-linear and had to be enlarged to accomodate unwelcome extra variables.

These deficiencies that the formulation still endures can be overcome by subjecting the finite description to the treatment of the usual techniques in perturbation analysis. To do so and simultaneously preserve a high accuracy it is necessary to replace the finite description of Statics and Kinematics by an incremental one. The incremental formulation will provide the natural transition from the finite to the perturbed descriptions of Statics and Kinematics and it will be used later as a basis when comparing the present formulation to similar ones presented in the literature.

Assume then, that the value of the static and kinematic variables (say variable y_m^k) describing the k-th state of stress and strain of member m are known and that a finite incremental procedure is to be relied on to find the deviations in the stress and strain fields (Δy_m^k) caused by varying a control parameter, allowing for the characterization of state k+1 by superposition of the incremental fields to those of the initial state k, i.e. ($y_m^{k+1} = y_m^k + \Delta y_m^k$).

As the finite descriptions of Statics and Kinematics, (2.1.24) and (2.1.25) respectively, are explicitly linear, their incremental versions can immediately be obtained just by replacing the variables by their increments, yielding

$$\Delta \underset{\sim}{R}_m^* = \left[\underline{A}^T \;\vdots\; \underline{A}^T \right]_m \begin{bmatrix} \Delta \underset{\sim}{X}' \\ --- \\ -\Delta \underset{\sim}{\pi} \end{bmatrix}_m \qquad (2.1.27a)$$

for Statics, and

$$\begin{bmatrix} \Delta \underset{\sim}{u}' + \Delta \underset{\sim}{u}'_{\pi} \\ -------- \\ \Delta \underset{\sim}{\delta}_{\pi} \end{bmatrix}_m = \begin{bmatrix} \underline{A} \\ ---- \\ \underline{A}_{\pi} \end{bmatrix}_m \Delta \underset{\sim}{r}_m^* \qquad \begin{matrix} (2.1.28a) \\ (2.1.28b) \end{matrix}$$

for Kinematics.

For simplicity of the presentation we will from now on relieve the variables involved in the derivation of some indices. For instance, we will write ρ instead of ρ_m^k to represent the chord rotations of member m at the k-th state; $\Delta\rho$ will represent the ensuing increment of the rotation.

Let h be the ratio between the chord length at stages k+1 and k

$$h = 1 - \frac{\Delta u_2'}{L_c} \qquad (2.1.29)$$

where $\Delta u_2'$ is the variation of the member chord shortening, and let us introduce another auxiliary variable z defined by

$$z = h^{-1} - 1 \qquad (2.1.30)$$

Assuming that

$$\left|\frac{\Delta u_2'}{L_c}\right| < 1$$

we may write

$$h^{-1} = \sum_{j=1}^{\infty} \left[\frac{\Delta u_2'}{L_c}\right]^j + 1 \qquad (2.1.31)$$

and therefore

$$z = \sum_{j=1}^{\infty} \left[\frac{\Delta u_2'}{L_c}\right]^j \qquad (2.1.32)$$

By definition

$$s = \sin\rho = \frac{\delta_t}{L_c} \qquad (2.1.33a)$$

and

$$c = \cos\rho = \frac{L-\delta_n}{L_c} \qquad (2.1.33b)$$

Hence

$$\sin(\rho+\Delta\rho) = s\cdot\cos\Delta\rho + c\cdot\sin\Delta\rho = \frac{\delta_t+\Delta\delta_t}{L_c-\Delta u_2'} = h^{-1}(s+\frac{\Delta\delta_t}{L_c})$$

$$\cos(\rho+\Delta\rho) = c\cdot\cos\Delta\rho - s\cdot\sin\Delta\rho = \frac{L-\delta_n-\Delta\delta_n}{L_c-\Delta u_2'} = h^{-1}(c-\frac{\Delta\delta_n}{L_c})$$

The identities above give

$$\Delta\rho = \arcsin\left[h^{-1}(\frac{c}{L_c}\Delta\delta_t + \frac{s}{L_c}\Delta\delta_n)\right]$$

which can be approximated to

$$\Delta\rho = - h^{-1}(\frac{c}{L_c}\Delta\delta_t + \frac{s}{L_c}\Delta\delta_n) + \frac{1}{6}h^{-3}(\frac{c}{L_c}\Delta\delta_t + \frac{s}{L_c}\Delta\delta_n)^3 + \cdots$$

if

$$\left|h^{-1}(\frac{c}{L_c}\Delta\delta_t + \frac{s}{L_c}\Delta\delta_n)\right| < 1$$

or, from definition (2.1.31) and segregating the non-linear terms from the linear ones

$$\Delta\rho = \frac{c}{L_c}\,\Delta\delta_t + \frac{s}{L_c}\,\Delta\delta_n + \Delta R_\rho \tag{2.1.34a}$$

$$\Delta R_\rho = \left(\frac{c}{L_c}\,\Delta\delta_t + \frac{s}{L_c}\,\Delta\delta_n\right)\left[\frac{1}{6}\left(\frac{s}{L_c}\,\Delta\delta_t + \frac{s}{L_c}\,\Delta\delta_n\right)^2 + \frac{\Delta u_2'}{L_c}\left(1+\frac{\Delta u_2'}{L_c}\right)\right] + O_4 \tag{2.1.34b}$$

where O_4 designates terms of order four and higher.

The member chord shortening and the additional force displacements are not independent as shown by equations (2.1.33), which give

$$L_c^2 = (L-\delta_n)^2 + \delta_t^2 \tag{2.1.35}$$

The corresponding incremental relationship

$$\Delta u_2' = L_c\left\{1-\left[1+\frac{2}{L_c}\,(s\,\Delta\delta_t - c\,\Delta\delta_n)+\frac{1}{L_c^2}\,(\Delta\delta_t^2+\Delta\delta_n^2)\right]^{\frac{1}{2}}\right\} \tag{2.1.36}$$

can be expressed, after a series expansion, as

$$\Delta u_2' = -s\,\Delta\delta_t + c\,\Delta\delta_n - \Delta R_{u_2} \tag{2.1.37a}$$

$$\Delta R_{u_2} = \frac{1}{2}\,(L_c - s\,\Delta\delta_t + c\,\Delta\delta_n)\left(\frac{c}{L_c}\,\Delta\delta_t + \frac{s}{L_c}\,\Delta\delta_n\right)^2 + O_4 \tag{2.1.37b}$$

From equations (2.1.23a, b) we find the increments on the additional rotations to be

$$\Delta u_{1\pi}'^{1} = -\,\Delta u_{1\pi}'^{2} = -\,\frac{\Delta\delta_t}{L} + \Delta\rho$$

or from (2.1.34)

$$\Delta u_{1\pi}'^{1} = -\,\Delta u_{1\pi}'^{2} = \left(-\frac{1}{L}+\frac{c}{L_c}\right)\Delta\delta_t + \frac{s}{L_c}\,\Delta\delta_n + \Delta R_\rho \tag{2.1.38a,b}$$

Similarly, starting now with equation (2.1.23c)

$$\Delta u'_{2\pi} = \Delta\delta_n - \Delta u'_2$$

or, from (2.1.37)

$$\Delta u'_{2\pi} = s\,\Delta\delta_t + (1-c)\,\Delta\delta_n + \Delta R_{u_2} \qquad (2.1.38c)$$

Equations (2.1.38) can be cast in a matrix form as

$$\boxed{\Delta \underset{\sim}{u}'_{\pi} = \underline{Q}\,\Delta \underset{\sim}{\delta}_{\pi} + \Delta \underset{\sim}{R}_{u_\pi}} \qquad (2.1.39a)$$

where

$$\underline{Q} = \begin{bmatrix} \frac{s}{L_c} & -\frac{1}{L}+\frac{c}{L_c} \\ -\frac{s}{L_c} & \frac{1}{L}-\frac{c}{L_c} \\ 1-c & s \end{bmatrix}, \quad \Delta \underset{\sim}{R}_{u_\pi} = \begin{bmatrix} \Delta R_\rho \\ -\Delta R_\rho \\ \Delta R_{u_2} \end{bmatrix} \qquad (2.1.39b,c)$$

The shear force at member m is defined by

$$X_3 = -\frac{1}{L_c}\,(X_1^1 - X_1^2) \qquad (2.1.40)$$

and using equations (2.1.29) and (2.1.30), its increment by

$$\Delta X_3 = X_3 \cdot z - \frac{1+z}{L_c}\,(\Delta X_1^1 - \Delta X_1^2)$$

which can be re-written as

$$\Delta X_3 = \frac{X_3}{L_c}\,\Delta u'_2 - \frac{1}{L_c}\,(\Delta X_1^1 - \Delta X_1^2) + \Delta R_3 \qquad (2.1.41a)$$

$$\Delta R_3 = \frac{u'_2}{L_c^2}\left(1+\frac{\Delta u'_2}{L_c}\right)(X_3\,\Delta u'_2 - \Delta X_1^1 + \Delta X_1^2) + 0_4 \qquad (2.1.41b)$$

The additional forces defined in (2.1.20) can be expressed as

$$\pi_n = -sX_3 + (1-c)\,X_2 \qquad (2.1.42a)$$

$$\pi_t = (\frac{L_c}{L} - c)\ X_3 + s\ X_2 \tag{2.1.42b}$$

and their increments as

$$\Delta\pi_n = -(sX_3+cX_2)z - \frac{X_3}{L_c}\ \Delta\delta_t + \frac{X_2}{L_c}\Delta\delta_n - s\Delta X_3 + (1-c)\ \Delta X_2$$

$$+(-\frac{X_3}{L_c}\ \Delta\delta_t + \frac{X_2}{L_c}\ \Delta\delta_n - s\Delta X_3 - c\Delta X_2)z - \frac{1}{L_c}(\Delta X_3\ \Delta\delta_t - \Delta X_2\ \Delta\delta_n)h^{-1}$$

$$\Delta\pi_t = -(cX_3-sX_2)z - \frac{X_3}{L}\ \Delta u_2' + \frac{X_3}{L_c}\ \Delta\delta_n + \frac{X_2}{L_c}\ \Delta\delta_t + s\Delta X_2 + (\frac{L_c}{L}-c)\Delta X_3$$

$$+(\frac{X_3}{L_c}\ \Delta\delta_n + \frac{X_2}{L_c}\ \Delta\delta_t + s\Delta X_2 - c\Delta X_3)z + \frac{1}{L_c}\ (\Delta X_3\Delta\delta_n + \Delta X_2\Delta\delta_t)h^{-1} - \frac{1}{L}\Delta u_2'\ \Delta X_3$$

Replacing above all the incremental variables by their series expansion approximation given in (2.1.31), (2.1.32), (2.1.37) and (2.1.41), we find

$$\boxed{\Delta\underline{\pi} = \underline{Q}^T\ \Delta\underline{X}' + \underline{P}\ \Delta\underline{\delta}_\pi + \Delta\underline{R}_\pi} \tag{2.1.43a}$$

where

$$\underline{P} = \left[\begin{array}{c|c} s^2\ \frac{X_2}{L_c} - 2sc\frac{X_3}{L_c} & sc\frac{X_2}{L_c} + (s^2-c^2)\frac{X_3}{L_c} \\ \hline sc\ \frac{X_2}{L_c} + (s^2-c^2)\frac{X_3}{L_c} & c^2\frac{X_2}{L_c} + 2sc\frac{X_3}{L_c} \end{array}\right]\ ,\quad \Delta\underline{R}_\pi = \left[\begin{array}{c} \Delta R_n \\ \hline \Delta R_t \end{array}\right]$$

(2.1.43b,c)

and

$$\Delta R_n = (1+\frac{\Delta u_2'}{L_c})\{-\frac{\Delta X_3}{L_c}\ \Delta\delta_t + \frac{\Delta X_2}{L_c}\ \Delta\delta_n + \frac{\Delta u_2'}{L_c}\ [\ (-\frac{X_3}{L_c}\ \Delta\delta_t + \frac{X_2}{L_c}\ \Delta\delta_n - s\Delta X_3 - c\Delta X_2)$$

$$-\frac{\Delta u_2'}{L_c}\ (sX_3+cX_2)]\ \} - sR_3 + (2s\frac{X_3}{L_c} + c\frac{X_2}{L_c})\ R_{u_2} + 0_4 \tag{2.1.44a}$$

$$\Delta R_t = (1+\frac{\Delta u_2'}{L_c})\{\frac{\Delta X_2}{L_c}\ \Delta\delta_t + \frac{\Delta X_3}{L_c}\ \Delta\delta_n + \frac{\Delta u_2'}{L_c}\ [\ (\frac{X_2}{L_c}\ \Delta\delta_t + \frac{X_3}{L_c}\ \Delta\delta_n - c\Delta X_3 + s\Delta X_2)$$

$$-\ \frac{\Delta u_2'}{L_c}\ (cX_3-sX_2)]\ \}\ + (\frac{L_c}{L}-c)R_3 + (2c\frac{X_3}{L_c} - s\frac{X_2}{L_c})R_{u_2} - \frac{1}{L}\ \Delta u_2'\ \Delta X_3 + 0_4 \tag{2.1.44b}$$

The incremental additional forces may now be eliminated from the nodal equilibrium equation (2.1.27), yielding

$$\underset{\sim}{0} = \underline{A}_m^T \Delta \underset{\sim}{X}'_m - \Delta \underset{\sim}{R}^*_m - \underline{A}^T_{\pi_m} (\underline{Q}^T_m \Delta \underset{\sim}{X}'_m + \underline{P}_m \Delta \underset{\sim}{\delta}_{\pi_m} + \Delta \underset{\sim}{R}_{\pi_m})$$

or from equation (2.1.28c)

$$\underset{\sim}{0} = (\underline{A}_m^T - \underline{A}^T_{\pi_m} \underline{Q}^T_m) \Delta \underset{\sim}{X}'_m - \Delta \underset{\sim}{R}^*_m - \underline{A}^T_{\pi_m} \underline{P}_m \underline{A}_{\pi_m} \Delta \underset{\sim}{r}^*_m - \underline{A}^T_{\pi_m} \Delta \underset{\sim}{R}_{\pi_m} \quad (2.1.45)$$

Let
$$\Delta \underset{\sim}{R}^*_{\pi_m} = \underline{A}^T_{\pi_m} \Delta \underset{\sim}{R}_{\pi_m} \quad (2.1.46)$$

and
$$\underline{\mathbb{K}}_{\pi_m} = \underline{A}^T_{\pi_m} \underline{P}_m \underline{A}_{\pi_m} \quad (2.1.47)$$

Carrying out the triple product in (2.1.47) we find that

$$\underline{\mathbb{K}}_{\pi_m} = \left[\begin{array}{c|c} \underline{K}'_\pi & -\underline{K}'_\pi \\ \hline -\underline{K}'_\pi & \underline{K}'_\pi \end{array}\right]_m \quad (2.1.48a)$$

where $\underline{K}'_{\pi_m}$ is the symmetric matrix

$$\underline{K}'_{\pi_m} = \left[\begin{array}{c|cc|cc} \cdot & & \cdot & & \cdot \\ \hline \cdot & \frac{X_2}{L_c}\cos^2(\alpha-\rho) & -\frac{X_3}{L_c}\sin(2\alpha-2\rho) & -\frac{1}{2}\frac{X_2}{L_c}\sin(2\alpha-2\rho) & -\frac{X_3}{L_c}\cos(2\alpha-2\rho) \\ \hline \cdot & -\frac{1}{2}\cdot\frac{X_2}{L_c}\sin(2\alpha-2\rho) & -\frac{X_3}{L_c}\cos(2\alpha-2\rho) & \frac{X_2}{L_c}\sin^2(\alpha-\rho) & +\frac{X_3}{L_c}\sin(2\alpha-2\rho) \end{array}\right]_m$$

(2.1.48b)

Furthermore, we also find that

$$\underline{\mathbb{A}}_m^T = \underline{A}_m^T - \underline{A}^T_{\pi_m} \underline{Q}^T_m = \underline{S}_m \quad (2.1.49)$$

The matrix $\underline{S}_m$ is defined in (2.1.14) where now ρ_m and L_{c_m} are the member chord rotation and length immediately before the incremental action takes place.

Eliminating the incremental additional deformations through equation (2.1.39), and making use of (2.1.49), the compatibility equation (2.1.28a) reduces to

$$\Delta \underset{\sim}{u}'_m + \Delta \underset{\sim}{R}_{u\pi_m} = \underline{A}_m \ \Delta \underset{\sim}{r}^*_m$$

The above equation and equation (2.1.45) together with equations (2.1.46) to (2.1.49), define the incremental description of Kinematics and Statics, respectively:

$$\text{statics:} \quad \begin{bmatrix} -\underline{\mathbb{K}}_{\pi} & \underline{A}^T \\ \underline{A} & \cdot \end{bmatrix}_m \cdot \begin{bmatrix} \Delta \underset{\sim}{r}^* \\ \Delta \underset{\sim}{X}' \end{bmatrix}_m = \begin{bmatrix} \Delta \underset{\sim}{R}^* \\ \Delta \underset{\sim}{u}' \end{bmatrix}_m + \begin{bmatrix} \Delta \underset{\sim}{R}^*_{\pi} \\ \Delta \underset{\sim}{R}_{u\pi} \end{bmatrix}_m \quad \text{:kinematics} \qquad \begin{matrix} (2.1.50) \\ (2.1.51) \end{matrix}$$

INCREMENTAL NODAL DESCRIPTION

2.1.4 Perturbation Analysis

The systems of equations (2.1.50) and (2.1.51) do not have a known closed form solution, in the sense that it is not possible to evaluate directly the incremental member deformations compatible with a given variation of the nodal displacements, which together with a given variation of the stress-resultants are not sufficient to directly evaluate the corresponding variation of the member nodal forces.

If an iterative numerical procedure is to be avoided, the original implicitly non-linear systems have to be replaced by (a set of) systems of known solution.

This can be achieved by expressing every incremental variable in the system as a power series of an arbitrary parameter ε

$$\Delta y = \sum_{i=1}^{\infty} y_i \frac{\varepsilon^i}{i!} \qquad (2.1.52)$$

and equating the terms of the same power of ε ; the original system is replaced by an infinite ($i=1,2...,\infty$) set of recursive systems (variables of order higher than the i-th are not involved in the i-th system) of solvable (for instance linear) equations.

If only $n<\infty$ systems are solved and the increments are evaluated by

$$\Delta y = \sum_{i=1}^{n} y_i \frac{\varepsilon^i}{i!}$$

the solution of the system is said to be of order n.

In general the convergence is fast and, depending on the amplitude of the control parameter ε , only a few terms of the series are required to satisfy the stipulated degree of accuracy.

Let us then expand the incremental variables present in equations (2.1.49) and (2.1.50) in power series of the form (2.1.52):

$$\Delta \underset{\sim}{X}'_m = \sum_{i=1}^{\infty} \underset{\sim}{X}'_{i_m} \frac{\varepsilon^i}{i!} \qquad (2.1.53a)$$

$$\Delta \underset{\sim}{R}^*_m = \sum_{i=1}^{\infty} \underset{\sim}{R}^*_{i_m} \frac{\varepsilon^i}{i!} \qquad (2.1.53b)$$

$$\Delta \underset{\sim}{u}'_m = \sum_{i=1}^{\infty} \underset{\sim}{u}'_{i_m} \frac{\varepsilon^i}{i!} \qquad (2.1.53c)$$

$$\Delta \underset{\sim}{r}^*_m = \sum_{i=1}^{\infty} \underset{\sim}{r}^*_{i_m} \frac{\varepsilon^i}{i!} \qquad (2.1.53d)$$

$$\Delta \underset{\sim}{R}^*_{\pi_m} = \sum_{i=1}^{\infty} \underset{\sim}{R}^*_{\pi_{i_m}} \frac{\varepsilon^i}{i!} \qquad (2.1.53e)$$

$$\Delta \underset{\sim}{R}_{u\pi_m} = \sum_{i=1}^{\infty} \underset{\sim}{R}_{u\pi_{i_m}} \frac{\varepsilon^i}{i!} \qquad (2.1.53f)$$

Substituting into the incremental description of Statics and Kinematics and collecting the same order terms, the incremental descriptions of Statics and Kinematics are replaced by the following equivalent infinite system of equations:

$$
\begin{array}{l}
\text{statics} \\
\text{kinematics}
\end{array}
\quad
\begin{bmatrix} -\mathbb{K}_{\pi} & \Lambda^{T} \\ \Lambda & \cdot \end{bmatrix}_m
\begin{bmatrix} r^*_i \\ X'_i \end{bmatrix}_m
=
\begin{bmatrix} R^*_i \\ u'_i \end{bmatrix}_m
+
\begin{bmatrix} R^*_{\pi_i} \\ R_{u\pi_i} \end{bmatrix}_m
\qquad
\begin{array}{r}
(2.1.54) \\
(2.1.55)
\end{array}
$$

PERTURBED NODAL DESCRIPTION

We will prove next that the above equations are recursive by demonstrating that the i-th order residue $R^*_{\pi_{i_m}}$ and $R_{u\pi_{i_m}}$ depend on coefficients of order lower than the i-th. As a consequence the first-order residue of those and similarly perturbed forms will always be zero.

Again, for simplicity of the presentation, we will be dropping from now onwards the member subscript m.

Replacing in (2.1.32) z and u'_2 by their power series approximations

$$z = \sum_{i=1}^{\infty} z_i \frac{\varepsilon^i}{i!} \qquad (2.1.53g)$$

$$\Delta u'_2 = \sum_{i=1}^{\infty} u'_{2_i} \frac{\varepsilon^i}{i!} \qquad (2.1.53h)$$

and after some simple operations on the series, we find the following relationship between the i-th order coefficients

$$z_i = \frac{u'_{2_i}}{L_c} + R_{z_i} \qquad (2.1.56a)$$

$$R_{z_2} = 2z_1^2 \qquad (2.1.56b)$$

$$R_{z_3} = 6z_1\,(z_2 - z_1^2) \qquad (2.1.56c)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Letting

$$\Delta\rho = \sum_{i=1}^{\infty} \rho_i \frac{\varepsilon^i}{i!} \qquad (2.1.53i)$$

$$\Delta\delta_n = \sum_{i=1}^{\infty} \delta_{n_i} \frac{\varepsilon^i}{i!} \qquad (2.1.53j)$$

and

$$\Delta\delta_t = \sum_{i=1}^{\infty} \delta_{t_i} \frac{\varepsilon^i}{i!} \qquad (2.1.53l)$$

in equation (2.1.34), solving the series and equating the same order coefficients we find

$$\rho_i = \frac{c}{L_c} \delta_{t_i} + \frac{s}{L_c} \delta_{n_i} + R_{\rho_i} \qquad (2.1.57a)$$

$$R_{\rho_2} = 2z_1 \rho_1 \qquad (2.1.57b)$$

$$R_{\rho_3} = 3z_1 \rho_2 + 3z_2 \rho_1 - 6z_1^2 \rho_1 + \rho_1^3 \qquad (2.1.57c)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Similarly, using now (2.1.53h,j,l), equation (2.1.37) can be replaced by the infinite system

$$u'_{2_i} = - s \delta_{t_i} + c \delta_{n_i} - R_{u_{2_i}} \qquad (2.1.58a)$$

$$R_{u2_2} = L_c \rho_1^2 \qquad (2.1.58b)$$

$$R_{u2_3} = 3 \rho_1^2 u'_{2_1} + 3L_c \rho_1 (\rho_2 - R_{\rho_2}) \qquad (2.1.58c)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Letting in the linear equation (2.1.38a)

$$\Delta \underset{\sim}{u}'_{\pi} = \sum_{i=1}^{\infty} \underset{\sim}{u}'_{\pi_i} \frac{\varepsilon^i}{i!} \qquad (2.1.53m)$$

together with (2.1.53f) and (2.1.53j,l), we find for the perturbed version of the incremental additional deformations

$$\boxed{\underset{\sim}{u}'_{\pi_i} = \underset{\sim}{Q}\, \underset{\sim}{\delta}_{\pi_i} + \underset{\sim}{R}_{u\pi_i}} \tag{2.1.59a}$$

where, from (2.1.38c)

$$R_{u\pi_i} = \begin{bmatrix} R_\rho \\ -R_\rho \\ R_{u2} \end{bmatrix}_i \tag{2.1.59b}$$

As R_{ρ_i} and R_{u2_i}, defined in (2.1.57) and (2.1.58), respectively, are recursive, $\underset{\sim}{R}_{u\pi_i}$ is also recursive.

Using the above results together with the incremental shear force in the form

$$\Delta X_3 = \sum_{i=1}^{\infty} X_{3_i} \frac{\varepsilon^i}{i!} \tag{2.1.53n}$$

in equation (2.1.41) and the results so obtained in equation (2.1.43) after replacing the incremental additional forces by

$$\Delta \underset{\sim}{\pi} = \sum_{i=1}^{\infty} \underset{\sim}{\pi}_i \frac{\varepsilon^i}{i!} \tag{2.1.53o}$$

equations (2.1.41) and (2.1.43) become, respectively

$$X_{3_i} = \frac{X_3}{L_c} u'_{2_i} - \frac{1}{L_c} (X^1_{1_i} - X^2_{1_i}) + R_{3_i} \tag{2.1.60a}$$

$$R_{3_2} = X_3 R_{z_2} - \frac{2}{L_c} z_1 (X^1_{1_1} - X^2_{1_1}) \tag{2.1.60b}$$

$$R_{3_3} = X_3 R_{z_3} - \frac{3}{L_c} [z_1(X^1_{1_2} - X^2_{1_2}) + z_2 (X^1_{1_1} - X^2_{1_1})] \tag{2.1.60c}$$

$$\vdots \qquad\qquad \vdots$$

and

$$\boxed{\underline{\pi}_i = \underline{Q}^T \underline{X}_i^! + \underline{P}\,\underline{\delta}_{\pi i} + \underline{R}_{\pi i}} \tag{2.1.61a}$$

where

$$\underline{R}_{\pi i} = \left[\begin{array}{c} R_n - sR_3 + (cX_2 + 2sX_3)\dfrac{R_{u_2}}{L_c} \\ \hline R_t + (\dfrac{L_c}{L} - c)R_3 - (sX_2 - 2cX_3)\dfrac{R_{u_2}}{L_c} \end{array}\right]_i \tag{2.1.61b}$$

$$R_{n_2} = 2(\pi_{n_1} - X_{2_1})\, z_1 + \frac{2}{L_c}(X_{2_1}\delta_{n_1} - X_{3_1}\delta_{t_1}) \tag{2.1.61c}$$

$$R_{t_2} = 2(\pi_{t_1} - 2\frac{L_c}{L}X_{3_1} + \frac{L_c}{L}X_3 z_1)z_1 + \frac{2}{L_c}(X_{2_1}\delta_{t_1} + X_{3_1}\delta_{n_1}) \tag{2.1.61d}$$

$$R_{n_3} = 3(\pi_{n_2} - X_{2_2})\, z_1 + 3(\pi_{n_1} - X_{2_1})\,(z_2 - 2z_1^2) + \frac{3}{L_c}(X_{2_1}\delta_{n_2} + X_{2_2}\delta_{n_1} - X_{3_1}\delta_{t_2} - X_{3_2}\delta_{t_1}) \tag{2.1.61e}$$

$$R_{t_3} = 3(\pi_{t_2} - 2\frac{L_c}{L}X_{3_2} + \frac{L_c}{L}X_3 z_2)z_1 + 3(\pi_{t_1} - 2\frac{L_c}{L}X_{3_1} + \frac{L_c}{L}X_3 z_1)(z_2 - 2z_1^2) + \frac{3}{L_c}(X_{3_1} - 2z_1 X_3)z_1^2 + \frac{3}{L_c}(X_{2_1}\delta_{t_2} + X_{2_2}\delta_{t_1} + X_{3_2}\delta_{n_1} + X_{3_1}\delta_{n_2}) \tag{2.1.61f}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

From (2.1.46)

$$\underline{R}^*_{\pi_i} = \underline{\Lambda}_\pi^T\, \underline{R}_{\pi_i} \tag{2.1.62}$$

and as $\underset{\sim}{R}_{\pi i}$ is recursive, as shown by (2.1.61b to f) and (2.1.58) and (2.1.60), $\underset{\sim}{R}^*_{\pi i}$ is also recursive since the above transformation is linear.

2.1.5 Asymptotic Analysis

The finite description of Statics and Kinematics will now be specialized for the analysis of the particular class of systems whose equilibrium paths branch from the original kinematically trivial path, as happens for rigid-plastic structures and axially undeformable elastoplastic structures, the latter under specific loading conditions.

The formulation will be presented in a form suitable for the application of the static perturbation method developed independently by Sewell (1965) and Thompson (1965).

Sewell (1965) considered only discrete systems and outlined the specific application of the perturbation method to buckling problems.

Thompson and Hunt (1973) published a detailed exposition of the results Thompson obtained by extensive application of this technique.

As we are specifically interested in deriving a formulation suitable for a fast estimate of the buckling load and of the initial tangent to the branching path, we will limit the presentation to a second-order formulation.

Within this limited scope, we may expand the generic variable y, and not its increment, in a power series

$$y = \sum_{i=0}^{\infty} y_i \frac{\varepsilon^i}{i!} \qquad (2.1.63)$$

As it is assumed that the initial path is kinematically trivial, then

$$y_0 = 0$$

whenever y represents a kinematic variable.

Expanding (2.1.35) in a power series

$$u'_2 = \delta_n - \frac{1}{2L}\,\delta_t^2 + \cdots$$

and letting above

$$u'_2 = \sum_{i=0}^{\infty} u'_{2_i}\,\frac{\varepsilon^i}{i!} \qquad (2.1.64a)$$

and

$$\underset{\sim}{\delta}_\pi = \sum_{i=0}^{\infty} \underset{\sim}{\delta}_{\pi_i}\,\frac{\varepsilon^i}{i!} \qquad (2.1.64b)$$

solving and equating the same order terms, we find

$$u'_{2_i} = \delta_{n_i} - R_{u2_i} \qquad (2.1.65a)$$

$$R_{u2_1} = 0 \qquad (2.1.65b)$$

$$R_{u2_2} = \frac{1}{L}\,\delta_{t_1}^2 \qquad (2.1.65c)$$

$$\vdots \qquad \vdots \qquad \vdots$$

From equations (2.1.33)

$$\rho = \arctan\left(\frac{\delta_t}{L-\delta_n}\right)$$

and expanding

$$\rho = \frac{\delta_t}{L}\left(1-\frac{\delta_n}{L}\right)^{-1} - \cdots$$

Letting above

$$\rho = \sum_{i=0}^{\infty} \rho_i\,\frac{\varepsilon^i}{i!} \qquad (2.1.64c)$$

together with (2.1.64b) and solving as previously, we find for the member chord rotation

$$\rho_i = \frac{\delta_{t_i}}{L} + R_{\rho i} \qquad (2.1.66a)$$

$$R_{\rho 1} = 0 \qquad (2.1.66b)$$

$$R_{\rho 2} = \frac{2}{L^2}\,\delta_{n_1}\,\delta_{t_1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Substituting (2.1.65) and (2.1.66) in (2.1.23) where now

$$\underset{\sim}{u}'_{\pi} = \sum_{i=0}^{\infty} \underset{\sim}{u}'_{\pi_i} \frac{\varepsilon^i}{i!} \qquad (2.1.64d)$$

the perturbed form of the additional deformations is found to be

$$\boxed{\underset{\sim}{u}'_{\pi_i} = \underset{\sim}{R}_{u\pi_i}} \qquad (2.1.67a)$$

where

$$\underset{\sim}{R}_{u\pi_i} = \begin{bmatrix} R_\rho \\ -R_\rho \\ R_{u2} \end{bmatrix}_i \qquad (2.1.67b)$$

The member shear force, as defined in (2.1.40), can be approximated to

$$X_3 = -\frac{1}{L}(X_1^1 - X_1^2)\left[1 + \frac{u_2'}{L} + \left(\frac{u_2'}{L}\right)^2 + \cdots\right]$$

Letting above

$$X_1^j = \sum_{i=0}^{\infty} X_{1_i}^j \frac{\varepsilon^i}{i!} \qquad j=1,2 \qquad (2.1.64e)$$

and

$$X_3 = \sum_{i=0}^{\infty} X_{3_i} \frac{\varepsilon^i}{i!} \qquad (2.1.64f)$$

together with the perturbed form (2.1.64) for the member chord shortening, the perturbed form of the member shear force turns out to be

$$X_{3_i} = -\frac{1}{L}(X_{1_i}^1 - X_{1_i}^2) - R_{3_i} \qquad (2.1.68a)$$

$$R_{3_1} = -\frac{1}{L^2}(X_{1_0}^1 - X_{1_0}^2)\, u_{2_1} \qquad (2.1.68b)$$

$$R_{3_2} = -\frac{1}{L^2}\left[(X_{1_0}^1 - X_{1_0}^2)(u_{2_2} - \frac{u_{2_1}^2}{L}) + 2(X_{1_1}^1 - X_{1_1}^2)\, u_{2_1}\right] \qquad (2.1.68c)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Expanding the additional forces definition (2.1.41) in a power series

$$\pi_n = -\rho X_3 + \tfrac{1}{2}\rho^2 X_2 + \cdots$$

$$\pi_t = \rho X_2 + \tfrac{1}{2}\rho^2 X_3 - \frac{u_2'}{L} X_3 + \cdots$$

and letting above

$$\underset{\sim}{\pi} = \sum_{i=0}^{\infty} \underset{\sim}{\pi}_i \frac{\varepsilon^i}{i!} \qquad (2.1.64g)$$

$$X_2 = \sum_{i=0}^{\infty} X_{2_i} \frac{\varepsilon^i}{i!} \qquad (2.1.64h)$$

together with the results previously obtained for the remaining variables the following matrix description emerges

$$\boxed{\underset{\sim}{\pi}_i = \underset{\sim}{P}\, \underset{\sim}{\delta}_{\pi_i} + \underset{\sim}{R}_{\pi_i}} \qquad (2.1.69a)$$

where

$$\underset{\sim}{R}_{\pi_1} = \underset{\sim}{0} \qquad (2.1.69b)$$

$$\underset{\sim}{R}_{\pi 2} = \begin{bmatrix} -X_{3_0} R_{\rho_2} + \rho_1^2 X_{2_0} - 2\rho_1 X_{3_1} \\ \hdashline X_{2_0} R_{\rho 2} + \dfrac{X_{3_0}}{L} R_{u2_2} + 2\rho_1 X_{2_1} - \dfrac{2}{L} u_{2_1} X_{3_1} \end{bmatrix} \qquad (2.1.69c)$$

Matrix $\underset{-}{P}$ can be found by specializing (2.1.43b) into the case of the initial kinematically trivial path

$$L_c = L \qquad (2.1.70a)$$

$$s = 0 \qquad (2.1.70b)$$

$$c = 1 \qquad (2.1.70c)$$

$$X_2 = X_{2_0} \qquad (2.1.70d)$$

$$X_3 = X_{3_0} \qquad (2.1.70e)$$

yielding

$$\underline{P} = \left[\begin{array}{c|c} \cdot & -\frac{X_{3_0}}{L} \\ \hline -\frac{X_{3_0}}{L} & \frac{X_{2_0}}{L} \end{array}\right] \qquad (2.1.69d)$$

Letting in equation (2.1.24)

$$\underset{\sim}{R}^* = \sum_{i=0}^{\infty} \underset{\sim}{R}_i^* \frac{\varepsilon^i}{i!} \qquad (2.1.64i)$$

together with (2.1.64e), (2.1.64g) and (2.1.64h), and equating the same order terms, the nodal equilibrium equation gives rise to the infinite system

$$\underset{\sim}{0} = \underline{A}^T \underset{\sim}{X}_i' - \underset{\sim}{R}_i^* - \underline{A}_\pi^T \underset{\sim}{\pi}_i$$

or

$$\underset{\sim}{0} = \underline{A}^T \underset{\sim}{X}_i' - \underline{R}_i^* - \underline{A}_\pi^T \underline{P}\, \underset{\sim}{\delta}_i - \underline{A}_\pi^T \underline{R}_{\pi_i} \qquad (2.1.71)$$

after the elimination of the additional forces through (2.1.69a).

Similarly, equation (2.1.25c) can be replaced by

$$\underset{\sim}{\delta}_{\pi_i} = \underline{A}_\pi \underset{\sim}{r}_i^* \qquad (2.1.72)$$

using (2.1.64b) and letting

$$\underset{\sim}{r}^* = \sum_{i=0}^{\infty} \underset{\sim}{r}_i^* \frac{\varepsilon^i}{i!} \qquad (2.1.64j)$$

Substituting (2.1.72) into (2.1.71) and noting

$$\underset{\sim}{R}_{\pi_i}^* = \underline{A}_\pi^T \underset{\sim}{R}_{\pi_i} \qquad (2.1.73)$$

the nodal equilibrium equation becomes

$$\underset{\sim}{R}_{\pi_i}^* + \underline{\mathbb{K}}_\pi \underset{\sim}{r}_i^* + \underset{\sim}{R}_i^* = \underline{A}^T \underset{\sim}{X}_i' \qquad (2.1.74)$$

Matrix $\underline{\mathbb{K}}_{\pi}$ is still defined by (2.1.48a), and matrix $\underline{K}'_{\pi}$ can be found by imposing (2.1.70) in (2.1.48b), yielding

$$\underline{K}'_{\pi} = \left[\begin{array}{c|c|c} \cdot & \cdot & \cdot \\ \hline \cdot & \frac{X_{2_0}}{L}\cos^2\alpha - \frac{X_{3_0}}{L}\sin 2\alpha & -\frac{1}{2}\,\frac{X_{2_0}}{L}\sin 2\alpha - \frac{X_{3_0}}{L}\cos 2\alpha \\ \hline \cdot & -\frac{1}{2}\,\frac{X_{2_0}}{L}\sin 2\alpha - \frac{X_{3_0}}{L}\cos 2\alpha & \frac{X_{2_0}}{L}\sin^2\alpha + \frac{X_{3_0}}{L}\sin 2\alpha \end{array}\right] \qquad (2.1.75)$$

Letting in equation (2.1.25a)

$$u'^{\,j}_1 = \sum_{i=0}^{\infty} u'^{\,j}_{1_i}\,\frac{\varepsilon^i}{i!} \qquad j=1,2 \qquad (2.1.641)$$

and substituting (2.1.64a), (2.1.67a) and (2.1.64j), the nodal compatibility equation becomes

$$\underset{\sim}{u}'_i + \underset{\sim}{R}_{u\pi_i} = \underline{A}\;\underset{\sim}{r}^{*}_i$$

The above equation together with equation (2.1.74) define the Kinematics and Statics descriptions in the desired format suitable to perform an asymptotic analysis

statics / kinematics

$$\left[\begin{array}{c|c} -\underline{\mathbb{K}} & \underline{\mathbb{A}}^T \\ \hline \underline{\mathbb{A}} & \cdot \end{array}\right]_m \left[\begin{array}{c} \underset{\sim}{r}^{*}_i \\ \hline \underset{\sim}{X}'_i \end{array}\right]_m = \left[\begin{array}{c} \underset{\sim}{R}^{*}_i \\ \hline \underset{\sim}{u}'_i \end{array}\right]_m + \left[\begin{array}{c} \underset{\sim}{R}^{*}_{\pi_i} \\ \hline \underset{\sim}{R}_{u\pi_i} \end{array}\right]_m \qquad \begin{array}{c} (2.1.76) \\ (2.1.77) \end{array}$$

ASYMPTOTIC NODAL DESCRIPTION

where now

$$\underline{\mathbb{A}}_m = \underline{A}_m$$

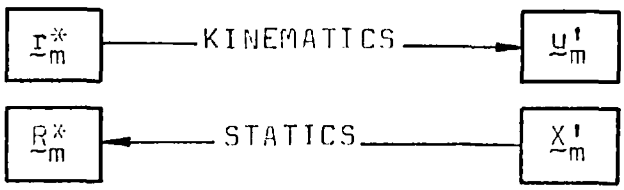
2.2 MESH DESCRIPTION OF STATICS AND KINEMATICS

The reasoning behind the Nodal Description of Statics and Kinematics, as presented in the previous section, can be summarized as follows:

1. Given a displacement field, find the associate (compatible) deformation field (KINEMATICS).

2. Given a stress field, find the associate (equilibrated) loading field (STATICS),

or diagrammatically



The present section is concerned with exploring the complementary process of describing compatibility and equilibrium, which in broad lines can be summarized as:

1. Given the deformation field (and the rigid-body displacements of a point) find the associate displacement field.

2. Given the loading field (and the stress-resultants at one point) find the associate stress field.

To do so it is necessary to found the static and kinematic analyses on a substructure characterized by a different, let us say complementary, connectivity, the mesh substructure.

2.2.1 The Exact Non-Linear Relations

Assume that a typical mesh, say mesh M, is disconnected from a structure, both in its initial and deformed configurations. We will be considering a (clockwise) directed mesh formed by four branches, the MESH MEMBERS, connecting four vertices, the MESH NODES; Fig.2.6 represents the disconnected mesh together with an arbitrary set of forces, the MESH FORCES $\underset{\sim}{R}^*_M$, not necessarily self-equilibrated.

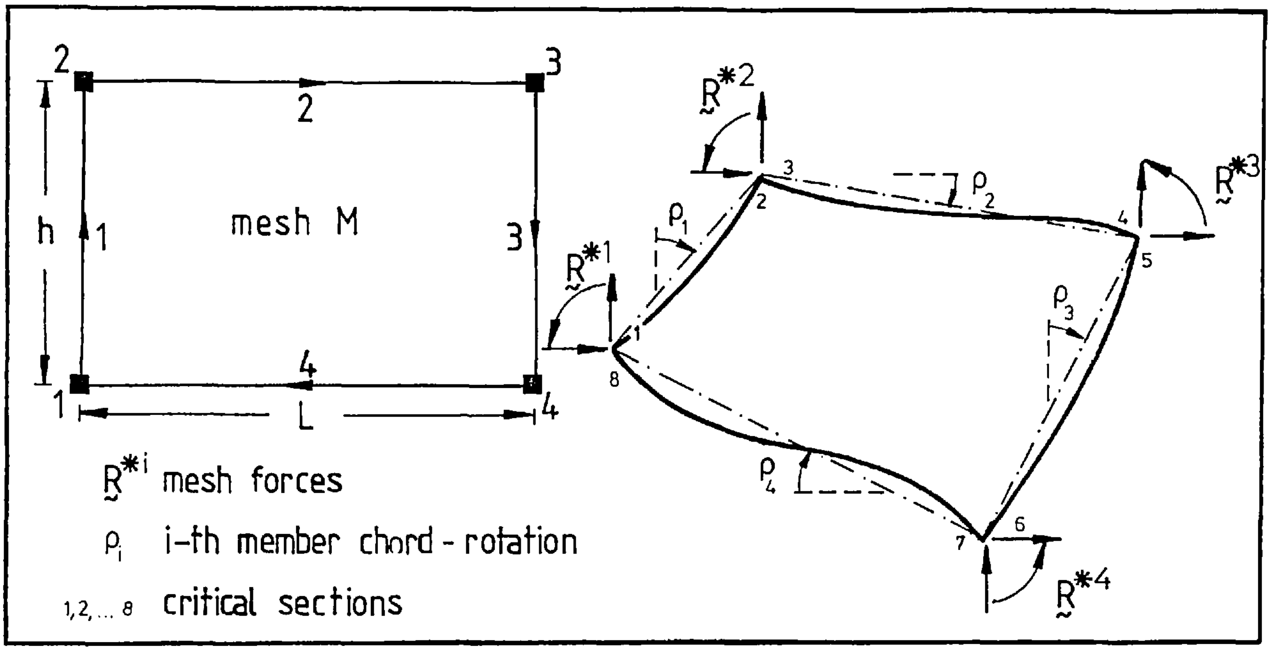


FIGURE 2.6

Let α be the mesh STATIC INDETERMINACY NUMBER. Freeing $\alpha = 3$ connections by introducing artificial RELEASES, the mesh is rendered statically determinate if the freed forces (or a statically equivalent set of forces) or BIACTIONS $\underset{\sim}{p}$ are selected as unknowns. In planar problems we can distinguish the three types of release represented in Fig.2.7; the associated biactions are

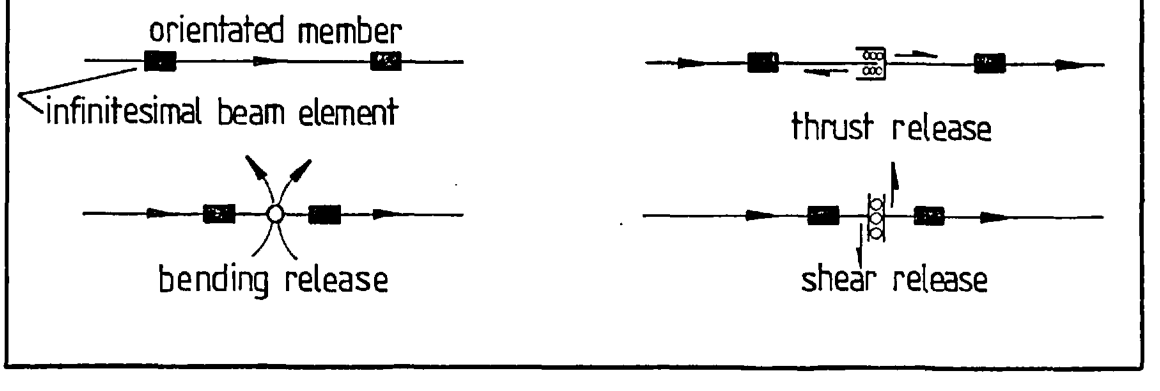


FIGURE 2.7

considered positive if they induce a positive stress distribution on the near-side face of the neighbouring infinitesimal element of the orientated member of the mesh.

If the mesh is rendered statically determinate (for instance by cutting member 4 at the immediate neighbourhood of critical section 8) the stress resultants at every critical section can be expressed as the superimposition of two stress fields, one induced by the biactions, the other by the loading. That is, the solution of Statics is decomposed into a complementary solution (a self-equilibrating stress field $\underset{\sim}{X}_M^c$ induced by the biactions) and a particular solution (a load equilibrating stress field $\underset{\sim}{X}_M^o$).

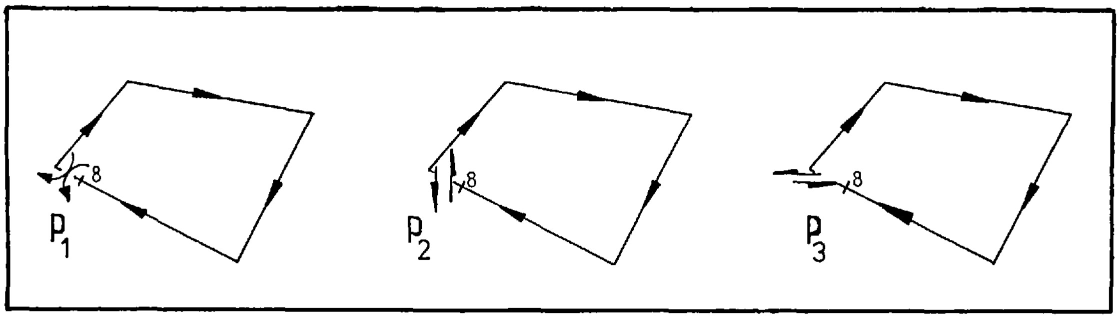


FIGURE 2.8

The figure above represents the adopted set of biactions and the associated complementary solution is defined by

$$\begin{bmatrix} X_1^1 \\ X_1^2 \\ X_2^1 \\ X_1^3 \\ X_1^4 \\ X_2^2 \\ X_1^5 \\ X_1^6 \\ X_2^3 \\ X_1^7 \\ X_1^8 \\ X_2^4 \end{bmatrix}_M^c = \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & -L_c^1 s_1 & L_c^1 c_1 \\ \cdot & -c_1 & -s_1 \\ 1 & -L_c^1 s_1 & L_c^1 c_1 \\ 1 & -L_c^4 c_4 - L_c^3 s_3 & -L_c^4 s_4 + L_c^3 c_3 \\ \cdot & s_2 & -c_2 \\ 1 & -L_c^4 c_4 - L_c^3 s_3 & -L_c^4 s_4 + L_c^3 c_3 \\ 1 & -L_c^4 c_4 & -L_c^4 s_4 \\ \cdot & c_3 & s_3 \\ 1 & -L_c^4 c4 & -L_c^4 s_4 \\ 1 & \cdot & \cdot \\ \cdot & -s_4 & c_4 \end{bmatrix}_M \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}_M \qquad (2.2.1)$$

where $\qquad s_i = \sin\rho_i \qquad$ (2.2.2a)

and $\qquad c_i = \cos\rho_i \qquad$ (2.2.2b)

or, in a more compact form

$$\underset{\sim}{X}^c_M = \underline{S}_M \underset{\sim}{\rho}_M \qquad (2.2.3)$$

Assume that one side of the cut is clamped, as in Fig.2.9, and let $\underset{\sim}{R}'^*_M$ be the reaction developed at the support of the cantilever. The mesh forces and the MESH REACTION FORCES $\underset{\sim}{R}'^*_M$ are

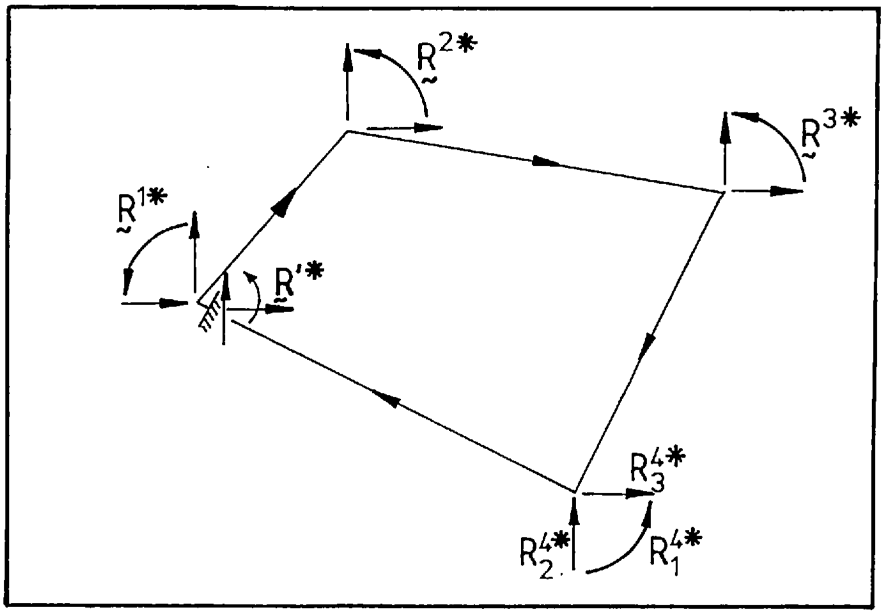


FIGURE 2.9

related through the following equilibrium condition

$$-\underset{\sim}{R}'^*_M = \underline{S}_{r_M} \underset{\sim}{R}^*_M \qquad (2.2.4)$$

where

$$\underline{S}_{r_M} = \left[\begin{array}{c|c|c:c|c|c:c|c|c:c|c|c} 1 & \bullet & \bullet & 1 & L_c^1 s_1 & -L_c^1 c_1 & 1 & L_c^4 c_4 + L_c^3 s_3 & -L_c^1 c_1 + L_c^2 s_2 & 1 & +L_c^4 c_4 & +L_c^4 s_4 \\ \hline \bullet & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 1 & \bullet \\ \hline \bullet & \bullet & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 1 & \bullet & \bullet & 1 \end{array}\right]_M$$

(2.2.5)

The particular solution defining the stresses caused by the loading and developing at the critical sections can be expressed as

$$\underset{\sim}{X}^{o}_{M} = \underset{\sim}{S}_{o_M} \underset{\sim}{R}^{*}_{M}$$

where

$$\underset{\sim}{S}_{o_M} = \left[\begin{array}{ccc|c|cc|c|cc|c|cc}
\cdot & \cdot & \cdot & 1 & L_c^1 s_1 & -L_c^1 c_1 & 1 & L_c^4 c_4 + L_c^3 s_3 & -L_c^1 c_1 + L_c^2 s_2 & 1 & L_c^4 c_4 & L_c^4 s_4 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & L_c^2 c_2 & L_c^2 s_2 & 1 & -L_c^3 s_3 + L_c^2 c_2 & L_c^3 c_3 + L_c^2 s_2 \\
\cdot & \cdot & \cdot & \cdot & -c_1 & -s_1 & \cdot & -c_1 & -s_1 & \cdot & -c_1 & -s_1 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & L_c^2 c_2 & L_c^2 s_2 & 1 & -L_c^3 s_3 + L_c^2 c_2 & L_c^3 c_3 + L_c^2 s_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & -L_c^3 s_3 & L_c^3 c_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s_2 & -c_2 & \cdot & s_2 & -c_2 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -L_c^3 s_3 & L_c^3 c_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & s_3 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]_M \qquad (2.2.6)$$

The mesh description of STATICS at element level can now be defined as

STATICS

$$\left[\begin{array}{c} \underset{\sim}{X}' \\ \hline -\underset{\sim}{R}'^{*} \end{array}\right]_M = \left[\begin{array}{c|c} \underset{\sim}{S} & \underset{\sim}{S}_o \\ \hline \cdot & \underset{\sim}{S}_r \end{array}\right]_M \left[\begin{array}{c} \underset{\sim}{p} \\ \hline \underset{\sim}{R}^{*} \end{array}\right]_M \qquad \begin{array}{r} (2.2.7a) \\ (2.2.7b) \end{array}$$

The derivation of the particular solution was based on the assumption that the faces of the cut did not suffer any relative displacements. Hence these DISCONTINUITIES $\underset{\sim}{v}$ at the releases must be expressed in terms of the mesh deformations and set to zero. A discontinuity is assumed positive, Fig.2.10, in the sense of the relative displacement of the faces of the release when acted upon by the corresponding biaction

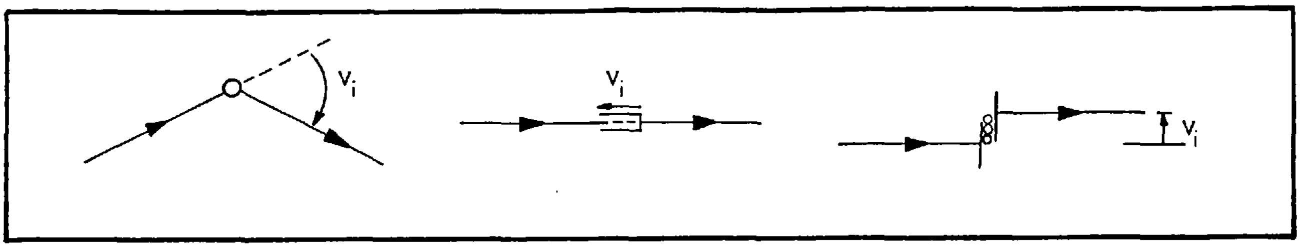


FIGURE 2.10

With help from Fig.2.11, the following relationship between the discontinuities and the MESH DEFORMATIONS $\underset{\sim}{u}'_M$ is found:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & hc'_2 & -1 & hc'_3 & hc'_4-Ls'_4 & \cdot & hc'_5-Ls'_5 & -Ls'_6 & 1 & -Ls'_7 & \cdot & \cdot \\ \cdot & hs'_2 & \cdot & hs'_3 & hs'_4+Lc'_4 & -1 & hs'_5+Lc'_5 & Lc'_6 & \cdot & Lc'_7 & \cdot & 1 \end{bmatrix} \begin{bmatrix} u'^1_1 \\ u'^2_1 \\ u'^1_2 \\ \hline u'^3_1 \\ u'^4_1 \\ u'^2_2 \\ \hline u'^5_1 \\ u'^6_1 \\ u'^3_2 \\ \hline u'^7_1 \\ u'^8_1 \\ u'^4_2 \end{bmatrix}_M \qquad (2.2.8)$$

where $$c'_i = \frac{1-\cos u'^i_1}{u'^i_1} \qquad (2.2.9a)$$

and $$s'_i = \frac{\sin u'^i_1}{u'^i_1} \qquad (2.2.9b)$$

or in a more compact form

$$\underset{\sim}{v}_M = \underset{\sim}{K}_M \, \underset{\sim}{u}'_M \qquad (2.2.10)$$

To enforce continuity we set

$$\underset{\sim}{v}_M = \underset{\sim}{0} \qquad (2.2.11)$$

When mesh M travels from the initial to the final position, as in Fig.2.12, the displacement of any point of the mesh, and for that matter the MESH FORCE DISPLACEMENTS $\underset{\sim}{r}^*_M$ can be understood as the sum of two parts; the contribution of the mesh deformation and the rigid body displacement of the mesh, described for instance by the MESH REACTION FORCE DISPLACEMENT $\underset{\sim}{r}'^*_M$. With help from the kinematic influence diagrams in Figs.2.11 and 2.13, it is found that

$$\underline{r}^*_M = \underline{K}_{o_M}\underline{u}'_M + \underline{K}_{r_M}\underline{r}'^*_M \qquad (2.2.12)$$

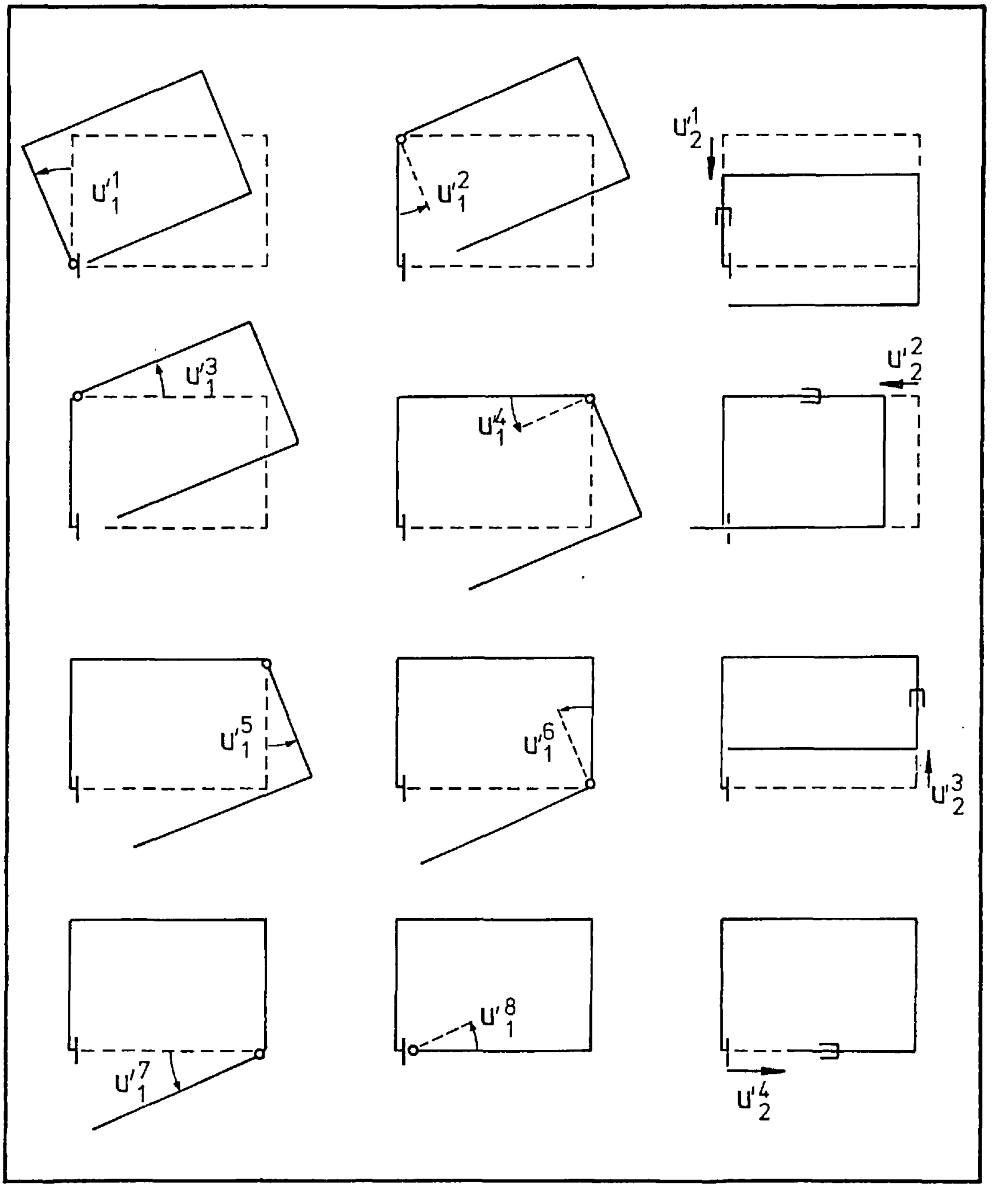


FIGURE 2.11

where

$$K_{n_M} = \left[\begin{array}{c|c|c:c|c|c:c|c|c:c|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-hc'_1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-hs'_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Ls'_1-hc'_1 & Ls'_2 & -1 & Ls'_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-Lc'_1-hs'_1 & -Lc'_2 & \cdot & -Lc'_3 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
1 & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
Ls'_1 & Ls'_2+hc'_2 & -1 & Ls'_3+hc'_3 & hc'_4 & \cdot & hc'_5 & \cdot & 1 & \cdot & \cdot & \cdot \\
-Lc'_1 & -Lc'_2+hs'_2 & \cdot & -Lc'_3+hs'_3 & hs'_4 & -1 & hs'_5 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]_M \qquad (2.2.13)$$

and

$$K_{r_M} = \left[\begin{array}{c|c|c}
1 & \cdot & \cdot \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1 \\
\hdashline
1 & \cdot & \cdot \\
-h\bar{c}_1 & 1 & \cdot \\
-h\bar{s}_1 & \cdot & 1 \\
\hdashline
1 & \cdot & \cdot \\
L\bar{s}_1-h\bar{c}_1 & 1 & \cdot \\
-L\bar{c}_1-h\bar{s}_1 & \cdot & 1 \\
\hdashline
1 & \cdot & \cdot \\
L\bar{s}_1 & 1 & \cdot \\
-L\bar{c}_1 & \cdot & 1
\end{array}\right]_M \qquad (2.2.14)$$

where

$$\bar{c}_i = \frac{1-\cos r'^{*}_i}{r'^{*}_i} \qquad (2.2.15a)$$

and

$$\bar{s}_i = \frac{\sin r'^{*}_i}{r'^{*}_i} \qquad (2.2.15b)$$

Equations (2.2.10), (2.2.11) and (2.2.12) represent the mesh description of kinematics, summarized below:

KINEMATICS

$$\begin{bmatrix} \underset{\sim}{0} \\ \hline \underset{\sim}{r}^{*} \end{bmatrix}_M = \begin{bmatrix} \underset{\sim}{K} & \cdot \\ \hline \underset{\sim}{K}_o & \underset{\sim}{K}_r \end{bmatrix}_M \cdot \begin{bmatrix} \underset{\sim}{u}' \\ \hline \underset{\sim}{r}'^{*} \end{bmatrix}_M \quad \begin{matrix} (2.2.16a) \\ (2.2.16b) \end{matrix}$$

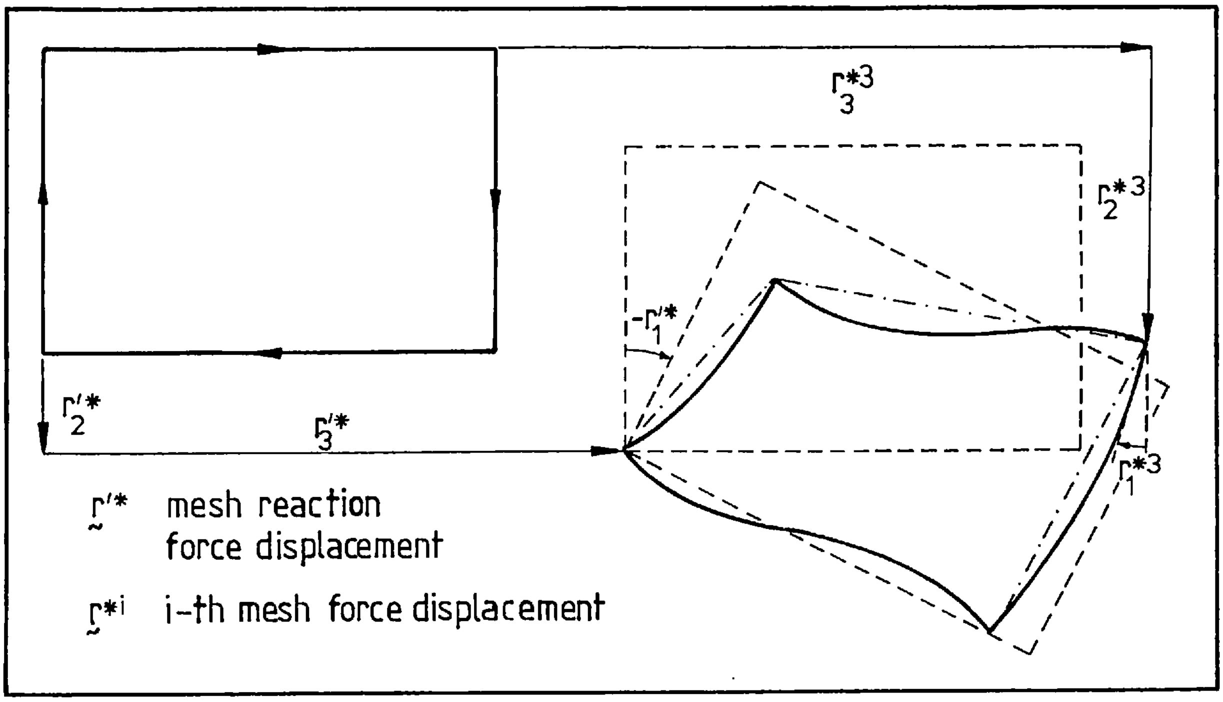


FIGURE 2.12

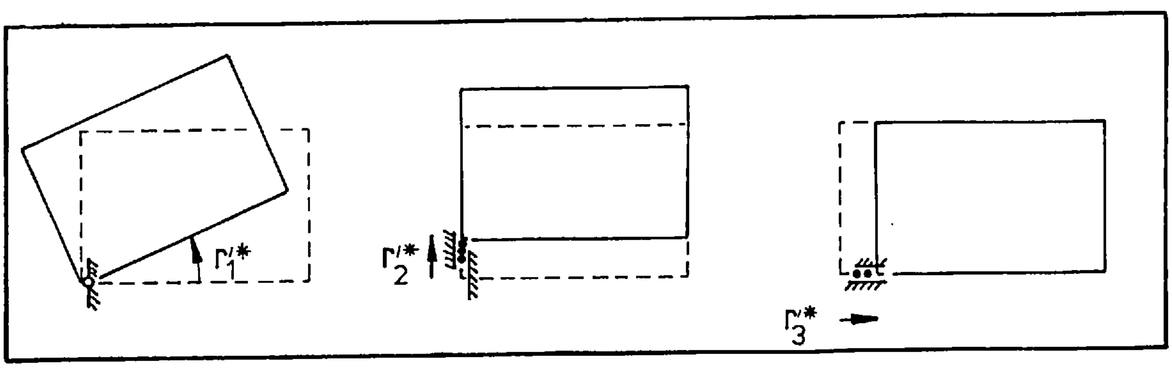


FIGURE 2.13

2.2.2 The Exact Explicitly Linear Dual Relations

Enforcing the linear analysis assumptions, definitions (2.2.2), (2.2.9) and (2.2.15) reduce, respectively, to:

$$s_i = 0 \qquad s'_i = 1 \qquad \bar{s}_i = 1$$
$$c_i = 1 \qquad c'_i = 0 \qquad \bar{c}_i = 0$$

and as in linear Statics and Kinematics where the chord length of a member is confused with its initial length, the linearized static and kinematic matrices reduce to

$$(\underline{S}_M^T)_{lin} = (\underline{K}_M)_{lin} = \underline{B}_M^T = \left[\begin{array}{c|c|c:c|c|c:c|c|c:c|c|c} 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -L & \cdot & -L & -L & 1 & -L & \cdot & \cdot \\ \cdot & h & \cdot & h & h & -1 & h & \cdot & \cdot & \cdot & \cdot & -1 \end{array}\right]_M \tag{2.2.17}$$

$$(\underline{S}_{o_M})_{lin} = (\underline{K}_{o_M}^T)_{lin} = \underline{B}_{o_M} = \left[\begin{array}{c|c|c:c|c|c:c|c|c:c|c|c} \cdot & \cdot & \cdot & 1 & \cdot & -h & 1 & L & -h & 1 & L & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & L & \cdot & 1 & L & h \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot \\ \hdashline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & L & \cdot & 1 & L & h \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & h \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & -1 \\ \hdashline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & h \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \hdashline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right]_M \tag{2.2.18}$$

$$(\underline{S}_{r_M})_{lin} = (\underline{K}_{r_M}^T)_{lin} = \underline{B}_{r_M} = \left[\begin{array}{c|c|c:c|c|c:c|c|c:c|c|c} 1 & \cdot & \cdot & 1 & \cdot & -h & 1 & L & -h & 1 & L & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \end{array}\right]_M \tag{2.2.19}$$

Hence, the linear description of Statics and Kinematics associated in the mesh M follow the dual transformations (2.2.20) and (2.2.21).

As in the previous section, let us forcibly introduce the linear operators in the non-linear Statics description (2.2.7) and condense the non-linear terms in the definition of two auxiliary variables:

$$\begin{bmatrix} \underset{\sim}{X}' - \underset{\sim}{\tilde{X}}' \\ \hline \underset{\sim}{R}'^{*} - \underset{\sim}{\tilde{R}}'^{*} \end{bmatrix}_M = \begin{bmatrix} \underline{B} & \underline{B}_o \\ \hline \cdot & \underline{B}_r \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{p} \\ \hline \underset{\sim}{R}^{*} \end{bmatrix}_M$$

LINEAR ANALYSIS			
	STATICS	KINEMATICS	
(2.2.20a) (2.2.20b)	$\begin{bmatrix} \underset{\sim}{X}' \\ \hline -\underset{\sim}{R}'_{*} \end{bmatrix}_M = \begin{bmatrix} \underline{B} & \underline{B}_o \\ \hline \cdot & \underline{B}_r \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{p} \\ \hline \underset{\sim}{R}^{*} \end{bmatrix}_M$	$\begin{bmatrix} \cdot \\ \hline \underset{\sim}{r}^{*} \end{bmatrix}_M = \begin{bmatrix} \underline{B}^T & \cdot \\ \hline \underline{B}_o^T & \underline{B}_r^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}' \\ \hline \underset{\sim}{r}'^{*} \end{bmatrix}_M$	(2.2.21a) (2.2.21b)
MESH DESCRIPTION			

The corrective stress resultant $\underset{\sim}{\tilde{X}}'_M$ and the corrective mesh force reaction $\underset{\sim}{\tilde{R}}'^{*}_M$ are therefore defined by

$$\underset{\sim}{\tilde{X}}'_M = (\underline{S}_M - \underline{B}_M)\underset{\sim}{p}_M + (\underline{S}_{o_M} - \underline{B}_{o_M})\ \underset{\sim}{R}^{*}_M \qquad (2.2.22a)$$

$$\underset{\sim}{\tilde{R}}'^{*}_M = \qquad (\underline{S}_{r_M} - \underline{B}_{r_M})\ \underset{\sim}{R}^{*}_M \qquad (2.2.22b)$$

Still possible, although now more cumbersome, the direct method adopted in the previous section could be used again in order to simplify the definition of the above auxiliary variables.

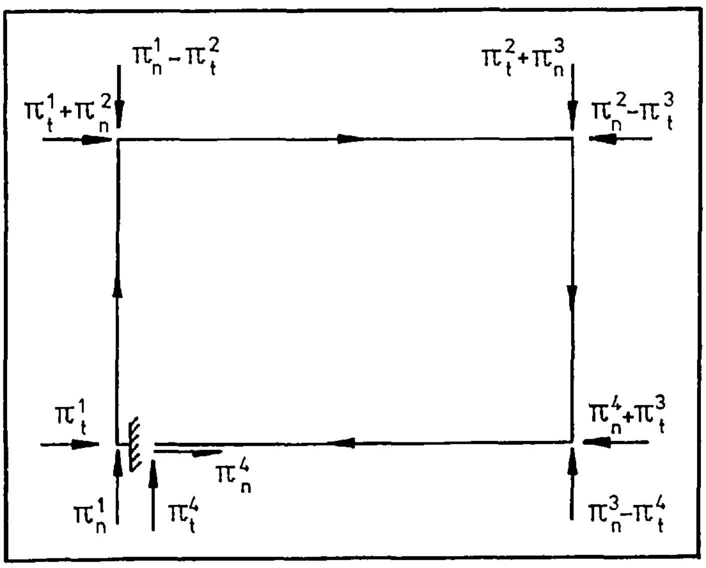


FIGURE 2.14

Instead let us build upon the knowledge gained in the direct method and introduce the ADDITIONAL MESH FORCES $\underset{\sim}{\pi}_M$ represented in Fig.2.14 and defined by equation (2.1.20); we may write then

$$\underset{\sim}{\pi}_M = \underline{\Pi}_M\ \underset{\sim}{X}'_m$$

or, from equation (2.2.7a)

$$\underset{\sim}{\pi}_M = \underline{\Pi}_M \underline{S}_M \underset{\sim}{p}_M + \underline{\Pi}_M \underline{S}_{o_M} \underset{\sim}{R}^{*}_M \qquad (2.2.23)$$

where

$$
\underset{\sim}{\Pi}_M = \left[\begin{array}{ccc|ccc|ccc|ccc}
\frac{s_1}{L_c^1} & & 1-c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\frac{1}{h}+\frac{c_1}{L_c^1} & \frac{1}{h}-\frac{c_1}{L_c^1} & s_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \frac{s_2}{L_c^2} & -\frac{s_2}{L_c^2} & 1-c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\frac{1}{L}+\frac{c_2}{L_c^2} & \frac{1}{L}-\frac{c_2}{L_c^2} & s_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{s_3}{L_c^3} & -\frac{s_3}{L_c^3} & 1-c_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{h}+\frac{c_3}{L_c^3} & \frac{1}{h}-\frac{c_3}{L_c^3} & s_3 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{s_4}{L_c^4} & -\frac{s_4}{L_c^4} & 1-c_4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{L}+\frac{c_4}{L_c^4} & \frac{1}{L}-\frac{c_4}{L_c^4} & s_4
\end{array}\right]_M
$$

The additional mesh forces induce in the (statically determined) mesh stress-resultants at the critical sections and reactions at the fixed face of the cut defined respectively by

$$\bar{\underset{\sim}{X}}'_M = \underline{B}_{o_{\pi_M}} \underset{\sim}{\pi}_M$$

and

$$\bar{\underset{\sim}{R}}'^{*}_M = \underline{B}_{r_{\pi_M}} \underset{\sim}{\pi}_M$$

or, from equation (2.2.23)

$$\bar{\underset{\sim}{X}}'_M = \underline{B}_{o_{\pi_M}} \underline{\Pi}_M \underline{S}_M \underset{\sim}{p}_M + \underline{B}_{o_{\pi_M}} \underline{\Pi}_M \underline{S}_{o_M} \underset{\sim}{R}^{*}_M \qquad (2.2.24a)$$

and

$$\bar{\underset{\sim}{R}}'^{*}_M = \underline{B}_{r_{\pi_M}} \underline{\Pi}_M \underline{S}_M \underset{\sim}{p}_M + \underline{B}_{r_{\pi_M}} \underline{\Pi}_M \underline{S}_{o_M} \underset{\sim}{R}^{*}_M \qquad (2.2.24b)$$

where

$$\underline{B}_{o\pi_M} = \left[\begin{array}{c|c:c|c:c|c:c|c}
\cdot & -h & \cdot & -L & \cdot & -h & \cdot & -L \\
\cdot & \cdot & \cdot & -L & \cdot & -h & \cdot & -L \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
\cdot & \cdot & \cdot & -L & \cdot & -h & \cdot & -L \\
\cdot & \cdot & \cdot & \cdot & \cdot & -h & \cdot & -L \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
\cdot & \cdot & \cdot & \cdot & \cdot & -h & \cdot & -L \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -L \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hdashline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -L \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{array}\right]_M \qquad (2.2.25)$$

$$\underline{B}_{r\pi_M} = \left[\begin{array}{c|c:c|c:c|c:c|c}
\cdot & -h & \cdot & -L & \cdot & -h & \cdot & -L \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]_M \qquad (2.2.26)$$

It is easy to assert the following relationships

$$\underline{B}_{o\pi_M} \underline{\Pi}_M \underline{S}_{o_M} = \underline{S}_{o_M} - \underline{B}_{o_M}$$

$$\underline{B}_{r\pi_M} \underline{\Pi}_M \underline{S}_{o_M} = \underline{S}_{r_M} - \underline{B}_{r_M}$$

$$\underline{B}_{o\pi_M} \underline{\Pi}_M \underline{S}_M = \underline{S}_M - \underline{B}_M$$

$$\underline{B}_{r\pi_M} \underline{\Pi}_M \underline{S}_M = \underline{0}$$

where $\underline{0}$ is the null matrix, which enable us to identify through equations (2.2.24) and (2.2.22) the stress resultants $\bar{\underline{X}}'_M$ and the mesh reactions $\bar{\underline{R}}'^*_M$ induced by the additional mesh forces with, respectively, the corrective stress resultants $\tilde{\underline{X}}'_M$ and the corrective mesh force reaction $\tilde{\underline{R}}'^*_M$, and therefore to re-write Statics as follows:

$$\left[\begin{array}{c} \underset{\sim}{X}' \\ \hdashline -\underset{\sim}{R}'^* \end{array}\right]_M = \left[\begin{array}{c:c:c} \underline{B} & \underline{B}_o & \underline{B}_{o\pi} \\ \hdashline \cdot & \underline{B}_r & \underline{B}_{r\pi} \end{array}\right]_M \left[\begin{array}{c} \underset{\sim}{p} \\ \hdashline \underset{\sim}{R}^* \\ \hdashline \underset{\sim}{\pi} \end{array}\right]_M \qquad \begin{array}{r} (2.2.27a) \\ (2.2.27b) \end{array}$$

If duality is to be preserved, the kinematic transformation (2.2.16) must be replaced by the following:

$$\begin{bmatrix} \underset{\sim}{k}^1 \\ \hline \underset{\sim}{k}^2 \\ \hline \underset{\sim}{k}^3 \end{bmatrix}_M = \begin{bmatrix} \underset{\sim}{B}^T & \cdot \\ \hline \underset{\sim}{B}_o^T & \underset{\sim}{B}_r^T \\ \hline \underset{\sim}{B}_{o\pi}^T & \underset{\sim}{B}_{r\pi}^T \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{k}^4 \\ \hline \underset{\sim}{k}^5 \end{bmatrix}_M \qquad \begin{matrix} (2.2.28a) \\ (2.2.28b) \\ (2.2.28c) \end{matrix}$$

The identification of the kinematic variables k_M^i may start by finding their dual static variables

$$\begin{aligned} \underset{\sim}{k}_M^1 &\diamond \underset{\sim}{p}_M \\ \underset{\sim}{k}_M^2 &\diamond \underset{\sim}{R}_M^* \\ \underset{\sim}{k}_M^3 &\diamond \underset{\sim}{\pi}_M \\ \underset{\sim}{k}_M^4 &\diamond \underset{\sim}{X}_M' \\ \underset{\sim}{k}_M^5 &\diamond \underset{\sim}{R}_M'^* \end{aligned}$$

The above relationships together with the dual correspondence summarized in (2.1.26) yield

$$\underset{\sim}{k}_M^2 = \underset{\sim}{r}_M^* \qquad (2.2.29a)$$

$$\underset{\sim}{k}_M^3 = \underset{\sim}{\delta}_{\pi_M} \qquad (2.2.29b)$$

$$\underset{\sim}{k}_M^4 = \underset{\sim}{u}_M' + \underset{\sim}{u}_{\pi_M}' \qquad (2.2.29c)$$

and we can antecipate that

$$\underset{\sim}{k}_M^5 = \underset{\sim}{r}_M'^* + \bar{\underset{\sim}{k}}_M^5 \qquad (2.2.29d)$$

The arrays

$$\underset{\sim}{\delta}_{\pi_M} = \begin{bmatrix} \underset{\sim}{\delta}_{\pi_1} \\ \hline \underset{\sim}{\delta}_{\pi_2} \\ \hline \underset{\sim}{\delta}_{\pi_3} \\ \hline \underset{\sim}{\delta}_{\pi_4} \end{bmatrix}_M, \text{ and } \underset{\sim}{u}_{\pi_M}' = \begin{bmatrix} \underset{\sim}{u}_{\pi_1}' \\ \hline \underset{\sim}{u}_{\pi_2}' \\ \hline \underset{\sim}{u}_{\pi_3}' \\ \hline \underset{\sim}{u}_{\pi_4}' \end{bmatrix}_M$$

collect the ADDITIONAL MESH FORCE DISPLACEMENTS $\underset{\sim}{\delta}_{\pi_{i_M}}$ and the

ADDITIONAL MESH STRAINS $\underset{\sim}{u}'_{\pi\, i_M}$ respectively, the latter being defined by equations (2.1.22) which give

$$\underset{\sim}{u}'_{\pi_M} = \underset{\sim}{K}_{\pi_M} \underset{\sim}{u}'_M + \underset{\sim}{K}'_{\pi_M} \underset{\sim}{\delta}_{\pi\, M} + \underset{\sim}{K}''_{\pi_M} \underset{\sim}{\rho}_M \qquad (2.2.30)$$

where the elements of $\underset{\sim}{\rho}_M$ are the MESH MEMBER CHORD ROTATIONS ρ_{i_M} and

$$\underset{\sim}{K}_{\pi_M} = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1
\end{bmatrix}$$

$$\underset{\sim}{K}'_{\pi_M} = \begin{bmatrix}
\cdot & -\frac{1}{h} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{h} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\frac{1}{L} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{1}{L} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{h} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{h} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{L} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{L} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{bmatrix}_M \quad \text{and} \quad \underset{\sim}{K}''_{\pi_M} = \begin{bmatrix}
1 & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot \\
\cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot
\end{bmatrix}_M$$

Substituting equations (2.2.29) back into equations (2.2.28a) and (2.2.28b) we find that the following relationships

$$\underset{\sim}{k}_M^1 = \underline{B}_M^T \, \underset{\sim}{u}'_{\pi_M} - (\underline{K}_M - \underline{B}_M^T)\underset{\sim}{u}'_M \tag{2.2.31a}$$

$$\underline{B}_{r_M}^T \, \underset{\sim}{\bar{k}}_M^5 = -\underline{B}_{o_M}^T \underset{\sim}{u}'_{\pi_M} + (\underline{K}_{o_M} - \underline{B}_{o_M}^T)\underset{\sim}{u}'_M + (\underline{K}_{r_M} - \underline{B}_{r_M}^T)\underset{\sim}{r}'^*_M \tag{2.2.31b}$$

must be identified if the updated equations (2.2.28) are to comply with the exact description of Kinematics summarized by equations (2.2.16).

The additional mesh force disolacements and the mesh chord rotations can be expressed as functions of the mesh deformations and the mesh reaction force displacements:

$$\underset{\sim}{\delta}_{\pi_M} = \underline{K}_{\delta_M} \underset{\sim}{u}'_M + \underline{K}'_{\delta_M} \underset{\sim}{r}'^*_M \tag{2.2.32a}$$

$$\underset{\sim}{\rho}_M = \underline{K}_{\rho_M} \underset{\sim}{u}'_M + \underline{K}'_{\rho_M} \underset{\sim}{r}'^*_M \tag{2.2.32b}$$

where

$$\underline{K}_{\delta_M} = \begin{bmatrix} hc'_1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -hs'_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Lc'_1 & Lc'_2 & \cdot & Lc'_3 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -Ls'_1 & -Ls'_2 & \cdot & -Ls'_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ hc'_1 & hc'_2 & \cdot & hc'_3 & hc'_4 & \cdot & hc'_5 & \cdot & 1 & \cdot & \cdot & \cdot \\ -hs'_1 & -hs'_2 & \cdot & -hs'_3 & -hs'_4 & \cdot & -hs'_5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ Lc'_1 & Lc'_2 & \cdot & Lc'_3 & Lc'_4 & \cdot & Lc'_5 & Lc'_6 & \cdot & Lc'_7 & \cdot & 1 \\ -Ls'_1 & -Ls'_2 & \cdot & -Ls'_3 & -Ls'_4 & \cdot & -Ls'_5 & -Ls'_6 & \cdot & -Ls'_7 & \cdot & \cdot \end{bmatrix}_M \qquad \underline{K}'_{\delta_M} = \begin{bmatrix} h\bar{c}_1 & \cdot & \cdot \\ -h\bar{s}_1 & \cdot & \cdot \\ L\bar{c}_1 & \cdot & \cdot \\ -L\bar{s}_1 & \cdot & \cdot \\ h\bar{c}_1 & \cdot & \cdot \\ -h\bar{s}_1 & \cdot & \cdot \\ L\bar{c}_1 & \cdot & \cdot \\ -L\bar{s}_1 & \cdot & \cdot \end{bmatrix}_M$$

$$\underline{K}_{\rho_M} = \begin{bmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & -1 & -1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & -1 & -1 & \cdot & -1 & -1 & \cdot & -1 & \cdot & \cdot \end{bmatrix} \qquad \underline{K}'_{\rho_M} = \begin{bmatrix} -1 & \cdot & \cdot \\ -1 & \cdot & \cdot \\ -1 & \cdot & \cdot \\ -1 & \cdot & \cdot \end{bmatrix}_M$$

The coefficients of the above matrices are found with help from the kinematic influence diagrams in Figs.2.11 and 2.13, respectively.

The additional mesh deformations may now be expressed too as functions of the mesh deformations and the mesh reaction force displacements by eliminating in equation (2.2.30) the additional mesh forces and the mesh chord member rotations through equations (2.2.32), yielding

$$\underset{\sim}{u}'_{\pi_M} = \underline{K}_{u_M}\underset{\sim}{u}'_M + \underline{K}'_{u_M}\underset{\sim}{r}'^*_M \qquad (2.2.32c)$$

where $\underline{K}_{u_M} = \underline{K}_{\pi_M} + \underline{K}'_{\pi_M}\underline{K}_{\delta_M} + \underline{K}''_{\pi_M}\ \underline{K}_{\rho_M}$

and $\underline{K}'_{u_M} = \underline{K}'_{\pi_M}\underline{K}'_{\delta_M} + \underline{K}''_{\pi_M}\ \underline{K}'_{\rho_M}$

After the elimination of the additional mesh deformations equations (2.2.31) transform, respectively, into

$$\underset{\sim}{k}^1_M = [\underline{B}^T_M\underline{K}_{u_M} - (\underline{K}_M - \underline{B}^T_M)]\,\underset{\sim}{u}'_M + [\underline{B}^T_M\underline{K}'_{u_M}]\,\underset{\sim}{r}'^*_M$$

$$\underline{B}^T_{r_M}\ \underset{\sim}{\bar{k}}^5_M = [-\underline{B}^T_{o_M}\underline{K}_{u_M} + (\underline{K}_{o_M} - \underline{B}^T_{o_M})]\,\underset{\sim}{u}'_M + [-\underline{B}^T_{o_M}\underline{K}'_{u_M} + (\underline{K}_{r_M} - \underline{B}^T_{r_M})]\,\underset{\sim}{r}'^*_M$$

or

$$\underset{\sim}{k}^1_M = \underset{\sim}{0}$$

$$\underline{B}^T_{r_M}\ \underset{\sim}{\bar{k}}^5_M = 0 \qquad (2.2.33)$$

since

$$\underline{B}^T_M\ \underline{K}_{u_M} = \underline{K}_M - \underline{B}^T_M\ .$$

$$\underline{B}^T_M\ \underline{K}'_{u_M} = \underline{0}$$

$$\underline{B}^T_{o_M}\underline{K}_{u_M} = \underline{K}_{o_M} - \underline{B}^T_{o_M}$$

$$\underline{B}^T_{o_M}\underline{K}'_{u_M} = \underline{K}_{r_M} - \underline{B}^T_{r_M}$$

The trivial solution

$$\underset{\sim}{\bar{k}}^5_M = \underset{\sim}{0} \qquad (2.2.34)$$

is the only solution of equation (2.2.33).

Equations (2.2.28c) and (2.2.29b) together with equation (2.2.34) require the additional mesh force displacements to be defined by

$$\underset{\sim}{\delta}_{\pi_M} = \underline{B}^T_{o\pi_M}(\underset{\sim}{u}'_M + \underset{\sim}{u}'_{\pi_M}) + \underline{B}^T_{r\pi_M}\underset{\sim}{r}'^*_M$$

or, substituting equation (2.2.32b) above

$$\underset{\sim}{\delta}_{\pi_M} = [\underline{B}^T_{o\pi_M}\underline{K}_{u_M} + \underline{B}^T_{o\pi_M}]\underset{\sim}{u}'_M + [\underline{B}^T_{o\pi_M}\underline{K}'_{u_M} + \underline{B}^T_{r\pi_M}]\underset{\sim}{r}'^*_M$$

which is the explicitly linear version of the exact definition (2.2.32a) since

$$\underline{B}^T_{o\pi_M}\underline{K}_{u_M} + \underline{B}^T_{o\pi_M} = \underline{K}_{\delta_M}$$

and

$$\underline{B}^T_{o\pi_M}\underline{K}'_{u_M} + \underline{B}^T_{r\pi_M} = \underline{K}'_{\delta_M}$$

The results obtained through the above process of identification may now be gathered and fed back into equations (2.2.28). The resulting set of equations is mathematically equivalent to the exact description of Kinematics (2.2.16), as well as equations (2.2.27) are mathematically equivalent to the exact description of Statics (2.2.7).

STATICS	KINEMATICS	
(2.2.35a), (2.2.35b): $\begin{bmatrix} \underset{\sim}{X}' \\ -\underset{\sim}{R}'^* \end{bmatrix}_M = \begin{bmatrix} \underline{B} & \underline{B}_o & \underline{B}_{o\pi} \\ \cdot & \underline{B}_r & \underline{B}_{r\pi} \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{p} \\ \underset{\sim}{R}^* \\ \underset{\sim}{\pi} \end{bmatrix}_M$	$\begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{r}^* \\ \underset{\sim}{\delta}_\pi \end{bmatrix}_M = \begin{bmatrix} \underline{B}^T & \cdot \\ \underline{B}^T_o & \underline{B}^T_r \\ \underline{B}^T_{o\pi} & \underline{B}^T_{r\pi} \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{u}' + \underset{\sim}{u}'_\pi \\ \underset{\sim}{r}'^* \end{bmatrix}_M$	(2.2.36a), (2.2.36b), (2.2.36c)
FINITE MESH DESCRIPTION		

Summarized below is the dual correspondence between Static and Kinematic variables:

DUAL CORRESPONDENCE		
STATIC VARIABLE	KINEMATIC VARIABLE	
$\underset{\sim}{X}'_M$	$\underset{\sim}{u}'_M + \underset{\sim}{u}'_{\pi_M}$	(2.2.37a)
$\underset{\sim}{R}'^*_M$	$\underset{\sim}{r}'^*_M$	(2.2.37b)
$\underset{\sim}{p}_M$	$\underset{\sim}{v}_M = \underset{\sim}{0}$	(2.2.37c)
$\underset{\sim}{R}^*_M$	$\underset{\sim}{r}^*_M$	(2.2.37d)
$\underset{\sim}{\pi}_M$	$\underset{\sim}{\delta}_{\pi_M}$	(2.2.37e)

In order to calculate automatically the structural coefficients and assemble them for the whole connected system, a fully automatic method of analysis requires data on Statics, Kinematics and Constitutive Relations of a generic unconnected substructure, together with its connectivity properties.

Chapter 4 deals with the system assemblage procedure and the causality relations at element level between static and kinematic variables are studied in Chapter 3.

In the previous section, the nodal description of Statics and Kinematics was derived and established in its full generality by equations (2.1.24) and (2.1.25), the nodal matrices being defined in (2.1.15) and (2.1.19).

If the mesh description of Statics and Kinematics is to be vested with similar scope and power, the definition of the mesh matrices involved in the (general) equations (2.2.35) and (2.2.36) must be extended to a mesh substructure of arbitrary geometry.

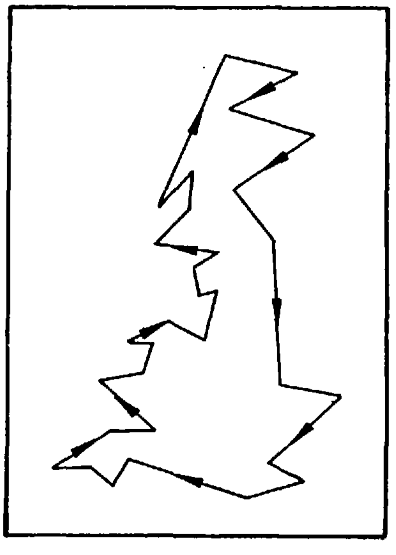


FIGURE 2.15

Consider then the clockwise directed polygonal mesh represented in Fig.2.15. Following the orientation of the mesh let its nodes and members be numbered from 1 to n; members i and i-1 connect at node i which intercalates critical sections 2i-2 and 2i-1. To transform the mesh into a tree, cut member n immediately after critical section 2n and let that point be the origin of the reference system $\underline{x}$, parallel to the global system $\underline{x}^*$.

The static analysis of the resulting polygonal cantilever is the simplest process of evaluating the coefficients of the mesh matrices. The elements of the cantilever relevant for that analysis were collected in Fig.2.16:

(a) The fixed face of the cut in the n-th member, the associate system of reference $\underline{x}$ and the mesh reaction forces $\underline{R}'^*$

(b) A typical member i with initial length L_i between critical sections 2i-1 and 2i and inclined α_i with respect to axis x_3; the angle is measured from the horizontal to the member in the sense of x_1.

(c) A node j where the j-th mesh force $\underset{\sim}{R}^{*j}$ is applied

and (d) A member k with the corresponding additional mesh forces $\underset{\sim}{\pi}^k$.

Equation (2.2.35a) shows that all statically possible stress resultants at member i

$$\underset{\sim}{X}'^i = \begin{bmatrix} X_1^{2i-1} \\ X_1^{2i} \\ X_2^i \end{bmatrix}$$

can be constructed by a linear combination of static influence sets $\underline{B}_i$, $\underline{B}_{o_{ij}}$ and $\underline{B}_{o\pi_{ij}}$, with multipliers $\underset{\sim}{p}$, $\underset{\sim}{R}^{*j}$ and $\underset{\sim}{\pi}^j$, respectively. From equation (2.2.35b) a similar conclusion can be drawn now in respect to the mesh reaction forces $\underset{\sim}{R}'^*$. Hence, the static influence submatrices may be interpreted as follows:

$\underline{B}_i$: stress-resultants at member i induced by a unit biaction

$\underline{B}_{o_{ij}}$ $(\underline{B}_{r_j})$: stress-resultants at member i (mesh reaction forces) induced by a unit mesh force applied at node j

$\underline{B}_{o\pi_{ij}}$ $(\underline{B}_{r\pi_j})$: stress-resultants at member i (mesh reaction forces) induced by unit additional forces applied on member j.

The general definition for the mesh submatrices may now be easily obtained with help from Fig.2.16 and 2.17.

In definitions (2.2.38) to (2.2.42), δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

and (x_2^k, x_3^k) the coordinates of the mesh vertex k.

Definitions (2.2.40) and (2.2.39) show that matrices $\underline{B}_{o\pi}$ and $\underline{B}_o$ are upper-triangular block-matrices, the diagonal sub-matrices of the latter one being always zero.

GENERALIZED MESH SUBMATRICES

$$\underline{B}_i = \begin{bmatrix} 1 & -x_3^{\,i} & x_2^{\,i} \\ 1 & -x_3^{\,i+1} & x_2^{\,i+1} \\ \cdot & -\sin\alpha_i & -\cos\alpha_i \end{bmatrix} \quad (2.2.38)$$

$$\underline{B}_{o_{ij}} \; (i<j) = \begin{bmatrix} 1 & x_3^{\,j}-x_3^{\,i} & -x_2^{\,j}+x_2^{\,i} \\ 1 & x_3^{\,j}-x_3^{\,i+1} & -x_2^{\,j}+x_2^{\,i+1} \\ \cdot & -\sin\alpha_i & -\cos\alpha_i \end{bmatrix} \quad (2.2.39)$$

$$\underline{B}_{o\pi_{ij}} \; (i\leq j) = \begin{bmatrix} \cdot & -L_j \\ \cdot & -(1-\delta_{ij})L_j \\ \delta_{ij} & \cdot \end{bmatrix} \quad (2.2.40)$$

$$\underline{B}_{r_j} = \begin{bmatrix} 1 & x_3^{\,j} & -x_2^{\,j} \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \quad (2.2.41)$$

$$\underline{B}_{r\pi_j} = \begin{bmatrix} \cdot & -L_j \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \quad (2.2.42)$$

if $i>j$, $\underline{B}_{o_{ij}} = \underline{0}$

if $i>j$, $\underline{B}_{o\pi_{ij}} = \underline{0}$

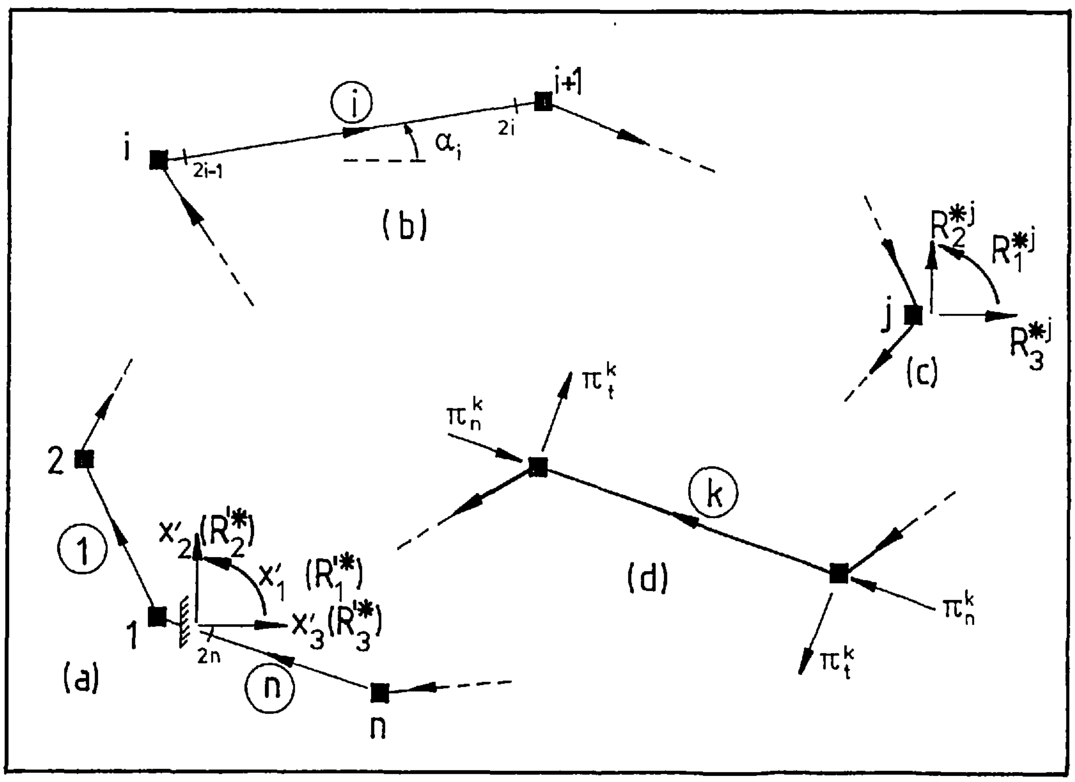


FIGURE 2.16

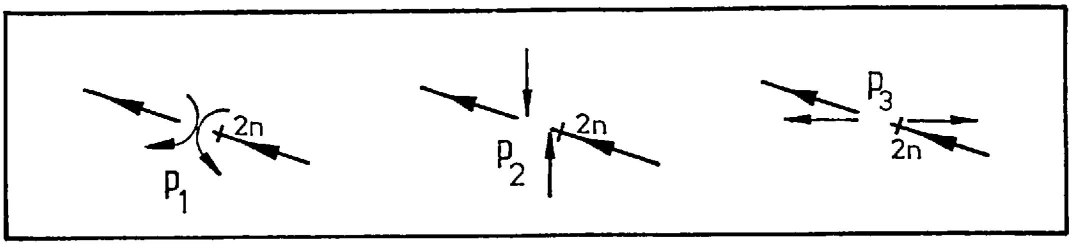


FIGURE 2.17

2.2.3 Incremental Analysis

The incremental mesh description of Statics and Kinematics is obtained by replacing in the linear equations (2.2.35) and (2.2.36) the static and kinematic variables by their increments, yielding

$$\begin{bmatrix} \Delta \underset{\sim}{X}' \\ -\Delta \underset{\sim}{R}'^{*} \end{bmatrix}_M = \begin{bmatrix} \underline{B} & \underline{B}_o & \underline{B}_{o\pi} \\ \cdot & \underline{B}_r & \underline{B}_{r\pi} \end{bmatrix} \begin{bmatrix} \Delta \underset{\sim}{p} \\ \Delta \underset{\sim}{R}^{*} \\ \Delta \underset{\sim}{\pi} \end{bmatrix}_M \qquad \begin{matrix} (2.2.43a) \\ (2.2.43b) \end{matrix}$$

for Statics, and

$$\begin{bmatrix} \cdot \\ \Delta \underset{\sim}{r}^{*} \\ \Delta \underset{\sim}{\delta}_{\pi} \end{bmatrix}_M = \begin{bmatrix} \underline{B}^T & \cdot \\ \underline{B}_o^T & \underline{B}_r^T \\ \underline{B}_{o\pi}^T & \underline{B}_{r\pi}^T \end{bmatrix}_M \begin{bmatrix} \Delta \underset{\sim}{u}' + \Delta \underset{\sim}{u}'_{\pi} \\ \Delta \underset{\sim}{r}'^{*} \end{bmatrix}_M \qquad \begin{matrix} (2.2.44a) \\ (2.2.44b) \\ (2.2.44c) \end{matrix}$$

for Kinematics.

The incremental mesh additional forces can be expressed as, and dropping the subscript M from now onwards

$$\Delta \underset{\sim}{\pi} = \underline{Q}^T \, \Delta \underset{\sim}{X}' + \underline{P} \, \Delta \underset{\sim}{\delta}_{\pi} + \Delta \underset{\sim}{R}_{\pi}$$

using equation (2.1.42a) where $\underline{Q}$, $\underline{P}$ and $\Delta \underset{\sim}{R}_{\pi}$ are now block-diagonal matrices with elements defined in (2.1.38b), (2.1.42b) and (2.1.43), respectively.

Eliminating the incremental stress-resultants through (2.2.43a)

$$-\underline{P}\,\Delta\underline{\delta}_{\pi}-\Delta\underline{R}_{\pi} = \underline{Q}^T\underline{B}\;\Delta\underline{p}+\underline{Q}^T\underline{B}_o\;\Delta\underline{R}^{*}+(\underline{Q}^T\underline{B}_{o\pi}-\underline{I})\,\Delta\underline{\pi}$$

and adding the above equation to the system (2.2.43), the incremental Statics description becomes

$$\begin{bmatrix} \Delta\underline{X}' \\ -\Delta\underline{R}^{*} \\ -\underline{P}\,\Delta\underline{\delta}_{\pi}-\Delta\underline{R}_{\pi} \end{bmatrix} = \begin{bmatrix} \underline{B} & \underline{B}_o & \underline{B}_{o\pi} \\ \cdot & \underline{B}_r & \underline{B}_{r\pi} \\ \underline{Q}^T\underline{B} & \underline{Q}^T\underline{B}_o & \underline{Q}^T\underline{B}_{o\pi}-\underline{I} \end{bmatrix} \cdot \begin{bmatrix} \Delta\underline{p} \\ \Delta\underline{R}^{*} \\ \Delta\underline{\pi} \end{bmatrix} \qquad \begin{matrix} (2.2.45a) \\ (2.2.45b) \\ (2.2.45c) \end{matrix}$$

Eliminating the incremental additional deformations through (2.1.38a) written now for the mesh element, the incremental description of Kinematics emerges in the form

$$\begin{bmatrix} -\underline{B}^T\Delta\underline{R}_{u\pi} \\ \Delta\underline{r}^{*}-\underline{B}^T\Delta\underline{R}_{u\pi} \\ -\underline{B}_{o\pi}^T\Delta R_{u\pi} \end{bmatrix} = \begin{bmatrix} \underline{B}^T & \cdot & \underline{B}^T\underline{Q} \\ \underline{B}_o^T & \underline{B}_r^T & \underline{B}_o^T\underline{Q} \\ \underline{B}_{o\pi}^T & \underline{B}_{r\pi}^T & \underline{B}_{o\pi}^T\underline{Q}-\underline{I} \end{bmatrix} \begin{bmatrix} \Delta\underline{u}' \\ \Delta\underline{r}'^{*} \\ \Delta\underline{\delta}_{\pi} \end{bmatrix} \qquad \begin{matrix} (2.2.46a) \\ (2.2.46b) \\ (2.2.46c) \end{matrix}$$

Together with the member constitutive relations and the laws of connectivity, the descriptions of Statics and Kinematics in the above are quite appropriate to perform the non-linear analysis of a structure.

However some theoretical insight is secured and some computational effort will be saved if the dual dependent variables $\Delta\underline{\delta}_{\pi}$ and $\Delta\underline{\pi}$ are eliminated from the formulation.

Definitions (2.1.49b) and (2.2.40) enable us to evaluate a typical element of the matrix operating on the incremental additional forces in equation (2.2.45c)

$$[\underline{I}-\underline{Q}^T\underline{B}_{o\pi}]_{kl} = \begin{bmatrix} \delta_{kl} & \cdot \\ \cdot & \delta_{kl} \end{bmatrix} - \begin{bmatrix} \dfrac{s_k}{L_{c_k}} & -\dfrac{s_k}{L_{c_k}} & 1-c_k \\ -\dfrac{1}{L_k}+\dfrac{c_k}{L_k L_{c_k}} & \dfrac{1}{L_k}-\dfrac{c_k}{L_k L_{c_k}} & s_k \end{bmatrix} \begin{bmatrix} \cdot & -L_l \\ \cdot & -(1-\delta_{kl})L_l \\ \delta_{kl} & \cdot \end{bmatrix}, \quad k\leq l$$

yielding

$$[\underline{I}-\underline{Q}^T\underline{B}_{o\pi}]_{kl} = \delta_{kl}\left[\begin{array}{c|c} c_k & \frac{L_l}{L_{c_k}}\, s_k \\ \hline s_k & 1-\frac{L_l}{L_k}+\frac{L_l}{L_{c_k}}\, c_k \end{array}\right], \quad [\underline{I}-\underline{Q}^T\underline{B}_{o\pi}]_{kl}=\underline{0}$$

$$k\leq l \qquad\qquad\qquad\qquad k>l$$

Hence, matrix $\underline{I}-\underline{Q}^T\underline{B}_o$ is a non-singular block-diagonal matrix, whose generic element is defined by

$$[\underline{I}-\underline{Q}^T\underline{B}_{o\pi}]_j = \left[\begin{array}{c|c} c & \frac{L}{L_c}\, s \\ \hline -s & \frac{L}{L_c}\, c \end{array}\right]_j \tag{2.2.47a}$$

Premultiplying the generic submatrices $\underline{B}_{kl}$ and $\underline{B}_{okl}$, defined in (2.2.38) and (2.2.39), respectively, by the transpose of the submatrix $\underline{Q}_k$, and noticing, from Fig.2.16, that

$$x_2^{j+1}-x_2^{j} = L_j \sin\alpha_j = L_j\, s'_j \tag{2.2.48a}$$

and

$$x_3^{j+1}-x_3^{j} = L_j \cos\alpha_j = L_j\, c'_j \tag{2.2.48b}$$

we obtain

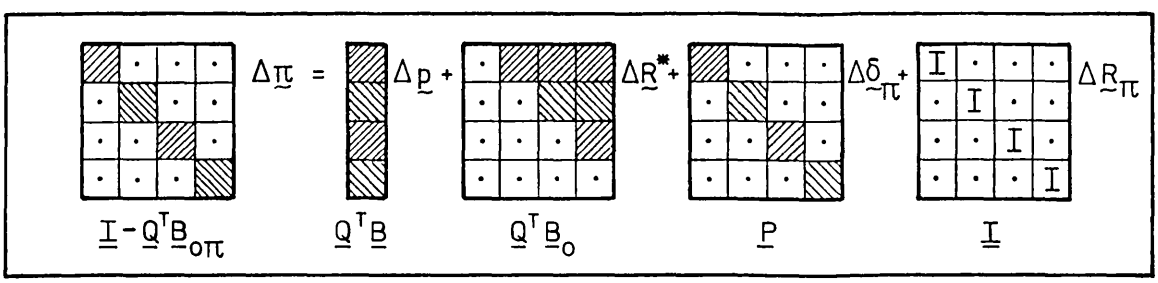
$$(\underline{Q}^T\underline{B})_j=(\underline{Q}^T\underline{B}_o)_{jk}= \left[\begin{array}{c|c|c} \cdot & \frac{L}{L_c}\, s\cdot c'-(1-c)s' & -\frac{L}{L_c}\, s\cdot s'-(1-c)c' \\ \hline \cdot & (-1+\frac{L}{L_c}c)c'-s\cdot s' & (1-\frac{L}{L_c}c)s'-s c' \end{array}\right]_j, \quad (\underline{Q}^T\underline{B}_o)_{jk}=0$$

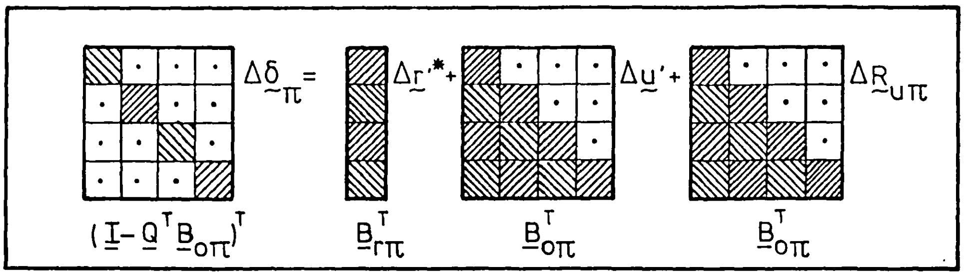
$$k>j \qquad\qquad\qquad\qquad k\leq j$$

(2.2.47b)

Hence, matrix $\underline{Q}^T\underline{B}_o$ is an upper-triangular matrix with zero block-diagonal elements; furthermore all the non-zero submatrices in the same row are equal.

Equations (2.2.45c) and (2.2.46c) can be represented diagramatically as follows





Let $\bar{B}_\pi$ be the inverse of matrix $I - Q^T B_{o\pi}$; matrix $\bar{B}_\pi$ is still block diagonal and its j-th element is defined by

$$\bar{B}_{\pi_j} = \begin{bmatrix} c & -s \\ \frac{L_c}{L}\,s & \frac{L_c}{L}\,c \end{bmatrix}_j \qquad (2.2.47c)$$

Equation (2.2.46c) may now be re-written as

$$\Delta \delta_\pi = \bar{B}^T_\pi [\, B^T_{o\pi}\,\Delta u' + B^T_{r\pi}\,\Delta r'^* + B^T_{o\pi}\,\Delta R_{u\pi}\,] \qquad (2.2.49a)$$

which together with equation (2.2.45c) gives

$$\Delta \pi = \bar{B}_\pi [\, Q^T B\,\Delta p + Q^T B_o\,\Delta R^* \,] + \bar{B}_\pi P \bar{B}^T_\pi [\, B^T_{o\pi}\,\Delta u' + B^T_{r\pi}\,\Delta r'^* \,] + [\, \bar{B}_\pi\,\Delta R_\pi + \bar{B}_\pi\, P B_\pi B^T_{o\pi}\,\Delta R_{u\pi} \,] \qquad (2.2.49b)$$

where

$$(\bar{B}_\pi Q^T B)_j = (\bar{B}_\pi Q^T B_o)_{jk} = \begin{bmatrix} \cdot & s'-s'' & c'-c'' \\ \cdot & c' - \frac{L_c}{L}c'' & -s' + \frac{L_c}{L}s'' \end{bmatrix}_j \;(k>j), \quad (\bar{B}_\pi Q^T B_o)_{jk} = 0 \;(k \leq j) \qquad (2.2.47d)$$

$$(\bar{B}_\pi P \bar{B}^T_\pi)_j = \begin{bmatrix} \cdot & -\frac{X_3}{L} \\ -\frac{X_3}{L} & -\frac{L_c}{L}\cdot\frac{X_2}{L} \end{bmatrix}_j \qquad (2.2.47e)$$

$$(B_{o\pi}\bar{B}_\pi)_{jk} = \begin{bmatrix} -L_{c_k} s_k & -L_{c_k} c_k \\ (\delta_{jk}-1) L_{c_k} s_k & (\delta_{jk}-1) L_{c_k} c_k \\ \delta_{jk} c_k & -\delta_{jk} s_k \end{bmatrix} \;(k \geq j) \qquad (B_{r\pi}\bar{B}_\pi)_j = \begin{bmatrix} -L_{c_j} s_j & -L_{c_j} c_j \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$(B_{o\pi}\bar{B}_\pi)_{jk} = 0$ if $k<j$ (summation convention inactive)

(2.2.47f,g)

and $$s''_j = \sin\,(\alpha_j - \rho_j) \qquad (2.2.50a)$$

$$c''_j = \cos\,(\alpha_j - \rho_j) \qquad (2.2.50b)$$

Eliminating the additional forces $\Delta\underset{\sim}{\pi}$ and their displacements $\Delta\underset{\sim}{\delta}_\pi$ in equations (2.2.45a,b) and (2.2.46a,b) through equations (2.2.49b) and (2.2.49a), respectively, the following descriptions emerge for Statics

$$-\begin{bmatrix}\Delta\underset{\sim}{X}'_\pi \\ \hline \Delta\underset{\sim}{R}'^*_\pi\end{bmatrix} - \left[\begin{array}{c|c}\underline{\mathbb{K}}_{uu} & \underline{\mathbb{K}}_{ur} \\ \hline \underline{\mathbb{K}}^T_{ur} & \underline{\mathbb{K}}_{rr}\end{array}\right]\begin{bmatrix}\Delta\underset{\sim}{u}' \\ \hline \Delta\underset{\sim}{r}'^*\end{bmatrix} + \begin{bmatrix}\Delta\underset{\sim}{X}' \\ \hline -\Delta\underset{\sim}{R}'^*\end{bmatrix} = \left[\begin{array}{c|c}\underline{\mathbb{B}} & \underline{\mathbb{B}}_o \\ \hline \underline{\mathbb{B}}_r & \underline{\mathbb{B}}_{or}\end{array}\right]\begin{bmatrix}\Delta\underset{\sim}{p} \\ \hline \Delta\underset{\sim}{R}^*\end{bmatrix} \qquad \begin{matrix}(2.2.51a)\\(2.2.51b)\end{matrix}$$

and Kinematics

$$-\begin{bmatrix}\Delta\underset{\sim}{v}_\pi \\ \hline \Delta\underset{\sim}{r}^*_\pi\end{bmatrix} + \begin{bmatrix}\cdot \\ \hline \Delta\underset{\sim}{r}^*\end{bmatrix} = \left[\begin{array}{c|c}\underline{\mathbb{B}}^T & \underline{\mathbb{B}}^T_r \\ \hline \underline{\mathbb{B}}^T_o & \underline{\mathbb{B}}^T_{or}\end{array}\right]\begin{bmatrix}\Delta\underset{\sim}{u}' \\ \hline \Delta\underset{\sim}{r}'^*\end{bmatrix} \qquad \begin{matrix}(2.2.52a)\\(2.2.52b)\end{matrix}$$

We note

$$\underline{\mathbb{B}} = \underline{B} + \underline{B}_{o\pi}\,\underline{\bar{B}}_\pi\,\underline{Q}^T\underline{B} \qquad (2.2.47h)$$

$$\underline{\mathbb{B}}_o = \underline{B}_o + \underline{B}_{o\pi}\,\underline{\bar{B}}_\pi\,\underline{Q}^T\underline{B}_o \qquad (2.2.47i)$$

$$\underline{\mathbb{B}}_r = \underline{B}_{r\pi}\,\underline{\bar{B}}_\pi\,\underline{Q}^T\underline{B}$$

$$\underline{\mathbb{B}}_{or} = \underline{B}_r + \underline{B}_{r\pi}\,\underline{\bar{B}}_\pi\,\underline{Q}^T\underline{B}_o \qquad (2.2.47j)$$

$$\underline{\mathbb{K}}_{uu} = \underline{B}_{o\pi}\,\underline{\bar{B}}_\pi\,\underline{P}\,\underline{\bar{B}}^T_\pi\,\underline{B}^T_{o\pi} = \underline{\mathbb{K}}^T_{uu} \qquad (2.2.47l)$$

$$\underline{\mathbb{K}}_{ur} = \underline{B}_{o\pi}\,\underline{\bar{B}}_\pi\,\underline{P}\,\underline{\bar{B}}^T_\pi\,\underline{B}^T_{r\pi} \qquad (2.2.47m)$$

$$\underline{\mathbb{K}}_{rr} = \underline{B}_{r\pi}\,\underline{\bar{B}}_\pi\,\underline{P}\,\underline{\bar{B}}^T_\pi\,\underline{B}^T_{r\pi} = \underline{\mathbb{K}}^T_{rr} \qquad (2.2.47n)$$

$$\Delta\underset{\sim}{X}'_\pi = \underline{\mathbb{K}}_{uu}\,\Delta\underset{\sim}{R}_{u\pi} + \underline{B}_{o\pi}\,\underline{\bar{B}}_\pi\,\Delta\underset{\sim}{R}_\pi \qquad (2.2.53a)$$

$$\Delta\underset{\sim}{R}'^*_\pi = \underline{\mathbb{K}}^T_{ur}\,\Delta\underset{\sim}{R}_{u\pi} + \underline{B}_{r\pi}\,\underline{\bar{B}}_\pi\,\Delta\underset{\sim}{R}_\pi \qquad (2.2.53b)$$

$$\Delta\underset{\sim}{v}_\pi = \underline{\mathbb{B}}^T\,\Delta\underset{\sim}{R}_{u\pi} \qquad (2.2.53c)$$

$$\Delta\underset{\sim}{r}^*_\pi = \underline{\mathbb{B}}^T_o\,\Delta\underset{\sim}{R}_{u\pi} \qquad (2.2.53d)$$

The typical elements of the above matrices are summarized in (2.2.47f,g) and (2.2.58); the latter were obtained through definitions (2.2.47) and (2.2.38) to (2.2.42).

In particular we found

$$\underline{\mathbb{B}}_r = \begin{bmatrix} \cdot & -L_\gamma c'_\gamma + L_{c_\gamma} c''_\gamma & L_\gamma s'_\gamma - L_{c_\gamma} s''_\gamma \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

the repeated indices indicating a summation in the range

$$1 \leq \gamma \leq n$$

where n is the total number of branches of the generalized mesh.

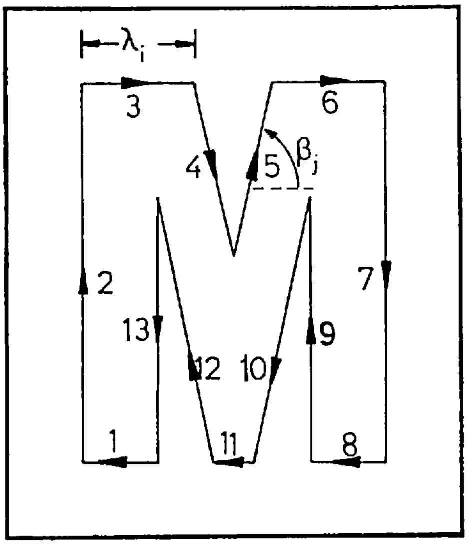


FIGURE 2.18

Consider the directed mesh illustrated in Fig.2.18 where λ_j and β_j represent respectively the j-th branch length and inclination with respect to the horizontal, and measured positively anti-clockwise; let $\bar{\beta}_j$ be the equivalent angle in the first quadrant.

Then

$$\lambda_\gamma \cos\beta_\gamma = \lambda_3+\lambda_6+\lambda_4\cos\bar{\beta}_4+\lambda_5\cos\bar{\beta}_5 - $$
$$-[\lambda_1+\lambda_8+\lambda_{11}+\lambda_{10}\cos\bar{\beta}_{10}+\lambda_{12}\cos\bar{\beta}_{12}]$$

That is, $\lambda_\gamma \cos\beta_\gamma$ represents the net horizontal projection of the connected mesh and consequently

$$\sum_{\gamma=1}^{13} \lambda_\gamma \cos\beta_\gamma = 0$$

Similarly, $\lambda_\gamma \sin\beta_\gamma$ represents the net vertical projection of the connected mesh and

$$\sum_{\gamma=1}^{13} \lambda_\gamma \sin\beta_\gamma = 0$$

Hence, whether the mesh is undeformed

$$\lambda_j = L_j \ , \ \beta_j = \alpha_j$$

or deformed $\qquad \lambda_j = L_{c_j}, \ \beta_j = \alpha_j - \rho_j$

we will always find that

$$\sum_{\gamma=1}^{n} L_{\gamma}\cdot c'_{\gamma} = \sum_{\gamma=1}^{n} L_{\gamma}\cdot s'_{\gamma} = \sum_{\gamma=1}^{n} L_{c_{\gamma}}\cdot c''_{\gamma} = \sum_{\gamma=1}^{n} L_{c_{\gamma}}\cdot s''_{\gamma} = 0$$

and therefore $$\underline{\mathbb{B}}_r = \underline{0} \qquad (2.2.54)$$

Furthermore it is possible to prove that the following identities hold for any mesh element M

$$\underline{\mathbb{B}}_M = \underline{S}_M \qquad (2.2.55a)$$

$$\underline{\mathbb{B}}_{orM} = \underline{S}_{rM} \qquad (2.2.55b)$$

$$\underline{\mathbb{B}}_{oM} = \underline{S}_{oM} \qquad (2.2.55c)$$

the matrices $\underline{S}_M$, $\underline{S}_{rM}$ and $\underline{S}_{oM}$ being respectively defined in (2.2.1), (2.2.5) and (2.2.6) for the four-branch rectangular mesh illustrated in Fig.2.6, where now L_{c_m} and ρ_m represent respectively the mesh members chord length and rotation.

Enforcing (2.2.54) in equations (2.2.51) and (2.2.52) we find the following incremental description for Statics and Kinematics:

$$\begin{bmatrix} \underline{\mathbb{K}}_{uu} & \underline{\mathbb{K}}_{ur} & \underline{\mathbb{B}} & \underline{\mathbb{B}}_o \\ \underline{\mathbb{K}}_{ur}^T & \underline{\mathbb{K}}_{rr} & \cdot & \underline{\mathbb{B}}_{or} \\ \underline{\mathbb{B}}^T & \cdot & \cdot & \cdot \\ \underline{\mathbb{B}}_o^T & \underline{B}_{or}^T & \cdot & \cdot \end{bmatrix}_M \begin{bmatrix} \Delta \underline{u}' \\ \Delta \underline{r}'^* \\ \Delta \underline{p} \\ \Delta \underline{R}^* \end{bmatrix}_M = \begin{bmatrix} \Delta \underset{\sim}{X}' \\ -\Delta \underset{\sim}{R}'^* \\ \cdot \\ \Delta \underset{\sim}{r}^* \end{bmatrix}_M - \begin{bmatrix} \Delta \underset{\sim}{X}'_{\pi} \\ \Delta \underset{\sim}{R}'^*_{\pi} \\ \Delta \underset{\sim}{v}_{\pi} \\ \Delta \underset{\sim}{r}^*_{\pi} \end{bmatrix}_M \qquad \begin{matrix} (2.2.56a) \\ (2.2.56b) \\ (2.2.57a) \\ (2.2.57b) \end{matrix}$$

STATICS (2.2.56a, b) — KINEMATICS (2.2.57a, b)

INCREMENTAL MESH DESCRIPTION

2.2.4 Perturbation Analysis

Replacing in the linear relations (2.2.56) and (2.2.57) the incremental variables by their expanded forms (2.1.51), the variation on the biactions, mesh reaction forces and their displacements being replaced respectively by

$$\Delta \underset{\sim}{p} = \sum_{i=1}^{\infty} \underset{\sim}{p}_i \frac{\varepsilon^i}{i!} \qquad (2.2.59a)$$

GENERALIZED MESH SUBMATRICES

$$\underline{\mathbb{B}}_j \underset{[j;n]}{=} \left[\begin{array}{c|c|c} 1 & -x_3^j - b_1 & x_2^j + b_2 \\ \hline 1 & -x_3^{j+1} - b_1' & x_2^{j+1} + b_2' \\ \hline \cdot & -s_j'' & -c_j'' \end{array}\right] \qquad (2.2.58a)$$

$$\underline{\mathbb{B}}_{or_k} \underset{[1;k]}{=} \left[\begin{array}{c|c|c} 1 & x_3^k - b_1 & -x_2^k + b_2 \\ \hline \cdot & 1 & \cdot \\ \hline \cdot & \cdot & 1 \end{array}\right] \qquad (2.2.58b)$$

$$\underline{\mathbb{B}}_{o_{jk}} \underset{\substack{j<k \\ [j;k]}}{=} \left[\begin{array}{c|c|c} 1 & x_3^k - x_3^j - b_1 & -x_2^k + x_2^j + b_2 \\ \hline 1 & x_3^k - x_3^{j+1} - b_1' & -x_2^k + x_2^{j+1} + b_2' \\ \hline \cdot & -s_j'' & -c_j'' \end{array}\right]$$

$$\underline{\mathbb{B}}_{o_{jk}} = \underline{0} \text{ if } j \geq k$$

(2.2.58c)

$b_1 = L_\gamma c_\gamma' - L_{c_\gamma} c_\gamma''$ $b_1' = b_1 - L_j c_j' + L_{cj} c_j''$
$b_2 = L_\gamma s_\gamma' - L_{c_\gamma} s_\gamma''$ $b_2' = b_2 - L_j s_j' + L_{cj} s_j''$

$$\underline{\mathbb{K}}_{uu_{jk}} = \left[\begin{array}{c|c|c} L_{c_\gamma} X_2^\gamma & (1-\delta_{k\gamma}) L_{c_\gamma} X_2^\gamma & x_3^k \\ \hline (1-\delta_{j\gamma}) L_{c_\gamma} X_2^\gamma & (1-\delta_{j\gamma})(1-\delta_{k\gamma}) L_{c_\gamma} X_2^\gamma & \delta_{k\gamma}(1-\delta_{j\gamma}) X_3^\gamma \\ \hline \delta_{j\gamma} X_3^\gamma & \delta_{j\gamma}(1-\delta_{k\gamma}) X_3^\gamma & \cdot \end{array}\right]$$

$[k;n]$ if $j \leq k$; $[j;n]$ if $k \leq j$

(2.2.58d)

$$\underline{\mathbb{K}}_{rr} \underset{[1;n]}{=} \left[\begin{array}{c|c|c} L_{c_\gamma} X_2^\gamma & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \end{array}\right] \qquad (2.2.58e)$$

Summation convention adopted over the repeated index γ (only) within the prescribed range[ ];   j: block row   k: block column   n: total number of branches of the generalized mesh

$$\underline{\mathbb{K}}_{ur_j} \underset{[j;n]}{=} \left[\begin{array}{c|c|c} L_{c_\gamma} X_2^\gamma & \cdot & \cdot \\ \hline (1-\delta_{j\gamma}) L_{c_\gamma} X_2^\gamma & \cdot & \cdot \\ \hline \delta_{j\gamma} X_3^\gamma & \cdot & \cdot \end{array}\right] \qquad (2.2.58f)$$

$$\Delta \underset{\sim}{R}'^* = \sum_{i=1}^{\infty} \underset{\sim}{R}_i'^* \frac{\varepsilon^i}{i!} \qquad (2.2.59b)$$

and

$$\Delta \underset{\sim}{r}'^* = \sum_{i=1}^{\infty} \underset{\sim}{r}_i'^* \frac{\varepsilon^i}{i!} \qquad (2.2.59c)$$

the perturbed mesh description of Statics and Kinematics is found to be

$$
\left[\begin{array}{c:c:c:c}
\underline{\mathbb{K}}_{uu} & \underline{\mathbb{K}}_{ur} & \underline{\mathbb{B}} & \underline{\mathbb{B}}_{o} \\ \hdashline
\underline{\mathbb{K}}_{ur}^{T} & \underline{\mathbb{K}}_{rr} & \cdot & \underline{\mathbb{B}}_{or} \\ \hdashline
\underline{\mathbb{B}}^{T} & \cdot & \cdot & \cdot \\ \hdashline
\underline{\mathbb{B}}_{o}^{T} & \underline{\mathbb{B}}_{or}^{T} & \cdot & \cdot
\end{array}\right]_{M}
\left[\begin{array}{c}
u'_{i} \\ \hdashline \underset{\sim}{r}{}'^{*}_{i} \\ \hdashline \underset{\sim}{p}_{i} \\ \hdashline \underset{\sim}{R}{}^{*}_{i}
\end{array}\right]_{M}
=
\left[\begin{array}{c}
\underset{\sim}{X}{}'_{i} \\ \hdashline -\underset{\sim}{R}{}'^{*}_{i} \\ \hdashline \cdot \\ \hdashline \underset{\sim}{r}{}^{*}_{i}
\end{array}\right]_{M}
-
\left[\begin{array}{c}
\underset{\sim}{X}{}'_{\pi_i} \\ \hdashline \underset{\sim}{R}{}'^{*}_{\pi_i} \\ \hdashline \underset{\sim}{v}_{\pi_i} \\ \hdashline \underset{\sim}{r}{}^{*}_{\pi_i}
\end{array}\right]_{M}
\begin{array}{l}
\text{(2.2.60a)} \\ \text{(2.2.60b)} \\ \text{(2.2.61a)} \\ \text{(2.2.61b)}
\end{array}
$$

STATICS (2.2.60a, b) — KINEMATICS (2.2.61a, b)

PERTURBED MESH DESCRIPTION

in which from (2.2.53) and (2.1.51e,f)

$$\underset{\sim}{X}{}'_{\pi_i} = \underline{\mathbb{K}}_{uu}\ \underset{\sim}{R}_{u\pi_i} + \underline{B}_{o\pi}\ \underline{\bar{B}}_{\pi}\ \underset{\sim}{R}_{\pi_i}$$

$$\underset{\sim}{R}{}'^{*}_{\pi_i} = \underline{\mathbb{K}}_{ur}^{T}\ \underset{\sim}{R}_{u\pi_i} + \underline{B}_{r\pi}\ \underline{\bar{B}}_{\pi}\ \underset{\sim}{R}_{\pi_i}$$

$$\underset{\sim}{v}_{\pi_i} = \underline{\mathbb{B}}^{T}\ \underset{\sim}{R}_{u\pi_i}$$

$$\underset{\sim}{r}{}^{*}_{\pi_i} = \underline{\mathbb{B}}_{o}^{T}\ \underset{\sim}{R}_{u\pi_i}$$

the elements of $\underset{\sim}{R}_{u\pi_i}$ and $\underset{\sim}{R}_{\pi_i}$ being defined in (2.1.58b) and (2.1.60b), respectively.

2.2.5 Asymptotic Analysis

The asymptotic mesh description of Statics and Kinematics can be defined by substituting in the finite descriptions (2.2.35) and (2.2.36) the variables by their expanded forms (2.1.63) together with the biactions, mesh reaction forces and their displacements in the form

$$\underset{\sim}{p} = \sum_{i=0}^{\infty} \underset{\sim}{p}_{i} \frac{\varepsilon^{i}}{i!} \tag{2.2.62a}$$

$$\underset{\sim}{R}{}'^{*} = \sum_{i=0}^{\infty} \underset{\sim}{R}{}'^{*}_{i} \frac{\varepsilon^{i}}{i!} \tag{2.2.62b}$$

and

$$\underset{\sim}{r}{}'^{*} = \sum_{i=1}^{\infty} \underset{\sim}{r}{}'^{*}_{i} \frac{\varepsilon^{i}}{i!} \tag{2.2.62c}$$

and eliminating the mesh additional forces and their displacements through (2.1.68) and (2.1.66), yielding

$$\left[\begin{array}{cc|c|c} \underline{\mathbb{K}}_{uu} & \underline{\mathbb{K}}_{ur} & \underline{\mathbb{B}} & \underline{\mathbb{B}}_{o} \\ \hline \underline{\mathbb{K}}_{ur}^{T} & \underline{\mathbb{K}}_{rr} & \cdot & \underline{\mathbb{B}}_{or} \\ \hline \underline{\mathbb{B}}^{T} & \cdot & \cdot & \cdot \\ \hline \underline{\mathbb{B}}_{o}^{T} & \underline{\mathbb{B}}_{or}^{T} & \cdot & \cdot \end{array}\right]_{M} \begin{bmatrix} \tilde{u}'_{i} \\ \tilde{r}'^{*}_{i} \\ \tilde{p}_{i} \\ \tilde{R}^{*}_{i} \end{bmatrix}_{M} = \begin{bmatrix} \tilde{X}'_{i} \\ -\tilde{R}'^{*}_{i} \\ \cdot \\ \tilde{r}^{*}_{i} \end{bmatrix}_{M} - \begin{bmatrix} \tilde{X}'_{\pi_i} \\ \tilde{R}'^{*}_{\pi_i} \\ \tilde{v}_{\pi_i} \\ \tilde{r}^{*}_{\pi_i} \end{bmatrix}_{M}$$

STATICS (2.2.63a) (2.2.63b)

KINEMATICS (2.2.64a) (2.2.64b)

ASYMPTOTIC MESH DESCRIPTION

where now

$$\underline{\mathbb{K}}_{uu} = \underline{B}_{o\pi}\ \underline{P}\ \underline{B}_{o\pi}^{T} = \underline{\mathbb{K}}_{uu}^{T}$$

$$\underline{\mathbb{K}}_{ur} = \underline{B}_{o\pi}\ \underline{P}\ \underline{B}_{r\pi}^{T}$$

$$\underline{\mathbb{K}}_{rr} = \underline{B}_{r\pi}\ \underline{P}\ \underline{B}_{r\pi}^{T} = \underline{\mathbb{K}}_{rr}^{T}$$

$$\tilde{X}'_{\pi_i} = \underline{\mathbb{K}}_{uu}\ \tilde{R}_{u\pi_i} + \underline{B}_{o\pi}\ \tilde{R}_{\pi_i}$$

$$\tilde{R}^{*}_{\pi_i} = \underline{\mathbb{K}}_{ur}^{T}\ \tilde{R}_{u\pi_i} + \underline{B}_{r\pi}\ \tilde{R}_{\pi_i}$$

$$\tilde{v}_{\pi_i} = -\underline{B}^{T}\ \tilde{R}_{u\pi_i}$$

$$\tilde{r}^{*}_{\pi_i} = \underline{B}_{o}^{T}\ \tilde{R}_{u\pi_i}$$

where $\underline{P}$, $\underline{R}_{u\pi_i}$ and $\underline{R}_{\pi_i}$ are now block matrices whose elements are respectively defined in (2.1.68d), (2.1.66b) and (2.1.68b,c).

It should be stressed again that the present formulation is designed to perform the asymptotic analysis of the special class of structures whose equilibrium paths branch from the original kinematically trivial path; hence in equations (2.2.63) and (2.2.64), the first term of the expanded form of any kinematic variable will always be zero, i.e.,

$$y_o = 0$$

where y_o represents a generic kinematic variable.

2.3 ALTERNATIVE FORMULATIONS

Most finite-element formulations, invariably in a nodal format, in geometrically non-linear analysis of structures develop from the application of certain variational principles in order to derive as directly as possible the associate load-displacement relationship.

As this implies the utilization of the structural material constitutive relations, we will postpone the comparison of the formulation presented herein with those in the literature, awaiting the derivation of the element constitutive relations that we will be adopting.

This section is primarily concerned with that class of formulations which make use of the so-called fictitious forces and deformations in order to extend the linear description of Statics and Kinematics to include some of the large displacements and deformations effects.

In general it is somewhat difficult to understand what in the literature is meant by a n-th order solution. A general trend is to describe as first order linear the solutions given by elementary mechanics; if the stability functions are included (in the constitutive relations), such solutions are often called first-order non-linear or second-order. If other non-linearities are taken into consideration, consistently or not, degrees are "added" to the order of the basic theory. Furthermore, as most formulations are not derived from first-principles, the specific influence on Statics and Kinematics of the assumptions made becomes ambiguous.

In order to compare the results obtained in the previous sections with other formulations, and bearing in mind, as shown by the definitions (2.1.20) and (2.1.23) of the additional forces and deformations which sinthetize the existing non-linearities, that the non-linearity of Statics and Kinematics is fundamentally related to the member chord rotation, the following definition, as artificial as any other, will be adopted herein: a finite formulation of Statics and Kinematics is said to be of order n if it is valid within a range of displacements such that the fundamental trignometric functions sine and cosine of the member chord rotation can be approximated by a power series of order not higher than the n-th.

Letting, in equation (2.1.20)

$$\sin\rho = 0\cdot\rho^0 + 1\cdot\rho + 0\cdot\rho^2 - \frac{1}{6}\cdot\rho^3 + \cdots$$

and

$$\cos\rho = 1\cdot\rho^0 + 0\cdot\rho - \frac{1}{2}\cdot\rho^2 + 0\cdot\rho^3 + \cdots$$

and replacing in equations (2.1.23) the additional forces displacements by

$$\delta_t = L_c\ [\ 0\cdot\rho^0 + 1\cdot\rho + 0\cdot\rho^2 - \frac{1}{6}\cdot\rho^3 + \cdots\ ]$$

$$L-\delta_n = L_c\ [\ 1\cdot\rho^0 + 0\cdot\rho - \frac{1}{2}\cdot\rho^2 + 0\cdot\rho^3 + \cdots\ ]$$

the following approximate solutions are found.

	APPROXIMATE FINITE FORMULATIONS		
	$u_2'=0$	$u_2'\neq 0$	
	ZEROTH-ORDER		
(2.3.1a,b)	$\pi_n = \pi_t = 0$		
(2.3.1c,e)	$u'^{1}_{1\pi} = u'^{2}_{1\pi} = u'_{2\pi} = 0$		
	FIRST-ORDER		
(2.3.2a)	$\pi_n=-\rho X_3$	$\pi_n=-\rho X_3$	(2.3.3a)
(2.3.2b)	$\pi_t=\rho X_2$	$\pi_t=(\frac{L_c}{L}-1)X_3+\rho X_2$	(2.3.3b)
(2.3.2c)	$u'^{1}_{1\pi}=-u'^{2}_{1\pi}=0$	$u'^{1}_{1\pi}=-u'^{2}_{1\pi}=\rho(1-\frac{L_c}{L})$	(2.3.3c)
(2.3.2d)	$u'_{2\pi}=0$	$u'_{2\pi}=0$	(2.3.3d)
	SECOND-ORDER		
(2.3.4a)	$\pi_n=-\rho X_3+\frac{1}{2}\rho^2X_2$	$\pi_n=-\rho X_3+\frac{1}{2}\rho^2X_2$	(2.3.5a)
(2.3.4b)	$\pi_t=\frac{1}{2}\rho^2X_3+\rho X_2$	$\pi_t=(\frac{1}{2}\rho^2-1+\frac{L_c}{L})X_3+\rho X_2$	(2.3.5b)
(2.3.4c)	$u'^{1}_{1\pi}=-u'^{2}_{1\pi}=0$	$u'^{1}_{1\pi}=-u'^{2}_{1\pi}=\rho(1-\frac{L_c}{L})$	(2.3.5c)
(2.3.4d)	$u'_{2\pi}=\frac{1}{2}\rho^2L$	$u'_{2\pi}=\frac{1}{2}\rho^2L_c$	(2.3.5d)

The Static and Kinematic descriptions presented previously, can be immediately specialized into the case of pin-jointed frames. As the bending moments X_1^i, and consequently the shear stress-resultant X_3, are zero, the corresponding columns (rows) of the nodal (mesh) static matrices can be removed; as the actual plus additional rotations $u_1'^{\,i}+u_{1\pi}'^{\,i}$, the dual variables of the moments, become superfluous, the corresponding rows (columns) of the nodal (mesh) description of Kinematics may also be removed, thus preserving Static-Kinematic Duality.

Hence, and from (2.1.20) and (2.1.23), the additional forces and deformations reduce respectively to

	PIN-JOINTED FRAMES		
	ADDITIONAL FORCES	ADDITIONAL DEFORMATIONS	
(2.3.6a)	$\pi_n=(1-\cos\rho)X_2$	$u'_{2\pi} = \delta_n - u'_2$	(2.3.7)
(2.3.6b)	$\pi_t=\sin\rho\cdot X_2$		

and the corresponding approximate solutions to

	APPROXIMATE SOLUTIONS: PIN-JOINTED FRAMES	
	$u'_2=0$	$u'_2\neq 0$
	ZEROTH-ORDER	
(2.3.8a,b)	$\pi_n=\pi_t=0$	
(2.3.8c)	$u'_{2\pi}=0$	
	FIRST-ORDER	
(2.3.9a-10a)	$\pi_n=0$	$\pi_n=0$
(2.3.9b-10b)	$\pi_t=\rho X_2$	$\pi_t=\rho X_2$
(2.3.9c-10c)	$u'_{2\pi}=0$	$u'_{2\pi}=0$
	SECOND-ORDER	
(2.3.11a-12a)	$\pi_n=\frac{1}{2}\rho^2 X_2$	$\pi_n=\frac{1}{2}\rho^2 X_2$
(2.3.11b-12b)	$\pi_t=\rho X_2$	$\pi_t=\rho X_2$
(2.3.11c-12c)	$u'_{2\pi}=\frac{1}{2}\rho^2 L$	$u'_{2\pi}=\frac{1}{2}\rho^2 L_c$

The concept of simulating a given effect by an artificial cause has had several applications in Mechanics.

In the field of non-linear structural analysis, Denke (1960) introduced the idea of replacing the actual non-linear equations by those derived from linear analysis together with corrective fictitious forces and deformations to simulate the effect of large displacements. Illustrating the principle for a pin-ended column, Denke (1964) found the following definitions for the fictitiuous forces and deformations

$$\pi_n = 0 \tag{2.3.13a}$$

$$\pi_t = \sin\rho\cdot X_2 \tag{2.3.13b}$$

$$u'_{2\pi} = L(1-\cos\rho) \tag{2.3.13c}$$

Lansing, Jones and Ratner (1961) and Warren (1962) developed (independently) a similar theory for the analysis of structures idealized into pin-jointed bars and constant shear panels.

The same concept was applied by Durret (1963) in the analysis of pin-jointed frameworks which were also studied by Griffin (1966); the latter checked the results obtained from first-principles by performing a parallel analysis based on the Principle of Virtual Work.

As referred to in Argyris (1964), Przemieniecki (1968) and Meek (1971), the concept of fictitious (also known in the literature as pseudo, initial, additional or Ersatz, and supplementary) forces and deformations has been generally understood as the natural way of extending the linear matrix force method of structural analysis to non-linear structures. Besides that fact, the formulations referred to in the above have also the following common-features:

i) Analysis of elastic pin-jointed frames

ii) Correction of Statics and Kinematics by means of first-order fictitious forces (2.3.9a,b) and second-order deformations (2.3.11c), respectively, and assuming

$$\delta_t = L\,\rho$$

iii) Elimination of the fictitious forces through their linear relationship to the stress-resultant X_2.

iv) Inversion, in the resulting equation, of the matrix affecting the stress-resultants by expanding it in a power series and neglecting terms of order higher than the second.

v) Iterative solution of the resulting non-linear governing equation.

Scarlat (1971) presented a relaxation method for the first-order analysis of elastic rigidly jointed frames in which joints are not allowed to displace (continuous beams). The analysis is performed on the undeformed structure subject to supplementary couples distributed along the axis, defined as the product of the axial force and the rotation at the section, evaluated from a preliminary linear analysis. Hence, Scarlat's fictitious forces are designed to simulate not the effect of large (rigid-body) displacements but the effect of (not too) large deformations; the outcome is an approximation of the stability functions, to which we will refer in section 3.1.

We mention next two formulations involving fictitious forces designed to analyze elastic systems under large displacements by the so-called displacement method; they are representative of a number of other works on the same subject, wherein the fictitious forces were used primarily as a tool in the numerical procedure of solution, rather than as a basis for further improvement of the theoretical formulation.

Haisler, Stricklin and Stebbins (1972) separate the strain energy in a linear (U_L) and a non-linear part (U_{NL}) and, after specializing for small strains the Principle of Virtual Work written in terms of the undeformed configuration, derived by Haisler (1970), obtain the following governing equation

$$\underline{K}\, \underset{\sim}{q} = \underset{\sim}{Q} - \underset{\sim}{Q}^*(\underset{\sim}{q})$$

where $\underline{K}$ is the usual linear stiffness matrix of the structure, obtained from the linear part of the strain energy U_L, $\underset{\sim}{Q}$ and $\underset{\sim}{q}$ are the generalized forces and displacements, respectively, and

$$\underset{\sim}{Q}^* = \frac{\partial}{\partial \underset{\sim}{q}} (U_{NL})$$

are pseudo generalized forces due to the non-linearities. The governing equation is then solved by the self-correcting initial-value formulation proposed by Stricklin, Haisler and Von Riesemann (1971) and Masset and Stricklin (1971).

Instead of separating the strain energy in its linear and non-linear parts, Oliveira (1974) starts by decomposing the displacement field $\underset{\sim}{u}$ of a finite element into two parts

$$\underset{\sim}{u} = \underset{\sim}{\tilde{u}} + \underset{\sim}{\hat{u}}$$

where $\underset{\sim}{\tilde{u}}$ represents a small displacement field. The element compatibility equation

$$\underset{\sim}{e} = \underline{D}\, \underset{\sim}{u}$$

where $\underset{\sim}{e}$ is the strain vector, is re-written as

$$\underset{\sim}{e} = \underline{D}_L \; (\underset{\sim}{u}-\hat{\underset{\sim}{u}}) + \underline{D}\hat{\underset{\sim}{u}}$$

where $\underline{D}_L$ is the linearized kinematic operator. Substituting the compatibility condition in the stress strain equation

$$\underset{\sim}{s} = \underline{H} \, [\, \underline{D}_L (\underset{\sim}{u}-\hat{\underset{\sim}{u}}) + \underline{D}\hat{\underset{\sim}{u}} \,]$$

and eliminating the stresses in the quilibrium condition, the equation governing the behaviour of the finite-element is found to be

$$\underset{\sim}{f} = \underline{E}\underline{H} \, [\, \underline{D}_L (\underset{\sim}{u}-\hat{\underset{\sim}{u}}) + \underline{D}\hat{\underset{\sim}{u}} \,]$$

where $\underset{\sim}{f}$ is the vector of external forces. Assuming that the strains are very small and that the dimensions are such that the variations of rotations are also very small within each element, then

$$\underline{D} \, \hat{\underset{\sim}{u}} \simeq \underset{\sim}{0} \qquad (2.3.14\,a)$$

and letting

$$\underline{E} = \underline{D}_L^T \qquad (2.3.14\,b)$$

the governing equation reduces to

$$\underline{K} \, \underset{\sim}{u} = \underset{\sim}{f} + \underset{\sim}{f}^*$$

where

$$\underline{K} = \underline{D}_L^T \, \underline{H} \, \underline{D}_L$$

is the element stiffness matrix and

$$\underset{\sim}{f}^* = \underline{K} \, \hat{\underset{\sim}{u}}$$

is the fictitious force vector. The problem of evaluating $\hat{\underset{\sim}{u}}$, a displacement field such that $\underset{\sim}{u}-\hat{\underset{\sim}{u}}$ is a small displacement field, is solved by assuming that the load varies by successive small increments producing variations on the displacements so small that the displacement field at state i-1 can be taken for field $\hat{\underset{\sim}{u}}$ of state i.

Smith (1974) presented an unified theory on the analysis and synthesis of linear structures; plastic limit analysis and synthesis, shakedown analysis and elastoplastic deformation analysis and their associate variational principles emerge naturally and in full generality when the relevant vectorial relations are encoded and interpreted by mathematical programming theory.

In order to extend the formulation to include the first-order effects of finite displacements, Smith (1975,1977) corrected the mesh and nodal descriptions of linear Statics by loading each member of the assembled structure with the additional forces

$$\pi_n = 0 \quad \text{and} \quad \pi_t = \frac{X_2}{L}\,\delta_t$$

and, by preserving Static-Kinematic Duality, found that the linear description of Kinematics had to include the additional force displacements δ_t; additional deformations were not considered. The formulation being suggested herein was decisively influenced by the above mentioned works of Smith; they provided the solid basis and the conceptual framework from which it developed. The four quadratic programs of first-order non-linear elastoplastic deformation analysis proposed by Smith (1975,1977), which include as special cases the so-called force and displacement methods of analysis, will be referred to in more detail later in the text.

Specifically on the matter of defining the additional forces, the approach presented in Oliveira (1974) suggested the process adopted in the previous sections of obtaining the general and exact definitions (2.1.20). However, instead of introducing the linear operator by imposing restrictions on the deformation and displacement fields in order to justify the assumptions (2.3.14) which in turn implied the utilization of an incremental procedure of solution with the inherent problems of convergence, we opted for the very basic approach of forcibly introducing the linear operator in the static and kinematic descriptions and collecting all the resulting non-linear terms in the broad definition of additional forces and deformations.

2.4 INTERNAL RELEASES

The discussion on Statics and Kinematics, as undertaken in the previous sections 2.1 and 2.2, was based on the assumption of continuity of the substructure.

The mesh and nodal descriptions must now be extended to include the effects of internal releases which can either represent mechanical devices actually existing in the structural system or be used to simulate special effects as, for instance, material and geommetric imperfections.

When referring to the discretization of the structure, one class of points where we insisted that a node had to be placed were those where the continuity of the structure was interrupted by the presence of internal releases; hence, to contemplate every possible arrangements of releases in a structure we must assume that a release may exist at the immediate neighbourhood of every critical section of the substructure.

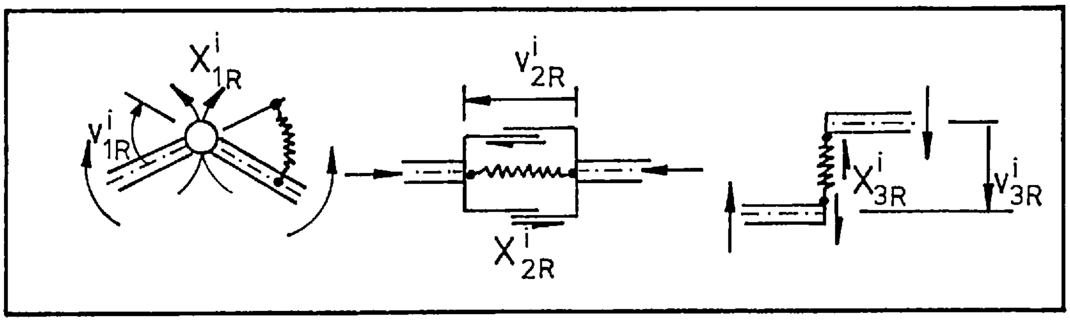


FIGURE 2.19

Illustrated in the figure above are the three types of release that may be encountered in planar frames. The forces developing at the i-th release, which we collect in the RELEASE FORCE vector $\underline{X}^i_R$

$$\underline{X}^i_R = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}^i_R$$

are considered positive if equilibrated by negative stress-resultants evolving at the neighbouring sections. By RELEASE DISLOCATIONS, which we group in the vector

$$\underline{v}^i_R = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^i_R$$

we understand the relative displacements of the faces of the release produced by the corresponding positive release force.

These three types of release devices can be combined in six different ways, which may be dissociated into three groups represented in Fig.2.20 depending on the relative position of the bending release. If the bending release is adjacent to the node, as in Fig.2.20(a), the rotation, when the structure deforms, of the shear and thrust release devices is controlled by the movement of the chord of the member. If the bending release is adjacent to

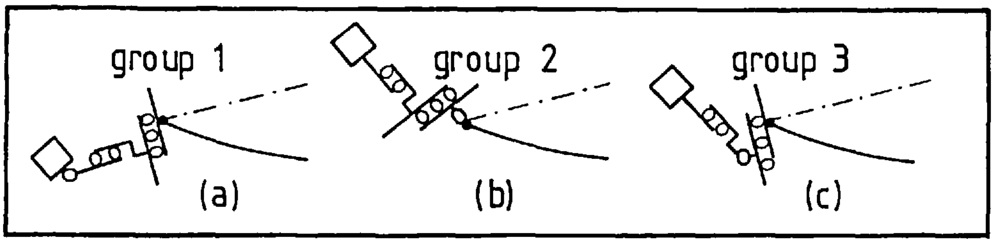


FIGURE 2.20

a critical section, as in Fig.2.20(b), the rotation of the shear and thrust releases is that of the neighbouring node. In the third type of combination, represented in Fig.2.20(c), the rotation of one of those force releases is controlled either by the rotation at the node or of the member chord. The results to be presented were derived considering combinations of the type of group 1; similar formulations would be obtained when considering the remaining combinations.

The release constitutive relations, which associate the release forces and dislocations through a causality condition, are given in section 4.1. An idealized elastoplastic behaviour covers a wide range of situations an analyst may wish to simulate, from static releases as the free-releases used in section 2.2, the faces of the release being connected by an elastic spring of infinite flexibility, to kinematic releases such as some safety devices controlling levels of stress which can be simulated by joining the faces of the release by a rigid-plastic spring.

2.4.1 The Extended Nodal Description

The existing releases at the ends of a typical member m can be incorporated in the kinematic description treating these releases as extra degrees of freedom besides the six degrees represented by the nodal displacements $\underset{\sim}{r}^{*}_{m}$. In other words, the member compatibility conditions may be defined by superimposing the effects of the member nodal displacements, assuming that the releases are fixed, as quantified in (2.1.7), to the effects of the release dislocations $\underset{\sim}{v}_{R_m}$ evaluated by imposing that the member nodes remain undisplaced.

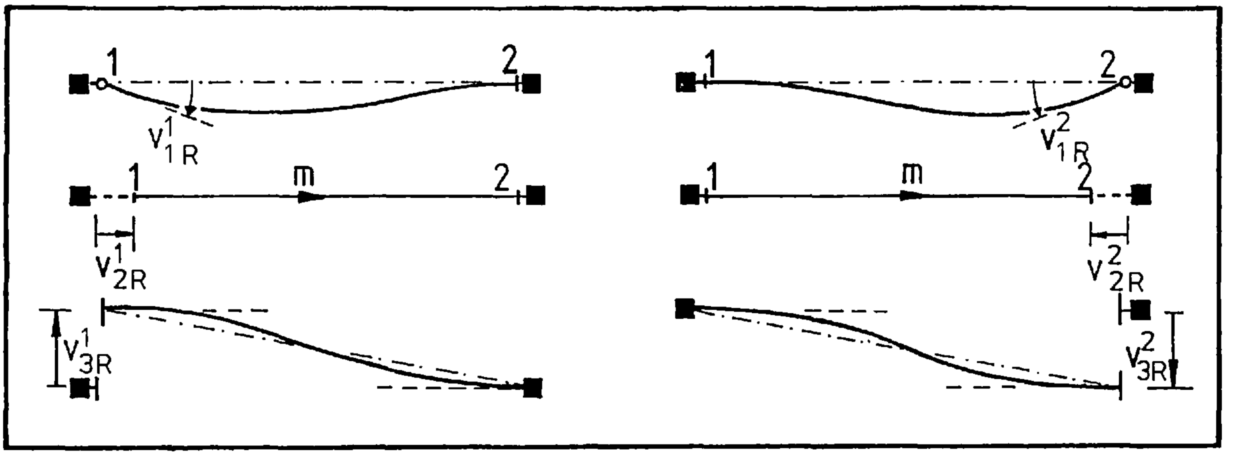


FIGURE 2.21

The member deformations due to the release dislocations are defined by

$$\underset{\sim}{u}'_m = \underset{\sim}{K}'_m \; \underset{\sim}{v}_{R_m} \tag{2.4.1}$$

where, from the figure above,

$$\underset{\sim}{K}'_m = \left[\begin{array}{ccc|ccc} 1 & \cdot & k_{13} & \cdot & \cdot & k_{16} \\ \cdot & \cdot & k_{23} & 1 & \cdot & k_{26} \\ \cdot & 1 & k_{33} & \cdot & 1 & k_{36} \end{array}\right]_m \tag{2.4.2a}$$

and

$$k_{1j} = -k_{2j} = -\frac{1}{v^i_{3R}} \text{ arc tan } \frac{v^i_{3R}}{L} \tag{2.4.2b}$$

$$k_{3j} = \frac{L}{v^i_{3R}} \left\{ 1 - \left[1+\left(\frac{v^i_{3R}}{L}\right)^2 \right]^{\frac{1}{2}} \right\} \tag{2.4.2c}$$

where $j = 3 \cdot i$ and $i = 1$ or 2.

The extended nodal description of Kinematics can now be defined by combining the compatibility conditions (2.1.7) and (2.4.1)

$$\underset{\sim}{u}'_m = \underline{K}_m \, \underset{\sim}{r}^*_m + \underline{K}'_m \, \underset{\sim}{v}_{R_m} \qquad (2.4.3)$$

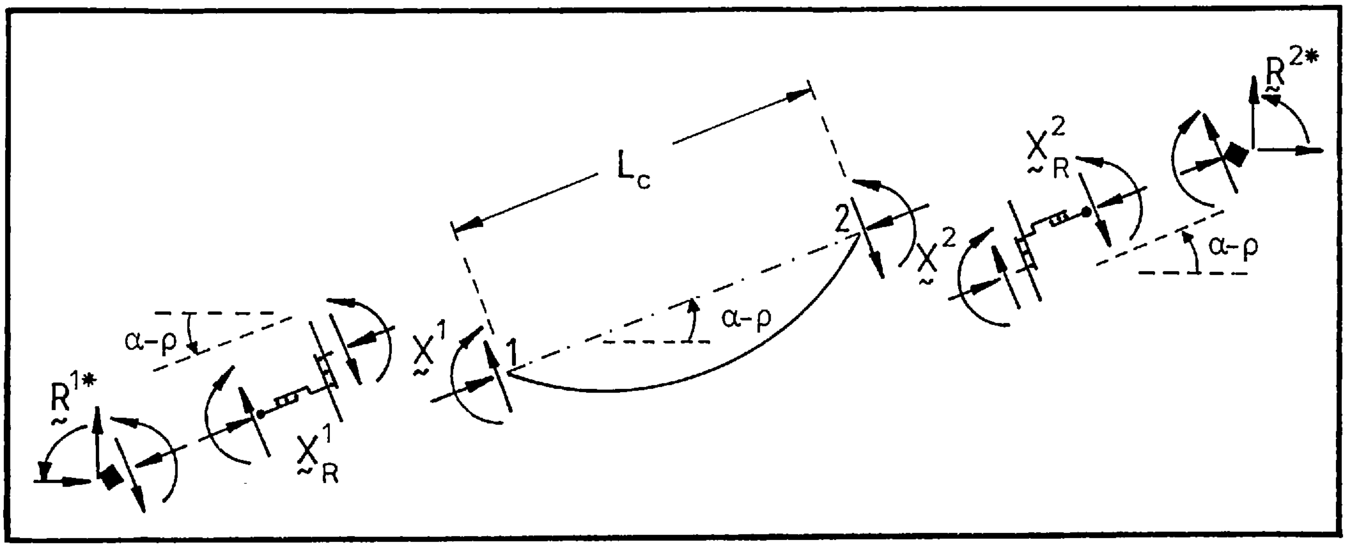


FIGURE 2.22

The derivation of the equilibrium equations based on the free-body diagram represented in Fig.2.22, can be performed in a manner similar to the one used in section 2.1, yielding

$$\underset{\sim}{R}^*_m = (\underline{S}_m + \underline{S}_{R_m}) \, \underset{\sim}{X}'_m \qquad (2.4.4)$$

where matrix $\underline{S}_m$ is defined in (2.1.14) and

$$\underline{S}_{R_m} = \left[\begin{array}{c|c|c} -v^1_{2R}/L_c & v^1_{2R}/L_c & -v_{3R} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \hdashline -v^2_{2R}/L_c & v^2_{2R}/L_c & -v^2_{3R} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}\right]_m$$

By definition, the (group 1) release forces are given by

$$X^i_{1R_m} = -(-1)^i \, R^{i*}_1 \qquad (i=1,2)$$

$$X^1_{2R_m} = X^2_{2R_m} = -X_{2_m}$$

$$X^1_{3R_m} = X^2_{3R_m} = \frac{1}{L_{c_m}}(X^1_1 - X^2_1)_m$$

or in terms of the independent stress-resultants

$$-\underset{\sim}{X}_{R_m} = \underline{S}'_m \ \underset{\sim}{X}'_m \qquad (2.4.5)$$

where

$$\underline{S}'_m = \left[\begin{array}{c|c|c} 1+v^1_{2R}/L_c & -v^1_{2R}/L_c & v^1_{3R} \\ \cdot & \cdot & 1 \\ -1/L_c & 1/L_c & \cdot \\ \hline -v^2_{2R}/L_c & 1+v^2_{2R}/L_c & -v^2_{3R} \\ \cdot & \cdot & 1 \\ -1/L_c & 1/L_c & \cdot \end{array}\right]_m$$

After enforcing the usual assumptions in linear Statics and Kinematics, matrix $\underline{K}'_m$ and the transpose of matrix $\underline{S}'_m$ reduce to

$$\underline{A}'_m = \left[\begin{array}{ccc|ccc} 1 & \cdot & -1/L & \cdot & \cdot & -1/L \\ \cdot & \cdot & 1/L & 1 & \cdot & 1/L \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \end{array}\right]_m \qquad (2.4.6)$$

and we re-write equation (2.4.5) as

$$-\underset{\sim}{X}_{R_m} = \underline{A}'^T_m \ \underset{\sim}{X}'_m - \underline{A}'^T_{R_m} \underset{\sim}{\pi}'_m \qquad (2.4.7)$$

where the linear operator $\underline{A}'_{R_m}$ is defined by

$$\underline{A}'_{R_m} = \left[\begin{array}{ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right] \qquad (2.4.8)$$

and the ADDITIONAL RELEASE FORCES, represented in Fig.2.23, by

$$\underset{\sim}{\pi}'_m = \left[\begin{array}{c} \underset{\sim}{\pi}'^1 \\ \hline \underset{\sim}{\pi}'^2 \end{array}\right]_m \qquad (2.4.9a)$$

where

ADDITIONAL RELEASE FORCES

$$\begin{bmatrix} \pi'_m \\ \hline \pi'_t \end{bmatrix}^i_m = \left[\begin{array}{c|c|c} (-1)^i \cdot \frac{V^i_{2R}}{L_c} & -(-1)^i \frac{V^i_{2R}}{L_c} & -(-1)^i V^i_{3R} \\ \hline -\frac{1}{L}+\frac{1}{L_c} & \frac{1}{L}-\frac{1}{L_c} & \cdot \end{array}\right]_m \begin{bmatrix} X^1_1 \\ \hline X^2_1 \\ \hline X_2 \end{bmatrix}_m \qquad \begin{matrix} (2.4.9b) \\ (2.4.9c) \end{matrix}$$

Treating equation (2.4.4) similarly, we find

$$\underset{\sim}{R}^*_m = \underset{\sim}{A}^T_m \, \underset{\sim}{X}'_m - \underset{\sim}{A}^T_{\pi m} \, \underset{\sim}{\pi}_m - \underset{\sim}{A}^T_{Rm} \, \underset{\sim}{\pi}'_m \qquad (2.4.10)$$

where

$$\underset{\sim}{A}_{R_m} = \left[\begin{array}{ccc|ccc} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \qquad (2.4.11)$$

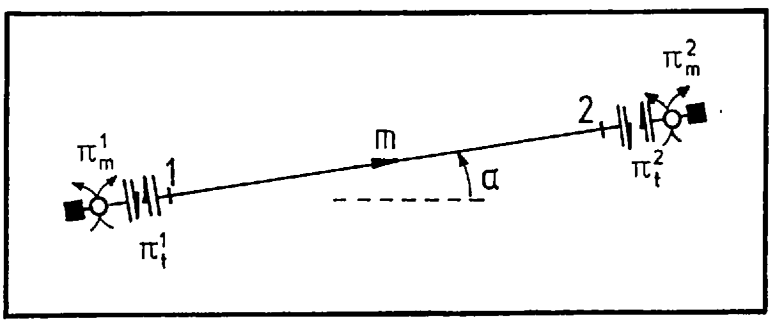


FIGURE 2.23

Matrices $\underset{\sim}{A}_m$ and $\underset{\sim}{A}_{\pi m}$ are given in (2.1.15) and (2.1.19), respectively and the additional forces $\underset{\sim}{\pi}_m$ are defined in (2.1.20); we emphasize that now the member chord does not coincide with the line joining the member nodes.

Equation (2.3.7) together with equation (2.3.10) represent the extended nodal description of Statics

$$\begin{bmatrix} \underset{\sim}{R}^* \\ \hline -\underset{\sim}{X}_R \end{bmatrix}_m = \left[\begin{array}{c|c|c} \underset{\sim}{A}^T & \underset{\sim}{A}^T_\pi & \underset{\sim}{A}^T_R \\ \hline \underset{\sim}{A}'^T & \cdot & \underset{\sim}{A}'^T_R \end{array}\right]_m \begin{bmatrix} \underset{\sim}{X}' \\ \hline -\underset{\sim}{\pi} \\ \hline -\underset{\sim}{\pi}' \end{bmatrix}_m \qquad \begin{matrix} (2.4.10) \\ (2.4.7) \end{matrix}$$

and if Static-Kinematic Duality is to be preserved, the extended nodal description of Kinematics must be expressed as follows

$$\begin{bmatrix} \underset{\sim}{k}^1 \\ \hline \underset{\sim}{k}^2 \\ \hline \underset{\sim}{k}^3 \end{bmatrix}_m = \begin{bmatrix} \underline{A} & \underline{A}' \\ \hline \underline{A}_\pi & \cdot \\ \hline \underline{A}_R & \underline{A}'_R \end{bmatrix}_m \begin{bmatrix} \underset{\sim}{k}^4 \\ \hline \underset{\sim}{k}^5 \end{bmatrix}_m \qquad \begin{matrix} (2.4.12a) \\ (2.4.12b) \\ (2.4.12c) \end{matrix}$$

Variables $\underset{\sim}{k}_m^2$ and $\underset{\sim}{k}_m^4$ can be identified just by confronting equations (2.4.12b) and (2.1.24c), yielding

$$\underset{\sim}{k}_m^2 = \underset{\sim}{\delta}_{\pi_m} \qquad (2.4.13a)$$

$$\underset{\sim}{k}_m^4 = \underset{\sim}{r}_m^* \qquad (2.4.13b)$$

Equation (2.4.3) can be written as

$$\underset{\sim}{u}'_m = \underline{A}_m \underset{\sim}{r}_m^* + \underline{A}'_m \underset{\sim}{v}_{R_m} - (\underline{A}_m - \underline{K}_m)\underset{\sim}{r}_m^* - (\underline{A}'_m - \underline{K}'_m)\underset{\sim}{v}_{R_m}$$

or, from equation (2.1.22) and noting

$$\underset{\sim}{u}'_{R_m} = (\underline{A}'_m - \underline{K}'_m)\ \underset{\sim}{v}_{R_m} \qquad (2.4.14)$$

$$\underset{\sim}{u}'_m + \underset{\sim}{u}'_{\pi_m} + \underset{\sim}{u}'_{R_m} = \underline{A}_m\ \underset{\sim}{r}_m^* + \underline{A}'_m\ \underset{\sim}{v}_{R_m}$$

which, confronted with equation (2.4.12a) allows the identification of the kinematic variables $\underset{\sim}{k}_m^1$ and $\underset{\sim}{k}_m^5$:

$$\underset{\sim}{k}_m^1 = \underset{\sim}{u}'_m + \underset{\sim}{u}'_{\pi_m} + \underset{\sim}{u}'_{R_m} \qquad (2.4.13c)$$

$$\underset{\sim}{k}_m^5 = \underset{\sim}{v}_{R_m} \qquad (2.4.13d)$$

Equation (2.4.12c) together with definitions (2.4.8), (2.4.11), (2.4.13b) and (2.4.13d) identify the remaining kinematic variable $\underset{\sim}{k}_m^3$ as the ADDITIONAL RELEASE FORCE DISLOCATIONS

$$\underset{\sim}{k}_m^3 = \underset{\sim}{\delta}'_\pi = \begin{bmatrix} \underset{\sim}{\delta}'^1_\pi \\ \hline \underset{\sim}{\delta}'^2_\pi \end{bmatrix}_m = \begin{bmatrix} -r_1'^{1*} + v_{1R}^1 \\ v_{3R}^1 \\ \hline r_1^{2*} + v_{1R}^2 \\ v_{3R}^2 \end{bmatrix}_m \qquad (2.4.13e)$$

Equations (2.4.7), (2.4.10) and (2.4.12), the latter with definitions (2.4.13) represent, respectively, the extended nodal description of Statics and Kinematics:

EXTENDED NODAL DESCRIPTION

STATICS	KINEMATICS
(2.4.15a), (2.4.15b) $\begin{bmatrix} \underset{\sim}{R}^* \\ -\underset{\sim}{X}_R \end{bmatrix}_m = \begin{bmatrix} \underset{\sim}{A}^T & \underset{\sim}{A}_\pi^T & \underset{\sim}{A}_R^T \\ \underset{\sim}{A}'^T & \cdot & \underset{\sim}{A}_R'^T \end{bmatrix}_m \begin{bmatrix} \underset{\sim}{X}' \\ -\underset{\sim}{\pi} \\ -\underset{\sim}{\pi}' \end{bmatrix}_m$	$\begin{bmatrix} \underset{\sim}{u}' + \underset{\sim}{u}'_\pi + \underset{\sim}{u}'_R \\ \underset{\sim}{\delta}_\pi \\ \underset{\sim}{\delta}'_\pi \end{bmatrix}_m = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{A}' \\ \underset{\sim}{A}_\pi & \cdot \\ \underset{\sim}{A}_R & \underset{\sim}{A}'_R \end{bmatrix}_m \begin{bmatrix} \underset{\sim}{r}^* \\ \underset{\sim}{v}_R \end{bmatrix}_m$ (2.4.16a), (2.4.16b), (2.4.16c)

Performing the matrix operations, the definition of the ADDITIONAL MEMBER DEFORMATIONS DUE TO THE RELEASE DISLOCATIONS as in (2.4.14) becomes:

ADDITIONAL DEFORMATIONS:RELEASE EFFECT

$$u'^{1}_{1R} = -u'^{2}_{1R} = \left[-\frac{v^1_{3R}}{L} + \arctan\frac{v^1_{3R}}{L}\right] + \left[-\frac{v^2_{3R}}{L} + \arctan\frac{v^2_{3R}}{L}\right] \qquad (2.4.17a,b)$$

$$u'_{2R} = L\left\{1-\left[1+\left(\frac{v^1_{3R}}{L}\right)^2\right]^{\frac{1}{2}}\right\} + L\left\{1-\left[1+\left(\frac{v^2_{3R}}{L}\right)^2\right]^{\frac{1}{2}}\right\} \qquad (2.4.17c)$$

The dual correspondence between static and kinematic variables can now be extended to include those describing the internal release effects

DUAL CORRESPONDENCE

STATIC VARIABLES	KINEMATIC VARIABLES	
$\underset{\sim}{X}'_m$	$\underset{\sim}{u}'_m + \underset{\sim}{u}'_{\pi_m} + \underset{\sim}{u}'_{R_m}$	(2.4.18a)
$\underset{\sim}{R}^*_m$	$\underset{\sim}{r}^*_m$	(2.4.18b)
$\underset{\sim}{X}_{R_m}$	$\underset{\sim}{v}_{R_m}$	(2.4.18c)
$\underset{\sim}{\pi}_m$	$\underset{\sim}{\delta}_{\pi_m}$	(2.4.18d)
$\underset{\sim}{\pi}'_m$	$\underset{\sim}{\delta}'_{\pi_m}$	(2.4.18e)

In the case of linear Statics and Kinematics equations

(2.4.15) and (2.4.16) reduce to

LINEAR ANALYSIS

STATICS	KINEMATICS
(2.4.19a) (2.4.19b) $\begin{bmatrix} \underset{\sim}{R}^* \\ \hline -\underset{\sim}{X}_R \end{bmatrix}_m = \begin{bmatrix} \underline{A}^T \\ \hline \underline{A}'^T \end{bmatrix}_m \underset{\sim}{X}'_m$	$\underset{\sim}{u}'_m = \left[\underline{A} \;\vdots\; \underline{A}' \right]_m \begin{bmatrix} \underset{\sim}{r}^* \\ \hline \underset{\sim}{v}_R \end{bmatrix}_m$ (2.4.20)

EXTENDED NODAL DESCRIPTION

2.4.2 The Extended Mesh Description

Instead of forcing the mesh member nodes to remain undisplaced, this time we must require the mesh member deformations $\underset{\sim}{u}'_M$ and the mesh reaction force displacements $\underset{\sim}{r}'^*_M$ to be zero while applying the release dislocations to the statically determinate mesh. By means of a procedure in every other aspect similar to the one used in the previous sub-section, the following are the relationships which were found to characterize the extended mesh description of Statics and Kinematics:

STATICS

$$\begin{bmatrix} \underset{\sim}{X}' \\ \hline \underset{\sim}{X}_R \\ \hline -\underset{\sim}{R}'^* \end{bmatrix}_M = \begin{bmatrix} \underline{B} & \underline{B}_o & \underline{B}_{o\pi} & \cdot \\ \underline{B}' & \underline{B}'_o & \underline{B}'_{o\pi} & \underline{B}'_R \\ \cdot & \underline{B}_r & \underline{B}_{r\pi} & \cdot \end{bmatrix}_M \cdot \begin{bmatrix} \underset{\sim}{p} \\ \hline \underset{\sim}{R}^* \\ \hline \underset{\sim}{\pi} \\ \hline \underset{\sim}{\pi}' \end{bmatrix}_M$$

KINEMATICS

$$\begin{bmatrix} \cdot \\ \hline \underset{\sim}{r}^* \\ \hline \underset{\sim}{\delta}_\pi \\ \hline \underset{\sim}{\delta}'_\pi \end{bmatrix}_M = \begin{bmatrix} \underline{B}^T & \underline{B}'^T & \cdot \\ \underline{B}_o^T & \underline{B}_o'^T & \underline{B}_r^T \\ \underline{B}_{o\pi}^T & \underline{B}_{o\pi}'^T & \underline{B}_{r\pi}^T \\ \cdot & \underline{B}_R'^T & \cdot \end{bmatrix}_M \cdot \begin{bmatrix} \underset{\sim}{u}' + \underset{\sim}{u}'_\pi + \underset{\sim}{u}'_R \\ \hline \underset{\sim}{v}_R \\ \hline \underset{\sim}{r}'^* \end{bmatrix}_M$$

(2.4.21a-22a)
(2.4.21b-22b)
(2.4.21c-22c)
(2.4.22d)

EXTENDED MESH DESCRIPTION

In the above equations, which besides being explicitly linear exhibit a dual transformation, the release forces and the release dislocations were collected in the vectors $\underset{\sim}{X}_{R_M}$ and $\underset{\sim}{v}_{R_M}$, respectively; the extended additional mesh forces,   the additional release forces and the additional mesh strains due to the release dislocations were collected in the vectors $\underset{\sim}{\pi}_M$, $\underset{\sim}{\pi}'_M$ and $\underset{\sim}{u}'_{R_M}$, their elements being defined by equations (2.1.20), (2.4.9) and (2.4.17), respectively.

Replacing the explicitly non-linear and non-dual, static and kinematic descriptions by an equivalent linear contragredient transformation, we accomplished the original purpose of performing the exact analysis by working on the undeformed substructure subject to the actual plus additional stresses and strains which condense the non-linearity of the problem and whose physical meaning is known.

Hence, and since in linear analysis the i-th release forces equal, by definition, the stress-resultants at critical section i, and bearing in mind that

$\underline{B}'_i$ $(\underline{B}'_{R_{ij}})$ is the stress- resultant at release i induced by a unit biaction (a unit additional release force j),

$\underline{B}'_{o_{ij}}$ $(\underline{B}'_{o\pi_{ij}})$ is the stress-resultant at release i induced by a unit mesh force (extended additional force) applied at node (member) j,

we may define the submatrices for a generalized mesh just by considering definitions (2.2.38) to (2.2.40) and (2.4.8), yielding

GENERALIZED MESH SUBMATRICES:EFFECT OF INTERNAL RELEASES

$$-\underline{B}'_i = \begin{bmatrix} 1 & -x_3^i & x_2^i \\ \cdot & -\sin\alpha_i & -\cos\alpha_i \\ \cdot & (x_3^i - x_3^{i+1})/L_i & (-x_2^i + x_2^{i+1})/L_i \\ \hdashline 1 & -x_3^{j+1} & x_2^{i+1} \\ \cdot & -\sin\alpha_i & -\cos\alpha_i \\ \cdot & (x_3^i - x_3^{i+1})/L_i & (-x_2^i + x_2^{i+1})/L_i \end{bmatrix} \quad (2.4.23)$$

$$-\underline{B}'_{o_{ij}} = \begin{bmatrix} 1 & x_3^j - x_3^i & -x_2^j + x_2^i \\ \cdot & -\sin\alpha_i & -\cos\alpha_i \\ \cdot & (x_3^i - x_3^{i+1})/L_i & (-x_2^i + x_2^{i+1})/L_i \\ \hdashline 1 & x_3^j - x_3^{i+1} & -x_2^j + x_2^{i+1} \\ \cdot & -\sin\alpha_i & -\cos\alpha_i \\ \cdot & (x_3^i - x_3^{i+1})/L_i & (-x_2^i + x_2^{i+1})/L_i \end{bmatrix} \quad (i<j) \quad (2.4.25)$$

$$-\underline{B}'_{o\pi_{ij}} = \begin{bmatrix} \cdot & -L_j \\ \delta_{ij} & \cdot \\ \cdot & \delta_{ij} \\ \hdashline \cdot & -(1-\delta_{ij})L_j \\ \delta_{ij} & \cdot \\ \cdot & \delta_{ij} \end{bmatrix} \quad (i \leq j) \quad (2.4.24)$$

$$-\underline{B}'_{R_{ij}} = \begin{bmatrix} \delta_{ij} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \delta_{ij} & \cdot & \cdot \\ \hdashline \cdot & \cdot & \delta_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta_{ij} \end{bmatrix} \quad (2.4.26)$$

if $i>j$ $\underline{B}'_{o\pi_{ij}} = \underline{0}$

if $i \geqslant j$ $\underline{B}'_{o_{ij}} = \underline{0}$

where δ_{ij} is the Kronecker delta, (x_2^k, x_3^k) the co-ordinates of vertex k, and L_j the length of member j of the generalized polygonal mesh illustrated in Figs.2.15 and 2.16 and described at the end of sub-section 2.2.2.

Equations (2.4.21) and (2.4.22) can be readily specialized into the case of linear Statics and Kinematics, yielding

LINEAR ANALYSIS	
STATICS	KINEMATICS
(2.4.27a) (2.4.27b) (2.4.27c) $\begin{bmatrix} \underset{\sim}{X}' \\ \underset{\sim}{X}_R \\ -\underset{\sim}{R}'^* \end{bmatrix}_M = \begin{bmatrix} \underline{B} & \underline{B}_o \\ \underline{B}' & \underline{B}'_o \\ \cdot & \underline{B}_R \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{p} \\ \underset{\sim}{R}^* \end{bmatrix}_M$	$\begin{bmatrix} \cdot \\ \underset{\sim}{r}^* \end{bmatrix}_M = \begin{bmatrix} \underline{B}^T & \underline{B}'^T_o & \cdot \\ \underline{B}^T_o & \underline{B}'^T_o & \underline{B}^T_R \end{bmatrix}_M \begin{bmatrix} \underset{\sim}{u}' \\ \underset{\sim}{v}_R \\ \underset{\sim}{r}'^* \end{bmatrix}_M$ (2.4.28a) (2.4.28b)
EXTENDED MESH DESCRIPTION	

2.4.3 Incremental Descriptions

The procedure adopted in sub-section 2.1.3 could now be applied again to obtain the following definitions for the incremental additional release forces

$$\Delta\underset{\sim}{\pi}'_m = \underline{Q}'^T_m \Delta\underset{\sim}{X}'_m + \underline{P}'_m \Delta\underset{\sim}{\delta}'_{\pi_m} + \Delta\underset{\sim}{R}''_{\pi_m} \qquad (2.4.29a)$$

and for the incremental additional deformations due to the release dislocations

$$\Delta\underset{\sim}{u}'_{R_m} = \underline{Q}'_m \Delta\underset{\sim}{\delta}'_{\pi_m} + \Delta\underset{\sim}{R}''_{u\pi_m} \qquad (2.4.29b)$$

We refrain from defining the elements of matrices $\underline{Q}'_m$ and $\underline{P}'_m$, the latter assumed to be symmetric, and of the non-linear residuals $\Delta\underset{\sim}{R}''_{\pi_m}$ and $\Delta\underset{\sim}{R}''_{u\pi_m}$, due to their direct dependence on the particular combination of releases they may refer to; their definitions would be particularly simple to derive for substructures free of shear and thrust internal releases, as the bending releases are not directly responsible for any new non-linear terms.

Substituting the incremental additional forces and deformations, defined respectively by (2.1.43a) and (2.4.29a), and (2.1.39a) and (2.4.29b), in the incremental forms of (2.4.15) and (2.4.16), the following extended nodal description of incremental Statics and Kinematics would emerge

$$\begin{bmatrix} -\mathbb{K}_{rr} & -\mathbb{K}_{rR} & \mathbb{A} \\ -\mathbb{K}_{rR}^T & -\mathbb{K}_{RR} & \mathbb{A}' \\ \mathbb{A}^T & \mathbb{A}'^T & \cdot \end{bmatrix}_m \begin{bmatrix} \Delta r^* \\ \Delta v_R \\ \Delta X' \end{bmatrix}_m = \begin{bmatrix} \Delta R^* \\ -\Delta X_R \\ \Delta u' \end{bmatrix}_m + \begin{bmatrix} \Delta R^*_{\pi} \\ \Delta R_{X\pi} \\ \Delta R'_{u\pi} \end{bmatrix}_m \qquad \begin{matrix} (2.4.30a) \\ (2.4.30b) \\ (2.4.31) \end{matrix}$$

EXTENDED INCREMENTAL NODAL DESCRIPTION

where

$$\mathbb{A}_m = A_m - Q'_m A_{R_m} - Q_m A_{\pi_m}$$

$$\mathbb{A}'_m = A'_m - Q'_m A'_{R_m}$$

$$\mathbb{K}_{rr_m} = A_{R_m}^T P'_m A_{R_m} + A_{\pi_m}^T P_m A_{\pi_m} = \mathbb{K}_{rr_m}^T$$

$$\mathbb{K}_{rR_m} = A_{R_m}^T P'_m A'_{R_m}$$

$$\mathbb{K}_{RR_m} = A'^T_{R_m} P'_m A'_{R_m} = \mathbb{K}_{RR_m}^T$$

and

$$\Delta R^*_{\pi_m} = A_{R_m}^T \Delta R''_{\pi_m} + A_{\pi_m}^T \Delta R_{\pi_m}$$

$$\Delta R_{X\pi_m} = A'^T_{R_m} \Delta R''_{\pi_m}$$

$$\Delta R'_{u\pi_m} = \Delta R_{u\pi_m} + \Delta R''_{u\pi_m}$$

Let the incremental mesh additional forces

$$\Delta \pi_M = Q_M^T \Delta X'_M + P_M \Delta \delta_{\pi_M} + \Delta R_{\pi_M} \qquad (2.4.32a)$$

$$\Delta \pi'_M = Q'^T_M \Delta X'_M + P'_M \Delta \delta'_{\pi_M} + \Delta R''_{\pi_M} \qquad (2.4.32b)$$

and deformations

$$\Delta u'_{\pi_M} = Q_M \Delta \delta_{\pi_M} + \Delta R_{u\pi_M} \qquad (2.4.33a)$$

$$\Delta u'_{R_M} = Q'_M \Delta \delta'_{\pi_M} + \Delta R''_{u\pi_M} \qquad (2.4.33b)$$

be defined by setting m to 1, 2,..., n, where n represents the number of branches of the generic mesh substructure M, in (2.1.43a), (2.4.29a), (2.1.39a) and (2.4.29b), and collecting the resulting relations, following the mesh members numbering sequence. Taking increments in (2.4.21) and (2.4.22), eliminating the additional forces and deformations through (2.4.32) and (2.4.33), and treating the resulting system in a manner in every aspect similar to the one used in sub-section 2.2.3, equations (2.4.34) and (2.4.35) below would be found to be the extended versions, to include the effects of the internal releases, of the incremental mesh description of Statics and Kinematics, (2.2.56) and (2.2.57), respectively.

STATICS / KINEMATICS

$$\left[\begin{array}{ccc:c:c} \mathbb{K}_{uu} & \mathbb{K}_{uR} & \mathbb{K}_{ur} & \mathbb{B} & \mathbb{B}_{o} \\ \mathbb{K}_{uR}^{T} & \mathbb{K}_{RR} & \mathbb{K}_{Rr} & \mathbb{B}' & \mathbb{B}'_{o} \\ \mathbb{K}_{ur}^{T} & \mathbb{K}_{Rr}^{T} & \mathbb{K}_{RR} & \bullet & \mathbb{B}_{or} \\ \hdashline \mathbb{B}^{T} & \mathbb{B}'^{T} & \bullet & \bullet & \bullet \\ \mathbb{B}_{o}^{T} & \mathbb{B}'^{T}_{o} & \mathbb{B}_{or}^{T} & \bullet & \bullet \end{array}\right]_{M} \left[\begin{array}{c} \Delta \mathbf{u}' \\ \Delta \mathbf{v}_{R} \\ \Delta \mathbf{r}'^{*} \\ \Delta \mathbf{p} \\ \Delta \mathbf{R}^{*} \end{array}\right]_{M} = \left[\begin{array}{c} \Delta \mathbf{X}' \\ \Delta \mathbf{X}_{R} \\ -\Delta \mathbf{R}'^{*} \\ \bullet \\ \Delta \mathbf{r}^{*} \end{array}\right]_{M} - \left[\begin{array}{c} \Delta \mathbf{X}'_{\pi} \\ \Delta \mathbf{X}_{R\pi} \\ \Delta \mathbf{R}'^{*}_{\pi} \\ \Delta \mathbf{v}_{\pi} \\ \Delta \mathbf{r}^{*}_{\pi} \end{array}\right]_{M} \quad \begin{array}{r} (2.4.34a) \\ (2.4.34b) \\ (2.4.34c) \\ (2.4.35a) \\ (2.4.35b) \end{array}$$

EXTENDED INCREMENTAL MESH DESCRIPTION

Matrices $\mathbf{B}_M$, $\mathbf{B}_{o_M}$, $\mathbf{B}_{or_M}$, $\mathbf{K}_{uu_M}$, $\mathbf{K}_{ur_M}$ and $\mathbf{K}_{rr_M}$ are defined in (2.2.47h-n), the remaining structural matrices and non-linear residuals being respectively defined by, and dropping the subscript M,

$$\mathbb{B}' = \mathbf{B}' + \mathbf{B}'_{o\pi}\,\bar{\mathbf{B}}_{\pi}\,\mathbf{Q}^{T}\,\mathbf{B} + \mathbf{B}'_{R}\,\mathbf{Q}'^{T}\,\mathbb{B}$$

$$\mathbb{B}'_{o} = \mathbf{B}'_{o} + \mathbf{B}'_{o\pi}\,\bar{\mathbf{B}}_{\pi}\,\mathbf{Q}^{T}\,\mathbf{B}_{o} + \mathbf{B}'_{R}\,\mathbf{Q}'^{T}\,\mathbb{B}_{o}$$

$$\mathbb{K}_{uR} = \mathbf{B}_{o\pi}\,\bar{\mathbf{B}}_{\pi}\,\mathbf{P}\,\bar{\mathbf{B}}_{\pi}^{T}\,\mathbb{B}_{\pi}^{T}$$

$$\mathbb{K}_{Rr} = \mathbb{B}_{\pi}\,\bar{\mathbf{B}}_{\pi}\,\mathbf{P}\,\bar{\mathbf{B}}_{\pi}^{T}\,\mathbf{B}_{r\pi}^{T}$$

$$\mathbb{K}_{RR} = \mathbb{B}_{\pi}\,\bar{\mathbf{B}}_{\pi}\,\mathbf{P}\,\bar{\mathbf{B}}_{\pi}^{T}\,\mathbb{B}_{\pi}^{T} + \mathbf{B}'_{R}\,\mathbf{P}'\,\mathbf{B}'^{T}_{R} = \mathbb{K}_{RR}^{T}$$

and

$$\Delta \mathbf{X}'_{\pi} = \mathbb{K}_{uu}(\Delta \mathbf{R}_{u\pi} + \Delta \mathbf{R}''_{u\pi}) + \mathbf{B}_{o\pi}\,\bar{\mathbf{B}}_{\pi}\,\Delta \mathbf{R}_{\pi}$$

$$\Delta \mathbf{X}_{R\pi} = \mathbb{K}_{uR}^{T}(\Delta \mathbf{R}_{u\pi} + \Delta \mathbf{R}''_{u\pi}) + \mathbb{B}_{\pi}\,\bar{\mathbf{B}}_{\pi}\,\Delta \mathbf{R}_{\pi} + \mathbf{B}'_{R}\,\Delta \mathbf{R}''_{\pi}$$

$$\Delta \mathbf{R}'^{*}_{\pi} = \mathbb{K}_{ur}^{T}(\Delta \mathbf{R}_{u\pi} + \Delta \mathbf{R}''_{u\pi}) + \mathbf{B}_{r\pi}\,\bar{\mathbf{B}}_{\pi}\,\Delta \mathbf{R}_{\pi}$$

$$\Delta \mathbf{v}_{\pi} = \mathbb{B}^{T}\,(\Delta \mathbf{R}_{u\pi} + \Delta \mathbf{R}''_{u\pi})$$

$$\Delta \mathbf{r}^{*}_{\pi} = \mathbb{B}_{o}^{T}\,(\Delta \mathbf{R}_{u\pi} + \Delta \mathbf{R}''_{u\pi})$$

where

$$\mathbb{B}_{\pi} = \mathbf{B}'_{o\pi} + \mathbf{B}'_{R}\,\mathbf{Q}'^{T}\,\mathbf{B}_{o\pi}$$

Matrix $\bar{\underline{B}}_{\pi}$ is defined in (2.2.47c).

The perturbed form of the nodal [mesh] description of Statics and Kinematics (2.1.54) and (2.1.55) [(2.2.60) and (2.2.61)] could now be easily extended to include the effects of the internal releases by replacing the incremental variables in (2.4.30) and (2.4.31) [(2.4.34) and (2.4.35)], respectively, by their power series expansions in the form (2.1.52) and solving the non-linear residuals as in sub-section 2.1.4 [2.2.4].

To derive the asymptotic nodal description of extended Statics and Kinematics, the total variables in the finite descriptions of equilibrium and compatibility (2.4.15) and (2.4.16), respectively, should be expanded in a power series of the form (2.1.63) and the same order terms equated next, under the assumption of a kinematically trivial initial equilibrium path. The resulting recursive linear systems would emerge in a format formally identical to their correspondents in the perturbation analysis formulation. The structural matrices would, however, differ quantitatively; in particular, matrix $\underline{Q}_m$, as shown in sub-section 2.1.5, comes to be a null matrix, and so would, one may expect, matrix $\underline{Q}'_m$.

The asymptotic mesh description of extended Statics and Kinematics could be derived, from the finite descriptions (2.4.21) and (2.4.22), in a similar way.

2.5 STATIC-KINEMATIC DUALITY

The energy methods have been extensively and successfully used in linear analysis and the Principle of Virtual Work proved to be the unifying element of the several proposed formulations, as shown by Argyris and Kelsey (1960).

Therefore, it was not surprising that when the research effort moved into the field of non-linear analysis, the energy theorems kept on being adopted as the basis from which the great majority of works on the subject were developed.

However, this time the energy theorems did not prove to be the unifying feature.

In spite of the efforts in that direction it is still difficult to relate the many formulations presented in the literature; different formulations tend to provide different descriptions, and the discussion continues on which forms of energy to use and on what order terms are neglegible.

Furthermore, and for procedural reasons, problems in elasticity and plasticity have grown more and more apart, to the detriment of the latter.

The philosophy behind the studies developed in the Systems and Mechanics Section at Imperial College has been quite the opposite.

The study of each problem is based upon the fundamental principles of mechanics: statics, kinematics and constitutive relations. This allows a continuous reference to be made to the physical nature of the problems, and simplifies the control of consistency in any hypotheses which may subsequently be made.

When those basic ingredients of the problem are brought together, the correctness of the formulation can be confirmed by the recovery of the associate variational principles.

So, while the energy-based formulations understand the contragredience in the static and kinematic transformations as a consequence of the Principle of Virtual Work, the first-principles-based formulations understand that principle as the variational interpretation of Static-Kinematic Duality.

For the formulation presented herein, the (two-dimensional) vectorial description of the Principle of Virtual Work at element level can be recovered as follows:

1. The NODAL DESCRIPTION of the PRINCIPLE OF VIRTUAL DISPLACEMENTS is defined by the internal product of equations (2.1.24) and (2.1.28)

EQUILIBRATED

$$\boxed{\underset{\sim}{R}{}^{*T}_m \, \Delta \underset{\sim}{r}{}^*_m = -\underset{\sim}{\pi}{}^T_m \, \Delta \underset{\sim}{\delta}_{\pi_m} + \underset{\sim}{X}{}'^T_m (\Delta \underset{\sim}{u}'_m + \Delta \underset{\sim}{u}'_{\pi_m})} \qquad (2.5.1)$$

COMPATIBLE

2. The NODAL DESCRIPTION of the PRINCIPLE OF VIRTUAL FORCES is defined by the internal product of equations (2.1.25) and (2.1.27)

EQUILIBRATED

$$\boxed{\mathbf{r}_m^{*T}\,\Delta\mathbf{R}_m^{*} = -\boldsymbol{\delta}_{\pi_m}^{T}\,\Delta\boldsymbol{\pi}_m + (\mathbf{u}'_m + \mathbf{u}'_{\pi_m})^{T}\,\Delta\mathbf{X}'_m} \qquad (2.5.2)$$

COMPATIBLE

Hence, the exact incremental work and complementary work, at element level, are respectively defined by

$$\Delta W_m = \mathbf{R}_m^{*T}\,\Delta\mathbf{r}_m^{*} \qquad (2.5.3)$$

and

$$\Delta W_m^{*} = \mathbf{r}_m^{*T}\,\Delta\mathbf{R}_m^{*} \qquad (2.5.4)$$

and, for infinitely near critical sections, the exact strain energy and complementary strain energy are defined as

$$\Delta U_m = \mathbf{X}_m'^{T}(\Delta\mathbf{u}'_m + \Delta\mathbf{u}'_{\pi_m}) - \boldsymbol{\pi}_m^{T}\,\Delta\boldsymbol{\delta}_{\pi_m} \qquad (2.5.5)$$

and

$$\Delta U_m^{*} = (\mathbf{u}'_m + \mathbf{u}'_{\pi_m})^{T}\Delta\mathbf{X}'_m - \boldsymbol{\delta}_{\pi_m}^{T}\,\Delta\boldsymbol{\pi}_m \qquad (2.5.6)$$

Similarly, we would find, through equations (2.2.20-21) and (2.2.35-36)

PRINCIPLE OF VIRTUAL DISPLACEMENTS	PRINCIPLE OF VIRTUAL FORCES
$\mathbb{R}_M^{*T}\,\Delta\mathbb{r}_M^{*} = -\boldsymbol{\pi}_M^{T}\,\Delta\boldsymbol{\delta}_{\pi_M} + \mathbf{X}_M'^{T}(\Delta\mathbf{u}'_M + \Delta\mathbf{u}'_{\pi_M})$ (2.5.7)	$\mathbb{r}_M^{*T}\,\Delta\mathbb{R}_M^{*} = -\boldsymbol{\delta}_{\pi_M}^{T}\,\Delta\boldsymbol{\pi}_M + (\mathbf{u}'_M + \mathbf{u}'_{\pi_M})\Delta\mathbf{X}'_M$ (2.5.8)
MESH DESCRIPTION	

where

$$\mathbb{R}_M^{*} = \begin{bmatrix} \mathbf{R}^{*} \\ \hline \mathbf{R}'^{*} \end{bmatrix}_M \quad \text{and} \quad \mathbb{r}_M^{*} = \begin{bmatrix} \mathbf{r}^{*} \\ \hline \mathbf{r}'^{*} \end{bmatrix}_M \qquad (2.5.9a,b)$$

The Principle of Virtual Displacements, but not that of the Virtual Forces, can be expressed exclusively in terms of the fundamental static and kinematic variables:

Multiplying by the member incremental displacements the transpose of the nodal description of Statics (2.1.11)

$$\underset{\sim}{R}_m^{*T}\,\Delta\underset{\sim}{r}_m^* = \underset{\sim}{X}_m'^{T}\,\underset{\sim}{S}_m^T\,\Delta\underset{\sim}{r}_m^*$$

and making use of identity (2.1.49) and of the incremental description of Kinematics (2.1.51), we find for the Principle of Virtual Displacements

PRINCIPLE OF VIRTUAL DISPLACEMENTS
$\underset{\sim}{R}_m^{*T}\,\Delta\underset{\sim}{r}_m^* = \underset{\sim}{X}_m'^{T}\,(\Delta\underset{\sim}{u}_m'+\Delta\underset{\sim}{R}_{u\pi_m})$
NODAL DESCRIPTION

(2.5.10)

Similarly, using now equations (2.2.7), (2.2.55) and (2.2.57), one would find

$$\underset{\sim}{X}_M'^{T}\,\Delta\underset{\sim}{u}_M' - \underset{\sim}{R}_M'^{*T}\Delta\underset{\sim}{r}_M'^{*} = -\underset{\sim}{p}_M^T\Delta\underset{\sim}{v}_{\pi_M} - \underset{\sim}{R}_M^{*T}\Delta\underset{\sim}{r}_{\pi_M}^{*} + \underset{\sim}{R}_M^{*T}\Delta\underset{\sim}{r}_M^* \qquad (2.5.11)$$

However, from (2.2.53d)

$$\underset{\sim}{R}_M^{*T}\,\Delta\underset{\sim}{r}_{\pi_M}^* = \underset{\sim}{R}_M^{*T}\,\underset{\sim}{\mathbb{B}}_{o_M}^T\,\Delta\underset{\sim}{R}_{u\pi_M}$$

or, from (2.2.55a,c) and (2.2.7b)

$$\underset{\sim}{R}_m^{*T}\,\Delta\underset{\sim}{r}_{\pi_M}^* = (\underset{\sim}{X}_M'-\underset{\sim}{B}_M\underset{\sim}{p}_M)^T\,\Delta\underset{\sim}{R}_{u\pi_M}$$

and using (2.2.53c)

$$\underset{\sim}{R}_M^{*T}\,\Delta\underset{\sim}{r}_{\pi_M}^* = \underset{\sim}{X}_M'^{T}\Delta\underset{\sim}{R}_{u\pi_M} - \underset{\sim}{p}_M^T\,\Delta\underset{\sim}{v}_{\pi_M}$$

Substituting the above relationship into (2.5.11) and regrouping, with help from (2.5.9), we find

PRINCIPLE OF VIRTUAL DISPLACEMENTS
$\underset{\sim}{\mathbb{R}}_M^{*T}\,\Delta\underset{\sim}{\mathbb{r}}_M^* = \underset{\sim}{X}_M'^{T}\,(\Delta\underset{\sim}{u}_M'+\Delta\underset{\sim}{R}_{u\pi_M})$
MESH DESCRIPTION

(2.5.12)

Equating the terms in the right-hand side of equations (2.5.1) and (2.5.10), or equations (2.5.7) and (2.5.12),

$$\underset{\sim}{X}'^{T}_{m}(\Delta\underset{\sim}{u}'_{m}+\Delta\underset{\sim}{u}'_{\pi_m})-\underset{\sim}{\pi}^{T}_{m}\,\Delta\underset{\sim}{\delta}_{\pi_m}=\underset{\sim}{X}'^{T}_{m}(\Delta\underset{\sim}{u}'_{m}+\Delta\underset{\sim}{R}_{u\pi_m})$$

or

$$\underset{\sim}{X}'^{T}_{m}(\Delta\underset{\sim}{u}'_{\pi_m}-\Delta\underset{\sim}{R}_{u\pi_m})=\underset{\sim}{\pi}^{T}_{m}\,\Delta\underset{\sim}{\delta}_{\pi_m}$$

Substituting above the definition (2.1.38) for the incremental additional deformations

$$\underset{\sim}{X}'^{T}_{m}\,\underset{\sim}{\Omega}_{m}\,\Delta\underset{\sim}{\delta}_{\pi_m}=\underset{\sim}{\pi}^{T}_{m}\,\Delta\underset{\sim}{\delta}_{\pi_m}$$

the definition for the additional member forces (2.1.20) is recovered

$$\underset{\sim}{\pi}_{m}=\underset{\sim}{\Omega}^{T}_{m}\,\underset{\sim}{X}'_{m} \qquad (2.5.13)$$

since $\Delta\underset{\sim}{\delta}_{\pi_m}$ represents an arbitrary set of displacements.

The nodal description of the Principle of Virtual Work in finite mechanics can be defined as the internal product of equations (2.1.24) and (2.1.25) yielding

$$\underset{\sim}{R}^{*T}_{m}\,\underset{\sim}{r}^{*}_{m}+\underset{\sim}{\pi}^{T}_{m}\,\underset{\sim}{\delta}_{\pi_m}=\underset{\sim}{X}'^{T}_{m}\,(\underset{\sim}{u}'_{m}+\underset{\sim}{u}'_{\pi_m}) \qquad (2.5.14)$$

The same principle can be expressed in terms of (finite) incremental variables by taking the scalar products of (2.1.49) and (2.1.50):

$$\Delta\underset{\sim}{R}^{*T}_{\pi_m}\,\Delta\underset{\sim}{r}^{*}_{m}+\Delta\underset{\sim}{r}^{*T}_{m}\,\underset{\sim}{\mathbb{K}}_{\pi_m}\,\Delta\underset{\sim}{r}^{*}_{m}+\Delta\underset{\sim}{R}^{*T}_{m}\,\Delta\underset{\sim}{r}^{*}_{m}=\Delta\underset{\sim}{X}'^{T}_{m}(\Delta\underset{\sim}{u}'_{m}+\Delta\underset{\sim}{R}_{u\pi_m}) \qquad (2.5.15)$$

Similar results would be obtained working on the mesh description or on the extended nodal and mesh descriptions presented in Sections 2.2 and 2.4, respectively.

C H A P T E R   T H R E E

CONSTITUTIVE RELATIONS AT ELEMENT LEVEL

The causality relations associating the member stress-resultants with the corresponding strain-resultants will be derived through a first-principle-based analysis of a three-degree of freedom elastoplastic finite element.

The analysis rests on a set of hypotheses, which will be referred to along the presentation, which reduce the fundamental constitutive relations to the association of axial stresses and strains, as illustrated in Fig.3.1. It is assumed that the elastic phase, characterized by REVERSIBLE strains, is linear; no assumption is made about the law of variation of the PERMANENT or plastic strains, except that it must represent a stable material in the sense of Drucker (1959).

The complexity of the behaviour of a simple three-degree of freedom elastoplastic beam element is primarily caused by the mechanics of the development of plasticity.

For relatively small deformations and under certain combinations of the end-loads, the maximum axial stress will occur at either of or both end-sections since the span of the beam is free of loads. If the stresses were able to increase for constant deformations, the stress at some fibres would reach the yield limit and yielding would start to spread, not only within the most highly stressed section but also through the neighbouring sections.

However, the increase in the stresses is accompanied by a variation on the deformations and consequently the most highly stressed section will not, in general, have a fixed position: the incidence of plasticity "travels" along the span, depending on the variation of the beam deformations as well as on the relative intensity of the combined end-forces. The problem becomes tractable only if restrictive assumptions, some of which quite severe, are introduced in the process of analysis.

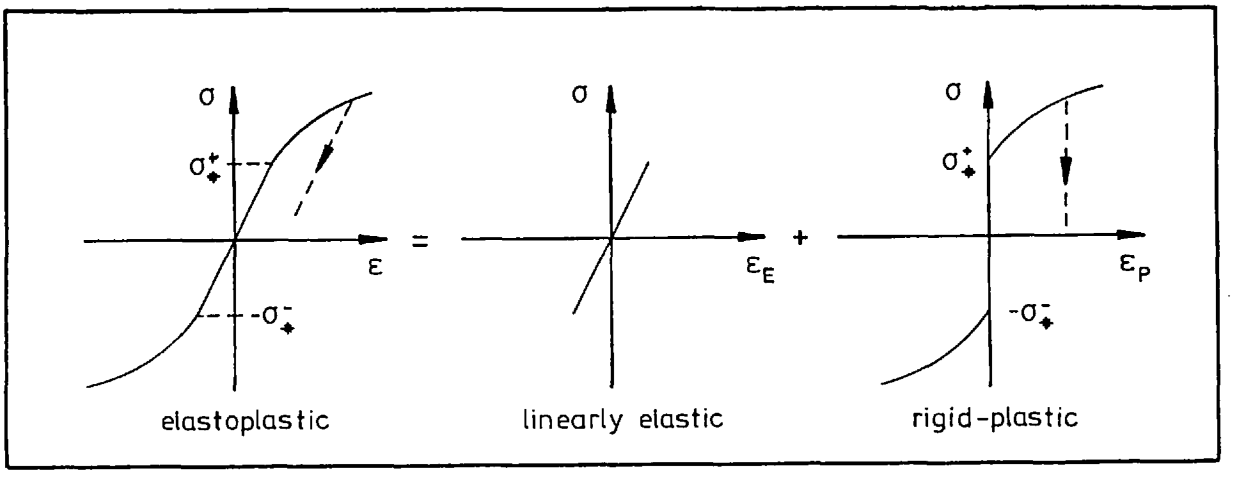


FIGURE 3.1

In this work the hypothesis of lumped plasticity holds, that is, the plastic strains are restricted to develop in discrete sections without spreading of plasticity between two critical sections, along the member.

Furthermore, we require the critical sections to coincide with the element end-sections; if a section in the span becomes plastic, the process of numerical analysis can be suspended and the element subdivided to transform the interior section into a limiting one. In any case, the critical sections are assumed to have a fixed position, that is, the plastically strained sections are not allowed to travel within the element.

As a consequence of the above mentioned hypotheses, the maximum axial stress at interior sections is required to remain within the elastic range

$$-\sigma_*^- < \sigma < +\sigma_*^+$$

enabling us to separate the strain field into a continuous field of elastic strains, developing along the beam, and a discrete field of plastic strains developing at its end-sections; in particular we may define the strain-resultants as the sum of their elastic and plastic components

$$\underset{\sim}{u} = \underset{\sim}{u}_E + \underset{\sim}{u}_p \tag{3.0.1}$$

The elastic constitutive relations are studied in section 3.1 and the plasticity relations in section 3.2.

The description of Statics and Kinematics of the three-degree of freedom elastic beam-column, represented in Fig.3.3, is fed into the constitutive relations, which are corrected to include a measure of shear deformation effects, and the differential governing equation so obtained is solved by a standard perturbation technique.

The elastic solutions are then interpreted and casted in formats suitable to perform the analysis of the structure by the finite-element method.

Section 3.1 ends with a brief reference to related formulations presented in the literature.

The formulation presented in section 3.2, concerned with the plastic constitutive relations, rests heavily on previous works by G. Maier and O. De Donato, in particular Maier (1969a, 1969b and 1970) and De Donato (1974).

As it will become apparent the analysis of the plastic behaviour will be very superficial. The primary objective is to utilize, within a limited scope, Maier's general and quite powerful matrix formulations of Koiter's theory of plasticity; the heart of the problem, the overriding difficulty which will be avoided herein, resides in the definition, for each particular case, of the elements of the matrix operators.

The formulation of Maier was originally designed to interpret through mathematical programming theory the elasto-plastic behaviour of structures for small deformations and displacements. Except for the unavoidably non-linear association condition, the description of the Static and Kinematic phases of plasticity are the only non-linear relations present in the formulation; for that reason, and at the cost of a rapidly increasing number of variables and constraints, those relations were consistently replaced by piecewise-linear approximations.

However, for large displacements and deformations every single equation in the formulation of the problem is, in principle, non-linear; consequently, herein the non-linear relations of plasticity are accepted as such and treated in a manner in every aspect similar to the one used when dealing with Statics and Kinematics of the fundamental substructures.

Duality is forced upon the description of the static and kinematic phases of plasticity, originally expressed in terms of total variables, in such a way that von Mises' theory of the plastic potential, as generalized by Koiter, is recovered when those descriptions are expressed in terms of infinitesimal incremental variables.

3.1 ELASTICITY

The present study deals with a slender prismatic beam of elastic, homogeneous and isotropic material, subject to in-plane terminal (conservative) loads, such that the following two basic assumptions are acceptable:

- Transverse sections, plane and normal to the centroidal axis before the deformation, remain plane and normal to the axis after deformation (Bernoulli-Euler hypothesis)

- The stress field is plane

The locus of the centroid of the cross-section of the beam is a straight line and the envelopes of the principal axes through the centroid are two orthogonal planes; the prismatic beam is initially straight with a symmetrical cross-section whose dimensions are small compared with the axial length.

The system of reference to be associated with the beam is so chosen as to form a right-handed rectangular Cartesian co-ordinate system; the x_I-axis is taken in the direction of the centroidal axis and the x_{II}-and x_{III}-axes are assumed to be parallel to the principal directions of the transverse sections.

Hence, and from Fig.3.2, the components of the displacement of any point in the plane of loading $x_{II}=0$ of the beam can be expressed as

$$\begin{cases} v_I = v_I^0(x_I)+\eta(x_I,x_{III})\cdot\xi(x_I) \\ v_{II} = 0 \\ v_{III} = v_{III}^0(x_I)+\eta(x_I,x_{III})\cdot\varsigma(x_I) \end{cases}$$

where v^0 represents the deformed middle line, ξ and ς are the components of the unit vector $\hat{n}$ normal to the deformed middle line, and η represents the deformation of the plane cross-section in the direction of the normal $\hat{n}$.

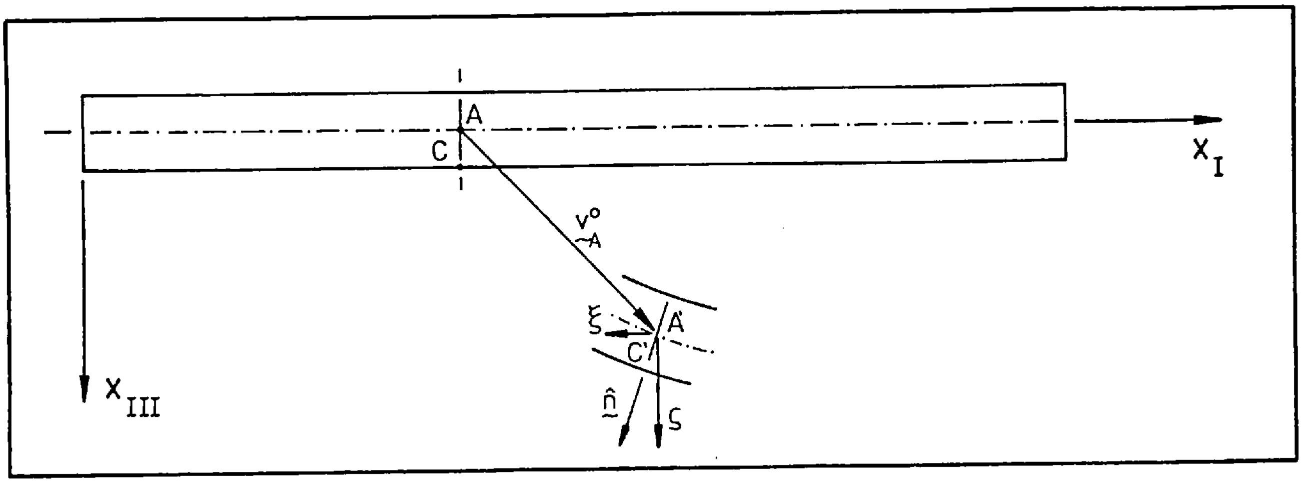


FIGURE 3.2

The search for a mathematical expression for the function η as well as for the stress distribution along the cross-section falls out of the scope of the present study; the objective is to derive the law relating the stress-and strain-resultants at the terminals of the constitutive element of a planar skeletal structure, and to do so it is sufficient to define the displacements of the points of the beam centroidal locus as well as the stress resultants at that point which embodies the mechanical properties of the associate cross-section, namely the (constant) cross-sectional area A and the flexural stiffness EI.

For simplicity of the presentation, NON-DIMENSIONAL PARAMETERS WILL BE USED THROUGHOUT; the corresponding variables may easily be regained by suitably affecting those parameters by the length scale factor L, the stress scale factor E, the force scale factor EI/L^2 and the moment scale factor EI/L.

For instance, dropping subscript m and with help from Fig.3.3, the stress-resultants at critical section i are defined by

$$X_1^i = m_i \frac{EI}{L} \qquad X_2^i = n \frac{EI}{L^2} \qquad X_3^i = -t \frac{EI}{L^2} \tag{3.0.1-3}$$

and the member elastic deformations by

$$u'_{1E} = \theta_1 \qquad u'^{2}_{1E} = \theta_2 \qquad u'_{2E} = L \cdot u \tag{3.0.4-6}$$

the non-dimensional variables being referred to the system axes

$$x_1 = x_I/L \;,\; x_2 = x_{II}/L \;\text{ and }\; x_3 = x_{III}/L$$

3.1.1 The Governing Equations

Once again the problem will be formulated by starting from the first-principles of mechanics.

Hence, KINEMATICS, the change in the distance between two arbitrary infinitely near points, STATICS, the equilibrium conditions for an arbitrary element of volume of the deformed body, and the CONSTITUTIVE RELATIONS implementing a causality condition between static and kinematic variables, will be treated separately and combined in the end to obtain the problem governing equations.

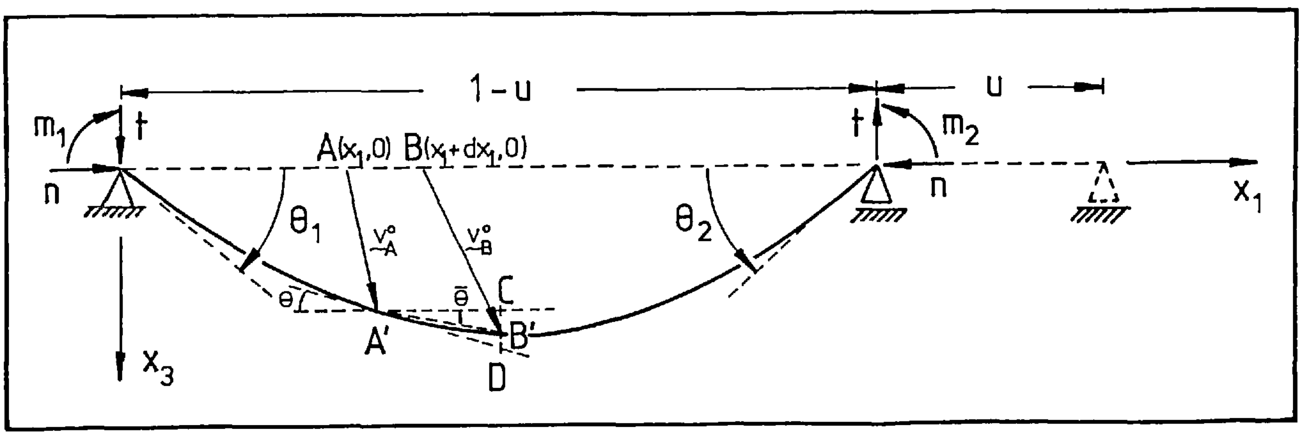


FIGURE 3.3

KINEMATICS:

Consider a generic point A of the beam centroidal locus and a point B in the neighbourhood of A, as represented in Fig.3.3.

Let $\underset{\sim}{r}$ ($\underset{\sim}{R}$) be the positional vector of point A before (after) the increase in the end-forces. The relative (non-dimensional) distance between points A and B prior to and after deformation are respectively

$$ds = (d\underset{\sim}{r}^T d\underset{\sim}{r})^{\frac{1}{2}} \qquad (3.1.1a)$$

$$ds_* = (d\underset{\sim}{R}^T d\underset{\sim}{R})^{\frac{1}{2}} \qquad (3.1.1b)$$

or, from Fig.3.3

$$\begin{aligned} ds_* &= [(d\underset{\sim}{r}+d\underset{\sim}{v}^o)^T(d\underset{\sim}{r}+d\underset{\sim}{v}^o)]^{\frac{1}{2}} \\ &= [2d\underset{\sim}{r}^T d\underset{\sim}{v}^o + d\underset{\sim}{v}^{oT} d\underset{\sim}{v}^o + ds^2]^{\frac{1}{2}} \end{aligned} \qquad (3.1.2)$$

where $\underset{\sim}{v}^o$ describes now the (non-dimensional) displacement field.

Let the parameter D define the ratio

$$D = \frac{ds_*}{ds} \qquad (3.1.3a)$$

or, from (3.1.2)

$$D = [(1+v^o_{1,1})^2 + v^{o2}_{3,1}]^{\frac{1}{2}} \qquad (3.1.3b)$$

where $v^o_{1,1}$ and $v^o_{3,1}$ are the first derivatives, with respect to x_1, of the axial and transverse components of the displacement field v^o.

The linear constraction suffered during deformation by the fiber joining the two points A and B can be represented by Cauchy's classical definition

$$e^o = 1-D \qquad (3.1.4)$$

The rotation suffered in the neighbourhood of point A is defined by the limit

$$\theta = \lim_{ds_*\to 0} \bar{\theta} = \text{arc tan}\, \frac{v^o_{3,1}}{1+v^o_{1,1}} \qquad (3.1.5)$$

where $\bar{\theta}$ represents the relative rotation between points A and B;

equation (3.1.5) together with equations (3.1.3) yield

$$1+v^{o}_{1,1} = D\cdot\cos\theta \qquad (3.1.6a)$$

$$v^{o}_{3,1} = D\cdot\sin\theta \qquad (3.1.6b)$$

The (non-dimensional) curvature is by definition

$$\varkappa = \frac{d\theta}{ds_{*}} \qquad (3.1.7a)$$

or, changing from curvilinear co-ordinates to the adopted system of reference

$$\varkappa = D^{-1}\theta_{,1} \qquad (3.1.7b)$$

In some cases, as is usually done in plate analysis and in the approximate beam theories, it is convenient to express the curvature in terms of the displacement components. After a set of suitable transformations based on equations (3.1.6) and (3.1.7b), the well-known expression

$$\varkappa = D^{-3}[v^{o}_{3,11}(1+v^{o}_{1,1})-v^{o}_{3,1}\cdot v^{o}_{1,11}] \qquad (3.1.7c)$$

where D is given by (3.1.3b), is obtained.

The exact expression for the curvature is rarely used. A literature survey shows that the most commonly used expressions are, following James et alia (1974),

1. The linear beam curvature of elementary mechanics

$$\varkappa = v^{o}_{3,11}$$

2. The classical non-linear beam curvature of elementary mechanics

$$\varkappa = v^{o}_{3,11}(1+v^{o\,2}_{3,1})^{-\frac{3}{2}} \quad v^{o}_{3,11}(1-\tfrac{3}{2}v^{o\,2}_{3,1})$$

3. The small strain-large displacement beam curvature of Novozhilov (1953)

$$\varkappa = v^{o}_{3,11}(1+v^{o}_{1,1})-v^{o}_{3,1}\cdot v^{o}_{1,1}$$

The exact curvature expression will be adopted herein when deriving the governing equation which will be successively approximated in such a way that a global consistency is maintained.

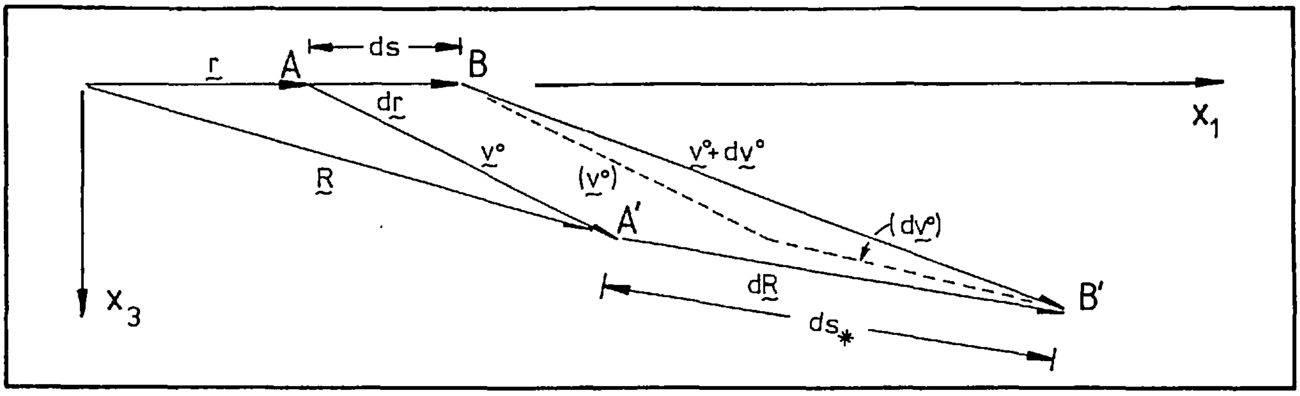


FIGURE 3.4

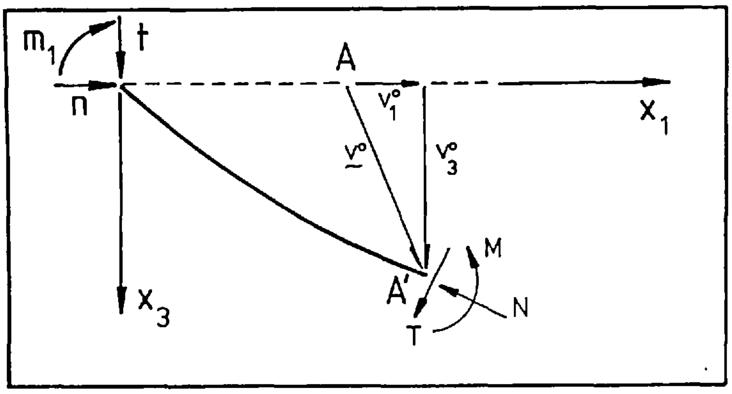


FIGURE 3.5

STATICS:

Let us consider now, with the help of Figs.3.3 and 3.5, the conditions for the equilibrium of the deformed body.

The moment equilibrium, expressed as

$$t = \frac{m_1 - m_2}{1-u} \qquad (3.1.8)$$

is the only non-trivial one of the three possible static boundary conditions; u represents the (non-dimensional) chord shortening.

The stress-resultants acting, after deformation, at a section at distance x_1 from the origin before the deformation are defined by

$$\begin{cases} M(x_1) = m_1 + n \cdot v_3^o - t(x_1 + v_1^o) & (3.1.9a) \\ N(x_1) = n \cdot \cos\theta + t\sin\theta & (3.1.9b) \\ T(x_1) = n \cdot \sin\theta - t\cos\theta & (3.1.9c) \end{cases}$$

where the usual sign convention of elementary mechanics is adopted.

The equilibrium conditions at a generic cross-section can be expressed, in a non-dimensional form, as

$$\begin{cases} M(x_1) = -\int (\sigma-\sigma^0)x_3 d\Omega & (3.1.10a) \\ N(x_1) = \int \sigma d\Omega & (3.1.10b) \\ T(x_1) = \int \tau d\Omega & (3.1.10c) \end{cases}$$

where σ^0 is the axial stress at the centroidal plane, σ is the axial stress at a distance x_3 from the neutral plane and τ is the tangential stress, all scaled by the modulus of elasticity, E.

The non-dimensional cross-sectional area parameter

$$\Omega = A\frac{L^2}{I} \qquad (3.1.11)$$

is the square of the member slenderness ratio

$$S_R = L\sqrt{\frac{A}{I}} \qquad (3.1.12)$$

As the Poisson effects have been neglected, A represents the cross-sectional area of the undeformed beam.

CONSTITUTIVE RELATIONS:

Assume, following Britvec (1973), that the axial force is applied before the end-couples are active and that at a certain level of axial compression the end-couples are increased from zero so that the member is deformed in a state of plane stress.

The (non-dimensional) elongation de of a fibre at distance $-x_3$ from the neutral plane can be expressed in the form

$$de = (\sigma-\sigma^0)ds \qquad (3.1.13a)$$

where both σ^0 and σ are assumed to be compressive stresses; the corresponding strain is defined by

$$\varepsilon = \frac{de}{ds_*}$$

or, from equations (3.1.13a), (3.1.3a) and (3.1.4)

$$\varepsilon = \frac{\sigma-\sigma^0}{1-e^0} \qquad (3.1.13b)$$

The Bernoulli-Euler assumption requires that

$$\varkappa = \frac{\varepsilon}{x_3}$$

or from (3.1.13b)

$$\sigma - \sigma^o = (1-e^o)\cdot x_3 \cdot \varkappa$$

which together with equation (3.1.10a) define the flexural aspect of the Constitutive Relations

$$\varkappa = - \frac{M}{1-e^o} \tag{3.1.14a}$$

The above equation ceases to be valid if the shear deformations are to be considered.

To include a crude correction following Timoshenko and Gere (1961), and since a definitive quantification of the shear deformation effects is yet to be established, equation (3.1.14a) can be replaced by the following:

$$\varkappa = - \frac{M}{1-e^o} + a'(n\cdot\cos\theta + t\sin\theta)\,\frac{d\theta}{ds_*} \tag{3.1.14b}$$

where $$a' = a\,K_s\,\frac{E}{G} \tag{3.1.15a}$$

and $$a = S_R^{-2} \tag{3.1.15b}$$

K_s being the usual shear coefficient and G the shear modulus.

It remains to define the aspect of the Constitutive Relations associating the axial deformations with the axial stress-resultant

$$e^o = a\,N \tag{3.1.16}$$

THE ELASTICA:

After the elimination of the curvature $\varkappa$ and the contraction e^o through equations (3.1.7) and (3.1.4), respectively, equation (3.1.14b) takes the form:

$$\theta_{,1} - a'(n\cdot\cos\theta + t\sin\theta)\theta_{,1} = -M$$

or $$\theta_{,1}-a'(n\cdot\sin\theta-t\cos\theta)_{,1}=-M \qquad (3.1.17)$$

Taking advantage of the fact that the first derivative of the bending moment, as defined in (3.1.9a), can be defined as a function of the rotation θ only, i.e.

$$M_{,1}=nv^{0}_{3,1}-t(1+v^{0}_{1,1})$$

and from (3.1.6)

$$M_{,1}=D(n\cdot\sin\theta-t\cos\theta)$$

after differentiation equation (3.1.17) can be re-written as

$$\theta_{,11}-a'(n\cdot\sin\theta-t\cos\theta)_{,11}+D(n\cdot\sin\theta-t\cos\theta)=0$$

or from (3.1.4) and (3.1.16) together with (3.1.9b)

$$\boxed{\theta_{,11}-a'(n\cdot\sin\theta-t\cos\theta)_{,11}+[1-a(n\cos\theta+t\sin\theta)](n\cdot\sin\theta-t\cos\theta)=0} \qquad (3.1.18)$$

In the above equation, the end-force t is defined by (3.1.8); The shortening parameter u can be found by integrating equation (3.1.6a) and eliminating the contraction e^{0} through equations (3.1.16) and (3.1.9b), yielding

$$u = 1-\int_{0}^{1}[1-a(n\cdot\cos\theta+t\sin\theta)]\cos\theta \; dx_{1} \qquad (3.1.19)$$

The Euler equation can be recovered by imposing in equation (3.1.18) the inextensibility condition $e^{0}=0$, or from (3.1.16)

$$a = 0$$

which, from (3.1.15a), implies that

$$a' = 0$$

and by restricting the end-loads to the axial forces:

$$\theta_{,11}+n\cdot\sin\theta = 0$$

On the other hand, if a' is set to zero, equation (3.1.18) reduces to the well known differential equation governing axially deformable beam-columns, as given for instance in Britvec (1973).

3.1.2 Solution of the Governing Equations

As summarized by Thompson and Hunt (1973), the several methods of solution proposed so far tend to fall into two classes: continuum approaches where one or a series of linear differential equations are generated; and discrete approaches which generate an ordered series of linear algebraic equations.

The closed form solutions are undoubtdly the most elegant of the continuum approaches which are, in general, mathematically more attractive than the discrete approaches.

Since, within a limited applicability, the complexity of the results provided by closed form solutions increases tremendously with the generality of the problem, its utilization in Structural Mechanics has been restricted to the study of very simple systems, such as, for instance, those analyzed by Mitchell (1959), Frisch-Fay (1962), Schille and Sierakowsky (1967) and Kerr (1964).

On the other hand, perturbation techniques, as presented for example by Bellman (1966), Nayfeh (1973) and Yakubovich and Starzhinskii (1975), have proved to be highly adaptable and have been extensively applied. Ames (1965), Thompson (1969), Hangai and Kawamata (1972,1973) and Gallagher (1975), to mention a few, refer to several problems in engineering mechanics solved by this technique. A standard perturbation technique, first applied by Linsted in 1883 according to Stocker (1950), will be used herein to solve the differential equation (3.1.18).

Assume then that variables θ, n, t, m_1 and m_2 (say y) can be expressed in a power series

$$y(x_1,\varepsilon) = \sum_{i=0}^{\infty} y_i \frac{\varepsilon^i}{i!} \qquad (3.1.20)$$

as functions of an arbitrary variable parameter ε, independent of x_1. The remaining variables, say u and e^o, and functions of variables, as $\sin\theta$ and $\cos\theta$, must be expressed in a power series

$$z = \sum_{i=0}^{\infty} z_i \frac{\varepsilon^i}{i!} \qquad (3.1.21)$$

where, in general, $z_i = z_i(\theta_j, n_j, m_{1_j}, m_{2_j})$, $j = 0,1,\ldots,i$

Let c_i (s_i) be that coefficient in the representation of $\cos\theta$ ($\sin\theta$). The products $n\cdot\sin\theta$ and $n\cdot\cos\theta$ may also be represented in the form (3.1.21) with coefficients

$$s_i^n = i! \sum_{j=0}^{i} \frac{n_j}{j!} \cdot \frac{s_{i-j}}{(i-j)!} \qquad (3.1.22a)$$

and

$$c_i^n = i! \sum_{j=0}^{i} \frac{n_j}{j!} \cdot \frac{c_{i-j}}{(i-j)!} \qquad (3.1.22b)$$

Equivalent forms can be written for the coefficients s_i^t and c_i^t for the representation of $t\cdot\sin\theta$ and $t\cdot\cos\theta$, respectively, just by replacing in equations (3.1.22) n_j by t_j.

Following several substitutions and after equating terms of the same power of ε, equation (3.1.18) gives rise to the infinite system of differential equations

$$\boxed{\theta_{i,11} - a'(s_i^n - c_i^t)_{,11} + (s_i^n - c_i^t) - a\cdot i! \sum_{j=0}^{i} \frac{c_j^n + s_j^t}{j!} \cdot \frac{s_{i-j}^n - c_{i-j}^t}{(i-j)!} = 0} \qquad (3.1.23a)$$

which, from (3.1.17), is subject to the boundary conditions

$$\theta_{i,1}(0) = -m_{1_i} + a'(s_i^n - c_i^t)_{,1}^{x_1=0} \qquad (3.1.23b)$$

$$\theta_{i,1}(1) = -m_{2_i} + a'(s_i^n - c_i^t)_{,1}^{x_1=1} \qquad (3.1.23c)$$

The coefficients u_i of the shortening parameter expansion are, from equation (3.1.19)

$$u_i = \delta_{oi} - \int_0^1 (c_i - e_i^!)\, dx_1 \qquad (3.1.24a)$$

where

$$e_i^! = a\cdot i! \sum_{j=0}^{i} \frac{c_j^n + s_j^t}{j!} \cdot \frac{c_{i-j}}{(i-j)!} \qquad (3.1.24b)$$

and δ_{rs} (r=0,s=i) is the Kronecker delta.

Replacing in equation (3.1.8) the variables m_1, m_2 and t in the expanded form (3.1.20) and the shortening parameter u in the form (3.1.21) with coefficients defined by (3.1.24), the static boundary conditions (3.1.8) generates the following infinite system of equations

$$i! \sum_{j=0}^{i} \frac{t_{i-j}}{j!(i-j)!} \int_0^1 (c_j - e'_j)\, dx_1 = m_{1_i} - m_{2_i} \tag{3.1.25}$$

If the trignometric functions are replaced by power series expansions on the rotation θ, the zero-th order problem, defined by setting i=0 in the previous equations, is, neglecting temporarily the shear deformation effects, defined by

$$\begin{cases} \theta_{o,11} + \theta(-t_o, n_o)\cdot[1-a\cdot\theta(n_o,t_o)] = 0 \\ t_o \cdot \int_0^1 \theta(\ 1,\ 0\ )\cdot[1-a\cdot\theta(n_o,t_o)]\ dx_1 = m_{1_o} - m_{2_o} \end{cases}$$

where $\theta(\xi, \eta) = \xi + \eta\theta_o - \frac{1}{2}\xi\theta_o^2 - \frac{1}{6}\eta\theta_o^3 + \frac{1}{24}\xi\theta_o^4$

To evade the situation of ending up with a problem apparently as difficult to solve as the initial one, it is necessary to particularize the initial state.

A systematic and orderly solution procedure is achieved if the perturbation parameter ε is such that it is possible to impose a flexurally unstrained initial state, reducing the zero-th order solution to the trivial axially loaded column:

$$\theta_o = v^o_{3_o} = 0, \quad v^o_{1_o} = -a\cdot n_o\cdot x_1, \quad m_{1_o} = m_{2_o} = t_o = 0 \tag{3.1.26a-f}$$

Furthermore, and notably, the infinite system (3.1.23a) becomes recursive and every constituent differential equation will take the general form

$$\theta_{i,11} + g_o^2\,\theta_i = b_o\,(m_{1_i} - m_{2_i}) + R_i(x_1) \tag{3.1.27}$$

where b_o and g_o are constants defined by

$$b_o = (1-a'\cdot n_o)^{-1}, \quad g_o = [(1-a\cdot n_o)\ b\cdot n_o]^{\frac{1}{2}} \tag{3.1.28a,b}$$

and $R_i(x_1)$ is a non-linear function of variables of order lower than the i-th; for instance for the first- and second-order equations $R_i(x_1)$ takes respectively the values

$$R_1(x_1) = 0$$

and
$$R_2(x_1) = 2b_o^2 n_1 [a'(m_{1_1} - m_{2_1}) - (1 - 2an_o + aa'n_o^2)\theta_1]$$

As the function $R_i(x_1)$ is known and integrable for each i-th order equation, the general solution of the differential equation (3.1.27) is defined by

$$\theta_i(x_1) = S_1 C_i^1 + S_2 C_i^2 + \frac{b_o}{g_o^2}(m_{1_i} - m_{2_i})(1 - S_2) + \frac{S_1}{g_o} Y_i^2 - S_2 Y_i^1 \qquad (3.1.29)$$

where
$$S_1(x_1) = \sin g_o x_1$$
$$S_2(x_1) = \cos g_o x_1$$
and
$$Y_i^j(x_1) = \int_o^{x_1} R_i S_j dx_1 \quad , \quad j=1,2$$

The constants of integration C_i^1 and C_i^2 can be found from the i-th order boundary conditions (3.1.23b,c), which will take the general form

$$\begin{cases} \theta_{i,_1}(0) = -b_o m_{1_i} + B_i(0) & (3.1.30a) \\ \theta_{i,_1}(0) = -b_o m_{2_i} + B_i(1) & (3.1.30b) \end{cases}$$

The auxiliary function $B_i(x_1)$ is structurally similar to the function $R_i(x_1)$ in the sense that it depends on variables of order lower than the i-th; for instance, for the first- and second-order boundary conditions B_i is defined by

$$\begin{cases} B_1(0) = 0 \\ B_1(1) = 0 \end{cases}$$

and
$$\begin{cases} B_2(0) = -2a'b_o^2\, n_i\, m_{1_1} \\ B_2(1) = -2a'b_o^2\, n_i\, m_{2_1} \end{cases}$$

respectively.

Differentiating once equation (3.1.29) with respect to

x_1, and noticing that

$$(S_1 Y_i^2 - S_2 Y_i^1)_{,1} = g_o[S_2 Y_i^2 + S_1 Y_i^1 + \frac{R_i}{g_o}(S_1 S_2 - S_2 \cdot S_1)]$$

we find

$$\theta_{i,1}(x_1) = S_2[g_o C_i^1 + Y_i^2] - S_1[g_o C_i^2 - \frac{b_o}{g_o}(m_{1_i} - m_{2_i}) - Y_i^1]$$

which together with the boundary conditions (3.1.30) give the following general expressions for the constants of integration

$$g_o\ C_i^1 = -b_o\ m_{1_i} + \bar{C}_i^1$$

and
$$g_o\ C_i^2 = \frac{b_o}{g_o}\left[1 - g_o\frac{S_2(1)}{S_1(1)}\right]m_{1_i} - \frac{b_o}{g_o}\left[1 - g_o\frac{1}{S_1(1)}\right]m_{2_i} + \bar{C}_i^2$$

where
$$\bar{C}_i^1 = B_i(0)$$

and
$$\bar{C}_i^2 = S_2(1)\frac{B_i(0)}{S_1(1)} + \frac{B_i(1)}{S_1(1)} + \frac{S_2(1)}{S_1(1)}Y_i^2(1) + Y_i^1(1)$$

After finding the general expression for the i-th order rotation θ_i, the same order shortening parameter u_i can be obtained through equations (3.1.24), the end-rotations being defined by

$$\begin{cases} \theta_{1_i} = \theta_i(0) \\ \theta_{2_i} = -\theta_i(1) \end{cases}$$

The i-th order stress-resultants can be recovered from

$$M_i(x_1) = -\theta_{i,1} + a'(s_i^n - c_i^t)_{,1} \qquad (3.1.31a)$$

$$N_i(x_1) = c_i^n + s_i^t \qquad (3.1.31b)$$

$$T_i(x_1) = s_i^n - c_i^t \qquad (3.1.31c)$$

and the same order components of the displacement field are defined by

$$v_{1_i}^o(x_1) = \int\left[c_i - ai!\sum_{j=0}^{i}\frac{c_j^n + s_j^t}{j!}\cdot\frac{c_{i-j}}{(i-j)!} - \delta_{oi}\right]dx_1 + V_i^1 \qquad (3.1.32a)$$

and $$v^o_{3_i}(x_1) = \int \left[s_i - a \cdot i! \sum_{j=0}^{i} \frac{c^n_j + s^t_j}{j!} \cdot \frac{s_{i-j}}{(i-j)!} \right] dx_1 + V^3_i \qquad (3.1.32b)$$

where the constants of integration can be found from the boundary conditions

$$v^o_{1_i}(0) = 0 \qquad \text{and} \qquad v^o_{3_i}(0) = 0$$

The solutions we obtained for the Elastica, up to the second-order and after a lengthy though elementary algebra, trough the application of this standard perturbation technique are summarized in the following sub-sections 3.1.3 to 3.1.5 in formats suitable for future application in the analysis of elastic and elastoplastic frames.

3.1.3 Asymptotic Analysis

The direct results provided by the perturbation analysis are summarized below, now in terms of the non-scaled variables

FLEXIBILITY FORMULATION
$\underset{\sim}{u}'_{E_i} = \underline{F}^o \, \underset{\sim}{X}'_i + \underset{\sim}{u}'_{E\pi_i}$

(3.1.33)

the zeroth-, first- and second-order (symmetric) flexibility matrices and additional elastic deformations being respectively defined by

ZEROTH-ORDER	FIRST-ORDER	SECOND-ORDER
$\underline{F}^o = \left[\begin{array}{c\|c} \underline{0} & \underset{\sim}{0} \\ \hline \underset{\sim}{0}^T & \frac{L}{EA} \end{array}\right]$	$\underline{F}^o = \left[\begin{array}{c\|c} \underline{f}^o & \underset{\sim}{0} \\ \hline \underset{\sim}{0}^T & \frac{L}{EA} \end{array}\right]$	$\underline{F}^o = \left[\begin{array}{c\|c} \underline{f}^o & \underset{\sim}{0} \\ \hline \underset{\sim}{0}^T & \frac{L}{EA} \end{array}\right]$
$\underset{\sim}{u}_{E\pi_0} = \underset{\sim}{0}$	$\underset{\sim}{u}_{E\pi_1} = \underset{\sim}{0}$	$\underset{\sim}{u}_{E\pi_2} = \left[\begin{array}{c} 2X'_{2_1} \underline{f}'^o_2 \underset{\sim}{X}'_{1_1} \\ \hline \underset{\sim}{X}'^T_{1_1} \underline{f}'^o_2 \underset{\sim}{X}'_{1_1} \end{array}\right]$

(3.1.34a,b) (3.1.35a,b) (3.1.36a,b)

where $\underline{f}^o$ is the (symmetric) flexural flexibility matrix, with elements

$$f^o_{11} = f^o_{22} = \frac{L}{EI} \cdot \frac{b_o}{g_o^2} \cdot (1 - g_o \text{ cotan } g_o) \qquad (3.1.37a)$$

$$f^o_{12} = f^o_{21} = -\frac{L}{EI} \cdot \frac{b_o}{g_o^2} \cdot (1 - g_o \text{ cosec } g_o) \qquad (3.1.37b)$$

and $\underline{f}^o_{,2}$ is the first derivative of $\underline{f}^o$ with respect to X'_{2_o} and characterizes the chord shortening due to bending.

The expressions for the flexibility coefficients as presented by Britvec and Chilver (1963) and Roorda and Chilver (1970) can be recovered by setting in the definitions (3.1.37) first a' to zero, neglecting the shear deformation effects, and then a to zero to simulate axially inextensible members; in this case the parameters b_o and g_o as defined in (3.1.28), reduce to

$$b_o = 1 \quad \text{and} \quad g_o = \sqrt{n_o}\ , \text{ where } n_o = \frac{L^2}{EI} X'_{2_o}$$

Letting $\underline{K}^o=(\underline{F}^o)^{-1}$, the stiffness version of the formulation (3.1.33) is found to be

STIFFNESS FORMULATION	
$\underset{\sim}{X}'_i = \underline{K}^o\ \underset{\sim}{u}'_{E_i} + \underset{\sim}{X}'_{\pi_i}$	(3.1.38)

where now

ZEROTH-ORDER	FIRST-ORDER	SECOND-ORDER
$\underline{K}^o=\begin{bmatrix} \underline{0} & \underset{\sim}{0} \\ \underset{\sim}{0}^T & \frac{EA}{L} \end{bmatrix}$	$\underline{K}^o=\begin{bmatrix} \underline{k}^o & \underset{\sim}{0} \\ \underset{\sim}{0}^T & \frac{EA}{L} \end{bmatrix}$	$\underline{K}^o=\begin{bmatrix} \underline{k}^o & \underset{\sim}{0} \\ \underset{\sim}{0}^T & \frac{EA}{L} \end{bmatrix}$
$\underset{\sim}{X}'_{\pi_o} = \underset{\sim}{0}$	$\underset{\sim}{X}'_{\pi_1} = \underset{\sim}{0}$	$\underset{\sim}{X}'_{\pi_2} = \begin{bmatrix} 2X'_{2_1}\underline{k}^o_{,2}\underset{\sim}{u}'_{1E_1} \\ \hline \frac{EA}{L}\underset{\sim}{u}'^T_{1E_1}\underline{k}^o_{,2}\underset{\sim}{u}'_{1E_1} \end{bmatrix}$
(3.1.39a,b)	(3.1.40a,b)	(3.1.41a,b)

since

$$\underline{k}^o_{,2} = -\underline{k}^{o^T}\ \underline{f}^o_{,2}\ \underline{k}^o \qquad (3.1.42)$$

where the elements of the first-order symmetric flexural stiffness matrix

$$\underline{k}^o = \underline{f}^{o^{-1}}$$

are given by

$$k^o_{11} = k^o_{22} = \frac{EI}{L}b_o^{-1}\left[\left(\frac{g_o}{2}\right)\text{cotan}\left(\frac{g_o}{2}\right)+\frac{\left(\frac{g_o}{2}\right)^2}{1-\left(\frac{g_o}{2}\right)\text{cotan}\left(\frac{g_o}{2}\right)}\right] \qquad (3.1.43a)$$

$$k_{12}^o = k_{21}^o = \frac{EI}{L}\, b_o^{-1}\left[(\frac{g_o}{2})\ \text{cotan}(\frac{g_o}{2}) - \frac{(\frac{g_o}{2})^2}{1-(\frac{g_o}{2})\text{cotan}(\frac{g_o}{2})}\right] \qquad (3.1.43b)$$

Their derivatives, defining the stiffness bowing functions $k_{ij,2}^o$, can be found either directly from (3.1.43) or through equation (3.1.42) if the flexibility bowing functions were already available.

Equations (3.1.37) and (3.1.43) were obtained assuming the beam subject to a compressive axial force; letting n' be the absolute value of the (non-dimensional) tensile axial force, and replacing n_i' by $-n_i'$ and using, where convenient, the relationships between trignometric and hyperbolic functions, definitions (3.1.37) and (3.1.43) reduce, respectively, to

$$f_{11}^o=f_{22}^o=-\frac{L}{EI}\cdot\frac{b_o'}{g_o'^2}\left(1-g_o'\text{cotanh}g_o'\right) \qquad (3.1.44a)$$

$$f_{12}^o=f_{21}^o=\frac{L}{EI}\cdot\frac{b_o'}{g_o'^2}\left(1-g_o'\text{cosech}g_o'\right) \qquad (3.1.44b)$$

and $$k_{11}^o=k_{22}^o=\frac{EI}{L}\, b_o'^{-1}\left[(\frac{g_o'}{2})\text{cotanh}(\frac{g_o'}{2}) - \frac{(\frac{g_o'}{2})^2}{1-(\frac{g_o'}{2})\text{cotanh}(\frac{g_o'}{2})}\right] \qquad (3.1.45a)$$

$$k_{12}^o=k_{21}^o=\frac{EI}{2}\, b_o'^{-1}\left[(\frac{g_o'}{2})\text{cotanh}(\frac{g_o'}{2}) + \frac{(\frac{g_o'}{2})^2}{1-(\frac{g_o'}{2})\text{cotanh}(\frac{g_o'}{2})}\right] \qquad (3.1.45b)$$

where now $$b_o' = 1+a'\cdot n_o' \qquad (3.1.46a)$$

$$g_o' = \left[(1+a\cdot n_o')b_o'\ n_o'\right]^{1/2} \qquad (3.1.46b)$$

and $$n_o' = \left|X_{2o}'\right| \frac{L^2}{EI} \qquad (3.1.46c)$$

3.1.4 Deformation Analysis

The beam element behaviour can be expressed in terms of total, finite variables, by reversing the perturbation procedure, i.e., we may reconstruct the generic variable y by substituting the solutions found for y_o, y_1, ..., y_∞ back in equation (3.1.20).

Take for instance the beam end-rotations $\underset{\sim}{u}'_{1E}$; by definition

$$\underset{\sim}{u}'_{1E} = \underset{\sim}{u}'_{1E_o} + \underset{\sim}{u}'_{1E_1}\varepsilon + \underset{\sim}{u}'_{1E_2}\frac{\varepsilon^2}{2!} + \cdots$$

or, from equations (3.1.33) to (3.1.36)

$$\underset{\sim}{u}'_{1E} = \underline{f}^o\ \underset{\sim}{X}'_{1_1}\varepsilon + (\underline{f}^o\underset{\sim}{X}'_{1_2} + 2X'_{2_1}\underline{f}^o_{,2}\ \underset{\sim}{X}'_{1_1})\ \frac{\varepsilon^2}{2!} + \cdots$$

Regrouping

$$\underset{\sim}{u}'_{1E} = \underline{f}^o(\underset{\sim}{X}'_{1_1}\varepsilon + \underset{\sim}{X}'_{1_2}\frac{\varepsilon^2}{2!} + \cdots) + (X'_{2_1}\varepsilon + \cdots)\underline{f}^o_{,2}(\underset{\sim}{X}'_{1_1}\varepsilon + \cdots) + \cdots$$

and considering the general solution (3.1.29) we may antecipate that

$$\underset{\sim}{u}'_{1E} = \underline{f}^o\ \underset{\sim}{X}'_1 + (X'_2 - X'_{2_o})\underline{f}^o_{,2}\ \underset{\sim}{X}'_1 + O_3 \qquad (3.1.47a)$$

where O_3 designates terms of order three and higher.

Similarly, for the axial shortening

$$u'_{2E} = u'_{2E_o} + u'_{2E_1}\varepsilon + u'_{2E_2}\frac{\varepsilon^2}{2!} + \cdots$$

$$= \frac{L}{EA}(X'_{2_o} + X'_{2_1}\varepsilon + X'_{2_2}\frac{\varepsilon^2}{2!} + \cdots) + \tfrac{1}{2}(\underset{\sim}{X}'_{1_1}\varepsilon + \cdots)^T\underline{f}^o_{,2}(\underset{\sim}{X}'_{1_1}\varepsilon + \ldots) + \ldots$$

or $$u'_{2E} = \frac{L}{EA}X'_2 + \tfrac{1}{2}\ \underset{\sim}{X}'^T_1\ \underline{f}^o_{,2}\ \underset{\sim}{X}'_1 + O_3 \qquad (3.1.47b)$$

Confronting the pattern developing in the above expressions (3.1.47) with that of the Taylor series expansion of matrices $\underline{f}$ and $\underline{f}_{,2}$ in the neighbourhood of point $X'_2 = X'_{2_o}$,

$$\underline{f}\ (X'_2) = \underline{f}^o + (X'_2 - X'_{2_o})\ \underline{f}^o_{,2} + \cdots$$

$$\underline{f}_{,2}(X'_2) = \underline{f}^o_{,2} + \cdots$$

we may write the following flexibility description of the elastic constitutive relations expressed in terms of the total variables

FLEXIBILITY FORMULATION
$\underset{\sim}{u}'_E = \underline{F}\,\underset{\sim}{X}' + \underset{\sim}{u}'_{E\pi}$

(3.1.48)

the zeroth-, first- and second-order symmetric flexibility matrices and additional elastic deformations being respectively defined by

ZEROTH-ORDER	FIRST-ORDER	SECOND-ORDER
$\underline{F} = \begin{bmatrix} \underline{0} & \underset{\sim}{0} \\ \underset{\sim}{0}^T & \frac{L}{EA} \end{bmatrix}$	$\underline{F} = \begin{bmatrix} \underline{f} & \underset{\sim}{0} \\ \underset{\sim}{0}^T & \frac{L}{EA} \end{bmatrix}$	$\underline{F} = \begin{bmatrix} \underline{f} & \underset{\sim}{0} \\ \underset{\sim}{0}^T & \frac{L}{EA} \end{bmatrix}$
$\underset{\sim}{u}'_{E\pi} = \underset{\sim}{0}$	$\underset{\sim}{u}'_{E\pi} = \underset{\sim}{0}$	$\underset{\sim}{u}'_{E\pi} = \begin{bmatrix} \underset{\sim}{0} \\ \frac{1}{2}\underset{\sim}{X}'^T_1\,\underline{f}_{,2}\,\underset{\sim}{X}'_1 \end{bmatrix}$
(3.1.49a,b)	(3.1.50a,b)	(3.1.51a,b)

where $\underline{f}$ is the symmetric flexural flexibility matrix, with elements

$$f_{11} = f_{22} = \frac{L}{EI}\,\frac{b}{g^2}\,(1-g\ \text{cotang}) \qquad (3.1.52a)$$

$$f_{12} = f_{21} = -\,\frac{L}{EI}\,\frac{b}{g^2}\,(1-g\ \text{cosecg}) \qquad (3.1.52b)$$

in which

$$b = (1-a'n)^{-1} \qquad (3.1.53a)$$

$$g = \left[(1-an)bn\right]^{\frac{1}{2}} \qquad (3.1.53b)$$

and

$$n = \frac{L^2}{EI}\,X'_2 \qquad (3.1.53c)$$

A similar procedure would yield the following stiffness description of the elastic constitutive relations

STIFFNESS FORMULATION
$\underset{\sim}{X}' = \underline{K}\,u'_E + \underset{\sim}{X}'_\pi$

(3.1.54)

where now

$$\underline{K} = \underline{F}^{-1}$$

and, for each approximate solution

ZEROTH-ORDER	FIRST-ORDER	SECOND-ORDER
$\underline{K} = \left[\begin{array}{c:c} \underset{\sim}{0} & \underset{\sim}{0} \\ \hdashline \underset{\sim}{0}^T & \frac{EA}{L} \end{array}\right]$	$\underline{K} = \left[\begin{array}{c:c} \underline{k} & \underset{\sim}{0} \\ \hdashline \underset{\sim}{0}^T & \frac{EA}{L} \end{array}\right]$	$\underline{K} = \left[\begin{array}{c:c} \underline{k} & \underset{\sim}{0} \\ \hdashline \underset{\sim}{0}^T & \frac{EA}{L} \end{array}\right]$
$\underset{\sim}{X}'_{\pi} = \underset{\sim}{0}$	$\underset{\sim}{X}'_{\pi} = \underset{\sim}{0}$	$\underset{\sim}{X}'_{\pi} = \left[\begin{array}{c} \underset{\sim}{0} \\ \hdashline \frac{1}{2}\frac{EA}{L}\underset{\sim}{u}'_{1E}\underline{K}_{,2}\underset{\sim}{u}'_{1E} \end{array}\right]$

(3.1.55a,b) (3.1.56a,b) (3.1.57a,b)

The coefficients of the symmetric flexural stiffness matrix

$$\underline{k} = \underline{f}^{-1}$$

are defined by

$$k_{11} = k_{22} = \frac{EI}{L}\, b^{-1}\left[\left(\frac{g}{2}\right)\cot an\left(\frac{g}{2}\right) + \frac{\left(\frac{g}{2}\right)^2}{1-\left(\frac{g}{2}\right)\cot an\left(\frac{g}{2}\right)}\right] \quad (3.1.58a)$$

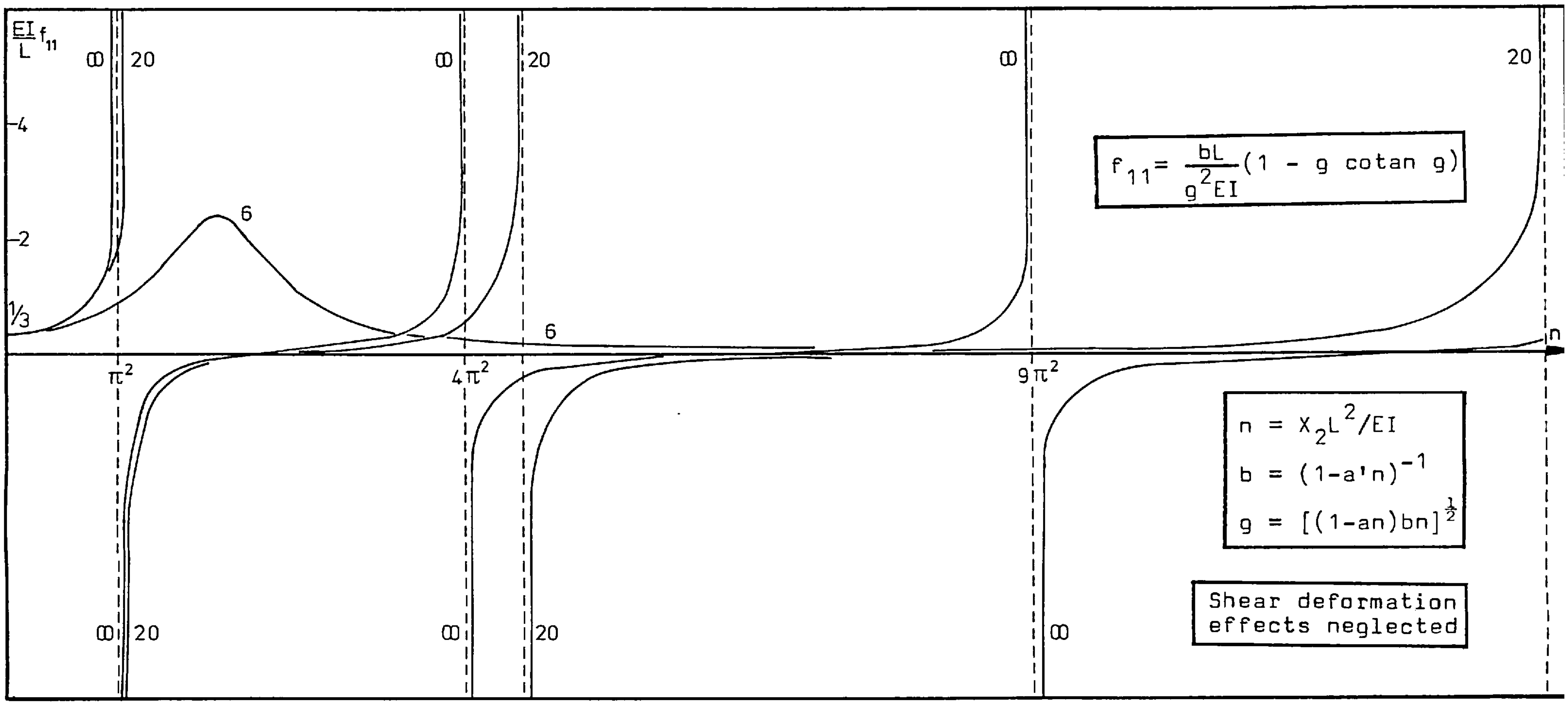
$$k_{12} = k_{21} = \frac{EI}{L}\, b^{-1}\left[\left(\frac{g}{2}\right)\cot an\left(\frac{g}{2}\right) - \frac{\left(\frac{g}{2}\right)^2}{1-\left(\frac{g}{2}\right)\cot an\left(\frac{g}{2}\right)}\right] \quad (3.1.58b)$$

The above formulations will be valid within a range of loading-deformation such that terms affected by powers of the parameter t higher than the i-th can be neglected in the perturbed form of the (unknown) exact formulations.

It could be easily proved that the first-order formulation is valid within a range of deformations such that the fundamental trignometric functions sine and cosine on the maximum rotation θ can be approximated by a power series of order not higher than the first.

The graphs in Figs.3.6 and 3.7 show that the effect of the member slenderness ratio on the flexibility coefficients is relevant only for short members (S_R=20, i.e. a=1/400); Figs. 3.8 and 3.9 illustrate the shear deformation effects.

Similar conclusions could be drawn for the stiffness coefficients (3.1.58); their sensitivity to the axial and shear deformability parameters is illustrated in Fig.3.10.



$\frac{EI}{L} f_{11}$
∞
20
4
2
1/3
6
π^2
$4\pi^2$
$9\pi^2$
n
$f_{11} = \frac{bL}{g^2 EI}(1 - g \cot an\ g)$
$n = X_2 L^2/EI$
$b = (1-a'n)^{-1}$
$g = [(1-an)bn]^{\frac{1}{2}}$
Shear deformation effects neglected

FIGURE 3.6

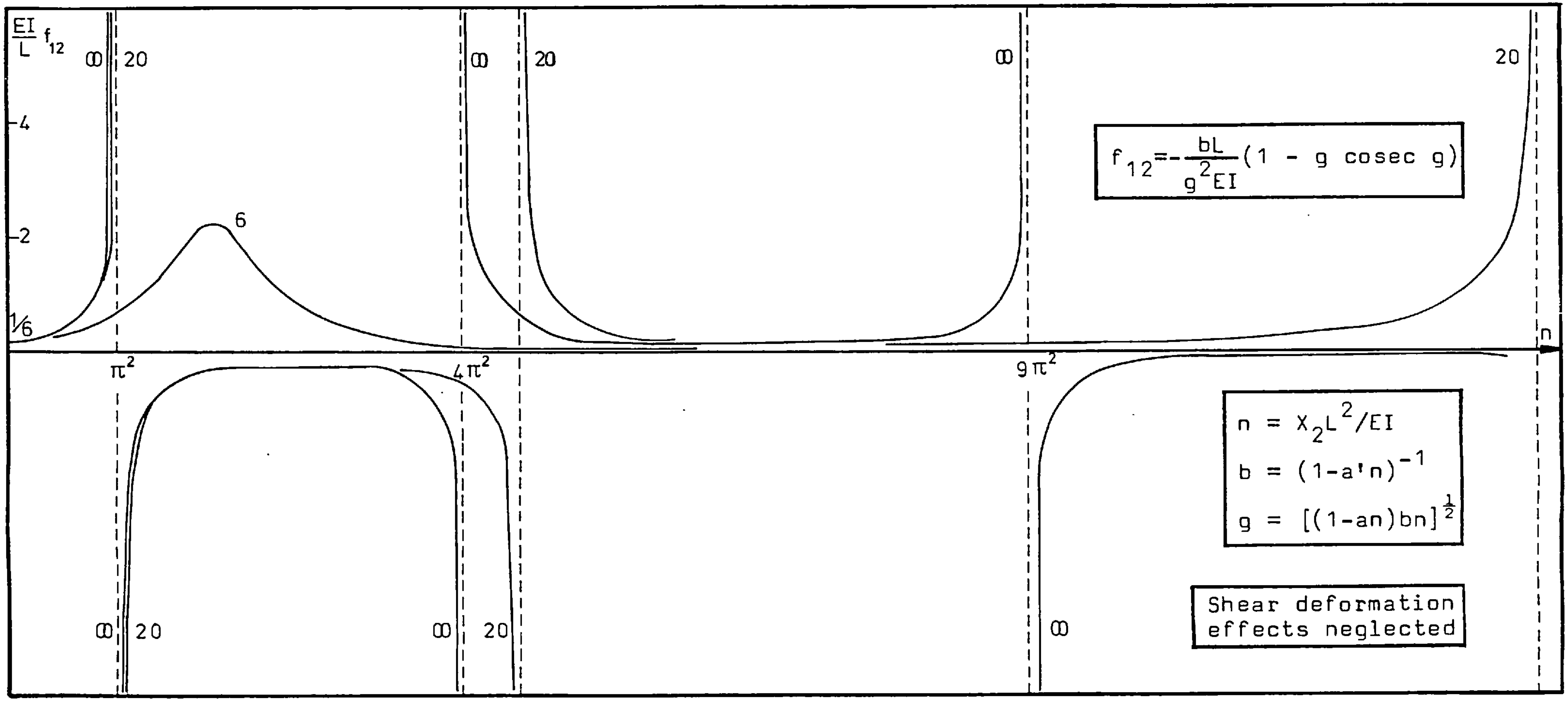

EI/L f12
∞
20
4
2
1/6
6
n
π²
4π²
9π²
f12 = - bL/(g²EI) (1 - g cosec g)
n = X2L²/EI
b = (1-a'n)^-1
g = [(1-an)bn]^½
Shear deformation effects neglected

FIGURE 3.7

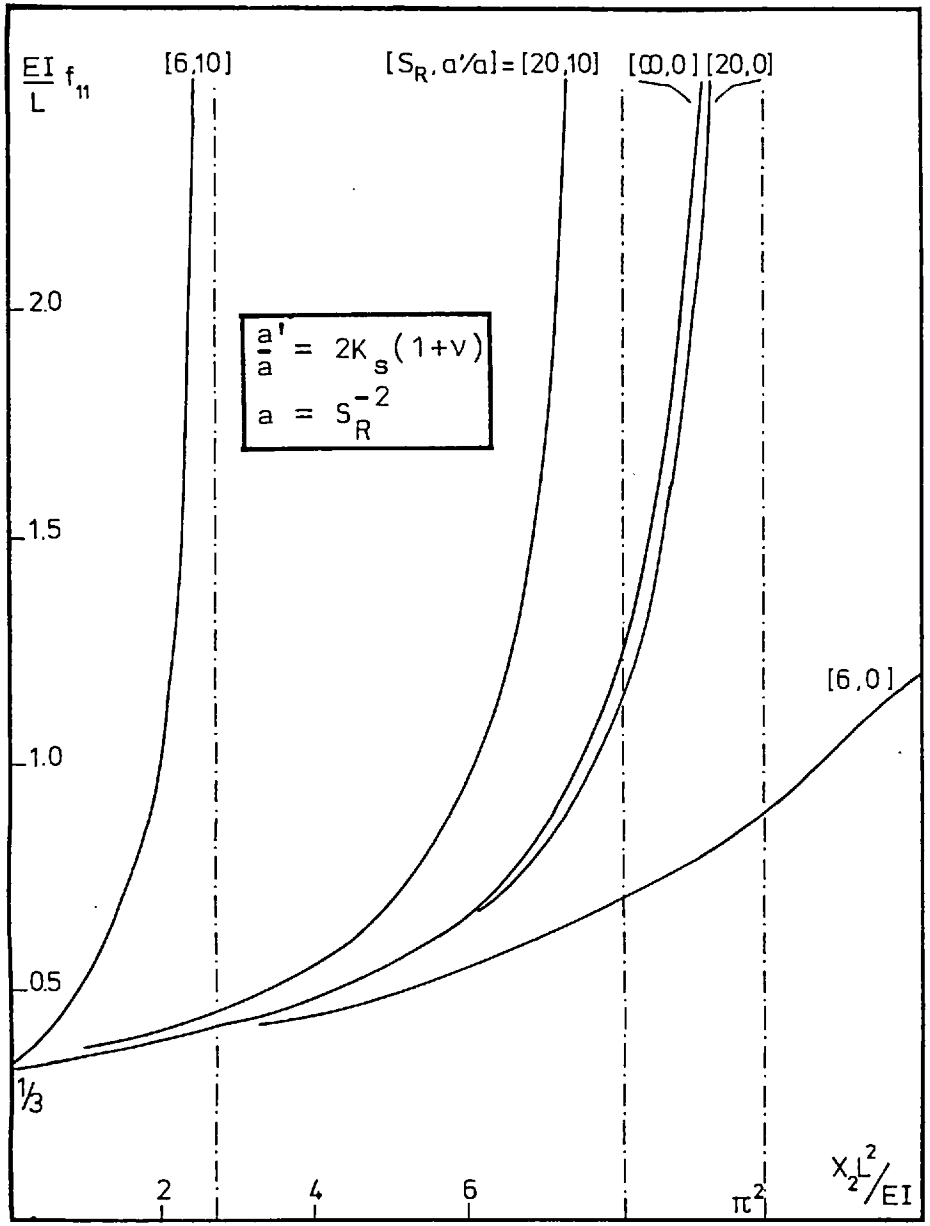


FIGURE 3.8

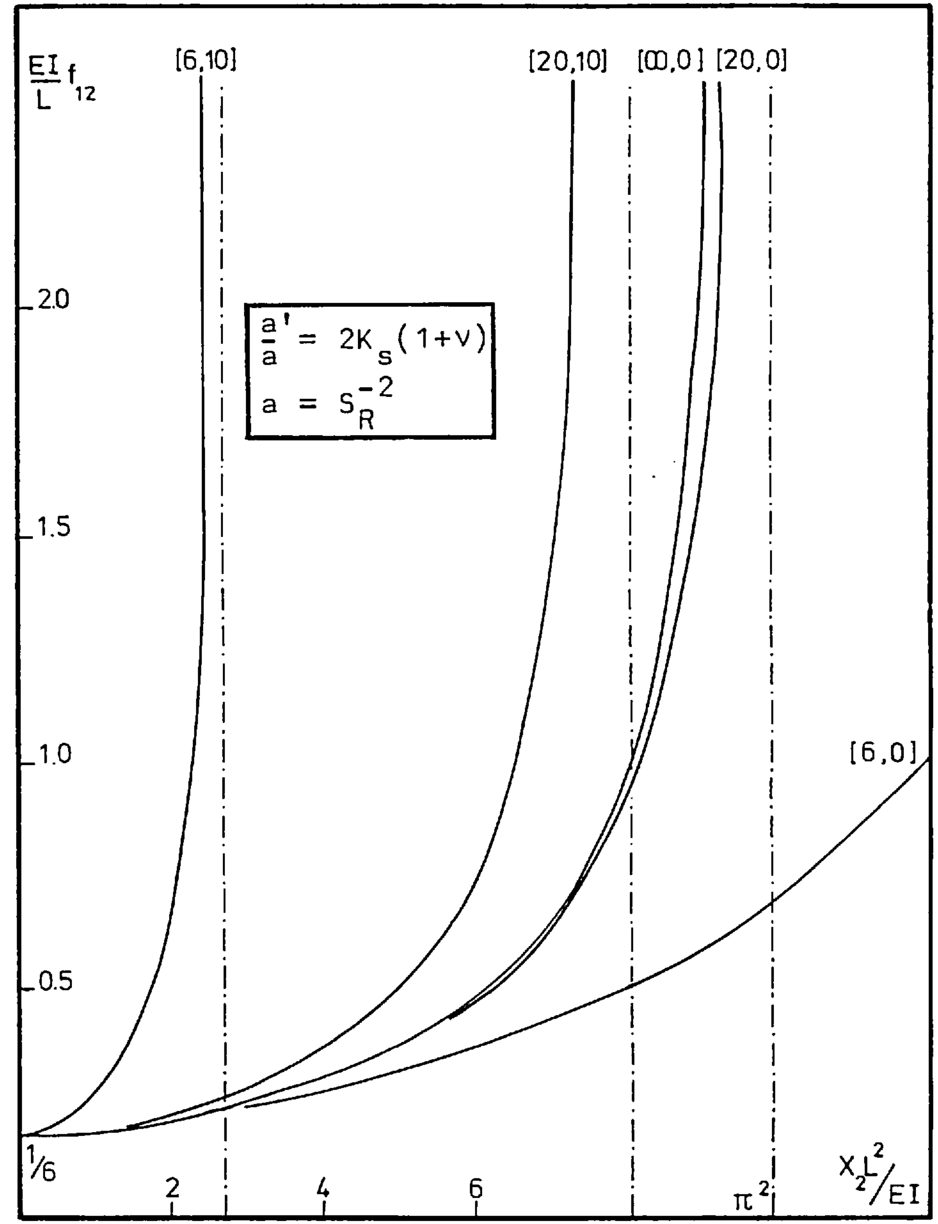


FIGURE 3.9

The flexibility bowing functions are defined by

$$f_{11,2}=f_{22,2}=\left(\frac{L}{EI}\right)^2\frac{L}{2g^2}\left[-2b_1+(b_1-b_2)g\,\mathrm{cotan}\,g+(b_1+b_2)g^2\mathrm{cosec}^2 g\right] \quad (3.1.59a)$$

$$f_{12,2}=f_{21,2}=-\left(\frac{L}{EI}\right)^2\frac{L}{2g^2}\left[-2b_1+(b_1-b_2)g\,\mathrm{cosec}\,g+(b_1+b_2)g^2\mathrm{cosec}\,g\,\mathrm{cotan}\,g\right] \quad (3.1.59b)$$

where $$b_1 = (1-2an)\left(\frac{b}{g}\right)^2 \quad (3.1.60a)$$

and $$b_2 = a'b^2 \quad (3.1.60b)$$

The above functions are represented in Figs.3.11 and 3.12, respectively.

The stiffness bowing functions can either be found from the equality

$$\underline{k}_{,2} = -\underline{k}^T\,\underline{f}_{,2}\,\underline{k} \quad (3.1.61)$$

or directly from (3.1.58).

The definitions for the flexibility and stiffness coefficients, and their derivatives, can be modified into the case of members subject to a tensile axial force through a procedure in every aspect similar to the one indicated in sub-section 3.1.3.

A j-th order approximation for the rotation θ at any point of the beam centroidal locus can be found by combining, in the manner of (3.1.20), all and up to the j-th expressions found for the components $\theta_i\ (x_1)$ through the application of the perturbation technique.

Similarly, general expressions can also be obtained for the stress resultants and for the components of the displacement if at every stage the i-th solution for the bending moment, axial and shear forces and the same order components of the axial and transverse displacements are derived, respectively, from equations (2.1.31) and (2.1.32).

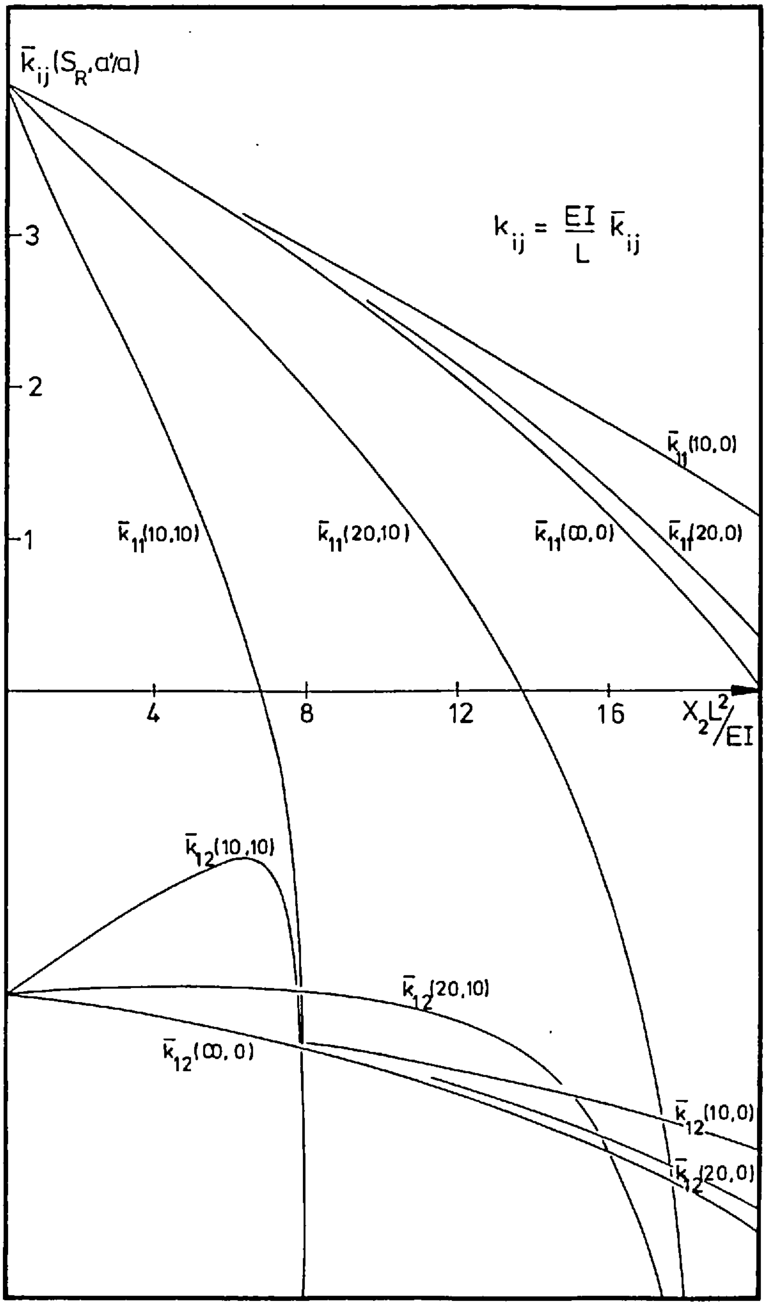


FIGURE 3.10

For instance, the beam-column theory widely presented in the literature can be found by combining the zeroth-order solution (3.1.26) with the first-order solution, defined, in terms of the non-dimensional variables, by

$$\theta_1(x_1) = \frac{b_o}{g_o^2}\left[-m_{1_1}g_o\sin g_o x_1+(-m_{1_1}\cot g_o+m_{2_1}\operatorname{cosec} g_o)g_o\cos g_o x_1+(m_{1_1}-m_{2_1})\right]$$

$$v^o_{3_1}(x_1) = \frac{1}{n_o}\Big[m_{1_1}(\cos g_o x_1 - 1) + (-m_{1_1}\operatorname{cotan} g_o + m_{2_1}\operatorname{cosec} g_o)\sin g_o x_1 + (m_{1_1} - m_{2_1})x_1\Big]$$

$$v^o_{1_1}(x_1) = -an_1 x_1$$

$$M_1\ (x_1) = m_{1_1}\cos g_o x_1 + (-m_{1_1}\operatorname{cotan} g_o + m_{2_1}\operatorname{cosec} g_o)\sin g_o x_1$$

$$N_1\ (x_1) = n_1$$

$$T_1\ (x_1) = n_o\theta_1 - t$$

yielding

$$\theta\ (x_1) = \frac{b}{g^2}\Big[-m_1 g\sin gx_1 + (-m_1\operatorname{cotan} g + m_2\operatorname{cosec} g)g\cos gx_1 + (m_1 - m_2)\Big] \quad (3.1.62a)$$

$$v^o_3\ (x_1) = \frac{1}{n}\Big[m_1(\cos gx_1 - 1) + (-m_1\operatorname{cotan} g + m_2\operatorname{cosec} g)\sin gx_1 + (m_1 - m_2)x_1\Big] \quad (3.1.62b)$$

$$v^o_1\ (x_1) = -anx_1 \quad (3.1.62c)$$

$$M\ (x_1) = m_1\cos gx_1 + (-m_1\operatorname{cotan} g + m_2\operatorname{cosec} g)\sin gx_1 \quad (3.1.62d)$$

$$N\ (x_1) = n \quad (3.1.62e)$$

$$T\ (x_1) = n\theta - t \quad (3.1.62f)$$

The buckling loads, which replace the Euler loads

$$n_E = (k\pi)^2 \qquad k=1,2,3,\ldots$$

can be found by various methods to be defined by

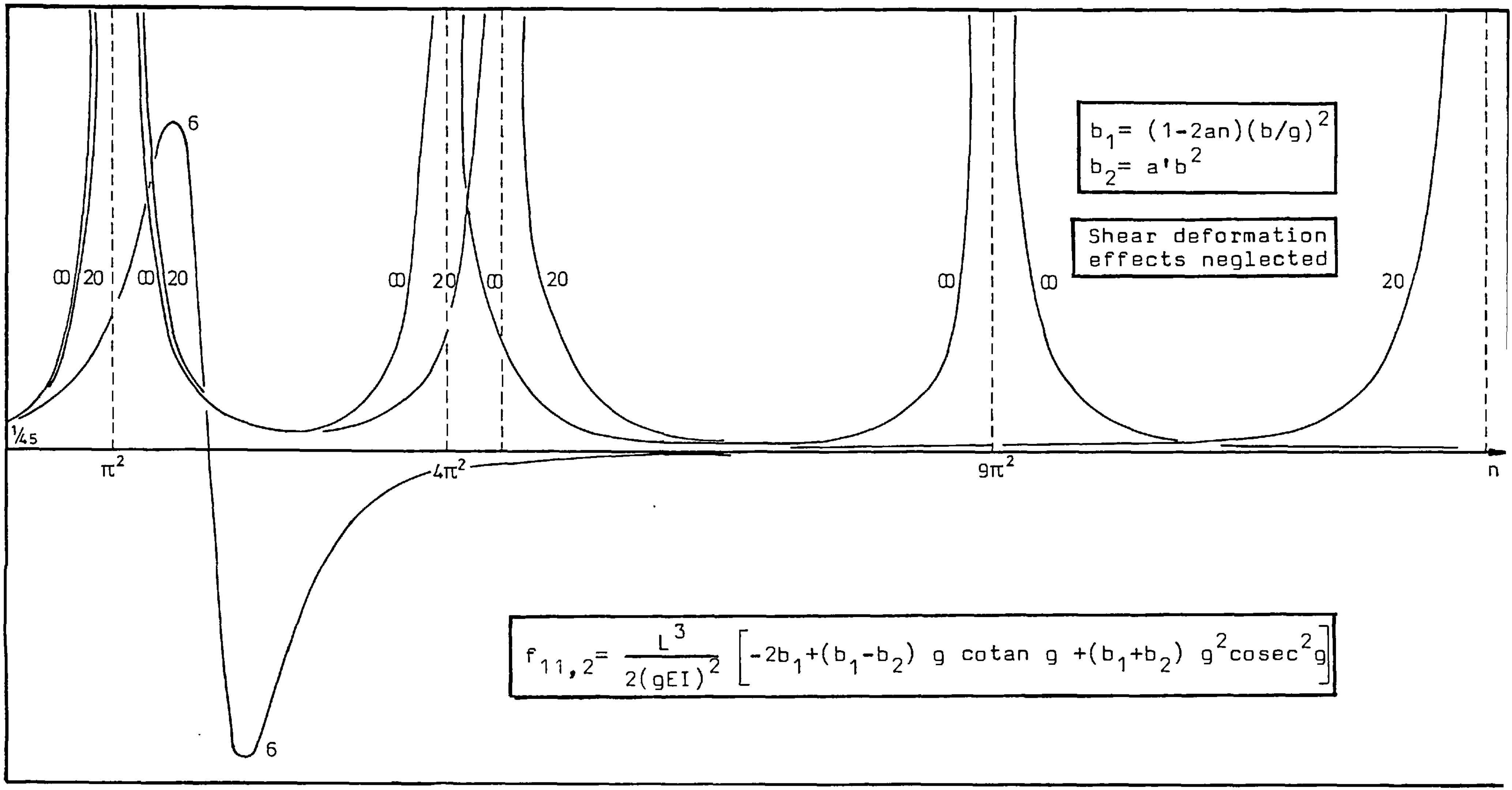
$$g_c = k\pi \qquad k=1,2,3,\ldots$$

or $$n_c = \{1+a'n_E - [(1+a'n_E)^2 - 4an_E]^{\frac{1}{2}}\}/2a \quad (3.1.63)$$

in total agreement with the results presented by Huddleston (1970).

Equation (3.1.63) shows that the buckling load converges to the Euler load when the member slenderness ratio tends to infinity; the buckling and Euler loads coincide in the particular case when

$$a' = a$$



b1= (1-2an)(b/g)2
b2= a'b2
Shear deformation effects neglected
f11,2= L3 / 2(gEI)2 [-2b1+(b1-b2) g cotan g +(b1+b2) g2cosec2g]
6
∞
20
∞
20
∞
20
∞
20
∞
∞
20
1/45
6
π2
4π2
9π2
n

FIGURE 3.11

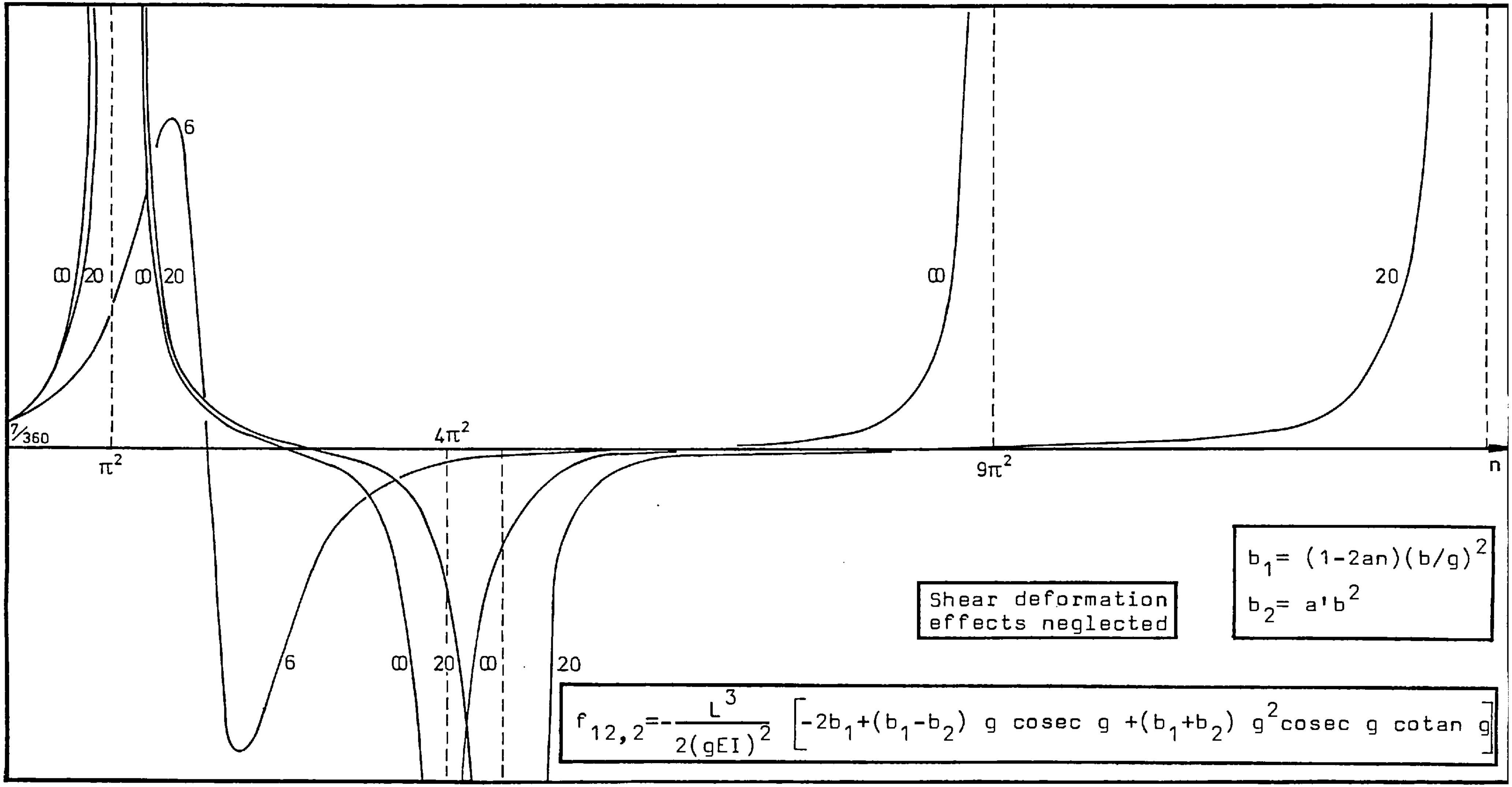
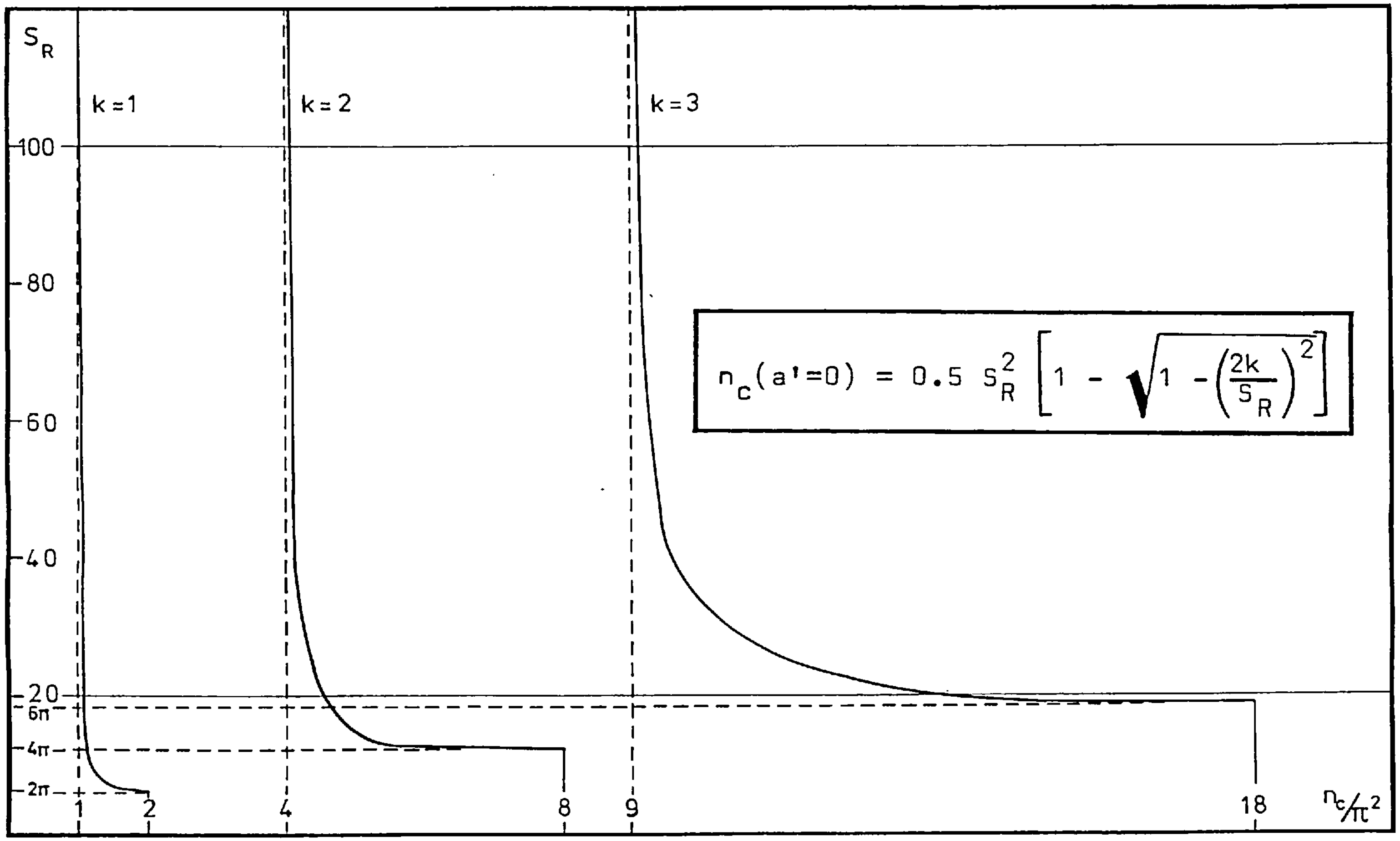

6
ꝏ
20
ꝏ
20
ꝏ
20
7/360
π²
4π²
9π²
n
6
ꝏ
20
ꝏ
20
Shear deformation effects neglected
b1= (1-2an)(b/g)²
b2= a'b²
f12,2 = -L³/2(gEI)² [-2b1+(b1-b2) g cosec g +(b1+b2) g² cosec g cotan g]

FIGURE 3.12



S_R
k = 1
k = 2
k = 3
100
80
60
40
20
6π
4π
2π
1
2
4
8
9
18
n_c/π^2
$n_c(a'=0) = 0.5\ S_R^2 \left[1 - \sqrt{1 - \left(\frac{2k}{S_R}\right)^2}\right]$

FIGURE 3.13

If the shear effect is neglected, equation (3.1.63) reduces to

$$n_c = 0.5S_R^2\left\{1-\left[1-\left(\frac{2k}{S_R}\right)^2\right]^{\frac{1}{2}}\right\} \qquad (3.1.64)$$

The graph of the above equation, illustrated in Fig.3.13, shows that the buckling loads diverge from the Euler loads for very short members; furthermore, it predicts that members with a slenderness ratio less than 2π can not buckle.

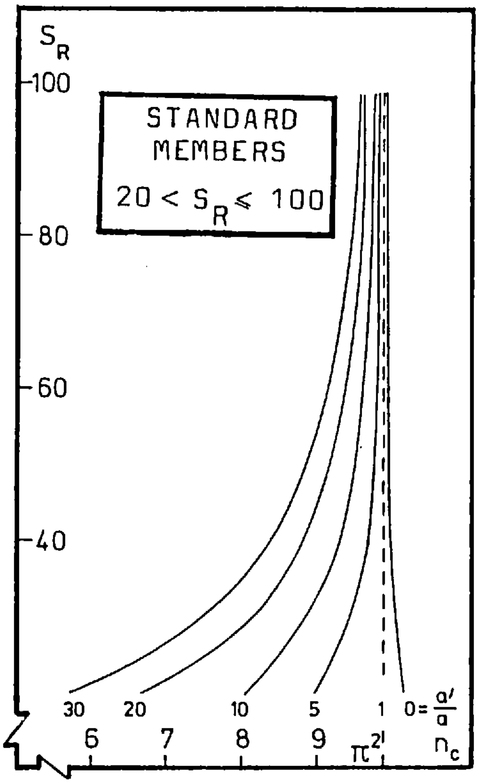


FIGURE 3.14

This anomaly is rectified by taking into consideration the shear effect which, as shown in Figs.3.14 and 3.15, is only relevant for short members.

The maximum bending moment, according to equations (3.1.9a) and (3.1.6), will occur at the section in the span where

$$\tan\theta = \frac{t}{n}$$

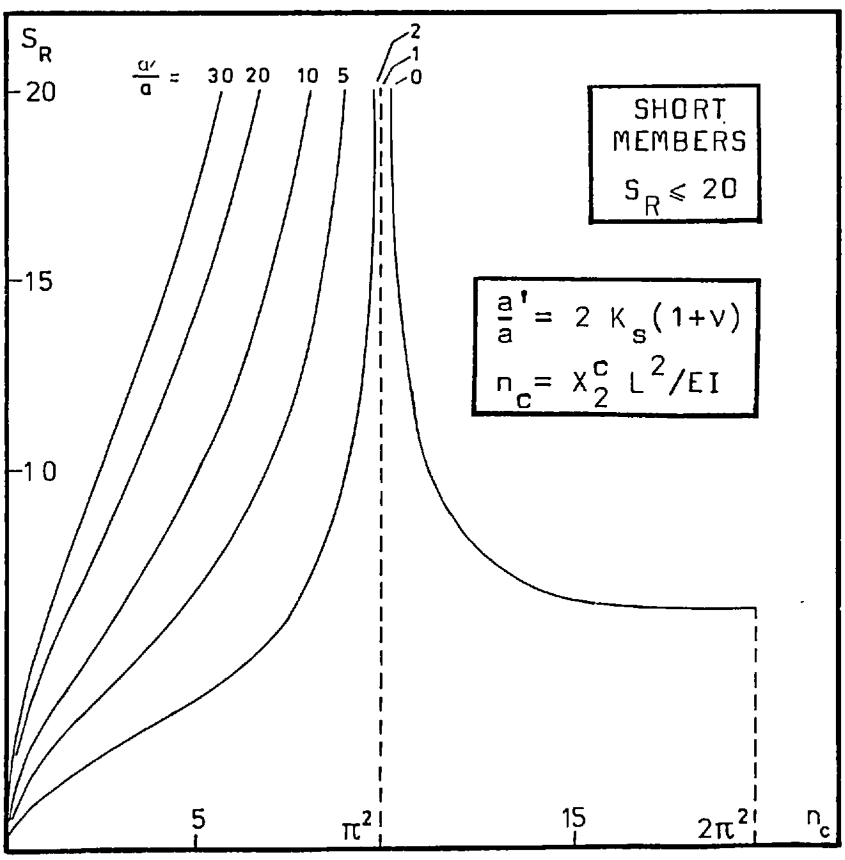


FIGURE 3.15

The maximum axial force will occur either at the same section or where the rotation is stationary:

$$\theta_{,1} = 0$$

The first-order theory gives a constant axial force along the span of the beam; the maximum bending moment will occur at

$$x_1^* = \frac{1}{g}\text{arc tan}\left(\frac{m_2}{m_1}\text{cosecg} - \text{cotang}\right) \qquad (3.1.65)$$

The graphs in Figs.3.16 and 3.17 combine the representation of the relative maximum bending moment M_{max}/m_i with the coordinate of the section where it occurs.

Assume that the bending moment at critical section 2 is greater than the moment at critical section 1 and let us investigate, within the limitations of the first-order theory, if the bending moment at any other section in the span of the beam may exceed that end-moment.

Equations (3.1.62d) and (3.1.65) show that only ratios within the range

$$0 \leq \frac{m_1}{m_2} \leq 1$$

are to be considered, since

$$-1 \leq \frac{m_1}{m_2} < 0 \qquad \Rightarrow \qquad x_1^* < 0$$

When the moment at critical section 1 increases from zero, x_1^* starts taking positive values within the allowable range

$$0 \leqslant x_1^* \leqslant 1 \qquad (3.1.66)$$

when $$g \geqslant \pi/2$$

In the limiting case

$$m_1 = m_2$$

the maximum bending moment will always occur, for positive values of the axial force, at the mid-span section.

The graphs in Fig.3.16 represent the curve

$$\frac{M_{max}}{m_2}$$

which, from (3.1.62d) and (3.1.65), is defined by

$$\frac{M(x_1^*)}{m_2} = \frac{m_1}{m_2} \cos g\, x_1^* (\operatorname{cosec} g - \frac{m_1}{m_2} \operatorname{cotan} g) \sin g\, x_1^*$$

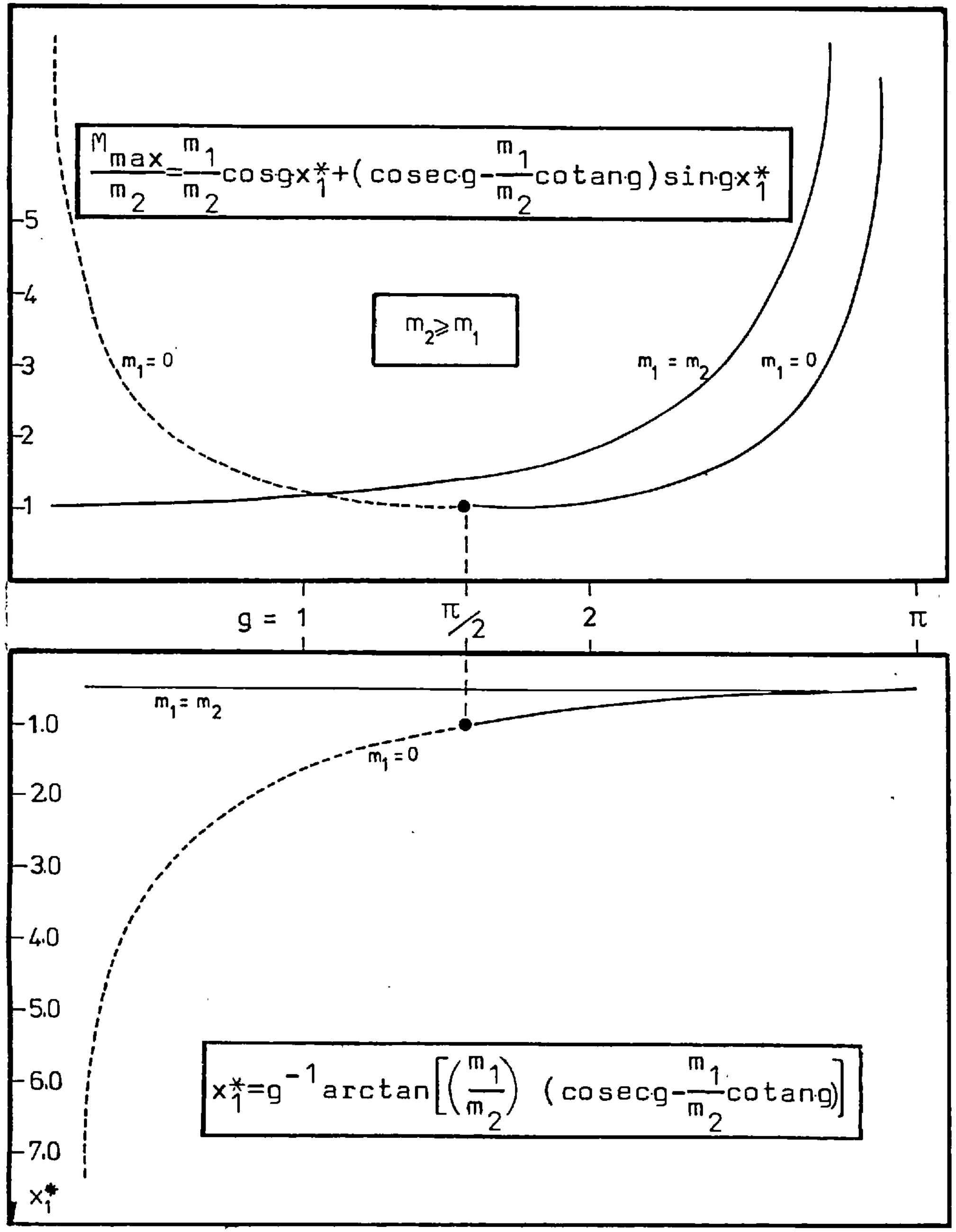


FIGURE 3.16

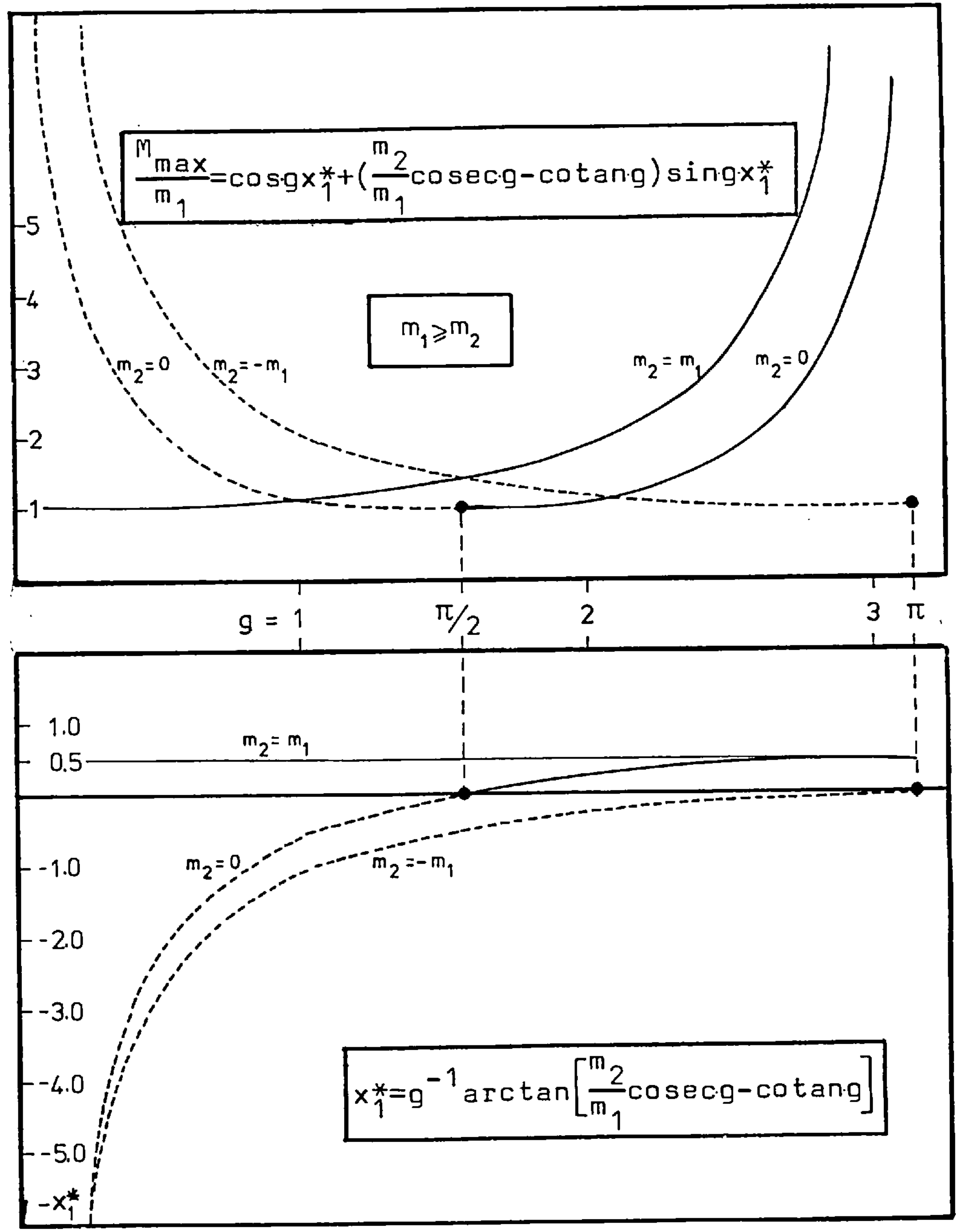


FIGURE 3.17

for the limiting values of the ratio m_1/m_2, together with the corresponding values for x_1^*, in order to select the cases satisfying the physical constraints (3.1.66).

Similar results are illustrated in Fig.3.17 which considers the complementary situation whereby

$$m_2 \leq m_1$$

The information provided by those graphs can be summarized as follows:

1. When $$0 \leq \frac{m_2}{m_1} \leq 1$$

   or $$0 \leq \frac{m_1}{m_2} \leq 1$$

   there will always exist at least one section in the span of beam where the bending moment exceeds the larger of the beam end-moments, provided that

   $$g \geqslant 0$$

   which, for practical values of the axial and shear deformability parameters, a and a', respectively, is equivalent to

   $$n \geqslant 0$$

   that is, provided the beam is under compression.

2. When $$-1 \leq \frac{m_1}{m_2} < 0$$

   the moment at critical section 2 will always be the maximum.

3. When $$-1 \leq \frac{m_2}{m_1} < 0$$

   the moment at a section in the span of the beam will exceed the larger end-moment (m_1) for very large values of the parameter g

   $$g > \pi/2$$

The asymmetry in the above conclusions is a reflection of the asymmetric support conditions of the beam element; complementary results would be obtained when considering a beam element free to displace horizontaly at critical section 1 instead of critical section 2.

If the end-moments are applied through an excentric axial load

$$m_1 = m_2 = n\ d$$

where d is the (non-dimensional) excentricity, the maximum bending moment and transverse displacement, given respectively by

$$M_{max} = n \cdot d \cdot \sec g/2$$

and

$$v^o_{3_{max}} = d(\sec g/2 - 1)$$

will occur at the mid-span section; the maximum axial stress is now defined by

$$\sigma_{max} = an\left[1 + \frac{d}{r} \sec \tfrac{1}{2}\sqrt{\frac{1-an}{1-a'n} \cdot n}\right]$$

where r represents the (non-dimensional) radius of the core, generalizing the well-known secant-formula

$$\sigma_{max} = an\left[1 + \frac{d}{r} \sec \tfrac{1}{2}\sqrt{n}\right]$$

to the case of axially and shearing deformable members.

3.1.5 Incremental Analysis

Let the flexural flexibility matrix be expanded in a Taylor series

$$\underline{f}(X_2' + \Delta X_2') = \underline{f}(X_2') + \sum_{n=1}^{\infty} \underline{f}^{(n)}(X_2') \frac{(\Delta X_2')^n}{n!}$$

so that its increment can be defined as

$$\Delta \underline{f} = \sum_{n=1}^{\infty} \underline{f}^{(n)} \frac{(\Delta X_2')^n}{n!} \qquad (3.1.67)$$

The flexural part of the system (3.1.48) can be expressed in an incremental form as

$$\Delta \underset{\sim}{u}'_{1E} = \underline{f}\, \Delta \underset{\sim}{X}'_1 + \Delta \underline{f}\, \underset{\sim}{X}'_1 + \Delta \underline{f}\, \Delta \underset{\sim}{X}'_1$$

or, from (3.1.67)

$$\Delta \underset{\sim}{u}'_{1E} = \underline{f}\, \Delta \underset{\sim}{X}'_1 + \underline{f}_{,2}\, \underset{\sim}{X}'_1\, \Delta X'_2 + \underset{\sim}{R}_{u_{1E}} \qquad (3.1.68a)$$

$$\text{where} \quad R_{u_{1E}} = \frac{(\Delta X'_2)^2}{2} \left[\underline{f}_{,22} + \frac{\Delta X'_2}{3}\, \underline{f}_{,222} \right] \underset{\sim}{X}'_1 + \Delta X'_2 \left[\underline{f}_{,2} + \frac{\Delta X'_2}{2}\, \underline{f}_{,22} \right] \Delta \underset{\sim}{X}'_1 + 0_4 \qquad (3.1.68b)$$

Let the variation in the length of the member chord be expressed as

$$\Delta u'_{2E} = \frac{L}{EA}\, \Delta X'_2 + \Delta u'_{2s} \qquad (3.1.68c)$$

where $\Delta u'_{2s}$ represents the increment on the chord shortening due to bending

$$u'_{2s} = \tfrac{1}{2}\, \underset{\sim}{X}'^T_1\, \underline{f}_{,2}\, \underset{\sim}{X}'_1$$

Hence

$$\Delta \underset{\sim}{u}'_{2s} = \underset{\sim}{X}'^T_1\, \underline{f}_{,2}\, \Delta \underset{\sim}{X}'_1 + \tfrac{1}{2}\, \underset{\sim}{X}'^T_1\, \Delta \underline{f}_{,2}\, \underset{\sim}{X}'_1 + \underset{\sim}{X}'^T_1\, \Delta \underline{f}_{,2}\, \Delta \underset{\sim}{X}'_1 + \tfrac{1}{2}\, \Delta \underset{\sim}{X}'^T_1\, \underline{f}_{,2}\, \Delta \underset{\sim}{X}'_1 + \tfrac{1}{2}\, \Delta \underset{\sim}{X}'^T_1\, \Delta \underline{f}_{,2}\, \Delta \underset{\sim}{X}'_1$$

or, and making use of (3.1.67)

$$\Delta u'_{2s} = \underset{\sim}{X}'^T_1\, \underline{f}_{,2}\, \Delta \underset{\sim}{X}'_1 + \tfrac{1}{2}\, \underset{\sim}{X}'^T_1\, \underline{f}_{,22}\, \underset{\sim}{X}'_1\, \Delta X'_2 + R_{u_{2s}} \qquad (3.1.68d)$$

where

$$R_{u_{2s}} = \tfrac{1}{2}\, \Delta \underset{\sim}{X}'^T_1\, \underline{f}_{,2}\, \Delta \underset{\sim}{X}'_1 + \frac{(\Delta X'_2)^2}{4}\, \underset{\sim}{X}'^T_1 \left[\underline{f}_{,222} + \frac{\Delta X'_2}{3}\, \underline{f}_{,2222} \right] \underset{\sim}{X}'_1 +$$

$$\Delta X'_2\, X'^T_1 \left[\underline{f}_{,22} + \frac{\Delta X'_2}{2}\, \underline{f}_{,222} \right] \Delta \underset{\sim}{X}'_1 + \frac{\Delta X'_2}{2}\, \Delta X'^T_1\, \underline{f}_{,22}\, \Delta \underset{\sim}{X}'_1 + 0_4 \qquad (3.1.68e)$$

Equations (3.1.68) can be expressed in matrix form as

FLEXIBILITY FORMULATION
$\Delta \underset{\sim}{u}'_E = \underline{F}\, \Delta \underset{\sim}{X}' + \underset{\sim}{R}_{uE}$

(3.1.69)

where the zeroth-, first- and second-order flexibility matrices and remainders are respectively defined by

$$\mathbb{F} = \left[\begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & f \end{array}\right] \qquad \mathbb{R}_{\tilde{u}E} = \left[\begin{array}{c} \cdot \\ \hline \cdot \end{array}\right] \qquad (3.1.70a,b)$$

$$\mathbb{F} = \left[\begin{array}{c|c} \underline{f} & \underline{f}_{,2}\ \tilde{X}'_1 \\ \hline \cdot & f \end{array}\right] \qquad \mathbb{R}_{\tilde{u}E} = \left[\begin{array}{c} \tilde{R}_{u_{1E}} \\ \hline \cdot \end{array}\right] \qquad (3.1.71a,b)$$

$$\mathbb{F} = \left[\begin{array}{c|c} \underline{f} & \underline{f}_{,2}\ \tilde{X}'_1 \\ \hline \tilde{X}'^{T}_1\ \underline{f}_{,2} & \bar{f} \end{array}\right] \qquad \mathbb{R}_{\tilde{u}E} = \left[\begin{array}{c} \tilde{R}_{u_{1E}} \\ \hline R_{u_{2s}} \end{array}\right] \qquad (3.1.72a,b)$$

in which

$$f = \frac{L}{EA} \qquad (3.1.73a)$$

and

$$\bar{f} = f + \tfrac{1}{2}\ \tilde{X}'^{T}_1\ \underline{f}_{,22}\ \tilde{X}'_1 \qquad (3.1.73b)$$

A similar procedure applied now to equations (3.1.54) to (3.1.57) would provide the following stiffness description of the incremental elastic constitutive relations

STIFFNESS FORMULATION
$\Delta \tilde{X}' = \mathbb{K}\ \Delta \tilde{u}'_E + \mathbb{R}_{\tilde{X}E}$

(3.1.74)

the zeroth-, first- and second-order stiffness matrices and remainders being respectively defined by

$$\mathbb{K} = \left[\begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & k \end{array}\right] \qquad \mathbb{R}_{\tilde{X}E} = \left[\begin{array}{c} \cdot \\ \hline \cdot \end{array}\right] \qquad (3.1.75a,b)$$

$$\mathbb{K} = \left[\begin{array}{c|c} \underline{k} & k\ \underline{k}_{,2}\tilde{u}'_{1E} \\ \hline \cdot & k \end{array}\right] \qquad \mathbb{R}_{\tilde{X}E} = \left[\begin{array}{c} \tilde{R}'_1 \\ \hline \cdot \end{array}\right] \qquad (3.1.76a,b)$$

$$\mathbb{K} = \left[\begin{array}{c|c} \bar{\underline{k}} & \bar{k}\ \underline{k}_{,2}\tilde{u}'_{1E} \\ \hline \bar{k}\ \tilde{u}'^{T}_{1E}\ \underline{k}_{,2} & \bar{k} \end{array}\right] \qquad \mathbb{R}_{\tilde{X}E} = \left[\begin{array}{c} \tilde{R}_1 \\ \hline \bar{k} R_2 \end{array}\right] \qquad (3.1.77a,b)$$

where

$$k = \frac{EA}{L} \qquad (3.1.78a)$$

$$\bar{k} = \left[\frac{1}{k} - \tfrac{1}{2}\ \tilde{u}'^{T}_{1E}\ \underline{k}_{,22}\ \tilde{u}'_{1E}\right]^{-1} \qquad (3.1.78b)$$

$$\bar{\underline{k}} = \underline{k} + \bar{k}\ \underline{k}_{,2}\ \underset{\sim}{u}'_{1E}\ \underset{\sim}{u}'^{T}_{1E}\ \underline{k}_{,2} \qquad (3.1.78c)$$

and
$$R'_1 = \frac{(\Delta X'_2)^2}{2}\ \underline{k}_{,22} + \frac{\Delta X'_2}{3}\ \underline{k}_{,222}\ \underset{\sim}{u}'_{1E} + X'_2\ \underline{k}_{,2} + \frac{X'_2}{2}\ \underline{k}_{,22}\ u'_{1E} + 0_4 \qquad (3.1.79a)$$

$$R_1 = R'_1 + R_2\ \underline{k}_{,2}\ \underset{\sim}{u}'_{1E} \qquad (3.1.79b)$$

$$R_2 = \frac{(\Delta X'_2)^2}{4}\ u'^{T}_{1E}\left[\underline{k}_{,222} + \frac{\Delta X'_2}{3}\ \underline{k}_{,2222}\right]\underset{\sim}{u}'_{1E} + \tfrac{1}{2}\ \Delta\underset{\sim}{u}'^{T}_{1E}\ \underline{k}_{,2}\ \Delta\underset{\sim}{u}'^{T}_{1E} + (\Delta X'_2)\ \underset{\sim}{u}'^{T}_{1E}\left[\underline{k}_{,22} + \frac{\Delta X'_2}{2}\ \underline{k}_{,222}\right]\Delta\underset{\sim}{u}'_{1E} + \frac{\Delta X'_2}{2}\ \Delta\underset{\sim}{u}'^{T}_{1E}\ \underline{k}_{,22}\ \Delta\underset{\sim}{u}'_{1E} + 0_4 \qquad (3.1.79c)$$

3.1.6 Perturbation Analysis

Letting in equations (3.1.69) and (3.1.74)

$$\Delta\underset{\sim}{u}'_E = \sum_{i=1}^{\infty} \underset{\sim}{u}'_{E_i}\ \frac{\varepsilon^i}{i!}$$

$$\Delta\underset{\sim}{X}' = \sum_{i=1}^{\infty} \underset{\sim}{X}'_i\ \frac{\varepsilon^i}{i!}$$

$$\underset{\sim}{\mathbb{R}}_{uE} = \sum_{i=1}^{\infty} \underset{\sim}{R}_{uE_i}\ \frac{\varepsilon^i}{i!}$$

$$\underset{\sim}{\mathbb{R}}_{XE} = \sum_{i=1}^{\infty} \underset{\sim}{R}_{XE_i}\ \frac{\varepsilon^i}{i!}$$

and equating the same order coefficients we find for the perturbed form of the elastic constitutive relations

	FLEXIBILITY FORMULATION	STIFFNESS FORMULATION	
(3.1.80)	$\underset{\sim}{u}'_{E_i} = \underline{\mathbb{F}}\ \underset{\sim}{X}'_i + \underset{\sim}{R}_{uE_i}$	$\underset{\sim}{X}'_i = \underline{\mathbb{K}}\ \underset{\sim}{u}'_{E_i} + \underset{\sim}{R}_{XE_i}$	(3.1.81)
	PERTURBED ELASTIC CONSTITUTIVE RELATIONS		

where now

$$\underset{\sim}{R}_{u1E_1} = \underset{\sim}{0} \tag{3.1.82a}$$

$$\underset{\sim}{R}_{u1E_2} = (X'_{2_1})[X'_{2_1}\underline{f}_{,22}\underset{\sim}{X}'_1 + 2\underline{f}_{,2}\underset{\sim}{X}'_{1_1}] \tag{3.1.82b}$$

$$\underset{\sim}{R}_{u1E_3} = 3(X'_{2_1})[X'_{2_2}\underline{f}_{,22}\underset{\sim}{X}'_1 + \underline{f}_{,2}\underset{\sim}{X}'_{1_2}] + 3X'_{2_2}\underline{F}_{,2}\underset{\sim}{X}'_{1_1} + (X'_{2_1})^2[X'_{2_1}\underline{f}_{,222}\underset{\sim}{X}'_1 + 3\underline{f}_{,22}\underset{\sim}{X}'_{1_1}] \tag{3.1.82c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$R_{u2s_1} = 0 \tag{3.1.83a}$$

$$R_{u2s_2} = \frac{1}{2}(X'_{2_1})[X'_{2_1}\underset{\sim}{X}'^{T}_1\,\underline{f}_{,222} + \underset{\sim}{X}'^{T}_{1_1}\underline{f}_{,22}]\underset{\sim}{X}'_1 + \underset{\sim}{X}'^{T}_{1_1}\underline{f}_{,2}\underset{\sim}{X}'_{1_1} \tag{3.1.83b}$$

$$R_{u2s_3} = \frac{3}{2}(X'_{2_1})[X'_{2_2}\underset{\sim}{X}'^{T}_1\,\underline{f}_{,222} + 2\underset{\sim}{X}'^{T}_{1_2}\underline{f}_{,22}]\underset{\sim}{X}'_1 + 3\underset{\sim}{X}'^{T}_{1_1}\underline{f}_{,2}\underset{\sim}{X}'_{1_2} + \frac{1}{2}(X'_{2_1})^2[X'_{2_1}\underset{\sim}{X}'^{T}_1\,\underline{f}_{,2222} + 6\underset{\sim}{X}'^{T}_{1_1}\,\underline{f}_{,222}]\underset{\sim}{X}'_1 + 3[X'_{2_2}\underset{\sim}{X}'^{T}_1 + X'_{2_1}\underset{\sim}{X}'^{T}_{1_1}]\underline{f}_{,22}\underset{\sim}{X}'_{1_1} \tag{3.1.83c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\underset{\sim}{R}'_{1_1} = \underset{\sim}{0} \tag{3.1.84a}$$

$$\underset{\sim}{R}'_{1_2} = X'_{2_1}[X'_{2_1}\underline{k}_{,22}\underset{\sim}{u}'_{1E} + \frac{1}{2}\underline{k}_{,2}\underset{\sim}{u}'_{1E_1}] \tag{3.1.84b}$$

$$\underset{\sim}{R}'_{1_3} = 3X'_{2_1}[X'_{2_2}\underline{k}_{,22}\underset{\sim}{u}'_{1E} + \underline{k}_{,2}\underset{\sim}{u}'_{1E_2}] + 3X_{2_2}\underline{K}_{,2}\underset{\sim}{u}'_{1E_1} + (X'_{2_1})^2[X_{2_1}\underline{k}_{,222}\underset{\sim}{u}'_{1E} + 3\underline{k}_{,22}\underset{\sim}{u}'_{1E_1}] \tag{3.1.84c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\underset{\sim}{R}_{1_i} = \underset{\sim}{R}'_{1_i} + R_{2_i}\underline{k}_{,2}\,\underset{\sim}{u}'_{1E} \tag{3.1.85}$$

$$\underset{\sim}{R}_{2_1} = \underset{\sim}{0} \tag{3.1.85a}$$

$$\underset{\sim}{R}_{2_2} = \frac{1}{2}(X'_{2_1})[X'_{2_1}\underset{\sim}{u}'^{T}_{1E}\,\underline{k}_{,222} + \underset{\sim}{u}'^{T}_{1E_1}\underline{k}_{,22}]\underset{\sim}{u}'_{1E} + \underset{\sim}{u}^{T}_{1E_1}\underline{k}_{,2}\underset{\sim}{u}'_{1E_1} \tag{3.1.85b}$$

$$\underset{\sim}{R}_{2_3} = \frac{3}{2}(X'_{2_1})[X'_{2_2}\underset{\sim}{u}'^{T}_{1E}\,\underline{k}_{,222} + 2\underset{\sim}{u}'^{T}_{1E_2}\underline{k}_{,22}]\underset{\sim}{u}'_{1E} + 3\underset{\sim}{u}'_{1E_1}\underline{k}_{,2}\underset{\sim}{u}'_{1E_2} + \frac{1}{2}(X'_{2_1})^2[X'_{2_1}\underset{\sim}{u}'^{T}_{1E}\underline{k}_{,2222} + 6\underset{\sim}{u}'^{T}_{1E_1}\underline{k}_{,222}]\,\underset{\sim}{u}'_{1E} + 3[X'_{2_2}\underset{\sim}{u}'^{T}_{1E_1} + X'_{2_1}\underset{\sim}{u}'^{T}_{1E_1}]\underline{k}_{,22}\,\underset{\sim}{u}'_{1E_1} \tag{3.1.85c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

3.1.7 Related Formulations

According to Chu (1959), the stability functions

$$k_{11}= k_{22}= \frac{EI}{L} \left\{ \left(\frac{n^{\frac{1}{2}}}{2}\right)\text{cotan}\left(\frac{n^{\frac{1}{2}}}{2}\right)+\left(\frac{n^{\frac{1}{2}}}{2}\right)^2 \left[1-\left(\frac{n^{\frac{1}{2}}}{2}\right)\text{cotan}\left(\frac{n^{\frac{1}{2}}}{2}\right)\right]^{-1} \right\} \tag{3.1.86a}$$

$$k_{12}= k_{21}= \frac{EI}{L} \left\{ \left(\frac{n^{\frac{1}{2}}}{2}\right)\text{cotan}\left(\frac{n^{\frac{1}{2}}}{2}\right)-\left(\frac{n^{\frac{1}{2}}}{2}\right)^2 \left[1-\left(\frac{n^{\frac{1}{2}}}{2}\right)\text{cotan}\left(\frac{n^{\frac{1}{2}}}{2}\right)\right]^{-1} \right\} \tag{3.1.86b}$$

were first derived by Manderla (1880). The above stability functions can be obtained by neglecting the axial and shear deformation effects in the stiffness coefficients k_{ij} defined in (3.1.58). If the same effects are neglected in the flexibility coefficients f_{ij}, defined in (3.1.53), yielding

$$f_{11}= f_{22}= \frac{L}{EI}\, n^{-1}(1-n^{\frac{1}{2}}\ \text{cotan}\ n^{\frac{1}{2}}) \tag{3.1.86c}$$

$$f_{12}= f_{21}= -\frac{L}{EI}\, n^{-1}(1-n^{\frac{1}{2}}\ \text{cosec}\ n^{\frac{1}{2}}) \tag{3.1.86d}$$

the stability functions derived by Berry (1916) are recovered. The stiffness stability functions (3.1.86a,b) were used by Manderla in the analysis of the effects of secondary bending moments in truss structures. Berry applied the flexibility stability functions (3.1.86c,d) in the analysis of the stresses developing in the wing spars of biplanes.

Baker (1934) modified Berry's functions in order to include the axial force effects in the slope-deflection method of structural analysis. To include the same effects in an extended version of the moment distribution method, James (1935) derived, independently, the expressions for the stiffness and carry over factors s and c

$$s = \frac{L}{EI}\, k_{11} \qquad \text{and} \qquad sc = -\frac{L}{EI}\, k_{12}$$

The introduction of the m, n and o functions, which can also be directly related with the stiffness stability functions (3.1.86a,b), and their use in deriving the sway critical loads of rigid-jointed frames is due to Merchant (1955).

The existing tables for the stability functions, as those provided by Lundquist and Kroll (1938, 1944) and Livesley and

Chandler (1956), can be readily used to include the axial and shear deformation effects just by entering the parameter g, as defined in (3.1.53b), instead of the non-dimensional force (3.1.53c), and dividing the result by the shear parameter b defined in (3.1.53a).

The stability functions, popularized in the 1960's by Livesley (1964) and Horne and Merchant (1965), have been extensively used, together with linear descriptions of Statics and Kinematics, in the buckling analysis of trusses and frames. Of the earlier studies we mention, besides those already referred to, the works of Chwalla (1938), Hoff (1941), Niles and Newell (1948), Winter et alia (1948), Wessman and Kavanagh (1950), Hoff et alia (1950) and Masur (1954).

Chu (1959) included in his formulation the effect of the chord shortening due to bending, in both flexibility

$$\frac{1}{2}\left[F_1(X_1^1+X_1^2)^2+F_2(X_1^1-X_1^2)^2\right] \qquad (3.1.87a)$$

and stiffness formats

$$\frac{1}{2}\left[S_1(u'^1_{1E}+u'^2_{1E})^2+S_2(u'^1_{1E}-u'^2_{1E})^2\right] \qquad (3.1.87b)$$

The bowing coefficients can be obtained by neglecting the axial and shear deformation effects in (3.1.59) and (3.1.61) and letting

$$\begin{cases} F_1+F_2=f_{11,2}(a'=a=0) \\ F_1-F_2=f_{12,2}(a'=a=0) \end{cases} \quad \text{and} \quad \begin{cases} S_1+S_2=k_{11,2}(a'=a=0) \\ S_1-S_2=k_{12,2}(a'=a=0) \end{cases}$$

The bowing functions have been expressed in terms of the stability functions but, apparently, they were never identified as their derivatives, probably because the scalar descriptions (3.1.87) of the chord shortening due to bending were preferred to their matrix description as a quadratic form.

In 1952, Bleich presented a systematic survey of the various stability theories. This survey, together with the advent of the digital computer, aroused further interest among the researchers working in this field and since the mid 1950's a multitude of more or less related formulations for the analysis of kinematically non-linear framed structures have been presented. From the earlier formulations, we will refer those which adopted a description for the elastic constitutive relations directly related to the formulation presented herein.

Among the authors using the stability functions (3.1.83) in their formulations, and besides those already mentioned, we refer Lu (1963), Saafan (1963), Renton (1964), Williams (1964), Connor et alia (1968) and Halldersson and Wang (1968).

Saafan (1963), Williams (1964) and Merchant and Brotton (1964) demonstrated that the chord shortening due to flexure could be, even for moderate displacements and deformations, of

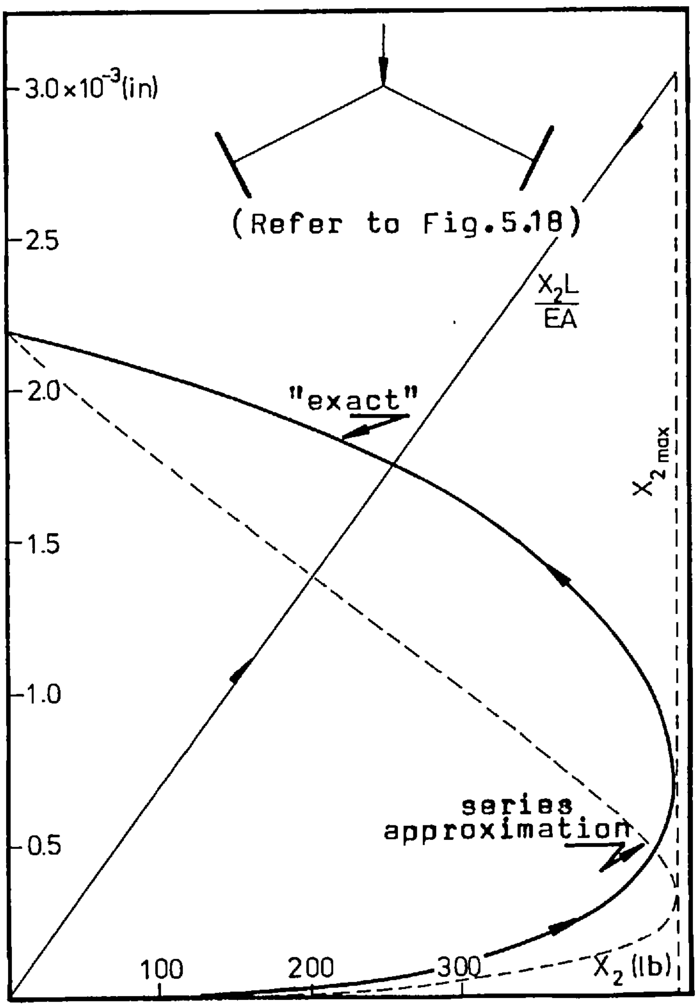


FIGURE 3.18

the same order of magnitude as the linear extensional term. William's toggle is analysed in section 5.5 and the variation we found for the linear extensional term and for the chord shortening due, primarily, to bending are plotted in Fig.3.18 versus the variation of the axial force.

The use of the stability and bowing functions declined when emphasis started to be layed on the finite-element formulations with the inherent tendency of approximating the displacement field by polynomial functions. After the pioneering work of Turner et alia (1960), many finite-element formulations, generally in a stiffnes format, have been proposed. Of these, and among the earlier ones, the formulations of Argyris (1964), Martin (1965), Jennings (1968), Mallet and Marçal (1968) and Powell (1969) appear to be the most significant since they involve the basic forms of the stiffness and bowing matrices generally used in finite-element formulations.

In order to relate them with the results presented herein, let, in the flexibility and stiffness coefficients definitions (3.1.52) and (3.1.58), respectively, the trignometric functions be replaced by power series expansions on their argument, yielding

$$\underline{f} = \frac{Lb}{EI} \sum_{i=1}^{\infty} g^{2(i-1)} \underline{f}_i \qquad (3.1.88)$$

and

$$\underline{k} = \frac{EI}{Lb} \sum_{i=1}^{\infty} g^{2(i-1)} \underline{k}_i \qquad (3.1.89)$$

where $\underline{f}_1 = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\underline{f}_2 = \frac{1}{360} \begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix}$,

$\underline{f}_3 = \frac{1}{15120} \begin{bmatrix} 32 & 31 \\ 31 & 32 \end{bmatrix}$, $\underline{f}_4 = \frac{1}{604800} \begin{bmatrix} 128 & 127 \\ 127 & 128 \end{bmatrix}$,...

and $\underline{k}_1 = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$, $\underline{k}_2 = -\frac{1}{30} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$,

$\underline{k}_3 = -\frac{1}{12600} \begin{bmatrix} 22 & 13 \\ 13 & 22 \end{bmatrix}$, $\underline{k}_4 = -\frac{1}{378000} \begin{bmatrix} 14 & 11 \\ 11 & 14 \end{bmatrix}$,...

Note that the above matrices would be those directly obtained through the application of the perturbation technique if the axial force was assumed strictly dependent on the perturbation parameter, that is, if n_o was set to zero in the perturbed governing equations.

Consider now the following assumptions commonly used in the literature:

1. The shear deformation effects are neglegible

$$b = 1 \;, \; b_1 = \frac{1-2an}{(1-an)n} \;, \quad b_2 = 0 \quad , \quad g = \sqrt{(1-an)n}$$

2. The effect of the axial extensibility has relevance only when evaluating the linear term of the axial deformation

$$b_1 = n^{-1} \qquad g = n^{\frac{1}{2}}$$

3. Terms of order higher than the second are neglegible.

Enforcing the above assumptions and substituting (3.1.88) and (3.1.89) in the flexural part of systems (3.1.48) and (3.1.54), the second-order formulations of the elastic constitutive relations reduce to

$$\begin{cases} \underline{\theta} = (\underline{f}_1 + n\underline{f}_2)\ \underline{m} & (3.1.90a) \\ u = an + \frac{1}{2}\ \underline{m}^T \underline{f}_2\ \underline{m} & (3.1.90b) \end{cases}$$

and

$$\begin{cases} \underline{m} = (\underline{k}_1 - n\underline{k}_2)\ \underline{\theta} & (3.1.91a) \\ n = a^{-1}u - \frac{1}{2}\ a^{-1}\underline{\theta}^T \underline{k}_2 \underline{\theta} & (3.1.91c) \end{cases}$$

which are expressed in terms of the non-dimensional variables for simplicity of the presentation.

The broken line in Fig.3.18 represents, for the structure illustrated there, the variation of the axial shortening due to bending when quantified by $\frac{1}{2}\ X_1'^T(\frac{Lb}{EI}\ \underline{f}_2)X_1'$. Besides showing a tendency to diverge, it compares poorly, even for relatively small deformations, with the more accurate description $\frac{1}{2}\ X_1'^T \underline{f}_{,2}\ X_1'$, represented by the solid curve in the same figure.

Matrix $(\frac{Lb}{EI}\,\underline{f}_2)$ represents the first term in the series expansion of matrix $\underline{f},_2$; matrix $\underline{f}_2$ is defined above, and matrix $\underline{f},_2$ in (3.1.59).

The axial force can be eliminated in (3.1.91a) through (3.1.91c), yielding

$$\underset{\sim}{m} = (\underline{k}_1 - a^{-1}u\,\underline{k}_2 + a^{-1}\underline{k}_*)\,\theta \qquad (3.1.91b)$$

where $\underline{k}_* = \frac{1}{2}\,\theta^T\underline{k}_2\theta\,\underline{k}_2$

or $\underline{k}_* = \frac{1}{30}\,(2\theta_1^2 + \theta_1\theta_2 + 2\theta_2^2)\,\underline{k}_2$

The incremental version of equations (3.1.90) and (3.1.91) is found to be

$$d\underset{\sim}{\theta} = (\underline{f}_1 + n\underline{f}_2)\,d\underset{\sim}{m} + dn\,\underline{f}_2\underset{\sim}{m} \qquad (3.1.92a)$$

$$du = a\,dn + \underset{\sim}{m}^T\underline{f}_2\,d\underset{\sim}{m} \qquad (3.1.92b)$$

and

$$d\underset{\sim}{m} = (\underline{k}_1 - n\underline{k}_2)\,d\underset{\sim}{\theta} - a^{-1}du\,\underline{k}_2\underset{\sim}{\theta} \qquad (3.1.93a)$$

or

$$d\underset{\sim}{m} = (\underline{k}_1 - a^{-1}u\,\underline{k}_2 + a^{-1}\underline{k}_*^i)\,d\underset{\sim}{\theta} - a^{-1}du\,\underline{k}_2\underset{\sim}{\theta} \qquad (3.1.93b)$$

$$dn = a^{-1}du - a^{-1}\underset{\sim}{\theta}^T\underline{k}_2\,d\underset{\sim}{\theta} \qquad (3.1.93c)$$

where

$$\underline{k}_*^i = \frac{1}{300}\left[\begin{array}{c|c} 8\theta_1^2+4\theta_1\theta_2+3\theta_2^2 & 2(\theta_1^2+3\theta_1\theta_2+\theta_2^2) \\ \hline 2(\theta_1^2+3\theta_1\theta_2+\theta_2^2) & 3\theta_1^2+4\theta_1\theta_2+8\theta_2^2 \end{array}\right]$$

Argyris (1964) starts by assuming that the finite-elements are initially in equilibrium under the action of the nodal forces. Then, for a small variation in geometry, the equilibrium is mantained by a suitable modification of those forces, the incremental stiffness matrix is generated by considering the changes due to the member deformation as well as to its rigid-body movement. The formulation of Argyris when specialized to the three-degree of freedom element gives

$$d\underset{\sim}{m} = \underline{k}_1\,d\underset{\sim}{\theta} \quad \text{and} \quad dn = a^{-1}du$$

Martin (1965) presented a unified view of the geometrically non-linear and stability analysis using the finite-element technique. The stiffness matrix is obtained from the strain energy via Castigliano's first theorem; when specialized for a three-degree of freedom element it reduces to

$$\underline{k}_1 + n\,\underline{k}_2$$

Martin does not take into consideration the member chord shortening due to bending.

The formulations of Argyris and Martin have obvious limitations even taking into consideration the fact that they were derived having in mind an incremental procedure as the method of solution.

The formulation of Jennings (1968), based on a three-degree of freedom element, uses theorems on the minimum of the potential energy, presented in Jennings (1963), in order to obtain the relationships between the member stress- and strain-resultants.

The direct and incremental formulations that Jennings found for the element flexural constitutive relations, using a cubic polynomial to describe the transverse displacement field, are identical to (3.1.91a) and (3.1.93a), respectively. The formulation was improved to include the effects of the chord shortening, so that the axial force becomes defined by (3.1.91c) and (3.1.93c).

The consistency of the formulation presented by Jennings has been questioned since no reference is made to the function describing the axial displacement field; the formulation is in fact consistent, the deficiency being an improper explanation of the assumption involved.

Mallet and Marçal (1968), using a quadratic strain and the principle of virtual displacements, established a load-displacement relationship in which the stiffness matrix accounts for up to second-order terms in the nodal displacements.

When specialized for a three-degree of freedom element, the resulting stiffness matrix for direct analysis results in

$$\underline{k} = \underline{k}_1 - a^{-1} n\, \underline{k}_2 + a^{-1} \underline{k}_*^M$$

where
$$\underline{k}_*^M = \frac{1}{840} \left[\begin{array}{c|c} 2(12\theta_1^2 + 3\theta_1\theta_2 + \theta_2^2) & (3\theta_1^2 + 4\theta_1\theta_2 + 3\theta_2^2) \\ \hline (3\theta_1^2 + 4\theta_1\theta_2 + 3\theta_2^2) & 2(\theta_1^2 + 3\theta_1\theta_2 + 12\theta_2^2) \end{array} \right]$$

which is flexurally equivalent to

$$\underset{\sim}{k}^{!}_{*} = \frac{1}{280}\left[\begin{array}{c|c} 2(4\theta_1^2 + \theta_1\theta_2 + 4\theta_2^2) & (\theta_1^2 - 6\theta_1\theta_2 + \theta_2^2) \\ \hline (\theta_1^2 - 6\theta_1\theta_2 + \theta_2^2) & 2(4\theta_1^2 + \theta_1\theta_2 + 4\theta_2^2) \end{array}\right] \quad (3.1.94a)$$

since $$\underset{\sim}{K}^{!}_{*}\,\underset{\sim}{\theta} = \underset{\sim}{K}^{M}_{*}\,\underset{\sim}{\theta}$$

The axial force is defined by (3.1.91c).

The incremental formulation is given by (3.1.93b,c) where the non-linear incremental stiffness matrix is now defined by

$$\underset{\sim}{k}^{!i}_{*} = \frac{1}{280}\left[\begin{array}{c|c} 2(12\theta_1^2 + 3\theta_1\theta_2 + \theta_2^2) & (3\theta_1^2 + 4\theta_1\theta_2 + 3\theta_2^2) \\ \hline (3\theta_1^2 + 4\theta_1\theta_2 + 3\theta_2^2) & 2(\theta_1^2 + 3\theta_1\theta_2 + 12\theta_2^2) \end{array}\right] \quad (3.1.94b)$$

Similarly to Jennings (1968), in order to dissociate the member deformations from its rigid-body displacements, Powell (1969) chooses the basic element to have three-degrees of freedom.

The formulation of Powell was developed through a procedure very similar to the one used by Mallet and Marçal (1968); the principle of virtual work was used to derive the non-linear stiffness matrix of the finite-element, whose transverse and axial displacements were approximated by cubic and linear polynomials, respectively.

Within the limitations of a clearly defined set of assumptions, the formulation of Powell is consistently derived and gives

$$\underset{\sim}{m} = (\underset{\sim}{k}_1 - a^{-1}u\underset{\sim}{k}_2 + a^{-1}\underset{\sim}{k}^{P}_{*})\,\underset{\sim}{\theta}$$

$$n = a^{-1}u - \frac{1}{2}\,a^{-1}\underset{\sim}{\theta}^{T}\underset{\sim}{k}_2\underset{\sim}{\theta}$$

for the direct analysis, and

$$d\underset{\sim}{m} = (\underset{\sim}{k}_1 - a^{-1}u\underset{\sim}{k}_2 + a^{-1}\underset{\sim}{k}^{!P}_{*})\,d\underset{\sim}{\theta} - u^{-1}dn\,\underset{\sim}{k}_2\underset{\sim}{\theta}$$

$$dn = a^{-1}du - a^{-1}\underset{\sim}{\theta}^{T}\underset{\sim}{k}_2\,d\underset{\sim}{\theta}$$

The non-linear stiffness matrix $\underset{\sim}{k}^{P}_{*}$, defined by Powell as

$$\underset{\sim}{k}_*^p = \frac{1}{280}\left[\begin{array}{c|c} 2(4\theta_1^2+\theta_1\theta_2+\theta_2^2) & (\theta_1^2+\theta_2^2) \\ \hline (\theta_1^2+\theta_2^2) & 2(\theta_1^2-\theta_1\theta_2+4\theta_2^2) \end{array}\right]$$

is flexurally equivalent to $\underset{\sim}{k}_*^!$, as defined in (3.1.94a); the non-linear incremental stiffness matrix $\underset{\sim}{k}_*^{!p}$ can also be expressed in the form (3.1.94b).

To facilitate a direct comparison, the above mentioned finite-element formulations are summarized in Tables 3.1. and 3.2.

	DIRECT FORMULATIONS	
MALLET AND MARÇAL (1968)	$\underset{\sim}{m} = (\underset{\sim}{k}_1 - a^{-1}u\underset{\sim}{k}_2 + a^{-1}\underset{\sim}{k}_*^!)\underset{\sim}{\theta}$	$n = a^{-1}u - \frac{1}{2}a^{-1}\underset{\sim}{\theta}^T\underset{\sim}{k}_2\underset{\sim}{\theta}$
POWELL (1969)		
JENNINGS (1968)	$\underset{\sim}{m} = (\underset{\sim}{k}_1 - a^{-1}u\underset{\sim}{k}_2 + a^{-1}\underset{\sim}{k}_*)\underset{\sim}{\theta}$	
(3.1.87)		

TABLE 3.1

	INCREMENTAL FORMULATIONS	
ARGYRIS (1964)	$d\underset{\sim}{m} = k_1 d\underset{\sim}{\theta}$	$dn = a^{-1}du$
MARTIN (1964)	$d\underset{\sim}{m} = (k_1 - a^{-1}uk_2)d\underset{\sim}{\theta}$	
MALLET AND MARÇAL (1968)	$d\underset{\sim}{m} = (\underset{\sim}{k}_1 - a^{-1}u\underset{\sim}{k}_2 + a^{-1}\underset{\sim}{k}_*^{!i})d\underset{\sim}{\theta} -$	$dn = a^{-1}du -$
POWELL (1969)	$a^{-1}dn\underset{\sim}{k}_2\underset{\sim}{\theta}$	
JENNINGS (1968)	$d\underset{\sim}{m} = (\underset{\sim}{k}_1 - a^{-1}u\underset{\sim}{k}_2 + a^{-1}\underset{\sim}{k}_*^{!i})d\underset{\sim}{\theta} -$	$a^{-1}\underset{\sim}{\theta}^T\underset{\sim}{k}_2 d\underset{\sim}{\theta}$
(3.1.89)	$a^{-1}dn\underset{\sim}{k}_2\underset{\sim}{\theta}$	

TABLE 3.2

We stress that, to perform the above comparisons, the formulations presented by Argyris, Martin and Mallet and Marçal had to be specialized for a three-degree of freedom element.

3.2 PLASTICITY

The study of natural processes should, ideally, develop in two consecutive phases, the first phase, aimed at understanding why and how the material phenomena occur, being seconded by the creation of a mathematical model describing the process as accurately as possible.

The mathematical theories of plasticity, designed to describe the stress- and strain-fields in a plastic body, have had, however, to develop from theoretical models based on results of simplified tests due to the overwhelmingly complex behaviour presented by ductile materials.

The material laws appear to be different for each material and the material behaviour, which depends on its previous history of deformation, is difficult to define, specially if it is subject to a complex multiaxial state of stress; because of this complexity, the physical theories of plasticity are yet unable to provide universally applicable laws explaining why and how materials flow plastically.

The several mathematical theories which have been proposed can be divided into two groups, according to whether the basic relationships connect stresses and strains or stresses and strain rates.

The results provided by these theories, respectively known as deformation theories and flow theories, will only coincide when the development of yielding is regularly progressive.

In either case, the first step is to decide on the yield criterion, that is the rule defining which combination of stresses will cause yield; the simplest situation is when yielding is controlled by just one stress component, for instance, and as in the present case, the axial stress.

In the general case the yield criterion will depend on the complete state of stress and strain and can be expressed as

$$\Phi(\underset{\sim}{X}, \underset{\sim}{u}_p) = 0$$

The function Φ is called the YIELD FUNCTION and the hypersurface, parametric in the plastic deformations $\underset{\sim}{u}_p$, that it represents in the stress-space $\underset{\sim}{X}$ is called YIELD-SURFACE. If, as we assume, the material is stable in the sense of Drucker (1959), the yield surface, although not necessarily continuous, is always convex.

In order to simplify further the problem, the continuous spreading of plasticity along the member cross-section will be neglected; it is assumed that the cross-section makes an abrupt transition from a completely rigid state to a state where all fibres are stressed to the yield level, and where unrestricted plastic deformation can occur.

Having defined the static equilibrium condition for a fully plastified cross-section, the next step is to impose its kinematic compatibility or, in the parlance of the theories of plasticity, to characterize the flow rule.

In the early works the yield condition (STATICS) and the flow rule (KINEMATICS) were treated independently. The possibility of deriving the latter from the yield condition is offered by the concept of plastic potential, introduced by von Mises (1928) for continuous yield functions and generalized by Koiter (1953) to include singular points. In the terminology adopted herein, this is understood as a relation of duality between the descriptions of the static and kinematic phases of plasticity.

The plasticity relations are completed when the static and kinematic variables are connected through an association condition; in this condition, is where the essential difference between the deformation and the flow theories of plasticity resides.

3.2.1 Deformation Analysis

In planar framed structures, the most general force acting on a cross-section comprises an axial force, a direct shear force and a flexural moment.

Only those stress-resultant components which contribute to strain energy have, following Hodge (1959), to be included as yield condition parameters. As shown for instance by Neal (1961), in planar frames the shear deformation effects are usually secondary and neglegible; therefore, the shear stress-resultant will not be included in the yield conditions presented herein.

As an introduction to the presentation of the plasticity relations, let us start by considering the simplest situation, that of one-dimensional perfect plasticity whereby one of the remaining stress-resultants, either the axial force or the bending moment, has unquestionable predominance.

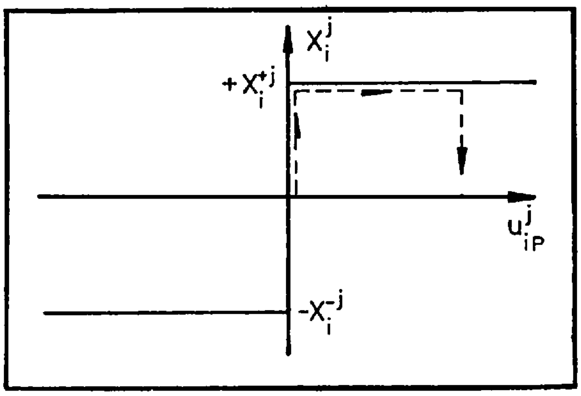


FIGURE 3.19

Let X_i^{+j} and $-X_i^{-j}$, as illustrated in Fig.3.19, be respectively the positive and negative plastic capacities with respect to the stress-resultant X_i^j at critical section j of member m; the static admissibility condition is defined by

$$-X_i^{-j} \leq X_i^j \leq +X_i^{+j}$$

or, in matrix form

$$\underset{\sim}{N}^T X_i^j \leq \underset{\sim}{X}_{*i}^j \qquad (3.2.1)$$

where we note

$$\underset{\sim}{N}^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{X}_{*i}^j = \begin{bmatrix} X^+ \\ X^- \end{bmatrix}_i^j \qquad (3.2.2a,b)$$

Let us introduce two yield functions Φ_i^{+j} and Φ_i^{-j} grouped in $\underset{\sim}{\Phi}_{*i}^j$ and defined by

$$\underset{\sim}{\Phi}_{*i}^j = \underset{\sim}{N}^T X_i^j - \underset{\sim}{X}_{*i}^j \qquad (3.2.3)$$

so that the static admissibility condition (3.2.1) can be replaced by

$$\underset{\sim}{\Phi}_{*i}^j \leq \underset{\sim}{0} \qquad (3.2.4)$$

The above condition defines a sub-space of the uni-dimensional stress space X_i^j, as diagramatically represented in Fig.3.20.

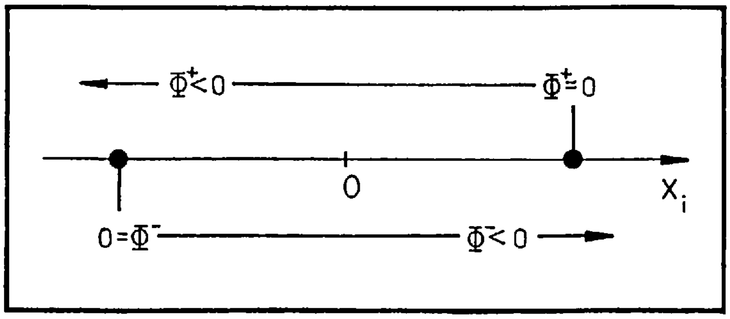


FIGURE 3.20

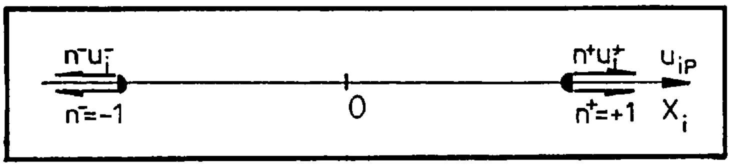


FIGURE 3.21

The stress-resultant acting on the cross-section is represented by a point in the stress-space; if that point is contained in the open sub-space

$$\underset{\sim}{\Phi}{}^{j}_{*i} < \underset{\sim}{0}$$

the cross-section is assumed to remain completely rigid and no plastic strain can develop. If the point lies on the boundary, where the yield function vanishes, $\underset{\sim}{\Phi}{}^{j}_{*i} = \underset{\sim}{0}$, every point of the cross-section is assumed to be stressed to the yield limit and the section is ready to deform plastically for constant stresses.

The plastic strain-resultant u^{j}_{ip}, unrestricted in sign, can be represented as the difference of two complementary and non-negative plastic multipliers u_i^{+j} and u_i^{-j}

$$u_i^{+j} \cdot u_i^{-j} = 0 \tag{3.2.5}$$

and

$$u_i^{+j} \,,\; u_i^{-j} \quad 0 \tag{3.2.6}$$

such that

$$u^{j}_{ip} = u_i^{+j} \quad \text{if} \quad X^{j}_{i} = +X_i^{+j}$$

or

$$u^{j}_{ip} = -u_i^{-j} \quad \text{if} \quad X^{j}_{i} = -X_i^{-j}$$

Hence

$$u^{j}_{ip} = u_i^{+j} - u_i^{-j}$$

or using (3.2.2a)

$$u^{j}_{ip} = \underset{\sim}{N}\ \underset{\sim}{u}{}^{j}_{*i} \tag{3.2.7}$$

where

$$\underset{\sim}{u}{}^{j}_{*i} = \begin{bmatrix} u^{+} \\ u^{-} \end{bmatrix}^{j}_{i} \tag{3.2.8}$$

Superimposing the strain- and stress-resultant spaces, as illustrated in Fig.3.21, the components of the vector $\underset{\sim}{N}$ can be interpreted as the external directional vectors of the subspace (3.2.4).

Considering general paths connecting typical stress-strain combinations, as for instance the one represented in Fig. 3.19, and consulting at every stage the previous relations (3.2.1) to (3.2.8), it could be easily concluded that the complementarity condition

$$\underset{\sim}{\Phi}^{jT}_{*i} \; \underset{\sim}{u}^{j}_{*i} = 0 \qquad (3.2.9)$$

associating the plastic potentials with the plastic multipliers will hold for every situation except those preceded by plastic unstressing, i.e., the transition from a plastic state into a rigid state.

The relations (3.2.1) to (3.2.9) can therefore be used to define situations of regularly progressive yielding (R.P.Y.). They are collected below

$$\left[\begin{array}{c|c} \cdot & \underset{\sim}{N}^T \\ \hline \underset{\sim}{N} & \cdot \end{array}\right] \left[\begin{array}{c} \underset{\sim}{u}_* \\ \hline \underset{\sim}{X} \end{array}\right]^j_i = \left[\begin{array}{c} \underset{\sim}{\Phi}_* \\ \hline \underset{\sim}{u}_p \end{array}\right]^j_i + \left[\begin{array}{c} \underset{\sim}{X}_* \\ \hline \cdot \end{array}\right]^j_i \qquad \begin{array}{r} (3.2.10a) \\ (3.2.10b) \end{array}$$

$$\underset{\sim}{\Phi}^{j}_{*i} \leq \underset{\sim}{0} \qquad \underset{\sim}{\Phi}^{jT}_{*i} \; \underset{\sim}{u}^{j}_{*i} = 0 \qquad \underset{\sim}{u}^{j}_{*i} \geq \underset{\sim}{0} \qquad (3.2.10c,d,e)$$

in the format Smith (1978) presents Maier's formulation, in order to emphasize the uncoupled and strictly dual relationship between the descriptions of the static and kinematic phases.

The complementarity condition (3.2.5) need not be included in the above summary because it is automatically taken into account by the Simplex-based algorithms.

Setting successively j to one and two in (3.2.10) and collecting, the following relations can be defined for member m

	RIGID-PERFECTLY PLASTIC MATERIALS:NO INTERACTION		
STATICS / KINEMATICS	$$\begin{bmatrix} \cdot & \tilde{N}^T \\ \tilde{N} & \cdot \end{bmatrix}_m \begin{bmatrix} \tilde{u}_* \\ \tilde{X}' \end{bmatrix}_m = \begin{bmatrix} \tilde{\Phi}_* \\ \tilde{u}'_p \end{bmatrix}_m + \begin{bmatrix} \tilde{X}_* \\ \cdot \end{bmatrix}_m$$		(3.2.11) (3.2.12)
$\tilde{\Phi}_{*_m} \leqslant \tilde{0}$	$\tilde{\Phi}^T_{*_m} \cdot \tilde{u}_{*_m} = 0$	$\tilde{u}_{*_m} \geqslant \tilde{0}$	(3.2.13-14-15)
YIELD RULE	ASSOCIATION (R.P.Y.)	FLOW RULE	

where now

$$\tilde{\Phi}^T_{*_m} = \left[\Phi_1^{+1} \quad \Phi_1^{+2} \quad \Phi_2^{+} \;\vdots\; \Phi_1^{-1} \quad \Phi_1^{-2} \quad \Phi_2^{-}\right]_m$$

$$\tilde{u}^T_{*_m} = \left[u_1^{+1} \quad u_1^{+2} \quad u_2^{+} \;\vdots\; u_1^{-1} \quad u_1^{-2} \quad u_2^{-}\right]_m$$

$$\tilde{X}^T_{*_m} = \left[X_1^{+1} \quad X_1^{+2} \quad X_2^{+} \;\vdots\; X_1^{-1} \quad X_1^{-2} \quad X_2^{-}\right]_m$$

and $\tilde{X}'_m$ are the member independent stress-resultants, $\tilde{u}'_{p_m}$ being the plastic components of the corresponding strain-resultants.

The incidence matrix $\underline{N}_m$, in the case of neglegible axial force effects, is defined by

$$\underline{N}_m = \begin{bmatrix} 1 & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \circ & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad (3.2.16a)$$

and by

$$\underline{N}_m = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & -1 \end{bmatrix} \qquad (3.2.16b)$$

when the axial force effects are predominant.

The yield condition (3.2.12) can be interpreted as a polytope in the $\tilde{X}_m$ - space, as represented in Fig.3.22, with non-interactive faces in the direction of X_{2_m} ; the columns of matrix $\underline{N}_m$ will then represent the outward normals of the polytope faces.

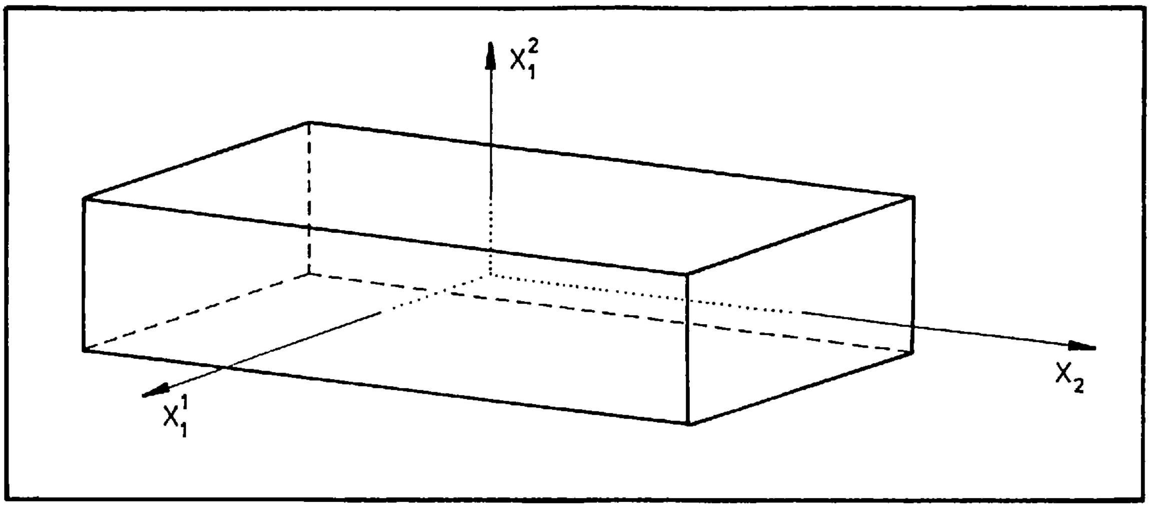


FIGURE 3.22

Let us consider now, still for rigid-perfectly plastic materials, the reduction in the bending (thrust) plastic capacity due to the presence of the axial force (bending moment).

In statically determined beam systems, the limiting state of strength is governed by the carrying capacity of the most highly stressed section and it is therefore natural that this problem, widely discussed in the educational literature, has received for long the attention of the researchers.

When the axial force is included in the analysis of beams and frames, the yield hinge retains its proeminent position as a basic mode of deformation. However, in this case we must deal with hinges that allow not only relative rotation but also relative axial displacement of adjacent cross-sections. Onat and Prager (1953,1954) called them extensible yield hinges.

As an illustration, and following for instance Prager (1959) or Hodge (1959), consider the solid rectangular cross-section, with depth d and breadth b, bent in the plane of symmetry, as represented in Fig.2.23.

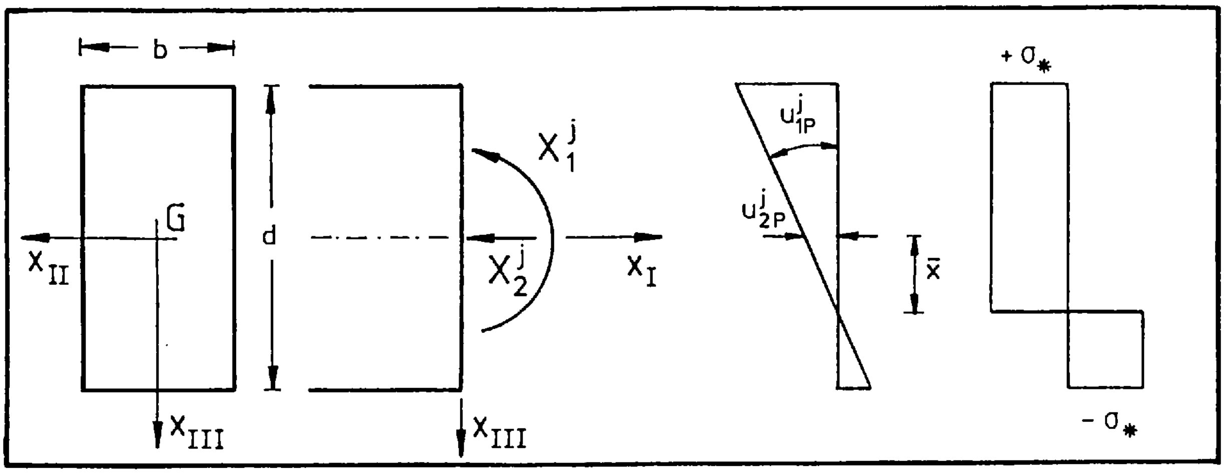


FIGURE 3.23

The internal equilibrium conditions reduce to

$$\pm X_1^j = X_{*1}^j \left[1-\left(\frac{2\bar{x}}{d}\right)^2\right] \qquad (3.2.17a)$$

$$X_2^j = X_{*2}^j \left(\frac{2\bar{x}}{d}\right) \qquad (3.2.17b)$$

where

$$X_{*1}^j = \frac{1}{4}\ \sigma_* b d^2 \qquad (3.2.18a)$$

and

$$X_{*2}^j = \sigma_* b d \qquad (3.2.18b)$$

are the cross-section plastic capacities when subject, respectively, to simple bending and axial thrust; it is assumed that the yield stress σ_* is the same both in tension ($\sigma_*^-=-\sigma_*$) and compression ($\sigma_*^+=+\sigma_*$).

The elimination in (3.2.17) of the variable $\bar{x}$, defining the distance of the neutral axis from the centroid, gives

$$\pm X_1^j + X_{*1}^j \left(\frac{X_2^j}{X_{*2}^j}\right)^2 - X_{*1}^j = 0$$

since, from (3.2.18a) and (3.2.18b)

$$d = 4\ \frac{X_{*1}^j}{X_{*2}^j} \qquad (3.2.18c)$$

If the plastic work dissipated up to when the last fibre starts to yield is neglected, the above conditions can be adopted as the yield criterion and, in the present case, two yield functions, represented in Fig.3.24, can be defined as

$$\Phi_i^j = (-1)^{i+1}\, X_1^j + X_{*1}^j \left(\frac{X_2^j}{X_{*2}^j}\right)^2 - X_{*1}^j \qquad i=1,2 \qquad (3.2.18d)$$

The intersection, in the stress-space, of the half-spaces $\Phi_i^j \leq 0$ defines the (convex) domain of statically admissible stress distributions. The outward vector normal to the yield locus $\Phi_i^j = 0$ has components

$$n_{i1}^j = \Phi_{i,1}^j = (-1)^{i+1} \quad \text{and} \quad n_{2i}^j = \Phi_{i,2}^j = 2\,\frac{X_{*1}^j}{(X_{*2}^j)^2}\, X_2^j \qquad (3.2.19a,b)$$

and the yield functions can be expressed as

$$\begin{bmatrix} n_{1i}^j & n_{2i}^j \end{bmatrix} \begin{bmatrix} X_1^j \\ X_2^j \end{bmatrix} = \Phi_i^j + X_{*1}^j + X_{*1}^j \left(\frac{X_2^j}{X_{*2}^j}\right)^2$$

or, setting above i = 1,2 and collecting, in matrix form as

$$\underline{N}^T \underset{\sim}{X}' = \underline{\underline{\Phi}}_*^j + \underset{\sim}{X}_*^j + \underset{\sim}{\pi}_\varphi^j$$

If the static and kinematic phases of plasticity are to be described by dual transformations, Kinematics has to be expressed as

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} n_{11}^j & n_{12}^j \\ n_{21}^j & n_{22}^j \end{bmatrix} \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} \qquad \begin{matrix} (3.2.20a) \\ (3.2.20b) \end{matrix}$$

The assumption that plane sections remain plane gives

$$u_{2p}^j = \bar{x}\tan u_{1p}^j \qquad (3.2.21)$$

where, from (3.2.17b), (3.2.18c) and (3.2.19b)

$$\bar{x} = 2 \frac{X^j_{*1}}{(X^j_{*2})^2} X^j_2 = n^j_{2i} \qquad (3.2.22)$$

Introducing the non-linear corrective term

$$u^j_{1\varphi} = -u^j_{1p} + \tan u^j_{1p} \qquad (3.2.23)$$

and identifying k_3 and k_4 as plastic multipliers

$$k_3 = u^j_{*1} \quad \text{and} \quad k_4 = u^j_{*2}$$

satisfying the non-negativity condition

$$u^j_{*1} \;,\; u^j_{*2} \geq 0 \qquad (3.2.24a)$$

and the complementarity condition

$$u^j_{*1} \cdot u^j_{*2} = 0 \qquad (3.2.24b)$$

with help from (3.2.21) to (3.2.23) we may identify the remaining kinematic variables as

$$k_1 = u^j_{1p} + u^j_{1\varphi} \quad \text{and} \quad k_2 = u^j_{2p} \qquad (3.2.25a,b)$$

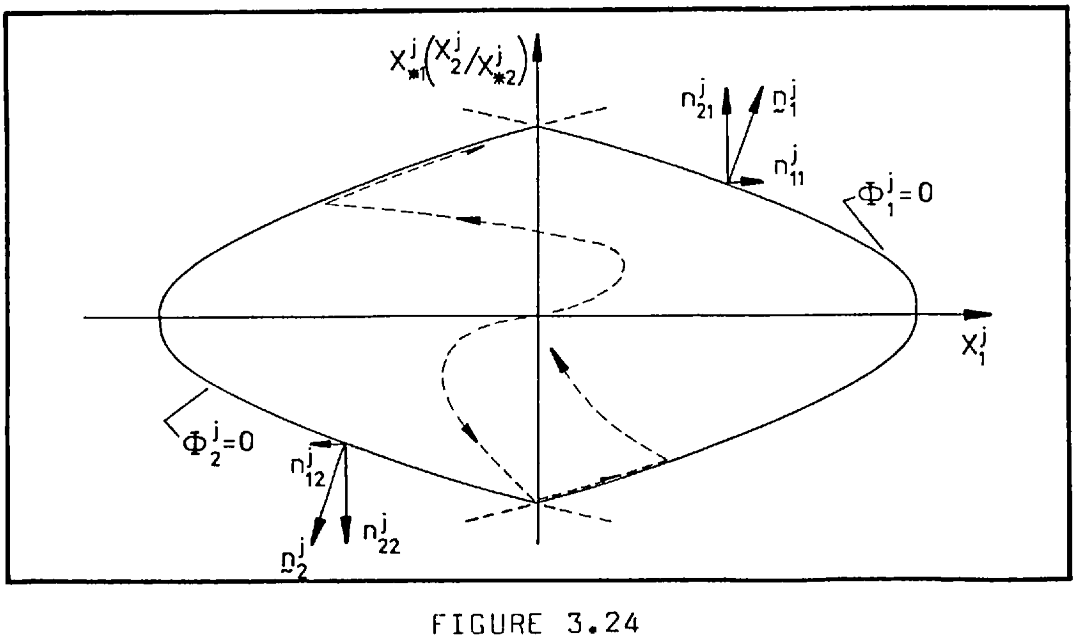


FIGURE 3.24

Superimposing the strain-space (u^j_{1p}, u^j_{2p}) upon the stress-space (X^j_1, X^j_2), it is easy to conclude from (3.2.25) and (3.2.20) that the deformations, affected by the non-linear corrective term, are proportional to the components of the outward normal to the yield locus; at the discontinuity points the corrected deformation vector will lie inside the cone formed by the normals to the incident loci.

Considering several stress-strain combinations, as for instance those connected by the paths indicated in Fig.3.24, the association condition

$$\underset{\sim}{\Phi}^{jT}_* \; \underset{\sim}{u}^j_* = 0 \qquad (3.2.26)$$

where $\underset{\sim}{u}^j_*$ collects the plastic parameters, subject to (3.2.24), will again prove valid only in the absence of plastic unstressing.

In general, and considering now both critical sections of member m, we would find

RIGID-PERFECTLY PLASTIC MATERIALS

STATICS / KINEMATICS

$$\begin{bmatrix} \cdot & \underline{N}^T \\ \underline{N} & \cdot \end{bmatrix}_m \begin{bmatrix} \underline{u}_* \\ \underline{X}' \end{bmatrix}_m = \begin{bmatrix} \underline{\Phi}_* \\ \underset{\sim}{u}_p \end{bmatrix}_m + \begin{bmatrix} \underline{X}'_* \\ \cdot \end{bmatrix}_m + \begin{bmatrix} \underline{\pi}_\varphi \\ \underset{\sim}{u}_\varphi \end{bmatrix}_m \qquad \begin{matrix} (3.2.27) \\ (3.2.28) \end{matrix}$$

$\underset{\sim}{\Phi}_{*_m} \leq \underset{\sim}{0}$	$\underset{\sim}{\Phi}^T_{*_m} \; \underset{\sim}{u}_{*_m} = 0$	$\underset{\sim}{u}_{*_m} \geq \underset{\sim}{0}$
YIELD RULE	ASSOCIATION(R.P.Y.)	FLOW RULE

(3.2.29-30-31)

The yield surface is defined by the intersection of n continuous and convex surfaces

$$\Phi_k = 0 \qquad , \; k = 1,2,\ldots,n$$

each of which is expressable in the form

$$\Phi_k = \frac{\partial \Phi_k}{\partial X^1_1} X^1_1 + \frac{\partial \Phi_k}{\partial X^2_1} X^2_1 + \frac{\partial \Phi_k}{\partial X_2} X_2 - X_{*_k} - \pi_{\varphi_k}$$

where X_{*_k} is a constant, depending on the geometry of the section and on the yield stress σ_*, and π_{φ_k} is either zero or a non-linear function of the stress-resultants, so that the k-th column of the incidence matrix $\underline{N}_m$

$$\underset{\sim}{n}_{k_m} = \begin{bmatrix} \dfrac{\partial}{\partial X_1^1} \\ \dfrac{\partial}{\partial X_1^2} \\ \dfrac{\partial}{\partial X_2} \end{bmatrix}_m \Phi_k$$

contains the components of the outward normal vector to the yield surface

$$\Phi_k = 0$$

The additional plastic deformations $\underset{\sim}{u}_{\varphi_m}$ are assumed to be non-linear functions of the plastic components of the strain-resultants, converging to zero as the plastic deformations become smaller.

The plasticity relations (3.2.28) to (3.2.31) are still valid for rigid-work hardening materials; the internal equilibrium condition will now be directly dependent on the kinematic variables, quantifying through the hardening matrix $\underline{H}_m$, which we assume symmetric, the interdependence of stress and strain:

RIGID-WORKHARDENING MATERIALS

STATICS / KINEMATICS

$$\begin{bmatrix} -\underline{H} & \underline{N}^T \\ \underline{N} & \cdot \end{bmatrix}_m \begin{bmatrix} \underset{\sim}{u}_* \\ \underset{\sim}{X}' \end{bmatrix}_m = \begin{bmatrix} \underset{\sim}{\Phi}_* \\ \underset{\sim}{u}_p \end{bmatrix}_m + \begin{bmatrix} \underset{\sim}{X}_* \\ \cdot \end{bmatrix}_m + \begin{bmatrix} \underset{\sim}{\pi}_\varphi \\ \underset{\sim}{u}_\varphi \end{bmatrix}_m \qquad \begin{matrix} (3.2.32) \\ (3.2.33) \end{matrix}$$

$\underset{\sim}{\Phi}_{*_m} \leqslant \underset{\sim}{0}$	$\underset{\sim}{\Phi}^T_{*_m} \underset{\sim}{u}_{*_m} = 0$	$\underset{\sim}{u}_{*_m} \geqslant \underset{\sim}{0}$	(3.2.34-35-36)
YIELD RULE	ASSOCIATION(R.P.Y.)	FLOW RULE	

If the above description was specialized for piecewise linear workhardening materials, Maier's formulation would then be recovered.

As an illustration let us assume now that the element, with the rectangular cross-section represented in Fig.3.23, is constituted by a rigid-linearly workhardening material, so that the stress-block associated with the axial deformations is that represented in Fig.3.25. The equilibrium conditions give, for positive strain-resultants, and dropping the superscript j

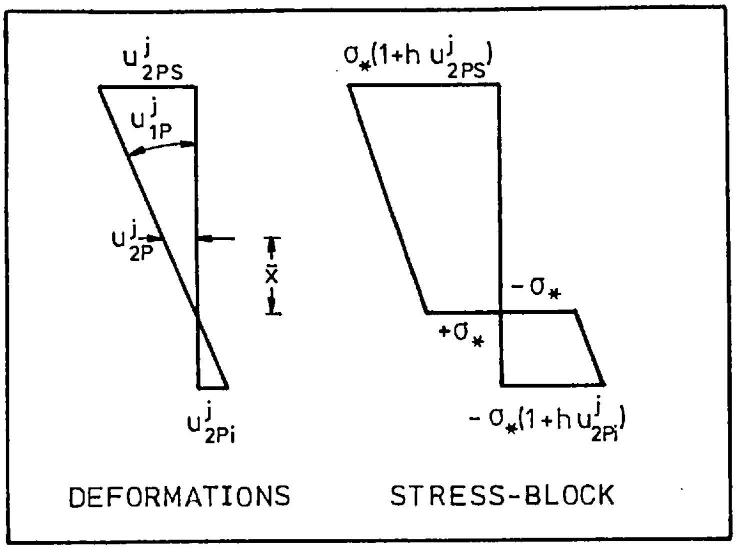


FIGURE 3.25

$$X_1 = X_{*1}\left[1-\left(\frac{2\bar{x}}{d}\right)^2+\frac{dh}{3}\tan u_{1p}\right]$$

$$X_2 = X_{*2}\left[\frac{2\bar{x}}{d}+h\ u_{2p}\right] \qquad (3.2.37)$$

the corresponding yield function being defined by

$$\Phi = X_1 + X_{*1}\left(\frac{X_2}{X_{*2}} - hu_{2p}\right)^2 - X_{*1}\frac{dh}{3}\tan u_{1p} - X_{*1} \qquad (3.2.38a)$$

or

$$\Phi = \frac{\partial\Phi}{\partial X_1}X_1 + \frac{\partial\Phi}{\partial X_2}X_2 - X_{*1}\frac{dh}{3}\tan u_{1p} - X_{*1} - X_{*1}\left[\left(\frac{X_2}{X_{*2}}\right)^2 - (hu_{2p})^2\right]$$

(3.2.38b)

where $\frac{\partial\Phi}{\partial X_1} = 1$

$$\frac{\partial\Phi}{\partial X_2} = 2\,\frac{X_{*1}}{X_{*2}}\left(\frac{X_2}{X_{*2}} - hu_{2p}\right) \qquad (3.2.39a)$$

which, using (3.2.37) and (3.2.18) reduces again to

$$\frac{\partial\Phi}{\partial X_2} = \bar{x} \qquad (3.2.39b)$$

Hence, we may still write Kinematics in the form (3.2.33), as:

$$\begin{bmatrix} u_{1p} + u_{1\varphi} \\ u_{2p} \end{bmatrix} = \begin{bmatrix} \frac{\partial \Phi}{\partial X_1} \\ \frac{\partial \Phi}{\partial X_2^j} \end{bmatrix} u_* \qquad (3.2.40)$$

the additional deformation $u_{1\varphi}$ being still defined by (3.2.23); confronting (3.2.40) with (3.2.21), backed by (3.2.39), the plastic multiplier is identified as

$$u_* = \tan u_{1p}$$

and, substituting the above in the yield function (3.2.38b), Statics can be expressed as

$$-\left[\frac{4}{3}\ \frac{(X_{*1})^2}{X_{*2}} h\right] u_* + \left[\frac{\partial \Phi}{\partial X_1}\ \ \frac{\partial \Phi}{\partial X_2}\right] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \Phi + X_{*1} + \pi_\varphi$$

with the same format of (3.2.32), where

$$\pi_\varphi = X_{*1}\left[\left(\frac{X_2}{X_{*2}}\right)^2 - (h\ u_{2p})^2\right]$$

The elements of the hardening matrix $\underline{H}_m$, as well as the corrective potential π_{φ_m}, will in general be non-linear functions of the strain- and stress-resultants, ultimately dependent on the stress-strain relationship.

The intersection of the yield functions $\underset{\sim}{\Phi}_{k_m}$ define a convex surface, parametric in

$$\underline{H}_m\ \underset{\sim}{u}_{*m}$$

which, as yield progresses, will vary in size, shape and orientation in the X_m-space; its movement is defined by the hardening coefficients h_{ij}.

The difficulty of characterizing accurately the hardening coefficients has been evaded by defining artificial hardening rules, their choice depending on the ease with which they can be applied in the chosen method of analysis as well as on their capability of representing the actual hardening behaviour of the material.

As the complementarity condition $\underset{\sim}{\Phi}^T_{*m} \underset{\sim}{u}_{*m} = 0$ does not allow for plastic unstressing, while the non-negativity condition $\underset{\sim}{u}_{*m} \geq \underset{\sim}{0}$ concedes for decreasing, i.e., reversible, plastic strains, the plasticity relations referred to in the above, termed holonomic by Maier (1969a), do in fact characterize a two-phase elastic material with an initial infinite stiffness followed, if $h_{ij} = 0$, by a phase of infinite flexibility; the only process of decreasing the stress-resultant at a strained cross-section is through a total, if $h_{ij} = 0$, or partial, if $h_{ij} \neq 0$, recovery of the developed strains. Smith and Munro (1978) proved wrong the generalized conviction that the behaviour of elastic-(holonomic) plastic systems was path-independent.

This inability to perform plastic unstressing is a common deficiency of every deformation theory.

For a long time it has been known that this difficulty could be overcome through the utilization of a convenient method of numerical analysis, that is an algorithm capable of detecting plastic unstressing and of separating the problem in its straining and unstressing parts. The mathematical programming algorithms have proven highly successful as shown by De Donato and Maier (1973, 1974) and Smith (1975, 1978). This problem will be dealt with in more detail in sub-section 5.4.4.

3.2.2 Incremental Analysis

The activation laws and the flow laws are the basic ingredients of the incremental theories of plasticity.

The activation laws, defining which yield modes are active at a given state of stress and strain, indicate at which points of the system plastic straining may develop; the flow laws distinguish among these points, those which will in fact be further strained from those where plastic unstressing will take place during the incremental action to which that system will be subject.

In the presentation to follow we will concentrate on the so-called flow laws.

The information they provide, insufficient per se, is assumed to be backed by a complete knowledge of the state of stress and strain just prior to the increment; in other words, the activation laws are assumed to be identically satisfied.

To start with, let us replace in the static and kinematic descriptions (3.2.11) and (3.2.12), respectively, which are valid for the k-th stage of loading, the static and kinematic variables by the sum of their components at stage k-1 and the subsequent increments:

$$(\underset{\sim}{\Phi}_{*_m} + \Delta\underset{\sim}{\Phi}_{*_m}) = \underset{-}{N}_m^T\ (\underset{\sim}{X}'_m + \Delta\underset{\sim}{X}'_m) - \underset{\sim}{X}_{*_m}$$

$$(\underset{\sim}{u}'_{p_m} + \Delta\underset{\sim}{u}'_{p_m}) = \underset{-}{N}_m\ (\underset{\sim}{u}_{*_m} + \Delta\underset{\sim}{u}_{*_m})$$

or, using the information provided by stage k-1

$$\Delta\underset{\sim}{\Phi}_{*_m} = \underset{-}{N}_m^T\ \Delta\underset{\sim}{X}'_m \qquad (3.2.41a)$$

$$\Delta\underset{\sim}{u}'_{p_m} = \underset{-}{N}_m\ \Delta\underset{\sim}{u}_{*_m} \qquad (3.2.41b)$$

where we assume that the increments on the plastic multipliers still satisfy the non-negativity and complementarity conditions

$$\Delta\underset{\sim}{u}^+_{*_m}\ ,\ \Delta\underset{\sim}{u}^-_{*_m} \geq \underset{\sim}{0} \qquad (3.2.42a)$$

$$\Delta\underset{\sim}{u}^{+T}_{*_m} \cdot \Delta\underset{\sim}{u}^-_{*_m} = 0 \qquad (3.2.43)$$

Let the incidence matrix $\underset{-}{I}_*$ collect the subset of activated yield modes, i.e.

$$\underset{\sim}{\Phi}^!_{*_m} = \underset{-}{I}_*\ \underset{\sim}{\Phi}_{*_m} \qquad (3.2.44a)$$

such that

$$\underset{\sim}{\Phi}^!_{*_m} = \underset{\sim}{0} \qquad (3.2.44b)$$

and consequently

$$\Delta\underset{\sim}{\Phi}^!_{*_m} \leq \underset{\sim}{0} \qquad (3.2.42b)$$

where $$\Delta \underset{\sim}{\Phi}'_{*_m} = \underset{\sim}{I}_* \Delta \underset{\sim}{\Phi}_{*_m}$$

or, from (3.2.41a) $$\Delta \underset{\sim}{\Phi}'_{*_m} = \underset{\sim}{N}'^{T}_{m} \, \Delta \underset{\sim}{X}'_{m} \qquad (3.2.42c)$$

where $$\underset{\sim}{N}'^{T}_{m} = \underset{\sim}{I}_* \underset{\sim}{N}^{T}_{m} \qquad (3.2.45)$$

To each activated yield mode, defined by (3.2.44) we associate the incremental plastic multiplier $\Delta \underset{\sim}{u}'_{*_m}$ defined by

$$\Delta \underset{\sim}{u}_{*_m} = \underset{\sim}{I}^{T}_{*} \, \Delta \underset{\sim}{u}'_{*_m}$$

or, from (3.2.41b) and (3.2.46)

$$\Delta \underset{\sim}{u}'_{p_m} = \underset{\sim}{N}'_{m} \, \Delta \underset{\sim}{u}'_{*_m} \qquad (3.2.42d)$$

The above relationship expresses the incremental plastic strains as a function of a subset of plastic parameters which includes only those which are potentially non-zero; the association condition

$$\Delta \underset{\sim}{\Phi}'_{*_m} \, \Delta \underset{\sim}{u}'_{*_m} = 0 \qquad (3.2.42e)$$

will distinguish between the yield modes activated at stage k, those which will remain active

$$\{\Delta \underset{\sim}{\Phi}'_{*_m}\} = 0 \;\Rightarrow\; \{\Delta \underset{\sim}{u}'_{*_m}\} \geq 0$$

from those which will cease to be active

$$\{\Delta \underset{\sim}{\Phi}'_{*_m}\} < 0 \;\Rightarrow\; \{\Delta \underset{\sim}{u}'_{*_m}\} = 0$$

where $\{\Delta \underset{\sim}{\Phi}'_{*_m}\}$ and $\{\Delta \underset{\sim}{u}'_{*_m}\}$ denote, respectively, subsets of $\Delta \underset{\sim}{\Phi}'_{*_m}$ and $\Delta \underset{\sim}{u}'_{*_m}$.

The incremental plasticity relations (3.2.42) were collected below, revealing the uncoupled, strictly dual relationship between the descriptions of Statics and Kinematics.

RIGID-PERFECTLY PLASTIC MATERIALS:NO INTERACTION		
STATICS / KINEMATICS: $\begin{bmatrix} \cdot & \underline{N}'^{T} \\ \underline{N}' & \cdot \end{bmatrix}_m \cdot \begin{bmatrix} \Delta \underline{u}'_* \\ \Delta \underset{\sim}{X}' \end{bmatrix}_m = \begin{bmatrix} \Delta \underset{\sim}{\Phi}'_* \\ \Delta \underset{\sim}{u}'_p \end{bmatrix}_m$ (3.2.46) (3.2.47)		
$\Delta \underset{\sim}{\Phi}'_{*m} \leqslant \underset{\sim}{0}$	$\Delta \underset{\sim}{\Phi}'^{T}_{*m} \; \Delta \underset{\sim}{u}'_{*m} = 0$	$\Delta \underset{\sim}{u}'_{*m} \geqslant \underset{\sim}{0}$ (3.2.48-49-50)
YIELD RULE	ASSOCIATION	FLOW RULE

Let us consider now the yield function (3.2.18c), obtained when analysing the solid rectangular section subject to a bending moment and to an axial force, as illustrated in Fig.3.21; its incremental form

$$\Delta\Phi_i^j = \left[(-1)^{i+1}\right] \Delta X_1^j + \left[2 \cdot \frac{X_{*1}^j X_2^j}{(X_{*2}^j)^2}\right] \Delta X_2^j - R_{\varphi_i}^j$$

immediately generates the components of outward normal to the yield mode Φ_i^j at point (X_1^j, X_2^j), as defined by (3.2.19). In the term $R_{\varphi_i}^j$ we collect all terms non-linear on the incremental static variables, which in the present case reduce to

$$R_{\varphi_i}^j = -X_{*1}^j \left(\frac{X_2^j}{X_{*2}^j}\right)^2$$

Considering now both yield functions and expressing the incremental static relationship as

$$+\underset{\sim}{R}_{\varphi}^j + \Delta \underset{\sim}{\Phi}_*^j = \begin{bmatrix} n_{11}^j & n_{21}^j \\ n_{12}^j & n_{22}^j \end{bmatrix} \begin{bmatrix} \Delta X_1^j \\ \Delta X_2^j \end{bmatrix} \qquad (3.2.51a)$$

we should expect to find the following description for incremental Kinematics:

$$\begin{bmatrix} \Delta u^j_{1p} + R^j_{u1} \\ \Delta u^j_{2p} + R^j_{u2} \end{bmatrix} = \begin{bmatrix} n^j_{11} & n^j_{12} \\ n^j_{21} & n^j_{22} \end{bmatrix} \Delta \underset{\sim}{u}^j_* \qquad (3.2.51b)$$

The corrective terms R^j_{u1} and R^j_{u2} have, not necessarily simultaneously, to be either zero or non-linear functions of the incremental kinematic variables.

This assertion is justified by the necessity of recovering the formulation of Koiter when, instead of finite increments, Kinematics is expressed in terms of infinitesimal increments; only when the corrective terms satisfy either of the above mentioned conditions, will the incremental description of Kinematics converge to

$$\begin{bmatrix} du^j_{1p} \\ du^j_{2p} \end{bmatrix} = \begin{bmatrix} n^j_{11} & n^j_{12} \\ n^j_{21} & n^j_{22} \end{bmatrix} d\underset{\sim}{u}^j_*$$

satisfying the condition of normality of the infinitesimal strain vector to the yield surface at the stress point.

However, and considering for instance the yield function Φ^j_1, the compatibility condition (3.2.21) when expressed in an incremental form yields

$$\Delta u^j_{2p} = n^j_{21}(\Delta u^j_{1p} + R'^j_{u1}) + s \left\{ s \, \Delta u^j_{2p} + c \, \Delta \bar{x} \right\} \qquad (3.2.52a)$$

where $$R'^j_{u1} = \frac{c}{\bar{x}}(s \, \Delta u^j_{2p} + c \, \Delta \bar{x}) \tan \Delta u^j_{1p} + (\tan \Delta u^j_{1p} - \Delta u^j_{1p}) \qquad (3.2.52b)$$

and $$s = \sin u^j_{1p} \text{ and } c = \cos u^j_{1p}$$

the linear term $\{ \quad \}$ in equation (3.2.52a) preventing us from expressing the compatibility condition in the desired parametric form

$$\begin{bmatrix} \Delta u^j_{1p} + R'^j_{u1} \\ \Delta u^j_{2p} \end{bmatrix} = \begin{bmatrix} n^j_{11} \\ n^j_{21} \end{bmatrix} \Delta u^j_{1*} \qquad \begin{matrix} (3.2.53a) \\ (3.2.53b) \end{matrix}$$

Detailed theoretical and experimental studies of yield hinge in solid plastic beams, as for instance those by Hundy (1954), show that the deformations on the neighbourhood of the hinge violates the basic assumptions of beam theory; on the other hand, and quoting from Palmer, Maier and Drucker (1967), "Normality may be proposed as a primitive postulate following von Mises (1928), or adopted as an expression of the results of experiment, or because of its strong and useful implications for variational and extremum theorems. Alternatively, it can be shown to be a property of certain wide classes of materials defined by postulated thermodynamic conditions".

Hence, at this stage and having in mind the significance of the normality condition, we either disregard the assumption that plane sections remain plane and derive the actual compatibility condition, which, if feasible, would be the correct approach, or we try to appease the inconsistencies created by the combination of the very basic assumptions in the technical theory of beams with those of the theory of lumped plasticity, by defining a law of the form

$$\Delta\bar{x} = -\tan u^j_{1p} \cdot \Delta u^j_{2p} + \frac{n^j_{21}}{sc} R_x \qquad (3.2.52c)$$

regulating the variation of the position of the neutral axis; R_x will either be zero or a non-linear function of the incremental kinematic variables.

If the above hypothesis (3.2.52c), which corresponds to the assumption in Prager (1959), page 52, that, for small strains, the infinitesimal strain increments are related through

$$du^j_{2p} = n^j_{21}\ du^j_{1p}$$

is found acceptable, the incremental compatibility condition (3.2.52a) will take the form

$$\Delta u^j_{2p} = n^j_{21}(\ \Delta u^j_{1p} + R^j_{u1}) \qquad (3.2.52d)$$

where, from (3.2.52b) and (3.2.52c)

$$R^j_{u1} = R_x(1+\tfrac{c}{s}\ \tan \Delta u^j_{1p}) + (\tan \Delta u^j_{1p} - \Delta u^j_{1p}) \qquad (3.2.52e)$$

which reduces to $$R^{j}_{u1} = \frac{1}{3}(\Delta u^{j}_{1p})^{3} + \cdots$$

if $$R_{x} = 0$$

The compatibility condition (3.2.52d) is now equivalent to the parametric form (3.2.53); considering both yield functions one would return to (3,2,51b).

In general and considering only the activated modes at both critical sections of member m, we would find the following description for the incremental behaviour of rigid-perfectly plastic materials:

RIGID-PERFECTLY PLASTIC MATERIALS

STATICS / KINEMATICS	$\begin{bmatrix} \cdot & \underline{N}'^{T} \\ \underline{N}' & \cdot \end{bmatrix}_{m} \cdot \begin{bmatrix} \Delta \underset{\sim}{u}'_{*} \\ \Delta \underset{\sim}{X}' \end{bmatrix}_{m} = \begin{bmatrix} \Delta \underset{\sim}{\Phi}'_{*} \\ \Delta \underset{\sim}{u}'_{p} \end{bmatrix}_{m} + \begin{bmatrix} \underset{\sim}{R}_{\varphi} \\ \underset{\sim}{R}_{u} \end{bmatrix}_{m}$	(3.2.54) (3.2.55)

$\Delta \underset{\sim}{\Phi}'_{*m} = \underset{\sim}{0}$	$\Delta \underset{\sim}{\Phi}'^{T}_{*m} \, \Delta \underset{\sim}{u}'_{*m} = 0$	$\Delta \underset{\sim}{u}'_{*m} = \underset{\sim}{0}$	(3.2.56-57-58)
YIELD RULE	ASSOCIATION	FLOW RULE	

Except for the description of Statics, similar results would be found for the description of rigid-workhardening materials:

RIGID-WORKHARDENING MATERIALS

STATICS / KINEMATICS	$\begin{bmatrix} -\underline{H}' & \underline{N}'^{T} \\ \underline{N}' & \cdot \end{bmatrix}_{m} \begin{bmatrix} \Delta \underset{\sim}{u}'_{*} \\ \Delta \underset{\sim}{X}' \end{bmatrix}_{m} = \begin{bmatrix} \Delta \underset{\sim}{\Phi}'_{*} \\ \Delta \underset{\sim}{u}'_{p} \end{bmatrix}_{m} + \begin{bmatrix} \underset{\sim}{R}_{\varphi} \\ \underset{\sim}{R}_{u} \end{bmatrix}_{m}$	(3.2.59) (3.2.60)

$\Delta \underset{\sim}{\Phi}'_{*m} = \underset{\sim}{0}$	$\Delta \underset{\sim}{\Phi}'^{T}_{*m} \, \Delta \underset{\sim}{u}'_{*m} = 0$	$\Delta \underset{\sim}{u}'_{*m} = \underset{\sim}{0}$	(3.2.61-62-63)
YIELD RULE	ASSOCIATION	FLOW RULE	

As an illustration of the modifications to introduce in the description of Statics, let us consider again a linearly-workhardening material and express the equilibrium condition (3.2.38a) in an incremental form:

$$\Phi+\Delta\Phi = (X_1+\Delta X_1)+X_{*1}\left[\left(\frac{X_2}{X_{*2}}-hu_{2p}\right)+\left(\frac{\Delta X_2}{X_{*2}}-h\,\Delta u_{2p}\right)\right]^2 -$$

$$X_{*1}\frac{dh}{3}\,\frac{\tan u_{1p}+\tan \Delta u_{1p}}{1-\tan u_{1p}\cdot\tan \Delta u_{1p}} - X_{*1}$$

or, after some simplifications

$$\Delta\Phi = \Phi,_1\,\Delta X_1+\Phi,_2\,\Delta X_2-X_{2*}hn_{21}\,\Delta u_{2p}+X_{*1}\left(\frac{X_{*2}}{2X_{*1}}\,\Delta\bar{x}\right)^2 -$$

$$\frac{4}{3}\left(\frac{X_{*1}}{X_{*2}}\right)^2\frac{h}{c^2}\tan \Delta u_{1p}\left[1+\frac{s}{c}\tan \Delta u_{1p}+\left(\frac{s}{c}\tan \Delta u_{1p}\right)^2+\cdots\right] \quad (3.2.64)$$

If assumption (3.2.52c) is valid with $R_x=0$, then from (3.2.52d) and (3.2.52e)

$$\Delta u_{2p} = \Phi,_2\,\tan \Delta u_{1p}$$

or, from (3.2.53b) $\quad \Delta u_* = \tan \Delta u_{1p}$

Hence, the incremental static condition (3.2.64) takes the form

$$-\mathrm{lh}\cdot\Delta u_*+\begin{bmatrix}\Phi,_1 & \Phi,_2\end{bmatrix}\begin{bmatrix}\Delta X_1\\ \Delta X_2\end{bmatrix} = \Delta\Phi + R_\varphi$$

where $\quad \mathrm{lh} = h\left\{X_{*2}\left(\frac{u_{2p}}{u_{*1}}\right)^2+\frac{4}{3}\,\frac{(X_{*1})^2}{X_{*2}}\left[1+(u_*)^2\right]\right\}$

and $\quad R_\varphi = \frac{4}{3}\,\frac{(X_{*1})^2}{X_{*2}}\,h\,u_*\left[1+(u_*)^2\right](\Delta u_*)^2\cdot(1+u_*\cdot\Delta u_*+\cdots) -$

$$X_{1*}\left(\frac{X_{*2}}{2X_{1*}}\cdot u_*\,\Delta u_{2p}\right)^2$$

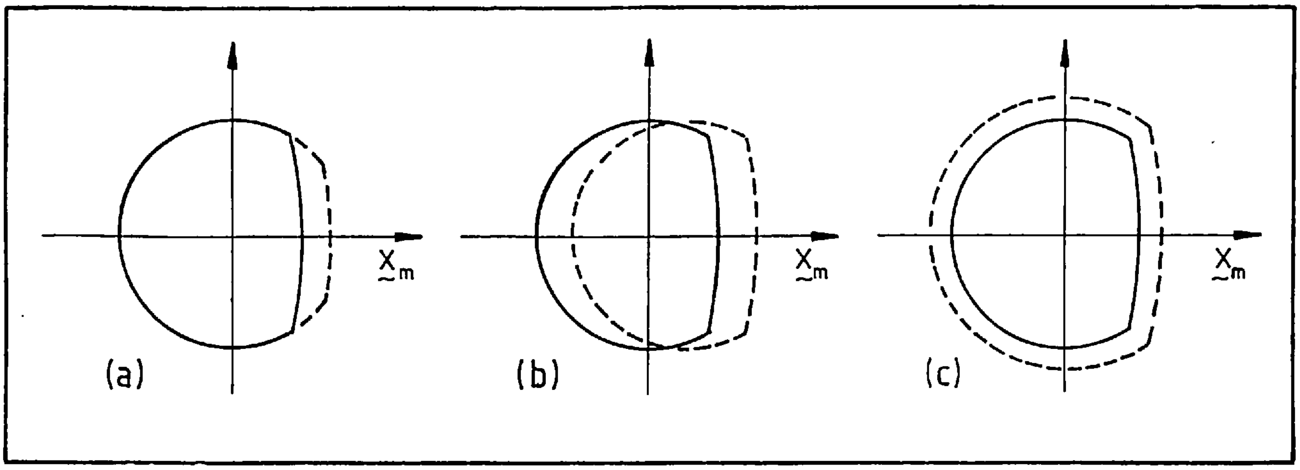


FIGURE 3.26

Illustrated in Fig.3.26 are the most commonly used hardening rules, namely:

(a) Non-interactive yield modes, Koiter (1960)
(b) Kinematic hardening, Prager (1955)
(c) Isotropic hardening, Hill (1950)

The hardening behaviour postulated in the theory of kinematic hardening assumes that during plastic deformation the yield surface translates as a rigid body in the stress-space. The primary aim of this theory is to provide a means of accounting for the ideal Bauschinger effect. Ziegler (1959) modified Prager's rule to overcome some inconsistencies arising when the kinematic hardening is applied to special stress fields, as, for instance, if some of the stress components are zero.

The theory of isotropic hardening assumes that during plastic straining the yield surface expands uniformely about the origin. Although in direct opposition to the concept of Bauschinger effect it has been frequently adopted and even found to be in general agreement with certain experiments.

Mróz (1967,1969) introduced the notion of a field of workhardening moduli and analysed the variation of this field during the course of plastic deformation.

The above mentioned hardening rules can be simulated through a convenient definition of the hardening matrix $\mathbb{H}_m$; we refer the works by G. Maier in this area, in particular Maier (1970).

3.2.3 Perturbation Analysis

Let us replace the incremental variables

$$\Delta \underset{\sim}{X}'_m = \sum_{i=1}^{\infty} \underset{\sim}{X}'_{i_m} \frac{\varepsilon^i}{i!}$$

$$\Delta \underset{\sim}{u}'_{p_m} = \sum_{i=1}^{\infty} \underset{\sim}{u}'_{pi_m} \frac{\varepsilon^i}{i!}$$

$$\Delta \underset{\sim}{\Phi}'_{*_m} = \sum_{i=1}^{\infty} \underset{\sim}{\Phi}'_{*i_m} \frac{\varepsilon^i}{i!}$$

$$\Delta \underset{\sim}{u}'_{*_m} = \sum_{i=1}^{\infty} \underset{\sim}{u}'_{*i_m} \frac{\varepsilon^i}{i!}$$

as well as the residuals

$$\underset{\sim}{R}_{\varphi_m} = \sum_{i=1}^{\infty} \underset{\sim}{R}_{\varphi i_m} \frac{\varepsilon^i}{i!}$$

$$\underset{\sim}{R}_{u_m} = \sum_{i=1}^{\infty} \underset{\sim}{R}_{ui_m} \frac{\varepsilon^i}{i!}$$

by their series expansion in the form (2.1.52); substituting into the incremental description of Statics and Kinematics and collecting the same order terms, equations (3.2.59) and (3.2.60) are replaced by the equivalent infinite system of linear equations

$$-\mathbb{H}'_m \, \underset{\sim}{u}'_{*i_m} + \underline{N}'^T_m \underset{\sim}{X}'_{i_m} = \underset{\sim}{\Phi}'_{*i_m} + \underset{\sim}{R}_{\varphi i_m} \qquad (3.2.65a)$$

$$\underline{N}'_m \, \underset{\sim}{u}'_{*i_m} = \underset{\sim}{u}'_{pi_m} + \underset{\sim}{R}_{ui_m} \qquad (3.2.65b)$$

Assuming that the perturbation parameter ε is positive, the yield and flow rules, (3.2.61) and (3.2.63) respectively, can be replaced by the sufficient conditions

$$\underset{\sim}{\Phi}'_{*m_i} \leq \underset{\sim}{0} \qquad \text{and} \qquad \underset{\sim}{u}'_{*m_i} \geq \underset{\sim}{0} \qquad (3.2.65c,d)$$

reducing the association condition (3.2.62) to

$$\underset{\sim}{\Phi}'^T_{*m_i} \, \underset{\sim}{u}'_{*m_i} = 0 \qquad (3.2.65e)$$

The perturbed plasticity relations (3.2.65), summarized below

RIGID-WORHARDENING MATERIALS

STATICS / KINEMATICS

$$\begin{bmatrix} -\mathbb{H}' & N'^{T} \\ N' & \cdot \end{bmatrix}_m \begin{bmatrix} u^{!}_{*_i} \\ X^{!}_{i} \end{bmatrix}_m = \begin{bmatrix} \Phi^{!}_{*_i} \\ u'_{p_i} \end{bmatrix}_m + \begin{bmatrix} R_{\varphi_i} \\ R_{u_i} \end{bmatrix}_m \qquad \begin{matrix} (3.2.66) \\ (3.2.67) \end{matrix}$$

$\Phi^{!}_{*_{i_m}} \leq 0$	$\Phi^{!\,T}_{*_{i_m}} \; u^{!}_{*_{i_m}} = 0$	$u^{!}_{*_{i_m}} \geq 0$	(3.2.68-69-70)
YIELD RULE	ASSOCIATION	FLOW RULE	

are sufficiently general to recover, by specialization, the particular cases of

a) perfect plasticity with interaction $\mathbb{H}'_m = 0$

b) perfect plasticity with no interaction $\mathbb{H}'_m = 0$

N_m defined either by (3.2.16a) or (3.2.16b)

c) workhardening with no interaction

$\mathbb{H}'_m$ defined accordingly to the actual or assumed hardening rule

N_m defined either by (3.2.16a) or (3.2.16b).

3.2.4 Asymptotic Analysis

We close this brief review of the plasticity relations be specializing the finite description (3.2.32) to (3.2.36) for the analysis of the particular class of structure whose equilibrium paths branch from the original kinematically trivial path.

Let us start with Statics and expand in a power series of the form (2.1.63) the variables involved in the definition (3.2.38a) of the plastic potential; hence

$$\Phi = \sum_{i=0}^{\infty} \Phi_i \frac{\varepsilon^i}{i!} \tag{3.2.71a}$$

$$X_j = \sum_{i=0}^{\infty} X_{j_i} \frac{\varepsilon^i}{i!} \qquad j = 1,2 \tag{3.2.71b,c}$$

$$u_{jp} = \sum_{i=1}^{\infty} u_{jp_i} \frac{\varepsilon^i}{i!} \qquad j = 1,2 \tag{3.2.71d,e}$$

Letting

$$\tan u_{1p} = u_{1p} + \frac{1}{3} u_{1p}^3 \tag{3.2.72}$$

in (3.2.38a), substituting the variables by their approximations (3.2.71) and solving, we find after equating the same order terms

$$\Phi_i = X_{1_i} + 2 \frac{X_{*1} X_{2_o}}{X_{*2}^2} X_{2_i} - h \frac{X_{*1}}{X_{*2}} \left(\frac{4}{3} X_{*1} u_{1p_i} + 2 X_{2_o} u_{2p_i}\right) - R'_{\varphi_i} \tag{3.2.73}$$

where $R'_{\varphi 0} = X_{*1} \left[1 + \left(\frac{X_{2_o}}{X_{*2}}\right)^2\right]$ (3.2.74a)

$$R'_{\varphi 1} = 0 \tag{3.2.74b}$$

$$R'_{\varphi 2} = -2X_{*1} \left(\frac{X_{2_1}}{X_{*2}}\right)^2 + 2hX_{*1}\left(2 \frac{X_{2_1}}{X_{*2}} - hu_{2p_1}\right) u_{2p_1} \tag{3.2.74c}$$

$$\vdots \qquad \qquad \vdots$$

Using (3.2.39) and (3.2.72), we may write for the kinematic compatibility condition (3.2.21)

$$u_{2p} = 2 \frac{X_{*1}}{X_{*2}} \left(\frac{X_2}{X_{*2}} - hu_{2p}\right)\left(u_{1p} + \frac{1}{3} u_{1p}^3 + \cdots\right)$$

generating, after substituting (3.2.71c) to (3.2.71e) above, solving and equating the same order terms, the infinite system

$$u_{2p_i} = 2 \frac{X_{*1}}{X_{*2}^2} X_{2_o} u_{1p_i} - R_{u_i} \tag{3.2.75}$$

where $R_{u_o} = 0$ (3.2.76a)

$$R_{u_1} = 0 \tag{3.2.76b}$$

$$R_{u_2} = -4 \cdot \frac{X_{*1}}{X_{*2}} \left(\frac{X_{2_1}}{X_{2*}} - hu_{2p_1}\right) u_{1p_1} \tag{3.2.76c}$$

$$\vdots \qquad \qquad \vdots$$

From (3.2.73) we find that

$$\frac{\partial \Phi_i}{\partial X_1^i} = 1 \tag{3.2.77a}$$

$$\frac{\partial \Phi_i}{\partial X_2^i} = 2 \frac{X_{*1} X_{2_o}}{X_{*2}^2} \tag{3.2.77b}$$

We may therefore define the plastic parameter

$$u_* = \sum_{i=1}^{\infty} u_{*_i} \frac{\varepsilon^i}{i!} \tag{3.2.71f}$$

such that $u_{1p_i} = \frac{\partial \Phi_i}{\partial X_{1_i}} u_{*_i}$

reducing (3.2.75) to $R_{u_i} + u_{2p_i} = \frac{\partial \Phi_i}{\partial X_{2_i}} u_{*_i}$

or, in matrix form

$$\begin{bmatrix} u_{1p_i} \\ u_{2p_i} + R_{u_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial \Phi_i}{\partial X_{1_i}} \\ \frac{\partial \Phi_i}{\partial X_{2_i}} \end{bmatrix} u_{*_i} \qquad \begin{matrix} (3.2.78a) \\ (3.2.78b) \end{matrix}$$

Substituting (3.2.77) and (3.2.78) in (3.2.73), the definition of the plastic potential becomes

$$\Phi_i = \frac{\partial \Phi_i}{\partial X_{1_i}} X_{1_i} + \frac{\partial \Phi_i}{\partial X_{2_i}} - \mathrm{lh}\, u_{*_i} - R_{\varphi_i} \tag{3.2.79a}$$

where $$\mathrm{lh} = 4\, h \frac{X_{*1}^2}{X_{*2}} \left[\frac{1}{3} + \left(\frac{X_{2_o}}{X_{*2}}\right)^2\right] \tag{3.2.79b}$$

and $$R_{\varphi_i} = R'_{\varphi_i} - 2h\, X_{*1} \frac{X_{2_o}}{X_{*2}} R_{u_i} \tag{3.2.79c}$$

In general, after considering for a typical member m of generic cross-section every yield surface enclosing the convex space of statically admissible combination of stress, together with the corresponding states of strain, we would find, for regularly progressive yielding the following description for the plasticity relations:

RIGID-WORKHARDENING MATERIALS			
STATICS / KINEMATICS	$$\begin{bmatrix} -\underline{H} & \underline{N}^T \\ \underline{N} & \cdot \end{bmatrix}_m \begin{bmatrix} \underline{u}_{*_i} \\ \underline{X}'_i \end{bmatrix}_m = \begin{bmatrix} \underline{\Phi}_{*_i} \\ \underline{u}_{p_i} \end{bmatrix}_m + \begin{bmatrix} \underline{R}_{\varphi_i} \\ \underline{R}_{u_i} \end{bmatrix}_m$$		(3.2.80) (3.2.81)
$\underline{\Phi}_{*_{i_m}} \leq \underline{0}$	$\underline{\Phi}^T_{*_{i_m}} \; \underline{u}_{*_{i_m}} = 0$	$\underline{u}_{*_{i_m}} \geq \underline{0}$	(3.2.82-83-84)
YIELD RULE	ASSOCIATION (R.P.Y.)	FLOW RULE	

where we assume that the yield and flow rules (3.2.82) and (3.2.84), as well as the association condition (3.2.83) were obtained through a process in every aspect similar to the one used in the previous sub-section when dealing with the corresponding relations in incremental form.

We note that in the above relations the infinite system (3.2.80) will only be recursive if the initial solution, obtained by setting i=0 in the above, is known; this is clearly shown by (3.2.79) together with (3.2.74a).

However, this fact will not in general represent a major problem since, in most cases, the very special structures for the analysis of which the above relations were designed, will be statically determinate for kinematically trivial configurations.

CHAPTER FOUR

STATICS, KINEMATICS AND CONSTITUTIVE RELATIONS OF THE STRUCTURE

The finite-element method by interpreting any structural system as an assembly of a finite number of building elements becomes a systematic procedure for formulating and solving problems in structural analysis. We distinguish three fundamental types of building elements; elastoplastic beams, rigid joints and elastoplastic mechanical releases. The beams or members interconnect at the joints or nodes and meet the medium supporting the structure at the FOUNDATION NODES. The continuity of the structure can be interrupted by any combination of the three fundamental types of internal releases shown in Fig.2.19 which may exist at either of the member ends. We simulate the deformability of the foundation by linking the foundation nodes to the ideally rigid foundation through elastoplastic EXTERNAL RELEASES. It is assumed herein that the external releases control displacements in directions parallel to those defined by the global system of axes $\underset{\sim}{x}^*$, as illustrated in Fig.4.1; the deformability of a foundation not complying directly with this requirement may always be simulated as a composition of up to three orthogonal releases with interdependent behaviour.

Chapter Two dealt with the conditions of equilibrium and compatibility of two different combinations of nodes and members to form two typical (fundamental) substructures tailored to fit

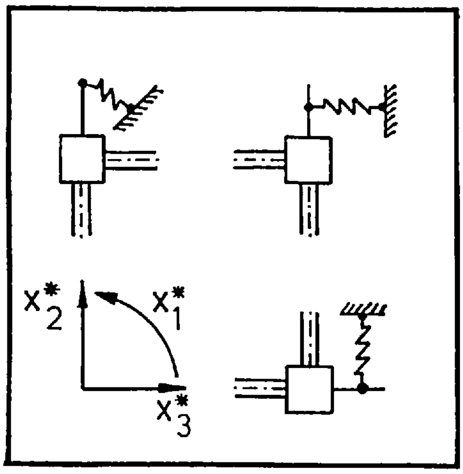


FIGURE 4.1

specific connectivity properties. In Chapter Three the beam constitutive relations were characterized by analyzing a three-degree of freedom elastoplastic finite-element, the idealization of a prismatic beam. The present Chapter is concerned with assembling that information in such a way that the resulting system of equations represents an appropriate mathematical model of the actual structure.

We start by defining the elastoplastic constitutive relations of the releases which are then grouped in accordance with the numbering given to the internal and external releases existing in the structure. The elastoplastic constitutive relations found in the previous chapter for the typical finite-element are also collected together following the labelling sequence of the structure members.

The process of assembling the substructures depends on the way in which the given structure has been substituted by the model and on the way in which their analysis has been performed.

Gallagher (1975) uses a congruent transformation method, based on the Principle of Virtual Work. Desai and Abel (1972) also use the Principle of Virtual Displacements applied to the entire structure. Both methods, requiring the use of the element (elastic) constitutive relations, are designed to form, in the end, the global stiffness matrix of the structure.

Maier et al (1972) assemble Kinematics by enforcing continuity of the displacements at each node and derive the description of Statics through the application of the Principle of Virtual Work.

Alexa (1976) deals with Kinematics and Statics independently by considering separately continuity and equilibrium at the nodes, thus recovering, through Static-Kinematic Duality, the Principle of Virtual Work, instead of using it.

The above mentioned methods, as well as the great majority of the finite-element formulations in kinematically non-linear structural analysis, were designed to perform the assemblage of nodal substructures. In the present work, wherein Statics and Kinematics are assembled independently, the process of assemblage is designed to suit the type of substructure one considers the structure to be formed of.

If the structure is interpreted as an assemblage of nodal substructures, we start by securing continuity of displacements at the nodes and, by resorting to the Principle of Work Invariance and thus automatically satisfying nodal equilibrium, we assemble next the nodal description of Statics.

The results to be presented in section 4.2 will show once again that, in general, it is simpler to assemble Statics and Kinematics, and therefore to FORMULATE the problem, using the nodal rather than the mesh description.

The mesh formulation may,however, be advantageous in what concerns the SOLUTION of the problem since, in general, it results in a better conditioned system with a smaller number of equations and unknowns, since the static indeterminancy of most engineering structures is significantly lower than their kinematic indeterminancy.

This is particularly relevant in non-linear analysis of framed structures wherein the finite-element mesh has, in many instances, to be refined in order to diminish the effects of the approximations built into the quantification of the finite-element behaviour; the finite-element mesh is refined by increasing the number of nodes, each of which adds three degrees to the kinematic indeterminancy of the planar frame, while its static indeterminancy remains unaltered.

The nodal formulation has received the unquestionable favour from most analysts, mainly because of the apparent incapability of the mesh description to be encoded in a format suitable for fully automated computer processing. The close relationship of the mesh description of Statics and Kinematics with the physical interpretation of the problem, so dear to the engineer's mind, made it pedagogically so attractive that its real potentialities remained concealed for a long period and were only perceived when the method was reinterpreted through graph theory; Henderson and Bickley (1955) pioneered the rehabilitation of the method which is still to gain the acceptance it deserves.

Let us refer to the structure represented in Fig.4.2(a) to illustrate briefly the two basic concepts involved.

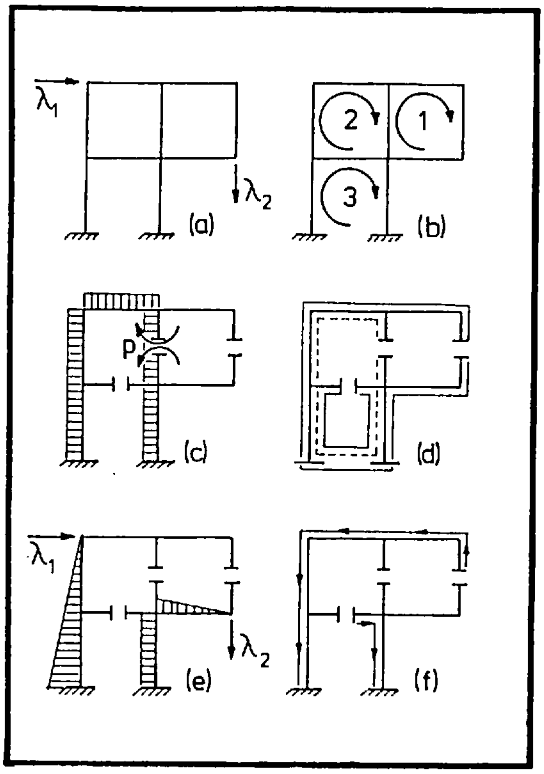


FIGURE 4.2

Introducing 9 releases concentrated in 3 sections, the frame, rendered statically determinate, is transformed into a tree-structure; from one side of the cut there is a unique circuit or cycle connecting it with the other side of the cut, as shown in Fig. 4.2(d), and from any point of the tree there is a unique continuous path to the foundation, Fig.4.2(f). From the structural mechanics point of view this means that an (indeterminate) stress-resultant produces a stress field flowing through the associated cut cycle, as illustrated in Fig.4.2(c), and that a point load applied at any section produces a stress field flowing through the only circuit connecting that point to the foundation, Fig.4.2(e).

Henderson (1960), Henderson and Maunder (1969) and Maunder (1971), concentrated their attention on the problem of automatic selection of cut-cycle bases, the change of bases and its influence in the conditioning of the structure flexibility matrix. The utility of regional cycle bases, Fig.4.2(b), was demonstrated by Edwards (1963,1964) and Munro (1963,1965a).

The above mentioned works relied on and fully explored the particularities existing in linear structural analysis. In order to extend the method to kinematically non-linear problems, instead of depending on algorithms to select an independent cycle basis to form the complementary solution of Statics (the self-equilibrating stress-field) and on shortest route algorithms to define the particular solution (load-equilibrating stress-field), or on the regional cycles and simple incidence to form the complementary solution, assuming that the particular solution is readily available, in order to derive a general formulation we opted to characterize simultaneously on the mesh substructure both particular and complementary solutions. The method uses a regional cycle basis and the complementary solution of Statics is easily assembled by superimposing the stresses flowing along branches common to incident meshes. To assemble the particular solution we start by assigning the applied loads (in which we include the forces developing at the supports and at the external releases) to the constituent mesh substructures and by transmiting the flow of stress they generate along a selected path of incident meshes. Using the Principle of Work Invariance we assemble Kinematics by satisfying continuity of the flow of strains.

The method is capable of assembling automatically the information on Statics and Kinematics of the mesh substructures to characterize the conditions of equilibrium and compatibility of the structure without resorting, in its basic format, as presented in subsection 4.2.2, to sophisticated algorithms; it is however our belief that its efficiency would be greatly improved by including the algorithms developed by the above mentioned authors.

In either of the formulations, nodal and mesh, Static-Kinematic Duality, at structural level, emerges as a direct consequence of the duality forced into the substructure relations through the use of the additional forces and deformations.

The Principle of Virtual Work is again interpreted as the variational representation of the relation of duality existing between the descriptions of Statics and Kinematics of the structure.

4.1 ELASTOPLASTIC CONSTITUTIVE RELATIONS

Let the structure members and the existing internal and external releases be numbered respectively from 1 to M, 1 to R and 1 to r. The numbering sequence, theoretically arbitrary, should follow a pattern designed to improve the efficiency of the numerical solution procedure; we refer to the works of Sabir (1976) on this subject.

In accordance with the chosen numbering sequence, it is convenient to define the STRUCTURE STRESS- and STRAIN-RESULTANT vectors $\underset{\sim}{X}'$ and $\underset{\sim}{u}'$

$$\underset{\sim}{X}'^T = \left[\underset{\sim}{X}_1'^T \; \underset{\sim}{X}_2'^T \; \ldots \; \underset{\sim}{X}_M'^T\right] \quad \text{and} \quad \underset{\sim}{u}'^T = \left[\underset{\sim}{u}_1'^T \; \underset{\sim}{u}_2'^T \; \ldots \; \underset{\sim}{u}_M'^T\right]$$

Similarly to (3.0.1), we dissociate the latter in its elastic and plastic components

$$\underset{\sim}{u}' = \underset{\sim}{u}_E' + \underset{\sim}{u}_p' \tag{4.1.1}$$

and define the flexibility and stiffness descriptions of the structure elastic constitutive relations as

$$\underset{\sim}{u}_E' = \underline{F}_u \underset{\sim}{X}' + \underset{\sim}{u}_{E\pi}' \quad \text{and} \quad \underset{\sim}{X}' = \underline{K}_u \underset{\sim}{u}_E' + \underset{\sim}{X}_{E\pi}' \tag{4.1.2-3}$$

where the block-diagonal matrices $\underline{F}_u$ and $\underline{K}_u$

$$\underline{F}_u = \begin{bmatrix} \underline{F}_1 & & & \\ & \underline{F}_2 & & \\ & & \ddots & \\ & & & \underline{F}_M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \underline{K}_1 & & & \\ & \underline{K}_2 & & \\ & & \ddots & \\ & & & \underline{K}_M \end{bmatrix} = \underline{K}_u$$

are the flexibility and stiffness matrices for the UNASSEMBLED members of the structure. Each of the (symmetric) sub-matrices are defined in subsection 3.1.4 as well as the corrective terms $\underset{\sim}{u}_{E\pi}'$ and $\underset{\sim}{X}_{E\pi}'$

$$\underset{\sim}{u}_{E\pi}'^T = \left[\underset{\sim}{u}_{E\pi_1}'^T \; \underset{\sim}{u}_{E\pi_2}'^T \; \ldots \; \underset{\sim}{u}_{E\pi_M}'^T\right], \quad \underset{\sim}{X}_{E\pi}'^T = \left[\underset{\sim}{X}_{E\pi_1}'^T \; \underset{\sim}{X}_{E\pi_2}'^T \; \ldots \; \underset{\sim}{X}_{E\pi_M}'^T\right]$$

which, for consistency of the formulation, have to satisfy the following relationships

$$\underset{\sim}{X}_{E\pi}' = -\underline{K}_u \underset{\sim}{u}_{E\pi}' \quad , \quad \underset{\sim}{u}_{E\pi}' = -\underline{F}_u \underset{\sim}{X}_{E\pi}' \tag{4.1.4a,b}$$

The results presented in section 3.2 can be grouped in a similar manner to represent the plastic behaviour of the unassembled members of the structure. For instance, we could adopt the following description to perform a deformation analysis of the structure:

$$\left[\begin{array}{c|c} -\underline{H}' & \underline{N}'^T \\ \hline \underline{N}' & \cdot \end{array}\right] \left[\begin{array}{c} \underset{\sim}{u}_*' \\ \hline \underset{\sim}{X}' \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{\Phi}_*' \\ \hline \underset{\sim}{u}' \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{X}_*' \\ \hline \cdot \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{\pi}_\varphi' \\ \hline \bar{\underset{\sim}{u}}_\varphi \end{array}\right] \qquad \begin{array}{r} (4.1.5a) \\ (4.1.5b) \end{array}$$

$$\underset{\sim}{\Phi}_*' \leq \underset{\sim}{0} \qquad \underset{\sim}{u}_*' \geq \underset{\sim}{0} \qquad \underset{\sim}{\Phi}_*'^T \underset{\sim}{u}_*' = 0 \tag{4.1.5c-e}$$

Conditions (4.1.5a-e) represent, respectively, the static and kinematic phases of plasticity, the yield and flow rules and the association condition for regularly progressive yielding. In the above relations the hardening and normality matrices are block diagonal matrices

$$\underline{Y} = \begin{bmatrix} \underline{Y}_1 & \underline{Y}_2 & \cdots & \underline{Y}_j \end{bmatrix} \qquad (4.1.6a)$$

and the plastic multipliers $\underset{\sim}{u}^{!}_{*}$, the plastic potential $\underset{\sim}{\Phi}^{!}_{*}$ and the plastic capacities $X^{!}_{*}$, as well as the corrective static and kinematic vectors $\underset{\sim}{\pi}'_{\varphi}$ and $\underset{\sim}{\bar{u}}_{\varphi}$ are defined as the generic super-vectors

$$\underset{\sim}{y}^T = \begin{bmatrix} \underset{\sim}{y}_1^T & \underset{\sim}{y}_2^T & \cdots & \underset{\sim}{y}_j^T \end{bmatrix} \qquad (4.1.6b)$$

where j is the number of structural members, M.

Implied in the plasticity relations (4.1.5) is that extensible plastic hinges may form at any of the 2M member critical sections.

However, by examining the forces acting on each node of the structure, the number and the relative plastic capacities of the members connecting on each node, the distribution and nature of the structural releases, both internal and external, the analyst can a priori select the STRUCTURE CRITICAL SECTIONS, that is the subset $c \leq 2M$ of sections where yielding may in fact occur. For instance, if all members of the structure represented in Fig.4.4 had the same plastic capacities, among the eight member critical sections we could select the following four as the structure critical sections; section 2 of member 1 (or section 1 of member 2) sections 2 of members 2 and 3 and section 2 of member 4 (or section 1 of member 3).

After numbering the structure critical sections and collecting in $\underset{\sim}{\bar{u}}_{*}$ the corresponding plastic multipliers, we define the incidence

$$\underset{\sim}{u}^{!}_{*} = \underline{J}_{*} \, \underset{\sim}{\bar{u}}_{*}$$

which when included in (4.1.5) gives:

$$\begin{bmatrix} -\underline{H}'\underline{J}_{*} & \underline{N}'^T \\ \underline{N}_u & \cdot \end{bmatrix} \begin{bmatrix} \underset{\sim}{\bar{u}}_{*} \\ \underset{\sim}{X}' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\Phi}^{!}_{*} \\ \underset{\sim}{u}'_p \end{bmatrix} + \begin{bmatrix} \underset{\sim}{X}^{!}_{*} \\ \cdot \end{bmatrix} + \begin{bmatrix} \underset{\sim}{\pi}'_{\varphi} \\ \underset{\sim}{u}'_{\varphi} \end{bmatrix} \qquad \begin{matrix} (4.1.7a) \\ (4.1.7b) \end{matrix}$$

$$\underset{\sim}{\Phi}^{!}_{*} \leq \underset{\sim}{0} \quad \underset{\sim}{\Phi}^{!T}_{*} \underline{J}_{*} \underset{\sim}{\bar{u}}_{*} = 0 \quad \underset{\sim}{\bar{u}}_{*} \geq \underset{\sim}{0} \qquad (4.1.7c\text{-}e)$$

in which $$\underset{\sim}{N}_u = \underset{\sim}{N}' \underset{\sim}{J}_* \tag{4.1.8a}$$

The relations in the system (4.1.7) corresponding to the trivial elements of $\underset{\sim}{u}'_*$ are irrelevant and they can be discarded of by pre-multiplying that system by $\underset{\sim}{J}_*^T$, implicitly satisfying the invariance in the descriptions of the plastic work:

$$\begin{bmatrix} -\underset{\sim}{H}_u & \underset{\sim}{N}_u^T \\ \underset{\sim}{N}_u & \cdot \end{bmatrix} \begin{bmatrix} \bar{\underset{\sim}{u}}_* \\ \underset{\sim}{X}' \end{bmatrix} = \begin{bmatrix} \bar{\underset{\sim}{\Phi}}_* \\ \underset{\sim}{u}'_p \end{bmatrix} + \begin{bmatrix} \bar{\underset{\sim}{X}}_* \\ \cdot \end{bmatrix} + \begin{bmatrix} \bar{\underset{\sim}{\pi}}_\varphi \\ \bar{\underset{\sim}{u}}_\varphi \end{bmatrix} \qquad \begin{matrix}(4.1.9a)\\(4.1.9b)\end{matrix}$$

$$\bar{\underset{\sim}{\Phi}}_* \leq \underset{\sim}{0} \qquad \bar{\underset{\sim}{\Phi}}_*^T \, \bar{\underset{\sim}{u}}_* = 0 \qquad \bar{\underset{\sim}{u}}_* \geq \underset{\sim}{0} \tag{4.1.9c-e}$$

in which

$$\underset{\sim}{H}_u = \underset{\sim}{J}_*^T \, \underset{\sim}{H}' \underset{\sim}{J}_* \quad , \quad \bar{\underset{\sim}{\Phi}}_* = \underset{\sim}{J}_*^T \, \underset{\sim}{\Phi}'_* \quad , \quad \bar{\underset{\sim}{X}}_* = \underset{\sim}{J}_*^T \, \underset{\sim}{X}'_* \quad , \quad \bar{\underset{\sim}{\pi}}_\varphi = \underset{\sim}{J}_*^T \, \underset{\sim}{\pi}'_\varphi \tag{4.1.8b-e}$$

The elastoplastic causality relations associating the force $\underset{\sim}{X}'^{\,i}_R$ ($\underset{\sim}{\lambda}^i_r$) developing at the i-th internal (external) release with the corresponding dislocation $\underset{\sim}{v}'^{\,i}_R$ ($\underset{\sim}{\delta}^i_r$) are presented next in a qualitative form which, we hope, is sufficiently general for many of the situations the analyst may wish to simulate. Although not considered, locking effects could also be included, for instance in the manner of Corradi and Maier (1969).

Let us then dissociate the dislocation vectors in their elastic and plastic components:

$$\underset{\sim}{v}'^{\,i}_R = \underset{\sim}{v}'^{\,i}_{RE} + \underset{\sim}{v}'^{\,i}_{Rp} \quad , \quad \underset{\sim}{\delta}^i_r = \underset{\sim}{\delta}^i_{rE} + \underset{\sim}{\delta}^i_{rp} \tag{4.1.10-11}$$

The elastic constitutive relations may now be expressed in a flexibility format as:

$$\underset{\sim}{v}'^{\,i}_{RE} = \underset{\sim}{F}^i_R \, \underset{\sim}{X}'^{\,i}_R + \underset{\sim}{v}'^{\,i}_{R\pi} \quad , \quad \underset{\sim}{\delta}^i_{rE} = \underset{\sim}{F}^i_r \, \underset{\sim}{\lambda}^i_r + \underset{\sim}{\delta}^i_{r\pi} \tag{4.1.12a-13a}$$

and in a stiffness format as

$$\underset{\sim}{X}'^{\,i}_R = \underset{\sim}{K}^i_R \, \underset{\sim}{v}'^{\,i}_{RE} + \underset{\sim}{X}'^{\,i}_{R\pi} \quad , \quad \underset{\sim}{\lambda}^i_r = \underset{\sim}{K}^i_r \, \underset{\sim}{\delta}^i_{rE} + \underset{\sim}{\lambda}^i_{r\pi} \tag{4.1.12b-13b}$$

where, for consistency

$$\underset{\sim}{X}'^{\,i}_{R\pi} = -\underset{\sim}{K}^i_R \, \underset{\sim}{v}'^{\,i}_{R\pi} \quad , \quad \underset{\sim}{v}'^{\,i}_{R\pi} = -\underset{\sim}{F}^i_R \, \underset{\sim}{X}'^{\,i}_{R\pi} \tag{4.1.14a,b}$$

and $$\underset{\sim}{\lambda}'^{\,i}_{r\pi} = -\underset{\sim}{K}^i_r \, \underset{\sim}{\delta}^i_{r\pi} \quad , \quad \underset{\sim}{\delta}^i_{r\pi} = -\underset{\sim}{F}^i_r \, \underset{\sim}{\lambda}^i_{r\pi} \tag{4.1.15a,b}$$

We assume that the flexibility and stiffness matrices, in general with functional coefficients, are symmetric but not necessarily diagonal, in order to simulate, if needed, the interaction between the release constituents. If the actual causality relations are non-symmetric, they may be forced to become symmetric by including the disturbing terms in the residuals $\underset{\sim}{v}'^{i}_{R\pi}$, $\underset{\sim}{\delta}^{i}_{r\pi}$ and $\underset{\sim}{X}'^{i}_{R\pi}$, $\underset{\sim}{\lambda}^{i}_{r\pi}$ the elements of which we assume, nevertheless, to be either zero or non-linear functions of the release forces and dislocations.

Setting i = 1,2, ... R (r) in (4.1.12(13)) and collecting in the manner of (4.1.6), the following flexibility and stiffness descriptions for the elastic constitutive relations are found

$$\underset{\sim}{v}'_{RE} = \underline{F}_R \underset{\sim}{X}'_R + \underset{\sim}{v}'_{R\pi} \quad , \quad \underset{\sim}{X}'_R = \underline{K}_R \underset{\sim}{v}'_{RE} + \underset{\sim}{X}'_{R\pi} \qquad (4.1.16\text{-}17)$$

for the internal

$$\underset{\sim}{\delta}_{rE} = \underline{F}_r \underset{\sim}{\lambda}_r + \underset{\sim}{\lambda}_{r\pi} \quad , \quad \underset{\sim}{\lambda}_r = \underline{K}_r \underset{\sim}{\delta}_{rE} + \underset{\sim}{\delta}_{r\pi} \qquad (4.1.18\text{-}19)$$

and external release systems.

We assume that the plastic behaviour of each of the R(r) internal (external) releases can be described by a set of relations qualitatively similar to (3.2.32-36); after collecting, we would find the following descriptions for the plastic constitutive relations

$$\left[\begin{array}{c|c} -\underline{H}_R & \underline{N}_R^T \\ \hline \underline{N}_R & \cdot \end{array}\right]\left[\begin{array}{c} \underset{\sim}{v}_{R*} \\ \hline \underset{\sim}{X}'_R \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{\Phi}_{R*} \\ \hline \underset{\sim}{v}'_{Rp} \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{X}_{R*} \\ \hline \cdot \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{\pi}_{R\varphi} \\ \hline \underset{\sim}{v}_{R\varphi} \end{array}\right] \qquad \begin{array}{c}(4.1.20a)\\(4.1.20b)\end{array}$$

$$\underset{\sim}{\Phi}_{R*} \leq \underset{\sim}{0} \qquad \underset{\sim}{\Phi}^T_{R*} \underset{\sim}{v}_{R*} = 0 \qquad \underset{\sim}{v}_{R*} \geq \underset{\sim}{0} \qquad (4.1.20c\text{-}e)$$

for the internal releases, and

$$\left[\begin{array}{c|c} -\underline{H}_r & \underline{N}_r^T \\ \hline \underline{N}_r & \cdot \end{array}\right]\left[\begin{array}{c} \underset{\sim}{\delta}_{r*} \\ \hline \underset{\sim}{\lambda}_r \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{\Phi}_{r*} \\ \hline \underset{\sim}{\delta}_{rp} \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{\lambda}_{r*} \\ \hline \cdot \end{array}\right] + \left[\begin{array}{c} \underset{\sim}{\pi}_{r\varphi} \\ \hline \underset{\sim}{\delta}_{r\varphi} \end{array}\right] \qquad \begin{array}{c}(4.1.21a)\\(4.1.21b)\end{array}$$

$$\underset{\sim}{\Phi}_{r*} \leq \underset{\sim}{0} \qquad \underset{\sim}{\Phi}^T_{r*} \underset{\sim}{\delta}_{r*} = 0 \qquad \underset{\sim}{\delta}_{r*} \geq \underset{\sim}{0} \qquad (4.1.21c\text{-}e)$$

for the external releases. We again require the symmetry of the hardening matrices $\underline{H}_R$ and $\underline{H}_r$ and the non-linearity of the residual terms and remind that the association conditions (4.1.20d-21d) will only hold in the absence of plastic unstressing.

For simplicity of the presentation we introduce the GENERALIZED VARIABLES summarized and defined in Table 4.1; regrouping the relations (4.1.2,3,16-19) and (4.1.9,20,21) according to the relevant generalized variables, the structure elastoplastic constitutive relations reduce to the synthetic forms (4.1.22) and (4.1.23) presented in Table 4.7; in Table 4.8 we collected the definitions for the corresponding generalized structural matrices.

In Tables 4.9 and 4.11 we summarize the incremental and perturbed descriptions of the generalized elastoplastic constitutive relations which were obtained by collecting, through a process in every aspect similar to the one just described, the member elastic and plastic constitutive relations, defined by (3.1.69,74), (3.1.80,81) and (3.2.59-63), (3.2.66-70), respectively, the formats of which we used to represent, qualitatively, the incremental elastoplastic constitutive relations of the structure release system.

The description of the structure constitutive relations in a format suitable to perform an asymptotic analysis would be found to be formally identical to (4.1.36) and (4.1.37). The asymptotic matrices $\underline{F}_i$, $\underline{K}_i$, $\underline{H}_i$ and $\underline{N}_i$, as well as the corrective terms $\underset{\sim}{R}_{uE_i}$, $\underset{\sim}{R}_{XE_i}$, $\underset{\sim}{R}_{\varphi_i}$ and $\underset{\sim}{R}_{p_i}$, will be quantitatively different from the corresponding ones in a perturbation analysis formulation, as shown in subsections 3.1.6 and 3.1.13 for the member elastic constitutive relations and illustrated for the rectangular cross-section in 3.2.3 and 3.2.4.

GENERALIZED VARIABLES

$$\underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u}' \\ \underset{\sim}{v}'_R \\ \underset{\sim}{\delta}_r \end{bmatrix} \quad \underset{\sim}{u}_E = \begin{bmatrix} \underset{\sim}{u}'_E \\ \underset{\sim}{v}'_{RE} \\ \underset{\sim}{\delta}_{ve} \end{bmatrix} \quad \underset{\sim}{u}_p = \begin{bmatrix} \underset{\sim}{u}'_p \\ \underset{\sim}{v}'_{Rp} \\ \underset{\sim}{\delta}_{rp} \end{bmatrix} \quad \underset{\sim}{u}_* = \begin{bmatrix} \bar{\underset{\sim}{u}}_* \\ \underset{\sim}{v}_{R*} \\ \underset{\sim}{\delta}_{r*} \end{bmatrix} \quad \underset{\sim}{q} = \begin{bmatrix} \bar{\underset{\sim}{q}} \\ \underset{\sim}{v}'_R \end{bmatrix}$$

TABLE 4.1a

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}' \\ \underset{\sim}{X}'_R \\ \underset{\sim}{\lambda}_r \end{bmatrix} \quad \underset{\sim}{X}_* = \begin{bmatrix} \bar{\underset{\sim}{X}}_* \\ \underset{\sim}{X}_{R*} \\ \underset{\sim}{\lambda}_{r*} \end{bmatrix} \quad \underset{\sim}{\Phi}_* = \begin{bmatrix} \bar{\underset{\sim}{\Phi}}_* \\ \underset{\sim}{\Phi}_{R*} \\ \underset{\sim}{\Phi}_{r*} \end{bmatrix} \quad \underset{\sim}{p} = \begin{bmatrix} \bar{\underset{\sim}{p}} \\ -\underset{\sim}{\lambda}_r \end{bmatrix}$$

GENERALIZED AUXILIARY VARIABLES

$$\underset{\sim}{u}_{\pi} = \begin{bmatrix} \underset{\sim}{u}'_{\pi} + \underset{\sim}{u}'_{R} \\ \cdot \\ \cdot \end{bmatrix} \quad \underset{\sim}{u}_{E\pi} = \begin{bmatrix} \underset{\sim}{u}'_{E\pi} \\ \underset{\sim}{u}'_{R\pi} \\ \underset{\sim}{\delta}_{r\pi} \end{bmatrix} \quad \underset{\sim}{u}_{\varphi} = \begin{bmatrix} \bar{\underset{\sim}{u}}'_{\varphi} \\ \underset{\sim}{v}'_{R\varphi} \\ \underset{\sim}{\delta}_{r\varphi} \end{bmatrix} \quad \underset{\sim}{\delta}_{\pi} = \begin{bmatrix} \bar{\underset{\sim}{\delta}}_{\pi} \\ \underset{\sim}{\delta}'_{\pi} \end{bmatrix}$$

$$\underset{\sim}{X}_{E\pi} = \begin{bmatrix} \underset{\sim}{X}'_{E\pi} \\ \underset{\sim}{X}'_{R\pi} \\ \underset{\sim}{\lambda}_{r\pi} \end{bmatrix} \quad \underset{\sim}{\pi}_{\varphi} = \begin{bmatrix} \bar{\underset{\sim}{\pi}}_{\varphi} \\ \underset{\sim}{\pi}_{R\varphi} \\ \underset{\sim}{\pi}_{r\varphi} \end{bmatrix} \quad \underset{\sim}{\pi} = \begin{bmatrix} \bar{\underset{\sim}{\pi}} \\ \underset{\sim}{\pi}' \end{bmatrix}$$

TABLE 4.1b

4.2 STATICS AND KINEMATICS

Before presenting the nodal and mesh assemblage procedures, let us define the degree of indeterminacy in the static and kinematic descriptions of a planar frame with M members, r external releases and R internal releases; by FUNDAMENTAL STRUCTURE we understand the same planar frame with neither constraints nor releases.

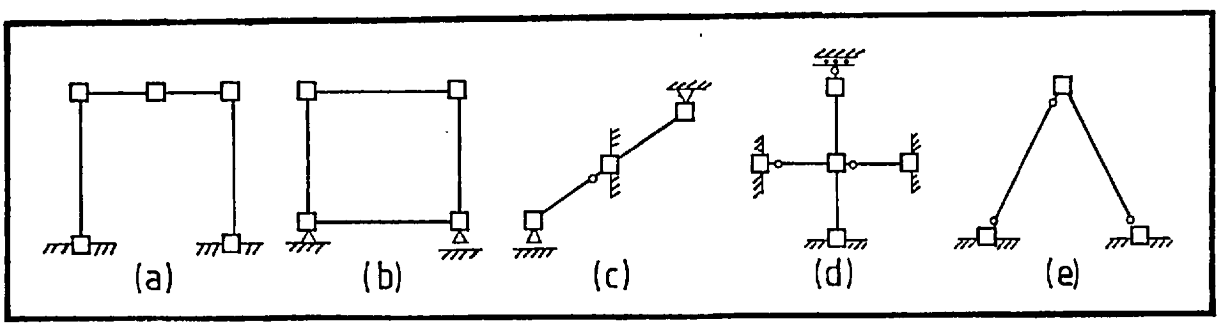


FIGURE 4.3

Let N represent the number of nodes of the structure when all foundation nodes are counted as but a single node; the structure KINEMATIC INDETERMINACY NUMBER β is given by

$$\beta = \beta_{*} + R + r \qquad (4.2.1a)$$

where

$$\beta_{*} = 3(N-1) \qquad (4.2.1b)$$

is the kinematic indeterminacy number of the corresponding fundamental structure. The STATIC INDETERMINACY NUMBER α is defined as $\alpha = \alpha_* - R - r$ (4.2.2a)

Henderson and Bickley (1955) define the static indeterminacy of the fundamental structure, α_*, as

$$\alpha_* = 3\mu = 3(M-N+1) \qquad (4.2.2b)$$

where μ represents the number of independent meshes or cycles in the graph model.

Table 4.2 summarizes the indeterminacy numbers found for the simple structures illustrated in Fig.4.3:

Structure	M	N	r	R	α_*	β_*	α	β
(a)	4	4	0	0	3	9	3	9
(b)	4	3	3	0	6	6	3	9
(c)	2	1	3	1	6	0	2	4
(d)	4	2	2	2	9	3	5	7
(e)	2	2	0	3	3	3	0	6

TABLE 4.2

4.2.1 Nodal Description

For each of the M nodal substructures we may write the following explicitly linear and dual relations

$$\begin{bmatrix} \underset{\sim}{R}^* \\ \hline -\underset{\sim}{X}_R \end{bmatrix}_m = \left[\begin{array}{c|c|c} \underline{A}^T & \underline{A}_\pi^T & \underline{A}_R^T \\ \hline \underline{A}'^T & \cdot & \underline{A}_R'^T \end{array}\right]_m \begin{bmatrix} \underset{\sim}{X}' \\ \hline -\underset{\sim}{\pi} \\ \hline -\underset{\sim}{\pi}' \end{bmatrix}_m \qquad \begin{bmatrix} \underset{\sim}{u}'' \\ \hline \underset{\sim}{\delta}_\pi \\ \hline \underset{\sim}{\delta}'_\pi \end{bmatrix}_m = \left[\begin{array}{c|c} \underline{A} & \underline{A}' \\ \hline \underline{A}_\pi & \cdot \\ \hline \underline{A}_R & \underline{A}'_R \end{array}\right]_m \begin{bmatrix} \underset{\sim}{r}^* \\ \hline \underset{\sim}{v}_R \end{bmatrix}_m$$

defining, respectively, the equilibrium and compatibility conditions, as derived in subsection 2.4.1; we note

$$\underset{\sim}{u}''_m = \underset{\sim}{u}'_m + \underset{\sim}{u}'_{\pi_m} + \underset{\sim}{u}'_{R_m}$$

Setting, in the above relations, m to 1,2, ..., M and collecting in the manner of (4.1.5) according to the sequence adopted for the member labeling, the following static and kinematic descriptions for the UNASSEMBLED M nodal substructures are found

$$
\begin{array}{l}
(4.2.3a) \\ (4.2.3b)
\end{array}
\quad
\left[\begin{array}{c} R^* \\ \hline -\tilde{X}_R \end{array}\right] = \left[\begin{array}{c:c:c} a^T & a_\pi^T & a_R^T \\ \hdashline a'^T & \cdot & a_R'^T \end{array}\right] \left[\begin{array}{c} X' \\ \hdashline -\bar{\pi} \\ \hdashline -\tilde{\pi}' \end{array}\right]
$$

$$
\left[\begin{array}{c} u'' \\ \hdashline \bar{\delta}_\pi \\ \hdashline \tilde{\delta}'_\pi \end{array}\right] = \left[\begin{array}{c:c} a & a' \\ \hdashline a_\pi & \cdot \\ \hdashline a_R & a'_R \end{array}\right] \left[\begin{array}{c} r^* \\ \hdashline \tilde{v}_R \end{array}\right]
\quad
\begin{array}{l}
(4.2.4a) \\ (4.2.4b) \\ (4.2.4c)
\end{array}
$$

where marked with a tilde are those variables which, at this stage, may contain superfluous information.

Assume that the structure is subject to n POINT LOADS λ_i which we group in vector λ ; let δ contain the corresponding POINT LOAD DISPLACEMENTS. The EXTERNAL RELEASE DISLOCATIONS δ_r and the point load displacements are subsets

$$
\begin{bmatrix} \delta \\ \delta_r \end{bmatrix} = \begin{bmatrix} J_o \\ J_r \end{bmatrix} r^* \qquad \begin{array}{l} (4.2.5a) \\ (4.2.5b) \end{array}
$$

of the supervector r^* gathering all possible member nodal displacements. The supervector $\tilde{\delta}'_\pi$ contains all possible additional release force dislocations and we define the array δ'_π collecting the structure ADDITIONAL INTERNAL RELEASE FORCE DISPLACEMENTS through

$$
\delta'_\pi = J_\pi \, \tilde{\delta}'_\pi \tag{4.2.6}
$$

Assume that the frame has a kinematic indeterminacy β' when all its R internal releases are blocked and let us arrange in $\bar{q}$ and v'_R the STRUCTURE NODAL DISPLACEMENTS and INTERNAL RELEASE DISLOCATIONS, respectively. The assemblage of Kinematics can now be performed by defining the incidence matrices J and J_R selecting, respectively, the non-zero nodal displacements and release dislocations:

$$
r_* = J \, \bar{q} \tag{4.2.7a}
$$

$$
\tilde{v}_R = J_R \, v'_R \tag{4.2.7b}
$$

Pre-multiplying equations (4.2.4c) by J_π , including (4.2.5) in the system, and making use of (4.1.7), the ASSEMBLED description of Kinematics emerges as

$$\left[\begin{array}{c} \underset{\sim}{u}'' \\ \hline \underset{\sim}{\delta} \\ \hline \underset{\sim}{\delta}_{r} \\ \hline \underset{\sim}{\bar{\delta}}_{\pi} \\ \hline \delta'_{\pi} \end{array}\right] = \left[\begin{array}{c|c} \underline{\bar{A}} & \underline{\bar{A}}' \\ \hline \underline{\bar{A}}_{o} & \cdot \\ \hline \underline{\bar{A}}_{r} & \cdot \\ \hline \underline{\bar{A}}_{\pi} & \cdot \\ \hline \underline{\bar{A}}_{R} & \underline{\bar{A}}'_{R} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{\bar{q}} \\ \hline \underset{\sim}{v}'_{R} \end{array}\right] \quad \begin{array}{r} (4.2.8a) \\ (4.2.8b) \\ (4.2.8c) \\ (4.2.8d) \\ (4.2.8e) \end{array}$$

where we note

$$\underline{\bar{A}} = \underline{a}\,\underline{J} \qquad \underline{\bar{A}}_{o} = \underline{J}_{o}\underline{J} \qquad \underline{\bar{A}}_{R} = \underline{J}_{\pi}\underline{a}_{R}\underline{J}_{R} \qquad (4.2.9a\text{-}c)$$

$$\underline{\bar{A}} = \underline{a}_{\pi}\underline{J} \qquad \underline{\bar{A}}' = \underline{a}'\underline{J} \qquad \underline{\bar{A}}_{r} = \underline{J}_{r}\underline{J} \qquad \underline{\bar{A}}'_{R} = \underline{J}_{\pi}\underline{a}'_{R}\underline{J}_{R} \qquad (4.2.9d\text{-}g)$$

Nodal equilibrium is ensured by pre-multiplying equations (4.2.3a) by the transpose of the incidence matrix $\underline{J}$; the structure INTERNAL RELEASE FORCES $\underset{\sim}{X}'_{R}$ are selected by pre-multiplying equations (4.2.3b) by the transpose of the incidence matrix $\underline{J}_{R}$:

$$\underset{\sim}{X}'_{R} = \underline{J}_{R}^{T}\,\underset{\sim}{\tilde{X}}_{R} \qquad (4.2.10)$$

Similarly, and considering (4.2.6) now, we define the structure ADDITIONAL RELEASE FORCES $\underset{\sim}{\pi}'$ as

$$\underset{\sim}{\pi}' = \underline{J}_{\pi}^{T}\,\underset{\sim}{\tilde{\pi}}' \qquad (4.2.11)$$

Thus, and using equations (4.2.9) to (4.2.11)

$$\left[\begin{array}{c} \underline{J}^{T}\underset{\sim}{R}^{*} \\ \hline -\underset{\sim}{X}'_{R} \end{array}\right] = \left[\begin{array}{c|c|c} \underline{\bar{A}}^{T} & \underline{\bar{A}}_{\pi}^{T} & \underline{\bar{A}}_{R}^{T} \\ \hline \underline{\bar{A}}'^{T} & \cdot & \underline{\bar{A}}'^{T}_{R} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{X}' \\ \hline -\underset{\sim}{\bar{\pi}} \\ \hline -\underset{\sim}{\pi}' \end{array}\right] \quad \begin{array}{r} (4.2.12a) \\ (4.2.12b) \end{array}$$

Among the member nodal forces grouped in the supervector $\underset{\sim}{R}^{*}$ we select those contributing to the internal work dissipated as the structure deforms; hence, and collecting in $\underset{\sim}{\lambda}_{r}$ the structure EXTERNAL RELEASE FORCES

$$\underset{\sim}{R}^{*} = \left[\begin{array}{c|c} \underline{J}_{o}^{T} & \underline{J}_{r}^{T} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{\lambda} \\ \hline -\underset{\sim}{\lambda}_{r} \end{array}\right] \qquad (4.2.13)$$

The ASSEMBLED description of Statics, obtained by substituting the above equation in (4.2.12a), using (4.2.9b) and (4.2.9f) and re-arranging, is defined below, together with the kinematics description (4.2.8).

STATICS	KINEMATICS
$\begin{bmatrix} \cdot \\ -\underline{X}'_R \end{bmatrix} = \begin{bmatrix} \underline{\bar{A}}^T & \underline{\bar{A}}_o^T & \underline{\bar{A}}_r^T & \underline{\bar{A}}_\pi^T & \underline{\bar{A}}_R^T \\ \underline{\bar{A}}'^T & \cdot & \cdot & \cdot & \underline{\bar{A}}'^T_R \end{bmatrix} \begin{bmatrix} \underline{X}' \\ -\underline{\lambda} \\ \underline{\lambda}_r \\ -\underline{\bar{\pi}} \\ -\underline{\pi}' \end{bmatrix}$	$\begin{bmatrix} \underline{u}' + \underline{u}'_\pi + \underline{u}'_R \\ \underline{\delta} \\ \underline{\delta}_r \\ \underline{\bar{\delta}}_\pi \\ \underline{\delta}'_\pi \end{bmatrix} = \begin{bmatrix} \underline{\bar{A}} & \underline{\bar{A}}' \\ \underline{\bar{A}}_o & \cdot \\ \underline{\bar{A}}_r & \cdot \\ \underline{\bar{A}}_\pi & \cdot \\ \underline{\bar{A}}_R & \underline{\bar{A}}'_R \end{bmatrix} \begin{bmatrix} \underline{\bar{q}} \\ \underline{v}'_R \end{bmatrix}$
NODAL DESCRIPTION	

(4.2.14a,b) (4.2.15a-e)

The above dual and explicitly linear transformations can be specialized to recover the static and kinematic descriptions in linear analysis, just by setting to zero the additional deformations $\underline{u}'_\pi$ and $\underline{u}'_R$ and the additional forces $\underline{\bar{\pi}}$ and $\underline{\pi}'$, eliminating next from the system the associate kinematic operators:

LINEAR ANALYSIS	
STATICS	KINEMATICS
$\begin{bmatrix} \cdot \\ -\underline{X}'_R \end{bmatrix} = \begin{bmatrix} \underline{\bar{A}}^T & \underline{\bar{A}}_o^T & \underline{\bar{A}}_r^T \\ \underline{\bar{A}}'^T & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \underline{X}' \\ -\underline{\lambda} \\ \underline{\lambda}_r \end{bmatrix}$	$\begin{bmatrix} \underline{u}' \\ \underline{\delta} \\ \underline{\delta}_r \end{bmatrix} = \begin{bmatrix} \underline{\bar{A}} & \underline{\bar{A}}' \\ \underline{\bar{A}}_o & \cdot \\ \underline{\bar{A}}_r & \cdot \end{bmatrix} \begin{bmatrix} \underline{\bar{q}} \\ \underline{v}'_R \end{bmatrix}$
NODAL DESCRIPTION	

(4.2.16a,b) (4.2.17a-c)

As an illustration of the process of assemblage consider the simple portal frame, referred to the global system of axes $\underline{x}^*$, represented in Fig.4.4(a). The frame has four members of length L_i

which are inclined of α_i with respect to the axes x_3^* , in the sense of x_1^* , according to the chosen orientation for each member, shown in Fig.4.4(b). Members 1 and 4 have bending releases and the deformability of the foundation is simulated by the external release at the right-hand side foundation node. If the internal releases were blocked, the structure would have ten degrees of kinematic freedom; referring to the corresponding displacements to the global system $\underset{\sim}{x}^*$ and numbering, we write

$$\bar{\underset{\sim}{q}}^T = \begin{bmatrix} q_1 & q_2 & \cdots & q_{10} \end{bmatrix}$$

Let the release dislocations be collected in

$$\underset{\sim}{v}_R'^T = \begin{bmatrix} v_R'^1 & v_R'^2 \end{bmatrix}$$

Equations (4.2.5) take the form:

$$\begin{bmatrix} \begin{bmatrix} r_1^{1*} \\ r_2^{1*} \\ r_3^{1*} \\ \hdashline r_1^{2*} \\ r_2^{2*} \\ r_3^{2*} \end{bmatrix}_1 \\ \underset{\sim}{r}_2^* \\ \hdashline \underset{\sim}{r}_3^* \\ \hdashline \underset{\sim}{r}_4^* \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hdashline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hdashline \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hdashline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} v_{1R}^1 \\ v_{2R}^1 \\ v_{3R}^1 \\ \hdashline v_{1R}^2 \\ v_{1R}^2 \\ v_{1R}^3 \end{bmatrix}_1 \\ \underset{\sim}{v}_{R_2} \\ \hdashline \underset{\sim}{v}_{R_3} \\ \hdashline \underset{\sim}{v}_{R_4} \end{bmatrix} = \begin{bmatrix} 1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \hdashline \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \hdashline \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \hdashline \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_R'^1 \\ v_R'^2 \end{bmatrix} \qquad (4.2.18a,b)$$

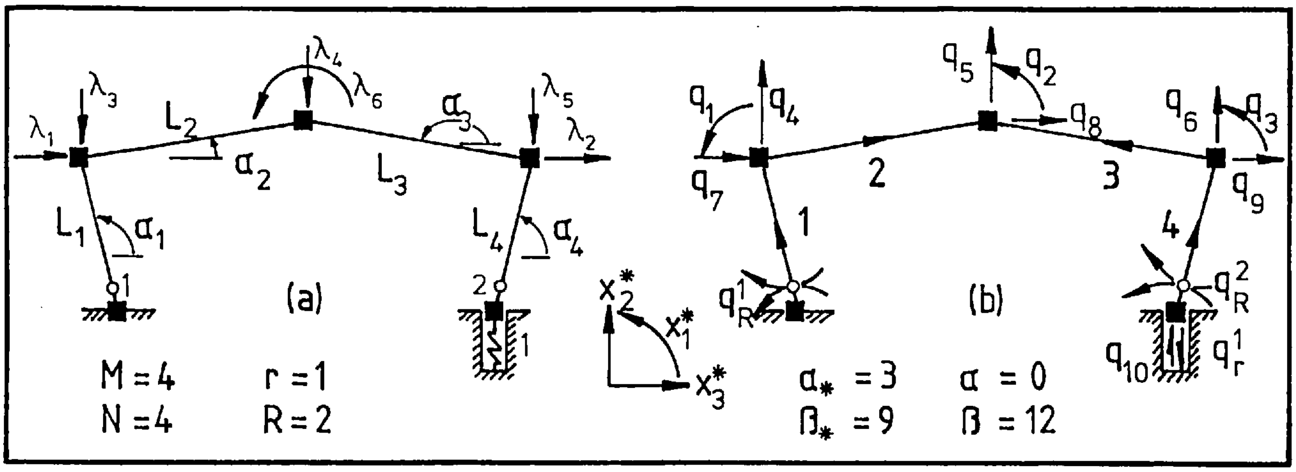


FIGURE 4.4

As all entries in the i-th row of the above incidence matrices are zero except, and in the maximum, one, say the j-th, which may take the values ±1, all the information they provide can be stored in just one array of the form

$\underline{J}^T \diamond$	•	•	•	1	4	7	1	4	7	2	5	8	3	6	9	2	5	8	•	10	•	3	6	9

(4.2.19a)

$\underline{J}_R^T \diamond$	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	2	•	•	•	•	•

(4.2.19b)

The remaining incidence matrices can be stored in a similar way, yielding

$\underline{J}_o \diamond$	•	•	•	•	-3	1	•	•	•	-6	-4	•	•	-5	2	•	•	•	•	•	•	•	•	•

(4.2.19c)

$\underline{J}_r \diamond$	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	-1	•	•	•	•

(4.2.19d)

$\underline{J}_\pi \diamond$	1	•	•	•	•	•	•	•	•	•	•	•	2	•	•	•

(4.2.19e)

which we read in the following way: Let $\underline{J}$ be the generic incidence matrix, with elements J_{kl}, and let i be the entry in the j-th column of the equivalent array; moreover, let n be the number of columns of this array and in the highest of its entries, taken in absolute value. Then

$$J_{kj} = 0 \qquad k = 1, 2, \ldots, m \; ; \quad j = 1, 2, \ldots, n$$

except when either $i > 0$ or $i < 0$, in which case

$$J_{ij} = +1 \quad \text{or} \quad J_{\iota j} = -1 \quad (\iota = -i)$$

respectively.

Performing the products in (4.2.9) based on (4.2.19) as well as the block-diagonal kinematic supermatrices with elements defined in (2.1.15), (2.1.19) and (2.4.6), (2.4.8) and (2.4.11), the Kinematics operator, the transpose of the Statics operator, given in Table 4.3, where we wrote

$$s_i = \sin \alpha_i \qquad s'_i = \frac{\sin\alpha_i}{L_i}$$

$$c_i = \cos \alpha_i \qquad c'_i = \frac{\cos\alpha_i}{L_i}$$

is finally obtained.

•	•	•	c'_1	•	•	$-s'_1$	•	•	•	1	•
1	•	•	$-c'_1$	•	•	s'_1	•	•	•	•	•
•	•	•	$-s_1$	•	•	$-c_1$	•	•	•	•	•
-1	•	•	$-c'_2$	c'_2	•	s'_2	$-s'_2$	•	•	•	•
•	1	•	c'_2	$-c'_2$	•	$-s'_2$	s'_2	•	•	•	•
•	•	•	s_2	$-s_2$	•	c_2	$-c_2$	•	•	•	•
•	•	-1	•	c'_3	$-c'_3$	•	$-s'_3$	s'_3	•	•	•
•	1	•	•	$-c'_3$	c'_3	•	s'_3	$-s'_3$	•	•	•
•	•	•	•	$-s_3$	s_3	•	$-c'_3$	c'_3	•	•	•
•	•	•	•	•	c'_4	•	•	$-s'_4$	$-c'_4$	•	1
•	•	1	•	•	$-c'_4$	•	•	s'_4	c'_4	•	•
•	•	•	•	•	$-s'_4$	•	•	$-c'_4$	s'_4	•	•
•	•	•	•	•	•	1	•	•	•		
•	•	•	•	•	•	•	•	1	•		
•	•	•	-1	•	•	•	•	•	•		
•	•	•	•	-1	•	•	•	•	•		
•	•	•	•	•	-1	•	•	•	•		
•	-1	•	•	•	•	•	•	•	•		
•	•	•	•	•	•	•	•	•	-1		
•	•	•	$-s_1$	•	•	$-c_1$	•	•	•		
•	•	•	$-c_1$	•	•	s_1	•	•	•		
•	•	•	s_2	$-s_2$	•	c_2	$-c_2$	•	•		
•	•	•	c_2	$-c_2$	•	$-s_2$	s_2	•	•		
•	•	•	•	$-s_3$	s_3	•	$-c_3$	c_3	•		
•	•	•	•	$-c_3$	c_3	•	s_3	$-s_3$	•		
•	•	•	•	•	$-s_4$	•	•	$-c_4$	s_4		
•	•	•	•	•	$-c_4$	•	•	s_4	c_4		
•	•	•	•	•	•	•	•	•	•	1	•
•	•	•	•	•	•	•	•	•	•	•	1

TABLE 4.3

The generalized finite description of Statics and Kinematics, obtained by introducing the generalized variables summarized in Table 4.1 in equations (4.2.14) and (4.2.15) and re-grouping is presented in Table 4.7.

We note that the generalized deformations vector $\underset{\sim}{u}$ is defined as the sum

$$\underset{\sim}{u} = \underset{\sim}{u}_E + \underset{\sim}{u}_p + \underset{\sim}{u}_D$$

where $\underset{\sim}{u}_E$ and $\underset{\sim}{u}_p$ are, respectively, the generalized elastic and plastic deformation vectors and $\underset{\sim}{u}_D$ a vector of PRESCRIBED DISLOCATIONS.

The corresponding incremental descriptions (4.2.65) and (4.2.66), shown in Table 4.9, can be obtained by taking increments in (4.2.61) and (4.2.62)

$$\underset{\sim}{0} = \left[\underline{A}^T \;\vdots\; \underline{A}_o^T \;\vdots\; \underline{A}_\pi^T\right] \begin{bmatrix} \Delta\underset{\sim}{X} \\ -\Delta\underset{\sim}{\lambda} \\ -\Delta\underset{\sim}{\pi} \end{bmatrix} , \quad \begin{bmatrix} \Delta\underset{\sim}{u}+\Delta\underset{\sim}{u}_\pi \\ \Delta\underset{\sim}{\delta} \\ \Delta\underset{\sim}{\delta}_\pi \end{bmatrix} = \begin{bmatrix} \underline{A} \\ \underline{A}_o \\ \underline{A}_\pi \end{bmatrix} \Delta\underset{\sim}{q}$$

and eliminating above the increments on the generalized additional forces and deformations through the following relations

$$\Delta\underset{\sim}{\pi} = \underline{\mathbb{Q}}^T \, \Delta\underset{\sim}{X} + \underline{\mathbb{P}} \, \Delta\underset{\sim}{\delta}_\pi + \Delta\underset{\sim}{\mathbb{R}}_\pi \qquad (4.2.20a)$$

and

$$\Delta\underset{\sim}{u}_\pi = \underline{\mathbb{Q}} \, \Delta\underset{\sim}{\delta}_\pi + \Delta\underset{\sim}{\mathbb{R}}_{u\pi} \qquad (4.2.20b)$$

which were obtained by grouping (2.1.43a) and (2.4.29a), and (2.1.39a) and (2.4.29b), respectively, setting then m to 1,2,...,M and collecting in the manner of (4.1.6).

An alternative way of obtaining relations (4.2.65) and (4.2.66) is to assemble the intended nodal descriptions of incremental Statics and Kinematics (2.4.30) and (2.4.31). Setting m = 1,2, ..., M, collecting as in (4.1.6)

$$\begin{bmatrix} -\underline{k}_{rr} & -\underline{k}_{rR} & \underline{\mathbb{a}}^T \\ -\underline{k}_{rR}^T & -\underline{k}_{RR} & \underline{\mathbb{a}}'^T \\ \underline{\mathbb{a}} & \underline{\mathbb{a}}' & \cdot \end{bmatrix} \begin{bmatrix} \Delta\underset{\sim}{r}^* \\ \Delta\underset{\sim}{\bar{v}}_R \\ \Delta\underset{\sim}{X}' \end{bmatrix} = \begin{bmatrix} \Delta\underset{\sim}{R}^* \\ -\Delta\underset{\sim}{X}_R \\ \Delta\underset{\sim}{u}' \end{bmatrix} + \begin{bmatrix} \Delta\underset{\sim}{R}^*_\pi \\ \Delta\underset{\sim}{\tilde{R}}_{X\pi} \\ \Delta\underset{\sim}{R}_{u\pi} \end{bmatrix}$$

and subjecting the above system to the assemblage procedure previously described; re-arranging that system in order to express the assembled relations in terms of the generalized variables, the structural matrices summarized in Table 4.10 emerge in the following (equivalent) forms:

$$\mathbb{K}_n = \left[\begin{array}{c|c} \underset{\sim}{J}^T & \cdot \\ \hline \cdot & \underset{\sim}{J}_R^T \end{array}\right] \left[\begin{array}{c|c} \underset{\sim}{k}_{rr} & \underset{\sim}{k}_{rR} \\ \hline \underset{\sim}{k}_{rR}^T & \underset{\sim}{k}_{RR} \end{array}\right] \left[\begin{array}{c|c} \underset{\sim}{J} & \cdot \\ \hline \cdot & \underset{\sim}{J}_R \end{array}\right]$$

$$\underset{\sim}{\mathbb{A}} = \left[\begin{array}{c|c} \mathbb{a}\,\underset{\sim}{J} & \mathbb{a}'\,\underset{\sim}{J}_R \\ \hline \cdot & I \\ \hline \underset{\sim}{J}_r\underset{\sim}{J} & \cdot \end{array}\right], \quad \underset{\sim}{\mathbb{A}}_o^T = \left[\begin{array}{c} \underset{\sim}{J}^T\underset{\sim}{J}_o^T \\ \hline \cdot \end{array}\right], \quad \Delta\underset{\sim}{\mathbb{R}}'_\pi = \left[\begin{array}{cc} \underset{\sim}{J}^T & \Delta\underset{\sim}{R}^*_\pi \\ \hline \underset{\sim}{J}_R^T & \tilde{\underset{\sim}{R}}_{X\pi} \end{array}\right] \quad \Delta\mathbb{R}_{u\pi} = \left[\begin{array}{c} \Delta\underset{\sim}{R}_{u\pi} \\ \hline \cdot \\ \hline \cdot \end{array}\right]$$

4.2.2 Mesh Description

Consider the structure graphically represented in Fig. 4.5(a) which can be interpreted as the graphic model of a crane composed by a load-receiving truss resting on a frame transmitting the load to the foundation. While the left-hand side leg of the frame rests on a foundation unable to resist to rotational movements, the right-hand side leg is founded on a medium with limited capability to absorb horizontal forces.

Let the structure members and releases be numbered and the members orientated, thus deciding the position of critical sections 1 and 2 for each member. The applied loads are also numbered and collected in $\underset{\sim}{\lambda}$ as well as, and according to the static boundary conditions, the structure REACTION FORCES developing at the (fixed) supports and the forces at the (deformable) external releases, which we group in $\underset{\sim}{\lambda}_s$ and $\underset{\sim}{\lambda}_r$, respectively.

We define the EQUIVALENT STRUCTURE, as in Fig.4.5(c) as the fundamental structure with additional members joining to neighbouring nodes every node of the structure where only one member connects. The equivalent structure is formed by M' mesh substructures, their equilibrium conditions being defined by

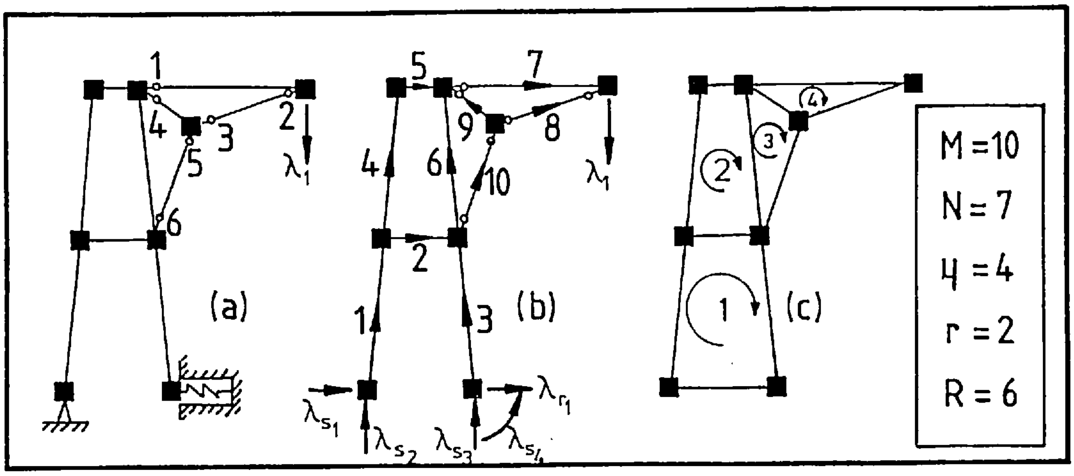


FIGURE 4.5

equations (2.4.21); setting in those equations M=1,2, ..., M' and collecting in the manner of (4.1.5), the following unassembled description of Statics is found:

$$\begin{bmatrix} \tilde{\underset{\sim}{X}}' \\ \hline \tilde{\underset{\sim}{X}}_R \\ \hline -\underset{\sim}{R}'^{*} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \underline{b} & \underline{b}_o & \underline{b}_{o\pi} & \cdot \\ \hline \underline{b}' & \underline{b}'_o & \underline{b}'_{o\pi} & \underline{b}'_R \\ \hline \cdot & \underline{b}_r & \underline{b}_{r\pi} & \cdot \end{array}\right] \begin{bmatrix} \underset{\sim}{p} \\ \hline \underset{\sim}{R}^{*} \\ \hline \underset{\sim}{\pi} \\ \hline \underset{\sim}{\pi}' \end{bmatrix} \qquad \begin{matrix} (4.2.21a) \\ (4.2.21b) \\ (4.2.21c) \end{matrix}$$

where marked again with a tilde are the supervectors which, at this stage, may contain superfluous information.

If the general expressions for the mesh matrices defined in (2.2.38) to (2.2.42) and in (2.4.23) to (2.4.26) are to be used, every constituent substructure has to be clock-wise orientated.

In section 2.2 the (hyperstatic) mesh substructure was replaced by an equivalent cantilever so that the solution of Statics could be defined as a statically determinate structure by adding the effects of the mesh forces $\underset{\sim}{R}^{*}_{M}$ (particular solution) to those of the indeterminate biactions $\underset{\sim}{p}_{M}$ (complementary solution).

A similar procedure is adopted herein to establish the equilibrium conditions of the deformed and displaced structure. Each of the constituent cantilevers are assembled to form the EQUIVALENT CANTILEVER, a tree-structure obtained by introducing M' cuts in the equivalent structure; one node, in preference belonging unequivocally to one of the mesh substructures, is chosen to be the foundation node of the equivalent cantilever.

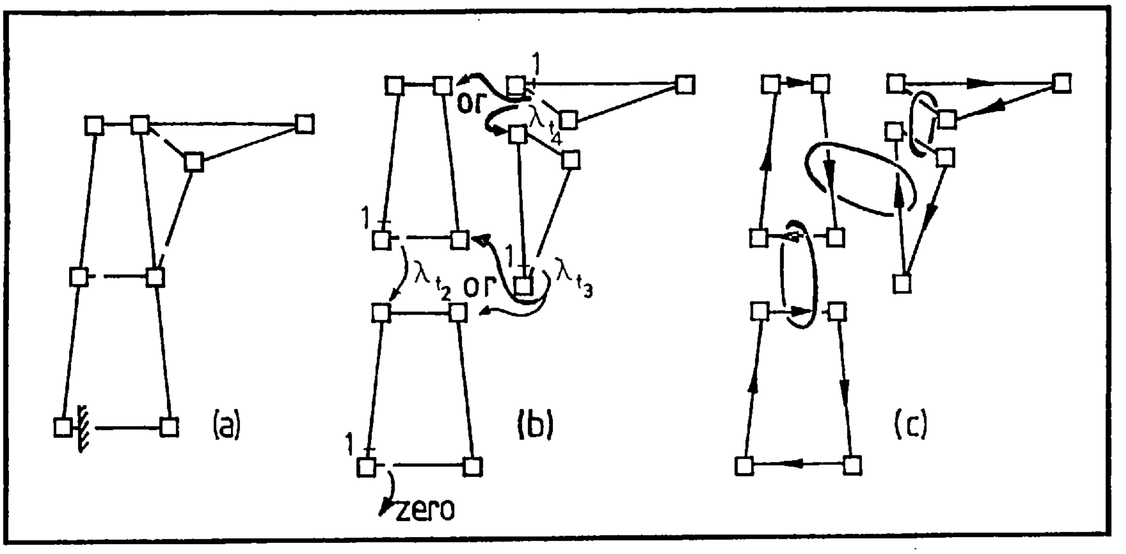


FIGURE 4.6

The mesh forces R^*_M of a generic substructure can be defined as the sum of the partial contributions of the applied loads λ and, if the mesh has foundation nodes, the forces at the fixed supports, λ_s, and at the external releases, λ_r, as well as the forces λ_t transmitted by neighbouring meshes; hence, for the unassembled M' meshes

$$R^* = J_o \lambda + J_s \lambda_s + J_r(-\lambda_r) + J_t \lambda_t \qquad (4.2.22)$$

As the mesh forces R^*_M need not be self-equilibrating, a generic force applied to a node can be assigned to any of the neighbouring meshes.

The mesh forces are equilibrated by the mesh reaction forces, collected in R'^*, as implied in (4.2.21c). We define the MESH TRANSMISSION FORCES as

$$\lambda_t = -J_t' R'^* \qquad (4.2.23a)$$

As illustrated in Fig.4.6(b), these forces can also be assigned to the corresponding node of any of the neighbouring meshes. For the mesh containing the foundation node of the equivalent cantilever, as, for instance, the bottom left-hand side node of mesh 1, Fig.4.6(a), we may write

$$0 = -J_t'' R'^* \qquad (4.2.23b)$$

since the external forces $\underset{\sim}{\lambda}$, $\underset{\sim}{\lambda}_s$ and $\underset{\sim}{\lambda}_r$ applied to the structure have to be in equilibrium.

Substituting (4.2.23a) in (4.2.22), pre-multiplying the resulting equation by $\underline{b}_R$ and using (4.2.21c) we find

$$-\underline{S}\ \underset{\sim}{R}'^{*} = \underline{b}_R\ \underset{\sim}{\Lambda} + \underline{b}_{R\pi}\ \underset{\sim}{\tilde{\pi}} \qquad (4.2.24)$$

where

$$\underline{S} = \underline{I} - \underline{b}_R\ \underline{J}_t\ \underline{J}_t' \qquad (4.2.25a)$$

and

$$\underset{\sim}{\Lambda} = \underline{J}_o\ \underset{\sim}{\lambda} + \underline{J}_s\ \underset{\sim}{\lambda}_s - \underline{J}_r\ \underset{\sim}{\lambda}_r \qquad (4.2.25b)$$

Matrix $\underline{S}$ can always be expressed as a triangular matrix with unit diagonal elements and hence unit determinant. Equation (4.2.24) can be resolved to give

$$-\underset{\sim}{R}'^{*} = \underline{T}\ \underline{b}_r\ \underset{\sim}{\Lambda} + \underline{T}\ \underline{b}_{r\pi}\ \underset{\sim}{\tilde{\pi}} \qquad (4.2.26)$$

where

$$\underline{T} = \underline{S}^{-1} \qquad (4.2.25c)$$

is still a triangular matrix of the same kind with coefficients defined by

$$t_{ii} = 1 \quad \text{and} \quad t_{ij} = -\sum_{k=i}^{j-1} t_{ik}\ s_{kj} \qquad (4.2.27a,b)$$

Substituting (4.2.26) back into (4.2.23) we find

$$\underset{\sim}{\lambda}_t = \underline{J}_t'\ \underline{T}\ \underline{b}_r\ \underset{\sim}{\Lambda} + \underline{J}_t'\ \underline{T}\ \underline{b}_{r\pi}\ \underset{\sim}{\tilde{\pi}} \qquad (4.2.28a)$$

and

$$\underset{\sim}{0} = \underline{J}_t''\ \underline{T}\ \underline{b}_r\ \underset{\sim}{\Lambda} + \underline{J}_t''\ \underline{T}\ \underline{b}_{r\pi}\ \underset{\sim}{\tilde{\pi}} \qquad (4.2.28b)$$

The transmission forces may now be eliminated in (4.2.22), yielding

$$\underset{\sim}{R}^{*} = \underline{I}'\ \underset{\sim}{\Lambda} + \underline{I}''\ \underset{\sim}{\tilde{\pi}} \qquad (4.2.29)$$

where

$$\underline{I}' = \underline{I} + \underline{J}_t\ \underline{J}_t'\ \underline{T}\ \underline{b}_r \qquad (4.2.30a)$$

and

$$\underline{I}'' = \underline{J}_t\ \underline{J}_t'\ \underline{T}\ \underline{b}_{r\pi} \qquad (4.2.30b)$$

The equilibrium equations

$$\underset{\sim}{\tilde{X}}' = \underline{b}\ \underset{\sim}{\tilde{p}} + \underline{b}_o\ \underline{I}'\ \underset{\sim}{\Lambda} + (\underline{b}_o \underline{I}'' + \underline{b}_{o\pi})\ \underset{\sim}{\tilde{\pi}} \qquad (4.2.31a)$$

$$\underset{\sim}{\tilde{X}}_R = \underline{b}'\underset{\sim}{\tilde{p}} + \underline{b}_o'\ \underline{I}'\ \underset{\sim}{\Lambda} + (\underline{b}_o'\ \underline{I}'' + \underline{b}_{o\pi}')\ \underset{\sim}{\tilde{\pi}} + \underline{b}_R'\ \underset{\sim}{\tilde{\pi}}' \qquad (4.2.31b)$$

were obtained substituting (4.2.29) in (4.2.21a) and (4.2.21b). The set of equations (4.2.21c), defining the external equilibrium of each of the constituent substructures, can be replaced by the equivalent equations (4.2.28b) which regulate the global equilibrium between the applied forces.

So far we have been solely concerned with assigning the loads to the different substructures and with transmitting them to the foundation of the equivalent cantilever; the mesh substructures are yet to be assembled.

Let us then collect in $\underset{\sim}{X}'$ and $\underset{\sim}{X}'_R$, respectively, the structure INDEPENDENT STRESS-RESULTANTS and RELEASE FORCES, obtainable through the simple incidence

$$\underset{\sim}{X}' = \underset{\sim}{J}\,\bar{\underset{\sim}{X}}' \quad \text{and} \quad \underset{\sim}{X}'_R = \underset{\sim}{J}_R\,\bar{\underset{\sim}{X}}_R \qquad (4.2.32a,b)$$

For instance, for member 2 of the structure represented in Fig.4.6(b), and considering Fig.4.6(c), we would write

$$\underset{\sim}{X}'_2 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} (\underset{\sim}{X}'_2)_{\text{mesh 1}} + \begin{bmatrix} \cdot & -1 & \cdot \\ -1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} (\underset{\sim}{X}'_4)_{\text{mesh 2}}$$

Let us collect in $\underset{\sim}{X}'_a$ the independent stress-resultants at the auxiliary members of the equivalent structure, as member 4 of mesh 1, Fig.4.6(c); then, for the actual structure

$$\underset{\sim}{X}'_a = \underset{\sim}{0} = \underset{\sim}{J}'\,\bar{\underset{\sim}{X}}' \qquad (4.2.33)$$

If the vectors $\bar{\underset{\sim}{\pi}}$ and $\underset{\sim}{\pi}'$ contain the structure additional forces and additional release forces, by stating

$$\tilde{\underset{\sim}{\pi}} = \underset{\sim}{J}_{\pi}\,\bar{\underset{\sim}{\pi}} \quad \text{and} \quad \tilde{\underset{\sim}{\pi}}' = \underset{\sim}{J}'_{\pi}\,\underset{\sim}{\pi} \qquad (4.2.32c,d)$$

we eliminate in the supervectors $\tilde{\underset{\sim}{\pi}}$ and $\tilde{\underset{\sim}{\pi}}'$ the information they provide for members and releases which do not exist in the actual structure.

Substituting (4.2.32) in (4.2.31) and (4.2.28b), adding (4.2.33) to the system, and using (4.2.25b) and re-grouping, the following description for the ASSEMBLED Statics relations is found

$$
\begin{bmatrix} X' \\ X'_R \\ \cdot \\ \cdot \end{bmatrix}
\begin{matrix} = \\ = \\ \\ \\ \end{matrix}
\begin{bmatrix}
\bar{b} & \bar{b}_o & \bar{b}_s & \bar{b}_r & \bar{b}_{o\pi} & \cdot \\
\bar{b}' & \bar{b}'_o & \bar{b}'_s & \bar{b}'_r & \bar{b}'_{o\pi} & \bar{B}'_R \\
\bar{b}'' & \bar{b}''_o & \bar{b}''_s & \bar{b}''_r & \bar{b}''_{o\pi} & \cdot \\
\cdot & \bar{b}_r & \bar{b}_{rs} & \bar{b}_{rr} & \bar{b}_{R\pi} & \cdot
\end{bmatrix}
\begin{bmatrix} p \\ \lambda \\ \lambda_s \\ -\lambda_r \\ \bar{\pi} \\ \pi' \end{bmatrix}
\begin{matrix} (4.2.34a) \\ (4.2.34b) \\ (4.2.34c) \\ (4.2.34d) \end{matrix}
$$

where we note

$$\bar{b} = J\,b \qquad \bar{b}_o = J\,b_o I' J_o \qquad \bar{b}_s = J\,b_o I' J_s \qquad \bar{b}_r = J\,b_o I' J_r \qquad \bar{b}_{o\pi} = J(b_o I'' + b_{o\pi})J_\pi \qquad (4.2.35a\text{-}e)$$

$$\bar{b}' = J_R b' \qquad \bar{b}'_o = J_R b'_o I' J_o \qquad \bar{b}'_s = J_R b'_o I' J_s \qquad \bar{b}'_r = J_R b'_o J' J_r \qquad \bar{b}'_{o\pi} = J_R(b'_o I'' + b'_{o\pi})J_\pi \qquad (4.2.35f\text{-}j)$$

$$\bar{b}'' = J' b \qquad \bar{b}''_o = J' b_o I' J_o \qquad \bar{b}'' = J' b_o I' J_s \qquad \bar{b}''_r = J' b_o I' J_r \qquad \bar{b}''_{o\pi} = J'(b_o I'' + b_{o\pi})J_\pi \qquad (4.2.35l\text{-}p)$$

$$\bar{B}'_R = J_R b'_R J'_\pi \quad \bar{b}_r = J''_t I\, b_R J_o \qquad \bar{b}_{rs} = J''_t I\, b_R J_s \qquad \bar{b}_{rr} = J''_t I\, b_R J_r \qquad \bar{b}_{r\pi} = J''_t I\, b_{R\pi} J_\pi \qquad (4.2.35q\text{-}u)$$

Equations (2.4.24) define the compatibility conditions for the generic mesh M; hence for the M' unconnected meshes forming the structure, the system

$$
\begin{bmatrix} \cdot \\ r^* \\ \bar{\delta}_\pi \\ \bar{\delta}'_\pi \end{bmatrix}
=
\begin{bmatrix}
b^T & b'^T & \cdot \\
b_o^T & b_o'^T & b_r^T \\
b_{o\pi}^T & b_{o\pi}'^T & b_{r\pi}^T \\
\cdot & b_R'^T & \cdot
\end{bmatrix}
\begin{bmatrix} \tilde{u}'' \\ \tilde{v}_R \\ r'^* \end{bmatrix}
\begin{matrix} (4.2.36a) \\ (4.2.36b) \\ (4.2.36c) \\ (4.2.36d) \end{matrix}
$$

in which

$$\tilde{u}'' = \tilde{u}' + \tilde{u}'_\pi + \tilde{u}'_R \qquad (4.2.37)$$

contains the necessary data to perform the assembly. Once again we marked with a tilde those variables which may contain superfluous information.

If a generic force was assigned, through (4.2.22), to one of the meshes sharing the node upon which that force acts, then its displacement must coincide with the associate mesh force displacement; hence

$$\begin{bmatrix} \underset{\sim}{\delta} \\ \underset{\sim}{\delta}_s \\ \underset{\sim}{\delta}_r \\ \underset{\sim}{\delta}_t \end{bmatrix} = \begin{bmatrix} \underline{J}_o^T \\ \underline{J}_s^T \\ \underline{J}_r^T \\ \underline{J}_t^T \end{bmatrix} \underset{\sim}{r}^* \qquad \begin{matrix} (4.2.38a) \\ (4.2.38b) \\ (4.2.38c) \\ (4.2.38d) \end{matrix}$$

In $\underset{\sim}{\delta}$ and $\underset{\sim}{\delta}_r$ we collect, respectively, the point load displacements and the displacements at the external releases; r^* contains all mesh force displacements and its elements are ordered according to the label ascribed to the mesh (from 1 to M'), the orientation of the mesh and the numbering of its members.

In $\underset{\sim}{\delta}_s$ we collect the fixed support displacements; hence

$$\underset{\sim}{\delta}_s = \underset{\sim}{0} \qquad (4.2.39)$$

The role of the TRANSMISSION FORCE DISPLACEMENTS $\underset{\sim}{\delta}_t$ is to transfer the rigid body displacements suffered by a given mesh to the incident meshes.

The reaction force displacements are defined by

$$\underset{\sim}{r}'^* = \left[\underline{J}_t'^T \mid \underline{J}_t''^T \right] \begin{bmatrix} \underset{\sim}{\delta}_t \\ \hline \underset{\sim}{\delta}_f \end{bmatrix} \qquad (4.2.40)$$

where $\underset{\sim}{\delta}_f$ represents the displacements of the node chosen to be the foundation of the equivalent cantilever. Substituting (4.2.38d) and (4.2.36b) in (4.2.40) and using (4.2.25a) and (4.2.25c) gives

$$\underset{\sim}{r}'^* = \underline{I}^T \underline{J}_t'^T \underline{J}_t^T (\underline{b}_o^T \underset{\sim}{\bar{u}}'' + \underline{b}_o^T \underset{\sim}{\bar{v}}_R) + \underline{I}^T \underline{J}_t''^T \underset{\sim}{\delta}_f \qquad (4.2.41a)$$

enabling us to eliminate the reaction for displacements in the definition (4.2.34b) of the mesh force displacements, yielding, with help from (4.2.28)

$$\underset{\sim}{r}^* = \underline{I}'^T (\underline{b}_o^T \underset{\sim}{\bar{u}}'' + \underline{b}_o^T \underset{\sim}{\bar{v}}_R) + \underline{b}_r^T \underline{I}^T \underline{J}''^T \underset{\sim}{\delta}_f \qquad (4.2.42)$$

Let us separate the generalized deformations $\tilde{\mathbf{u}}''$ into two vectors, one containing the generalized deformations of the given structure, the other containing similar information about the auxiliary members of the equivalent structure:

$$\tilde{\mathbf{u}}'' = \mathbf{J}^T \mathbf{u}'' + \mathbf{J}'^T \mathbf{u}''_a \qquad (4.2.41b)$$

Similarly, let us define which elements of $\tilde{\mathbf{v}}_R$, $\bar{\boldsymbol{\delta}}_\pi$ and $\boldsymbol{\delta}'_\pi$, through

$$\tilde{\mathbf{v}}_R = \mathbf{J}_R^T \, \mathbf{v}'_R \qquad (4.2.41c)$$

$$\boldsymbol{\delta}'_\pi = \mathbf{J}'^T_\pi \, \bar{\boldsymbol{\delta}}'_\pi \qquad (4.2.41d)$$

$$\bar{\boldsymbol{\delta}}_\pi = \mathbf{J}^T_\pi \, \bar{\boldsymbol{\delta}}_\pi \qquad (4.2.41e)$$

correspond to the actual release dislocations, additional release force displacements and additional force displacements existing at each of the M members of the given structure, respectively collected in $\mathbf{v}'_R$, $\boldsymbol{\delta}'_\pi$ and $\bar{\boldsymbol{\delta}}_\pi$.

Substituting (4.2.41) in (4.2.36) as well as in (4.2.42) and eliminating in turn the mesh force displacements in (4.2.38) and using (4.2.30), (4.2.35) and (4.2.39), the following description for Kinematics of the assembled structure is found

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\delta} \\ \mathbf{0} \\ \boldsymbol{\delta}_r \\ \bar{\boldsymbol{\delta}}_\pi \\ \boldsymbol{\delta}'_\pi \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{b}}^T & \bar{\mathbf{b}}'^T & \bar{\mathbf{b}}''^T & \cdot \\ \bar{\mathbf{b}}_o^T & \bar{\mathbf{b}}'^T_o & \bar{\mathbf{b}}''^T_o & \bar{\mathbf{b}}^T \\ \bar{\mathbf{b}}_s^T & \bar{\mathbf{b}}'^T_s & \bar{\mathbf{b}}''^T_s & \bar{\mathbf{b}}^T_{rs} \\ \bar{\mathbf{b}}_r^T & \bar{\mathbf{b}}'^T_r & \bar{\mathbf{b}}''^T_r & \bar{\mathbf{b}}^T_{rr} \\ \bar{\mathbf{b}}_{o\pi}^T & \bar{\mathbf{b}}'^T_{o\pi} & \bar{\mathbf{b}}''^T_{o\pi} & \mathbf{b}^T_{r\pi} \\ \cdot & \bar{\mathbf{B}}'^T_R & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \mathbf{u}'' \\ \mathbf{v}'_R \\ \mathbf{u}''_a \\ \boldsymbol{\delta}_f \end{bmatrix} \qquad \begin{array}{r} (4.2.43a) \\ (4.2.43b) \\ (4.2.43c) \\ (4.2.43d) \\ (4.2.43e) \\ (4.2.43f) \end{array}$$

which, as expected, is the dual transformation of (4.2.34).

Both systems are explicitly linear and although able to support any theoretical or numerical structural analysis they contain unwanted auxiliary variables which should be eliminated.

Let α' be the static indeterminacy of the structure without internal releases and let us create a vector $\tilde{\mathbf{p}}$ of α' INDETERMINATE FORCES. The objective now is to eliminate the variables $\tilde{p}$ and λ_s from the description of Statics by using equations (4.2.36c) and (4.2.36d)

$$\begin{bmatrix} 0 \\ \hline 0 \end{bmatrix} = \left[\begin{array}{c|c} \bar{b}'' & \bar{b}''_s \\ \hline \cdot & \bar{b}_{RS} \end{array}\right] \begin{bmatrix} \tilde{p} \\ \hline \lambda_s \end{bmatrix} + \left[\begin{array}{c|c|c} \bar{b}''_o & \bar{b}''_r & \bar{b}''_{o\pi} \\ \hline \bar{b}_R & \bar{b}_{Rr} & \bar{b}_{R\pi} \end{array}\right] \begin{bmatrix} \lambda \\ \hline -\lambda_r \\ \hline \bar{\bar{\pi}} \end{bmatrix} \qquad \begin{matrix}(4.2.44a)\\(4.2.44b)\end{matrix}$$

It is always possible to define any α' independent variables $\bar{p}$, for instance

$$\bar{p} = \left[\begin{array}{c|c} I_p & I_s \end{array}\right] \begin{bmatrix} p \\ \hline \lambda_s \end{bmatrix} \qquad (4.2.45)$$

such that the enlarged system (4.2.44)

$$\left[\begin{array}{c|c} I_p & I_s \\ \hline -\bar{b}'' & -\bar{b}''_s \\ \hline \cdot & -\bar{b}_{Rs} \end{array}\right] \begin{bmatrix} \tilde{p} \\ \hline \lambda_s \end{bmatrix} = \left[\begin{array}{c|c|c|c} I & \cdot & \cdot & \cdot \\ \hline \cdot & \bar{b}''_o & \bar{b}''_r & \bar{b}''_{o\pi} \\ \hline \cdot & \bar{b}_R & \bar{b}_{Rr} & \bar{b}_{R\pi} \end{array}\right] \begin{bmatrix} \bar{p} \\ \hline \lambda \\ \hline -\lambda_r \\ \hline \bar{\pi} \end{bmatrix} \qquad \begin{matrix}(4.2.46a)\\(4.2.46b)\\(4.2.46c)\end{matrix}$$

can be solved to give

$$\begin{bmatrix} p \\ \hline \lambda_s \end{bmatrix} = \left[\begin{array}{c|c|c|c} C & C_o & C_r & C_\pi \\ \hline C' & C'_o & C'_r & C'_\pi \end{array}\right] \begin{bmatrix} \bar{p} \\ \hline \lambda \\ \hline -\lambda_r \\ \hline \pi \end{bmatrix} \qquad \begin{matrix}(4.2.47a)\\(4.2.47b)\end{matrix}$$

The various forms the above system may take, extends to kinematically non-linear analysis the use by Argyris and Kelsey (1960) of standard sets of self-equilibrating stress-resultants to replace those derived from a physical release system, as well as Jenkins' idea of equilibrating each independent load on a different system of releases, Jenkins (1953,1954).

The substitution of (4.2.47) in (4.2.34a,b) gives rise to the more concise description of Statics (4.2.50), where now

$\bar{B} = \bar{b}\ C + \bar{b}_s C'$ $\quad \bar{B}_o = \bar{b}\ C_o + \bar{b}_s C'_o + \bar{b}_o$ $\quad \bar{B}_r = \bar{b}\ C_r + \bar{b}_s C'_r + \bar{b}_r$ $\quad \bar{B}_\pi = \bar{b}\ C_\pi + \bar{b}_s C'_\pi + \bar{b}_{o\pi}$

$\bar{B}' = \bar{b}' C + \bar{b}'_s C'$ $\quad \bar{B}'_o = \bar{b}' C_o + \bar{b}'_s C'_o + \bar{b}'_o$ $\quad \bar{B}'_r = \bar{b}' C_r + \bar{b}'_s C'_r + \bar{b}_r$ $\quad \bar{B}'_\pi = \bar{b}' C_\pi + \bar{b}'_s C'_\pi + \bar{b}'_{o\pi}$

(4.2.48a,b) (4.2.48c,d) (4.2.48e,f) (4.2.48g,h)

Substituting (4.2.47) into (4.2.44) and re-grouping, we find

$$\left[\begin{array}{c|c|c|c} \bar{B}''C+\bar{B}''_s C' & \bar{B}''C_o+\bar{B}''_s C'_o+\bar{B}''_o & \bar{B}''C_r+\bar{B}''_s C'_r+\bar{B}''_r & \bar{B}''C_\pi+\bar{B}''_s C'_\pi+\bar{B}''_{o\pi} \\ \hline \bar{B}_{Rs}C' & \bar{B}_{Rs}C'_o+\bar{B}_R & \bar{B}_{Rs}C'_r+\bar{B}_{Rr} & \bar{B}_{Rs}C'_\pi+\bar{B}_{R\pi} \end{array}\right] \left[\begin{array}{c} \bar{p} \\ \hline \lambda \\ \hline -\lambda_r \\ \hline \bar{\pi} \end{array}\right] = 0$$

implying that

$$\bar{B}''C+\bar{B}''_s C'=0,\quad \bar{B}''C_o+\bar{B}''_s C'_o+\bar{B}''_o=0,\quad \bar{B}''C_r+\bar{B}''_s C'_r+\bar{B}''_r=0,\quad \bar{B}''C_\pi+\bar{B}''_s C'_\pi+\bar{B}''_{o\pi}=0$$

$$\bar{B}_{Rs}C'=0,\quad \bar{B}_{Rs}C'_o+\bar{B}_R=0,\quad \bar{B}_{Rs}C'_r+\bar{B}_{Rr}=0,\quad \bar{B}_{Rs}C'_\pi+\bar{B}_{R\pi}=0$$

(4.2.49a,b) (4.2.49c,d) (4.2.49e,f) (4.2.49g,h)

since $\bar{p}$, λ, λ_r and $\bar{\pi}$ are independent variables. Hence, the following (trivial) transformation can be written

$$\left[\begin{array}{c|c} (\bar{B}''C+\bar{B}'_s C')^T & (\bar{B}_{Rs}C')^T \\ \hline (\bar{B}''C_o+\bar{B}''_s C'+\bar{B}''_o)^T & (\bar{B}_{Rs}C'_o+\bar{B}_R)^T \\ \hline (\bar{B}''C_r+\bar{B}''_s C'_r+\bar{B}''_r)^T & (\bar{B}_{Rs}C'_r+\bar{B}_{r\pi})^T \\ \hline (\bar{B}''C_\pi+\bar{B}''_s C'_\pi+\bar{B}''_{o\pi})^T & (\bar{B}_{Rs}C'_\pi+\bar{B}_{R\pi})^T \end{array}\right] \left[\begin{array}{c} u''_a \\ \hline \delta_f \end{array}\right] = 0$$

and used in (4.2.43) together with (4.2.40) enabling us to express Kinematics as the contragredient transformation of (4.2.50), thus recovering Static-Kinematic Duality:

STATICS	KINEMATICS
$\left[\begin{array}{c} X' \\ \hline X'_R \end{array}\right] = \left[\begin{array}{c\|c\|c\|c\|c} \bar{B} & \bar{B}_o & \bar{B}_r & \bar{B}_\pi & \cdot \\ \hline \bar{B}' & \bar{B}'_o & \bar{B}'_r & \bar{B}'_\pi & \bar{B}'_R \end{array}\right] \left[\begin{array}{c} \bar{p} \\ \hline \lambda \\ \hline -\lambda_r \\ \hline \bar{\pi} \\ \hline \pi' \end{array}\right]$	$\left[\begin{array}{c} 0 \\ \hline \delta \\ \hline \delta_r \\ \hline \bar{\delta}_\pi \\ \hline \delta'_\pi \end{array}\right] = \left[\begin{array}{c\|c} \bar{B}^T & \bar{B}'^T \\ \hline \bar{B}_o^T & \bar{B}'^T_o \\ \hline \bar{B}_r^T & \bar{B}'^T_r \\ \hline \bar{B}_\pi^T & \bar{B}'^T_\pi \\ \hline \cdot & \bar{B}'^T_R \end{array}\right] \left[\begin{array}{c} u'+u'_\pi+u'_R \\ \hline v'_R \end{array}\right]$
MESH DESCRIPTION	

(4.2.50a,b) (4.2.51a,e)

The mesh description of linear Statics and Kinematics is obtained by setting in the above relations the additional deformations and forces to zero and removing from the system (4.2.50) the kinematic duals of the latter:

LINEAR ANALYSIS		
STATICS	KINEMATICS	
$\begin{bmatrix} \underset{\sim}{X}' \\ \underset{\sim}{X}'_R \end{bmatrix} = \begin{bmatrix} \bar{\underset{\sim}{B}} & \bar{\underset{\sim}{B}}_o & \bar{\underset{\sim}{B}}_r \\ \bar{\underset{\sim}{B}}' & \bar{\underset{\sim}{B}}'_o & \bar{\underset{\sim}{B}}'_r \end{bmatrix} \begin{bmatrix} \bar{\underset{\sim}{p}} \\ \underset{\sim}{\lambda} \\ -\underset{\sim}{\lambda}_r \end{bmatrix}$	$\begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{\delta} \\ \underset{\sim}{\delta}_r \end{bmatrix} = \begin{bmatrix} \bar{\underset{\sim}{B}}^T & \bar{\underset{\sim}{B}}'^T \\ \bar{\underset{\sim}{B}}_o^T & \bar{\underset{\sim}{B}}'^T_o \\ \bar{\underset{\sim}{B}}^T & \bar{\underset{\sim}{B}}'^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}' \\ \underset{\sim}{v}'_R \end{bmatrix}$	(4.2.52a,53a) (4.2.52b,53b) (4.2.53c)
MESH DESCRIPTION		

We will illustrate next the process of assemblage of Statics for the kinematically linear analysis of the simple structure shown in Fig.4.7(a); the description of Statics for large displacements, although lengthier, is conceptually identical.

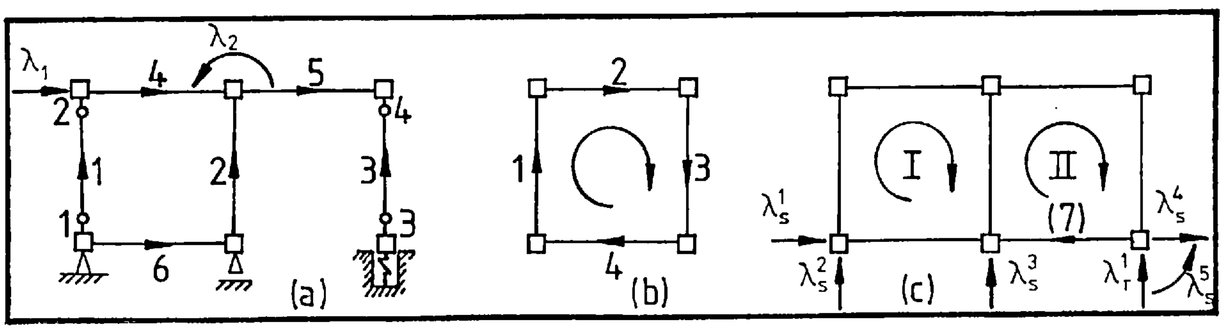


FIGURE 4.7

The frame has six members of length L which we orientate and number; members 1 and 3 have internal bending releases, numbered from 1 to 4. The loading and the kinematic boundary conditions are shown in Fig.4.7(a) and the forces developing at the fixed supports and at the external releases are numbered as indicated in Fig.4.7(c).

An auxiliary member, member 7, is added to the frame so that the equivalent structure, Fig.4.7(c), is formed by two square meshes each of which we number and orientate as in Fig. 4.7(b), in order to use the static influence coefficients derived in section 2.2 when studying the rectangular mesh shown in Fig. 2.6. For meshes with a different numbering sequence and/or geometry, the coefficients should be obtained from (2.2.38-42) and (2.4.23-26).

The elements of the block-diagonal matrices $\underline{b}$, $\underline{b}_o$ and $\underline{b}_R$ are respectively defined in (2.2.17) to (2.2.19), where we replace h bq L to conform with the mesh geometry. The elements of matrices $\underline{b}'$ and $\underline{b}'_o$ have to be derived from (2.4.23) and (2.4.25), respectively.

Stored in one array, in the manner indicated in the previous sub-section, the incidence matrices summarized in Table 4.4 were used to find, successively, matrices $\underline{S}$, $\underline{T}$, $\underline{I}'$ and $\underline{I}''$, respectively defined in (4.2.25a), (4.2.27) and (4.2.30a,b).

$\underline{J}$ ◇	1 2 3 10 11 12 -5 -4 6 -17 -16 18 4 5 6 13 14 15 -8 -7 9 • • •
$\underline{J}'$ ◇	• 1 2 3
$\underline{J}_R$ ◇	1 • • 2 • • 30 ZEROS -4 • • -3 • • • • • • • •
$\underline{J}_t$ ◇	10 11 12
$\underline{J}'_t$ ◇	• • • 1 2 3
$\underline{J}''_t$ ◇	1 2 3 • • •
$\underline{J}_o$ ◇	6 16
$\underline{J}_r$ ◇	23
$\underline{J}_s$ ◇	3 2 14 24 22

TABLE 4.4

Matrices $\bar{\underline{b}}$, $\bar{\underline{b}}_o$, $\bar{\underline{b}}_s$ and $\bar{\underline{b}}_r$, $\bar{\underline{b}}'_r$, $\bar{\underline{b}}'$, $\bar{\underline{b}}'_o$, $\bar{\underline{b}}'_s$ and $\bar{\underline{B}}'_R$, $\bar{\underline{b}}''$, $\bar{\underline{b}}''_o$, $\bar{\underline{b}}''_s$ and $\bar{\underline{b}}''_r$, and $\bar{\underline{b}}_r$, $\bar{\underline{b}}_{Rs}$ and $\bar{\underline{b}}_{Rr}$, respectively defined in (4.2.35l-o) and (4.2.35r-t) were then formed and grouped in Table 4.5 following the layout of the Statics description (4.2.34), in its linear version (i.e. $\bar{\underline{\pi}}=\underline{0}, \underline{\pi}'=\underline{0}$). The corresponding influence diagrams, presented in Fig.4.8, prove the equivalent cantilever, shown in the same figure, to be the adopted (statically determinate) basis for the analysis.

For simplicity of its physical interpretation, we selected the following $\alpha'=5$ variables $\bar{\underline{p}}$

$$\bar{\underline{p}}^T = \begin{bmatrix} p_1^1 & p_2^1 & p_3^1 & \lambda_s^5 & \lambda_s^4 \end{bmatrix}^T$$

which can be expressed in the form (4.2.45) by letting

$$\underline{I}_p = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \underline{I}_s = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

The solution (4.2.47) of the enlarged system (4.2.46) gives:

$$\underline{C} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \underline{C}_o = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \quad \underline{C}_r = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \underline{C}' = \begin{bmatrix} \cdot & \cdot & \cdot & & -1 \\ \cdot & \cdot & \cdot & \frac{1}{L} & \cdot \\ \cdot & \cdot & \cdot & -\frac{1}{L} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix} \quad \underline{C}'_o = \begin{bmatrix} -1 & \cdot \\ -1 & \frac{1}{L} \\ 1 & -\frac{1}{L} \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \quad \underline{C}'_r = \begin{bmatrix} \cdot \\ 1 \\ -2 \\ \cdot \\ \cdot \end{bmatrix}$$

which enable us to find matrices $\underline{\bar{B}}$, $\underline{\bar{B}}_o$, $\underline{\bar{B}}_r$, $\underline{\bar{B}}'$, $\underline{\bar{B}}'_o$ and $\underline{\bar{B}}'_r$, defined in (4.2.48). The new description of Statics is given in Table 4.6, following the layout of (4.2.52), and the corresponding influence diagrams are shown in Fig.4.9. Equalities (4.2.49) can be easily confirmed.

The concise description of finite Statics and Kinematics (4.2.59) and (4.2.60), presented in Table 4.7 was obtained by introducing the generalized variables defined in Table 4.1 into (4.2.50) and (4.2.51), respectively, and re-arranging the resulting system; the corresponding generalized mesh structural matrices are given in Table 4.8.

The incremental mesh descriptions of Statics and Kinematics, (4.2.63) and (4.2.64), shown in Table 4.9, can be derived either directly from the assembled finite descriptions (4.2.59) and (4.2.60) or by assembling their incremental descriptions (2.4.34) and (2.4.35) for the generic mesh sub-structure M.

Setting in (2.4.34) and (2.4.35) M = 1,2, ..., M' and grouping as in (4.1.6)

$$
\left[\begin{array}{c:c:c:c:c}
\underline{k}_{uu} & \underline{k}_{uv} & \underline{k}_{ur} & \underline{\mathbb{b}} & \underline{\mathbb{b}}_{o} \\ \hdashline
\underline{k}_{uv}^{T} & \underline{k}_{vv} & \underline{k}_{vr} & \underline{\mathbb{b}}' & \underline{\mathbb{b}}' \\ \hdashline
\underline{k}_{ur}^{T} & \underline{k}_{vr}^{T} & \underline{k}_{rr} & \cdot & \underline{\mathbb{b}}_{or} \\ \hdashline
\underline{\mathbb{b}}^{T} & \underline{\mathbb{b}}'^{T} & \cdot & \cdot & \cdot \\ \hdashline
\underline{\mathbb{b}}_{o}^{T} & \underline{\mathbb{b}}_{o}'^{T} & \underline{\mathbb{b}}_{or}^{T} & \cdot & \cdot
\end{array}\right]
\left[\begin{array}{c}
\Delta\underline{u}' \\ \hdashline
\Delta\underline{v}_{R} \\ \hdashline
\Delta\underline{r}'^{*} \\ \hdashline
\Delta\underline{p} \\ \hdashline
\Delta\underline{R}^{*}
\end{array}\right]
=
\left[\begin{array}{c}
\Delta\underline{\tilde{X}}' \\ \hdashline
\Delta\underline{\tilde{X}}_{R} \\ \hdashline
-\Delta\underline{R}'^{*} \\ \hdashline
\cdot \\ \hdashline
\Delta\underline{r}^{*}
\end{array}\right]
-
\left[\begin{array}{c}
\Delta\underline{\tilde{X}}_{\pi}' \\ \hdashline
\Delta\underline{\tilde{X}}_{R\pi} \\ \hdashline
\Delta\underline{R}_{\pi}'^{*} \\ \hdashline
\Delta\underline{\tilde{v}}_{\pi} \\ \hdashline
\Delta\underline{r}_{\pi}^{*}
\end{array}\right]
\qquad
\begin{array}{r}
(4.2.54a) \\ (4.2.54b) \\ (4.2.54c) \\ (4.2.55a) \\ (4.2.55b)
\end{array}
$$

1	•	•	•	•	•	-L	1	•	•	L	•	1	2L
1	•	L	•	•	•	•	1	•	•	L	L	1	2L
•	-1	•	•	•	•	•	•	•	•	-1	•	•	-1
-1	L	•	1	•	•	•	•	•	•	•	•	•	•
-1	L	-L	1	•	L	•	•	•	•	•	•	•	•
•	1	•	•	-1	•	•	•	•	•	1	•	•	-1
•	•	•	-1	L	•	•	•	•	•	•	•	-1	•
•	•	•	-1	L	-L	•	•	•	•	•	-L	-1	•
•	•	•	•	1	•	•	•	•	•	•	•	•	1
1	•	L	•	•	•	•	1	•	•	L	L	1	2L
1	-L	L	•	•	•	•	1	•	•	•	L	1	L
•	•	-1	•	•	•	•	•	•	•	•	-1	•	•
•	•	•	1	•	L	•	•	•	•	•	L	1	L
•	•	•	1	-L	L	•	•	•	•	•	L	1	•
•	•	•	•	•	-1	•	•	•	•	•	-1	•	•
-1	•	•	•	•	•	•	•	•	•	•	•	•	•
-1	L	•	•	•	•	•	•	•	•	•	•	•	•
•	•	1	•	•	•	•	•	•	•	•	•	•	•
-1	•	•	•	•	•	L	-1	•	•	-L	•	-1	-2L
-1	•	-L	•	•	•	•	-1	•	•	-L	-L	-1	-2L
•	•	•	1	-L	•	•	•	•	•	•	•	1	•
•	•	•	1	-L	L	•	•	•	•	•	L	1	•
•	•	•	1	-L	•	•	•	•	•	•	•	•	•
•	•	•	1	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	1	•	•	•	•	•	•	•	•
•	•	•	•	•	•	-L	1	•	•	L	•	1	2L
•	•	•	•	•	•	•	•	•	1	1	•	•	1
•	•	•	•	•	•	1	•	1	•	•	1	•	•

TABLE 4.5

1	•	•	•	•	•	•	•
1	•	L	•	L	L	•	•
•	-1	•	1/L	•	-1	1/L	1
-1	L	•	•	•	•	•	•
-1	L	-L	•	•	•	•	•
•	1	•	-1/L	•	1	-1/L	-2
•	•	•	-1	•	•	•	•
•	•	•	-1	-L	•	•	•
•	•	•	•	•	•	•	1
1	•	L	•	L	L	•	•
1	-L	L	1	L	•	1	L
•	•	-1	•	-1	•	•	•
•	•	•	1	L	•	•	L
•	•	•	1	L	•	•	•
•	•	•	•	-1	•	•	•
-1	•	•	•	•	•	•	•
-1	L	•	•	•	•	•	•
•	•	1	•	•	•	•	•
-1	•	•	•	•	•	•	•
-1	•	-L	•	-L	-L	•	•
•	•	•	1	•	•	•	•
•	•	•	1	L	•	•	•

TABLE 4.6

Subjecting system (4.2.54) to a treatment very similar to the one used to assemble the corresponding finite relations (4.2.21) and (4.2.36), the following incremental description of Statics and Kinematics is found:

$$
\left[\begin{array}{cccc|cccc}
\bar{\mathbb{K}}_{uu} & \bar{\mathbb{K}}_{ur} & \bar{\mathbb{K}}_{ua} & \bar{\mathbb{K}}_{uf} & \bar{\mathbb{B}} & \bar{\mathbb{B}}_{o} & \bar{\mathbb{B}}_{s} & \bar{\mathbb{B}}_{r} \\
\bar{\mathbb{K}}_{uv}^{T} & \bar{\mathbb{K}}_{vv} & \bar{\mathbb{K}}_{va} & \bar{\mathbb{K}}_{vf} & \bar{\mathbb{B}}' & \bar{\mathbb{B}}'_{o} & \bar{\mathbb{B}}'_{s} & \bar{\mathbb{B}}'_{r} \\
\bar{\mathbb{K}}_{ua}^{T} & \bar{\mathbb{K}}_{va}^{T} & \bar{\mathbb{K}}_{aa} & \bar{\mathbb{K}}_{af} & \bar{\mathbb{B}}'' & \bar{\mathbb{B}}''_{o} & \bar{\mathbb{B}}''_{s} & \bar{\mathbb{B}}''_{r} \\
\bar{\mathbb{K}}_{uf}^{T} & \bar{\mathbb{K}}_{vf}^{T} & \bar{\mathbb{K}}_{af}^{T} & \bar{\mathbb{K}}_{ff} & \cdot & \bar{\mathbb{B}}_{R} & \bar{\mathbb{B}}_{Rs} & \bar{\mathbb{B}}_{Rv} \\
\hline
\bar{\mathbb{B}}^{T} & \bar{\mathbb{B}}'^{T} & \bar{\mathbb{B}}''^{T} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\bar{\mathbb{B}}_{o}^{T} & \bar{\mathbb{B}}_{o}'^{T} & \bar{\mathbb{B}}_{o}''^{T} & \bar{\mathbb{B}}_{R} & \cdot & \cdot & \cdot & \cdot \\
\bar{\mathbb{B}}^{T} & \bar{\mathbb{B}}'^{T} & \bar{\mathbb{B}}''^{T} & \bar{\mathbb{B}}^{T} & \cdot & \cdot & \cdot & \cdot \\
\bar{\mathbb{B}}_{r}^{T} & \bar{\mathbb{B}}_{r}'^{T} & \bar{\mathbb{B}}_{r}''^{T} & \bar{\mathbb{B}}_{Rr}^{T} & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\left[\begin{array}{c}
\Delta \underline{u}' \\ \Delta \underline{v}'_{R} \\ \Delta \underline{u}'_{a} \\ \Delta \underline{\delta}_{f} \\ \Delta \tilde{\underline{p}} \\ \Delta \underline{\lambda} \\ \Delta \underline{\lambda}_{s} \\ \Delta \underline{\lambda}_{r}
\end{array}\right]
=
\left[\begin{array}{c}
\Delta \underline{X}' \\ \Delta \underline{X}'_{R} \\ \cdot \\ \cdot \\ \cdot \\ \Delta \underline{\delta} \\ \cdot \\ \Delta \underline{\delta}_{r}
\end{array}\right]
-
\left[\begin{array}{c}
\Delta \bar{\underline{R}}_{X\pi} \\ \Delta \bar{\underline{R}}_{R\pi} \\ \Delta \bar{\underline{R}}_{a\pi} \\ \Delta \bar{\underline{R}}_{f\pi} \\ \Delta \bar{\underline{R}}_{v\pi} \\ \Delta \bar{\underline{R}}_{o\pi} \\ \Delta \bar{\underline{R}}_{s\pi} \\ \Delta \bar{\underline{R}}_{r\pi}
\end{array}\right]
\qquad
\begin{array}{r}
(4.2.56a) \\ (4.2.56b) \\ (4.2.56c) \\ (4.2.56d) \\ (4.2.57a) \\ (4.2.57b) \\ (4.2.57c) \\ (4.2.57d)
\end{array}
$$

Matrix $\bar{\mathbb{K}}$, present in (4.2.56), and representing the Statics dependence on Kinematics is defined by

$$\bar{\mathbb{K}} = \tilde{\underline{J}}\ \underline{I}\ \underline{k}\ \underline{I}^{T}\tilde{\underline{J}}^{T} \qquad (4.2.58a)$$

where $\underline{k}$ is its correspondent in the unassembled description (4.2.54); matrix $\bar{\mathbb{B}}$ relating the equilibrated static variables in (4.2.56) is defined by

$$\bar{\mathbb{B}} = \tilde{\underline{J}}\ \bar{\mathbb{b}}\ \tilde{\underline{J}}_{o} \qquad (4.2.58b)$$

In (4.2.58) we note

$$
\underline{J} = \left[\begin{array}{c|c|c}
\underline{J} & \cdot & \cdot \\ \hline
\cdot & \underline{J}_{R} & \cdot \\ \hline
\underline{J}' & \cdot & \cdot \\ \hline
\cdot & \cdot & \underline{I}
\end{array}\right]
\quad
\underline{I} = \left[\begin{array}{c|c|c}
\underline{I} & \cdot & \mathbb{b}_{o}\underline{J}'_{*} \\ \hline
\cdot & \underline{I} & \mathbb{b}'_{o}\underline{J}'_{*} \\ \hline
\cdot & \cdot & \underline{J}''_{*}
\end{array}\right]
\quad
\underline{b} = \left[\begin{array}{c|c}
\mathbb{b} & \mathbb{b}_{o}\underline{I}' \\ \hline
\mathbb{b}' & \mathbb{b}'_{o}\underline{I}' \\ \hline
\cdot & \underline{J}''_{*}\mathbb{b}_{or}
\end{array}\right]
\quad
\underline{J}_{o}^{T} = \left[\begin{array}{c|c}
\underline{I} & \cdot \\ \hline
\cdot & \underline{J}_{o}^{T} \\ \hline
\cdot & \underline{J}_{s}^{T} \\ \hline
\cdot & \underline{J}_{r}^{T}
\end{array}\right]
$$

where $\quad \underline{J}'_{*} = \underline{J}_{t}\underline{J}'_{t}\underline{I} \quad$ and $\quad \underline{J}''_{*} = \underline{J}''_{t}\ \underline{I}$

Matrices $\underline{I}$ and $\underline{I}'$ are defined in (4.2.27) and (4.2.30a), respectively. The static and kinematic non-linear residuals present in (4.2.56) and (4.2.57) are given by:

$$\tilde{\underline{J}}\left[\begin{array}{c}(\underline{k}_{ur}+\underline{\mathbb{b}}_o\underline{J}'_*\underline{k}_{rr})\underline{J}'^T_*\,\Delta\underline{r}^*_\pi + \underline{\mathbb{b}}_o\underline{J}'_*\,\Delta\underline{R}'^*_\pi + \Delta\tilde{\underline{X}}'_\pi \\ \hline (\underline{k}_{vr}+\underline{\mathbb{b}}'_o\underline{J}'_*\underline{k}_{rr})\underline{J}'^T_*\,\Delta\underline{r}^*_\pi + \underline{\mathbb{b}}'_o\underline{J}'_*\,\Delta\underline{R}'^*_\pi + \Delta\tilde{\underline{X}}_{R\pi} \\ \hline \underline{J}''_*\underline{k}_{rr}\,\underline{J}'^T_*\,\Delta\underline{r}^*_\pi + \underline{J}''_*\,\Delta\underline{R}'^*_\pi\end{array}\right] \quad \text{and} \quad \tilde{\underline{J}}^T_o\left[\begin{array}{c}\Delta\tilde{\underline{v}}_\pi \\ \hline \underline{I}'^T\,\Delta\underline{r}^*_\pi\end{array}\right]$$

respectively.

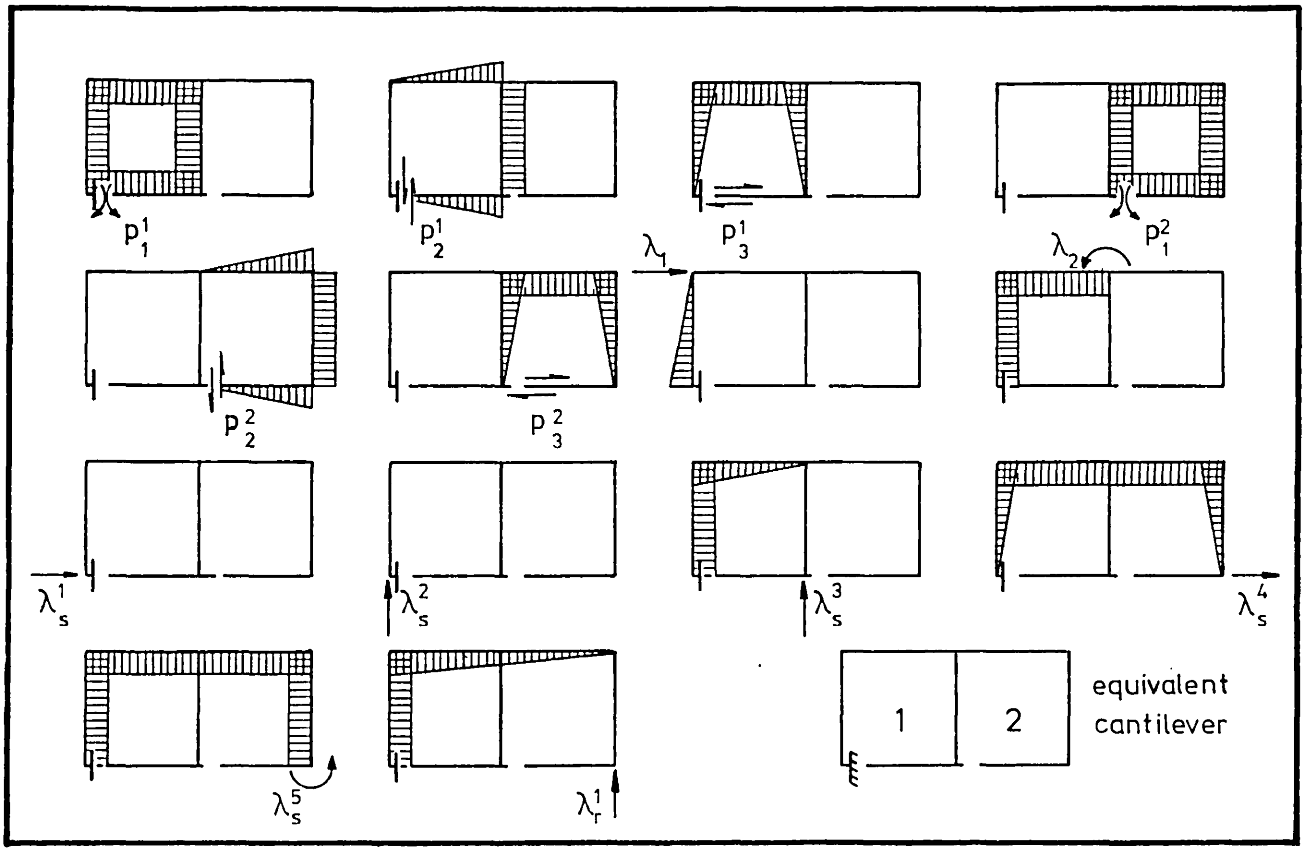


FIGURE 4.8

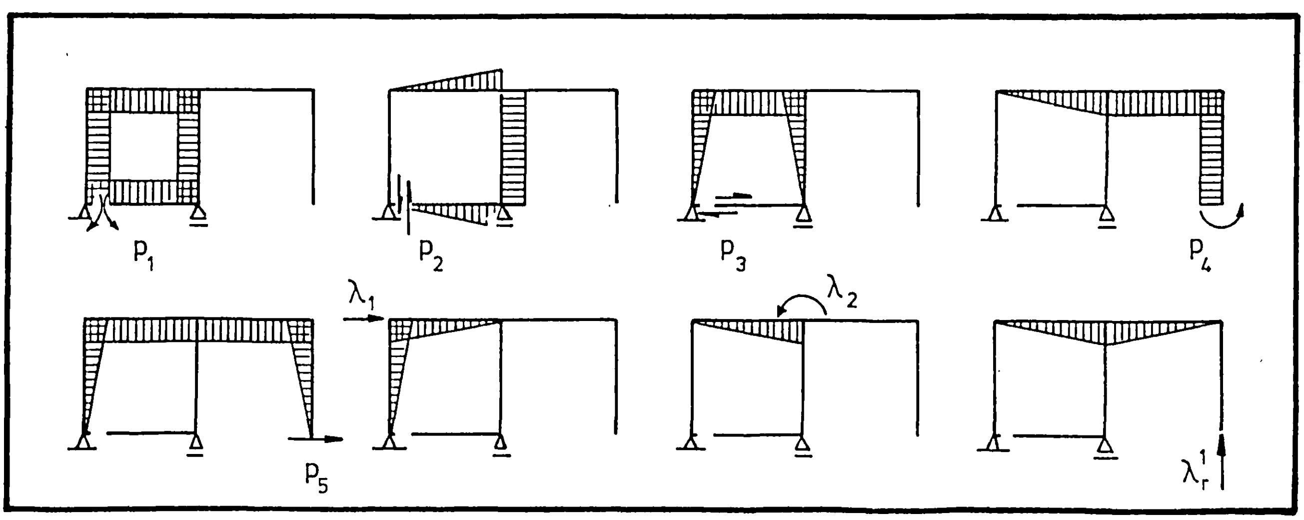


FIGURE 4.9

Using the equilibrium equation (4.2.56c) and (4.2.56d) and introducing α' indeterminate forces $\bar{\underline{p}}$, obtained by taking increments in (4.2.45), system (4.2.56-57) can be simplified through a procedure similar, although more cumbersome now, to the one adopted in the treatment of their finite descriptions (4.2.34) and (4.2.43), yielding the following mesh description of Statics and Kinematics:

$$\left[\begin{array}{c|c|c|c|c} \mathbb{K}_{uu} & \mathbb{K}_{uv} & \mathbb{B} & \mathbb{B}_o & \mathbb{B}_r \\ \hline \mathbb{K}_{uv}^T & \mathbb{K}_{uv} & \mathbb{B}' & \mathbb{B}'_o & \mathbb{B}'_r \\ \hline \mathbb{B}^T & \mathbb{B}'^T & \cdot & \cdot & \cdot \\ \hline \mathbb{B}_o^T & \mathbb{B}'^T_o & \cdot & \cdot & \cdot \\ \hline \mathbb{B}_r^T & \mathbb{B}'^T_r & \cdot & \cdot & \cdot \end{array}\right] \left[\begin{array}{c} \Delta \underline{u}' \\ \hline \Delta \underline{v}'_R \\ \hline \Delta \bar{\underline{p}} \\ \hline \Delta \underline{\lambda} \\ \hline -\Delta \underline{\lambda}_r \end{array}\right] = \left[\begin{array}{c} \Delta \underline{X}' \\ \hline \Delta \underline{X}'_R \\ \hline \cdot \\ \hline \Delta \underline{\delta} \\ \hline \Delta \underline{\delta}_r \end{array}\right] - \left[\begin{array}{c} \Delta \underline{R}_{X\pi} \\ \hline \Delta \underline{R}_{R\pi} \\ \hline \Delta \underline{R}_{v\pi} \\ \hline \Delta \underline{R}_{\delta\pi} \\ \hline \Delta \underline{R}_{r\pi} \end{array}\right]$$

Introducing above the generalized variables in incremental form and re-arranging the resulting system, the concise description (4.2.63) and (4.2.64), presented in Table 4.9, of incremental Statics and Kinematics is thus obtained.

An alternative and apparently much simpler way of deriving the incremental equilibrium and compatibility conditions (4.2.63) and (4.2.64) is to take increments in the finite description (4.2.59-60) and eliminate next the generalized incremental additional forces and deformations through equations (4.2.20). The corresponding structural matrices are given in Table 4.10 which reveals the necessity of inverting matrix $[\underline{I}-\mathbb{Q}^T\underline{B}_\pi]$, similarly to what happened when deriving the incremental relations (2.2.56-57) regulating the behaviour of the generalized mesh sub-structure M. As shown in sub-section 2.2.3, matrix

$$\bar{\underline{B}}_{\pi_M} = (\underline{I} - \underline{Q}_M^T \, \underline{B}_{o\pi_M})^{-1}$$

and consequently the structural matrix

$$\bar{\mathbb{B}}_\pi = (\underline{I} - \mathbb{Q}^T \, \underline{B}_\pi)^{-1}$$

is a block-diagonal matrix for generalized cantilevers; hence care should be taken when identifying the structure constituent mesh sub-structures, and their pattern of incidence, so that matrix $\bar{\mathbb{B}}_\pi$

can be derived with as little effort as possible.

If the two processes of deriving the incremental mesh description of Statics and Kinematics (4.2.63) and (4.2.64) mentioned in the above are rejected, the first due to the necessity of following the lengthy process of mesh assemblage at every new increment, the second due to the compulsive matrix inversion, a third and last possibility exists;it consists in, after taking increments in the finite descriptions (4.2.59) and (4.2.60), eliminating the generalized additional deformations through (4.2.20b) and to relate in Statics the additional forces with their displacements through (4.2.20a). The resulting system would be similar to (2.2.45-46) found in sub-section 2.2.3 for the generalized mesh substructure M; the matrix inversion is thus avoided at the cost of a formulation involving additional, and unwelcome , variables, the dual and dependent variables $\Delta\bar{\underset{\sim}{u}}$ and $\Delta\underset{\sim}{\delta}_{\pi}$.

The nodal [mesh] perturbation analysis description of Statics and Kinematics, (4.2.69) and (4.2.70)[(4.2.67) and (4.2.68)] respectively, is obtained by replacing the incremental generalized variables, as well as the residuals, in a power series, as defined in (2.1.52), and equating next the terms affected by the same power of the perturbation parameter ε. The original system , non-linear in the residuals $\Delta\underset{\sim}{\mathbb{R}}$, is thus replaced by an infinite set of recursive, linear systems.

The nodal [mesh] asymptotic description is obtained by expanding the total variables in the finite description of Statics and Kinematics (4.2.61) and (4.2.63) [(4.2.59) and (4.2.60)] in a power series of the form (2.1.63) and equating next the same order terms. The resulting recursive linear systems are formally identical to (4.2.69) and (4.2.70) [(4.2.67) and (4.2.68)] although the structural matrices differ quantitatively; matrix $\mathbb{Q}$, in particular, is now a null matrix which is particularly relevant in the mesh formulation since matrix $\underset{\sim}{\mathbb{B}}_{\pi}$ becomes the identity matrix.

Similar results would be obtained by assembling the nodal [mesh] substructure static and kinematic asymptotic descriptions.

DEFORMATION ANALYSIS	
STATICS	
MESH	NODAL
$\mathbf{X} = \begin{bmatrix} \mathbf{B} & \mathbf{B}_o & \mathbf{B}_\pi \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \boldsymbol{\lambda} \\ \boldsymbol{\pi} \end{bmatrix}$ (4.2.59)	$\mathbf{0} = \begin{bmatrix} \mathbf{A}^T & \mathbf{A}_o^T & \mathbf{A}_\pi^T \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ -\boldsymbol{\lambda} \\ -\boldsymbol{\pi} \end{bmatrix}$ (4.2.61)
KINEMATICS	
MESH	NODAL
(4.2.60a) (4.2.60b) (4.2.60c) $\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\delta} \\ \boldsymbol{\delta}_\pi \end{bmatrix} = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{B}_o^T \\ \mathbf{B}_\pi^T \end{bmatrix} (\mathbf{u} + \mathbf{u}_\pi)$	$\begin{bmatrix} \mathbf{u} + \mathbf{u}_\pi \\ \boldsymbol{\delta} \\ \boldsymbol{\delta}_\pi \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}_o \\ \mathbf{A}_\pi \end{bmatrix} \mathbf{q}$ (4.2.62a) (4.2.62b) (4.2.62c)
ELASTICITY	
FLEXIBILITY	STIFFNESS
$\mathbf{u}_E = \mathbf{F}\,\mathbf{X} + \mathbf{u}_{E\pi}$ $(\mathbf{u}_{E\pi} = -\mathbf{F}\mathbf{X}_{E\pi})$ (4.1.22)	$\mathbf{X} = \mathbf{K}\,\mathbf{u}_E + \mathbf{X}_{E\pi}$ $(\mathbf{X}_{E\pi} = -\mathbf{K}\mathbf{u}_{E\pi})$ (4.1.23)
PLASTICITY	

(4.1.24)
(4.1.25)

$$\begin{bmatrix} -\mathbf{H} & \mathbf{N}^T \\ \mathbf{N} & \cdot \end{bmatrix} \begin{bmatrix} \mathbf{u}_* \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_* \\ \mathbf{u}_p \end{bmatrix} + \begin{bmatrix} \mathbf{X}_* \\ \cdot \end{bmatrix} + \begin{bmatrix} \boldsymbol{\pi}_\varphi \\ \mathbf{u}_\varphi \end{bmatrix}$$

$\boldsymbol{\Phi}_* \leq \mathbf{0}$ (4.1.26)	$\boldsymbol{\Phi}_*^T \mathbf{u}_* = 0$ (4.1.27)	$\mathbf{u}_* \geq \mathbf{0}$ (4.1.28)
YIELD RULE	ASSOCIATION (R.P.Y.)	FLOW RULE

SYMMETRIC MATRICES: $\mathbf{F}$, $\mathbf{K}$, $\mathbf{H}$

TABLE 4.7

	GENERALIZED STRUCTURAL MATRICES		
MESH	$\mathbf{B} = \begin{bmatrix} \bar{B} & \bar{B} \\ \bar{B}' & \bar{B}'_r \\ \cdot & -\mathbf{I} \end{bmatrix}$ $\mathbf{B}_\pi = \begin{bmatrix} \bar{B}_\pi & \cdot \\ \bar{B}'_\pi & \bar{B}'_R \\ \cdot & \cdot \end{bmatrix}$ $\mathbf{B}_o = \begin{bmatrix} \bar{B}_o \\ \bar{B}'_o \\ \cdot \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} \mathbf{F}_u & \cdot & \cdot \\ \cdot & \mathbf{F}_R & \cdot \\ \cdot & \cdot & \mathbf{F}_r \end{bmatrix}$ $\mathbf{K} = \begin{bmatrix} \mathbf{K}_u & \cdot & \cdot \\ \cdot & \mathbf{K}_R & \cdot \\ \cdot & \cdot & \mathbf{K}_r \end{bmatrix}$	ELASTICITY
NODAL	$\mathbf{A} = \begin{bmatrix} \bar{A} & \bar{A}' \\ \cdot & \mathbf{I} \\ \bar{A}_r & \cdot \end{bmatrix}$ $\mathbf{A}_\pi = \begin{bmatrix} \bar{A}_\pi & \cdot \\ \bar{A}_R & \bar{A}'_R \end{bmatrix}$ $\mathbf{A}_o^T = \begin{bmatrix} \bar{A}_o^T \\ \cdot \end{bmatrix}$	$\mathbf{H} = \begin{bmatrix} \mathbf{H}_u & \cdot & \cdot \\ \cdot & \mathbf{H}_R & \cdot \\ \cdot & \cdot & \mathbf{H}_r \end{bmatrix}$ $\mathbf{N} = \begin{bmatrix} \mathbf{N}_u & \cdot & \cdot \\ \cdot & \mathbf{N}_R & \cdot \\ \cdot & \cdot & \mathbf{N}_r \end{bmatrix}$	PLASTICITY

TABLE 4.8

INCREMENTAL ANALYSIS		
STATICS		
MESH	NODAL	
$-\mathbb{R}'_{\pi} - \mathbb{K}_m \Delta u + \Delta X = \begin{bmatrix}\mathbb{B} & \mathbb{B}_o\end{bmatrix} \begin{bmatrix}\Delta p \\ \Delta \lambda\end{bmatrix}$ (4.2.63)	$\mathbb{R}'_{\pi} + \mathbb{K}_n \Delta q = \begin{bmatrix}\mathbb{A}^T & \mathbb{A}_o^T\end{bmatrix} \begin{bmatrix}\Delta X \\ -\Delta \lambda\end{bmatrix}$ (4.2.65)	
KINEMATICS		
MESH	NODAL	
(4.2.64a) (4.2.64b) $-\begin{bmatrix}\mathbb{R}'_{v\pi} \\ \mathbb{R}'_{\delta\pi}\end{bmatrix} + \begin{bmatrix}\cdot \\ \Delta\delta\end{bmatrix} = \begin{bmatrix}\mathbb{B}^T \\ \mathbb{B}_o^T\end{bmatrix} \Delta u$	$\begin{bmatrix}\mathbb{R}_{u\pi} \\ \cdot\end{bmatrix} + \begin{bmatrix}\Delta u \\ \Delta\delta\end{bmatrix} = \begin{bmatrix}\mathbb{A} \\ \mathbb{A}_o\end{bmatrix} \Delta q$ (4.2.66a) (4.2.66b)	
ELASTICITY		
FLEXIBILITY	STIFFNESS	
(4.1.29) $\Delta u_E = F\, \Delta X + \mathbb{R}_{uE}$ $(\mathbb{R}_{uE} = -\mathbb{F}\ \mathbb{R}_{XE})$	$\Delta X = K\, \Delta u_E + \mathbb{R}_{XE}$ (4.1.30) $(\mathbb{R}_{XE} = -\mathbb{K}\ \mathbb{R}_{uE})$	
PLASTICITY		
(4.1.31) (4.1.32) $\begin{bmatrix}-\mathbb{H} & \mathbb{N}^T \\ \mathbb{N} & \cdot\end{bmatrix} \begin{bmatrix}\Delta u'_* \\ \Delta X\end{bmatrix} = \begin{bmatrix}\Delta \Phi'_* \\ \Delta u_p\end{bmatrix} + \begin{bmatrix}\mathbb{R}_{\varphi} \\ \mathbb{R}_p\end{bmatrix}$		
$\Delta \Phi'_* \leq 0$ (4.1.33)	$\Delta \Phi'^{T}_* \Delta u'_* = 0$ (4.1.34)	$\Delta u_* \geq 0$ (4.1.35)
YIELD RULE	ASSOCIATION	FLOW RULE

SYMMETRIC MATRICES: $\mathbb{K}_n$, $\mathbb{K}_m$, $\mathbb{F}$, $\mathbb{K}$, $\mathbb{H}$

TABLE 4.9

	GENERALIZED STRUCTURAL MATRICES		
MESH	$\bar{\mathbb{B}}_{\pi} = (I - \mathbb{Q}^T B_{\pi})^{-1}$, $I'_{\pi} = I + B_{\pi} \bar{\mathbb{B}}_{\pi} \mathbb{Q}^T_{\pi}$ $\mathbb{B} = I'_{\pi} B$, $\mathbb{B}_o = I'_{\pi} B_o$, $\mathbb{K}_m = B_{\pi} \bar{\mathbb{B}}_{\pi} \mathbb{P} \bar{\mathbb{B}}^T_{\pi} B^T_{\pi}$ $\mathbb{R}'_{\pi} = B_{\pi} \bar{\mathbb{B}}_{\pi}\ \mathbb{R}_{\pi} + \mathbb{K}_m\ \mathbb{R}_{u\pi}$ $\mathbb{R}'_{v\pi} = \mathbb{B}^T\ \mathbb{R}_{u\pi}$, $\mathbb{R}'_{\delta\pi} = \mathbb{B}_o^T\ \mathbb{R}_{u\pi}$	$\mathbb{F} = \begin{bmatrix}\mathbb{F}_u & \cdot & \cdot \\ \cdot & \mathbb{F}_R & \cdot \\ \cdot & \cdot & \mathbb{F}_r\end{bmatrix}$ $\mathbb{K} = \begin{bmatrix}\mathbb{K}_u & \cdot & \cdot \\ \cdot & \mathbb{K}_R & \cdot \\ \cdot & \cdot & \mathbb{K}_r\end{bmatrix}$ FLEXIBILITY STIFFNESS	ELASTICITY
NODAL	$\mathbb{A} = A - \mathbb{Q}\ A_{\pi}$ $\mathbb{A}_o = A_o$ $\mathbb{K}_n = A^T_{\pi}\ \mathbb{P}\ A_{\pi}$ $\mathbb{R}'_{\pi} = A^T_{\pi}\ \mathbb{R}_{\pi}$	$\mathbb{H} = \begin{bmatrix}\mathbb{H}'_u & \cdot & \cdot \\ \cdot & \mathbb{H}'_R & \cdot \\ \cdot & \cdot & \mathbb{H}'_r\end{bmatrix}$ $\mathbb{N} = \begin{bmatrix}\mathbb{N}'_u & \cdot & \cdot \\ \cdot & \mathbb{N}'_R & \cdot \\ \cdot & \cdot & \mathbb{N}'_r\end{bmatrix}$ HARDENING NORMALITY	PLASTICITY

TABLE 4.10

<table>
<tr><th colspan="6">PERTURBATION ANALYSIS</th></tr>
<tr><th colspan="6">STATICS</th></tr>
<tr><th colspan="3">MESH</th><th colspan="3">NODAL</th></tr>
<tr><td colspan="3">$-\mathbb{R}'_{\pi_i} - \mathbb{K}_m u_i + X_i = \begin{bmatrix} B & B_o \end{bmatrix} \begin{bmatrix} p \\ \lambda \end{bmatrix}_i$
(4.2.67)</td><td colspan="3">$\mathbb{R}'_{\pi_i} + \mathbb{K}_n q_i = \begin{bmatrix} \mathbb{A}^T & \mathbb{A}_o^T \end{bmatrix} \begin{bmatrix} X \\ -\lambda \end{bmatrix}_i$
(4.2.69)</td></tr>
<tr><th colspan="6">KINEMATICS</th></tr>
<tr><th colspan="3">MESH</th><th colspan="3">NODAL</th></tr>
<tr><td colspan="3">(4.2.68a)
(4.2.68b) $-\begin{bmatrix} \mathbb{R}'_{u\pi} \\ \mathbb{R}'_{\delta\pi} \end{bmatrix}_i + \begin{bmatrix} \cdot \\ \delta_i \end{bmatrix} = \begin{bmatrix} \mathbb{B}^T \\ \mathbb{B}_o^T \end{bmatrix} u_i$</td><td colspan="3">$\begin{bmatrix} \mathbb{R}_{u\pi} \\ \cdot \end{bmatrix}_i + \begin{bmatrix} u \\ \delta \end{bmatrix}_i = \begin{bmatrix} \mathbb{A} \\ \mathbb{A}_o \end{bmatrix} q_i$ (4.2.70a)
(4.2.70b)</td></tr>
<tr><th colspan="6">ELASTICITY</th></tr>
<tr><th colspan="3">FLEXIBILITY</th><th colspan="3">STIFFNESS</th></tr>
<tr><td colspan="3">(4.1.36) $u_{E_i} = \mathbb{F} X_i + \mathbb{R}_{uE_i}$
$(\mathbb{R}_{uE_i} = -\mathbb{F}\, \mathbb{R}_{XE_i})$</td><td colspan="3">$X_i = \mathbb{K}\, u_{E_i} + \mathbb{R}_{XE_i}$ (4.1.37)
$(\mathbb{R}_{XE_i} = -\mathbb{K}\, \mathbb{R}_{uE_i})$</td></tr>
<tr><th colspan="6">PLASTICITY</th></tr>
<tr><td colspan="6">(4.1.38)
(4.1.39) $\begin{bmatrix} -\mathbb{H} & \mathbb{N}^T \\ \mathbb{N} & \cdot \end{bmatrix} \begin{bmatrix} u_* \\ X \end{bmatrix}_i = \begin{bmatrix} \Phi_* \\ u_p \end{bmatrix}_i + \begin{bmatrix} \mathbb{R}_\varphi \\ \mathbb{R}_p \end{bmatrix}_i$</td></tr>
<tr><td colspan="2">$\Phi_{*_i} \leqslant 0$
(4.1.40)</td><td colspan="2">$\Phi_{*_i}^T u_{*_i} = 0$
(4.1.41)</td><td colspan="2">$u_{*_i} \geqslant 0$
(4.1.42)</td></tr>
<tr><th colspan="2">YIELD RULE</th><th colspan="2">ASSOCIATION</th><th colspan="2">FLOW RULE</th></tr>
</table>

SYMMETRIC MATRICES: $\mathbb{K}_n$, $\mathbb{K}_m$, $\mathbb{F}$, $\mathbb{K}$, $\mathbb{H}$

TABLE 4.11

4.2.3 Static-Kinematic Duality

The treatment to which the exact explicitly non-linear descriptions of Statics and Kinematics were subject in sections 2.1 and 2.2 was aimed at finding new, artificial variables so that the study of the conditions of equilibrium and compatibility of the nodal and mesh substructures could be performed on the undeformed and undisplaced substructure.

After assembling the constituent substructures, the nodal and mesh descriptions of the structure equilibrium and compatibility conditions remained linear, although still exact, and Static-Kinematic Duality was secured; the additional forces and deformations, enabled us to simulate the non-linear behaviour by analysing the structure on its original state.

The results presented in section 2.5, wherein the Principle of Virtual Work, for the unassembled substructure, was interpreted as the variational description of Static-Kinematic Duality, can easily be extended to the system relations. To avoid profitless repetitions we will restrict the derivation of the Principle of Virtual Work from the nodal descriptions of Statics and Kinematics; for simplicity of the presentation generalized variables will be used throughout.

The PRINCIPLE OF VIRTUAL WORK in finite mechanics is recovered by performing the internal product of (4.2.61) and (4.2.62), yielding:

$$\boxed{\underset{\sim}{\lambda}^T \underset{\sim}{\delta} = \underset{\sim}{\chi}^T (\underset{\sim}{u}+\underset{\sim}{u}_\pi) - \underset{\sim}{\pi}^T \underset{\sim}{\delta}_\pi} \qquad (4.2.71)$$

The same principle can be expressed in terms of (finite) incremental variables by multiplying internally the incremental forms of (4.2.65) and (4.2.66):

$$\boxed{\Delta\underset{\sim}{\lambda}^T \Delta\underset{\sim}{\delta} = \Delta\underset{\sim}{\chi}^T (\Delta\underset{\sim}{u}+\Delta\underset{\sim}{u}_\pi) - \Delta\underset{\sim}{\pi}^T \Delta\underset{\sim}{\delta}_\pi} \qquad (4.2.72)$$

The PRINCIPLE OF VIRTUAL FORCES is defined by the internal product of (4.2.62) and the incremental version of (4.2.61)

$$\boxed{\underset{\sim}{\delta}^T \Delta\underset{\sim}{\lambda} = (\underset{\sim}{u}+\underset{\sim}{u}_\pi)^T \Delta\underset{\sim}{\chi} - \underset{\sim}{\delta}_\pi^T \Delta\underset{\sim}{\pi}} \qquad (4.2.73)$$

and the PRINCIPLE OF VIRTUAL DISPLACEMENTS by the internal product of (4.2.61) and the incremental form of (4.2.62):

$$\boxed{\underset{\sim}{\lambda}^T \Delta\underset{\sim}{\delta} = \underset{\sim}{X}^T (\Delta\underset{\sim}{u}+\Delta\underset{\sim}{u}_{\pi}) - \underset{\sim}{\pi}^T \Delta\underset{\sim}{\delta}_{\pi}} \qquad (4.2.74)$$

We may therefore define the INCREMENTAL WORK and the COMPLEMENTARY WORK respectively as

$$\Delta W = \underset{\sim}{\lambda}^T \Delta\underset{\sim}{\delta} \qquad (4.2.75)$$

and

$$\Delta W^* = \underset{\sim}{\delta}^T \Delta\underset{\sim}{\lambda} \qquad (4.2.76)$$

and the STRAIN ENERGY and the COMPLEMENTARY STRAIN ENERGY as

$$\Delta U = \underset{\sim}{X}^T (\Delta\underset{\sim}{u}+\Delta\underset{\sim}{u}_{\pi}) - \underset{\sim}{\pi}^T \Delta\underset{\sim}{\delta}_{\pi} \qquad (4.2.77)$$

and

$$\Delta U^* = (\underset{\sim}{u}+\underset{\sim}{u}_{\pi})^T \Delta\underset{\sim}{X} - \underset{\sim}{\delta}_{\pi}^T \Delta\underset{\sim}{\pi} \qquad (4.2.78)$$

In section 2.5, considering the typical mesh and nodal substructures, we have shown how the Principle of Virtual Displacements, but not that of the Virtual Forces, could be exclusively expressed in terms of the fundamental static and kinematic variables. We will now extend the conclusion for the assembled structure. However, instead of following the strictly first-principle based method adopted in section 2.5, we will use now a simpler method based on equation (4.2.20b).

Let us then eliminate, through (4.2.20b), the generalized additional deformations $\Delta\underset{\sim}{u}_{\pi}$ in the description (4.2.74) of the Principle of Virtual Displacements, yielding, after simple regrouping:

$$\underset{\sim}{\lambda}^T \Delta\underset{\sim}{\delta} = \underset{\sim}{X}^T (\Delta\underset{\sim}{u}+\Delta\underset{\sim}{\mathbb{R}}_{u\pi}) - \Delta\underset{\sim}{\delta}^T (\underset{\sim}{\pi}-\underset{\sim}{\mathbb{Q}}^T\underset{\sim}{X})$$

Substituting above the definition of the generalized additional forces

$$\underset{\sim}{\pi} = \underset{\sim}{\mathbb{Q}}^T \underset{\sim}{X} \qquad (4.2.79)$$

which corresponds to the generalization of (2.5.13), and then of (2.1.20), to include the additional forces due to the internal release effects, the PRINCIPLE OF VIRTUAL DISPLACEMENTS reduces to

$$\underset{\sim}{\lambda}^T \Delta\underset{\sim}{\delta} = \underset{\sim}{X}^T (\Delta\underset{\sim}{u}+\Delta\underset{\sim}{\mathbb{R}}_{u\pi}) \qquad (4.2.80)$$

the definition of the structure strain energy becoming

$$\Delta U = \underset{\sim}{X}^T (\Delta \underset{\sim}{u} + \Delta \underset{\sim}{R}_{u\pi}) \qquad (4.2.81)$$

In the above results, (4.2.71) to (4.2.81), the static variables are related by an equilibrium condition and the kinematic variables define a compatible state of displacements and deformations; the stress- and strain-resultant fields were not, however, associated through any causality relationship.

The relation of duality between the descriptions of linear Statics and Kinematics, inherent in the works of Clebsch (1862) and Maxwell (1864), was recognised and explored by Jenkins (1947, 1953) and popularized through the post-graduation courses delivered at Imperial College by J. C. de C. Henderson in the 1950's and by J. Munro in the 1960's.

Munro (1965a) has offered a proof of duality in the mesh description of linear Statics and Kinematics which is founded on the transformation of static bases.

The results in (2.4.19-20) and (2.4.27-28) may also be accepted as a general proof of duality between the linear description of Statics and Kinematics of the nodal and mesh sub-structures since the exact Statics and Kinematics were derived independently and specialized next to the linear case. A similar statement can be made about the nodal and mesh descriptions of Statics and Kinematics of the assembled structure, summarized in (4.2.16-17) and (4.2.50-51), respectively.

In kinematically linear structural mechanics and within a mathematical programming formalism, Static-Kinematic Duality has been extensively used in plastic limit analysis and synthesis by Munro and Smith (1972) and Smith (1974), in shakedown analysis by Smith (1974) and in elastoplastic deformation analysis by Maier (1968) and Smith (1974).

In kinematically non-linear elastoplastic analysis, the concept was used by Maier (1971), Corradi and Maier (1975), Alexa (1976), Contro et alia (1977) in nodal formats and by Smith (1975, 1977) in both mesh and nodal descriptions of first-order non-linear Statics and Kinematics.

C H A P T E R   F I V E

ELASTIC-PLASTIC STRUCTURES

Although the solution of problems in structural mechanics by optimization techniques can be traced back to Fourier, it was only in the early 1950's that Charnes and Greenberg (1951) and Foulkes (1953) formally identified plastic limit analysis and synthesis as linear programming problems.

After a dormant period during which structural discretization techniques were being developed, in the late 1960's Maier in a brilliant series of papers proved mathematical programming to be the ideal mathematical formalism for the discrete representation of the mechanics of elastoplastic and non-linear elastic continuum problems. A number of subsequent contributions have developed an extensive range of solutions for skeletal structures, plates and shells, for a variety of loadings and materials; besides plastic limit analysis and synthesis, they include applications in shakedown analysis and linear and non-linear elastoplastic analysis, with or without limited deformations, efforts being now oriented into the implementation of commercially viable computer program packages.

Attesting the important role that mathematical programming plays in modern engineering, a NATO Advanced Studies Institute on Engineering Plasticity by Mathematical Programming was held in August 2-12, 1977, at the University of Waterloo, Canada, the proceedings of which present an excelent review of the different techniques and approaches, applied to a multiplicity of problems in structural mechanics.

As was pointed out by Maier (1973), the role of mathematical programming in structural mechanics is two-fold:- "To provide a unified theoretical framework for the study of discrete or discretized structures and to supply computer-suited algorithms for the numerical solution of engineering structures". Although acknowledging the practical importance of the latter, the presentation to follow places more emphasis in the first of these two aspects.

The present Chapter starts with a summary of some results in mathematical programming theory required in the applications to follow; they include the symmetric quadratic programs of Cottle (1963), the associated Kuhn-Tucker Conditions and Equivalence requirements, and the theorems on duality and on uniqueness and multiplicity of optimal solutions.

The fundamental conditions characterizing the behaviour of elastoplastic structures undergoing large displacements, presented in the previous Chapters, are then combined in a consistent way following an approach first proposed by Smith (1974); in this manner, the governing system is expressed in four distinct and alternative formats, the nodal-stiffness, nodal-flexibility, mesh-stiffness and mesh-flexibility formulations. The governing systems are then treated following the usual procedure in the mathematical programming theory of structural analysis, e.g. Maier (1968) and Smith (1974); the structure governing system is identified as the Kuhn-Tucker Conditions and, through Kuhn-Tucker Equivalence, the corresponding mathematical programs derived. Section 5.2 ends with a brief reference to related formulations presented in the literature.

Although this alternative is not to be explored, reference should be made to the mixed nodal-mesh formulations which in some circumstances can be advantageously used in the derivation of a governing system involving a number of variables inferior to both the static and kinematic indeterminacy of the structure; we refer the article by Araujo (1972) on this matter.

After processing the alternative governing systems through Kuhn-Tucker Equivalence, the corresponding mathematical programs are physically interpreted and analyzed through mathematical programming theory.

Following Maier (1971), conditions for uniqueness of solution are established and multiple solutions qualitatively investigated.

As the structure governing system defines configurations which are simultaneously statically and kinematically admissible, the role of the Kuhn-Tucker Equivalence will prove to be to separate that system into two distinct problems wherein static and kinematic admissibility are enforced independently. The extremization of the objective functions of the associated mathematical programs become the criteria of selecting among all statically (kinematically) admissible states the correct static (kinematic) field or fields; the variational principles of kinematically non-linear elastoplastic analysis are thus recovered.

An interpretation, in the manner of Corradi (1977a) of Drucker's stability criteria completes section 5.3.

After a brief description of the algorithms used in the solution of illustrative examples, two special occurrences in the behaviour of elastoplastic structures, namely plastic unstressing and limit and bifurcation points, are analyzed and numerical procedures for identifying and solving such situations presented.

Essential in the development of modern plastic buckling theory was Shanley's (1947) interpretation of Engesser's tangent modulus load. Post-buckling and imperfection-sensitivity aspects of plastic buckling are reviewed in Sewell (1972) and Hutchinson (1974). After extensive experimental and theoretical studies of the column problem, the research effort has, apparently due to and for engineering practical purposes, moved into the analysis of plates and shells, with considerably less attention being paid to skeletal structures.

Instead of adopting directly Hill's (1956, 1958, 1961) bifurcation criteria for elastoplastic solids, we opted to adapt Thompson's perturbation procedures in elastic stability theory to the formulation being suggested. The adaptation of perturbation methods in finite-element representations of structures is due to

Thompson and Croll (1968); in the context of elastic systems, extensive additional work has since been done by these authors and their collaborators, much of which can be found in Croll and Walker (1972) and Thompson and Hunt (1973). Of particular interest is the article by Thompson and Hunt (1975) correlating the authors' bifurcation theory with the catastrophe theory of Thom (1972).

As was shown by Drucker (1950), plastic unstressing, also referred to as "local unloading", a common occurrence in (elastic-) plastic systems subject to discontinuous loading programs, may also occur in structures under proportional loading. Various studies have been published since, by Finzi (1956) and Hodge (1959) for instance, but it was only two decades later, with the utilization of mathematical programming theory, that a unified treatment of the problem emerged.

Using some results on the parametric linear complementarity problem due to Cottle (1972a, 1972b, 1974), De Donato and Maier (1974, 1976) proposed a set of criteria systematizing the conditions required for regular progression of yielding. The physical interpretation of the adopted mathematical programming algorithm steps had already proved to be an efficient process of tackling numerically the problem of plastic unstressing. As a consequence, the numerical procedures mentioned in De Donato and Maier (1972, 1973) and presented by Maier et alia (1976) are conceptually identical to the procedure proposed independently by Smith (1975, 1978), from which were developed the techniques to detect and solve situations of multiple plastic unstressing and apparent locking presented in the latter part of section 5.4.

The alternative descriptions for the elastoplastic governing system are specialized in section 5.5 for the analysis of elastic structures and in section 5.6 for rigid-plastic analysis. The associated mathematical programs are then derived and interpreted following the procedure adopted in section 5.3.

Chapter Five ends with a brief comparative study of the behaviour a structure presents when elastic, elastoplastic and rigid-plastic constitutive relations are assumed; emphasis is given to "non-typical" responses as they are commonly ignored by textbooks and other publications on kinematically non-linear structural analysis.

5.1 CONSTRAINED OPTIMIZATION

The problem of constrained optimization consists in finding the minimum (or maximum) of a function z, the OBJECTIVE FUNCTION, on n variables $\underset{\sim}{u}$, the DESIGN VARIABLES, which must satisfy certain relations, the m CONSTRAINTS:

$$\text{Min } z(\underset{\sim}{u}) \qquad (5.1.1a)$$

subject to:-

$$\underset{\sim}{g}_1(\underset{\sim}{u}) \geq \underset{\sim}{0} \qquad (5.1.1b)$$

$$\underset{\sim}{g}_2(\underset{\sim}{u}) = \underset{\sim}{0} \qquad (5.1.1c)$$

In the above MATHEMATICAL PROGRAM, for convenience, the m constraints were separated into m_1 inequality constraints (5.1.1b), which may include SIGN CONSTRAINTS $u_j \geq 0$ on some of the variables, and m_2 strict equality constraints (5.1.1c). A FEASIBLE SOLUTION is a vector $\underset{\sim}{u}$ which satisfies all the problem constraints and the set of all such points defines the FEASIBLE REGION in the design space $\underset{\sim}{u}$; a feasible solution is said to be OPTIMAL if it satisfies the OPTIMALITY CRITERION (5.1.1a).

The question arises as to which further conditions one should add to the set of constraints (5.1.1b,c) in order to replace the constrained optimization problem (5.1.1) by an equivalent system of equations. Let the Lagrangian of problem (5.1.1) be defined as

$$L(\underset{\sim}{u};\underset{\sim}{v}) = z(\underset{\sim}{u}) - \underset{\sim}{v}_1^T \underset{\sim}{g}_1(\underset{\sim}{u}) - \underset{\sim}{v}_2^T \underset{\sim}{g}_2(\underset{\sim}{u})$$

By constraining the lagrange multiplier $\underset{\sim}{v}_1 \in R^{m_1}$ to be non-negative, we guarantee that $L \leq z$ since $\underset{\sim}{g}_1$ and $\underset{\sim}{g}_2$ are, respectively, positive and null for a feasible solution $\underset{\sim}{u}$; by further imposing a complementarity condition between $\underset{\sim}{v}_1$ and $\underset{\sim}{g}_1$, the Lagrangian is brought into coincidence with the objective function, L=z:

THEOREM (KUHN-TUCKER CONDITIONS): If $\underset{\sim}{u}^*$ is a feasible solution of the mathematical program (5.1.1) and is also a minimizing point, then there exist $\underset{\sim}{v}_1 \in R^{m1}$ and $\underset{\sim}{v}_2 \in R^{m2}$ such that

$$\underset{\sim}{\nabla}_u[z(\underset{\sim}{u}^*) - \underset{\sim}{v}_1^T\, \underset{\sim}{g}_1(\underset{\sim}{u}^*) - \underset{\sim}{v}_2^T\, \underset{\sim}{g}_2(\underset{\sim}{u}^*)] = \underset{\sim}{0} \qquad (5.1.2a)$$

$$\underset{\sim}{g}_1(\underset{\sim}{u}^*) \geqq \underset{\sim}{0} \qquad (5.1.2b)$$

$$\underset{\sim}{g}_2(\underset{\sim}{u}^*) = \underset{\sim}{0} \qquad (5.1.2c)$$

$$\underset{\sim}{v}_1^T \cdot \underset{\sim}{g}_1(\underset{\sim}{u}^*) = 0 \qquad (5.1.2d)$$

$$\underset{\sim}{v}_1 \geqq \underset{\sim}{0} \qquad (5.1.2e)$$

$$\underset{\sim}{v}_2 \gtreqless \underset{\sim}{0} \qquad (5.1.2f)$$

A constrained maximization problem

$$\text{Max } w(\underset{\sim}{u}) : \underset{\sim}{g}_1(\underset{\sim}{u}) \geqq \underset{\sim}{0} \quad , \quad \underset{\sim}{g}_2(\underset{\sim}{u}) = \underset{\sim}{0}$$

can be treated similarly by studying the equivalent minimization problem

$$\text{Min } z = -w(\underset{\sim}{u}) : \underset{\sim}{g}_1(\underset{\sim}{u}) \geqq \underset{\sim}{0} \quad , \quad \underset{\sim}{g}_2(\underset{\sim}{u}) = \underset{\sim}{0}$$

The necessary conditions (5.1.2) for the constrained optimality of z were established by Kuhn and Tucker in 1951 and were subsequently termed KUHN-TUCKER OPTIMALITY CONDITIONS; the Lagrangian stationarity conditions (5.1.2a) are known as KUHN-TUCKER CONSTRAINTS.

The Kuhn-Tucker Conditions essentially replace the original mathematical program by sets of equations and inequalities in terms of z, $\underset{\sim}{g}$, $\underset{\sim}{u}$ and a new vector $\underset{\sim}{v}$, the Lagrange multipliers, and their gradients.

If however the fundamental conditions of a problem are originally expressed in the form (5.1.2), the mathematical program (5.1.1) will represent its equivalent constrained optimization problem if it satisfies the following sufficient conditions:

COROLLARY (KUHN-TUCKER EQUIVALENCE): If $z(\underset{\sim}{u})$ is convex, the inequality constraints concave and the equality constraints linear (i.e. if the problem is a CONVEX PROGRAMMING PROBLEM) and the Kuhn-Tucker Conditions hold, then $\underset{\sim}{u}^*$ is a solution to the problem.

A proof for the above Corollary can be found in Mangasarian (1969).

Of particular importance is the following

THEOREM (UNIQUENESS): Let the feasible region of a minimization (maximization) problem be non-empty and convex, and let $\bar{\underset{\sim}{u}}$ be a feasible solution. If the objective function is strictly convex (concave) at $\bar{\underset{\sim}{u}}$, then the problem has a unique optimal solution $\underset{\sim}{u}^* = \bar{\underset{\sim}{u}}_*$. (5.1.3)

We refer next to a relationship which exists between any given convex program, here called the PRIMAL program, and a second program, called the DUAL. Wolfe (1961) framed this dual relationship with the following properties:

i) one problem, the primal, is a constrained minimization problem and the other, the dual, is a constrained maximization problem,

ii) the existence of an optimal solution to one of these problems ensures the existence of the same solution to the other,

iii) if one problem is feasible while the other is not, there is a sequence of points satisfying the constraints of the first on which its objective functions tend to infinity,

and stated the following

THEOREM (DUALITY): Let (5.1.4) be the primal problem, where z is a convex function, $\underset{\sim}{u} \in R^n$ and $\underset{\sim}{g}$ is a vector of m concave functions, and define the dual problem (5.1.5) where

$$w = L(\underset{\sim}{u};\underset{\sim}{v}) = z - \underset{\sim}{v}^T \underset{\sim}{g} \quad , \quad \underset{\sim}{v} \in R^m$$

Then, if $\underset{\sim}{u}^*$ is a solution to the primal problem, there is a $\underset{\sim}{v}^*$ such that $(\underset{\sim}{u}^*, \underset{\sim}{v}^*)$ solves the dual problem and

$$z(\underset{\sim}{u}^*) = w(\underset{\sim}{u}^*, \underset{\sim}{v}^*)$$

	PRIMAL PROGRAM	DUAL PROGRAM	
(5.1.4)	Min $z(\underset{\sim}{u}) : \underset{\sim}{g}(\underset{\sim}{u}) \geq \underset{\sim}{0}$	Max $w(\underset{\sim}{u},\underset{\sim}{v}) : \underset{\sim}{\nabla}_u w = \underset{\sim}{0},\ v \geq \underset{\sim}{0}$	(5.1.5)

The constrained optimization problem (5.1.1) is said to be a QUADRATIC PROGRAMMING problem if the objective function is QUADRATIC in the variables $\underline{u} \in R^n$

$$z = \frac{1}{2}\, \underline{u}^T \underline{Q}\, \underline{u} + \underline{u}^T \underline{q} \qquad (5.1.6a)$$

and the m constraints (5.1.2b,c) are LINEAR

$$\underline{G}\, \underline{u} \begin{Bmatrix} \geq \\ = \end{Bmatrix} \underline{g} \qquad (5.1.6b)$$

thus defining a CONVEX feasible region.

The convexity of the objective function is ensured if matrix $\underline{Q}$, which, and without loss of generality, can be assumed symmetric, is positive semi-definite, i.e. if

$$\underline{u}^T \underline{Q}\, \underline{u} \geq 0 \quad \text{for all (real) } \underline{u} \neq \underline{0}$$

If matrix $\underline{Q}$ is positive definite, i.e. if

$$\underline{u}^T \underline{Q}\, \underline{u} > 0 \quad \text{for all (real) } \underline{u} \neq \underline{0}$$

the objective function becomes STRICTLY CONVEX.

It will prove convenient to re-arrange the data in (5.1.6) in order to obtain the following problem

PRIMAL PROGRAM (5.1.7)
Min $z(\underline{x},\underline{y}) = \frac{1}{2}\, \underline{y}^T \underline{D}\, \underline{y} + \frac{1}{2}\, \underline{x}^T \underline{C}\, \underline{x} + \underline{x}^T \underline{c}$
subject to:- $\underline{D}\, \underline{y} + \underline{A}\, \underline{x} \begin{Bmatrix} = \\ \geq \end{Bmatrix} \underline{b}$

which corresponds to the quadratic program analyzed by Cottle (1963) with the non-negativity condition on variables $\underline{x} \in R^n$ relaxed. Matrices $\underline{C}$ (nxn) and $\underline{D}$ (mxm) are assumed symmetric and positive semi-definite.

The Lagrangian function of the minimization problem (5.1.7) is defined by

$$L(\underline{x},\underline{y};\underline{v}) = z - \underline{v}^T \left[\underline{D}\, \underline{y} + \underline{A}\, \underline{x} - \underline{b} \right] \qquad (5.1.8)$$

Within the elements of the lagrange multipliers $\underline{v}^T = \left[\underline{v}_1^T \;\; \underline{v}_2^T \right]$ we distinguish those associated with the strict equality constraints, collected in $\underline{v}_1$, from the ones associated with the inequality constraints, grouped in $\underset{\sim}{v}_2$; while the m_2 elements of $\underset{\sim}{v}_2$ have to be non-negative, the m_1 elements of $\underset{\sim}{v}_1$ may be unrestricted in sign.

The Kuhn-Tucker Conditions for program (5.1.6) are defined below:

$$\underset{\sim}{\nabla}_x L = \underline{C}\,\underset{\sim}{x} - \underline{A}^T \underset{\sim}{v} + \underset{\sim}{c} = \underset{\sim}{0} \qquad (5.1.9a)$$

$$\underset{\sim}{\nabla}_y L = \underline{D}\,\underset{\sim}{y} - \underline{D}\,\underset{\sim}{v} = \underset{\sim}{0} \qquad (5.1.9b)$$

$$\underline{D}\,\underset{\sim}{y} + \underline{A}\,\underset{\sim}{x} - \underset{\sim}{b} \left\{ \begin{matrix} = \\ \geq \end{matrix} \right\} \underset{\sim}{0} \qquad (5.1.9c)$$

$$\underset{\sim}{v}^T [\underline{D}\,\underset{\sim}{y} + \underline{A}\,\underset{\sim}{x} - \underset{\sim}{b}] = 0 \qquad (5.1.9d)$$

$$\underset{\sim}{v} \left\{ \begin{matrix} \gtreqless \\ \geq \end{matrix} \right\} \underset{\sim}{0} \qquad (5.1.9e)$$

Substituting (5.1.9a,b) into (5.1.8), the Lagrangian simplifies to

$$L(\underset{\sim}{x},\underset{\sim}{y};\underset{\sim}{v}) = -\tfrac{1}{2}\, \underset{\sim}{y}^T \underline{D}\, \underset{\sim}{y} - \tfrac{1}{2}\, \underset{\sim}{x}^T \underline{C}\, \underset{\sim}{x} + \underset{\sim}{v}^T \underset{\sim}{b} \qquad (5.1.10)$$

According to Wolfe's theory, the dual program of the minimization problem (5.1.7) is conceived with the maximization of the Lagrangian (5.1.10) subject to the Kuhn-Tucker constraints (5.1.9a,b):

$$\text{Max } w(\underset{\sim}{x},\underset{\sim}{y};\underset{\sim}{v}) = -\tfrac{1}{2}\, \underset{\sim}{y}^T \underline{D}\, \underset{\sim}{y} - \tfrac{1}{2}\, \underset{\sim}{x}^T \underline{C}\, \underset{\sim}{x} + \underset{\sim}{v}^T \underset{\sim}{b} \qquad (5.1.11a)$$

$$\text{subject to:-} \quad \underline{C}\,\underset{\sim}{x} - \underline{A}^T \underset{\sim}{v} = -\underset{\sim}{c} \qquad (5.1.11b)$$

$$\underline{D}\,\underset{\sim}{y} - \underline{D}\,\underset{\sim}{v} = \underset{\sim}{0} \qquad (5.1.11c)$$

$$\underset{\sim}{v} \left\{ \begin{matrix} \gtreqless \\ \geq \end{matrix} \right\} \underset{\sim}{0} \qquad (5.1.11d)$$

If matrix $\underline{D}$ is positive definite the Kuhn-Tucker constraint (5.1.9b) identifies univocally the lagrange multiplier $\underset{\sim}{v}$ with variable $\underset{\sim}{y}$; otherwise, the particular solution $\underset{\sim}{v}=\underset{\sim}{y}$ is always a possible solution. Enforcing this identification, the Kuhn-Tucker Conditions (5.1.9) reduce to system (5.1.12), and the dual quadratic program (5.1.11) can be expressed exclusively in terms of the primal variables as in (5.1.13).

It is worth reversing the procedure to find the Kuhn-Tucker Conditions directly associated with program (5.1.13); the Lagrangian function is now

	KUHN-TUCKER CONDITIONS (5.1.12)	
PRIMAL CONSTRAINTS	$\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} + \begin{bmatrix}A_{11}\\ A_{21}\end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2\end{bmatrix} \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix}0\\ 0\end{bmatrix} \begin{matrix}(a)\\ (b)\end{matrix}$	MAIN CONSTRAINTS
DUAL CONSTRAINTS	$-\begin{bmatrix}A_{11}^T & A_{21}^T\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} + C x + c = 0 \quad (c)$	KUHN-TUCKER CONSTRAINTS
	$y_2 \geq 0 \quad (d)$	SIGN CONSTRAINTS
	$y_2^T\left[D_{12}^T y_1 + D_{22} y_2 + A_{21} x - b_2\right] = 0 \quad (e)$	COMPLEMENTARITY

DUAL PROGRAM	(5.1.13)
$\text{Max } w = -\frac{1}{2} y^T D y - \frac{1}{2} x^T C x + y^T b$	
subject to: $-A^T y + C x = -c$	
$y_2 \geq 0$	

$$L(x, y; u, r) = -w - u^T\left[-A^T y + C x + c - r^T y_2\right]$$

and the Kuhn-Tucker Conditions become

$$\nabla_x L = C x - C u = 0 \qquad (5.1.14a)$$

$$\nabla_y L = D y + A u - b = \begin{bmatrix} 0 \\ r \end{bmatrix} \qquad \begin{matrix}(5.1.14b)\\ (5.1.14c)\end{matrix}$$

$$-A^T y + C x + c = 0 \qquad (5.1.14d)$$

$$u^T\left[-A^T y + C x + c\right] = 0 \qquad (5.1.14e)$$

$$r \geq 0 , \quad y_2 \geq 0 , \quad r^T y_2 = 0 \qquad (5.1.14f\text{-}h)$$

The complementarity condition (5.1.12e) can be recovered by substituting (5.1.14c) into (5.1.14h). Interpreting the non-negative variable as a slack variable, the stationarity condition (5.1.14b,c) identifies with the main constraints in (5.1.12) if $u=x$. Then, and as the complementarity condition (5.1.14e) is rendered trivial by feasible solutions (x, y), systems (5.1.14) and (5.1.12) become equivalent. Hence, under the assumption that a JOINT SOLUTION $u=x$, $v=y$ exists, programs (5.1.7) and (5.1.13) share the SAME Kuhn-Tucker Conditions, defined by (5.1.12); programs (5.1.7) and (5.1.13) are said to constitute a pair of SYMMETRIC dual programs in the sense that

i) program (5.1.13) is the direct dual of program (5.1.7)
and ii) the dual of program (5.1.13) when written as a minimization problem is program (5.1.7) written as a maximization problem.

The identification of the lagrange multiplier $\underline{v}$ with variable $\underline{y}$, although ALWAYS valid, may not be the ONLY possible solution for $\underline{v}$. If $(\underline{x}^*,\underline{y}^*)$ is an optimal solution for the primal program (5.1.7), $(\underline{x}^*,\underline{y}^*,\underline{v}=\underline{y}^*)$ is always a possible optimal solution for the dual program (5.1.11); however,it will be the unique solution with respect to $\underline{v}$ (and $\underline{y}$) if matrix $\underline{D}$ is positive definite. Similar considerations would apply to variables $\underline{x}$ and $\underline{u}$ with regard to matrix $\underline{C}$.

Since , under the restrictions imposed by the joint solution, the Kuhn-Tucker Conditions (5.1.12) are shared by both programs (5.1.7) and (5.1.13), the feasible regions of which are convex and assumed non-empty, the Kuhn-Tucker Equivalence requirements reduce to:-

KUHN-TUCKER EQUIVALENCE [System (5.1.12)] : If matrices $\underline{C}$ and $\underline{D}$ are (at least) positive semi-definite, every solution of (5.1.12) is a solution of programs (5.1.7) and (5.1.13); otherwise, there may exist solutions of (5.1.12) which do not minimize (maximize) the primal (dual) objective function z (w).

Wolfe's duality theorem, when specialized to programs (5.1.7) and (5.1.13), reduces to the following

DUALITY THEOREM [Cottle (1963)] : (5.1.16)

1. Weak Duality: if $(\underline{x}',\underline{y}')$ and $(\underline{x}'',\underline{y}'')$ are feasible solutions of the primal and dual quadratic programs (5.1.7) and (5.1.13), then

$$z(\underline{x}',\underline{y}') = w(\underline{x}'',\underline{y}'')$$

2. Duality: if $(\underset{\sim}{x}', \underset{\sim}{y}')$ is an optimal solution of the primal program (5.1.7), then there exists a $\underset{\sim}{v}'$ satisfying $\underline{D}\ \underset{\sim}{v}' = \underline{D}\ \underset{\sim}{y}'$ such that $(\underset{\sim}{x}', \underset{\sim}{y}')$ is an optimal solution of the dual program (5.1.13), and

$$\text{Max } w = \text{Min } z$$

3. Converse Duality: if $(\underset{\sim}{x}'', \underset{\sim}{y}'')$ is an optimal solution of the dual program (5.1.13), then there exists a $\underset{\sim}{u}''$ satisfying $\underline{C}\ \underset{\sim}{u}'' = \underline{C}\ \underset{\sim}{x}''$ such that $(\underset{\sim}{x}'', \underset{\sim}{y}'')$ is an optimal solution of the primal program (5.1.7), and

$$\text{Min } z = \text{Max } w$$

4. Unbounded Primal Program: if the dual program (5.1.13) has no feasible solution, then if the primal program is feasible its objective function is unbounded in the direction of the extremization

$$\text{Min } z \rightarrow -\infty$$

5. Unbounded Dual Program: if the primal program (5.1.7) has no feasible solution, then if the dual program (5.1.13) is feasible its objective function is unbounded in the direction of the extremization

$$\text{Max } w \rightarrow +\infty$$

6. Joint Solution: if either the primal or the dual programs has an optimal solution, then there exists an $(\underset{\sim}{x}, \underset{\sim}{y})$ which is optimal for both the primal and dual programs

7. Existence: if both the primal and the dual programs are feasible then both have optimal solutions.

The uniqueness theorem (5.1.3) when specialized to programs (5.1.7) and (5.1.13) becomes

UNIQUENESS THEOREM [programs (5.1.7) and (5.1.13)]: If the primal [dual] feasible region is non-empty and $(\underset{\sim}{x}', \underset{\sim}{y}')$ $[(\underset{\sim}{x}'', \underset{\sim}{y}'')]$ is a feasible solution of program (5.1.7) [(5.1.13)], then

i) If matrix $\underline{C}$ is positive definite $\underset{\sim}{x}' = \underset{\sim}{x}^* = \underset{\sim}{x}''$ is the unique optimal solution with respect to $\underline{x}$, in general $\underline{y}' \neq \underline{y}''$ at optimality.

ii) If matrix $\underline{D}$ is positive definite $\underline{y}' = \underline{y}^* = \underline{y}''$ is the unique optimal solution with respect to $\underline{y}$; in general $\underline{x}' \neq \underline{x}''$ at optimality. (5.1.17)

If the objective function is not strictly convex, the following theorem, demonstrated for instance in Kunzi et alia (1966), can be used to define the totality of relevant solutions.

THEOREM ON MULTIPLICITY: The entire set of optimal solutions of a convex quadratic programming problem is the intersection of the feasible domain with the linear manifold obtained by adding to any optimal vector all vectors which make zero the quadratic form of the objective function and, simultaneously, are orthogonal to the constant vector of the linear part of the objective function. (5.1.18)

Following Cottle (1963), consider the COMPOSITE program (5.1.19) consisting of minimizing the difference of the primal and dual objective functions in (5.1.7) and (5.1.13) over the set of jointly feasible solutions:

COMPOSITE PROGRAM (5.1.19)

$$\text{Min } Q = \underline{y}^T \underline{D}\, \underline{y} + \underline{x}^T \underline{C}\, \underline{x} + \underline{x}^T \underline{c} - \underline{y}^T \underline{b}$$

subject to:-

$$\left[\begin{array}{c|c} \underline{D} & \underline{A} \\ \hline -\underline{A}^T & \underline{C} \end{array}\right] \left[\begin{array}{c} \underline{x} \\ \hline \underline{y} \end{array}\right] \begin{array}{c} \left\{ = \atop \geq \right\} \\ = \end{array} \left[\begin{array}{c} \underline{b} \\ \hline \underline{c} \end{array}\right]$$

$$\underline{y}_2 = \underline{0}$$

Cottle (1963) showed the composite program is SELF-DUAL, in the sense the dual of program (5.1.19) is identical to program (5.1.19) itself; from Cottle's theorem (5.1.16), one may conclude that if the composite program (5.1.19) has an optimal solution, then at optimality Q=0.

Let $(\underline{x}', \underline{y}')$ be a solution for program (5.1.19); the theorem on multiplicity of solutions states that $(\underline{x}'', \underline{y}'')$ defined by

$$\begin{bmatrix} \underset{\sim}{x}'' \\ \hdashline \underset{\sim}{y}'' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{x}' \\ \hdashline \underset{\sim}{y}' \end{bmatrix} + \alpha \begin{bmatrix} \Delta \underset{\sim}{x} \\ \hdashline \Delta \underset{\sim}{y} \end{bmatrix} \qquad (5.1.20)$$

will also be optimal provided

$$\Delta \underset{\sim}{y}^T \underset{\sim}{D} \; \Delta \underset{\sim}{y} + \Delta \underset{\sim}{x}^T \underset{\sim}{C} \; \Delta \underset{\sim}{x} = 0 \qquad (5.1.21a)$$

and

$$-\underset{\sim}{b}^T \Delta \underset{\sim}{y} + \underset{\sim}{c}^T \Delta \underset{\sim}{x} = 0 \qquad (5.1.21b)$$

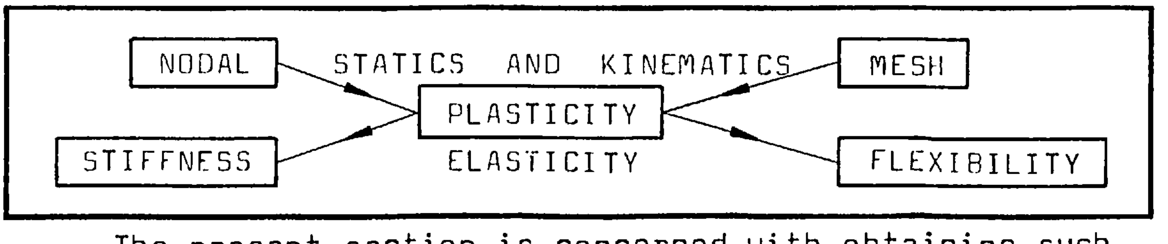
5.2 THE QUADRATIC PROGRAMS OF ELASTOPLASTIC ANALYSIS

In the preceding Chapters, the fundamental (vectorial) conditions (equilibrium, compatibility and constitutive relations) characterizing the problem of kinematically non-linear structural analysis were derived from first-principles of mechanics.

By interpreting the structure as an assemblage of either MESH or NODAL substructures, Statics and Kinematics were expressed in two fundamental alternative descriptions.

Each member forming a typical substructure was then assumed to behave elastoplastically. The elastic association conditions were studied in some detail and care was taken in expressing these conditions in the alternative FLEXIBILITY and STIFFNESS descriptions. As stressed before, the model adopted for the plasticity relations rests heavily on Maier's matrix formulation of Koiter's theory of plasticity.

The system of equations governing the elastoplastic response of a frame under large displacements and/or deformations may therefore, as shown by Smith (1975, 1977), be expressed in either of the FOUR alternative ways diagramatically represented below:



The present section is concerned with obtaining such systems in a consistent way and arranging their constituent equations so that each system may be identified with the Kuhn-Tucker Conditions (5.1.12); under the assumption that the Kuhn-Tucker Equivalence requirements are fulfilled, pairs of primal-dual programs (5.1.7-13) are then derived.

5.2.1 Perturbation Analysis

Let us assume that the static and kinematic fields prevailing in an elastoplastic frame subject to a given load $\underset{\sim}{\lambda}$ are known; the objective now is to determine the change in those fields when the load is varied by a GIVEN quantity $\Delta\underset{\sim}{\lambda}$ which can be expressed as a non-linear but continuous function of a load-path dependent parameter ε

$$\Delta\underset{\sim}{\lambda} = \sum_{i=1}^{\infty} \underset{\sim}{\lambda}_i \frac{\varepsilon^i}{i!}$$

The variations on the problem variables have been already defined in a similar form and Table 4.11 summarizes how those variables should be related in order to ensure static equilibrium and kinematic compatibility, subject to the elastoplastic association conditions, while parameter ε is increased from zero.

Except for the plasticity association condition (4.1.41), the fundamental conditions of mechanics were expressed through linear relations which, and notably, proved to be recursive; as the i-th order generic residuals $\mathbb{R}_i$ depend on static and kinematic variables of order lower than the i-th, they behave as (known) constants.

Table 4.11 shows that of the problem variables only two are sign-constrained, the non-negative plastic multipliers $\underset{\sim}{u}_{*_i}$ and the non-negative plastic potentials $\underset{\sim}{\Phi}_{*_i}$. The latter are dependent variables defined by the static phase of plasticity (4.1.38) which enables us to re-write the flow rule (4.1.40) as

$$\underset{\sim}{\mathbb{H}}\ \underset{\sim}{u}_{*_i} - \underset{\sim}{\mathbb{N}}^T \underset{\sim}{X}_i + \underset{\sim}{\mathbb{R}}_{\varphi_i} \geq \underset{\sim}{0} \qquad (5.2.1a)$$

transforming the association condition (4.1.41) into

$$\underset{\sim}{u}_{*_i}^T [\underset{\sim}{\mathbb{H}}\ \underset{\sim}{u}_{*_i} - \underset{\sim}{\mathbb{N}}^T \underset{\sim}{X}_i + \underset{\sim}{\mathbb{R}}_{\varphi_i}] = 0 \qquad (5.2.1b)$$

The fundamental relations in kinematically non-linear analysis of elastoplastic planar frames when expressed in a perturbed form prove, therefore, to be LINEAR [except for the plasticity association condition (4.1.41) or (5.2.1b)] and of STRICT EQUALITY type [except for the yield conditions (4.1.40) or (5.2.1a)], involving SIGN-UNRESTRICTED variables [except for the non-negative plastic multipliers $\underset{\sim}{\mu}_{*i}$]; the structure of the problem is therefore qualitatively IDENTICAL to that of the Kuhn-Tucker problem (5.1.12).

The main inequality constraints (5.1.12b) can be identified with the yield rule in the form (5.2.1a) so that the complementarity condition (5.2.12e) becomes the association condition (5.2.1b); variables $\underset{\sim}{y}_2$ identify with the plastic multipliers $\underset{\sim}{\mu}_{*i}$ and the sign-constraint (5.2.12d) stands for the flow rule (4.1.42). The identification of variables $\underset{\sim}{x}$ and $\underset{\sim}{y}_1$ as well as of the (constant) entries of matrices $\underline{A}$, $\underline{C}$ and $\underline{D}$ and vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ will depend on the way the alternative descriptions for the fundamental conditions of mechanics involved are inter-combined.

AUXILIARY RESIDUAL VARIABLES
$\Delta\bar{\omega}_o = \Delta\bar{\omega}_1 + \Delta\mathbb{R}_{uE}$ $\Delta\bar{\omega}_1 = \Delta\mathbb{R}_{u\pi} - \Delta\mathbb{R}_p$
$\Delta\bar{\omega}_2 = \underline{B}_\pi \underline{\bar{B}}_\pi \Delta\mathbb{R}_\pi + \underline{\mathbb{K}}_M \Delta\bar{\omega}_1$ $\Delta\bar{\omega}_3 = \Delta\bar{\omega}_2 - \Delta\mathbb{R}_{XE}$
Nodal-Stiffness Formulation
$\Delta\omega_1 = \underline{A}_\pi^T \Delta\mathbb{R}_\pi + \underline{\mathbb{A}}^T \underline{\mathbb{K}} \Delta\bar{\omega}_o$, $\Delta\omega_2 = -\Delta\mathbb{R}_\varphi - \underline{\mathbb{N}}^T \underline{\mathbb{K}} \Delta\bar{\omega}_o$
Nodal-Flexibility Formulation
$\Delta\omega_o = \Delta\bar{\omega}_o$, $\Delta\omega_1 = \underline{A}_\pi^T \Delta\mathbb{R}_\pi$, $\Delta\omega_2 = -\Delta\mathbb{R}_\varphi$
Alternative Mesh-Stiffness Formulation
$\Delta\omega_o = \underline{\mathbb{B}}^T \Delta\bar{\omega}_1$, $\Delta\omega_1 = \Delta\bar{\omega}_3$, $\Delta\omega_2 = -\Delta\mathbb{R}_\varphi + \underline{\mathbb{N}}^T \Delta\bar{\omega}_2$
Alternative Mesh-Flexibility Formulation
$\Delta\omega_o = \underline{\mathbb{N}}^T(\underline{I}_*^T \Delta\bar{\omega}_3 + \Delta\mathbb{R}_{XE}) - \Delta\mathbb{R}_\varphi$, $\Delta\omega_1 = \underline{\mathbb{B}}^T(\underline{\mathbb{F}}\Delta\bar{\omega}_3 + \Delta\bar{\omega}_1)$, $\Delta\omega_2 = \underline{\mathbb{K}}_M \underline{\mathbb{F}} \Delta\bar{\omega}_3$
Mesh-Stiffness Formulation
$\Delta\omega_1 = \underline{\mathbb{K}} \Delta\mathbb{R}_{uE}$, $\Delta\omega_2 = -\Delta\mathbb{R}_\varphi$, $\Delta\omega_3 = \underline{\mathbb{F}}_p' \Delta\mathbb{R}_\pi + \underline{B}_*^T \Delta\bar{\omega}_1$
Mesh-Flexibility Formulation
$\Delta\omega_o = -\Delta\mathbb{R}_\varphi$, $\Delta\omega_1 = \underline{F}_p' \Delta\mathbb{R}_\pi + \underline{B}_*^T \Delta\bar{\omega}_o$

TABLE 5.1

NODAL-STIFFNESS (5.2.2)

$$\begin{bmatrix} \mathbb{K}_* & -\mathbb{A}^T\mathbb{K}\mathbb{N} \\ -\mathbb{N}^T\mathbb{K}\,\mathbb{A} & \mathbb{H}+\mathbb{N}^T\mathbb{K}\mathbb{N} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix}_i \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \mathbb{A}_o^T\lambda+\mathbb{A}^T\mathbb{K}\underline{u}_D+\underline{\omega}_1 \\ -\mathbb{N}^T\mathbb{K}\underline{u}_D+\underline{\omega}_2 \end{bmatrix}_i \qquad \begin{matrix} (a) \\ (b) \end{matrix}$$

$$\underline{u}_{*_i} \geq \underline{0} \qquad (d)$$

$$\underline{u}_{*_i}^T\left[-\mathbb{N}^T\mathbb{K}\mathbb{A}\underline{q} + (\mathbb{H}+\mathbb{N}^T\mathbb{K}\mathbb{N})\,\underline{u}_* + \mathbb{N}^T\mathbb{K}\underline{u}_D - \underline{\omega}_2\right]_i = 0 \qquad (e)$$

NODAL-FLEXIBILITY (5.2.3)

$$\begin{bmatrix} -\mathbb{K}_N & \cdot \\ \cdot & \mathbb{H} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix}_i + \begin{bmatrix} \mathbb{A}^T \\ -\mathbb{N}^T \end{bmatrix} \underline{X}_i \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \mathbb{A}_o^T\underline{\lambda}+\underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}_i \qquad \begin{matrix} (a) \\ (b) \end{matrix}$$

$$-\begin{bmatrix} \mathbb{A} & -\mathbb{N} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix}_i + \mathbb{F}\,\underline{X}_i = -(\underline{u}_D+\underline{\omega}_o)_i \qquad (c)$$

$$\underline{u}_{*_i} \geq \underline{0} \qquad (d)$$

$$\underline{u}_{*_i}^T\left[\mathbb{H}\,\underline{u}_* - \mathbb{N}^T\,\underline{X} - \underline{\omega}_2\right]_i = 0 \qquad (e)$$

(ALTERNATIVE) MESH-STIFFNESS (5.2.4)

$$\begin{bmatrix} \mathbb{K}-\mathbb{K}_M & -\mathbb{K}_M\mathbb{N} \\ -\mathbb{N}^T\,\mathbb{K}_M & \mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N} \end{bmatrix} \begin{bmatrix} \underline{u}_E \\ \underline{u}_* \end{bmatrix}_i + \begin{bmatrix} -\mathbb{B} \\ -\mathbb{N}^T\mathbb{B} \end{bmatrix} \underline{p}_i \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \mathbb{B}_o\underline{\lambda}+\mathbb{K}_M\underline{u}_D+\underline{\omega}_1 \\ \mathbb{N}^T(\mathbb{B}_o\underline{\lambda}+\mathbb{K}_M\underline{u}_D)+\underline{\omega}_2 \end{bmatrix}_i \qquad \begin{matrix} (a) \\ (b) \end{matrix}$$

$$-\begin{bmatrix} -\mathbb{B}^T & -\mathbb{B}^T\mathbb{N} \end{bmatrix} \begin{bmatrix} \underline{u}_E \\ \underline{u}_* \end{bmatrix}_i = -(\mathbb{B}^T\underline{u}_D+\underline{\omega}_o)_i \qquad (c)$$

$$\underline{u}_{*_i} \geq \underline{0} \qquad (d)$$

$$\underline{u}_{*_i}^T\left[-\mathbb{N}^T\mathbb{K}_M\underline{u}_E + (\mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N})\underline{u}_* - \mathbb{N}^T\mathbb{B}\,\underline{p} - \mathbb{N}^T(\mathbb{B}_o\underline{\lambda}+\mathbb{K}_M\underline{u}_D)-\underline{\omega}_2\right]_i = 0 \qquad (e)$$

(ALTERNATIVE) MESH-FLEXIBILITY (5.2.5)

$$\mathbb{H}_*\,\underline{u}_{*_i} + \begin{bmatrix} -\mathbb{N}^T\underline{I}_*\mathbb{B} & -\mathbb{N}^T\mathbb{K}_M\mathbb{F}\,\mathbb{K}_M \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{u}_E \end{bmatrix}_i \geq \mathbb{N}^T\underline{I}_*^T(\mathbb{B}_o\underline{\lambda}+\mathbb{K}_M\underline{u}_D)_i + \underline{\omega}_{o_i} \qquad (b)$$

$$-\begin{bmatrix} -\mathbb{B}^T\underline{I}_*^T\mathbb{N} \\ -\mathbb{K}_M\mathbb{F}\,\mathbb{K}_M\mathbb{N} \end{bmatrix} \underline{u}_{*_i} + \begin{bmatrix} \mathbb{B}^T\mathbb{F}\,\mathbb{B} & \mathbb{B}^T\mathbb{F}\,\mathbb{K}_M \\ \mathbb{K}_M\mathbb{F}\,\mathbb{B} & \mathbb{K}_M\mathbb{F}\,\mathbb{K}_M-\mathbb{K}_M \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{u}_E \end{bmatrix}_i = -\begin{bmatrix} \mathbb{B}^T(\mathbb{F}\,\mathbb{B}_o\underline{\lambda}+\underline{I}_*^T\underline{u}_D)+\underline{\omega}_1 \\ \mathbb{K}_M\,\mathbb{F}(\mathbb{B}_o\underline{\lambda}+\mathbb{K}_M\underline{u}_D)+\underline{\omega}_2 \end{bmatrix}_i \qquad (c)$$

$$\underline{u}_{*_i} \geq \underline{0} \qquad (d)$$

$$\underline{u}_{*_i}^T\left[\mathbb{H}_*\underline{u}_{*_i} - \mathbb{N}^T\underline{I}_*\mathbb{B}\,\underline{p} - \mathbb{N}^T\mathbb{K}_M\mathbb{F}\mathbb{K}_M\underline{u}_E - \mathbb{N}^T\underline{I}_*^T(\mathbb{B}_o\underline{\lambda}+\mathbb{K}_M\underline{u}_D)-\underline{\omega}_o\right]_i = 0 \qquad (e)$$

TABLE 5.2

Summarized in Table 5.2 are the four alternative formulations for the system of relations governing the behaviour of elastoplastic structures under large displacements.

The NODAL-STIFFNESS [NODAL-FLEXIBILITY] formulation (5.2.2) [(5.2.3)] is based on the nodal description of Statics and Kinematics (4.2.69) and (4.2.70) and uses the stiffness [flexibility] description of Elasticity (4.1.37) [(4.1.36)] together with the Plasticity relations (4.1.38) to (4.1.42); the latter, together with the flexibility [stiffness] description of Elasticity were combined with the mesh description of Statics and Kinematics (4.2.67) and (4.2.68) to form the MESH-FLEXIBILITY [MESH-STIFFNESS] formulation (5.2.5) [(5.2.4)]. The residuals $\underset{\sim}{\omega}_{ij}$ are the coefficients in the series expansion (2.1.52) of the auxiliary residual variables $\Delta\underset{\sim}{\omega}_i$ defined in Table 5.1.

In each formulation, equilibrium [compatibility] is ensured by conditions (a) [(c)], the yield [flow] rule and the association condition being represented by relations (b) [(d)] and (e), respectively. The compatibility (equilibrium) condition is implicitly stated in the equilibrium (compatibility) yield and association conditions of the nodal-stiffness (mesh-flexibility) formulation. In the mesh-flexibility formulation we note

$$\underline{I}_* = \underline{I} + \underline{\mathbb{K}}_M \underline{\mathbb{F}} \quad \text{and} \quad \underline{\mathbb{H}}_* = \underline{\mathbb{H}} - \underline{\mathbb{N}}^T (\underline{\mathbb{K}}_M + \underline{\mathbb{K}}_M \underline{\mathbb{F}}\ \underline{\mathbb{K}}_M) \underline{\mathbb{N}} \qquad (5.2.6a,b)$$

and in the nodal-stiffness formulation

$$\underline{\mathbb{K}}_* = \underline{\mathbb{A}}^T \underline{\mathbb{K}}\ \underline{\mathbb{A}} - \underline{\mathbb{K}}_N \qquad (5.2.6c)$$

Each of the four alternative governing systems may now be identified with the Kuhn-Tucker Conditions (5.1.12); their layout in Table 5.2 allows for an immediate identification of the variables and operators they involve with those present in the Kuhn-Tucker problem. Assuming that such systems satisfy the Kuhn-Tucker Equivalence requirements, those identifications can be enforced into the primal-dual pair of programs (5.1.7-13) thus obtaining the four pairs of quadratic programs (5.2.7) to (5.2.14) of kinematically non-linear elastoplastic analysis.

A superficial analysis of system (5.2.4) shows that

THE NODAL-STIFFNESS FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i^T \left[\begin{array}{c|c} \mathbb{K}_* & -\mathbb{A}^T\mathbb{K}\mathbb{N} \\ \hline -\mathbb{N}^T\mathbb{K}\,\mathbb{A} & \mathbb{H}+\mathbb{N}^T\mathbb{K}\mathbb{N} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i$$

$$\text{subject to:-} \quad \left[\begin{array}{c|c} \mathbb{K}_* & -\mathbb{A}^T\mathbb{K}\mathbb{N} \\ \hline -\mathbb{N}^T\mathbb{K}\,\mathbb{A} & \mathbb{H}+\mathbb{N}^T\mathbb{K}\mathbb{N} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i \begin{array}{c} = \\ \geq \end{array} \begin{bmatrix} \mathbb{A}_o^T\lambda+\mathbb{A}^T\mathbb{K}\,u_D+\omega_1 \\ \hline -\mathbb{N}^T\mathbb{K}\,u_D+\omega_2 \end{bmatrix}_i$$

PRIMAL PROGRAM (5.2.7)	DUAL PROGRAM (5.2.8)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i^T \left[\begin{array}{c|c} \mathbb{K}_* & -\mathbb{A}^T\mathbb{K}\mathbb{N} \\ \hline \mathbb{N}^T\mathbb{K}\,\mathbb{A} & \mathbb{H}+\mathbb{N}^T\mathbb{K}\mathbb{N} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i + \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i^T \begin{bmatrix} \mathbb{A}_o^T\lambda+\mathbb{A}^T\mathbb{K}\,u_D+\omega_1 \\ \hline -\mathbb{N}^T\mathbb{K}\,u_D+\omega_2 \end{bmatrix}_i$$

$$\text{subject to:-} \qquad u_{*i} \geq 0$$

THE NODAL-FLEXIBILITY FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i^T \left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i + \frac{1}{2}X_i^T\mathbb{F}\,X_i + X_i^T(u_D+\omega_o)_i$$

$$\text{subject to:-} \quad \left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i + \begin{bmatrix} \mathbb{A}^T \\ \hline -\mathbb{N}^T \end{bmatrix} X_i \begin{array}{c} = \\ \geq \end{array} \begin{bmatrix} \mathbb{A}_o^T\lambda+\omega_1 \\ \hline \omega_2 \end{bmatrix}_i$$

PRIMAL PROGRAM (5.2.9)	DUAL PROGRAM (5.2.10)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i^T \left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i - \frac{1}{2}X_i^T\mathbb{F}\,X_i + \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i^T \begin{bmatrix} \mathbb{A}_o^T\lambda+\omega_1 \\ \hline \omega_2 \end{bmatrix}_i$$

$$\text{subject to:-} \quad -\left[\begin{array}{c|c} \mathbb{A} & -\mathbb{N} \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}_i + \mathbb{F}\,X_i = -(u_D+\omega_o)_i$$

$$u_{*i} \geq 0$$

wherever the generalized elastoplastic deformations can be isolated, they appear constantly coupled with the mesh static-kinematic interdependence matrix $\mathbb{K}_M$. The product $\mathbb{K}_M(u_E+u_p)_i$ has the dimensions of a static variable, suggesting that the adopted policy of keeping Kinematics exclusively in terms of kinematic variables and concentrating static-kinematic interdependence in the description of Statics may not be particularly well suited for the mesh formulations; as the mesh formulation is essentially a static procedure, in the sense that it chooses static variables for unknowns, it is perhaps more convenient, and consistent, to have Statics expressed exclusively in terms of static variables

THE (ALTERNATIVE) MESH-STIFFNESS FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i^T \begin{bmatrix} \mathbb{K}-\mathbb{K}_M & -\mathbb{K}_M\mathbb{N} \\ -\mathbb{N}^T\mathbb{K}_M & \mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N} \end{bmatrix} \begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i + \mathbf{p}_i^T(\mathbb{B}^T\mathbf{u}_D+\boldsymbol{\omega}_o)_i$$

$$\text{subject to:-} \quad \begin{bmatrix} \mathbb{K}-\mathbb{K}_M & -\mathbb{K}_M\mathbb{N} \\ -\mathbb{N}^T\mathbb{K}_M & \mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N} \end{bmatrix} \begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i + \begin{bmatrix} -\mathbb{B} \\ -\mathbb{N}^T\mathbb{B} \end{bmatrix} \mathbf{p}_i \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D+\boldsymbol{\omega}_1 \\ \mathbb{N}^T(\mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D)+\boldsymbol{\omega}_2 \end{bmatrix}_i$$

PRIMAL PROGRAM (5.2.11) | DUAL PROGRAM (5.2.12)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i^T \begin{bmatrix} \mathbb{K}-\mathbb{K}_M & -\mathbb{K}_M\mathbb{N} \\ -\mathbb{N}^T\mathbb{K}_M & \mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N} \end{bmatrix} \begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i + \begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i^T \begin{bmatrix} \mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D+\boldsymbol{\omega}_1 \\ \mathbb{N}^T(\mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D)+\boldsymbol{\omega}_2 \end{bmatrix}_i$$

$$\text{subject to:-} \quad -\begin{bmatrix} -\mathbb{B}^T & -\mathbb{B}^T\mathbb{N} \end{bmatrix} \begin{bmatrix} \mathbf{u}_E \\ \mathbf{u}_* \end{bmatrix}_i = -(\mathbb{B}^T\mathbf{u}_D+\boldsymbol{\omega}_o)_i$$

$$\mathbf{u}_{*\,i} \geq \mathbf{0}$$

THE (ALTERNATIVE) MESH-FLEXIBILITY FORMULATION

$$\text{Min } z = \frac{1}{2}\mathbf{u}_{*i}^T\mathbb{H}_*\mathbf{u}_{*i} + \frac{1}{2}\begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i^T \begin{bmatrix} \mathbb{B}^T\mathbb{F}\mathbb{B} & \mathbb{B}^T\mathbb{F}\mathbb{K}_M \\ \mathbb{K}_M\mathbb{F}\mathbb{B} & \mathbb{K}_M\mathbb{F}\mathbb{K}_M-\mathbb{K}_M \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i + \begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i^T \begin{bmatrix} \mathbb{B}^T(\mathbb{F}\mathbb{B}_o\boldsymbol{\lambda}+\mathbf{I}_*^T\mathbf{u}_D)+\boldsymbol{\omega}_1 \\ \mathbb{K}_M\mathbb{F}(\mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D)+\boldsymbol{\omega}_2 \end{bmatrix}_i$$

$$\text{subject to:-} \quad \mathbb{H}_*\mathbf{u}_{*i} + \begin{bmatrix} -\mathbb{N}^T\mathbf{I}_*^T\mathbb{B} & -\mathbb{N}^T\mathbb{K}_M\mathbb{F}\mathbb{K}_M \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i \geq \begin{bmatrix} \mathbb{N}^T\mathbf{I}_*^T(\mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D)+\boldsymbol{\omega}_o \end{bmatrix}_i$$

PRIMAL PROGRAM (5.2.13) | DUAL PROGRAM (5.2.14)

$$\text{Max } w = -\frac{1}{2}\mathbf{u}_{*i}^T\mathbb{H}_*\mathbf{u}_{*i} - \frac{1}{2}\begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i^T \begin{bmatrix} \mathbb{B}^T\mathbb{F}\mathbb{B} & \mathbb{B}^T\mathbb{F}\mathbb{K}_M \\ \mathbb{K}_M\mathbb{F}\mathbb{B} & \mathbb{K}_M\mathbb{F}\mathbb{K}_M-\mathbb{K}_M \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i + \mathbf{u}_{*i}^T\begin{bmatrix} \mathbb{N}^T\mathbf{I}_*^T(\mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D)+\boldsymbol{\omega}_o \end{bmatrix}_i$$

$$\text{subject to:-} \quad \begin{bmatrix} \mathbb{B}^T\mathbf{I}_*^T\mathbb{N} \\ \mathbb{K}_M\mathbb{F}\mathbb{K}_M\mathbb{N} \end{bmatrix}\mathbf{u}_{*i} + \begin{bmatrix} \mathbb{B}^T\mathbb{F}\mathbb{B} & \mathbb{B}^T\mathbb{F}\mathbb{K}_M \\ \mathbb{K}_M\mathbb{F}\mathbb{B} & \mathbb{K}_M\mathbb{F}\mathbb{K}_M-\mathbb{K}_M \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u}_E \end{bmatrix}_i = -\begin{bmatrix} \mathbb{B}^T(\mathbb{F}\mathbb{B}_o\boldsymbol{\lambda}+\mathbf{I}_*^T\mathbf{u}_D)+\boldsymbol{\omega}_1 \\ \mathbb{K}_M\mathbb{F}(\mathbb{B}_o\boldsymbol{\lambda}+\mathbb{K}_M\mathbf{u}_D)+\boldsymbol{\omega}_2 \end{bmatrix}_i$$

$$\mathbf{u}_{*\,i} \geq \mathbf{0}$$

and transfer the static-kinematic interdependence onto the description of Kinematics.

An immediate consequence is that matrix

$$\mathbf{I}_\pi = \mathbf{I} - \mathbb{Q}^T\mathbf{B}_\pi \qquad (5.2.15)$$

has not to be inverted at the beginning of EACH new increment.

Let us then re-write the perturbed form of equation (4.2.20a) as

$$\boldsymbol{\delta}_{\pi_i} = \mathbb{\Pi}\ (\mathbb{Q}^T\mathbf{X} - \boldsymbol{\pi} + \mathbb{R}_\pi)_i \qquad (5.2.16)$$

thus expressing the generalized additional force displacements exclusively in terms of static variables. In (5.2.16) we note

$$\underline{\Pi} = -\underline{P}^{-1} \tag{5.2.17}$$

Matrix $\underline{P}$ is block-diagonal and therefore easily invertible; for instance, and from (2.1.43b), the m-th diagonal block of matrix $\underline{\Pi}$ corresponding to the additional force displacements associated with the structure m-th member is found to be

$$\underline{\Pi}_m = \frac{L_c}{X_3}_m \left[\begin{array}{c|c} c^2 \frac{X_2}{L_c} + 2sc \frac{X_3}{L_c} & -sc \frac{X_2}{L_c} - (s^2-c^2) \frac{X_3}{L_c} \\ \hline -sc \frac{X_2}{L_c} -(s^2-c^2) \frac{X_3}{L_c} & s^2 \frac{X_2}{L_c} - 2sc \frac{X_2}{L_c} \end{array}\right]$$

Substituting (5.2.16) into the perturbed form of (4.2.20b) the following definition for the perturbed generalized additional deformations is obtained:

$$\underline{u}_{\pi_i} = \underline{Q}\,\underline{\Pi}\,(\underline{Q}^T\underline{X}-\underline{\pi})_i + (\underline{R}_{u\pi}+\underline{Q}\,\underline{\Pi}\,\underline{R}_{\pi})_i \tag{5.2.18}$$

Letting

$$\underline{B}_* = \left[\underline{B} \mid \underline{B}_\pi\right] \quad \text{and} \quad \underline{p}_* = \left[\begin{array}{c} \underline{p} \\ \hline \underline{\pi} \end{array}\right] \tag{5.2.19a,b}$$

the perturbed description of Statics

$$\underline{X}_i = \left[\underline{B}_* \mid \underline{B}_o\right] \left[\begin{array}{c} \underline{p}_* \\ \hline \underline{\lambda} \end{array}\right]_i \tag{5.2.20}$$

is found after re-grouping and equating the same order terms in the incremental form of (4.2.59), wherein each of the incremental variables was replaced in a power series; the super-vector $\underline{p}_*$ is termed EXTENDED GENERALIZED BI-ACTIONS.

Treating similarly the finite description of Kinematics (4.2.60) the following perturbed description is found

$$\left[\begin{array}{c} \underline{v}_* \\ \hline \underline{\delta} \end{array}\right]_i = \left[\begin{array}{c} \underline{B}_*^T \\ \hline \underline{B}_o^T \end{array}\right] (\underline{u}+\underline{u}_\pi)_i \tag{5.2.21a,b}$$

where $\underline{v}_{*_i}$ is the i-th coefficient in the series expansion of the EXTENDED GENERALIZED BI-ACTION DISCONTINUITIES

$$\underline{v}_* = \left[\begin{array}{c} \underline{0} \\ \hline \underline{\delta}_\pi \end{array}\right] \tag{5.2.19c}$$

Using (5.2.15) and substituting in (5.2.16) the equilibrated generalized stress-resultants (5.2.20), the definition for the generalized additional force displacements becomes

$$\underline{\delta}_{\pi_i} = \left[\underline{\mathbb{F}}_p'^T \,\vdots\, \underline{\mathbb{F}}_o'^T \right] \begin{bmatrix} \underline{p}_* \\ \underline{\lambda} \end{bmatrix}_i + \underline{\mathbb{\Pi}}\ \underline{\mathbb{R}}_{\pi_i} \qquad (5.2.22)$$

where $\underline{\mathbb{F}}_p'^T = \underline{\mathbb{\Pi}} \left[\underline{\mathbb{Q}}^T \underline{B} \,\vdots\, -\underline{I}_\pi \right]$ and $\underline{\mathbb{F}}_o'^T = \underline{\mathbb{\Pi}}\ \underline{\mathbb{Q}}^T \underline{B}_o$ (5.2.23a,b)

Substituting (5.2.18) and (5.2.22) in (5.2.21) and re-arranging, the following description of Kinematics in terms of both kinematic and static variables is finally obtained:

$$\begin{bmatrix} \underline{0} \\ \underline{\delta}_i \end{bmatrix} = \begin{bmatrix} \underline{B}_*^T \\ \underline{B}_o^T \end{bmatrix} (\underline{u} + \underline{R}_{u\pi})_i + \begin{bmatrix} \underline{\mathbb{F}}_M & \underline{\mathbb{F}}_o \\ \underline{\mathbb{F}}_o^T & \underline{\mathbb{F}}_\lambda \end{bmatrix} \begin{bmatrix} \underline{p}_* \\ \underline{\lambda} \end{bmatrix}_i + \begin{bmatrix} \underline{\mathbb{F}}_p' \\ \underline{\mathbb{F}}_o' \end{bmatrix} \underline{\mathbb{R}}_{\pi_i} \qquad \begin{matrix} (5.2.24a) \\ (5.2.24b) \end{matrix}$$

In the above we note

$$\underline{\mathbb{F}}_\lambda = \underline{B}_o^T\ \underline{\mathbb{Q}}\ \underline{\mathbb{\Pi}}\ \underline{\mathbb{Q}}^T \underline{B}_o \quad , \quad \underline{\mathbb{F}}_o = \underline{\mathbb{F}}_p'\ \underline{\mathbb{Q}}^T \underline{B}_o \qquad (5.2.25a,b)$$

$$\underline{\mathbb{F}}_M = \begin{bmatrix} \underline{B}^T\ \underline{\mathbb{Q}}\ \underline{\mathbb{\Pi}}\ \underline{\mathbb{Q}}^T \underline{B} & -\underline{B}^T \underline{\mathbb{Q}}\ \underline{\mathbb{\Pi}}\ \underline{I}_\pi \\ -\ \underline{I}_\pi^T \underline{\mathbb{\Pi}}\ \underline{\mathbb{Q}}^T \underline{B} & \underline{I}_\pi^T \underline{\mathbb{\Pi}}\ \underline{I}_\pi \end{bmatrix} \qquad (5.2.25c)$$

The mesh-stiffness and mesh-flexibility formulations (5.2.26) and (5.2.27), respectively, were obtained using the alternative mesh descriptions of Statics (5.2.20) and Kinematics (5.2.24) in the above; their identification with the Kuhn-Tucker Conditions (5.1.12) is immediate and the corresponding pairs of primal-dual quadratic programs are given in (5.2.28-29) and (5.2.30-31).

If the generalized additional forces $\underline{\pi}$ are interpreted as generalized indeterminate forces in addition to $\underline{p}$, as in (5.2.19b), matrix $\underline{\mathbb{F}}_*$, defined by the sum

$$\underline{\mathbb{F}}_* = \underline{B}_*^T\ \underline{\mathbb{F}}\ \underline{B}_* + \underline{\mathbb{F}}_M \qquad (5.2.32)$$

will then represent the SYSTEM INCREMENTAL FLEXIBILITY MATRIX; it determines the generalized discontinuities and the generalized additional force displacements corresponding to a set of

THE MESH-STIFFNESS FORMULATION (5.2.26)

$$\left[\begin{array}{c|c}\mathbb{K} & \cdot \\ \hline \cdot & \mathbb{H}\end{array}\right]\left[\begin{array}{c}\underset{\sim}{u}_E \\ \hline \underset{\sim}{u}_*\end{array}\right]_i + \left[\begin{array}{c}-\underset{\sim}{B}_* \\ \hline -\mathbb{N}^T\underset{\sim}{B}_*\end{array}\right]\underset{\sim}{p}_{*_i} \begin{array}{c} = \\ \geq \end{array} \left[\begin{array}{c}\underset{\sim}{B}_o\underset{\sim}{\lambda}+\underset{\sim}{\omega}_1 \\ \hline \mathbb{N}^T\underset{\sim}{B}_o\underset{\sim}{\lambda}+\underset{\sim}{\omega}_2\end{array}\right]_i \qquad \begin{array}{c}(a)\\(b)\end{array}$$

$$-\left[\begin{array}{c|c}-\underset{\sim}{B}_*^T & -\underset{\sim}{B}_*^T\mathbb{N}\end{array}\right]\left[\begin{array}{c}\underset{\sim}{u}_E \\ \hline \underset{\sim}{u}_*\end{array}\right]_i + \mathbb{F}_M\,\underset{\sim}{p}_{*_i} = -\left[\mathbb{F}_{po}\,\underset{\sim}{\lambda} + \underset{\sim}{B}_*^T\underset{\sim}{u}_D + \underset{\sim}{\omega}_3\right]_i \qquad (c)$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0} \qquad (d)$$

$$\underset{\sim}{u}_{*_i}^T\left[\mathbb{H}\,\underset{\sim}{u}_* - \mathbb{N}^T\underset{\sim}{B}_*\underset{\sim}{p}_* - \mathbb{N}^T\underset{\sim}{B}_o\underset{\sim}{\lambda} - \underset{\sim}{\omega}_2\right]_i = 0 \qquad (e)$$

THE MESH-FLEXIBILITY FORMULATION (5.2.27)

$$\mathbb{H}\,\underset{\sim}{u}_{*_i} + \left[-\mathbb{N}^T\underset{\sim}{B}_*\right]\underset{\sim}{p}_{*_i} \geq \left[\mathbb{N}^T\underset{\sim}{B}_o\underset{\sim}{\lambda}+\underset{\sim}{\omega}_o\right]_i \qquad (b)$$

$$-\left[-\underset{\sim}{B}_*^T\mathbb{N}\right]\underset{\sim}{u}_{*_i} + \mathbb{F}_*\,\underset{\sim}{p}_{*_i} = -\left[(\underset{\sim}{B}_*^T\mathbb{F}\,\underset{\sim}{B}_o + \mathbb{F}_{po})\underset{\sim}{\lambda} + \underset{\sim}{B}_*^T\underset{\sim}{u}_D + \underset{\sim}{\omega}_1\right]_i \qquad (c)$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0} \qquad (d)$$

$$\underset{\sim}{u}_{*_i}^T\left[\mathbb{H}\,\underset{\sim}{u}_* - \mathbb{N}^T\underset{\sim}{B}_*\underset{\sim}{p}_* - \mathbb{N}^T\underset{\sim}{B}_o\underset{\sim}{\lambda} - \underset{\sim}{\omega}_o\right]_i = 0$$

indeterminate forces $\underset{\sim}{p}$ and $\underset{\sim}{\pi}$ for which the structure responds entirely elastically. The presence of the generalized additional forces in the mesh-flexibility system (5.2.27) and programs (5.2.30-31) is a direct consequence of establishing the condition of equilibrium on the deformed and displaced structure; matrix $\mathbb{F}_M$, defined in (5.2.25c), which may well be interpreted as the SYSTEM "GEOMETRIC" FLEXIBILITY MATRIX, quantifies the effects of such finite displacements in the subsequent deformability of the structure. The effects of the axial forces on the flexibility of the structure are accounted for by the component $\underset{\sim}{B}_*^T\mathbb{F}\underset{\sim}{B}_*$, which represents the flexibility matrix of structures able to deform without displacing.

Matrix $\mathbb{K}_*$, defined in (5.2.6c), is the SYSTEM INCREMENTAL STIFFNESS MATRIX and determines the generalized nodal forces corresponding to a set of generalized nodal displacements for which the structures respond entirely elastically; while the stiffness stability functions present in the unassembled stiffness matrix $\mathbb{K}$ reduce the member stiffness for increasing axial compressions, the corrective term $-\mathbb{K}_N$, which can be interpreted as the SYSTEM "GEOMETRIC" STIFFNESS MATRIX, tends to decrease the overall stability of the structure.

The essential difference between programs (5.2.11) and (5.2.28) [(5.2.12) and (5.2.29)] is that the former can only be set up after the inversion of matrix $\underline{I}_\pi$, while in the latter that matrix is inverted during the solution procedure. As the effectiveness of the existing mathematical programming algorithms depends on the existing number of variables and, and specially, as the number of constraints, program (5.2.11) [(5.2.12)] is from a numerical point of view, more adequate than program (5.2.28) [(5.2.29)].

Programs (5.2.13) and (5.2.30) [(5.2.14) and (5.2.31)], although structurally very similar, differ essentially in the variables they select to characterize the static-kinematic interdependence, the generalized elastic deformations in the former and the generalized additional forces in the latter. Letting M, R and r represent the number of structural members, internal and external releases, it is easy to conclude that the dimension of these variables, defined in Table 4.1, are given by

$$d_{u_E} = 3M+R+r \quad \text{and} \quad d_\pi = 2M+R'$$

where $R' \leq R$. The number of variables in program (5.2.30) [(5.2.31)] can therefore be significantly lower than the number of variables present in the alternative program (5.2.13) [(5.2.14)]. Although programs (5.2.13) and (5.2.30) have the same number of constraints, program (5.2.14) will always involve more constraints than program (5.2.31). The primal-dual pair of programs associated with the mesh-flexibility formulation (5.2.27) enjoy the further advantage of not requiring the explicit inversion of matrix $\underline{I}_\pi$ prior to every load increment.

Programs (5.2.11) and (5.2.12) can be recovered by eliminating in programs (5.2.28) and (5.2.29) the generalized additional forces through (the second sub-set of) condition (5.2.26c). Similar considerations could be made involving the mesh-flexibility systems (5.2.5) and (5.2.27) and, consequently, their associated programs.

THE MESH-STIFFNESS FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i^T \begin{bmatrix}\mathbb{K} & \cdot\\ \cdot & \mathbb{H}\end{bmatrix}\begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i + \frac{1}{2}\,\underline{p}_{*i}^T\,\mathbb{F}_M\,\underline{p}_{*i} + \underline{p}_{*i}^T\left[\mathbb{F}_{po}\underline{\lambda}+\underline{B}_*^T\underline{u}_D+\underline{\omega}_3\right]_i +$$

$$\text{subject to: } \begin{bmatrix}\mathbb{K} & \cdot\\ \cdot & \mathbb{H}\end{bmatrix}\begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i + \begin{bmatrix}-\underline{B}_*\\ -\mathbb{N}^T\underline{B}_*\end{bmatrix}\underline{p}_{*i} \begin{matrix}=\\ \geq\end{matrix} \begin{bmatrix}\underline{B}_o\underline{\lambda}+\underline{\omega}_1\\ \mathbb{N}^T\underline{B}_o\underline{\lambda}+\underline{\omega}_2\end{bmatrix}_i$$

PRIMAL PROGRAM (5.2.28)	DUAL PROGRAM (5.2.29)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i^T \begin{bmatrix}\mathbb{K} & \cdot\\ \cdot & \mathbb{H}\end{bmatrix}\begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i - \frac{1}{2}\,\underline{p}_{*i}^T\,\mathbb{F}_M\,\underline{p}_{*i} + \begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i^T\begin{bmatrix}\underline{B}_o\underline{\lambda}+\underline{\omega}_1\\ \mathbb{N}^T\underline{B}_o\underline{\lambda}+\underline{\omega}_2\end{bmatrix}_i$$

$$\text{subject to:} - \left[-\underline{B}_*^T \;\middle|\; -\underline{B}_*^T\mathbb{N}\right]\begin{bmatrix}\underline{u}_E\\ \underline{u}_*\end{bmatrix}_i + \mathbb{F}_M\,\underline{p}_{*i} = -\left[\mathbb{F}_{po}\underline{\lambda}+\underline{B}_*^T\underline{u}_D+\underline{\omega}_3\right]_i$$

$$\underline{u}_{*i} \geq \underline{0}$$

THE MESH-FLEXIBILITY FORMULATION

$$\text{Min } z = \frac{1}{2}\,\underline{u}_{*i}^T\,\mathbb{H}\,\underline{u}_{*i} + \frac{1}{2}\,\underline{p}_{*i}^T\,\mathbb{F}_*\,\underline{p}_{*i} + \underline{p}_{*i}^T\left[(\underline{B}_*^T\mathbb{F}\,\underline{B}_o+\mathbb{F}_{po})\underline{\lambda}+\underline{B}_*^T\underline{u}_D+\underline{\omega}_1\right]_i$$

$$\text{subject to: } \mathbb{H}\,\underline{u}_{*i} - \mathbb{N}^T\,\underline{B}_*\underline{p}_{*i} \geq \left[\mathbb{N}^T\underline{B}_o\underline{\lambda}+\underline{\omega}_o\right]_i$$

PRIMAL PROGRAM (5.2.30)	DUAL PROGRAM (5.2.31)

$$\text{Max } w = -\frac{1}{2}\,\underline{u}_{*i}^T\,\mathbb{H}\,\underline{u}_{*i} - \frac{1}{2}\,\underline{p}_{*i}^T\,\mathbb{F}_*\,\underline{p}_{*i} + \underline{u}_{*i}^T\left[\mathbb{N}^T\underline{B}_o\underline{\lambda}+\underline{\omega}_o\right]_i$$

$$\text{subject to: } \underline{B}_*^T\mathbb{N}\,\underline{u}_{*i} + \mathbb{F}_*\,\underline{p}_{*i} = -\left[(\underline{B}_*^T\mathbb{F}\,\underline{B}_o+\mathbb{F}_{po})\underline{\lambda}+\underline{B}_*^T\underline{u}_D+\underline{\omega}_1\right]_i$$

$$\underline{u}_{*i} \geq \underline{0}$$

5.2.2 Asymptotic Analysis

If the asymptotic analysis relations were to be combined as the perturbation analysis relations have just been, four pairs of quadratic programs qualitatively identical to (5.2.7-8), (5.2.9-10), (5.2.11-12) or (5.2.28-29), and (5.2.13-14) or (5.2.30-31), would thenceforth be obtained. However, as the asymptotic relations were derived by expanding in a power series the total variables and not their increments, those programs would only be valid in the absence of plastic unstressing.

5.2.3 Incremental Analysis

The procedure followed in subsection 5.2.1 can also be applied to the incremental system relations defined in Table 4.9; the alternative descriptions of Statics and Kinematics, the perturbed forms of which are given by (5.2.19) and (5.2.23), respectively, should also be considered.

The governing systems for the elastoplastic incremental analysis are structurally identical to systems (5.2.2) to (5.2.5) and (5.2.26) and (5.2.27), it being only necessary to replace there each variable, say x_i, by its corresponding increment Δx. As the residuals $\Delta\omega_i$, defined in Table 5.1, are non-linear functions of the system variables, such systems may only be identified with the Kuhn-Tucker Conditions (5.1.12) under the assumption that the actual values taken by those residuals are known a priori. Under this assumption, and supposing that the governing systems satisfy the Kuhn-Tucker Equivalence requirements, the associated pairs of primal-dual (iterative) quadratic programs could then be derived. Such programs would emerge in formats structurally identical to their counterparts in the perturbation analysis programs; in fact, the (iterative) quadratic programs of incremental analysis can be obtained just by replacing in programs (5.2.7) to (5.2.14) and (5.2.28) to (5.2.31) each perturbation coefficient by the incremental variable itself.

Once a program is solved, based on a first estimate of the non-linear residuals $\Delta\omega_i$, in lieu of repeating the solution procedure using an improved estimation, one should make use of the post-optimal (or sensitivity) analysis techiniques which enable quantification of the variation in the optimal solution caused by changes in the program data. These techniques are explained in most works dealing with mathematical programming algorithms; we refer to Boot (1964), Fiacco and McCormick (1968), Orchard-Hays (1968) and Gass (1969) for further information on this subject.

The same comment applies to the quadratic programs of perturbation analysis. After solving the first-order program, wherein the residuals are zero, as the programs are recursive, the j-th order optimal solution can then be obtained using a post-optimal analysis based exclusively on the information provided by the previous j-1 sets of lower-order optimal solutions.

5.2.4 Deformation Analysis

Most of the works in non-linear structural analysis employing fictitious or additional forces choose to eliminate these forces from the formulation by exploring their linear dependence on the stress-resultants. It proves however more rewarding to assume that there exist (known) matrices $\underline{P}$ and $\underline{\Pi}$ which relate the (finite) generalized additional forces with their displacements through

$$\underset{\sim}{\pi} = \underline{P}\,\underset{\sim}{\delta}_{\pi} \quad \text{and} \quad \underset{\sim}{\delta}_{\pi} = \underline{\Pi}\,\underset{\sim}{\pi} \qquad (5.2.33a,b)$$

The four alternative formulations in non-linear elasto-plastic deformation analysis presented in Table 5.4 were obtained by combining, as previously, the alternative descriptions of Statics and Kinematics (4.2.59) to (4.2.62) given in Table 4.7, with either the Elasticity descriptions (4.1.22) or (4.1.23), together with the Plasticity relations (4.1.24) to (4.1.28), and relating the generalized additional forces with their displacements through (5.2.33a) or, as in the mesh-flexibility formulation, through (5.2.33b). In the governing systems (5.2.34) to (5.2.35) we note

$$\underline{K}_N = \underline{A}_{\pi}^T \underline{P}\,\underline{A}_{\pi} \quad \text{and} \quad \underline{K}_M = \underline{B}_{\pi} \underline{P}\,\underline{B}_{\pi}^T \qquad (5.2.33c,d)$$

the auxiliary variables $\underset{\sim}{\omega}_i$ being defined in Table 5.3.

In order to identify each of the alternative formulations with the Kuhn-Tucker Conditions (5.1.12) it is necessary to assume that for a GIVEN loading configuration $\underset{\sim}{\lambda}$ the elements of the functional matrices $\underline{F}$, $\underline{K}$, $\underline{H}$ and $\underline{N}$ and of the functional vectors $\underset{\sim}{u}_{\pi}$, $\underset{\sim}{u}_{E\pi}$, $\underset{\sim}{X}_{E\pi}$, $\underset{\sim}{\pi}_{\varphi}$ and $\underset{\sim}{u}_{\varphi}$ have KNOWN, fixed values. The four pairs of primal-dual programs (5.2.37) to (5.2.44) were obtained

AUXILIARY VARIABLES	
$\hat{\underset{\sim}{\omega}}_o = \hat{\underset{\sim}{\omega}}_1 + \underset{\sim}{u}_{E\pi}$	$\hat{\underset{\sim}{\omega}}_1 = \underset{\sim}{u}_{\pi} - \underset{\sim}{u}_{\varphi}$
Nodal-Stiffness Formulation	
$\underset{\sim}{\omega}_1 = \underline{A}^T \underline{K}\, \hat{\underset{\sim}{\omega}}_o$	$\underset{\sim}{\omega}_2 = -\underset{\sim}{\pi}_{\varphi} - \underline{N}^T \underline{K}\, \hat{\underset{\sim}{\omega}}_o$
Nodal-Flexibility Formulation	
$\underset{\sim}{\omega}_o = \hat{\underset{\sim}{\omega}}_o$	$\underset{\sim}{\omega}_1 = -\underset{\sim}{\pi}_{\varphi}$
Mesh-Stiffness Formulation	
$\underset{\sim}{\omega}_o = \underline{B}^T \hat{\underset{\sim}{\omega}}_1$, $\underset{\sim}{\omega}_1 = \underline{B}_{\pi} \underline{P}\, \underline{B}_{\pi}^T \hat{\underset{\sim}{\omega}}_1 - \underset{\sim}{X}_{E\pi}$, $\underset{\sim}{\omega}_2 = -\underset{\sim}{\pi}_{\varphi} + \underline{N}^T \underline{B}_{\pi} \underline{P}\, \underline{B}_{\pi}^T \hat{\underset{\sim}{\omega}}_1$	
Mesh-Flexibility Formulation	
$\underset{\sim}{\omega}_o = -\underset{\sim}{\pi}_{\varphi}$ $\underset{\sim}{\omega}_1 = \underline{B}^T \underset{\sim}{\omega}_o$	$\underset{\sim}{\omega}_2 = \underline{B}_{\pi}^T \hat{\underset{\sim}{\omega}}_o$

TABLE 5.3

after enforcing those identifications into the pair (5.1.7-13).

In general, the values the above mentioned functionals (as well as those of the elements of matrices $\underline{P}$ and $\underline{\Pi}$) take for a given loading $\underset{\sim}{\lambda}$ can not be known a priori. The post-optimal analysis techniques we referred to previously need not be confined to the study of the effects caused by changes in the stipulation vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$, as required in the incremental analysis programs. It is also possible to study the effects of varying one or more, as will be the case in the deformation analysis programs, of the coefficients of the structural matrices $\underline{A}$, $\underline{C}$ and $\underline{D}$. Although the rationale is similar the technique can become so involved that an iterative procedure coupled with prediction techniques may well be more practicable.

5.2.5 Related Formulations

Let the RATE OF VARIATION of a generic variable y be defined as

$$\dot{y} = \lim_{\varepsilon \to 0} \frac{\Delta y}{\varepsilon} = \lim_{\varepsilon \to 0} \sum_{i=1}^{\infty} y_i \frac{\varepsilon^{i-1}}{i!} = y_1 \qquad (5.2.45)$$

THE FORMULATIONS OF DEFORMATION ANALYSIS

NODAL-STIFFNESS (5.2.34)

$$\begin{bmatrix} -K_N + A^T K A & -A^T K N \\ -N^T K A & H + N^T K N \end{bmatrix} \begin{bmatrix} q \\ u_* \end{bmatrix} \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} A_o^T \lambda + A^T K u_D + \omega_1 \\ -X_* - N^T K u_D + \omega_2 \end{bmatrix}$$

$$u_* \geq 0$$

$$u_*^T \left[-N^T K A q + (H + N^T K N) u_* + X_* + N^T K u_D - \omega_2 \right] = 0$$

NODAL-FLEXIBILITY (5.2.35)

$$\begin{bmatrix} -K_N & \cdot \\ \cdot & H \end{bmatrix} \begin{bmatrix} q \\ u_* \end{bmatrix} + \begin{bmatrix} A^T \\ -N^T \end{bmatrix} X \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} A_o^T \lambda \\ -X_* + \omega_1 \end{bmatrix}$$

$$-\begin{bmatrix} A & -N \end{bmatrix} \begin{bmatrix} q \\ u_* \end{bmatrix} + F X = -(u_D + \omega_o)$$

$$u_* \geq 0$$

$$u_*^T \left[H u_* - N^T X + X_* - \omega_1 \right] = 0$$

MESH-STIFFNESS (5.2.36)

$$\begin{bmatrix} K - K_M & -K_M N \\ -N^T K_M & H - N^T K_M N \end{bmatrix} \begin{bmatrix} u_E \\ u_* \end{bmatrix} + \begin{bmatrix} -B \\ -N^T B \end{bmatrix} p \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} B_o \lambda + K_M u_D + \omega_1 \\ -X_* + N^T (B_o \lambda + K_M u_D) + \omega_2 \end{bmatrix}$$

$$-\begin{bmatrix} -B^T & -B^T N \end{bmatrix} \begin{bmatrix} u_E \\ u_* \end{bmatrix} = -B^T u_D - \omega_o$$

$$u_* \geq 0$$

$$u_*^T \left[-N^T K_M u_E + (H - N^T K_M N) u_* - N^T B p + X_* - N^T (B_o \lambda + K_M u_D - \omega_2) \right] = 0$$

MESH-FLEXIBILITY (5.2.37)

$$H u_* + \begin{bmatrix} -N^T B & -N^T B_\pi \end{bmatrix} \begin{bmatrix} p \\ \pi \end{bmatrix} \geq -X_* + N^T B_o \lambda + \omega_o$$

$$-\begin{bmatrix} -B^T N \\ -B_\pi^T N \end{bmatrix} u_* + \begin{bmatrix} B^T F B & B^T F B_\pi \\ B_\pi^T F B & B_\pi^T F B_\pi - \Pi \end{bmatrix} \begin{bmatrix} p \\ \pi \end{bmatrix} = -\begin{bmatrix} B^T F B_o \lambda + B^T u_D + \omega_1 \\ B_\pi^T F B_o \lambda + B_\pi^T u_D + \omega_2 \end{bmatrix}$$

$$u_* \geq 0$$

$$u_*^T \left[H u_* - N^T B p - N^T B_\pi \pi + X_* - N^T B_o \lambda - \omega_o \right] = 0$$

TABLE 5.4

THE NODAL-STIFFNESS FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} -K_N + A^TKA & -A^TKN \\ \hline -N^TKA & H+N^TKN \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix}$$

subject to:-

$$\left[\begin{array}{c|c} -K_N + A^TKA & -A^TKN \\ \hline -N^TKA & H+N^TKN \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} \begin{array}{c} = \\ \geq \end{array} \begin{bmatrix} A_o^T\lambda + A^TKu_D + \omega_1 \\ \hline -X_* - N^TKu_D + \omega_2 \end{bmatrix}$$

PRIMAL PROGRAM (5.2.38)	DUAL PROGRAM (5.2.39)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} -K_N + A^TKA & -A^TKN \\ \hline -N^TKA & H+N^TKN \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + \begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \begin{bmatrix} A_o^T\lambda + A^TKu_D + \omega_1 \\ \hline -X_* - N^TKu_D + \omega_2 \end{bmatrix}$$

subject to:-

$$u_* \geq 0$$

THE NODAL-FLEXIBILITY FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} -K_N & \cdot \\ \hline \cdot & H \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + \frac{1}{2} X^TFX + X^T(u_D + \omega_o)$$

subject to:-

$$\left[\begin{array}{c|c} -K_N & \cdot \\ \hline \cdot & H \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + \begin{bmatrix} A^T \\ \hline -N^T \end{bmatrix} X \begin{array}{c} = \\ \geq \end{array} \begin{bmatrix} A_o^T\lambda \\ \hline -X_* + \omega_1 \end{bmatrix}$$

PRIMAL PROGRAM (5.2.40)	DUAL PROGRAM (5.2.41)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} -K_N & \cdot \\ \hline \cdot & H \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} - \frac{1}{2} X^TFX + \begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \begin{bmatrix} A_o^T\lambda \\ \hline -X_* + \omega_1 \end{bmatrix}$$

subject to:-

$$-\left[\begin{array}{c|c} A & -N \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + FX = -(u_D + \omega_o)$$

$$u_* \geq 0$$

THE MESH-STIFFNESS FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} u_E \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} K-K_M & -K_M N \\ \hline -N^T K_M & H - N^T K_M N \end{array}\right] \begin{bmatrix} u_E \\ \hline u_* \end{bmatrix} + p^T(B^T u_D + \omega_0)$$

subject to:-

$$\left[\begin{array}{c|c} K-K_M & -K_M N \\ \hline -N^T K_M & H - N^T K_M N \end{array}\right] \begin{bmatrix} u_E \\ \hline u_* \end{bmatrix} + \begin{bmatrix} -B \\ \hline -N^T B \end{bmatrix} p \begin{array}{c} = \\ \geq \end{array} \begin{bmatrix} B_0 \lambda + K_M u_D + \omega_1 \\ \hline -X_* + N^T(B_0 \lambda + K_M u_D) + \omega_2 \end{bmatrix}$$

PRIMAL PROGRAM (5.2.42)	DUAL PROGRAM (5.2.43)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} u_E \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} K-K_M & -K_M N \\ \hline -N^T K_M & H - N^T K_M N \end{array}\right] \begin{bmatrix} u_E \\ \hline u_* \end{bmatrix} + \begin{bmatrix} u_E \\ \hline u_* \end{bmatrix}^T \begin{bmatrix} B_0 \lambda + K_M u_D + \omega_1 \\ \hline -X_* + N^T(B_0 \lambda + K_M u_D + \omega_2) \end{bmatrix}$$

subject to:-

$$\left[\begin{array}{c|c} -B^T & -B^T N \end{array}\right] \begin{bmatrix} u_E \\ \hline u_* \end{bmatrix} = -B^T u_D - \omega_0$$

$$u_* \geq 0$$

THE MESH-FLEXIBILITY FORMULATION

$$\text{Min } z = \frac{1}{2} u_*^T H u_* + \frac{1}{2}\begin{bmatrix} p \\ \hline \pi \end{bmatrix}^T \left[\begin{array}{c|c} B^T F B & B^T F B_\pi \\ \hline B_\pi^T F B & B_\pi^T F B_\pi - \Pi \end{array}\right] \begin{bmatrix} p \\ \hline \pi \end{bmatrix} + \begin{bmatrix} p \\ \hline \pi \end{bmatrix}^T \begin{bmatrix} B^T F B_0 \lambda + B^T u_D + \omega_1 \\ \hline B_\pi^T F B_0 \lambda + B_\pi^T u_D + \omega_2 \end{bmatrix}$$

subject to:- $H u_* + \left[\begin{array}{c|c} -N^T B & -N^T B_\pi \end{array}\right] \begin{bmatrix} p \\ \hline \pi \end{bmatrix} \geq -X_* + N^T B_0 \lambda + \omega_0$

PRIMAL PROGRAM (5.2.44)	DUAL PROGRAM (5.2.45)

$$\text{Max } w = -\frac{1}{2} u_*^T H u_* - \frac{1}{2}\begin{bmatrix} p \\ \hline \pi \end{bmatrix}^T \left[\begin{array}{c|c} B^T F B & B^T F B_\pi \\ \hline B_\pi^T F B & B_\pi^T F B_\pi - \Pi \end{array}\right] \begin{bmatrix} p \\ \hline \pi \end{bmatrix} + u_*^T(-X_* + N^T B_0 \lambda + \omega_0)$$

subject to:- $\begin{bmatrix} -B^T N \\ \hline -B_\pi^T N \end{bmatrix} u_* + \left[\begin{array}{c|c} B^T F B & B^T F B_\pi \\ \hline B_\pi^T F B & B_\pi^T F B_\pi - \Pi \end{array}\right] \begin{bmatrix} p \\ \hline \pi \end{bmatrix} = -\begin{bmatrix} B^T F B_0 \lambda + B^T u_D + \omega_1 \\ \hline B_\pi^T F B_0 \lambda + B_\pi^T u_D + \omega_2 \end{bmatrix}$

$$u_* \geq 0$$

Enforcing the above identification in the nodal formulation of Statics and Kinematics (4.2.69) and (4.2.70a), respectively, the conditions of equilibrium and compatibility during infinitesimal changes of configuration emerge as

$$\underline{\mathbb{K}}_N \dot{\underline{q}} = \underline{\mathbb{A}}^T \dot{\underline{X}} - \underline{\mathbb{A}}_o^T \dot{\underline{\lambda}} \quad \text{and} \quad \dot{\underline{u}}_E + \dot{\underline{u}}_p + \dot{\underline{u}}_D = \underline{\mathbb{A}} \, \dot{\underline{q}}$$

thus recovering the corresponding descriptions adopted by Maier (1971); the definition of matrix $\underline{\mathbb{K}}_N$, known as the structure "geometric" stiffness matrix, is referred to by Argyris (1965a, 1965b).

The stiffness description of Elasticity (4.1.37) becomes $\dot{\underline{X}} = \underline{\mathbb{K}} \, \dot{\underline{u}}_E$; Maier assumes that the elastic stiffness matrix is positive definite thus ignoring **the effect of the** axial forces on the stability of the structure members. Lack of normality is allowed for in the plasticity relations which one defined as follows:

$$\left[\begin{array}{c|c} -\underline{\mathbb{H}} & \underline{\mathbb{N}}^T \\ \hline \underline{\mathbb{V}} & \cdot \end{array}\right] \left[\begin{array}{c} \dot{\underline{u}}_* \\ \hline \dot{\underline{X}} \end{array}\right] = \left[\begin{array}{c} \underline{\Phi}_* \\ \hline \dot{\underline{u}}_p \end{array}\right]$$

Furthermore, reciprocity of interaction between yielding modes is relaxed, i.e. $\underline{\mathbb{H}} \neq \underline{\mathbb{H}}^T$ in general. Otherwise, that is if $\underline{\mathbb{V}} = \underline{\mathbb{N}}$ and $\underline{\mathbb{H}} = \underline{\mathbb{H}}^T$, the plasticity relations in the above coincide, as must be expected, with relations (4.1.38) to (4.1.42) when specialized for infinitesimal changes; for this particular description of Plasticity, Maier (1971) presents two pairs of quadratic programs bearing a direct correspondence with the (first-order) nodal-stiffness programs (5.2.7-8) and nodal-flexibility programs (5.2.9-10). A parallelism is then drawn between the information provided by those programs and the results presented by Hill (1958) and Capurso (1969).

Corradi and Maier (1975) presented a formulation in terms of finite incremental variables. Kinematics is left in the implicit form

$$\Delta\underline{u} = \Delta\underline{u}\ (\Delta\underline{q}) \qquad (5.2.47)$$

and Statics is defined through the Principle of Virtual Work

$$(\partial\Delta\underline{u}^T/\partial\Delta\underline{q})(\underline{\bar{X}}+\Delta\underline{X}) = \underline{\bar{f}}+\Delta\underline{f} \qquad (5.2.48)$$

in which $\Delta\underline{f}$ are (body and surface) equivalent forces; the barred quantities refer to the structural configuration prior to the incremental action. Elasticity is expressed in either of the formats (4.1.29) or (4.1.30) but the axial shortening due to bending, as well as the instabilizing effects due to the axial forces, are neglected; consequently matrices $\underline{F}$ and $\underline{K}$ are rendered positive definite and the non-linear corrective terms $\underset{\sim}{\mathbb{R}}_{uE}$ and $\underset{\sim}{\mathbb{R}}_{XE}$ come to be null. The yield surface is approximated to a polytope so that the plasticity relations become piecewise linear

$$\left[\begin{array}{c|c} -\underline{H} & \underline{N}^T \\ \hline \underline{N} & \cdot \end{array}\right]\left[\begin{array}{c} \Delta\underset{\sim}{u}_* \\ \hline \Delta\underset{\sim}{X} \end{array}\right] = \left[\begin{array}{c} \Delta\underset{\sim}{\Phi}_* \\ \hline \Delta\underset{\sim}{u}_p \end{array}\right] + \left[\begin{array}{c} \bar{\underset{\sim}{X}}_* \\ \hline \cdot \end{array}\right]$$

$$\Delta\underset{\sim}{\Phi}_* \leq \underset{\sim}{0} \quad \Delta\underset{\sim}{\Phi}_*^T \, \Delta\underset{\sim}{u}_* = 0 \quad \Delta\underset{\sim}{u}_* \geq \underset{\sim}{0}$$

where
$$\bar{\underset{\sim}{X}}_* = \underset{\sim}{X}_* - \underline{N}^T\bar{\underset{\sim}{X}} + \underline{H}\,\bar{\underset{\sim}{u}}_*$$

Corradi and Maier (1975) present the following programs:

$$\text{Min } z = \frac{1}{2}\,\Delta\underset{\sim}{X}^T\underline{F}\,\Delta\underset{\sim}{X} + \frac{1}{2}\,\Delta\underset{\sim}{u}_*^T\underline{H}\,\Delta\underset{\sim}{u}_* + (\bar{\underset{\sim}{f}}+\Delta\underset{\sim}{f})^T\Delta\underset{\sim}{q} - (\bar{\underset{\sim}{X}}+\Delta\underset{\sim}{X})^T\left[\Delta\underset{\sim}{u}(\Delta\underset{\sim}{q}) - \Delta\underset{\sim}{u}_D\right]$$

$$\text{subject to:} \quad \underline{N}^T\Delta\underset{\sim}{X} - \underline{H}\,\Delta\underset{\sim}{u}_* \leq \bar{\underset{\sim}{X}}_*\ , \quad (\partial\Delta\underset{\sim}{u}^T/\partial\Delta\underset{\sim}{q})(\bar{\underset{\sim}{X}}+\Delta\underset{\sim}{X}) = \bar{\underset{\sim}{f}}+\Delta\underset{\sim}{f}$$

$$\text{Max } w = -\frac{1}{2}\,\Delta\underset{\sim}{u}_E^T\,\underline{K}\,\Delta\underset{\sim}{u}_E - \frac{1}{2}\,\Delta\underset{\sim}{u}_*^T\,\underline{H}\,\Delta\underset{\sim}{u}_* - \bar{\underset{\sim}{X}}^T(\Delta\underset{\sim}{u}_E + \underline{N}\Delta\underset{\sim}{u}_*) + \bar{\underset{\sim}{X}}_*^T\,\Delta\underset{\sim}{u}_* + (\bar{\underset{\sim}{f}}+\Delta\underset{\sim}{f})^T\Delta\underset{\sim}{q}$$

$$\text{subject to:} \quad \Delta\underset{\sim}{u}_E + \underline{N}\,\Delta\underset{\sim}{u}_* + \Delta\underset{\sim}{u}_D = \Delta\underset{\sim}{u}(\Delta\underset{\sim}{q})\ , \quad \Delta\underset{\sim}{u}_* \geq \underset{\sim}{0}$$

They relate the minimization program with the generalizations of the complementary energy principle developed by Langhaar (1953) and Fraeijs de Veubeke (1972); the maximization program, applied by Contro et alia (1974) to the solution of plane cable systems, is regarded as an extension of the minimum potential energy principle to the case of combined physical and kinematic non-linearities.

Alternative proofs of the statements implicit in the above programs, based on direct arguments rather than on mathematical programming concepts, can be found in Contro et alia (1977).

Establishing a correspondence between the nodal description of incremental Statics and Kinematics (4.2.65) and (4.2.66) respectively with (5.2.48) and (5.2.47), and taking into consideration the assumptions present in the elastoplastic constitutive relations adopted by Corradi and Maier (1975), the programs they present can easily be identified with the incremental versions of the nodal-stiffness programs (5.2.8-9) and nodal-stiffness programs(5.2.9-10).

Alexa (1976), following Jennings (1968), derives from first-principles of mechanics the incremental equilibrium and compatibility conditions of nodal substructures; Static-Kinematic Duality is preserved at both substructure and structure levels. The material is assumed linear elastic-perfect plastic; stress interaction is not accounted for. The element elastic constitutive relations are derived after the usual procedure of approximating by cubic and linear polynomials the transverse and axial displacement fields. The adopted plasticity relations, based on those established by Maier (1968), correspond to simple plastic bending for regular progressive yielding. Including in the governing equations adopted by Corradi and Maier (1975) the set of assumptions adopted by Alexa (1976), the programs derived by the former can be brought to coincide with those presented by the latter.

Abdel-Baset et alia (1973) presented an iterative procedure for the analysis of elastoplastic frames to determine the failure load accounting for first-order effects due to axial forces and member deformations. The formulation is nodal and in terms of total variables, not their increments. The material is assumed linear elastic-perfectly plastic. The adopted elastic constitutive relations are identical to (3.1.90a); axial shortening due to bending is neglected. The influence the axial forces have on the plastic moment capacities is accounted for. The analysis procedure is based on the assumption that the failure load provided by a plastic limit analysis is an upper-bound estimate of the actual failure load. The procedure involves the iterative performance of a series of limit and deformation analysis as linear programming problems.

Smith (1974) presented a unified theory on the elastoplastic analysis of structures under small displacements. In order to extend the formulation to include the first-order effects of finite displacements, Smith (1975,1977) corrected the mesh and nodal descriptions of linear Statics by loading each member of the assembled structure with the additional forces

$$\pi_n = 0 \quad \text{and} \quad \pi_t = \frac{X_2}{L} \delta_t$$

and, by preserving Static-Kinematic Duality, found that the linear description of Kinematics had to include the additional force displacements δ_t. The formulation, in terms of total variables, is semi-automatic as the influence coefficients (both mesh and nodal) have to be derived by direct inspection of the particular structure under analysis. The material is assumed linear elastic-perfectly plastic. The instabilizing effect of the axial forces is accounted for in the elastic constitutive relations; axial shortening due to bending is neglected. The effect of stress interaction on the member plastic capacities are not explicitly considered and although the adopted plasticity relations, atributed to Maier (1969b), presuppose the regular progression of yielding, the adopted numerical procedure of solution, described in Smith (1978), is capable of identifying and perform plastic unstressing.

The works of Smith provided the basis from which the deformation analysis formulation presented here was developed. It is therefore only natural that the first-order formulation suggested by Smith (1975,1977) can be recovered by specialization of the formulation presented in Table 4.7. To do so it is sufficient to replace the generalized variables by the first variable in their definitions in Table 4.1a and to set to zero every auxiliary variable defined in Table 4.16, except $\bar{\underline{\pi}}$ and $\bar{\underline{\delta}}_\pi$ which should be replaced by $\underline{\pi}_t$ and $\underline{\delta}_t$, respectively; the hardening matrix $\underline{H}$ should be set to zero in (4.1.24) and the entries of the normality matrix $\underline{N}$ assumed path-independent.

Enforcing the same simplifications in programs (5.2.38) to (5.2.45), the programs proposed by Smith are thus recovered; Smith chooses to eliminate the additional forces $\underline{\pi}_t$ from the mesh-flexibility programs by inverting a matrix the role of which is similar to that of matrix $[\underline{B}_\pi^T \underline{F}\, \underline{B}_\pi - \underline{\Pi}]$.

References to alternatives formulations can be found in Hofmeister et alia (1970), Marçal (1971), Yamada (1971), Stricklin et alia (1973), and McMeeking and Rice (1975).

5.3 GENERAL CONSIDERATIONS IN ELASTOPLASTIC ANALYSIS

In the theory of elastoplastic systems, as in any other theory, of fundamental importance are the theorems on extremum properties, uniqueness and existence.

In the previous section the governing systems for the analysis of elastoplastic structures, after being expressed in four (alternative) formats, were identified as Kuhn-Tucker problems and processed through Kuhn-Tucker Equivalence; the objective of the present section is to interpret physically the programs so derived and analyze them through mathematical programming theory.

The physical interpretation of the programs will show once again that an extremely important advantage of this use of Kuhn-Tucker Equivalence is that it leads to a formalism for the automatic generation of the variational principles for the class of structural problems under analysis.

The application of the mathematical programming theorems summarized in section 5.1 will establish which are the sufficient conditions for an elastoplastic solution to exist and be unique; they will also provide a theoretical framework for the study of critical configurations such as multiplicity of solutions and stability problems.

5.3.1 Existence and Uniqueness of Optimal Solutions

A structural configuration is said to be KINEMATICALLY ADMISSIBLE if (at least) the compatibility condition and the flow rule of plasticity are satisfied; it is said to be STATICALLY ADMISSIBLE if (at least) the equilibrium condition and the yield rule are satisfied.

When setting up the quadratic programs of perturbation analysis it was indicated which of the fundamental conditions (Statics, Kinematics and Constitutive Relations) were used to form the sets of relationships which were to be identified with the primal and dual constraints in the Kuhn-Tucker Conditions (5.1.12); according to the definitions in the above, the PRIMAL (DUAL) CONSTRAINTS REPRESENT STATIC (KINEMATIC) ADMISSIBILITY. As it is always possible to define a kinematically admissible configuration, the DUAL PROGRAMS WILL ALWAYS BE FEASIBLE. However, THE PRIMAL PROGRAMS MAY NOT BE FEASIBLE as the (prescribed) load variations may exceed, locally or globally, the structure load-carrying capacity; if so, according to Cottle's Theorem (Unbounded Dual Program) the dual objective function is unbounded in the direction of the extremization. Otherwise, and following the same theorem, it can be stated that IF, FOR A GIVEN LOAD INCREMENT, STATICALLY AND KINEMATICALLY ADMISSIBLE CONFIGURATIONS EXIST, THEN BOTH PRIMAL AND DUAL PROGRAMS HAVE OPTIMAL SOLUTIONS.

Having specified the conditions for existence of optimal solutions, the next step is to investigate whether the optimal solution is unique or multiple.

We note that for a given perturbation ε, the uniqueness of the generic perturbation coefficient y_i is a sufficient condition for the uniqueness of the finite increment Δy.

In the following no reference will be made either to the mesh-stiffness programs (5.2.11-12) or to the mesh-flexibility programs (5.2.13-14), as they require the inversion of matrix $\underline{I}_{\pi}$, the regularity of which can not be a priori guaranteed.

Consider the nodal-stiffness programs (5.2.7-8):

(I) If matrix $\left[\begin{array}{c|c} \underline{\mathbb{K}}_* & -\underline{\mathbb{A}}^T\underline{\mathbb{K}}\,\underline{\mathbb{N}} \\ \hline -\underline{\mathbb{N}}^T\underline{\mathbb{K}}\,\underline{\mathbb{A}} & \underline{\mathbb{H}}+\underline{\mathbb{N}}^T\underline{\mathbb{K}}\,\underline{\mathbb{N}} \end{array}\right]$ is positive definite, the generalized nodal displacements $\Delta\underline{q}$ and the generalized plastic multipliers $\Delta\underline{u}_*$ are uniquely defined.

If $|\underline{\mathbb{K}}_*|\neq 0$, the nodal displacements can be eliminated from system (5.2.2). After setting up the new programs, exclusively in terms of $\underline{u}_{*_i}$, it can be concluded that

(II) If matrix $\underline{\mathbb{K}}_*$ is non-singular, a unique solution for the generalized plastic multipliers exists if matrix $\underline{\mathbb{H}}+\underline{\mathbb{N}}^T[\underline{\mathbb{K}}+\underline{\mathbb{K}}\,\underline{\mathbb{A}}\,\underline{\mathbb{K}}_*^{-1}\underline{\mathbb{A}}^T\underline{\mathbb{K}}]\underline{\mathbb{N}}$ is positive definite.

Consider now the nodal-flexibility programs (5.2.9-10):

(III) If matrix $\underline{\mathbb{F}}$ is positive definite, the generalized stress-resultants $\Delta\underline{X}$ are unique.

and (IV) If matrix $\underline{\mathbb{H}}$ is positive definite and matrix $\underline{\mathbb{K}}_N$ negative definite, the generalized plastic multipliers and nodal displacements are unique.

For elastic-perfectly-plastic materials ($\underline{\mathbb{H}}=\underline{0}$), the generalized plastic multipliers disappear from the primal program (5.2.9); hence,

(V) For elastic-perfectly-plastic materials if matrix $\underline{\mathbb{K}}_N$ is negative definite, the generalized nodal displacements are unique; the generalized plastic multipliers need not be unique.

The mesh-stiffness programs (5.2.28-29) give:

(VI) If matrix $\underline{\mathbb{F}}_M$ is positive definite, the generalized indeterminate forces $\Delta\underline{p}$ and the generalized additional forces $\Delta\underline{\pi}$ are unique.

In particular, if $\left|\underline{I}_\pi^T\underline{\mathbb{\Pi}}\ \underline{I}_\pi\right| \neq 0$, then

(VII) If matrix $\underline{B}^T\underline{\mathbb{Q}}\left\{\underline{\mathbb{\Pi}}-\underline{\mathbb{\Pi}}\ \underline{I}_\pi[\underline{I}_\pi^T\underline{\mathbb{\Pi}}\ \underline{I}_\pi]^{-1}\underline{I}_\pi^T\underline{\mathbb{\Pi}}\right\}\underline{\mathbb{Q}}^T\underline{B}$ (5.3.1)
is positive definite, the generalized undeterminate forces are unique,

and (VIII) The generalized elastoplastic deformations will be unique if matrix

$$\left[\begin{array}{c|c} \underline{\mathbb{K}} + \underline{B}_\pi(\underline{I}_\pi^T\underline{\mathbb{\Pi}}\ \underline{I}_\pi)^{-1}\underline{B}_\pi^T & \underline{B}_\pi(\underline{I}_\pi^T\underline{\mathbb{\Pi}}\ \underline{I}_\pi)^{-1}\underline{B}_\pi^T\underline{\mathbb{N}} \\ \hline \underline{\mathbb{N}}^T\ \underline{B}_\pi(\underline{I}_\pi^T\underline{\mathbb{\Pi}}\ \underline{I}_\pi)^{-1}\underline{B}_\pi^T & \underline{\mathbb{H}}+\underline{\mathbb{N}}^T\underline{B}_\pi(\underline{I}_\pi^T\underline{\mathbb{\Pi}}\ \underline{I}_\pi)^{-1}\underline{B}_\pi^T\underline{\mathbb{N}} \end{array}\right]$$

is positive definite.

Finally, from the mesh-flexibility programs (5.2.30-31):

(IX) If matrix $\underline{\mathbb{H}}$ is positive definite, the generalized plastic multipliers are unique.

(X) If matrix $\underline{\mathbb{F}}_*$ is positive definite, the generalized indeterminate and additional forces are unique.

Assume that $\left| \underline{B}_\pi^T \underline{\mathbb{F}} \underline{B}_\pi + \underline{I}_\pi^T \underline{\Pi} \underline{I}_\pi \right| \neq 0$; then

(XI) If matrix $[\underline{\mathbb{H}} + \underline{\mathbb{N}}^T \underline{B}_\pi (\underline{B}_\pi^T \underline{\mathbb{F}} \underline{B}_\pi + \underline{I}_\pi^T \underline{\Pi} \underline{I}_\pi)^{-1} \underline{B}_\pi^T \underline{\mathbb{N}}]$ is positive definite, the generalized plastic multipliers are unique.

and (XII) If the matrix defined below is positive definite, the

$$\underline{B}^T [\underline{\mathbb{F}} + \underline{\mathbb{Q}} \underline{\Pi} \underline{\mathbb{Q}}^T - (\underline{\mathbb{F}} \underline{B}_\pi - \underline{\mathbb{Q}} \underline{\Pi} \underline{I}_\pi)(\underline{B}_\pi^T \underline{\mathbb{F}} \underline{B}_\pi + \underline{I}_\pi^T \underline{\Pi} \underline{I}_\pi)^{-1} (\underline{B}_\pi^T \underline{\mathbb{F}} - \underline{I}_\pi^T \underline{\Pi} \underline{\mathbb{Q}}_\pi^T)] \underline{B} \qquad (5.3.2)$$

generalized indeterminate forces will be unique.

Among all possible multiple optimal solutions, the relevant ones are those which are simultaneously statically and kinematically admissible; as the feasible regions of the COMPOSITE forms of the previously presented quadratic programs are formed by the intersection of the primal and dual feasible regions, if the composite program has an optimal solution, that solution has to be both statically and kinematically admissible.

Consider, for instance, the nodal-flexibility programs (5.2.9-10) and let $(\underline{q}_i', \underline{u}_{*i}', \underline{X}_i')$ be the i-th order optimal solution, to which a set of plastic potentials $\underline{\Phi}_{*i}'$ is associated. From Theorem (5.1.18)

$$(\underline{q}'', \underline{u}_*'', \underline{X}'')_i = (\underline{q}', \underline{u}_*', \underline{X}')_i + \alpha (\delta\underline{q}, \delta\underline{u}_*, \delta\underline{X})_i \qquad (5.3.3)$$

will also be an optimal solution, with plastic potentials $\underline{\Phi}_{*i}'' = \underline{\Phi}_{*i}' + \alpha \delta \underline{\Phi}_{*i}$ provided, and dropping the subscript i,

$$-\delta\underline{q}^T \underline{\mathbb{K}}_N \delta\underline{q} + \delta\underline{u}_*^T \underline{\mathbb{H}} \delta\underline{u}_* + \delta\underline{X}^T \underline{\mathbb{F}} \delta\underline{X} = 0 \qquad (5.3.4)$$

and, for first-order solutions,

$$\delta\underline{X}^T \underline{u}_D = \delta\underline{q}^T \underline{\mathbb{A}}_o^T \underline{\lambda} \qquad (5.3.5)$$

As the solution $(\underline{q}'', \underline{u}_*'', \underline{X}'')$ is also a feasible solution, the following conditions must be satisfied:

$$\begin{bmatrix} -\underline{\mathbb{K}}_N & \cdot \\ \cdot & \underline{\mathbb{H}} \end{bmatrix} \begin{bmatrix} \delta\underline{q} \\ \delta\underline{u}_* \end{bmatrix} + \begin{bmatrix} \underline{\mathbb{A}}^T \\ -\underline{\mathbb{N}}^T \end{bmatrix} \delta\underline{X} = - \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} \delta\underline{\Phi}_* \qquad \begin{matrix} (5.3.6) \\ (5.3.7) \end{matrix}$$

$$- \begin{bmatrix} \underline{\mathbb{A}} & -\underline{\mathbb{N}} \end{bmatrix} \begin{bmatrix} \delta\underline{q} \\ \delta\underline{u}_* \end{bmatrix} + \underline{\mathbb{F}} \quad \delta\underline{X} = \underline{0} \qquad (5.3.8)$$

$$\underline{u}_*' + \alpha \delta\underline{u}_* \geq \underline{0} \ , \quad \underline{\Phi}_*' + \alpha \delta\underline{\Phi}_* \leq \underline{0} \qquad (5.3.9a,b)$$

$$\underset{\sim}{u}_*^{!T}\,\delta\underset{\sim}{\Phi}_* + \underset{\sim}{\Phi}_*^{!T}\,\delta\underset{\sim}{u}_* + \alpha\,\delta\underset{\sim}{u}_*^{T}\,\delta\underset{\sim}{\Phi}_* = 0 \qquad (5.3.10)$$

The sign constraint (5.3.9) limits the parameter α to exist within an interval $[\alpha_{min}, \alpha_{max}]$, depending on the sign of $\delta\underset{\sim}{u}_*$ and $\delta\underset{\sim}{\Phi}_*$. If this interval is non-empty and there exists a configuration $(\delta\underset{\sim}{q}, \delta\underset{\sim}{u}_*, \delta\underset{\sim}{X})$ satisfying system (5.3.4-8), then solutions (5.3.3) characterize the BIFURCATION of the equilibrium path $\alpha = 0$, an occurrence first recognized by Shanley (1947).

Summarized in Table 5.5 are the forms system (5.3.6-10) takes when specialized for the cases of unique nodal displacements, plastic multipliers and stress-resultants. In the latter case, if the material is elastic-perfectly plastic, the system reduces further to

$$\delta\underset{\sim}{q}^T\underset{\sim}{\mathbb{A}}_0^T\underset{\sim}{\lambda} = 0 \;,\; \underset{\sim}{\mathbb{K}}_N\,\delta\underset{\sim}{q} = \underset{\sim}{0} \;,\; -\underset{\sim}{\mathbb{A}}\,\delta\underset{\sim}{q}+\underset{\sim}{\mathbb{N}}\,\delta\underset{\sim}{u}_* = \underset{\sim}{0}$$

$$\underset{\sim}{u}_*^! + \alpha\,\delta\underset{\sim}{u}_* \geq \underset{\sim}{0} \;,\; \underset{\sim}{\Phi}_*^{!T}\,\delta\underset{\sim}{u}_* = 0$$

The above system extends to kinematically non-linear analysis $(\underset{\sim}{\mathbb{K}}_N \neq \underset{\sim}{0})$ Smith and Munro's (1978) justification of lack of uniqueness of the kinematic solution of holonomic elastoplastic structures under small displacements; the pseudo mechanism concept, first employed by Munro (1963b) to explain a mode of plastic unstressing, is used in their physical interpretation of the phenomenon.

Another situation of interest is when

$$\underset{\sim}{q}' = \underset{\sim}{0} \;,\; \underset{\sim}{u}_*' = \underset{\sim}{0} \;\text{ and }\; \underset{\sim}{X}' = \underset{\sim}{0}$$

implying $\qquad \underset{\sim}{\Phi}_*^! = \underset{\sim}{0} \;,\; \underset{\sim}{A}_0^T\,\underset{\sim}{\lambda} = \underset{\sim}{0} \;\text{ and }\; \underset{\sim}{u}_0 = \underset{\sim}{0}$

Then, the only non-trivial conditions that the configuration $(\delta\underset{\sim}{q}, \delta\underset{\sim}{u}_*, \delta\underset{\sim}{X})$ has to satisfy are (5.3.6) to (5.3.8) together with

$$\alpha\,\delta\underset{\sim}{u}_* \geq \underset{\sim}{0} \;,\; \alpha\,\delta\underset{\sim}{\Phi}_* \leq \underset{\sim}{0} \;,\; \delta\underset{\sim}{u}_*^T\,\delta\underset{\sim}{\Phi}_* = 0 \qquad (5.3.11\text{-}13)$$

If $\delta\underset{\sim}{u}_* \geq \underset{\sim}{0}$ $(\delta\underset{\sim}{u}_* \leq \underset{\sim}{0})$ and $\delta\underset{\sim}{\Phi}_* \leq \underset{\sim}{0}$ $(\delta\underset{\sim}{\Phi}_* \geq \underset{\sim}{0})$ there exists an unbounded set of solutions $\alpha>0$ $(\alpha<0)$ defining a neutral state which Maier (1971) identifies with the "eigenstate" of Hill (1958). If however $\delta\underset{\sim}{u}_*$ [or $\delta\underset{\sim}{\Phi}_*$] is unrestricted in sign, condition (5.3.11) [or (5.3.13)] can only be satisfied for $\alpha = 0$ and the solution is rendered unique and trivial.

$\delta q = 0$	$\delta u_* = 0$	$\delta X = 0$
$\delta u_*^T \delta \Phi_* = 0 \quad \delta X^T u_D = 0$	$\delta X^T u_D = \delta q^T \underline{\mathbb{A}}_o^T \lambda$	$\delta q^T \underline{\mathbb{A}}_o^T \lambda = 0$, $\delta u_*^T \delta \Phi_* = 0$
$\underline{\mathbb{A}}^T \delta X = 0$	$-\underline{\mathbb{K}}_N \delta q + \underline{\mathbb{A}}^T \delta X = 0$	$\underline{\mathbb{K}}_N \delta q = 0$
$\underline{\mathbb{H}} \delta u_* - \underline{\mathbb{N}}^T \delta X = -\delta \Phi_*$	$\underline{\mathbb{N}}^T \delta X = \delta \Phi_*$	$\underline{\mathbb{H}} \delta u_* = -\delta \Phi_*$
$\underline{\mathbb{N}} \delta u_* + \underline{\mathbb{F}} \delta X = 0$	$-\underline{\mathbb{A}}\ \delta q + \underline{\mathbb{F}}\ \delta X = 0$	$-\underline{\mathbb{A}}\ \delta q + \underline{\mathbb{N}} \delta u_* = 0$
$u_*^! + \alpha \delta u_* \geq 0$, $\Phi_*^! + \alpha \delta \Phi_* \leq 0$	$\Phi_*^! + \alpha\ \delta \Phi_* \leq 0$	$u_*^! + \alpha \delta u_* \geq 0$, $\Phi_*^! + \alpha \delta \Phi_* \leq 0$
$u_*^{!T}\ \delta \Phi_* + \Phi_*^{!T}\ \delta u_* = 0$	$u_*^{!T}\ \delta \Phi_* + \alpha \delta u_*^T \delta \Phi_* = 0$	$u_*^{!T}\ \delta \Phi_* + \Phi_*^{!T} \delta u_* = 0$

TABLE 5.5

5.3.2 Bounding Theorems

After simple substitutions, particularly easy for the nodal-flexibility formulation programs, the primal and dual objective functions of the programs previously presented, and IRRESPECTIVELY to the formulation they may be concerned with, can be reduced to the following forms:

- Deformation analysis programs

$$z^D = \frac{1}{2}\left\{X^T u_E + \frac{1}{2}\ u_*^T \underline{H}\ u_* - \frac{1}{2}\ \pi^T \delta_\pi\right\} - \left\{-(u_D + \frac{1}{2} u_{E\pi} + u_\pi - u_\varphi)^T X\right\} + O_4 \qquad (5.3.14)$$

$$-\omega^D = \frac{1}{2}\left\{X^T u_E + \frac{1}{2}\ u_*^T \underline{H}\ u_* - \frac{1}{2}\ \pi^T \delta_\pi\right\} + \left\{X_*^T u_*\right\} - \left\{\lambda^T \delta - \frac{1}{2}\ X_{E\pi}^T u_E - \pi_\varphi^T u_*\right\} - O_4 \qquad (5.3.15)$$

- Incremental analysis programs

$$z^I = \left\{\frac{1}{2}\Delta X^T (\Delta u_E + \Delta u_p + \Delta u_\pi) - \frac{1}{2}\ \Delta\pi^T \Delta\delta_\pi \ + \frac{1}{2}\left[\Delta X^T (\mathbb{R}_{uE} + \mathbb{R}_{u\pi} - \mathbb{R}_p) + \Delta\delta_\pi^T \mathbb{R}_\pi - \Delta u^T{}_* \mathbb{R}_\varphi\right]\right\} - \left\{-\Delta X^T \Delta u_D\right\} + O_4 \qquad (5.3.16)$$

$$-\omega^I = \left\{\frac{1}{2}\Delta X^T (\Delta u_E + \Delta u_p + \Delta u_\pi) - \frac{1}{2}\ \Delta\pi^T \Delta\delta_\pi - \frac{1}{2}\left[\Delta X^T (\mathbb{R}_{uE} + \mathbb{R}_{u\pi} - \mathbb{R}_p) + \Delta\delta_\pi^T \mathbb{R}_\pi - \Delta u_*^T \mathbb{R}_\varphi\right]\right\} - \left\{-\Delta\lambda^T \Delta\delta\right\} - O_4 \qquad (5.3.17)$$

- Perturbation analysis programs:

$$z_i^P=\left\{\tfrac{1}{2}\underset{\sim}{X}_i^T(\underset{\sim}{u}_E+\underset{\sim}{u}_P+\underset{\sim}{u}_\pi)_i-\tfrac{1}{2}\underset{\sim}{\pi}_i^T\cdot\underset{\sim}{\delta}_{\pi_i}\cdot+\tfrac{1}{2}\left[\underset{\sim}{X}_i^T(\underset{\sim}{R}_{uE}+\underset{\sim}{R}_{u\pi}-\underset{\sim}{R}_P)_i+\underset{\sim}{\delta}_{\pi i}^T\underset{\sim}{R}_{\pi_i}-\underset{\sim}{u}_{*i}^T\underset{\sim}{R}_{\varphi_i}\right]\right\}-\left\{-\underset{\sim}{X}_i^T\underset{\sim}{u}_{Di}\right\}+\text{Const.} \quad (5.3.18)$$

$$-w_i^P=\left\{\tfrac{1}{2}\underset{\sim}{X}_i^T(\underset{\sim}{u}_E+\underset{\sim}{u}_P+\underset{\sim}{u}_\pi)_i-\tfrac{1}{2}\underset{\sim}{\pi}_i^T\underset{\sim}{\delta}_{\pi_i}-\tfrac{1}{2}\left[\underset{\sim}{X}_i^T(\underset{\sim}{R}_{uE}+\underset{\sim}{R}_{u\pi}-\underset{\sim}{R}_P)_i+\underset{\sim}{\delta}_{\pi_i}^T\underset{\sim}{R}_{\pi_i}-\underset{\sim}{u}_{*i}^T\cdot\underset{\sim}{R}_{\varphi_i}\right]\right\}-\left\{-\underset{\sim}{\lambda}_i^T\underset{\sim}{\delta}_i\right\}-\text{Const.} \quad (5.3.19)$$

In the expressions for z^D and w^D (z^I and w^I), O_4 represents terms of fourth- and higher-order on the total (incremental) corrective variables, which were treated as constants when applying Kuhn-Tucker Equivalence. The (genuinely) constant terms in the expression of z_i^P and w_i^P do not affect the extremization of these functionals and can therefore be disregarded.

From Static-Kinematic Duality it is found that, for each of the three formulations under consideration,

$$\underset{\sim}{X}^T(\underset{\sim}{u}+\underset{\sim}{u}_\pi)-\underset{\sim}{\lambda}^T\underset{\sim}{\delta}-\underset{\sim}{\pi}^T\underset{\sim}{\delta}_\pi=z^D-w^D=0$$

$$\Delta\underset{\sim}{X}^T(\Delta\underset{\sim}{u}+\Delta\underset{\sim}{u}_\pi)-\Delta\underset{\sim}{\lambda}^T\Delta\underset{\sim}{\delta}-\Delta\underset{\sim}{\pi}^T\Delta\underset{\sim}{\delta}_\pi=z^I-w^I=0$$

$$\underset{\sim}{X}_i^T(\underset{\sim}{u}+\underset{\sim}{u}_\pi)_i-\underset{\sim}{\lambda}_i^T\underset{\sim}{\delta}_i-\underset{\sim}{\pi}_i^T\underset{\sim}{\delta}_{\pi i}=z_i^P-w_i^P=0$$

thus confirming that at optimality the primal and dual objective functions attain the same value, $z(\underset{\sim}{x}^*,\underset{\sim}{y}^*)=w(\underset{\sim}{x}^*,\underset{\sim}{y}^*)$.

Consider now the first-order primal and dual objective functions of the perturbation analysis programs and assume that an optimal solution $(\underset{\sim}{x}_1,\underset{\sim}{y}_1)$ exists; then, from (5.3.18) and (5.3.19)

$$z_1^P+w_1^P=\underset{\sim}{X}_1^T\underset{\sim}{u}_{D_1}+\underset{\sim}{\lambda}_1^T\underset{\sim}{\delta}_1=2z_1^P=2w_1^P$$

and let us define the (dimensionally inconsistent) quantity

$$\varkappa=\frac{\underset{\sim}{X}_1^T\underset{\sim}{u}_{D_1}+\underset{\sim}{\lambda}_1^T\underset{\sim}{\delta}_1}{\underset{\sim}{u}_{D_1}^T\underset{\sim}{u}_{D_1}+\underset{\sim}{\lambda}_1^T\underset{\sim}{\lambda}_1}$$

Let $(\underset{\sim}{x}_1^S,\underset{\sim}{y}_1^S)$ $\left[(\underset{\sim}{x}_1^K,\underset{\sim}{y}_1^K)\right]$ be a(first-order) statically [kinematically] admissible solution and $z_1^p(\underline{x}_1^S,\underline{y}_1^S)$ $[w_1^p(\underline{x}_1^K,\underline{y}_1^K)]$ be the value taken by the primal [dual] objective function at that point. Cottle's Theorem (Weak Duality) states that

$$2w_1^p(\underline{x}_1^K,\underline{y}_1^K) \leq 2z_1^p = 2w_1^p \leq 2z_1^p(\underline{x}_1^S,\underline{y}_1^S)$$

or, since $\underline{u}_D^T \underline{u}_D+\underline{\lambda}_1^T\underline{\lambda}_1 > 0$

$$\frac{2}{\underline{u}_D^T\underline{u}_D+\underline{\lambda}_1^T\underline{\lambda}_1} w_1^p(\underline{x}_1^K,\underline{y}_1^K) \leq \varkappa \leq \frac{2}{\underline{u}_D^T\underline{u}_D+\underline{\lambda}_1^T\underline{\lambda}_1} z_1^p(\underline{x}_1^S,\underline{y}_1^S) \qquad (5.3.20a)$$

Following Maier (1971), let us define the flexibility and stiffness parameters $\varkappa_j^\lambda$ and $\varkappa_j^D$

$$\varkappa_j^\lambda=\left(\frac{\delta_1}{\lambda_1}\right)_j \quad \text{and} \quad \varkappa_j^D=\left(\frac{X_1}{u_{D_1}}\right)_j$$

which estimate the variation of the j-th nodal displacement (stress-resultant) caused by a unit load (dislocation) applied at the same point. The bounding theorems in Maier (1971) can be recovered after specializing in (5.3.20a) the parameter $\varkappa$ into the flexibility and stiffness parameters

$$\varkappa_j^\lambda = \varkappa(\underline{u}_{D_1} = \underline{0}, \underline{\lambda}_1^T = [\cdot\cdot\cdot\lambda_{1_j}\cdot\cdot\cdot])$$

$$\varkappa_j^D = \varkappa(\underline{\lambda}_1 = \underline{0}, \underline{u}_{D_1}^T = [\cdot\cdot\cdot u_{D_{1_j}}\cdot\cdot\cdot])$$

yielding $$\frac{2}{\lambda_{1_j}^2} w_1^p(\underline{x}_1^K,\underline{y}_1^K) = \varkappa_j^\lambda = \frac{2}{\lambda_{1_j}^2} z_1^p(\underline{x}_1^S,\underline{y}_1^S) \qquad (5.3.20b)$$

and $$\frac{2}{u_{D_{1_j}}^2} w_1^p(\underline{x}_1^K,\underline{y}_1^K) = \varkappa_j^D = \frac{2}{u_{D_{1_j}}^2} z_1^p(\underline{x}_1^S,\underline{y}_1^S) \qquad (5.3.20c)$$

The above inequalities are useful estimates of local flexibilities since they only require the identification of feasible solutions which are easier to find than optimal solutions; if $(\underline{x}_1^S,\underline{y}_1^S)$ or $(\underline{x}_1^K,\underline{y}_1^K)$ coincide with a (first-order) optimal solution the correspondent inequalities in the above become strict equalities.

5.3.3 Extremum Theorems

As the primal (dual) constraints of the programs previously presented, the objective functions of which can be expressed as in (5.3.14-19), represent static (kinematic) admissibility, these programs can be read as follows:

(I) Among all statically admissible stress fields, the actual stress field(s) make the functional z a minimum,

and (II) Among all kinematically admissible stress fields, the actual strain field(s) make the functional -w a minimum.

The above statements have obvious similarities with the principles of minimum complementary potential energy and minimum potential energy, respectively. These principles can be recovered if the incremental strain energy and complementary strain energy are defined respectively as

$$\Delta U = \frac{1}{2}\Delta \underset{\sim}{X}^T(\Delta \underset{\sim}{u}_E+\Delta \underset{\sim}{u}_p+\Delta \underset{\sim}{u}_\pi)-\frac{1}{2}\Delta \underset{\sim}{\pi}^T \Delta \underset{\sim}{\delta}_\pi -\frac{1}{2}\Big[\Delta \underset{\sim}{X}^T(\underset{\sim}{\mathbb{R}}_{uE}+\underset{\sim}{\mathbb{R}}_{u\pi}-\underset{\sim}{\mathbb{R}}_p)+\Delta \underset{\sim}{\delta}_\pi^T \underset{\sim}{\mathbb{R}}_\pi - -\Delta \underset{\sim}{u}_*^T\ \underset{\sim}{\mathbb{R}}_\varphi\Big]$$

$$\Delta U* = \frac{1}{2}\ \Delta \underset{\sim}{X}^T(\Delta \underset{\sim}{u}_E+\Delta \underset{\sim}{u}_p+\Delta \underset{\sim}{u}_\pi)-\frac{1}{2}\Delta \underset{\sim}{\pi}^T \Delta \underset{\sim}{\delta}_\pi +\frac{1}{2}\Big[\Delta \underset{\sim}{X}^T(\underset{\sim}{\mathbb{R}}_{uE}+\underset{\sim}{\mathbb{R}}_{u\pi}-\underset{\sim}{\mathbb{R}}_p)+\Delta \underset{\sim}{\delta}_\pi^T\ \underset{\sim}{\mathbb{R}}_\pi- -\Delta \underset{\sim}{u}_*^T\ \underset{\sim}{\mathbb{R}}_\varphi\Big]$$

and the incremental work performed by the prescribed forces and dislocations respectively as

$$\Delta W = \Delta \underset{\sim}{\lambda}^T \Delta \underset{\sim}{\delta} \qquad \text{and} \qquad \Delta W^* = -\ \Delta \underset{\sim}{u}_D^T\ \Delta \underset{\sim}{X}$$

Then, and neglecting the fourth- and higher-order terms, we may write

$$z^I = \Delta E* = \Delta U* - \Delta W*$$

and $$-w^I = \Delta E = \Delta U - \Delta W$$

where ΔE and ΔE^* represent the variation on the potential energy and complementary potential energy, respectively. The functionals $-w_i^p$ and z_i^p would then represent the non-linear terms (the only terms relevant in the minimization procedure) in the series expansion of ΔE and ΔE^*, respectively.

The definitions given above for the incremental work and complementary work only include the contributions of the loading and prescribed dislocations; those definitions could however be easily extended to include the effects of any other prescribed forces and displacements, for instance in the manner of Smith (1974).

The deformation analysis programs were obtained by imposing two sets of (severe) assumptions; they presuppose the absence of plastic unstressing (thus rendering the elastoplastic constitutive relations undistinguishable from those of non-linear elasticity) and treat every non-linear corrective variable as constants, that is as PRESCRIBED forces and displacements. The structure is therefore assumed to behave linearly for each set of prescribed forces and displacements and, as a consequence, the strain energy and complementary strain energy present an identical form

$$U = U^* = \frac{1}{2}\underset{\sim}{X}^T \underset{\sim}{u}_E + \frac{1}{2}\underset{\sim}{u}_*^T \underset{\approx}{H}\, \underset{\sim}{u}_* - \frac{1}{2}\underset{\sim}{\pi}^T \underset{\sim}{\delta}_\pi$$

The expressions for the total work and complementary work becoming

$$W = \underset{\sim}{\lambda}^T \underset{\sim}{\delta} - \frac{1}{2}\underset{\sim}{X}_{E\pi}^T \underset{\sim}{u}_E - \underset{\sim}{\pi}_\varphi^T \underset{\sim}{u}_*$$

and

$$W^* = -\underset{\sim}{u}_D^T \underset{\sim}{X} - \left(\frac{1}{2}\underset{\sim}{u}_{E\pi} + \underset{\sim}{u}_\pi - \underset{\sim}{u}_\varphi\right)^T \underset{\sim}{X}$$

respectively. Letting $D = \underset{\sim}{X}_*^T \underset{\sim}{u}_*$ represent the plastic dissipation, the following correspondence between the dual (primal) objective function and the (complementary) potential energy is found:

$$E = -W^D = (U + D) - W\ , \quad E^* = Z^D = U^* - W^*$$

The above identifications reduce statements (I) and (II) in the above to the Haar-Karman and Kachanov-Hodge principles, respectively.

Maier (1971), Contro and Maier (1973), Contro et alia (1974) and Corradi and Maier (1975) have presented variational theorems, the proofs of which are based on mathematical programming theory; Contro et alia (1977) recovered the theorems presented by Corradi and Maier (1975) using direct arguments rather than mathematical programming concepts.

5.3.4 Stability Criteria

Stability is regarded as an intrinsically dynamical subject. Several definitions of stability have been proposed, some of which are briefly discussed in Langhaar (1978).

The kinetic definitions of stability, although simple in appearance, are of difficult application and statical criteria have been suggested in recurrence.

As a consequence of the inexistence of a unique definition, some ambiguities and paradoxes have arisen on the theory of stability of elastoplastic systems; critical considerations can be found, for instance, in Sewell (1972). In the context of conservative systems with associated flow laws, Drucker's statical criterion of positive second-order work

$$\dot{W} = \dot{\underline{\lambda}}^T \dot{\underline{\delta}} \qquad (5.3.21)$$

is generally accepted as a valid stability definition. According to this criterion, Drucker (1964), the state of equilibrium is said to be STABLE if $\dot{W}$ is positive for any (infinitesimal) transition $\dot{\underline{\delta}}$ into a neighbouring configuration, the equilibrium of which is ensured by a variation $\dot{\underline{\lambda}}$ of the external loads; the equilibrium state is said to be CRITICAL if $\dot{W} = 0$ for some paths and positive for others, and UNSTABLE if there are some paths for which $\dot{W}$ becomes negative.

Cottle's Theorem (Duality) states that at optimality the objective function of programs (5.1.7.) and (5.1.13) will attain an identical value; in the absence of prescribed dislocations and adopting the notation in (5.2.46), expressions of the form

$$\dot{\underline{x}}\,\underline{A}\,\dot{\underline{x}} = \dot{\underline{\lambda}}^T\,\dot{\underline{\delta}} \qquad (5.3.22)$$

are found after equating the first-order objective functions of programs (5.2.8-9), (5.2.9-10), (5.2.28-29) and (5.2.30-31). The different forms matrix $\underline{A}$ presents for each of the four previously considered formulations are summarized in Table 5.6; using (5.3.18) and (5.3.19) equality (5.3.22) can be expressed alternatively as

$$\dot{\underline{u}}_E^T\,\mathbb{K}\,\dot{\underline{u}}_E + \dot{\underline{u}}_*^T\,\underline{\mathbb{H}}\,\dot{\underline{u}}_* - \dot{\underline{\delta}}_\pi^T\,\underline{\mathbb{P}}\,\dot{\underline{\delta}}_\pi = \dot{\underline{\lambda}}^T\,\dot{\underline{\delta}} \qquad (5.3.23)$$

Considering matrix $\underline{A}$ defined in either of the formats shown in Table 5.6 or as in (5.3.23), Drucker's stability criterion can now be read as follows:

(I) If matrix $\underline{A}$ is positive definite the equilibrium state is stable.

(II) If matrix $\underline{A}$ is positive semi-definite the equilibrium state is non-unstable.

When matrix $\underline{A}$ is defined in the nodal-stiffness format, the above statements reduce to the stability conditions in Maier (1971) when specialized for elastoplastic materials with plastic strain-rates normal to the reciprocally interactive yield modes.

A stronger stability requirement was proposed by Drucker (1964); applying and subsequently removing $\dot{\underline{\lambda}}$, the net work must vanish if no plastic deformations occur, and must be positive otherwise, i.e.

$$\dot{W}_p = \dot{\underline{\lambda}}^T\,\dot{\underline{\delta}}_p > 0 \quad \text{if} \quad \dot{\underline{\delta}}_p \neq \underline{0} \qquad (5.3.24)$$

$\dot{\underline{\delta}}_p$ being the permanent displacements caused by $\dot{\underline{\lambda}}$. It can be easily shown that

$$\dot{W} = \dot{\underline{\lambda}}^T\,\dot{\underline{\delta}} = \dot{\underline{\lambda}}^T\,\underline{L}\,\dot{\underline{\lambda}} + \dot{\underline{\lambda}}^T\,\dot{\underline{\delta}}_p \qquad (5.3.25)$$

where

$$\underline{L} = \underline{B}_o^T \,\mathbb{F}\, \underline{B}_o + \mathbb{F}_\lambda - (\underline{B}_*^T \,\mathbb{F}\, \underline{B}_o + \mathbb{F}_o)^T \,\mathbb{F}_*^{-1} (\underline{B}_*^T \,\mathbb{F} \underline{B}_o + \mathbb{F}_o) \qquad (5.3.26)$$

$$\dot{\underline{\delta}}_p = \left[\underline{B}_o^T \quad -(\underline{B}_*^T \,\mathbb{F}\, \underline{B}_o + \mathbb{F}_o)^T \mathbb{F}_*^{-1} \; \underline{B}_*^T \right] \; \dot{\underline{u}}_p \qquad (5.3.27)$$

for the mesh-flexibility formulation, and, and similarly to Corradi (1977),

$$\underline{L} = \mathbb{A}_o \, \mathbb{K}_*^{-1} \, \mathbb{A}_o^T \qquad \text{and} \qquad \dot{\underline{\delta}}_p = \mathbb{A}_o \, \mathbb{K}_*^{-1} \mathbb{A}^T \mathbb{K} \, \dot{\underline{u}}_p \qquad (5.3.28,29)$$

for the nodal-stiffness formulation. As pointed out by Corradi (1977), who quotes Mandel (1966), condition (5.2.24) and the positive definiteness of matrix $\underline{L}$ are sufficient to ensure the positivity of the incremental work $\dot{W}$, the converse being not necessarily true since the positivity of $\dot{W}$ does not ensure the positivity of the net plastic work $\dot{W}_p$, as shown by (5.3.25).

NODAL-STIFFNESS	NODAL-FLEXIBILITY
$\begin{bmatrix} \mathbb{K}_* & -\mathbb{A}^T \mathbb{K} \, \mathbb{N} \\ -\mathbb{N}^T \mathbb{K} \, \mathbb{A} & \mathbb{H} + \mathbb{N}^T \mathbb{K} \, \mathbb{N} \end{bmatrix}$	$\begin{bmatrix} \mathbb{F} & \cdot & \cdot \\ \cdot & -\mathbb{K}_N & \cdot \\ \cdot & \cdot & \mathbb{H} \end{bmatrix}$
$\begin{bmatrix} \mathbb{K} & \cdot & \cdot & \cdot \\ \cdot & \mathbb{H} & \cdot & \cdot \\ \cdot & \cdot & \mathbb{F}_M & \mathbb{F}_o \\ \cdot & \cdot & \mathbb{F}_o^T & \mathbb{F}_\lambda \end{bmatrix}$	$\begin{bmatrix} \mathbb{H} & \cdot & \cdot \\ \cdot & \mathbb{F}_* & \underline{B}_*^T \mathbb{F} \, \underline{B}_o + \mathbb{F}_o \\ \cdot & \underline{B}_o^T \mathbb{F} \, \underline{B}_* + \mathbb{F}_o^T & \underline{B}_o^T \mathbb{F} \, \underline{B}_o + \mathbb{F}_\lambda \end{bmatrix}$
MESH-STIFFNESS	MESH-FLEXIBILITY

Table 5.6

5.4 THE BEHAVIOUR OF ELASTOPLASTIC STRUCTURES

In the present section a brief description is given of the algorithms used to solve the quadratic programs derived in section 5.2 after processing through Kuhn-Tucker Equivalence the alternative governing systems of elastoplastic structures undergoing large displacements.

First considered are the structures presenting a normal behaviour, that is, a history of deformation that, from the virgin state to the mobilization of a collapse mechanism, does not include the de-activation of yield modes (plastic unstressing) nor the occurrence of critical points (limit and bifurcation points); not excluded however is the possibility of such structures becoming unstable after the formation of fewer plastic hinges than those required to mobilize a mechanism.

Critical configurations are considered next. The techniques of elastic instability analysis are borrowed and adapted to the formulation being proposed and procedures to identify and solve situations of plastic unstressing presented.

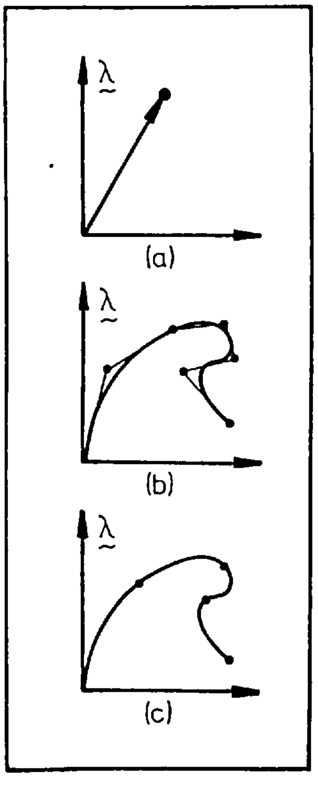


FIGURE 5.1

In the following it is assumed that the loads applied to the structure vary proportionally to a parameter λ.

Linear loading programs, as shown in Fig. 5.1(a), are represented thus

$$\underset{\sim}{\lambda} = \underset{\sim}{\Lambda}\, \lambda$$

where $\underset{\sim}{\Lambda}$ is the directional vector in the λ-space.

If the loading program is non-linear, it can be replaced by a piecewise-linear approximation

$$\underset{\sim}{\lambda} = \underset{\sim}{\lambda}_0 + \underset{\sim}{\Lambda}\, \lambda$$

the directional vector $\underset{\sim}{\Lambda}$ changing now with the phase $\underset{\sim}{\lambda}_o$. When using incremental analysis formulations, we note in either case

$$\Delta\underset{\sim}{\lambda} = \underset{\sim}{\Lambda}\ \Delta\lambda$$

However, if a perturbation analysis formulation is to be used instead, the loading program can be represented in the form

$$\underset{\sim}{\lambda}_i = \underset{\sim}{\Lambda}\ \lambda_i + \underset{\sim}{R}'_{\lambda_i}$$

where

$$\underset{\sim}{R}'_{\lambda\ i} = \underset{\sim}{R}'_{\lambda\ i}\ (\lambda_1, \lambda_2, \ldots, \lambda_{i-1})$$

allowing the analysis to follow exactly the non-linear load program, if, as illustrated in Fig. 5.1(c), it can be subdivided into segments with known analytical representation.

5.4.1 Computational Aspects

The selection of an algorithm depends essentially on the degree (and type) of non-linearity of the Kuhn-Tucker Conditions of the program to be solved.

When, except for the complementarity conditions, the Kuhn-Tucker Conditions are linear, as happens for linear and quadratic programming problems, the utilization of simplex-based algorithms is advisable; extensive research has been done and efficient algorithms are already available.

If the Kuhn-Tucker Conditions are non-linear and linearization is not advisable, approximating procedures, such as the method of the feasible directions, have to be adopted. Selection of a method to solve a particular problem should be based on a judgement of the characteristics of convergence and rate of convergence of the particular algorithm for the particular geometry of the problem.

Descriptions of many of the existing solution procedures can be found in Kunzi et alia (1966) and Avriel (1976). A particularly efficient non-linear programming algorithm is reported by Pierre and Lowe (1975). FORTRAN codes for several mathematical programming algorithms are given in Kuester and Mize (1973).

Substantial gains in computational terms can be achieved through a suitable physical interpretation of the algorithm operations. Examples of successful adaptations of available algorithms to the behaviour of elastoplastic structures are the gradient method of De Donato and Franchi (1973) and the restricted basis linear programming technique first proposed by Maier (1970) and extensively applied ever since by Maier and his collaborators.

The determination of the collapse configuration is, from the engineering point of view, the central objective in a kinematically non-linear elastoplastic analysis.

As the variables involved are strongly history-dependent, the only process of attaining the exact solution is to follow the sequence of formation of plastic hinges, checking constantly for the occurrence of critical points and de-activation of yield modes; such is the objective of the procedure described in the following subsection.

The computer time required for such an analysis is large and may soon become prohibitive for design (rather than research) purposes; instead of following the consistent but excessively long path dictated by the non-linear system governing the problem, preference has to be given to "short-cut" approaches leading to near-optimal solutions, molded by the knowledge gained by experience and constrained by the requirements of the relevant codes of practice.

An improved consistency and a better rate of convergence can be achieved if such design-orientated methods, some of which are referred to by Horne (1972), are treated within a framework and processed through procedures provided by a mathematical programming approach.

The most usual design preconditions are minimum load factors and an associated distribution of plastic hinges at the beams and columns of the structure. Let $\underset{\sim}{\lambda}$ be the required load-level and $\underset{\sim}{X}$ a (trial) stress-distribution (nearly) equilibrating the loading and satisfying the yield conditions for the pre-selected yield pattern. As the stress distribution $\underset{\sim}{X}$ is

PRESCRIBED the elastic strain-field becomes automatically determined and distinction between stiffness and flexibility descriptions of the elastic constitutive relations is thus made irrelevant. If the stress-distribution $\underset{\sim}{X}$ is, as assumed, statically admissible it defines a (local) optimum rendering trivial the primal programs of elastoplastic deformation analysis; the dual programs become LINEAR when conditioned to a prescribed stress distribution.

Performing the internal product of the deformation analysis descriptions of statics and kinematics and substituting in the resulting virtual-work equation

$$\underset{\sim}{X}^T(\underset{\sim}{u}_E+\underset{\sim}{u}_P+\underset{\sim}{u}_D+\underset{\sim}{u}_\pi) = \underset{\sim}{\lambda}^T\underset{\sim}{\delta} + \underset{\sim}{\pi}^T\underset{\sim}{\delta}_\pi$$

the plasticity relations (4.1.24-28), the following relationship results after a simple re-arrangement of terms:

$$\begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix}^T \begin{bmatrix} -\underset{\sim}{K}_N & \cdot \\ \cdot & \underset{\sim}{H} \end{bmatrix} \begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix} = -\underset{\sim}{X}^T(\underset{\sim}{u}_E+\underset{\sim}{u}_P+\underset{\sim}{u}_\pi-\underset{\sim}{u}_\varphi) - \begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix}^T \begin{bmatrix} -\underset{\sim}{A}_o^T\underset{\sim}{\lambda} \\ \underset{\sim}{X}_*+\underset{\sim}{\pi}_\varphi \end{bmatrix}$$

The objective function of the dual nodal-stiffness program (5.2.41) may now be expressed in the form

$$\omega = -\frac{1}{2} \begin{bmatrix} -\underset{\sim}{A}_o^T\underset{\sim}{\lambda} \\ \underset{\sim}{X}_* + \underset{\sim}{\pi}_\varphi \end{bmatrix}^T \begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix} + \frac{1}{2}(\underset{\sim}{u}_D + \underset{\sim}{\omega}_o)^T\underset{\sim}{X}$$

the maximization of which is equivalent to the minimization of

$$z = \begin{bmatrix} -\underset{\sim}{A}_o^T\underset{\sim}{\lambda} \\ \underset{\sim}{X}_* + \underset{\sim}{\pi}_\varphi \end{bmatrix}^T \begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix}$$

as the entries of $\underset{\sim}{X}$ and $\underset{\sim}{u}_D$ have prescribed values and those of $\underset{\sim}{\omega}_o$ are constant. Following a similar procedure, the objective function of the dual mesh-stiffness program (5.2.43) can be reduced to the linear form

$$z = \left[-\underset{\sim}{N}^T\,\underset{\sim}{B}_o\underset{\sim}{\lambda} + \underset{\sim}{X}_* + \underset{\sim}{\pi}_\varphi\right]^T \underset{\sim}{u}_*$$

The corresponding linear programs (5.4.1.) and (5.4.2.) represent the alternative formulations, nodal and mesh, for the problem of kinematically non-linear elastoplastic analysis under a prescribed stress-field. The specialized programs for kinematically linear analyses can be obtained just by setting to zero the non-linear corrective terms $\underline{\pi}_{\varphi}$, $\underline{u}_{\pi}$ and $\underline{u}_{\varphi}$; the programs proposed by Smith (1974) would be thus regained.

DEFORMATION ANALYSIS UNDER A PRESCRIBED STRESS-FIELD	
$\text{Min } z = \begin{bmatrix} -\underline{A}_o^T \underline{\lambda} \\ \underline{X}_* + \underline{\pi}_{\varphi} \end{bmatrix}^T \begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix} \; : \; \begin{bmatrix} \underline{A} & \underline{N} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix} = \underline{u}_E + \underline{u}_D + \underline{u}_{\pi} - \underline{u}_{\varphi}$; $\underline{u}_* \geq \underline{0}$	
NODAL PROGRAM (5.4.1)	MESH PROGRAM (5.4.2)
$\text{Min } z = \left[-\underline{N}^T \underline{B}_o \underline{\lambda} + \underline{X}_* + \underline{\pi}_{\varphi}\right]^T \underline{u}_*$:	$\underline{B}^T \underline{N}\, \underline{u}_* = -\underline{B}^T(\underline{u}_E + \underline{u}_D + \underline{u}_{\pi} - \underline{u}_{\varphi})$; $\underline{u}_* \geq \underline{0}$

Of the 2c possible yield modes, c being the number of critical sections of the structure, only the c' activated modes $\underline{\Phi}'_* = \underline{0}$ fitting the pre-selected field pattern need be considered. The corresponding c' columns of the normal matrix $\underline{N}$ can be assembled into a matrix $\underline{N}'$ and the c' possible non-zero components of $\underline{u}_*$ into $\underline{u}'_*$; the relevant sub-set of corrected plastic capacities are collected in $\underline{X}'_* + \underline{\pi}_{\varphi}$. If the linear programs (5.4.1) and (5.4.2) are condensed employing $\underline{X}'_*$, $\underline{\pi}_{\varphi}$, $\underline{N}'$ and $\underline{u}'_*$, then their only feasible solutions are optimal solutions, and the optimization procedure may now be implemented. First estimates of $\underline{u}_E$, $\underline{u}_{\pi}$, $\underline{u}_{\varphi}$ and $\underline{\pi}_{\varphi}$ are determined from the trial stress-field (and an approximation to the strain-field) and the selected linear program solved. A post-optimality analysis procedure, rather than a repetitive solution of the up-dated linear programs, should then be applied until convergence is guaranteed. If, at convergence, some of the plasticity conditions are found to have been contravened, the plastic capacities of the offending members are modified accordingly and the analysis repeated.

An adaptation of Smith's (1975, 1978) interpretation of the Wolfe-Markowitz algorithm was used to solve the deformation analysis programs (5.2.38) to (5.2.45). As every other simplicial method, it involves a finite sequence of pivot steps. Unlike the simplex method for linear programming, the pivot steps do not generate a sequence of adjacent extreme points of the primal (dual) feasible region; instead, they are concerned with satisfying the Kuhn-Tucker Conditions for the primal (dual) problem. For the programs under consideration, and as a consequence of this symmetry and the adoption of joint solutions, both primal and dual programs (5.1.7) and (5.1.13) share the same system (5.1.12) as their Kuhn-Tucker Conditions.

The question arises on which of the previously presented formulations should be adopted in numerical applications, the decision having to be based on two basic considerations; ease of formulating and solving the problem governing systems.

With regard to the derivation of the structural matrices, the nodal formulations appear to enjoy a significant advantage over the mesh formulations. As was shown in the preceding Chapter, the automatic assemblage of the mesh matrices will not in general be as straightforward as the nodal assemblage. The nodal-flexibility and the mesh-stiffness formulations require less matrix operations than the nodal-stiffness and mesh-flexibility formulations.

The more compact nature of the latter two formulations is the result of a more economic use of variables and (equality) constraints, on the number of which the speed of solution of most algorithms depends upon. Summarized in Table 5.7 are the number of variables and constraints involved in the quadratic programs presented in subsection 5.2.1; α, β, γ and x represent the dimensions of vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{\pi}$ and $\underset{\sim}{X}$ (or $\underset{\sim}{u}_E$) respectively, c' being the number of currently active yield modes.

PROGRAM	NO. OF VARIABLES	NO. OF CONSTRAINTS	NO. OF CONSTRAINTS	NO. OF VARIABLES	PROGRAM
(5.2. 7)	$\beta+c$	$\beta+c$	c	$\beta+c$	(5.2. 8)
(5.2. 9)	$\beta+c+x$	$\beta+c$	$c+x$	$\beta+c+x$	(5.2.10)
(5.2.11)	$\alpha+c+x$	$c+x$	$\alpha+c$	$c+x$	(5.2.12)
(5.2.13)	$\alpha+c+x$	c	$\alpha+c+x$	$\alpha+c+x$	(5.2.14)
(5.2.28)	$\gamma+\alpha+c+x$	$c+x$	$\gamma+\alpha+c$	$\gamma+\alpha+c+x$	(5.2.29)
(5.2.30)	$\alpha+\gamma+c$	c	$\gamma+\alpha+c$	$\alpha+\gamma+c$	(5.2.31)

TABLE 5.7

The results summarized in Table 5.7 show that the nodal-stiffness primal program and the mesh-flexibility dual program are the ones involving fewer constraints; the latter will in general involve significantly fewer variables than the former, since for most practical skeletal frames the number of generalized nodal displacements far exceeds the combined number $\alpha+\gamma$ of generalized indeterminate forces $\underset{\sim}{p}$ and additional forces $\underset{\sim}{\pi}$.

Non-linear programming applications to kinematically non-linear elastoplastic analysis are described in Alexa (1976) and Corradi (1977b); traditional methods are reviewed in Horne (1972) and Argyris and Scharpf (1972).

5.4.2 Normal Behaviour

For later convenience, let us regroup the variables in system (5.1.12) and collect in $\underset{\sim}{u}^T = \left[\underset{\sim}{y}_1^T \quad \underset{\sim}{x}^T\right]$ all the unrestricted variables, and introduce the non-negative slack variables $\underset{\sim}{y}_*$ so that system (5.1.12) becomes

$$\begin{bmatrix} \bar{\underset{\sim}{D}}_{11} & \bar{\underset{\sim}{D}}_{12} \\ \bar{\underset{\sim}{D}}_{12}^T & \bar{\underset{\sim}{D}}_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u} \\ \underset{\sim}{y}_2 \end{bmatrix} - \begin{bmatrix} \cdot \\ \underset{\sim}{I} \end{bmatrix} \underset{\sim}{y}_* = \begin{bmatrix} \bar{\underset{\sim}{b}}_1 \\ \bar{\underset{\sim}{b}}_2 \end{bmatrix}$$

$$\underset{\sim}{y}_2 \geq \underset{\sim}{0}, \quad \underset{\sim}{y}_* \geq \underset{\sim}{0}, \quad \underset{\sim}{y}_2^T \underset{\sim}{y}_* = 0$$

The comparison of the above system with the deformation analysis formulations (5.2.34) to (5.2.37) immediately identifies $\underset{\sim}{y}_2$ with

$\underset{\sim}{u}_*$ and $\underset{\sim}{y}_*$ with $-\underset{\sim}{\Phi}_*$. In vectors $\bar{\underset{\sim}{b}}_1$ and $\bar{\underset{\sim}{b}}_2$ two terms can be distinguished, one involving the fixed loading $\underset{\sim}{\lambda}_0$ and prescribed dislocations $\underset{\sim}{u}_D$ together with the non-linear corrective variables $\underset{\sim}{\omega}$, the other dependent on the load parameter. Introducing these specializations in the system above and continuing to treat the elements of $\underset{\sim}{\omega}$ as constants, the resulting system emerges as a PARAMETRIC (in λ) LINEAR COMPLEMENTARITY PROBLEM; the objective of the solution procedure is to trace the sequence of values the variables $\underset{\sim}{u}$, $\underset{\sim}{u}_*$ and $\underset{\sim}{y}_*$ take as parameter $\underset{\sim}{\lambda}$ is increased from zero. The quadratic programs (5.1.7) and (5.1.13) are replaced by the LINEAR PROGRAMMING problem

Max λ , subject to:-

$$\left[\begin{array}{c|c} \bar{\underset{\sim}{D}}_{11} & \bar{\underset{\sim}{D}}_{12} \\ \hline \bar{\underset{\sim}{D}}_{12}^T & \bar{\underset{\sim}{D}}_{22} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{u} \\ \hline \underset{\sim}{u}_* \end{array}\right] - \left[\begin{array}{c} \cdot \\ \hline \underset{\sim}{I} \end{array}\right] \underset{\sim}{y}_* - \left[\begin{array}{c} \underset{\sim}{a}_1 \\ \hline \underset{\sim}{a}_2 \end{array}\right] \lambda = \left[\begin{array}{c} \underset{\sim}{w}_1 \\ \hline \underset{\sim}{w}_2 \end{array}\right]$$

$$\underset{\sim}{u}_* \geq \underset{\sim}{0} \qquad \underset{\sim}{y}_* \geq \underset{\sim}{0}$$

together with the complementarity condition over $\underset{\sim}{u}_*$ and $\underset{\sim}{y}_*$ which is enforced by preventing the simultaneous presence of u_{*_i} and y_{*_i} in the pivoting basis. The adaptation of the Wolfe-Markowitz algorithm to elastoplastic deformation (linear) analysis problems developes as follows:

STEP 1: Pivot the unrestricted variables into the basis and update the remaining constraints:

$$\underset{\sim}{u} = \bar{\underset{\sim}{D}}^{-1} \left[-\bar{\underset{\sim}{D}}_{12}\underset{\sim}{u}_* + \underset{\sim}{a}_1 \lambda + \underset{\sim}{w}_1 \right], \quad \underset{\sim}{D}'\underset{\sim}{u}_* - \underset{\sim}{y}_* - \underset{\sim}{a}'\lambda = \underset{\sim}{w}' \quad (5.4.3a,b)$$

STEP 2: Collect in $\underset{\sim}{u}_*^B$ the plastic multipliers associated with the yield modes already activated and re-write system (5.4.3b) as

$$\left[\begin{array}{c|c} \underset{\sim}{D}'_{11} & \underset{\sim}{D}'_{12} \\ \hline \underset{\sim}{D}'^{T}_{12} & \underset{\sim}{D}'_{22} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{u}_*^B \\ \hline \underset{\sim}{u}_*^N \end{array}\right] - \left[\begin{array}{c} \underset{\sim}{y}_*^N \\ \hline \underset{\sim}{y}_*^B \end{array}\right] - \left[\begin{array}{c} \underset{\sim}{a}'_1 \\ \hline \underset{\sim}{a}'_2 \end{array}\right] \lambda = \left[\begin{array}{c} \underset{\sim}{w}'_1 \\ \hline \underset{\sim}{w}'_2 \end{array}\right]$$

$\underset{\sim}{u}_*^N$ may contain information on previously developed plastic strains at sections currently behaving elastically; that information is passed on into $\underset{\sim}{w}'$.

STEP 3: Pivot $\underset{\sim}{u}_*^B$ into the basis. Then

$$\underset{\sim}{u}_*^B = \underset{\sim}{D}_{11}'^{-1}\left[-\underset{\sim}{D}_{12}'\,\underset{\sim}{u}_*^N + \underset{\sim}{y}_*^N + \underset{\sim}{a}_1'\,\lambda + \underset{\sim}{u}_1'\right] \qquad (5.4.4a)$$

and

$$\underset{\sim}{D}_u\,\underset{\sim}{u}_*^N + \underset{\sim}{D}_y\,\underset{\sim}{y}_*^N + \underset{\sim}{y}_*^B + \bar{\underset{\sim}{a}}\,\lambda = \bar{\underset{\sim}{w}} \qquad (5.4.4b)$$

STEP 4: Check for plastic unstressing and modify the basis if necessary.

STEP 5: Select outgoing basic variable. As $\underset{\sim}{u}_*^N$ and $\underset{\sim}{y}_*^N$ are non-basic, and thus null, system (5.4.4b) identifies $\bar{\underset{\sim}{w}}=\underset{\sim}{y}_*^B$ for $\lambda=0$; hence, by definition, $\bar{\underset{\sim}{w}} \geqslant \underset{\sim}{0}$. If λ is to increase from zero it has to become a basic variable by replacing one of the $\underset{\sim}{y}_*^B$. Say that y_{*i}^B is to leave the basis; then y_{*i}^B becomes zero meaning that the i-th yield mode is about to be activated. The load parameter takes the place of y_{*i}^B on the basis with the value

$$\lambda = \frac{\bar{w}_i}{\bar{a}_i} - \frac{y_{*i}^B}{\bar{a}_i} \qquad (5.4.5a)$$

where $\bar{a}_i > 0$ if λ is to increase ($\bar{w}_i > 0$ and $y_{*i}^B = 0$). If y_{*i}^B leaves the basis, y_{*j}^B takes the value

$$y_{*j}^B = \bar{w}_j - \bar{a}_j\,\frac{\bar{w}_i}{\bar{a}_i}. \qquad (5.4.5b)$$

As $\bar{a}_i$, $\bar{w}_i$ and $\bar{w}_j$ are non-negative, if $\bar{a}_j$ is non-positive y_{*j}^B may increase further; otherwise it tends to decrease and the non-negativity condition gives $\bar{w}_j/\bar{a}_j \geq \bar{w}_i/\bar{a}_i$. This inequality provides the rule to select the outgoing variable y_{*i}^B; among all positive elements of $\bar{\underset{\sim}{a}}$ select that which minimizes the ratio $\bar{w}_i/\bar{a}_i$, thus guaranteeing that the remaining y_{*j}^B do not become negative.

STEP 6: Pivot λ into the basis, replacing y_{*i}^B. The equation of system (5.4.4b) associated with the basic λ reads

$$d_{i\alpha}\,u_{*\alpha}^N + d_{i\alpha}'\,y_{*\alpha}^N + y_{*i}^B + \bar{a}_i\,\lambda = \bar{w}_i$$

where the $d_{i\alpha}$ and $d_{i\alpha}'$ are elements of $\underset{\sim}{D}_u$ and $\underset{\sim}{D}_y$, respectively.

STEP 7: Update the program variables using (5.4.5), (5.4.4a) and (5.4.3a).

STEP 8: The plastic multiplier u^N_{*i} may now enter the basis. As $\bar{a}_i$ and u^N_{*i} are non-negative, either of the following situations may arise: a) if $d_{ij} \leq 0$, λ will not decrease and the maximization procedure may proceed after pivoting u^N_{*i} into the basis and returning to step 4, or b) if $d_{ij} > 0$, λ may not be further increased; λ_{max} has been attained. Note that if some of the d_{ij} are null, the corresponding u^N_{*j} can be brought into the basis with a non-trivial value (if the corresponding y^B_{*j} happened to be zero) without affecting the solution found for λ , thus revealing a kinematically multiple solution.

This algorithm was applied to obtain the discontinuity points (the program BASIC SOLUTIONS) of the piecewise-linear graph shown in Fig. 5.2 which represents the behaviour of the simple portal frame illustrated in Fig. 5.3, as predicted by the simple bending theory of elastoplastic linear analysis. Each basic solution represents the activation of a new plastic hinge, the formation sequence of which is shown in Table 5.8 under "CS", standing for activated critical section.

(*) Estimated from graph, Horne (1971)
(1) $\pi_n \neq 0$, & tangent stability functions
(2) $\pi_n = 0$, & "exact" stability functions
(3) $\pi_n \neq 0$, & "exact" stability functions

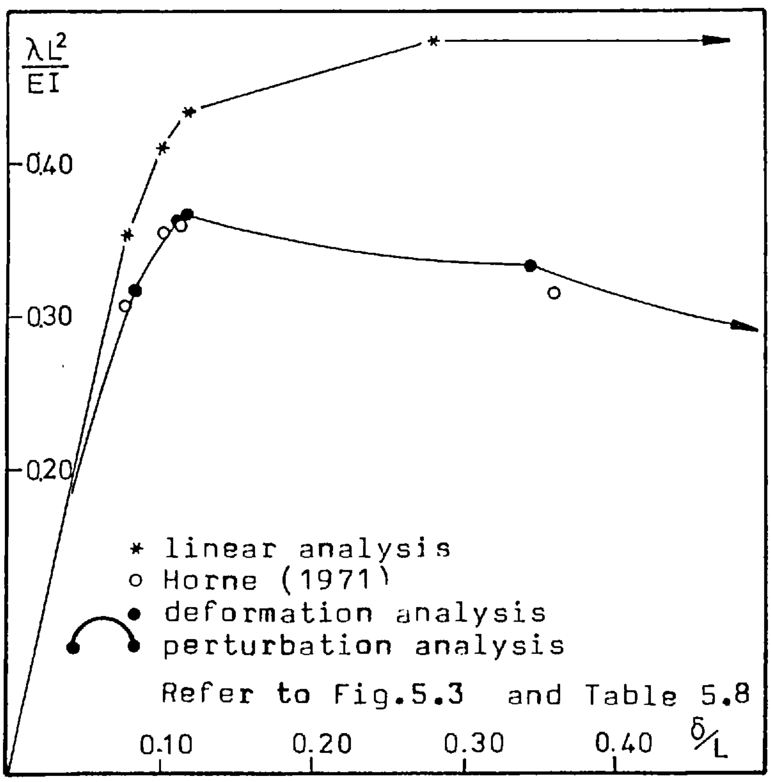
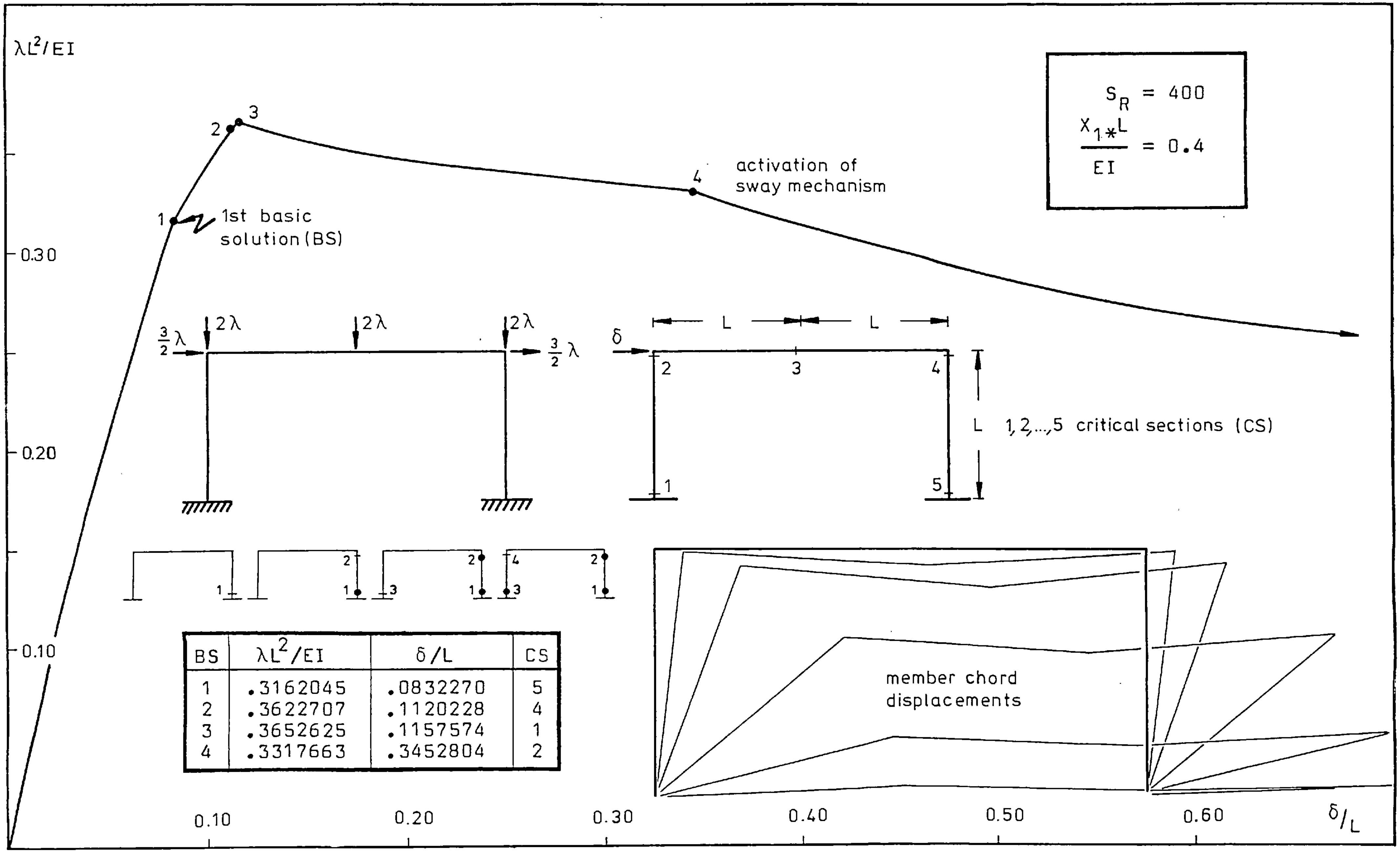


FIGURE 5.2

	BS	$\lambda L^2/EI$	δ/L	CS
ZERO-TH ORDER	1	.3516	.0769	5
	2	.4096	.10	4
	3	.4333	.1167	1
	4	.48	.28	3
HORNE (1971)	1	.306	.077*	5
	2	.352	.104*	4
	3	.360	.114*	1
	4	.314	.360	2
FIRST-ORDER (1)	1	.3110	.0806	5
	2	.3574	.1105	4
	3	.3613	.1159	1
	4	.3145	.3480	2
FIRST-ORDER (2)	1	.3111	.0807	5
	2	.3572	.1106	4
	3	.3613	.1162	1
	4	.3145	.3480	2
FIRST-ORDER (3)	1	.3128	.0806	5
	2	.3606	.1099	4
	3	.3657	.1162	1
	4	.3498	.3623	2
"EXACT"	1	.3162	.0832	5
	2	.3623	.1120	4
	3	.3653	.1158	1
	4	.3318	.3453	2

TABLE 5.8



BS	$\lambda L^2/EI$	δ/L	CS
1	.3162045	.0832270	5
2	.3622707	.1120228	4
3	.3652625	.1157574	1
4	.3317663	.3452804	2

FIGURE 5.3

In many situations the analyst may not be interested in the history of the structure deformations but only in the variation (in general a reduction) of the structure loading capacity when the effects of finite displacements, as well as those of the developing axial forces, are accounted for. The elastoplastic failure load can be obtained by repetitive application of the algorithm just described, updating recurrently the structural matrix $\underline{D}$ and the stipulation vector $\underline{w}$ using the information provided by the previous iteration; the structure, in its original state, is "loaded" with the axial forces and finite displacements effects formed when attaining λ_{max} in the previous iteration. As the sequence of plastic hinge formation thus formed can differ from the actual sequence provided by an historical analysis, the check for plastic unstressing is deprived of its significance and step 4 should therefore be disregarded. Eventually, in general 3 to 4 iterations prove to be sufficient, the solutions associated with two consecutive failure loads are found to converge, satisfying a pre-established accuracy requirement. A similar procedure has been discussed by Corradi et alia (1973).

If however the analyst wishes to obtain the correct static and kinematic configurations developing along an equilibrium path, the following modifications have to be introduced in the above algorithm:

STEP 7:(a) Update variables $\underline{u}$ and $\underline{u}_*$. (b) Check convergence; if convergence achieved go to 7f). (c) Recalculate λ using the Principle of Virtual Work in the form

$$\lambda = (\underline{u}^T \bar{\underline{D}}_{11} \underline{u} - \underline{u}_*^T \bar{\underline{D}}_{22} \underline{u}_* - \underline{u}^T \underline{w}_1 + \underline{u}_*^T \underline{w}_2)/(\underline{u}^T \underline{a}_1 - \underline{u}_*^T \underline{a}_2)$$

or an equivalent one. (d) Check convergence for two successive load parameters; if convergence is achieved go to 7f). (e) Recalculate static variables using new λ and return to 7c). (f) Update $\bar{\underline{D}}$ and $\underline{w}$. If coming from 7d) return to 1.

STEP 8: Start new basic solution: a) If $d_{ij} \leq 0$ make u_{*i}^N basic and return to 1, b) If $d_{ij} > 0$ start minimization of λ if the post-buckling path is to be investigated. Otherwise stop.

Step 7c) proved very efficient for the improvement of the algorithm rate of convergence. It reduced the number of iterations per basic solution from more than 20 in some cases, to 2 to 5 depending on the "length" of the load increment between basic solutions.

The efficiency of this algorithm can be significantly improved if, instead of updating repeatedly the arrays $\bar{\underset{\sim}{D}}$ and $\bar{\underset{\sim}{w}}$ until convergence of the basic solution is achieved, all correction terms are concentrated in $\bar{\underset{\sim}{w}}$, leaving $\bar{\underset{\sim}{D}}$ unchanged, and procuring convergence using sensitivity analysis techniques.

The incremental procedures, by their very nature, allow for a direct control of which of the yield modes are currently activated. As a consequence, the Kuhn-Tucker Conditions of the incremental analysis programs can be expressed as a strict equality system:

$$\left[\begin{array}{c|c} \underset{\sim}{E}_{11} & \underset{\sim}{E}_{12} \\ \hline \underset{\sim}{E}_{12}^{T} & \underset{\sim}{E}_{22} \end{array}\right] \left[\begin{array}{c} \Delta \underset{\sim}{u} \\ \hline \Delta \underset{\sim}{u}_{*}^{B} \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{e}_{1} \\ \hline \underset{\sim}{e}_{2} \end{array}\right] \lambda + \left[\begin{array}{c} \Delta \underset{\sim}{w}_{1} \\ \hline \Delta \underset{\sim}{w}_{2} \end{array}\right] \qquad \begin{array}{r} (5.4.6a) \\ (5.4.6b) \end{array}$$

The complementarity condition between the plastic multipliers $\Delta \underset{\sim}{u}_{*}$ and the plastic potentials $\Delta \underset{\sim}{\Phi}_{*} = -\Delta \underset{\sim}{y}_{*}$ is automatically accounted for; the absence of $\Delta \underset{\sim}{u}_{*}^{N}$ from the pivoting basis makes $\Delta \underset{\sim}{u}_{*}^{N} = \underset{\sim}{0}$, while equation (5.4.6b) implies $\Delta \underset{\sim}{y}_{*}^{N} = \underset{\sim}{0}$.

The detection of activation of a new yield mode can be performed in a number of ways. The crudest process is to maintain a fixed step length and check the yield rule after each increment; if contravened, a sequence of trial step lengths is attempted until incipient yielding is exposed. The method has obvious disadvantages and other more refined techniques have been proposed; procedures to reduce the violation of the plasticity condition by predictor and corrector schemes can be found in Zienkiewicz et alia (1969) and Argyris and Scharpf (1972). The linear prediction techniques, as for instance the one suggested by Jennings and Majid (1965), are conceptually very similar to the rule in the simplicial methods for determining which variable is to leave the pivoting basis (step 5 in the algorithm described in the above).

The numerical solution of system (5.4.6) can be performed using any of the many procedures available. As the value for the corrective terms $\Delta \underline{w}$ is not known a priori, the "exact" solution has to be gained after successive approximations; sensitivity analysis techniques should be preferred to strictly iterative procedures.

The essential advantage of the perturbation techniques is their ability to provide a regular pattern for successive improvements of the solution. For the perturbation analysis formulations, system (5.4.6) generates an infinite set of strict equality recursive systems, each of which can be expressed in the form

$$\left[\begin{array}{c|c} \underline{E}_{11} & \underline{E}_{12} \\ \hline \underline{E}_{12}^T & \underline{E}_{22} \end{array}\right] \left[\begin{array}{c} \underline{u} \\ \hline \underline{u}_*^B \end{array}\right]_i - \left[\begin{array}{c} \cdot \\ \hline \underline{I} \end{array}\right] \underline{y}_{*i}^N = \left[\begin{array}{c} \underline{e}_1 \\ \hline \underline{e}_2 \end{array}\right] \lambda_i + \left[\begin{array}{c|c} \underline{F}_{11} & \underline{F}_{12} \\ \hline \underline{F}_{21} & \underline{F}_{22} \end{array}\right] \left[\begin{array}{c} \underline{f}_1 \\ \hline \underline{f}_2 \end{array}\right]_i \qquad \begin{array}{c}(5.4.7a)\\(5.4.7b)\end{array}$$

where, for later convenience, the plastic potentials $-y_{*i}^N = 0$ were included; the i-th order corrective term f_{j_i} is a function of the system variables of order lower than i-th.

The detection of plastic straining becomes particularly simple. Let Φ_{*ri} be the i-th coefficient in the series expansion of the r-th plastic potential $\Delta\Phi_{*r}$; then, by definition

$$\Phi_{*r} = \Phi_{*r}^o + \sum_{i=1}^{\infty} \Phi_{*ri} \frac{\varepsilon^i}{i!} \leq 0 \qquad (5.4.8)$$

Let $\bar{\varepsilon}$ be the adopted step length; if for $\varepsilon = \bar{\varepsilon}$ the above condition is contravened, we may select the step length ε_{max} responsible for the activation of the r-th yield mode using the following procedure; the onset of plastic straining is characterized by

$$\Phi_{*r}^o + \sum_{i=1}^{\infty} \Phi_{*ri} \frac{(\varepsilon_{max})^i}{i!} = 0$$

which can be reversed to give

$$\varepsilon_{max} = \sum_{i=1}^{\infty} \varphi_i \left(- \frac{\Phi_{*r}^o}{\Phi_{*r_1}} \right)^i \qquad (5.4.9)$$

where $\varphi_1 = 1$, $\varphi_2 = -\frac{1}{2!} \frac{\Phi_{*r2}}{\Phi_{*r1}}$, $\varphi_3 = 2\varphi_2^2 - \frac{1}{3!} \frac{\Phi_{*r3}}{\Phi_{*r1}}$, ...

The solution procedure is straightforward and can be summarized as follows:

STEP 1: Set up matrices $\underset{\sim}{E}$, $\underset{\sim}{F}$ and vector $\underset{\sim}{e}$.

STEP 2: Solve system (5.4.7) for $\underset{\sim}{u}_i$ and $\underset{\sim}{u}^B_{*i}$, to give

$$\underset{\sim}{u}_i + \bar{\underset{\sim}{A}}\, \underset{\sim}{y}^N_{*i} = \bar{\underset{\sim}{a}}\, \lambda_i + \bar{\underset{\sim}{R}}_i \qquad (5.4.10a)$$

$$\underset{\sim}{u}^B_{*i} - \underset{\sim}{A}\, \underset{\sim}{y}^N_{*i} = \underset{\sim}{a}\, \lambda_i + \underset{\sim}{R}_i \qquad (5.4.10b)$$

STEP 3: Check for plastic unstressing and change the basis if necessary.

STEP 4: For load-control programs $\lambda_i = \delta_{1i}$ since $\Delta\lambda = \varepsilon$.

1) The first-order solution is $\underset{\sim}{u}_1 = \bar{\underset{\sim}{a}}$, $\underset{\sim}{u}^B_{*1} = \underset{\sim}{a}$; evaluate $\underset{\sim}{R}_2$ and $\bar{\underset{\sim}{R}}_2$.

⋮

i) The i-th order solution is $\underset{\sim}{u}_i = \bar{\underset{\sim}{R}}_i$, $\underset{\sim}{u}^B_{*i} = \underset{\sim}{R}_i$; evaluate $\underset{\sim}{R}_{i+1}$ and $\bar{\underset{\sim}{R}}_{i+1}$.

⋮

STEP 5 : Determine the step length from $\varepsilon^n = \mathcal{E}.n!/y_n$ where $\mathcal{E}$ is the allowable error, y a generic program variable and n the order of the highest solution considered (n=3, in general).

STEP 6: Evaluate the plastic potential coefficients Φ^B_{*i} and check the yield rule (5.4.8) for each of the yield modes yet to be activated. If the yield rule is contravened re-calculate the step-length through (5.4.9).

STEP 7: Evaluate the generic incremental variable by using (2.1.52).

STEP 8: Update the problem variables and return to step 1.

The above procedure can be readily adapted to allow any other variable, contained either in $\Delta\underset{\sim}{u}$ or in $\Delta\underset{\sim}{u}^B_*$, to be used as a control parameter.

The fixed-base portal frame of Fig. 5.3 was first analyzed by Horne (1971). The dimensions of the frame and the applied loads are shown in the same figure. The flexural rigidity is $EI=2.5X_{1*}L$, the shape factor 1.15 and the ratio of radius of gyration to the cross-section depth 0.4; these values, together with the typical ratio $E/\sigma_*=900$ for mild steel, give an unrealistically high slenderness ratio of 400 for the frame members.

The discontinuity points of the non-linear graph in Fig. 5.2 represent the "basic solutions" provided by the (iterative) Wolfe-Markowitz algorithm for solving deformation analysis programs. Table 5.8 summarizes the results obtained using first-order formulations. The results presented by Horne (1971) appear to be equivalent to those provided by the first-order formulation with zero π_n forces. Most direct stiffness methods use tangent approximations for the stability functions; as the results in the same table show, this approximation makes little difference in elastoplastic sway frames where the axial forces are usually small.

The non-linear graph in Fig. 5.2, shown in more detail in Fig. 5.3, representing the equilibrium path associated with the prescribed proportional loading, was obtained using the perturbation analysis numerical procedure. The identification of the formation

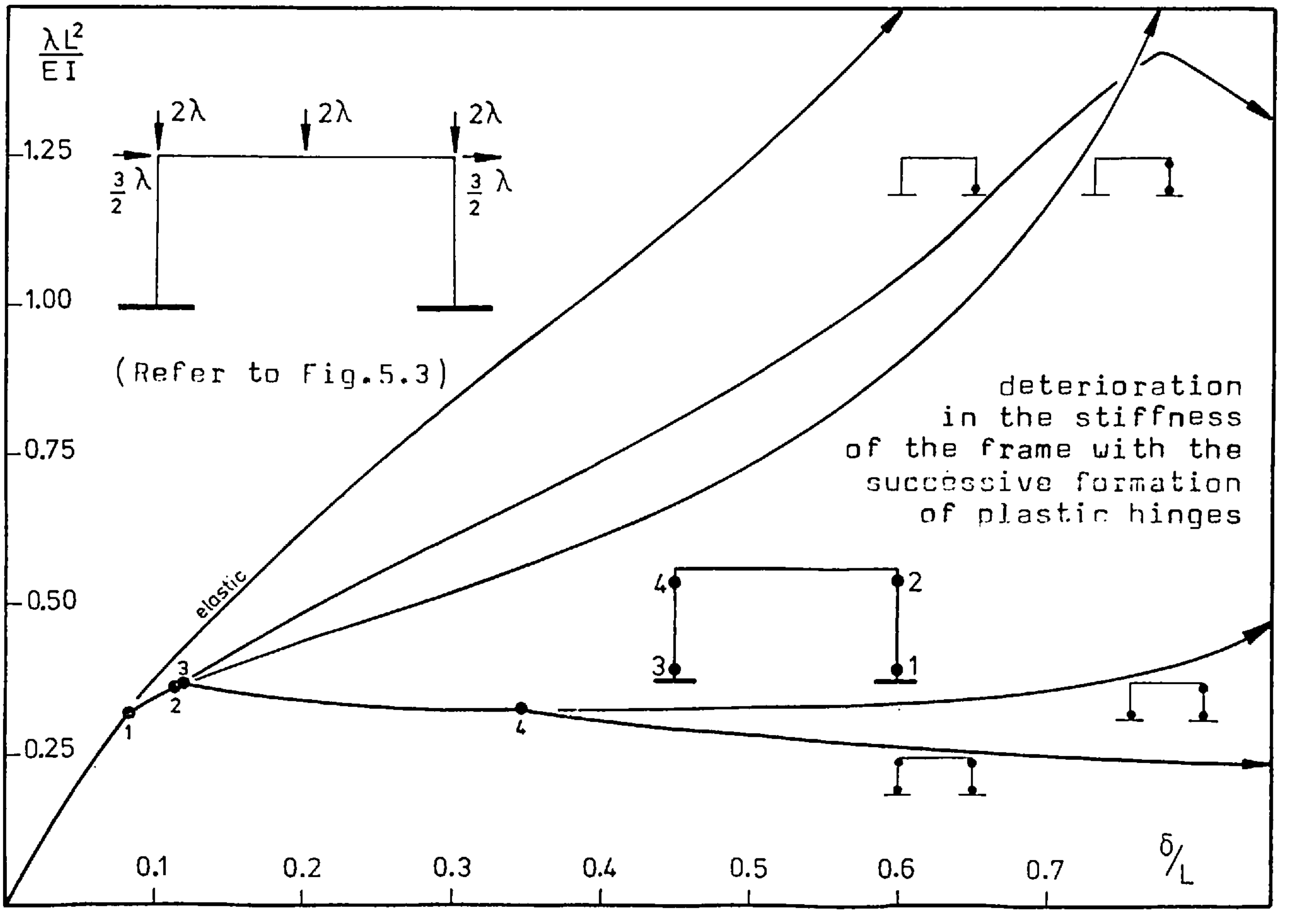


FIGURE 5.4

of a new plastic hinge was performed using the series reversion procedure described in the above. The results summarized in the table in Fig. 5.3 show an excellent agreement with those provided by the "exact" deformation analysis programs, the latter being given in Table 5.8.

The effect on the frame deformability caused by the formation of a new plastic hinge is shown in Fig. 5.4.

The progressive deterioration in the stiffness of elasto-plastic structures has been discussed by Wood (1958) who introduced the concept of "deteriorated critical load". This concept has no physical basis in exact calculations, as Fig. 5.4 shows; the deterioration of the frame stiffness immediately after the activation of a new yield mode can be recovered after increasing deformations. The deteriorated critical load concept has been used, among others, by Horne (1960, 1963) in the development of methods for practical estimation of the failure load of elastoplastic frames.

5.4.3 Critical Points

Let system (5.4.5) be expressed in the form

$$\underset{\sim}{E}\, \underset{\sim}{u}_i' = \underset{\sim}{e}\, \lambda_i + \underset{\sim}{R}_{u_i} + \underset{\sim}{R}_{\lambda_i} \tag{5.4.11}$$

where $\underset{\sim}{R}_{u_i} = \underset{\sim}{R}_{u_i}(\underset{\sim}{u}_1', \underset{\sim}{u}_2', \ldots, \underset{\sim}{u}_{i-1}')$ and $\underset{\sim}{R}_{\lambda_i} = \underset{\sim}{R}_{\lambda_i}(\lambda_1, \lambda_2, \ldots, \lambda_{i-1})$, the latter vanishing for piecewise-linear paths. When matrix $\underset{\sim}{E}$ becomes singular, the values taken by $\underset{\sim}{u}_i'$ for a given load increment become indefinite thus revealing the occurrence of a CRITICAL POINT. If the analysis is to be extended beyond this point, it is necessary to determine its nature; in general, a critical point is a limit point or/and a bifurcation point.

This problem has been intensively and extensively discussed in the context of elastic systems and the objective of the present subsection is to bring that knowledge into the framework of the formulation being proposed. Equations (5.4.11) may well be interpreted as the governing system of an elastic structure with additional kinematic indeterminacies $\underset{\sim}{u}_*^B$; the theory and techniques of elastic stability can thus be applied if further constrained by the yield and flow rules of plasticity. The yield rule poses no special problems; the identification and solution of plastic unstressing may not be a complex procedure but can easily become a laborious one, as shown in the following subsection where a technique for the identification and solution of plastic unstressing is presented.

A detailed discussion was made by Thompson (1963, 1969) on the properties, classification and solution of critical points. Schemes of diagonalization, although theoretically very attractive, are in general numerically inefficient and tend to be abandoned, e.g. Thompson and Hunt (1971). Of the remaining numerical solution procedures, Gaussian elimination is still favoured by most analysts; a detailed discussion on this matter can be found in Wilkinson (1965).

Let us assume then that system (5.4.11) is to be solved by sequential pivoting; if a pivot, say e_{kk}, is found to be null (or nearly so) a new pivot $e_{jk} \neq 0$ $(k < j \leqslant n)$ is procured and, if found, the k-th and j-th rows of $\underline{E}$ are interchanged; otherwise, pivoting is allowed to proceed to stage k+1. Collecting in $\underline{v}_i^!$ the N' variable associated with non-zero pivots (N' being the rank of matrix $\underline{E}$) and in $\underline{v}_i$ the N variables associated with zero pivots, system (5.4.11) can be solved to give

$$\underline{I} \cdot \underline{v}_i^! = \underline{a}' \lambda_i + \underline{R}_i^! \quad , \quad \underline{0} \cdot \underline{v}_i = \underline{a} \lambda_i + \bar{\underline{R}}_i \qquad (5.4.12a,b)$$

In the terminology of stability analysis,a N-fold compound critical point has been found; from that point 2^N-1 (post-buckling) paths may emerge, of which only those satisfying the plasticity conditions are relevant.

Variables $\underline{v}_i^!$, termed PASSIVE, can be eliminated in $\bar{\underline{R}}_i$ so that system (5.4.12b) can be expressed exclusively in terms of the load parameter λ_i and of the ACTIVE variables $\underline{v}_i$:

$$a_j \lambda_i + R_{j_i} = 0 \quad , \quad j = 1,2, \ldots, N \quad (5.4.13)$$

where $R_{j_1} = 0$ (5.4.14a)

$$R_{j_2} = A_{j\alpha\beta} v_{\alpha_1} v_{\beta_1} + A_{j\alpha} v_{\alpha_1} \lambda_1 + A_j \lambda_1^2 \qquad (5.4.14b)$$

$$R_{j_3} = B_{j\alpha\beta\gamma}\, v_{\alpha_1}\, v_{\beta_1}\, v_{\gamma_1} + B_{j\alpha\beta}\, v_{\alpha_1}\, v_{\beta_1}\, \lambda_1 + B'_{j\alpha\beta}\, v_{\alpha_1}\, v_{\beta_2} +$$
$$B_{j\alpha} v_{\alpha_1}\, \lambda_1^2 + B'_{j\alpha} v_{\alpha_1}\, \lambda_2 + B''_{j\alpha}\, v_{\alpha_2} \lambda_1 + B_j \lambda_1^3 + B'_j\, \lambda_1\, \lambda_2 \qquad (5.4.14c)$$

⋮

in which the summation convention over α, β, and γ is adopted. The N (non-linear) equations (5.4.13) and the fixing of the path parameter ε are sufficient to determine the i-th order N+1 unknowns λ_i and $\underset{\sim}{v}_i$, although recurrence to higher-order equations is in general needed. The treatment of distinct critical points (N=1) is particularly simple and the two following cases may arise:

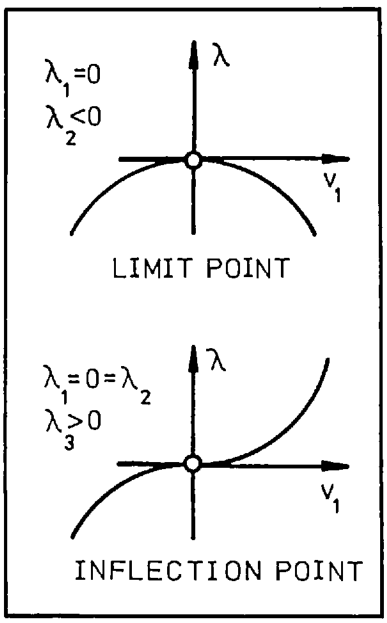


FIGURE 5.5

DISTINCT STATIONARY POINT: If $a_1 \neq 0$, equation (5.4.13) gives $\lambda_i = -R_{ji}/a_j$. As $\lambda_1 = 0, \Delta v_1$ has to be selected for path parameter, giving $v_{1_i} = \delta_{1i}$. The two possible situations for paths emerging from the third quadrant are represented in Fig. 5.5.

DISTINCT BRANCHING POINT: If $a_1 = 0$ (and assuming that $A_1 \neq 0$), the first-order equation (5.4.13) becomes trivial; in principle, either $\Delta\lambda$ or Δv_1 may be selected as a path parameter. In the latter case, the higher-order equations (5.4.13), together with (5.4.14), give

$$\lambda_1 = \frac{-A_{11} \pm \sqrt{A_{11}^2 - 4A_1 A_{11}}}{2A_1}$$

$$\lambda_2 = \frac{B_{1111} + B_{111}\, \lambda_1 + B_{11}\, \lambda_1^2 + B_1\, \lambda_1^3}{B'_{11} + B'_1\, \lambda_1}$$

⋮

The three cases of interest are represented in Fig. 5.6. The ASYMMETRIC point of bifurcation is characterized by a non-zero slope at the critical point ($\lambda_1 \neq 0$). When the slope vanishes, the point of bifurcation is said to be STABLE-SYMMETRIC if the curvature, controlled by λ_2, is positive, and UNSTABLE-SYMMETRIC otherwise.

The forms of the post-buckling equilibrium paths and their influence on the stability behaviour of structures were first established by Koiter (1945).

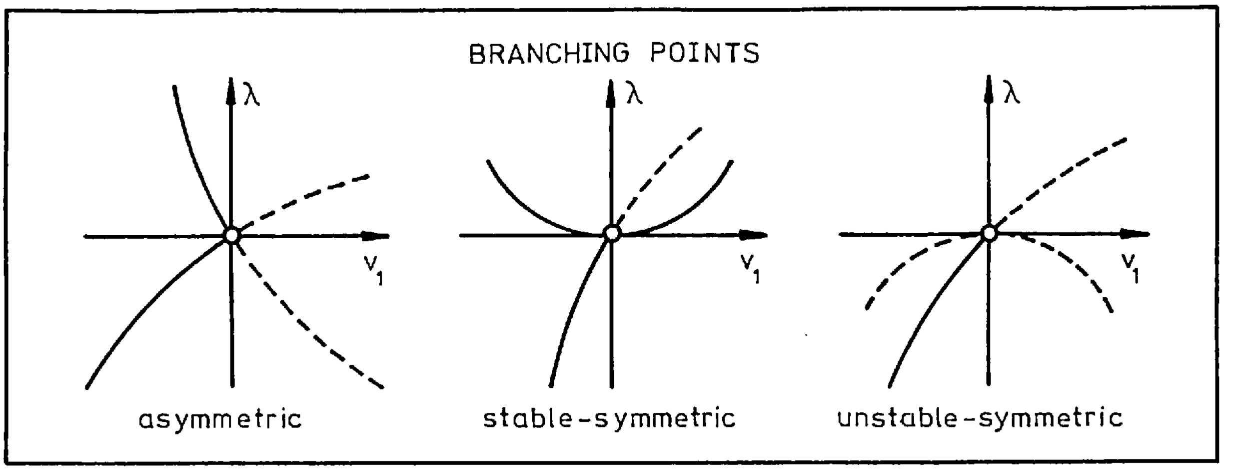


FIGURE 5.6

The (elastic) post-buckling analysis technique proposed by Ecer (1973) can be readily adapted to the incremental formulation suggested herein.

Compound critical points ($N > 1$) can be studied in a similar way, the solution procedure becoming however rather more involved. In the context of elastic systems, reference to the main aspects of the problem can be found in Sewell (1969, 1970), Chilver and Johns (1971), Thompson and Hunt (1971, 1973, 1975), Supple (1973) and Koiter and Pignataro (1976). Chilver (1973) has shown that in situations of nearly simultaneous branching points, the equilibrium path may suffer contortions too severe to be handled satisfactorily by a perturbation technique; the general practice in such situations is to induce the artificial coalescence of such points into a compound critical point.

The formulation being suggested is adaptable to studies in imperfection-sensitivity. Loading imperfections can be easily simulated, as well as geometric imperfections if use is made of the internal and external release devices; some types of material imperfections can however be more difficult to simulate in a realistic manner.

5.4.4 Plastic Unstressing

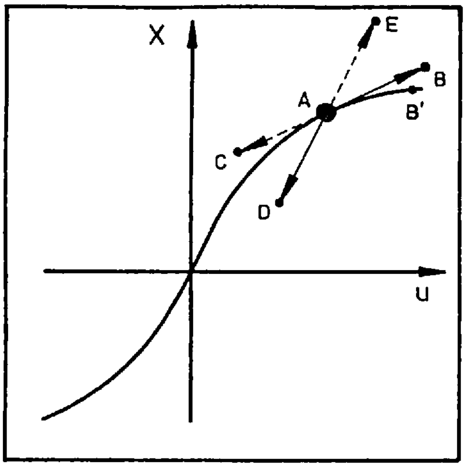


FIGURE 5.7

After pivoting the unrestricted variables into the basis, system (5.4.7) can be solved for the olastic multipliers associated with the currently active yield modes; for load-controlled programs and using (5.4.10b), the first-order solution is defined by

$$u^B_{*j_1} - a_{j\alpha} y^N_{*\alpha_1} = a_j \qquad (5.4.15)$$

Let us consider the k-th yield mode and isolate in the k-th equation of system (5.4.15) the complementary variable of u^B_{*k}:

$$u^B_{*k_1} - a_{k\alpha} y^N_{*\alpha_1} - a_{kk} y^N_{*k_1} = a_k, \qquad \alpha \neq k \qquad (5.4.16)$$

Depending on the sign of coefficient a_k and a_{kk}, the following four situations, illustrated in Fig. 5.7, may arise:

(1)	$a_k > 0$,	$-a_k/a_{kk} < 0$	: paths AB or AE	(5.4.17a)
(2)	$a_k < 0$,	$-a_k/a_{kk} > 0$	: paths AC or AD	(5.4.17b)
(3)	$a_k < 0$,	$-a_k/a_{kk} < 0$	: paths AC or AE	(5.4.17c)
(4)	$a_k > 0$,	$-a_k/a_{kk} > 0$	: paths AB or AD	(5.4.17d)

Whenever a_k is positive, $u^B_{*k_1}$ can remain basic and a further increase of the plastic strains will take place; if however the ratio $-a_k/a_{kk}$ is positive, $y^N_{*k_1}$ can be brought into the basis replacing $u^B_{*k_1}$, thus satisfying complementarity. Among the four paths indicated in Fig. 5.7, only paths AB and AD satisfy the plasticity relations:

$$\text{Path AB} \Rightarrow u^B_{*k_1} > 0 \quad \text{and} \quad y^N_{*k_1} = 0$$

$$\text{Path AC} \Longrightarrow u^B_{*k_1} < 0 \quad \text{although} \quad y^N_{*k_1} = 0$$
$$\text{Path AD} \Longrightarrow u^P_{*k_1} = 0 \quad \text{and} \quad y^N_{*k_1} = -\Phi^N_{*k_1} > 0$$
$$\text{Path AE} \Longrightarrow u^B_{*k_1} = 0 \quad \text{but} \quad y^N_{*k_1} = -\Phi^N_{*k_1} < 0$$

If a_k is positive and $-a_k/a_{kk}$ negative, the plastic strains developed at the critical section associated with the k-th yield mode will increase further if, for the remaining yield modes the following condition is satisfied:

$$a_j > 0 \quad \text{and} \quad -a_j/a_{jj} < 0 \quad \text{for all } j \neq k \qquad (5.4.18)$$

The first-order path is represented by AB, in Fig. 5.7, the higher-order solutions forcing B to coincide with B'.

Situation (2) also associates two paths, of which only one complies with the plasticity conditions. Path AC contravenes the flow rule and, assuming that condition (5.4.18) is fulfilled, path AB has to be followed instead, revealing the occurrence of PLASTIC UNSTRESSING, or de-activation of the yield mode; in the space of the stress components point A loses contact with the k-th yield mode and moves inwards. The plastic multiplier u^B_{*k} has to be removed from the basis and its complement y^N_{*k} brought into it, transforming system (5.4.15) into

$$y^N_{*k_1} - \frac{1}{a_{kk}} u^B_{*k_1} + \frac{1}{a_{kk}} a_{j\alpha} y^N_{*\alpha_1} = \frac{-a_k}{a_{kk}} \qquad (5.4.19a)$$

$$u^B_{*j_1} - \frac{a_{jk}}{a_{kk}} u^B_{*k_1} - \left(1 - \frac{a_{jk}}{a_{kk}}\right) a_{j\alpha} y^N_{*\alpha_1} = a_j + a_{jk} \frac{-a_k}{a_{kk}}, \quad j \neq k \qquad (5.4.19b)$$

The above equations show that if a_{jk} is negative and sufficiently large in absolute value, plastic unstressing may occur at the generic section j if

$$a_j - \frac{a_{jk}}{a_{kk}} a_k < 0 \quad \text{and} \quad \left(1 - \frac{a_{jk}}{a_{kk}}\right) a_{jj} > 0$$

Hence, even if all sections $j \neq k$ satisfy condition (5.4.18), the occurrence of plastic unstressing at section k and the subsequent change in the system basis, can modify the system equation sufficiently to open the four options (5.4.17) for the remaining sections $j \neq k$.

Before studying situations (3) and (4), let us consider the possibility of MULTIPLE UNSTRESSING. Assume that modes $k = p, q, r, \ldots$ were found to be associated with negative coefficients a_k and let mode q satisfy condition (5.4.17b). Then, from (5.4.19b)

$$u^B_{*j_1} = a_j + a_{jq}\, y^N_{*q_1} \quad , \quad j \neq k \qquad (5.4.20a)$$

and

$$u^B_{*k_1} = a_k + a_{kq}\, y^N_{*q_1} \quad , \quad k \neq q \qquad (5.4.20b)$$

where, from (5.4.19a)

$$y^N_{*q_1} = -\frac{a_q}{a_{qq}} > 0$$

The ratio a_q/a_{qq} is a measure of the tendency of mode q to unstress. If a_{jq} is positive and a_{kq} negative, the value taken by y^N_{*q1} does not affect qualitatively (5.4.20); the tendency of modes $j \neq k$ to strain plastically and of modes $k \neq q$ to unstress is magnified. If however a_{jq} is negative and a_{kq} positive, system (5.4.20) can be altered qualitatively, the disruption increasing with the value of the ratio $-a_q/a_{qq}$; the higher this ratio is, the higher is the possibility of sections j to unstress ($a_j + a_{jq}\, y^N_{*q_1}$ may become negative) and of further plastic strains to develop at sections $k \neq q$ ($a_k + a_{kq}\, y^N_{*q_1}$ may become positive). Hence the following procedure of identifying those sections where plastic unstressing and selecting those which in fact do so:

STEP 1 : Identify which of the yield modes are in situation (2)
STEP 2 : Select, among them, the one associated with the highest ratio $-a_q/a_{qq}$; remove u^B_{*q} from the basis and bring $Y^N_{*q_1}$ into it.
STEP 3 : Return to step 1 and repeat until the list of all modes in situation (2) is exhausted.

Both paths offered in situation (3) contravene the plasticity conditions. If all the remaining active yield modes comply with

(5.4.18) no changes can be introduced in the system basis and the load parameter cannot be further increased; the structure locks in the direction λ but it may well be able to deforme further if the control parameter is changed. Assume that variable u_c is chosen to replace λ as a control parameter; then $u_{c_i} = \delta_{1i}$ and from (5.4.10)

$$\lambda_1 = \frac{1}{\bar{a}_c} (1 + \bar{a}_{c\alpha} y^N_{*\alpha_1})$$

and

$$u^B_{*j_1} - (a_{j\alpha} + \frac{a_j}{\bar{a}_c} \bar{a}_{c\alpha}) y^N_{*\alpha_1} = \frac{a_j}{\bar{a}_c}$$

and any of options (5.4.17) is re-opened for every yield mode j.

Situation (4) opens the possibility for the generation of a multiple solution since both paths AB and AD satisfy the plasticity relations. A multiple solution will in fact exist if the following situations may occur simultaneously:

1) Path $u^B_{*k} > 0$, $y^N_{*k} = 0$: when u^B_{*k} is left in the basis all the remaining modes j satisfy condition (5.4.18).
2) Path $u^B_{*k} = 0$, $y^N_{*k} > 0$: when u^B_{*k} is replaced by y^N_{*k} in the basis (plastic unstressing), after the necessary re-adjustment in the system basis (which may induce plastic unstressing to occur elsewhere) all the (ultimately) active yield modes $j \neq k$ satisfy condition (5.4.18).

The above considerations can easily be adapted for the analysis of similar situations exposed when using either a deformation or an incremental analysis formulation.

Computer codes to detect, distinguish and solve situations (1) to (4) are of simple implementation; the routine we used starts by selecting and exhausting situations (2), following, in that order, by situations (4) and (3). In our limited experience, situation (4) has always occurred coupled with situation (3). After applying the multiple unstressing routine to every mode k in situation (4) it was always found that no yield mode remained in situation (3), of "apparent locking"; instead, plastic unstressing occurred at modes j initially in situation (1).

As an illustration, consider the single-bay, two-storey frame shown in Fig. 5.9, first studied by Horne (1963). All members are of the same symmetrical I-section with the web in the plane of the frame. The modulus of elasticity is 30 x 10^6 lb/in^2 and the yield stress (at which indefinite plastic deformations may develop) is 36 x 10^3 lb/in^2. The members slenderness ratio is 100, the variation of the full plastic moment with the axial force being negligible in frames with that order of slenderness ratio.

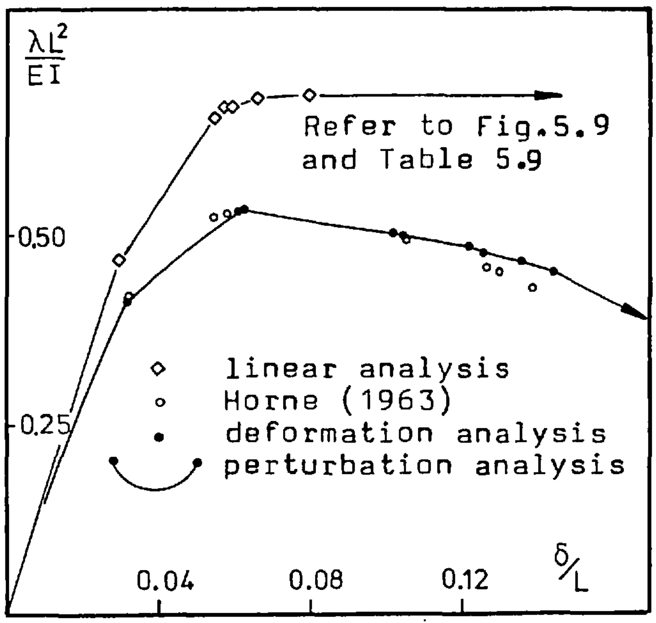
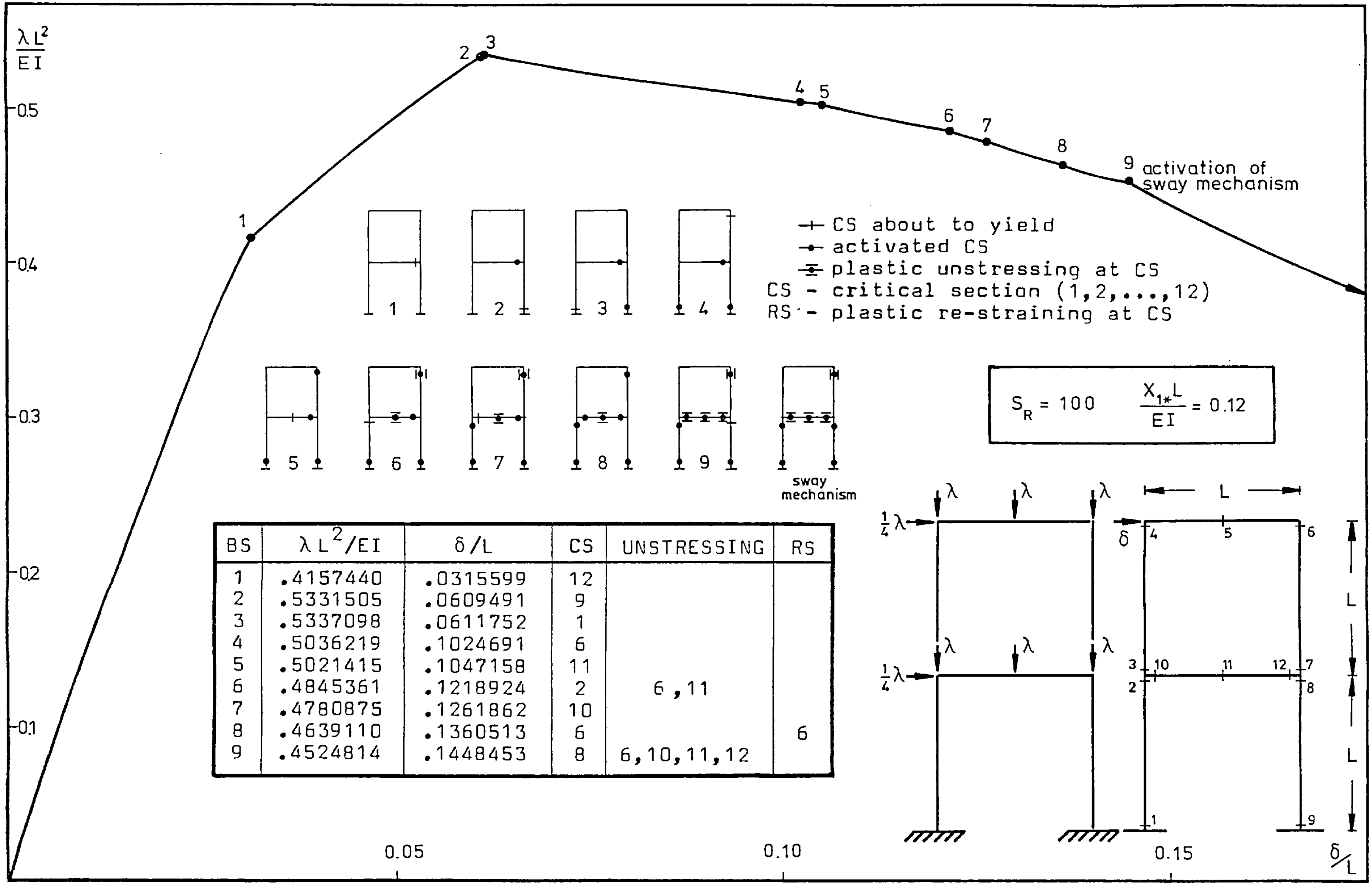


FIGURE 5.8

Horne has found that the frame collapses due to instability after the formation of the third plastic hinge. The solution presented by Horne appears to be equivalent to the results provided by the proposed first-order formulation with zero π_n forces, as shown in Table 5.9; Jenkins and Majid (1965) and Corradi (1977a), using first-order formulations, report values for the collapse load similar to Horne's.

The non-linear graph in Fig. 5.8, shown in more detail in Fig. 5.9, representing the equilibrium path followed by the frame when subject to the proportional loading indicated, was obtained using the perturbation analysis formulation; the solutions found at the onset of plastic straining at a new critical section, summarized



BS	$\lambda L^2/EI$	δ/L	CS	UNSTRESSING	RS
1	.4157440	.0315599	12		
2	.5331505	.0609491	9		
3	.5337098	.0611752	1		
4	.5036219	.1024691	6		
5	.5021415	.1047158	11		
6	.4845361	.1218924	2	6 ,11	
7	.4780875	.1261862	10		
8	.4639110	.1360513	6		6
9	.4524814	.1448453	8	6,10,11,12	

FIGURE 5.9

	BS	$\lambda L^2/EI$	δ/L	CS	UNSTRESSING
ZEROTH-ORDER	1	.4676	.0298	12	
	2	.6588	.0554	11	
	3	.6669	.0578	9	
	4	.6687	.0585	6	
	5	.6799	.0664	1	
	6	.6857	.0798	5	
HORNE(1963)	1	.4203	.033*	12	
	2	.5266	.055*	9	
	3	.5328	.060*	1	
	4	.5109	.097*	6	
	5	.496*	.106*	11	
	6	.459*	.127*	2	11
	7	.452*	.130*	10	
	8	.430*	.139*	8	6,10,12
FIRST-ORDER(1)	1	.4161	.0315	12	
	2	.5305	.0597	9	
	3	.5332	.0608	1	
	4	.5032	.1023	6	
	5	.5015	.1048	11	
	6	.4839	.1219	2	6,11
	7	.4782	.1257	10	
	8	.4652	.1347	6	
	9	.4533	.1437	8	6,10,11,12
FIRST-ORDER(2)	1	.4156	.0315	12	
	2	.5321	.0606	9	
	3	.5336	.0612	1	
	4	.5028	.1030	6	
	5	.5016	.1048	11	
	6	.4839	.1219	2	6,11
	7	.4781	.1257	10	
	8	.4650	.1347	6	
	9	.4532	.1436	8	6,10,11,12
FIRST-ORDER(3)	1	.4157	.0315	12	
	2	.5325	.0606	9	
	3	.5340	.0612	1	
	4	.5051	.1020	6	
	5	.5038	.1040	11	
	6	.4843	.1235	2	6,11
	7	.4788	.1274	10	
	8	.4685	.1346	6	
	9	.4544	.1458	8	6,10,11,12
"EXACT"	1	.4157	.0316	12	
	2	.5331	.0609	9	
	3	.5337	.0612	1	
	4	.5036	.1025	6	
	5	.5022	.1047	11	
	6	.4845	.1219	2	6,11
	7	.4781	.1262	10	
	8	.4640	.1360	6	
	9	.4525	.1449	8	6,10,11,12

(*) Estimated from graph
(1) $\pi_n=0$, tangent approx. of stability functions
(2) $\pi_n=0$ (3) $\pi_n\neq 0$

TABLE 5.9

in the table in Fig. 5.9, show a good agreement with the "basic solutions" provided by the deformation analysis using the proposed formulation, given in Table 5.9.

The solution found differs from Horne's from the moment critical section 2 becomes active, well into the post-buckling phase. Horne identifies plastic unstressing at critical section 11, followed by the activation of sections 10 and 8, the sway mechanism being then mobilized with sections 6, 10 and 12 unstressing simultaneously. When yielding started at section 2 we found that sections 6, 11 and 12 were in situation (2). Section 11 was found to show the highest tendency to unstress and the corresponding plastic multiplier was then removed from the basis; section 6, but not section 12, continued showing a tendency, to be confirmed, to unstress. The activation of sections 10 and 8, consecutive in Horne's solution, was separated by the re-activation of section 6 which re-unstressed later together with sections 10 and 12 to mobilize the same sway mechanism found by Horne. A sequence of the frame displacements in the sway mechanism phase is shown in Fig. 5.10.

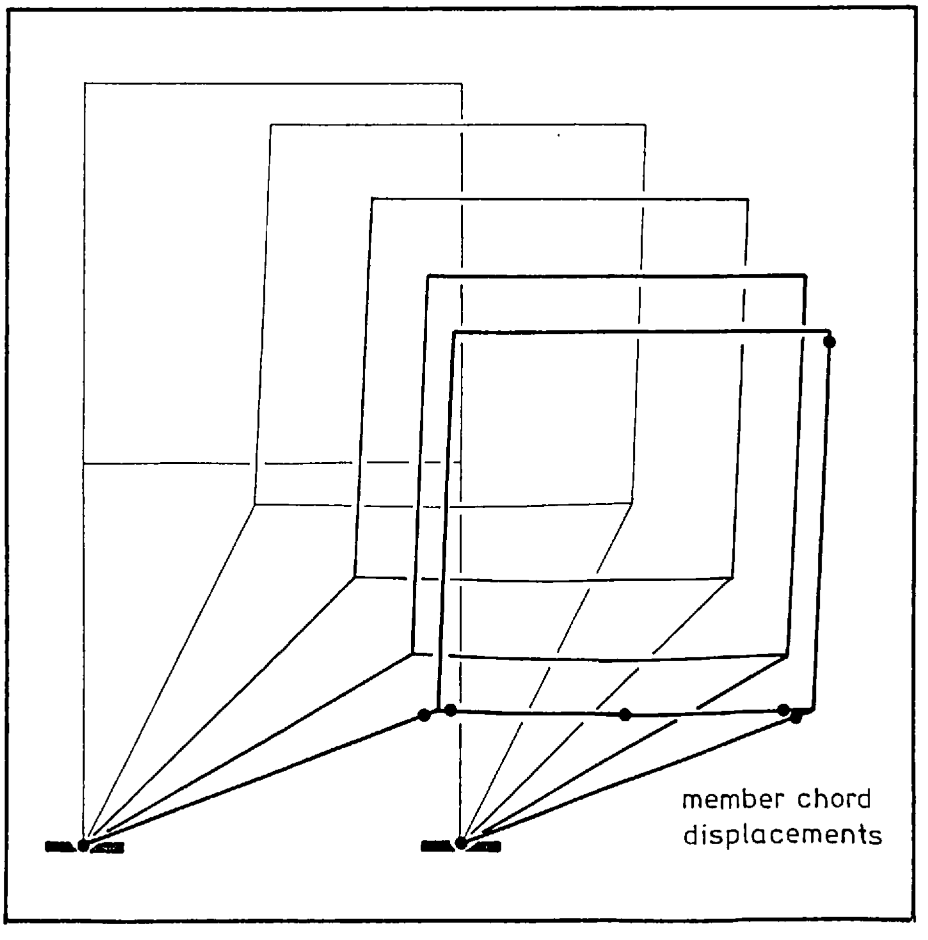
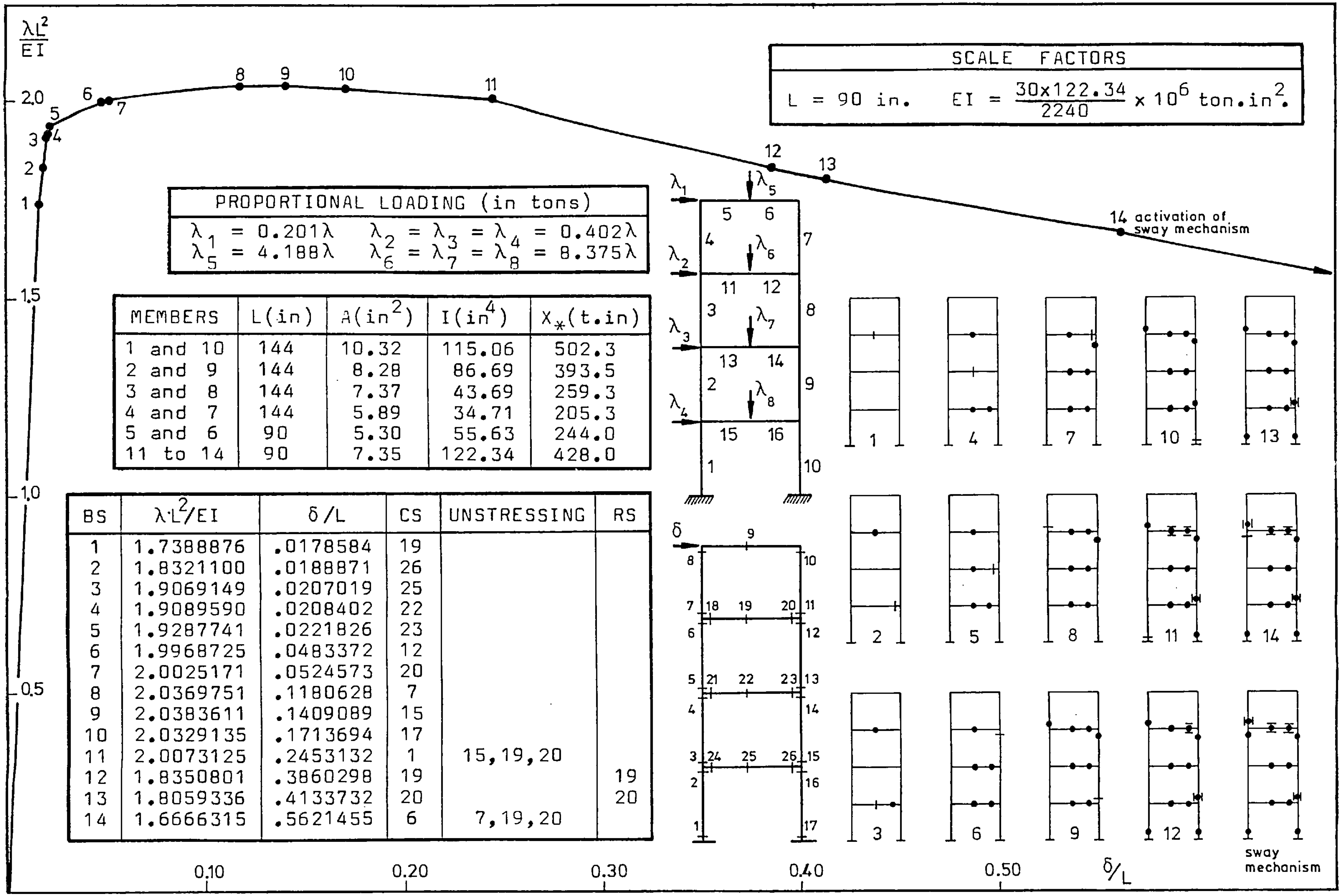


FIGURE 5.10



PROPORTIONAL LOADING (in tons)	
$\lambda_1 = 0.201\lambda$	$\lambda_2 = \lambda_3 = \lambda_4 = 0.402\lambda$
$\lambda_5 = 4.188\lambda$	$\lambda_6 = \lambda_7 = \lambda_8 = 8.375\lambda$

MEMBERS	L(in)	A(in^2)	I(in^4)	X_*(t.in)
1 and 10	144	10.32	115.06	502.3
2 and 9	144	8.28	86.69	393.5
3 and 8	144	7.37	43.69	259.3
4 and 7	144	5.89	34.71	205.3
5 and 6	90	5.30	55.63	244.0
11 to 14	90	7.35	122.34	428.0

BS	$\lambda L^2/EI$	δ/L	CS	UNSTRESSING	RS
1	1.7388876	.0178584	19		
2	1.8321100	.0188871	26		
3	1.9069149	.0207019	25		
4	1.9089590	.0208402	22		
5	1.9287741	.0221826	23		
6	1.9968725	.0483372	12		
7	2.0025171	.0524573	20		
8	2.0369751	.1180628	7		
9	2.0383611	.1409089	15		
10	2.0329135	.1713694	17		
11	2.0073125	.2453132	1	15,19,20	
12	1.8350801	.3860298	19		19
13	1.8059336	.4133732	20		20
14	1.6666315	.5621455	6	7,19,20	

FIGURE 5.11

When section 8 became active and the corresponding plastic multiplier was brought into the basis, sections 1, 2, 8 and 9 were found to be in situation (3) and section 12 in situation (4); when plastic unstressing of section 12 was performed, sections 1, 2, 8 and 9 went back to situation (1) (of further plastic straining) while sections 6 and 10 moved from situation (1) into situation (2); their plastic multipliers were then removed from the basis in that order since section 6 showed an higher tendency to unstress.

The frame in Fig. 5.11 is a four-storey, single-bay frame bent about the strong axis. This frame was first analysed by Wood (1958), who obtained a collapse configuration with the loads increased by a factor of approximately 1.90. At collapse Wood found that 5 plastic hinges were fully formed and partial yielding had also occurred at four other points, as shown in Fig. 5.12; in Wood's analysis spread of plasticity is accounted for. Jennings and Majid (1965) verified that the frame was still stable after the formation of 5 hinges; they found a collapse load of 1.93 when allowing a sixth hinge to form, after an astonishing increase in the displacements.

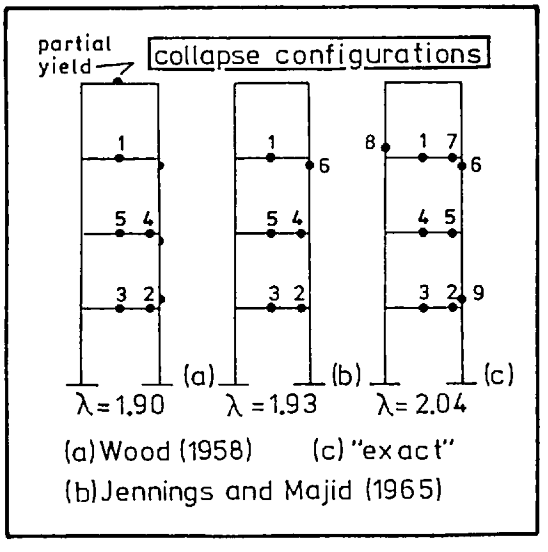


FIGURE 5.12

The solution we obtained using a perturbation analysis formulation showed that after the formation of the sixth hinge only 98% of the frame load-carrying capacity had already been used; the frame remained stable up to the formation of the ninth plastic hinge. Table 5.10 compares this solution with that of Jennings and Majid. The almost simultaneous activation of sections 22, 23 and 25 and the approximations in the first-order non-linear formulation adopted by Jennings and Majid can explain the different sequence of yielding found by them.

After collapse the frame is still able of accepting 98.5% of the collapse load in a range of displacements of about 74% of those

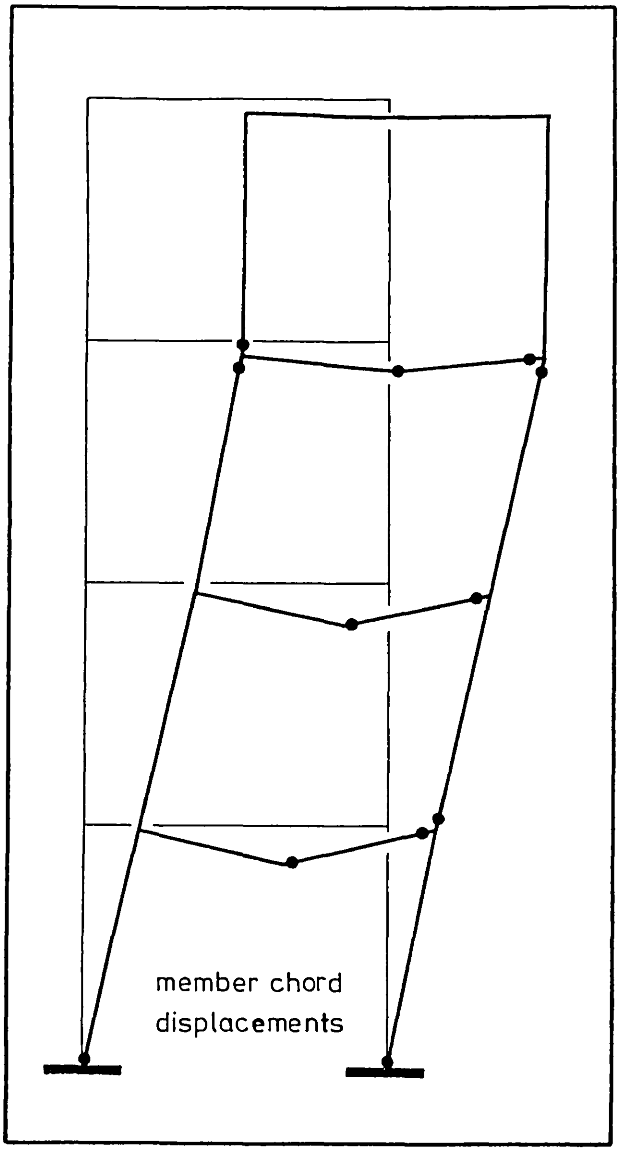


FIGURE 5.13

	BS	$\lambda L^2/EI$	δ/L	CS
JENNINGS & MAJID(1965)	1	1.74*	.019*	19
	2	1.79*	.020*	26
	3	1.88*	.023*	25
	4	1.90*	.024*	23
	5	1.91*	.025*	22
	6	1.93	.060*	12
"EXACT" (Refer to Fig.5.11)	1	1.739	.0179	19
	2	1.832	.0189	26
	3	1.907	.0207	25
	4	1.909	.0208	22
	5	1.929	.0221	23
	6	1.997	.0483	12
	7	2.003	.0525	20
	8	2.037	.1181	7
	9	2.038	.1409	15

(*) Estimated from graph

TABLE 5.10

developed up to collapse; of these, 84% are developed after the formation of the fifth plastic hinge. The multiple unstressing at the formation of the eleventh plastic hinge marks a significant loss of load-carrying capacity. At this stage sections 7, 15, 19 and 20 were found to be in situation (2). Among them, it was section 19 which showed the highest tendency to unstress and after removing the associated plastic multiplier from the system basis, section 7 was allowed to develop further plastic strains; plastic unstressing did occur though at sections 15 and 20. Sections 19 and 20 were then re-activated but, when section 6 yielded and a sway mechanism was mobilized, they re-unstressed again, together with section 7. Fig. 5.13 represents the frame displacements, well in the mechanism phase.

Most engineering structures attain the maximum load-carrying capacity for relatively small displacements, the post-failure behaviour being in most cases irrelevant for practical purposes. The first-order deformation analysis programs should perform successfully in such situations. The solutions they provide are fairly accurate, it being always possible to decrease the margin of error by relaxing the approximations as much as desired. The rate of convergence is good, with 3 to 4 iterations per basic solution; difficulties have however been experienced in the post-failure phase, in particular if it is highly sensitive to the type and degree of approximation introduced. Although able to detect the onset of a mechanism motion, the simplicial algorithm is, due to its very nature, incapable of following the associated equilibrium path; it is also unable to identify the occurrence of critical points between the activation of two consecutive yield modes. On the other hand, situations of multiple plastic unstressing can be efficiently dealt with.

Critical points and the post-buckling path(s) are, from the theoretical point of view, the most interesting phases of the structural response. Hence, for research purposes the utilization of the incremental and, in particular, the perturbation analysis programs is highly advisable, despite their tendency to require more computer storage and, in general, greater solution times.

5.5 ELASTIC STRUCTURES

The classical approach to the large displacements analysis of flexible members is to solve a certain set of governing non-linear differential equations in terms of elliptic integrals. Exact closed form solutions have thus been obtained for cantilever beams [Barten (1944, 1945), Bisshopp and Drucker (1945), Rohde (1953), Frisch-Fay (1961)] , simply-supported beams [Connway (1947), Scott and Carver (1955), Mitchell (1959), Frisch-Fay (1962), Sliter and Boresi (1964), Schille and Sierakowsky (1967)] , curved beams [Connway (1956), Gospodnetic (1959)] , and frames formed by very simple combinations of bars [Kerr (1964), Jenkins et alia (1966)].

This very elegant method is however of limited applicability and approximate methods had to be considered and developed in order to implement the analysis of more complex systems.

Despite its successful application in the analysis of plates and shells, the finite-difference method has only been occasionally used in the analysis of discrete systems; reference should however be made to the works of Merchant (1955), Pisanty and Tene (1972), Yamada (1972) and Bunce and Brown (1976) in kinematically non-linear and stability analyses.

In the last ten to fifteen years, a vast number of contributions to large displacement elastic analysis by the finite-element method have appeared in the literature. The mathematical theory of the finite-element method was born in the engineering literature in the works of Melosh (1963), Fraeijs de Veubeke (1964), Oliveira (1968), Johnson and McLay (1968) and Oden (1969); a concise description of the method can be found in Oden (1975) and more detailed expositions in various textbooks which have been published in the intervening period.

In his valuable pioneering work of formulating structural analysis in matrix form, Livesley (1956) treated kinematical non-linearities for the purpose of stability analysis.

Turner et alia (1960) applied for the first time a direct displacement method in the analysis of structures under large displacements. The method they propose adopts a step-by-step procedure with linearized relationships within each step.

Another important contribution was that of Argyris (1965b) and Argyris et alia (1964), the main feature of his technique lying in the determination of the necessary forces to maintain the equilibrium of a given displacement configuration; this technique has been recently adopted by Besseling (1974, 1975), Oliveira (1974) and Oliveira and Pires (1976).

Oden (1966) extended the treatment of kinematic non-linearities to very general and complex systems. Przemieniecki (1968) incorporated both kinematic and material non-linearities and presented a comparison between analytical and experimental results.

The surveys on the proposed finite-element formulations, e.g. Ueda (1971), Stricklin et alia (1973) and Felippa (1974) tend to reveal an absence of a synthetic, unified approach to the problem of kinematically non-linear and stability analysis. In general, the different approaches are classified according to either the adopted formulation, Eulerian or Lagrangian, or the numerical implementation procedures they use.

Examples of Lagrangian approaches are the works of Hibbitt et alia (1970), Needleman (1972) and Felippa and Sharifi (1973); Eulerian approaches have been used, among others, by Yaghmai and Popov (1971) and Benedetti and Fontana (1974). A general discussion on these alternative formulations which refer the static and kinematic variables to the initial and updated configurations, respectively, is given in Almoroth and Felippa (1974).

According to the adopted solution procedure, most of the proposed formulations fall into either of the following (not entirely independent) groups; conventional incremental methods, e.g. Marçal (1970), Newton-Raphson iteration procedures, as Stricklin et alia (1968) and Kawai and Yoshimura (1969), and self-correcting incremental methods, of which the method of Haisler et alia (1972) is an example.

A rather special self-correcting technique is the perturbation method of analysis.

In the field of non-linear and stability analysis by matrix methods, increasing attention has been directed to the perturbation method as one of the most powerful tools of analysis that can be employed to follow the structural response in the highly non-linear range.

The systematic analysis of the buckling and post-buckling behaviour of structures is a relatively recent development of the theory of elastic stability. Koiter in 1945 presented a comprehensive high-order theory describing the stability and immediate post-buckling behaviour of structures and the effect of imperfections.

Before the translation of Koiter's work in the mid-sixties, only parts of it were available in English, Koiter (1962, 1963). In the meantime parallel work was being done by Thompson and by Sewell who developed, independently, an asymptotic technique based on the perturbation method.

Thompson's method of deriving the perturbation analysis governing systems, as presented for instance in Thompson and Hunt (1973), is implicit differentiation of the energy function with respect to a path parameter. The alternative method adopted herein and, among others, by Connor and Morin (1970), Hangai and Kawamata

(1972) and Mau and Gallagher (1972), is to establish the governing system in incremental form and to substitute into it the series expansions of the relevant problem variables, collecting afterwards the terms in the like powers of the perturbation parameter. The notion of "sliding co-ordinates", which Thompson uses in his post-buckling method of analysis, has been adopted by several other authors to establish a set of (nodal-stiffness) governing equations containing the solution to the fundamental path as knowns, and the relative displacements of the post-buckling path as unknowns.

Koiter's method, which was further developed by Budiansky and Hutchinson (1966), has been almost exclusively used for continuum problems, while the Thompson-Sewell method was, on the other hand, originally developed and ever since applied to discrete systems; Haftka et alia (1971) formally related the two methods and proved them equally suited to both continuum and discrete problems.

A perturbation technique was used by Roorda and Chilver (1970) for the analysis of frame buckling. The technique is applied to a two-bar frame hinged at the foundation and rigidly jointed at right angles at the apex; the first- and second-order solutions they obtained proved to be in agreement with experimental and theoretical results previously published by Roorda (1965) and Koiter (1967), respectively. The formulation for asymptotic analysis about to be presented is essentially identical to Roorda's; it is however believed to be more synthetic and systematic, and consequently better tailored to computer analysis.

Lang and Hartz (1970), in a direct extension to the finite-element format of Koiter's perturbation method, including the assumption of a linear pre-buckling state, presented a matrix formulation for the perturbation of total potential energy; buckling and post-buckling response of a shallow arch and a thin flat plate are calculated and the influence of various levels and types of geometric imperfections on the load-displacement response is assessed.

Connor and Morin (1970) applied the perturbation technique to obtain the buckling load accompanying non-linear pre-buckling deformations and also to trace the post-buckling equilibrium path. Analyses were performed of a circular cylindrical shell under uniform normal pressure.

The essential concept in the "modified structure" method of Haftka et alia (1971) and Mallet and Haftka (1972) is that a structure with pre-buckling non-linearity is an imperfect version of another hypothetical structure which has a linear pre-buckling path; pre-buckling non-linearities are incorporated as special imperfections. Cohen and Haftka (1972) discuss the limitations on the applicability of the method.

In Gallagher et alia (1971) and Mau and Gallagher (1972) the calculation of the fundamental path is conducted on the basis of an iterative solution of the algebraic equations; the intensities of the critical load at bifurcation and limit points are determined by interpolation and extrapolation, respectively, of solution points of the fundamental path. Numerical solutions are given for a beam on an elastic foundation, the shallow arch and a flat plate.

Hangai and Kawamata (1971, 1972, 1973) used the perturbation method to solve the non-linear governing equations in matrix form, and obtained the complete equilibrium paths for reticulated domes formed by unidimensional members.

Ecer (1973) presents a thorough exposition of the perturbation method which he applies in the analysis of arches, the Euler column, and a rectangular plate.

Endo et alia (1974) adopt a perturbation approach in the analysis of the critical behaviour of spherical shells.

Glaum et alia (1975) specialized into a discrete model analysis a method previously developed by Masur and Schreyer (1967) for incorporating directly the effects of pre-buckling displacements, by expanding in a power series the pre-buckling state, as well as the buckling parameter and mode. A computer program was developed for the analysis of planar structures and the results are compared with the exact solution for the buckling of shallow circular arches.

The role of mathematical programming in structural analysis in general and in kinematically non-linear analysis in particular, has been understood in two quite different ways. Some researchers constrained themselves to extract from mathematical programming the numerical algorithms developed in the recent past, which they then use in the solution of programs established by direct application of certain energy principles; such is the approach followed by

Mallet and Schmit (1967) and Dawe (1973). Others, and among them Maier and his colleagues have to be mentioned, have realized that mathematical programming is a particularly well suited theory not only for the implementation of solutions but also, and specially, for the encodement of problems in structural mechanics. After establishing the problem governing system, preferably from first-principles, the analyst only has to process it through mathematical programming equivalence theory in order to derive the associated mathematical programs; the natural and consistent transition between the discrete and variational descriptions of the problem is thus materialized. The physical interpretation of such programs will either confirm, correct or extend previously known results, the application of specially designed algorithms allowing for their efficient numerical implementation.

5.5.1 The Quadratic Programs of Elastic Analysis

Assume that the stress-strain curve represented in Fig. 3.1 characterizes the behaviour of a two-phase, linear-non linear elastic material, that is, a material wherein reversible strains are allowed for in both linear and non-linear phases.

Structures formed by a combination of members and internal and external release devices constituted of such non-linear elastic materials can still be analyzed adopting exactly the same procedures presented in the previous solutions. The plastic potentials may now be interpreted as non-linear elastic potentials, the plastic strains representing the non-linear components of the (total) elastic strains. The non-linear elastic components of strain must now be allowed to be reversible and, in the non-linear phase, the reduction of stress levels accompanied by constant values of the non-linear elastic strain component prevented.

It must be recalled that developments in the non-linear phase of the stress-strain relationship have been allowed to occur only at a pre-selected and restricted number of cross-sections, the structure critical sections, while all the remaining sections were assumed to behave inside the initial linear elastic phase. If the non-linear elastic structure is to be conveniently modelled, the number of critical sections (and consequently the number of nodes) has to be increased significantly.

The present section is concerned with the analysis of structures constituted by materials (and external and internal release devices) presenting a linear elastic behaviour. Such structures can be interpreted as elastoplastic structures with infinite plastic capacities; the alternative descriptions for the governing system and the associated quadratic programs of the former can thus be obtained by specialization of the corresponding systems and programs characterizing the behaviour of the latter, it being sufficient to set to zero the plastic multipliers, thus trivializing the flow rule present in the dual programs, and removing from the primal programs the yield rule as it is now never satisfied as a strict equality.

THE QUADRATIC PROGRAMS OF ELASTIC (PERTURBATION) ANALYSIS

THE NODAL-STIFFNESS FORMULATION

$$\left(\text{Min } z = \tfrac{1}{2}\, q^T\, \mathbb{K}_*\, q_i \;:\right)\quad \mathbb{K}_*\, q_i = \left[\mathbb{A}_o^T \lambda + \mathbb{A}^T \mathbb{K}\, u_D + \omega_1\right]_i$$

PRIMAL PROGRAM (5.5.1)	DUAL PROGRAM (5.5.2)

$$\text{Max } w = -\tfrac{1}{2}\, q_i^T\, \mathbb{K}_*\, q_i + q_i^T \left[\mathbb{A}_o^T \lambda + \mathbb{A}^T \mathbb{K}\, u_D + \omega_1\right]_i$$

THE NODAL-FLEXIBILITY FORMULATION

$$\text{Min } z = \tfrac{1}{2}\, q_i^T \left[-\mathbb{K}_N\right] q_i + \tfrac{1}{2}\, X_i^T \mathbb{F}\, X_i + X_i^T \left[u_D + \omega_o\right]_i : \left[-\mathbb{K}_N\right] q_i + \mathbb{A}^T X_i = \left[\mathbb{A}_o^T \lambda + \omega_1\right]_i$$

PRIMAL PROGRAM (5.5.3)	DUAL PROGRAM (5.5.4)

$$\text{Max } w = -\tfrac{1}{2}\, q_i^T \left[-\mathbb{K}_N\right] q_i - \tfrac{1}{2}\, X_i^T \mathbb{F}\, X_i + q_i^T \left[\mathbb{A}_o^T \lambda + \omega_1\right]_i : -\mathbb{A}\, q_i + \mathbb{F}\, X_i = -\left[u_D + \omega_o\right]_i$$

THE MESH-STIFFNESS FORMULATION

$$\text{Min } z = \tfrac{1}{2}\, u_{E_i}^T \mathbb{K} u_{E_i} + \tfrac{1}{2}\, p_{*_i}^T \mathbb{F}_M p_{*_i} + p_{*_i}^T \left[\mathbb{F}_{po} \lambda + B_*^T u_D + \omega_3\right]_i : \mathbb{K} u_{E_i} - B_* p_{*_i} = \left[B_o \lambda + \omega_1\right]_i$$

PRIMAL PROGRAM (5.5.5)	DUAL PROGRAM (5.5.6)

$$\text{Max } w = -\tfrac{1}{2}\, u_{E_i}^T \mathbb{K} u_{E_i} - \tfrac{1}{2}\, p_{*_i}^T \mathbb{F}_M p_{*_i} + u_{E_i}^T \left[B_o \lambda + \omega_1\right]_i : B_*^T u_{E_i} + \mathbb{F}_M p_{*_i} = -\left[\mathbb{F}_{po} \lambda + B_*^T u_D + \omega_3\right]_i$$

THE MESH-FLEXIBILITY FORMULATION

$$\text{Min } z = \tfrac{1}{2}\, p_{*_i}^T\, \mathbb{F}_* p_{*_i} + p_{*_i}^T \left[\left(B_*^T \mathbb{F}\, B_o + \mathbb{F}_{po}\right)\lambda + B_*^T\, u_D + \omega_1\right]_i$$

PRIMAL PROGRAM (5.5.7)	DUAL PROGRAM (5.5.8)

$$\left(\text{Max } w = -\tfrac{1}{2}\, p_{*_i}^T\, \mathbb{F}_* p_{*_i} \;:\right)\quad \mathbb{F}_* p_{*_i} = -\left[\left(B_*^T \mathbb{F}\, B_o + \mathbb{F}_{po}\right)\lambda + B_*^T u_D + \omega_1\right]_i$$

Programs (5.5.1-8) were obtained by enforcing this set of specializations on the corresponding quadratic programs of elastoplastic perturbation analysis; the asymptotic, incremental and deformation programs of elastic analysis can be derived in a similar manner.

It will be noticed that, for any of the four formats of analysis under consideration, two (three, considering the alternative mesh formulation) of the original quadratic programs of elastoplastic analysis decompose into linear equation sets; these are the PRIMAL NODAL-STIFFNESS PROGRAMS and the DUAL MESH-FLEXIBILITY PROGRAMS, the former corresponding to the familiar DISPLACEMENT (or STIFFNESS) METHOD of analysis, the latter to the much less favoured FORCE (or FLEXIBILITY) METHOD. The duals of these programs require the extremization of unconstrained quadratic functions of the indeterminate generalized displacements and forces, respectively.

The primal nodal-stiffness and dual mesh-flexibility, formulation programs are not however more convenient than the other formulations when the structure under analysis exhibits unilateral releases or constraints. The remaining quadratic programs may easily be modified to include these effects and non-negative constraints may be added to ensure unilateral constraints.

The considerations in the above are a direct extension to kinematically non-linear analysis of similar results presented by Smith (1974); the eight alternative formulations suggested by Smith can be regained by specializing the elastic deformation programs for the analysis of structures in the range of small displacements and deformations.

5.5.2 General Considerations in Elastic Analysis

As the programs of elastic analysis were obtained by simple specialization of the corresponding programs for the analysis of elastoplastic systems, the conditions for the existence of optimal (elastoplastic) solutions presented in subsection 5.3.1 are still valid in the context of elastic systems.

Under the assumption that optimal solutions exist for both primal and dual quadratic programs of elastic analysis, their uniqueness can now be investigated; the physical interpretation of theorem (5.1.17) when applied to programs (5.5.1) to (5.5.8) results in the following conclusions:

(I) If the stiffness matrix $\underline{\mathbb{K}}_*$ is positive definite, the generalized nodal displacements $\Delta\underset{\sim}{q}$ are unique.

(II) If the unassembled flexibility matrix $\underline{\mathbb{F}}$ and the "geometric" stiffness matrix $-\underline{\mathbb{K}}_N$ are positive definite, the generalized stress-resultants $\Delta\underset{\sim}{X}$ and the generalized nodal displacements $\Delta\underset{\sim}{q}$ are unique.

(III) If the unassembled stiffness matrix $\underline{\mathbb{K}}$ and the "geometric" flexibility matrix $\underline{\mathbb{F}}_M$ are positive definite, the generalized elastic deformations $\Delta\underset{\sim}{u}_E$ and the generalized indeterminate forces $\Delta\underset{\sim}{p}_*$ are unique.

and (IV) If the flexibility matrix $\underline{\mathbb{F}}_*$ is positive definite, the generalized indeterminate forces $\Delta\underset{\sim}{p}_*$ are unique.

In particular,

(V) If $|\underline{I}_\pi^T \,\underline{\square}\, \underline{I}_\pi| \neq 0$ and matrices (5.3.1) and $[\underline{\mathbb{K}} + \underline{B}_\pi (\underline{I}_\pi^T \underline{\square}\,\underline{I}_\pi)^{-1} \underline{B}_\pi^T]$ are positive definite, the generalized indeterminate forces $\Delta\underset{\sim}{p}$ and the generalized elastic deformations $\Delta\underset{\sim}{u}_E$ will be unique.

and (VI) If $|\underline{B}_\pi^T \,\underline{\mathbb{F}}\, \underline{B}_\pi + \underline{I}_\pi^T \,\underline{\square}\, \underline{I}_\pi| \neq 0$ and matrix (5.3.2) is positive definite, the generalized indeterminate forces $\Delta\underset{\sim}{p}$ will be unique.

Let $(\dot{\underset{\sim}{q}}, \dot{\underset{\sim}{X}}')$ be a first-order optimal solution to the composite form of the nodal-flexibility programs (5.5.3-4); according to theorem (5.1.18), the solution

$$\begin{bmatrix} \dot{q}'' \\ \dot{X}'' \end{bmatrix} = \begin{bmatrix} \dot{q}' \\ \dot{X}' \end{bmatrix} + \alpha \begin{bmatrix} \delta q \\ \delta X \end{bmatrix}$$

will also be optimal if system (5.5.9) admits a non-trivial solution $(\delta q, \delta X)$; systems (5.5.10) and (5.5.11) specialize system (5.5.9) into the cases of uniquely defined fields of stress-resultants and displacements, respectively.

(5.5.9)	(5.5.10)	(5.5.11)
$\delta X^T u_D = \delta q^T \mathbb{A}_0^T \lambda$	$\lambda^T \mathbb{A}_0 \ \delta q = 0$	$u_D^T \ \delta X = 0$
$-\mathbb{K}_N \ \delta q + \mathbb{A}^T \delta X = 0$	$-\mathbb{K}_N \ \delta q = 0$	$\mathbb{A}^T \ \delta X = 0$
$-\mathbb{A} \ \delta q + \mathbb{F} \ \delta X = 0$	$-\mathbb{A} \ \delta q = 0$	$\mathbb{F} \ \delta X = 0$

Systems (5.5.9) to (5.5.11) can be read as follows:

(VII) If the members of a structure are locally stable, i.e. if $\mathbb{F} = |\mathbb{K}|^{-1} \neq 0$, a kinematically multiple solution $q'' = q' + \alpha\, \delta q$ will exist if the stiffness matrix $\mathbb{K}_*$ is singular, the associated stress field being defined by $X'' = X' + \alpha \mathbb{K}\, \mathbb{A}\, \delta q$; the energy dissipated at the prescribed dislocations by the difference stress-field δX equals the work realized by the difference displacement field δq.

(VIII) If the stress-resultant field is uniquely defined, a kinematically multiple solution will exist if the "geometric" stiffness matrix $-\mathbb{K}_N$ is non-definite, the difference displacement field δq being self-compatible and orthogonal to the loading $\mathbb{A}_0^T \lambda$.

(IX) If the displacement field is uniquely defined, a statically multiple solution will exist if the unassembled flexibility matrix $\mathbb{F}$ is non-definite, the difference stress-resultant field δX being self-equilibrated and orthogonal to the prescribed dislocations.

Similar conclusions could be drawn by applying the same theorem to the pairs of primal-dual programs associated with the nodal-stiffness mesh-flexibility and mesh-stiffness formulations; they would involve the remaining variables of elastic analysis, afected by the alternative structural matrices.

The objective functions of the quadratic programs of elastic analysis can be obtained by suppressing the plastic components from the elastoplastic objective functions (5.3.14-19), yielding

$$z^D = \left\{ \frac{1}{2}\, \underline{X}^T \underline{u}_E - \frac{1}{2}\, \underline{\pi}^T \underline{\delta}_\pi \right\} - \left\{ -\,(\underline{u}_D + \underline{u}_\pi + \frac{1}{2}\underline{u}_{E\pi})^T \underline{X} \right\} + O_4 \qquad (5.5.12)$$

$$-w^D = \left\{ \frac{1}{2}\, \underline{X}^T \underline{u}_E - \frac{1}{2}\, \underline{\pi}^T \underline{\delta}_\pi \right\} - \left\{ \underline{\lambda}^T \underline{\delta} - \frac{1}{2}\, \underline{X}^T_{E\pi} \underline{u}_E \right\} - O_4 \qquad (5.5.13)$$

for the deformation analysis programs and

$$z^I = \left\{ \frac{1}{2}\, \Delta\underline{X}^T (\Delta\underline{u}_E + \Delta\underline{u}_\pi) - \frac{1}{2}\, \Delta\underline{\pi}^T \Delta\underline{\delta}_\pi + \frac{1}{2} \left[\Delta\underline{X}^T (\underline{\mathbb{R}}_{uE} + \underline{\mathbb{R}}_{u\pi}) + \Delta\underline{\delta}^T_\pi \underline{\mathbb{R}}_\pi \right] \right\} - \left\{ -\Delta\underline{X}^T \Delta\underline{u}_D \right\} + O_4 \qquad (5.5.14)$$

$$-w^I = \left\{ \frac{1}{2}\, \Delta\underline{X}^T (\Delta\underline{u}_E + \Delta\underline{u}_\pi) - \frac{1}{2}\, \Delta\underline{\pi}^T \Delta\underline{\delta}_\pi - \frac{1}{2} \left[\Delta\underline{X}^T (\underline{\mathbb{R}}_{uE} + \underline{\mathbb{R}}_{u\pi}) + \Delta\underline{\delta}^T_\pi \underline{\mathbb{R}}_\pi \right] \right\} - \left\{ \Delta\underline{\lambda}^T \Delta\underline{\delta} \right\} - O_4 \qquad (5.5.15)$$

for the incremental analysis programs; the objective functions of the i-th order programs of perturbation analysis can be obtained by replacing in (5.5.14-15) each incremental variable by the i-th order component of its series expansion, and replacing the fourth- and higher-order terms O_4 by constants.

From Static-Kinematic Duality, it can be easily concluded that, at optimality, the primal and dual objective functions attain an identical value, thus confirming Cottle's Theorem (Duality).

Maier's bounding theorems can easily be specialized for elastic systems, it being sufficient to let in (5.3.20b) and (5.3.20c z_1^p and w_1^p represent the objective functions of the first-order programs of elastic perturbation analysis.

Let the incremental strain energy and complementary strain energy be defined by

$$\Delta U = \frac{1}{2}\,\Delta \underset{\sim}{X}^T(\,\Delta \underset{\sim}{u}_E + \Delta \underset{\sim}{u}_\pi) - \frac{1}{2}\,\Delta \underset{\sim}{\pi}^T\,\Delta \underset{\sim}{\delta}_\pi - \frac{1}{2}\;\;\Delta \underset{\sim}{X}^T(\underset{\sim}{\mathbb{R}}_{uE} + \underset{\sim}{\mathbb{R}}_{u\pi}) + \Delta \underset{\sim}{\delta}_\pi^T \underset{\sim}{\mathbb{R}}_\pi$$

and $$\Delta U^* = \frac{1}{2}\,\Delta \underset{\sim}{X}^T(\,\Delta \underset{\sim}{u}_E + \Delta \underset{\sim}{u}_\pi) + \frac{1}{2}\,\Delta \underset{\sim}{\pi}^T\,\Delta \underset{\sim}{\delta}_\pi + \frac{1}{2}\;\;\Delta \underset{\sim}{X}^T(\underset{\sim}{\mathbb{R}}_{uE} + \underset{\sim}{\mathbb{R}}_{u\pi}) + \Delta \underset{\sim}{\delta}_\pi^T \underset{\sim}{\mathbb{R}}_\pi$$

respectively, and the incremental work performed by the prescribed forces and dislocations respectively as

$$\Delta W = \Delta \underset{\sim}{\lambda}^T\,\Delta \underset{\sim}{\delta} \qquad \text{and} \qquad \Delta W^* = -\;\Delta \underset{\sim}{u}_D^T\;\;\Delta \underset{\sim}{X}$$

Neglecting fourth- and higher-order terms, the primal and dual objective functions of the quadratic programs of elastic incremental analysis can be expressed as

$$z^I = \Delta E^* = \Delta U^* - \Delta W^*$$

and $$-w^I = \Delta E = \Delta U - \Delta W$$

where ΔE and ΔE^* represent, respectively, the variation of the potential energy and complementary potential energy.

It could be similarly concluded that the primal and dual objective functions, z_i^p and w_i^p, of analysis formulations represent, except for the linear terms which are irrelevant in the extremization procedures, the i-th order terms of the series expansion of ΔE^* and $-\;\Delta E$, respectively.

As the deformation analysis programs are derived assuming that the (non-linear) corrective variables are a priori known, i.e. that the structure acted upon by additional prescribed forces and dislocations behaves linearly, the strain energy and the complementary strain energy take an identical form

$$U = U^* = \frac{1}{2}\,\underset{\sim}{X}^T \underset{\sim}{u}_E - \frac{1}{2}\,\underset{\sim}{\pi}^T\,\underset{\sim}{\delta}_\pi$$

the expressions for the total work and complementary work becoming

$$W = \underset{\sim}{\lambda}^T \underset{\sim}{\delta} \;\; - \frac{1}{2}\,\underset{\sim}{X}_{E\pi}^T\,\underset{\sim}{u}_E$$

and $$W^* = -\underset{\sim}{u}_D^T\,\underset{\sim}{X} - (\frac{1}{2}\,\underset{\sim}{u}_{E\pi} + \underset{\sim}{u}_\pi)^T\,\underset{\sim}{X}$$

respectively, thus establishing the following correspondence between the dual (primal) objective function and the (complementary) potential energy:

$$E = -w^D = U - W \quad , \quad E^* = z^D = U^* - W^*$$

As the primal and dual constraints of the programs of elastic analysis enforce, respectively, the static and kinematic admissibility conditions, the identifications in the above allow us to interpret those programs as the discrete representation of the PRINCIPLES OF MINIMUM COMPLEMENTARY POTENTIAL ENERGY and MINIMUM POTENTIAL ENERGY, respectively.

The principle of minimum potential energy seems to have been initiated for linear elastic media by Kirchhoff (1850). The canonical transformation of this principle into that of minimum complementary potential energy was carried out by Friedrichs (1929), however the essential role of strain energy and complementary strain energy in the canonical theory was known to Crotti (1888); in 1889, Engesser applied a complementary energy based method in the analysis of non-linear elastic structures. In recent years the dual role of energy and complementary energy has been demonstrated by Westergaard (1941) and Argyris and Kelsey (1960). Zubov (1972), Fraeijs de Veubeke (1972) and Koiter (1973) devoted important work to extend the complementary energy principles to kinematically non-linear analysis.

Statements (I) and (II) in subsection 5.3.4, concerning Drucker's stability criterion, can be directly applied to elastic systems, it being sufficient to re-define matrix $\underline{A}$ as

$$\underline{A} = \underline{\mathbb{K}}_* \qquad \text{and} \qquad \underline{A} = \left[\begin{array}{c|c} \underline{\mathbb{F}} & \cdot \\ \hline \cdot & -\underline{\mathbb{K}}_N \end{array}\right]$$

for the nodal-stiffness and nodal-flexibility formulations, respectively, and by

$$\underline{A} = \left[\begin{array}{c|c|c} \underline{\mathbb{K}} & \cdot & \cdot \\ \hline \cdot & \underline{\mathbb{F}}_M & \underline{\mathbb{F}}_o \\ \hline \cdot & \underline{\mathbb{F}}_o^T & \underline{\mathbb{F}} \end{array}\right] \qquad \text{and} \quad \underline{A} = \left[\begin{array}{c|c} \underline{\mathbb{F}}_* & \underline{B}_*^T \underline{\mathbb{F}} \, \underline{B}_o + \underline{\mathbb{F}}_o \\ \hline \underline{B}^T \underline{\mathbb{F}} \, \underline{B}_* + \underline{\mathbb{F}}_o^T & \underline{B}_o \, \underline{\mathbb{F}} \, \underline{B}_o + \underline{\mathbb{F}}_\lambda \end{array}\right]$$

for the mesh-stiffness and mesh-flexibility formulations, respectively.

An alternative statement can be set forth if use is made of the second-order work in the form (5.3.24) where now $\underset{\sim}{\delta}_p = \underset{\sim}{0}$:

(X) If matrix $\underset{\sim}{L}$, defined in either of the formats (5.3.26) or (5.3.28), is positive definite (semi-definite) the equilibrium state is stable (non-unstable).

Identical would be the results obtained when using Wiessman's (1965) stability criterion instead of the specialized version of Drucker's.

Croll and Walker (1972), using a dynamic interpretation of stable equilibrium, present a proof of the sufficiency, for defining a stable equilibrium configuration, of the relative minimum of the total potential energy, a requirement already emphasized by Horne (1960, 1961); Pian and Tong (1970) showed how the stationarity principle of potential energy can be substituted by that of the stationarity of its variation. Detailed expositions of the use of energy principles in stability analysis can be found in Britvec (1973) and Thompson and Hunt (1973).

Concluded in the above is what so often is used axiomatically in alternative formulations of problems in kinematically non-linear and stability analysis. Namely, that the minimum of the (complementary) potential energy ensures (kinematic compatibility) static equilibrium, while the quality of the definiteness of the second variation of the energy characterizes the stability of the configuration.

Most of the proposed formulations develop from pre-established energy functions; first-principle based formulations yielding to the associated energy principles have, on the other hand, been significantly less favoured. Examples of these two distinct approaches are the formulations proposed by Brebbia and Connor (1969) and Alexa (1976), respectively.

When judging the unbalanced popularity enjoyed by each approach,it would be sensible to remember, and it has gone unrecognized for long, that mathematical programming purveys the link between first-principles, the most natural way of formulating a problem, and the energy principles, which synthesize its characteristics.

5.5.3 Numerical Applications

In selecting a set of numerical examples, the emphasis was placed on their suitability for testing the numerical accuracy of the perturbation analysis formulation being suggested, not its computational efficiency; thus the academic nature of the problems being considered, some of which of known exact solution.

The computer execution times per incremental step are given, in decimal seconds, in Table 5.11 for each of the examples to be presented; for the given number of finite-elements into which the structure is discretized, β represents the resulting kinematic indeterminacy and β' the number of axially inextensible members, so that the dimension of the governing system becomes $\beta + \beta'$. The computer time required to solve Example 6 was not printed during execution.

EXAMPLE	1	2	3 and 4				5	7	8
No. of F.E.	1	2	1	2	4	8	1	9	12
β	1	2	3	6	12	24	1	8	30
β'	0	1	1	2	4	8	0	0	0
Time (sec.)	0.005	0.013	0.053	0.194	0.520	5.87	0.013	0.110	4.190

Table 5.11

As the author's knowledge on numerical analysis techniques and on the implementation of efficient computer codes is fairly superficial, the execution times being given should be taken as very conservative "upper bounds"; numerical analysts should however find them of easy improvement.

The deformation analysis formulations have also been tested. The perturbation analysis formulation did prove better in accuracy of solution, sensibly as good as in execution times, but requiring significantly more computer storage.

EXAMPLE 1: Von Mises' Truss

The symmetric arch formed by two deformable members pinned to each other and to rigid supports is often used to illustrate the "snap through" phenomenon, typical in the response of shallow arches and domes.

Because of the symmetry of the loading (2λ) and the frame geometry only half of the structure, shown in Fig. 5.14, need be analyzed. The exact relationship between the loading and its displacements can be found in various ways to be

$$\begin{cases} \lambda/EA = \sin\beta - \cos\alpha \cdot \tan\beta \\ \delta/L = \sin\alpha - \cos\alpha \cdot \tan\beta \end{cases}$$

where L, E and A are the member length, modulus of elasticity and cross-sectional area, respectively and α and β the horizontal inclination of the member in its initial and displaced configurations.

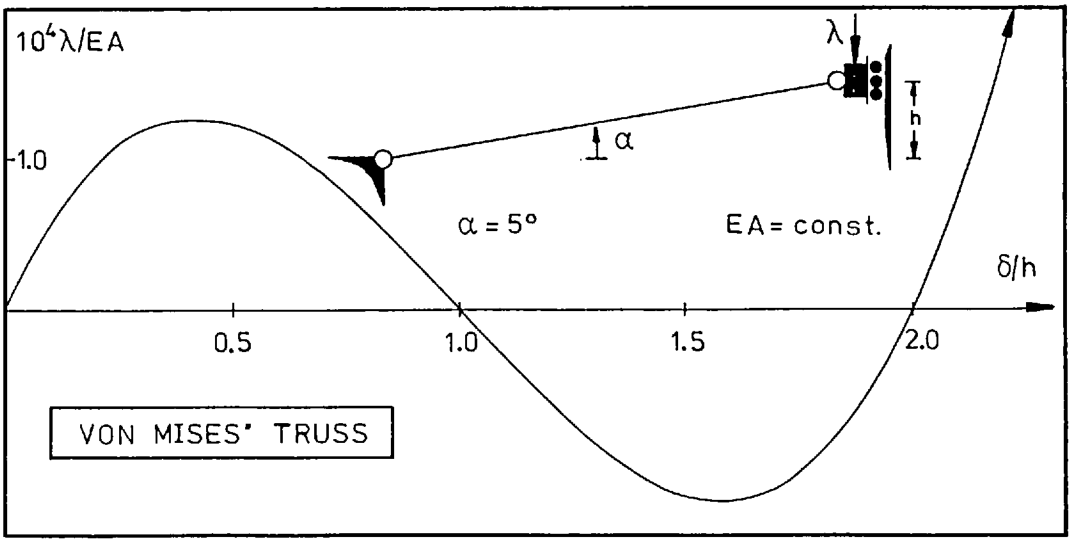


FIGURE 5.14

Summarized in Table 5.12, and confronted with the exact solution in the above, are the results found when using, in the perturbation analysis formulation being suggested, one, two and three forms in the series expansion. The structure was first solved without using the corrective technique mentioned in subsection 5.4.2 (STEP 7c in the generic algorithm), and it proved necessary

δ /L	10^4x λ/EA				
	EXACT SOLUTION	WITHOUT PVW			WITH PVW
		3	2	1	1
0.0174311	0.955	0.955	0.949	1.137	0.955
0.0435779	1.247	1.247	1.231	1.627	1.247
0.0697246	0.639	0.639	0.613	1.130	0.639
0.0958713	-0.330	-0.330	-0.366	0.181	-0.330
0.1220180	-1.118	-1.118	-1.164	-0.677	-1.118
0.1481649	-1.185	-1.185	-1.241	-0.905	-1.185
0.1743115	-0.000	-0.000	-0.066	0.333	-0.000
0.2004582	2.958	2.958	2.883	2.657	2.958
0.2266049	8.190	8.190	8.106	7.474	8.190
0.2527516	16.175	16.175	16.083	14.966	16.175
0.2788984	27.368	27.368	27.267	25.592	27.368
0.3050451	42.189	42.189	42.081	39.779	42.189

TABLE 5.12

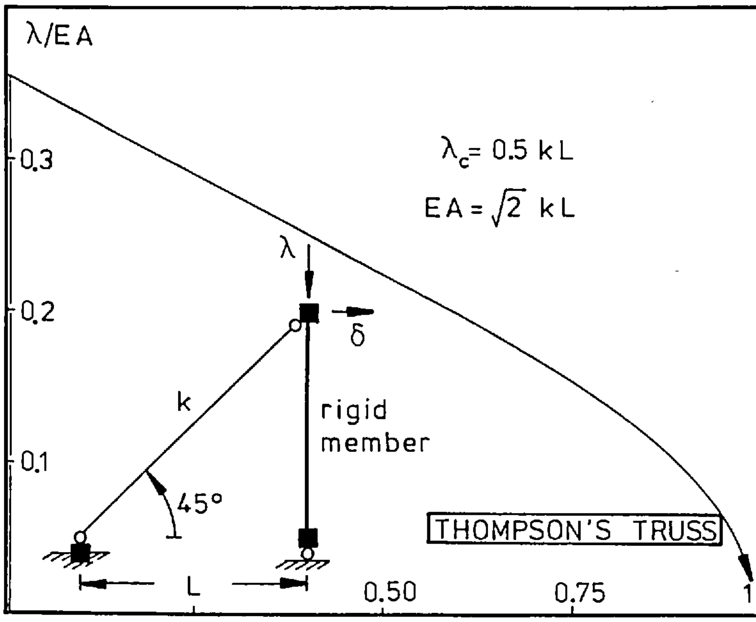
to take three terms of the series in order to achieve a seventh decimal accuracy in the loading for a displacement control sequence; due to the very nature of the structure only one term of the series is required if the corrective technique, based on the Principle of Virtual Work (i.e. Static-Kinematic Duality) is used.

Oliveira (1974) reports good results in the analysis of a similar truss ($\alpha = 10^o$) using his fictitious forces formulation; conceptually identical to Oliveira's is the formulation proposed by Khonke (1978).

EXAMPLE 2: Thompson's Truss

Thompson and Hunt (1973) used the structural system shown in Fig. 5.15 to illustrate the occurrence of asymmetric points of bifurcation. It comprises a rigid link of length L supported by an articulated bar of stiffness $k = EA/\sqrt{2}L$ in both tension and compression and inclined initially at 45^o. The exact relationship between the vertical load λ and the side-sway δ of the structure can be found to be

$$\lambda/EA = \left(\sqrt{1-(\delta/L)^2} - \sqrt{1-\delta/L}\right)/(\sqrt{2}\,\delta/L)$$



δ/L	λ/EA	
	EXACT	3 TERMS
0.	.35355	
0.1	.32742	.32742
0.2	.30182	.30182
0.3	.27643	.27643
0.4	.25088	.25088
0.5	.22474	.22474
0.6	.19745	.19745
0.7	.16811	.16811
0.8	.13505	.13504
0.9	.09402	.09401
0.95	.06598	.06596
0.995	.02073	.02065

FIGURE 5.15

The results summarized in the Table in Fig. 5.15 show a good agreement between the exact and proposed solutions.

EXAMPLE 3: <u>The Transversely Loaded Cantilever</u>

In 1945, Bisshopp and Drucker solved, using elliptic integrals, the non-linear differential equations governing the behaviour of an axially inextensible cantilever under the action of a transverse point load. The exact solution they provided, re-worked and recorded in Timoshenko and Gere (1972), has been used extensively as a basis of comparison for several proposed formulations.

The results obtained after solving the exact governing equations are set out in Table 5.13 in comparison with those provided by the present formulation; the cantilever was discretized into one, two, four and eight finite-elements.

Walker and Hall (1968) applied the Raylaigh-Ritz finite-element method in the analysis of a simply supported beam, discretized into eight finite-elements, acted upon by a transverse load at mid-span and solved the associated non-algebraic equations by three distinct techniques; a perturbation method, the Newton-Raphson method and a step-by-step method. Shown in Fig. 5.16 is the sequence

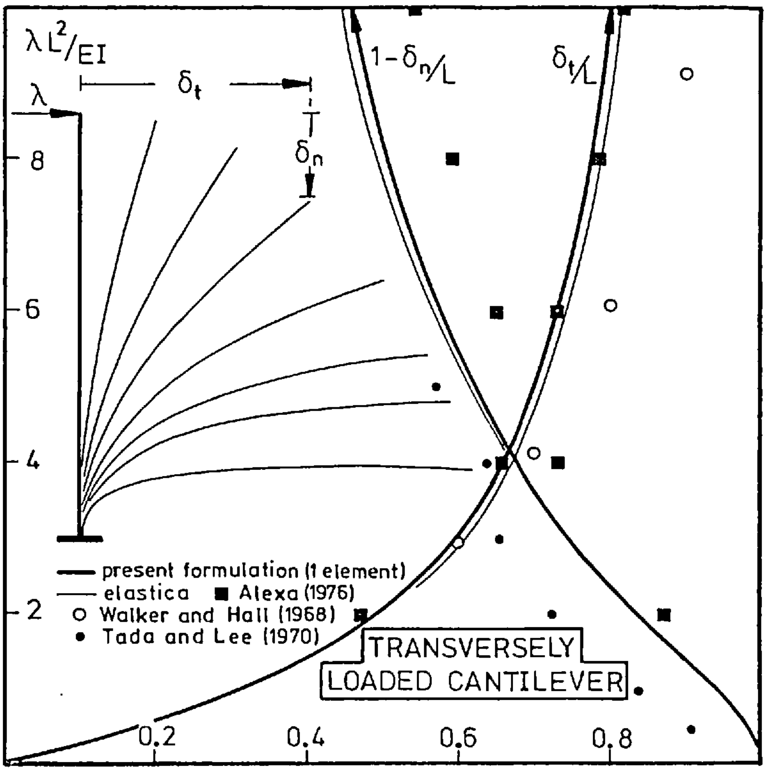


FIGURE 5.16

Nodal Rotation	δ_t/L		δ_n/L		$\lambda L^2/EI$		No. of F.E.
	ELASTICA	PRESENT	ELASTICA	PRESENT	ELASTICA	PRESENT	
5	.0581	.0581	.0020	.0020	.1750	.1750	1
10	.1160	.1160	.0081	.0081	.3530	.3531	1
15	.1734	.1734	.0182	.0182	.5371	.5375	1
20	.2302	.2300	.0324	.0324	.7306	.7316	1
25	.2859	.2856	.0505	.0504	.9376	.9396	1
30	.3406	.3400	.0726	.0724	1.1626	1.1665	1
35	.3938	.3937	.0985	.0985	1.4117	1.4121	2
40	.4455	.4454	.1284	.1284	1.6923	1.6930	2
45	.4955	.4953	.1621	.1621	2.0145	2.0157	2
50	.5436	.5433	.1997	.1996	2.3922	2.3941	2
55	.5898	.5893	.2412	.2411	2.8456	2.8487	2
60	.6340	.6340	.2868	.2868	3.4054	3.4057	4
65	.6762	.6762	.3368	.3368	4.1214	4.1219	4
70	.7167	.7166	.3918	.3918	5.0812	5.0821	4
75	.7558	.7557	.4531	.4531	6.4595	6.4612	4
80	.7968	.7948	.5275	.5236	8.6788	8.6789	8
85	.8397	.8373	.6173	.6120	13.2333	13.2335	8
90	DIVERGED	.9315	DIVERGED	.8348	DIVERGED	68.6113	8

of transverse displacements they found for an equivalent cantilever; the axial displacements were not recorded.

Tada and Lee (1970) used a very fine mesh of twenty elements and obtained a very accurate solution for the transverse displacements; the values they found for the axial displacements, shown in Fig. 5.16, are seen to contain quite noticeable errors.

Shown in the same figure are the results obtained by Alexa (1976) using a non-linear programming algorithm developed by Sargent and Murtagh (1973); only one finite-element was used.

The finite-difference method, rarely applied in the analysis of plane bars and frames, was used by Pisanti and Tene (1972) and Bunce and Brown (1976) in the solution of the cantilever problem. While the former, using a two hundred and one point grid, solved the resulting non-linear algebraic equations by the Newton-Raphson's iterative procedure, the latter also report very accurate results, for a wider range of displacements, employing a dynamic relaxation method on a twenty-one point grid.

Using an integral equation approach, Reeves (1975) has also analyzed the transversely loaded cantilever, based on complementary energy principles; the solution procedure, although of proved accuracy, appears to be of limited practical applicability.

EXAMPLE 4: <u>The Axially Loaded Cantilever</u>

The axially loaded cantilever is a classic example in buckling analysis; the exact solution for this problem is known and can be found in most of the works dealing with beam-columns, as for instance in Timoshenko and Gere (1961).

Summarized in Table 5.14 are the results we found for the exact solution and after analyzing the cantilever when discretized in one, two, four and eight finite-elements.

This problem, seldomly used to test finite-element formulations, has also been solved by Oliveira and Pires (1976) as an illustration of the direct discrete formulation of kinematically non-linear analysis they propose; the non-linear equations are solved iteratively based on the direct derivation of the non-linear

Nodal Rotation	δ_t/L		δ_n/L		$\lambda L^2/EI$		No.of F.E.
	ELASTICA	PRESENT	ELASTICA	PRESENT	ELASTICA	PRESENT	
20	.2194	.2192	.0303	.0302	2.5054	2.5074	1
40	.4222	.4222	.1188	.1188	2.6245	2.6244	2
60	.5932	.5930	.2590	.2590	2.8418	2.8416	2
80	.7195	.7190	.4406	.4406	3.1925	3.1924	2
100	.7915	.7902	.6510	.6509	3.7465	3.7469	2
120	.8032	.8031	.8768	.8768	4.6506	4.6500	4
140	.7504	.7502	1.1069	1.1069	6.2728	6.2719	4
160	.6246	.6240	1.3403	1.3403	9.9440	9.9427	4
180	.1457	.2125	1.8543	1.7657	188.47	85.71	4

TABLE 5.14

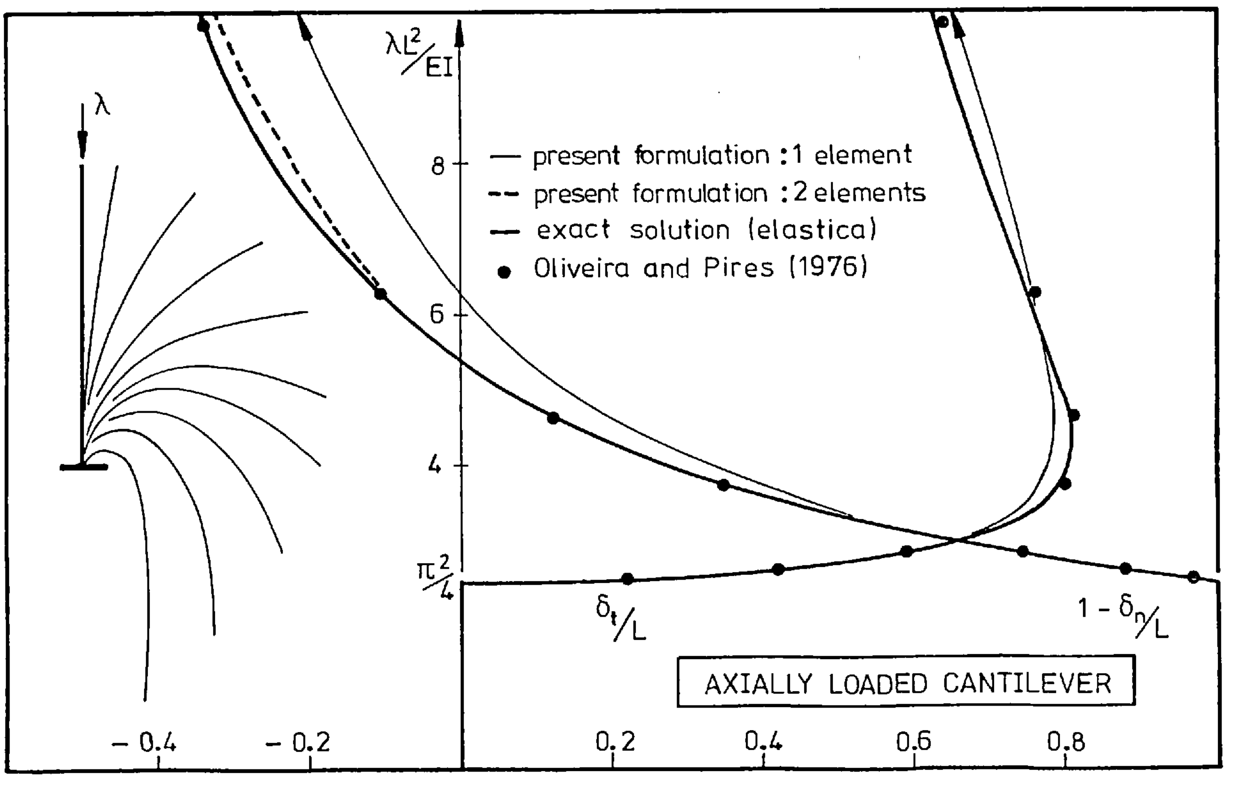
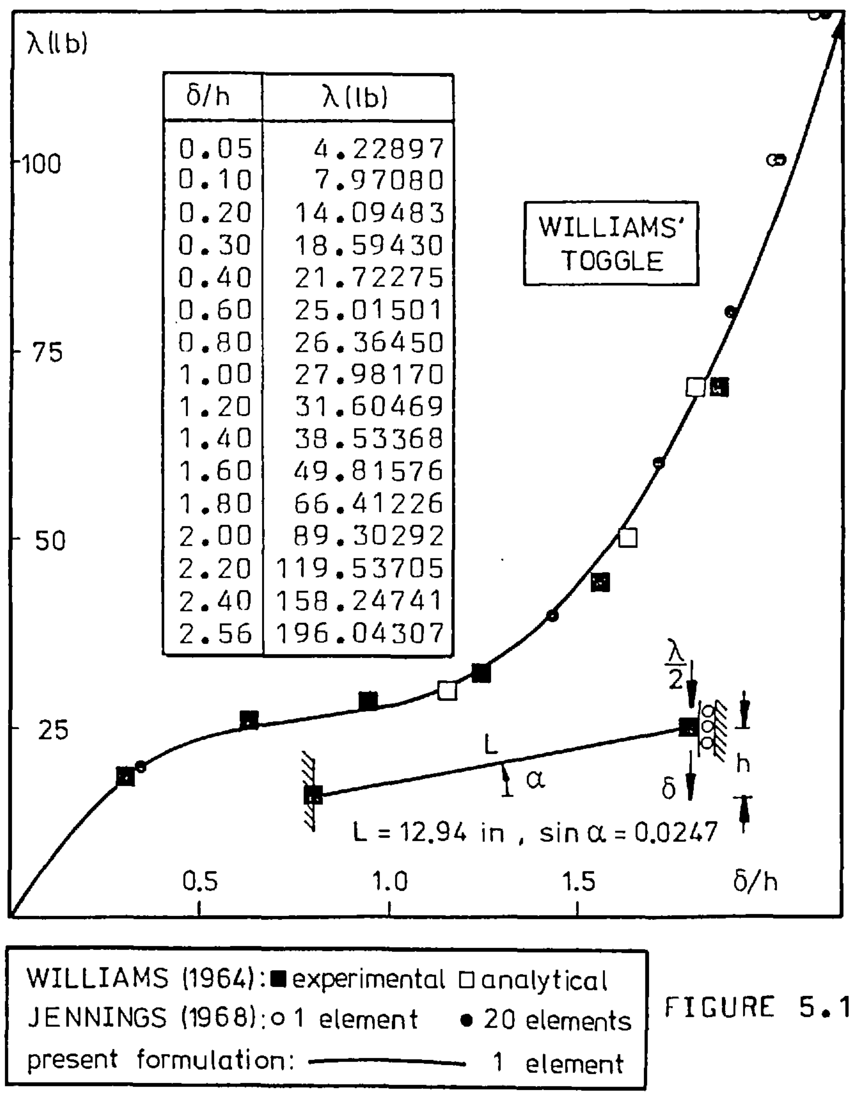


FIGURE 5.17

discrete equations from the linear discrete ones. The solution they obtained by decomposing the strut into five elements is shown in Fig. 5.17.

EXAMPLE 5: <u>Williams' Toggle</u>

Williams, in 1964, developed a formulation for the analysis, by the so-called displacement method, of elastic planar frames undergoing large displacements. The effects of change of member



δ/h	λ(lb)
0.05	4.22897
0.10	7.97080
0.20	14.09483
0.30	18.59430
0.40	21.72275
0.60	25.01501
0.80	26.36450
1.00	27.98170
1.20	31.60469
1.40	38.53368
1.60	49.81576
1.80	66.41226
2.00	89.30292
2.20	119.53705
2.40	158.24741
2.56	196.04307

FIGURE 5.18

flexural stiffness, finite displacements of the joints and axial shortening due to bending are considered separately. In order to assess the accuracy of the results predicted by this formulation, Williams tested the symmetric rigid jointed toggle shown in Fig. 5.18; the flexural and axial stiffnesses are $9.27 \times 10^3 lb/in^2$ and 1.885×10^6 lb, respectively.

Compared in the same figure are the results found by Jennings (1968) and those provided by the present formulation, which are recorded in the inset; Alexa (1976) reports a good agreement with Jennings' solution.

EXAMPLE 6: Chwalla's Frame

The sidesway buckling of the symmetric frame shown in Fig. 5.19 was first investigated by Chwalla (1938); he found that the equilibrium path exhibits a bifurcation at a load slightly lower than that of the same frame when axially loaded.

Masur et alia (1961) using stability functions and an extension of the moment distribution technique of Winter et alia (1948) reached the same governing equations of Chwalla and thence the same value for the critical load, $\lambda_c = 1.775\ EI/L^2$.

Horne's (1962) approach, similar to Ariaratnam's (1959), is based on the expression of an arbitrary deformation of the frame as an infinite series in terms of the critical modes of the same structure when axially loaded; the bifurcation load he thus found for Chwalla's frame was $1.780\ EI/L^2$.

The aforementioned assumed that the displacements remained small and their effects in the equilibrium equation were negligible.

Lee et alia (1968) proposed a finite-element formulation for analyzing large deflections of elastic planar frames subject to discrete conservative loads. The system of simultaneous non-linear equations, obtained after assembling the general solution of the non-linear finite-element, are solved by a modified Newton-Raphson iteration procedure. The stability analysis is based on a method proposed by Horne (1961). The values they obtained, in a step-by-step sequence of load increments, for the symmetrical and sidesway buckling loads of Chwalla's frame are 14.9 and $1.7507\ EI/L^2$, respectively.

Illustrated in Fig. 5.20 is the behaviour we found for this simple frame when deforming in the symmetrical and sidesway modes, the latter being shown in more detail in Figs. 5.19 and 5.21; the associated sequence of displacements are presented in Figs. 5.22 and 5.23.

The value we obtained for the symmetrical mode limit load, $\lambda_c \simeq 13.3\ EI/L^2$ is significantly lower than Lee's, although the

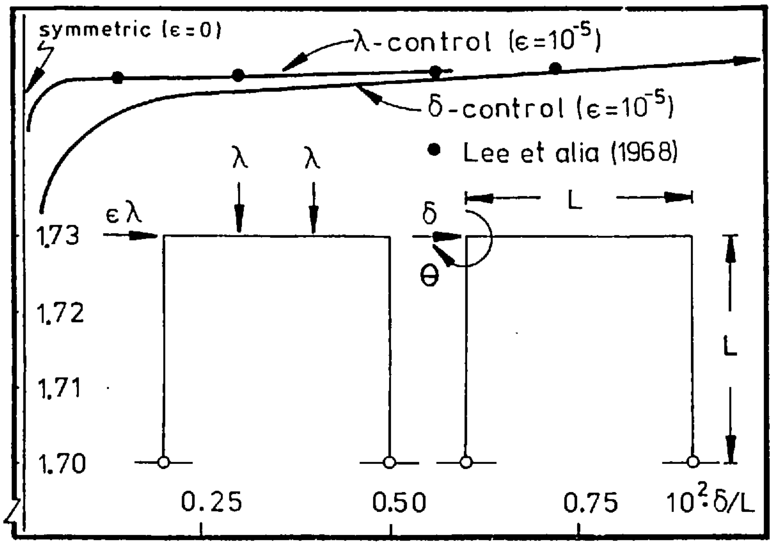


FIGURE 5.19

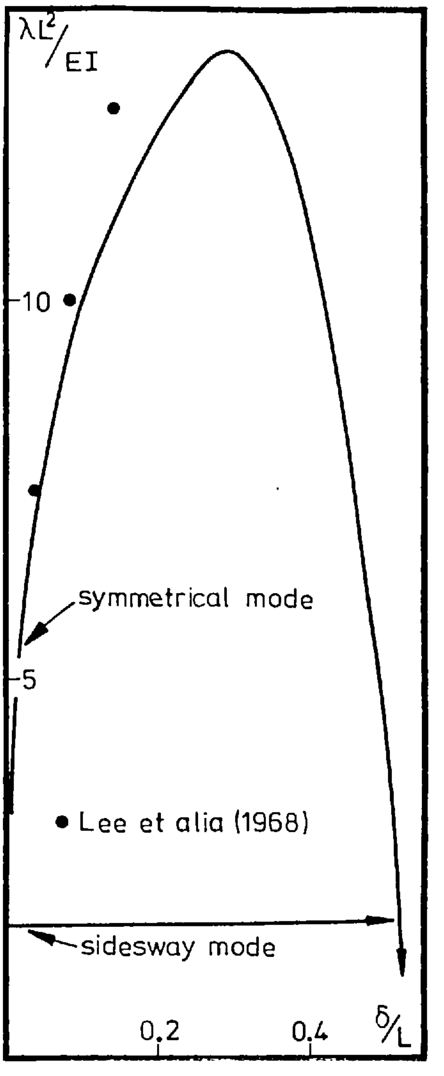


FIGURE 5.20

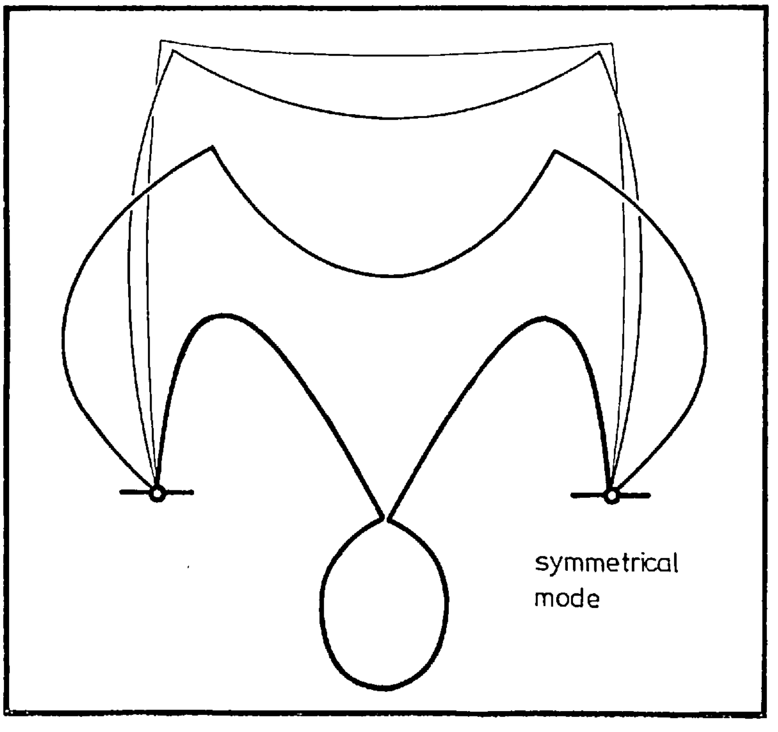
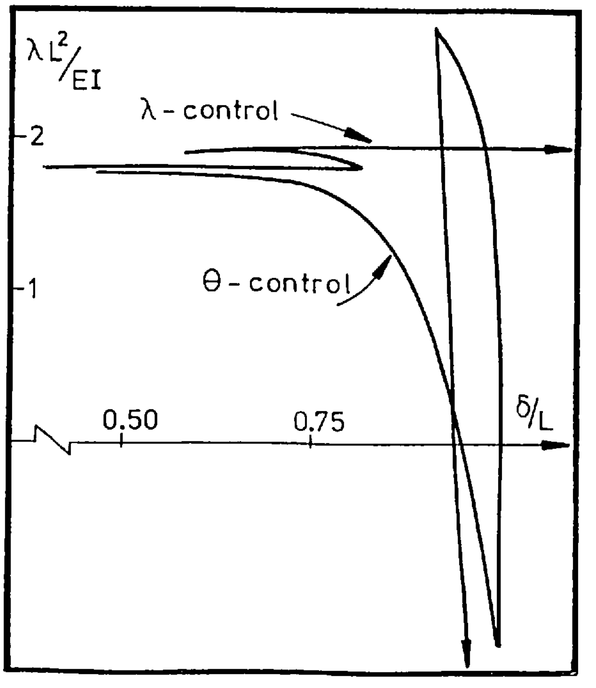


FIGURE 5.22

CHWALLA'S FRAME



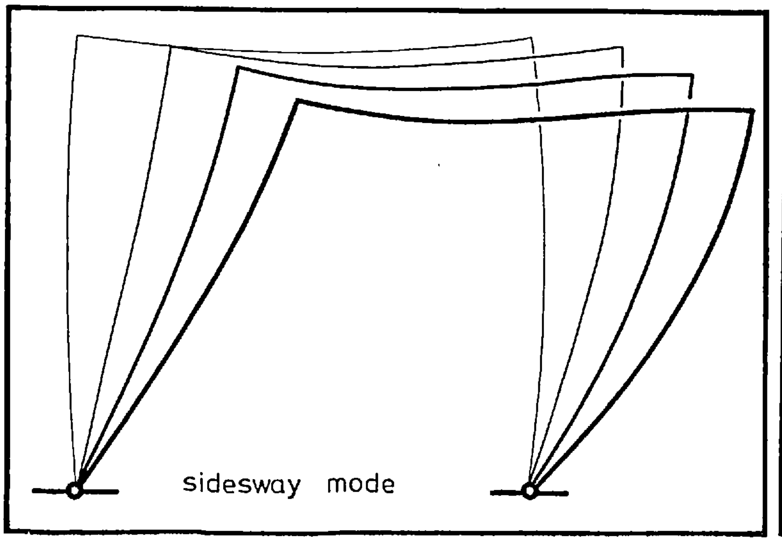


FIGURE 5.23

FIGURE 5.21

SYMMETRICAL MODE (*)		
$\lambda L^2/EI$	δ/L	θ
1.6470	0.0013	0.0800
4.1792	0.0099	0.2352
6.3607	0.0273	0.4135
7.8859	0.0481	0.5772
8.9556	0.0693	0.7228
9.7337	0.0899	0.8523
10.3310	0.1097	0.9697
10.8180	0.1288	1.0784
11.2411	0.1477	1.1817
11.6336	0.1669	1.2827
12.0216	0.1869	1.3844
12.4228	0.2088	1.4905
12.8325	0.2340	1.6056
13.1782	0.2644	1.7352
13.2720	0.2886	1.8321
12.9371	0.3312	1.9929
11.7305	0.3788	2.1641
9.6350	0.4255	2.3301
6.9788	0.4663	2.4779
4.1577	0.4986	2.6010
1.4498	0.5226	2.6981

SIDESWAY MODE: δ-CONTROL (■)		
$\lambda L^2/EI$	δ/L	θ
0.4396	0.0001	0.0200
0.8597	0.0003	0.0400
1.2617	0.0007	0.0600
1.6467	0.0013	0.0800
1.7390	0.0063	0.0864
1.7473	0.0163	0.0900
1.7491	0.0263	0.0933
1.7499	0.0363	0.0965
1.7505	0.0463	0.0998
1.7510	0.0563	0.1030
1.7515	0.0663	0.1063
1.7520	0.0763	0.1096
1.7525	0.0863	0.1130
1.7531	0.0963	0.1163

(*) $\varepsilon=0$ (■) $\varepsilon=10^{-5}$

Refer to Fig. 5.19

SIDESWAY MODE: λ-CONTROL (■)		
$\lambda L^2/EI$	δ/L	θ
1.6467	0.0013	0.0800
1.7367	0.0019	0.0849
1.7467	0.0029	0.0858
1.7487	0.0040	0.0862
1.7502	0.0071	0.0873
1.7507	0.0106	0.0884
1.7512	0.0218	0.0920
1.7517	0.0425	0.0987
1.7522	0.0579	0.1037
1.7622	0.2050	0.1545
1.7722	0.2877	0.1863
1.7822	0.3554	0.2145
1.7922	0.4186	0.2430
1.8022	0.4846	0.2759
1.8122	0.8124	0.4723
1.9022	0.7211	0.3839
1.9122	0.5802	0.2825
1.9322	1.0977	0.6419
1.9522	1.0968	0.6384
1.9722	1.0958	0.6348
1.9922	1.0948	0.6312

SIDESWAY MODE: θ-CONTROL (■)		
$\lambda L^2/EI$	δ/L	θ
0.4396	0.0001	0.0200
0.8597	0.0003	0.0400
1.2617	0.0007	0.0600
1.6467	0.0013	0.0800
1.6080	0.1609	0.1300
1.7304	0.3989	0.2300
1.7716	0.5832	0.3300
1.5409	0.8056	0.5300
-1.2882	0.9979	1.4300
-1.1848	0.9997	1.5300
-0.2551	1.0000	1.6300
1.3621	0.9901	1.7300
2.0718	0.9728	1.8300
2.4070	0.9516	1.9300
2.7315	0.9116	2.0800
0.9743	0.9265	2.1800
-0.9478	0.9489	2.2800
-2.6296	0.9674	2.3300
-2.9227	0.9242	2.3800
-2.7904	0.9277	2.4300
-2.7466	0.9301	2.4800

lateral displacement at collapse is in both cases accepted to be $\delta \simeq 0.285L$. The solutions appear to agree only for relatively small displacements, as shown in Fig. 5.20.

At the time this frame was analyzed, as a computer routine for the detection and solution of critical points was yet to be developed, the sidesway buckling had to be induced by the application of a small disturbing lateral load, as shown in Fig. 5.19; the value we estimate for the buckling load; for a load-control program, is $\lambda_c \simeq 1.7505\ EI/L^2$.

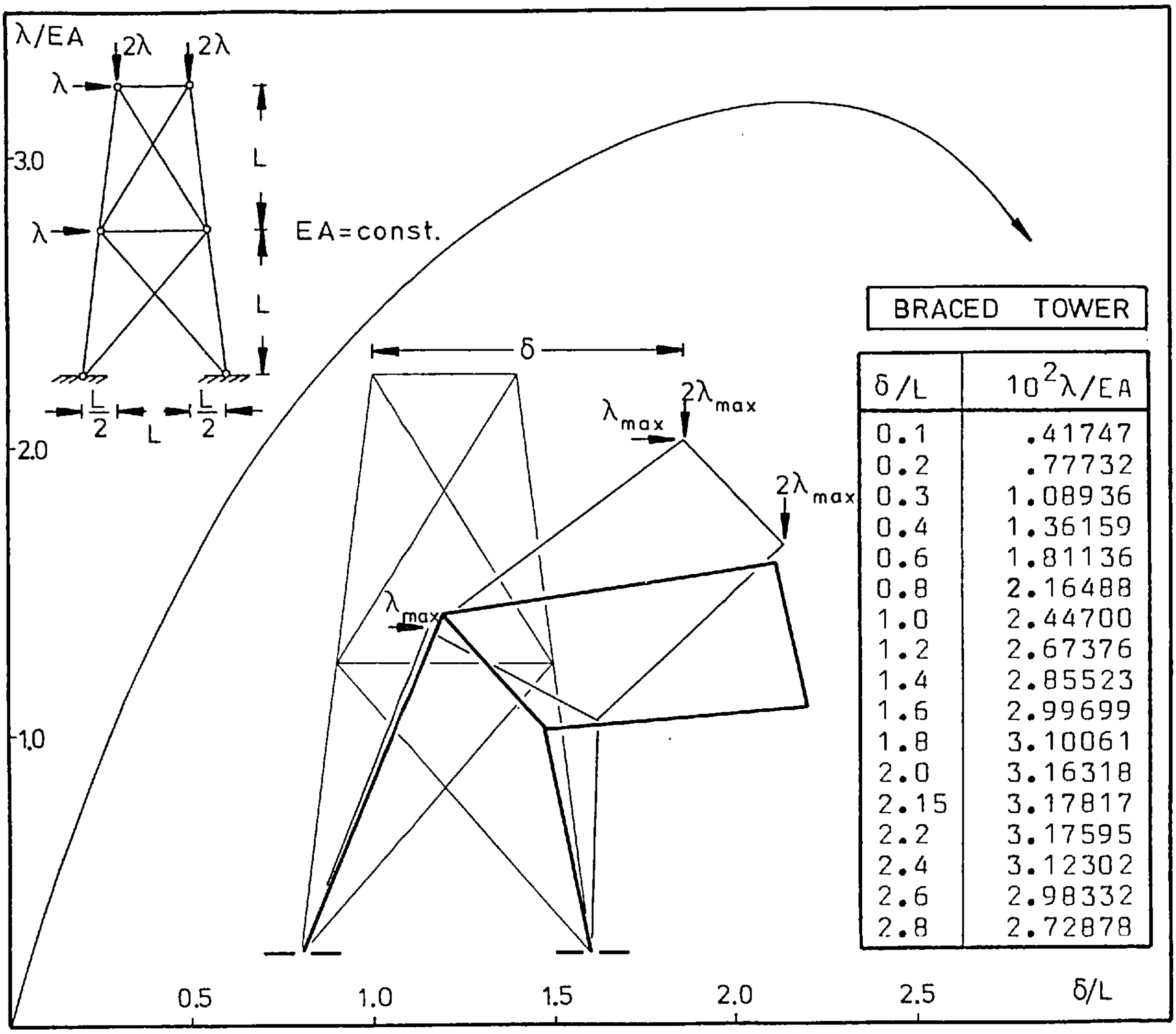
Chwalla and Masur did not discuss the stability of the equilibrium path; according to Horne's analysis the frame is marginally unstable immediately after bifurcation. The results we obtained confirm Lee's conclusion that a very small but decidedly non-zero stiffness remained after bifurcation. Figs. 5.19 and 5.21 show that the frame remains stable (with respect to direction δ) for a significant range of displacements. The sharp discontinuities in the graphs of Fig. 5.21 translate the drastic accommodations the frame has to endure when the variation of a control parameter, with tendency to decrease, is prescribed to increase monotonically.

As, in the present case, the displacements at bifurcation are very small, $\delta \simeq 1.5 \times 10^{-3}L$, the values for the critical load predicted by small displacement stability analyses should not differ substantially in magnitude from that obtained by large displacement analyses; the lower value predicted by the latter, 1.751 against 1.775 of Chwalla and Masur and 1.780 of Horne, is explained by Lee as the result of the additional eccentricity to the column load produced by the flexural shortening of the beam, a contribution neglected in small displacement analysis.

Brown (1970) has also analyzed Chwalla's frame. Although the effects of finite displacements are taken into consideration in the formulation he proposes, the value he found for the bifurcation load, $\lambda_c = 1.8196\ EI/L^2$, appears to be rather high; the stability of the equilibrium path is not discussed.

EXAMPLE 7: Braced Tower

Illustrated in Fig. 5.24 is the response of the truss structure shown when subject to a monotonically increasing sidesway; the response is quantified in the inset.



δ/L	$10^2\lambda/EA$
0.1	.41747
0.2	.77732
0.3	1.08936
0.4	1.36159
0.6	1.81136
0.8	2.16488
1.0	2.44700
1.2	2.67376
1.4	2.85523
1.6	2.99699
1.8	3.10061
2.0	3.16318
2.15	3.17817
2.2	3.17595
2.4	3.12302
2.6	2.98332
2.8	2.72878

FIGURE 5.24

EXAMPLE 8: Two-Storey Portal Frame

The two-storey portal frame analyzed in the previous section is herein assumed to be constituted of an ideal linear elastic material. The frame was discretized into twelve finite-elements and the solution we obtained is given in Table 5.15 and illustrated in Fig. 5.25; the loading and a sequence of displace-

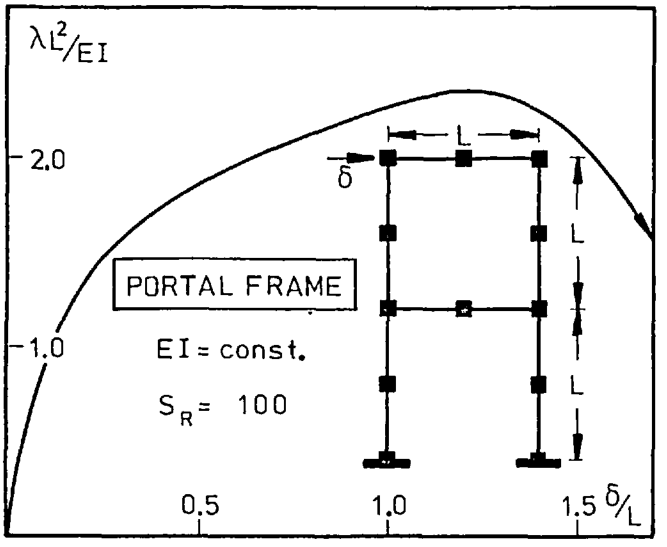


FIGURE 5.25

δ/L	$\lambda L^2/EI$ (S_R=100)
0.10	0.9548525
0.20	1.3636201
0.30	1.5954081
0.40	1.7505655
0.50	1.8674266
0.60	1.9637561
0.70	2.0488045
0.80	2.1275415
1.00	2.2717489
1.20	2.3686215
1.40	2.2768004
1.60	1.8621526
1.80	1.3479834

TABLE 5.15

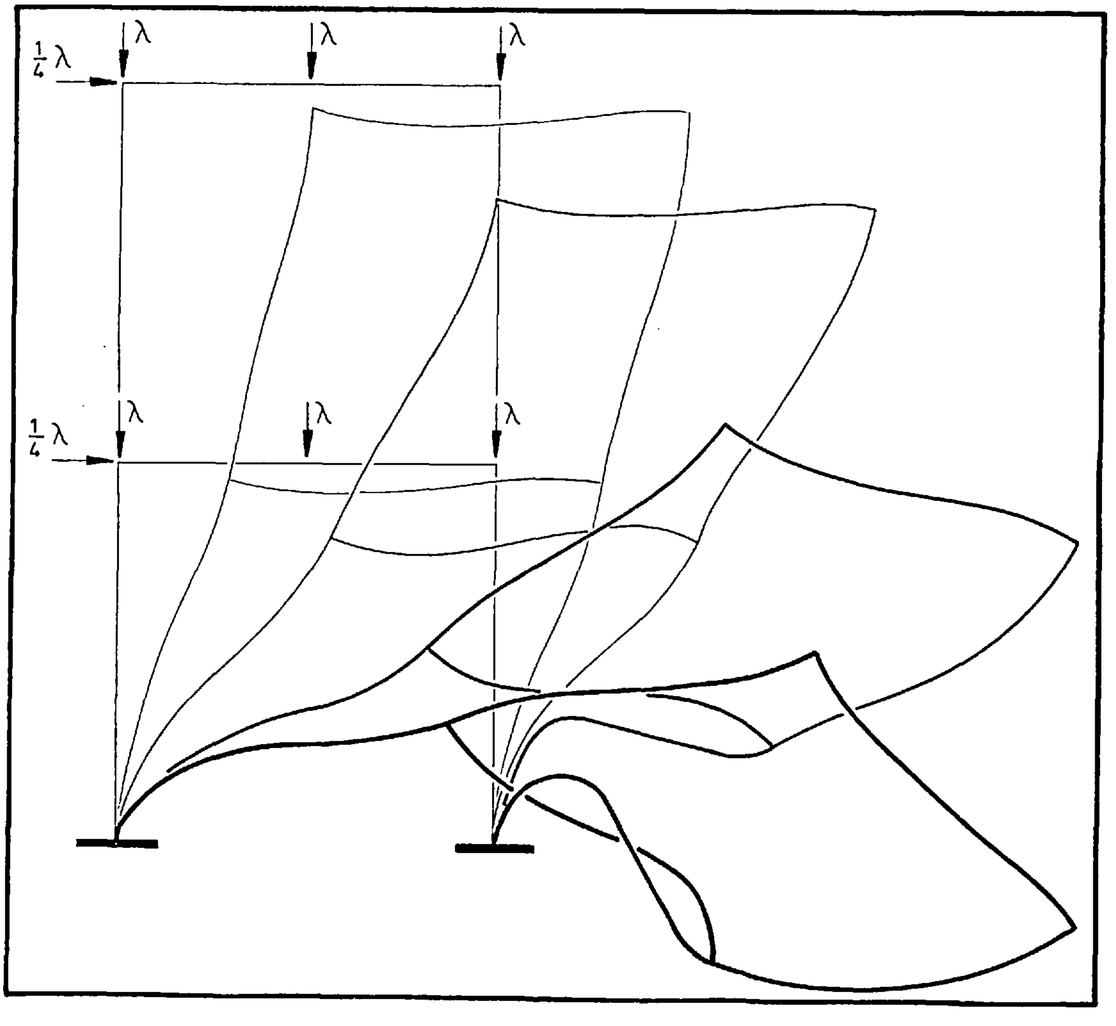


FIGURE 5.26

ments is shown in Fig. 5.26. For the same computational cost, circa 3.5 decimal seconds per step, a more accurate solution would have been obtained if the finite-element mesh had been more refined in the first level columns and rendered more coarse in the second level ones.

5.6 RIGID-PLASTIC STRUCTURES

Several methods used in the eighteenth and nineteenth centuries to assess the safety of arches, domes, retaining walls and earth structures may be regarded as the precursors of limit analysis. Prager (1974), in a most engaging presentation on the evolution of concepts in limit analysis, traces its origins back to the seventeenth century, to Galileo's study of the transversely loaded cantilever.

After Cauchy laid the foundations of the theory of elasticity, in 1822, the substantial majority of methods in structural analysis and design developed during the subsequent decades adopted the linear elastic model.

From the late 1930's onwards, the wish and necessity of designing structures of ever increasing complexity was not immediately accompanied by a parallel development in sophistication of the available processes of numerical implementation. Elastic analysis was proving too cumbersome and revealing an inadequacy for estimating realistically the load-carrying capacity of steel structures, as demonstrated in the pioneering investigations of Kazinczy (1914) and Kist (1917). Practical necessity, rather than theoretical consistency with experiment, was the main motivation in the search for alternative idealizations of the structures behaviour. The plastic theory of structures, which has been developed at Cambridge by Professor Sir John Baker and his co-workers, Baker (1954) has proved in its simplest form to be a suitable conceptual tool for the analysis and design of many beam and frame structures.

Attempts to formulate the limit analysis problem and establish the uniqueness of the collapse surface accompanied attempts to formulate the limit theorems, first enunciated by Gvozdev (1936). When Gvozdev's work became known to Western researchers, in the aftermath of the last World War, the fundamental theorems had already been derived by Greenberg and Prager (1944, 1952), Horne (1950) and

Drucker et alia (1952). According to Neal (1977), the static theorem was first suggested by Kist (1917) as an intuitive axiom; the corollary concerning the effect of strengthening the structure by increasing the plastic capacities was first stated, but not proved, by Feinberg (1948).

The extremum theorems form the basis from which the proposed algorithms of structural plastic analysis developed. The static theorem is used in the method of inequalities of Neal and Symonds (1950) to select the collapse configuration among a set of statically admissible stress distributions; a quite different technique based upon the same theorem is that of moment distribution developed simultaneously by Horne (1954) and English (1954). The kinematic theorem is the criterion adopted in Neal and Symond's (1952) method of combining mechanisms. A tabular procedure for developing simultaneously upper and lower bounds was developed by Munro (1965b).

The application of the simple plastic theory grows improper as the structures become more sensitive to stress-interaction effects to overcome this problem formulations and modified algorithms have been proposed, based on more truthful yield criteria.

The answer to questions on the quality of the stability at incipient collapse is totally beyond the realm of limit analysis theory, as it requires a (kinematically non-linear) post-collapse analysis. Influence of change in the geometry in the behaviour of rigid-plastic structures have been studied by Haythornthwaite (1956, 1957) and Murray (1956), for built-in beams and triangulated frames, respectively; Onat (1960) considered the post-collapse behaviour of arches and Horne (1963) that of planar frames. The early researchers recognized that, while the application of simple plastic theory to beam and frame structures could be justified by experiment, the behaviour of other kinds of structures was not patient of such simple analysis; accordingly, the emphasis on experimental and theoretical research moved from skeletal structures to plates slabs and shells, from as early as the late 1950's.

The static theorem of plastic limit analysis was first recognized as a standard linear programming problem by Charnes and Greenberg (1951). Duality theory was then used by Dorn and Greenberg (1957) and Charnes et alia (1959) to formulate kinematic procedures;

a consistent discussion of the relationships between the safe and unsafe theorems and the primal and dual linear programs of plastic limit analysis, long overdue, was finally presented by Munro and Smith (1972).

Extensive and fairly well-known research work has been done on mathematical programming applications in plastic limit analysis (and synthesis); representative contributions are those of Prager (1962), Hodge (1964), Gavarini (1966), Grierson and Gladwell (1971), Cohn et alia (1972), Maier (1973) and Smith (1974).

Gavarini's (1969) step-by-step procedure is, apparently, the sole application of mathematical programming in the post-collapse analysis of rigid-plastic structures; a sequence of limit analysis by linear programming is performed to obtain a corresponding number of points defining a piecewise-linear approximation of the load-displacement curve.

5.6.1 The Quadratic Programs of Rigid-Plastic Analysis

Instead of removing the elastic phase components from the governing systems (and associated quadratic programs) of elastoplastic analysis, the rigid-plastic analysis systems and programs will be derived from first-principles of mechanics. This approach is justified by the desire of following as closely as possible the usual procedures in plastic limit analysis, wherein the loading is treated as an analysis variable; the external work expended per unit variation of the loading will now be used as a control variable. The formulations of elastic and elastoplastic analysis may also be adapted into variable loading programs following a procedure in every aspect similar to the one to be presented; this approach is particularly relevant for the mesh formulations wherein generalized displacements do not appear explicitly and may not therefore be used directly as alternatives to loading-control sequences.

For simplicity of the presentation, prescribed dislocations are not to be considered and the variation of the loading will be assumed proportional to a parameter λ. General loading descriptions can however be included, for instance in the manner of Smith (1974). The treatment of prescribed dislocations, which may be used as imperfection parameters, should not present special difficulties.

The specialization of the fundamental conditions of mechanics for structures formed by rigid-plastic members is straightforward, it being only necessary to remove the elastic components from the generalized deformation variables present in the descriptions of Statics and Kinematics.

If a rigid-plastic member is axially rigid, the potential plastic hinges being inextensible, equation (2.1.36) provides a further compatibility condition of the form

$$c \cdot \Delta\delta_n = s \cdot \Delta\delta_t + \Delta R_{u2}(\,\Delta u'_2 = 0) \qquad (5.6.1)$$

The above equation can be used to eliminate either $\Delta\delta_t$ or $\Delta\delta_n$ from the system kinematic equations, it being necessary however to replace $\Delta\delta_t(\,\Delta\delta_n)$ by $\Delta\delta_n(\,\Delta\delta_t)$ as a passive variable when s (c) approaches zero.

Theoretically more attractive is an alternative approach wherein both variables $\Delta\delta_n$ and $\Delta\delta_t$ are allowed to remain explicitly present in the system compatibility equations while condition (5.6.1) is implicitly satisfied. If the axial rigidity condition

$$\Delta u'_2 = 0 \qquad (5.6.2)$$

is directly enforced, the general definition (2.1.38) for the additional deformations becomes

$$\Delta \underline{u}'_{\pi} = \underline{Q}'\,\Delta\underline{\delta}_{\pi} + \Delta\underline{R}_{u\pi}\,(\,\Delta u'_2 = 0) \qquad (5.6.3)$$

and, as the shear force reduces to

$$\Delta X_3 = -\frac{1}{L}\,(\,\Delta X_1^1 - \Delta X_1^2) + \Delta R_3(\,\Delta u'_2 = 0)$$

the additional forces become characterized by

$$\Delta\underline{\pi} = \underline{Q}^T\,\Delta\underline{X}' + \underline{P}'\,\Delta\underline{\delta}_{\pi} + \Delta\underline{R}_{\pi}\,(\,\Delta u'_2 = 0) \qquad (5.6.4)$$

In the above we note

$$\underline{Q}' = \begin{bmatrix} \frac{s}{L} & -\frac{1-c}{L} \\ -\frac{s}{L} & \frac{1-c}{L} \\ 1 & \cdot \end{bmatrix} \quad \text{and} \quad \underline{P}' = \begin{bmatrix} \frac{X_2}{L} & -\frac{X_3}{L} \\ \frac{X_3}{L} & \frac{X_2}{L} \end{bmatrix} \qquad (5.6.5a,b)$$

Matrix $\underline{Q}$ is defined in (2.1.39b), where now the member chord length L_c coincides with its initial length L.

Static-Kinematic Duality is destroyed because matrices $\underline{Q}'$ and $\underline{Q}$ are not identical; furthermore, as matrix $\underline{P}'$ is non-symmetric, the static-kinematic interdependence operators (mesh and nodal) cease being symmetric.

This most undesirable situation can however be righted if use is made of the local compatibility equation (5.6.1); let us re-write definitions (5.6.3) and (5.6.4) as

$$\Delta \underset{\sim}{u}'_{\pi} = \underline{Q}\ \Delta \underset{\sim}{\delta}_{\pi} + (\underline{Q}'-\underline{Q})\ \Delta \underset{\sim}{\delta}_{\pi} + \Delta \underset{\sim}{R}_{u\pi}(\Delta u'_2 = 0) \qquad (5.6.6a)$$

and $$\Delta \underset{\sim}{\pi} = \underline{Q}^T \Delta \underset{\sim}{X}' + \underline{P}\ \Delta \underset{\sim}{\delta}_{\pi} + (\underline{P}'-\underline{P})\ \Delta \underset{\sim}{\delta}_{\pi} + \Delta \underset{\sim}{R}_{\pi}(\Delta u'_2 = 0) \qquad (5.6.6b)$$

respectively. From definitions (5.6.5), (2.1.39b) and (2.1.43b), and using condition (5.6.1) it can be found that

$$(\underline{Q}'-\underline{Q})\Delta \underset{\sim}{\delta}_{\pi} = \begin{bmatrix} \cdot \\ \cdot \\ 1 \end{bmatrix} \Delta \underset{\sim}{R}_{u2}(\Delta u'_2 = 0) \quad \text{and} \quad (\underline{P}'-\underline{P})\Delta \underset{\sim}{\delta}_{\pi} = \begin{bmatrix} c\frac{X_2}{L} + 2s\frac{X_3}{L} \\ -s\frac{X_2}{L} + 2c\frac{X_3}{L} \end{bmatrix} \Delta \underset{\sim}{R}_{u2}(\Delta u'_2 = 0)$$

thus reducing (5.6.6) to

$$\Delta \underset{\sim}{u}'_{\pi} = \underline{Q}\, \Delta \underset{\sim}{\delta}_{\pi} + \Delta \underset{\sim}{R}_{u\pi}$$

$$\Delta \underset{\sim}{\pi} = \underline{Q}^T \Delta \underset{\sim}{X}' + \underline{P}\ \Delta \underset{\sim}{\delta}_{\pi} + \Delta \underset{\sim}{R}_{\pi}$$

where now $$\Delta \underset{\sim}{R}_{u\pi} = \Delta \underset{\sim}{R}_{u\pi}(\Delta u'_2 = 0) + (\underline{Q}'-\underline{Q})\ \Delta \underset{\sim}{\delta}_{\pi}$$

and $\Delta \underset{\sim}{R}_{\pi} = \Delta \underset{\sim}{R}_{\pi} (\Delta u'_2=0)+ (\underline{P}'-\underline{P})\, \Delta \underset{\sim}{\delta}_{\pi}$

The derivation of the corresponding description suitable for a perturbation analysis is immediate. The asymptotic analysis definitions, given in subsection 2.1.4, can be treated similarly. The apparent loss of Static-Kinematic Duality and of symmetry of the static-kinematic interdependence matrix does not occur when enforcing the axial-rigidity condition $u'_2=0$ in the deformation analysis fundamental conditions.

The above procedure should be applied in the solution of elastic and elastoplastic structure containing axially-rigid members.

AUXILIARY RESIDUAL VARIABLES	
$\Delta \underset{\sim}{\omega}_1 = \underline{\mathbb{A}}_{\pi}^T \underset{\sim}{\mathbb{R}}_{\pi}$, $\Delta \underset{\sim}{\omega}_2 = -\underset{\sim}{\mathbb{R}}_{\varphi}$, $\Delta \underset{\sim}{\omega}_3 = \underline{M}^T(\underset{\sim}{\mathbb{R}}_p - \underset{\sim}{\mathbb{R}}_{u\pi})$	
Nodal Formulation	
$\Delta \underset{\sim}{\omega}_o = \underset{\sim}{\mathbb{R}}_{\varphi} + \underline{\mathbb{N}}^T\left[\underset{\sim}{B}_{\pi}\underset{\sim}{\bar{\mathbb{B}}}_{\pi}\underset{\sim}{\mathbb{R}}_{\pi} - \underset{\sim}{\mathbb{K}}_M\ (\underset{\sim}{\mathbb{R}}_p - \underset{\sim}{\mathbb{R}}_{u\pi})\right]$	
$\Delta \underset{\sim}{\omega}_1 = \underline{\mathbb{B}}^T(\underset{\sim}{\mathbb{R}}_p - \underset{\sim}{\mathbb{R}}_{u\pi})$, $\Delta \omega_2 = \underset{\sim}{\Lambda}^T \underset{\sim}{\mathbb{B}}_o^T\ (\underset{\sim}{\mathbb{R}}_p - \underset{\sim}{\mathbb{R}}_{u\pi})$	
Alternative Mesh Formulation	
$\Delta \underset{\sim}{\omega}_o = -\underset{\sim}{\mathbb{R}}_{\varphi}$	$\Delta \underset{\sim}{\omega}_1 = \underline{B}_*^T\ (\underset{\sim}{\mathbb{R}}_p - \underset{\sim}{\mathbb{R}}_{u\pi}) - \underset{\sim}{\mathbb{F}}'_p\ \underset{\sim}{\mathbb{R}}_{\pi}$
	$\Delta \omega_2 = \underset{\sim}{\Lambda}^T \underset{\sim}{B}_o^T(\underset{\sim}{\mathbb{R}}_p - \underset{\sim}{\mathbb{R}}_{u\pi}) - \underset{\sim}{\mathbb{F}}'_o\ \underset{\sim}{\mathbb{R}}_{\pi}$
Mesh Formulation	

TABLE 5.16

The alternative descriptions for the governing system of rigid-plastic perturbation analysis are summarized in (5.6.7-9), wherein the residuals $\underset{\sim}{\omega}_{j_i}$ are the coefficients in the series expansion (2.1.52) of the auxilliary residual variables $\Delta\underset{\sim}{\omega}_j$ defined in Table 5.16.

As the loading is assumed proportional to a VARIABLE parameter, in order to preserve the symmetry of each governing system, the displacement compatibility conditions (4.2.70b),(4.2.68b) and (5.2.24b) have to be included in the systems, after being pre-multiplied by the transpose of the loading proportionality vector $\underset{\sim}{\Lambda}$. It can be easily concluded that

PERTURBATION ANALYSIS FORMULATIONS

NODAL FORMULATION (5.6.7)

$$\left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{array}\right]_i + \left[\begin{array}{c|c} \mathbb{A}^T\mathbb{M} & -\underset{\sim}{a}_o \\ \hline -\mathbb{N}^T\mathbb{M} & \cdot \end{array}\right] \left[\begin{array}{c} \underset{\sim}{X}_c \\ \hline \lambda \end{array}\right]_i \begin{array}{c} = \\ \geq \end{array} \left[\begin{array}{c} \underset{\sim}{\omega}_1 \\ \hline \underset{\sim}{\omega}_2 \end{array}\right]_i \quad \begin{array}{c} (a) \\ (b) \end{array}$$

$$-\left[\begin{array}{c|c} \mathbb{M}^T\mathbb{A} & -\mathbb{M}^T\mathbb{N} \\ \hline -\underset{\sim}{a}_o^T & \cdot \end{array}\right] \left[\begin{array}{c} \underset{\sim}{q} \\ \hline \underset{\sim}{v}_* \end{array}\right]_i = -\left[\begin{array}{c} -\underset{\sim}{\omega}_3 \\ \hline -\bar{W} \end{array}\right]_i \quad (c)$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0} \quad (d)$$

$$\underset{\sim}{u}_{*_i}^T \left[\mathbb{H} \quad \underset{\sim}{u}_{*_i} \quad -\mathbb{N}^T\mathbb{M} \ \underset{\sim}{X}_c \quad -\underset{\sim}{\omega}_2 \right]_i = 0 \quad (e)$$

ALTERNATIVE MESH FORMULATION (5.6.8)

$$\left[\mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N}\right] \underset{\sim}{u}_{*_i} + \left[\begin{array}{c|c} -\mathbb{N}^T\mathbb{B} & -\mathbb{N}^T\underset{\sim}{b}_o \end{array}\right] \left[\begin{array}{c} \underset{\sim}{p} \\ \hline \lambda \end{array}\right]_i \geq \underset{\sim}{\omega}_{o_i} \quad (b)$$

$$-\left[\begin{array}{c} -\mathbb{B}^T\mathbb{N} \\ \hline -\underset{\sim}{b}_o^T\mathbb{N} \end{array}\right] \underset{\sim}{u}_{*_i} = -\left[\begin{array}{c} -\underset{\sim}{\omega}_1 \\ \hline -\bar{W}-\omega_2 \end{array}\right]_i \quad (c)$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0} \quad (d)$$

$$\underset{\sim}{u}_{*_i}^T \left[(\mathbb{H}-\mathbb{N}^T\mathbb{K}_M\mathbb{N}) \ \underset{\sim}{u}_{*_i} \quad -\mathbb{N}^T\mathbb{B} \ \underset{\sim}{p} \ -\mathbb{N}^T\underset{\sim}{b}_o\lambda \ -\underset{\sim}{\omega}_o \right]_i = 0 \quad (e)$$

MESH FORMULATION (5.6.9)

$$\mathbb{H} \quad \underset{\sim}{u}_{*_i} + \left[\begin{array}{c|c} -\mathbb{N}^T\underset{\sim}{B}_* & -\mathbb{N}^T\underset{\sim}{b}_o \end{array}\right] \left[\begin{array}{c} \underset{\sim}{p}_* \\ \hline \lambda \end{array}\right]_i \geq \underset{\sim}{\omega}_i \quad (b)$$

$$-\left[\begin{array}{c} -\underset{\sim}{B}_*^T\mathbb{N} \\ \hline -\underset{\sim}{b}_o^T\mathbb{N} \end{array}\right] \underset{\sim}{u}_{*_i} + \left[\begin{array}{c|c} \mathbb{F}_M & \mathbb{F}_o \\ \hline \mathbb{F}_o^T & \mathbb{F}_\lambda \end{array}\right] \left[\begin{array}{c} \underset{\sim}{p}_* \\ \hline \lambda \end{array}\right]_i = -\left[\begin{array}{c} -\underset{\sim}{\omega}_1 \\ \hline -\bar{W}-\omega_2 \end{array}\right]_i \quad (c)$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0} \quad (d)$$

$$\underset{\sim}{u}_{*_i}^T \left[\mathbb{H} \ \underset{\sim}{u}_* \ -\mathbb{N}^T\underset{\sim}{B}_*\underset{\sim}{p}_* - \mathbb{N}^T\underset{\sim}{b}_o\lambda - \underset{\sim}{\omega}_o \right]_i = 0 \quad (e)$$

TABLE 5.17

$$\overline{W}_i = \underset{\sim}{\Lambda}^T \underset{\sim}{\delta}_i$$

which is to be identified with the perturbation parameter ε (thus $\overline{W}_i = \delta_{1i}$), represents the (perturbed form of the) external work expended during a unit variation of the load parameter λ.

Consider a generic node k and the adjoining members, as shown in Fig. 5.27, and assume that members with end-sections i and j have identical bending plastic capacities. The equilibrium condition gives $X_1^i = X_1^j = X_1^c$ where X_1^c is the bending moment at critical section c, chosen in the present case to coincide with section j. Collecting in $\underset{\sim}{X}_{c_i}$ the generalized stress-resultants at all critical sections, the incidence $\underset{\sim}{X}_i = \underline{M}\ \underset{\sim}{X}_{c_i}$ can be defined and substituted into (4.2.69) thus reducing in the nodal system the number of generalized stress-resultants from x to c, where x > c is the dimension of $\underset{\sim}{X}_i$. To preserve Static-Kinematic Duality, the nodal compatibility condition (4.2.70a) has to be pre-multiplied by $\underline{M}^T$, thus enforcing the summation of the rotations developing at the sections neighbouring node k; as a consequence, the rotation at nodes k where only two members connect becomes zero and may also be removed from the system. This procedure is also applicable to the nodal-flexibility programs of elastic and elastoplastic analysis.

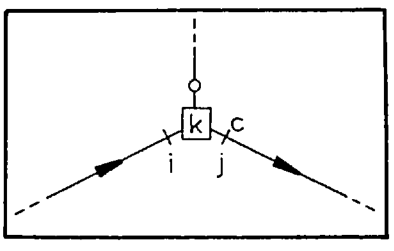


FIGURE 5.27

The three pairs of quadratic programs (5.6.10) to (5.6.15) were obtained after confronting each of the governing systems summarized in Table 5.17 with the Kuhn-Tucker Conditions (5.1.12) and, assuming the Kuhn-Tucker Equivalence requirements satisfied, enforcing the resulting identifications in the pair of primal-dual programs (5.1.7-13).

The asymptotic analysis governing systems and the associated quadratic programs are qualitatively identical to systems (5.6.7-9) and programs (5.6.10-15), respectively.

For non-workhardening plastic materials, the zeroth-order

NODAL FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i^T \left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i + \begin{bmatrix} \underset{\sim}{X}_c \\ \hline \lambda \end{bmatrix}_i^T \begin{bmatrix} -\underset{\sim}{\omega}_3 \\ \hline -\bar{W} \end{bmatrix}_i$$

$$\text{subject to:-} \quad \left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i + \left[\begin{array}{c|c} \mathbb{A}^T\mathbb{M} & -\mathbb{a}_0 \\ \hline -\mathbb{N}^T\mathbb{M} & \cdot \end{array}\right] \begin{bmatrix} \underset{\sim}{X}_c \\ \hline \lambda \end{bmatrix}_i \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \underset{\sim}{\omega}_1 \\ \hline \underset{\sim}{\omega}_2 \end{bmatrix}_i$$

PRIMAL PROGRAM (5.6.10)	DUAL PROGRAM (5.6.11)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i^T \left[\begin{array}{c|c} -\mathbb{K}_N & \cdot \\ \hline \cdot & \mathbb{H} \end{array}\right] \begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i + \begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i^T \begin{bmatrix} \underset{\sim}{\omega}_1 \\ \hline \underset{\sim}{\omega}_2 \end{bmatrix}_i$$

$$\text{subject to:-} \quad -\left[\begin{array}{c|c} \mathbb{M}^T\mathbb{A} & -\mathbb{M}^T\mathbb{N} \\ \hline -\mathbb{a}_0^T & \cdot \end{array}\right] \begin{bmatrix} \underset{\sim}{q} \\ \hline \underset{\sim}{u}_* \end{bmatrix}_i = -\begin{bmatrix} \underset{\sim}{\omega}_3 \\ \hline -\bar{W} \end{bmatrix}_i$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0}$$

ALTERNATIVE MESH FORMULATION

$$\text{Min } z = \frac{1}{2}\,\underset{\sim}{u}_{*_i}^T \left[\mathbb{H} - \mathbb{N}^T\mathbb{K}_M\mathbb{N}\right] \underset{\sim}{u}_{*_i} + \begin{bmatrix} \underset{\sim}{p} \\ \hline \lambda \end{bmatrix}_i^T \begin{bmatrix} -\underset{\sim}{\omega}_1 \\ \hline -\bar{W}-\omega_2 \end{bmatrix}_i$$

$$\text{subject to:-} \quad \left[\mathbb{H} - \mathbb{N}^T\mathbb{K}_M\mathbb{N}\right] \underset{\sim}{u}_{*_i} + \left[\begin{array}{c|c} -\mathbb{N}^T\mathbb{B} & -\mathbb{N}^T\mathbb{b}_0 \end{array}\right] \begin{bmatrix} \underset{\sim}{p} \\ \hline \lambda \end{bmatrix}_i \geq \underset{\sim}{\omega}_{o_i}$$

PRIMAL PROGRAM (5.6.12)	DUAL PROGRAM (5.6.13)

$$\text{Max } w = -\frac{1}{2}\,\underset{\sim}{u}_{*_i}^T \left[\mathbb{H} - \mathbb{N}^T\mathbb{K}_M\mathbb{N}\right] \underset{\sim}{u}_{*_i} + \underset{\sim}{u}_{*_i}^T\,\underset{\sim}{\omega}_{o_i}$$

$$\text{subject to:-} \quad -\begin{bmatrix} -\mathbb{B}^T\mathbb{N} \\ \hline -\underset{\sim}{\mathbb{b}}_0^T\mathbb{N} \end{bmatrix} \underset{\sim}{u}_{*_i} = -\begin{bmatrix} -\underset{\sim}{\omega}_1 \\ \hline -\bar{W}-\omega_2 \end{bmatrix}_i$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0}$$

programs become LINEAR. Consider for instance the primal and dual nodal programs

MESH DESCRIPTION

$$\text{Min } z = \frac{1}{2}\, \underset{\sim}{u}_{*_i}^T\, \underline{\mathbb{H}}\, \underset{\sim}{u}_{*_i} + \frac{1}{2} \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i^T \begin{bmatrix} \mathbb{F}_M & \underset{\sim}{\mathbb{F}}_o \\ \underset{\sim}{\mathbb{F}}_o^T & \mathbb{F}_\lambda \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i + \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i \begin{bmatrix} -\underset{\sim}{\omega}_1 \\ -\bar{W}-\omega_2 \end{bmatrix}_i$$

$$\text{subject to:- } \underline{\mathbb{H}}\, \underset{\sim}{u}_{*_i} + \begin{bmatrix} -\underline{\mathbb{N}}^T \underline{B}_* & -\underline{\mathbb{N}}^T \underset{\sim}{b}_o \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i \geq \underset{\sim}{\omega}_{o_i}$$

PRIMAL PROGRAM (5.6.14)	DUAL PROGRAM (5.6.15)

$$\text{Max } w = -\frac{1}{2}\, \underset{\sim}{u}_{*_i}^T\, \underline{\mathbb{H}}\, \underset{\sim}{u}_{*_i} - \frac{1}{2} \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i^T \begin{bmatrix} \mathbb{F}_M & \underset{\sim}{\mathbb{F}}_o \\ \underset{\sim}{\mathbb{F}}_o^T & \mathbb{F}_\lambda \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i + \underset{\sim}{u}_{*_i}^T\, \underset{\sim}{\omega}_{o_i}$$

$$\text{subject to:- } -\begin{bmatrix} -\underline{B}_*^T \underline{\mathbb{N}} \\ -\underset{\sim}{b}_o^T \underline{\mathbb{N}} \end{bmatrix} \underset{\sim}{u}_{*_i} + \begin{bmatrix} \mathbb{F}_M & \underset{\sim}{\mathbb{F}}_o \\ \underset{\sim}{\mathbb{F}}_o^T & \mathbb{F}_\lambda \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}_* \\ \lambda \end{bmatrix}_i = \begin{bmatrix} -\underset{\sim}{\omega}_1 \\ -\bar{W}-\omega_2 \end{bmatrix}_i$$

$$\underset{\sim}{u}_{*_i} \geq \underset{\sim}{0}$$

$$\text{Max } -z_o = \begin{bmatrix} \underset{\sim}{X}_c \\ \lambda \end{bmatrix}_o^T \begin{bmatrix} \cdot \\ \bar{W} \end{bmatrix}_o : \begin{bmatrix} \mathbb{A}^T \underline{M} & -\underset{\sim}{a}_o \\ -\underline{\mathbb{N}}^T \underline{M} & \cdot \end{bmatrix} \begin{bmatrix} \underset{\sim}{X}_c \\ \lambda \end{bmatrix}_o \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \cdot \\ -\underset{\sim}{X}_* \end{bmatrix}$$

and

$$\text{Min } -w_o = \begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix}_o^T \begin{bmatrix} \cdot \\ \underset{\sim}{X}_* \end{bmatrix} : \begin{bmatrix} -\underline{M}^T \mathbb{A} & -\underline{M}^T \underline{\mathbb{N}} \\ -\underset{\sim}{a}_o^T & \cdot \end{bmatrix} \begin{bmatrix} \underset{\sim}{q} \\ \underset{\sim}{u}_* \end{bmatrix}_o = \begin{bmatrix} \cdot \\ \bar{W} \end{bmatrix}, \quad \underset{\sim}{u}_{*_o} \geq \underset{\sim}{0}$$

respectively. The unit work $\bar{W}$ can still be identified with the perturbation parameter ε, yielding $\bar{W}_i = \delta_{1i}$ and consequently $\bar{W}_o = 0$. As by definition the zeroth-order kinematic variables y_o are also null, the primal objective function becomes trivial and the dual feasible region empty. However, as $-z_o$, $\underset{\sim}{q}_o$, $\underset{\sim}{u}_{*_o}$ and $\bar{W}_o$ are all null, the indeterminacies

$$z_o' = -z_o/\bar{W}_o \quad , \quad \underset{\sim}{q}' = \underset{\sim}{q}_o/\bar{W}_o \; , \quad \underset{\sim}{u}_{*_o}' = \underset{\sim}{u}_{*_o}/\bar{W}_o \quad (5.6.16a\text{-}c)$$

can be artificially introduced, reducing the programs in the above to the following:

$$\text{Max } \lambda_o \quad : \begin{bmatrix} -\underline{A}^T\underline{M} & \underline{a}_o \\ \underline{N}^T\underline{M} & \cdot \end{bmatrix} \begin{bmatrix} \underline{X}_c \\ \lambda \end{bmatrix}_o \begin{matrix} = \\ \leq \end{matrix} \begin{bmatrix} \cdot \\ \underline{X}_* \end{bmatrix} \tag{5.6.17}$$

$$\text{and} \quad \text{Min } \underline{X}_*^T \underline{u}^!_{*_o} \quad : \begin{bmatrix} -\underline{M}^T\underline{A} & \underline{M}^T\underline{N} \\ \underline{a}_o^T & \cdot \end{bmatrix} \begin{bmatrix} \underline{o}' \\ \underline{u}_* \end{bmatrix}_o \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} \cdot \\ 1 \end{bmatrix}, \quad \underline{u}^!_{*_o} \geq \underline{0} \tag{5.6.18}$$

The zeroth-order mesh programs can be similarly reduced to

$$\text{Max } \lambda_o \quad : \begin{bmatrix} \underline{N}^T\underline{B} & \underline{N}^T\underline{b}_o \end{bmatrix} \begin{bmatrix} \underline{p} \\ \lambda \end{bmatrix} \leq \underline{X}_* \tag{5.6.19}$$

$$\text{and} \quad \text{Min } \underline{X}_*^T \underline{u}^!_{*_o} \quad : \begin{bmatrix} \underline{B}^T\underline{N} \\ \underline{b}_o^T\underline{N} \end{bmatrix} \underline{u}^!_{*_o} = \begin{bmatrix} \cdot \\ 1 \end{bmatrix}, \quad \underline{u}^!_{*_o} \geq \underline{0} \tag{5.6.20}$$

Setting, in the nodal programs (5.6.17-18), the incidence matrix $\underline{M}$ to the identity matrix $\underline{I}$ (implying that $\underline{X}_c = \underline{X}$ and increasing the dual constraints by x-c) the programs in the above become the NODAL AND MESH LINEAR PROGRAMS OF PLASTIC LIMIT ANALYSIS proposed by Smith (1974); equivalent programs in alternative formats are widely presented in the literature, wherein the scaling operation (5.6.16) is known as the "normalization of the collapse mechanism".

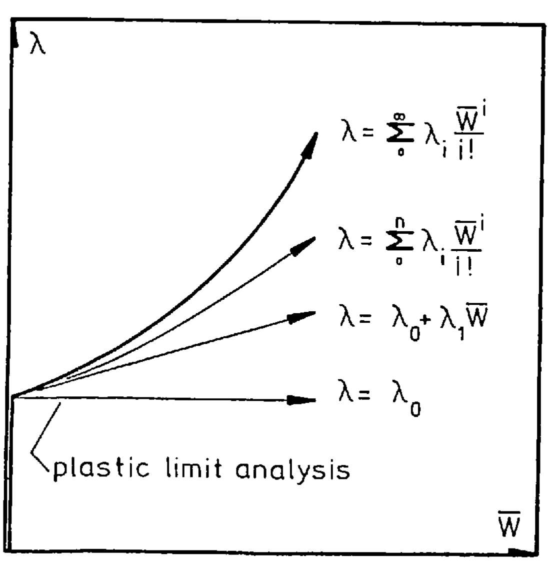


FIGURE 5.28

Illustrated in Fig.5.28 is a typical sequence of solutions as provided by an asymptotic analysis which essentially approximates by an infinite series

$$\lambda = \lambda_o + \lambda_1 \bar{\omega} + \lambda_2 \frac{\bar{\omega}^2}{2!} + \lambda_3 \frac{\bar{\omega}^3}{3!} + \cdots$$

the actual equilibrium path $\lambda = \lambda(\bar{\omega})$.

The alternative descriptions of the governing systems for the rigid-plastic incremental analysis are structurally identical to systems (5.6.7-9), it being only necessary to replace in there each variable by its corresponding increment.

As the residuals $\Delta \underset{\sim}{\omega}_i$, defined in Table 5.16, are non-linear functions of the system variables, such systems may only be identified with the Kuhn-Tucker Conditions (5.1.12) under the assumption that the actual values taken by the residuals are known a priori. Under this assumption, and supposing that the governing systems satisfy the Kuhn-Tucker Equivalence requirements, the associated pairs of primal-dual (iterative) quadratic programs could then be derived, emerging in formats similar to those of the perturbation analysis programs.

The alternative governing systems of kinematically non-linear rigid-plastic deformation analysis are presented in Table 5.19; Table 5.18 summarizes the definitions of the relevant auxiliary variables $\underset{\sim}{\omega}_i$. Matrices $\underline{K}_N$ and $\underline{K}_M$ are defined in (5.2.32c) and (5.2.32d), respectively.

For a compatible kinematic configuration associated with a unit work $\overline{W}$, assuming known the elements of functionals $\underline{K}_N$, $\underline{K}_M$, $\underline{H}$ and $\underset{\sim}{\omega}_i$, systems (5.6.21) and (5.6.22) can be identified with the Kuhn-Tucker Conditions (5.1.12) thus generating the quadratic programs (5.6.23-24) and (5.6.25-26), respectively, to which they become equivalent if the Kuhn-Tucker Equivalence requirements are satisfied.

Specializing the plasticity relations for the case of perfectly plastic materials and setting to zero all coefficients associated with the kinematic non-linear effects, programs (5.6.23-26), after enforcing a normalization condition similar to (5.6.16), are reduced to two pairs of primal-dual linear programs equivalent to programs (5.6.17-18) and (5.6.19-20).

AUXILIARY VARIABLES
$\underline{\omega}_1 = -\underline{\pi}_\varphi \quad \underline{\omega}_2 = \underline{M}^T(\underline{u}_\varphi - \underline{u}_\pi)$
Nodal Formulation
$\underline{\omega}_o = -\underline{\pi}_\varphi - \underline{N}^T\underline{K}_M\,(\underline{u}_\varphi - \underline{u}_\pi)$, $\underline{\omega}_1 = \underline{B}^T(\underline{u}_\varphi - \underline{u}_\pi)$, $\omega_2 = \underline{\Lambda}^T\underline{B}_o^T(\underline{u}_\varphi - \underline{u}_\pi)$
Mesh Formulation

TABLE 5.18

THE FORMULATIONS OF DEFORMATION ANALYSIS

NODAL FORMULATION (5.6.21)

$$\begin{bmatrix} -\underline{K}_N & \cdot \\ \cdot & \underline{H} \end{bmatrix}\begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix} + \begin{bmatrix} \underline{A}^T\underline{M} & -\underline{a}_o \\ -\underline{N}^T\underline{M} & \cdot \end{bmatrix}\begin{bmatrix} \underline{X}_c \\ \lambda \end{bmatrix} \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} \cdot \\ -\underline{X}_* + \underline{\omega}_1 \end{bmatrix}$$

$$-\begin{bmatrix} \underline{M}^T\underline{A} & -\underline{M}^T\underline{N} \\ -\underline{a}_o^T & \cdot \end{bmatrix}\begin{bmatrix} \underline{q} \\ \underline{u}_* \end{bmatrix} = -\begin{bmatrix} -\underline{\omega}_2 \\ -\bar{W} \end{bmatrix}$$

$$\underline{u}_*^T\left[\underline{H}\;\underline{u}_* \quad -\underline{N}^T\underline{M}\;\underline{X}_c \quad +\underline{X}_* - \underline{\omega}_1\right] = 0$$

$$\underline{u}_* \geq \underline{0}$$

MESH FORMULATION (5.6.22)

$$\left[\underline{H} \; -\underline{N}^T\underline{K}_M\underline{N}\right]\underline{u}_* + \left[-\underline{N}^T\underline{B} \;\vdots\; -\underline{N}^T\underline{b}_o\right]\begin{bmatrix} \underline{p} \\ \lambda \end{bmatrix} \geq -\underline{X}_* + \underline{\omega}_o$$

$$-\begin{bmatrix} -\underline{B}^T\underline{N} \\ -\underline{b}_o^T\underline{N} \end{bmatrix}\underline{u}_* = -\begin{bmatrix} -\underline{\omega}_1 \\ -\bar{W} - \omega_2 \end{bmatrix}$$

$$\underline{u}_*^T\left\{\left[\underline{H} \; -\underline{N}^T\underline{K}_M\underline{N}\right]\underline{u}_* \quad -\underline{N}^T\underline{B}\;\underline{p} \; -\underline{N}^T\underline{b}_o\lambda + \underline{X}_* - \underline{\omega}_o\right\} = 0$$

$$\underline{u}_* \geq \underline{0}$$

TABLE 5.19

NODAL FORMULATION

$$\text{Min } z = \frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} -K_N & \cdot \\ \hline \cdot & H \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + \begin{bmatrix} X_c \\ \hline \lambda \end{bmatrix}^T \begin{bmatrix} -\omega_2 \\ \hline -\bar{\omega} \end{bmatrix}$$

$$\text{subject to:-} \quad \left[\begin{array}{c|c} -K_N & \cdot \\ \hline \cdot & H \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + \left[\begin{array}{c|c} A^T M & -a_o \\ \hline -N^T M & \cdot \end{array}\right] \begin{bmatrix} X_c \\ \hline \lambda \end{bmatrix} \begin{array}{c} = \\ \geq \end{array} \begin{bmatrix} \cdot \\ \hline -X_* + \omega_1 \end{bmatrix}$$

PRIMAL PROGRAM (5.6.23)	DUAL PROGRAM (5.6.24)

$$\text{Max } w = -\frac{1}{2}\begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \left[\begin{array}{c|c} -K_N & \cdot \\ \hline \cdot & H \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} + \begin{bmatrix} q \\ \hline u_* \end{bmatrix}^T \begin{bmatrix} \cdot \\ \hline -X_* + \omega_1 \end{bmatrix}$$

$$\text{subject to:-} \quad -\left[\begin{array}{c|c} M^T A & -M^T N \\ \hline -a_o^T & \cdot \end{array}\right] \begin{bmatrix} q \\ \hline u_* \end{bmatrix} = \begin{bmatrix} -\omega_2 \\ \hline -\bar{\omega} \end{bmatrix}$$

$$u_* \geq 0$$

MESH FORMULATION

$$\text{Min } z = \frac{1}{2} u_*^T \left[H - N^T K_M N \right] u_* + \begin{bmatrix} p \\ \hline \lambda \end{bmatrix}^T \begin{bmatrix} -\omega_1 \\ \hline -\bar{\omega} - \omega_2 \end{bmatrix}$$

$$\text{subject to:-} \left[H - N^T K_M N \right] u_* + \left[\begin{array}{c|c} -N^T B & -N^T b_o \end{array}\right] \begin{bmatrix} p \\ \hline \lambda \end{bmatrix} \geq -X_* + \omega_o$$

PRIMAL PROGRAM (5.6.25)	DUAL PROGRAM (5.6.26)

$$\text{Max } w = -\frac{1}{2} u_*^T \left[H - N^T K_M N \right] u_* + u_*^T(-X_* + \omega_o)$$

$$\text{subject to:-} \quad -\begin{bmatrix} -B^T N \\ \hline -b_o^T N \end{bmatrix} u_* = -\begin{bmatrix} -\omega_1 \\ \hline -\bar{\omega} - \omega_2 \end{bmatrix}$$

$$u_* \geq 0$$

5.6.2 General Considerations in Rigid-Plastic Analysis

The alternative descriptions of the system governing rigid-plastic behaviour were arranged in such a manner that when identified with the Kuhn-Tucker problem (5.1.12) the primal constraints [constraints of type (a) and (b)] represented the static admissibility conditions, while the dual constraints [constraints of the type (c) and (d)] represented the kinematic admissibility conditions.

As all static quantities were treated as variables, the primal programs will always be feasible; although non-empty, their feasible regions may reduce to a point only satisfied by a trivial solution. The kinematic variables were, on the other hand, constrained to dissipate a prescribed amount of work per unit variation of the loading; if the value attributed to $\Delta\bar{W}$ is unattainable, the dual programs will not be feasible, thus rendering the primal objective functions unbounded. Otherwise, it can be stated that IF, FOR A PRESCRIBED INCREMENT $\Delta\bar{W}$, STATICALLY AND KINEMATICALLY ADMISSIBLE CONFIGURATIONS EXIST, THEN BOTH PRIMAL AND DUAL PROGRAMS WILL HAVE OPTIMAL SOLUTIONS.

The application of Theorem (5.1.17) to the perturbation analysis programs presented in the previous subsection generates the following statements regulating the sufficient conditions for uniqueness at optimality:

- Consider the nodal formulation programs (5.6.10-11):

(I) If the "geometric" stiffness matrix $-\underline{\mathbb{K}}_N$ and the hardening matrix $\underline{\mathbb{H}}$ are positive definite, the generalized nodal displacements $\Delta\underline{q}$ and the generalized plastic multipliers $\Delta\underline{u}_*$ are unique

and in particular

(II) If matrix $\underline{\mathbb{K}}_N$ is non-singular, and matrix $[\underline{A}^T\underline{M} - \underline{a}_o]^T(-\underline{\mathbb{K}}_N)^{-1}$ $[\underline{A}^T\underline{M} - \underline{a}_o]$ is positive definite, the generalized stress-resultants $\Delta\underline{X}$ and the loading parameter $\Delta\lambda$ are unique.

- From the mesh formulation programs (5.6.14-15):

(III) If matrix $\begin{bmatrix} \underline{\mathbb{F}}_M & \underline{\mathbb{F}}_o \\ \underline{\mathbb{F}}_o^T & \mathbb{F}_\lambda \end{bmatrix}$ is positive definite the stress-resultant field $\Delta\underline{X} = \Delta\underline{X}(\Delta\underline{p}_*, \Delta\lambda)$ is unique.

and, in particular,

(IV) If matrices $\begin{bmatrix} \underline{\mathbb{F}}_M & \underline{\mathbb{F}}_o \\ \underline{\mathbb{F}}_o^T & \mathbb{F}_\lambda \end{bmatrix}$ and $\underline{\mathbb{H}} + \begin{bmatrix} -\underline{B}_*^T\underline{\mathbb{N}} \\ -\underline{b}_o^T\underline{\mathbb{N}} \end{bmatrix} \begin{bmatrix} \underline{\mathbb{F}}_M & \underline{\mathbb{F}}_o \\ \underline{\mathbb{F}}_o^T & \mathbb{F}_\lambda \end{bmatrix}^{-1} \begin{bmatrix} -\underline{B}_*^T\underline{\mathbb{N}} \\ -\underline{b}_o^T\underline{\mathbb{N}} \end{bmatrix}^T$

are non-singular and positive definite, respectively, the generalized plastic multipliers $\Delta \underset{\sim}{u}_*$ are unique.

The characterization of the conditions which a multiple optimal solution must satisfy, should be based on the composite form of the programs to which it refers if it is to be guaranteed explicitly that the configurations defined by such a solution are simultaneously statically and kinematically admissible. The application of Theorem (5.1.18) to the composite form of the nodal programs (5.6.10-11) generates system (5.6.27) which characterizes the possible (first-order) multiple solutions

$$(\dot{\underset{\sim}{q}}'', \dot{\underset{\sim}{u}}''_*, \dot{\underset{\sim}{X}}'', \dot{\lambda}'') = (\dot{\underset{\sim}{q}}', \dot{\underset{\sim}{u}}'_*, \dot{\underset{\sim}{X}}', \dot{\lambda}') + \alpha(\delta \underset{\sim}{q}, \delta \underset{\sim}{u}_*, \delta \underset{\sim}{X}, \delta\lambda)$$

where $(\dot{\underset{\sim}{q}}', \dot{\underset{\sim}{u}}'_*, \dot{\underset{\sim}{X}}', \dot{\lambda}')$ represents a known optimal solution. System (5.6.27) could now be treated and interpreted following a procedure identical to the one used on system (5.3.4-9) which regulates multiple elastoplastic optimal solutions.

MULTIPLE SOLUTIONS (5.6.27)

$$\begin{bmatrix} \delta \underset{\sim}{q} \\ \delta \underset{\sim}{u}_* \end{bmatrix}^T \begin{bmatrix} -\underset{\sim}{\mathbb{K}}_N & \cdot \\ \cdot & \mathbb{H} \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{q} \\ \delta \underset{\sim}{u}_* \end{bmatrix} = 0 \qquad \delta\lambda\, \dot{\overline{W}} = 0$$

$$\begin{bmatrix} -\underset{\sim}{\mathbb{K}}_N & \cdot \\ \cdot & \underset{\sim}{\mathbb{H}} \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{q} \\ \delta \underset{\sim}{u}_* \end{bmatrix} + \begin{bmatrix} \underset{\sim}{\mathbb{A}}^T & -\underset{\sim}{a}_0 \\ -\underset{\sim}{\mathbb{N}}^T & \cdot \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{X} \\ \delta\lambda \end{bmatrix} = -\begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{I} \end{bmatrix} \delta \underset{\sim}{\Phi}_*$$

$$-\begin{bmatrix} \underset{\sim}{\mathbb{A}} & -\underset{\sim}{\mathbb{N}} \\ -\underset{\sim}{a}_0^T & \cdot \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{q} \\ \delta \underset{\sim}{u}_* \end{bmatrix} = \underset{\sim}{0} \qquad \begin{array}{l} \dot{\underset{\sim}{u}}_* + \alpha\, \delta \underset{\sim}{u}_* \geq \underset{\sim}{0} \\ \dot{\underset{\sim}{\Phi}}'_* + \alpha\, \delta \underset{\sim}{\Phi}_* \leq \underset{\sim}{0} \end{array}$$

$$\dot{\underset{\sim}{u}}'^T_*\, \delta \underset{\sim}{\Phi}_* + \dot{\underset{\sim}{\Phi}}'^T_*\, \delta \underset{\sim}{u}_* + \alpha\, \delta \underset{\sim}{u}^T_*\, \delta \underset{\sim}{\Phi}_* = 0$$

After a sequence of substitutions, particularly simple for the nodal formulation programs, the primal and dual objective functions of the programs previously presented can be reduced to the following forms:

- Deformation analysis programs

$$z^D = \left\{\frac{1}{2}\ \underline{u}_*^T\ \underline{H}\ \underline{u}_* - \frac{1}{2}\ \underline{\pi}^T \underline{\delta}_\pi\right\} - \left\{\underline{\lambda}^T \underline{\delta}\right\} + \left\{\underline{X}^T(\underline{u}_\pi - \underline{u}_\varphi)\right\} + O_4 \qquad (5.6.28)$$

$$-w^D = \left\{\frac{1}{2}\ \underline{u}_*^T\ \underline{H}\ \underline{u}_* - \frac{1}{2}\ \underline{\pi}^T \underline{\delta}_\pi\right\} + \left\{\underline{X}_*^T\ \underline{u}_*\right\} + \left\{\ \underline{u}_*^T\ \underline{\pi}_\varphi\right\} - O_4 \qquad (5.6.29)$$

- Incremental analysis programs

$$z^I = \left\{\frac{1}{2}\ \Delta\underline{X}^T(\ \Delta\underline{u}_p + \Delta\underline{u}_\pi) - \frac{1}{2}\ \Delta\underline{\pi}^T \Delta\underline{\delta}_\pi + \frac{1}{2}\left[\Delta\underline{X}^T(\underline{\mathbb{R}}_{u\pi} - \underline{\mathbb{R}}_p) + \Delta\underline{\delta}_\pi^T\ \underline{\mathbb{R}}_\pi - \Delta\underline{u}_*^T \underline{\mathbb{R}}_\varphi\right]\right\} - \left\{\Delta\underline{\lambda}^T \Delta\underline{\delta}\right\} + O_4 \qquad (5.6.30)$$

$$-w^I = \left\{\frac{1}{2}\ \Delta\underline{X}^T(\ \Delta\underline{u}_p + \Delta\underline{u}_\pi) - \frac{1}{2}\ \Delta\underline{\pi}^T \Delta\underline{\delta}_\pi - \frac{1}{2}\left[\Delta\underline{X}^T(\underline{\mathbb{R}}_{u\pi} - \underline{\mathbb{R}}_p) + \Delta\underline{\delta}_\pi^T \underline{\mathbb{R}}_\pi - \Delta\underline{u}_*^T \underline{\mathbb{R}}_\varphi\right]\right\} - O_4 \qquad (5.6.31)$$

The objective functions of the i-th order program of perturbation analysis can be obtained by replacing in (5.3.30-31) each incremental variable by the i-th order component of its series expansion, and replacing the fourth- and higher-order terms O_4 by constants.

Combining through the Principle of Virtual Work, i.e., using Static-Kinematic Duality, two configurations, one statically the other kinematically admissible, and substituting in the above the results so obtained, it can be verified that the primal and dual objective functions attain the same value at optimality.

In particular, the deformation analysis formulations give, for perfectly plastic materials

$$\underline{X}_*^T\ \underline{u}_* = \underline{\pi}^T \underline{\delta}_\pi + \lambda\ \overline{W} - \underline{X}^T(\underline{u}_\pi - \underline{u}_\varphi) + \underline{u}_*^T\ \underline{\pi}_\varphi$$

Introducing the assumptions of the simple bending theory of plasticity ($\underline{u}_\varphi = \underline{0}$, $\underline{\pi}_\varphi = \underline{0}$) and neglecting all the second- and higher-order kinematic effects ($\pi_n = -\rho X_3$, $\pi_t = \rho X_2$, $\delta_n = 0$, $\delta_t = \rho L$, $\underline{u}_\pi = \underline{0}$) the above equation reduces to

$$\underline{X}_*^T\ \underline{u}_* = \sum_i X_2^i\ L_i\ \rho_i^2 + \lambda\ \overline{W} \qquad (5.6.32)$$

The load at incipient collapse ($\rho_i=0$) is

$$\lambda_o = D/\overline{W} \qquad (5.6.33)$$

where $D=\underline{X}_*^T \underline{u}_*$ represents the plastic dissipation; assuming that, at any load factor λ during the deformation of the collapse mechanism the axial forces are proportional to the axial forces at incipient collapse, i.e.

$$X_2^i = \frac{\lambda}{\lambda_o} X_{2_o}^i$$

equation (5.6.32) reduces to

$$\lambda = \lambda_o \left[1 + D^{-1} \sum_i X_{2_o}^i L_i \rho_i^2 \right]^{-1} \qquad (5.6.34)$$

The above equation, first proposed by Horne (1963), characterizes the equilibrium path of a (predetermined) collapse mechanism undergoing very small but finite deformations.

The application of the Principle of Virtual Work to the asymptotic analysis formulations gives for the load at incipient collapse

$$\lambda_o = \frac{1}{\overline{W}_1} \left[\underline{X}_o^T \underline{u}_{p_1} \right] \qquad (5.6.35a)$$

re-generating (5.6.33) and for its first variation

$$\lambda_1 = \frac{1}{2\overline{W}_1} \left[\underline{X}_o^T \underline{u}_{p_2} + 2\underline{X}_1^T \underline{u}_{p_1} - \lambda_o \overline{W}_2 - \sum_i \left(2X_{3_o}^2 \delta_{n_{1_i}} - X_{2_o}^i \delta_{t_{1_i}} \right) \frac{\delta_{t_{1_i}}}{L_i} \right]$$

(5.6.35b)

Higher-order coefficients in the series expansion of the load parameter λ could be obtained in a simpler way. If the mechanism under consideration has only one degree of freedom, all kinematic variables present in the set of definitions (5.6.35a,b,...) can be expressed in terms of the parameter selected to describe the mechanism motion; this parameter could then be identified with the perturbation parameter ε. If, on the other hand, the mechanism has two or more degrees of freedom, a series expansion in as many parameters should be used.

Let $\varkappa_j = (\lambda_1 / \delta_1)_j$ be a stiffness parameter measuring the (first-order) variation of the j-th applied load, λ_j, during a unit increment of the corresponding displacement; following the procedure adopted in subsection 5.3.2 it may be concluded that the local stiffness $\varkappa_j$ is bounded by

$$-\frac{2}{\delta_{1_j}^2} z_1^p (\underline{x}_1^s, \underline{y}_1^s) \leq \varkappa_j \leq -\frac{2}{\delta_{1_j}^2} w_1^p(\underline{x}_1^k, \underline{y}_1^k)$$

where $(\underline{x}_1^s, \underline{y}_1^s)$ and $(\underline{x}_1^k, \underline{y}_1^k)$ represent (first-order) non-associated statically and kinematically admissible fields, respectively. If either of such fields coincides with a (first-order) optimal solution, the corresponding inequality in the above becomes a strict equality, characterizing therefore the actual value taken by the local stiffness $\varkappa_j$.

As the primal and dual constraints of the programs given in the previous subsection represent, respectively, the static and kinematic admissibility conditions, the primal and dual programs, which read

(V) Among all statically admissible stress fields, the actual stress field(s) make the functional z a minimum.

and (VI) Among all kinematically admissible strain fields, the actual strain field(s) make the functional -w a minimum.

can be interpreted as representing the PRINCIPLES OF MINIMUM COMPLEMENTARY POTENTIAL ENERGY AND MINIMUM POTENTIAL ENERGY, respectively, if the incremental strain energy and complementary strain energy are defined as

$$\Delta U = \frac{1}{2}\Delta\underline{X}^T(\Delta\underline{u}_p + \Delta\underline{u}_\pi) - \frac{1}{2}\Delta\underline{\pi}^T\Delta\underline{\delta}_\pi - \frac{1}{2}[\Delta\underline{X}^T(\underline{\mathbb{R}}_{u\pi} - \underline{\mathbb{R}}_p) + \Delta\underline{\delta}_\pi^T\underline{\mathbb{R}}_\pi - \Delta\underline{u}_*\underline{\mathbb{R}}_\varphi]$$

and $$\Delta U^* = \frac{1}{2}\Delta\underline{X}^T(\Delta\underline{u}_p + \Delta\underline{u}_\pi) - \frac{1}{2}\Delta\underline{\pi}^T\Delta\underline{\delta}_\pi + \frac{1}{2}[\Delta\underline{X}^T(\underline{\mathbb{R}}_{u\pi} - \underline{\mathbb{R}}_p) + \Delta\underline{\delta}_\pi^T\underline{\mathbb{R}}_\pi - \Delta\underline{u}_*\underline{\mathbb{R}}_\varphi]$$

and the incremental complementary work by $\Delta W^* = \Delta\underline{\lambda}^T\Delta\underline{\delta}$; as in the present case prescribed stresses have not been included, the incremental work ΔW is null. Then, and neglecting fourth- and higher-order terms, we may write

$$z^I = \Delta E^* = \Delta U^* - \Delta W^* \quad \text{and} \quad -w^I = \Delta E = \Delta U - \Delta W$$

where ΔE and ΔE^* represent the variations of the potential energy and complementary potential energy, respectively.

In the perturbation analysis formulation, the functionals $-w_j^p$ and z_j^p represent the non-linear terms, the only relevant terms in the minimization procedure, in the series expansion of ΔE and ΔE^*, respectively.

In the derivation of the deformation analysis programs, by treating every non-linear corrective variable as a known constant, the rigid-plastic structure is assumed to behave linearly within each iteration; as a consequence, during each step, and therefore at convergence, the strain and complementary strain energies present identical values

$$U = U^* = \frac{1}{2} \underset{\sim}{u}_*^T \underset{\sim}{H} \underset{\sim}{u}_* - \frac{1}{2} \underset{\sim}{\pi}^T \underset{\sim}{\delta}_\pi$$

The expressions for the total work and complementary work become

$$W = -\underset{\sim}{u}_*^T \underset{\sim}{\pi}_\varphi \quad \text{and} \quad W^* = \underset{\sim}{\lambda}^T \underset{\sim}{\delta} - \underset{\sim}{X}^T(\underset{\sim}{u}_\pi - \underset{\sim}{u}_\varphi)$$

respectively, since the corrective variables $\underset{\sim}{\pi}_\varphi$ are interpreted as prescribed forces and $\underset{\sim}{u}_\pi$ and $\underset{\sim}{u}_\varphi$ as prescribed dislocations. Letting $D = \underset{\sim}{X}_*^T \underset{\sim}{u}_*$ represent the plastic dissipation, the following correspondence is found

$$z^D = E^* = U^* - W^* \; , \; -w^D = E = (U+D) - W$$

thus reducing statements (V) and (VI) in the above to the specialization of the Haar-Karman and Hodge-Kachanov principles to rigid-plastic systems undergoing large displacements.

The zeroth-order primal (5.6.17-19) and dual (5.6.18-20) programs of asymptotic analysis represent, respectively, the SAFE (Static or Lower Bound) AND UNSAFE (Kinematic or Upper Bound)

THEOREMS OF PLASTIC LIMIT ANALYSIS, first enunciated by Gvozdev (1936).

The uniqueness theorem, first proposed by Horne (1950), was established by Hill (1951) for regular yield loci, and the corollaries given by Bishop et alia (1956) extended by Haythornthwaite and Schield (1958) to yield loci with singularities; the uniqueness theorem can be recovered from the composite forms of programs (5.6.17-18) and (5.6.19-20). The theorems and corollaries of plastic limit analysis can be found in most textbooks dealing with this subject, as for instance Horne (1971) and Neal (1977); the proposed proofs are invariably based on direct arguments rather than on mathematical programming concepts.

Instead of following closely Hill's (1957, 1959) stability theory for rigid-plastic solids, allow us, for the sake of unity of the presentation, use of Drucker's stability criteria once again.

Applying, in the manner of subsection 5.3.4, Cottle's theorem on Duality to the (first-order) programs of rigid-plastic perturbation analysis, expressions of the form (5.3.21) are again encountered; statements (I) and (II) in subsection 5.3.4, concerning Drucker's weak stability criterion (5.3.20), can be applied to rigid-plastic structures, it being necessary however to re-define matrix $\underline{A}$ as

$$\underline{A} = \left[\begin{array}{c|c} -\underline{\mathbb{K}}_N & \cdot \\ \hline \cdot & \underline{\mathbb{H}} \end{array}\right] \qquad \text{and} \qquad \underline{A} = \left[\begin{array}{c|c|c} \underline{\mathbb{H}} & \cdot & \cdot \\ \hline \cdot & \underline{\mathbb{F}}_M & \underline{\mathbb{F}}_o \\ \hline \cdot & \underline{\mathbb{F}}_o^T & \underline{\mathbb{F}}_\lambda \end{array}\right]$$

for the nodal and mesh descriptions, respectively. Specialized for linear behaviour, those statements reduce to the following well-known results: at incipient collapse, the equilibrium configuration of rigid-workhardening (-perfectly) plastic structures is stable (critical).

The dissipated work $\dot{W}$ can still be expressed in the form (5.3.24), where now matrix $\underline{L}$ and the plastic (and only) component of the load displacements are defined by

$$\underline{L} = \underline{\mathbb{F}}_\lambda - \underline{\mathbb{F}}_o^T \, \underline{\mathbb{F}}_M^{-1} \underline{\mathbb{F}}_o$$

and $\dot{\underline{\delta}}_p = (\underline{B}_o^T - \underline{F}_o^T \underline{F}_M^{-1}) \dot{\underline{u}}_p$

for the mesh formulation and

$$\underline{L} = \underline{A}_o \left\{ -(-\underline{K}_N)^{-1} \underline{A}^T \left[\underline{A}(-\underline{K}_N)^{-1} \underline{A}^T \right]^{-1} A(-\underline{K}_N)^{-1} + (-\underline{K}_N)^{-1} \right\} \underline{A}_o^T$$

and

$$\dot{\underline{\delta}}_p = \underline{A}_o (-\underline{K}_N)^{-1} \underline{A}^T \left[\underline{A} (-\underline{K}_N)^{-1} \underline{A}^T \right]^{-1} \dot{\underline{u}}_p$$

for the nodal formulation.

The reasoning followed in subsection 5.3.4 in the interpretation of Drucker's strong stability criterion (5.3.23) and its comparison with the weak criterion in the form (5.3.24) could now be applied and specialized for rigid-plastic systems. It is worth noting that, as shown by equation (5.3.24), if matrix $\underline{L}$ is positive definite the work $\dot{W}$ can still be positive even if the plastic strain rates are zero, that is, even if the displacements of the rigid-plastic structure remain unchanged.

An approximate method outlined in Sawczuk (1971) is developed by Duszek and Sawczuk (1976) for the determination of the stability of rigid-plastic frames at incipient collapse. Positivity of the loading rate $\dot{\lambda}$ at the onset of the mechanism motion is the adopted stability criterion; the definition they use for the loading rate

$$\dot{\lambda} = \frac{1}{\dot{W}} \left[\underline{X}_o^T \ddot{\underline{u}}_p - \lambda_o \ddot{W} \right] \qquad (5.6.36)$$

is the discretized version of an expression previously proposed by Duszek (1973). The method is illustrated in the analysis of simple portal frames, the regions of stable and unstable load combinations being specified on the load interaction surface. The material is assumed rigid-perfectly plastic, and provisions are made to take into consideration bending and axial force interaction.

Specializing the loading rate as defined in (5.6.35b) for perfectly-plastic materials ($\dot{\underline{X}}^T \dot{\underline{u}}_p = 0$) and confronting the resulting expression with (5.6.36), one is led to the conclusion that Duszek neglects the contribution

$$-\sum_i \left(2\, X^i_{3_o} \; \dot{\delta}_{n_i} - X^i_{2_o} \; \dot{\delta}_{t_i} \right) \frac{\dot{\delta}_{t_i}}{L_i}$$

of the rigid-body displacement components $\dot{\underset{\sim}{\delta}}_t$ and $\dot{\underset{\sim}{\delta}}_n$. The axial component $\dot{\delta}_{n_i}$ will only be zero if member i is axially undeformable, or if it moves without rotating; the latter is also the necessary and sufficient condition for the transverse component $\dot{\delta}_{t_i}$ to vanish.

5.6.3 An Illustration of Lack of Uniqueness: Pseudomechanisms

The behaviour, in linear regime, of the simple portal frame with pinned bases shown in Fig. 5.29 has been studied in detail by Smith (1974, 1975). The rigid-perfectly plastic response of this frame to the proportional loading $\underset{\sim}{\lambda}^T = \lambda\,[10 \quad 1]$ is analyzed herein to illustrate a situation characterized by the existence of two optimal solutions, the kinematic configurations of which differ by that of a pseudomechanism.

The concept of a pseudomechanism was originally introduced by Munro (1963b) as an aid in explaining one of the modes of plastic unstressing. The concept was later used by Smith (1974) and Smith and Munro (1978) to illustrate, and regulate through appropriate theorems, situations of kinematic multiplicity in the response of elastoplastic structures under small displacements.

By definition, a PSEUDOMECHANISM is a device associating with a kinematically admissible mechanism, a statically admissible stress distribution contravening the plasticity association condition for at least one, but not all, critical sections; the plastic hinges where parity is contravened are termed PSEUDOHINGES.

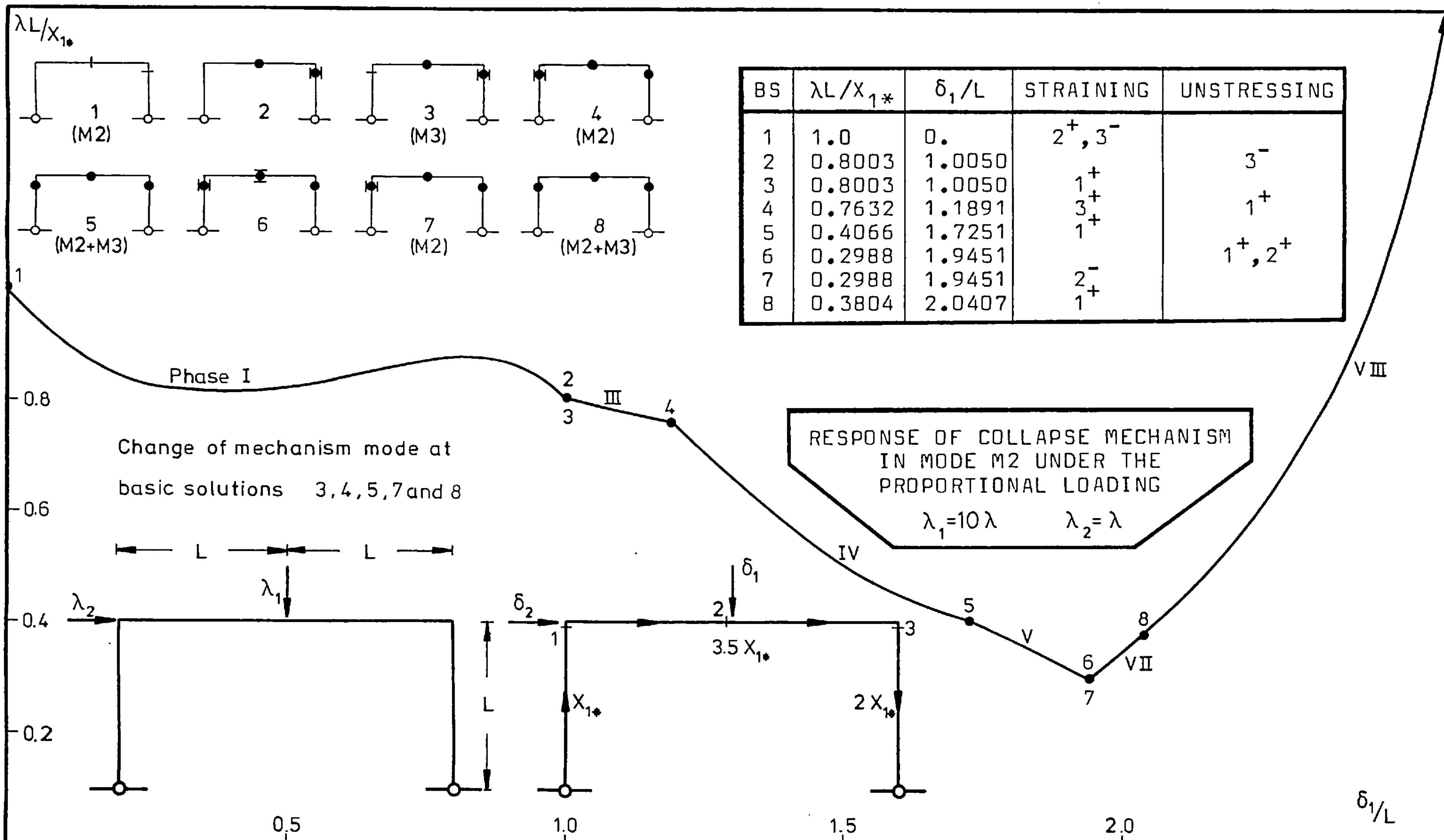
In the derivation of system (5.6.35) only Static-Kinematic Duality, i.e., the Principle of Virtual Work, was used; let it be expressed in the form

$$\underset{\sim}{\lambda}_o^T \; \underset{\sim}{\delta}_1 = \underset{\sim}{X}_o^T \; \underset{\sim}{u}_{p_1} \tag{5.6.37a}$$

$$2\, \underset{\sim}{\lambda}_1^T \; \underset{\sim}{\delta}_1 = \underset{\sim}{X}_o^T \; \underset{\sim}{u}_{p_2} + 2\underset{\sim}{X}_1^T \; \underset{\sim}{u}_{p_1} - \underset{\sim}{\lambda}_o^T \; \underset{\sim}{\delta}_2 - \sum_i \left(2X^i_{3_o} \delta_{n_{1_i}} - X^i_{2_o} \delta_{t_{1_i}} \right) \frac{\delta_{t_{1_i}}}{L_i}$$

$$\vdots$$

(5.6.37b)



BS	$\lambda L/X_{1*}$	δ_1/L	STRAINING	UNSTRESSING
1	1.0	0.	$2^+, 3^-$	
2	0.8003	1.0050		3^-
3	0.8003	1.0050	1^+	
4	0.7632	1.1891	3^+	1^+
5	0.4066	1.7251	1^+	
6	0.2988	1.9451		$1^+, 2^+$
7	0.2988	1.9451	2^-	
8	0.3804	2.0407	1^+	

FIGURE 5.29

The plasticity relations give

$$\underset{\sim}{X}_j^T \underset{\sim}{u}_{p_i} = \underset{\sim}{u}_{*_j}^T \underset{=}{\mathbb{H}} \underset{\sim}{u}_{*_i} + \underset{\sim}{\Phi}_{*_j}^T \underset{\sim}{u}_{*_i} + \delta_{oj} \underset{\sim}{X}_*^T \underset{\sim}{u}_{*_i} + \underset{\sim}{\mathbb{R}}_{\varphi_j}^T \underset{\sim}{u}_{*_i} - \underset{\sim}{\mathbb{R}}_{p_i}^T \underset{\sim}{X}_j^T \quad (5.6.38)$$

Let the incidence matrices $\underset{=}{J}^+$ and $\underset{=}{J}^-$ collect from $\underset{\sim}{u}_{*_i}^+$ and $\underset{\sim}{u}_{*_i}^-$, respectively, those critical sections where the association condition is contravened; after separating $\underset{\sim}{\Phi}_*$ into two sets, one containing the sections wherein the association condition is satisfied, the other the (remaining) sections where it is infringed, it can be easily found that equation (5.6.38) can be re-written as

$$\underset{\sim}{X}_j^T \underset{\sim}{u}_{p_i} = \underset{\sim}{u}_{*_j}^T \underset{=}{\mathbb{H}}_p \underset{\sim}{u}_{*_i} + \delta_{oj} \underset{\sim}{X}_{*_p}^T \underset{\sim}{u}_{*_i} + \underset{\sim}{\mathbb{R}}_{\varphi_j}^T \underset{\sim}{u}_{*_i} - \underset{\sim}{\mathbb{R}}_{p_i}^T \underset{\sim}{X}_j \quad (5.6.39)$$

where $\underset{=}{\mathbb{H}}_p = \underset{=}{I}_*^T \underset{=}{\mathbb{H}} \underset{=}{I}_*$ and $\underset{\sim}{X}_{*_p} = \underset{=}{I}_*^T \underset{\sim}{X}_*$

and

$$\underset{=}{I}_* = \left[\begin{array}{c|c} \underset{=}{I} - \underset{=}{J}^+ & -\underset{=}{J}^- \\ \hline -\underset{=}{J}^+ & \underset{=}{I} - \underset{=}{J}^- \end{array}\right]$$

Substituting (5.6.39) into system (5.6.37) in the above, the following (asymptotic) definition for the pseudomechanism equation is found:

$$\underset{\sim}{\lambda}_o^T \underset{\sim}{\delta}_1 = \underset{\sim}{X}_{*_p}^T \underset{\sim}{u}_{*_1} - \underset{\sim}{X}_o^T \underset{\sim}{\mathbb{R}}_{p_1} \quad (5.6.40a)$$

$$2 \underset{\sim}{\lambda}_1^T \underset{\sim}{\delta}_1 = \underset{\sim}{X}_{*_p}^T \underset{\sim}{u}_{*_2} + 2\underset{\sim}{u}_{*_1}^T \underset{=}{\mathbb{H}}_p \underset{\sim}{u}_{*_1} - \underset{\sim}{\lambda}_o^T \underset{\sim}{\delta}_2 - \sum_i \left(2X_{3_o}^i \delta_{n_{1_i}} - X_{2_o}^i \delta_{t_{1_i}}\right) \frac{\delta_{t_{1_i}}}{L_i} -$$

$$\underset{\sim}{X}_o^T \underset{\sim}{\mathbb{R}}_{p_2} - 2\underset{\sim}{X}_1^T \underset{\sim}{\mathbb{R}}_{p_1} + 2\underset{\sim}{u}_{*_1}^T \underset{\sim}{\mathbb{R}}_{\varphi_1} \quad (5.6.40b)$$

$$\vdots \qquad\qquad \vdots$$

When $\underset{=}{J}^+$ and $\underset{=}{J}^-$ are null matrices, i.e. when parity is satisfied at every critical section, matrix $\underset{=}{I}_*$ becomes the identity matrix and system (5.6.40) the asymptotic definition of a collapse mechanism.

The natural process for generating pseudomechanisms is to consider the relevant collapse mechanisms and associate next to it statically admissible stress distributions contravening parity at $1 \leqslant \gamma < c'$ of the $c' \leqslant c$ activated critical sections, c being the number of existing critical sections in the structure; if the mechanism has d-degrees of freedom, a d-degree of freedom pseudo-mechanism of multiplicity γ is thus generated.

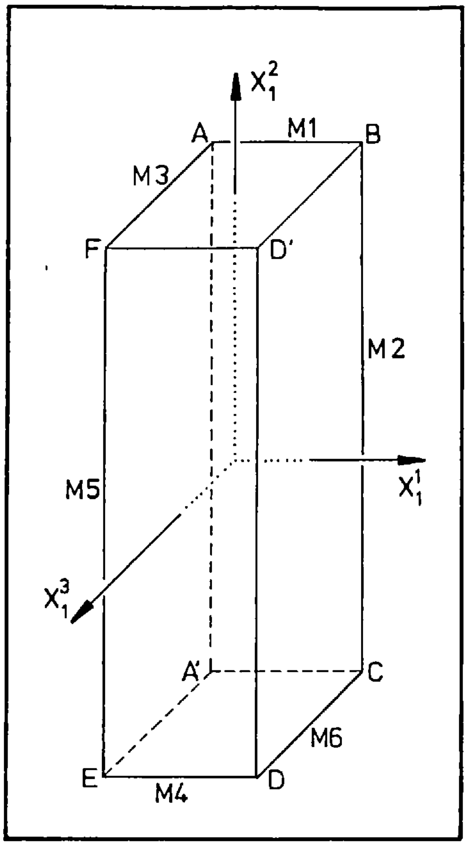


FIGURE 5.30

For structures formed by members of a rigid-perfectly plastic, non-interactive, material the yield polytope is an orthotope, the dimension of which is the number of critical sections of the structure, c. The yield polytope has

$$N_k = 2^{c-k} C_k^c \qquad (5.6.41)$$

elements of dimension k. The orthotope shown in Fig. 5.30 is the yield polytope of the simple frame under consideration; it has six, twelve and eight elements of dimension two, one and zero, respectively its faces, edges and vertices.

A mechanism with d-degrees of freedom has $\alpha+d \leqslant c$ activated critical sections, α being the static indeterminacy number of the structure. The static configuration associated with such a mechanism is represented in the stress-space $\underset{\sim}{X}$ by the intersection of $\alpha+d$ hyperplanes of the yield polytope; the number of elements of the yield polytope in such condition is, from (5.6.41)

$$N_{c-(\alpha+d)} = 2^{\alpha+d} C^c_{c-(\alpha+d)} = 2^{\alpha+d} C^c_{\alpha+d}$$

The number of test mechanisms, i.e. one-degree of freedom mechanisms, is, in the absence of partial collapse mechanisms,

$$M = 2\, C^c_{\alpha+1}$$

As from each test mechanism $\sum_{j=1}^{\alpha} C_j^{\alpha+1}$ pseudomechanisms can be generated, the total number of one-degree of freedom pseudomechanisms is

$$P = 2C^{c}_{\alpha+1}\left(\sum_{j=1}^{\alpha} C_{j}^{\alpha+1}\right)$$

The yield polytope has

$$N_{c-(\alpha+1)} = 2^{\alpha+1} C^{c}_{\alpha+1}$$

elements of dimension $\alpha+1$, representing static configurations which may be associated with one-degree of freedom mechanisms. The difference

$$D = 2(2^{\alpha}-1)\, C^{c}_{\alpha+1}$$

must therefore represent the number of stress distributions that although statically admissible at collapse cannot be associated with collapse test mechanisms; they represent the number of distinct static configurations which may be associated with the possible P one-degree of freedom pseudomechanisms. In general $P \geqslant D$ as the pseudomechanism hyperplanes may coincide.

Analogous considerations can be made concerning multi-degree of freedom Prager mechanisms and the corresponding pseudomechanisms; Smith (1974) defines a PRAGER MECHANISM as an n-degree of freedom mechanism formed by a non-negative combination of n one-degree of freedom constituent mechanisms such that at any critical section the constituent mechanism rotations are all of like sense. Pseudo-mechanisms generated from non-Prager mechanisms are not of relevance.

The connected form shown in Fig. 5.31(a) represents the mapping, defined by the equilibrium conditions, of the yield polytope, which materializes the yield rule, onto the load-space $\underset{\sim}{\lambda}$. The distinctive attribute of this mapping is of separating the c-(α +1) dimensional elements of the X-space into two sets each of n-1 dimensional elements, n being the dimension of the λ-space; one set contains the test mechanisms which bound the λ-space (collapse mechanisms) defining an interior safe region (hence the Safe Theorem of Plastic Limit Analysis); the other groups the one-degree of freedom pseudomechanism hyperplanes which become interior to that collapse polytope.

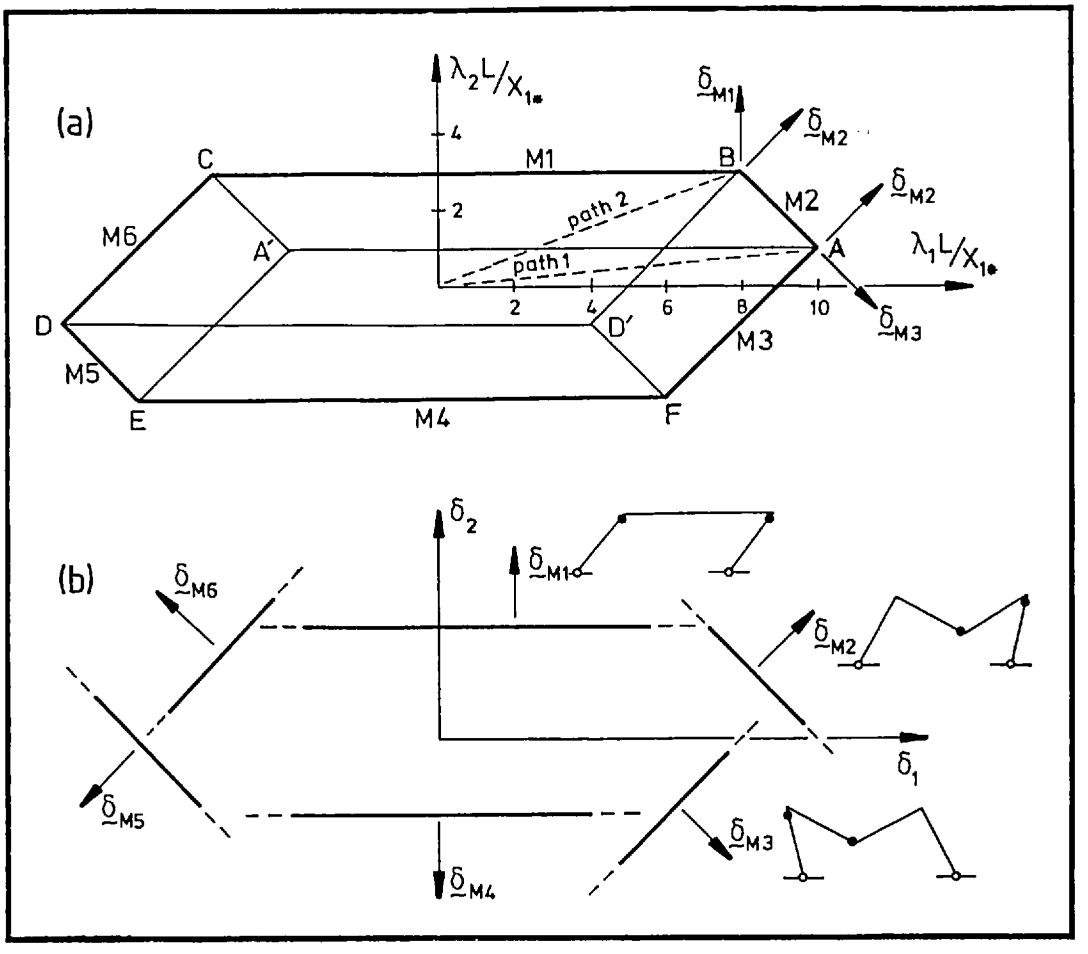


FIGURE 5.31

If the elements of the X-space associated with the test mechanisms form a connected figure, as ABCDEF in Fig. 5.30, the (convex) λ-space will be bounded; otherwise it may be unbounded.

The three test mechanisms (the remaining three being their negatives) are shown in Fig. 5.32 and their relative positions in the strain-resultant space in Fig. 5.33; the tangential vectors at the origin represent the kinematic configurations at incipient collapse.

These three first-order solutions were mapped, through the displacement compatibility condition, onto the load-point displacement space $\underset{\sim}{\delta}$, defining the (first-order) displacement vectors shown in Fig. 5.31(b); the supporting hyperplanes may displace freely in the δ-space being only constrained to maintain the same orientation. Their relative position is determined, apart from a scaling parameter, thus defining a polytope in the δ-space, by minimizing the associated plastic dissipation, as required by the Unsafe Theorem of Plastic Limit Analysis. Superimposing the λ- and δ-spaces, the polytopes are brought into coincidence; thus the Uniqueness Theorem.

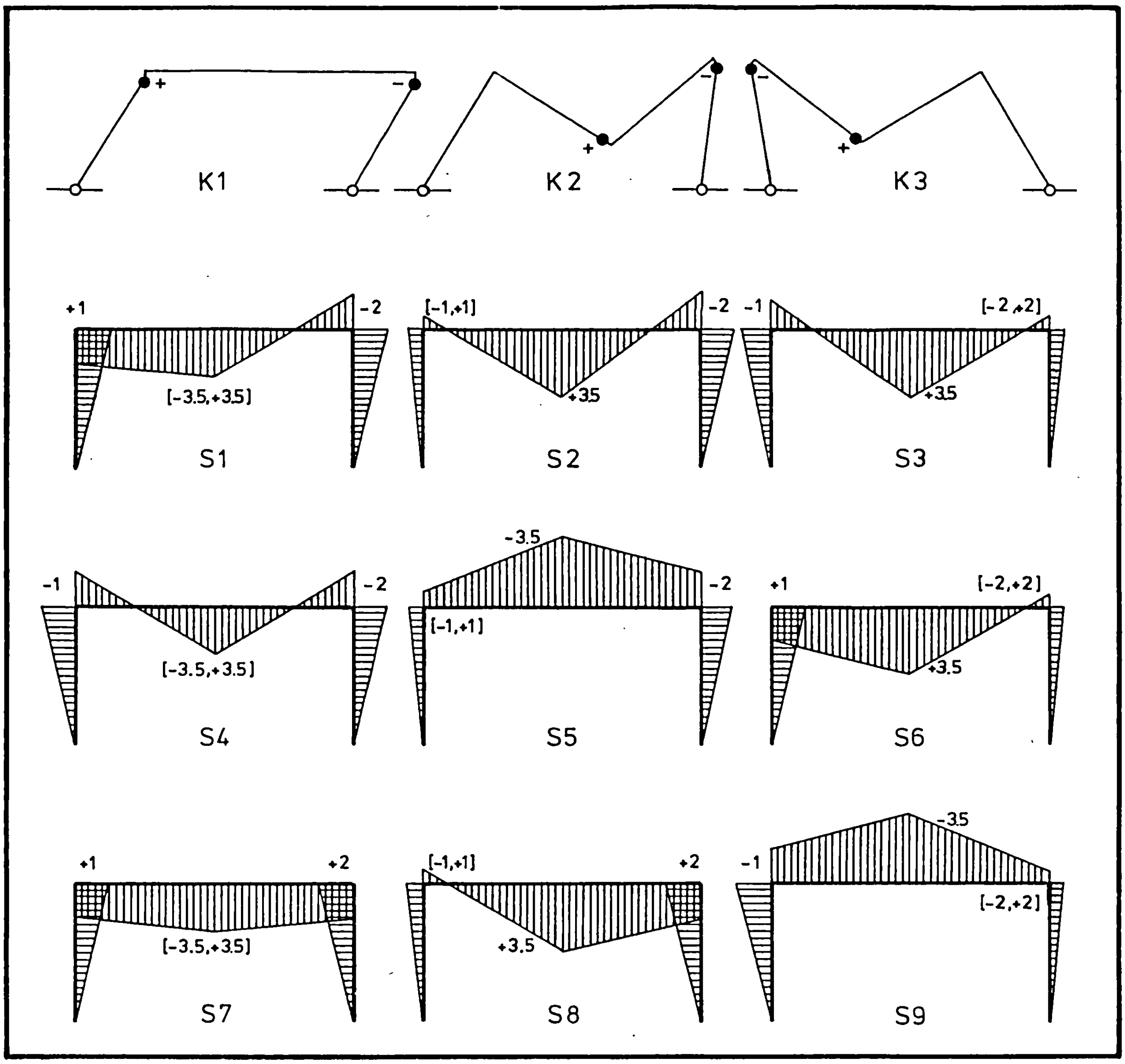


FIGURE 5.32

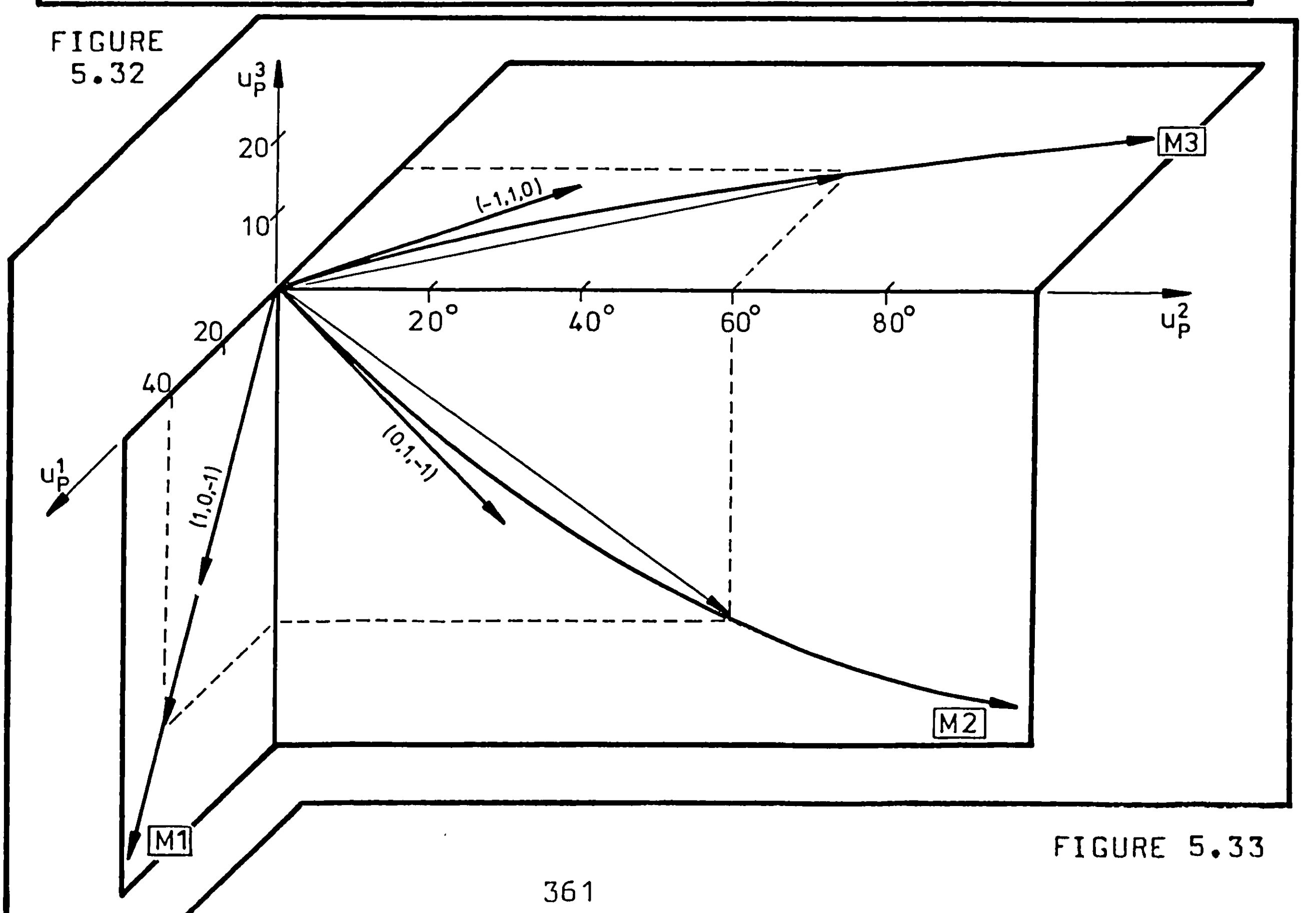


FIGURE 5.33

If, when implementing the minimization of the plastic dissipation, infringement of the association condition at some of the activated critical sections is allowed for, the position of the (pseudomechanism) supporting hyperplanes in the δ-space is determined so that when the δ- and λ-spaces are superimposed, those hyperplanes coincide with the corresponding pseudomechanism hyperplanes in the λ-space.

Summarized in Table 5.20 are the static and kinematic configurations corresponding to the vertices and edges of the collapse polytope for the simple portal frame under consideration. Table 5.21 shows how to combine these configurations for each of the one-degree of freedom collapse mechanisms and pseudomechanisms; a pseudomechanism generated from the test mechanism collapsing in mode Mi by considering a pseudohinge at critical section j, is represented thus Pi(j), where $1 \leqslant i \leqslant 6$ and $1 \leqslant j \leqslant 3$.

	A	B	C	D	E	F	A'	D'
X_1^1/X_{1*}	-1.	1.	1.	1.	-1.	-1.	-1.	1.
X_1^2/X_{1*}	3.5	3.5	-3.5	-3.5	-3.5	3.5	-3.5	3.5
X_1^3/X_{1*}	-2.	-2.	-2.	2.	2.	2.	-2.	2.
$\lambda_1 L/X_{1*}$	10.	8.	-6.	-10.	-8.	6.	-4.	4.
$\lambda_2 L/X_{1*}$	1.	3.	3.	-1.	-3.	-3.	1.	-1.

	AB	BC	CD	DE	EF	FA
$u_{1P_1}^1$	.	1.	1.	.	-1.	-1.
$u_{1P_1}^2$	1.	.	-1.	-1.	.	1.
$u_{1P_1}^3$	-1.	-1.	.	1.	1.	.
δ_{1_1}/L	1.	.	-1.	-1.	.	1.
δ_{2_1}/L	1.	1.	1.	-1.	-1.	-1.

TABLE 5.20

TABLE 5.21

	S1	S2	S3	-S1	-S2	-S3	S4	S5	S6	S7	S8	S9
K1	M1						P1(1)			P1(3)		
K2		M2						P2(2)			P2(3)	
K3			M3						P3(1)			P3(2)
-K1				M4			P4(3)			P4(1)		
-K2					M5			P5(3)			P5(2)	
-K3						M6			P6(2)			P6(1)

Consider now the proportional load path $\underset{\sim}{\lambda}^T = \frac{\lambda X_{1*}}{L}$ [10 1]. When $0 < \lambda < 1$, the portal frame remains undeformed under an undetermined stress distribution. When $\lambda = 1$, the (bending) stress-resultant distribution S2(S3), with critical section 1(3) about to yield negatively, is found to equilibrate the loading $\underset{\sim}{\lambda} = \underset{\sim}{\lambda}(\lambda = 1)$. Plastic Limit Analysis would then predict that either of the collapse mechanisms M2 or M3, or any linear combination of them, could then be mobilized for collapse, as it would predict that either of the modes M1 or M2, or any linear combination of them, could be mobilized for a second load path $\lambda^T = \frac{\lambda X_{1*}}{L}$ [8 3] at $\lambda = 1$.

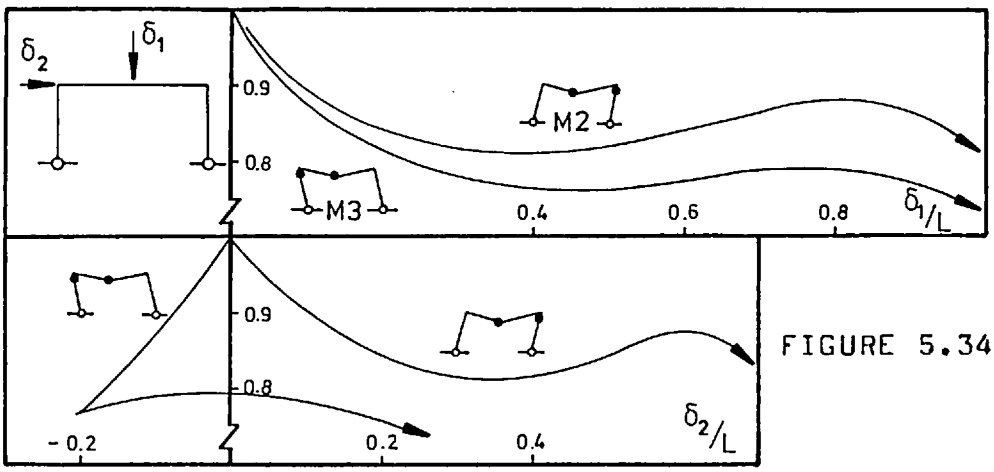


FIGURE 5.34

A kinematically non-linear rigid-plastic analysis shows that the solution associated with the second load path is in fact unique; if collapse in mode M2 is attempted, it is found that, at incipient collapse, plastic unstressing occurs at critical section 2 while plastic straining develops at critical section 1, thus reverting to the collapse configuration of mode M1.

Associated with the first load path there are however two optimal solutions corresponding to collapse modes M2 and M3. The corresponding load-displacement and load-bending moment relationships are illustrated in Figs. 5.34 and 5.35, respectively. For the same load path, the load-displacement relationship found for the (physically inadmissible) pseudomechanism P1(1) is shown in Fig. 5.36.

The kinematic configurations associated with these three modes of collapse are represented in Figs. 5.33 and 5.37. As illustrated, any M2 solution can be obtained as the VECTORIAL sum of M1 [and therefore P1(1)] and M3 solutions. For structures undergoing small displacements, Smith (1974) has shown that when kinematically

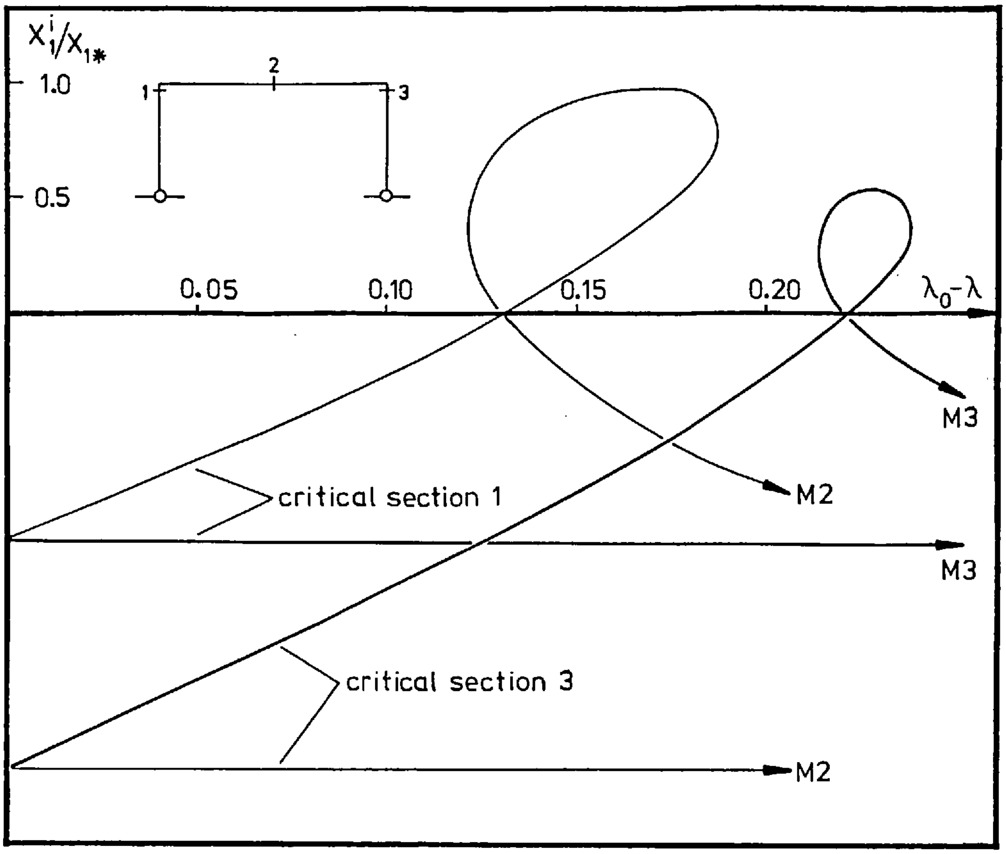


FIGURE 5.35

multiple optimal solutions exist, any of the extreme (or basic) solutions can be determined by adding (scalarly) a mechanism displacement to another of the existing extreme solutions; thus, alternative reference may be made to pseudomechanisms as DIFFERENCE MECHANISMS.

The response of the frame when collapsing in mode M2 was followed up for increasing values of the point load displacement δ_1, up to the limit $\delta_1 = \sqrt{3}$ L, as shown in Fig. 5.38.

The load-displacement, moment-displacement and moment-rotation relationships thus found are presented in Figs. 5.29, 5.39 and 5.40, respectively.

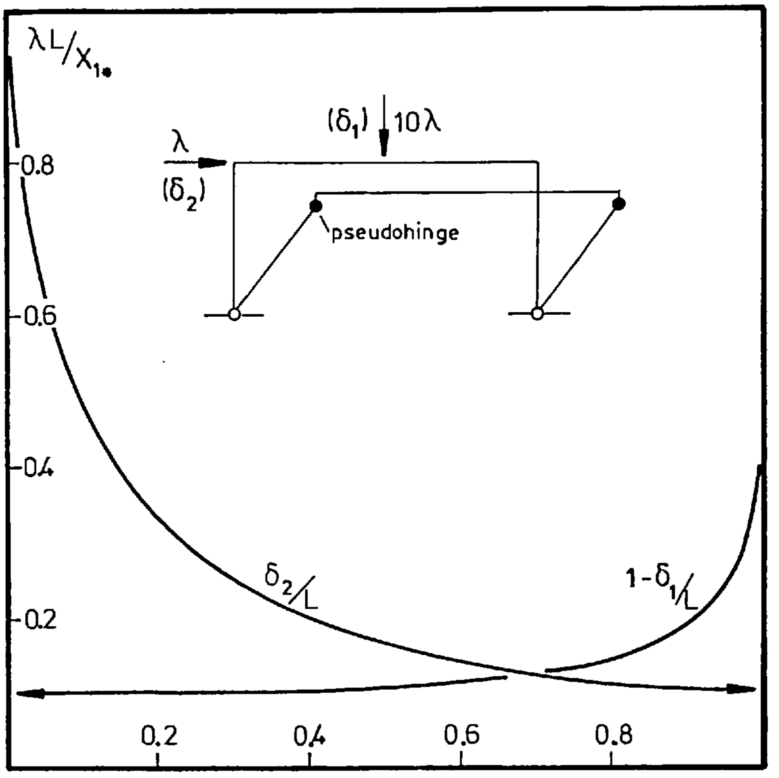


FIGURE 5.36

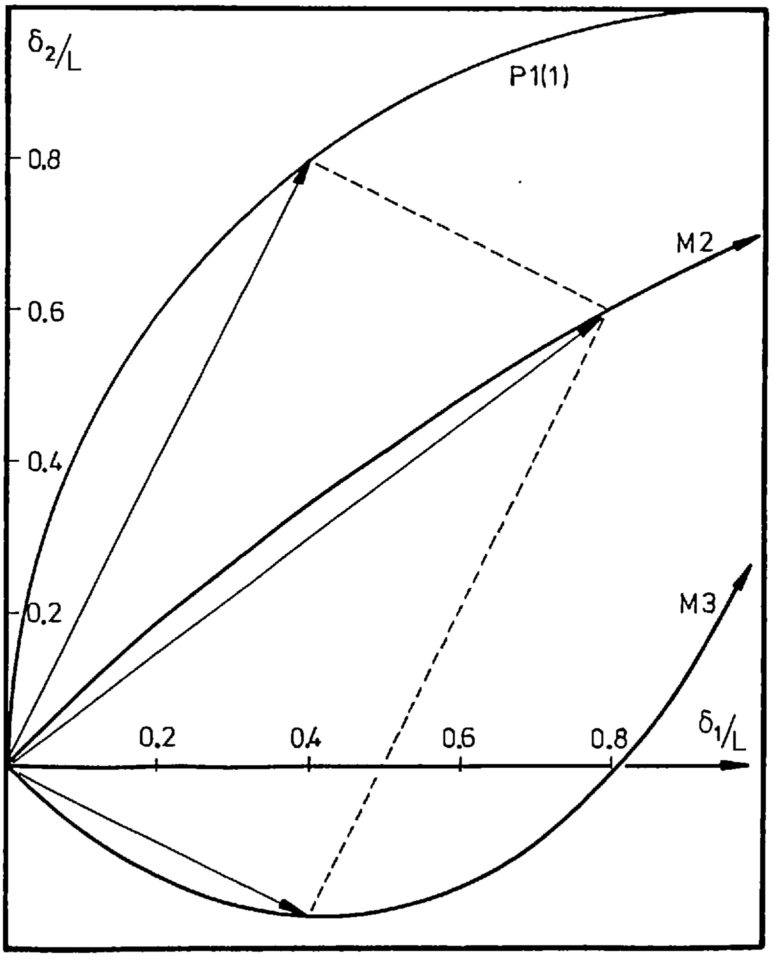


FIGURE 5.37

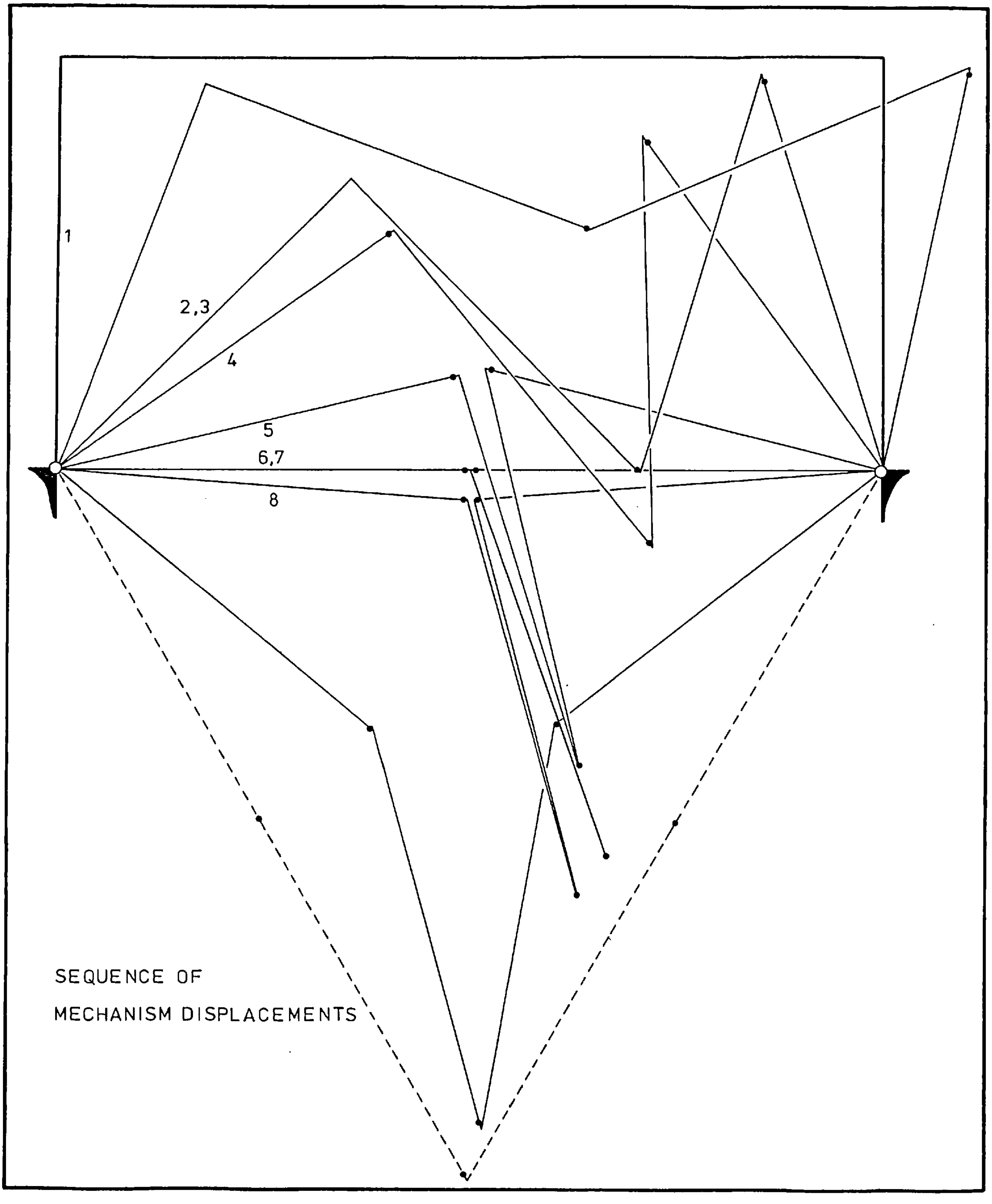
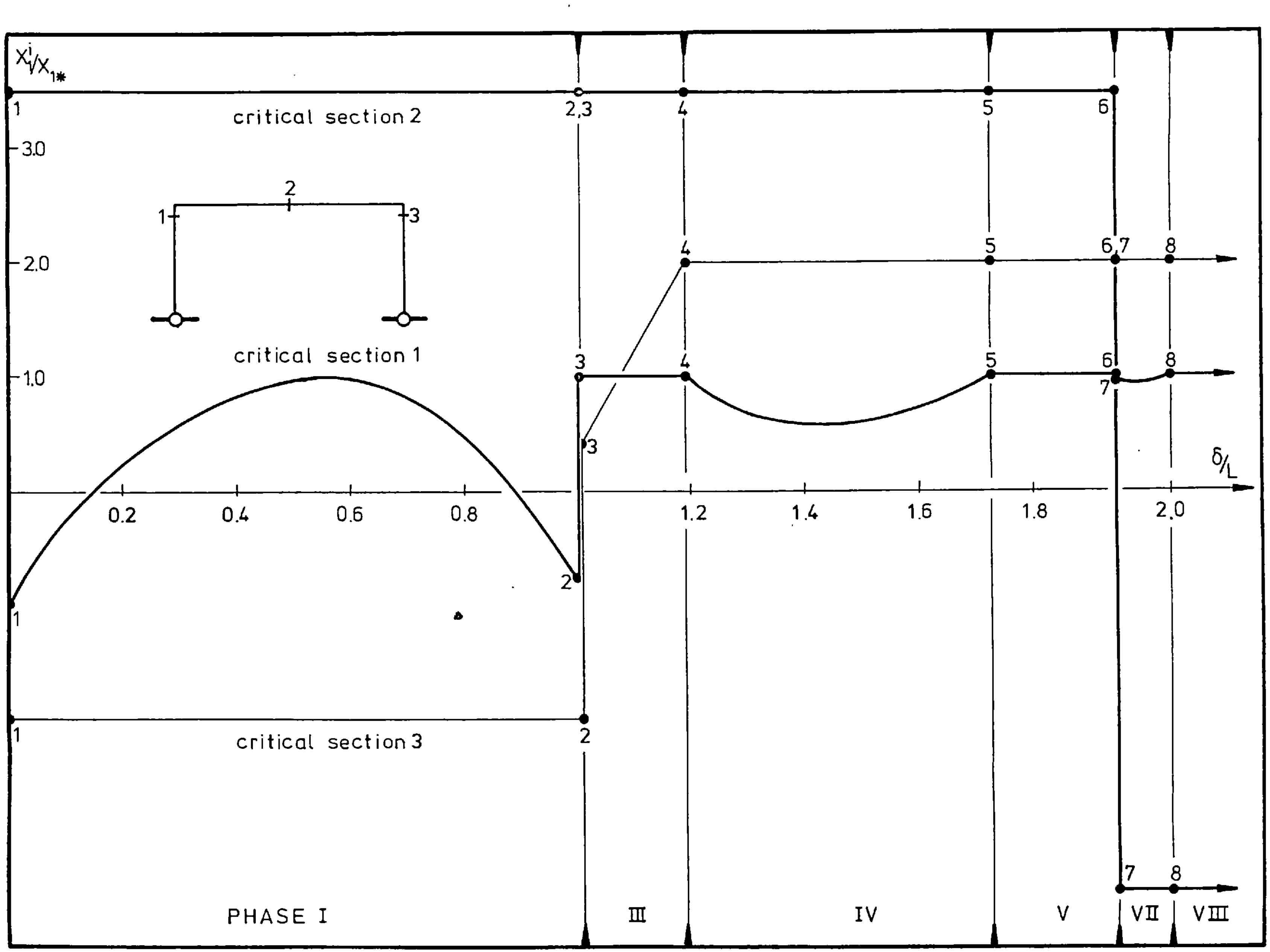


FIGURE 5.38



X_1^i/X_{1*}
critical section 2
critical section 1
critical section 3
3.0
2.0
1.0
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8
2.0
δ/L
1
2
3
2,3
4
5
6
6,7
7
8
PHASE I
III
IV
V
VII
VIII

FIGURE 5.39

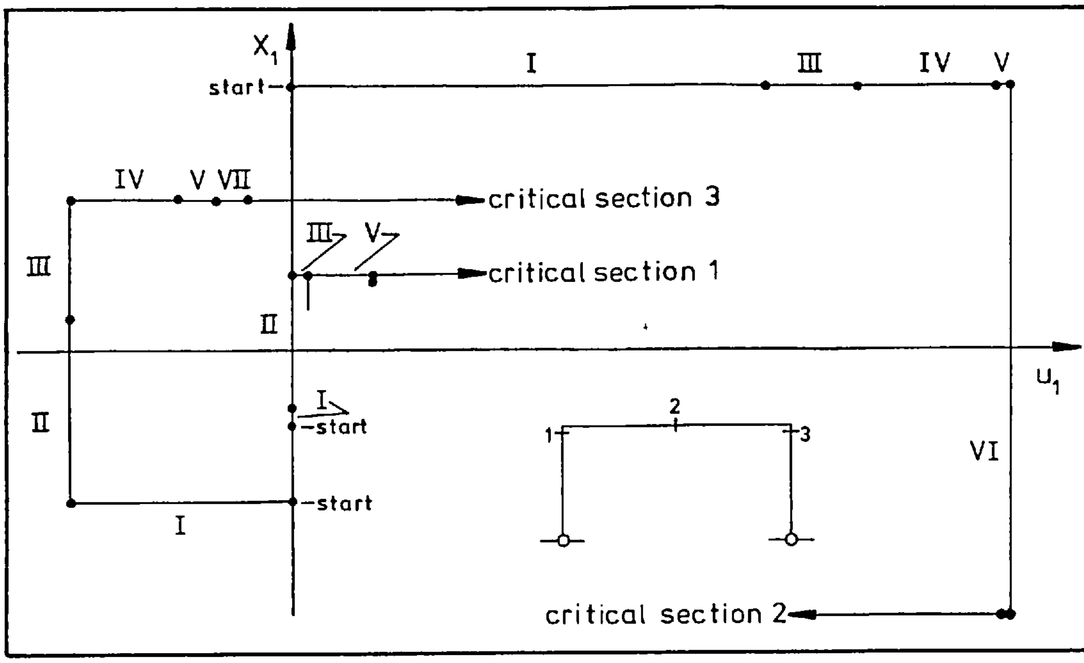


FIGURE 5.40

The following phases, indicated in the above mentioned illustrations, can be distinguished in the frame response:

PHASE I: After the mobilization of the collapse mode M2, critical section 1 unstresses and the bending moment is increased, almost to the point of activating the positive yield mode. Then it decreases but not sufficiently to re-activate the negative yield mode. In the meantime, plastic straining is developing at critical sections 2 and 3; this progression of yielding is interrupted in section 2 when $\delta_1 = 1.0050$ L, when plastic unstressing occurs at section 3.

PHASE II: As only one plastic hinge remains active, if the frame is to displace further, an INSTANTANEOUS readjustment of the stress distribution has to take place. While plastic unstressing is implemented at critical section 3, the positive yield mode is activated at section 1, the kinematic field remaining unchanged during the rearrangement of stresses.

PHASE III: The collapse mode changes from mode M2 into mode M3, and the displacements may now progress.

PHASE IV: When the positive yield mode at section 3 is activated, instead of the development of a beam mechanism mode M2+M3, the collapse mode reverts into mode M2, as plastic unstressing occurs at critical section 1. After an initial fall, the bending moment at this section increases again, reactivating the positive yield mode and subsequently mobilizing the combined mode M2+M3.

PHASE V: The frame displacements proceed to the point where the frame columns reach a rotation of 90 degrees. Then, plastic unstressing is revealed to be about to occur simultaneously at critical sections 1 and 2.

PHASE VI: Once again only one plastic hinge remains active, the frame becoming a rigid system. If the displacements are to evolve further, yet another instantaneous rearrangement of stresses must take place; at critical section 2, where positive plastic strains had been developing, the negative yield mode is suddenly made active. As marginal plastic unstressing occurs at critical section 1, the collapse mode reverts once again into mode M2.

PHASE VII: From this point onwards the frame manifests a stiffening behaviour. In the meantime, the instantaneous loss of stress at critical section 1 is recovered and the associated positive yield mode reactivated.

PHASE VIII: A combined collapse mode M2+M3 is thus mobilized and no further mode changes can be encountered up to the locking position $\delta_1=\sqrt{3}L$.

5.7 THE COMPARATIVE BEHAVIOUR OF ELASTIC, ELASTOPLASTIC AND RIGID-PLASTIC STRUCTURES

In most textbooks and co-related publications, readers are referred to diagrams of the kind shown in Fig.5.41 which are used to illustrate the "typical" responses a structure will present when idealized elastic, elastoplastic and rigid-plastic behaviours are assumed.

The straight line 1 is the linear elastic solution in which changes of geometry effects are ignored. If such effects are

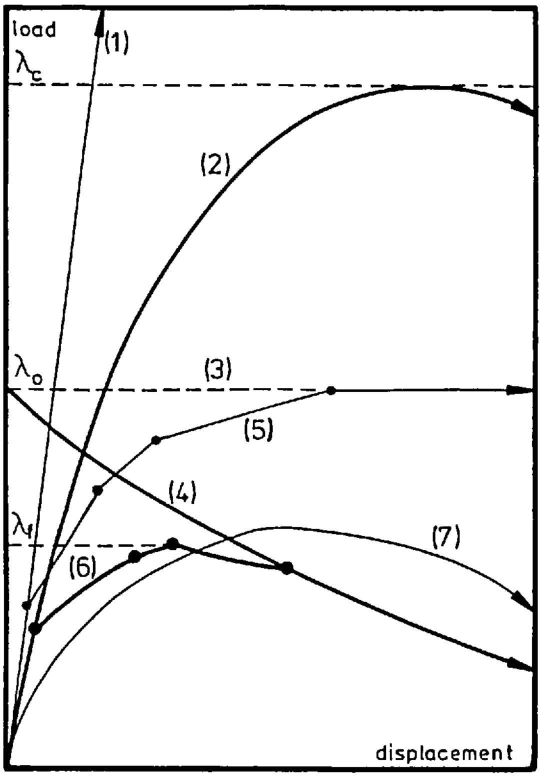


FIGURE 5.41

accounted for and/or a non-linear elastic constitutive relation is adopted a non-linear load-deflection relationship, represented by curve 2, is obtained. In the present case, instability is characterized by the attainment of a peak load, the elastic critical load λ_c.

The straight line 3 represents the rigid-plastic response obtained when assuming that the collapse mechanism displacements remain infinitesimally small; λ_o is the plastic collapse load. Curve 4 shows the result of a kinematically non-linear rigid-plastic analysis. It assumes that the collapse mode does not change in the range of displacements being shown.

The piecewise linear curve 5 represents the elastoplastic response when assuming unit shape factors and neglecting change of geometry effects. Plastic hinges successively decrease the stiffness of the structure until finally the plastic collapse load is attained, with the ensuing mobilization of a collapse mechanism displacing in the mode predicted by the rigid-plastic analysis. Curve 6 represents the kinematically non-linear elastoplastic solution; it is assumed in the present case that the structure fails due to global instability as the loading reaches the elastoplastic failure load λ_f. As the load-carrying capacity diminishes, a number of new plastic hinges are activated with the subsequent mobilization of an elastoplastic collapse mechanism which may have the same mode as the rigid-plastic collapse configuration; thus the coalescence of curves 4 and 6 is tacitly assumed.

Finally, when spreading of plastic zones, residual stresses, initial imperfections and strain-hardening are allowed for, curve 7 is obtained.

The examples being presented in the following have been deliberately chosen to illustrate alternative modes of behaviour, which are rarely, if at all mentioned in the engineering literature. No attempt is or should be made to extrapolate quantatively into more realistic structural systems the types of response to be presented; the objective of the exercise is essentially qualitative, a mere call for caution, an invitation not to accept tempting generalization based on an expected, but possibly unconfirmed, "typical" behaviour, for which the practical methods for estimation of the failure load have been specifically devised.

5.7.1 Illustrative Examples

The structures to be examined are Horne's one-and two-storey frames shown in Fig.5.3 and 5.9, to be referred to as Examples 1 and 2, respectively, and Smith's simple portal frame with pinned bases, shown in Fig.5.29; for the latter, the different combinations of loading and mechanical properties summarized in Table 5.22 will be considered.

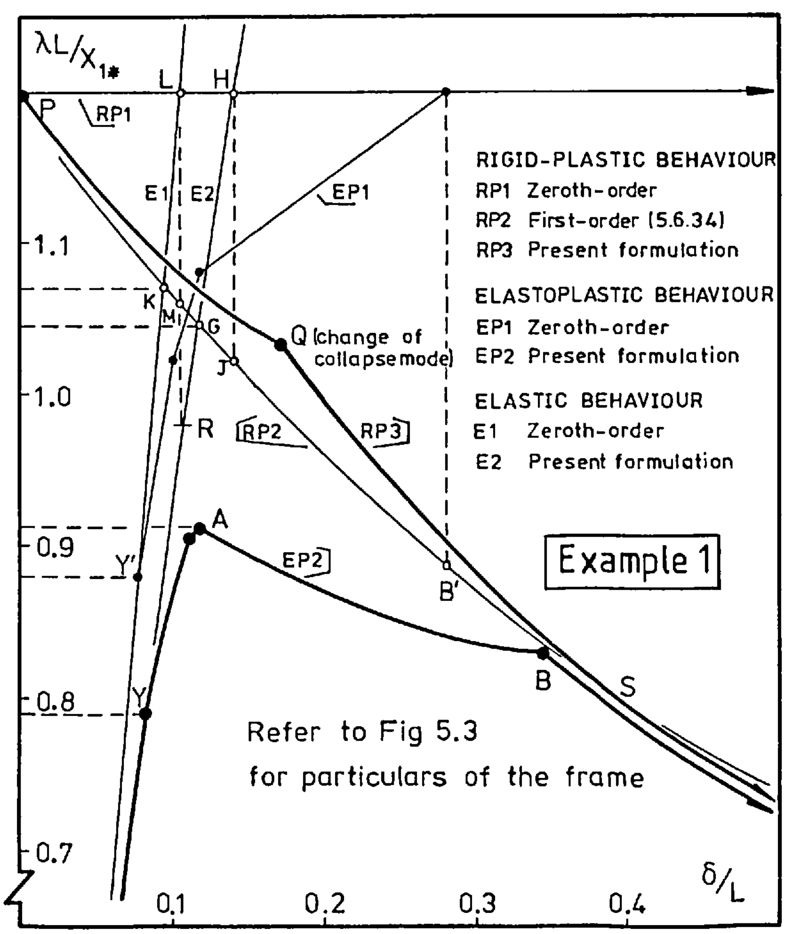


FIGURE 5.42

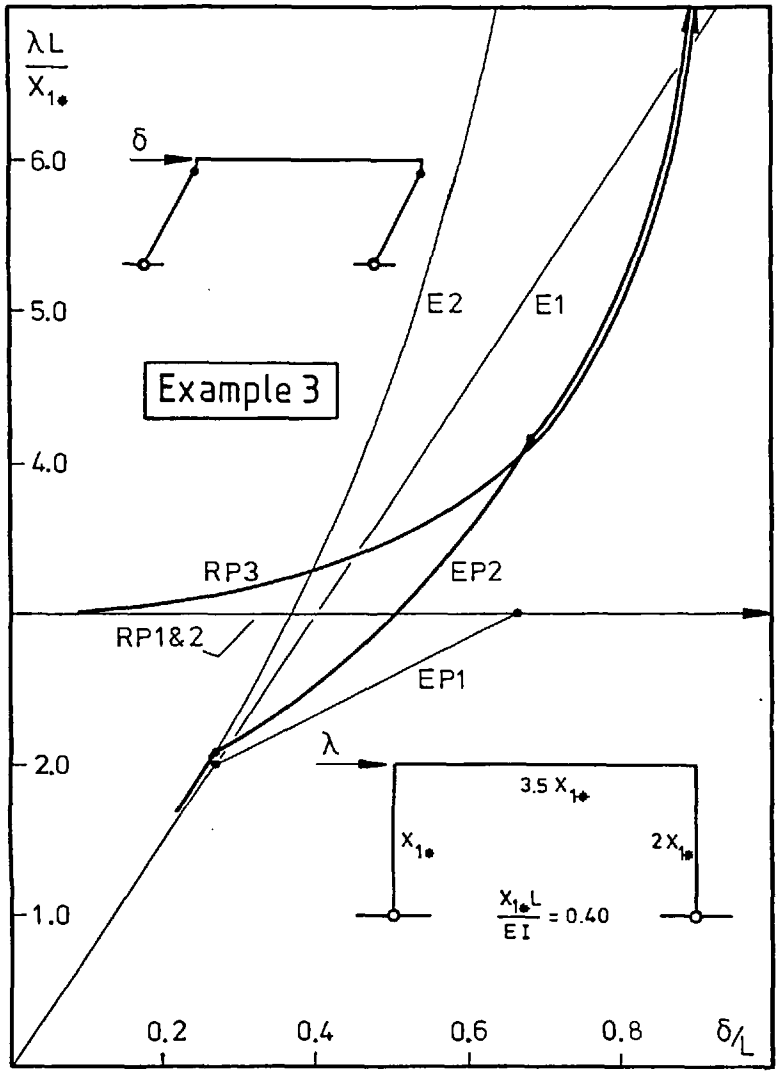


FIGURE 5.43

EXAMPLE	3	4	5	6	7	8
$X_{1*}L/EI$	0.40	0.12	0.12	0.40	1.	1.
λ_1	0	9λ	10λ	10λ	9λ	10λ
λ_2	λ	0	λ	λ	0	λ

TABLE 5.22

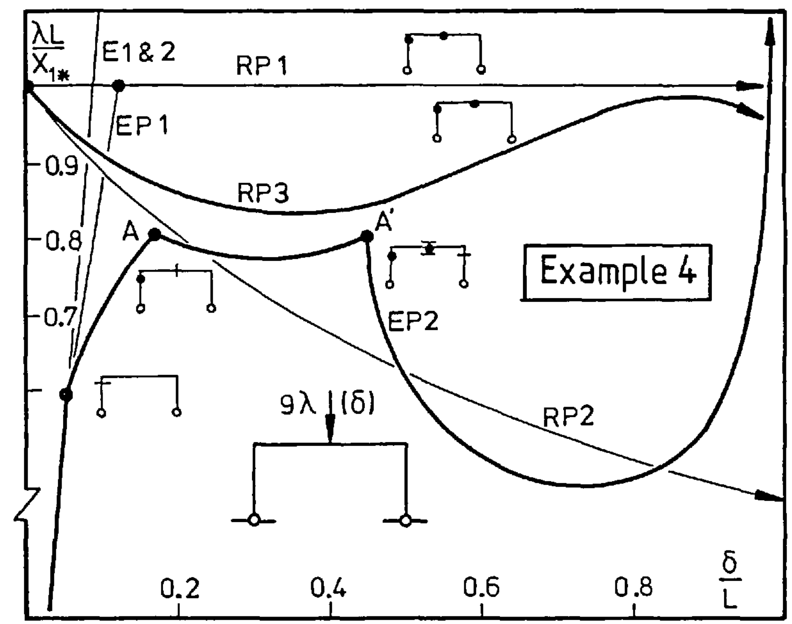


FIGURE 5.44

The elastic, elastoplastic and rigid-plastic responses of Examples 2.5 and 8 are not graphically represented as the quality of information they provide is already contained in either of the remaining examples, the comparative behaviour of which is illustrated in Fig.5.42 to 5.46.

Examples 1 and 3 illustrate the two extreme cases of unstable and stable responses. In the first case, although unable to expose a change of the collapse mode, the first-order solution provided by equation (5.6.34) results in a good approximation to the real rigid-plastic response; in the second case, and in contrast, equation (5.6.34) is quite unable to introduce any improvement in the limit analysis solution.

Example 4 illustrates the possibility of elastoplastic failure going undetected if the behaviour of the structure between the activation of two successive yield modes is not carefully investigated. If the plastic capacity of member 4 happened to be higher, the (positive) gradient between points A and A' could be significant thus inciting the analyst to contemplate a convex, and therefore stable equilibrium path joining those two points.

The possibility that multiple solutions may exist (and go undetected) is recalled in Example 6.

Example 7 shows that when an elastoplastic structure fails due to global instability, the first-order rigid-plastic solution will not necessarily provide an upper-bound containing the elastoplastic response.

Attempts were made to illustrate branching of the elastoplastic equilibrium path using Chwalla's frame; they did however fail.

The elastoplastic response of Horne's simple portal frame with fixed bases has been described in subsection 5.4.2.

The elastic response, shown in Fig.5.4 , is characterized by the recovery, as the displacements increase, of stiffness initially lost. The elastic frame is still stable for sway displacements of the order of the columns length.

Plastic limit analysis predicts collapse through a mechanism displacing under constant loading, in a combined mode involving the activation of the positive yield modes at critical sections 3 and 5 and the negative yield modes at critical sections 1 and 4. The rigid-plastic response defined by equation (5.6.34), which only accounts for first-order kinematic effects, is represented by curve PJB, in Fig.5.42. An exact analysis, to which the equilibrium path PQS corresponds, shows that as the displacements progress, the bending moment at critical section 2 increases, leading to the activation of the positive yield mode; at this stage, represented by point Q, plastic unstressing occurs at section 3 and the collapse mode changes into a sway mode.

After failure due to overall instability, the sway mode is also the collapse mode which the frame mobilizes when behaving elastoplastically. The rigid-plastic and elastoplastic mechanism lines, although manifesting a tendency to, do not in the present case and will not in general coalesce, due to the different histories of deformation with which each of the models is associated.

The downward sloping curve PQ indicates instability of the rigid-plastic structure at the collapse load, λ_0, resulting most often in a catastrophic complete collapse, unless important material work-hardening has been neglected. When the rigid-plastic model is unstable at incipient collapse, the elastoplastic failure load is likely not to reach the plastic collapse load as prior elastoplastic deformations may induce and accelerate the onset of a global instability phenomenon, as at point A in Fig.5.42.

Example 3, to which Fig.5.43 refers, is a typical case of stable, stiffning behaviour; the linear formulations underestimate the real load-carrying capacity of the structure, irrespectively of the assumed type of behaviour, elastic, elastoplastic or rigid-plastic. The raising, concave equilibrium paths indicate favourable changes of geometry, the so-called "geometrical work-hardening".

The rigid-plastic structure remains stable under the limit and higher loads. Unloading will however result in large permanent deflections that most often will render the structure unstable.

The set of responses illustrated in Fig.5.44 are qualitatively distinct from those discussed so far.

For the range of loading and displacements being shown, the elastic solutions proved only marginally sensitive to changes in the geometry of the frame.

Despite being unstable at incipient collapse, the rigid-plastic frame is able to recover a certain degree of stability as the displacements increase, almost recovering its maximum load-carrying capacity. The first-order non-linear solution is incapable of foretelling such behaviour.

The elastoplastic model presents certain interesting features. The first plastic hinge forms, at critical section 1, sensibly at the same load and displacement levels for the linear and non-linear formulations. The linear formulation proceeds by detecting the activation of critical section 2 and the ensuing mobilization of a collapse mechanism. The non-linear formulation shows that this collapse mechanism although initially unstable is capable of recovering the lost stiffness, thus mirroring the behaviour of the rigid-plastic model. As the elastoplastic failure load is regained (exceeded by 0.04% according to the computer results) plastic straining starts developing at critical section 3 while plastic unstressing occurs at critical section 2. The elastoplastic collapse mechanism changes into a sway mode and from then onwards the concurrence between the elastoplastic and rigid-plastic responses ceases to exist.

Examples 6 and 7 were prepared to illustrate situations wherein the elastic critical load is of the order of the rigid-plastic collapse load and of the elastoplastic failure load, respectively.

The optional rigid-plastic solutions that frame 6 may exhibit have been already discussed in the previous section. The associated equilibrium paths are shown in Fig.5.45 together with the corresponding first-order approximations.

The elastoplastic frame fails as the first plastic hinge forms, at critical section 1. The next hinge forms at critical section 3 and as a sway collapse mechanism is mobilized the elastoplastic collapse phase does not even vaguely correspond, within the range of displacements being shown, to either of the rigid-plastic mechanism lines.

The rigid-plastic response of Example 7 is identical to that of Example 4.

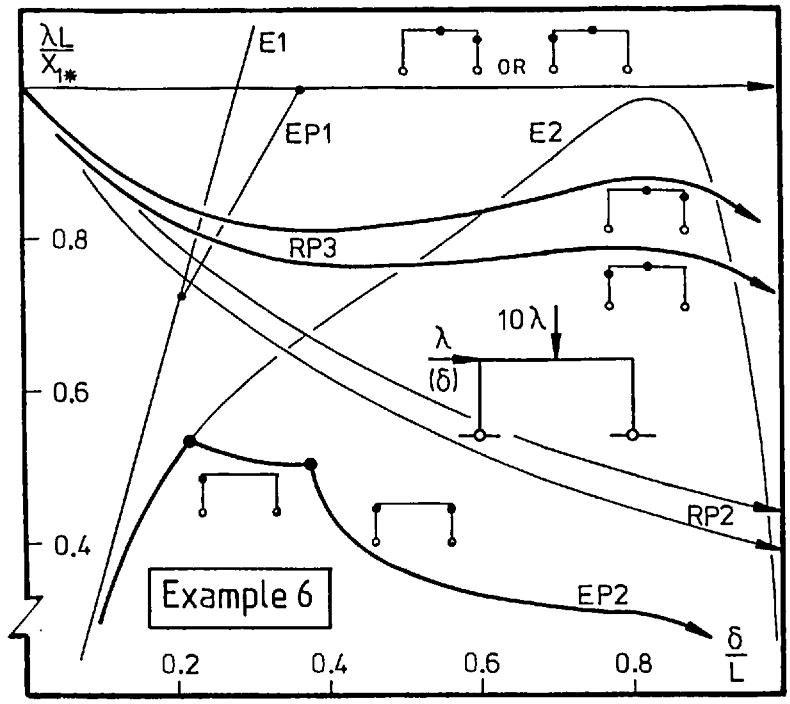


FIGURE 5.45

The elastic and elastoplastic failure loads differ by less than 0.5%.

The non-linear elastoplastic analysis shows that first yield, at critical section 1, occurs at a load level 9.8% higher than the predicted by a linear analysis; the opposite is the usual situation for non-stiffning responses. Elastoplastic failure occurs when plastic straining starts developing at critical section 2. A collapse mechanism is not mobilized though, as section 1 unstresses plastically

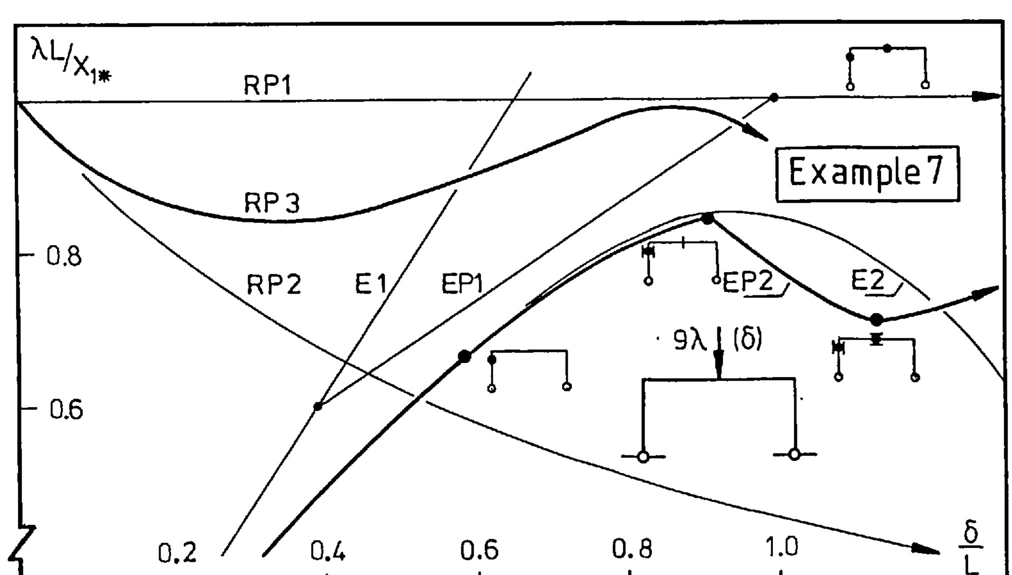


FIGURE 5.46

at that moment. Instead, and after a temporary loss of load-carrying capacity, plastic unstressing occurs at critical section 2 and the elastoplastic frame reverts into an elastic, stable response.

5.7.2 The Merchant-Rankine Formula and Related Methods for Estimating the Elastoplastic Failure Load

The collapse load λ_f is of prime importance from the engineering point of view. Its evaluation does however require considerable computation as the governing system is highly non-linear, involving history-dependent variables; it is not only necessary to follow the sequence of formation of plastic hinges but also to investigate the occurence of critical (limit and bifurcation) points at and between the activation of plastic hinges. Undetected plastic unstressing is likely not to affect substantially the elastoplastic failure load; it is however pf prime importance when the structure displaced configuration and the correct sequence and location of yielding are of relevance.

The computer time required for such an analysis is large and soon becomes prohibitive when sizeable frames are being considered. Simplified methods of analysis, perhaps still too involved to be used on an every-day basis at design offices, have already been mentioned. In the following reference is made to a class of procedures characterized by their common objective of estimating the elastoplastic failure load through a convenient combination of the two distinct responses the structure presents when assumed to behave elastically and rigid-plastically.

Merchant (1954) generalized Rankine's formula for frames and established the following relationship between the rigid-plastic collapse load λ_o and the elastic critical load λ_c.

$$1/\lambda_f^M = 1/\lambda_o + 1/\lambda_c \qquad (5.7.1)$$

thus obtaining a rough estimate, the Merchant-Rankine load λ_f^M, of the elastoplastic failure load λ_f.

Merchant's empirical formula CAN NOT be theoretically demonstrated. In an attempt to justify it, Horne (1963) was forced to introduce a set of quite drastic assumptions; the exercise does however indicate that the Merchant-Rankine load is not totally disaffected from the actual elastoplastic failure load.

The second reason for the undeniable success of such a basic approach to the problem is the insensitivity to error of the formula for practical values of its entries. Code-based designs, in which servicibility constraints and other requirements play a major (limitative) role, tend to produce structures for which the elastoplastic failure load tends not to exceed substantially 20% of the elastic critical load; as illustrated in Fig.5.47, if the elastic critical load is overestimated by as much as 100%, the Merchant-Rankine load is affected by less than 10%. Except when the elastic critical load is grossly underestimated, a substantial margin of error is permissible in its evaluation without affecting significantly the Merchant-Rankine load, its effect declining as the rigidity of frame response increases.

Based on Wood's (1974) proposals, the European Convention for Constructional Steelwork, ECCS (1977), has adopted the following criterion, a demarcation of the frontier between rigid and flexible frames:

- If $\lambda_c/\lambda_o>10$, the frame can be designed using a linear formulation ($\lambda_f\approx\lambda_o$).

- If $4\leq\lambda_c/\lambda_o\leq10$, particular consideration must be given to stability; the elastoplastic failure load can be evaluated using Wood's modification of the Merchant-Rankine formula

$$\lambda_f\approx\lambda_f^W=\lambda_c/(0.90+\lambda_o/\lambda_c) \qquad (5.7.2)$$

- If $\lambda_c/\lambda_o<4$, calculations of λ_f from λ_c and λ_o are not allowed; a (first-order) non-linear analysis should be used to design the frame.

The shaded area in Fig.5.47 contains the class of frames which are allowed to be designed using Wood's formula (5.7.2).

As the evaluation of the rigid-plastic collapse load poses no special problems, the emphasis has long ago been shifted into the research of reliable procedures for estimating the elastic critical load, many of which are collected by Lightfoot (1961) in his textbook; further references on alternative methods can be found in Horne and Merchant (1965), Wood (1974), Horne (1975) and Williams (1977).

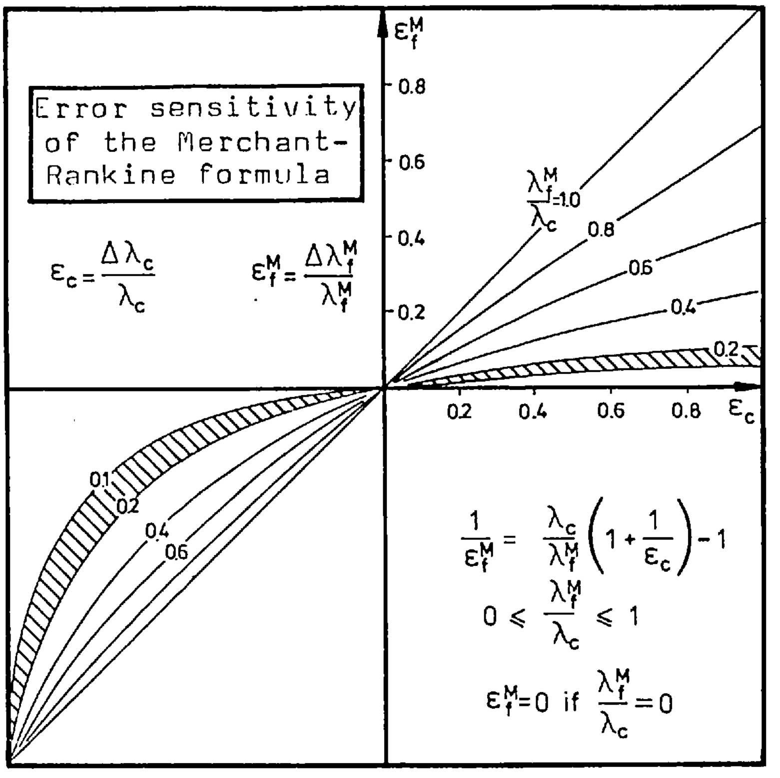


FIGURE 5.47

Except for Horne's simple portal frame, all the remaining examples under consideration fall into the category for which a kinematically non-linear analysis is recommended.

Summarized in Table 5.23 are the errors

$$\varepsilon = 100\left(\frac{\lambda_{approx.}}{\lambda_{exact}} - 1\right)$$

that would have been perpetrated if a Rankine-type approximation had been adopted.

In the summary of the characteristics of the frames, σ_y is the yield stress and $\bar{\nu} = \nu\, r/d$ the cross-section shape factor affected by the ratio of the radius of gyration r to the depth d.

Given in the same Table are the rigid-plastic collapse factor λ_0, as well as the elastoplastic failure load λ_f^E and the elastic critical load λ_c^E factors, obtained through the formulation being proposed; λ_c^H is the elastic critical load factor as provided by Horne's (1975) approximate method, and λ_c^i is the value it should ideally take in order to produce the exact elastoplastic failure

EXAMPLE	1	2	4	5	6	7	8
S_R	400	100	100	100	300	300	300
$E/\sigma_y\bar{v}$	2000	1667	1667	1667	1500	600	600
$X_{1*}L/EI$	0.40	0.12	0.12	0.12	0.40	1.0	1.0
$\lambda_o L^2/EI$	0.48	0.69	0.12	0.12	0.40	1.0	1.0
$\lambda_c^H L^2/EI$	2.06	2.25	0.30	0.27	0.27	0.30	0.27
$\lambda_c^E L^2/EI$	(∞)	2.37	0.85	0.39	0.39	0.85	0.39
$\lambda_f^E L^2/EI$	0.37	0.53	0.10	0.10	0.22	0.85	0.37
$\lambda_c^i L^2/EI$	1.16	1.78	0.36	0.37	0.42	3.53	0.55
ε_f^M (%)	6.58	-1.50	-11.34	-15.31	-25.81	-72.66	-42.12
ε_f^m (%)	31.51	-0.37	8.25	-6.12	-8.76	-45.68	-23.37
ε_f^W (%)	16.16	6.55	-5.15	-9.18	-22.58	-72.07	-41.03
ε_f^w (%)	46.03	8.05	18.56	2.04	-3.69	-43.08	-21.20
ε_c^H (%)	//////	-5.02	-64.66	-31.30	-31.30	-64.66	-31.30
λ_c^E / λ_o	(∞)	3.45	7.08	3.28	0.98	0.85	0.39
λ_c^H / λ_o	4.29	3.28	2.50	2.25	0.68	0.30	0.27

TABLE 5.23

EXAMPLE	1	2	4	5(M2)	5(M3)	6(M2)	6(M3)	7
ε_y (%)	-13.4	-22.1	-26.6	-6.8	-6.8	0.0	0.0	-21.9
$\varepsilon_{y'}$ (%)	-3.7	-12.4	-26.6	-11.7	-11.7	36.5	36.5	-29.6
ε_K (%)	17.2	16.2	12.8	11.7	9.2	45.8	40.2	-22.5
ε_G (%)	14.5	13.0	12.8	10.4	8.6	36.5	29.0	-27.8
ε_M (%)	16.1	14.9	12.1	10.4	8.0	27.1	21.5	-46.2
ε_J (%)	12.3	10.6	12.1	9.2	6.1	/////	/////	/////
ε_R (%)	5.7	3.1	23.9	11.7	11.7	14.0	14.0	-14.8
ε_B (%)	6.6	24.4	7.8	9.2	4.9	34.7	26.7	/////
$\varepsilon_{B'}$ (%)	-3.1	5.5	7.8	9.2	4.9	27.1	19.6	-46.8

TABLE 5.24

load factor using the Merchant-Rankine formula (5.7.1).

The errors involved when using Horne's estimate of the elastic critical load in the Merchant-Rankine formula and in Wood's modification (5.7.2) are defined by ε_f^M and ε_f^W, respectively; ε_f^m and ε_f^w represent the same errors when the actual elastic critical load λ_c^E is used instead.

The examples under consideration confirm Horne's prediction that the method leads to conservative estimates of the elastic critical load; however, with the exception of Example 2, in all the remaining examples the 20% limit he sets for the margin of error, ε_c^H, is largely exceeded.

As a result of many analytical and experimental studies, the general consensus is that the Merchant-Rankine load tends to be a lower bound on the value of the elastoplastic failure load, although for simple portal frames the latter can sometimes fall slightly below the Merchant-Rankine load. Horne's frames, not Smith's though, confirm this generalization.

Referred to lastly are alternative methods for estimating the elastoplastic failure load from the elastic and plastic responses of the frame which utilize the intersection points of the rigid-plastic and elastic equilibrium paths shown in Fig.5.42.

Points K and G are the intersection of the (first-order) non-linear rigid-plastic equilibrium path with the linear and non-linear elastic equilibrium paths; the projection on the former of the intersection of the latter with the zeroth-order rigid-plastic mechanism line, points L and H, define points M and J, respectively suggested by Murray (1956) and Majid (1968).

Point R is the projection of point L on the non-linear elastic equilibrium path; Majid's (1967) claim that the load associated with point R is the Merchant-Rankine load could not be confirmed.

The loads associated with points Y' and Y are the first-yield loads defined by linear and non-linear elastoplastic analysis, respectively.

Point B' is used in Horne's (1961) "last hinge method" to estimate not the elastoplastic failure load but the incipient collapse load, λ_B.

For structures with unstiffning behaviour, the loads associated with points K, M, G, J and R (Y' and B') tend to be upper (lower)-bounds on the elastoplastic failure load; $\lambda_{B'}$ should in such cases overestimate the actual incipient collapse load.

Summarized in Table 5.24 are the errors found when using the loads associated with points Y, Y', K, G, M, J, R and B' to estimate the elastoplastic failure load; ε_B is the error involved when using the "last hinge method" to estimate the incipient collapse load. In Examples 5 and 6 a distinction is made between the results obtained using either of the possible to rigid-plastic solutions, the collapse modes M2 and M3 described in subsection 5.6.3. Herein use was made of the exact non-linear elastic equilibrium path to define points R, G and J; as the objective of the method is to estimate the elastoplastic failure load utilizing easy-to-obtain approximations on the plastic and elastic responses, simplified methods, as for instance, the amplification factor method described in Horne and Merchant (1965), leading to close approximations of the non-linear elastic load-deflection curve should be used instead.

The results summarized in Tables 5.23 and 5.24 show a remarkable improvement when, instead of the Merchant-Rankine formula, the loads associated with the intersection points K, M, G and J are used to estimate the elastoplastic failure loads of very flexible frames.

Both methods require the separate implementation of elastic and limit analyses, in comparison to which the supplementary calculations involved in the latter method represent a minor additional effort, amply justified by the quality of the estimates it generates.

CHAPTER SIX

CLOSURE

Among the varied extensions and many improvements to which the proposed formulation for problems in kinematically non-linear structural analysis is open, the following are thought to be prominent.

6.1 NECESSARY IMPROVEMENTS

The objectives set for the study on Statics and Kinematics of the mesh and nodal fundamental substructures were accomplished in essence.

A more efficient codification can be given to the (non-linear) effects which the internal releases have on the conditions of equilibrium and compatibility at substructure level.

The process suggested in Chapter Two to eliminate the additional forces from the mesh description of Statics is justifiable only when leading to a significantly more compact governing system; the alternative and simpler process for accounting for static-kinematic interdependence described in Chapter Five should be used otherwise.

Although not devised for that particular purpose, the mathematical model for the simply supported elastic beam element, presented in Chapter Three, is thought not to be out-performed by those used in most of the proposed beam-column theories. The degree of accuracy achieved seems quite appropriate for finite-element based formulations in structural analysis.

Distributed loads should be included to complete the study of the elastic beam element. Besides introducing further inaccuracies in the representation of the causality relations, the discretization of the distributed loads increases considerably the dimension of the problem if a nodal formulation is to be adopted in the analysis of the structure; to increase the number of nodes affects only marginally the mesh description, as the static indeterminacy number is left unchanged.

Shear deformation effects can become relevant in many situations which should certainly not be served conveniently by the crude correction introduced in the elastic governing system.

The format adopted for the representation of the plasticity causality relations had already been proved to be particularly suitable for discrete formulations to be processed through mathematical programming.

The proposed approach of forcing the emergence of symmetric operators with constant entries by concentrating the disturbing non-linearities in corrective vectors, seems to be more rewarding than a piecewise linearization of the static and kinematic phases of yielding, at the cost of increasing immensely the dimension of the problem.

Improvements should be concentrated upon the determination of normality and hardening coefficient functions, and on devising a practicable corrective procedure to include the more relevant of the spreading of plasticity effects.

The ideal would be to substantiate a model preserving the piecewise continuity in the flow of elastoplastic finite strains, amenable to a discrete, vectorial representation; the propositions contained in the doctoral thesis on "Elastoplastic Deformation Analysis by Mathematical Programming" my colleague Mr. J. Appleton is submitting will certainly facilitate the development of such a formulation.

The procedure for assembling the nodal description of Statics and Kinematics described in Chapter Four, appears to be generally adequate. It may be improved through the use of connectivity theory algorithmic procedures; such improvement would be particularly beneficial for the proposed mesh assemblage procedure.

The derivation of mesh formulations from nodal formulations, and vice-versa, is from a pedagogical standpoint a most unadvisable exercise, as it totally destroys the essential concepts characterizing the alternative and fundamental mesh and nodal connectivities. The derivation of a mesh formulation from a nodal formulation, may however be advantageous for practical, numerical implementation purposes, as it combines the easiness of codifying an efficient nodal assemblage with the compactness in the governing system a mesh description generally propitiates.

Further improvements in the efficiency of implementation of solutions can be achieved through the use of mixed,simultaneously mesh and nodal, formulations, an avenue deserving more attention than it has received to date.

As stressed in several occasions throughout the presentation, the role of mathematical programming in structural mechanics is two-fold: to encode and synthetize the basic vectorial relations and to supply efficient algorithmic procedures for numerical implementation of solutions.

None of the many algorithms that mathematical programmers have developed is a universal-purpose routine. They were often suggested by specific problems, with particular geometries, encountered in the many fields which the applied sciences embrace; their adaptation to structural analysis problems is a necessity users have to face and overcome by learning and interpreting physically the way such algorithms operate. Their use as "black boxes" is an inadvisable practice, as it implies sub-utilization of capabilities and submission to situations of failure.

The numerical procedures described in Chapter Five are suitable only for research purposes; their direct application to practical engineering problems will often be prohibitive. The lack of sophistication and the consequent simplicity for interpreting physically each of the algorithm operations, did however prove to be highly rewarding when extricating the many computing failures that a first acquaintance with any problem always involves (caused by a faulty or defficient programming of the alternative and complex responses a non-linear system may present). The size of the problem could have been reduced, and computing costs thus saved, if, instead of solving directly the associated Kuhn-Tucker problem, an algorithm operating on either the primal or dual programs had been adopted.

The linear programs for elastoplastic analysis under a prescribed stress field should provide a sound basis for developing economically feasible computer codes for the design (or analysis) by re-analysis of large scale structural systems. A valid experience in engineering code-based design warrants the reduction of the multiple options open in structural mechanics to a set of realistic possibilities, within which mathematical programming algorithmic procedures will select the best path to optimality.

At the cost of high computer execution times, the formulation suggested herein is capable of substantiating a degree of accuracy and a completeness of analysis dispensable in most practical applications. However, as a consequence of being derived from, and presented in the form of first-principles of mechanics, the proposed formulation is amenable to simple and CONSISTENT adaptations suiting the designer's particular needs. Knowledge gained from practical experience, together with constructional and serviceability constraints and other code requirements, enable the designer to establish bounds for the variation of displacements and strain- and stress-resultants; in order to obtain a consistent approximate formulation, the designer is only requested to take as many terms in the series expansion of the additional forces and deformations, of the stability and bowing functions and of the normality and hardening functions, as required by the accuracy of the problem thus bounded. Those simplifications should be reflected in the time consuming routines for identifying and solving situations of multiple unstressing and branching, inessential in the preliminary phases of design.

The theoretical considerations summarized in Chapter Five, regarding the more relevant aspects in structural analysis, namely extremum principles, conditions for uniqueness and stability criteria, represent a simple delineation of the possibilities mathematical programming opens up when applied to non-linear structural analysis.

Despite the superficiality of the exploration undertaken, a certain degree of unification was achieved in the treatment of the aforementioned aspects within, and enclosing, the analysis of elastoplastic, elastic and rigid-plastic systems.

6.2 FORESEEABLE EXTENSIONS

An exhaustive study on extremum principles, static and kinematic uniqueness theorems and stability, leading to an ordered qualification of alternatives offered within each, and to the clarification of the inter-relationships between them, and from which efficient criteria to identify, classify and solve situations of multiple unstressing and/or bifurcation, can certainly be accomplished through a more effective use of mathematical programming system analysis theory; the immediate extension of the present work, is in the development of a unified theory of kinematically non-linear elastoplastic analysis, based on the proposed formulation ameliorated to include at least the improvements specified in the previous section, from which problems of elastic and rigid-plastic structures may result by simple specialization.

The scope of such theory could be subsequently widened after extending the proposed formulation into the analysis of spatial frames and continuum structures, subject to quasi-static or dynamic deterministic actions.

The procedure adopted herein for preserving duality in the exact, mesh and nodal, finite descriptions of Statics and Kinematics can be easily developed into the study of deformed skeletal sub-structures undergoing large rigid-body movements in three-dimensional space; difficulties can however be anticipated in the derivation of the causality relations associating stress-resultants with finite elastoplastic strain-resultants, based on a spatial beam element.

The common interpretation of the fundamentals of the mechanics of structures would certainly facilitate a fruitful combination of the procedures adopted herein to deal with kinematic and material non-linearities, with the approach and representation of the (piecewise) linear response of plates and shells my colleague Mr.

A. da Fonseca proposes in the doctoral thesis he is submitting, entitled "Plastic Analysis and Synthesis of Plates and Shells by Mathematical Programming".

The incorporation of deterministic dynamic actions in a discrete formulation is not particularly problematic; in the qualification and implementation of the many new options of structural response it opens up is where the implied difficulties reside.

Structural stochastic analysis has already received the attention of many researchers. It is sincerely hoped that, as a result of a productive and unifying use of entropy, the research work my colleague Mr. E. Mello is undertaking will render more manageable the formidable task of implementing a kinematically non-linear elastoplastic stochastic analysis.

Structural analysis has to be complemented by structural synthesis if a general and complete portrayal of structural mechanics is to be accomplished.

In recent years significant advances have been made in using mathematical programming in structural synthesis; many aspects are still untouched, others not yet fully explored.

Kinematically non-linear structural synthesis promises to be a most interesting research topic, as it will require the generalization, and even the replacement of some of the fundamental concepts from which the existing formulations in linear structural synthesis have been developed. A rewarding use of the proposed additional forces and deformations, treated by a perturbation technique in the context of a mathematical programming approach can be forecast.

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