# Linear and Nonlinear Dissipative Dynamics

B. Zegarliński

Imperial College London
180 Queens Gate, London SW7 2AZ, UK
b.zegarlinski@imperial.ac.uk

**Abstract**: In this paper we introduce and study new dissipative dynamics for large interacting systems.

**Keywords**: Dunkl type generators, infinite dimensions, noncommutative and non-linear dynamics, ergodicity.

## 1 Introduction

The theory of large dissipative systems has a long and growing mathematical history. Some of the classical literature one could find e.g. in [24] and [37]; see also references there in. In this paper we focus on dissipative dynamics with non-compact configuration space and their counterparts in noncommutative algebras.

A construction of Markov semigroups on the space of continuous functions with an infinite dimensional underlying space well suited to study strong ergodicity problems can be found in [51] in case of fully elliptic generators. More recently it was extended to subelliptic situation in [16], [31] and Lévy type generators [35]. An interesting approach via stochastic differential equations one can find in [15] and some recent extension to subelliptic generators in [50] (see also [4], [3] and references therein). Another approach via Dirichlet forms theory which is well adapted to $L_2$ theory, can be found e.g. in [1], [45] and reference therein.

For symmetric semigroups, after a recent progress in proving the log-Sobolev inequality for infinite dimensional Hörmander type generators $\mathcal{L}$ symmetric in $L_2(\mu)$ defined with a suitable nonproduct measure $\mu$ ([32], [25], [28], [26], [27],

[43]), one can expect an extension of the established strategy ([51]) for proving strong pointwise ergodicity for the corresponding Markov semigroups $P_t \equiv e^{t\mathcal{L}}$, (respectively in the uniform norm in case of the compact spaces as in [24] and refs therein). One could obtain more results in this direction, including configuration spaces given by infinite products of general noncompact nilpotent Lie groups other than Heisenberg type groups, by conquering a (finite dimensional) problem of sub-Laplacian bounds (of the corresponding control distance) which for a moment remains still very hard.

The ergodicity theory in case when an invariant measure is not given in advance, in noncompact subelliptic setup is an interesting and challenging problem which was initially studied in [16] and was extended in new directions in [31] developing further strategy based on generalised gradient bounds. We remark that in fully elliptic case a strategy based on classical Bakry-Emery arguments involving restricted class of interactions can be achieved. In case of the stochastic strategy of [15], the convexity assumption enters via dissipativity condition in a suitable Hilbert space and does not improve the former one as far as ergodicity is concerned; (on the other hand it allows to study a number of stochastically natural models). In subelliptic setup involving subgradient this strategy faces serious obstacles, see e.g. comments in [6].

In noncommutative setup the development of mathematical description of infinite dissipative systems is much less developed. Some description of infinite dimensional dissipative dynamics of jump type which are not symmetric with respect to a given Gibbs state as well some results on theirs ergodicity can be found in [54]; see also references therein and [23], [38], [14] on constructions associated to classical Gibbs states (where interaction potential is classical). In [40] a construction and ergodicity results were provided for an interesting class where generator of jumps part corresponds to a classical potential, but additionally the generator contains a conservative part corresponding to a different possibly nonclassical potential. In general for an infinite dimensional system still no construction of jump type dynamics exists which would be symmetric for a Gibbs state associated to a generic nonclassical potential. Some interesting general constructions, based on application of Dirichlet form theory [13], are provided in [44], [14] (see also references there in).

A study of diffusion type dynamics providing a construction and ergodicity results were given in [34], including generators associated to a family of noncommuting fields, but not apriori symmetric with respect to an $\mathbb{L}_2$ scalar product associated to a given state.

Another recent examples of dissipative dynamics for infinite boson systems can be found in [41] , [7].

One of the important techniques developed to study ergodicity of dissipative dynamics of infinite classical interacting systems is based on use of hypercontractivity property or its infinitesimal form encoded in Log-Sobolev inequality ([24] and references therein). A noncommutative basis for such theory was introduced in [42]. Since then, in noncommutative setup some progress was achieved in studying certain directions ([9], [12], [11], [2]) with interesting new results emerging in connection to quantum information theory ([29], [30]). Still many important technical aspects necessary to effective implementation of the theory remain elusive in noncommutative world. (This includes e.g. product and perturbation properties of Log-Sobolev inequality.)

In Section 2 and 3, we study finite and infinite dimensional systems for which we construct dissipative dynamics described by Dunkl type generators and provide certain basic ergodicity results. In section 4 we give an example of such dissipative dynamics in noncommutative setup. In section 5 we discuss some nonlinear classical dissipative dynamics and theirs noncommutative counterparts. In Appendix we provide some discussion of monotone convergence in noncommutative $\mathbb{L}_p$ spaces.

# 2 Dunkl type Markov Generators and Semigroups

In this section we discuss linear dissipative dynamics associated to Markov generators of the following form

$$\mathcal{L} \equiv \sum_{i \in \mathcal{R}} \mathcal{L}_i$$

defined on a dense domain of the space of bounded continuous functions $\mathcal{C}(\Omega)$ on a product space $\Omega \equiv \times_{i\in\mathcal{R}}\Omega_i$ with $\Omega_i \sim \Omega_0$ is a smooth manifold of finite dimension $n$, where the indices $i$ form a countable, possibly infinite, set $\mathcal{R}$, and

$$\mathcal{L}_i \equiv \mathbb{T}_i^2 - \beta_i \cdot \mathbb{T}_i$$

with $\mathbb{T}_i \equiv \nabla_i + A_i$ , where $\nabla_i$ denotes the gradient operator and

$$(A_i f)_l \equiv \frac{\kappa}{x_{i,l}} (f - f \circ \sigma_{i,l})$$

with $\sigma_{i,l} \circ \sigma_{i,l} = \mathrm{id} \quad \sigma_{i,l}(x_{i,l}) = -x_{i,l}$ , $l = 1, \dots, n$, are both acting on $i$-th coordinate, while $\beta_i$'s are dependent possibly on many coordinates and are continuously differentiable. First of all we notice that we have

$$\Gamma_{\mathcal{L}_i}(f) \equiv \frac{1}{2}(\mathcal{L}_i f^2 - 2f\mathcal{L}_i f) = |\nabla_i f|^2 + \frac{1}{2\kappa}(A_i f)^2 - \beta_i \cdot \frac{1}{2}(A_i f^2 - 2fA_i f)$$

We note that, unlike as in the diffusion case, the first order term gives a nontrivial contribution. Since for $A$ type component, we have

$$A_{i,l}f^2 - 2fA_{i,l}f = \frac{\kappa}{x_{i,l}}((f^2 - f^2 \circ \sigma_{i,l}) - 2f(f - f \circ \sigma_{i,l}))$$

$$= -\frac{\kappa}{x_{i,l}}(f - f \circ \sigma_{i,l})^2 = -\frac{x_{i,l}}{\kappa}(A_{i,l}f)^2,$$

so we get

$$\Gamma_{\mathcal{L}}(f) \equiv \sum_i \left( |\nabla_i f|^2 + \sum_l \frac{1}{2\kappa_l}(1 + \beta_{i,l} \cdot x_{i,l})(A_{i,l}f)^2 \right)$$

which is nonnegative if for all $i, l$ we have

$$1 + \beta_{i,l} \cdot x_{i,l} \geq 0$$

Next we note that at a minimum point $\tilde{\omega}$ for which components are outside reflection set, we have

$$-\beta_{i,l} \cdot A_{i,l}f = +\beta_{i,l} \cdot x_{i,l}\kappa_l \left( \frac{f \circ \sigma_{i,l}(\tilde{\omega}) - f(\tilde{\omega})}{x_{i,l}^2} \right)$$

Thus, assuming $\nabla_{i,l}x_{i,l} = 1$, we have

$(\mathbb{T}_{i,l}^2 - \beta_{i,l}\mathbb{T}_{i,l})f = (\tilde{\omega})$

$$\nabla_{i,l}^2 f(\tilde{\omega}) + \frac{2\kappa_l}{x_{i,l}}\nabla_{i,l}f(\tilde{\omega}) + \frac{\kappa_l}{x_{i,l}^2}(f \circ \sigma_{i,l}(\tilde{\omega}) - f(\tilde{\omega})) - \beta_{i,l} \cdot \nabla_{i,l}f(\tilde{\omega}) + \beta_{i,l} \cdot x_{i,l}\frac{\kappa_l}{x_{i,l}^2}(f \circ \sigma_{i,l}(\tilde{\omega}) - f(\tilde{\omega}))$$

$$= \nabla_{i,l}^2 f(\tilde{\omega}) + \frac{\kappa_l}{x_{i,l}^2}(1 + \beta_{i,l} \cdot x_{i,l})(f \circ \sigma_{i,l}(\tilde{\omega}) - f(\tilde{\omega})) \quad \geq \quad 0$$

under the same condition for the coefficients as before. Using suitable limiting procedure, one obtains similar result if any component of the minimum point belongs to the reflection invariant set.
Hence we get the following condition for $\mathcal{L}$ being a Markov generator.

**Theorem 1**
*Suppose for all $i, l$ we have*

$$1 + \beta_{i,l}x_{i,l} \geq 0$$

Then

$$\Gamma_{\mathcal{L}}(f) \geq 0$$

and $\mathcal{L}$ satisfies the minimum principle, i.e. at a minimum point $\tilde{\omega} \in \Omega$

$$(\mathcal{L}f)(\tilde{\omega}) \geq 0$$

**Remark 1**
Note that positivity of canonical quadratic form implies minimum principle for functions $f$ for which $(f - \min f)^{1/2}$ is in the domain of the generator.

**Example 1**
*Suppose* $\Omega \equiv \mathbb{R}^{\mathcal{R}}$ *and*

$$A_i f = \frac{\kappa}{\omega_i}(f - f \circ \sigma_i)$$

with $\kappa > 0$ and $\sigma_i(\omega)_j = (-1)^{\delta_{ij}}\omega_j$. Supose

$$\beta_i = a_{2n+1}\, \omega_i^{2n+1} + \sum_{m=2,..,2n} a_m\, \omega_i^m + \tilde{M}\omega_i + \sum_{O:O\ni i} b_O \prod_{k\in O} \varsigma(\omega_k)$$

where $a_{2n+1} > 0$, $a_m \in \mathbb{R}$, $n \geq 1$, and $b_O \in \mathbb{R}^+$, with finite sets $O$, and $\sup_j \sum_{O:O\ni j} |b_O| < \infty$, where $\varsigma(x) = x\chi_{x\in[-1,+1]} + \chi_{x\in[+1,\infty]} - \chi_{x\in(-\infty,-1]}$ and with $\tilde{M} > 0$. Then conditions of the above theorem are satisfied provided the coefficients $a_m$, $m = 2,..,2n$, are sufficiently small in absolute value. (It should be clear that one can add to such $\beta$'s a sufficiently small continuous bounded functions without harming the conditions of the theorem.)

Since $\mathcal{L}$ is densely defined and vanishes on constants, it is a Markov (pre-)generator. Thus one can expect that, the corresponding semigroup $P_t \equiv e^{t\mathcal{L}}$ can be well defined $C_0$-Markov semigroup on the space of bounded (uniformly-)continuous functions. If the dimension of the space is finite this is fine; in infinite dimensions this requires more arguments which will be discussed later.

# 3 Generalised Gradient Bounds

Given a Markov semigroup introduced in the previous section and assuming that it provides some mild smoothing properties, it would be interesting to consider a problem when the following generalised gradient type bounds can be satisfied

$$\tilde{\Gamma}(P_t f)^q \leq Ce^{-mt} P_t \tilde{\Gamma}(f)^q$$

where $\tilde{\Gamma}$ is a quadratic form involving first order operators, $C \in \mathbb{R}^+$, $m \in \mathbb{R}$ and $q \in [\frac{1}{2}, 1]$ are constants independent of $f$ and $t \in \mathbb{R}^+$. In particular one could ask this question for the canonical $\Gamma$ form associated to the Markov generator or a form $|\mathbb{T}f|^2 \equiv \sum_i |\mathbb{T}_i f|^2$. Similar bounds involving differential operators may have a variety of applications including ergodicity theory (cf. [16]) or certain smoothing properties of the semigroup (see e.g. [5], [17], [36], [6], [27] and references therein). Even in the case of diffusion operators in finite dimensions it is a hard problem for which a relatively satisfactory solution currently only exists in case of (products of) Heisenberg type groups; for $q = \frac{1}{2}$ the other groups constitute a formidable challenge. Therefore one can expect that our case is even more challenging. Thus, to gain at least some intuition, we discuss here a simplified situations starting from a case of single field and one reflection.

With a function $\eta$ such that $\eta \circ \sigma = -\eta$ and $X\eta = \varepsilon$, for some constant $\varepsilon \in (0, \infty)$, we set

$$A_\sigma(f) \equiv A(f) \equiv \frac{f - f \circ \sigma}{\eta} \qquad T \equiv X + A$$

and

$$\mathcal{L} \equiv T^2 - \beta\eta T, \qquad \text{with } \beta > 0.$$

Then one has

$$(Tf) \circ \sigma = -T(f \circ \sigma), \qquad (\mathcal{L}f) \circ \sigma = \mathcal{L}(f \circ \sigma).$$

Now for $f_s \equiv P_s f$, we have

$$\partial_s P_{t-s}|Tf_s|^2 = P_{t-s}(-\mathcal{L}|Tf_s|^2 + 2Tf_s \cdot T\mathcal{L}f_s)$$
$$= P_{t-s}(-2\Gamma(Tf_s) + 2Tf_s \cdot [T, \mathcal{L}]f_s) \leq P_{t-s}(2Tf_s \cdot [T, \mathcal{L}]f_s)$$

with use of $-2\Gamma(Tf_s) \equiv -\mathcal{L}|Tf_s|^2 + 2Tf_s \cdot \mathcal{L}Tf_s \leq 0$. Next note that

$$[T, \mathcal{L}]g = [T, T^2 - \beta\eta T]g = -\beta[T, \eta]Tg = -\beta(\varepsilon Tg + 2(Tg) \circ \sigma)$$

Thus

$$\partial_s P_{t-s}|Tf_s|^2 \leq -2\beta P_{t-s}(Tf_s \cdot (\varepsilon Tf_s + 2(Tf_s) \circ \sigma)) \tag{1}$$

Repeating our computation for $f_s \circ \sigma \equiv (P_s f) \circ \sigma$,

$$\partial_s P_{t-s}(|(Tf_s)|^2 \circ \sigma) = P_{t-s}(-\mathcal{L}(|Tf_s|^2 \circ \sigma) + 2(Tf_s) \circ \sigma(T\mathcal{L}f_s) \circ \sigma) \tag{2}$$
$$= P_{t-s}(-2\Gamma((Tf_s) \circ \sigma) + 2(Tf_s) \circ \sigma((T\mathcal{L}f_s) \circ \sigma - \mathcal{L}(Tf_s \circ \sigma)))$$
$$= P_{t-s}(-2\Gamma((Tf_s) \circ \sigma) + 2(Tf_s) \circ \sigma(([T, \mathcal{L}]f_s) \circ \sigma))$$
$$\leq -2\beta P_{t-s}((Tf_s) \circ \sigma \cdot (2(Tf_s) + \varepsilon(Tf_s) \circ \sigma)))$$

Adding (1) & (2), we obtain

$$\partial_s P_{t-s}(|(Tf_s)|^2 + |(Tf_s)|^2 \circ \sigma) \leq -2(2+\varepsilon)\beta P_{t-s}(|(Tf_s)|^2 + |(Tf_s)|^2 \circ \sigma).$$

Integrating this differential inequality, yields

$$(|(Tf_s)|^2 + |(Tf_s)|^2 \circ \sigma) \leq e^{-2(2+\varepsilon)\beta t}\, P_t(|Tf|^2 + |Tf|^2 \circ \sigma).$$

Next, (although there is no doubt that what follows below can be done for general case of classical (finite) Coxeter groups of Dunkl theory), to focus our attention we consider the case of products of real lines each with a single natural reflection. That is we consider

$$T_i f \equiv (\nabla_i + A_i) f$$

with $\nabla_i$ denoting partial derivative with respect to $i$-th coordinate and

$$A_i f \equiv \kappa \, \frac{f - f \circ \sigma_i}{\omega_i}$$

with a reflection defined by

$$(\sigma_i \omega)_j \equiv (-1)^{\delta_{ij}} \omega_j$$

In this setup we note the following relation, in which we set $f_s \equiv P_s f$,

$$\begin{aligned} \partial_s P_{t-s}|T_i f_s|^2 &= P_{t-s}(-\mathcal{L}|T_i f_s|^2 + 2T_i f_s \cdot T_i \mathcal{L} f_s) \\ &= P_{t-s}(-2\Gamma(T_i f_s) + 2T_i f_s \cdot [T_i, \mathcal{L}] f_s) \\ &\leq P_{t-s}(2T_i f_s \cdot [T_i, \mathcal{L}] f_s) \end{aligned}$$

where in the last step we have used the fact that

$$-2\Gamma(T_i f_s) \equiv -\mathcal{L}|T_i f_s|^2 + 2T_i f_s \cdot \mathcal{L} T_i f_s \leq 0.$$

We remark that in the current setup where all directions in the tangent space are represented in the generator, we can afford to disregard otherwise vital nonpositive term $-2\Gamma(T_i f_s)$. Next we note that, by our current assumption

$$\begin{aligned} [T_i, \mathcal{L}_j] g &= [T_i, T_j^2 - \beta_j T_j] g = -[T_i, \beta_j] T_j g \\ &= -(\nabla_i \beta_j) T_j g - A_i(\beta_j)(T_j g) \circ \sigma_i. \end{aligned}$$

Combining this with our previous bounds, we obtain the following relation

$$\partial_s P_{t-s}|T_i f_s|^2 \leq -2P_{t-s}((\nabla_i \beta_i)|T_i f_s|^2) - 2P_{t-s}\left(A_i(\beta_i)\, T_i f_s \cdot (T_i f_s) \circ \sigma_i\right)$$

$$-2\sum_{j\neq i} P_{t-s}\left((\nabla_i\beta_j)\, T_i f_s \cdot T_j f_s\right) - 2\sum_{j\neq i} P_{t-s}\left(A_i(\beta_j)\, T_i f_s \cdot (T_j f_s)\circ\sigma_i\right)$$

As compared to a conventional situation, where reflections are not in the game, we have now got a trouble in the form of terms containing reflected factors. In case when $\beta_j = \sum_k G_{jk}\omega_k + \eta_j$ with $G_{ii} > 0$ and $G_{jk}$ sufficiently small, and $\eta_j$ are sufficiently small cylinder functions, at this point we could use quadratic inequality to separate terms containing $|T_i f_s|^2$ and get the following bound

$$\partial_s P_{t-s}|T_i f_s|^2 \le -2\alpha P_{t-s}|T_i f_s|^2 + P_{t-s}\left(A_i(\beta_i)\, |T_i f_s \circ \sigma_i|^2\right)$$
$$+\sum_{j\neq i} P_{t-s}\left(|\nabla_i\beta_j|\, |T_j f_s|^2\right) + \sum_{j\neq i} P_{t-s}\left(|A_i(\beta_j)|\, |(T_j f_s)\circ\sigma_i|^2\right)$$

with a constant

$$\alpha \le \inf_i \left(\nabla_i\beta_i - \frac{1}{2}\sum_{j\neq i}|\nabla_i\beta_j| - \frac{1}{2}\sum_j |A_i(\beta_j)|\right)$$

Solving this inequality with respect to $P_{t-s}|T_i f_s|^2$, after integration with respect to $s \in [0,t]$ and using supremum bounds for the coefficients, we arrive at

$$|T_i f_t|^2 \le e^{-\alpha t} P_t |T_i f|^2 + \|A_i(\beta_i)\|_\infty \int_0^t ds\, e^{-\alpha(t-s)} P_{t-s}|T_i f_s \circ \sigma_i|^2$$
$$+\sum_{j\neq i} \|A_i(\beta_j)\|_\infty \int_0^t ds\, e^{-\alpha(t-s)}\, P_{t-s}|(T_j f_s)\circ\sigma_i|^2$$

At this stage, if $P_t$ is a Markov semigroup, one can pass to the following supremum bounds

$$\|T_i f_t\|_\infty^2 \le e^{-\alpha t}\|T_i f\|_\infty^2 + \|A_i(\beta_i)\|_\infty \int_0^t ds\, e^{-\alpha(t-s)} \|T_i f_s\|_\infty^2$$
$$+\sum_{j\neq i} \|A_i(\beta_j)\|_\infty \int_0^t ds\, e^{-\alpha(t-s)}\, \|T_j f_s\|_\infty^2$$

This relation allows us to show existence of a semigroup in infinite dimensions as well as uniform ergodicity in sup norm if additionally $\alpha > 0$ ([52], [16]).

**Unbounded Drifts.**

In what follows we would like to improve on that above by allowing nonlinear unbounded drifts $\beta_i$'s as well as getting suitable pointwise bounds. To this end we will keep on an assumption that symmetric parts $(\beta_j + \beta_j \circ \sigma_i)$ are zero or sufficiently small. Now we propose to consider simultaneously reflected terms, as follows

$$\partial_s P_{t-s}|T_i f_s \circ \sigma_i|^2 = P_{t-s}(-\mathcal{L}|T_i f_s \circ \sigma_i|^2 + 2(T_i f_s) \circ \sigma_i \cdot (T_i \mathcal{L} f_s) \circ \sigma_i) =$$

$$P_{t-s}(-2\Gamma(T_i f_s \circ \sigma_i) + 2(T_i f_s) \circ \sigma_i \cdot ([T_i, \mathcal{L}] f_s) \circ \sigma_i + 2(T_i f_s) \circ \sigma_i \cdot ((\mathcal{L} T_i f_s) \circ \sigma_i - \mathcal{L}(T_i f_s \circ \sigma_i)))$$

$$= P_{t-s}(-2\Gamma(T_i f_s \circ \sigma_i) + 2(T_i f_s) \circ \sigma_i \cdot \{(-(\nabla_i \beta_i) T_i f_s - A_i(\beta_i)(T_i f_s) \circ \sigma_i) \circ \sigma_i\})$$

$$+ \sum_{j \neq i} 2P_{t-s}((T_i f_s) \circ \sigma_i \cdot \{(-(\nabla_i \beta_j) T_j f_s - A_i(\beta_j)(T_j f_s) \circ \sigma_i) \circ \sigma_i\})$$

$$+ P_{t-s} 2 \left( (T_i f_s) \circ \sigma_i \cdot \left( (\beta_i + \beta_i \circ \sigma_i) T_i((T_i f_s) \circ \sigma_i) + \sum_{j \neq i} (\beta_j + \beta_j \circ \sigma_i) T_j((T_i f_s) \circ \sigma_i) \right) \right).$$

Since with some constant $C \in (0, \infty)$, one has

$$|T_i g|^2 \leq C \Gamma_i(g),$$

as long as $\gamma \equiv \sup_i \sum_j ||\beta_j + \beta_j \circ \sigma_i||_\infty^2 < \infty$, with the use of quadratic inequality we see that

$$-2\Gamma(T_i f_s \circ \sigma_i)$$

$$+2 \left( (T_i f_s) \circ \sigma_i \cdot \left( (\beta_i + \beta_i \circ \sigma_i) T_i((T_i f_s) \circ \sigma_i) + \sum_{j \neq i} (\beta_j + \beta_j \circ \sigma_i) T_j((T_i f_s) \circ \sigma_i) \right) \right)$$

$$\leq \frac{C}{2} \gamma |T_i f_s \circ \sigma_i|^2$$

This allows us to get

$$\partial_s P_{t-s}|T_i f_s \circ \sigma_i|^2 \leq -2P_{t-s} \left( \left( (\nabla_i \beta_i) \circ \sigma_i - \frac{C}{4}\gamma \right) |T_i f_s \circ \sigma_i|^2 \right)$$

$$-2P_{t-s}((A_i(\beta_i) \circ \sigma_i)(T_i f_s) \circ \sigma_i \cdot (T_i f_s))$$

$$-2 \sum_{j \neq i} P_{t-s} \left( (T_i f_s) \circ \sigma_i \cdot \{(((\nabla_i \beta_j) \circ \sigma_i) \, (T_j f_s) \circ \sigma_i + A_i(\beta_j) \circ \sigma_i \, (T_j f_s))\} \right)$$

This together with similar bound for $\partial_s P_{t-s}(T_i f_s)$ obtained before, yields

$$\partial_s P_{t-s}(|T_i f_s|^2 + |T_i f_s \circ \sigma_i|^2)$$

$$\leq -2P_{t-s}((\nabla_i \beta_i)|T_i f_s|^2) - 2P_{t-s} \left( \left( (\nabla_i \beta_i) \circ \sigma_i - \frac{C}{4}\gamma \right) |T_i f_s \circ \sigma_i|^2 \right)$$

$$-2P_{t-s}\left(\left(A_i(\beta_i)+A_i(\beta_i)\circ\sigma_i\right) T_i f_s\cdot (T_i f_s)\circ\sigma_i\right)$$
$$-2\sum_{j\neq i} P_{t-s}\left(\left(\nabla_i\beta_j\right) T_i f_s\cdot T_j f_s\right) -2\sum_{j\neq i} P_{t-s}\left(A_i(\beta_j)\ T_i f_s\cdot (T_j f_s)\circ\sigma_i\right)$$
$$-2\sum_{j\neq i} P_{t-s}\left(\left(\left(\nabla_i\beta_j\right)\circ\sigma_i\right) (T_i f_s)\circ\sigma_i\cdot (T_j f_s)\circ\sigma_i\right) -2\sum_{j\neq i} P_{t-s}\left(A_i(\beta_j)\circ\sigma_i\ (T_i f_s)\circ\sigma_i\cdot (T_j f_s)\right)$$

We can simplify that by using the quadratic inequality to have

$$\partial_s P_{t-s}(|T_i f_s|^2+|T_i f_s\circ\sigma_i|^2)\leq -2MP_{t-s}(|T_i f_s|^2+|T_i f_s\circ\sigma_i|^2)$$
$$+\sum_{j\neq i}\gamma_{ij}P_{t-s}(|T_j f_s|^2+|(T_j f_s)\circ\sigma_i|^2)$$

provided that

$$(\nabla_i\beta_i)+(\nabla_i\beta_i)\circ\sigma_i-\frac{1}{2}|A_i(\beta_i)+A_i(\beta_i)\circ\sigma_i|-\frac{1}{2}\sum_{j\neq i}\gamma_{ij}-\frac{C}{4}\gamma\geq M$$

and where we set

$$\gamma_{ij}\equiv\|\nabla_i\beta_j\|_\infty+\|A_i(\beta_j)\|_\infty$$

Now we are in much better shape than before. This is because the first condition allows for $\beta_i$ other than linear, for example including

$$\beta_i=a_{2n+1}\omega_i^{2n+1}+\sum_{l=2,..,2n}a_l\ \omega_i^l+\tilde{M}\omega_i+\sum_{k\neq i}G_{ik}\omega_k+\sum_{O:O\ni i}b_O\prod_{k\in O}\varsigma(\omega_k)$$

where $a_{2n+1}>0$, $n\geq 1$, and $a_l, b_O\in\mathbb{R}$, with finite sets $O$, and $\sup_j\sum_{O:O\ni j}|b_O|<\infty$, where $\varsigma(x)=x\chi_{x\in[-1,+1]}+\chi_{x\in[+1,\infty]}-\chi_{x\in(-\infty,-1]}$ and finally with $\tilde{M}>0$. Thus for such drift coefficients $\beta_i$, integration with respect to $s$ of our differential inequality yields the following.

$$|T_i f_t|^2+|T_i f_t\circ\sigma_i|^2$$
$$\leq e^{-2Mt}P_t(|T_i f|^2+|T_i f\circ\sigma_i|^2)$$
$$+\sum_{j\neq i}\gamma_{ij}\int_0^t ds\ e^{-2M(t-s)}P_{t-s}(|T_j f_s|^2+|(T_j f_s)\circ\sigma_i|^2)$$

From this we get the the following bound as a simple implication.

**Lemma 1**

$$\|T_i f_t\|_\infty^2\leq 2e^{-2Mt}\|T_i f\|_\infty^2+\sum_{j\neq i}2\gamma_{ij}\int_0^t ds\ e^{-2M(t-s)}\|T_j f_s\|_\infty^2$$

With this inequality via standard arguments, (see e.g. [16], [49], [51] and references therein), one obtains finite speed of propagation of information which allows to show the existence of the semigroup in infinite dimensions and under additional assumptions existence of invariant measure and strong ergodicity. That is one has the following result.

**Theorem 2**
*Suppose $M, \gamma_{ij} \in \mathbb{R}$ with $\gamma_{ij} > 0$ and $\sup_i \sum_j \gamma_{ij} < \infty$. Then the Markov semigroup $P_t$ is well defined in infinite dimensions. Moreover, if $M > 0$ and $\sup_i \sum_j \gamma_{ij} > 0$ is sufficiently small, then there exists $m \in (0, \infty)$ such that*

$$\|\mathbb{T} f_t\|_\infty^2 \leq 2e^{-2mt} \|\mathbb{T} f\|_\infty^2$$

with

$$\|\mathbb{T} g\|_\infty^2 \equiv \sum_i \|T_i g\|_\infty^2$$

*In this case there exists a unique measure $\mu$ with finite moments such that*

$$\|f_t - \int f d\mu\|_\infty^2 \leq e^{-2mt} C(\|\mathbb{T} f\|_\infty)$$

*for any cylinder function $f$ with bounded $\|T_i f\|_\infty^2$ with some constant $C(\|\mathbb{T} f\|_\infty) \in (0, \infty)$ independent of $t \in (0, \infty)$.*

Now we get back to our symmetrised with respect to $\sigma_i$ inequality in our claim and notice that, at least when our Coxeter group generated by reflections is finite, one could consider full symmetrisation to get after resummation the following Gronwal type inequality

$$\|\mathbb{T} f_t\|_{\mathcal{C}ox}^2 \leq e^{-2\hat{M}t} P_t \|\mathbb{T} f\|_{\mathcal{C}ox}^2 + \sum_{j \neq i} \hat{\gamma}_{ij} \int_0^t ds\, e^{-2\hat{M}(t-s)} P_{t-s} \|\mathbb{T} f_s\|_{\mathcal{C}ox}^2$$

with

$$\|\mathbb{T} g\|_{\mathcal{C}ox}^2 \equiv \sum_i \sum_{c \in \mathcal{C}ox} |T_i g \circ c|^2$$

A simple application of this yields the following bound.

**Claim** *With some $\hat{m} \in \mathbb{R}$*

$$\|\mathbb{T} f_t\|_{\mathcal{C}ox}^2 \leq e^{-2\hat{m}t} P_t \|\mathbb{T} f\|_{\mathcal{C}ox}^2$$

One may expect that similar bound could be possible for square of a seminorm in which we sum over $i$ and composition with $c$ is replaced by projections (on subspaces obtained via symmetrisation subordinated to $Cox$). One may hope that the last could possibly survive also in the case when the Coxeter group is infinite (at least on some smaller class of functions which are sufficiently quickly decreasing to zero with the size of $c \in Cox$). This is for a moment an interesting, challenging and widely open problem.

**Remark 2** *A theory of dissipative semigroups generated by Dunkl type operators associated to noncommutative groups was recently developed in [52] and [53] .*

# 4 Quantum Dunkl Type Generators.

In this section we provide a description of linear dissipative semigroup with Dunkl type generators in a noncommutative algebra $\mathcal{A}$. While the principal objective here is to provide a new noncommutative model, one could also potentially hope for a possible application of such models to quantum information theory.
Let $\sigma_j \in \mathcal{A}$, $j \in \mathcal{I}$, be such that $\sigma_j^* = \sigma_j$, $\sigma_j^2 = 1$ and $\{\sigma_j, \sigma_k\} = 0$. Define maps

$$\mathcal{A} \ni f \to \mathfrak{S}_{jk}(f) \equiv \sigma_j f \sigma_k \in \mathcal{A}.$$

Then we have

$$\mathfrak{S}_{jk}^2 = I \quad \text{and} \quad \mathfrak{S}_{jk}(fg) = \mathfrak{S}_{jk}(f)\mathfrak{S}_{kk}(g) = \mathfrak{S}_{jj}(f)\mathfrak{S}_{jk}(g).$$

Define

$$A_{jk}^L(f) \equiv \kappa_{jk}(f - \mathfrak{S}_{jk}(f)) \text{ and } A_{jk}^R(f) \equiv (f - \mathfrak{S}_{jk}(f))\tilde{\kappa}_{jk}$$

with $\mathfrak{S}_{jj}(\kappa_{jk}) = -\kappa_{jk}$ and $\mathfrak{S}_{kk}(\tilde{\kappa}_{jk}) = -\tilde{\kappa}_{jk}$ . Then we have

$$\begin{aligned} A_{jk}^L(A_{jk}^L(f)) &= A_{jk}^L(\kappa_{jk}(f - \mathfrak{S}_{jk}(f))) \\ &= \kappa_{jk}(\kappa_{jk}(f - \mathfrak{S}_{jk}(f)) - \mathfrak{S}_{jk}(\kappa_{jk}(f - \mathfrak{S}_{jk}(f)))). \end{aligned}$$

Since

$$\begin{aligned} \mathfrak{S}_{jk}(\kappa_{jk}(f - \mathfrak{S}_{jk}(f))) &= -\kappa_{jk}(\mathfrak{S}_{jk}(f) - \mathfrak{S}_{jk}^2(f)) = -\kappa_{jk}(\mathfrak{S}_{jk}(f) - f) \\ &= \kappa_{jk}(f - \mathfrak{S}_{jk}(f)) \end{aligned}$$

we obtain

$$A_{jk}^L(A_{jk}^L(f)) = 0.$$

Similarly we have

$$A^R_{jk}(A^R_{jk}(f)) = 0.$$

We also note that

$$A^R_{jk}(A^L_{jk}(f)) = A^R_{jk}(\kappa_{jk}(f - \mathfrak{S}_{jk}(f))) = (\kappa_{jk}(f - \mathfrak{S}_{jk}(f)) - \mathfrak{S}_{jk}(\kappa_{jk}(f - \mathfrak{S}_{jk}(f))))\tilde{\kappa}_{jk}$$

$$= (\kappa_{jk}(f - \mathfrak{S}_{jk}(f)) - (\kappa_{jk}(f - \mathfrak{S}_{jk}(f))))\tilde{\kappa}_{jk} = 0$$

and similarly

$$A^L_{jk}(A^R_{jk}(f)) = 0.$$

Next consider a derivation $\delta_l(f) \equiv [\sigma_l, f]$,which satisfies

$$\delta_l(\sigma_j) = 2\sigma_l\sigma_j(1 - \delta_{lj}).$$

Then, for $l \neq j, k$, we have

$$\begin{aligned}\delta_l(\mathfrak{S}_{jk}(f)) = \delta_l(\sigma_j f \sigma_k) = \delta_l(\sigma_j) f \sigma_k + \sigma_j \delta_l(f)\sigma_k + \sigma_j f \delta_l(\sigma_k)\\ = -2\sigma_j\sigma_l f\sigma_k + \sigma_j\delta_l(f)\sigma_k + \sigma_j f 2\sigma_l\sigma_k = -\sigma_j\delta_l(f)\sigma_k\\ = -\mathfrak{S}_{jk}(\delta_l(f)).\end{aligned}$$

That is $\mathfrak{S}_{jk}$is a reflection in the sense of [52], [53] (in the direction of "tangent vector" $\delta_l$).
Using this we can introduce the following generalised derivations

$$\mathbb{T}f \equiv \nabla f + \mathbb{A}(f)$$

with components $\mathbb{T}_l \equiv \nabla_l + \mathbb{A}_l$ , $l \in \mathcal{I}$, defined by $\nabla_l = \delta_l$ and

$$\mathbb{A}_l \equiv A^L_{jk} + A^R_{jk}$$

We define an operator

$$\mathcal{L}_l f \equiv \mathbb{T}^2_l f = (\delta^2_l + \delta_l\mathbb{A}_l + \mathbb{A}_l\delta_l)f \equiv \mathcal{L}_0 f + \{\nabla_l, \mathbb{A}_l\}f$$

and its associated quadratic form

$$\Gamma_{\mathcal{L}_l}(f) \equiv \tfrac{1}{2}(\mathcal{L}_l(f^*f) - \mathcal{L}_l(f^*)f - f^*\mathcal{L}_l(f)).$$

Note that

$$\Gamma_{\mathcal{L}_l}(f) \equiv -(\delta_l(f))^*\delta_l(f) + \Gamma_{\{\delta_l,\mathbb{A}_l\}}(f)$$

where

$$\Gamma_{\{\delta_l,\mathbb{A}_l\}}(f) \equiv \frac{1}{2}(\{\delta_l, \mathbb{A}_l\}(f^*f) - \{\delta_l, \mathbb{A}_l\}(f^*)f - f^*\{\delta_l, \mathbb{A}_l\}(f))$$
$$= \Gamma_{\{\delta_l,\mathbb{A}_l^L\}}(f) + \Gamma_{\{\delta_l,\mathbb{A}_l^R\}}(f).$$

Since, using reflection property $\delta_l(\mathfrak{S}_{jk}(f)) = -\mathfrak{S}_{jk}(\delta_l(f))$, we have

$$\{\delta_l, \mathbb{A}_l^L\}(f) = 2\kappa_{jk}\delta_l(f) + \delta_l(\kappa_{jk})(f - \mathfrak{S}_{jk}(f))$$

so

$$\frac{1}{2}(\{\delta_l, \mathbb{A}_l^L\}(f^*f) - \{\delta_l, \mathbb{A}_l^L\}(f^*)f - f^*\{\delta_l, \mathbb{A}_l^L\}(f)) = 2[\kappa_{jk},f^*]\delta_l(f)+$$

$$\frac{1}{2}\left(\delta_l(\kappa_{jk})(f^*f - \mathfrak{S}_{jk}(f^*f)) - \delta_l(\kappa_{jk})\left(f^* - \mathfrak{S}_{jk}(f^*)\right)f - f^*\delta_l(\kappa_{jk})(f - \mathfrak{S}_{jk}(f))\right).$$

The second part on the right hand side can be represented as follows

$$\frac{1}{2}\left(\delta_l(\kappa_{jk})(f^*f - \mathfrak{S}_{jk}(f^*f)) - \delta_l(\kappa_{jk})\left(f^* - \mathfrak{S}_{jk}(f^*)\right)f - f^*\delta_l(\kappa_{jk})(f - \mathfrak{S}_{jk}(f))\right)$$
$$= -\frac{1}{2}\delta_l(\kappa_{jk})(f^* - \mathfrak{S}_{jk}(f^*)) \cdot (f - \mathfrak{S}_{jk}(f))$$
$$+\frac{1}{4}\delta_l(\kappa_{jk})(\mathfrak{S}_{jk}(f^*)(\mathfrak{S}_{jk}(f) - \mathfrak{S}_{kk}(f)) + (\mathfrak{S}_{jk}(f^*) - \mathfrak{S}_{jj}(f^*))\mathfrak{S}_{jk}(f))$$
$$+\frac{1}{2}([\delta_l(\kappa_{jk}), f^*](f - \mathfrak{S}_{jk}(f))).$$

In particular we see that for a special case $j = k$, we obtain

$$\frac{1}{2}\left(\delta_l(\kappa_{jk})(f^*f - \mathfrak{S}_{jk}(f^*f)) - \delta_l(\kappa_{jk})\left(f^* - \mathfrak{S}_{jk}(f^*)\right)f - f^*\delta_l(\kappa_{jk})(f - \mathfrak{S}_{jk}(f))\right)$$
$$= -\frac{1}{2}\delta_l(\kappa_{jj})(f - \mathfrak{S}_{jj}(f))^* \cdot (f - \mathfrak{S}_{jj}(f)) + \frac{1}{2}([\delta_l(\kappa_{jj}), f^*](f - \mathfrak{S}_{jj}(f)))$$

and hence, we have

$$\Gamma_{\{\delta_l,\mathbb{A}_l^L\}}(f) = -2([\kappa_{jj},f])^*\delta_l(f) - \frac{1}{2}\delta_l(\kappa_{jj})(f - \mathfrak{S}_{jj}(f))^* \cdot (f - \mathfrak{S}_{jj}(f))$$
$$+\frac{1}{2}([\delta_l(\kappa_{jj}), f^*](f - \mathfrak{S}_{jj}(f))).$$

Similarly, we have

$$\Gamma_{\{\delta_l,\mathbb{A}_l^R\}}(f) = 2\delta_l(f^*)[\tilde{\kappa}_{jk}, f]+$$
$$\frac{1}{2}\left((f^*f - \mathfrak{S}_{jk}(f^*f))\delta_l(\tilde{\kappa}_{jk}) - (f^* - \mathfrak{S}_{jk}(f^*))\,\delta_l(\tilde{\kappa}_{jk})f - f^*(f - \mathfrak{S}_{jk}(f))\delta_l(\tilde{\kappa}_{jk})\right)$$

and

$$\frac{1}{2}\left((f^*f - \mathfrak{S}_{jk}(f^*f))\delta_l(\tilde{\kappa}_{jk}) - (f^* - \mathfrak{S}_{jk}(f^*))\,\delta_l(\tilde{\kappa}_{jk})f - f^*(f - \mathfrak{S}_{jk}(f))\delta_l(\tilde{\kappa}_{jk})\right)$$

$$= -\frac{1}{2}((f^* - \mathfrak{S}_{jk}(f^*))(f - \mathfrak{S}_{jk}(f)))\delta_l(\tilde{\kappa}_{jk})$$

$$+\frac{1}{4}\left((\mathfrak{S}_{jk}(f^*)(\mathfrak{S}_{jk}(f) - \mathfrak{S}_{kk}(f)) + (\mathfrak{S}_{jk}(f^*) - \mathfrak{S}_{jj}(f^*))\,\mathfrak{S}_{jk}(f))\,\delta_l(\tilde{\kappa}_{jk})\right)$$

$$+\frac{1}{2}(f^* - \mathfrak{S}_{jk}(f^*))[\delta_l(\tilde{\kappa}_{jk}), f].$$

Again, for $j = k \neq l$, we can simplify this expression as follows

$$\frac{1}{2}\left((f^*f - \mathfrak{S}_{jk}(f^*f))\delta_l(\tilde{\kappa}_{jk}) - (f^* - \mathfrak{S}_{jk}(f^*))\,\delta_l(\tilde{\kappa}_{jk})f - f^*(f - \mathfrak{S}_{jk}(f))\delta_l(\tilde{\kappa}_{jk})\right)$$
$$= -\frac{1}{2}((f^* - \mathfrak{S}_{jj}(f^*))(f - \mathfrak{S}_{jj}(f)))\delta_l(\tilde{\kappa}_{jj}) + \frac{1}{2}(f^* - \mathfrak{S}_{jj}(f^*))[\delta_l(\tilde{\kappa}_{jj}), f].$$

Hence we get

$$\Gamma_{\{\delta_l,\mathbb{A}_l^R\}}(f) = -2(\delta_l(f))^*[\tilde{\kappa}_{jj}, f] - \frac{1}{2}(f - \mathfrak{S}_{jj}(f))^* \cdot (f - \mathfrak{S}_{jj}(f))\delta_l(\tilde{\kappa}_{jj})$$
$$+\frac{1}{2}(f^* - \mathfrak{S}_{jj}(f^*))[\delta_l(\tilde{\kappa}_{jj}), f]$$

Assuming

$$\kappa_{jj} = \kappa\sigma_l \quad \text{and} \quad \tilde{\kappa}_{jj} = \tilde{\kappa}\sigma_l,$$

combining our calculations we arrive at

$$\Gamma_{\mathcal{L}_l}(f) = -(1 - 2\kappa - 2\tilde{\kappa})(\delta_l(f)^*)\delta_l(f)$$

which is nonpositive provided $2\kappa + 2\tilde{\kappa} \leq 1$. Thus an operator

$$\mathcal{L}f \equiv \mathbb{T}^2 f \equiv \textstyle\sum_l \mathbb{T}_l^2 f$$

is Markovian. We remark that in general the operators $\mathbb{T}_l$ may not commute (and thus we are in general setup of [53]).

# 5 On Nonlinear Dissipative Dynamics.

To begin we mention first that in [48] an interesting nonlinear dissipative dynamics of jump type was introduced and studied for infinite interacting systems of classical spins on a lattice. The generator of this dynamics is formally given by

$$\mathcal{L}f \equiv \sum_{l\in\mathbb{Z}^d}(\mathbb{E}_l - \mathbb{I})(f)$$

where

$$\mathbb{E}_l f \equiv \tfrac{1}{\beta}\log E_{X+l}e^{\beta f}$$

with $E_{X+l}$ denotes a conditional expectation given a configuration of the system in $\mathbb{Z}^d \setminus \{X+l\}$ associated to a Gibbs measure and $\beta \in \mathbb{R}\setminus\{0\}$. (The elementary operator in the sum can be understood as a Glauber type generator corrected by the relative entropy part.) One can show that the corresponding semigroup $\mathcal{P}_t \equiv e^{t\mathcal{L}}$ preserves unit and positivity and it was demonstrated there that ,under suitable mixing condition, the corresponding dynamics is exponentially ergodic ([48]. Without getting into more detail, (a more extensive description can be found in [55]), such kind of dynamics could prove to be interesting in relation to certain optimization problems, (see also a work [39] for some other application of nonlinear averages to economy).

A desire to construct and understand nonlinear noncommutative dissipative dynamics led to the paper [33] where in particular the following result was proved. For $E_i, i = 1, ..., n$, being linear, positive and unital operators on a $C^*$ algebra $\mathcal{A}$, we define $\mathcal{L} : D(F) \to \mathcal{A}$,

$$\mathcal{L}(x) = \sum_{i=1}^n \alpha_i \log E_i(e^x) - x,$$

with
$D(F) = \mathcal{A}_{\mathrm{sa}} \cap K(x,r) \equiv \{y \in \mathcal{A} : \| x - y \| < r\}$, $r > 0$,
and $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$. Note that $\mathcal{L} - (e^r - 1)I$ is strictly dissipative, because

$$\| \log E_i(e^{x_2}) - \log E_i(e^{x_1}) \| \leq e^r \| x_2 - x_1 \|,$$

and so,

$$\forall\varphi \in J(x_2 - x_1) \equiv (\text{tangent functionals at } x_2 - x_1)$$

$$\Re\langle\varphi, F(x_2) - F(x_1)\rangle = \sum_{i=1}^n \alpha_i \Re\langle\varphi, \log E_i(e^{x_2}) - \log E_i(e^{x_1})\rangle - \| x_2 - x_1 \|$$

$$\leq \sum_{i=1}^{n} \alpha_i \parallel \log E_i(e^{x_2}) - \log E_i(e^{x_1}) \parallel - \parallel x_2 - x_1 \parallel \leq (e^r - 1) \parallel x_2 - x_1 \parallel .$$

Moreover, one point dissipativity also holds

$\forall x \in D(F) \setminus \{0\} \forall \varphi \in J(x)$

$$\Re\langle \varphi, F(x) \rangle = \sum_{i=1}^{n} \alpha_i \Re\langle \varphi, \log E_i(e^x) \rangle - \parallel x \parallel \leq \sum_{i=1}^{n} \alpha_i \parallel \log E_i(e^x) \parallel - \parallel x \parallel \leq 0.$$

Hence we have the following result (see [33] for details).

**Theorem 3** *The operator*

$$\mathcal{L}(x) = \sum_{i=1}^{n} \alpha_i \log E_i(e^x) - x,$$

*generates a Lipschitz semigroup $S_t : D(F) \to D(F)$ which is contractive and preserves unit and positivity*, i.e. $(S_t)_{t \geq 0}$ is a *conservative Markov semigroup.*

It is a challenging problem to obtain an infinite dimensional extension of this result and ergodicity theory for the corresponding semigroup.

**Remark 3** *It is also an interesting open question, if it could be possible to extend a classical nonlinear annealing algorithm of [55] to study a challenging problem of determining ground states for large interacting quantum systems.*

A theory of nonlinear dissipation for infinite dimensional interacting systems has been developed over time in [21], [18] and recently in [19]. In particular in the last work we have used log-Sobolev inequality to provide a solution of Reaction-Diffusion type problem when, first of all the underlying space is infinite dimensional and secondly, when one can have different type of mixing. That is we have studied a system

$$\partial_t u_i = L_i u_i + (\beta_i - \alpha_i) \left( k \prod_{j=1}^{q} u_j^{\alpha_j} - l \prod_{j=1}^{q} u_j^{\beta_j} \right),$$

where $i = 1, \ldots, q$; $\alpha_i, \beta_i \in \mathbb{R}^+$, $\beta_i \neq \alpha_i$;and $L_i$ an operator which models how the $i$th substance diffuses, with a key assumption being that these generators satisfy log-Sobolev inequality

$$\mu \left( f^2 \log \frac{f^2}{\mu f^2} \right) \leq c_i \mu(f(-L_i f))$$

with a given probability measure $\mu$ and a constant $c_i \in (0, \infty)$ independent of a function $f$.

This inequality played in the past an essential role in development of ergodicity theory for infinite spin systems on a lattice, (see e.g. [47], [24]), and it is expected that it will be similar in the discussed case of R-D systems ([20]).

As we mentioned in the introduction a general theory for log-Sobolev inequality and associated hypercontractivity property for corresponding linear dissipative semigroups in noncommutative algebras was introduced and initially studied in [42]. In general there is still a number of elements well known for classical case, but hard to get in the noncommutative case. One of them, the equivalence of log-Sobolev inequality to Sobolev-Orlicz type inequalites (as introduced in [8]), was recently obtained in [2], but still there are many other (including perturbation and product property) awaiting to be understood. One of possibly promising direction of the progress should be the one including the systems with classical potentials for which jump type dynamics can be well defined for the infinite system. In this case one can expect that for any local observable $f$ we have the following limit

$$\lim_{n \to \infty} E_{i_n} \dots E_{i_1} f = \omega(f)$$

where $E_j$ denotes a completely positive map given by a generalised conditional expectation which is symmetric in $\mathbb{L}_{2,1/2}(\omega)$ space, with suitable sequence $(i_k)_{k \in \mathbb{N}}$ "going infinitely many times through each site of a lattice" in the sense of [51]. (In the Appendix at the end of the paper we discuss briefly some matters related to this and other type of limits involving generalised conditional expectation given by completely positive map.)

**Appendix. Towards the Martingale Convergence Theorem in Noncommutative $\mathbb{L}_p$ Spaces:**

At this point it is interesting to notice the joint monotonicity inequalities for $\mathbb{L}_{p,1/2}(\omega)$ norms obtained in [2], with $\omega \equiv \mathrm{Tr}(\rho \cdot) \equiv \mathrm{Tr}(P^{-1} \cdot)$ where $P = P^* > 0$ with $\mathrm{Tr} P^{-1} = 1$.

**Theorem 4** : $\forall \alpha \in [0,1], \forall r = 2n, n \in \mathbb{N}$

$$\mathrm{Tr}|\varphi(P)^{-(1-\alpha)/r} \varphi(f) \varphi(P)^{-\alpha/r}|^r \leq \mathrm{Tr}|P^{-(1-\alpha)/r} f P^{-\alpha/r}|^r \equiv \|f\|_{P^{-1},\alpha,r}$$

where $\varphi$ is a Completely Positive Mapping.

Let

$\langle f,g\rangle_{P,\alpha} \equiv \mathrm{Tr}(P^{-(1-\alpha)}f^*P^{-\alpha}g) = \mathrm{Tr}((P^{-\alpha/2}fP^{-(1-\alpha)/2})^*(P^{-\alpha/2}gP^{-(1-\alpha)/2}))$
and
$E_{X,\alpha}(f) \equiv \mathrm{Tr}_X(\gamma^*_{X,\alpha,L} f \gamma_{X,\alpha,R})$
with
$\gamma_{X,\alpha,R} \equiv P^{-(1-\alpha)}(\mathrm{Tr}_X P^{-1})^{-(1-\alpha)} \equiv \gamma^*_{X,1-\alpha,L}$ . Then we have
$\langle E_{X,\alpha}(f), E_{X,\alpha}(g)\rangle_{E_{X,\alpha}(P),\alpha} = \mathrm{Tr}(E_{X,\alpha}(P)^{-(1-\alpha)}E_{X,\alpha}(f)^*E_{X,\alpha}(P)^{-\alpha}E_{X,\alpha}(g))$
with

$$\begin{aligned} E_{X,\alpha}(P) &= \mathrm{Tr}_X(((\mathrm{Tr}_X P^{-1})^{-\alpha})P^{-\alpha}PP^{(1-\alpha)}(\mathrm{Tr}_X P^{-1})^{-(1-\alpha)}) = \\ &= \mathrm{Tr}_X((\mathrm{Tr}_X P^{-1})^{-1}) = (\mathrm{Tr}_X P^{-1})^{-1}. \end{aligned}$$

In particular for $\alpha = \frac{1}{2}$, we have that $E_{X,\alpha}(\cdot)$ is a completely positive map.

**The Product Case**.

We consider first a product state given by

$$P \equiv \otimes_{k=1}^n P_k$$

where $P_k \equiv P_{X_k} \in \mathcal{A}_{X_k}$, $k = 1,\ldots,n$, are commuting positive matrices s.t. $\mathrm{Tr}_{X_k}P_k^{-1} = 1$, and for $n > j \in \mathbb{N}$ set $P_{\geq j} \equiv \otimes_{k=1}^j I_k \otimes_{k=j+1}^n P_k$ and $P_{\geq n} \equiv I$. Then, we have

$$\prod_{k=1}^j E_{X_k,\alpha}(P) \equiv E_{\geq j,\alpha}(P) = P_{\geq j}$$

In the current situation $\gamma_{X_k,\alpha,R} \equiv P^{-(1-\alpha)}(\mathrm{Tr}_{X_k}P^{-1})^{-(1-\alpha)} = P_{X_k}^{-(1-\alpha)}$

$$E_{X_k,\alpha}(f) = \mathrm{Tr}_{X_k}(P_{X_k}^{-\alpha} f P_{X_k}^{-(1-\alpha)}) = \mathrm{Tr}_{X_k}(P_{X_k}^{-1} f).$$

In a special case $\alpha = \frac{1}{2}$, we will omit the index $\alpha$ writing $E_{X_k}(f) \equiv E_{X_k,1/2}(f)$ and $\|f\|_r \equiv \|f\|_{1/2,r}$. The monotonicity result above, yields

$$\begin{aligned} \|E_{\geq j}(f)\|_{E_{\geq j}(P)^{-1},r} &= \|E_{X_j}E_{\geq j-1}(f)\|_{E_{X_j}E_{\geq j-1}(P)^{-1},r} \\ &\leq \|E_{\geq j-1}(f)\|_{E_{\geq j-1}(P)^{-1},r} \end{aligned}$$

For $j = n$, we have

$$\|E_{\geq n}(f)\|_{E_{\geq n}(P)^{-1},r} = \|E_{\geq n}(f)\|_{I,r}$$

and

$$E_{\geq n}(f) = \mathrm{Tr}(P^{-1}f) \equiv \omega(f).$$

Naturally this can be generalised to infinite product states with the claim that

$$\lim_{j\to\infty} \|E_{\geq j}(f)\|_{E_{\geq j}(P)^{-1},r} = |\omega(f)|$$

for any local observable $f$.
Next consider a family of completely positive operator of the form

$$E_X(f) = \mathrm{Tr}_X(\gamma_X^* f \gamma_X),\ X \subset\subset \mathfrak{R}$$

which are symmetric in $\mathbb{L}_2(\omega) \equiv \mathbb{L}_{2,\frac{1}{2}}(\omega)$ and unital. Let us assume that there exists a commutative subalgebra $\mathcal{A}_c$ such that $\gamma_X \in \mathcal{A}_c$ and $E_X(\mathcal{A}_c) \subseteq \mathcal{A}_c$ . Suppose a family
$\{E_X\}_{X\in\mathfrak{R}_0}$, for some countable $\mathfrak{R}_0 \subsetneq \mathfrak{R}$, is ergodic in the sense that

$$\forall f \in \mathcal{A}_c \qquad \lim_{n\to\infty} E_{X_n} \dots E_{X_1}(f) = \omega(f) \tag{3}$$

and

$\forall g \in \mathcal{A}_0$ , with a dense subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, $\exists n \in \mathbb{N} \qquad E_{X_n} \dots E_{X_1}(g) \in \mathcal{A}_c \cap \mathcal{A}_0$.

Then, for $f = E_{X_m} \dots E_{X_1}(g) \in \mathcal{A}_c \cap \mathcal{A}_0$ given by $g \in \mathcal{A}_0$ with some $m \in \mathbb{N}$, we have

$\lim_{n\to\infty} E_{X_n} \dots E_{X_1}(f) = \omega(f) = \omega(E_{X_m} \dots E_{X_1}(g)) = \langle \mathbb{1}, E_{X_m} \dots E_{X_1}(g)\rangle_{2,\omega}$.

Since by our assumption $E_X$ are symmetric and unital, by induction we get

$\langle E_{X_m}(\mathbb{1}), E_{X_{m-1}} \dots E_{X_1}(g)\rangle_{2,\omega} = \langle \mathbb{1}, E_{X_{m-1}} \dots E_{X_1}(g)\rangle_{2,\omega} = \langle \mathbb{1}, g\rangle_{2,\omega} = \omega(g)$.

In particular this idea can be used for system with classical interaction, i.e. when for $f \in \mathcal{A}_0$

$\omega(f) \equiv \lim_{\Lambda\to\mathfrak{R}} \mathrm{Tr}(e^{-U_\Lambda} f)/\mathrm{Tr}(e^{-U_\Lambda})$

with

$U_\Lambda \equiv \sum_{X\cap\Lambda\neq\emptyset} \Phi_X$, and $\Phi_X \in \mathcal{A}_c \cap \mathcal{A}_0$ with $\sup_{i\in\mathfrak{R}} \sum_{X\subset\mathfrak{R},\, X\ni i} \|\Phi_X\|_{\mathcal{A}} < \infty$;

and one is given a family

$\{\mathrm{Tr}_X : \ X \subset \mathfrak{R}, |X| < \infty | \mathrm{Tr}(\mathrm{Tr}_X(f)) = \mathrm{Tr}(f),\ \mathrm{Tr}_X \mathrm{Tr}_X(f) = \mathrm{Tr}_X(f), \mathrm{Tr}_X(\mathbb{1}) = \mathbb{1}\}$ .

When restricted to $\mathcal{A}_c$, the corresponding structure reduces to the one known in the classical Gibbs measure theory. In particular all $E_X$ act as the classical conditional expectations and one can formulate for them conditions which assure the ergodicity (3) holds (cf. [24]).

In similar spirit one can also discuss more general sequences $(E_{\Lambda_n} : \Lambda_n \subset \Lambda_{n+1})$.

$\circ$

**Acknowledgements** : During this research the author was supported by the Royal Society Wolfson Research Merit Award.

# References

[1] S. Albeverio, Y.G. Kondratiev and M. Röckner, Symmetrizing measures for infinite dimensional diffusions: an analytic approach, "Geometric analysis and nonlinear partial differential equations" (Stefan Hildebrandt et al., eds.), Springer, Berlin, 2003, pp. 475-486.

[2] M. Al-Rashed and B. Zegarliński, Monotone Norms and Finsler Structure in Noncommutative Spaces, Infinite Dimensional Analysis, Quantum Probability and Related Topics **17**, No. 4 (2014) 1450029; `http://dx.doi.org/10.1142/S0219025714500295`

[3] F. Baudoin, M. Hairer and J. Teichmann, Ornstein-Uhlenbeck processes on Lie groups, J. Func. Analysis **255** (2008), 877–890.

[4] F. Baudoin and J. Teichmann, Hypoellipticity in infinite dimensions and an application in interest rate theory, Ann. Appl. Probab. **15** (2005), 1765-1777.

[5] D. Bakry and M. Emery, Diffusions hypercontractives, pp. 177-206 in Sém. de Probab. **XIX**, Lecture Notes in Math., **1123**, Springer-Verlag, Berlin, 1985.

[6] D. Bakry, F. Baudoin, M. Bonnefont and D. Chafai, On gradient bounds for the heat kernel on the Heisenberg group, J. Func. Analysis **255** (2008), 1905-1938.

[7] Ph. Blanchard, M. Hellmich, P. Ługiewicz, R. Olkiewicz, Quantum Dynamical Semigroups for Finite and Infinite Bose Systems, J. Math. Phys. 48 (2007), Continuity and generators of dynamical semigroups for infinite Bose systems, J. Func. Analysis **256** (2009) 1453-1475.

[8] S. G. Bobkov and F. Goetze, Exponential Integrability and Transportation Cost Related to Logarithmic Sobolev Inequalities, J. Func. Analysis **163** (1999) 1-28, `http://www.idealibrary.com`

[9] T. Bodineau and B. Zegarliński, Hypercontractivity via Spectral Theory, Infinite Dimensional Analysis, Quantum Probability and Related Topics Vol. 3, No. 1 (2000) 15-31.

[10] L. Bouten, Applications of Quantum Stochastic Processes in Quantum Optics, pp 277-308 in Quantum Potential Theory, Michael Schuermann & Uwe Franz (Eds)., Lec. Notes in Math. **1954**, Springer 2008.

[11] R. Carbone and A. Martinelli, Logarithmic Sobolev inequalities in non-commutative algebras, Infin. Dimens. Anal. Quantum. Probab. Relat. Top. **18** (2015) 1550011; `http://dx.doi.org/10.1142/S0219025715500113`

[12] R. Carbone, E.Sasso; Hypercontractivity for a Quantum Ornstein-Uhlenbeck Semi-group, Probab. Theory and Relat. Fields **140** (2008), no.3-4, 505-522.

[13] F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, J. Funct. Anal. **147** (1997), no. 2, 259-300; *Dirichlet Forms on Noncommutative Spaces*, pp 161-276 in Quan-

tum Potential Theory, Michael Schuermann & Uwe Franz (Eds)., Lec. Notes in Math. **1954**, Springer 2008.

[14] Veni Choi, Chul-Ki Ko and Yong-Moon Park, Quantum Markovian Semigroups On Quantum Spin System: Glauber Dynamics, J. the Korean Math. Soc. **45** (2008) 1075-1087; http://dx.doi.org/10.4134/JKMS.2008.45.4.1075

[15] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions. Cambridge Univ. Press (1992).

[16] F. Dragoni, V. Kontis, B. Zegarliński, Ergodicity of Markov Semigroups with Hörmander type generators in Infinite Dimensions, J. Potential Analysis **37** (2011) 199–227; arXiv:1012.0257v1.

[17] B.K. Driver and T. Melcher, Hypoelliptic heat kernel inequalities on Lie groups, Stoch. Process. Appl. **118** (2008), 368-388 ; Hypoelliptic heat kernel inequalities on the Heisenberg group, J. Func. Analysis **221** (2005), 340-365.

[18] P. Fougéres, C. Roberto and B. Zegarliński, Sub-gaussian measures and associated semilinear problems, Rev. Mat. Iberoam. **28** (2012), no. 2, 305-350. http://rmi.rsme.es/index.php?option=com_contentview=sectionlayout=blogid=7Itemid=95lang=en

[19] P. Fougéres, I. Gentil and B. Zegarliński, Application of logarithmic Sobolev inequality to Cauchy problems of reaction-diffusion equations, [hal-00987370]

[20] P. Fougéres, I. Gentil and B. Zegarliński, Ergodicity and smoothing for reaction-diffusion equations via application of logarithmic Sobolev inequality, work in progress.

[21] P. Fougéres and B. Zegarliński,Semi-linear Problems in Infinite Dimensions, J. Func. Analysis **228** (2005) 39-88.

[22] L. Gallardo & M. Yor, Some Remarkable Properties of the Dunkl Martingales, in Lecture Notes in Mathematics, 2006, **1874/2006**, 337-356, http://dx.doi.org/doi:10.1007/978-3-540-35513-7_21

[23] D. Goderis and C. Maes: Constructing quantum dissipations and their reversible states from classical interacting spin systems, Ann. Inst. Henri Poincaré **55** (1991) 805-828.

[24] A. Guionnet and B. Zegarliński, Lectures on logarithmic Sobolev inequalities. *Séminaire de Probabilités, XXXVI,* 1–134, Lecture Notes in Math., **1801**, *Springer, Berlin,* 2003.

[25] W. Hebisch and B. Zegarliński, Coercive inequalities on metric measure spaces, J. Func. Analysis **258** (2010) 814-851, `http://dx.doi.org/10.1016/j.jfa.2009.05.016`

[26] J. Inglis, Coercive inequalities for generators of Hormander type. PhD Thesis, Imperial College 2010.

[27] J. Inglis, V. Kontis, B. Zegarliński, From U-bounds to isoperimetry with applications to H-type groups, J. Func. Analysis **260** (2011), 76-116.

[28] J. Inglis, I. Papageorgiou, Logarithmic Sobolev inequalities for infinite dimensional Hoermander type generators on the Heisenberg group. J. Pot. Analysis **31** (2009) 79-102.

[29] K. Temme, F. Pastawski, and M. J. Kastoryano Hypercontractivity of quasi-free quantum semigroups, J. Phys. A: Math. Theor. **47** (2014) 405303; arXiv:1403.5224

[30] M. J. Kastoryano and K. Temme, Quantum logarithmic Sobolev inequalities and rapid mixing, J. Math. Phys. **54** (2013) 052202, arXiv:1207.3261

[31] V. Kontis, M.Ottobre, B. Zegarliński, Markov semigroups with hypocoercive-type generator in Infinite Dimensions I: Ergodicity and Smoothing, to appear J. Func. Analysis **270** (2016), 3173 - 3223, `http://arxiv.org/abs/1306.6452`arXiv:1306.6452;
Markov semigroups with hypocoercive-type generator in Infinite Dimensions II: Applications, Infin. Dimens. Anal. Qu. 2016 to appear, `http://arxiv.org/abs/1306.6453`arXiv:1306.6453

[32] P. Ługiewicz and B. Zegarliński, Coercive inequalities for Hörmander type generators in infinite dimensions, J. Func. Analysis, **247** (2007), 438–476.

[33] R. Olkiewicz, P. Ługiewicz and B. Zegarliński, Nonlinear Markov semigroups on C*-algebras, Infinite Dimensional Analysis, Quantum Probability and Related Topics **16**, No. 1 (2013) 1350004; `http://dx.doi.org/10.1142/S0219025713500045`

[34] P. Lugiewicz, R. Olkiewicz, B. Zegarliński, Ergodic properties of diffusion-type quantum dynamical semigroups, J. Phys. A: Math. Theor. **43** (2010) 425207.

[35] Xu Lihu and B. Zegarliński, Existence and Exponential mixing of infinite white $\alpha$-stable Systems with unbounded interactions, Electronic J. Probab. 15 (2010) 1994-2018; Ergodicity of finite and infinite dimensional $\alpha$-stable systems, Stoch. Anal. Appl. **27** (2009) 797-824.

[36] H.-Q. Li, Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg, J. Func. Analysis **236** (2006), 369-394.

[37] T.M. Liggett, (1985) *Interacting Particle Systems*; (1999) *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Springer

[38] Taku Matsui, Interacting Particle Systems on Non-Commutative Spaces, pp.115-124 in M. Fannes, C. Maes and A. Verbeure (Eds) On three levels: micro-, meso and macro-phenomena in physics, Plenum (1994), Proceedings of the ASI/ARW Workshop in Leuven, 19-23 July 1993; Purification and uniqueness of quantum Gibbs states, Comm. Math. Phys.**162** (1994) 321-332.

[39] V.P. Maslov, Nonlinear averages in economics, Mathematical Notes, 2005 - Springer; Nonlinear averaging axioms in financial mathematics and stock price dynamics, Theory of Probability & Its Applications, 2004 - SIAM ; Quantum economics, Russian J. Math. Phys, 2006.

[40] A.W. Majewski , R. Olkiewicz and B. Zegarliński, Dissipative dynamics for quantum spin systems on a lattice, J. Phys. A: Math. Gen. **31** (1998) 2045. `http://dx.doi.org/10.1088/0305-4470/31/8/015`

[41] R. Olkiewicz and M. Zaba Dynamics of microcavity polaritions in the Markovian limit Phys. Lett. **A 372** (2008) 3176-3183, Dynamics of a degenerate parametric oscillator in a squeezed reservoir, Phys. Lett. **A 372** (2008) 4985-4989.

[42] R. Olkiewicz and B. Zegarliński, Hypercontractivity in Non-commutative $\mathbb{L}_p$ spaces, J. Func. Analysis **161** (1999) 246-285.

[43] I. Papageorgiou, The Logarithmic Sobolev Inequality for Gibbs measures on infinite product of Heisenberg groups, Markov Processes Relat. Fields **20** 705-749 (2014); A note on the modified Log-Sobolev inequality. J. Potential Anal., **35**, 3, 275-286 (2011); Concentration inequalities for Gibbs measures. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14**, 1, 79-104 (2011); The Logarithmic Sobolev Inequality in Infinite dimensions for Unbounded Spin systems on the Lattice with non Quadratic Interactions, Markov Processes Relat. Fields **16**, 447-484 (2010).

[44] Y. M. Park, Construction of Dirichlet forms on standard forms of von Neumann algebras, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), no. 1, 1-14; Ergodic property of Markovian semigroups on standard forms of von Neumann algebras. J. Math. Physics 46 (2005) 113507.

[45] M. Röckner, $L_p$-analysis of finite and infinite dimensional diffusion operators, Stochastic PDE's and Kolmogorov's equations in infinite dimensions (Giuseppe Da Prato, ed.), Lect. Notes Math., vol. 1715, Springer, Berlin, 1999, pp. 65-116; An analytic approach to Kolmogorov's equations in infinite dimensions and probabilistic consequences, XIVth International Congress on Mathematical Physics, World Scientific, 2005, invited talk at the International Congress on Mathematical Physics (ICMP) 2003, pp. 520-526.

[46] M. Rösler, Dunkl Operators: Theory and Applications, in Lecture Notes in Math. **1817**, pp. 93–136. (Lecture Notes of the 2002 SIAM Euro Summer School Orthogonal Polynomials and Special functions; eds. E. Koelink, W. van Assche); Springer-Verlag, 2003.

[47] D.W.Stroock and B. Zegarliński, The Equivalence of the Logarithmic Sobolev Inequality and the Dobrushin – Shlosman Mixing Condition, Commun. Math. Phys. **144** (1992) 303-323; The Logarithmic Sobolev Inequality for Discrete Spin Systems on a Lattice , Commun. Math. Phys. **149** (1992) 175-193.

[48] Lihu Xu, R. Olkiewicz and B. Zegarliński, Nonlinear problems in infinite interacting particle systems, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.* **11**, 179 (2008). http://dx.doi.org/10.1142/S0219025708003063

[49] Lihu Xu and B. Zegarliński, Ergodicity of the finite and infinite dimensional $\alpha$-stable Systems, J. of Stoch. Analysis and Appl. 27 (2009) 797 - 824., Existence and Exponential mixing of infinite white $\alpha$-stable Systems with unbounded interactions, Electronic Journal of Probability **15** (2010), Paper no. 65, pages 1994–2018.

[50] F. Zak, Exponential ergodicity of infinite system of interacting diffusions, preprint.

[51] B. Zegarliński, On log-Sobolev Inequalities for Infinite Lattice Systems, Lett. Math. Phys. **20** (1990) 173-182; Log-Sobolev Inequalities for Infinite One Dimensional Lattice Systems, Commun. Math. Phys.

**133** (1990) 147-162; Dobrushin Uniqueness Theorem and Logarithmic Sobolev Inequalities, J. Funct. Anal. **105** (1992) 77-111; The Strong Decay to Equilibrium for the Stochastic Dynamics of an Unbounded Spin System on a Lattice, Commun. Math. Phys. **175** (1996) 401-432

[52] B. Zegarliński, Analysis on Extended Heisenberg Group, Annales de la Faculté des Sciences de Toulouse, **XX** Nr 2 (2011) 379-405.

[53] B. Zegarliński, Crystallographic Groups of Hörmander Fields. I , http://arxiv.org/abs/1306.6453hal-01160736

[54] B. Zegarliński, Analysis of classical and quantum interacting particle systems., Quantum interacting particle systems, River Edge, NJ, Publisher: World Sci. Publ., 2002, Pages: 241-336.

[55] B. Zegarliński, Linear and Nonlinear Concentration Phenomena, Markov Processes Relat. Fields **16** (2010) 753-782.