# SOME SAMPLE PATH PROPERTIES OF GAUSSIAN PROCESSES

by

JOAQUIN ORTEGA-SANCHEZ

THESIS

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Department of Mathematics

Imperial College

A Isbelia, Titica y Madraina

# ABSTRACT

The aim of this thesis is to study the asymptotic behaviour of certain Gaussian processes.

Let $X = \{X(t), t \in \mathbb{R}\}$ be a centred Gaussian process with $EX^2(t) = 1$ for all $t \in \mathbb{R}$ and $\sigma(t,s) = [E(X(t) - X(s))^2]^{1/2}$. Then $\sigma(t,s)$ is a pseudometric in $\mathbb{R}$ which we use for all metric considerations.

Let $Z(t) = \sup\{X(s), s \in [0,t]\}$. Sufficient conditions in terms of the metric entropy of the process are obtained for the following result:

$$\limsup_{t\to\infty} (Z(t) - (2 \log t)^{1/2}) \leq 0 \qquad \text{a.s.}$$

It is shown that all continuous, $\sigma$-separable and stationary Gaussian processes satisfy these conditions.

As more restrictions are imposed on $\sigma(t,s)$ it is possible to obtain sharper results. In Chapter 3 we obtain upper and lower bounds for $P\{Z(t) > x\}$ as $x \to \infty$ for stationary processes that satisfy certain local conditions. These inequalities are used in Chapter 4 to obtain bounds for the rate of convergence of $(Z(t) - (2 \log t)^{1/2})$ to zero as $t \to \infty$, for a wide class of continuous, stationary Gaussian processes which satisfy the mixing condition $r(s) = O(s^{-\lambda})$ as $s \to \infty$ for some $\lambda > 0$, where $r(s) = EX(t+s)X(t)$. This class is a subclass of the stationary processes having $\sigma(t,s) = \sigma(|t-s|)$ a slowly varying function at zero.

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# Chapter I

## Introduction

In this work we shall be interested in the asymptotic behaviour of the maximum of certain Gaussian processes.

Consider a separable, continuous Gaussian process $X = \{X(t,\omega); t \in \mathbb{R}, \omega \in \Omega\}$ on a probability space $(\Omega,F,P)$. All the processes we shall deal with satisfy:

$$EX(t) = 0 \qquad EX^2(t) = 1 \tag{1.1.1}$$

for all $t \in \mathbb{R}$, unless we specify otherwise. We denote the covariance by $r(t,s) = EX(t)X(s)$. Define

$$Z(t,\omega) = \sup_{0 \le s \le t} X(s,\omega)$$

since X is separable Z(t) is well defined. We say that X is stable if

$$Z(t) - (2 \log t)^{1/2} \to 0$$

as $t \to \infty$, and relatively stable if

$$\frac{Z(t)}{(2 \log t)^{1/2}} \to 1$$

as $t \to \infty$, the types of convergence considered being almost sure (a.s.) and in probability (i.p.).

We shall be interested in the following two problems:

1. Under what conditions do we have a.s. stability.
2. If we have a.s. stability what is the rate of convergence to this limit.

A considerable amount of attention has been given to both problems. Concerning the first, Gnedenko [13] in 1943, showed that if $Zn = \max_{1 \le i \le n} X_i$ where $X_i$ is a sequence of independent identically distributed Gaussian variables satisfying (1.1.1) then

$$Zn - (2 \log n)^{1/2} \to 0 \qquad \text{i.p.}$$

Berman [2,3] extended this result to stationary Gaussian sequences with $EX_iX_j = r(i,j) = r(|i-j|)$, and also considered the problem of relative stability i.p. The type of condition he introduced is known as a mixing condition and concerns the limit behaviour of the covariance as $|i-j| \to \infty$.

Pickands [29] proved that $r(n) \to 0$ as $n \to \infty$ implies a.s. relative stability but is not enough for a.s. stability. Moreover, he establishes that a stationary Gaussian sequence is always a.s. upper stable, i.e.

$$\limsup_{n\to\infty} (Zn - (2 \log n)^{1/2}) \le 0 \qquad \text{a.s.}$$

and that either

$$r(n) \log n \to 0 \qquad \text{as } n \to \infty$$

or

$$\sum_{n=1}^{\infty} r^2(n) < \infty$$

are sufficient for a.s. stability.

The study of continuous processes requires local as well as mixing conditions on the covariance. The first to have obtained results in this case seem to have been Cramèr [5] and Shur [37]. Let $f(\lambda)$ be the spectral density of the stationary covariance $r(t)$. Assume $f(\lambda)$ is of bounded variation in $(-\infty,\infty)$ and satisfies

$$\int_0^\infty \lambda^2 (\log(1+\lambda))^a f(\lambda) d\lambda < \infty$$

for some $a > 1$. Then Cramer shows

$$\lim_{t\to\infty} P\{|Z(t) - (2 \log t)^{1/2}| < \frac{\log \log t}{(\log t)^{1/2}}\} = 1$$

and Shur proves that there is a $t_0(\omega)$ a.s. such that for $t > t_0$, $\varepsilon > 0$

$$|Z(t) - (2 \log t)^{1/2}| < \frac{(1+\varepsilon) \log \log t}{(2 \log t)^{1/2}}$$

These spectral conditions imply the local condition $r(t) = 1 - ct^2 + o(t^2)$ as $t \to 0$ and the mixing condition $\limsup_{t\to\infty} t|r(t)| < \infty$ .

Pickands [29] considered stationary processes with

$$\sigma(h) \equiv [E(X(t+h) - X(t))^2]^{1/2} = \frac{1}{\sqrt{2}} (1 - r(h))^{1/2} \leq ch^\alpha \qquad (1.1.2)$$

as $h \to 0$ for $0 < \alpha \leq 1$. X is then a.s. upper stable:

$$\limsup_{t\to\infty} (Z(t) - (2 \log t)^{1/2}) \leq 0 \quad \text{a.s.}$$

and if either

$$r(t) \log t \to 0 \qquad \text{as } t \to \infty \qquad (1.1.3)$$

or

$$\int_0^\infty r^2(t) dt < \infty \qquad (1.1.4)$$

then

$$\liminf_{t\to\infty} (Z(t) - (2 \log t)^{1/2}) \geq 0 \quad \text{a.s.}$$

so that (1.1.2) with either (1.1.3) or (1.1.4) gives a.s. stability.

Nisio [26] returned to the problem of a.s. relative stability. If X is a process with

$$E(X(t) - X(s))^2 \leq \psi^2(|t-s|)$$

where $\psi(h)$ is nondecreasing and continuous on $[0,\infty)$ with

$$\int_0^\infty \psi(e^{-u^2})du < \infty$$

then

$$\limsup_{t\to\infty} \frac{Z(t)}{(2 \log t)^{1/2}} \leq 1 \quad \text{a.s.} \qquad (1.1.5)$$

and if $\lim_{T\to\infty} \sup_{|t-s|>T} r(s,t) \leq 0$

then the converse is true

$$\limsup_{t\to\infty} \frac{Z(t)}{(2 \log t)^{1/2}} \geq 1 \quad \text{a.s.}$$

The first half of this result was also proved by Marcus [20] using Fernique's Inequality. By improving this inequality, in [21] he weakened Pickand's local condition (1.1.2) for a.s. upper stability to

$$\sigma(h) = O\left(\frac{1}{(\log 1/h)^\alpha}\right), \qquad \alpha > 1$$

(There is a mistake in the statement of theorem 1.1 in [21]. The proof only works for $\alpha > 1$ and not for $\alpha > \frac{1}{2}$). In the same paper he obtained a general result for the continuous parameter case: all continuous stationary Gaussian processes satisfy (1.1.5). He uses an upper bound for the probability of $\{Z(t) > x\}$ as $x \to \infty$ due to Fernique, Landau, Shepp and Marcus [10,19,23].

We tackle the problem of a.s. upper stability by means of metric entropy methods. This type of method has been used with some success by Dudley [6,7] and Fernique [11] to study the problem of sample path continuity. $\sigma(t,s)$ is taken as a pseudometric on the parameter space $\mathbb{R}$ and all metric considerations are made with respect to it.

The main tool used is an inequality for the tail of the distribution of $Z(t)$ (section 2.3) which is more general than Fernique's Inequality when the process has constant variance. We show that a.s. upper stability holds for a class of Gaussian processes that includes all those that are stationary, continuous and $\sigma$-separable.

Using the results obtained by Pickands [29] quoted above we get that all stationary continuous $\sigma$-separable processes satisfying (1.1.3) or (1.1.4) are a.s. stable.

Not all continuous, stationary Gaussian processes are a.s. stable, however. Marcus [22] has shown that if X has discrete spectrum then

$$\limsup_{t\to\infty} \frac{Z(t)}{(2 \log t)^{1/2}} = 0 \quad \text{a.s.}$$

He gives examples of processes in this class having various growth rates ranging from $(\log \log t)^{1/2}$ to $(\log t)^{(1-\varepsilon)/2}$, $\varepsilon > 0$.

Let us look now at the second question. To do this define a new process $Y = \{Y(t),\ t \in \mathbb{R}\}$ by

$$Y(t) = Z(t) - (2 \log t)^{1/2}$$

Then $Y(t) \to 0$ as $t \to \infty$ a.s. We are interested in obtaining continuous curves $\nu_1(t) < \nu_2(t)$, $\nu_i(t) \to 0$ as $t \to \infty$ for $i = 1,2$, such that the process $Y(t)$ will always stay between the two curves, i.e. we want Y(t) to cross either curve only a finite number of times but to get arbitrarily close to both infinitely often as $t \to \infty$.

The functions $\pi_i = (2 \log t)^{1/2} + \nu_i(t)$ will be called the envelopes of the process. This question as it stands is much too general. Very specific assumptions on the covariance of the process are required in order to obtain precise results. Historically, there has been more interest in obtaining these conditions for the upper envelope problem. The first efforts were directed towards obtaining functions $\pi_1(t)$ satisfying some or all the conditions [30,32]. Later on, the solution was sharpened by dividing the class of positive, continuous nondecreasing functions $\theta(t)$ with $\theta(t) \to \infty$ as $t \to \infty$ into an upper and a lower class in a manner analogous to the classical law of the iterated logarithm results of Feller [8,9] and the similar results for Brownian Motion (Petrovsky [28]; Chung, Erdos & Sirao [4]; Sirao and Nisida [39]).

We say that $\theta(t)$ belongs to the upper class U(X) if there is a $t_0(\omega)$ with probability 1 such that for all $t \geq t_0$

$$Z(t) < \theta(t)$$

If no such $t_0(\omega)$ exists, with probability one, we say that $\theta(t)$ belongs to the lower class L(X).

The tests for deciding whether a function $\theta(t)$ belongs to either class are in the form of integrals $I(\theta)$ of the function whose convergence properties determine whether $\theta \in L(X)$ or $\theta \in U(X)$. The first such tests for Gaussian processes other than Brownian Motion were obtained by Watanabe [40] under the local conditions

$$c_1 h^\alpha \leq \sigma(h) \leq c_2 h^\alpha \qquad (1.1.6)$$

as $h \to 0$, $0 < \alpha \leq 1$, $c_1 \leq c_2$ and the mixing condition

$$r(t,t+s)s \to 0 \qquad (1.1.7)$$

as $s \to \infty$ uniformly in t.

His method is based on that of Sirao [38]. Qualls and Watanabe [33] used an exact asymptotic result for the tail of the distribution of Z(t) obtained by Pickands [31] to relax (1.1.7) in the stationary case to

$$r(t) = O(t^{-\lambda})$$

as $t \to \infty$ for some $\lambda > 0$.

They were later able to obtain similar results for all stationary Gaussian processes with

$$\sigma(h) = h^{\alpha}G(h) \qquad (1.1.8)$$

as $h \to 0$ for $0 < \alpha \leq 1$, where G(h) is a slowly varying function at zero, i.e.

$$\lim_{h \downarrow 0} \frac{G(th)}{G(h)} = 1$$

for any $t > 0$. They did this by extending Pickands' asymptotic result to this case (see theorem 3.1.1).

The best result available was given by Pathak and Qualls [27]. Suppose $\sigma(h)$ satisfies the following condition:

$$\sigma(h) = h^{\alpha}G(h) + o(h^{\alpha}G(h))$$

as $h \to 0$ for $0 < \alpha \leq 1$, where G(h) is as before and that

$$r(t) = o\left(\frac{1}{\log t}\right) \tag{1.1.9}$$

as $t \to \infty$. Then

$$\theta(t) \in U(X) \quad (L(X))$$

according as

$$I(\theta) = \int^{\infty} \frac{\dot{g}(\theta(t))}{\theta(t)} \exp\left\{-\frac{\theta^2(t)}{2}\right\} < \infty \qquad (= \infty),$$

where $g(x) = \dfrac{1}{\tilde{\sigma}^{-1}(1/x)}$

and $\tilde{\sigma}(h) = \sqrt{2}\, h^{\alpha} G(h)$.

They also give a similar result for stationary Gaussian sequences. Mittal [24,25] obtained independently results that can be deduced from these theorems.

By means of a time transformation [33,34,40], some of these results have been used to obtain similar theorems for non-stationary processes.

The lower envelope has been studied by Pickands [30,32] and Mittal [24,25]. In this case no division into upper and lower classes by means of an integral test has been obtained except in the case of Brownian motion (Jain & Taylor [15]). The best results presently known are those of Mittal who shows that (1.1.6) and (1.1.9) imply

$$\liminf_{t\to\infty} \left(Z(t) - (2 \log t)^{1/2}\right) \frac{(2 \log t)^{1/2}}{\log \log t} = \frac{1}{\alpha} - \frac{1}{2} \quad \text{a.s.}$$

Our efforts are directed in both cases to weakening the local conditions assumed by previous authors. In Chapter 3 we obtain upper and lower bounds for the tail of the distribution of $Z(t)$, for a class of Gaussian processes having $\sigma(h)$ a slowly varying function, i.e. $\alpha = 0$

in (1.1.8). A crucial role is played by the structure function $a(h)$ of $\sigma(h)$. (For definitions see section 3.1.) The inequalities are not as accurate as Qualls and Watanabe's in the case $\alpha > 0$ but are enough to obtain sharp results in many cases. We show in theorem 3.4.1 how our work can be combined with theirs to give a more general set of inequalities.

The results of Chapter 3 are the main tool used in Chapter 4 to gain some insight about the general form of the envelopes for Gaussian processes. The proofs are based on methods used by other authors.

Corollaries 4.2.3 and 4.3.1 give precise estimates for the envelopes of some of these processes.

We have been unable to obtain sharp integral tests for the upper envelope in the case $\alpha = 0$ because the inequalities we obtain in Chapter 3 are not asymptotically exact. Therefore, even though the results are stated in the form of integral tests in theorems 4.2.1 and 4.2.2, there is still a gap that leaves a class of functions for which we obtain no information.

In section 4.4 we use the time transformation mentioned before to get theorems for some non-stationary Gaussian processes.

Some of the results reviewed have been extended to Gaussian Random Fields by Kôno [18] and Qualls and Watanabe [35].

We end this chapter with some notation that will be used repeatedly.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\psi(x) = \frac{1}{x}\phi(x)$$

If X is a Gaussian random variable with mean 0 and variance 1 we write $X \sim N(0,1)$. It is well known that in this case

$$P\{X > x\} \leq \psi(x) \qquad \text{for } x > 0$$

and

$$\frac{P\{X > x\}}{\psi(x)} \to 1 \qquad \text{as } x \to \infty$$

Let f(x) and g(x) be two functions defined near the origin. We shall write $f >> g$ $(x \downarrow 0)$ if there exists a $u_0 > 0$ such that $f(x) \geq g(x)$ for any x in $(0, u_0)$.

We shall put $f(x) \div g(x)$ if there are constants A and B, $0 < A \leq B < \infty$ and a $u_0 > 0$ such that

$$Af(x) \leq g(x) \leq Bf(x)$$

for all $x \in (0,u_0)$. If $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$ we say $f \sim g$.

c and const. denote unknown constants which may change from line to line.

Theorem a.b.c. denotes theorem c of section b, chapter a. Equations will be denoted by (a.b.c) with a similar convention.

# Chapter 2

# Stability of Gaussian Processes

## 2.1. Introduction

The purpose of this chapter is to obtain in section 2.3 an upper bound for the tail of the distribution of the supremum of a Gaussian process, subject to certain conditions that are weaker than those assumed by Fernique's Inequality ([11,14] or Lemma 3.2.2), and with it to show, in section 2.4, that a wide class of Gaussian processes, including all stationary and continuous processes, satisfy

$$\limsup_{t\to\infty} \; (Z(t) - (2\log t)^{1/2}) \le 0 \quad \text{a.s.}$$

In section 2.5 we comment on the conditions under which the lim inf is also zero and give conditions for stability a.s.

Let $X = \{X(t,\omega),\ t \in T,\ \omega \in \Omega\}$, $T \subset \mathbb{R}$ an arbitrary index set, be a Gaussian Process with covariance $r(t,s) = EX(t)X(s)$ continuous on $T\times T$. To study X we shall use the function defined on $T\times T$ by

$$\sigma(t,s) \;=\; [E(X(t) - X(s))^2]^{1/2}$$

This function is not necessarily a metric on T since

$$\sigma(t,s) \;=\; 0 \;\not\Rightarrow\; t = s$$

We shall call it the pseudometric induced on T by X. It generates a topology on T: for every $t \in T$ and every real $h > 0$ let $B(t,h)$ denote the open ball having centre t and radius h defined by

$$B(t,h) \;=\; \{s \in T\colon \sigma(t,s) < h\}$$

We assume that X is $\sigma$-separable, i.e. there is a countable set $S \subset T$ called the separating set and a null set $N_s$ such that for all $\omega \notin N_s$, all $t \in T$, all $\varepsilon > 0$

$$X(\omega,t) \in Cl\{X(\omega,s), \quad s \in S \cap B(t,\varepsilon)\}$$

where Cl denotes the closure.

Our assumptions imply that

$$\begin{aligned}\sigma(t,s) &= [E(X(t) - X(s))^2]^{1/2} \\ &= [E(X^2(t) + X^2(s) - 2X(t)X(s))]^{1/2} \\ &= [EX^2(t) + EX^2(s) - 2EX(t)X(s)]^{1/2} \\ &= [2(1 - r(t,s))]^{1/2}\end{aligned}$$

and the continuity of $r(t,s)$ on $T{\times}T$ implies the continuity of $\sigma(t,s)$.

Suppose now that for fixed t, $\sigma(t,s)$ is a monotone increasing function of s. Then

$$\sigma(t,s) = 0 \Rightarrow t = s$$

and $\sigma$ is a metric. It is topologically equivalent to the usual metric $d(s,t) = |t-s|$. But the fact that X is $\sigma$-separable is equivalent to $(T,\sigma)$ being separable. Since $(T,\sigma)$ and $(T,d)$ are topologically equivalent, $(T,d)$ is also separable and this implies, in turn, that X has a d-separable version $\tilde{X}$.

In Chapters 3 and 4 we shall assume that this is true and so will only speak of separability.

Theorem 2.1.1. ([12], theorem 5, page 155)

Let $X = \{X(t,\omega),\ t \in T,\ \omega \in \Omega\}$ be a $\sigma$-separable process that is continuous in probability, when the continuity is taken with respect to $\sigma$.

Then every countable subset dense in T separates X.

Let $S \subseteq T$ be any subset of T and $\varepsilon$ any positive real number. The minimal number of $\sigma$-balls $B(t,h)$ with $t \in S$ and $h \leq \varepsilon$ needed to cover S is denoted by $N(s,\varepsilon)$. The function

$$H(s,\varepsilon) = \log N(s,\varepsilon)$$

is called the metric entropy of the set S. When S = T these functions will be denoted by $N(\varepsilon)$ and $H(\varepsilon)$ respectively.

Note that $N(\varepsilon)$ and $H(\varepsilon)$ are non-decreasing as $\varepsilon$ decreases.

Theorem 2.1.2. (Dudley [6,7], Jain and Marcus [14])

Let $X = \{X(t,\omega),\ t \in T,\ \omega \in \Omega\}$ be a centred, $\sigma$-separable Gaussian process with covariance $r(t,s) = EX(t)X(s)$ continuous on $T \times T$. Assume that for some $\nu > 0$

$$\int_0^\nu H^{1/2}(\varepsilon)d\varepsilon < \infty$$

then there is a version $\tilde{X} = \{\tilde{X}(t,\omega),\ t \in T,\ \omega \in \Omega\}$ of X with continuous sample paths.

## 2.2. Preliminary Results

Lemma 2.2.1. ([38] lemma 2)

Let X and Y be jointly Gaussian random variables with means 0, variances 1 and correlation r. Then $P\{X > b, Y \leq a\}$ is a nonincreasing function of r for a,b fixed, $0 < a < b$.

Lemma 2.2.2

Let X and Y be as in Lemma 2.2.1. For any $a > 0$, $h > 0$, if

$$rh - a(1-r) > 0 \tag{2.2.1}$$

$$P\{X > a+h, Y \leq a\} \leq \psi(a) \frac{(1-r^2)^{1/2}}{rh-a(1-r)} \exp\left\{\frac{-h^2}{2(1-r^2)} - \frac{a^2(1-r)}{2(1+r)} - \frac{ah}{1+r}\right\} \tag{2.2.2}$$

Proof

$$P\{X > a+h, Y \leq a\} =$$

$$\int_{a+h}^{\infty} \int_{-\infty}^{a} \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left\{-\frac{x^2-2rxy+y^2}{2(1-r^2)}\right\} dydx$$

$$= \int_{a+h}^{\infty} \int_{-\infty}^{a} \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left\{-\frac{x^2-x^2r^2+x^2r^2-2rxy+y^2}{2(1-r^2)}\right\} dydx$$

$$= \int_{a+h}^{\infty} \int_{-\infty}^{a} \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left\{-\frac{x^2}{2} - \frac{(y-xr)^2}{2(1-r^2)}\right\} dydx$$

$$= \int_{a+h}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{(a-xr)/(1-r^2)^{1/2}} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz dx$$

$$\leq \int_{a+h}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{(a-r(a+h))/(1-r^2)^{1/2}} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz dx$$

$$= \int_{a+h}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx \int_{\frac{hr-a(1-r)}{(1-r^2)^{1/2}}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

If (2.2.1) is satisfied

$$\leq \psi(a+h) \; \psi\left(\frac{hr-a(1-r)}{(1-r^2)^{1/2}}\right)$$

$$\leq \frac{(1-r^2)^{1/2}}{2\pi(a+h)(hr-a(1-r))} \exp\left\{-\frac{a^2+2ah+h^2}{2} - \frac{(hr-a(1-r))^2}{2(1-r^2)}\right\}$$

$$\leq \psi(a) \frac{(1-r^2)^{1/2}}{hr-a(1-r)} \exp\left\{-ah - \frac{h^2}{2} - \frac{h^2r^2-2rha(1-r)+a^2(1-r)^2}{2(1-r^2)}\right\}$$

$$= \psi(a) \frac{(1-r^2)^{1/2}}{hr-a(1-r)} \exp\left\{-\frac{h^2}{2(1-r^2)} - \frac{a^2(1-r)}{2(1+r)} - \frac{ah}{1+r}\right\}$$

□

## 2.3. Main Inequality

We now proceed to obtain the main tool needed to get the major result of this chapter. It is an inequality for the supremum of the process X over an arbitrary interval $T \subset \mathbb{R}$. The method used in the proof is a combination of the well-known procedure of Sirao [38], which has been employed by many authors [16,17,18,40], with the more recent methods of metric entropy introduced by Dudley [6] and also used in other works [11,14].

### Theorem 2.3.1

Let $X = \{X(t),\ t \in T\}$ be a $\sigma$-separable Gaussian process with $r(t,s) = EX(t)X(s)$ continuous on $T \times T$. Assume that for some $\nu > 0$

$$\int_0^{\nu} H^{1/2}(u)\,du < \infty \tag{2.3.1}$$

Then

$$P\{\sup_{t \in T} X(t) > x + A_1 \int_0^{\varepsilon_0} H^{1/2}(u)\,du\} \leq \text{const}\ \psi(x) \tag{2.3.2}$$

where $\varepsilon_0(x)$ is defined by

$$\varepsilon_0 = \inf\{\varepsilon : \frac{H^{1/2}(\varepsilon)}{\varepsilon} \leq x\} \tag{2.3.3}$$

and $A_1$ is a constant.

### Proof

We start by defining a monotone decreasing sequence $\varepsilon_n$, $n = 1,2,\ldots$ in terms of x and the metric entropy of T, $H(\varepsilon)$. Let

$$\delta_n = 2 \inf\{\varepsilon : H(\varepsilon) \leq 2H(\varepsilon_n)\} \tag{2.3.4}$$

$$\varepsilon_{n+1} = \min\left( \frac{\varepsilon_n}{3}, \delta_n \right) \tag{2.3.5}$$

for $n = 0,1,\ldots$

Note that $\varepsilon_0 \to 0$ as $x \to \infty$, $\varepsilon_n \to 0$ as $n \to \infty$ and that unless $H(\varepsilon) \uparrow \infty$ as $\varepsilon \to 0$, $\delta_n$ and $\varepsilon_n$ will be identically zero for all n greater than a certain $n_0$. We assume, for the time being, that

$$H(\varepsilon) \uparrow \infty \quad \text{as} \quad \varepsilon \downarrow 0 \tag{2.3.6}$$

Let

$$T_n = \{t_i^n;\ 1 \le i \le N(\varepsilon_n)\} \tag{2.3.7}$$

for $n = 0,1,\ldots$ be a minimal $\varepsilon_n$-net, i.e. the set of centres of balls of radii $\le \varepsilon_n$ that constitute a minimal covering of T, and

$$Z_n = \sup_{t \in T_n} X(t)$$

$$Z = \sup_{t \in T} X(t)$$

$$A = \{\omega : Z(\omega) > x + \sum_{i=0}^{\infty} x_i\}$$

$$A_n = \{\omega : Z_n(\omega) > x + \sum_{i=0}^{n} x_i\}$$

$$Y_n = x + \sum_{0}^{n} x_i \tag{2.3.8}$$

where

$$x_0 = \frac{\log N(\varepsilon_0)}{x} \tag{2.3.9}$$

and

$$x_i = 2\varepsilon_{i-1} H^{1/2}(\varepsilon_i) \qquad i \ge 1 \tag{2.3.10}$$

Since we have assumed (2.3.6) we have

$$x_i > 0 \quad \text{all } i \tag{2.3.11}$$

Since $\Sigma = \bigcup_{i=0}^{\infty} T_i$ is a countable $\sigma$-dense subset of T and we have assumed X to be $\sigma$-separable and continuous in probability, theorem 2.1.1 implies that $\Sigma$ is a separating set for X. From

$$A \subseteq A_0 \cup (A_1 \setminus A_0) \cup (A_2 \setminus A_1) \cup \ldots$$

we get

$$P(A) \leq P(A_0) + \sum_{n=1}^{\infty} P(A_n \setminus A_{n-1}) \tag{2.3.12}$$

Clearly, for $X \sim N(0,1)$,

$$\begin{aligned} P(A_0) &= P\{Z_0 > x + \frac{\log N(\varepsilon_0)}{x}\} \\ &\leq N(\varepsilon_0) P\{X > x + \frac{\log N(\varepsilon_0)}{x}\} \\ &\leq \psi(x) \end{aligned} \tag{2.3.13}$$

On the other hand, using (2.3.8)

$$\begin{aligned} P(A_n \setminus A_{n-1}) &= P\{Z_n > Y_n, Z_{n-1} \leq Y_{n-1}\} \\ &= P\left\{\begin{array}{ll} X(t) > Y_n & \text{for some } t \in T_n \\ X(s) \leq Y_{n-1} & \text{for all } s \in T_{n-1} \end{array}\right\} \\ &\leq \sum_{i=1}^{N(\varepsilon_n)} P\left\{\begin{array}{l} X(t_i^n) > Y_n \\ X(s) \leq Y_{n-1} \text{ for all } s \in T_{n-1} \end{array}\right\} \end{aligned}$$

$$= \sum_{i=1}^{N(\varepsilon_n)} P\left\{ \begin{array}{l} X(t_i^n) > Y_n \\ X(t_j^{n-1}) \le Y_{n-1} \quad \text{for all } j, \quad 1 \le j \le N(\varepsilon_{n-1}) \end{array} \right\}$$

$$\le \sum_{i=1}^{N(\varepsilon_n)} P\left\{ \begin{array}{l} X(t_i^n) > Y_n \\ X(t_{j(i)}^{n-1}) \le Y_{n-1} \end{array} \right\} \tag{2.3.14}$$

where $t_{j(i)}^{n-1} \in T_{n-1}$ and $\sigma(t_i^n, t_{j(i)}^{n-1}) \le \varepsilon_{n-1}$ (since $T_{n-1}$ is a $\varepsilon_{n-1}$-net there is such a $t_{j(i)}^{n-1}$).

From

$$\sigma^2(t,s) = E(X(t) - X(s))^2$$

$$= 2(1 - r(t,s))$$

we have

$$1 - r(t_i^n, t_{j(i)}^{n-1}) = \frac{1}{2}\sigma^2(t_i^n, t_{j(i)}^{n-1})$$

$$\le \frac{\varepsilon_{n-1}^2}{2}$$

Define $r_n$ by

$$r_n = 1 - \frac{\varepsilon_{n-1}^2}{2} \tag{2.3.15}$$

then

$$r_n \le r(t_i^n, t_{j(i)}^{n-1}) \tag{2.3.16}$$

for $1 \le i \le N(\varepsilon_n)$, and since $\varepsilon_n < \varepsilon_0$ for $n \ge 1$ and $\varepsilon_0 \to 0$ as $x \to \infty$, for x large

$$r_n > \frac{1}{2} \quad , \qquad n \ge 1 \tag{2.3.17}$$

Let $\xi$ and $\eta$ be centred Gaussian random variables with $E\xi^2 = E\eta^2 = 1$ and $E\xi\eta = r_n$. Then (2.3.11), (2.3.16) and Lemma 2.2.1 imply

$$P\left\{\begin{array}{l} X(t_i^n) > Y_n \\ X(t_{j(i)}^{n-1}) \le Y_{n-1} \end{array}\right\} \le P\left\{\begin{array}{l} \xi > Y_n \\ \eta \le Y_{n-1} \end{array}\right\}$$

and (2.3.14) becomes

$$P\{A_n \setminus A_{n-1}\} \le N(\varepsilon_n) P\{\xi > Y_n; \quad \eta \le Y_{n-1}\} \tag{2.3.18}$$

At this point we want to use Lemma 2.2.2 to get an upper bound for the probability on the right hand side. We have to check that (2.2.1) is satisfied, i.e.

$$r_n x_n - (1-r_n)Y_{n-1} > 0 \tag{2.3.19}$$

for n = 1,2,... and x large. We begin by looking at $\sum_1^\infty x_i$ .

<u>Lemma 2.3.1</u>

$$\sum_{i=0}^{\infty} x_i \le 17 \int_0^{\varepsilon_0} H^{1/2}(u)du$$

<u>Proof</u>

$$\begin{aligned} \sum_{i=0}^{\infty} x_i &= \frac{\log N(\varepsilon_0)}{x} + 2 \sum_{i=1}^{\infty} \varepsilon_{i-1} H^{1/2}(\varepsilon_i) \\ &= \frac{H(\varepsilon_0)}{x} + 2 \sum_{i=1}^{\infty} \varepsilon_{i-1} H^{1/2}(\varepsilon_i) \end{aligned} \tag{2.3.20}$$

From (2.3.5) we get

$$\varepsilon_i \leq \frac{\varepsilon_{i-1}}{3}$$

$$\Rightarrow \quad \frac{3}{2}\,\varepsilon_i \leq \frac{1}{2}\,\varepsilon_{i-1}$$

$$\Rightarrow \quad 0 \leq \frac{1}{2}\,\varepsilon_{i-1} - \frac{3}{2}\,\varepsilon_i$$

$$\Rightarrow \quad \varepsilon_{i-1} \leq \frac{3}{2}\,(\varepsilon_{i-1} - \varepsilon_i) \tag{2.3.21}$$

Also from (2.3.3)

$$\frac{H(\varepsilon_0)}{x} \leq \varepsilon_0 H^{1/2}(\varepsilon_0)$$

therefore

$$(2.3.20) \leq \varepsilon_0 H^{1/2}(\varepsilon_0) + 2 \sum_{i=1}^{\infty} \varepsilon_{i-1}\, H^{1/2}(\varepsilon_i)$$

and using (2.3.21)

$$\leq \frac{3}{2}\,(\varepsilon_0 - \varepsilon_1) H^{1/2}(\varepsilon_0) + 3 \sum_{i=1}^{\infty} (\varepsilon_{i-1} - \varepsilon_i) H^{1/2}(\varepsilon_i)$$

Using (2.3.5)

$$\leq \frac{3}{2}\,(\varepsilon_0 - \varepsilon_1) H^{1/2}(\varepsilon_0) + 3 \sum_{i=1}^{\infty} (\varepsilon_{i-1} - \varepsilon_i)\left(H^{1/2}\left(\frac{\varepsilon_{i-1}}{3}\right) + H^{1/2}(\delta_{i-1})\right)$$

Since (2.3.4) implies that $H(\delta_n) \leq 2H(\varepsilon_n)$

$$\leq \frac{3}{2}(\varepsilon_0-\varepsilon_1)H^{1/2}(\varepsilon_0) + 9\sum_{i=1}^{\infty}\left(\frac{\varepsilon_{i-1}}{3} - \frac{\varepsilon_i}{3}\right)H^{1/2}\left(\frac{\varepsilon_{i-1}}{3}\right)$$

$$+ 6\sum_{i=1}^{\infty}(\varepsilon_{i-1}-\varepsilon_i)H^{1/2}(\varepsilon_{i-1})$$

$$\leq \frac{3}{2}\int_{\varepsilon_1}^{\varepsilon_0} H^{1/2}(u)du + 9\sum_{i=1}^{\infty}\int_{\varepsilon_i/3}^{\varepsilon_{i-1}/3} H^{1/2}(u)du$$

$$+ 6\sum_{i=1}^{\infty}\int_{\varepsilon_i}^{\varepsilon_{i-1}} H^{1/2}(u)du$$

$$< 17\int_0^{\varepsilon_0} H^{1/2}(u)du$$

□

Since $\varepsilon_0 \to 0$ as $x \to \infty$ this lemma implies that

$$\sum_{i=0}^{\infty} x_i \to 0 \quad \text{as } x \to \infty$$

but by definition

$$Y_n = x + \sum_0^n x_i$$

Therefore, for x large

$$Y_n \leq \frac{3}{2}x, \qquad \text{all } n,$$

and to prove (2.3.19) it is enough to show that

$$r_n x_n - \frac{3}{2}(1 - r_n)x > 0$$

for n = 1,2,... and x large.

Using (2.3.15) and (2.3.17), for any $n \geq 1$ and x large

$$r_n x_n - \frac{3}{2}(1-r_n)x > \frac{x_n}{2} - \frac{3}{4}\varepsilon_{n-1}^2 x$$

$$= \varepsilon_{n-1}H^{1/2}(\varepsilon_n) - \frac{3}{4}\varepsilon_{n-1}^2 x$$

$$= \varepsilon_{n-1}(H^{1/2}(\varepsilon_n) - \frac{3}{4}\varepsilon_{n-1} x)$$

Since, by definition, $\varepsilon_n$ is monotone decreasing and $H(\varepsilon)$ is non-decreasing as $\varepsilon$ decreases

$$\geq \varepsilon_{n-1}(H^{1/2}(\varepsilon_1) - \frac{3}{4}\varepsilon_0 x)$$

From (2.3.5) we get $\varepsilon_1 \leq \frac{\varepsilon_0}{3}$, and (2.3.3) implies

$$H^{1/2}(\varepsilon_1) \geq \varepsilon_0 x$$

then

$$r_n x_n - \frac{3}{2}(1-r_n)x > \frac{\varepsilon_{n-1}\varepsilon_0 x}{4} \qquad (2.3.22)$$

$$> 0$$

by (2.3.6). Hence (2.3.19) is satisfied and we can use Lemma 2.2.2 in (2.3.18)

$$(2.3.18) \leq N(\varepsilon_n)\psi(Y_n)\frac{(1-r_n^2)^{1/2}}{r_n x_n-(1-r_n)Y_n} \exp\left\{\frac{-x_n^2}{2(1-r_n)} - \frac{Y_n(1-r_n)}{2(1+r_n)} - \frac{Y_n x_n}{1+r_n}\right\} \qquad (2.3.23)$$

Using (2.3.15) and (2.3.22) we get

$$\frac{(1-r_n^2)^{1/2}}{r_n x_n - (1-r_n)Y_n} \leq \frac{c\varepsilon_{n-1}}{\varepsilon_{n-1}\varepsilon_0 x} = \frac{c}{\varepsilon_0 x}$$

using (2.3.3)

$$\leq \frac{c}{H^{1/2}(\varepsilon_0)}$$

$$\to 0$$

as $\varepsilon_0 \to 0$ $(x \to \infty)$.

Since we also have

$$\frac{Y_n(1-r_n)}{2(1+r_n)} \geq 0$$

and

$$\frac{Y_n x_n}{1 + r_n} \geq 0$$

and by Lemma 2.3.1

$$Y_n \sim x \qquad \text{as } x \to \infty$$

and

$$\psi(Y_n) \leq \psi(x)$$

we get

$$\text{(2.3.23)} \quad \leq \psi(x)N(\varepsilon_n) \exp\{-\frac{x_n^2}{2(1-r_n)}\}$$

using (2.3.15)

$$= \psi(x) \exp\{\log N(\varepsilon_n) - \frac{x_n^2}{\varepsilon_{n-1}^2}\}$$

using (2.3.10)

$$= \psi(x) \exp\{H(\varepsilon_n) - 4H(\varepsilon_n)\}$$

$$= \psi(x) \exp\{-3H(\varepsilon_n)\}$$

so that

$$\sum_{n=1}^{\infty} P\{A_n \setminus A_{n-1}\} \leq \psi(x) \sum_{n=1}^{\infty} \exp\{-3H(\varepsilon_n)\}$$

$$\leq \psi(x) \left[ \sum_{n=1}^{\infty} \exp\{-3H(\varepsilon_{2n})\} + \sum_{n=1}^{\infty} \exp\{-3H(\varepsilon_{2n-1})\} \right] \qquad (2.3.24)$$

From (2.3.5)

$$\varepsilon_{n+2} \leq \frac{\varepsilon_{n+1}}{3} \leq \frac{\delta_n}{3} < \frac{\delta_n}{2}$$

and using (2.3.4) and the monotonicity of $H(\varepsilon)$

$$H(\varepsilon_{n+2}) \geq H(\frac{\delta_n}{3}) > 2H(\varepsilon_n)$$

therefore

$$(2.3.24) \leq \psi(x) \left[ \sum_{n=1}^{\infty} \exp\{-3.2^n H(\varepsilon_0)\} + \sum_{n=0}^{\infty} \exp\{-3.2^n H(\varepsilon_1)\} \right]$$

$$\leq 2\psi(x) \sum_{n=0}^{\infty} \exp\{-32^n H(\varepsilon_0)\}$$

and since we have assumed that $H(\varepsilon_0) \to \infty$ as $\varepsilon_0 \to 0$ $(x \to \infty)$, this series is convergent and uniformly bounded as $x \to \infty$. Hence

$$\sum_{n=1}^{\infty} P\{A_n \setminus A_{n-1}\} \leq c\psi(x)$$

Combining this with (2.3.12) and (2.3.13)

$$P(A) \leq P(A_0) + \sum_{n=1}^{\infty} P(A_n \setminus A_{n-1})$$

$$\leq \text{const } \psi(x)$$

and using Lemma 2.3.1

$$P\{\sup_{t \in T} X(t) > x + 17 \int_0^{\varepsilon_0} H^{1/2}(u)du\} \leq P(A)$$

$$\leq \text{const } \psi(x)$$

We have shown that if (2.3.6) holds then (2.3.2) is true with $A_1 = 17$. Now suppose (2.3.6) is not true. Then for some constant K

$$H(\varepsilon) < K \qquad \text{as } \varepsilon \to 0.$$

this means

$$N(\varepsilon) < e^K \qquad \text{as } \varepsilon \to 0$$

and therefore

$$P\{Z(\omega) > x\} \leq e^K P\{X > x\}$$

$$\leq c\psi(x)$$

since K is independent of x. In this case $A_1 = 0$.

□

## 2.4. Stability

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a $\sigma$-separable Gaussian process.

In this section we give sufficient conditions for upper stability to hold, i.e. for

$$\limsup_{t\to\infty} (Z(t) - (2 \log t)^{1/2}) \leq 0 \qquad \text{a.s.} \tag{2.4.1}$$

**Theorem 2.4.1**

Let X be a Gaussian process as above with $r(t,s) = EX(t)X(s)$ continuous on $\mathbb{R}\times\mathbb{R}$. Let $I_K = [K,K+1]$ for $K = 0,1,\ldots$ Assume that there exists a non-increasing positive function $G(\varepsilon)$ and a $\nu > 0$ such that for all K and all $\varepsilon \leq \nu$,

$$H(I_K,\varepsilon) \leq G(\varepsilon) \tag{2.4.2}$$

and

$$\int_0^{\nu} G^{1/2}(u)du < \infty \tag{2.4.3}$$

then (2.4.1) holds.

**Proof**

For each interval $I_K$ we define $\varepsilon_0(K)$ by

$$\varepsilon_0(K) = \inf \{\varepsilon : \frac{H^{1/2}(I_K,\varepsilon)}{\varepsilon} \leq x \} \tag{2.4.4}$$

Theorem 2.3.1 implies

$$P \{\sup_{t\in I_K} X(t) > x + A_1 \int_0^{\varepsilon_0(K)} H^{1/2}(I_K,u)du\} \leq \psi(x)$$

Define $\varepsilon_0(G)$ by

$$\varepsilon_0(G) = \inf\{\varepsilon : \frac{G^{1/2}(\varepsilon)}{\varepsilon} \le x\}$$

then, for any K,

$$\frac{H^{1/2}(I_K,\varepsilon_0(G))}{\varepsilon_0(G)} \le \frac{G^{1/2}(\varepsilon_0(G))}{\varepsilon_0(G)} \le x$$

and (2.4.4) implies

$$\varepsilon_0(K) \le \varepsilon_0(G)$$

Therefore

$$\int_0^{\varepsilon_0(K)} H^{1/2}(I_K,u)du \le \int_0^{\varepsilon_0(G)} H^{1/2}(I_K,u)du$$

Using (2.4.2)

$$\le \int_0^{\varepsilon_0(G)} G^{1/2}(u)du$$

$$\to 0 \quad \text{as } \varepsilon_0(G) \to 0 \quad (x \to \infty)$$

and is independent of K.

Let

$$g(x) = A_1 \int_0^{\varepsilon_0(G,x)} G^{1/2}(u)du$$

then $g(x)$ is independent of K and tends to zero as $x \to \infty$. Also

$$P\{\sup_{t \in I_K} X(t) > x + g(x)\} \le c\psi(x)$$

Let

$$x = z_K = (2 \log K)^{1/2} + \frac{\log \log K}{(2 \log K)^{1/2}}$$

then

$$P_K \equiv P\{ \sup_{I_K} X(t) > z_K + g(z_K)\} \leq c\psi(z_K)$$

and

$$\sum_{K=K_0}^{\infty} P_K \leq c \sum_{K=K_0}^{\infty} \frac{1}{K(\log K)^{3/2}} < 1$$

and the Borel-Cantelli lemma implies that given any $\delta > 0$

$$\sup_{t \in I_K} X(t) > (2 \log K)^{1/2} + \delta$$

only a finite number of times as $K \to \infty$ with probability one. But since the processes we are considering are continuous this implies that there is, with probability one, a $t_0(\omega)$ such that for $t \geq t_0$

$$Z(t) < (2 \log t)^{1/2} + \delta$$

and (2.4.1) holds.

□

## 2.5. Comments

Let X be a stationary process. This means that $\sigma(t,s) = \sigma(|t-s|)$ is invariant under translations along the real line. Hence the metric entropy is also invariant and we can choose

$$H(I_K,\varepsilon) = G(\varepsilon)$$

for all K.

By a theorem of Fernique ([11] theorem 8.1.1), for stationary processes continuity is equivalent to

$$\int_0^{\nu} G^{1/2}(u)du < \infty \qquad (2.4.3)$$

Therefore theorem 2.4.1 includes all stationary and continuous Gaussian processes. This extends results of Marcus [21] where (2.4.1) is proved under the stronger condition:

$$\sigma(h) = O\left(\frac{1}{(\log 1/h)^{\alpha}}\right), \quad \alpha > 1$$

As we mentioned in Chapter 1, Pickands [29] has shown that either

$$r(t) \log t \to 0 \quad \text{as } t \to \infty$$

or

$$\int_0^{\infty} r^2(t)dt < \infty$$

imply

$$\liminf_{t\to\infty} (Z(t) - (2 \log t)^{1/2}) \geq 0 \quad \text{a.s.}$$

So we have

<u>Theorem 2.5.1</u>

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a stationary continuous Gaussian process satisfying either

$$r(t) \log t \to 0 \quad \text{as } t \to \infty \tag{2.5.1}$$

or

$$\int_0^\infty r^2(t)\,dt < \infty$$

then

$$Z(t) - (2 \log t)^{1/2} \to 0 \quad \text{a.s.}$$

as $t \to \infty$.

Recently Mittal [41] has weakened (2.5.1) to

$$\lambda\{t\colon 0 \le t \le s;\ \ |r(t)| > \frac{f(t)}{\log t}\} = o(s^\beta)$$

for some $0 \le \beta < 1$ and some $f(t) = o(1)$, and $r(t) = o(1)$ where $\lambda$ denotes Lebesgue measure.

# Chapter 3

## Bounds for the Tail of the Distribution of the Maximum

### 3.1. Introduction and Definitions

Let us turn now to the second question stated in the introduction. We shall consider in the next two chapters real, separable, continuous, stationary Gaussian processes $X = \{X(t), t \in \mathbb{R}\}$ satisfying

$$EX(t) = 0, \qquad EX^2(t) = 1, \qquad \text{all } t.$$

We shall denote such processes by the letters G.P. In this case the structure of the process is determined by its covariance function $r(h) = EX(t+h)X(t)$, or equivalently by the increments variance

$$\begin{aligned} \sigma^2(h) &= E(X(t+h) - X(t))^2 \\ &= 2(1 - r(h)) \end{aligned} \tag{3.1.1}$$

We shall assume in all cases that this function is monotone increasing near 0. As we pointed out in the comments before theorem 2.1.1. this implies that separability with respect to the usual metric and $\sigma$-separability are equivalent. Henceforth we shall only speak of separability.

In order to obtain detailed information about the rate of convergence of $Z(t) - (2 \log t)^{1/2}$ to zero as $t \to \infty$ we need more knowledge of the distribution of $Z(t)$ than one can get from theorem 2.3.1. Ideally one would like to get an exact asymptotic estimate of the tail of the distribution, i.e., a function $R(x)$ which depends on the process, and a function of the increments variance $S(\sigma)$, independent of $x$ and $t$, such that, for fixed $t$,

$$\lim_{x\to\infty} \frac{P\{Z(t) > x\}}{t\psi(x)R(x)} = S(\sigma)$$

This type of result has been obtained for G.P. having $\sigma(h)$ a regularly varying function of order $\alpha$, $0 < \alpha \leq 1$, by Pickands [31] and Qualls and Watanabe [34] (see also theorem 3.1.1 below). We investigate the case of G.P. having rougher sample paths, in particular those whose increments variance $\sigma(h)$ belongs to a subclass of the slowly varying functions. Unfortunately, we have been unable to obtain an asymptotically exact result. We can only obtain bounds for the function $R(x)$ in terms of certain functions derived from $\sigma(h)$. When $\sigma(h)$ is small these bounds are good and they will enable us to obtain accurate estimates of the rate of convergence in the next chapter. As $\sigma(h)$ increases, i.e. as we get nearer to the discontinuous case, the bounds get worse, so that our results depend on the smoothness of the paths.

The fact that we have not solved this problem completely means that we will not be able to decide in all cases whether a positive continuous, nondecreasing function which tends to $\infty$ as $t \to \infty$ belongs to the upper or the lower class.

<u>Definitions</u>

<u>Regularly Varying Function</u>

Let $F(h)$ be a positive function defined on $(0,\tau]$ for some $\tau$, $0 < \tau < \infty$. We shall say that $F(h)$ is a regularly varying function at zero with exponent $\alpha > 0$ (r.v.f. $(\alpha)$) if, for any $t > 0$

$$\lim_{h\to 0} \frac{F(th)}{F(h)} = t^{\alpha}$$

Slowly Varying Function

Let $G(h)$ be a positive function defined on $(0,\tau]$ for some $\tau$, $0 < \tau < \infty$. We shall say that $G(h)$ is a slowly varying function at zero (s.v.f.) if, for any $t > 0$,

$$\lim_{h \to 0} \frac{G(th)}{G(h)} = 1.$$

Regularly varying functions and slowly varying functions at $\infty$ are similarly defined. All the functions of this kind that we shall consider will have the relevant property at zero unless otherwise specified.

The results we shall now quote about these functions can be found in [34,36].

(3.1.a) $F(h)$ is a r.v.f. $(\alpha)$ iff $F(h) = h^{\alpha}G(h)$ with $G(h)$ a s.v.f.

(3.1.b) Representation theorem

If $G(h)$ is a s.v.f. then it can be expressed as

$$G(h) = \eta(h) \exp\left( - \int_h^{c_0} \frac{a(y)}{y}\, dy\right) \qquad (3.1.2)$$

for some $c_0$, $0 < c_0 \leq \tau$, where $\eta$ is a bounded function such that $\eta(h) \to A$ as $h \to 0$, $A$ being a constant, and $a(h)$ is a continuous function on $(0, c_0)$ with $a(h) \to 0$ as $h \to 0$. $a(h)$ is said to be the structure function of $G(h)$. It can be chosen to be differentiable and we shall always do so.

If $\eta(h) \equiv A$, $G(h)$ is said to be a normalized slowly varying function. (n.s.v.f.)

(3.1.c) If $G(h)$ is a n.s.v.f. then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$t^{\varepsilon} \leq \frac{G(th)}{G(h)} \leq t^{-\varepsilon}$$

for all $0 < t < 1$ and all $h$ such that $th < \delta$ .

(3.1.d) If $G(h)$ is a s.v.f. then for any $\varepsilon > 0$

$$\lim_{h \to 0} h^{-\varepsilon} G(h) = \infty$$

$$\lim_{h \to 0} h^{\varepsilon} G(h) = 0$$

(3.1.e) If $G(h)$ is a n.s.v.f. then, for any $\alpha > 0$, $h^{\alpha}G(h)$ is monotone increasing near 0.

(3.1.f) If $L(x)$ is a s.v.f. at $\infty$ then

$$\frac{\log L(x)}{\log x} \to 0 \qquad \text{as } x \to \infty.$$

(3.1.g) If $F(h)$ is a monotone increasing r.v.f. $(\alpha)$ then its inverse $F^{-1}(h)$ is a r.v.f. $(1/\alpha)$.

Let $G(h)$ be a n.s.v.f. Using (3.1.2)

$$\frac{d}{dh} G(h) = G(h) \frac{d}{dh} \int_{c_0}^{h} \frac{a(y)}{y} dy$$

$$= \frac{G(h)a(h)}{h} \qquad (3.1.3)$$

so that in this case

$$a(h) = \frac{hG'(h)}{G(h)} \qquad (3.1.4)$$

For the rest of this chapter and Chapter 4 all the slowly varying functions we shall deal with will be normalized and continuous.

The case of G.P.'s with $\sigma(h)$ a r.v.f. $(\alpha)$, $0 < \alpha \leq 1$ has been pointed out in the introductory remarks. Pickands [31] considered the case

$$\sigma(h) = ch^{\alpha} + o(h^{\alpha})$$

and obtained an asymptotic expression for the tail of the distribution of the supremum. Later Qualls and Watanabe [34] extended this result to processes with

$$\sigma(h) = h^{\alpha}G(h) + o(h^{\alpha}G(h))$$

where $G(h)$ is a n.s.v.f.

Theorem 3.1.1 ([34])

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. with

$$\sigma(h) = h^{\alpha}G(h) + o(h^{\alpha}G(h))$$

as $h \to 0$ where $0 < \alpha \leq 1$ and $G(h)$ is a n.s.v.f.

Let

$$\tilde{\sigma}(h) = \sqrt{2}\, h^{\alpha}G(h)$$

If

$$\sigma^2(h) > 0 \quad \text{for } h \neq 0$$

then

$$\lim_{x\to\infty} \frac{P\{Z(t) > x\}}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = H\alpha$$

where $\tilde{\sigma}^{-1}$ denotes the inverse function of $\tilde{\sigma}$ and $0 < H\alpha < \infty$.

We concentrate in the next two sections on G.P.'s having $\sigma(h)$ a n.s.v.f.

In [16] Kôno, in order to study the modulus of continuity of G.P.'s with $\sigma(h)$ continuous, nondecreasing and satisfying Fernique's condition:

$$\int^{\infty} \sigma(e^{-u^2})du < \infty$$

proposes a classification of G.P.'s.

Let

$$I(\sigma,h) = \int_0^{\infty} \sigma(he^{-u^2})du$$

and

$$\bar{\sigma}(h) = \frac{I(\sigma,h)}{\sigma(h)}$$

The classification is as follows:

I. X belongs to Class I if $\bar{\sigma}(h)$ is bounded

II. X belongs to Class II if $\bar{\sigma}(h)$ is not bounded but

$$\lim_{x \to 0} \frac{\bar{\sigma}(h)}{\sqrt{\log 1/h}} = 0$$

III. X belongs to Class III if $\bar{\sigma}(h) \div \sqrt{\log 1/h}$

IV. The rest of G.P.'s.

Kôno ([16,17] prop. 1) has shown that if $\sigma(h)$ is a r.v.f $(\alpha)$ then the process is of Class I and if $\sigma(h)$ is a s.v.f. it is not. We deal with processes belonging to Classes II and III. As examples of the sort of processes considered in the theorems of the next section we have in Class II those with

$$\sigma(h) = A\, e^{-(\log 1/h)^{\gamma}}$$

for $0 < \gamma < 1$ and A constant, and in Class III those with

$$\sigma(h) = \frac{A}{(\log 1/h)^{\alpha}}$$

for $\alpha > \frac{1}{2}$ and A constant.

## 3.2. Preliminary Results

In this section we quote some results that will be useful in the proofs of the next section.

Lemma 3.2.1 ([31] Lemma 2.3)

Let X and Y be jointly normally distributed random variables with mean 0, variance 1 and covariance r. Then, for $x > 0$

$$P\{X > x,\ Y > x\} \leq \frac{c}{x(1-r)^{1/2}} \exp\{-\frac{(1-r)x^2}{4}\}\ P\{X > x\} \tag{3.2.1}$$

Lemma 3.2.2 (Fernique's Inequality)

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a centred, separable Gaussian process with

$$E(X(t) - X(s))^2 \leq \sigma^2(|t-s|)$$

where $\sigma(h)$ is a nondecreasing continuous function satisfying

$$\int^{\infty} \sigma(e^{-u^2})du < \infty \tag{3.2.2}$$

Then, for any bounded interval $K \subset \mathbb{R}$ with diameter $d(K)$ we have

$$P\{\sup_{s\in K} |X(s)| \geq u(\Gamma_K^{1/2} + \frac{4I(\sigma,d(K))}{\sqrt{\log P}})\} \leq c\,p^2\psi(u) \tag{3.2.3}$$

for any $p > 1$ and $u \geq \sqrt{1 + 4 \log p}$ where

$$\Gamma_K = \sup_{t,s\in K} |r(s,t)| \tag{3.2.4}$$

There are several versions of this lemma (Fernique [11], Marcus [20], Jain and Marcus [14]). The present one can be found in Kôno [16,17].

<u>Lemma 3.2.3</u> ([16,17] Lemma 11)

Assume that $\sigma(h)$ is a s.v.f. with structure function $a(h)$ satisfying

(i) $$a(h) \gg \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

as $h \to 0$ for $\gamma > 1/2$

(ii) There exists a constant $D_1 > 0$ such that

$$a(h^{1+\varepsilon}) \gg D_1 a(h) \tag{3.2.6}$$

as $h \to 0$ uniformly for any $0 \leq \varepsilon \leq 1$.

Then

$$\bar{\sigma}(h) = \frac{I(\sigma,h)}{\sigma(h)} \leq \frac{D_2}{\sqrt{a(h)}}$$

for some constant $D_2 > 0$.

## 3.3. Inequalities

Theorem 3.3.1

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$E(X(t) - X(s))^2 = \sigma^2(|t-s|)$$

where $\sigma(h)$ is a continuous positive, monotone increasing for $0 \le h \le \tau$, some $\tau > 0$, n.s.v.f. with structure function $a(h)$ and satisfying

$$\inf_{h \ge \tau} \sigma(h) = \zeta > 0 \tag{3.3.1}$$

We assume that $a(h)$ satisfies

$$a(h) >> \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

as $h \to 0$ for $\gamma > 1/2$, and that there exists a constant $D_1 > 0$ such that

$$a(h^{1+\varepsilon}) >> D_1 a(h) \tag{3.2.6}$$

as $h \to 0$ uniformly for any $0 \le \varepsilon \le 1$. Also $a(h)$ is nondecreasing for $0 \le h \le \tau$, so that the continuous function

$$F(h) \equiv \sigma(h)\sqrt{a(h)} \tag{3.3.2}$$

is monotone increasing for $0 \le h \le \tau$. Define $N = N(x)$ implicitly by

$$F(1/N) = \frac{2}{x} \tag{3.3.3}$$

and $P(x)$ by

$$P(x) = \exp\left\{\frac{1}{a(1/N(x))}\right\} \tag{3.3.4}$$

Then there are constants $0 < c_1 < c_2$ such that

$$\frac{1}{2} P(x)^{c_1} \leq \frac{P\{Z(t) > x\}}{t\psi(x)N(x)} \leq \frac{3}{2} P(x)^{c_2} \quad (3.3.5)$$

as $x \to \infty$.

This theorem is a consequence of theorems 3.3.2 and 3.3.3 where the lower and upper inequalities are proved separately. The purpose of (3.3.1) is to avoid the periodic case. However, if X is periodic with period $t_1$ then (3.3.5) holds with t replaced by $t_2 = \min(t, t_1)$ in the denominator.

Since F(h) is continuous and monotone increasing for $0 \leq h \leq \tau$, it has an inverse $F^{-1}$ which is also continuous and monotone increasing. Therefore we can write

$$N(x) = \frac{1}{F^{-1}(2/x)}$$

for x sufficiently large and it is possible to see that N(x) is monotone increasing as $x \uparrow \infty$.

Because a(h) is nondecreasing as h increases for $0 \leq h \leq \tau$ and $a(h) \to 0$ as $h \to 0$, we have that

$$P(x) = \exp\left\{ \frac{1}{a(1/N(x))} \right\}$$

is nondecreasing and tends to $\infty$ as $x \to \infty$.

<u>Theorem 3.3.2</u>

Let $X = \{X(t), t \in \mathbb{R}\}$ be a stationary G.P. satisfying all the conditions of theorem 3.3.1. Let $M = M(x)$ be defined by

$$M(x) = [P(x)^{c_1} N(x)] \quad (3.3.6)$$

where [ ] denotes the integer part and $c_1$ is independent of x. Define

$$Z_X(t) = \max\left\{X\left(\frac{i}{M(x)}\right); \quad 0 \leq i \leq [tM(x)]\right\} \quad (3.3.7)$$

then

$$P\{Z_X(t) > x\} \geq \frac{1}{2}\, t\, P^{c_1}(x)\; N(x)\psi(x) \tag{3.3.8}$$

as $x \to \infty$.

Proof

Note that the remarks before the theorem imply that $M(x)$ is a non-decreasing function of $x$ as $x \to \infty$. Define

$$t_i = \frac{i}{M(x)} \qquad i = 0,1,\ldots,[tM]$$

$$A_i = \{X(t_i) > x\}$$

we have

$$1 - r(t_i,t_j) \doteq \frac{1}{2}\,\sigma^2(|t_i-t_j|)$$

$$= \frac{1}{2}\,\sigma^2\left(\frac{|i-j|}{M}\right)$$

We start with

$$P\{Z_X(t) > x\} = P\{\bigcup_{i=0}^{[tM]} A_i\}$$

$$\geq \sum_{i=0}^{[tM]} P(A_i) - \sum_{0\leq i<j\leq[tM]}\sum P(A_i \cap A_j)$$

Let $X \sim N(0,1)$. Using Lemma 3.2.1

$$\geq tMP\{X > x\}$$

$$- \sum_{i<j}\sum \frac{c}{x(1-r(t_i,t_j))^{1/2}} \exp\{-\frac{1}{4}(1-r(t_i,t_j))x^2\}P(A_i)$$

$$= tMP\{X > x\}$$

$$- \sum_{i=0}^{[tM]-1} P(A_i) \sum_{j=i+1}^{[tM]} \frac{c}{x\sigma(\frac{j-i}{M})} \exp\{ - \frac{x^2}{8} \sigma^2 (\frac{j-i}{M})\}$$

$$\geq tMP\{X > x\} - \sum_{i=0}^{[tM]-1} P(A_i) \sum_{K=1}^{[tM]} \frac{c}{x\sigma(K/M)} \exp\{ - \frac{x^2}{8} \sigma^2 (\frac{K}{M})\}$$

$$= tMP\{X > x\} \{ 1 - \sum_{K=1}^{[tM]} \frac{c}{x\sigma(K/M)} \exp\{ - \frac{x^2}{8} \sigma^2 (\frac{K}{M})\} \} \qquad (3.3.9)$$

We concentrate now on obtaining a bound for the sum on the right hand side.

Let $\chi$ be the value of h in the interval $[0,\tau]$ that satisfies $\sigma(h) = \zeta$. Since, by definition, $\zeta \leq \sigma(\tau)$ this is well defined. If $\zeta = \sigma(\tau)$ then $\tau = \chi$, otherwise $\tau > \chi$ and we have that $\sigma(h)$ is monotone increasing in $[0,\chi]$ and if $h > \chi$ then $\sigma(h) \geq \sigma(\chi)$. Using this

$$\sum_{K=1}^{[tM]} \frac{c}{x\sigma(K/M)} \exp \{- \frac{1}{8} x^2\sigma^2 (\frac{K}{M})\}$$

$$\leq \frac{ct}{\chi} \sum_{K=1}^{[\chi M]} \frac{1}{x\sigma(K/M)} \exp \{- \frac{1}{8} x^2\sigma^2 (\frac{K}{M})\} \qquad (3.3.10)$$

Since $\sigma(h)$ is increasing in the range of summation the terms decrease as K increases and so

$$\sum_{K=2}^{[\chi M]} \frac{1}{x\sigma(K/M)} \exp \{- \frac{1}{8} x^2\sigma^2 (\frac{K}{M})\}$$

$$\leq \int_1^{\chi M-1} \frac{1}{x\sigma(u/M)} \exp \{- \frac{x^2\sigma^2(u/M)}{8} \} du$$

and

$$\frac{1}{x\sigma(1/M)} \exp\{-\frac{1}{8} x^2\sigma^2(1/M)\} \leq 2 \int_{1/2}^{1} \frac{1}{x\sigma(u/M)} \exp\{-\frac{x^2\sigma^2(u/M)}{8}\}du$$

Hence

$$\frac{ct}{\chi} \sum_{K=1}^{[\chi M]} \frac{1}{x\sigma(K/M)} \exp\{-\frac{1}{8} x^2\sigma^2(\frac{K}{M})\}$$

$$\leq \frac{ct}{\chi} \int_{1/2}^{\chi M} \frac{1}{x\sigma(u/M)} \exp\{-\frac{1}{8} x^2\sigma^2(\frac{u}{M})\} du \qquad (3.3.11)$$

Let

$$u = Mw, \qquad du = Mdw$$

then

$$(3.3.11) = \frac{cMt}{\chi x} \int_{1/2M}^{\chi} \frac{1}{\sigma(w)} \exp\{-\frac{x^2\sigma^2(w)}{8}\}dw \qquad (3.3.12)$$

and the rest of the proof is devoted to showing that this expression tends to zero as $x \to \infty$ for $c_1$ suitably chosen in the definition of M.

We make the following change of variables

$$\sigma(w) = \frac{1}{v}$$

$$\sigma'(w)dw = \frac{-1}{v^2} dv$$

$$dw = \frac{-1}{v^2\sigma'(w)} dv$$

and using (3.1.3)

$$dw = \frac{-w}{v^2\sigma(w)a(w)} dv$$

$$= \frac{-w}{va(w)} dv$$

Since we have assumed $\sigma(h)$ to be continuous and increasing in the range of integration it has an inverse $\sigma^{-1}(h)$ which is also continuous and increasing. Therefore

$$w = \sigma^{-1}(\frac{1}{v})$$

and

$$dw = \frac{-\sigma^{-1}(1/v)}{va(\sigma^{-1}(1/v))} dv \tag{3.3.13}$$

With this transformation

$$(3.3.12) = \frac{cMt}{\chi x} \int_{1/\sigma(\chi)}^{1/\sigma(1/2M)} \frac{\sigma^{-1}(1/v)}{a(\sigma^{-1}(1/v))} \exp\{-\frac{x^2}{8v^2}\}dv$$

$$= \frac{cMt}{\chi x} \int_{1/\sigma(\chi)}^{1/\sigma(1/2M)} \frac{1}{a(\sigma^{-1}(1/v))} \exp\{\log \sigma^{-1}(\frac{1}{v}) - \frac{x^2}{8v^2}\}dv \tag{3.3.14}$$

We now look for the maximum of the exponent in (3.3.14). Using (3.3.13)

$$\frac{d}{dv}(\log \sigma^{-1}(1/v) - \frac{x^2}{8v^2}) = \frac{-\sigma^{-1}(1/v)}{\sigma^{-1}(1/v)va(\sigma^{-1}(1/v))} + \frac{x^2}{4v^3}$$

$$= \frac{1}{v}(\frac{-1}{a(\sigma^{-1}(1/v))} + \frac{x^2}{4v^2}) = 0$$

and the exponent has a maximum at $v = \xi$ where $\xi$ is defined by

$$\frac{\xi}{[a(\sigma^{-1}(1/\xi))]^{1/2}} = \frac{x}{2} \tag{3.3.15}$$

Hence

$$(3.3.14) < \frac{cMt}{\chi x} \exp\{\log \sigma^{-1}(1/\xi) - \frac{x^2}{8\xi^2}\} \int_{1/\sigma(\chi)}^{1/\sigma(1/2M)} \frac{1}{a(\sigma^{-1}(1/v))} dv$$

and using the monotonicity of a(h),

$$\leq \frac{cMt}{\chi x} \left(\frac{1}{\sigma(\frac{1}{2M})} - \frac{1}{\sigma(\chi)}\right) \frac{1}{a(\frac{1}{2M})} \exp\{\log \sigma^{-1}(\frac{1}{\xi}) - \frac{x^2}{8\xi^2}\}$$

$$< \frac{cMt\sigma^{-1}(1/\xi)}{\chi x \sigma(\frac{1}{2M}) a(\frac{1}{2M})} \exp\{-\frac{x^2}{8\xi^2}\}$$

using (3.3.15)

$$= \frac{cMt\sigma^{-1}(1/\xi)}{\chi x \sigma(\frac{1}{2M}) a(\frac{1}{2M})} \exp\{\frac{-1}{2a(\sigma^{-1}(1/\xi))}\} \qquad (3.3.16)$$

By comparing (3.3.15) with the definition of N(x), (3.3.3), we get

$$\sigma(1/N)[a(1/N)]^{1/2} = \frac{[a(\sigma^{-1}(1/\xi))]^{1/2}}{\xi}$$

which gives

$$N = \frac{1}{\sigma^{-1}(1/\xi)} \qquad (3.3.17)$$

or equivalently

$$\frac{1}{\xi} = \sigma(1/N)$$

Hence

$$(3.3.16) = \frac{cMt}{\chi x \sigma(\frac{1}{2M}) a(\frac{1}{2M}) N} \exp\{\frac{-1}{2a(1/N)}\}$$

using the definition of P

$$= \frac{cMtP^{-1/2}}{\chi x\sigma(\frac{1}{2M})a(\frac{1}{2M})N}$$

and using the definition of M

$$\leq \frac{ctP^{c_1-1/2}}{\chi x\sigma(\frac{1}{2M})\ a(\frac{1}{2M})} \qquad (3.3.18)$$

Using property (3.1.c) for $\varepsilon > 0$ arbitrary, x large

$$\sigma(\frac{1}{2M}) \geq \sigma(\frac{1}{2NP^{c_1}})$$

$$\geq \frac{1}{(2P^{c_1})^{\varepsilon}}\ \sigma(\frac{1}{N})$$

and so

$$(3.3.18) \leq \frac{ctP^{c_1-(1/2)+\varepsilon c_1}}{\chi x\sigma(1/N)\ a(\frac{1}{2M})}$$

Using the fact that we have assumed

$$a(h) >> \frac{\gamma}{\log 1/h}$$

as $h \to 0$ for $\gamma > \frac{1}{2}$ we get

$$2M \leq 2NP^{c_1} = 2Ne^{c_1/a(1/N)}$$

$$\leq 2N\ e^{c_1 \log N/\gamma}$$

$$< N^{1+2c_1}$$

for x large. So

$$a(\frac{1}{2M}) \geq a(1/N^{1+2c_1})$$

If $c_1 \leq \frac{1}{2}$ we can use (3.2.6) and get

$$a(\tfrac{1}{2M}) \geq D_1\, a(1/N)$$

which implies

$$(3.3.18) \leq \frac{ctP^{c_1(1+\varepsilon)-1/2}}{D_1\chi x\sigma(1/N)a(1/N)}$$

and since, by definition,

$$x\sigma(1/N) = \frac{2}{\sqrt{a(1/N)}} \qquad (3.3.19)$$

$$(3.3.18) \leq \frac{ctP^{c_1(1+\varepsilon)-1/2}}{D_1\chi\ \sqrt{a(1/N)}}$$

Choose $c_1$ so that $c_1 < \frac{1}{2(1+\varepsilon)}$. Then the expression above tends to zero as $x \to \infty$ since $t$ and $\chi$ are fixed and $P(x) \to \infty$ as $x \to \infty$. This, together with (3.3.9), (3.3.10) and (3.3.11) show that

$$P\{Z_X(t) > x\} \geq \frac{1}{2}\, tP^{c_1}(x)N(x)\psi(x)$$

where $0 < c_1 < \frac{1}{2}$.

□

<u>Corollary 3.3.1</u>

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. satisfying all the conditions of theorem 3.3.1. Then

$$P\{Z(t) > x\} \geq \frac{1}{2}\, t\, P^{c_1}(x)\ N(x)\psi(x)$$

as $x \to \infty$.

Theorem 3.3.3

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. satisfying all the conditions of theorem 3.3.1. Then, as $x \to \infty$

$$P\{Z(t) > x\} \leq \frac{3}{2}\, t\, P(x)^{c_2}\, N(x)\psi(x)$$

where $c_2$ is independent of x.

Proof

We start by showing that

$$a(h) \gg \frac{\gamma}{\log 1/h} \qquad (3.2.5)$$

as $h \to 0$ for $\gamma > 1/2$ implies that $\sigma(h)$ satisfies Fernique's condition:

$$\int^{\infty} \sigma(e^{-u^2})du < \infty \qquad (3.2.2)$$

Using the Representation theorem, (3.1.b),

$$\sigma(h) = A \exp\{-\int_h^{c_0} \frac{a(y)}{y}\}dy$$

$$\leq A \exp\{-\int_h^{c_0} \frac{\gamma}{y \log 1/y} dy\}$$

$$= A \exp\{-\gamma \log\log 1/h + \gamma \log\log 1/c_0\}$$

$$= \frac{\text{const.}}{(\log 1/h)^{\gamma}} \qquad (3.3.20)$$

Therefore

$$\int^{\infty} \sigma(e^{-u^2})du < \int \frac{\text{const.}}{u^{2\gamma}} du < \infty$$

since $\gamma > \frac{1}{2}$.

To study the process we subdivide the interval [0,t] into subintervals of length 1/N. Let x be large enough so that $1/N < \tau$ and define $Q = \frac{D_3}{a(1/N)}$, $D_3 > 0$ a constant to be specified later. Using stationarity

$$P\{Z(t) > x + \frac{Q}{x}\} \leq Nt\, P\{Z(1/N) > x + \frac{Q}{x}\}$$

$$\leq NtP\{(X(0) > x) \cup (Z(1/N) > x + \frac{Q}{x},\ X(0) \leq x)\}$$

$$\leq NtP\{X(0) > x\} + NtP\begin{Bmatrix} Z(1/N) > x + \frac{Q}{x} \\ X(0) \leq x \end{Bmatrix} \tag{3.3.21}$$

The probability in the second term can be expressed as

$$\int_{-\infty}^{x} P\{Z(1/N) > x + \frac{Q}{x}\,|X(0) = u\}\,\phi(u)du \tag{3.3.22}$$

We make the following changes of variables

$$u = x - \frac{z}{x} \qquad du = \frac{-1}{x}dz$$

$$(3.3.22) = \psi(x)\int_{0}^{\infty} P\{Z(1/N) > x + \frac{Q}{x}|X(0) = x - \frac{z}{x}\}$$

$$e^{z-z^2/2x^2}\,dz$$

The parameters of the conditional distribution of Gaussian random variables are well known (see e.g. [1])

$$E(X(s)\,|X(0) = x - \frac{z}{x}) = r(s)\,(x - \frac{z}{x}) \tag{3.3.23}$$

and

$$\mathrm{cov}(X(s)X(v)\,|X(0) = x - \frac{z}{x}) = r(|s-v|) - r(s)r(v) \tag{3.3.24}$$

Let $\{Y_1(s),\ s \in [0, 1/N]\}$ be a Gaussian process having mean (3.3.23) and covariance (3.3.24). Then

$$(3.3.22) \quad = \quad \psi(x) \int_0^\infty P\{ \sup_{[0,1/N]} Y_1(s) > x + \frac{Q}{x} \} e^{z-z^2/2x^2} dz$$

$$\leq \psi(x) \int_0^\infty P\{ \sup_{[0,1/N]} Y_1(s) > x + \frac{Q}{x} \} e^{z}\, dz \qquad (3.3.25)$$

Since

$$x + \frac{Q}{x} - r(s) \ (x - \frac{z}{x})$$

$$= \ x(1 - r(s)) + \frac{Q}{x} + \frac{r(s)z}{x}$$

$$\geq \frac{Q}{x} + \frac{rz}{x}$$

for all $s \in [0, 1/N]$, where $r = r(1/N)$, and letting $\{Y(s), s \in [0, 1/N]\}$ be a centred Gaussian process with covariance (3.3.24) we get

$$(3.3.25) \quad \leq \quad \psi(x) \int_0^\infty P\{ \sup_{[0,1/N]} Y(s) > \frac{Q}{x} + \frac{rz}{x} \} e^{z}\, dz \qquad (3.3.26)$$

The increments variance of $Y(s)$ is

$$E(Y(s) - Y(v))^2 \ = \ EY^2(s) + EY^2(v) - 2EY(s)Y(v)$$

$$= \ 1 - r^2(v) + 1 - r^2(s) - 2r(|s-v|) + 2r(s)r(v)$$

$$= \ 2(1 - r(|s-v|)) - (r(v) - r(s))^2$$

$$\leq 2(1 - r(|s-v|)) \ = \ \sigma^2(|s-v|)$$

for all $s,v$ in $[0, 1/N]$ and we have shown at the beginning of the proof that $\sigma(h)$ satisfies Fernique's condition. Therefore we can use Fernique's Inequality to obtain an upper bound for the probability in (3.3.26). In this case we have $K = [0, 1/N]$ and $d(K) = 1/N$. From (3.3.24) we get

$$\Gamma_K = \sup_{s\in[0,1/N]} (1 - r^2(s))$$

$$= \sigma^2(1/N)$$

therefore

$$u\ (\Gamma_K^{1/2} + \frac{4I(\sigma,1/N)}{(\log p)^{1/2}}) = u\sigma(1/N)\ (1 + \frac{4I(\sigma,\ 1/N)}{(\log p)^{1/2}\sigma(1/N)})$$

$$= u\sigma(1/N)\quad (1 + \frac{4\bar{\sigma}(1/N)}{(\log p)^{1/2}})$$

and using lemma 3.2.3

$$\leq u\sigma(1/N)\quad (1 + \frac{4D_2}{(\log p\ \ a(1/N))^{1/2}})$$

Let $\log p = \frac{1}{a(1/N)}$ and $D_4 = 1 + 4D_2$. Then Fernique's Inequality is

$$P\{\sup_{[0,1/N]} |Y(s)| > D_4 u\sigma(1/N)\} \leq P^2(x)\psi(u) \tag{3.3.27}$$

$$\text{provided}\quad u \geq (1 + \frac{4}{a(1/N)})^{1/2} \tag{3.3.28}$$

To use this in (3.3.26) we want

$$u = \frac{1}{D_4\sigma(1/N)}\ (\frac{Q+rz}{x}) \tag{3.3.29}$$

and we have to check that (3.3.28) is satisfied

$$u \geq \frac{Q}{D_4 x\sigma(1/N)}$$

using (3.3.19)

$$= \frac{Q\sqrt{a(1/N)}}{2D_4}$$

and using the definition of Q

$$= \frac{D_3}{2D_4 \sqrt{a(1/N)}}$$

Let

$$D_3 > 4D_4^{\,2} \tag{3.3.30}$$

then (3.3.28) is satisfied for x large and we can use (3.3.27) with u as in (3.3.29). Therefore

$$(3.3.26) \quad \leq \quad \psi(x) \int_0^\infty c\, P^2(x)\psi(u)e^z\, dz$$

$$\leq c\psi(x)P^2(x) \int_0^\infty \frac{D_4 x\sigma(1/N)}{Q + rz} \exp\{-\frac{(Q + rz)^2}{2D_4^2x^2\sigma^2(1/N)} + z\}\, dz$$

$$\leq \frac{c\psi(x)P^2(x)x\sigma(1/N)}{Q} \int_0^\infty \exp\{-\frac{Q^2+r^2z^2+2rQz}{2D_4^2x^2\sigma^2(1/N)} + z\}\, dz$$

Using (3.3.19)

$$= \frac{c\psi(x)P^2(x)}{Q\sqrt{a(1/N)}} \int_0^\infty \exp\{-\frac{a(1/N)}{8D_4^2}(Q^2+r^2z^2+2rQz) + z\}\, dz$$

using the definitions of Q and P

$$= c\psi(x)\sqrt{a(1/N)}\, P(x)^{2-D_3^2/8D_4^2} \int_0^\infty \exp\{-\frac{a(1/N)}{8D_4^2}(r^2z^2+2rQz) + z\}\, dz$$

$$= c\psi(x)\sqrt{a(1/N)}\, P(x)^{2-D_3^2/8D_4^2}$$

$$\int_0^\infty \exp\{-\frac{a(1/N)r^2z^2}{8D_4^2} - z(\frac{rD_3}{4D_4^2} - 1)\}\, dz \tag{3.3.31}$$

Since $D_3 > 4D_4^2$ and $r \equiv r(1/N) \to 1$ as $x \to \infty$ we have, for x large enough

$$(3.3.31)\quad < c\psi(x)\ \sqrt{a(1/N)}\ P(x)^{2-2D_4^2} \int_0^\infty \exp\{-\frac{a(1/N)r^2}{8D_4^2} z^2\}\, dz$$

$$= c\psi(x)\ \sqrt{a(1/N)}\ \frac{\sqrt{\pi}}{2} \frac{2\sqrt{2}\, D_4}{r\sqrt{a(1/N)}}\ P(x)^{2-2D_4^2}$$

but $D_4 = 1 + 4D_2$. Therefore

$$< c\psi(x)\ P(x)^{-8D_2}\ .$$

Using this in (3.3.21)

$$P\{Z(t) > x + \frac{Q}{x}\} \le t\ N(x)\psi(x)\ (1 + cP(x)^{-8D_2})$$

$$\le (1+\delta)\ t\ N(x)\psi(x) \qquad (3.3.32)$$

for $\delta > 0$, x large, where $Q = \dfrac{D_3}{a(1/N)}$ .

Let $w = x + \dfrac{D_3}{xa(1/N)}$ . Then (3.3.32) becomes

$$P\{Z(t) > w\} \le (1+\delta)tN(x)\psi(x)$$

$$= (1+\delta)tN(w - \frac{D_3}{xa(1/N)})\ \psi(w - \frac{D_3}{xa(1/N)})$$

Since N(x) increases with x

$$\le (1+\delta)tN(w)\ \psi(w - \frac{D_3}{xa(1/N)}) \qquad (3.3.33)$$

(3.3.19) implies

$$\frac{D_3}{xa(1/N)} = \frac{D_3\sigma(1/N)}{2\sqrt{a(1/N)}}$$

and using (3.2.5), for x large

$$\leq \frac{D_3}{2\gamma}\sqrt{\log N}\quad \sigma(1/N)$$

Finally, using (3.3.20) we get

$$\frac{D_3}{xa(1/N)} \leq \frac{\text{const.}}{(\log N)^{\gamma-1/2}} \to 0 \quad \text{as } x \to \infty$$

since $\gamma > \frac{1}{2}$.

Hence,

$$(3.3.33) \quad \leq \frac{(1+\delta)tN(w)}{(w - \frac{D_3}{xa(1/N)})\sqrt{2\pi}} \exp\{-\frac{1}{2}(w - \frac{D_3}{xa(1/N)})^2\}$$

$$\leq (1+2\delta)\frac{tN(w)}{w\sqrt{2\pi}} \exp\{-\frac{w^2}{2} + \frac{D_3 w}{xa(1/N(x))}\}$$

and using again the fact that N(x) is increasing and a(h) is nondecreasing

$$\leq (1+2\delta)tN(w)\psi(w) \exp\{\frac{D_3 w}{xa(1/N(w))}\}$$

but by definition

$$\frac{w}{x} = 1 + \frac{D_3}{x^2 a(1/N)} \to 1 \quad \text{as } x \to \infty$$

Let $c_2 > D_3$. Then

$$P\{Z(t) > w\} \le \frac{3}{2} tN(w) \; P^{c_2}(w)\psi(w)$$

as $w \to \infty$.

□

## 3.4. Comments

3.4.1. From the proofs of the theorems it is easy to see that the constants $\frac{1}{2}$ and $\frac{3}{2}$ in (3.3.5) can be replaced by $(1-\delta)$ and $(1+\delta)$ respectively, for any $\delta > 0$ and $x$ large. Also $c_1 < \frac{1}{2}$. The value of $c_2$, however, is not so clear. We need

$$c_2 > 4(1 + 4D_2)^2$$

but $D_2$ depends on the product $\sqrt{a(h)}\,\bar{\sigma}(h)$ which may depend on the particular process being considered. It is evident that $c_2 > 4$ in all cases.

3.4.2. It seems interesting to point out that in theorem 3.3.3 the definition of $N(x)$ can be changed to

$$F(1/N) = \frac{\nu}{x}$$

for any $\nu > 0$. In this case $N(x,\nu)$ decreases as $\nu$ increases but $c_2$ is now an increasing function of $\nu$ since it must satisfy

$$c_2 > \nu^2(1 + 4D_2)^2$$

It is not clear what is the value of $\nu$ which makes

$$N(x,\nu)\ P(x)^{c_2(\nu)}$$

a minimum.

In theorem 3.3.2 we need $\nu = 2$ for the proof to work. Since the method used in this proof is rather rough it is possible that a more accurate method for estimating this probability would yield a different value.

3.4.3 Suppose X is a G.P. satisfying the conditions of theorem 3.3.1. If we have that

$$a(h) \gg \frac{\delta}{\log 1/h}$$

as $h \to 0$ holds for any $\delta > 0$ then ([16], proposition 2) X belongs to Class II, and from the definition of P(x) we get that for any $c \geq 0$

$$\frac{P^c(x)}{N(x)} \to 0$$

as $x \to \infty$. On the other hand, if we have constants $A_2$ and $A_3$, $\frac{1}{2} < A_2 < A_3$ such that

$$\frac{A_3}{\log 1/h} \gg a(h) \gg \frac{A_2}{\log 1/h}$$

as $h \to 0$ then ([16], proposition 2) the process is in Class III and in this case

$$N^{A_2}(x) \leq P(x) \leq N^{A_3}(x)$$

We see that, in the first case, the difference between the bounds in (3.3.5), in terms of N(x), is of a smaller order of magnitude than in the second case, which is the process with the larger increments variance.

This illustrates how the precision of our result decreases as we approach the discontinuous case.

3.4.4 Let us now consider the case of a G.P. having $\sigma(h)$ a r.v.f. $(\alpha)$, $0 < \alpha \leq 1$, i.e. $\sigma(h) = h^\alpha G(h)$ where G(h) is a n.s.v.f. with structure function b(h). We propose to show how our results relate to theorem 3.1.1 and how, by a suitable extension of the definitions of the functions a(h), N(x) and P(x), we can obtain from this theorem and from theorem 3.3.1 a set of inequalities covering both cases.

Using the Representation theorem

$$h^{\alpha}G(h) = A \exp\{-\int_h^1 \frac{\alpha+b(y)}{y}\,dy\}$$

for any $\alpha \geq 0$. Define

$$a(h) = \max(\alpha, b(h)) \qquad (3.4.1)$$

In the case $\alpha = 0$ we get that $a(h)$ is, as before, the structure function of the n.s.v.f. $\sigma(h)$. In the case $\alpha > 0$ we may, abusing the language, call $\alpha + b(h)$ the structure function of $\sigma(h)$, and we are taking as our definition of $a(h)$ the dominant part of the structure function since $b(h) \to 0$ as $h \to 0$, i.e.

$$a(h) \sim \alpha + b(h)$$

We use (3.4.1) to define $N(x)$ in the case $\alpha > 0$:

$$F(1/N) = \sigma(1/N)\sqrt{a(1/N)} = \sigma(1/N)\sqrt{\alpha} = \frac{2}{x}$$

$$\sigma(1/N) = \frac{2}{\sqrt{\alpha}\,x}$$

or

$$\frac{1}{N} = \sigma^{-1}\left(\frac{2}{\sqrt{\alpha}\,x}\right)$$

but by (3.1.g) if $\sigma$ is a r.v.f. $(\alpha)$ then $\sigma^{-1}$ is a r.v.f. $(1/\alpha)$. Hence

$$\frac{1}{N} \sim \left(\frac{2}{\sqrt{\alpha}}\right)^{1/\alpha} \sigma^{-1}(1/x)$$

and

$$N(x) \sim \frac{D_5}{\sigma^{-1}(1/x)} \tag{3.4.2}$$

for some constant $D_5$ depending on $\alpha$. Since

$$\sigma^{-1}(1/x) \doteq \tilde{\sigma}^{-1}(1/x)$$

we get

$$N(x) \doteq \frac{1}{\tilde{\sigma}^{-1}(1/x)}$$

On the other hand $P(x)$ is defined by

$$P(x) = \exp\left\{ \frac{1}{a(1/N(x))} \right\}$$

and since $a(h) \equiv \alpha$ for all $h$, $P(x)$ is simply $e^{1/\alpha}$ in this case.

Hence, for properly chosen constants $c_1$ and $c_2$ (3.3.5) is a consequence of theorem 3.1.1 when $\alpha > 0$. In this sense theorem 3.3.1 can be said to be an extension of theorem 3.1.1. Taking into account these comments we can state:

<u>Theorem 3.4.1</u>

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. with $\sigma(h)$ a positive, continuous, monotone increasing function for $0 \leq h \leq \tau$ for some $\tau > 0$ which can be expressed as

$$\sigma(h) = h^{\alpha} G(h)$$

for $0 \leq h \leq \tau$ where $0 \leq \alpha \leq 1$ and $G(h)$ is a n.s.v.f. with structure function $b(h)$. Assume

$$\inf_{h \geq \tau} \sigma(h) = \zeta > 0$$

and if $\alpha = 0$ we also assume that $b(h)$ satisfies the following conditions:

a) $b(h)$ is nondecreasing for $0 \le h \le \tau$

b) $b(h) >> \frac{\gamma}{\log 1/h}$ as $h \to 0$ for $\gamma > \frac{1}{2}$

c) there is a constant $D_1 > 0$ such that

$$b(h^{1+\varepsilon}) >> D_1 b(h)$$

as $h \to 0$ uniformly for any $0 \le \varepsilon \le 1$.

Define

$$a(h) = \max(\alpha, b(h)) \tag{3.4.1}$$

$$P(x) = \exp\{\frac{1}{a(1/N(x))}\}$$

where $N(x)$ is defined implicitly by

$$\sigma(1/N) \sqrt{a(1/N)} = \frac{2}{x}$$

then there are constants $0 < c_1 \le c_2$, $0 < c_3 < 1 < c_4$ such that

$$c_3 P^{c_1}(x) \le \frac{P\{Z(t) > x\}}{t\psi(x) N(x)} \le c_4 P^{c_2}(x) \tag{3.4.3}$$

as $x \to \infty$.

In the case $\alpha > 0$, $c_1 = c_2$. If $\alpha = 0$ then $c_1 < 1/2$ and $c_2 > 4$.

3.4.5. Property (3.1.3) is satisfied asymptotically by $\sigma(h)$ as $h \to 0$, with (3.4.1) as the definition of $a(h)$, when $\alpha > 0$

$$\begin{aligned} \sigma'(h) &= \frac{d}{dh}(h^\alpha G(h)) \\ &= \alpha h^{\alpha-1} G(h) + h^\alpha G'(h) \\ &= h^{\alpha-1} G(h) \ (\alpha + b(h)) \\ &\sim \frac{\sigma(h)\ a(h)}{h} \qquad \text{as } h \to 0. \end{aligned} \tag{3.4.4}$$

3.4.6. Finally, to end this chapter we give a partial discrete version of theorem 3.4.1. The case $\alpha = 0$ follows from theorem 3.3.2 and the case $\alpha > 0$ from Qualls and Watanabe ([34] Lemma 2.3) by choosing a appropriately.

Theorem 3.4.2

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. satisfying all the conditions of theorem 3.4.1. Define

$$M(x) = [P^{c_1}(x)\ N(x)] \tag{3.3.6}$$

and

$$Z_X(t) = \max \{X(\frac{i}{M(x)});\quad 0 \leq i \leq [t\ M(x)]\} \tag{3.3.7}$$

then

$$P\{Z_X(t) > x\} \geq \text{const. } t\ P^{c_1}(x) N(x) \psi(x) \tag{3.4.5}$$

as $x \to \infty$.

# Chapter 4

## The Upper and Lower Envelopes

### 4.1 Introduction

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. The results of Chapter 2 imply that

$$Z(t) - (2 \log t)^{1/2} \to 0$$

as $t \to \infty$ with probability one, as long as X satisfies a not too restrictive mixing condition like

$$r(t) \log t \to 0 \tag{4.1.1}$$

as $t \to \infty$. In this Chapter we shall be interested in estimating the rate at which this convergence takes place.

We shall state the problem in a different form. We are interested in obtaining continuous curves $\pi_1(t)$ and $\pi_2(t)$, $\pi_1(t) < \pi_2(t)$ such that the process $Z(t)$ gets arbitrarily near to both infinitely often as $t \to \infty$ but only crosses either of them a finite number of times. Such functions are called the lower and upper envelopes respectively.

Let $\theta(t)$ be a nondecreasing continuous function with $\theta(t) \to \infty$ as $t \to \infty$ and consider the following events

$$S(\theta) = \left\{ \begin{array}{l} \omega : \text{there is a } t_0(\omega) \text{ with } Z(t,\omega) < \theta(t) \\ \qquad \text{for all } t > t_0(\omega) \end{array} \right\}$$

$$T(\theta) = \left\{ \begin{array}{l} \omega : \text{there is a } t_0(\omega) \text{ with } Z(t,\omega) > \theta(t) \\ \qquad \text{for all } t > t_0(\omega) \end{array} \right\}$$

Note that $P\{S(\theta)\} = 1$ (or 0) is equivalent to $\theta \in U(X)$ $(L(X))$.

Our objective is to obtain conditions for these events to have probability 0 or 1. If one is able to characterize these events with sufficient precision then one can obtain information about the envelopes by investigating functions of the type

$$\theta(t) = (2 \log t)^{1/2} + g(t)$$

with $g(t) \to 0$ as $t \to \infty$.

To carry this out we have to restrict the class of processes under investigation to those for which we have some detailed information about the distribution of the maximum. This means that we shall concentrate on those processes that satisfy the conditions of theorem 3.3.1 plus a standard mixing condition stronger than (4.1.1).

This type of investigation has been done for G.P.'s having $\sigma(h)$ a r.v.f. $(\alpha)$. The event $S(\theta)$ has been characterized by an integral $I(\theta)$ of the function whose convergence properties determine whether $P\{S(\theta)\} = 0$ or 1. This result is quoted in theorem 4.2.3. About $T(\theta)$, less is known; in this case no sharp division in terms of integrals has been obtained for G.P.'s and only specific functions $\theta(t)$ for which $P\{T(\theta)\} = 0$ or 1 are known ([25,32]).

In section 4.2 we give integral tests for deciding whether $P\{S(\theta)\} = 0$ or 1. These tests, however, are not sharp enough to decide in all cases which alternative takes place, since the results depend crucially on theorem 3.3.1 and we have $c_1 < c_2$. Nevertheless the results obtained are sufficiently good to give precise estimates of the upper envelope. This is illustrated in a general way in Corollary 4.2.3 and for particular processes in the comments we make in 4.2.4.

As regards $T(\theta)$ we obtain in section 4.3 a function $\theta_2(t,c,\varepsilon)$ such that, for any $\varepsilon > 0$

$$P\{T(\theta_2(t,c_2,-\varepsilon)\} = 0 \qquad P\{T(\theta_2(t,c_1,\varepsilon))\} = 1$$

This result gives an estimate of the lower envelope that complements the results of 4.2. It is given in Corollary 4.3.1 and the comments of 4.3.4.

Define the functions $N(x)$ and $P(x)$ in terms of $\sigma(h)$ as in theorem 3.3.1. These functions will be used repeatedly throughout this Chapter and play a very important role. By combining Corollaries 4.2.3 and 4.3.1 we get, for X having $\sigma(h)$ a n.s.v.f. satisfying the conditions of theorem 3.3.1.,

$$r(s) = O(s^{-\lambda}) \quad \text{as } s \to \infty \quad \text{for } \lambda > 0$$

and

$$a(1/N(x)) = o(\log x) \quad \text{as } x \to \infty,$$

the following result about the asymptotic behaviour of $Z(t)$:

$$\limsup_{t\to\infty} \left(Z(t) - (2 \log t)^{1/2} - \frac{\log N((2 \log t)^{1/2})}{(2 \log t)^{1/2}}\right) \frac{(2 \log t)^{1/2}}{\frac{1}{2} \log \log t} = 1$$

$$\liminf_{t\to\infty} \left(Z(t) - (2 \log t)^{1/2} - \frac{\log N((2 \log t)^{1/2})}{(2 \log t)^{1/2}}\right) \frac{(2 \log t)^{1/2}}{\frac{1}{2} \log \log t} = -1$$

with probability one.

In 4.2.4 and 4.3.4 we also comment on how the results obtained in each section relate to the known results for processes with $\sigma(h)$ a r.v.f. $(\alpha)$, $0 < \alpha \leq 1$. For the upper envelope we summarize all these results in theorem 4.2.4.

Finally, we point out that in the case $\sigma(h)$ a r.v.f. $(\alpha)$ the results can be obtained under a weaker mixing condition [25,27].

## 4.2. The Upper Envelope

Let $\theta(t)$ be a positive, continuous, nondecreasing function with $\theta(t) \to \infty$ as $t \to \infty$. In this section we shall denote $N(\theta(t))$ by $N_t$ and $P(\theta(t))$ by $P_t$.

### 4.2.1 Lower Bound

**Theorem 4.2.1**

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. with

$$E(X(t) - X(s))^2 = \sigma^2(|t-s|)$$

where $\sigma(h)$ is continuous, positive, monotone increasing for $0 \leq h \leq \tau$, some $\tau > 0$, n.s.v.f. with structure function $a(h)$. We assume that $a(h)$ is nondecreasing for $0 \leq h \leq \tau$, satisfies

$$a(h) >> \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

as $h \to 0$ for $\gamma > \frac{1}{2}$ and that there exists a constant $D_1 > 0$ such that

$$a(h^{1+\varepsilon}) >> D_1 a(h) \tag{3.2.6}$$

as $h \to 0$ uniformly for any $0 \leq \varepsilon \leq 1$.

We also assume that

$$r(s) = O(s^{-\lambda}) \tag{4.2.1}$$

as $s \to \infty$ for some $\lambda > 0$.

If

$$I_1(\theta) = \int_c^\infty P_t^{c_1} N_t \psi(\theta(t))dt = \infty \tag{4.2.2}$$

then

$$P\{S(\theta)\} = 0$$

where $0 < c_1 < \frac{1}{2}$ is the constant in theorem 3.3.1.

The proof follows the method of Qualls and Watanabe [33]. The following lemmas will be needed in the proof. The first one gives an idea of how large N(x) is in the cases under consideration.

Lemma 4.2.1

For any $\beta \geq 0$

$$\frac{N(x)}{x^{\beta}} \to \infty \qquad \text{as } x \to \infty \tag{4.2.3}$$

Also

$$\frac{\log N(x)}{x} \to 0 \qquad \text{as } x \to \infty \tag{4.2.4}$$

Proof

Since $F(h) \equiv \sigma(h)\sqrt{a(h)}$ is continuous and monotone increasing in the interval $[0,\tau]$, it has a continuous and increasing inverse $F^{-1}$ and, for x sufficiently large, we have

$$N(x) = \frac{1}{F^{-1}(2/x)} \tag{4.2.5}$$

If G(x) is also a continuous monotone increasing function with $F(h) \gg G(h)$ as $h \to 0$ then, for all x large

$$\frac{1}{F^{-1}(2/x)} \geq \frac{1}{G^{-1}(2/x)} \tag{4.2.6}$$

But from property (3.1.d) we see that if $\sigma(h)$ is a n.s.v.f. then, for any $\varepsilon > 0$,

$$\sigma(h) \gg h^{\varepsilon/2}$$

as $h \to 0$, and since we have assumed that

$$a(h) \gg \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

$\gamma > \frac{1}{2}$, $h \to 0$, we get, for any $\varepsilon > 0$

$$\sqrt{a(h)} \gg h^{\varepsilon/2}$$

as $h \to 0$. Therefore

$$F(h) \gg h^{\varepsilon}$$

as $h \to 0$. Choose $\varepsilon < \frac{1}{\beta}$. Then (4.2.5) and (4.2.6) imply

$$N(x) \geq (\frac{x}{2})^{1/\varepsilon}$$

which gives

$$\frac{N(x)}{x^{\beta}} \geq \frac{1}{2^{1/\varepsilon}} x^{(1/\varepsilon)-\beta}$$

and (4.2.3) follows.

To obtain an upper bound for $N(x)$ we choose the largest possible $F(h)$, which by (3.2.5) above is

$$F(h) = \frac{A}{(\log 1/h)^{\gamma}} \frac{\gamma}{(\log 1/h)^{1/2}}$$

$$= \frac{c}{(\log 1/h)^{(1/2)+\gamma}}$$

for $\gamma > \frac{1}{2}$. Therefore

$$\log N(x) \leq (\frac{c}{2} x)^{2/(1+2\gamma)}$$

and (4.2.4) follows.

□

The next lemma shows that $\psi(x)$ is the dominant function in the integrand of $I_1(x)$.

Lemma 4.2.2

$P^c(x)\ N(x)\psi(x)$ is a decreasing function of x, for all large x and any $c > 0$.

Proof

$$P^c(x)N(x)\psi(x) = \frac{1}{\sqrt{2\pi x}} \exp\{-\frac{x^2}{2} + \log N(x) + \frac{c}{a(1/N)}\}$$

let $$f(x) = \frac{x^2}{2} - \log N(x) - \frac{c}{a(1/N)}$$

It is sufficient to show that this is increasing and for this it is enough that $f'(x) > 0$ for all x large.

$N(x)$ is defined by $$N(x) = \frac{1}{F^{-1}(2/x)}$$

where $F(h) = \sigma(h)\ \sqrt{a(h)}$ . and from (3.1.3) we get

$$\sigma'(h) = \frac{\sigma(h)a(h)}{h} \tag{4.2.7}$$

This, together with the fact that $a(h)$ is nondecreasing and has a positive lower bound given by (3.2.5) imply that for some $c_0 > 0$, $F'(h) > 0$ for $h \in (0,c_0]$, and then, by the inverse function theorem, $N(x)$ has a positive derivative for all x large. We have

$$N'(x) = \frac{d}{dx}\left(\frac{1}{F^{-1}(2/x)}\right)$$

$$= \frac{2N^2(x)}{x^2} \frac{d}{dh} F^{-1}(h)\Big|_{h=2/x} \tag{4.2.8}$$

and

$$\frac{d}{dh} F^{-1}(h) = \frac{1}{F'(F^{-1}(h))} \tag{4.2.9}$$

but

$$F'(h) = \sigma'(h)\sqrt{a(h)} + \frac{\sigma(h)a'(h)}{2\sqrt{a(h)}}$$

$$\geq \sigma'(h)\sqrt{a(h)}$$

because a(h) and σ(h) are positive and nondecreasing. (4.2.7) implies

$$F'(h) \geq \frac{\sigma(h)(a(h))^{3/2}}{h}$$

$$= \frac{F(h)a(h)}{h}$$

for h small. Therefore (4.2.9) gives

$$\frac{d}{dh} F^{-1}(h) \leq \frac{F^{-1}(h)}{ha(F^{-1}(h))}$$

and

$$\left.\frac{d}{dh} F^{-1}(h)\right|_{h=2/x} \leq \frac{xF^{-1}(2/x)}{2a(F^{-1}(2/x))}$$

$$= \frac{x}{2N(x)a(1/N(x))}$$

Hence (4.2.8) gives

$$N'(x) \leq \frac{N(x)}{xa(1/N(x))} \tag{4.2.10}$$

Also, from the definition of N(x) we see that

$$\frac{c}{a(1/N)} = \frac{c}{4} x^2\sigma^2 \ (1/N)$$

so that we have

$$f'(x) = x - \frac{N'(x)}{N(x)} - \frac{c}{2} x\sigma^2(\tfrac{1}{N}) + \frac{cx^2\sigma(\frac{1}{N})\sigma'(\frac{1}{N})N'(x)}{2N^2(x)}$$

and since the last term is nonnegative

$$\geq x - \frac{N'(x)}{N(x)} - \frac{c}{2} x\sigma^2(\tfrac{1}{N})$$

$$\geq x - \frac{1}{xa(1/N)} - \frac{c}{2} x\sigma^2 (\tfrac{1}{N})$$

using (3.2.5)

$$\geq x - \frac{2 \log N(x)}{x} - \frac{c}{2} x\sigma^2(\tfrac{1}{N})$$

$$> 0$$

for all x large by (4.2.4) and the fact that $\sigma^2(1/N) \to 0$ as $x \to \infty$.

□

<u>Lemma 4.2.3</u>

If theorem 4.2.1 is true under the additional restriction that for large t

$$(2 \log t)^{1/2} \leq \theta(t) \leq (3 \log t)^{1/2} \tag{4.2.11}$$

then it is true without it.

<u>Proof</u>

Let $\theta(t)$ satisfy (4.2.2). Define

$$\hat{\theta}(t) = \min\left(\max(\theta(t); (2 \log t)^{1/2}); (3 \log t)^{1/2}\right)$$

Suppose first that there is an infinite sequence $\{t_n\}$ such that $\theta(t_n) < (2 \log t_n)^{1/2}$ and $t_n \to \infty$ as $n \to \infty$. Then

$$\int^{\infty} \hat{P}_t^{c_1} \hat{N}_t \, \psi(\hat{\theta}(t))dt = \infty$$

where $\hat{P}_t = P(\hat{\theta}(t))$ and $\hat{N}_t = N(\hat{\theta}(t))$, because in this case

$$\hat{\theta}(t_n) = (2 \log t_n)^{1/2}$$

and

$$\int_{t_1}^{\infty} \hat{P}_t^{c_1} \hat{N}_t \psi(\hat{\theta}(t))dt \geq \int_{t_1}^{t_n} \hat{P}_t^{c_1} \hat{N}_t \psi(\hat{\theta}(t))dt$$

$$\geq (t_n - t_1) \hat{P}_{t_n}^{c_1} \hat{N}_{t_n} \psi(\hat{\theta}(t_n))$$

$$= \frac{(t_n - t_1)}{t_n} \frac{\hat{P}_{t_n}^{c_1} \hat{N}_{t_n}}{(2 \log t_n)^{1/2}}$$

and by (4.2.3) this tends to $\infty$ as $n \to \infty$.

Next suppose $\theta(t) > (2 \log t)^{1/2}$ for all $t$ large. Then

$$\hat{\theta}(t) = \min(\theta(t), (3 \log t)^{1/2})$$

therefore, for all $t$ large

$$P_t^{c_1} N_t \psi(\theta(t)) \leq \hat{P}_t^{c_1} \hat{N}_t \psi(\hat{\theta}(t))$$

and (4.2.2) implies

$$\int_{c}^{\infty} \hat{P}_t^{c_1} \hat{N}_t \psi(\hat{\theta}(t))dt = \infty$$

So that in both cases we have $I_1(\hat{\theta}) = \infty$. Furthermore

$$(2 \log t)^{1/2} \leq \hat{\theta}(t) \leq (3 \log t)^{1/2}$$

and by the assumptions of the lemma there is with probability one, a sequence $T_1 < T_2 < \ldots < T_n$, $T_n \to \infty$ as $n \to \infty$ such that

$$X(T_n) > \hat{\theta}(T_n)$$

the results of Chapter 2 imply that

$$(3 \log T_n)^{1/2} > X(T_n) \qquad \text{for all } n \geq n_0 \text{ a.s.}$$

therefore

$$(3 \log T_n)^{1/2} > \hat{\theta}(T_n) \qquad \text{for all } n \geq n_0 \text{ a.s.}$$

Hence

$$\hat{\theta}(T_n) = \max(\theta(T_n), (2 \log T_n)^{1/2})$$

$$\geq \theta(T_n)$$

for all $n \geq n_0$ a.s. and thus

$$X(T_n) > \theta(T_n)$$

for all $n \geq n_0$ a.s.

□

<u>Lemma 4.2.4</u> ([33] Lemma 1.5)

Let $\{X(t), t \in \mathbb{R}\}$ be a G.P. and define

$$H_K = \{X(t_{K,\nu}) \leq x_{K,\nu} : \nu = 0.1, \ldots, m_K\}$$

$K = 1,2,\ldots,n$ and all $t_{K,\nu}$ are distinct. Then

$$|P\{\bigcap_{1}^{n} H_K\} - \prod_{1}^{n} P\{H_K\}|$$

$$\leq \sum_{1\leq i<j\leq n} \sum_{\mu=0}^{m_j} \sum_{\nu=0}^{m_i} |r| \int_0^1 g(x_{i\nu}, x_{j\mu}; zr)dz$$

where $g(x,y; zr)$ is the standard bivariate normal density with correlation coefficient

$$zr = zr(t_{i\nu}, t_{j\mu})$$

Proof of theorem 4.2.1

We want to show that there is an increasing sequence $t_i$, $t_i \to \infty$ as $i \to \infty$, such that

$$Z(t_i) > \theta(t_i)$$

for all i.

To do this we consider a sequence of intervals of fixed length and investigate the behaviour of the process in a set of points within each interval. These points get closer together and their number increases as the intervals get further away from 0.

Assume that (4.2.11) holds. We start by giving some definitions:

$$I_n = [n\Delta, n\Delta + \delta] \qquad 0 < \delta < \Delta, \qquad n = 1,2,\ldots$$

both $\Delta$ and $\delta$ constants.

$$M_t = P_t^{c_1} N_t \equiv P^{c_1}(\theta(t))\ N(\theta(t))$$

$$G_K = \{ t_{K,\nu} = K\Delta + \frac{\nu}{M_{K\Delta+\delta}} \ ; \quad \nu = 1,2,\ldots [\delta M_{K\Delta+\delta}]\}$$

$$H_K = \{ \sup_{s \in G_K} X(s) \leq \theta(K\Delta+\delta) \}$$

Suppose $\delta \leq \tau$. Theorem 3.3.2 gives, with $H_K^c = \Omega \sim H_K$

$$P\{H_K^c\} \geq c\, \delta M_{K\Delta+\delta}\, \psi(\theta(K\Delta+\delta))$$

and using Lemma 4.2.2

$$\infty = \int_{c_0}^{\infty} M_t\, \psi(\theta(t))dt \leq \sum_{K=K_0}^{\infty} M_{K\Delta} \psi(\theta(K\Delta))$$

where $K_0 = [\frac{c_0}{\Delta}] - 1$, and this implies that

$$\sum_{K=K_0}^{\infty} P\{H_K^c\} \geq c \sum_{K=K_0}^{\infty} M_{K\Delta+\delta}\, \psi(\theta(K\Delta+\delta)) = \infty \tag{4.2.12}$$

Since

$$P\{H_K^c \text{ i.o.}\} = P\left\{ \bigcap_{m=1}^{\infty} \bigcup_{K=m}^{\infty} H_K^c \right\}$$

$$= \lim_{m\to\infty} P\left\{ \bigcup_{K=m}^{\infty} H_K^c \right\}$$

$$= 1 - \lim_{m\to\infty} P\left\{ \bigcap_{K=m}^{\infty} H_K \right\}$$

we have

$$1 - P\{H_K^c \text{ i.o.}\} = \lim_{m\to\infty} P\left\{ \bigcap_{K=m}^{\infty} H_K \right\}$$

$$= \lim_{m\to\infty} \left[ P\left\{ \bigcap_{K=m}^{\infty} H_K \right\} - \prod_{m}^{\infty} P(H_K) + \prod_{m}^{\infty} P\{H_K\} \right]$$

$$= \lim_{m\to\infty} \prod_m^\infty P\{H_K\} + \lim_{m\to\infty}\left[P\left\{\bigcap_m^\infty H_K\right\} - \prod_m^\infty P\{H_K\}\right] \tag{4.2.13}$$

and the first term is zero by (4.2.12).

Using Lemma 4.2.4

$$A_{m,n} = \left|P\left\{\bigcap_m^n H_K\right\} - \prod_m^n P\{H_K\}\right|$$

$$\leq \sum_{m\leq i<j\leq n} \sum_{\mu=0}^{[\delta M_j^*]} \sum_{\nu=0}^{[\delta M_i^*]} |r| \int_0^1 g(\theta_i,\theta_j; zr)\, dz$$

where $M_j^* = M_{j\Delta+\delta}$ and $\theta_j = \theta(j\Delta+\delta)$.

Since

$$t_{j\mu} - t_{i\nu} = j\Delta + \frac{\mu}{M_j^*} - i\Delta - \frac{\nu}{M_i^*}$$

$$\geq (j-i)\Delta - \frac{\nu}{M_i^*}$$

$$\geq \Delta - \delta$$

by (4.2.1) we can choose $\Delta$ so that

$$|r(t_{j\mu}, t_{i\nu})| \leq \varrho_0((j-i)\Delta - \delta)^{-\lambda}$$

for all $j,i \geq m$ and some constant $\varrho_0 > 0$, and such that for some $w > 0$

$$|r| < w$$

Now

$$\frac{\varrho_0}{((j-i)\Delta-\delta)^\lambda} \leq \frac{\varrho_0}{((j-i)\Delta - (j-i)\delta)^\lambda}$$

$$= \frac{\varrho_0}{(j-i)^\lambda \Delta^\lambda (1 - \frac{\delta}{\Delta})^\lambda}$$

$$= \frac{Q}{(j-i)^{\lambda}\Delta^{\lambda}} \quad .$$

where $Q = Q_0(1 - \frac{\delta}{\Delta})^{-\lambda}$

Also

$$g(\theta_i,\theta_j;\ zr) = \frac{1}{2\pi(1-z^2r^2)^{1/2}} \exp\left\{-\frac{\theta_i^2 + \theta_j^2 - 2zr\theta_i\theta_j}{2(1 - z^2r^2)}\right\}$$

$$\leq \frac{1}{2\pi(1-w^2)^{1/2}} \exp\left\{-\frac{1}{2}(\theta_i^2 + \theta_j^2 - 2w\theta_j^2)\right\}$$

Using (4.2.11)

$$\leq \frac{1}{2\pi(1-w^2)^{1/2}} \exp\{-\log(i\Delta+\delta) - \log(j\Delta+\delta) + 3w\log(j\Delta+\delta)\}$$

$$= \frac{\text{const.}}{(1-w^2)^{1/2}(i\Delta+\delta)(j\Delta+\delta)^{1-3w}}$$

Therefore

$$A_{m,\infty} \leq c \sum_{m\leq i<j<\infty}\sum \frac{M_i^* M_j^* Q}{(j-i)^{\lambda}\Delta^{\lambda}(i\Delta+\delta)(j\Delta+\delta)^{1-3w}}$$

$$\leq c \sum_{m\leq i<j<\infty}\sum \frac{(M_j^*)^2}{(j-i)^{\lambda}(i\Delta+\delta)(j\Delta+\delta)^{1-3w}} \qquad (4.2.14)$$

but for some constant D (3.2.5) implies

$$(M_j^*)^2 = (M_{j\Delta+\delta})^2 = N_{j\Delta+\delta}^2 P_{j\Delta+\delta}^{2c_1}$$

$$\leq N_{j\Delta+\delta}^D = N_{(\theta(j\Delta+\delta))}^D$$

and for j large, (4.2.4) implies

$$\leq \exp\{ \frac{D}{2} \theta (j\Delta+\delta) \}$$

using (4.2.11)

$$\leq \exp\{ \frac{D}{2} (3 \log j\Delta+\delta)^{1/2} \}$$

$$< (j\Delta + \delta)^{w}$$

Hence

$$(4.2.14) \quad \leq c \sum_{m \leq i < j < \infty} \sum \frac{1}{(j-i)^{\lambda} (i\Delta+\delta) (j\Delta+\delta)^{1-4w}}$$

$$< \text{const.} \sum_{m \leq i < j < \infty} \sum \frac{1}{(j-i)^{\lambda} i \, j^{1-4w}}$$

Putting K = j - i

$$= c \sum_{i=m}^{\infty} \sum_{K=1}^{\infty} \frac{1}{K^{\lambda} i (i+K)^{1-4w}}$$

$$\leq c \sum_{i=m}^{\infty} \sum_{K=1}^{\infty} \frac{1}{K^{\lambda} i^{1+w} (i+K)^{1-5w}}$$

$$\leq c \sum_{i=m}^{\infty} \frac{1}{i^{1+w}} \sum_{K=1}^{\infty} \frac{1}{K^{1+\lambda-5w}} < \infty$$

if $5w < \lambda$ and then

$$\lim_{m \to \infty} A_{m\infty} = 0$$

using this in (4.2.13) the theorem is proved.

□

### 4.2.2 Upper Bound

**Theorem 4.2.2**

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$E(X(t) - X(s))^2 = \sigma^2(|t-s|)$$

where $\sigma(h)$ is continuous, positive, monotone increasing for $0 \le h \le \tau$, some $\tau > 0$, n.s.v.f. with structure function $a(h)$. We assume that $a(h)$ is nondecreasing for $0 \le h \le \tau$, satisfies

$$a(h) \gg \frac{\gamma}{\log 1/h} \qquad (3.2.5)$$

as $h \to 0$ for $\gamma > \frac{1}{2}$ and that there exists a constant $D_1 > 0$ such that

$$a(h^{1+\varepsilon}) \gg D_1 a(h) \qquad (3.2.6)$$

as $h \to 0$ uniformly for any $0 \le \varepsilon \le 1$.

If

$$I_2(\theta) = \int_c^\infty P_t^{c_2} N_t \psi(\theta(t))dt < \infty$$

then

$$P\{S(\theta)\} = 1 \qquad (4.2.15)$$

**Proof**

By theorem 3.3.1

$$\sum_{n=n_0}^{\infty} P\{\sup_{[n,n+1]} X(s) > \theta(n)\}$$

$$\le c \sum_{n=n_0}^{\infty} P_n^{c_2} N_n \psi(\theta(n))$$

If $n_0$ is large enough the terms of the sum are decreasing so

$$\leq c \int_{n_0-1}^{\infty} P_t^{c_2} N_t \psi(\theta(t))dt < \infty$$

The Borel-Cantelli lemma yields

$$P\left\{ \begin{array}{l} \omega : \text{there is a } n^*(\omega) \text{ such that } \sup_{[n,n+1]} X(s) \leq \theta(n) \\ \text{for all } n \geq n^* \end{array} \right\} = 1$$

and this implies (4.2.15).

□

### 4.2.3 Corollaries and Special Cases

In this section we intend to investigate in more detail some consequences of the previous theorems. The first two corollaries give functions satisfying the conditions of theorems 4.2.1 and 4.2.2 for all the processes being considered. Using them we obtain, in Corollary 4.2.3, a more interesting result for a reduced class of processes, namely those with smaller increments variance. The estimate we obtain for the upper envelope in this corollary is reasonably precise and will be complemented by a similar estimate for the lower envelope in Corollary 4.3.1. More detailed results for specific processes will be given in section 4.2.4.

Definitions

$$\beta_t = (2 \log t)^{1/2}$$

$$\gamma_1(t,c,\varepsilon) = \frac{\log N(\beta_t)}{\beta_t} + \frac{c}{a(1/N(\beta_t))\beta_t} + \frac{(1+\varepsilon)\log \beta_t}{\beta_t}$$

$$\theta_1(t,c,\varepsilon) = \beta_t + \gamma_1(t,c,\varepsilon).$$

These functions will be denoted sometimes by $\gamma_1$ and $\theta_1$ to simplify the expressions.

Since we are assuming

$$a(h) >> \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

as $h \to 0$ for $\gamma > \frac{1}{2}$, and we have shown in Lemma 4.2.1 that

$$\frac{\log N(x)}{x} \to 0 \tag{4.2.4}$$

as $x \to \infty$ we have that $\gamma_1(t,c,\varepsilon) \to 0$ as $t \to \infty$ for any c.

The next two lemmas show that $N(\theta_1)$ and $P(\theta_1)$ are asymptotically equal to $N(\beta_t)$ and $P(\beta_t)$ respectively.

<u>Lemma 4.2.5</u>

$$N(\beta_t) \sim N(\theta_1(t,c,\varepsilon))$$

as $t \to \infty$ for any $c > 0$.

<u>Proof</u>

Since $N(x)$ is increasing we only have to show that for any $\varepsilon_1 > 0$

$$\frac{N(\theta_1(t,c,\varepsilon))}{N(\beta_t)} < 1 + \varepsilon_1$$

for t sufficiently large. The mean value theorem gives

$$\begin{aligned} N(\theta_1) &= N(\beta_t + \gamma_1) \\ &= N(\beta_t) + \gamma_1 N'(\beta_t + \xi\gamma_1) \end{aligned}$$

for $\xi \in (0,1)$. Using (4.2.10)

$$N(\theta_1) \le N(\beta_t) + \frac{\gamma_1 N(\beta_t + \xi\gamma_1)}{(\beta_t+\xi\gamma_1)a(1/N(\beta_t+\xi\gamma_1))}$$

$$\le N(\beta_t) + \frac{\gamma_1 N(\theta_1)}{(\beta_t+\xi\gamma_1)a(1/N(\beta_t+\xi\gamma_1))}$$

and

$$\frac{N(\theta_1)}{N(\beta_t)} \le 1 + \frac{\gamma_1 N(\theta_1)}{(\beta_t+\xi\gamma_1)a(1/N(\beta_t+\xi\gamma_1))N(\beta_t)}$$

therefore

$$\frac{N(\theta_1)}{N(\beta_t)} \left(1 - \frac{\gamma_1}{(\beta_t+\xi\gamma_1)a(1/N(\beta_t+\xi\gamma_1))}\right) \le 1$$

and

$$\frac{N(\theta_1)}{N(\beta_t)} \le \frac{1}{1 - \dfrac{\gamma_1}{(\beta_t+\xi\gamma_1)a(1/N(\beta_t+\xi\gamma_1))}}$$

using (3.2.5)

$$\le \frac{1}{1 - \dfrac{2\gamma_1 \log N(\beta_t+\xi\gamma_1)}{\beta_t+\xi\gamma_1}}$$

and using

$$\frac{\log N(x)}{x} \to 0 \qquad (4.2.4)$$

as $x \to \infty$ we get that, for $t$ large,

$$\frac{N(\theta_1)}{N(\beta_t)} < 1 + \varepsilon_1$$

□

Lemma 4.2.6

$$\frac{1}{a(1/N(\theta_1))} - \frac{1}{a(1/N(\beta_t))} \to 0 \qquad (4.2.16)$$

as $t \to \infty$.

Proof

Since we have assumed

$$a(h) >> \frac{\gamma}{\log 1/h} \qquad (3.2.5)$$

as $h \to 0$ for $\gamma > \frac{1}{2}$, and using Lemma 4.2.1 we have, for some constant $D > 0$,

$$\theta_1(t,c,\varepsilon) \leq \beta_t + \frac{D \log N(\beta_t)}{\beta_t}$$

Hence

$$\theta_1^2(t,c,\varepsilon) \leq \beta_t^2 + 3D \log N(\beta_t)$$

From the definition of $N(x)$ we get

$$\frac{1}{a(1/N(x))} = \frac{1}{4}\sigma^2(1/N(x))\, x^2$$

and the left hand side of (4.2.16) can be expressed as

$$\frac{1}{4}\sigma^2(1/N(\theta_1))\, \theta_1^2 - \frac{1}{4}\sigma^2(1/N(\beta_t))\, \beta_t^2$$

$$\leq \frac{1}{4}[\sigma^2(1/N(\theta_1))(\beta_t^2 + 3D \log N(\beta_t)) - \sigma^2(1/N(\beta_t))\beta_t^2]$$

and since

$$N(\theta_1) > N(\beta_t) \qquad (4.2.17)$$

this is

$$\leq \frac{3D}{4}\, \sigma^2(1/N(\theta_1)) \log N(\beta_t) \tag{4.2.18}$$

In Chapter 3 we showed that (3.2.5) implies

$$\sigma(h) \leq \frac{\text{const.}}{(\log 1/h)^{\gamma}} \tag{3.3.20}$$

for $\gamma = \frac{1}{2} + \delta$, $\delta > 0$. Using this

$$(4.2.18) \leq \frac{3D}{4} \frac{\log N(\beta_t)}{(\log N(\theta_1))^{1+2\delta}}$$

$$< \frac{\text{const.}}{(\log N(\theta_1))^{2\delta}} \to 0 \quad \text{as } t \to \infty.$$

Note that this lemma implies that for any $c > 0$

$$P^c(\theta_1) \sim P^c(\beta_t)$$

as $t \to \infty$.

□

We are now ready to give some corollaries to theorems 4.2.1 and 4.2.2.

Corollary 4.2.1

Let $X = \{X(t)\ t \in \mathbb{R}\}$ be a G.P. as in theorem 4.2.1. Then, for any $\varepsilon \geq 0$

$$Z(t) > \theta_1(t, c_1, -\varepsilon) \quad \text{i.o.}$$

as $t \to \infty$ with probability one.

Proof

$$I_1(\theta_1(t,c_1,-\varepsilon)) = \int_c^\infty P^{c_1}(\theta_1)\, N(\theta_1)\psi(\theta_1)dt$$

using (4.2.17)

$$\geq \int_c^\infty P^{c_1}(\beta_t)\, N(\beta_t)\psi(\theta_1)dt$$

$$\geq \int_c^\infty \frac{1}{t(\log t)^{1-\varepsilon}}\,dt = \infty$$

and the result follows from theorem 4.2.1.

□

Corollary 4.2.2

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. as in theorem 4.2.2. Then, for any $\varepsilon > 0$,

$$Z(t) > \theta_1(t,\ c_2,\varepsilon) \quad \text{i.o.} \tag{4.2.19}$$

as $t \to \infty$ with probability zero.

Proof

$$I_2(\theta_1(t,c_2,\varepsilon)) = \int_c^\infty P^{c_2}(\theta_1)\, N(\theta_1)\psi(\theta_1)dt$$

using lemmas 4.2.5 and 4.2.6

$$\sim \int_c^\infty P^{c_2}(\beta_t)\, N(\beta_t)\psi(\theta_1)dt$$

$$\leq c \int_c^\infty \frac{1}{t(\log t)^{1+\varepsilon}}\,dt < \infty$$

and (4.2.19) follows from theorem 4.2.2.

□

Corollary 4.2.3

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. satisfying all the conditions of theorem 4.2.1. Suppose $\sigma(h)$ is such that

$$\frac{1}{a(1/N(x))} = o(\log x) \qquad \text{as } x \to \infty \tag{4.2.20}$$

then

$$\limsup_{t\to\infty} (Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}) \frac{\beta_t}{\log \beta_t} = 1$$

with probability one.

Proof

The process satisfies the conditions of cor. 4.2.1. and 4.2.2. The first yields, for any $\varepsilon > 0$

$$Z(t) > \theta_1(t, c_1, -\varepsilon) \quad \text{i.o.}$$

$$> \beta_t + \frac{\log N(\beta_t)}{\beta_t} + (1-\varepsilon) \frac{\log\beta_t}{\beta_t} \quad \text{i.o.}$$

as $t \to \infty$ with probability one. This implies

$$\limsup_{t\to\infty} (Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}) \frac{\beta_t}{\log \beta_t} \geq 1$$

with probability one.

For the converse inequality note that (4.2.20) implies that for any constant $c$ and $\varepsilon > 0$ given,

$$\frac{c}{a(1/N(\beta_t))\beta_t} < \varepsilon \frac{\log \beta_t}{\beta_t}$$

and by cor. 4.2.2 there is, with probability one, a $t_0(\omega)$ such that for all $t > t_0$

$$Z(t) < \theta_1(t,c_2,\varepsilon)$$

$$< \beta_t + \frac{\log N(\beta_t)}{\beta_t} + (1+2\varepsilon)\frac{\log \beta_t}{\beta_t}$$

which implies

$$\limsup_{t\to\infty} \left(Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}\right)\frac{\beta_t}{\log \beta_t} \leq 1$$

with probability one.

□

4.2.4 Comments

We give below examples of processes included in Corollary 4.2.3 and of processes included in Corollaries 4.2.1 and 4.2.2 but not in 4.2.3. We assume that all processes satisfy the mixing condition (4.2.1).

a. Let $\sigma(h) = Ae^{-B(\log 1/h)^\gamma}$ for $0 < \gamma < 1$ where A and B are positive constants. For this function

$$a(h) = \frac{B\gamma}{(\log 1/h)^{1-\gamma}}$$

and the conditions of theorem 4.2.1 hold so that the conclusions of Corollaries 4.2.1 and 4.2.3 are valid in this case. $N(x)$ is defined by

$$\frac{A(B\gamma)^{1/2}\, e^{-B(\log N)^\gamma}}{(\log N)^{(1-\gamma)/2}} = \frac{2}{x}$$

or

$$B(\log N)^{\gamma} = \log x - \frac{1-\gamma}{2} \log \log N + \log \frac{A(B\gamma)^{1/2}}{2}$$

$$\log N = \frac{1}{B^{1/\gamma}} (\log x - \frac{1-\gamma}{2} \log \log N + \log \frac{A(B\gamma)^{1/2}}{2})^{1/\gamma} \qquad (4.2.21)$$

therefore

$$\log N \sim (\frac{\log x}{B})^{1/\gamma}$$

as $x \to \infty$ and

$$\frac{1}{a(1/N(x))} = \frac{(\log N)^{1-\gamma}}{B\gamma}$$

$$\sim \frac{1}{B\gamma} (\frac{\log x}{B})^{(1-\gamma)/\gamma} \qquad \text{as } x \to \infty$$

and (4.2.20) is only satisfied for $\frac{1}{2} < \gamma < 1$. Let's consider this case in more detail. From (4.2.21)

$$\log N(x) = (\frac{1}{B} \log x)^{1/\gamma} (1 - \frac{1-\gamma}{2} \frac{\log \log N}{\log x} + \frac{\text{const}}{\log x})^{1/\gamma}$$

$$= (\frac{1}{B} \log x)^{1/\gamma} (1 - \frac{1-\gamma}{2\gamma} \frac{\log \log N}{\log x} + o(\frac{\log \log N}{\log x}))$$

$$= (\frac{1}{B} \log x)^{1/\gamma} - \frac{1-\gamma}{2\gamma B^{1/\gamma}} (\log x)^{1-\gamma/\gamma} \log \log N$$

$$+ (\log x)^{1/\gamma} o(\frac{\log \log N}{\log x})$$

$$= (\frac{1}{B} \log x)^{1/\gamma} + o(\log x)$$

since $\frac{1-\gamma}{\gamma} < 1$ for $\frac{1}{2} < \gamma < 1$.

Therefore, for these processes we get

$$\limsup_{t\to\infty} \left(Z(t) - \beta_t - \left(\frac{1}{B}\log \beta_t\right)^{1/\gamma}\frac{1}{\beta_t}\right)\frac{\beta_t}{\log \beta_t} = 1 \qquad (4.2.22)$$

with probability one.

For the case $0 < \gamma \leq \frac{1}{2}$ the best general result we can get from corollaries 4.2.1 and 4.2.2 is

$$Z(t) > \beta_t + \frac{1}{B^{1/\gamma}}\frac{(\log \beta_t)^{1/\gamma}}{\beta_t} - \frac{1-\gamma}{2\gamma^2 B^{1/\gamma}}\frac{(\log \beta_t)^{1-\gamma/\gamma}\log\log \beta_t}{\beta_t}$$
$$+ \frac{c_1(1-\varepsilon)(\log \beta_t)^{1-\gamma/\gamma}}{B^{1/\gamma}\gamma\ \beta_t}$$

infinitely often as $t \to \infty$ with probability one, and there is, with probability one as well, a $t_0(\omega)$ such that for all $t > t_0$

$$Z(t) < \beta_t + \frac{1}{B^{1/\gamma}}\frac{(\log \beta_t)^{1/\gamma}}{\beta_t} - \frac{1-\gamma}{2\gamma^2 B^{1/\gamma}}\frac{(\log \beta_t)^{1-\gamma/\gamma}\log\log \beta_t}{\beta_t}$$
$$+ \frac{(1+\varepsilon)c_2}{B^{1/\gamma}\ \gamma}\frac{(\log\beta_t)^{1-\gamma/\gamma}}{\beta_t}$$

Here we can see clearly how the fact that $c_1 > c_2$ affects the precision of the results.

b. Next we look at processes with less regular paths. Let

$$\sigma(h) = \frac{A}{(\log 1/h)^{\gamma}}, \qquad \gamma = \frac{1}{2} + \delta, \quad \delta > 0$$

and A constant. For this function

$$a(h) = \frac{\gamma}{\log 1/h}$$

and N(x) is defined by

$$\frac{A\sqrt{\gamma}}{(\log N)^{1+\delta}} = \frac{2}{x}$$

or

$$(\log N)^{1+\delta} = \frac{A\sqrt{\gamma}}{2}\, x$$

$$\log N = \left(\frac{A\sqrt{\gamma}}{2}\, x\right)^{1/1+\delta}$$

and (4.2.20) is clearly not satisfied. However the conditions of theorem 4.2.1 are and we get from the corollaries

$$Z(t) > \beta_t + \frac{(\gamma+c_1)}{\gamma}\left(\frac{A\sqrt{\gamma}}{2}\right)^{1/1+\delta}\beta_t^{-\delta/1+\delta} + (1-\varepsilon)\,\frac{\log \beta_t}{\beta_t}$$

infinitely often a.s. as $t \to \infty$. Also, with probability one there is a $t_0(\omega)$ such that for all $t > t_0$

$$Z(t) < \beta_t + \frac{(\gamma+c_2)}{\gamma}\left(\frac{A\sqrt{\gamma}}{2}\right)^{1/1+\delta}\beta_t^{-\delta/1+\delta} + (1+\varepsilon)\,\frac{\log \beta_t}{\beta_t}$$

c. To end this series of examples consider any increasing n.s.v.f. $\sigma(h)$ with a nondecreasing structure function a(h) satisfying

$$a(h)(\log 1/h)^{1/2} \to \infty \qquad \text{as } h \to 0$$

Then, for $\varepsilon > 0$ arbitrary and $x$ large

$$\frac{1}{a(1/N(x))} < \varepsilon(\log N(x))^{1/2}$$

but then the Representation theorem implies

$$\sigma(h) \leq e^{-(\log 1/h)^{1/2}}$$

and by the argument used to prove Lemma 4.2.1 $N(x) \leq Q(x)$, where the latter is defined by

$$e^{-(\log Q(x))^{1/2}} = \frac{c}{x}$$

for some $c > 1$. Therefore

$$\log Q(x) = (\log x - \log c)^2$$

$$< (\log x)^2$$

and

$$\frac{1}{a(1/N)} \leq \frac{1}{a(1/Q(x))} < \varepsilon \log x$$

and since $\varepsilon$ is arbitrary (4.2.20) is satisfied. Case a. is included in this case. Also functions like

$$\sigma(h) = \exp\{-\log\frac{1}{h}(\log\log\frac{1}{h})^{\gamma}\} \qquad \gamma > 0$$

$$\sigma(h) = \exp\{-(\log 1/h)^{\gamma}(\log\log 1/h)^{\beta}\} \quad \frac{1}{2} < \gamma < 1$$

$$-\infty < \beta < \infty$$

$$\sigma(h) = \exp\{-(\log 1/h)^{\gamma}(\log\log 1/h)^{\beta}(\log\log\log 1/h)^{\alpha}\}$$

$$\frac{1}{2} < \gamma < 1, \qquad -\infty < \alpha,\beta < \infty$$

are included.

Theorems 4.2.1 and 4.2.2 were obtained under the assumption of stationarity but by using Slepian's lemma [14] this condition can be relaxed somewhat. We shall not develop this fully. Another way of obtaining results for non-stationary processes by means of a time transformation is given in section 4.4.

The definitions and results obtained in 3.4.3 of the last chapter can be used to get a link between the theorems of this section and the integral tests for processes having $\sigma(h)$ a r.v.f. $(\alpha)$, $0 < \alpha \leq 1$, given by Qualls and Watanabe in [34].

Theorem 4.2.3 ([34], theorem 3.1)

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$\sigma(h) = h^{\alpha}G(h) + o(h^{\alpha}G(h))$$

as $h > 0$ where $0 < \alpha \leq 1$ and $G(h)$ is a n.s.v.f. Let

$$\tilde{\sigma}(h) = \sqrt{2}\, h^{\alpha}G(h)$$

If

$$r(s) = O(s^{-\lambda})$$

as $s \to \infty$ for some $\lambda > 0$ then, for any positive, nondecreasing function $\theta(t)$ on some interval $[c,\infty)$

$$P\{S(\theta)\} = 1 \text{ or } 0$$

as the integral

$$I(\theta) = \int_{c}^{\infty} \frac{\psi(\theta(t))}{\tilde{\sigma}^{-1}(1/\theta(t))}\, dt$$

converges or diverges.

In 3.4.3 we defined $a(h) = \alpha$ for these processes and showed that

$$N(x) \div \frac{1}{\tilde{\sigma}^{-1}(1/x)}$$

$$P(x) = e^{1/\alpha}$$

therefore we get that in this case

$$I(\theta) \div I_1(\theta) \div I_2(\theta)$$

We can now put together all these results.

Theorem 4.2.4

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. with

$$E(X(t) - X(s))^2 = \sigma^2(|t-s|)$$

where $\sigma(h)$ is continuous, positive, monotone increasing for $0 \leq h \leq \tau$, some $\tau > 0$, and can be expressed as

$$\sigma(h) = h^\alpha G(h)$$

where $0 \leq \alpha \leq 1$ and $G(h)$ is a n.s.v.f. with structure function $b(h)$. If $\alpha = 0$ we assume that $b(h)$ is nondecreasing for $0 \leq h \leq \tau$, satisfies

$$b(h) >> \frac{\gamma}{\log 1/h} \qquad (3.2.5)$$

as $h \to 0$ for $\gamma > \frac{1}{2}$, and that there is a constant $D_1$ such that

$$b(h^{1+\varepsilon}) >> D_1 b(h) \qquad (3.2.6)$$

as $h \to 0$ uniformly for any $0 \leq \varepsilon \leq 1$.

Assume also that

$$r(s) = O(s^{-\lambda})$$

as $t \to \infty$ for some $\lambda > 0$. Define $a(h)$, $N(x)$ and $P(x)$ as in theorem 3.4.1. Let $\theta(t)$ be a positive, continuous nondecreasing function on some interval $[c,\infty)$. Denote $N(\theta(t))$ by $N_t$ and $P(\theta(t))$ by $P_t$. Then

$$I_1(\theta) = \infty \Rightarrow P\{S(\theta)\} = 0$$

$$I_2(\theta) < \infty \Rightarrow P\{S(\theta)\} = 1$$

From Chapter 3 we have that $c_1 < c_2$. Also, for a n.s.v.f., $a(h) \to 0$ as $h \to 0$ and this means that $P(x) \to \infty$ as $x \to \infty$. This implies that there is a set of functions $\theta(t)$ for which we cannot decide whether $P\{S(\theta)\}$ is 0 or 1 in the n.s.v.f. case. The results in theorems 4.2.1 and 4.2.2 were stated in the form of integral tests to emphasize their relation with previously known results, as explained above.

## 4.3. The Lower Envelope

We turn to the lower envelope problem. We deal with the same processes considered in 4.2 and obtain in 4.3.1 and 4.3.2 upper and lower bounds for this envelope.

In 4.3.3 we get, as a consequence of the results mentioned above, a more precise estimate of the lower envelope for the same class of processes that we considered in Corollary 4.2.3. It is interesting to compare these corollaries since they give a good evaluation of the rate of convergence of $Z(t) - (2 \log t)^{1/2}$ to zero as $t \to \infty$ for this class of processes.

Comments along the lines of 4.2.4 about specific processes are made in 4.3.4.

Definitions:

$$\beta_t = (2 \log t)^{1/2}$$

$$\gamma_2(t_1,c,\varepsilon) = \frac{\log N(\beta_t)}{\beta_t} + \frac{c}{a(1/N(\beta_t))\beta_t} - \frac{(1+\varepsilon)\log \beta_t}{\beta_t}$$

$$\theta_2(t,c,\varepsilon) = \beta_t + \gamma_2(t,c,\varepsilon) \qquad (4.3.1)$$

Since

$$N(\beta_t) \leq N(\theta_2(t,c,\varepsilon)) \leq N(\theta_1(t,c,\varepsilon))$$

lemmas 4.2.5 and 4.2.6 hold with $\theta_1$ replaced by $\theta_2$.

### 4.3.1 Upper Bound

**Theorem 4.3.1**

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. with

$$E(X(t) - X(s))^2 = \sigma^2(|t-s|)$$

where $\sigma(h)$ is a continuous, positive n.s.v.f. which is monotone increasing for $0 \leq h \leq \tau$, some $\tau > 0$, and has structure function $a(h)$. We assume that $a(h)$ is nondecreasing for $0 \leq h \leq \tau$, satisfies

$$a(h) >> \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

as $h \to 0$ for $\gamma > \frac{1}{2}$, and that there is a constant $D_1 > 0$ such that

$$a(h^{1+\varepsilon}) >> D_1 a(h) \tag{3.2.6}$$

as $h \to 0$ uniformly for any $0 \leq \varepsilon \leq 1$.

Then, for any $\varepsilon > 0$

$$Z(t) < \theta_2(t, c_2, -\varepsilon) \quad \text{i.o.} \tag{4.3.2}$$

as $t \to \infty$ with probability one.

**Proof**

We adapt Mittal's method [24].

To prove (4.3.2) it is enough to show that

$$P\{Z(t) \leq \theta_2(t, c_2, -\varepsilon) \text{ finitely often}\} = 0$$

This probability is

$$\leq P\{\bigcup_n \bigcap_{K=n}^{\infty} (Z(K+1) > \theta_2(K, c_2, -\varepsilon))\}$$

$$\leq \sum_n P\{ \bigcap_{K=n}^{\infty} (Z(K+1) > \theta_2(K, c_2, -\varepsilon))\}$$

$$\leq \sum_n \lim_{L\to\infty} P\{ \bigcap_{K=n}^{L} (Z(K+1) > \theta_2(K , c_2, -\varepsilon))\}$$

$$\leq \sum_n \lim_{L\to\infty} P\{ Z(L+1) > \theta_2(L, c_2, -\varepsilon) \}$$

and it is sufficient to prove that

$$P\{Z(L+1) > \theta_2(L, c_2, -\varepsilon) \} \to 0 \quad \text{as } L \to \infty \tag{4.3.3}$$

This probability is equal to

$$P\{ \bigcup_{K=0}^{L_\tau - 1} ( \sup_{[K\tau, (K+1)\tau]} X(s) > \theta_2(L, c_2, -\varepsilon)) \}$$

where $L_\tau = [\frac{L+1}{\tau}]$ ,

$$\leq \sum_{K=0}^{L_\tau - 1} P\{ \sup_{[K\tau, (K+1)\tau]} X(s) > \theta_2(L, c_2, -\varepsilon) \}$$

and using theorem 3.3.1

with

$$N_L = N(\theta_2(L, c_2, -\varepsilon))$$

and

$$P_L = P(\theta_2(L, c_2, -\varepsilon))$$

$$\leq L_\tau c\tau P_L^{c_2} N_L \, \psi(\theta_2(L, c_2, -\varepsilon))$$

Using Lemmas 4.2.5 and 4.2.6

$$\leq c \, L \, P^{c_2}(\beta_L) \, N(\beta_L)\psi(\theta_2(L, c_2, -\varepsilon))$$

$$= c \, \beta_L^{-\varepsilon} \to 0 \quad \text{as } L \to \infty$$

and (4.3.3) is proved.

□

## 4.3.2 Lower Bound

**Theorem 4.3.2**

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$E(X(t) - X(s))^2 = \sigma^2(|t-s|)$$

where $\sigma(h)$ is a continuous, positive, n.s.v.f. which is monotone increasing for $0 \leq h \leq \tau$, some $\tau > 0$, and has structure function $a(h)$. We assume that $a(h)$ is nondecreasing for $0 \leq h \leq \tau$ and satisfies

$$a(h) >> \frac{\gamma}{\log 1/h} \tag{3.2.5}$$

as $h \to 0$ for $\gamma > \frac{1}{2}$, and that there is a constant $D_1 > 0$ such that

$$a(h^{1+\varepsilon}) >> D_1 a(h) \tag{3.2.6}$$

as $h \to 0$ uniformly for any $0 \leq \varepsilon \leq 1$. Assume also that

$$r(s) \to 0 \tag{4.3.4}$$

as $s \to \infty$. Define

$$L_t \equiv N(\beta_t)P^{c_1}(\beta_t)$$

and suppose that for every $\varepsilon > 0$ small there is a $\nu > 1$ and a $\delta$, $0 < \delta < 1$ such that

$$\lim_{t\to\infty} (\log t)^\nu D(\varepsilon,t) = 0 \tag{4.3.5}$$

where

$$D(\varepsilon,t) = t\, L_t \sum_{n=[\delta L_t]}^{[tL_t]} r\left(\frac{n}{L_t}\right) \exp\left\{-\frac{\theta_2^2(t,c_1,\varepsilon)}{1 + r(n/L_t)}\right\} \tag{4.3.6}$$

then there exists a $t_0(\omega)$, with probability one, such that for all $t > t_0$

$$Z(t) > \theta_2(t, c_1, \varepsilon) \tag{4.3.7}$$

This theorem will be proved after a series of lemmas. We shall follow Pickand's method [32].

<u>Lemma 4.3.1</u>

Let $t(\varepsilon,m)$ be a real valued function of $\varepsilon > 0$ and the integer m. Assume that for every $\varepsilon > 0$ sufficiently small

$$\theta_2(t(\varepsilon,m), c_1, \varepsilon) \geq \theta_2(t(\varepsilon, m+1), c_1, 2\varepsilon) \tag{4.3.8}$$

for all m sufficiently large. If, for $\varepsilon > 0$ small

$$Z(t(\varepsilon,m)) \leq \theta_2(t(\varepsilon,m), c_1, \varepsilon) \tag{4.3.9}$$

only a finite number of times with probability one, then (4.3.7) holds.

<u>Proof</u>

Let $\varepsilon > 0$ be arbitrary but so that (4.3.8) is satisfied. There exists $m_0$ such that for all $m > m_0$

$$Z(t(\varepsilon,m)) > \theta_2(t(\varepsilon,m), c_1, \varepsilon)$$

But $Z(t)$ and $\theta_2(t, c_1, \varepsilon)$ are nondecreasing for large t. Let $t > t(\varepsilon, m_0)$ and m be such that

$$t(\varepsilon,m) \leq t \leq t(\varepsilon,m+1)$$

Then

$$Z(t) \geq Z(t(\varepsilon,m))$$

$$> \theta_2(t(\varepsilon,m), c_1, \varepsilon)$$

$$\geq \theta_2(t(\varepsilon,m+1),\ c_1,\ 2\varepsilon)$$

$$\geq \theta_2(t,\ c_1,\ 2\varepsilon)$$

□

Lemma 4.3.2

(4.3.8) is satisfied if $t(\varepsilon,m) = e^{\varepsilon m}$.

Proof

We want to show that for m large, $\varepsilon > 0$

$$\theta_2(t(\varepsilon,m),\ c_1,\varepsilon) - \theta_2(t(\varepsilon,m+1),\ c_1,\ 2\varepsilon) \geq 0$$

Let $\beta(t(\varepsilon,m))$ be denoted by $\beta(\varepsilon,m)$, $N(\beta(\varepsilon,m))$ by $N(\varepsilon,m)$ and $a(1/N(\varepsilon,m))$ by $a(\varepsilon,m)$. Then we want

$$(\beta(\varepsilon,m) - \beta(\varepsilon,m+1)) + \left(\frac{\log N(\varepsilon,m)}{\beta(\varepsilon,m)} - \frac{\log N(\varepsilon,m+1)}{\beta(\varepsilon,m+1)}\right)$$

$$+ c_1 \left(\frac{1}{a(\varepsilon,m)\beta(\varepsilon,m)} - \frac{1}{a(\varepsilon,m+1)\beta(\varepsilon,m+1)}\right)$$

$$+ (1+\varepsilon) \left(\frac{\log \beta(\varepsilon,m+1)}{\beta(\varepsilon,m+1)} - \frac{\log \beta(\varepsilon,m)}{\beta(\varepsilon,m)}\right)$$

$$+ \frac{\varepsilon \log \beta(\varepsilon,m+1)}{\beta(\varepsilon,m+1)}$$

$$\equiv \sum_{i=1}^{5} R_i \geq 0 \qquad (4.3.10)$$

Since $t(\varepsilon,m) = e^{\varepsilon m}$, $\beta(\varepsilon,m) = \sqrt{2\varepsilon m}$.

a. $R_1 = \beta(\varepsilon,m) - \beta(\varepsilon,m+1)$

$$= \sqrt{2\varepsilon m} - \sqrt{2\varepsilon(m+1)}$$

$$= \sqrt{2\varepsilon m} - \sqrt{2\varepsilon m}\ (1 + \frac{1}{m})^{1/2}$$

$$\sim - (\frac{\varepsilon}{2m})^{1/2}$$

b. $$R_2 = \frac{\log N(\varepsilon,m)}{\beta(\varepsilon,m)} - \frac{\log N(\varepsilon,m+1)}{\beta(\varepsilon,m+1)}$$

$$= \frac{\log N(\sqrt{2\varepsilon m})}{\sqrt{2\varepsilon m}} - \frac{\log N(\sqrt{2\varepsilon(m+1)})}{\sqrt{2\varepsilon(m+1)}}$$

$$\geq \frac{1}{\sqrt{2\varepsilon m}} (\log N(\sqrt{2\varepsilon m}) - \log N(\sqrt{2\varepsilon(m+1)}\ ))$$

$$= \frac{-1}{\sqrt{2\varepsilon m}} \log \frac{N(\sqrt{2\varepsilon(m+1)})}{N(\sqrt{2\varepsilon m})}$$

$$> \frac{-c}{\sqrt{\varepsilon m}}$$

The last step follows from an argument similar to that of Lemma 4.2.5.

c. Since, by definition,

$$\frac{1}{a(1/N(x))} = \frac{x^2\sigma^2(1/N(x))}{4}$$

$$R_3 = c_1 \left[\frac{1}{a(\varepsilon,m)\beta(\varepsilon,m)} - \frac{1}{a(\varepsilon,m+1)\beta(\varepsilon,m+1)}\right]$$

$$= \frac{c_1}{4} [\beta(\varepsilon,m)\sigma^2(1/N(\varepsilon,m)) - \beta(\varepsilon,m+1)\sigma^2(1/N(\varepsilon,m+1))]$$

$$= \frac{c_1}{4} [\sqrt{2\varepsilon m}\ \sigma^2(1/N(\sqrt{2\varepsilon m}) - \sqrt{2\varepsilon(m+1)}\ \sigma^2(1/N(\sqrt{2\varepsilon(m+1)})\ )]$$

$$= \frac{c_1}{4} [\sqrt{2\varepsilon m}\ \sigma^2(\frac{1}{N(\sqrt{2\varepsilon m})}) - \sqrt{2\varepsilon m}\ (1 + \frac{1}{m})^{1/2}\ \sigma^2(\frac{1}{N(\sqrt{2\varepsilon(m+1)})})\ ]$$

$$\geq \frac{c_1}{4}\sigma^2\left(\frac{1}{N(\sqrt{2\varepsilon m})}\right)\left(\sqrt{2\varepsilon m} - \sqrt{2\varepsilon m}\left(1 + \frac{1}{2m} - \ldots\right)\right)$$

$$\sim \frac{c_1}{4}\left(\frac{\varepsilon}{2m}\right)^{1/2}\sigma^2\left(\frac{1}{N(\sqrt{2\varepsilon m})}\right)$$

and

$$\sigma^2\left(\frac{1}{N(\sqrt{2\varepsilon m})}\right) \to 0 \qquad \text{as } m \to \infty$$

d. $$R_4 = (1+\varepsilon)\left(\frac{\log \beta(\varepsilon,m+1)}{\beta(\varepsilon,m+1)} - \frac{\log \beta(\varepsilon,m)}{\beta(\varepsilon,m)}\right)$$

$$= (1+\varepsilon)\left(\frac{\log\sqrt{2\varepsilon(m+1)}}{\sqrt{2\varepsilon(m+1)}} - \frac{\log\sqrt{2\varepsilon m}}{\sqrt{2\varepsilon m}}\right)$$

$$\geq (1+\varepsilon)\log\sqrt{2\varepsilon m}\left(\frac{1}{\sqrt{2\varepsilon m}\,(1+\frac{1}{m})^{1/2}} - \frac{1}{\sqrt{2\varepsilon m}}\right)$$

$$= (1+\varepsilon)\frac{\log\sqrt{2\varepsilon m}}{\sqrt{2\varepsilon m}}\left(\left(1+\frac{1}{m}\right)^{-1/2} - 1\right)$$

$$\sim -c\frac{\log 2\varepsilon m}{(\varepsilon m^3)^{1/2}} \qquad \text{as } m \to \infty$$

e. $$R_5 = \frac{\varepsilon \log \beta(\varepsilon,m+1)}{\beta(\varepsilon,m+1)}$$

$$= \frac{\varepsilon \log 2\varepsilon(m+1)}{2\sqrt{2\varepsilon(m+1)}}$$

$$\sim c\left(\frac{\varepsilon}{m}\right)^{1/2}\log 2\varepsilon m \qquad \text{as } m \to \infty$$

Of all the terms in the sum the dominant one as $m \to \infty$ is the last one, $R_5$, which is positive. Hence (4.3.8) is satisfied.

□

Lemma 4.3.3

In order that (4.3.7) hold, it is sufficient that for every $\varepsilon > 0$ small enough there exists a $\nu > 1$ such that

$$\lim_{t \to \infty} (\log t)^{\nu} P\{Z(t) \leq \theta_2(t, c_1, \varepsilon)\} = 0 \qquad (4.3.11)$$

Proof

By Lemmas 4.3.1 and 4.3.2 it is sufficient to show that for every $\varepsilon > 0$ sufficiently small

$$\sum_{m=1}^{\infty} P\{Z(t(\varepsilon,m)) \leq \theta_2(t(\varepsilon,m), c_1, \varepsilon)\} < \infty$$

and for this it is sufficient that for any $\varepsilon > 0$ and some $\nu > 1$

$$\lim_{m \to \infty} m^{\nu} P\{Z(t(\varepsilon,m)) \leq \theta_2(t(\varepsilon,m), c_1, \varepsilon)\} = 0$$

this is equivalent to (4.3.11).

□

Lemma 4.3.4

Let P( ) and P'( ) be two multivariate Gaussian measures assigning means 0, variances 1 and covariances $r_{ij}$ and $r'_{ij}$ respectively. Then, for any $d < \infty$ real

$$\left| P\{\max_{1 \leq i \leq n} X_i \leq d\} - P'\{\max_{1 \leq i \leq n} X_i \leq d\} \right| \leq W_n \qquad (4.3.12)$$

with

$$W_n = \sum_{i,j=1}^{n} \frac{|r_{ij} - r'_{ij}|}{[1 - (r''_{ij})^2]^{1/2}} \exp\{ \frac{-d^2}{1 + |r''_{ij}|} \} \qquad (4.3.13)$$

where $r''_{ij} = \max(r_{ij}, r'_{ij})$.

This lemma is another version of Lemma 4.2.4. For a proof see [29].

Proof of Theorem 4.3.2

Let $0 < \delta < 1$ be a constant. Define

$$Z_1(K) = \max \{X(\frac{i}{L_t}); \; [(K+\delta)L_t] \le i \le [(K+1)L_t]\}$$

Let $\varepsilon > 0$ be arbitrarily chosen and assume that (4.3.5) is satisfied. Let

$$Z_2(t) = \max \{Z_1(K); \; 0 \le K \le [t-1] \}$$

Clearly, for any x,

$$P\{Z(t) \le x\} \le P\{Z_2(t) \le x\}$$

therefore by Lemma 4.3.3 it is enough to show that for some $\nu > 1$

$$\lim_{t\to\infty} (\log t)^{\nu} P\{Z_2(t) \le \theta_2(t, c_1,\varepsilon)\} = 0$$

We first show that

$$\lim_{t\to\infty} (\log t)^{\nu} P'\{Z_2(t) \le \theta_2(t, c_1,\varepsilon)\} = 0 \tag{4.3.14}$$

where P' is the measure which confers independence among successive half closed intervals (K,K+1] but is otherwise identical to P. The second part consists in showing that

$$\lim_{t\to\infty} (\log t)^{\nu}\left|P\{Z_2(t) \le \theta_2(t,c_1,\varepsilon)\} - P'\{Z_2(t) \le \theta_2(t,c_1,\varepsilon)\}\right| = 0 \tag{4.3.15}$$

First.

$$P'\{Z_2(t) \le \theta_2(t,c_1,\varepsilon)\} \le \prod_{K=0}^{[t-1]} P\{Z_1(K) \le \theta_2(t,c_1,\varepsilon)\}$$

so

$$-\log P' \{Z_2(t) \leq \theta_2(t, c_1, \varepsilon)\}$$

$$\geq - \sum_{K=0}^{[t-1]} \log P\{Z_1(K) \leq \theta_2(t, c_1, \varepsilon)\} \qquad (4.3.16)$$

but if F(x) is a distribution function

$$- \log F(x) = -\log (1 - (1 - F(x)))$$

$$= 1 - F(x) + O((1 - F(x))^2)$$

as $F(x) \to 1$. Therefore

$$- \log P\{Z_1(K) \leq \theta_2(t, c_1, \varepsilon)\}$$

$$\geq B\, P\{Z_1(K) > \theta_2(t, c_1, \varepsilon)\}$$

for some $B < 1$ and t large.

We have assumed that $r(s) \to 0$ as $s \to \infty$. Since X is stationary this implies that

$$r(s) < 1 \quad \text{for } s > 0 \qquad (4.3.17)$$

This is equivalent to $\sigma(s) > 0$ for $s > 0$. Therefore X satisfies all the conditions of theorem 3.3.1 and

$$- \log P\{Z_1(K) \leq \theta_2(t, c_1, \varepsilon)\}$$

$$\geq \text{const. } N(\theta_2) P^{c_1}(\theta_2) \psi(\theta_2)$$

and by lemmas 4.2.5 and 4.2.6

$$\geq \text{const } N(\beta_t) P^{c_1}(\beta_t) \psi(\theta_2)$$

Using the definition of $\theta_2$

$$\geq \text{const.}\ \frac{(\log t)^{\varepsilon}}{t}$$

Therefore (4.3.16) gives

$$-\log P'\{Z_2(t) \leq \theta_2(t, c_1, \varepsilon)\} \geq c(\log t)^{\varepsilon}$$

and

$$P'\{Z_2(t) \leq \theta_2(t, c_1, \varepsilon)\} \leq e^{-c(\log t)^{\varepsilon}}$$

as $t \to \infty$ and (4.3.14) holds.

We now prove (4.3.15). Looking at (4.3.12) and (4.3.13) we see that the terms of the sum are zero whenever we consider i and j such that $i/L_t$ and $j/L_t$ belong to the same interval $[K+\delta, K+1]$ since, in this case, $r_{ij} = r'_{ij}$.

If they belong to different intervals $r'_{ij} = 0$ and $r''_{ij} = r_{ij}$. But we know that (4.3.17) holds and we are "chopping off" a piece of each interval of length $\delta$. Therefore $r''_{ij}$ has an upper bound and we can replace $(1 - (r''_{ij})^2)^{1/2}$ by a positive constant. Therefore

$$|P\{Z_2(t) \leq \theta_2(t, c_1, \varepsilon)\} - P'\{Z_2(t) \leq \theta_2(t, c_1, \varepsilon)\}|$$

$$\leq c \sum_{K=0}^{[t-1]} \sum_{i=[(K+\delta)L_t]}^{[(K+1)L_t]} \sum_{\ell=K+1}^{[t-1]} \sum_{j=[(\ell+\delta)L_t]}^{[(\ell+1)L_t]} r(\frac{i}{L_t}, \frac{j}{L_t})$$

$$\exp\left\{-\frac{\theta_2^2(t, c_1, \varepsilon)}{1 + r(\frac{i}{L_t}, \frac{j}{L_t})}\right\}$$

$$\leq c \sum_{i=0}^{[tL_t]} \sum_{j=i+[\delta L_t]}^{[tL_t]} r(\frac{j-i}{L_t}) \exp\left\{ - \frac{\theta_2^2(t, c_1, \varepsilon)}{1 + r(\frac{j-i}{L_t})} \right\}$$

$$\leq c\ t\ L_t \sum_{n=[\delta L_t]}^{[tL_t]} r(\frac{n}{L_t}) \exp\left\{ - \frac{\theta_2^2(t, c_1, \varepsilon)}{1 + r(n/L_t)} \right\}$$

$$= c\ D(\varepsilon,t)$$

and (4.3.6) implies (4.3.15).

□

<u>Theorem 4.3.3</u>

Let $X = \{X(t), t \in \mathbb{R}\}$ be a G.P. satisfying all the conditions of theorem 4.2.1. Then (4.3.7) holds.

<u>Proof</u>

Clearly (4.2.1) implies (4.3.4) so we need only show that under the assumptions of theorem 4.2.1 (4.3.5) holds. Note that

$$e^{-x^2/1+r} = (e^{-x^2/2})^{2/1+r}$$

so that

$$e^{-\theta_2^2/1+r} \div (\frac{\beta_t^{1+\varepsilon}}{tL_t})^{2/1+r}$$

$$\div \frac{(\log t)^{(1+\varepsilon)/1+r}}{(tL_t)^{2/(1+r)}}$$

Define

$$\Delta(q) = \sup_{q \leq s < \infty} \frac{2r(s)}{1 + r(s)}$$

Note also that $\Delta(\delta) < 1$,

$$r_n = r\left(\frac{n}{L_t}\right)$$

and for some $t_0$ fixed

$$D_1(\varepsilon,t) = (\log t)^{\nu}\, tL_t \sum_{n=[\delta L_t]}^{[t_0 L_t]} \frac{r_n (\log t)^{(1+\varepsilon)/1+r_n}}{(tL_t)^{2/1+r_n}}$$

$$D_2(\varepsilon,t) = (\log t)^{\nu}\, tL_t \sum_{n=[t_0 L_t]+1}^{[tL_t]} \frac{r_n (\log t)^{(1+\varepsilon)/1+r_n}}{(tL_t)^{2/1+r_n}}$$

In the first case since $n \geq [\delta L_t]$ we have $r_n \leq \Delta(\delta)$ for all n so

$$D_1(\varepsilon,t) \leq \frac{(\log t)^{(1+\nu+\varepsilon)} tL_t^2\, t_0 \Delta(\delta)}{(tL_t)^{2/1+\Delta(\delta)}}$$

$$\leq c\, (\log t)^{(1+\nu+\varepsilon)} L_t^{2\Delta(\delta)/1+\Delta(\delta)}\, t^{(\Delta(\delta)-1)/1+\Delta(\delta)}$$

We defined $L_t$ by

$$L_t = N(\beta_t) P^{c_1}(\beta_t) = N(\beta_t) \exp\left\{ \frac{c_1}{a(1/N(\beta_t))} \right\}$$

and we have assumed that

$$a(h) >> \frac{\gamma}{\log 1/h}$$

as $h \to 0$ for $\gamma > \frac{1}{2}$, and have shown in lemma 4.2.1 that

$$\frac{\log N(x)}{x} \to 0$$

as $x \to \infty$. Hence for $\varepsilon_1 > 0$ arbitrary and t large

$$L_t < e^{\varepsilon_1 \beta_t}$$

and

$$D_1(\varepsilon,t) \leq \frac{c(\log t)^{1+\nu+\varepsilon}}{t^{(1-\Delta(\delta))/1+\Delta(\delta)}} \exp\{ \frac{2\Delta(\delta)\varepsilon_1}{1+\Delta(\delta)} (2 \log t)^{1/2} \}$$

$$\to 0$$

as $t \to \infty$ since $\Delta(\delta) < 1$.

It only remains to show that $D_2(\varepsilon,t)$ also tends to zero as $t \to \infty$

$$D_2(\varepsilon,t) \leq (\log t)^{1+\nu+\varepsilon} tL_t \sum_{n=[t_0L_t]+1}^{[tL_t]} \frac{r_n}{(tL_t)^{2/1+r_n}}$$

$$\leq \frac{(\log t)^{1+\nu+\varepsilon}}{tL_t} \sum_{n=[t_0L_t]+1}^{[tL_t]} r_n (tL_t)^{2r_n/1+r_n}$$

$$\leq \frac{(\log t)^{1+\nu+\varepsilon} (tL_t)^{\Delta(t_0)}}{tL_t} \sum_{n=[t_0L_t]+1}^{[tL_t]} r_n$$

Let $t_0$ be so large that $r(s) < B s^{-\lambda_1}$, for $s > t_0$, some $0 < \lambda_1 < 1$ and $B > 0$. Then

$$r_n = r(\frac{n}{L_t}) < B(\frac{L_t}{n})^{\lambda_1}$$

and

$$D_2(\varepsilon,t) \leq \frac{B(\log t)^{(1+\nu+\varepsilon)} t^{\Delta(t_0)} L_t^{\lambda_1+\Delta(t_0)}}{tL_t} \sum_{n=[t_0L_t]+1}^{[tL_t]} n^{-\lambda_1}$$

$$\leq B(\log t)^{(1+\nu+\varepsilon)}\, t^{(\Delta(t_0)-1)}\, L_t^{(\lambda_1+\Delta(t_0)-1)} \int_{[t_0L_t]}^{[tL_t]} y^{-\lambda_1}\, dy$$

$$\leq B(\log t)^{(1+\nu+\varepsilon)}\, t^{(\Delta(t_0)-1)}\, L_t^{\Delta(t_0)}\, \frac{(t^{1-\lambda_1} - t_0^{1-\lambda_1})}{1-\lambda_1}$$

$$= B(\log t)^{(1+\nu+\varepsilon)}\, t^{(\Delta(t_0)-\lambda_1)}\, L_t^{\Delta(t_0)}$$

Choose $t_0$ so large that $\Delta(t_0) < \lambda_1$. Then $D_2(\varepsilon,t) \to 0$ as $t \to \infty$ and the theorem is proved.

□

### 4.3.3 Special Cases

We look at the same processes we considered in Corollary 4.2.3. The proof will follow along the same lines.

#### Corollary 4.3.1

Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a G.P. satisfying all the conditions of theorem 4.2.1. Suppose $\sigma(h)$ is such that

$$\frac{1}{a(1/N(x))} = o(\log x) \quad \text{as } x \to \infty \tag{4.2.20}$$

then

$$\liminf_{t\to\infty} \left(Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}\right) \frac{\beta_t}{\log \beta_t} = -1 \tag{4.3.18}$$

with probability one.

#### Proof

The process satisfies all the conditions of theorems 4.3.1 and 4.3.3. Hence, for $\varepsilon > 0$

$$Z(t) < \theta_2(t, c_2, -\varepsilon) \quad \text{infinitely often a.s.}$$

$$= \beta_t + \frac{\log N(\beta_t)}{\beta_t} + \frac{c_2}{a(1/N(\beta_t))\beta_t} - \frac{(1-\varepsilon)\log \beta_t}{\beta_t}$$

and, as in Corollary 4.2.3

$$\frac{c_2}{a(1/N(\beta_t))\beta_t} < \frac{\varepsilon \log \beta_t}{\beta_t}$$

for t large. Therefore

$$Z(t) < \beta_t + \frac{\log N(\beta_t)}{\beta_t} - \frac{(1-2\varepsilon)\log \beta_t}{\beta_t} \quad \text{i.o.}$$

as $t \to \infty$ with probability one and this gives

$$\liminf_{t\to\infty} \left(Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}\right) \frac{\beta_t}{\log \beta_t} \leq -1$$

with probability one.

From Theorem 4.3.3 we know that there is, with probability one, a $t_0(\omega)$ such that for all $t > t_0$

$$Z(t) > \theta_2(t, c_1, \varepsilon)$$

$$= \beta_t + \frac{\log N(\beta_t)}{\beta_t} + \frac{c_2}{a(1/N(\beta_t))\beta_t} - \frac{(1+\varepsilon)\log \beta_t}{\beta_t}$$

$$> \beta_t + \frac{\log N(\beta_t)}{\beta_t} - \frac{(1+\varepsilon)\log \beta_t}{\beta_t}$$

and so

$$\liminf_{t\to\infty} \left(Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}\right) \frac{\beta_t}{\log \beta_t} \geq -1$$

with probability one.

□

For comparison we quote the result of Corollary 4.2.3.

$$\limsup_{t\to\infty} \left(Z(t) - \beta_t - \frac{\log N(\beta_t)}{\beta_t}\right) \frac{\beta_t}{\log \beta_t} = 1$$

with probability one.

### 4.3.4 Comments

We consider here the same processes we looked at in 4.2.4. The details are very similar so we shall only give the results.

a. Let $\sigma(h) = A\, e^{-B(\log 1/h)^{\gamma}}$ for $\frac{1}{2} < \gamma < 1$

where A and B are positive constants. In this case (4.2.20) is satisfied and

$$\liminf_{t\to\infty} \left(Z(t) - \beta_t - \left(\frac{1}{B}\log \beta_t\right)^{1/\gamma} \frac{1}{\beta_t}\right) \frac{\beta_t}{\log \beta_t} = -1$$

with probability one. c.f. (4.2.22)

b. If $\sigma(h) = \dfrac{A}{(\log 1/h)^{\gamma}}, \quad \gamma = \frac{1}{2} + \delta, \quad \delta > 0$

and A constant then (4.2.20) is not satisfied. From theorem 4.3.1 and 4.3.3 we get

$$Z(t) > \beta_t + \frac{(\gamma+c_1)}{\gamma} \left(\frac{A\sqrt{\gamma}}{2}\right)^{1/1+\delta} \beta_t^{-\delta/1+\delta} + \frac{(1+\varepsilon)\log \beta_t}{\beta_t}$$

for all $t > t_0(\omega)$ with probability one and

$$Z(t) < \beta_t + \frac{(\gamma+c_2)}{\gamma} \left(\frac{A\sqrt{\gamma}}{2}\right)^{1/1+\delta} \beta_t^{-\delta/1+\delta} + \frac{(1-\varepsilon)\log \beta_t}{\beta_t}$$

infinitely often as $t \to \infty$ with probability one.

c. The conclusion of Corollary 4.3.1 also applies to the class of processes considered in c of 4.2.4.

We pointed out in 4.2.4 that the assumption of stationarity can be relaxed. The same applies for the theorems of 4.3. Moreover, in theorems 4.2.1, 4.3.2 and 4.3.3 it is enough to have

$$E(X(t) - X(s))^2 \geq \sigma^2(|t-s|)$$

and in theorems 4.2.2 and 4.3.1

$$E(X(t) - X(s))^2 \leq \sigma^2(|t-s|)$$

is sufficient.

We shall make use of this in the next section.

It is also possible to show, as we did for theorem 4.2, that the conclusions of theorems 4.3.1 and 4.3.3 can be deduced, in the case of a process having $\sigma(h) = ch^\alpha$, $0 < \alpha \leq 1$, from the results of Mittal [25]. It is not difficult to extend this to $\sigma(h)$ a r.v.f. ($\alpha$).

## 4.4. Extension to Other Processes

In this section we make use of a time transformation and the results of the previous two sections to obtain information about some nonstationary processes.

**Theorem 4.4.1**

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$E(Y(t+h) - Y(t))^2 \geq \sigma^2 \left(\frac{h}{t}\right) \tag{4.4.1}$$

for all $t \in \mathbb{R}$ and $h \leq \tau$ for some $\tau > 0$, where $\sigma(h)$ satisfies all the conditions of theorem 4.2.1. Assume also that

$$r(v,\ vs) \;=\; O((\log s)^{-\lambda}) \tag{4.4.2}$$

as $s \to \infty$ for some $\lambda > 0$, uniformly in $v$. Let $\Gamma(t)$ be a nondecreasing continuous function with $\lim_{t\to\infty} \Gamma(t) = \infty$. Then if

$$I_3(\Gamma) \;=\; \int_c^\infty \frac{1}{t}\, P_t^{c_1} N_t \psi(\Gamma(t))dt \;=\; \infty$$

$$P\{S(\Gamma)\} \;=\; 0 \tag{4.4.3}$$

where all these functions are defined as in theorem 4.2.1 with $\Gamma(t)$ instead of $\theta(t)$.

**Proof**

Define a new process $\{X(s),\ s \in \mathbb{R}\}$ by

$$X(s) \;=\; Y(e^s)$$

Then, if $t \geq s$,

$$E(X(t) - X(s))^2 \geq \sigma^2 \left(\frac{e^t - e^s}{e^s}\right)$$

$$= \sigma^2 (e^{t-s} - 1)$$

$$\geq \sigma^2 (t-s)$$

for $t,s \in \mathbb{R}$ with $|t-s| < \tau$. Also, for $s,v \in \mathbb{R}$

$$E(X(v+s)X(v)) = r(e^{v+s}, e^v)$$

$$= O(s^{-\lambda})$$

as $s \to \infty$ for some $\lambda > 0$ uniformly in v. Therefore $\{X(s), s \in \mathbb{R}\}$ satisfies all the conditions of theorem 4.2.1.

Suppose $\Gamma(t)$ is such that $I_3(\Gamma) = \infty$. We make the following change of variables in the integral

$$t = e^s \qquad dt = e^s ds$$

$$I_3(\Gamma) = \int_{e^c}^{\infty} P_{e^s}^{c_1} N_{e^s} \psi(\Gamma(e^s))\, ds = \infty$$

let

$$\theta(s) \equiv \Gamma(e^s)$$

then

$$I_1(\theta) = \infty$$

and theorem 4.2.1 implies that, with probability one

$$\sup_{[0,t]} X(s) > \theta(t) \quad \text{i.o.}$$

i.e.

$$\sup_{[0,t]} Y(e^s) > \Gamma(e^t) \quad \text{i.o.}$$

and this implies (4.4.3)

□

Corollary 4.4.1

Let $\{Y(t),\ t \in \mathbb{R}\}$ be as above. Define:

$$\alpha_t = (2 \log \log t)^{1/2}$$

$$\Gamma_1(t, c, \varepsilon) = \alpha_t + \frac{\log N(\alpha_t)}{\alpha_t} + \frac{c}{a(1/N(\alpha_t))\alpha_t} + \frac{(1+\varepsilon)\log \alpha_t}{\alpha_t}$$

then, for $\varepsilon > 0$

$$\sup_{[0,t]} Y(s) > \Gamma_1(t, c_1, -\varepsilon) \quad \text{i.o.} \quad \text{as } t \to \infty$$

with probability one.

c.f. Corollary 4.2.1.

Theorem 4.4.2

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$E(Y(t+h) - Y(t))^2 \leq \sigma^2(\tfrac{h}{t})$$

for all $t \in \mathbb{R}$ and $h \leq \tau$ for some $\tau > 0$, where $\sigma(h)$ satisfies all the conditions of theorem 4.2.2. Then if

$$I_4(\Gamma) = \int_c^\infty \frac{1}{t} P_t^{c_2} N_t \psi(\Gamma(t))dt < \infty$$

$$P\{S(\Gamma)\} = 1$$

The proof is similar to that of theorem 4.4.1 and will not be given.

Corollary 4.4.2

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a G.P. as above. Then, for $\varepsilon > 0$

$$\sup_{[0,t]} Y(s) > \Gamma_1(t, c_2, \varepsilon) \quad \text{i.o.}$$

as $t \to \infty$ with probability zero.

c.f. Corollary 4.2.2.

Theorem 4.4.3

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a G.P. as in theorem 4.4.1. Define

$$\Gamma_2(t,c_1,\varepsilon) = \alpha_t + \frac{\log N(\alpha_t)}{\alpha_t} + \frac{c}{a(1/N(\alpha_t))\alpha_t} - \frac{(1+\varepsilon)\log \alpha_t}{\alpha_t}$$

then

$$P\{T(\Gamma_2(t,c_1,\varepsilon))\} = 1$$

Once again the proof follows along the lines of theorem 4.4.1.

c.f. Theorem 4.3.3.

Theorem 4.4.4

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a G.P. as in theorem 4.4.2. Then

$$P\{T(\Gamma_2(t,\ c_2,\ -\varepsilon))\} = 0$$

c.f. theorem 4.3.1.

Corollary 4.4.3

Let $\{Y(t),\ t \in \mathbb{R}\}$ be a G.P. with

$$E(Y(t+h) - Y(t))^2 = \sigma^2(\tfrac{h}{t})$$

where $\sigma(h)$ satisfies all the conditions of theorem 4.4.1. Assume that (4.4.2) and (4.2.20) are also satisfied. Then, with probability one

$$\limsup_{t\to\infty} \left( \sup_{[0,t]} Y(s) - \alpha_t - \frac{\log N(\alpha_t)}{\alpha_t} \right) \frac{\alpha_t}{\log \alpha_t} = 1$$

$$\liminf_{t\to\infty} \left( \sup_{[0,t]} Y(s) - \alpha_t - \frac{\log N(\alpha_t)}{\alpha_t} \right) \frac{\alpha_t}{\log \alpha_t} = -1$$

c.f. Corollaries 4.2.3 and 4.3.1.

## 4.5. Final Remarks

Although we have not been able to solve the problem we dealt with in this chapter completely, it seems to us that the results are interesting and encouraging in that they point towards a possible general solution for continuous stationary Gaussian processes with $\sigma(h)$ slowly varying, and indicate the relation with the known solution for other processes.

It seems clear that the problem will not be solved until asymptotically exact estimates for the tail of the distribution of $Z(t)$ are available, since the accuracy of the results of this chapter depends crucially on the precision of these estimates. Such results could be used for tackling other problems and would therefore be of independent interest.

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