NORMAL SUBGROUPS OF MULTIPLY TRANSITIVE PERMUTATION GROUPS

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THESIS

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ABSTRACT

The aim of this thesis is to study multiply transitive permutation groups and their normal subgroups. In this abstract we shall use the following notation: G is a permutation group on a finite set Ω of size n, G is not the symmetric group on Ω and acts t-fold transitively on Ω. H is a normal subgroup of G, different from the identity, and $x(H)$ denotes the number of H-orbits on $\Omega(t)$ where $\Omega(t)$ is the set of all ordered sequences of t distinct points in Ω.

Our main objective is the following question going back to C. JORDAN: When is H t-fold transitive as well?

In Chapter II and III we extend some known results and prove the following theorems:

THEOREM A (2.8)

With the above notation, the following three statements hold:

(i) If $t = 2$, H is generously transitive if and only if H has even order. Therefore H is generously transitive if G contains no regular normal subgroup.

(ii) If $t = 3$, H is generously doubly transitive except if H is regular <u>or</u> if H is a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ in their usual representation on the projective line, $n = q+1$.

(iii) If t is at least 4, H is generously $(t-1)$-fold transitive.

THEOREM B (3.6 & 3.7)

Let p be a prime, $p < t$, and let r be the smallest non-negative integer with $r \equiv (n-t+1)/x(H) \bmod p$.

(i) If p does not divide $n-t+1$, we have $0 < rx(H) < p$.

(ii) If p divides $n-t+1$, then p also divides $(n-t+1/x(H)$ except if $t = 3$ and H is either regular or a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ where $n = q + 1$.

As a corollary to Theorem B we obtain the following generalisation of results by WAGNER and ITO:

THEOREM C (3.11)

Let $3 \leqslant t \leqslant 6$ and let Γ' be a subset of Ω of size $t-1$. Suppose there are primes p and q, $p < t$, such that a Sylow q-subgroup of $H_{\Gamma'}$ fixes exactly k points where $k-t+1 \not\equiv 0 \bmod p$. Then H is t-fold transitive on Ω.

In Chapter IV we prove some results in the case of doubly transitive groups.

THEOREM D (4.4)

Let G be a doubly transitive permutation group of degree n and $H \neq 1$ a normal subgroup of G. Suppose G contains an involution i with $i^G = i^H$ and let $f = |Fix(i)|$. Let $y(H)$ be the number of H-orbits on $\Omega\{2\}$.

Then $y(H)$ divides $(n-1, f-1)$. In particular, H is doubly transitive if $(n-1, f-1) = 1$.

THEOREM E (4.5)

Let G be a doubly transitive permutation group on Ω of even degree n and $H \neq 1$ a normal subgroup of G. Let α, β be two distinct points in Ω and S a Sylow 2-subgroup of $H_{\alpha,\beta}$. Put $N = N_G(S)$ and $N' = N \cap H$.

Suppose there is some subgroup I, $S < I < H$, with $[I:S] = 2$ and $I^N = I^{N'}$. Then H is doubly transitive.

THEOREM F (4.6)

Let G be doubly transitive of degree n and H normal in G with index $d = [G:H]$ in G. Let p be some prime dividing n exactly to the j^{th} power. Suppose either

(i) $(d, n-1) = 1$ or

(ii) $(d, p^i - 1) = 1$ for all $i \leqslant j$ and further that G/H is solvable if $p = 2$ and $j \geqslant 2$.

Then H is doubly transitive on Ω.

In Chapter V we investigate normal subgroups of triply transitive permutation groups.

THEOREM G (5.4)

Let G be a triply transitive group of degree $n \equiv 0 \bmod 4$ and let H be a non-regular normal subgroup of G.

Suppose no involution in H fixes $2 \cdot |Fix(S)|$ points where S is a Sylow 2-subgroup of $H_{\alpha,\beta,\gamma}$ (α, β and γ distinct).

Then H is either triply transitive or $PSL(2,q) \leqslant H \leqslant G \leqslant P\Gamma L(2,q)$ with $n = q+1$.

THEOREM H (5.5)

Let G be a triply transitive of degree $n \equiv 0 \bmod 3$. Suppose G_α contains a normal subgroup $M \neq 1$ such that $|M_{\beta,\gamma}|$ is prime to 3 for three distinct points α, β and γ.

Then G is isomorphic to a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ for some prime power $q = n-1$.

THEOREM I (5.7)

Let G be triply transitive on Ω of degree $n \equiv 0 \bmod 3$. Then every normal subgroup of G has at most two orbits on $\Omega(3)$.

THEOREM J (5.8)

Let G be triply transitive of degree $n \equiv 0 \bmod 12$. Then either every normal subgroup of G is triply transitive or $PSL(2,q) \leqslant G \leqslant P\Gamma L(2,q)$ where $q+1 = n$.

PREFACE

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CHAPTER 1 INTRODUCTION

In 1861 Emile Mathieu discovered the first non-trivial quintuply transitive permutation groups M_{12} and M_{24}. Ever since multiply transitive permutation groups have been in the centre of interest of group theorists. Many problems about these groups have been solved such as the existence of M_{24} for instance which was in doubt for more than 75 years, and these investigations have led to many important discoveries. Yet we are far from understanding these groups completely.

Mathieu and Jordan observed *une sorte de fossé entre le groupe alterné et les autres groupes des substitutions* (p.41 in [13]) in so far as the degree of transitivity of non-alternating groups is much smaller than that of the alternating group. And in fact, Mathieu's groups M_{12} and M_{24}, together with their subgroups M_{11} and M_{23}, are the only presently known groups operating quadruply transitively without containing the alternating group of the same degree. It has therefore been conjectured that 6 or even 5 is the highest degree of transitivity occurring for non-trivial multiply transitive permutation groups. A number of attempts to prove this very fundamental property could only show that the degree of transitivity t is bounded by a logarithmic function of the representation degree (Wielandt, 1934 in [26]) and an absolute bound $t \leqslant 6$ has been given subject to Schreier's conjecture on the automorphisms of simple groups. (Wielandt [28], Nagao [18] and Suzuki [22]).

In his paper [13] Jordan posed another question which was much more accessible: Given a t-fold transitive permutation group G on a finite set Ω of degree n. What is the degree of transistivity t' of a subgroup H normal in G? This problem has found a number of interesting answers and is the theme of this thesis.

Jordan proved the classical result that t' is, apart from two obvious exceptions, at least $t - 1$. In 1955 Wielandt and Huppert [27] introduced the concepts of multiple primitivity and half-transitivity. Using these descriptions they were able to show $t' \geqslant t - \frac{1}{2}$ and Ito (1958 in [11]) showed that H is $(t - 1)$-fold primitive if t is at least 3. A decade later Livingstone and Wagner [15] introduced the notion of multiple homogeneity and Wagner [24] uses a description somehow complementary to multiple homogeneity to prove in an entirely elementary way that t' equals t if $t \geqslant 3$ and $n - t$ is even. This new concept later became known as multiple generosity in an article by Neumann [19]. Here Neumann develops a theory of multiple generosity similar to the general theory of multiple transitivity and establishes the natural link to the character theory of multiply transitive groups. He also considers Jordan's problem under the generosity aspect. A number of other authors have contributed to the normal subgroup problem and most of their results will be mentioned as we go along with our own discussion.

Turning to the available evidence in terms of known examples, one observes that there are only few. A list of doubly transitive groups, given in Kantor's survey in [7] on doubly transitive designs, could roughly be summarized in the following 4 sections:

(1) Symmetric and alternating groups.

(2) Groups with regular normal subgroups.

(3) Groups of Suzuki type containing normal subgroups $PSL(n,q)$, $PSU(3,q)$, $Sz(2^{2r+1})$ and groups of Ree type. Also the symplectic group $Sp(2m,2)$ and

(4) Sporadic examples: unusual representations of some groups in the above sections and representations of the sporadic simple

groups M_{11}, M_{12}, M_{24}, the Higman-Sims group HS and Conway's group .3.

Let G be one of the above groups in a t-fold transitive representation on a set Ω of size n and $H \neq 1$ a subgroup normal in G. The groups in section (1) and (2) are the exceptions in Jordan's result: $G = \mathrm{Sym}\,\Omega$ is n-fold transitive and $H = \mathrm{Alt}\,\Omega$ is sharply (n-2)-fold transitive. In section (2) t is at most 3 and Ω can be identified with a vector space such that G_0 is a group of linear transformations on Ω. Here H may be regular or $(t-\frac{1}{2})$-fold transitive. If G is a group in section (4), t has values 2, 3, 4 or 5 but normal subgroups H always have the same degree of transitivity t.

The groups in section (3) are geometrical groups acting on certain subsets of projective spaces and apart from groups containing PSL(2,q) as a normal subgroup, they are all exactly doubly transitive and the same is true for their normal subgroups. So groups containing PSL(2,q) are the only exceptions and very interesting ones they are: The groups $PSL(2,q) \leqslant H \leqslant G \leqslant P\Gamma L(2,q)$ operate on the projective line $\Omega = PG_1(q)$ and $|\Omega| = q + 1$. $P\Gamma L(2,q)$ is triply transitive on this set and if q is an odd prime power, PSL(2,q) is only doubly transitive. The stabilizer in PSL of the points 0 and ∞ has two orbits on $GF(q)^*$, squares and non-squares.

A suitable measure to determine the drop in transitivity from G to H seems to be the following: By definition G is transitive on the set of all ordered t-tuples $\Omega(t)$ and H is t-fold transitive on Ω if and only if $\Omega(t)$ is one orbit under H. Let therefore $\chi(H)$ be the number of H-orbits on $\Omega(t)$. This definition allows us to sum up the list of known examples. In section (1) we obtain examples for every value of t and here $\chi(H)$ is at most 2. In section (2) the consideration of the one-dimensional affine groups shows that $\chi(H)$ can take every value dividing

$|\Omega| - 1$. In section (3) we have $t = 2$ and $\chi(H) = 1$ or $t = 3$, $H = PSL(2,q)$ and $\chi(H) = 2$ if q is odd. Finally in section (4) we obtain for every example $\chi(H) = 1$.

There is a number of conjectures to fit these findings. The first conjecture,

(C1) Normal subgroups H of quadruply transitive groups are quadruply transitive, i.e. $\chi(H) = 1$ if $|\Omega| - 2 \geqslant t \geqslant 4$,

is fairly well established and has been proved by Ito [12] for $|\Omega| \not\equiv 0 \bmod 3$ and for $|\Omega| \leqslant 10^6$ by Saxl [20].

A second conjecture,

(C2) Non-regular normal subgroups H of triply transitive groups have at most 2 orbits on $\Omega(t)$, i.e. $\chi(H) \leqslant 2$ if $t \geqslant 3$ and H is not regular on Ω,

takes account of the symmetric and one-dimensional projective groups. (C2) seems reasonable since, as we shall see, these groups always occur inductively as constituents of certain subgroups of G.

The strongest conjecture, implying both C2 and C1 would be

(C3) Normal subgroups of triply transitive groups are either regular, triply transitive or contain $PSL(2,q)$ as a characteristic subgroup.

We are able to show that (C2) implies (C3) if $|\Omega| \not\equiv 2 \bmod 4$ (Theorem 3.8). In chapter V we prove that (C2) holds if $|\Omega|$ is a multiple of 3 and also for the case $|\Omega| \not\equiv 2 \bmod 4$ under some additional assumptions on involutions in H. Therefore (C3) holds for $|\Omega| \equiv 0 \bmod 12$ and for $|\Omega| \not\equiv 2 \bmod 4$ subject to some restrictions and these results give some evidence that

C3 could be true in general.

This thesis is organized in the following way:

Chapter II gives the usual definitions, a list of results used and a proof of a very essential tool, the generosity theorem for normal subgroups which was initially proposed in Ito's paper [12]. In chapter III we develop a concept which allows a homogeneous treatment of many known theorems leading to new proofs and extensions of these results. This analysis is continued in Chapter V and leads to the above mentioned conclusion. In chapter IV we follow a different line of argument investigating some symmetries of Ω. The methods there are mainly of a combinatorial nature and give some arithmetic conditions for the doubly transitive case.

CHAPTER II PRELIMINARIES

In this section we list most of the group theoretic results we shall use throughout this thesis. While most of the results are quoted from the literature, some are extensions of known results like for instance a variation of a theorem by Huppert (lemma 2.6), lemma 2.2 and the generosity theorem for normal subgroups 2.8.

2.1 NOTATION AND DEFINITIONS

For completeness we develop the notations and definitions to be used here. They are standard definitions and Wielandt's book [29] is an excellent reference for most of them.

All groups and sets considered are finite. Let Ω be a set consisting of n elements also called points and denoted by Greek symbols. The symmetric group on Ω is denoted by *Sym* Ω or by *Sym(n)* when there is no emphasis on the set Ω; *Alt* Ω or *Alt(n)* stands for the alternating group on Ω.

If k is a positive integer not exceeding n, $\Omega\{k\}$ denotes the set of all subsets of Ω containing k distinct points. Members of $\Omega\{k\}$ will sometimes be called *k-blocks* or simply *blocks*. Similarly, $\Omega(k)$ denotes the set of all ordered sequences $(\alpha_1,\alpha_2, \ldots\alpha_k)$ where the α_i are distinct points in Ω. Note that Sym Ω induces permutation groups on both $\Omega\{k\}$ and $\Omega(k)$, i.e. there are canonical homomorphisms from Sym Ω to Sym $\Omega\{k\}$ and to Sym $\Omega(k)$. In general, if G is a permutation group on Ω, Ω' some other set and h a homomorphism from G to Sym Ω', we shall say 'G acts on Ω' by h' and omit the mentioning of h if the reference is clear. Let g be an element of G, α a point and Δ a subset of Ω. Then α^g and Δ^g denote the images of α and Δ under g. Put $\alpha^G = \{\alpha^g | g\epsilon G\} \leqslant \Omega$ and $\Delta^G = \{\Delta^g | g\epsilon G\} \leqslant \Omega\{|\Delta|\}$. The subgroup of G fixing Δ pointwise or setwise is denoted by G_Δ and $G_{\{\Delta\}}$ respectively. The largest subgroup of G acting on Δ is $G_{\{\Delta\}}$ and the kernel of this action is G_Δ. Therefore, $G_{\{\Delta\}}/G_\Delta$ is a permutation group on Δ, denoted by G^Δ.

A permutation group G on Ω is said to be *half transitive* on Ω if $|G:G_\alpha| \neq 1$ is independent of the choice of α in Ω. G is called *transitive* on Ω if $|G:G_\alpha| = |\Omega|$ or equivalently if $\alpha^G = \Omega$; if G is transitive on Ω with $|G_\alpha| = 1$, G is called *sharply transitive* or also *regular*. G is said to be *primitive* on Ω if G is transitive and if G_α is a maximal subgroup of G. There are similar definitions for higher degrees of transitivity: If t is a positive integer not exceeding n, G is called *t-fold transitive* on Ω if G acts transitively on $\Omega(t)$, and *(t + ½)-fold transitive*, if G acts also half transitively on $\Omega(t+1)$. G is said to be *sharply t-fold transitive* on Ω if G acts regularly on $\Omega(t)$. Finally, G is called *t-fold primitive* on Ω if G is t-fold transitive on Ω and if $G_{\Gamma'}$ is primitive on $\Omega\setminus\Gamma'$ for every Γ' in $\Omega\{t-1\}$.

There are two further concepts related to multiple transitivity I would like two introduce. G is said to be *t-fold homogeneous* on Ω if G acts transitively on $\Omega\{t\}$ and G is called *t-fold generously transitive* on Ω if G^Δ = Sym Δ for all blocks Δ contained in $\Omega\{t+1\}$.

Let $B \leqslant \Omega\{k\}$ be a non-empty collection of k-blocks for some k, $1 \leqslant k \leqslant n = |\Omega|$, and suppose B has the following property: For some t, $1 \leqslant t \leqslant k$, there is a number ℓ such that every Γ in $\Omega\{t\}$ is contained in exactly ℓ blocks of B. Then the pair (Ω,B) is called a *design with parameters t, n, k and* ℓ or shortly a *t-(n,k,*ℓ*) design*. This definition is slightly different from the usual one where trivial cases are not included in the definition. An *automorphism group of a design* (Ω,B) is a permutation group G on Ω with the additional property $\Delta^g \in B$ for every Δ in B and every g in G.

Some further notation and symbols.

$\|m\|_p$, $\|m\|_{p'}$	:	Let m be a positive integer and p a prime. $\|m\|_p$ is the largest power of p dividing m and $\|m\|_{p'} = m/\|m\|_p$.
X^Y, $CC\ell_Y(X)$	:	X and Y are subgroups of some group G. $X^Y = CC\ell_Y(X) = \{X^y \mid y \in Y\}$.
$CC\ell_{Y:\tilde{Y}}(X)$	:	$\tilde{Y} \leq Y$ and X are subgroups of some group G. $CC\ell_{Y:\tilde{Y}}(X) = \{CC\ell_{\tilde{Y}}(X^y) \mid y \in Y\}$.
'X is G-weakly closed in Y'	:	$X \leq Y$ are subgroups of some group G. If X^g is contained in Y for some $g \in G$, then $X^g \in CC\ell_Y(X)$.
$Sy\ell_p(X)$	:	The set of all Sylow p-subgroups of the group X.
Fix(U)	:	U is a subgroup of some permutation group G on Ω. Fix U = $\{\alpha \mid \alpha \in \Omega$ and $\alpha^u = \alpha$ for all u in U$\}$.
$x(H)$, $x(H^\Delta)$, $y(H)$, $y(H^\Delta)$	:	See beginning of section 3.1.
AUT(G), AUT(D)	:	Automorphism group of the group G or design D.
$I(G) \leq AUT(G)$	:	The group of inner automorphisms of G.
PG(1,q)	:	The projective line over the field with q elements.

2.2 PRELIMINARY RESULTS

LEMMA 2.1 (WITT, [31])

Let G be a t-fold transitive permutation group on Ω and let U be a subgroup of G_Γ where $|\Gamma| = t$. Then $N_G(U)$ acts t-fold transitively on $Fix(U)$ if and only if for every $g \in G$ with $U^g \leqslant G_\Gamma$ there is some $h \in G_\Gamma$ such that $U^g = U^h$.

LEMMA 2.2

Let G be a permutation group on Ω of degree n and let t, k be integers with $1 \leqslant t \leqslant k \leqslant n$. Let B be an orbit of G in $\Omega\{k\}$ and suppose B satisfies

I: For every Γ in $\Omega\{t\}$ there is some Δ in B with $\Gamma \leqslant \Delta$, and

II: For some Δ in B, G^Δ is t-fold transitive (t-fold homogeneous) on Δ.

Then G is t-fold transitive (t-fold homogeneous) on Ω.

Proof: Let Δ be a member of B such that II holds. Let $\alpha = (\alpha_1,\ldots\alpha_t)$ be an arbitrary element of $\Omega(t)$, $\alpha^* = \{\alpha_1,\ldots\alpha_t\}$ an arbitrary element of $\Omega\{t\}$, $\beta = (\beta_1,\ldots\beta_t)$ some element in $\Delta(t)$ and $\beta^* = \{\beta_1,\ldots\beta_t\}$ some element in $\Delta\{t\}$. It suffices to show, that there is some g in G with $\alpha^g = \beta$, $(\alpha^{*g} = \beta^*)$.

The first condition (I) implies, that there is some Δ' in B containing $\alpha_1,\ldots\alpha_t$. Since B is an orbit under G there is some $h \in G$ such that $\Delta'^h = \Delta$, and so α^h is contained in $\Delta(t)$, $(\alpha^{*h}$ in $\Delta\{t\})$. By the second condition there is some k in $G_{\{\Delta\}}$ with $(\alpha^h)^k = \beta$, $((\alpha^{*h})^k = \beta^*)$. $\diamond$

LEMMA 2.3. (WIELANDT & HUPPERT, [27]).

Let G be a permutation group on Ω of degree n. Let H be a normal subgroup of G and suppose H is regular on Ω.

(i) If G is 2-fold transitive on Ω then H is elementary Abelian and n is a power of a prime.

(ii) If G is 2-fold primitive or $2\frac{1}{2}$-fold transitive on Ω, n is a power of 2 or $n = 3$.

(iii) If G is 3-fold primitive on Ω, $n = 3$ or 4.

(iv) If G is $3\frac{1}{2}$-fold transitive on Ω, $n = 4$.

LEMMA 2.4 (WIELANDT & HUPPERT, [27]).

Let G be a permutation group on Ω, $G \neq \mathrm{Sym}\,\Omega$. Let $H \neq 1$ be a normal subgroup of G, not regular on Ω.
If G is t-fold transitive, then H is $(t-\frac{1}{2})$-fold transitive on Ω.

LEMMA 2.5 (ITO,[11]).

Let G be a t-fold transitive permutation group on Ω, $G \neq \mathrm{Sym}\,\Omega$ and $t \geq 3$. Then every non-regular normal subgroup $H \neq 1$ of G is $(t-1)$-fold primitive.

LEMMA 2.6 (see HUPPERT, II 3.13 in [10]).

Let G be triply transitive on Ω. Suppose G has a solvable normal subgroup $H \neq 1$ which is not regular on Ω. Then $|\Omega| = 3$ or 4 and $G = \mathrm{Sym}\,\Omega$.

Proof: Let M be a minimal normal subgroup of G contained in H. Since H is solvable, M is elementary abelian of order p^m, transitive on Ω and hence regular, see for instance 11.5 in [29]. Therefore $|\Omega| = p^m = 3$ or 2^m by lemma 2.3. Suppose $|\Omega| = 2^m$. Similarly, let F be a minimal normal subgroup of G_α contained in $H_\alpha \neq 1$. Then F is elementary abelian of order q^r, transitive on $\Omega\setminus\{\alpha\}$ and hence regular on $\Omega\setminus\{\alpha\}$, $|F| = q^r = 2^m - 1$.

The group $M \circ F$ is therefore sharply doubly transitive on Ω, i.e. $M \circ F$ is a Frobenius group with kernel M and complement F. By Burnside's theorem (V.8.7 in [10]), F is cyclic. Thus $|F| = q$. $G_{\alpha,\beta}$ is a complement of F in $G_\alpha = F \circ G_{\alpha,\beta}$ and $G_{\alpha,\beta}$ is also cyclic as a group of automorphisms of F. Hence $G_{\alpha,\beta}$ is regular on $\Omega \setminus \{\alpha,\beta\}$ and this implies that G is sharply 3-fold transitive on Ω, $|G| = 2^m \cdot (2^m-1) \cdot (2^m-2)$ where $2^m-1 = q$ is a prime.

Since $|G|$ is divisible by 3, we either have $2^m-1 = 3$ and $|\Omega| = 4$, or otherwise 3 divides 2^m-2. We show that the latter can not happen.

Suppose $3 | 2^m-2$ and let $S \neq 1$ be the maximal subgroup of $G_{\alpha,\beta}$ with order prime to 3. (S exists since $G_{\alpha,\beta}$ is cyclic of even order). The group $(M \cdot F) \cdot S$ is a normal subgroup of G since $G = G_{\alpha,\beta} \cdot (M \cdot F)$ and $S \triangleleft G_{\alpha,\beta}$. The group $(M \cdot F) \cdot S$ has order prime to 3 by this construction. Since S has even order, there is some involution i in S with $i = (\alpha)(\beta)(\gamma\delta)\ldots$. Conjugating i by an appropriate $g \in G$ also $i' = (\delta)(\beta\gamma)(\ldots)\ldots$ is contained in $(M \cdot F) \cdot S$. But this implied that $i \cdot i' = (\beta\gamma\delta)\ldots$ was contained in $(M \cdot F) \circ S$, a contradiction. Hence $|\Omega| = 4$. $\diamond$

With elementary tools one can prove the following lemma on the generosity of normal subgroups.

<u>LEMMA 2.7</u>

Let G be a t-fold transitive permutation group on Ω of degree n and $H \neq 1$ be a normal subgroup of G with $t \geq 3$ and $n > 3$. Let Γ' be in $\Omega\{t-1\}$ and x the number of $H_{\Gamma'}$-orbits on $\Omega \setminus \Gamma'$.

Suppose $(x, (t-1)!) = 1$. Then H is $(t-1)$-fold generously transitive on Ω.

<u>Proof:</u> We can assume $G \neq \mathrm{Sym}\,\Omega$. Also, if G is triply transitive and H regular, then $n = 2^m$ (see 2.3) and $x = n-2$, i.e. $(x, t-1) = 2$. Therefore

assume by 2.4 that H is $(t-1)$-fold transitive. Let Γ' be any member of $\Omega\{t-1\}$ and let $\{T_1,\ldots,T_x\}$ be the set of all $H_{\Gamma'}$-orbits on $\Omega\setminus\Gamma'$. Since $H_{\Gamma'} \trianglelefteq H_{\{\Gamma'\}} \trianglelefteq G_{\{\Gamma'\}}$, $H_{\{\Gamma'\}}$ and $G_{\{\Gamma'\}}$ induce permutation groups $\overline{H}$ and $\overline{G}$ on $\{T_1,\ldots,T_x\}$. The group $\overline{G}$ is transitive on this set since $G_{\{\Gamma'\}}$ is transitive on $\Omega\setminus\Gamma'$ and since $\overline{H} \trianglelefteq \overline{G}$, either $\overline{H} = 1$ or $\overline{H}$ is half transitive on $\{T_1,\ldots,T_x\}$. Assume $\overline{H} \neq 1$ and let $\overline{T} = \{T_1,\ldots T_s\}$ be an orbit of $\overline{H}$ on $\{T_1,\ldots,T_x\}$. Then s divides x and also of course $|\overline{H}|$. Since $H_{\{\Gamma'\}}/H_{\Gamma'} \cong \mathrm{Sym}(\Gamma')$, $|\overline{H}|$ divides $|\mathrm{Sym}(\Gamma')| = (t-1)!$. Hence s divides $(x,(t-1)!) = 1$ and so $s = 1$ and $|\overline{H}| = 1$, i.e. $H_{\Gamma'}$ and $H_{\{\Gamma'\}}$ have the same orbits on $\Omega\setminus\Gamma'$ for any Γ' in $\Omega\{t-1\}$.

Let Γ be in $\Omega\{t\}$. We have to show that $H_{\{\Gamma\}}$ acts on Γ like $\mathrm{Sym}(\Gamma)$. Let γ be a point in Γ and put $\Gamma' = \Gamma\setminus\{\gamma\}$. Since by the above argument $H_{\Gamma'}$ is transitive on the $H_{\{\Gamma'\}}$-orbit containing γ, we have $H_{\{\Gamma'\}} = H_{\Gamma'}\cdot H_{\{\Gamma'\},\gamma}$. This means that $H_{\{\Gamma'\},\gamma}$ operates on Γ' like Sym Γ' and fixes Γ as a set. Since $|\Gamma| = t \geq 3$, Sym Γ is generated by $\{\mathrm{Sym}(\Gamma\setminus\gamma) \mid \gamma \in \Gamma\}$ and therefore $H_{\{\Gamma\}}$ acts on Γ like Sym Γ. Since Γ was arbitrary in $\Omega\{t\}$, H is $(t-1)$-fold generously transitive on Ω. $\diamond$

The following main result on the generosity of normal subgroups has been proved by various authors in the case of quadruply transitive groups. See for instance Ito, lemma b in [12], Saxl, lemma 1 in [20], or Neumann, theorem 9.1 in [19].

GENEROSITY THEOREM FOR NORMAL SUBGROUPS 2.8

Let G be a t-fold transitive permutation group on Ω of degree n, $G \neq \mathrm{Sym}\ \Omega$ and $t \geq 2$. Suppose $H \neq 1$ is a normal subgroup of G.

(i) If $t = 2$, H is generously transitive if and only if H has even order. In particular H is generously transitive if G contains no regular normal subgroup.

(ii) If $t = 3$, H is generously 2-fold transitive except if H is regular or if $PSL(2,q) \leq H \leq G \leq P\Gamma L(2,q)$ in their usual representation on the projective line, $q + 1 = n \equiv 0 \bmod 4$.

(iii) If $t \geq 4$, H is generously $(t-1)$-fold transitive.

Proof: Let Γ be a member of $\Omega\{t\}$. Since G is t-fold transitive, $G_{\{\Gamma\}}$ acts on Γ like Sym Γ. To show that H is $(t-1)$-fold generously transitive, we have to prove that the same is true for $H_{\{\Gamma\}}$. Since $H_{\{\Gamma\}}$ is normal in $G_{\{\Gamma\}}$, it suffices to show, that $H_{\{\Gamma\}}$ contains an element h which acts on Γ like a transposition.

On the other hand, G is also t-fold homogeneous on Ω. This means that $H_{\{\Gamma\}}$ is conjugate in G to $H_{\{\Gamma^*\}}$ for any Γ^* in $\Omega\{t\}$. So one may choose a particular Γ to show the required property.

Let therefore Δ be a subset of Ω with $|\Delta| = t - 2$. Then H is $(t-1)$-fold generously transitive on Ω if and only if H_Δ has even order. For assume that $|H_\Delta|$ is even. Then there is some element h in H_Δ interchanging two points α and β in $\Omega \setminus \Delta$. Take $\Gamma = \Delta \cup \{\alpha,\beta\}$ and h acts on Γ like a transpostion. The converse implication is trivial.

(a) If $t = 2$, $\Delta = \emptyset$ and (i) will be proved if we can show, that H has even order if G contains no regular normal subgroup. Let M be a minimal normal subgroup of G contained in H. By a result of Burnside (page 202 in [5]) M is simple and by the Feit-Thompson Theorem H has even order.

(b) Now let $t = 3$. Then by lemmata 2.3 and 2.4 H is either regular or doubly transitive. If H is regular, $H_\alpha = 1$ and by the above remark H is certainly not generously doubly transitive. So suppose H is doubly transitive. Put $\{\alpha\} = \Delta$. Then we have to show that H_α has even order if G is not contained in $P\Gamma L(2,n-1)$. Since H_α is transitive on $\Omega \setminus \{\alpha\}$, $|H_\alpha|$ is even if n is odd for $n-1$ divides $|H_\alpha|$.

Hence assume n is even and H_α has odd order. Then a theorem of Bender, lemma 2.12, applies and H is either solvable or otherwise contains $PSL(2,q)$ as a normal subgroup for some prime power q. Lemma 2.6 implies that H cannot be solvable unless $G = Sym(3)$ or $Sym(4)$ which contradicts our assumption. Hence $PSL(2,q) \trianglelefteq H$ and using Burnside's results (page 202 and chapter 153 in [5]) one concludes that $PSL(2,q)$ is characteristic in H and hence normal in G. Therefore $PSL(2,q)$ is doubly transitive on Ω and checking through Dickson's list of subgroups of $PSL(2,q)$, (see for instance 11.8.27 in [10]), we find that $PSL(2,q)$ acts on the projective line $PG(1,q)$ in its usual representation. Hence $q + 1 = n$ and $PSL(2,n-1) \leqslant H \leqslant G \leqslant P\Gamma L(2,n-1)$.

(c) Now suppose $t \geqslant 4$. Since $G \neq Sym\, \Omega$, H is at least $(t-1)$-fold transitive on Ω. Let Γ' be a subset of Ω with $|\Gamma'| = t - 3$. Then $G_{\Gamma'}$ is triply transitive on $\Omega \setminus \Gamma'$ and at the beginning of the proof we saw that H is $(t-1)$ fold generously transitive on Ω if and only if $H_{\Gamma'}$ is doubly generously transitive on $\Omega \setminus \Gamma'$. This is the case if $H_{\Gamma'}$ is not one of the exceptions in (ii). But $H_{\Gamma'}$ cannot be regular on $\Omega \setminus \Gamma'$, because then H could only be $t-3+1 = t-2$ transitive on Ω. Similarly if $PSL(2,n-|\Gamma'|-1) \leqslant H_{\Gamma'} \leqslant G_{\Gamma'} \leqslant P\Gamma L(2,n-|\Gamma'|-1)$, as permutation groups on $\Omega \setminus \Gamma'$, let Γ^* be a subset of Γ' with $|\Gamma^*| = |\Gamma'| - 1$. Then G_{Γ^*} and H_{Γ^*} are transitive extensions of $G_{\Gamma'}$ and $H_{\Gamma'}$ on $\Omega \setminus \Gamma^*$. Since we can assume $n > t + 2$, $n-|\Gamma'|-1 > 4$ and therefore by lemma 2.10, $G_{\Gamma^*} = M_{11}$, the Mathieu group on 11 points. Since M_{11} is simple, also $M_{11} = H_{\Gamma^*}$ and H_{Γ^*} is quadruply transitive on $\Omega \setminus \Gamma^*$. In particular H_{Γ^*} is 3-fold generously transitive on $\Omega\setminus\Gamma^*$ and hence $H_{\Gamma'}$ is doubly generously transitive on $\Omega\setminus\Gamma'$. $\diamond$

LEMMA 2.9 (HERING, [8])

Let G be a doubly transitive permutation group on Ω of degree n. Suppose the stabilizer in G of two distinct points has even order and the stabilizer of three distinct points has odd order.

Then either $PSL(2,q) \trianglelefteq G \leqslant P\Gamma L(2,q)$ with $n = q + 1$ or else $G = Alt(6)$, $n = 6$.

The one-dimensional projective groups belong to an important class of doubly transitive permutation groups:

Definition: A doubly transitive permutation group G on Ω is said to be of *Suzuki type* if the stabilizer in G of a point α contains a characteristic p-subgroup which is regular on $\Omega\setminus\{\alpha\}$.

The known groups of Suzuki type are the sharply triply transitive groups, the groups $PSL(2,q)$, $PSU(3,q^2)$ and $PGU(3,q^2)$, the Suzuki groups $Sz(2^{2r+1})$ and the Ree groups $R(3^{2r+1})$.

Suzuki type groups have the following extension property:

LEMMA 2.10 (SUZUKI, [22])

If G is a group of Suzuki type on Ω having no regular normal subgroup then G has no transitive extension unless $|\Omega| = 5$, $G \geqslant Alt(5)$ or $|\Omega| = 10$ and G is a sharply triply transitive group with extension M_{11}, the Mathieu group on eleven points.

A result of Hering, Kantor, Seitz and Shult classifies all Suzuki type groups:

LEMMA 2.11 ([9])

Let G be a finite doubly transitive permutation group on Ω. Suppose that, for $\alpha \in \Omega$, G_α has a normal subgroup Q regular on $\Omega \setminus \{\alpha\}$. Then G has a normal subgroup M such that $M \leqslant G \leqslant \mathrm{Aut}\, M$ and M acts on Ω as one of the following groups in its usual doubly transitive representation: a sharply doubly transitive group, $PSL(2,q)$, $Sz(2^{2r+1})$, $PSU(3,q)$ or a group of Ree type.

LEMMA 2.12 (BENDER [4])

Let G be a doubly transitive permutation group on Ω of degree n. Suppose the stabilizer in G of one point has odd order. Then G is either solvable or else G contains a normal subgroup isomorphic to $PSL(2,q)$.

LEMMA 2.13 (MARTINEAU [17], THOMPSON [23])

Let G be a non-abelian finite simple group and assume $|G|$ is not divisible by 3. Then G is isomorphic to a Suzuki group $Sz(2^{2r+1})$.

CHAPTER III. HOMOGENEITY AND GENEROSITY

As an introduction to this chapter I would like to describe some of the basic concepts involved in the normal subgroup problem.

We consider a finite t-fold transitive permutation group G with normal subgroup H and ask: Is H t-fold transitive as well? From the definitions it is clear that the answer is positive if and only if H is both t-fold homogeneous and (t-1)-fold generously transitive.

The only known examples where G is at least triply transitive but H only (t-1)-fold transitive are provided by the one-dimensional projective linear groups over finite fields $GF(q)$: $PGL(2,q) = G$ is triply transitive on the projective line $PG_1(q)$ and $PSL(2,q) = H$ is only doubly transitive if q is an odd prime power. Hence H fails to be both triply homogeneous and doubly generously transitive on $\Omega = PG_1(q)$. It is not difficult to see that H is triply homogeneous if and only if $q \equiv 3 \bmod 4$ and doubly generously transitive if and only if $q \equiv 1 \bmod 4$. In either case H has exactly 2 orbits on the set of all 3-tupels $\Omega(3)$.

The Generosity theorem 2.8 now implies that this is the only example of an at least triply transitive group G where H fails to be (t-1)-fold generously transitive. So the question really becomes: Is H t-fold homogeneous? Or equivalently, if Γ in $\Omega\{t\}$ is uniquely determined by $H_{\{\Gamma\}}$: Is the set of subgroups $\{H_{\{\Gamma\}} | \Gamma \in \Omega\{t\}\}$ a class of H-conjugate groups?

The aim of this chapter mainly is to give a variation of the homogeneity – generosity concept. A key observation for this is lemma 2.2. Suppose Δ is a subset of Ω of size $k \geqslant t$ such that G induces a t-fold transitive group G^{Δ} on Δ. Call such a subset 'inductive'. Then clearly the set of all G-images Δ^G has the property that every Γ in $\Omega\{t\}$ is contained in at least one block of Δ^G, i.e. Δ^G 'covers' $\Omega\{t\}$. Lemma 2.2 now gives a transitivity criterion for H : If (I) Δ^H also covers $\Omega\{t\}$ and if (II) H^Δ is t-fold transitive on Δ, then H is t-fold transitive on Ω. The covering property (I) is closely related to the homogeneity of H, in fact, (I) holds if H is t-fold homogeneous and the converse is true if Δ has size $k = t$. Similarly, in this case (II) holds if and only if H is (t-1)-fold generously transitive. So it seems natural to consider the covering property as a generalisation of homogeneity and property (II) as a generalized generosity property. Lemma 2.2 now implies that H is t-fold transitive on Ω if there exists some inductive subset Δ such that H is homogeneous and generous in this wider sense. And, of course, it is the choice of a suitable Δ that makes this lemma useful for our problem. Apart from the obvious possibilities $\Delta = \Gamma \in \Omega\{t\}$ and $\Delta = \Omega$ the only known general way of producing inductive subsets uses certain subgroups: call a subgroup U 'inductive' if the set of points fixed by U is an inductive set. The significance of Witt's Lemma is that weakly closed subgroups of G_Γ are inductive.

But it should be mentioned that inductive subsets do not necessarily originate from inductive subgroups. If, for instance, G is an automorphism group of a t-$(|\Omega|, k, 1)$ design, the blocks of this design are inductive without necessarily being sets of points fixed by some subgroups of G.

Throughout this thesis inductive sets Δ arise from Sylow subgroups of G_Γ or H_Γ. The discussion so far then suggests studying two questions: Does Δ^H cover $\Omega\{t\}$? and Is H^Δ also t-fold transitive on Δ?

We shall see that the first question refers to the fusion of $Syl_p(H_\Gamma)$ in H and in this chapter we use simple arguments to show that

$\{Syl_p(H_\Gamma) \mid \Gamma \in \Omega\{t\}\}$ is a class of

H-conjugate subgroups if p does not divide $|\Omega| - t + 1$. The second question can be dealt with inductively: $H^{\Delta} \lhd G^{\Delta}$ are groups of smaller degree reflecting many properties of the original situation $H \lhd G$. In H^{Δ} $Syl_p((H^{\Delta})_{\Gamma}) = \{1\}$ and this fact can be used in counting arguments. More importantly: For $p = 2$ and $t \geqslant 2$ G^{Δ} is a known group.

3.1 INDUCTIVE SUBSETS AND SUBGROUPS

Throughout this chapter we shall use the following notation: G usually denotes a t-fold transitive permutation group of degree n acting on the finite set Ω for some fixed integer $t \geqslant 1$. H is a non-identity normal subgroup of G. The group G acts canonically on both $\Omega\{t\}$ and $\Omega(t)$ and is transitive on these sets. $\Omega(t)$ is a disjoint union of H-orbits O_i, $\Omega(t) = O_1 \cup \ldots \cup O_x$, and similarly $\Omega\{t\}$ is split up into H-orbits U_i, $\Omega\{t\} = U_1 \cup \ldots \cup U_y$.

We put $x = x(H)$ and $y = y(H)$ and bear in mind that both $x(H)$ and $y(H)$ refer to the given value of t. So $x(H) = 1$ is the same as saying H is t-fold transitive and $y(H) = 1$ means that H is t-fold homogeneous. However, to avoid repeated consideration of special cases, $x(H)$ and $y(H)$ shall have no meaning if G is triply transitive and H is regular.

Similarly, if Δ is a subset of Ω with $|\Delta| \geqslant t$ such that $G_{\{\Delta\}}$ acts t-fold transitively on Δ, let U_i^{Δ}, $i = 1, \ldots, y'$, and O_i^{Δ}, $i = 1, \ldots, x'$ be the orbits of $H_{\{\Delta\}}$ on $\Delta\{t\}$ and $\Delta(t)$ respectively. In analogy with the above notation we define $x(H^{\Delta}) := x'$ and $y(H^{\Delta}) := y'$.

The results of Wielandt and Huppert (2.3 and 2.4) imply that if $G \neq Sym(\Omega)$ and if Γ' is a member of $\Omega\{t-1\}$, then $x(H)$ is the number of orbits of $H_{\Gamma'}$ on the remaining points $\Omega \setminus \Gamma'$. Since all these orbits have equal length, $x(H)$ is a divisor of $n-t+1$.

PROPOSITION 3.1

Let t,k be integers with $1 \leq t \leq k \leq n = |\Omega|$, let B be some G-orbit in $\Omega\{k\}$, $\emptyset \neq B \leq \Omega\{k\}$ and let $B_1, \ldots, B_z$ be the orbits of H on B, $B = B_1 \cup \ldots \cup B_z$.

If G is t-fold homogeneous on Ω and if for some Γ in $\Omega\{t\}$ $H_{\{\Gamma\}}$ is transitive on the set of all blocks in B containing Γ, then G^Δ is t-fold homogeneous on Δ for every Δ in B and

$$y(H) = y(H^\Delta) \cdot z .$$

If G is t-fold transitive on Ω and if H_Γ is transitive on the set of all blocks in B containing Γ, then G^Δ is t-fold transitive on for every Δ in B and

$$x(H) = x(H^\Delta) \cdot z .$$

Proof: Let Γ_1, Γ_2 be members of $\Omega\{t\}$ and define $\Gamma_1 \sim \Gamma_2$ if there is some B_i, $i = 1, \ldots z$, and blocks Δ_1, Δ_2 both contained in B_i with $\Gamma_1 \leq \Delta_1$, $\Gamma_2 \leq \Delta_2$. This relation is reflexive since $B \neq \emptyset$ and G is t-fold homogeneous, and is also transitive: Let $\Gamma_1 \leq \Delta_1$, $\Gamma_2 \leq \Delta_2$ with $\Delta_1, \Delta_2 \in B_1$ and let $\Gamma_2 \leq \Delta_2'$, $\Gamma_3 \leq \Delta_3$ with Δ_2', $\Delta_3 \in B_2$. Then Δ_2, $\Delta_2' \geq \Gamma_2$ and by assumption Δ_2 and Δ_2' belong to the same H-orbit, i.e. $B_1 = B_2$ and so $\Gamma_1 \sim \Gamma_3$. Therefore $\sim$ is an equivalence relation splitting $\Omega\{t\}$ into precisely z equivalence classes. H fixes each of these classes while G permutes them transitively.

Consider for any Δ in B the H^Δ -orbits U_i^Δ and 0_i^Δ on $\Delta\{t\}$ and $\Delta(t)$ respectively: $\Delta\{t\} = U_1^\Delta \cup \ldots \cup U_{y'}^\Delta$ and $\Delta(t) = 0_1^\Delta \cup \ldots \cup 0_{x'}^\Delta$. Then clearly x' and y' are independent of the choice of Δ since G acts transitively on B.

Now define $\Gamma_1 \sim_u \Gamma_2$ if there is some Δ in B and some $h \in H$ such that Γ_1^h, $\Gamma_2 \leq \Delta$ are contained in the same U^Δ_i.

If $\Gamma_1 \sim u\ \Gamma_2$, i.e. $\Gamma_1^h, \Gamma_2 \leq \Delta$ and $\Gamma_1^h = \Gamma_2^f$ for some f in $H_{\{\Delta\}}$, then $\Gamma_2^{h^{-1}}, \Gamma_1 \leq \Delta^{h^{-1}} = \bar{\Delta}$ and $\Gamma_2^{h^{-1}} = \Gamma_1^{hf^{-1}h^{-1}}$ with $h f^{-1} h^{-1}$ in $H_{\{\bar{\Delta}\}}$. So $\sim_u$ is symmetric and also transitive: Let $\Gamma_1 \sim u\ \Gamma_2$ and $\Gamma_2 \sim u\ \Gamma_3$, i.e. Γ_1^h, $\Gamma_2 \leq \Delta$, $\Gamma_1^h = \Gamma_2^f$, $f \in H_{\{\Delta\}}$ and $\Gamma_2^{\bar{h}}, \Gamma_3 \leq \bar{\Delta}$, $\Gamma_2^{\bar{h}} = \Gamma_3^{f'}$, $f' \in H_{\{\bar{\Delta}\}}$. Then $\Gamma_1^{hf^{-1}\bar{h}}, \Gamma_3 \leq \bar{\Delta}$ and $\Gamma_1^{hf^{-1}\bar{h}} = \Gamma_3^{f'}$, that is $\Gamma_1 \sim u\ \Gamma_3$.

Hence $\sim u$ is an equivalence relation on $\Omega\{t\}$ and clearly $\sim u$ implies $\sim$.

Suppose $\Gamma_1 \sim u\ \Gamma_2$ are both contained in some $\bar{\Delta}$. Let $\Gamma_1^h, \Gamma_2 \leq \Delta$ and $\Gamma_1^h = \Gamma_2^f$ with $f \in H_{\{\Delta\}}$. Then both $\bar{\Delta}^{hf^{-1}}$ and $\bar{\Delta}$ contain Γ_2 and so by assumption there is some $\bar{h}$ in $H_{\{\Gamma_2\}}$ mapping $\bar{\Delta}^{hf^{-1}}$ onto $\bar{\Delta}$. Hence $\bar{f} = h \cdot f^{-1}\bar{h}$ belongs to $H_{\{\bar{\Delta}\}}$ and $\Gamma_1^{\bar{f}} = \Gamma_2$, i.e. Γ_1 and Γ_2 belong to the same $U_i^{\bar{\Delta}}$. From this remark we conclude that every $\sim$ class splits into y' $\sim u$ classes and altogether we obtain $z \cdot y'$ $\sim u$-classes. Obviously $\Gamma_1 \sim u \Gamma_2$ if and only if Γ_1 and Γ_2 belong to the same H-orbit in $\Omega\{t\}$.
So we have $y(H) = z \cdot y'$. Evaluate this equation for $H = G$ and $1 = y(G) = 1 \cdot y'$ shows that $y' = 1$ and G^Δ acts t-fold homogeneously on Δ for every Δ in B. So put $y' = y(H^\Delta)$ and the first part of 3.1 is proved.

Similarly, if (Γ_1) and (Γ_2) are members of $\Omega(t)$ with underlying sets Γ_1 and Γ_2, define $(\Gamma_1) \sim o (\Gamma_2)$ if there is some Δ in B and some h in H such that $\Gamma_1^h, \Gamma_2 \leq \Delta$ and (Γ_1), (Γ_2) are contained in the same O_i^Δ. By the same arguments as above we get $x(H) = z \cdot x(H^\Delta)$ and G^Δ is t-fold transitive on Δ. $\diamond$

<u>LEMMA 3.2</u>

Let Γ be a subset of Ω with $|\Gamma| = t$, let p be a prime and let S' be in $Syl_p(G_\Gamma)$, $S = S' \cap H$. Put $N := N_G(S')$, $N_H := N_H(S')$, $M := N_G(S)$ and $M_H := N_H(S)$. Let $\Delta' = Fix\ S'$ and $\Delta = Fix\ S$.

Then $G^{\Delta'} \cong N^{\Delta'}$, $G^{\Delta} \cong M^{\Delta}$, $H^{\Delta'} \cong N_H^{\Delta'}$, $H^{\Delta} \cong M_H^{\Delta}$ and all these isomorphisms are permutation isomorphisms. In particular $(G^{\Delta'})_{\Gamma}$, $(H^{\Delta'})_{\Gamma}$ and $(H^{\Delta})_{\Gamma}$ are p'-groups.

$H^{\Delta'}$ and H^{Δ} are permutation isomorphic to normal subgroups of $G^{\Delta'}$ and G^{Δ} respectively and hence we write by abuse of notation $H^{\Delta'} \trianglelefteq G^{\Delta'}$ and $H^{\Delta} \trianglelefteq G^{\Delta}$.

<u>Proof:</u> We use the Frattini argument. $N^{\Delta'} = N/N_{\Delta'}$ since $N_{\{\Delta'\}} = N$. Since S' is also a Sylow p-subgroup of $G_{\Delta'}$, we have $G_{\{\Delta'\}} = N_{G\{\Delta'\}}(S') \cdot G_{\Delta'}$, i.e. $G_{\{\Delta'\}} = N \cdot G_{\{\Delta'\}}$. Therefore $G^{\Delta'} = G_{\{\Delta'\}}/G_{\Delta'} = N \cdot G_{\Delta'}/G_{\Delta'} \cong N/N \cap G_{\Delta'} = N/N_{\Delta'} = N^{\Delta'}$. Obviously the isomorphism involved is a permutation isomorphism i.e. $N^{\Delta'}$ and $G^{\Delta'}$ act on Δ' in the same way. Similarly one proves $H^{\Delta} \cong M_H^{\Delta}$ and $G^{\Delta} \cong M^{\Delta}$. To prove $N_H^{\Delta'} \cong H^{\Delta'}$ let $A = H \cdot S'$. Then $A^{\Delta'} = A_{\{\Delta'\}}/A_{\Delta'} = (H \cdot S')_{\{\Delta'\}}/(H \cdot S')_{\Delta'} = (H_{\{\Delta'\}} \cdot S')/(H_{\Delta'} \cdot S') = (H_{\{\Delta'\}} \cdot (H_{\Delta'} \cdot S'))/H_{\Delta'} \cdot S') \cong H_{\{\Delta'\}}/(H_{\{\Delta'\}} \cap H_{\Delta'} \cdot S) = H_{\{\Delta'\}}/H_{\Delta'} = H^{\Delta'}$. Now, since S' is a Sylow p-subgroup of $A_{\Delta'}$, $A_{\{\Delta'\}} = N_{A_{\{\Delta'\}}}(S') \cdot A_{\Delta'} = N_H(S') \cdot S' \cdot A_{\Delta'} = \quad = N_H(S') \cdot A_{\Delta'}$ and therefore $A^{\Delta'} = N_H(S') \cdot A_{\Delta'}/A_{\Delta'} \cong N_H(S')/N_H(S') \cap A_{\Delta'} = N_H{}^{\Delta'}$. Together we have $N_H^{\Delta'} \cong H^{\Delta'}$.

Since $(G^{\Delta'})_{\Gamma} = N_{\Gamma}/N_{\Delta'}$ and S' is a Sylow p-subgroup of both N_{Γ} and $N_{\Delta'}$, $(G^{\Delta'})_{\Gamma}$ is not divisible by p. Similarly $(H^{\Delta})_{\Gamma}$ is a p'-group.
For the remainder note that $H^{\Delta'} = H_{\{\Delta'\}}/(H_{\{\Delta'\}} \cap G_{\Delta'}) \cong (H_{\{\Delta'\}} \cdot G_{\Delta'})/G_{\Delta'} \trianglelefteq G_{\{\Delta'\}}/G_{\Delta'} = G^{\Delta'}$. In the same way we obtain $H^{\Delta} \trianglelefteq G^{\Delta}$. $\Diamond$

<u>Definition:</u> Let Δ be a subset of Ω of size k, $t \leq k \leq n$, and let H be a subgroup of G. Then Δ is said to be *inductive with respect to H* if the following implication holds: If H is t'-fold transitive on Ω, $1 \leq t' \leq t$, then $H_{\{\Delta\}}$ (or H^{Δ}) acts t'-fold transitively on Δ.

A subgroup U of G is called *inductive with respect to H* if Fix U is inductive with respect to H.

PROPOSITION 3.3

Let G be a t-fold transitive permutation group on Ω, Γ a subset of Ω with $|\Gamma| = t$. Let S' be a Sylow p-subgroup of G_Γ for some prime p and let H be a normal subgroup of G.

Then (i) S' and $S = S' \cap H$ are inductive with respect to G and (ii) S' and S are inductive with respect to H.

Proof: (i) Let $\Delta' = Fix(S')$, $\Delta = Fix(S)$ and put $B' = \Delta'^G$ and $B = \Delta^G$. Let Γ be in $\Omega\{t\}$ and suppose $\Gamma \leqslant \Delta'_1, \Delta'_2$, $\Gamma \leqslant \Delta_1, \Delta_2$ where Δ'_i are members of B' and Δ_i members of B. Then $\Delta'_i = Fix(S'_i)$ and $\Delta_i = Fix(S_i)$ for some conjugates of S' and S respectively. Since S'_i are contained in $Syl_p(G_\Gamma)$, they are conjugate in G_Γ. This implies $\Delta'^s_1 = \Delta'_2$ for some s in G_Γ and so G_Γ is transitive on the blocks of B' containing Γ. For a similar reason G_Γ is also transitive on the blocks of B containing Γ. Hence proposition 3.1 applies (taking H = G there) and $G^{\Delta'}$, G^Δ are t-fold transitive on Δ', Δ respectively.

(ii) First show that S is inductive with respect to H. Clearly this is true if S = 1 and $\Delta = Fix(S) = \Omega$. Hence assume $S \neq 1$ and in particular H is not regular. Then by 2.4 either $G = Sym(\Omega)$, $t = |\Omega|$, or else H is at least (t-1)-fold transitive on Ω. If $G = Sym\,\Omega$, and $t = |\Omega|$, then S = 1, contrary to our assumption. So assume the latter. If H is actually t-fold transitive it follows from part (i) applied to H that S is inductive with respect to H. Hence let H be exactly (t-1)-fold transitive and prove that H^Δ acts (t-1)-fold transitively on $Fix(S) = \Delta$.

Let $\Gamma' \subset \Gamma$, $|\Gamma'| = t - 1$. If S is also in $Syl_p(H_{\Gamma'})$, again the property required follows by (i), (replacing G by H and t by t-1). Hence assume that S is not a Sylow p-subgroup of $H_{\Gamma'}$. Let T be in $Syl_p(H_{\Gamma'})$ containing S . Then T fixes all points of Γ' but no further point in Δ (in fact not even in Ω). The same is true for every subgroup U, $S < U \leq T$, and hence $|\Delta| = (t-1) + \lambda\cdot p$, $\lambda \geq 1$. Since $U := N_T(S) \neq S$, we obtain that there are nontrivial elements in $U^\Delta \leq H^\Delta$ fixing t-1 points.

By part (i) we already know that G^Δ is t-fold transitive on Δ and by 3.2 $H^\Delta \trianglelefteq G^\Delta$. So by 2.4 and 2.3 H^Δ is either (a) regular, (b) (t-1)-fold transitive or (c) $G^\Delta = \mathrm{Sym}\,\Delta$. By the above remark H^Δ cannot be regular, so it remains to show that in case (c) H^Δ is still (t-1)-fold transitive, i.e. $(t-1) \geq |\Delta| - 2$. But this is also clear since $\lambda\cdot p \geq 2$. Thus S is inductive with respect to H.

Finally we prove that S' is inductive with respect to H. Suppose H is t'-fold transitive on Ω. Then the same is true for $A := H\cdot S'$. Let Γ' be a subset of Γ with $|\Gamma'| = t'$. If S' is a Sylow p-subgroup of $A_{\Gamma'}$, then $N_A(S')$ is t'-fold transitive on $\Delta' = \mathrm{Fix}(S')$ by part (i) (putting $G = A$, $t = t'$). Obviously $N_A(S') = N_H(S')\cdot S'$ and since S' fixes all points of Δ', $N_H(S')$ acts on Δ' in the same way as $N_A(S')$ does. Therefore S' is inductive with respect to H if S' is a Sylow p-subgroup of $A_{\Gamma'}$. This is true in particular if H is regular since then $A_{\Gamma'} = H_{\Gamma'}\cdot S'$ $= S'$, or if H is $t' = t$-fold transitive because then $\Gamma' = \Gamma$ and $S' \leq A_{\Gamma'} = A_\Gamma \leq G$ is even in $\mathrm{Syl}_p(G_\Gamma)$, and also if $G = \mathrm{Sym}\,\Omega$, $t' = n-2$.

For the remainder of the proof therefore assume that H is (t-1)-fold transitive and that S' is not a Sylow p-subgroup of $A_{\Gamma'}$, where $\Gamma' \subset \Gamma \subset \Delta'$ and $|\Gamma'| = t-1$. Let T be in $\mathrm{Syl}_p(A_{\Gamma'})$ containing S'. Again, $\mathrm{Fix}(T) = \Gamma'$, $|\Delta'| = |\Gamma'| + \lambda\cdot p$, $(\lambda \geq 1)$, and $N_T(S')$ contains an element fixing Γ' point-wise and Δ' setwise but not pointwise. Thus $N_A(S')$ acts neither trivially nor regularly on Δ' and the same is true for $N_H(S')$ since $N_A(S') = N_H(S')\cdot S'$. Therefore by 3.2 $1 \neq H^{\Delta'}$ is a normal, non-regular subgroup of $G^{\Delta'}$ and as in part (i) of the proof $H^{\Delta'}$ is (t-1)-fold transitive on Δ' and thus S' is inductive with respect to H. $\diamond$

Proposition 3.3 partly overlaps with Witt's lemma and also with a result of Livingstone and Wagner ([15], proof of theorem 3).

Definition: Let U be a subgroup of G fixing $k \geq t$ points. Put $B :=$ $(\text{Fix } U)^G = \{\text{Fix } U^g | g \in G\} \leq \Omega \{k\}$ and let $B' \leq B$ be an orbit of H on B.

Define $D(U) := (\Omega, B)$, $D_H(U) := (\Omega, B')$. Let Γ be a subset of Fix U with $|\Gamma| = t$. If U is a Sylow p-subgroup of G_Γ for some prime p, we also write $D(U) =: D(p,G)$ and $D_H(U) =: D_H(p,G)$. Similarly, if U is a Sylow p-subgroup of H_Γ, put $D(U) =: D(p,H)$ and $D_H(U) =: D_H(p,H)$.

As an immediate consequence of this definition we have the following lemma:

LEMMA 3.4

Let G be t-fold transitive on Ω and H be a normal subgroup of G which is t'-fold transitive on Ω, $1 \leq t' \leq t$. Let Γ be a subset of Ω and let p be some prime.

Put Δ = Fix S for some S in $\mathrm{Syl}_p(H_\Gamma)$ and Δ' = Fix S' for some S' in $\mathrm{Syl}_p(G_\Gamma)$. Then we have:

(i) D(p,G) is a t-(n,k',ℓ) design where $k' = |\Delta'|$ and $\ell = |G_\Gamma| / |G_\Gamma \cap G_{\{\Delta'\}}|$.

(ii) D(p,H) is a t-(n,k,ℓ) design where $k = |\Delta|$ and $\ell = |G_\Gamma| / |G_\Gamma \cap G_{\{\Delta\}}|$

G is a group of automorphisms of both D(p,G) and D(p,H), transitive on blocks. If Δ' and Δ are blocks of D(p,G) and D(p,H) respectively, then $G^{\Delta'}$ is t-fold transitive on Δ' and G^{Δ} is t-fold transitive on Δ.

(iii) $D_H(p,G)$ is a t'-(n,k',ℓ) design where $k' = |\Delta'|$ and $\ell = |H_{\Gamma'}|/|H_{\Gamma'} \cap H_{\{\Delta'\}}|$, Γ' a subset of Γ with $|\Gamma'| = t'$.

(iv) $D_H(p,H)$ is a t'-(n,k,ℓ) design where $k = |\Delta|$ and $\ell = |H_{\Gamma'}| / |H_{\Gamma'} \cap H_{\{\Delta\}}|$.

H is a group of automorphisms of both $D_H(p,G)$ and $D_H(p,H)$, transitive on blocks. If Δ' and Δ are blocks of $D_H(p,G)$ and $D_H(p,H)$ respectively, then $H^{\Delta'}$ is t'-fold transitive on Δ' and H^{Δ} is t'-fold transitive on Δ.

Proof: Let A be a u-fold transitive permutation group on Ω and let Θ be a subset of Ω with $|\Theta| \geqslant u$ and $|\Omega| = n$.

We note, that then (Ω,Θ^A) is always a u-design. To prove this we only have to show that for any Γ in $\Omega\{u\}$ the number of blocks in Θ^A containing Γ is independent of the choice of Γ. Let $[\Gamma]$ be this number.

Suppose $\Theta_1,\ldots,\Theta_x$, $x = [\Gamma]$, are all blocks containing Γ and $\overline{\Theta}_1,\ldots,\overline{\Theta}_{\overline{x}}$, $\overline{x} = [\overline{\Gamma}]$ are all blocks containing $\overline{\Gamma}$ where $\overline{\Gamma}$ is some other member of $\Omega\{u\}$. Since A is u-fold transitive on Ω, there are elements a and $\overline{a}$ in A with $\overline{\Gamma}^{\overline{a}} = \Gamma$ and $\Gamma^a = \overline{\Gamma}$. Hence $\overline{\Theta}_1^{\overline{a}},\ldots,\overline{\Theta}_{\overline{x}}^{\overline{a}}$ are blocks in Θ^A containing Γ and $\Theta_1^a,\ldots,\Theta_x^a$ are blocks in Θ^A containing $\overline{\Gamma}$. Thus $\overline{x} \leqslant x \leqslant \overline{x}$ and $[\Gamma] \geqslant 1$ is independent of Γ. Therefore (Ω,Θ^A) is a u-$(n,|\Theta|,[\Gamma])$ design.

Now express $\ell = [\Gamma]$ in terms of A. Since A operates transitively on Θ^A, the total number ℓ_0 of blocks in Θ^A equals $|A| / |A_{\{\Theta\}}|$. On the other hand, there are $\binom{n}{u}$ members of $\Omega\{u\}$ each of them contained in ℓ blocks. Let $|\Theta| = k$. Since every block Θ contains $\binom{k}{u}$ members of $\Omega\{u\}$, we count in all $\ell_0 = \ell.\binom{n}{u}/\binom{k}{u}$ different blocks in Θ^A. Hence $|A|/|A_{\{\Theta\}}| = \ell\cdot n\cdot(n-1)\cdot\ldots\cdot(n-u+1)/k\cdot(k-1)\cdot\ldots\cdot(k-u+1)$. Since A is u-fold transitive on Ω, we have $n\cdot(n-1)\cdot\ldots\cdot(n-u+1)\cdot|A_\Gamma| = |A|$. Now suppose $A_{\{\Theta\}}$ acts u-fold transitively on Θ. Then also $k\cdot(k-1)\cdot\ldots\cdot(k-u+1)\cdot|(A_{\{\Theta\},\Gamma})| = |A_{\{\Theta\}}|$ for some Γ contained in Θ, $|\Gamma| = u$. Therefore $\ell = |A_\Gamma|/|A_{\{\Theta\},\Gamma}|$.

Note that the assumption about $A_{\{\Theta\}}$ is true for $G_{\{\Delta'\}}$, $G_{\{\Delta\}}$, $H_{\{\Delta'\}}$ and $H_{\{\Delta\}}$ by proposition 3.3. Substitute for A, u and Θ the various groups and parameters to prove lemma 3.4. $\diamond$

THEOREM 3.5

Let G be a t-fold transitive permutation group on Ω of degree n and $H \neq 1$ a normal subgroup of G. Let Γ be a subset of Ω, $|\Gamma| = t$, p some prime and let S' be a Sylow p-subgroup of G_Γ and S a Sylow p-subgroup of H_Γ. Put $\Delta' = Fix(S')$ and $\Delta = Fix(S)$.

Then $x(H) = |cc\ell_{G:H}(S)| \cdot x(H^\Delta)$.

Therefore H is t-fold transitive if $|cc\ell_{G:H}(S)| = 1$ and if H^Δ is t-fold transitive on Δ. If p does not divide n-t+1, then $|cc\ell_{G:H}(S)| = 1$.

Also H is t-fold transitive if $|cc\ell_{G:H}(S')| = 1$ and if $H^{\Delta'}$ is t-fold transitive on Δ'.

Proof: We apply proposition 3.1. Put $B = \Delta^G = \{Fix\ S^g | g \in G\}$ and observe that if $\Delta_1 = Fix(S_1)$ and $\Delta_2 = Fix(S_2)$ contain Γ, then S_1, S_2 are Sylow p-subgroups of H_Γ and therefore there is some h in H_Γ with $S_1{}^h = S_2$ and $\Delta_1^h = \Delta_2$. Let $B_1, \ldots, B_z$ be the H-orbits on B. Then $\Delta_1 = Fix(S_1)$ and $\Delta_2 = Fix(S_2) \in B_i$ if and only if S_1 is conjugated to S_2 in H. Hence $z = |cc\ell_{G:H}(S)|$ and by 3.1 we obtain $x(H) = |cc\ell_{G:H}(S)| \cdot x(H^\Delta)$. Now suppose H^Δ is t-fold transitive on Δ and p does not divide n-t+1. If $S = 1$, then $\Delta = \Omega$ and clearly $H = H^\Delta$ is t-fold transitive on Ω. This is the case if (a) H is regular on Ω or (b) if $G = Sym(\Omega)$, $H = Alt(\Omega)$ and $t = n-1$. Hence by lemma 2.4 we can assume that H is at least

$(t-\frac{1}{2})$-fold transitive on Ω. Let γ be a point in Γ and put $\Gamma' = \Gamma \setminus \{\gamma\}$. Then $[H_{\Gamma'} : H_\Gamma]$ is the length of the $H_{\Gamma'}$-orbit that contains γ. Since all $H_{\Gamma'}$-orbits on $\Omega \setminus \Gamma'$ have equal length, $[H_{\Gamma'} : H_\Gamma]$ divides $n-t+1$ $= |\Omega \setminus \Gamma'|$ and since p is prime to $n-t+1$, S is also a Sylow p-subgroup of $H_{\Gamma'}$. Hence by the Frattini argument, $G_{\Gamma'} = N_{G_{\Gamma'}}(S) \cdot H_{\Gamma'}$ and since H is $(t-1)$-fold transitive on Ω, $G = G_{\Gamma'} \cdot H = N_{G_{\Gamma'}}(S) \cdot H = N_G(S) \cdot H$. Thus $G/H = N_G(S) \cdot H/H \cong N_G(S)/N_H(S)$. Now calculate $|cc\ell_{G:H}(S)|$; $|cc\ell_{G:H}(S)| = |cc\ell_G(S)|/|cc\ell_H(S)| = (|G|/|N_G(S)|)/(|H|/|N_G(S)|) = |G/H|/|N_G(S)/N_H(S)|$ $= 1$. Hence $x(H) = 1 \cdot x(H^\Delta) = 1$ and H is t-fold transitive on Ω.

To prove the second statement, put $B = \Delta'^G = \{\text{Fix } S'^g | g \in C\}$ and observe that H operates transitively on B since $|cc\ell_{G:H}(S')| = 1$. Clearly B has the property that any Γ in $\Omega\{t\}$ is contained in at least one block of B and so by lemma 2.2 H is t-fold transitive on Ω. $\Diamond$

In the following theorem we use elementary counting arguments to obtain a bound~~ary~~ for $x(H)$. This result as we shall see later on can be used to prove that H is t-fold transitive in all cases mentioned in the theorem for $t \leqslant 6$.

<u>THEOREM 3.6</u>

Let G be a t-fold transitive permutation group on Ω of degree n, $G \neq \text{Sym}(\Omega)$, and let $H \neq 1$ be a normal subgroup of G.

Suppose p is a prime, $p < t$, not dividing $n-t+1$. Let x be the number of H-orbits on $\Omega(t)$ and r the smallest positive integer with $r \equiv (n-t+1)/x \mod p$. Then $0 < r \cdot x < p < t$.

Proof: Of course $t \geq 3$. If $t = 3$, $p = 2$, H cannot be regular on Ω because this would imply that n is a power of 2 (see 2.3) and hence $p = 2$ divides $n - 3 + 1$. Hence by 2.3 and 2.4 H is at least $(t-\frac{1}{2})$-fold transitive on Ω and therefore $x = x(H)$ is the number of $H_{\Gamma'}$-orbits on $\Omega\setminus\Gamma'$ where Γ' is a member of $\Omega\{t-1\}$. Since all these orbits have equal length, x divides $n-t+1$ and $r \equiv (n-t+1)/x \bmod p$ is a well-defined number $0 < r < p$.

Let $\Gamma > \Gamma'$ be in $\Omega\{t\}$ and let S be in $Syl_p(H_\Gamma)$ with $\Delta = \mathrm{Fix}\, S$. By 3.5 we have $x(H) = |ccl_{G:H}(S)| \cdot x(H^\Delta)$ and as in the proof of 3.5 one shows that $|ccl_{G:H}(S)| = 1$ since p is prime to $n-t+1$. Therefore $x(H) = x(H^\Delta)$. By 3.3 G^Δ is t-fold transitive on Δ, H^Δ is $(t-1)$-fold transitive on Δ and by 3.2 $H^\Delta \trianglelefteq G^\Delta$. Since $|\Delta| =: n' \equiv n \bmod p$, p does not divide $n'-t+1$. Hence the hypotheses of 3.6 are also satisfied for $G^\Delta \trianglelefteq H^\Delta$ except if $G^\Delta = \mathrm{Sym}(\Delta)$.

So we deal with the case $G^\Delta = \mathrm{Sym}(\Delta)$ first. If $H^\Delta (\trianglelefteq G^\Delta)$ is t-fold transitive on Δ, then by 3.5 H is t-fold transitive on Ω, that is $x = 1$ and $r \cdot x = r < p$. Hence assume H^Δ is $(t-1)$-fold transitive but not t-fold transitive on Δ, i.e. $H^\Delta = \mathrm{Alt}(\Delta)$, $|\Delta| = n' = t + 1$ and $x(H^\Delta) = 2$. Then by the above remark $x(H) = x(H^\Delta) = 2$. Since $n - t + 1 \equiv n' - t + 1 \equiv t + 1 - t + 1 \equiv 2 \bmod p$, $r \equiv 2/x \equiv 1 \bmod p$ and in particular $p \neq 2$. Hence $r \cdot x = 2 < p$.

By induction we therefore may assume that $H^\Delta = H$, $\Delta = \Omega$ and $S = 1$.

Let Γ' be in $\Omega\{t-1\}$. Then $H_{\Gamma'}$ has $t-1$ fixed points orbits and x orbits $T_1, \ldots, T_x$ on $\Omega\setminus\Gamma'$ of equal length $(n-t+1)/x$. Since H is $(t-1)$-fold transitive on Ω, there is an element h in $H_{\{\Gamma'\}}$ of order p^m $(p \leq t-1)$, fixing Γ' as a set such that h consists of a single p cycle and $t - 1 - p$

fixed points inside Γ'. Clearly h normalizes $H_{\Gamma'}$ and therefore h induces a permutation of the orbits $T_1,\ldots,T_x$, i.e. T_i^h is again an orbit of $H_{\Gamma'}$.

By hypothesis p does not divide n-t+1 and so p does not divide x. Thus, h fixes at least one orbit, say $T_1^h = T_1$. Assume h does not fix all orbits $T_1,\ldots,T_x$. Let $T_2^h = T_3$. Choose g_1 in $G_{\{\Gamma'\}}$ such that h^{g_1} acts on Γ' like h^{-1}, i.e. $h^{g_1}|_{\Gamma'} = h^{-1}|_{\Gamma'}$. Clearly such a g_1 exists and again, since $H_{\Gamma'} \lhd G_{\{\Gamma'\}}$, g_1 preserves $\{T_1,\ldots,T_x\}$. Since G is t-fold transitive on Ω, $G_{\Gamma'} (\leqslant G_{\{\Gamma'\}})$ permutes $\{T_1,\ldots,T_x\}$ transitively. Hence choose some g_2 in $G_{\Gamma'}$ such that $T_2^{g_1 g_2} = T_1$.

Then, if $g = g_1 \cdot g_2$, still $h^g|_{\Gamma'} = h^{-1}|_{\Gamma'}$. Therefore $h \cdot h^g$ is an element of $H_{\Gamma'}$ and so $T_1^{hh^g} = T_1$. On the other hand $T_1^{hh^g} = T_1^{h^g} = T_1^{g^{-1}hg} = T_2^{hg} = T_3^g$. Hence $T_1 = T_3^g = T_2^g$, a contradiction unless h fixes T_2. Therefore h fixes all $H_{\Gamma'}$-orbits $T_1,\ldots,T_x$.

Now we count the minimal number of points fixed by h : In Γ' h has t-1-p fixed points and in every T_i, i = 1,..., X, at least r fixed points. Therefore $|\mathrm{Fix}\ h| \geqslant t-1-p + r \cdot x$. But we assumed that S = 1 and so no element of p-order in H fixes as much as t distinct points. Hence $t > t-1-p+r \cdot x$ or $p \geqslant r \cdot x$. Since neither x nor r equal p, we finally have $p > r \cdot x$. $\diamond$

In the following theorem 3.7 we extend the arguments of 3.6 for the situation $1 < |n-t+1|_p$ for primes $p < t$. The proof of 3.7 uses counting arguments but also involves the generosity Theorem 2.8.

It is worth noting that 3.6 is independent of the Generosity Theorem.

THEOREM 3.7

Let G be a t-fold transitive permutation group on Ω of degree n, $G \neq \mathrm{Sym}(\Omega)$, and let $H \neq 1$ be a non-regular normal subgroup of G having x orbits on $\Omega(t)$. Suppose $p < t$ is a prime dividing $n-t+1$.

Then either (i): $|n-t+1|_p > |x|_p$ or else (ii): $t = 3$ and $PSL(2,q) \leqslant H \leqslant G \leqslant P\Gamma L(2,q)$ with $n = q + 1 \equiv 0 \bmod 4$.

Proof: Since $t \geqslant 3$, H is at least $(t - \frac{1}{2})$-fold transitive on Ω by 2.4. If H is actually t-fold transitive, clearly $x = 1$ and hence $|n-t+1|_p \geqslant p > 1$. Hence assume $x > 1$. Then x is the number of orbits of $H_{\Gamma'}$ $(\Gamma' \in \Omega\{t-1\})$ on $\Omega\setminus\Gamma'$ and so x divides $n-t+1$, i.e. $|n-t+1|_p \geqslant |x|_p$. Therefore it suffices to show that $|n-t+1|_p = |x|_p$ leads to a contradiction if G is not a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ with $n = q + 1 \equiv 0 \bmod 4$.

Let $\Gamma = \Gamma' \cup \{\gamma\}$ be in $\Omega\{t\}$ and let S be a Sylow p-subgroup of H_Γ with $\Delta = \mathrm{Fix}(S)$ and assume $|n-t+1|_p = |x|_p$, i.e. $p \nmid (n-t+1)/x$. Then S is also in $\mathrm{Syl}_p(H_{\Gamma'})$ since $[H_{\Gamma'}:H_\Gamma] = (n-t+1)/x$ is not divisible by p. Hence by the Frattini argument $G_{\Gamma'} = N_{G_{\Gamma'}}(S)\cdot H_{\Gamma'}$ and since H is $(t-1)$-fold transitive, also $G = G_{\Gamma'}\cdot H$, i.e. $G = N_G(S)\cdot H$. Therefore $|\mathrm{ccl}_G(S)| = [G:N_G(S)]$ $= [N_G(S)\cdot H : N_G(S)] = [H:N_H(S)] = |\mathrm{ccl}_H(S)|$ and so $|\mathrm{ccl}_{G:H}(S)| = 1$.

Hence by 3.5 we have $x = x(H) = x(H^\Delta)$ and from now on we look at H^Δ, G^Δ to produce a contradiction.

By 3.2 and 3.3 H^Δ is $(t-1)$-fold transitive on Δ and is normal in G^Δ where G^Δ is t-fold transitive on Δ. Also $(H^\Delta)_\Gamma$ is a p'-group. But if H and G are not the groups under (ii), H^Δ is even $(t-1)$ generously transitive on Δ. We see this in the following way: Let Γ^* be any member of $\Delta\{t\}$ and let s be an element of $\mathrm{Sym}(\Gamma^*)$. Since H is $(t-1)$-fold generously transitive on Ω by 2.8, H contains some element h with $h|_{\Gamma^*} = s$. In particular $\Gamma^{*h} = \Gamma^* \subset \Delta$ and S^h, S are both contained in H_{Γ^*}. Since $|H_{\Gamma^*}| = |H_\Gamma|$, S^h and S are elements of $Syl_p(H_{\Gamma^*})$ and therefore there is some k in H_{Γ^*} with $(S^h)^k = S$. Especially $\Delta^{h\cdot k} = \Delta$ and also $h\cdot k|_{\Gamma^*} = s$. This implies that H^Δ is $(t-1)$-fold generously transitive on Δ.

Consider $(H^\Delta)_{\{\Gamma'\}}$. Since H^Δ is $(t-1)$-fold transitive on Δ, there is some element h of p-order in $(H^\Delta)_{\{\Gamma'\}}$ consisting of a p cycle and $t-1-p$ fixed points inside Γ'. Let $T_1,\dots,T_{x'}$ be the orbits of $(H^\Delta)_{\Gamma'}$ on $\Delta\setminus\Gamma'$ where $x' = x(H^\Delta) = x(H) = x$.

Since h normalizes $(H^\Delta)_{\Gamma'}$, h induces a permutation of the set $\{T_1,\dots,T_x\}$; suppose h fixes f orbits T_i, $i = 1, \dots f$. Then h has in each of these f orbits at least r fixed points where r is the smallest non-negative number with $r \equiv |T_1| \equiv (|\Delta| - t+1)/x' \equiv (n-t+1)/x \not\equiv 0 \bmod p$ since $|\Delta| \equiv n \bmod p$. In particular $r \neq 0$. Therefore h fixes altogether at least $t - p - 1 + f\cdot r$ points in Δ. But since H_Γ is a p'-group and $[(H^\Delta)_{\Gamma'} : (H^\Delta)_\Gamma] = (|\Delta| - t + 1)/x \not\equiv 0 \bmod p$, also $(H^\Delta)_{\Gamma'}$ is a p'-group. Therefore $t - 1 > |\mathrm{Fix}\ h| \geq t - p - 1 + f\cdot r$, i.e. $f\cdot r < p$.

Our assumption implies in particular that p divides x. Therefore we will arrive to a final contradiction if we can show that h fixes all T_i's or in other words, that $f = x$.

For this purpose it suffices to prove that $(H^\Delta)_{\Gamma'}$ and $(H^\Delta)_{\{\Gamma'\}}$ have the same orbits on $\Delta\setminus\Gamma'$. Clearly an orbit of $(H^\Delta)_{\{\Gamma'\}}$ on $\Delta\setminus\Gamma'$ is a union of $(H^\Delta)_{\Gamma'}$-orbits since $(H^\Delta)_{\Gamma'} \leq (H^\Delta)_{\{\Gamma'\}}$. Show therefore that $(H^\Delta)_{\Gamma'}$ is transitive on every $(H^\Delta)_{\{\Gamma'\}}$-orbit. Let γ be some point in $\Delta\setminus\Gamma'$ and put $\Gamma = \Gamma' \cup \{\gamma\}$. Since H^Δ is $(t-1)$-fold generously transitive on Δ, $(H^\Delta)_{\{\Gamma\}}$ acts on Γ like $\mathrm{Sym}(\Gamma)$ and therefore $(H^\Delta)_{\{\Gamma\},\gamma}$ acts on Γ' like $\mathrm{Sym}(\Gamma')$. Thus we have: $(H^\Delta)_{\{\Gamma'\}} = (H^\Delta)_{\{\Gamma\},\gamma}\cdot(H^\Delta)_{\Gamma'}$. But this implies that

$(H^{\Delta})_{\Gamma'}$ is transitive on the $(H^{\Delta})_{\{\Gamma'\}}$-orbit which contains γ. Since γ was arbitrary, $(H^{\Delta})_{\Gamma'}$ and $(H^{\Delta})_{\{\Gamma'\}}$ have the same orbits on $\Delta \setminus \Gamma'$. $\diamond$

We mention one important consequence of 3.7 separately:

THEOREM 3.8

Let G be a triply transitive permutation group on Ω of degree n, $4 < n \equiv 0 \bmod 4$.

Suppose $H \neq 1$ is a non-regular normal subgroup of G and let x be the number of $H_{\alpha,\beta}$-orbits on $\Omega \setminus \{\alpha, \beta\}$ for two distinct points α and β in Ω. Then either x is odd or else x = 2 and $PSL(2,q) \trianglelefteq H \trianglelefteq G \trianglelefteq P\Gamma L(2,q)$ with $n = q + 1 \equiv 0 \bmod 4$.

At this stage we note that x is in general odd if $t \geq 6$. This is a result of E. Bannai:

THEOREM 3.9 (BANNAI, theorem 1 in [3])

Let G be a 6-fold transitive permutation group on Ω, $G \neq \mathrm{Sym}\,\Omega$, and let $H \neq 1$ be a normal subgroup of G. If Γ is a subset of Ω with $|\Gamma| = 5$, then H_{Γ} has an odd number of orbits on $\Omega \setminus \Gamma$.

3.2 THE THEOREM OF WAGNER

In 1966 Wagner proved in [24] that normal subgroups $\neq 1$ of triply transitive permutation groups of odd degree are also triply transitive. (With the obvious exception $\mathrm{Alt}(3) \triangleleft \mathrm{Sym}(3)$). The proof of this theorem given in [24] asserts the 2-generosity of the normal subgroup and uses this property to bound the orbits on $\Omega\{3\}$. All this only involves Sylow's Theorem and in fact Wagner's proof is most elementary. This makes his theorem one of the most important results on multiply transitive permutation groups.

Six years later, Ito published his paper [12] on quadruply transitive groups of degree $n \not\equiv 0 \bmod 3$ in an attempt to extend Wagner's theorem. Ito's proof, however, is much less basic: To secure the 3-generosity of the normal subgroup Ito uses Bender's theorem [4] and 4-homogeneity is shown by some character theoretical arguments.

Wagner's theorem is of course already contained in theorem 3.6 and so is Ito's result in so far as H is quadruply transitive if $n \equiv 1 \bmod 3$ and has at most 2 orbits on $\Omega(4)$ if $n \equiv 2 \bmod 3$.

In the following theorem we will show that under the hypotheses of 3.6 H is t-fold transitive if t does not exceed 6. This is the highest degree of transitivity for which one could reasonably expect to find groups not containing the alternating group $\mathrm{Alt}(\Omega)$. (See for instance Nagao's paper [18], theorem 3). The proofs are given at the end of this section.

THEOREM 3.10

Let G be a t-fold transitive permutation group on Ω of degree n, $2 < t \leq 6$, $G \neq \mathrm{Sym}\,\Omega$, and let $H \neq 1$ be a normal subgroup of G.

Suppose $n - t + 1$ is not divisible by some prime p, $p < t \leq 6$.

Then H is t-fold transitive on Ω.

As generalisation of 3.10 we have:

THEOREM 3.11

Let G be a t-fold transitive permutation group on Ω, $G \neq \mathrm{Sym}\,\Omega$ and $3 \leq t \leq 6$. Let $H \neq 1$ be a normal subgroup of G.
Suppose there are primes p and q with $p < t$ such that $S' \in \mathrm{Syl}_q(G_{\Gamma'})$ fixes exactly k' points and $S = S' \cap H \in \mathrm{Syl}_q H_{\Gamma'}$ fixes k points in

Ω where Γ' is a member of $\Omega\{t-1\}$.

Assume either (i): $k' - t + 1 \not\equiv 0 \bmod p$ and $N_G(S')\cdot H = G$ or (ii): $k - t + 1 \not\equiv 0 \bmod p$. Then H is t-fold transitive on Ω.

Note that 3.11 (ii) is theorem 3.10 for p = q: since $k \equiv n \bmod q$, $k - t + 1 \not\equiv 0 \bmod p$ implies that $n - t + 1$ is not divisible by p.

A result which has some similarity to theorem 3.11 is due to Atsumi [2], who proved that in the case $t = 4$, $n \equiv 4$ (5) H also is quadruply transitive. With his method one can prove

<u>THEOREM 3.12</u>

Let G be a t-fold transitive permutation group on Ω of degree n, $4 \leqslant t$ and $G \neq \mathrm{Sym}\,\Omega$, and let $H \neq 1$ be a normal subgroup of G.

Assume $t + 1 = p$ is a prime and $n \equiv t$ (p).

Then H is t-fold transitive on Ω.

For $t = 2$ and $n \equiv 2(3)$, the above statement is not true, doubly transitive groups with regular normal subgroups are counter examples. But apart from this exception, theorem 3.12 probably also holds for $t = 2$ and $n \equiv 2(3)$. In Chapter V we will see that H has at most 2 orbits on $\Omega(2)$ if G has a transitive extension.

<u>Proof of 3.10:</u> We divide the proof into steps according to the four possibilities $t = 3, 4, 5$ and 6.

(i) Let $t = 3$ and $p = 2$. Then by 3.6 $x = x(H) < 2$, i.e. $x(H) = 1$ and H is triply transitive on Ω.

(ii) Let $t = 4$ and $p = 2$ or 3. If $n - 4 + 1 \not\equiv 0(2)$, by 3.6 again $x < 2$,
quadruply
i.e. H is triply transitive. Hence assume $n - 4 + 1 \equiv 0(2)$ and $n - 4 + 1 \not\equiv 0(3)$. The inequality $0 < r.x < 3$ only has solutions for

$x = 1$ or $x = 2$, $r = 1$ and $n \equiv 2(3)$. Let therefore $1 \neq H \lhd G \neq Sym(\Omega)$ be a counter example with $t = 4$, $x = 2$, $n \equiv 1(2)$ and $n \equiv 2(3)$, minimal with respect to the degree n and the index $[G:H]$.

Let $\alpha,\beta,\gamma,\delta$ be distinct points in Ω. The group $G_{\alpha,\beta,\gamma}$ permutes the two $H_{\alpha,\beta,\gamma}$-orbits. Let $K \subset G_{\alpha,\beta,\gamma}$ be the kernel of this operation. Then $[G_{\alpha,\beta,\gamma}:K] = [G:K\cdot H] = 2$ and since $K\cdot H \lhd G_{\alpha,\beta,\gamma}\cdot H = G$ we obtain by minimality $K\cdot H = H$. Therefore $[G:H] = 2$ and $G_{\alpha,\beta,\gamma,\delta} = H_{\alpha,\beta,\gamma,\delta}$.

Let S_3 be in $Syl_3(H_{\alpha,\beta,\gamma,\delta})$ and put $\Delta_3 = Fix\ S_3$. Then $1 \neq H^{\Delta_3} \lhd G^{\Delta_3}$ are groups of degree $|\Delta_3| \equiv n \equiv 2(3)$ and by 3.5 H^{Δ_3} cannot be quadruply transitive on Δ_3. Hence, by the minimality of the counter example, we either have $S_3 = 1$, $\Delta_3 = \Omega$ and $G^{\Delta_3} = G$ or else $G^{\Delta_3} = Sym(5)$ and $H^{\Delta_3} = Alt(5)$. The latter is impossible. Let $\Delta_3 = \{\alpha,\beta,\gamma,\delta,\delta^*\}$. If $H^{\Delta_3} = Alt(5)$, $N_H(S_3)$ acts on Δ_3 like $Alt(5)$ and therefore $(N_H(S_3))_{\delta^*}$ acts on $\{\alpha,\beta,\gamma,\delta\}$ like $Alt(4)$. By the Frattini argument $H_{\{\alpha,\beta,\gamma,\delta\}} = (N_H(S_3))_{\delta^*}\cdot H_{\alpha,\beta,\gamma,\delta}$ and hence $H^{\{\alpha,\beta,\gamma,\delta\}} = Alt(4)$. This contradicts theorem 2.8. Thus we have $S_3 = 1$ and every element of 3-order fixes exactly 2 points.

Now let S_2 be in $Syl_2(H_{\alpha,\beta,\gamma,\delta})$ and put $\Delta_2 = Fix\ S_2$. Then G^{Δ_2} is quadruply transitive on Δ_2 of odd degree $|\Delta_2| \equiv n \equiv 1\ (2)$ with $|(G^{\Delta_2})_{\alpha,\beta,\gamma,\delta}| \equiv 1(2)$. Therefore G^{Δ_2} is a transitive extension of one of the groups in lemma 2.9 and therefore by 2.10 $G^{\Delta_2} = Sym(5)$, $Alt(7)$ or M_{11}. $G^{\Delta_2} = Alt(7)$ is impossible since $Alt(7)$ contains 3-elements fixing 4 points. If $G^{\Delta_2} = Sym(5)$, by the same argument as above, also $H^{\Delta_2} = Sym(5)$ and if $G^{\Delta_2} = M_{11}$, then $H^{\Delta_2} = M_{11}$ since M_{11} is simple and in both cases H^{Δ_2} is at least quadruply transitive on Δ_2. This implies that $\Delta_2 \setminus \{\alpha,\beta,\gamma\}$ is contained in the same $H_{\alpha,\beta,\gamma}$-orbit T_1. In both cases $N_H(S_2)$ contains an element h consisting of a 3-cycle $(\alpha,\beta,\gamma)\ldots$ and at least 2 fixed points (δ) and (δ^*) with δ,δ^* in $\Delta_2\setminus\{\alpha,\beta,\gamma\}$. Since h is contained in $H_{\{\alpha,\beta,\gamma\}}$, h preserves the $H_{\alpha,\beta,\gamma}$-orbits T_1 and T_2 and hence h has at least 4 fixed points, a contradiction.

(iii) Let $t = 5$. This case can be reduced to (ii): Since there are no new primes below 5, $n - t + 1 \not\equiv 0 \bmod 5$ implies $(n - 1) - (t - 1) + 1 \not\equiv 0 \bmod 5$ with $p < t - 1 = 4$. The groups $\overline{G} = G_\alpha$ and $\overline{H} = H_\alpha$ have degree $n - 1$ and are 4-transitive by part (ii). Hence H is 5-fold transitive on Ω.

(iv) Let $t = 6$. If $p = 2$, $n - 6 + 1 \not\equiv 0(2)$ implies $(n-3) - (6-3) + 1 \not\equiv 0(2)$ and the groups $\overline{G} := G_{\alpha,\beta,\gamma} \trianglerighteq \overline{H} := H_{\alpha,\beta,\gamma}$ are triply transitive on $\Omega \setminus \{\alpha,\beta,\gamma\}$ by part (i). Hence H is 6-fold transitive on Ω. Similarly one uses part (iii) of the proof to eliminate the case $p = 3$.

Hence let $p = 5$ and n satisfy the congruences $n - 6 + 1 \not\equiv 0(5)$, $n - 5 \equiv 0(2)$ and $n - 5 \equiv 0(3)$. Then 3.6 implies $0 < r \cdot x < 5$ and this inequality has solutions for either $x = 1$ (in this case H is 6-fold transitive) or else for $r = 1$, $x = 2$, 3 or 4 and $r = 2$, $x = 2$. The congruences for n modulo 5 are then given by

	$x = 2$	$x = 3$	$x = 4$
$r = 1$	$n \equiv 2(5)$	$n \equiv 3(5)$	$n \equiv 4(5)$
$r = 2$	$n \equiv 4(5)$	—	—

Let $1 \neq H \triangleleft G \neq \mathrm{Sym}(\Omega)$ be a counter example, i.e. $n \not\equiv 0(5)$, $n \equiv 1(2)$ and $n \equiv 2(3)$, G in 6-fold transitive on Ω and H is 5-fold transitive. Suppose this counter example is minimal with respect to the degree $n = |\Omega|$ and the index $[G:H] = d$. Then r, x and n are given by the above table.

If $x = 2$ or 3, one shows as under (i) that $d = x$. Suppose $x = 4$. Let Γ' be in $\Omega\{5\}$ and $\Gamma = \Gamma' \cup \{\gamma\}$ in $\Omega\{6\}$. Then $G_{\Gamma'}$ permutes the 4 orbits of $H_{\Gamma'}$ on $\Omega \setminus \Gamma'$ transitively and by minimality we can assume that the kernel of this operation equals $H_{\Gamma'}$. Therefore $G_{\Gamma'}/H_{\Gamma'}$ is a transitive subgroup of $\mathrm{Sym}(4)$ containing a transitive group C of order 4 where C is either a Klein group or cyclic. Let L, $H_{\Gamma'} < L \leq G_{\Gamma'}$ be the inverse image of C. Then L is transitive on $\Omega \setminus \Gamma'$ and therefore $H \cdot L$ is 6-fold transitive on Ω and again we have $H \cdot L = G$. Let C^* be a subgroup in C of order 2, $C^* \lhd C$, and $L^* \lhd L = G_{\Gamma'}$ the inverse image of C^*. Then $H \neq H \cdot L^*$ is not 6-fold transitive on Ω and $H \cdot L^*$ is normal in G with index 2. This shows that H was not maximal in G, a contradiction to our assumption about the minimality of $d = [G : H]$. Hence the case $x = 4$ cannot occur and we only need to consider the remaining cases $x = 2$ or 3.

Let S_5 be in $\mathrm{Syl}_5(H_\Gamma) = \mathrm{Syl}_5(G_\Gamma)$ and put $\Delta_5 = \mathrm{Fix}\, S_5$. Then $1 \neq H^{\Delta_5} \trianglelefteq G^{\Delta_5}$ where G^{Δ_5} is 6-fold transitive on Δ_5 of degree $|\Delta_5| \not\equiv 0 \bmod 5$ and H^{Δ_5} is 5-fold transitive on Δ_5 but not 6-fold transitive by 3.5. Having assumed that G and H are minimal, we either have $G^{\Delta_5} = G$, $\Delta_5 = \Omega$ and $S_5 = 1$ or else $G^{\Delta_5} = \mathrm{Sym}(\Delta_5)$. Since H^{Δ_5} cannot be 6-transitive the latter possibility implies $|\Delta_5| = 7$ and $H^{\Delta_5} = \mathrm{Alt}(7)$. But this contradicts 2.8 just as in part (ii) of the proof. Hence $S_5 = 1$ and every element of 5-order fixes exactly 1, 2, 3 or 4 points according as $n \equiv 1, 2, 3$ or 4 modulo 5.

Now let S_2 be in $\mathrm{Syl}_2(H_\Gamma)$ and put $\Delta_2 = \mathrm{Fix}\, S_2$. Then $1 \neq H^{\Delta_2} \trianglelefteq G^{\Delta_2}$ and G^{Δ_2} is a 6-fold transitive group of degree $|\Delta_2| \equiv n \equiv 1(2)$ with $|(G^{\Delta_2})_\Gamma| \equiv 1(2)$. Hence G^{Δ_2} is a 3-fold transitive extension of one of the groups in lemma 2.9. Since M_{11} extends only once (to M_{12}), $G^{\Delta_2} = \mathrm{Sym}(7)$ or $G^{\Delta_2} = \mathrm{Alt}(9) = H^{\Delta_2}$. If $G^{\Delta_2} = \mathrm{Sym}(7)$, then $H^{\Delta_2} = \mathrm{Sym}(7)$ also because $H^{\Delta_2} = \mathrm{Alt}(7)$ would contradict 2.8. In either case $N_H(S_2) \cap H_{\{\Gamma'\}}$ contains some element h of 5-order consisting of a 5-cycle in Γ' fixing $\Delta_2 \setminus \Gamma'$ pointwise. Since $N_H(S_2)$ acts 6-fold transitively on Δ_2, the points of $\Delta_2 \setminus \Gamma'$ must be contained in the same $H_{\Gamma'}$-orbit T_1. Since

h fixes T_1 as a set, we have $|T_1| = (n-6+1)/x \equiv r = |\Delta_2| - 5$ modulo 5. This remark excludes the possibility $G^{\Delta_2} = \mathrm{Alt}(9)$, since $r = 4$ and $r \cdot x < 5$ implies $x = 1$, and it also excludes the possibility $x = 3$. Hence $G^{\Delta_2} = \mathrm{Sym}(7) = H^{\Delta_2}$, $r = 2$, $x = 2$ (from the table) and $n \equiv 4(5)$ is the only remaining possibility. But $x = 2$ contradicts Bannai's theorem 3.9 and so the assumption of a minimal counter example leads to a contradiction.

◇

<u>PROOF OF 3.11.</u> Let S' be in $\mathrm{Syl}_p(G_{\Gamma'})$ and $S = S' \cap H$ in $\mathrm{Syl}_p(H_{\Gamma'})$ with Fix $S' = \Delta'$ and Fix $S = \Delta$. Since $k' = |\Delta'| \geq t \leq k = |\Delta|$, S' is also in $\mathrm{Syl}_p(G_\Gamma)$ and S in $\mathrm{Syl}_p(H_\Gamma)$ where $\Gamma' \leq \Gamma \leq \Delta'$, $\Gamma \in \Omega\{t\}$.

By 2.3, 2.4 H is either $(t-1)$-fold transitive on Ω or else regular, $t = 3$ and $|H| = |\Omega| = 2^m$. This latter possibility cannot occur: since S and S' are inductive in H (3.3), $H_{\{\Delta\}}$ and $H_{\{\Delta'\}}$ are regular on Δ and Δ' respectively and $|\Delta| \equiv 0 \equiv |\Delta'|$ mod 2 violates the condition $|\Delta| - 3 + 1 \not\equiv 0 \not\equiv |\Delta'| - 3 + 1$ mod 2.

Hence H is at least $(t-1)$ fold transitive on Ω and therefore $G = G_{\Gamma'} \cdot H$. By the Frattini argument and the condition under (ii) we have $G = N_G(S) \cdot H$ and $G = N_G(S') \cdot H$. This implies $[\mathrm{ccl}_{G:H}(S)] = [G:N_G(S)] : [H:N_H(S)] = 1$ and also $[\mathrm{ccl}_{G:H}(S')] = [G:N_G(S')] : [H:N_H(S')] = 1$. Hence by theorem 3.5 we obtain $x(H) = 1 \cdot x(H^\Delta)$ and $x(H) = 1 \cdot x(H^{\Delta'})$.

The groups H^Δ and $H^{\Delta'}$ are $(t-1)$-fold transitive normal subgroups of the t-fold transitive groups G^Δ and $G^{\Delta'}$ of degree k and k' respectively where $k - t + 1 \not\equiv 0$ mod p and $k' - t + 1 \not\equiv 0$ mod p for some prime p less than t. If $G^\Delta \neq \mathrm{Sym}\ \Delta$ and $G^{\Delta'} \neq \mathrm{Sym}\ \Delta'$ we can apply 3.10 to show that then also H^{Δ} and $H^{\Delta'}$ are t-fold transitive. This then implies $1 = x(H^{\Delta}) = x(H)$, $1 = x(H^{\Delta'}) = x(H)$ and H is t-fold transitive on Ω.

So assume for the remainder of the proof that $G^{\Delta} = \mathrm{Sym}(\Delta)$ or $G^{\Delta'} = \mathrm{Sym}\ \Delta'$. If $G^{\Delta} = \mathrm{Sym}\ \Delta$ and H^{Δ} is not t-fold transitive, then necessarily $|\Delta| = k = t + 1$ and $H^{\Delta} = \mathrm{Alt}(\Delta)$, and the condition $k - t + 1 = 2 \not\equiv 0 \bmod p$ implies $p \neq 2$ and therefore $t > 3$. Since S is a Sylow p-subgroup of H_{Γ}, the Frattini argument gives $(N_H(S))_{\{\Gamma\}} \cdot H_{\Gamma} = H_{\{\Gamma\}}$ and, taking $\Delta = \Gamma \cup \{\gamma\}$ we obtain that $(N_G(S))_{\delta} \cdot H_{\Gamma} = H_{\{\Gamma\}}$ acts on Γ like $\mathrm{Alt}(\Delta \setminus \gamma)$, i.e. $H^{\Gamma} = \mathrm{Alt}(\Gamma)$. This is a contradiction to theorem 2.8 since $t > 3$. Similarly one proves that if $G^{\Delta'} = \mathrm{Sym}\ (\Delta')$ then also $H^{\Delta'} = \mathrm{Sym}(\Delta')$.

$\diamond$

<u>PROOF OF 3.12</u> Let $H \trianglelefteq G$ be a counterexample to theorem 3.12. By 3.6 we can assume that $n - t + 1 \equiv 0 \bmod 2$. Let S_2 be in $\mathrm{Syl}_2 G_{\Gamma}$, $\Gamma \in \Omega\{t\}$, and put $\Delta_2 = \mathrm{Fix}\ S_2$. Then G^{Δ_2} is t-fold transitive on Δ_2 and $|(G^{\Delta_2})_{\Gamma}| \equiv 1(2)$. Therefore G^{Δ_2} is a (t-3)-fold transitive extension of one of the groups in lemma 2.9. If $t = 4$, then $G^{\Delta_2} = \mathrm{Sym}(5)$, $\mathrm{Alt}(7)$ and M_{11} are the only possibilities by lemma 2.10. Hence $H^{\Delta_2} \geqslant \mathrm{Alt}(5)$, $\mathrm{Alt}(7)$ or M_{11}. If $t \geqslant 6$, only $G^{\Delta_2} = \mathrm{Sym}\ \Delta_2$, $|\Delta_2| = t + 1$ and $G^{\Delta_2} = \mathrm{Alt}\ \Delta_2$, $|\Delta_2| = t + 3$ remain since M_{11} is not extendible twice. Therefore $H^{\Delta_2} \geqslant \mathrm{Alt}(\Delta_2)$ and $|H^{\Delta_2}|$, $|H|$ are divisible by $t + 1 = p$ which is a prime by assumption.

Let therefore $1 \neq S_p$ be a Sylow p-subgroup of H. Since $|\mathrm{Fix}\ S_p| \equiv n \equiv t \bmod p$, $p > t$, S_p fixes at least t points and therefore $S_p \in \mathrm{Syl}_p(H_{\Gamma})$ for some Γ contained in $\mathrm{Fix}\ S_p$, $|\Gamma| = t$. Put $\Delta_p = \mathrm{Fix}\ S_p$. Then $H^{\Delta p} \trianglelefteq G^{\Delta p}$ are groups of degree $n' = |\Delta_p|$ where $G^{\Delta p}$ is t-fold transitive and $H^{\Delta p}$ is at least (t-1)-fold transitive on Δ^p. $(H^{\Delta}p)_{\Gamma}$ is a p'-group by construction and since $|H^{\Delta p}| = \left(n' \cdot (n'-1) \ldots . (n'-t+1)/x(H^{\Delta p})\right) \cdot |(H^{\Delta p})_{\Gamma}|$,

$|H^{\Delta p}|$ is not divisible by p. Therefore $H^{\Delta p} \trianglelefteq G^{\Delta p}$ is no counterexample and hence $H^{\Delta p}$ is either t-fold transitive on Δ_p or else $G^{\Delta p} = \mathrm{Sym}(\Delta_p)$, $H^{\Delta p} = \mathrm{Alt}(\Delta_p)$ with $|\Delta_p| = n' = t + 1$. But both is impossible: $n' = t + 1$ implies $n \equiv 0 \mod t + 1$ and in the first case H is t-fold transitive on Ω by theorem 3.5.

◇

CHAPTER IV HOMOGENEITY AND SYMMETRIC SETS

In the present chapter we will concentrate on the homogeneity aspect of the normal subgroup problem.

We suppose again that G is a t-fold transitive group on Ω and that $H \neq 1$ is a normal subgroup of G. Then we can define an equivalence relation on $\Omega\{t\}$ by $\Gamma \sim \Gamma'$ if $\Gamma^h = \Gamma'$ for some h in H and so the $\sim$-classes are exactly the orbits of H on $\Omega\{t\}$.

This definition can easily be extended to equivalence relations $\sim_k$ on $\Omega\{k\}$ for every $k \leqslant |\Omega|$: $\Delta \sim_k \Delta'$ if both $\Delta\{t\}$ and $\Delta'\{t\}$ contain the same number of t-subsets in each H-orbit on $\Omega\{t\}$, that is, if $|\{\Gamma|\Gamma \in \Delta\{t\}, \Gamma \sim \Gamma^*\}| = |\{\Gamma'|\Gamma' \in \Delta'\{t\}, \Gamma' \sim \Gamma^*\}|$ for each Γ^* in $\Omega\{t\}$. Then, of course, $\sim_t$ is the original $\sim$-relation and we observe that for $\Delta \in \Omega\{k\}$, $\Delta \sim_k \Delta^h$ for any h in H. If this should be true for every h in G, necessarily $|\{\Gamma|\Gamma\in\Delta\{t\}, \Gamma \sim \Gamma^*\}|$ must be independent of Γ^* and therefore a constant.

For some values of k $\sim_k$ is the trivial relation, i.e. $\Delta \sim_k \Delta'$ for every pair of members of $\Omega\{k\}$. This is certainly true for $k < t$, and also, as we shall see later, for $k > n-t$ if H is (t-1)-fold transitive on Ω. Let y_k be the number of $\sim_k$-classes on $\Omega\{k\}$. Then $y_k = y_{n-k} = 1$ for $k < t$ and $y_t = y_{n-t} = y(H)$, the number of H-orbits on $\Omega\{t\}$. An interesting question would be to determine all values k for which $\sim_k$ is a trivial relation. Or more generally, how is y_k related to y_{k+1}?

Of special interest are subsets Δ of Ω with the property $\Delta \sim_k \Delta^g$ for the appropriate k and every g in G. For obvious reasons we will call such a subset 'symmetric'. As we have seen above, a subset Δ of Ω is symmetric if and only if $|\{\Gamma | \Gamma \in \Delta\{t\}, \Gamma \sim \Gamma^*\}|$ is independent of Γ^* in $\Omega\{t\}$. So, for instance, if $y_k = 1$ for some particular k, every k-subset is symmetric. Therefore the sets Θ and $\Omega \backslash \Theta$ are examples of symmetric sets where Θ is any set of size at most $t - 1$. Obviously a set of size t is symmetric if and only if $y(H) = 1$, i.e. H is t-fold homogeneous. In general one derives from the above condition for symmetric sets Δ various numerical restrictions on $|\Delta|$ and $y(H)$ which one can use to show that symmetric sets of certain sizes imply the t-fold homogeneity of H.

In this chapter we describe canonical ways to produce more symmetric sets. In chapter III, for instance, the points fixed by group S in $Syl_p(H_\Gamma)$ where symmetric sets if $S^G = S^H$. More generally we observe that any subset of G with $X^G = X^H$ leads to symmetric sets and collections associated with its orbits and their transversals. This last remark shows that the above symmetry concept basically compares the conjugation properties of G with its permutation properties (e.g. homogeneity) and we study some of these aspects for doubly transitive groups containing a normal subgroup which is not doubly homogeneous.

We adopt the notation of Chapter III. G is t-fold transitive on Ω, $H \neq 1$ is a normal subgroup of G which is not t-fold homogeneous, i.e. $y(H) > 1$. Therefore $\Omega\{t\}$ is a disjoint union of H-orbits U_i, $1 \leq i \leq y = y(H)$,

$$\Omega\{t\} = U_1 \cup \ldots \cup U_y,$$

and G acts transitively on the set $\{U_1, \ldots U_y\}$.

Assume H is (t-1)-fold transitive on Ω and let Γ be a member of $\Omega\{t\}$. Then

(1) $x(H) = x(H^{\Gamma}) \cdot y(H)$ and so $x(H) = y(H)$ if and only if H is (t-1)-fold generously transitive.

Proof: Proposition 3.1. ◇

Let $V = \mathbb{Q}^y$ be the vector space over the rational numbers of dimension y. Define for every subset M of Ω a vector c^M in V by the following rule:

$$(c^M)_i = |M\{t\} \cap U_i| \qquad i = 1, \ldots, y$$

$$= |\{\Gamma | \Gamma \in U_i,\ \Gamma \subset M\}|,\ i = 1, \ldots, y.$$

Then, obviously, we obtain:

(2) $\sum_{i=1}^{y} (c^M)_i = \binom{|M|}{t}$ for every $M \subset \Omega$.

Let Γ_j be a member of U_j, $j = 1, \ldots, y$. Then the vectors $c^j := c^{\Gamma_j}$ form a basis C of V which we will fix. Define a G-action on V by

$$g: c^j \to c^{jg} = c^{(\Gamma_j{}^g)}, \qquad j = 1, \ldots, y.$$

Clearly c^{jg} is contained in C and G acts transitively on C.

By this definition V becomes a G-module and .

(3) $\qquad (c^M)^g = c^{(M^g)}$ for every $M \subset \Omega$.

and every g in G.

Proof: Let $c^M = \Sigma(c^M)_j c^j$ and hence $(c^M)^g = \Sigma(c^M)_j c^{jg}$. Then the i-th component of $(c^M)^g$, $(c^M)^g_i$, equals $(c^M)_j$ for some j such that $c^{jg} = c^i$, i.e. $c^{\Gamma_j g} = c^{\Gamma_i}$ and $\Gamma_j{}^g$ belongs to U_i. But $\Gamma_j{}^g$ also is a member of U_{j-1}^g and so $U_i = U_j{}^g$. Therefore $(c^M)_i{}^g = (c^M)_j = |M\{t\} \cap U_j| = |M\{t\} \cap U_j^g| = |(M\{t\})^g \cap U_i| = |(M^g)\{t\} \cap U_i| = (c^{(M^g)})_i$. Thus $(c^M)^g = c^{(M^g)}$. ◇

Let H* be the kernel of the G-action on V. Then

(4) $\qquad H \leq H^* < G$

Proof: If h is contained in H and if Γ_i is a member of U_i, also $\Gamma_i{}^h$ is in U_i. Therefore $c^{ih} = c^i$ for all i, $i = 1, \ldots, y$. ◇

Let X be a subset of G and $X^G = \{X^g \mid g \in G\}$ the set of all G-conjugates of X. Let $P(\Omega)$ be the set whose elements are all finite collections of subsets of Ω, $P(\Omega) = \{\{M_1,\ldots,M_r\} \mid M_i \leqslant \Omega, \ |1 \leqslant i \leqslant r < \infty\}$.
G acts on $P(\Omega)$ in the usual way: $\{M_1,\ldots,M_r\}^g = \{M_1{}^g,\ldots,M_r^g\} \in P(\Omega)$.
A function $\mathcal{M} : X^G \to P(\Omega)$ is a G-function if $\mathcal{M}(X^g) = (\mathcal{M}(X))^g$.

Call a collection $\overline{M} = \{M_1,\ldots,M_r\}$ in $P(\Omega)$ *symmetric* if $\{c^{M_i} \mid M_i \in \overline{M}\} = \{(c^{M_i})^g \mid M_i \in \overline{M}\}$ for all g in G. Call a subset M of Ω *symmetric* if $\{M\}$ is symmetric.

LEMMA 4.1

Let X be a subset of G with $X^G = X^H$ and let $\mathcal{M} : X^G \to P(\Omega)$ be a G-function.

Then $\mathcal{M}(X^*)$ is symmetric for every X^* in X^G and $c^{\Sigma\mathcal{M}} := \Sigma_{M \in \mathcal{M}(X^*)} c^M$ is independent of X^* in X^G and invariant under G.

Proof: Let g be in G and M in $\mathcal{M}(X^*)$. There is some h in H such that $X^{*g} = X^{*h}$ and so $M^g \in \mathcal{M}(X^*)^g = \mathcal{M}(X^{*g}) = \mathcal{M}(X^{*h}) = \mathcal{M}(X^*)^h$, i.e. there is some M' in $\mathcal{M}(X^*)$ with $M^g = M'^h$. Therefore $(c^M)^g = c^{M^g} = c^{M'^h} = c^{M'}$ and so $\mathcal{M}(X^*)$ is symmetric. Let X', X^* be in X^G and h in H with $X'^h = X^*$. Then $\Sigma_{M \in \mathcal{M}(X')} c^M = \Sigma_{M \in \mathcal{M}(X')} (c^M)^h = \Sigma_{M \in \mathcal{M}(X')^h} c^M = \Sigma_{M \in \mathcal{M}(X^*)} c^M$ shows that $c^{\Sigma\mathcal{M}}$ is independent of X^* in X^G. This also implies that $c^{\Sigma\mathcal{M}}$ is a G-invariant vector. $\diamond$

LEMMA 4.2

Let X be a subset of G with $X^G = X^H$ and let M be a G-function on X^G.

Then $N_G(X^*)$ preserves $M(X^*)$ for every X^* in X^G and if T: = $\{M_1,\dots,M_s \mid M_i \in M(X^*)\}$ is a $N_G(X^*)$-orbit or a union of orbits on $M(X^*)$, then T is a symmetric collection.

In particular, if $N_H(X^*)$ is transitive on T, then every M in T is a symmetric set.

Proof: $N_G(X^*)$ acts on $M(X^*)$: Let $\bar{g}$ be in $N_G(X^*)$ and M in $M(X^*)$. Then $M^{\bar{g}} \in M(X^*)^{\bar{g}} = M(X^{*\bar{g}}) = M(X^*)$. Now show that T is symmetric. Since $X^G = X^H$ we have $G = N_G(X^*)\cdot H$. So let $g = h\cdot\bar{g}$ be an element in G where h is in H and $\bar{g}$ in $N_G(X^*)$. Then for any M in T we obtain $(c^M)^g = (c^M)^{h\cdot\bar{g}} = (c^M)^{\bar{g}} = c^{(M^{\bar{g}})}$. Since $\bar{g}$ is in $N_G(X^*)$, $M^{\bar{g}}$ is also contained in T and hence T is symmetric.

If T is even an $N_H(X^*)$-orbit, $M^{\bar{g}} = M^{\bar{h}}$ for some $\bar{h}$ in $N_H(X^*)$ and so $(c^M)^g = c^{(M^{\bar{g}})} = c^{M^{\bar{h}}} = (c^M)^{\bar{h}} = c^M$. Hence every M in T is a symmetric set.

◇

The following lemma gives some divisibility conditions for symmetric sets:

LEMMA 4.3

Let $M^0, M^1, \ldots, M^j, \ldots, M^r$ be symmetric subsets of Ω with size $|M^j| = m + j$, $0 \leqslant j \leqslant r \leqslant t - 1$.

Then y divides $\binom{m + (j-i)}{t-i}$ for all i and j with $0 \leqslant i \leqslant j \leqslant r$.

Proof: Let $M := M^j$ be a symmetric subset of Ω. This implies $(c^M)^g = c^M$ for all $g \in G$. Since G operates transitively on $C = \{c^1, \ldots, c^y\}$, we have $(c^M)_\ell = (c^N)_1$, $1 \leqslant \ell \leqslant y$. Therefore

$\binom{|M|}{t} = \binom{m+j}{t} = \sum_{\ell=1}^{y} (c^M)_\ell = y \cdot (c^M)_1$ and y divides $\binom{m+j}{t}$ for all j, $0 \leqslant j \leqslant r$.

Now suppose the lemma is true for some i and all $j \geqslant i$. Then, by a general formula for combinations, we obtain

$$\binom{m+j - (i+1)}{t - (i+1)} = \binom{m+j - i}{t - i} - \binom{m+(j-1) - i}{t - i} .$$

If $j \geqslant i + 1$, both terms on the right hand side are divisible by y and thus the lemma is proved.

$\diamond$

Suppose now that H is $(t-1)$-fold transitive on Ω and let θ_i be a subset of Ω with $0 \leqslant |\theta_i| = i \leqslant t - 1$. Then

(5) $\Omega \setminus \theta_i$ is a symmetric set and y divides $n - t + 1$.

Proof: If $\theta = \theta_i$ has size at most $t - 1$ and if g is in G, then there exists some h in H such that $\theta^g = \theta^h$ since H is $(t-1)$-fold transitive. Therefore $(H_\theta)^g = H_{(\theta^g)} = H_{(\theta^h)} = (H_\theta)^h$ and H_θ^G is a class of H-conjugate subgroups. Observe that $\mathrm{Fix}(H_\theta) = \theta$: this is clear if $|\theta| < t - 1$ since then H_θ is transitive on $\Omega \setminus \theta$. If $|\theta| = t - 1$ and $G \neq \mathrm{Sym}\,\Omega$, the H_θ-orbits on $\Omega \setminus \theta$ have length at least 2 by lemma 2.4. If $G = \mathrm{Sym}\Omega$, $n = t + 1$ and H is t-fold homogeneous, i.e. $y = y(H) = 1$. Therefore $\mathcal{M}(H_{\theta^*}) := \{\Omega \setminus \theta^*\}$ for every θ^* in $\Omega\{i\}$ is a well-defined G-function and $\Omega\setminus\theta$ is a symmetric set by 4.1. By lemma 4.3 y divides $n - t + 1$. $\diamond$

With these preparations we turn to some applications. The existence of subgroups U (or subsets) of G with $U^G = U^H$ provides symmetric collections which in some cases lead to symmetric subsets of Ω. As we have seen above, the size of symmetric sets imposes restrictions on y so that one hopes to be able to show $y = 1$, that is, H is t-fold homogeneous.

As an illustration consider the following case: G is doubly transitive on Ω having a regular normal subgroup H. If n is even, by lemma 2.3, H is an elementary Abelian 2-group of order $|\Omega|$. Choose a base point α in Ω and identify $H \leftrightarrow \Omega$ in the usual way, i.e. $h \leftrightarrow \beta$ if $\alpha^h = \beta$. Then it is clear that G acts (by conjugation) transitively on the involutions $H \setminus \{1\}$. But also the converse is true: Any subgroup A of G containing H is doubly transitive on Ω if $H \setminus \{1\}$ is a class of A-conjugate involutions.

We show that this observation can be generalized to

THEOREM 4.4

Let G be a doubly transitive permutation group of degree n and $H \neq 1$ a normal subgroup of G. Suppose G contains an involution i with $i^G = i^H$ and let $f = |Fix(i)|$.

Then y(H) divides (n-1, f-1). In particular, H is doubly transitive if (n-1, f-1) = 1.

Proof: H is at least transitive and so y(H) divides n-1 by remark (5). Let i^* be an involution in i^G and $i^* = (\alpha_1)(\alpha_2)\ldots(\alpha_f)\ (\alpha_{f+1},\alpha_{f+2})\ldots(\alpha_{n-1},\alpha_n)$ its cycle decomposition. Define a G-function M: $i^G \to P(\Omega)$ by $M(i^*) = \{\{\alpha_{f+1},\alpha_{f+2}\},\ldots,\{\alpha_{n-1},\alpha_n\}\}$. By lemma 4.1, $c^{\Sigma M}$ is a G-invariant vector, this implies that the y(H) components of $c^{\Sigma M}$ are all equal, i.e. y(H) divides their sum which equals (n-f)/2. Therefore $0 \equiv n-f \equiv (n-1) + 1 - f \equiv (f-1)$ mod y and thus y(H) divides (n-1, f-1). If (n-1, f-1) = 1, H is therefore doubly homogeneous but also generously transitive: Since $n \equiv f \bmod 2$, (n-1, f-1) = 1 implies that n is even and trivially (or theorem 2.8) H is generously transitive and so doubly transitive.

$\diamond$

Remark: We mention that the above proof can be equally used for arbitrary elements i of order p (p a prime) to prove that y divides $(n-1, ((f-1)\cdot(p-1))/2)$. Hence H is doubly homogeneous if G contains some element i of order 3 with $i^G = i^H$ such that (n-1, f-1) = 1.

As an application of theorem 4.4 we obtain the following result relating properties of some 2-local subgroup of H to the transitivity of H.

THEOREM 4.5

Let G be a doubly transitive permutation group on Ω of even degree n and $H \neq 1$ a normal subgroup of G. Let α, β be two distinct points in Ω and S a Sylow 2-subgroup of $H_{\alpha,\beta}$. Put $N = N_G(S)$ and $N' = N \cap H$.

Suppose there is some subgroup I, $S < I < H$, with $[I:S] = 2$ and $I^N = I^{N'}$. Then H is doubly transitive.

Proof: Let $\Delta = Fix(S)$, $2 \leqslant |\Delta| \equiv n \bmod 2$; by theorem 3.5 we have $x(H) = |CCl_{G:H}(S)| \cdot x(H^\Delta)$ and since n-1 is odd $x(H) = x(H^\Delta)$. We recall that G^Δ is doubly transitive on Δ, H^Δ is a normal subgroup of G^Δ and involutions in H^Δ fix no point at all. It suffices to show that H^Δ is doubly transitive.

The subgroup I of H normalizes S and so I fixes Δ as a set. I corresponds to some involution $i = I \cdot H_\Delta / H_\Delta$ in H^Δ fixing no point of Δ. We show $i^{(G^\Delta)} = i^{(H^\Delta)}$. By the Frattini argument we have $G_{\{\Delta\}} = N_G(S) \cdot H_\Delta$. Let g be in $G_{\{\Delta\}}$, $g = \bar{g} \cdot h$ with $\bar{g} \in N_G(S)$ and $h \in H_\Delta$; by assumption there is some $\bar{h}$ in $N_H(S)$ such that $I^g = I^{\bar{h} \cdot h}$, $\bar{h} \cdot h \in H_{\{\Delta\}}$. This implies $i^{(G^\Delta)} = i^{(H^\Delta)}$. Now theorem 4.4 applies for G^Δ and i (f = 0) and so H^Δ is doubly transitive on Δ and by the above remark, H is doubtly transitive.

◇

Remark 1: Candidates for I are subgroups of a Sylow 2-subgroup T of $N_H(S)$. In particular every characteristic subgroup I of T with $2=[I:S]$ satisfies the hypothesis of theorem 4.5. (More precisely: I only needs to be invariant under all automorphisms of T normalizing S). Since $[T:S]$ is bounded by $|n|_2$, one arrives immediately to the following conclusions:

(a) Aschbacher [1] : If $n \equiv 2 \bmod 4$, H is doubly transitive (take $I = T$).

(b) If $n \equiv 4 \bmod 8$, either H is doubly transitive or else $n \equiv 1 \bmod 3$, S is normal in some Sylow 2-subgroup U of H, U/S is a Klein group and the 3 involutions in U/S correspond to $y(H) = 3$ H-orbits on $\Omega\{2\}$.

Remark 2: As we said earlier (in the remark after theorem 4.4) a similar version of theorem 4.4 holds for elements j of order p ($p \neq 2$) with $j^G = j^H$: $y(H)$ divides $(n-1, ((f-1)\cdot(p-1))/2)$ where $f = |Fix(j)|$.

We can use this in the situation of theorem 4.5 : Let $p \neq 2$ be a prime dividing n, S in $Syl_p(H_{\alpha,\beta})$ and J a subgroup of H with $[J:S] = p$ and $J^N = J^{N'}$ where $N = N_G(S)$ and $N' = N \cap H$. Then $y(H)$ divides $(n-1, {}^{(p-1)}/2)$.

(We omit the proof of this statement which can be given in almost the same way as the above proof.) Similar to the conclusions in Remark 1 we obtain:

(a) If $n \equiv p \bmod p^2$ and $(n-1, {}^{(p-1)}/_2) = 1$ H is doubly homogeneous and H is also doubly transitive if and only if H has even order. In particular we obtain a version of Aschbacher's result for p = 3: If $n \equiv 3 \bmod 9$, H is doubly homogeneous on Ω and H is doubly transitive for $n > 3$ since in this case G cannot contain a regular normal subgroup. (See theorem 2.8).

(b) If $n \equiv p^2 \bmod p^3$ and $(n-1, {}^{(p-1)}/_2) = 1$, then either H is doubly homogeneous or else S is normal in some Sylow p-subgroup U of H, U/S is elementary Abelian of order p^2, $|Fix(S)| \equiv p^2 \bmod p^3$ and $y(H)$ divides $p+1$.

Using similar methods we obtain the following result about doubly transitive groups whose degree contains some prime only in small powers:

THEOREM 4.6

Let G be a doubly transitive group of degree n and H a normal subgroup of G with index $[G:H] = d$ in G.

Let p be a prime dividing n to the j-th power, i.e. $|n|_p = p^j$, $j \geqslant 1$.

Suppose either (i) $(d, n-1) = 1$ or (ii) $(d, p^i - 1) = 1$ for all $i \leqslant j$ and assume that G/H is solvable if $p = 2$ and $j \geqslant 2$.

Then H is doubly transitive.

Proof: We assume that G and H are groups of smallest degree n satisfying the hypothesis but not the conclusion of the theorem. Furthermore, we may also assume that there is no normal subgroup of G between G and H, that is, G/H is simple. We will see that H must be doubly homogeneous and generously transitive, contrary to our assumption.

The first hypothesis (i) is clear: G/H acts transitively on $\{U_1, \ldots, U_y\}$ and so y divides $|G/H| = d$. But y also divides n-1 by (5). Hence $y = y(H) = 1$ and H is doubly homogeneous. If n is even, also $|H|$ is even and by theorem 2.8 H is generously transitive and so H is doubly transitive. If n is odd, also d is odd and since $|G|$ is even, also $|H|$ is even, i.e. H is doubly transitive.

Now let H,G be a counterexample as above and p a prime with $|n|_p = p^j$, $j \geqslant 1$. Let T be in $Syl_p(H_\alpha)$ and $\Delta = \mathrm{Fix}\ T$. Put $n' = |\Delta|$; then $n' \equiv n \bmod p$ and $|n'|_p \leqslant [S:T] = |n|_p$ for some S in $Syl_p(H)$ containing T. G^Δ is doubly transitive on Δ (proposition 3.3), H^Δ is normal in G^Δ and transitive but not doubly transitive on Δ by theorem 3.5. Observe also that $d' = |G^\Delta/H^\Delta|$ divides $d = |G/H|$. Thus G^Δ and H^Δ satisfy the hypotheses of the theorem and by minimality we have $n = n'$ and therefore $T = 1$, i.e. H_α is a p'-group. Hence p^j is the largest power of p dividing the order of H and Sylow p-subgroups of H are semiregular having $|n|_{p'}$ orbits of equal length $|n|_p$.

Now we deal first with the case $p = 2$, $j = 1$, that is $n \equiv 2 \bmod 4$. Since $|H|$ is divisible by 2 but not by 4, H contains a normal 2-complement $\bar{H}$. $\bar{H}$ is normal in G and cannot be transitive since $2 \nmid |\bar{H}|$.

Thus $\bar{H} = 1$ and so $H = G = Sym\ (2)$, a contradiction to our assumption.

In all other cases G/H is solvable: This follows in the case $p \neq 2$ by the Feit-Thompson Theorem. Hence G/H is a simple abelian group, i.e. $d = |G/H| = q$ for some prime q. By assumption q is odd and so $|H|$ is even and H is generously transitive. Hence H is not doubly homogeneous. Let $y = y(H) = q$ be the number of H-orbits on $\Omega\{2\}$. Then $y = q$ divides $n-1$ by (5).

Define a G-map M: $Syl_p(H) \to P(\Omega)$ by $M(S) = \{O \mid O$ is an orbit of S. $S \in Syl_p(H)\}$. Then $M(S)$ is a symmetric collection of $|n|_{p'}$ sets of size $|n|_p$ by lemma 4.1. Let T be an $N_G(S)$-orbit on $M(S)$. Since $[N_G(S):N_H(S)] = q$, T is either an $N_H(S)$-orbit or a union of q $N_H(S)$-orbits. Not all $N_G(S)$-orbits can split over $N_H(S)$, otherwise q would divide $|M(S)| = |n|_{p'}$, which is impossible since q divides $n - 1$. Hence $M(S)$ contains some symmetric set of size $|n|_p = p^j$ by lemma 4.2. Hence q divides $\binom{p^j}{2}$, i.e. $q \mid p^j-1$. This contradicts the assumption $(d, p^j-1) = 1$ and H is doubly homogeneous on Ω, a final contradiction.

$\diamond$

CHAPTER V TRIPLY TRANSITIVE GROUPS

This chapter is a continuation of Chapter III. As we have seen there the transitivity of a normal subgroup H in G may be discussed in terms of its transitivity properties on designs D(p,H). In fact, proposition 3.1 tells us that H is t-fold transitive on Ω if and only if H is transitive on the blocks of D(p,H) and t-fold transitive on each block. In the present chapter we will investigate these two conditions for triply transitive groups.

Section 5.1 deals with the design D(2,H). Here one uses a result of Hering to show that G^{Δ} and H^{Δ}, Δ a block of D(2,H), are one-dimensional projective groups and so the second condition above does not pose any problem. The transitivity of H on blocks of D(2,H) is equivalent to $S^G = S^H$ where S is a Sylow 2-subgroup of $H_{\alpha,\beta,\gamma}$. So, whenever one can show $S^G = S^H$, we obtain that H has at most 2 orbits on $\Omega\{3\}$. In proposition 5.3 we prove this in the case $n \equiv 0 \bmod 4$ under the assumption that involutions in H do not fix too many points. Using more elaborate fusion arguments one should be able to extend proposition 5.3. considerably.

In section 5.2 we suppose that the degree of G is divisible by 3 and we investigate the design D(3,H). In this situation one observes easily that H is transitive on the blocks of D(3,H) and so the emphasis lies on the determination of G^{Δ}. We are able to show that also in this case G^{Δ} is a projective group and this implies that H has at most 2 orbits on $\Omega\{3\}$.

We mention that the proofs in this section use a number of very deep results on abstract finite groups like the Feit-Thompson theorem, Glauberman's Z*-theorem and Suzuki's characterization of ZT-groups.

5.1 TRIPLY TRANSITIVE GROUPS OF EVEN DEGREE

G is a triply transitive group on Ω of even degree n and H is a normal subgroup of G. In Chapter III we have defined designs D(p,H) whose points are the points in Ω and whose blocks are the G-images of $Fix(S_p)$ where S_p is a Sylow p-subgroup of $H_{\alpha,\beta,\gamma}$. If H is at least doubly transitive and if $H_{\alpha,\beta,\gamma}$ has odd order, D(2,H) has only one block Ω and Hering's theorem (lemma 2.9) implies that G is a projective group acting on the projective line Ω in the usual way or $Alt(6) = H \leqslant G \leqslant Sym(6)$ with n = 6.

The following proposition is a consequence of Hering's result and shows in the general case that G preserves a 3-design whose blocks are projective lines:

PROPOSITION 5.1

Let G be a triply transitive permutation group of even degree n and $H \neq 1$ a non-regular normal subgroup of G.

Then D(2,H) is a 3-design, G is contained in the automorphism group of D(2,H) and is transitive on its blocks.

For any block Δ of D(2,H), H^Δ and G^Δ are isomorphic to either

(i) subgroups of $P\Gamma L(2,q)$ containing $PSL(2,q)$ where $|\Delta| - 1 = q$ is some odd prime power, or

(ii) $H^\Delta \cong Alt(6) \leqslant G^\Delta \leqslant Sym(6)$ and n = 6.

Proof: We only need to show that the statement about the isomorphisms holds.

The group H^{Δ} is normal in G^{Δ} and is at least doubly transitive on Δ where $|\Delta|$ is even. This follows from 2.4 and 3.3. By construction $|(H^{\Delta})_{\alpha,\beta,\gamma}|$ is odd for any three distinct points in Δ and to apply Hering's theorem we need to show that $|(H^{\Delta})_{\alpha,\beta}|$ is even. Let S be in $Syl_2(H_{\alpha,\beta,\gamma})$ such that Fix(S) = Δ and let T be in $Syl_2(H_{\alpha,\beta})$ containing S. If t is an element of $N_T(S)\setminus S$, then t fixes Δ as a set but not pointwise and so $(H^{\Delta})_{\alpha,\beta}$ has even order. So it remains to show that $N_T(S) \setminus S$ is not empty, i.e. $T \neq S$. Now T = S if and only if $[H_{\alpha,\beta}:H_{\alpha,\beta,\gamma}] = n-2/x(H)$ is odd. But this contradicts theorem 3.8 except in the case $PSL(2,n-1) \leqslant H \leqslant G \leqslant P\Gamma L(2,n-1)$.

Therefore there are two possibilities: either $|(H^{\Delta})_{\alpha,\beta}|$ is even and by Lemma 2.9 we get (i) $PSL(2,q) \leqslant H^{\Delta} \leqslant P\Gamma L(2,q)$, $|\Delta| = q+1$, or (ii) $H^{\Delta} \cong Alt(6)$, $|\Delta| = 6$, or otherwise H itself is a subgroup of $P\Gamma L(2,n-1)$ containing $PSL(2,n-1)$. In the first case the statement of the proposition holds since $PSL(2,q)$ is characteristic in H^{Δ} and so $G^{\Delta} \leqslant P\Gamma L(2,q)$ or $G^{\Delta} \leqslant Sym(6)$. In the second case, H^{Δ} and G^{Δ} must be one of the groups under (i) since these are the only triply transitive groups involved in $P\Gamma L(2,n-1)$. See for instance II.8.27 in [10].

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A well-known type of 3-designs arises in this way from one-dimensional projective groups: Let K be a finite field with p^f, $f > 1$, elements. Then $P\Gamma L(2,p^f) = PGL(2,p^f)\cdot F$ is triply transitive on the projective line $PG_1(p^f)$ where F is the group of Frobenius automorphisms of K of order $|F| = f$. Let p^* be a prime dividing f and F^* a p^*-subgroup of F. Put $f^* = [F:F^*]$; then the subfield K^* of K fixed by F^* has p^{f^*} elements Put $H := PGL(2,p^f)\cdot F^*$. Then $D(p^*,H)$ is a $3\text{-}(p^f+1,\ p^{f^*}+1,\ 1)$ design and H^{Δ}, G^{Δ} satisfy $PSL(2,p^{f^*}) \lessapprox H^{\Delta} \trianglelefteq G^{\Delta} \lessapprox P\Gamma L(2,p^{f^*})$ for any block Δ of $D(p^*,H)$. The block through 0, 1, ∞ is $K^* \cup \{\infty\}$ and so $D(p^*,H)$ consists of 'sublines' $PG_1(p^{f^*})$ of the projective line $PG_1(p^f)$.

This example also includes the Miquelian type of finite inversive planes for $f = 2\cdot f^*$.

Other examples of designs where blocks are in some sense projective lines one should mention in this context are the 2-designs associated with $PSU(3,q)$ and the groups of Ree type: Let G be either $AUT(PSU(3,q))$, q odd, or a group of Ree type with characteristic $q = 3^{2a+1}$ in their usual representation on a set Ω of q^3+1 points. (See for instance Ward [25] and lemmata 3.2, 3.3 in [9]).

In either case G contains some involution i fixing precisely $q + 1$ points. Arguments similar to those in the proof of 5.1 show that G induces a doubly transitive group G^Δ on $\Delta = Fix(i)$ and $G^\Delta \geqslant PSL(2,q)$. Therefore $D = (\Omega, \Delta^G)$ is a 2-design whose blocks are projective lines.

In general, however, it seems to be quite hopeless to recognise $D(p,H)$ as some specific geometrical object. Yet for the question about the transitivity of H, this design concept is to some extent useful. So, for instance the question arises, under which circumstances is H necessarily transitive on the blocks of $D(p,H)$?

Before we come to this question, we first prove the following lemma:

<u>LEMMA 5.2</u>

Let $B \lhd A$ be groups and S a p-subgroup of B for some prime p. Suppose $ccl_A(S) \neq ccl_B(S)$. Then there exists some $a \in A$ such that $S \neq S^a$, $S^a \leqslant N_B(S)$ and $S \leqslant N_B(S^a)$.

Proof: Assume first there is some $T \in Syl_p(B)$ such that $S \ntriangleleft T$. Then $N_T(N_T(S)) \setminus N_T(S)$ is not empty and for any a in this set we get $S \neq S^a \leqslant (N_T(S))^a = N_T(S) = N_T(S^a)$. Thus $S^a \leqslant N_B(S)$ and $S \leqslant N_B(S^a)$, $S^a \neq S$.

Now we may assume that S is normal in every Sylow p-subgroup of B containing S and the same is true for every conjugate of S. Let c be in A such that S and S^c are not conjugate in B. Let T be in $Syl_p(B)$ containing S. Then there is some $b \in B$ such that $(S^c)^b$ is also contained in T. Put $cb = a$. Then $S^a \neq S$ are by assumption normal subgroups of T and hence normalize each other.

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Now suppose H is not transitive on the blocks of $D(p,H)$. This implies that $S^G(S \in Syl_2(H_{\alpha,\beta,\gamma}))$ is not a class of H-conjugate subgroups of H and by 5.2 every S^* in S^G determines a set of conjugates $S_0 = S^*, S_1, \ldots, S_r$ such that S_0 and S_i normalise each other.

Therefore 5.2 implies that $D(p,H)$ also carries a graph structure preserved by G by defining edges (Δ_0,Δ_i) if $\Delta_0 = Fix(S_0)$ and $\Delta_i = Fix(S_i)$. This graph has some interesting properties.

PROPOSITION 5.3

Let G be a triply transitive permutation group of degree n with normal subgroup H.

Suppose $n \not\equiv 2 \bmod 4$ and if $n \equiv 0 \bmod 4$, that no involution in H fixes as much as $2\cdot|\mathrm{Fix}(S)|(\geq 8)$ points where S is a Sylow 2-subgroup of the stabilizer in H of three distinct points.

Then H has at most two orbits on the blocks of D(2,H).

Proof. We may assume that $H_{\alpha,\beta,\gamma}$ has even order, otherwise D(2,H) is degenerate and consists of only one block Ω. Hence $1 \neq H$ is not regular and so doubly transitive. If n is odd, $S \in Syl_2(H_{\alpha,\beta,\gamma})$ is also a Sylow subgroup of $H_{\alpha,\beta}$ and so S^G is a class of H-conjugate subgroups of H. This implies that H is transitive on the blocks of D(2,H).

Hence assume for the remainder of the proof that n is divisible by 4 and that H is not transitive on the blocks of D(2,H). Let Δ and Δ' be two blocks in D(2,H) belonging to different H-orbits. Since H is doubly transitive on Ω, we may assume that Δ and Δ' have two points α,β in common. Let S be in $Syl_2(H_{\alpha,\beta,\gamma})$ and S' in $Syl_2(H_{\alpha,\beta,\gamma'})$ such that $\Delta = \mathrm{Fix}(S)$ and $\Delta' = \mathrm{Fix}(S')$. S and S' are not conjugate in $B = H_{\alpha,\beta}$ but conjugate in $A = G_{\alpha,\beta}$; $G_{\alpha,\beta}$ is transitive on $\Omega \setminus \{\alpha,\beta\}$ and so there is some g in $G_{\alpha,\beta}$ with $\gamma^g = \gamma'$, i.e. $S^g \leq H_{\alpha,\beta,\gamma'}$ and therefore S and S' are conjugate in A.

By lemma 5.2, there is some conjugate S^* of S in B such that S and S^* normalize each other. Let $\Delta^* = \mathrm{Fix}\, S^*$. Then $\Delta \cap \Delta^* = \{\alpha,\beta\}$ because a further point δ in $\Delta \cap \Delta^*$ would imply $S \trianglelefteq S\cdot S^* < H_{\alpha,\beta,\delta}$ which is impossible since S is a Sylow subgroup of $H_{\alpha,\beta,\gamma}$.

Suppose i is an involution in $S \cap S^*$. Then i fixes all $2\cdot|\Delta| - 2$ points in $\Delta \cup \Delta^*$ and by assumption these are the only points fixed by i. So i has degree $n - 2(|\Delta| - 1) \equiv 2 \bmod 4$ since $|\Delta|$ is even and $n \equiv 0 \bmod 4$. Thus H contains an odd permutation and so there is a normal subgroup H^* of H not containing i, $\langle i\rangle \cdot H^* = H$; clearly H^* is also normal in G. If U in $Syl_2(H^*_{\alpha,\beta,\gamma})$ is contained in S, we get $S = \langle i\rangle \cdot U$ and since every involution in $S \cap S^*$ is an odd permutation, we obtain $U \cap S^* = 1$ and so $S \cap S^* = \langle i\rangle$. This shows that $S \cap S^*$ has order at most 2. On the other hand, since $[H_{\alpha,\beta}:H_{\alpha,\beta,\gamma}]$ divides $n-2$, $|[H_{\alpha,\beta}:H_{\alpha,\beta,\gamma}]|_2 \leqslant 2$. Thus $T := S\cdot S^*$ is a Sylow 2-subgroup of $H_{\alpha,\beta}$ and $|S\cdot S^*| = 2\cdot|S|$. This gives two possibilities:

(a) $S \cap S^* = 1$, $|S| = |S^*| = 2$ and T is a four group with involutions s, s^* and $s\cdot s^*$,

(b) $S \cap S^* = \langle i\rangle$, $|S| = |S^*| = 4$, $|T| = 8$; $\langle u\rangle = U = S \cap H^*$, $\langle u^*\rangle = U^* = S^* \cap H^*$. Since u^* is the only even permutation in S^*, S centralizes $S^* = \langle u^*, i\rangle$. So $T = S\cdot S^*$ is elementary abelian containing the involutions u, u^*, $u\cdot u^*$ (even permutations) and i, $i\cdot u$, iu^*, iuu^* (odd permutations).

Obviously we can assume that S' is also contained in T and we recall that S and S' are not conjugate under H. This means that in case (a) not all involutions in T are H-conjugate and hence there is some involution j in T such that $j^H \cap T = j$.

In case (b) we claim that either the same fact is true or otherwise H has at most 2 block orbits on $D(2,H)$: If u, u^*, uu^* are not all H-conjugate, we take j in $\{u, u^*, u \cdot u^*\}$. Hence suppose they are all conjugate in H. By Burnside's theorem (p.155 in [5]) this is equivalent to conjugacy in $N_H(T)$. Therefore we can assume that S' is not only contained in T but also contains (say) u. But apart from S there is only one subgroup of T containing u and some odd transposition and therefore $S' = \langle u, iu^* \rangle$. Thus every block orbit of H on $D(2,H)$, different from the one containing Δ, leads to the same subgroup S' clearly H has at most two orbits on the blocks of $D(2,H)$.

Hence assume for the remaining part of the proof that $j \in T \in Syl_2(H_{\alpha,\beta})$ has the property $j^H \cap T = \{j\}$. Note that T is also a Sylow 2-subgroup of H_α. By Glauberman's Z^*-theorem [6] j is in $Z^*(H_\alpha)$. If $O(H_\alpha) = 1$, $Z^*(H_\alpha) = Z(H_\alpha)$, the centre of H_α, is a normal subgroup of G_α and therefore transitive on $\Omega \setminus \{\alpha\}$. But since $Z(H_\alpha)$ is abelian, $Z(H_\alpha)$ is indeed regular on $\Omega \setminus \{\alpha\}$, a contradiction, $n-1$ is odd. Thus $O(H_\alpha) \neq 1$. Let M be a minimal normal subgroup of G_α contained in $O(H_\alpha)$. Then M is elementary abelian and transitive on $\Omega \setminus \{\alpha\}$ since $O(H_\alpha)$ is solvable and G_α doubly transitive on $\Omega \setminus \{\alpha\}$. Lemma 2.11 lists all groups with this property and the only possibility is $PSL(2,n-1)$ $H \leqslant G \leqslant P\Gamma L(2,n-1)$. Now it is easy to see that H has at most 2 block orbits on $D(2,H)$: Let Δ, Δ' be two blocks and, since H is doubly transitive, assume that Δ and Δ' both contain $0, \infty$. If Δ and Δ' both contain some (non-)squares (as elements of the field $GF(n-1)$, then Δ and Δ' belong to the same $(PSL)_{0,\infty}$-orbit, since $(PSL)_{0,\infty}$ is transitive on (non-)squares. So, if Δ and Δ' do not belong to the same H-orbit, necessarily $\Delta \setminus \{0,\infty\}$ consists of squares and $\Delta' \setminus \{0,\infty\}$ consists of non-squares. Hence H has at most 2 block orbits on $D(2,H)$.

$\diamond$

THEOREM 5.4

Let G be a triply transitive permutation group on Ω of degree n, $n \equiv 0 \bmod 4$.

Suppose H is a nonregular normal subgroup of G such that no involution in H fixes $2\cdot|Fix(S)|(\geq 8)$ points where S is a Sylow 2-subgroup of the stabilizer in H of three distinct points in Ω.

Then either H is triply transitive on Ω or $PSL(2,q) \leq H \leq G \leq P\Gamma L(2,q)$ for some odd prime power q with $q + 1 = n$.

Proof: Let $H \triangleleft G$ be a counterexample of minimal degree n and let S be in $Syl_2(H_{\alpha,\beta,\gamma})$. By theorem 3.8 and lemma 2.9, S is not the identity subgroup and by proposition 3.1 we obtain $x(H) = z\cdot x(H^{\Delta})$ where z is the number of block orbits of H in D(2,H) and $\Delta = Fix(S)$. Proposition 5.3 implies that z is at most 2.

G^{Δ} is a triply transitive group of degree $d = |\Delta|$ with normal subgroup H^{Δ} for which the theorem is true: By construction involutions in H^{Δ} fix at most 2 points and d is divisible by 4: $|d-2|_2 =$
$=|[(G^{\Delta})_{\alpha,\beta}:(G^{\Delta})_{\alpha,\beta,\gamma}]|_2 \leq |[G_{\alpha,\beta}: G_{\alpha,\beta,\gamma}]|_2 = |n-2|_2 = 2$. Therefore $x(H^{\Delta}) \leq 2$ and so $x(H) = 1, 2, 4$. $x(H) = 4$ is impossible since 4 does not divide n-2 and $x(H) = 2$ contradicts theorem 3.8. So only $x(H) = 1$ remains, a contradiction to our assumption.

$\diamond$

5.2 TRIPLY TRANSITIVE GROUPS OF DEGREE DIVISIBLE BY 3.

In this section G is a triply transitive group on Ω of degree n, $n \equiv 0 \bmod 3$ and H is a normal subgroup of G.

We begin with a classification theorem similar to Hering's theorem.

THEOREM 5.5

Let G be a triply transitive permutation group of degree n, $n \equiv 0 \bmod 3$.

Suppose G_α contains a normal subgroup $M \neq 1$ such that $|M_{\beta,\gamma}|$ is prime to 3 for distinct points α,β,γ.

Then G is isomorphic to a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ for some prime power q with $q = n-1$.

Proof: Let M^* be a minimal normal subgroup of G_α contained in M. Since G_α is doubly transitive on $\Omega \setminus \{\alpha\}$, a result of Burnside (page 202 in [5]) implies that M^* is either simple and primitive on $\Omega \setminus \{\alpha\}$ or else is an elementary abelian p-group, regular on $\Omega \setminus \{\alpha\}$.

Consider the simple case first. It is not difficult to see that the hypotheses of the theorem imply that M^* is a 3'-group: M^* is 3/2-fold transitive on $\Omega\setminus\{\alpha\}$ and so $|M^*|$ divides $|M^*_{\beta,\gamma}|\cdot(n-1)\cdot(n-2)$ which is prime to 3. Therefore M^* is a Suzuki group $Sz(2^{2r+1})$, $r \geqslant 1$, by the Martineau-Thompson result Lemma 2.13.

Since M^* is primitive on $\Omega \setminus \{\alpha\}$, $C_{G_\alpha}(M^*) \leqslant M^*$, which means that G_α/M^* is a subgroup of $OUT(M^*)$. And so $[G_\alpha : M^*]$ divides $2r+1$ (see

Suzuki's paper [21], theorem 11), i.e. the order of G_α divides $(2r+1)\cdot q^2\cdot(q-1)\cdot(q^2+1)$ where $q = 2^{2r+1}$. The primitivity of M^* also implies that M^*_β is a maximal subgroup of M^*. From this information and the list of subgroups of M^* (theorem 9 in [21]) we conclude that M^* operates on $\Omega\backslash\alpha$ in its usual representation on $q^2 + 1$ points.

We show that this leads to a contradiction, a Suzuki group of degree $q^2 + 1$ does not possess a transitive extension. For suppose G existed such that G_α contains a normal subgroup $M^* \cong Sz(q)$ and G_α/M^* (as we have seen above) has odd order. Since G is transitive its degree is $q^2 + 2 = 2(2^{4r+1} + 1)$. Let i be an involution in G. If i fixes a point of Ω, i lies in M^* since we then can assume that $\alpha \in Fix(i)$, and so i fixes at most 2 points. But i must have at least one fixed point, otherwise i is an odd permutation and G has a normal subgroup of index 2 which is impossible. Hence every involution in G fixes exactly 2 points. Suppose $j = (\alpha,\beta)(\gamma)(\delta)\ldots$ is an involution normalizing $G_{\alpha,\beta}$ and let S be a Sylow 2-subgroup of M^*_β. S is characteristic in $G_{\alpha,\beta}$ and so j normalizes S. According to the structure of M^*, S is regular on $\Omega\backslash\{\alpha,\beta\}$ and fixes α,β. $C_S(i) \neq 1$ fixes $\{\gamma,\delta\}$ as a set and hence by regularity $|C_S(i)| = 2$.
Thus $C_{S\cdot\langle i\rangle}(i)$ has order 4 and by a well-known lemma of Suzuki $S\cdot\langle i\rangle$ is either dihedral or semidihedral and the same is true for S. This finally is a contradiction, S has exponent 4 and is not dihedral, see lemma 1 in [21].

This shows that M^* is not a simple group and therefore M^* is an elementary abelian p-group, regular on $\Omega\backslash\{\alpha\}$.

Let $\bar{M}$ be a minimal normal subgroup of G. $\bar{M}$ is solvable only if $G = \mathrm{Sym}\ (3)$ (lemma 2.6) and, by lemma 2.11, $\bar{M}$ is one of the following groups:

(a) $\bar{M} \cong PSL(2,q)$, $n = q + 1$

(b) $\bar{M} \cong Sz(q)$, $n = q^2 + 1$, $q = 2^{2r+1}$

(c) $\bar{M} \cong PSU(3,q)$, $n = q^3 + 1$

(d) $\bar{M}$ is isomorphic to a group of Ree type,

$$n = q^3 + 1,\ q = 3^{2r+1}.$$

Possibilities (b) and (d) do not occur since 3 divides n. In case (c) we prove that PSU(3,q) is not a normal subgroup of a triply transitive group: $(PSU)_{\alpha,\beta}$ is cyclic and contains a subgroup U with $\{\alpha,\beta\} \neq \mathrm{Fix}\ U \neq \Omega$. (See II.10.12 in [10]). U is normal in $G_{\alpha,\beta}$ and so $\Delta \setminus \{\alpha,\beta\}$ contains an orbit of $G_{\alpha,\beta}$, i.e. $G_{\alpha,\beta}$ is not transitive on $\Omega \setminus \alpha,\beta$. Therefore only $PSL(2,q) = \bar{M} \leqslant G \leqslant P\Gamma L(2,q)$, $q = n-1$ remains and the theorem is proved.

$\diamond$

Theorem 5.5 allows us to state the following analog to proposition 5.1:

<u>PROPOSITION 5.6</u>

Let G be a triply transitive permutation group of degree n, $n \equiv 0 \bmod 3$, and $H \neq 1$ a normal subgroup of G.

Then D(3,H) is a 3-design, G is contained in the automorphism group of D(3,H) and is transitive on its blocks. For any block Δ in D(3,H) we have $PSL(2,q) \leqslant H^{\Delta} \leqslant G^{\Delta} \leqslant P\Gamma L(2,q)$ for some prime power q, $q \equiv 2 \bmod 3$.

<u>Proof</u>: G^{Δ} and H^{Δ} satisfy the hypotheses of theorem 5.5.

◇

Compared to the situation in section 5.1 this time the question about the transitivity of H on the blocks of D(3,H) finds an easy answer: If $G \neq Sym(3)$, H is at least doubly transitive and $[H_{\alpha,\beta} : H_{\alpha,\beta,\gamma}] = (n-2)/x(H) \not\equiv 0 \bmod 3$ shows that $S \in Syl_3(H_{\alpha,\beta,\gamma})$ is also a Sylow 3-subgroup of $H_{\alpha,\beta}$ and therefore S^G is a class of H-conjugate subgroups, i.e. H is transitive on the blocks of D(3,H). By theorem 3.5 we obtain $x(H) = x(H^{\Delta})$ where $\Delta = Fix(S)$ and 5.6 implies $x(H^{\Delta}) \leqslant 2$. Hence:

<u>THEOREM 5.7</u>

Let G be a triply transitive permutation group on Ω of degree divisible by 3.

Then every normal subgroup $H \neq 1$ of G has at most 2 orbits on $\Omega\{3\}$.

In Chapter III, theorem 3.8, we have seen that the projective groups are the only groups of degree $\equiv 0 \bmod 4$ for which x(H) is even. Therefore, as a corollary of 3.8 and 5.7 we have:

THEOREM 5.8

Let G be a triply transitive group of degree n, $n \equiv 0 \bmod 12$.

Then either every normal subgroup $H \neq 1$ of G is triply transitive or $PSL(2,q) \leqslant H \leqslant G \leqslant P\Gamma L(2,q)$ with $q = n - 1$.

◇ ◇ ◇

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